The ω-Categories Associated With Products of Infinite-Dimensional Globes

by

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To My Wife and Son

Cuiping and Tony

Summary

The results in this thesis are organised in four chapters.

Chapter 1 is preliminary. We state the necessary definitions and results in ω-complexes, atomic complexes and products of ω-complexes. Some definitions are restated to meet the requirement for the following chapters. There is a new proof for the existence of *'natural homomorphism'* (Theorem 1.3.6) and a new result for the decomposition of molecules in loop-free ω-complexes (Theorem 1.4.13).

In Chapter 2, we study the product of three infinite dimensional globes. The main result in this chapter is that a subcomplex in the product of three infinite dimensional globes is a molecule if and only if it is pairwise molecular (Theorem 2.1.6). The definition for pairwise molecular subcomplexes is given in section 1. One direction of the main theorem, molecules are necessarily pairwise molecular, is proved in section 2. Some properties of pairwise molecular subcomplexes are studied in section 3. These properties are the preparation for a more explicit description of pairwise molecular subcomplexes, which is given in section 4. The properties for the sources and targets of pairwise molecular subcomplexes are studied in section 5, where we prove that the class of pairwise molecular subcomplexes is closed under source and target operation; there are also algorithms to calculate the sources and targets of a pairwise molecular subcomplex. Section 6 deals with the composition of pairwise molecular subcomplexes. The proof of the main theorem is completed in section 7, where an algorithm for decomposing molecules into atoms is implied in the proof.

The construction of molecules in the product of three infinite dimensional globes is studied in Chapter 3. The main result is that any molecule can be constructed inductively by a systematic approach. Section 1 gives another description for molecules in the product

of three infinite dimensional globes which is the theoretical basis for the construction. Section 2 states the inductive process of constructing molecules. The justification for the construction is given in section 3.

The main result in Chapter 4 is that a subcomplex in the product of four infinite dimensional globes is a molecule if and only if it is pairwise molecular (Theorem 4.1.4). In the first four sections, some basic concepts and properties have to be reestablished to suit more general case. The organisation for the last three sections is parallel to that in Chapter 2. The corresponding results for sources, targets, composition and decomposition of pairwise molecular subcomplexes are also achieved.

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Statement

This thesis is submitted according to the regulations for the degree of Doctor of Philosophy in the University of Glasgow. It presents part of research results carried out by the author during the academic years 1997-2000.

All the results of this thesis are the original work of the author except for the instances indicated within the text. Some results are obtained jointly with Doctor R. J. Steiner.

Contents

Introduction

In this work, we study the ω-complexes of products of infinite-dimensional globes.

An n-category is an algebraic structure consisting of objects, morphisms between objects, 2-morphisms between morphisms, and so on up to n-morphisms, subject to various composition rules.

The study of n-categories started from 2-categories which generalise the idea that natural transformations can certainly be thought as morphisms between morphisms. The theory of bicategories (a generalisation of 2-categories) has successfully been established by the wonderful work of, for example, Eilenberg and Kelly [10], Kelly [16], Kelly and Street [17], and Mac Lane and Paré [18].

The concept of ω-category or ∞-category ([6, 22]) is a generalisation of n-category with no restriction of 'up to n'. It was originated by Brown and Higgins in [6], in connection with homotopy theory. It was not long after the concept was introduced that the researchers realised that a sort of pasting diagrams representing compositions in multiple categories should be introduced. There are several approaches in the study of such pasting diagrams with different names such as parity complexes, pasting schemes, directed complexes or ω-complexes. See Al-Agl and Steiner [1], Johnson [12], Kapranov and Voevodsky [14], Power [19], Steiner [20, 21] and Street [23, 24]. We follow the approach in Steiner [21] because the concept of ω-complex is certainly the most general one.

There is a concept of products of ω-complexes defined in Steiner [21]. It is natural because the products of ω-complexes give the tensor product of the underlying ω-categories. (For the study of tensor products of multiple categories, see the work of Gray [11], Al-Agl and Steiner [1], Crans [9], Joyal and Street [13], and Brown and Higgins [7]). It is shown in paper [21] that the products of ω-complexes are still ω-complexes. Since the definition for the product is given by generators and relations, it is natural to seek explicit descriptions for the products of ω-complexes. This problem is difficult in general, since the molecules, which are representatives of elements in the underlying ω-categories, in the products are difficult to recognise. We thus consider the solution for the products of the simplest ω-complexes, globes.

An n-dimensional globe u is the an ω-complex representing the n-category with exactly one n-morphism and two m-morphisms u_m^- and u_m^+ for every non-negative integer $m < n$ such that the l-source $d_l^- u_m^\gamma$ and l-target $d_l^+ u_m^\gamma$ of u_m^γ are u_l^- and u_l^+ respectively for $l < m \leq n$ and $\gamma = \pm$. The infinite dimensional globe is the obvious generalisation of n-globes. The globes are basic ω-complexes because they serves as the generators in the standard representation of ω-categories. (See Crans [8].) The product of, for example, three infinite dimensional globes $u \times v \times w$ is generated by elements of the form $u_i^\alpha \times v_j^\beta \times w_k^\varepsilon$ (called atoms) with relations resembling those in homological algebra. Thus an element (called molecule) in the product of three globes is a union of atoms (called subcomplex). One of the main result in this thesis characterises molecules in the product of three infinite dimensional globes, in terms of such subcomplexes.

The study for the product of infinite dimensional globes is important not only because infinite dimensional globe is a basic ω-complex, but also because it may help to understand the products of general ω-complexes. According to the approaches used in paper [21], it seems that the product of infinite dimensional globes has a sort of universal property which may be used to study product of general ω-complexes, although we have not yet been able to describe this universal property precisely. Moreover, the explicit description of products of infinite dimensional globes may also help in better understanding some work in weak n-categories. (See Baez and Neuchl [4], and Kapranov and Voevodsky [15].)

For the product of two infinite dimensional globes, there are descriptions in paper [23] and [21]. The description in [21] is more explicit in the way that the molecules are easily recognised and constructed, and there are explicit algorithms to calculate sources and targets of a molecule and the composites of molecules, there is also an algorithm to decompose a molecule into atoms. The main work in this thesis is to extend these results to products of three and four infinite dimensional globes.

As stated above, the construction for the product of two infinite dimensional globes is fairly clear. So it is natural to reduce the problem for the product of three infinite dimensional globes to that of two infinite dimensional globes. This consideration leads to the idea of describing a molecule in product of three infinite dimensional globes by projecting it to the (twisted) products of two infinite dimensional ones. This results in the definition of *pairwise molecular subcomplexes* in the product of three infinite dimensional globes. It is proved that molecules are exactly pairwise molecular subcomplexes.

A more explicit description for molecules in the product of three infinite dimensional globes is influenced by [23] and [21]. Some conditions in this description come from the requirement that a molecule should be *well-formed*, while some come from the comparison with the description of molecules in the product of two infinite dimensional globes. A crucial concept is the *adjacency* of maximal atoms in a subcomplex. This description has some new features distinguished from that for the product of two infinite dimensional globes. Some restrictions must be given because of the middle factor. For example, if there is a pair of distinct maximal atoms $u_{i_1}^{\alpha_1} \times v_{j_1}^{\beta_1} \times w_{k_1}^{\varepsilon_1}$ and $u_{i_2}^{\alpha_2} \times v_{j_2}^{\beta_2} \times w_{k_2}^{\varepsilon_2}$ in a pairwise molecular subcomplex such that $i_1 > i_2$, $\min\{j_1, j_2\} > 0$ and $k_1 < k_2$, it is required that there is a maximal atom $u_i^{\alpha} \times v_j^{\beta} \times w_k^{\varepsilon}$ such that $i > i_2$, $j \geq \min\{j_1, j_2\} - 1$ and $k > k_1$.

After the descriptions of molecules in the product of three infinite dimensional globes are proposed, we have to prove that pairwise molecular subcomplexes are closed under source and target operations, and they are also closed under composition operations. The algorithms for calculating the sources, targets and composites of a pairwise molecular subcomplex are also studied.

To prove that pairwise molecular subcomplexes are molecules, we have to show that they can be decomposed into atoms. To do this, a total order, called *natural order*, on the set of atoms in the product of three infinite dimensional globes is introduced.

The natural order is designed so that the maximal atoms of dimensions greater than the *frame dimension* p (see paper [20]) in a pairwise molecular subcomplex can be listed as $\lambda_1, \lambda_2, \ldots, \lambda_S$ with $\lambda_s \cap \lambda_t \subset d_p^+\lambda_s \cap d_p^-\lambda_t$ for all $1 \leq s < t \leq S$. This means that the decomposition approach in paper [20] applies. In the proof that pairwise molecular subcomplexes are molecules, there is also an explicit algorithm to decompose a molecule into atoms.

At this stage, we have satisfactory descriptions for the product of three infinite dimensional globes. However, these are still descriptive. From these descriptions, it is fairly easy to check whether a given subcomplex is a molecule. But we still cannot construct all the molecules. Our next goal is to find a systematic way to construct all molecules. The approach is based on the middle factor. According our results, we can construct any molecule, inductively, by listing its maximal atoms as $\lambda_1, \lambda_2, \ldots, \lambda_R$ with $\lambda_r = u_{i_r}^{\alpha_r} \times v_{j_r}^{\beta_r} \times w_{k_r}^{\varepsilon_r}$ such that $j_1 \geq \cdots \geq j_R$ and such that $i_r > i_{r+1}$ when $1 \leq r < R$ and $j_r = j_{r+1}$. In more detail, let maximal atoms $\lambda_1, \ldots, \lambda_r$ be an initial segment of the list. We can easily determine whether $\lambda_1 \cup \cdots \cup \lambda_r$ is already a molecule and determine the set of possible next maximal atoms λ_{r+1}, so that all the molecules can be constructed inductively.

Up to now, we have a completely satisfactory theory for the product of three infinite dimensional globes.

Our discussion for the product of four infinite dimensional globes is roughly parallel to that for the product of three infinite dimensional globes. Since the construction of the product of three infinite dimensional globes, by our results, is thought to be clear, we propose that the molecules in the product of four infinite dimensional globes should be the subcomplexes such that they are projected to the molecules in the (twisted) products of three infinite dimensional globes. This results in the basic definition for pairwise molecular subcomplexes in the product of four infinite dimensional globes.

To work out the more explicit description (the one without using projection), some basic concepts, for example, the definition of *adjacency* and *projection maximal* must be reestablished because of another middle factor. Compared with the description for the molecules in the product of three infinite dimensional globes, this description is less explicit. However, it is good enough to check whether a given subcomplex in the product of four infinite dimensional globes is a molecule. Best of all, both descriptions for the molecules in the product of four infinite dimensional globes can easily be stated for those in the product of more infinite dimensional globes. This may lead us to further study the product of more infinite dimensional globes.

After the basic concepts and tools are properly established, the rest of the work very much resembles that for dealing with the product of three infinite dimensional globes: the closedness of molecular subcomplexes under the source, target and composition operations are proved, algorithms for the calculations of sources, targets and composites are given, and in the proof that molecular subcomplexes are exactly molecules, an algorithm for decomposing molecules into atoms is also established.

Unfortunately, we have not been able to work out the construction of molecules in the product of four infinite dimensional globes. The difficulty remains that there are two 'middle' factors. Thus our theory for the product of four infinite dimensional globes is not as satisfactory as that for the product of three infinite dimensional globes.

We end the introduction by raising some questions following this work.

1. What are the explicit descriptions for the product of n infinite dimensional globes with $n > 4$?

We have proposed some fairly reasonable explicit descriptions for the molecules in the product of n infinite dimensional globes which resembles very much that for the product of four infinite dimensional globes. Some proofs in the study of the product of four infinite dimensional globes are already quite complicated, and the problem is how to generalise them. We feel pretty confident about working this out.

2. How can one construct the molecules in the product of four infinite dimensional globes?

As stated above, the construction of the product of three infinite dimensional globes is satisfactory because there is systematic way to construct any molecule in the product of three infinite dimensional globes. However, we have not yet worked out the analogue for the product of four infinite dimensional globes. The difficulty remains how to handle the two 'middle' factors. We still have no idea of what the construction should look like.

3. What about the explicit descriptions for the product of general ω-complexes.

As stated at the beginning of the introduction, the study of the product of infinite dimensional globes may help to understand products of general ω-complexes. Following this idea, for example, the construction of such ω-complexes as $(u_1 \#_p u_2) \times v \times w$, where u_1, u_2, v and w are infinite dimensional globes, must firstly be studied before one can carry on the study for the general problem.

The following are two questions which we have not had time to think of deeply.

4. What about the construction of the joins of infinite dimensional globes (simplexes)?

5. What about the product of globes in the weak n-categories or weak ω-categories? (for the definition of weak n-categories and weak ω-categories, see [2], [3], [5] [22] and [25]).

Chapter 1

Preliminaries

In this chapter, we give some basic definitions and discuss some properties of ω-complexes and products of ω-complexes which are used throughout the thesis. All the results are based on papers [21] and [20], although some treatments are different from those in these two papers. In the last section, we give a new decomposition theorem which will be used later in the thesis.

Throughout the thesis, non-negative integers are denoted by i, j, k, l, m, n, p, q, etc. We also use α, β, γ, σ, τ, ε, ω, etc, to denote signs $\pm$.

1.1 ω-complexes

In this section, we define ω-complexes and give some local descriptions of ω-complexes.

It is well known that a small category can be described purely by its morphism set by regarding objects as identities.

Informally, an ω-category is a set X which forms the morphism set of a small category C_n for every non-negative integer n such that every element x in X is an identity in some C_n and $ob(C_0) \subset ob(C_1) \subset \dots$, where $ob(C_n)$ denote the set of objects (identities) of C_n. We also require that the categorical structures commute for every pair of non-negative integers. The formal definition is as follows.

Definition 1.1.1. A partial ω-category is a set X together with unary operations d_0^-, d_0^+, d_1^-, d_1^+, $\dots$ and not everywhere defined binary operations $\#_0$, $\#_1$, $\dots$ on X such that the following conditions hold for all elements x, x', y, y' and z in X, all non-negative integers m and n, and all signs α and β:

1. if $x\#_n y$ is defined, then $d_n^+ x = d_n^- y$;

2. $$d_m^\beta d_n^\alpha x = \begin{cases} d_m^\beta x & \text{if } m < n, \\ d_n^\alpha x & \text{if } m \geq n; \end{cases}$$

3. $d_n^- x\#_n x = x\#_n d_n^+ x = x$;

4. if $x\#_n y$ is defined, then
$$d_m^\alpha(x\#_n y) = d_m^\alpha x = d_m^\alpha y \text{ for } m < n,$$
$$d_n^-(x\#_n y) = d_n^- x,\ \ d_n^+(x\#_n y) = d_n^+ y,$$
$$d_m^\alpha(x\#_n y) = d_m^\alpha x\#_n d_m^\alpha y \text{ for } m > n;$$

5. $(x\#_n y)\#_n z = x\#_n(y\#_n z)$ if either side is defined;

6. $(x\#_n y)\#_m(x'\#_n y') = (x\#_m x')\#_n(y\#_m y')$ if $m < n$ and the left side is defined;

7. for every $x \in X$ there is a non-negative integer p such that $d_n^\alpha x = x$ if and only if $n \geq p$.

The unique non-negative integer p in condition 7 is called the *dimension* of x, denoted by $\dim x$.

Example 1.1.2. There is a partial ω-category $X = \{a, b, x, y\}$ such that $\dim a = \dim b = 0$, $\dim x = \dim y = 1$, $d_0^- x = d_0^+ y = a$ and $d_0^+ x = d_0^- y = b$.

Definition 1.1.3. Let X be a partial ω-category. If $d_n^+ x = d_n^- y$ implies that $x\#_n y$ is defined for all elements x, y in X and for all non-negative integers n, then X is an *ω-category.*

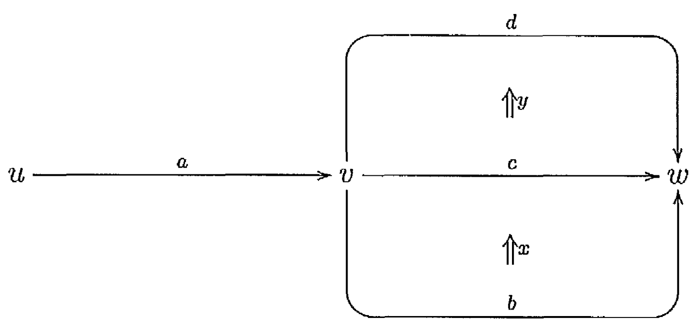
From Example 1.1.2, a partial ω-category is not necessarily an ω-category.

It is natural to consider representing a partial ω-category X by a suitable 'pasting diagram'. The 'pasting diagram' is a sort of cell complex such that the indecomposable elements of X are represented by atoms, the operations d_n^α are represented by parts of boundaries, composites are represented by well behaved unions, elements in the ω-category are represented by subcomplexes which are well-behaved unions of atoms.

Example 1.1.4. There is an ω-category X with the following presentation: there are generators a, x, y and relations

$$\dim a = 1,\ \dim x = \dim y = 2,\ d_1^+ x = d_1^- y,\ d_0^+ a = d_0^- x = d_0^- y.$$

Then X has 16 elements which can be represented by subcomplexes of the diagram in the following figure:



There are three cells a, x, y representing the generators; three additional 0-cells u, v, w representing $d_0^- a$, $d_0^+ a = d_0^- x = d_0^- y$ and $d_0^+ x = d_0^+ y$; three additional 1-cells b, c, d representing $d_1^- x$, $d_1^+ x = d_1^- y$ and $d_1^+ y$; and the seven subcomplexes

$$x \cup y,\ a \cup b,\ a \cup c,\ a \cup d,\ a \cup x,\ a \cup y,\ a \cup x \cup y$$

representing

$$x\#_1 y,\ a\#_0 b,\ a\#_0 c,\ a\#_0 d,\ a\#_0 x,\ a\#_0 y,\ a\#_0(x\#_1 y).$$

In this figure, d_0^-, d_0^+, d_1^-, d_1^+ are represented by left end, right end, bottom and top respectively; for example, $d_1^+ a = a$ because $\dim a = 1$, and

$$d_1^+[a\#_0(x\#_1 y)] = d_1^+ a\#_0 d_1^+(x\#_1 y) = a\#_0 d_1^+ y = a\#_0 d.$$

Suppose that $x\#_n y$ is a composite in a partial ω-category, and suppose that x and y are represented by complexes in a pasting diagram. We then have $d_n^+ x = d_n^- y = z$, say, and z must be represented by a subcomplex of the intersection $x \cap y$. In fact our intuition requires z to be the whole of $x \cap y$. For we want z to be at one extreme of x and at the opposite extreme of y, so $x \setminus z$ and $y \setminus z$ should be on opposite sides of z, and therefore disjoint.

For an example of what can go wrong if this requirement is not satisfied, let X be the partial ω-category in Example 1.1.2. This partial ω-category can be represented by the following diagram

$$\begin{tikzcd} a \arrow[r, bend left, "x"] & b \arrow[l, bend left, "y"] \end{tikzcd}$$

where the composites $x\#_0 y$ and $y\#_0 x$ do not exist. We argue that the composites like these, if exist, would lead to an unsatisfactory behaviour in such pasting diagrams. Suppose otherwise that the composites $x\#_0 y$ and $y\#_0 x$ both exist. They are distinct because $d_0^-(x\#_0 y) \neq d_0^-(y\#_0 x)$, so it is not satisfactory to have them both represented by the union $x \cup y$. This unsatisfactory behaviour arises because $x \cap y$ strictly contains a and strictly contains b.

These considerations lead to the following definition.

Definition 1.1.5. An *ω-complex* is a set K together with a family of subsets called atoms and a family of subsets called molecules such that the following conditions hold.

1. The molecules form a partial ω-category.

2. Let x and y be molecules. Then $x\#_n y$ is defined if and only if $x \cap y = d_n^+ x = d_n^- y$; if $x\#_n y$ is defined, then $x\#_n y = x \cup y$.

3. Every atom is an molecule; every molecule is generated from some atoms by applying composition operations $\#_0$, $\#_1$, ...

4. The set K is the union of its atoms.

5. For an atom a and a sign α, let $\partial^\alpha a$ be given by

$$\partial^\alpha a = \begin{cases} d_{p-1}^\alpha a & \text{if } \dim a = p > 0, \\ \emptyset & \text{if } \dim a = 0; \end{cases}$$

let the *interior* of a be the subset $\operatorname{Int} a$ given by

$$\operatorname{Int} a = a \setminus (\partial^- a \cup \partial^+ a).$$

Then interiors of atoms are non-empty and disjoint.

Example 1.1.6. There is an ω-complex u_2 called 2-dimensional globe. It is a closed disk. The boundary of the disk consists of two semicircles u_1^- and u_1^+ intersecting at two distinct points u_0^- and u_0^+. The atoms are u_2 itself, the two semicircles u_1^- and u_1^+, and the two distinct points u_0^- and u_0^+. The operators d_m^β are such that $d_m^\beta u_2 = u_m^\beta$ for $m < 2$ and $d_0^\beta u_1^\alpha = u_0^\beta$. It is easy to see that all the molecules in u_2 are atoms, and they form an ω-category. The ω-complex u_2 can be represented by the following diagram.

$$\begin{tikzcd} u_0^- \arrow[r, bend left, "u_1^-"] \arrow[r, bend right, "u_1^+"'] \arrow[r, phantom, "\Downarrow u_2"] & u_0^+ \end{tikzcd}$$

Similarly, there is an ω-complex u_3 called 3-dimensional globe. It is a closed 3-dimensional ball. The boundary sphere consists of two hemispheres u_2^- and u_2^+ intersecting in a circle, and the circle consists of two semicircles u_1^- and u_1^+ intersecting in two distinct points u_0^- and u_0^+. The atoms are the ball u_3 itself, the two hemispheres u_2^- and u_2^+, the two semicircles u_1^- and u_1^+, and the two distinct points u_0^- and u_0^+. The operators d_m^β are such that $d_m^\beta u_3 = u_m^\beta$ for $m < 3$ and $d_m^\beta u_n^\alpha = u_m^\beta$ for $m < n < 3$. It is easy to see that all the molecules in u_3 are atoms, and they form an ω-category.

As another example of ω-complex, let K be a 7 element set $\{e_3, e_2^-, e_2^+, e_1^-, e_1^+, e_0^-, e_0^+\}$. The atoms are $\bar{e}_3 = \{e_3, e_2^-, e_2^+, e_1^-, e_1^+, e_0^-, e_0^+\}$, $\bar{e}_2^- = \{e_2^-, e_1^-, e_1^+, e_0^-, e_0^+\}$, $\bar{e}_2^+ = \{e_2^+, e_1^-, e_1^+, e_0^-, e_0^+\}$, $\bar{e}_1^- = \{e_1^-, e_0^-, e_0^+\}$, $\bar{e}_1^+ = \{e_1^+, e_0^-, e_0^+\}$, $\bar{e}_0^- = \{e_0^-\}$, and $\bar{e}_0^+ = \{e_0^+\}$. The operators d_m^β are such that $d_m^\beta \bar{e}_3 = \bar{e}_m^\beta$ for $m < 3$ and $d_m^\beta \bar{e}_n^\alpha = \bar{e}_m^\beta$ for $m < n < 3$. It turns out that all the molecules in K are atoms, and they indeed form an ω-category.

Example 1.1.7. There is an ω-complex u called p-dimensional globe such that the atoms in u can be listed as u_p, u_{p-1}^-, u_{p-1}^+, $\ldots$, u_0^-, u_0^+ such that $d_m^\beta u_p = u_m^\beta$ for $m < p$ and $d_m^\beta u_n^\alpha = u_m^\beta$ for $m < n < p$. It is easy to check that all the molecules are atoms in p-dimensional globes. We also denote the p-dimensional globe by u_p.

For instance, both of the subcomplexes u_3 and K described in Example 1.1.6 represent the 3-dimensional globe. We are going to see that they are equivalent.

Similarly, there is an ω-complex u called *infinite dimensional globe* with exactly two n-dimensional atoms u_n^- and u_n^+ for every non-negative integer n, such that $d_m^\beta u_n^\alpha = u_m^\beta$ for $m < n$. It is easy to see that all the molecules in a globe are atoms, and they form an ω-category.

In the thesis, an atom u_n^α in an infinite dimensional globe u is also denoted by $u[n, \alpha]$.

We now state some results about local description of ω-complexes in [21].

Proposition 1.1.8.

1. *Let x be a molecule in an ω-complex. Then $d_n^\alpha x \subset x$ for every sign α and every non-negative integer n.*

2. *Let a be an atom in an ω-complex. If $\partial^\alpha a \neq \emptyset$, then $\partial^\alpha a$ is a molecule and $\dim \partial^\alpha a < \dim a$ for every sign α.*

Proposition 1.1.9. *Let ξ be an element in an ω-complex. If a is an atom of minimal dimension such that $\xi \in a$, then $\xi \in \operatorname{Int} a$.*

Proposition 1.1.10. *Let x be a molecule and a be an atom in an ω-complex. Then $a \subset x$ if and only if $\operatorname{Int} a \cap x \neq \emptyset$.*

Proposition 1.1.11. *Let x be a molecule in an ω-complex. Then*

$$d_n^\alpha x = \bigcup\{a : \ a \subset x \ \text{and} \ \dim a \leq n\} \setminus \bigcup\{b \setminus \partial^\alpha b : \ b \subset x \ \text{and} \ \dim b = n+1\},$$

where a and b are atoms.

According to this proposition, we can see that an element $\xi \in d_n^\alpha x$ if and only if (1) $\xi \in a$ for some atom $a \subset x$ with $\dim a \leq n$; and (2) for every atom $b \subset x$ with $\dim b = n+1$, if $\xi \in b$, then $\xi \in d_n^\alpha b$.

As an example, we use Proposition 1.1.11 to verify that, in Example, 1.1.4 $d_1^-(a\#_0(x\#_1 y)) = a\#_0 b$. By the above theorem, $d_1^-(a\#_0(x\#_1 y))$ is the difference of the union $a \cup b \cup c \cup d$ and $\operatorname{Int} d \cup \operatorname{Int} c$. Thus $d_1^-(a\#_0(x\#_1 y)) = a \cup b = a\#_0 b$.

Corollary 1.1.12. *An ω-complex is determined by its atoms , their dimensions and the functions ∂^- and ∂^+.*

Definition 1.1.13. Let X and Y be partial ω-categories. A *homomorphism* $f : X \to Y$ is a map such that

1. $f(d_n^\gamma x) = d_n^\gamma f(x)$ for all $x \in X$, all non-negative integers n and all signs γ;

2. $f(x \#_n y) = f(x) \#_n f(y)$ whenever $x \#_n y$ is defined.

Example 1.1.14. Let u be a infinite dimensional globes. Let u_p be a p dimensional globe. It is evident that there is a homomorphism $f_p^u : \mathcal{M}(u) \to \mathcal{M}(u_p)$ of ω-categories such that for all atom $u_i^\alpha \in \mathcal{M}(u)$

$$f_p^u(u_i^\alpha) = \begin{cases} u_i^\alpha & \text{when } i < p, \\ u_p & \text{when } i \geq p. \end{cases}$$

We end this section by introducing a definition of equivalence of ω-complexes.

Let K be an ω-complex. A *subcomplex* is a subset of K which can be written as a union of atoms. The set of all subcomplexes of K is denoted by $\mathcal{C}(K)$; the set of all atoms of K is denoted by $\mathcal{A}(K)$; The set of all molecules of K is denoted by $\mathcal{M}(K)$.

Definition 1.1.15. Let K and L be ω-complexes. We say K and L are *equivalent* if there exists a map $f : \mathcal{C}(K) \to \mathcal{C}(L)$ called an *equivalence of ω-complexes* such that the following conditions hold:

1. If $a \in \mathcal{A}(K)$, then $f(a) \in \mathcal{A}(L)$. Moreover, $f|_{\mathcal{A}(K)} : \mathcal{A}(K) \to \mathcal{A}(L)$ is a bijection.

2. If A is a set of atoms, then $f(\bigcup A) = \bigcup\{f(a) : \ a \in A\}$.

3. If $a \in \mathcal{A}(K)$, then $\dim f(a) = \dim a$.

4. If $a \in \mathcal{A}(K)$, then $f(\partial^\alpha a) = \partial^\alpha f(a)$ for $\alpha = \pm$.

It is easy to check that the geometric description and combinatorial descriptions for 3-dimensional globes in Example 1.1.7 are equivalent. From this, we may use the geometric model to understand the combinatorial model and vice versa.

We are going to prove that an equivalence of ω-complexes preserves molecules. We need several technical lemmas.

Lemma 1.1.16. *Let K be an ω-complex. If $c_1, c_2 \in \mathcal{C}(K)$, then $c_1 \cap c_2 \in \mathcal{C}(K)$.*

Proof. It suffices to prove that $a \cap b$ is a subcomplex of K for every pair a and b of atoms in K.

Let $\xi \in a \cap b$. Let a_ξ be the atom of the minimal dimension with $\xi \in a_\xi$. According to Propositions 1.1.9 and 1.1.10, we have $a_\xi \subset a \cap b$. It follows that $a \cap b = \bigcup\{a_\xi : \ \xi \subset a \cap b\}$. Thus $a \cap b$ is a subcomplex of K, as required. $\square$

Lemma 1.1.17. *Let $f : \mathcal{C}(K) \to \mathcal{C}(L)$ be an equivalence of ω-complexes. If $c_1, c_2 \in \mathcal{C}(K)$, then $f(c_1 \cap c_2) = f(c_1) \cap f(c_2)$.*

Proof. Since f preserves unions of subcomplexes, we have $f(c_1) \subset f(c_2)$ for a pair of subcomplexes c_1 and c_2 in K with $c_1 \subset c_2$. Note that $f : \mathcal{C}(K) \to \mathcal{C}(L)$ is a bijection, it follows easily from Lemma 1.1.16 that $f(c_1 \cap c_2) = f(c_1) \cap f(c_2)$, as required. $\square$

Lemma 1.1.18. *Let $f : \mathcal{C}(K) \to \mathcal{C}(L)$ be an equivalence of ω-complexes. If $x \in \mathcal{M}(K)$ and $f(x) \in M(L)$, then $f(d_p^\gamma x) = d_p^\gamma f(x)$.*

Proof. Suppose that $b \in \mathcal{A}(L)$ and $\operatorname{Int} b \subset f(d_p^\gamma x)$. Then there exists $a \in \mathcal{A}(K)$ with $b = f(a)$ such that $a \subset d_p^\gamma x$. Thus $a \subset x$ and $\dim a \leq p$. It follows that $b = f(a) \subset f(x)$ and $\dim b = \dim f(a) = \dim a \leq p$. Now suppose that $b' \in \mathcal{A}(L)$ with $\dim b' = p+1$ such that $\operatorname{Int} b \subset b'$. Then there exists $a' \in \mathcal{A}(K)$ such that $f(a') = b'$. It is evident that $\dim a' = p+1$ and $a \subset a'$. So we have $a \subset \partial^\gamma a'$. This implies that $b = f(a) \subset f(\partial^\gamma a') = \partial^\gamma f(a') = \partial^\gamma b'$. According to Proposition 1.1.11, we have $\operatorname{Int} b \subset d_p^\gamma f(x)$. It follows that $f(d_p^\gamma x) \subset d_p^\gamma f(x)$.

By a similar argument, we can prove the reverse inclusion.

This completes the proof. $\square$

Proposition 1.1.19. *Let $f : \mathcal{C}(K) \to \mathcal{C}(L)$ be an equivalence of ω-complexes. If $x \in \mathcal{M}(K)$, then $f(x) \in \mathcal{M}(L)$. Moreover, $f|_{\mathcal{M}(K)} : \mathcal{M}(K) \to \mathcal{M}(L)$ is a homomorphism of partial ω-categories.*

Proof. We give the proof by induction.

Firstly, if $a \in \mathcal{M}(K)$, then $f(a) \in \mathcal{M}(L)$ by the definition of equivalence.

Suppose that $m > 1$ and $f(x) \in \mathcal{M}(K)$ if x can be written as a composite of less then m atoms. Let x be an atom which can be written as a composite of m atoms. We must prove that $f(x) \in \mathcal{M}(L)$.

Indeed, it is evident that x has a proper decomposition $x = y\#_p z$ into molecules such that y and z are molecules which can be written as a composite of less than p atoms. By the inductive hypothesis, we have $f(y) \in \mathcal{M}(L)$ and $f(z) \in \mathcal{M}(L)$. To prove $f(x) \in \mathcal{M}(L)$, it suffices to show that the composite $f(y)\#_p f(z)$ exists and that $f(y\#_p z) = f(y)\#_p f(z)$.

Since $x = y\#_p z$, we have $d_p^+ y = d_p^- z = y \cap z$. By the previous lemmas, we get $d_p^+ f(y) = f(d_p^+ y) = f(d_p^- z) = d_p^- f(z) = f(y) \cap f(z)$. Therefore $f(y)\#_p f(z)$ is defined and $f(y\#_p z) = f(y \cup z) = f(y) \cup f(z) = f(y)\#_p f(z)$, as required.

By a similar argument, we can show that f preserves composition operation. Thus $f : \mathcal{M}(K) \to \mathcal{M}(L)$ is a homomorphism of partial ω-complexes.

This completes the proof. $\square$

1.2 Atomic Complexes

Corollary 1.1.12 shows that it is possible to describe an ω-complex by its atoms and the boundary operators ∂^- and ∂^+. This leads us to a concept consisting of atoms and boundary operators which we call an atomic complex.

In this section, we first define atomic complexes and state some properties of atomic complexes. Then we state a necessary and sufficient condition for an atomic complex to be an ω-complex. From this theorem, we will see that the results in paper [20] for loop-free directed complexes can be generalised to loop-free ω-complexes. We shall discuss this in section 1.4.

Definition 1.2.1. An *atomic complex* is a set K together with a family of subsets $\mathcal{A}(K)$ called atoms and functions dim, ∂^- and ∂^+ defined on $\mathcal{A}(K)$ such that the following conditions hold.

1. For every atom a, $\dim a$ is an non-negative integer called the *dimension* of a.

2. If a is an atom and α is a sign, then $\partial^\alpha a$ is a subset of a such that $\partial^\alpha a$ is a union of *atoms* of dimensions less than $\dim a$.

3. $K = \bigcup \mathcal{A}(K)$.

4. For an atom a, let $\operatorname{Int} a = a \backslash (\partial^- a \cup \partial^+ a)$. Then the interiors of atoms are non-empty and disjoint.

Proposition 1.2.2. *ω-complexes are atomic complexes.*

To give the necessary and sufficient conditions for an atomic complex to be an ω-complex, we need to define operators d_n^α on an arbitrary subset of K. This can be given by generalising Proposition 1.1.11.

Definition 1.2.3. Let K be an atomic complex.

- If $x \subset K$ and $\alpha = \pm$, then

  $$d_n^\alpha x = \bigcup \{a : \ a \subset x \text{ and } \dim a \leq n\} \setminus \bigcup \{b \setminus \partial^\alpha b : \ b \subset x \text{ and } \dim b = n+1\},$$

  where a and b are atoms.

- If $x \subset K$ and $y \subset K$, then the composite $x \#_n y$ is defined if and only if $x \cap y = d_n^+ x = d_n^- y$; If $x \#_n y$ is defined, then $x \#_n y = x \cup y$.

- A molecule is a subset generated from atoms by finitely applying the composition operations $\#_n$ $(n = 0, 1, \ldots)$.

With the definition of the operators d_n^α on an arbitrary subsets of ω-complexes, we can define *finite dimensional subcomplexes.*

Definition 1.2.4. Let K be an atomic complex.

- If x is a union of atoms in K, then x is a *subcomplex* of K.

- Let x be a subcomplex of K. If there exists an integer n such that $x = d_n^- x = d_n^+ x$, then x is *finite dimensional.*

Proposition 1.2.5. *Let a and c be distinct atoms in an atomic complex. If $\operatorname{Int} a \cap c \neq \emptyset$, then $\dim a \leq \dim c$ and $a \subset \partial^\alpha c$ for some sign α.*

Proposition 1.2.6. *Let x and y be subcomplexes of an atomic complex. If $y \subset x$, then $d_n^\alpha x \cap y \subset d_n^\alpha y$.*

Proposition 1.2.7. *Let x and y be subcomplexes of an atomic complex. Then $d_n^\alpha(y \cup z) = (d_n^\alpha y \cap d_n^\alpha z) \cup (d_n^\alpha y \setminus z) \cup (d_n^\alpha z \setminus y)$.*

Now we can state the necessary and sufficient conditions for an atomic complex to be an ω-complex.

Theorem 1.2.8. *Let K be an atomic complex. Then K is an ω-complex if and only if the following conditions hold.*

1. *If a is an atom and $\dim a > 0$, then $\partial^\alpha a$ is a molecule for every sign α.*

2. *If a is an atom and $\dim a = p > 1$, then $d_{p-2}^\beta d_{p-1}^\alpha a = d_{p-2}^\beta a$ for every pair of signs α and β.*

Example 1.2.9. Let w^J be the atomic complex with atoms $w^J[k,\varepsilon]$ ($k = 0, 1, \cdots$ and $\varepsilon = \pm$) such that dim $w^J[k,\varepsilon] = k$ and $d_{k-1}^\gamma w^J[k,\varepsilon] = w^J[k-1, (-)^J\gamma]$ for $k > 0$. It is clear that w^J satisfies conditions in Theorem 1.2.8. Thus it is an ω-complex. It is also easy to see that the ω-complex w^J is equivalent to infinite dimensional globe w under an obvious equivalence of ω-complexes sending $w^J[k, (-)^J\varepsilon]$ to $w[k, \varepsilon]$.

Lemma 1.2.10. *In an ω-complex, if x is a subcomplex, then $d_n^\alpha x$ can be written as a union of interior of atoms.*

Proof. Suppose that $\xi \in d_n^\alpha x$. Let a_ξ be the atom in x such that $\xi \in \operatorname{Int} a_\xi$. Then $\dim a_\xi \leq p$. We claim $\operatorname{Int} a_\xi \subset d_n^\alpha x$. Indeed, for every $\eta \in \operatorname{Int} a_\xi$, we have $\eta \in a_\xi$. Moreover, suppose that $\eta \in b$ for an atom $b \subset x$ with $\dim b = p+1$, then $\xi \in a_\xi \subset b$. Hence $\xi \in d_n^\alpha b$ by Definition 1.2.3. Since $d_n^\alpha b$ is a molecule, we have $a_\xi \subset d_n^\alpha b$. Therefore

$\eta \in d_n^\alpha b$. It follows from Definition 1.2.3 that $\eta \in d_n^\alpha x$ for every $\eta \in \operatorname{Int} a_\xi$. Therefore $\operatorname{Int} a_\xi \subset d_n^\alpha x$.

Now it is evident that

$$d_n^\alpha x = \bigcup_{\xi \in d_n^\alpha x} \operatorname{Int} a_\xi$$

which shows that $d_n^\alpha x$ is a union of interiors of atoms.

This completes the proof. $\square$

Lemma 1.2.11. *In an ω-complex, let x be a subcomplex and a be an atom. Then* $\operatorname{Int} a \subset d_n^\alpha x$ *if and only if*

1. *$a \subset x$ and* $\dim a \leq p$*;*

2. *If $a \subset b \subset x$ for an atom $b \subset x$ with* $\dim b = p+1$*, then $a \subset d_n^\alpha b$.*

Proof. This is an direct consequence of Lemma 1.2.10. $\square$

1.3 Products of ω-complexes

In this section, we give the product construction for ω-complexes. Some treatments are different from (but of course equivalent to) those in paper [21]. This applies in particular to structures of products of two infinite dimensional globes.

Proposition 1.3.1. *Let K and L be atomic complexes. Then the product $K \times L$ of sets is made into an atomic complex as follows. The atom set $\mathcal{A}(K \times L)$ is given by*

$$\mathcal{A}(K \times L) = \{a \times b : \ a \in \mathcal{A}(K) \ \textit{and}\ b \in \mathcal{A}(L)\};$$

the structure functions are given by

- $\dim(a \times b) = \dim a + \dim b$;

- $\partial^\gamma(a \times b) = (\partial^\gamma a \times b) \cup (a \times \partial^{(-)^{\dim a}\gamma} b)$.

The atomic complex $K \times L$ is called the *product* of K and L.

Theorem 1.3.2. *Let K and L be ω-complexes. Then $K \times L$ is an ω-complex.*

Example 1.3.3. Let u and v be infinite dimensional globes. Then the product $u \times v$ of sets is made into an atomic complex as follows: the atoms are of the form $u_i^\alpha \times v_j^\beta$ with i and j run over all non-negative integers, and α and β run over all the signs; the structure functions are given by

- $\dim(u_i^\alpha \times v_j^\beta) = i + j$;
- $\partial^\gamma(u_i^\alpha \times v_j^\beta) = (u_{i-1}^\gamma \times v_j^\beta) \cup (u_i^\alpha \times v_{j-1}^{(-)^i\gamma})$.

By Theorem 1.3.2, $u \times v$ is actually an ω-complex.

It is straightforward to verify that the product construction is associative. Therefore we can write a product of three ω-complexes K, L and M as $K \times L \times M$. By Theorem 1.3.2, The product $K \times L \times M$ is still an ω-complex. In particular, the atom set $\mathcal{A}(K \times L \times M)$ is given by

$$\mathcal{A}(K \times L \times M) = \{a \times b \times c\colon\ a \in \mathcal{A}(K) \text{ and } b \in \mathcal{A}(L) \text{ and } b \in \mathcal{A}(M)\};$$

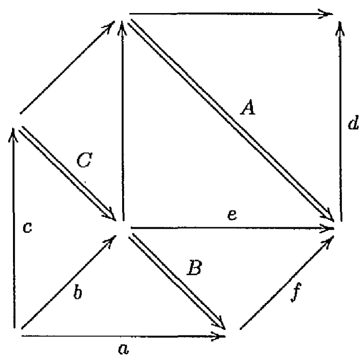
the structure functions are given by

- $\dim(a \times b \times c) = \dim a + \dim b + \dim c$;
- $\partial^\gamma(a \times b \times c) = (\partial^\gamma a \times b \times c) \cup (a \times \partial^{(-)^{\dim a}\gamma} b \times c) \cup (a \times b \times \partial^{(-)^{\dim a + \dim b}\gamma} c)$.

Example 1.3.4. We now consider the product of three 1-dimensional globes $u_1 \times v_1 \times w_1$. Since the 1-dimensional globe is represented by the closed interval, the product $u_1 \times v_1 \times w_1$ is a cube. Recall that the 1-dimensional globe consists of 3 atoms. So the product $u_1 \times v_1 \times w_1$ consists of 27 atoms. The following figure illustrates the source boundary

$$\partial^-(u_1 \times v_1 \times w_1) = (u_1 \times v_1 \times w_0^-) \cup (u_0^- \times v_1 \times w_1) \cup (u_1 \times v_0^+ \times w_1)$$

of the cube, where $A = u_1 \times v_0^+ \times w_1$, $B = u_1 \times v_1 \times w_0^-$ and $C = u_0^- \times v_1 \times w_1$.



We can identify edges and vertices. For example, the edge

$$b = B \cap C = (u_1 \times v_1 \times w_0^-) \cap (u_0^- \times v_1 \times w_1) = u_0^- \times v_1 \times w_0^-$$

and the vertex

$$a \cap b = (u_1 \times v_0^- \times w_0^-) \cap (u_0^- \times v_1 \times w_0^-) = u_0^- \times v_0^- \times w_0^-.$$

We can then check that the directions of the edges and vertices are as shown in the figure. For example, since

$$\partial^- b = \partial^-(u_0^- \times v_1 \times w_0^-) = u_0^- \times v_0^- \times w_0^- = a \cap b$$

and

$$\partial^+ b = \partial^+(u_0^- \times v_1 \times w_0^-) = u_0^- \times v_0^+ \times w_0^- = b \cap e,$$

the direction of b is as shown in the figure. Similarly, since

$$\partial^- B = \partial^-(u_1 \times v_1 \times w_0^-) = (u_0^- \times v_1 \times w_0^-) \cup (u_1 \times v_0^+ \times w_0^-) = b \cup e$$

and

$$\partial^+ B = \partial^+(u_1 \times v_1 \times w_0^-) = (u_0^+ \times v_1 \times w_0^-) \cup (u_1 \times v_0^- \times w_0^-) = a \cup f$$

Thus the direction of $B = u_1 \times v_1 \times w_0^-$ is as shown in the figure.

From the diagram of $u_1 \times v_1 \times w_1$, we can see that all the subcomplexes in the following list are molecules.

1. $u_1 \times v_1 \times w_1$,
2. $u_1 \times v_1 \times w_0^-$,
3. $u_1 \times v_1 \times w_0^+$,
4. $u_1 \times v_0^- \times w_1$,
5. $u_1 \times v_0^+ \times w_1$,
6. $u_0^- \times v_1 \times w_1$,
7. $u_0^+ \times v_1 \times w_1$,
8. $u_1 \times v_0^- \times w_0^-$,
9. $u_1 \times v_0^- \times w_0^+$,

10. $u_1 \times v_0^+ \times w_0^-$,

11. $u_1 \times v_0^+ \times w_0^+$,

12. $u_0^- \times v_1 \times w_0^-$,

13. $u_0^- \times v_1 \times w_0^+$,

14. $u_0^+ \times v_1 \times w_0^-$,

15. $u_0^+ \times v_1 \times w_0^+$,

16. $u_0^- \times v_0^- \times w_1$,

17. $u_0^- \times v_0^+ \times w_1$,

18. $u_0^+ \times v_0^- \times w_1$,

19. $u_0^+ \times v_0^+ \times w_1$,

20. $u_0^- \times v_0^- \times w_0^-$,

21. $u_0^- \times v_0^- \times w_0^+$,

22. $u_0^- \times v_0^+ \times w_0^-$,

23. $u_0^- \times v_0^+ \times w_0^+$,

24. $u_0^+ \times v_0^- \times w_0^-$,

25. $u_0^+ \times v_0^- \times w_0^+$,

26. $u_0^+ \times v_0^+ \times w_0^-$,

27. $u_0^+ \times v_0^+ \times w_0^+$,

28. $(u_1 \times v_1 \times w_0^+) \cup (u_0^+ \times v_1 \times w_1) \cup (u_1 \times v_0^- \times w_1)$,

29. $(u_1 \times v_1 \times w_0^-) \cup (u_0^- \times v_1 \times w_1) \cup (u_1 \times v_0^+ \times w_1)$,

30. $(u_1 \times v_1 \times w_0^+) \cup (u_1 \times v_0^- \times w_1)$,

31. $(u_1 \times v_1 \times w_0^-) \cup (u_1 \times v_0^+ \times w_1)$,

32. $(u_1 \times v_1 \times w_0^+) \cup (u_0^- \times v_0^- \times w_1)$,

33. $(u_1 \times v_1 \times w_0^-) \cup (u_0^+ \times v_0^+ \times w_1)$,

34. $(u_0^+ \times v_1 \times w_1) \cup (u_1 \times v_0^- \times w_1)$,

35. $(u_0^- \times v_1 \times w_1) \cup (u_1 \times v_0^+ \times w_1)$,

36. $(u_0^+ \times v_1 \times w_1) \cup (u_1 \times v_0^- \times w_0^-)$,

37. $(u_0^- \times v_1 \times w_1) \cup (u_1 \times v_0^+ \times w_0^+)$,

38. $(u_0^+ \times v_1 \times w_0^+) \cup (u_1 \times v_0^- \times w_1)$,

39. $(u_0^- \times v_1 \times w_0^-) \cup (u_1 \times v_0^+ \times w_1)$,
40. $(u_0^+ \times v_1 \times w_0^+) \cup (u_1 \times v_0^- \times w_0^+)$,
41. $(u_0^- \times v_1 \times w_0^+) \cup (u_1 \times v_0^+ \times w_0^+)$,
42. $(u_0^+ \times v_1 \times w_0^-) \cup (u_1 \times v_0^- \times w_0^-)$,
43. $(u_0^- \times v_1 \times w_0^-) \cup (u_1 \times v_0^+ \times w_0^-)$,
44. $(u_0^- \times v_1 \times w_0^+) \cup (u_0^- \times v_0^- \times w_1)$,
45. $(u_0^- \times v_1 \times w_0^-) \cup (u_0^- \times v_0^+ \times w_1)$,
46. $(u_0^+ \times v_1 \times w_0^+) \cup (u_0^+ \times v_0^- \times w_1)$,
47. $(u_0^+ \times v_1 \times w_0^-) \cup (u_0^+ \times v_0^+ \times w_1)$,
48. $(u_0^+ \times v_1 \times w_0^-) \cup (u_1 \times v_0^- \times w_0^-) \cup (u_0^+ \times v_0^+ \times w_1)$,
49. $(u_0^+ \times v_1 \times w_0^+) \cup (u_1 \times v_0^- \times w_0^-) \cup (u_0^+ \times v_0^- \times w_1)$,
50. $(u_0^- \times v_1 \times w_0^-) \cup (u_1 \times v_0^+ \times w_0^+) \cup (u_0^- \times v_0^+ \times w_1)$,
51. $(u_0^- \times v_1 \times w_0^+) \cup (u_1 \times v_0^+ \times w_0^+) \cup (u_0^- \times v_0^- \times w_1)$,
52. $(u_0^+ \times v_1 \times w_0^+) \cup (u_1 \times v_0^- \times w_0^+) \cup (u_0^- \times v_0^- \times w_1)$,
53. $(u_0^- \times v_1 \times w_0^-) \cup (u_1 \times v_0^+ \times w_0^-) \cup (u_0^+ \times v_0^+ \times w_1)$,
54. $(u_1 \times v_0^- \times w_0^+) \cup (u_0^- \times v_0^- \times w_1)$,
55. $(u_1 \times v_0^- \times w_0^-) \cup (u_0^+ \times v_0^- \times w_1)$,
56. $(u_1 \times v_0^+ \times w_0^+) \cup (u_0^- \times v_0^+ \times w_1)$,
57. $(u_1 \times v_0^+ \times w_0^-) \cup (u_0^+ \times v_0^+ \times w_1)$,

For example, from the figure, one can see that the 31st subcomplex $A \cup B = (u_1 \times v_1 \times w_0^-) \cup (u_1 \times v_0^+ \times w_1)$ in the list can be decomposed into atoms as $(b\#_0 A)\#_1(B\#_0 d) = [(u_0^- \times v_1 \times w_0^-)\#_0(u_1 \times v_0^+ \times w_1)]\#_1[(u_1 \times v_1 \times w_0^-)\#_0(u_0^+ \times v_0^+ \times w_1)]$.

One can show that every composite of molecules in the list is still a molecule in the list. So we have a complete list of the molecules in $u_1 \times v_1 \times w_1$.

In chapter 3, we will show how this list is compiled and how to compile such lists for the molecules in the products of any three finite dimensional globes.

Theorem 1.3.5. *Let K and L be ω-complexes. Let x and y be molecules in K and L respectively. Then $x \times y$ is a molecule in $K \times L$ and*

$$d_n^\gamma(x \times y) = (d_n^\gamma x \times d_0^{(-)^n\gamma} y) \cup (d_{n-1}^\gamma x \times d_1^{(-)^{n-1}\gamma} y) \cup \cdots \cup (d_0^\gamma x \times d_n^\gamma y).$$

The following Theorem is implicit in Paper [1]. To avoid introducing more concepts, we give an independent proof.

Theorem 1.3.6. *Let K_i and L_i be ω-complexes such that $\mathcal{M}(K_i)$ and $\mathcal{M}(L_i)$ are ω-categories for $1 \le i \le r$. Let $f_i : \mathcal{M}(K_i) \to \mathcal{M}(L_i)$ be homomorphisms of partial ω-categories for $1 \le i \le r$. If $\mathcal{M}(K_1 \times \cdots \times K_r)$ and $\mathcal{M}(L_1 \times \cdots \times L_r)$ are ω-categories, then there is a* natural homomorphism $f : \mathcal{M}(K_1 \times \cdots \times K_r) \to \mathcal{M}(L_1 \times \cdots \times L_r)$ *of partial ω-categories such that $f(a_1 \times \cdots \times a_r) = f_1(a_1) \times \cdots \times f_r(a_r)$ for all atoms $a_1 \times \cdots \times a_r$ in $K_1 \times \cdots \times K_r$.*

Proof. The arguments for different choices of r are similar. We give the proof for $r = 2$.

Let $F : \mathcal{C}(K_1 \times K_2) \to \mathcal{C}(L_1 \times L_2)$ be the union-preserving map such that $F(a_1 \times a_2) = f_1(a_1) \times f_2(a_2)$ for all atoms a_1 and a_2. To prove the theorem, it suffices to show that $F(\mathcal{M}(K_1 \times K_2)) \subset \mathcal{M}(L_1 \times L_2)$ and $F|_{\mathcal{M}(K_1 \times K_2)} : \mathcal{M}(K_1 \times K_2) \to M(L_1 \times L_2)$ is a homomorphism of partial ω-categories.

Firstly, we verify inductively that $F(x)$ is a molecule and $F(d_n^\gamma x) = d_n^\gamma F(x)$ for all non-negative integers n, all signs γ and all molecules x in $K_1 \times K_2$.

To begin the induction, let $a_1 \times a_2$ be an atom in $K_1 \times K_2$. Then $F(a_1 \times a_2) = f_1(a_1) \times f_2(a_2)$ by the definition of F. Since $f_i(a_i) \subset \mathcal{M}(K_i)$, we have $F(a_1 \times a_2) \subset \mathcal{M}(K_1 \times K_2)$ by Theorem 1.3.5. Moreover, by Theorem 1.3.5, we have

$$\begin{aligned}
& F(d_n^\gamma(a_1 \times a_2)) \\
= \ & F((d_n^\gamma a_1 \times d_0^{(-)^n\gamma} a_2) \cup (d_{n-1}^\gamma a_1 \times d_1^{(-)^{n-1}\gamma} a_2) \cup \cdots \cup (d_0^\gamma a_1 \times d_n^\gamma a_2)) \\
= \ & (f_1(d_n^\gamma a_1) \times f_2(d_0^{(-)^n\gamma} a_2)) \cup (f_1(d_{n-1}^\gamma a_1) \times f_2(d_1^{(-)^{n-1}\gamma} a_2)) \cup \cdots \cup (f_1(d_0^\gamma a_1) \times f_2(d_n^\gamma a_2)) \\
= \ & (d_n^\gamma f_1(a_1) \times d_0^{(-)^n\gamma} f_2(a_2)) \cup (d_{n-1}^\gamma f_1(a_1) \times d_1^{(-)^{n-1}\gamma} f_2(a_2)) \cup \cdots \cup (d_0^\gamma f_1(a_1) \times d_n^\gamma f_2(a_2)) \\
= \ & d_n^\gamma(f_1(a_1) \times f_2(a_2)) \\
= \ & d_n^\gamma F(a_1 \times a_2).
\end{aligned}$$

Thus $F(d_n^\gamma x) = d_n^\gamma F(x)$ holds when x is an atom in $K_1 \times K_2$.

Next, suppose that $F(x')$ is a molecule and that $F(d_n^\gamma x') = d_n^\gamma F(x')$ for every molecule x' in $K_1 \times K_2$ which can be written as a composite of less than q atoms. Suppose also that x is a molecule in $K_1 \times K_2$ which can be written as a composite of q atoms. We verify that $F(x)$ is a molecule in $L_1 \times L_2$ and that $F(d_n^\gamma x) = d_n^\gamma F(x)$. It is evident that

x can be decomposed into molecules $x = y\#_p z$ such that y and z can be written as less than q atoms.

We first verify that $F(x)$ is a molecule. Indeed, since $d_p^+ F(y) = F(d_p^+ y) = F(d_p^- z) = d_p^- F(z)$, the composite $F(y)\#_p F(z)$ is defined. Moreover, since F preserves unions of atoms, we have $F(x) = F(y \cup z) = F(y) \cup F(z) = F(y)\#_p F(z)$. This shows that $F(x)$ is a molecule in $L_1 \times L_2$.

We next verify that $F(d_n^\gamma x) = d_n^\gamma F(x)$. Indeed, if $n = p$, then

$$F(d_n^- x) = F(d_n^-(y\#_n z)) = F(d_n^- y) = d_n^- F(y) = d_n^- F(x)$$

and, similarly, $F(d_n^+ x) = d_n^+ F(x)$. If $n > p$, then

$$\begin{aligned}
& F(d_n^\gamma x) \\
= \; & F(d_n^\gamma (y\#_p z)) \\
= \; & F(d_n^\gamma y\#_p d_n^\gamma z) \\
= \; & F(d_n^\gamma y \cup d_n^\gamma z) \\
= \; & F(d_n^\gamma y) \cup F(d_n^\gamma z) \\
= \; & d_n^\gamma F(y) \cup d_n^\gamma F(z)
\end{aligned}$$

and

$$d_p^+ d_n^\gamma F(y) = d_p^+ F(y) = d_p^- F(z) = d_p^- d_n^\gamma F(z);$$

thus

$$F(d_n^\gamma x) = d_n^\gamma F(y)\#_p d_n^\gamma F(z) = d_n^\gamma (F(y)\#_p F(z)) = d_n^\gamma F(x).$$

If $n < p$, then

$$F(d_n^\gamma x) = F(d_n^\gamma (y\#_p z)) = F(d_n^\gamma y) = d_n^\gamma F(y) = d_n^\gamma F(x).$$

Therefore, $F(x)$ is a molecule and $F(d_n^\gamma x) = d_n^\gamma F(x)$.

This shows that $F(x)$ is a molecule and $F(d_n^\gamma x) = d_n^\gamma F(x)$ for all molecules x in $K_1 \times K_2$ by induction.

Finally, by arguments similar to that in the proof of $F(x)$ being a molecule above, we can see that $F|_{\mathcal{M}(K_1 \times K_2)}$ preserves composites of molecules. This completes the proof that $F|_{\mathcal{M}(K_1 \times K_2)}$ is a homomorphism of partial ω-categories, as required.

$\square$

Let x be a subcomplex in an ω-complex. An atom a is a *maximal* atom in x if $a \subset x$ and $a \subset b \subset x$ implies $a = b$ for every atom b.

The following result characterises molecules in the product of two infinite dimensional globes.

Theorem 1.3.7. *Let u and v be infinite dimensional globes. Then a subcomplex Λ of $u \times v$ is a molecule if and only if the following conditions hold.*

- *There are no distinct maximal atoms $u[i_1, \alpha_1] \times v[j_1, \beta_1]$ and $u[i_2, \alpha_2] \times v[j_2, \beta_2]$ such that $i_1 \leq i_2$ and $j_1 \leq j_2$, so that the maximal atom in Λ can be listed as $\lambda_1, \ldots, \lambda_S$ with $\lambda_s = u[i_s, \alpha_s] \times v[j_s, \beta_s]$ for $1 \leq s \leq S$ such that $i_1 > \cdots > i_S$ and $j_1 < \cdots < j_S$.*
- *If λ_{s-1} and λ_s are a pair of consecutive maximal atoms in the above list, then $\beta_{s-1} = -(-)^{i_s}\alpha_s$.*

Now we give the construction of $d_p^\gamma \Lambda$ for a molecule Λ in $u \times v$.

Theorem 1.3.8. *Let Λ be a molecule in $u \times v$. Then the set of maximal atoms in $d_p^\gamma \Lambda$ consists of all the maximal atoms $u[i', \alpha'] \times v[j', \beta']$ in Λ with $i' + j' < p$ and all the atoms $u[i, \alpha] \times v[j, \beta]$ with $i + j = p$ such that $i \leq i''$ and $j \leq j''$ for some maximal atom $u[i'', \alpha''] \times v[j'', \beta'']$ of Λ, where the signs α and β are determined as follows:*

1. *If $u[i'', \alpha''] \times v[j'', \beta'']$ can be chosen such that $i'' > i$, then $\alpha = \gamma$; otherwise, $\alpha = \alpha''$.*
2. *If $u[i'', \alpha''] \times v[j'', \beta'']$ can be chosen such that $j'' > j$, then $\beta = (-)^i\gamma$; otherwise, $\beta = \beta''$.*

The composition of molecules in $u \times v$ is characterised as follows.

Theorem 1.3.9. *Let Λ^- and Λ^+ be molecules of $u \times v$. If $d_p^+\Lambda^- = d_p^-\Lambda^+$, then the composite $\Lambda^- \#_p \Lambda^+$ is defined and the maximal atoms in $\Lambda^- \#_p \Lambda^+$ consists of all the q-dimensional common maximal atoms of Λ^- and Λ^+ with $q \leq p$ together with all the r-dimensional atoms in either Λ^- and Λ^+ with $r > p$.*

Corollary 1.3.10. *The molecules in $u \times v$ form an ω-category.*

Example 1.3.11. Let the subcomplexes

$$\begin{aligned}\Lambda^- = & \ u_5^- \times v_0^+ \\ \cup & \ u_4^- \times v_2^+ \\ \cup & \ u_2^- \times v_3^- \\ \cup & \ u_1^- \times v_4^+ \\ \cup & \ u_0^- \times v_5^+\end{aligned}$$

and

$$\begin{aligned}\Lambda^+ = & \ u_6^+ \times v_0^- \\ \cup & \ u_5^- \times v_1^+ \\ \cup & \ u_3^+ \times v_2^+ \\ \cup & \ u_2^- \times v_4^+ \\ \cup & \ u_0^- \times v_5^+.\end{aligned}$$

By Theorem 1.3.7, it is easy to see that Λ^- and Λ^+ are molecules of $u \times v$. Moreover, by Theorem 1.3.8, we have $d_5^+\Lambda^-$ and $d_5^-\Lambda^+$ are both equal to the molecule

$$\begin{aligned} & \ u_5^- \times v_0^+ \\ \cup & \ u_4^- \times v_1^+ \\ \cup & \ u_3^+ \times v_2^+ \\ \cup & \ u_2^- \times v_3^- \\ \cup & \ u_1^- \times v_4^+ \\ \cup & \ u_0^- \times v_5^+.\end{aligned}$$

Therefore, by Theorem 1.3.9, the composite $\Lambda^- \#_5 \Lambda^+$ is defined and the composite is the following molecule.

$$\begin{aligned} & \ u_6^+ \times v_0^- \\ \cup & \ u_5^- \times v_1^+ \\ \cup & \ u_4^- \times v_2^+ \\ \cup & \ u_2^- \times v_4^+ \\ \cup & \ u_0^- \times v_5^+.\end{aligned}$$

Example 1.3.12. Let u_2 be the 2-dimensional globe and let v_1 be a 1-dimensional globe. Geometrically, u_2 is a closed disk and v_1 is a closed interval. Therefore the product $u_2 \times v_1$ is a cylinder. Since u_2 has 5 atoms and v_1 has 3 atoms, the product $u_2 \times v_1$ has 15 atoms.

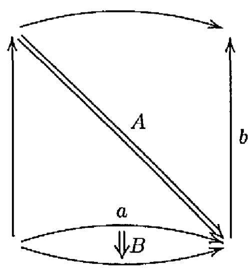
We calculate the boundaries $\partial^-(u_2 \times v_1)$ and $\partial^+(u_2 \times v_1)$. By definition, we have

$$\partial^-(u_2 \times v_1) = (u_1^- \times v_1) \cup (u_2 \times v_0^-)$$

and

$$\partial^+(u_2 \times v_1) = (u_1^+ \times v_1) \cup (u_2 \times v_0^+).$$

If we put the disk u_2 in a horizontal plane and put the interval u_1 in a vertical line and represent d_0^- and d_0^+ on v_1 by bottom and top respectively, then $\partial^-(u_2 \times v_1)$ is the union of the bottom disk and half of the curved part of the boundary of the product $u_1^- \times v_1$, as shown in the following figure, where $a = u_1^- \times v_0^-$, $b = u_0^+ \times v_1$, $A = u_1^- \times v_1$ and $B = u_2 \times v_0^-$.



Since

$$\partial^-(u_1^- \times v_1) = u_0^- \times v_1 \cup u_1^- \times v_0^+$$

and

$$\partial^+(u_1^- \times v_1) = u_0^+ \times v_1 \cup u_1^- \times v_0^-,$$

one can easily see that the direction of $A = u_1^- \times v_1$ is as indicated in the figure. Similarly, we can get the direction for $B = u_2 \times v_0^-$. Moreover, it is easy to check that

$$\partial^-(u_2 \times v_1) = (u_1^- \times v_1) \cup (u_2 \times v_0^-) = (u_1^- \times v_1)\#_1[(u_2 \times v_0^-)\#_0(u_0^+ \times v_1)].$$

One can also see this graphically from the figure.

Similarly, $\partial^+(u_2 \times v_1)$ is the union of the top disk and the other half of the curved part of the boundary of the product $u_1^+ \times v_1$, and we have

$$\partial^+(u_2 \times v_1) = (u_1^+ \times v_1) \cup (u_2 \times v_0^+) = [(u_0^- \times v_1)\#_0(u_2 \times v_0^+)]\#_1(u_1^+ \times v_1).$$

Therefore $\partial^-(u_2 \times v_1)$ and $\partial^+(u_2 \times v_1)$ are indeed molecules.

We can similarly workout the boundaries of other atoms.

Example 1.3.13. Let u be a infinite dimensional globes. Let u_p be a p dimensional globe. Recall that there is a homomorphism $f_p^u : \mathcal{M}(u) \to \mathcal{M}(u_p)$ of ω-categories such that for all atom $u_i^\alpha \in \mathcal{M}(u)$

$$f_p^u(u_i^\alpha) = \begin{cases} u_i^\alpha & \text{when } i < p, \\ u_p & \text{when } i \geq p. \end{cases}$$

It follows from the Theorem 1.3.6 and Corollary 1.3.10 that there is a natural homomorphism $f : \mathcal{M}(u \times v) \to \mathcal{M}(u_p \times v_q)$ of ω-categories such that $f(u_i^\alpha \times v_j^\beta) = f_p^u(u_i^\alpha) \times f_q^v(v_j^\beta)$ for all atoms $u_i^\alpha \times v_j^\beta$ in $u \times v$.

1.4 Decomposition of Molecules in Loop-Free ω-Complexes

In this section, we prove a decomposition theorem for molecules in an loop-free ω-complex. This theorem will be used later in the thesis.

Firstly, we need to generalise some concepts and results from loop-free directed complexes in paper [20] to loop-free ω-complexes.

Definition 1.4.1. A *directed precomplex* is a set K together with functions dim, ∂^- and ∂^+ on K satisfying the following conditions.

1. If $\sigma \in K$, then $\dim \sigma$ is an non-negative integer, called *dimension of* σ.
2. If $\sigma \in K$ and $\dim \sigma > 0$, then $\partial^- \sigma$ and $\partial^+ \sigma$ are subsets of K consisting of $\dim \sigma - 1$ dimensional elements of K.

Let K be a directed precomplex. A subset x of K is *closed* if $\partial^\alpha \sigma \subset x$ for every $\sigma \in x$ with $\dim \sigma > 0$ and every sign α. For a subset y of K, the closure $\mathrm{Cl}(y)$ of y is the smallest closed subset of K containing y. The closure $\mathrm{Cl}\{\sigma\}$ of a singleton $\{\sigma\}$, denoted by $\bar{\sigma}$, is called an *atom*.

Definition 1.4.2. Let K be a directed precomplex.

If $x \subset K$ and $\alpha = \pm$, then

$$d_n^\alpha x = \bigcup\{\sigma : \sigma \in x \text{ and } \dim \sigma \leq n\} \setminus \bigcup\{\bar{\tau} \setminus \mathrm{Cl}(\partial^\alpha \tau) : \tau \in x \text{ and } \dim \tau = n+1\}.$$

If x and y are closed subsets of K, then the *composite* $x \#_n y$ is defined if and only if $x \cap y = d_n^+ x = d_n^- y$; If $x \#_n y$ is defined, then $x \#_n y = x \cup y$.

A *molecule* is a subset generated from atoms by finitely applying the composition operations $\#_n$ $(n = 0, 1, \dots)$.

Definition 1.4.3. A *directed complex* is a directed precomplex satisfying the following conditions.

1. If $\bar{\sigma}$ is an atom with $\dim \sigma = p > 0$, then $d_{p-1}^\alpha \bar{\sigma}$ is a molecule for $\alpha = \pm$.

2. If $\bar{\sigma}$ is an atom with $\dim \sigma = p > 1$, then $d_{p-2}^\beta d_{p-1}^\alpha \bar{\sigma} = d_{p-2}^\beta \bar{\sigma}$ for $\alpha = \pm$ and $\beta = \pm$.

Definition 1.4.4. Let K be a directed complex and n be a non-negative integer. Let a and b be elements in K.

- An *n-path* of length k from a to b is a sequence $a = a_0, \dots, a_k = b$ of elements in K such that for $1 \leq i \leq k$ either

  $$\dim a_{i-1} \leq n \text{ and } \dim a_i > n \text{ and } a_{i-1} \in d_n^- \bar{a}_i \setminus (d_{n-1}^- \bar{a}_i \cup d_{n-1}^+ \bar{a}_i)$$

  or

  $$\dim a_{i-1} > n \text{ and } \dim a_i \leq n \text{ and } a_i \in d_n^+ \bar{a}_{i-1} \setminus (d_{n-1}^- \bar{a}_{i-1} \cup d_{n-1}^+ \bar{a}_{i-1}).$$

- A *total path* of length k from a to b is a sequence $a = a_0, \dots, a_k = b$ of elements in K such that for $1 \leq i \leq k$ either $a_{i-1} \in \partial^- a_i$ or $a_i \in \partial^+ a_{i-1}$.

- An *n-loop* is an n-path of positive length from some element of K to itself; A *total loop* is a total path of positive length from some element of K to itself.

- A subset of K is *loop-free* if it does not contain n-loops for any n; A subset of K is *total loop-free* if it does not contain total loops.

We can now generalise the concept of loop-freeness to ω-complexes, as follows.

Definition 1.4.5. Let K be an ω-complex and n be a non-negative integer. Let a and b be atoms.

- An *n-path* of length k from a to b is a sequence $a = a_0, \ldots, a_k = b$ of atoms such that for $1 \le i \le k$ either

$$\dim a_{i-1} \le n \text{ and } \dim a_i > n \text{ and } \operatorname{Int} a_{i-1} \subset d_n^- a_i \setminus (d_{n-1}^- a_i \cup d_{n-1}^+ a_i)$$

  or

$$\dim a_{i-1} > n \text{ and } \dim a_i \le n \text{ and } \operatorname{Int} a_i \subset d_n^+ a_{i-1} \setminus (d_{n-1}^- a_{i-1} \cup d_{n-1}^+ a_{i-1}).$$

- A *total path* of length k from a to b is a sequence $a = a_0, \ldots, a_k = b$ of atoms such that for $1 \le i \le k$ either $\dim a_{i-1} = \dim a_i - 1$ and $a_{i-1} \subset \partial^- a_i$, or $\dim a_i = \dim a_{i-1} - 1$ and $a_i \subset \partial^+ a_{i-1}$.

- An *n-loop* is an n-path of positive length from some atom to itself; A *total loop* is a total path of positive length from some atom to itself.

- A subcomplex of K is *loop-free* if it does not contain n-loops for any n; A subcomplex of K is *total loop-free* if it does not contain total loops.

For example, the 0-path and total path in a 1-dimensional ω-complex is a directed path, regarded as a sequence of alternate vertices and edges. In Example 1.1.4, the sequence u, a, v, b, x, c is a total path; the sequence b, x, c, y, d is a 1-path.

Lemma 1.4.6. *Let K be a loop-free ω-complex. Let a be an atom in K with* $\dim a = p > 0$. *Then $\partial^\gamma a$ is a union of its $p-1$ dimensional atoms.*

Proof. Suppose otherwise that there is a p-dimensional atom a of K such that $\partial^\gamma a$ is not a union of its $p-1$ dimensional atoms for some γ. Then $\partial^\gamma a$ has a maximal atom b of dimension less than $p-1$. Let $q = \dim b$. By Lemma 1.2.3, we can see that b is an maximal atom in both $d_q^- \partial^\gamma a = d_q^- a$ and $d_q^+ \partial^\gamma a = d_q^+ a$ which implies that a, b, a is a q-loop. This contradicts the assumption of the loop-freeness of K. $\square$

Let K be a directed complex. According to the definition of directed complexes and Theorem 1.2.8, set K is also an ω-complex such that the atoms in the ω-complex K are exactly those in the directed complex K. Note that $\operatorname{Int}\bar{\sigma}$ is a singleton $\{\sigma\}$ for every atom $\bar{\sigma}$. By Theorem 1.2.3, Definition 1.1.15 and Lemma 1.4.6, it is easy to see that every loop-free ω-complex is equivalent to an ω-complex associated with a loop-free directed complex. Thus all results for loop-free directed complexes can be generalised to loop-free ω-complexes. In particular, we have the following definitions and theorems.

Definition 1.4.7. Let x be a non-empty finite subcomplex of an ω-complex which is not an atom. Then the non-negative integer

$$\max\{\dim(a \cap b) : a \text{ and } b \text{ are distinct maximal atoms in } x\}$$

is called *frame dimension* of x, denoted by $\mathrm{fr}\ \dim x$.

Definition 1.4.8. A molecule x in an ω-complex is *split* if the following conditions hold.

- Let a be a p-dimensional atom in x. If b is a $p-1$ dimensional atom in $\partial^{-}a$ and if c is a $p-1$ dimensional atom in $\partial^{+}a$, then b and c are distinct.
- If y is a factor in some expression of x as an iterated composite, then there exists an expression of y as an iterated composite of atoms using the operations $\#_n$ only for $n \leq \mathrm{fr}\ \dim y$.

Proposition 1.4.9. *If a subcomplex of an ω-complex is total loop-free, then it is loop-free.*

Theorem 1.4.10. *In a loop-free ω-complex, all molecules are split.*

Theorem 1.4.11. *If the atoms in an ω-complex are all total loop-free, then the molecules are all total loop-free, so that all the molecules are split.*

Theorem 1.4.12. *Let K and L be ω-complexes. If both K and L are total loop-free, then so is $K \times L$.*

According to this theorem, the products of infinite-dimensional globes are total loop-free. Hence all molecules in products of infinite-dimensional globes are split.

Now we can state the main theorem in this section.

Theorem 1.4.13. *Let x be a molecule in a loop-free ω-complex and $p = \mathrm{fr}\ \dim x$. Let q be an integer with $q \geq p$. If there is a maximal atom a_1 in x with $\dim a_1 > p$ such that $a_1 \cap a' \subset d_q^+ a_1 \cap d_q^- a'$ for every other maximal atom a' in x with $\dim a' > q$, then x can be decomposed into molecules*

$$x = x^- \#_q x^+,$$

where $x^- = d_q^- x \cup a_1$ and $x^+ = d_q^+ x \cup \bigcup\{a'' : a'' \text{ is a maximal atom in } x \text{ with } a'' \neq a_1\}$.

The decomposition for $\partial^-(u_2 \times v_1)$ and $\partial^-(u_2 \times v_1)$ in Example 1.3.12 is actually obtained by using this theorem.

The proof is separated into several lemmas.

Lemma 1.4.14. *Let x be a subcomplex and $y = y_1 \cup \cdots \cup y_n$ be a union of subcomplexes. If $x \subset y$ and $x \cap y_i \subset d_p^\gamma y_i$ for all $1 \leq i \leq n$, then $x \subset d_p^\gamma y$.*

Proof. We give the proof only for $n = 2$. The general case can be shown by induction.

By Proposition 1.2.7, we have $d_p^\gamma y = (d_p^\gamma y_1 \cap d_p^\gamma y_2) \cup (d_p^\gamma y_1 \setminus y_2) \cup (d_p^\gamma y_2 \setminus y_1)$.

Suppose that $\xi \in x$. Then $\xi \in y_1$ or $\xi \in y_2$. If $\xi \in y_1$ and $\xi \in y_2$, then $\xi \in d_p^\gamma y_1 \cap d_p^\gamma y_2 \subset d_p^\gamma y$. If $\xi \in y_1$ but $\xi \notin y_2$, then $\xi \in d_p^\gamma y_1$ but $\xi \notin y_2$; this implies that $\xi \in d_p^\gamma y$. If $\xi \in y_2$ but $\xi \notin y_1$, then $\xi \in d_p^\gamma y_2$ but $\xi \notin y_1$; this implies that $\xi \in d_p^\gamma y$. This completes the proof that $x \subset d_p^\gamma y$.

□

Corollary 1.4.15. *Let $x = x_1 \cup \cdots x_m$ and $y = y_1 \cup \cdots \cup y_n$ be a union of subcomplexes. If $x \subset y$ and $x_i \cap y_j \subset d_p^\gamma y_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, then $x \subset d_p^\gamma y$.*

Lemma 1.4.16. *Let x be a subcomplex of an ω-complex and a be a maximal atom with* $\dim a \leq n$. *If $d_n^\alpha x$ is a subcomplex, then a is a maximal atom in $d_n^\alpha x$ for every sign α.*

Proof. This follows straightforwardly from Definition 1.2.3. □

Lemma 1.4.17. *Let x be a subcomplex of an ω-complex and $\xi \in x$. Then $\xi \in d_p^\gamma x$ if and only if $\xi \in d_p^\gamma a$ for every maximal atom a in x with $\xi \in a$.*

Proof. Suppose that $\xi \in d_p^\gamma x$. According to Proposition 1.2.6, we have $\xi \in a \cap d_p^\gamma x \subset d_p^\gamma a$ for every maximal atom a in x with $\xi \in a$.

Conversely, suppose that $\xi \in d_p^\gamma a$ for every maximal atom a in x with $\xi \in a$. Then it is evident that $\xi \in a'$ for some atom a' in x with $\dim a' \leq q$. Moreover, for every $p+1$-dimensional atom $b \subset x$ with $\xi \in b$, we have $\xi \in b \cap d_p^\gamma b' \subset d_p^\gamma b$ by Proposition 1.2.6, where b' is a maximal atom containing b . It follows from Definition 1.2.3 that $\xi \in d_p^\gamma x$, as required.

This completes the proof. □

Lemma 1.4.18. *Let x, x^- and x^+ be as in the statement of Theorem 1.4.13. Then $x = x^- \#_q x^+$.*

Proof. Recall that

$$x^- = d_q^- x \cup a_1$$

and

$$x^+ = d_q^+ x \cup \bigcup \{a'' : a'' \text{ is a maximal atom in } x \text{ with } a'' \neq a_1\}.$$

Thus $x = x^- \cup x^+$.

Since x is a molecule, the set $\{a'' : a'' \text{ is a maximal atom in } x \text{ with } a'' \neq a_1\}$ is finite. Moreover, it is evident that x^- and x^+ are subcomplexes.

Now we trivially have $x^+ \cap d_q^- x \subset d_q^+ d_q^- x$; since $a_1 \subset x$, we have $d_q^+ x \cap a_1 \subset d_q^+ a_1$ by Proposition 1.2.6; according to the assumption, we also have $a_1 \cap a'' \subset d_q^+ a_1 \cap d_q^- a'' \subset d_q^+ a_1$ for every maximal atom a'' in x with $a'' \neq a_1$. It follows from Corollary 1.4.15 that $x^- \cap x^+ \subset d_q^+ x^-$.

On the other hand, suppose that $\xi \in d_q^+ x^-$ and $\xi \notin a''$ for every atom a'' distinct from a_1 such that $\dim a'' > q$. We claim that $\xi \in d_q^+ x$ so that $\xi \in x^+$ and hence $d_q^+ x^- \subset x^- \cap x^+$. Indeed, let a be a maximal atom in x with $\xi \in a$. If $a = a_1$, then, by Proposition 1.2.6, $\xi \in a_1 \cap d_q^+ x^- \subset d_q^+ a_1$; if $a \neq a_1$, then $\dim a \leq q$ by the assumption; hence $\xi \in a = d_q^+ a$. It follows from Lemma 1.4.17 that $\xi \in d_q^+ x$, as required.

We have now shown that $x^- \cap x^+ = d_q^+ x^-$. By a similar argument, we can also get $x^- \cap x^+ = d_q^- x^+$. This implies that $x^- \#_q x^+$ is defined and hence $x = x^- \#_q x^+$, as required.

This completes the proof.

□

Lemma 1.4.19. *Let x^- and x^+ be as in the statement of Theorem 1.4.13. Then x^- and x^+ are molecules.*

Proof. Since x is a molecule in a loop-free ω complex, it is split by Theorem 1.4.10. Hence x^- and x^+ are molecules, as required. □

We have now completed the proof of Theorem 1.4.13.

Chapter 2

Molecules in the Product of Three Infinite-Dimensional Globes

In this chapter, we study molecules in the product of three infinite dimensional globes. We are going to give two equivalent descriptions for the molecules in the product of three infinite dimensional globes.

Throughout this chapter, infinite dimensional globes are denoted by u, v or w. An atom u_i^α is denoted by $u[i,\alpha]$. All subcomplexes refer to finite and non-empty subcomplexes in the ω-complex $u \times v \times w$.

2.1 The Definition of Pairwise Molecular Subcomplexes

In this section, we first define 'projection maps' and prove some of their basic properties. Then we state one of the main results in this chapter which says that a subcomplex in products of three infinite dimensional globes is a molecule if and only if it is 'projected' to molecules in (twisted) products of two infinite dimensional globes together with a natural requirement. This leads to the definition of pairwise molecular subcomplexes of $u \times v \times w$.

Let w^J be the atomic complex with atoms $w^J[k,\varepsilon]$ ($k = 0, 1, \cdots$ and $\varepsilon = \pm$) such that dim $w^J[k,\varepsilon] = k$ and $d_{k-1}^\gamma w^J[k,\varepsilon] = w^J[k-1, (-)^J\gamma]$ for $k > 0$. It is clear that

w^J satisfies conditions in Theorem 1.2.8. Thus it is an ω-complex. It is also easy to see that the ω-complex w^J is equivalent to infinite dimensional globe w under an obvious equivalence of ω-complexes sending $w^J[k,(-)^J\varepsilon]$ to $w[k,\varepsilon]$. Moreover, it is evident that this induces a equivalence of ω-complexes $u \times w$ and $u \times w^J$ sending $u[i,\alpha] \times w[k,\varepsilon]$ to $u[i,\alpha] \times w^J[k,\varepsilon]$. By this equivalence, all the results for products of two infinite dimensional globes in Section 1.3 can be generalised to $u \times w^J$. In particular, we have $d_p^\gamma(u[i,\alpha] \times w^J[k,\varepsilon]) = u[i,\alpha] \times w^J[k,\varepsilon]$ if $i+k \le p$, while, if $i+k > p$, the maximal atoms in $d_p^\gamma(u[i,\alpha] \times w^J[k,\varepsilon])$ consists of $u[l,\sigma] \times w^J[n,\omega]$ such that $l \le i$, $n \le k$ and $l+n = p$; the signs σ and ω are determined as follows:

1. if $l = i$, then $\sigma = \alpha$; if $l < i$, then $\sigma = \gamma$;
2. if $n = k$, then $\omega = \varepsilon$; if $n < k$, then $\omega = (-)^{l+J}\gamma$.

For an atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ in $u \times v \times w$, let

$$F_J^v(\operatorname{Int}\lambda) = \begin{cases} \operatorname{Int}(u[i,\alpha] \times w^J[k,\varepsilon]), & \text{when } j \ge J; \\ \emptyset, & \text{when } j < J. \end{cases}$$

This gives a map sending interiors of atoms in $u \times v \times w$ to interiors of atoms in $u \times w^J$ or the empty set.

Since interiors of atoms are disjoint, it is clear that the map F_J^v can be extended uniquely to a map sending unions of interiors of atoms in $u \times v \times w$ to unions of interiors of atoms in $u \times w^J$ by requiring it preserves unions.

We can similarly define a map F_I^u sending unions of interiors of atoms in $u \times v \times w$ to unions of interiors of atoms in $v^I \times w^I$ and a map F_K^w sending unions of interiors of atoms in $u \times v \times w$ to unions of interiors of atoms in $u \times v$.

It is easy to see that every atom can be written as a union of interiors of atoms. It follows that F_I^u, F_J^v and F_k^w are defined on subcomplexes of $u \times v \times w$ and preserve unions.

We next prove that F_I^u, F_J^v and F_k^w send atoms to atoms or the empty set so that they send subcomplexes to subcomplexes. We need two preliminary results.

Lemma 2.1.1.

$$d_p^\gamma(u[i,\alpha]\times v[j,\beta]\times w[k,\varepsilon]) = \bigcup\{d_l^\gamma u[i,\alpha]\times d_m^{(-)^l\gamma} v[j,\beta]\times d_n^{(-)^{l+m}\gamma} w[k,\varepsilon] \ : \ l+m+n=p\}$$

Proof. According to Proposition 1.3.5,

$$\begin{aligned} & d_p^\gamma(u[i,\alpha]\times v[j,\beta]\times w[k,\varepsilon]) \\ = \ & d_p^\gamma[(u[i,\alpha]\times v[j,\beta])\times w[k,\varepsilon]] \\ = \ & \bigcup\{d_s^\gamma(u[i,\alpha]\times v[j,\beta])\times d_t^{(-)^s\gamma} w[k,\varepsilon] \ : \ s+t=p\}. \end{aligned}$$

Then the result follows easily by applying Proposition 1.3.5 again. □

Proposition 2.1.2. *Let $\lambda = u[i,\alpha]\times v[j,\beta]\times w[k,\varepsilon]$ be an atom.*

- *If $i+j+k\le p$, then $d_p^\gamma\lambda = \lambda$.*
- *If $i+j+k > p$, then the set of maximal atoms in $d_p^\gamma\lambda$ consists of all the atoms $u_l^\sigma\times v_m^\tau\times w_n^\omega$ such that $l+m+n=p$ and $l\le i$, $m\le j$ and $n\le k$, where the signs σ, τ and ω are determined as follows:*

  1. *If $l=i$, then $\sigma=\alpha$; if $l<i$, then $\sigma=\gamma$.*
  2. *If $m=j$, then $\tau=\beta$; if $m<j$, then $\tau=(-)^l\gamma$.*
  3. *If $n=k$, then $\omega=\varepsilon$; if $l<i$, then $\omega=(-)^{l+m}\gamma$.*

Proof. It is evident that $d_p^\gamma\lambda=\lambda$ when $i+j+k\le p$. We may assume in the following proof that $i+j+k>p$.

Let Λ_1 denote the union of the atoms described in this lemma. We must show that $d_p^\gamma\lambda = \Lambda_1$.

By the formation of Λ_1, it is easy to see that every maximal atom $\mu = u[l,\sigma]\times v[m,\tau]\times w[n,\omega]$ in Λ_1 can be expressed as $\mu = d_l^\gamma u[i,\alpha]\times d_m^{(-)^l\gamma} v[j,\beta]\times d_n^{(-)^{l+m}\gamma} w[k,\varepsilon]$. By Lemma 2.1.1, we can see that $\mu\subset d_p^\gamma\lambda$, and hence $\Lambda_1\subset d_p^\gamma\lambda$.

To prove the reverse inclusion, by Lemma 2.1.1, it suffices to prove that $d_l^\gamma u[i,\alpha]\times d_m^{(-)^l\gamma} v[j,\beta]\times d_n^{(-)^{l+m}\gamma} w[k,\varepsilon]\subset\Lambda_1$ for every triple (l,m,n) with $l+m+n=p$. By the formation of Λ_1, this inclusion is obvious when $l\le i$, $m\le j$ and $n\le k$. So it suffices to prove that $d_l^\gamma u[i,\alpha] \times d_m^{(-)^l\gamma} v[j,\beta] \times d_n^{(-)^{l+m}\gamma} w[k,\varepsilon]$ is not a maximal atom in $d_p^\gamma \lambda$ when $l > i$, $m > j$ or $n > k$.

Suppose that $l > i$. Then $m < j$ or $n < k$. If $m < j$, then $d_l^\gamma u[i,\alpha] \times d_m^{(-)^l\gamma} v[j,\beta] \times d_n^{(-)^{l+m}\gamma} w[k,\varepsilon] \subsetneqq d_{l-1}^\gamma u[i,\alpha] \times d_{m+1}^{(-)^{l-1}\gamma} v[j,\beta] \times d_n^{(-)^{l+m}\gamma} w[k,\varepsilon] \subset d_p^\gamma \lambda$. If $m \geq j$, then $n < k$, so $d_l^\gamma u[i,\alpha] \times d_m^{(-)^l\gamma} v[j,\beta] \times d_n^{(-)^{l+m}\gamma} w[k,\varepsilon] \subsetneqq d_{l-1}^\gamma u[i,\alpha] \times d_m^{(-)^{l-1}\gamma} v[j,\beta] \times d_{n+1}^{(-)^{l+m-1}\gamma} w[k,\varepsilon] \subset d_p^\gamma \lambda$. This shows that, in both cases, $d_l^\gamma u[i,\alpha] \times d_m^{(-)^l\gamma} v[j,\beta] \times d_n^{(-)^{l+m}\gamma} w[k,\varepsilon]$ is not a maximal atom in $d_p^\gamma \lambda$. Similarly, the above statement is true if $m > j$ or $n > k$.

This completes the proof of the lemma. $\square$

Proposition 2.1.3. *Let $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ be an atom in $u \times v \times w$. Then*

1. $$F_I^u(u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]) = \begin{cases} v^I[j,\beta] \times w^I[k,\varepsilon], & \text{when } i \geq I; \\ \emptyset, & \text{when } i < I; \end{cases}$$

2. $$F_J^v(u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]) = \begin{cases} u[i,\alpha] \times w^J[k,\varepsilon], & \text{when } j \geq J; \\ \emptyset, & \text{when } j < J; \end{cases}$$

3. $$F_K^w(u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]) = \begin{cases} u[i,\alpha] \times v[j,\beta], & \text{when } k \geq K; \\ \emptyset, & \text{when } k < K. \end{cases}$$

In particular, F_I^u, F_J^v and F_K^w send atoms to atoms or the empty set so that they send subcomplexes to subcomplexes.

Proof. The argument for the three cases are similar. We only prove the second one. The proof is given by induction on dimension of atoms.

For an atom $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ in $u \times v \times w$, if $\dim \lambda = 0$, then $i = j = k = 0$;

hence

$$
\begin{aligned}
& F_J^v(\lambda) \\
= \ & F_J^v(\operatorname{Int}\lambda) \\
= \ & \begin{cases} \operatorname{Int}(u[i,\alpha] \times w^J[k,\varepsilon]), & \text{when } J = 0; \\ \emptyset, & \text{when } J > 0 \end{cases} \\
= \ & \begin{cases} u[i,\alpha] \times w^J[k,\varepsilon], & \text{when } J = 0; \\ \emptyset, & \text{when } J > 0, \end{cases}
\end{aligned}
$$

as required.

Suppose that the proposition holds for every atom of dimension less then p. Suppose also that $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ is a p-dimensional atom. If $j < J$, then it is easy to see that $F_J^v(\lambda) = \emptyset$, as required. If $j > J$, then we have

$$
\begin{aligned}
& F_J^v(\lambda) \\
= \ & F_J^v(\operatorname{Int}\lambda \cup \partial^-\lambda \cup \partial^+\lambda) \\
\supset \ & F_J^v(\partial^+\lambda) \\
\supset \ & F_J^v(u[i,\alpha] \times v[j-1,(-)^i] \times w[k,\varepsilon]) \\
= \ & u[i,\alpha] \times w^J[k,\varepsilon]
\end{aligned}
$$

since $u[i,\alpha] \times v[j-1,(-)^i] \times w[k,\varepsilon]$ is an atom of dimension $p-1$; the reverse inclusion holds automatically; so $F_J^v(\lambda) = u[i,\alpha] \times w^J[k,\varepsilon]$, as required. Now suppose that $j = J$. Then, by Lemma 2.1.2, $\partial^\gamma\lambda$ is the union of atoms $u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ with $i'+j'+k' = p-1$ such that

1. if $i' = i$, then $\alpha' = \alpha$; if $i' = i-1$, then $\alpha' = \gamma$;

2. if $j' = j$, then $\beta' = \beta$; if $j' = j-1$, then $\beta' = (-)^i\gamma$;

3. if $k' = k$, then $\varepsilon' = \varepsilon$; if $k' = k-1$, then $\varepsilon' = (-)^{i+J}\gamma$.

It follows easily from the induction hypothesis that $F_J^v(\partial^\gamma\lambda) = \partial^\gamma(u[i,\alpha] \times w^J[k,\varepsilon])$ for every sign γ. Therefore

$$\begin{aligned}
& F_J^v(\lambda) \\
= \; & F_J^v(\operatorname{Int}\lambda) \cup F_J^v(\partial^-\lambda) \cup F_J^v(\partial^-\lambda) \\
= \; & \operatorname{Int}(u[i,\alpha] \times w^J[k,\varepsilon]) \cup \partial^-(u[i,\alpha] \times w^J[k,\varepsilon]) \cup \partial^+(u[i,\alpha] \times w^J[k,\varepsilon]) \\
= \; & u[i,\alpha] \times w^J[k,\varepsilon],
\end{aligned}$$

as required.

This completes the proof of the proposition. $\square$

Now we can state one of the main results in this chapter which says that a subcomplex in $u \times v \times w$ is a molecule if and only if it is *pairwise molecular,* i.e., it is 'projected' to molecules in (twisted) products of two infinite dimensional globes together with a natural condition (condition 1).

Definition 2.1.4. Let Λ be a subcomplex in $u \times v \times w$. Then Λ is *pairwise molecular* if the following conditions hold:

1. there are no distinct maximal atoms $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ and $u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ in Λ such that $i \leq i'$, $j \leq j'$ and $k \leq k'$;
2. $F_I^u(\Lambda)$ is a molecule in $v^I \times w^I$ or the empty set for every integer I;
3. $F_J^v(\Lambda)$ is a molecule in $u \times w^J$ or the empty set for every integer J;
4. $F_K^w(\Lambda)$ is a molecule in $u \times v$ or the empty set for every integer K.

Example 2.1.5. It is easy to check that the following subcomplex of $u \times v \times w$ is pairwise molecular.

$$
\begin{aligned}
& u_8^+ \times v_2^+ \times w_1^- \\
\cup \quad & u_5^- \times v_2^+ \times w_5^- \\
\cup \quad & u_1^- \times v_2^+ \times w_8^+ \\
\cup \quad & u_9^+ \times v_1^- \times w_2^+ \\
\cup \quad & u_4^- \times v_1^- \times w_6^+ \\
\cup \quad & u_0^+ \times v_1^+ \times w_9^+ \\
\cup \quad & u_8^- \times v_0^- \times w_5^- \\
\cup \quad & u_5^- \times v_0^+ \times w_6^+ \\
\cup \quad & u_4^- \times v_0^- \times w_7^+ \\
\cup \quad & u_2^- \times v_0^- \times w_9^+
\end{aligned}
$$

Theorem 2.1.6. *A subcomplex of $u \times v \times w$ is a molecule if and only if it is pairwise molecular.*

We end this section with a property of 'projection' maps which is used later in the thesis.

Proposition 2.1.7. *Let Λ and Λ' be subcomplexes of $u \times v \times w$ satisfying condition 1 for pairwise molecular subcomplexes. If $F_I^u(\Lambda) = F_I^u(\Lambda')$, $F_J^v(\Lambda) = F_J^v(\Lambda')$ and $F_K^w(\Lambda) = F_K^w(\Lambda')$ for all I, J and K, then $\Lambda = \Lambda'$.*

Proof. It suffices to prove that Λ and Λ' consists of the same maximal atoms.

Let $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ be a maximal atom in Λ. It is easy to see that $v^i[j,\beta] \times w^i[k,\varepsilon]$ is a maximal atom in $F_i^u(\Lambda) = F_i^u(\Lambda')$. Thus Λ' has a maximal atom $u[i',\alpha'] \times v[j,\beta] \times w[k,\varepsilon]$ with $i' \geq i$. Since $v^i[j,\beta] \times w^i[k,\varepsilon] \not\subset F_{i+1}^u(\Lambda) = F_{i+1}^u(\Lambda')$, we have $i' = i$. One can similarly get a maximal atom $u[i,\alpha] \times v[j,\beta'] \times w[k,\varepsilon]$ in Λ'. It follows from condition 1 for pairwise molecular subcomplexes that $\alpha' = \alpha$ and $\beta' = \beta$. This shows that $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ is a maximal atom in Λ'.

Symmetrically, we can see that every maximal atom in Λ' is a maximal atom in Λ.

This completes the proof that $\Lambda = \Lambda'$. $\square$

Remark 2.1.8. The above proposition does not holds without Condition 1 for pairwise molecular subcomplexes. This can be seen from the following subcomplexes of $u \times v \times w$:

$$\begin{aligned}\Lambda = \ & u_1^+ \times v_1^+ \times w_1^+ \\ \cup \ & u_1^+ \times v_1^- \times w_1^- \\ \cup \ & u_1^- \times v_1^+ \times w_1^- \\ \cup \ & u_1^- \times v_1^- \times w_1^+\end{aligned}$$

and

$$\begin{aligned}\Lambda' = \ & u_1^+ \times v_1^- \times w_1^+ \\ \cup \ & u_1^+ \times v_1^+ \times w_1^- \\ \cup \ & u_1^- \times v_1^- \times w_1^- \\ \cup \ & u_1^- \times v_1^+ \times w_1^+.\end{aligned}$$

2.2 Molecules Are Pairwise Molecular

In this section, we prove that molecules in $u \times v \times w$ are pairwise molecular.

Proposition 2.2.1. *Let Λ be a molecule in $u \times v \times w$. Then there are no distinct maximal atoms $u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ in Λ such that $i_1 \leq i_2$, $j_1 \leq j_2$ and $k_1 \leq k_2$.*

Proof. Suppose otherwise that there are distinct maximal atoms $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ in Λ such that $i_1 \leq i_2$, $j_1 \leq j_2$ and $k_1 \leq k_2$. Then we have $u[i_2, \alpha_2] = u[i_1, -\alpha_1]$, $v[j_2, \beta_2] = v[j_1, -\beta_1]$ or $w[k_2, \varepsilon_2] = w[k_1, -\varepsilon_1]$. The arguments for various cases are similar, we only give the proof for the case $u[i_2, \alpha_2] = u[i_1, -\alpha_1]$, $j_1 < j_2$ and $k_1 < k_2$. In this case, it is easy to see that there is a natural homomorphism $f : \mathcal{M}(u \times v \times w) \to \mathcal{M}(u_{i_1} \times v \times w)$ of ω-categories such that $f(u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]) = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ for $i < i_1$ and $f(u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]) = u_{i_1} \times v[j, \beta] \times w[k, \varepsilon]$ for $i \geq i_1$. We are going to use this homomorphism to get a contradiction.

Since λ_1 and λ_2 are maximal in the molecule Λ, it is easy to see that there is a composite of molecules $\Lambda_1 \#_n \Lambda_2$ or $\Lambda_2 \#_n \Lambda_1$ such that λ_1 is a maximal atom in Λ_1 and λ_2 is a maximal atom in Λ_2 and $\lambda_1 \not\subset \Lambda_2$ and $\lambda_2 \not\subset \Lambda_1$. We may assume that $\Lambda_1 \#_n \Lambda_2$ is defined. In this case, we have $d_n^+ \Lambda_1 = d_n^- \Lambda_2 = \Lambda_1 \cap \Lambda_2$. It follows from Lemma 1.4.16 that

$n < \dim \lambda_1 = i_1 + j_1 + k_1$. On the other hand, since $f : \mathcal{M}(u \times v \times w) \to \mathcal{M}(u_{i_1} \times v \times w)$ is a homomorphism, the composite $f(\Lambda_1)\#_n f(\Lambda_2)$ is defined; since $f(\lambda_1) \subset f(\Lambda_1)$ and $f(\lambda_1) = u_{i_1} \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1] \subset u_{i_1} \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2] = f(\lambda_2) \subset f(\Lambda_2)$, we have $f(\lambda_1) \subset f(\Lambda_1) \cap f(\Lambda_2)$; this implies that $n \geq \dim d_n^+ f(\Lambda_1) = \dim(f(\Lambda_1) \cap f(\Lambda_2)) \geq \dim f(\lambda_1) = i_1 + i_2 + i_3$, a contradiction.

This completes the proof.

□

We have now proved that a molecule satisfies condition 1 for pairwise molecular subcomplexes. We next prove that F_I^u, F_J^v and F_K^w send molecules to molecules or $\emptyset$. The arguments for the three maps are similar. We only give the proof that F_J^v sends molecules in $u \times v \times w$ to molecules in $u \times w^J$ or the empty set.

Let v_J be a J-dimensional globe. For Λ a subcomplex in $u \times v_J \times w$, let $g_J^v(\Lambda) = \mathrm{pr}[\Lambda \cap (u \times \{\eta\} \times w)]$, where $\eta \in \mathrm{Int}(v_J)$ and pr is projection onto the first and third factors. Then $g_J^v(\Lambda) \subset u \times w^J$. We are going to show that $g_J^v(\Lambda)$ is a molecule in $M(u \times w^J)$ or the empty set for every molecule Λ.

We first investigate the image of $d_p^\gamma \lambda$ for an atom λ in $u \times v \times w$ under the map g_J^v.

Lemma 2.2.2. *Let $\lambda = u[i, \alpha] \times v_J[j, \beta] \times w[k, \varepsilon]$ be an atom in the ω-complex $u \times v_J \times w$ and $\Lambda, \Lambda' \in \mathcal{M}(u \times v_J \times w)$. Then*

1. *$g_J^v(\lambda) \in \mathcal{A}(u \times w^J) \cup \{\emptyset\}$;*
2. *If $\Lambda \#_n \Lambda'$ is defined, then $g_J^v(\Lambda \#_n \Lambda') = g_J^v(\Lambda) \cup g_J^v(\Lambda')$;*
3. *$g_J^v(\Lambda) \neq \emptyset$ if and only if there is a maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ in Λ such that $j = J$;*
4. $g_J^v(d_p^\gamma \lambda) = \begin{cases} d_{p-J}^\gamma g_J^v(\lambda) & \text{when } p \geq J \text{ and } j = J, \\ \emptyset & \text{when } p < J \text{ or } j < J; \end{cases}$

Proof. The proofs of the first three conditions are trivial verification from the definition of g_J^v. we now verify condition 4.

If $p < J$ or $j < J$, then it is evident that $g_J^v(d_p^\gamma \lambda) = \emptyset$ by the definition of g_J^v.

Now, suppose that $p \geq J$ and $j = J$. Then $g_J^v(\lambda) = u[i, \alpha] \times w^J[k, \varepsilon]$. The set of all maximal atoms in $d_p^\gamma \lambda$ consists of all $u[l, \sigma] \times v[m, \tau] \times w[n, \omega]$ with $l \leq i$, $m \leq J$ and $n \leq k$ by proposition 2.1.2, where the signs σ, τ and ω are determined as follows:

1. if $l = i$, then $\sigma = \alpha$; if $l < i$, then $\sigma = \gamma$;
2. if $m = j$, then $\tau = \beta$; if $m < j$, then $\tau = (-)^l \gamma$;
3. if $n = k$, then $\omega = \varepsilon$; if $n < k$, then $\omega = (-)^{l+m} \gamma$.

From this description and the formation of $d_{p-J}^\gamma(u[i, \alpha] \times w^J[k, \varepsilon])$ in $u \times w^J$, it is easy to see that $g_J^v(d_p^\gamma \lambda) = d_{p-J}^\gamma g_J^v(\lambda)$, as required.

$\square$

Now we can prove that g_J^v sends molecules to molecules or the empty set.

Theorem 2.2.3. *Let $g_J^v : \mathcal{M}(u \times v_J \times w) \to \mathcal{P}(u \times w^J)$ be the map as above. Then*

1. *$g_J^v(\mathcal{M}(u \times v_J \times w)) \subset \mathcal{M}(u \times w^J) \cup \{\emptyset\}$;*
2. *For every molecule Λ in $u \times v_J \times w$, we have*

$$g_J^v(d_p^\gamma \Lambda) = \begin{cases} d_{p-J}^\gamma g_J^v(\Lambda) & \text{when } p \geq J \text{ and } g_J^v(\Lambda) \neq \emptyset, \\ \emptyset & \text{when } p < J \text{ or } g_J^v(\Lambda) = \emptyset. \end{cases}$$

3. *If $\Lambda \#_n \Lambda'$ is defined, then*

$$g_J^v(\Lambda \#_n \Lambda') = \begin{cases} g_J^v(\Lambda) \#_{n-J} g_J^v(\Lambda') & \text{when } g_J^v(\Lambda) \neq \emptyset \text{ and } g_J^v(\Lambda') \neq \emptyset, \\ g_J^v(\Lambda') & \text{when } g_J^v(\Lambda) = \emptyset, \\ g_J^v(\Lambda) & \text{when } g_J^v(\Lambda') = \emptyset. \end{cases}$$

Proof. We are going to prove the first two conditions by induction and then prove the third condition.

By Lemma 2.2.2, it is evident that the first two conditions hold when Λ is an atom.

Now suppose that $q > 1$ and the first two conditions hold for every molecule which can be written as a composite of less than q atoms. Suppose also that Λ is a molecule which can be written as a composite of q atoms. Since $q > 1$, we have a proper decomposition $\Lambda = \Lambda' \#_n \Lambda''$ such that Λ' and Λ'' are molecules. According to the induction hypothesis, we know that the first two conditions hold for Λ' and Λ''. We must show that the first two conditions in the proposition hold for Λ. There are two cases, as follows.

1. Suppose that $g^v_J(\Lambda') = \emptyset$ or $g^v_J(\Lambda'') = \emptyset$. We may assume that $g^v_J(\Lambda') = \emptyset$. We have $g^v_J(\Lambda) = g^v_J(\Lambda'')$. Thus $g^v_J(\Lambda) \in \mathcal{M}(u \times w^J) \cup \{\emptyset\}$ as required by the first condition. Moreover, if $p \neq n$, then

$$\begin{aligned}
& g^v_J(d^\gamma_p \Lambda) \\
= \; & g^v_J(d^\gamma_p \Lambda' \#_n d^\gamma_p \Lambda'') \\
= \; & g^v_J(d^\gamma_p \Lambda') \cup g^v_J(d^\gamma_p \Lambda'') \\
= \; & \begin{cases} d^\gamma_{p-J} g^v_J(\Lambda'') & \text{when } p \geq J \text{ and } g^v_J(\Lambda'') \neq \emptyset, \\ \emptyset & \text{when } g^v_J(\Lambda'') = \emptyset \text{ or } p < J, \end{cases} \\
= \; & \begin{cases} d^\gamma_{p-J} g^v_J(\Lambda) & \text{when } p \geq J \text{ and } g^v_J(\Lambda) \neq \emptyset, \\ \emptyset & \text{when } g^v_J(\Lambda) = \emptyset \text{ or } p < J, \end{cases}
\end{aligned}$$

as required by the second condition. Suppose that $p = n \geq J$. Then $g^v_J(d^+_p \Lambda') = \emptyset$. So $g^v_J(d^-_p \Lambda'') = \emptyset$. Hence, by the hypothesis, one gets $g^v_J(\Lambda'') = \emptyset$. Therefore $g^v_J(\Lambda) = \emptyset$ and $g^v_J(d^\gamma_p \Lambda) = \emptyset$, as required by the second condition.

2. Suppose that $g^v_J(\Lambda') \neq \emptyset$ and $g^v_J(\Lambda'') \neq \emptyset$. Then there is a maximal atom $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ in Λ' and a maximal atom $\lambda'' = u[i'', \alpha''] \times v[j'', \beta''] \times w[k'', \varepsilon'']$ in Λ'' such that $j' = j'' = J$. We claim that $n \geq J$. There are two cases, as follows:

a. Suppose that both λ' and λ'' are maximal in Λ. By Proposition 2.2.1, we have $i' \neq i''$ and $k' \neq k''$. So $\lambda' \cap \lambda'' \subset \Lambda' \cap \Lambda'' = d^+_n \Lambda' = d^-_n \Lambda''$; Since $\lambda' \cap \lambda'' \neq \emptyset$ and $j = j' = J$ and $\dim(d^+_n \Lambda') \leq n$, we can see that $J \leq n$, as required.

b. Suppose that λ' is not maximal in Λ. Then Λ has a maximal atom $\lambda'_1 = u[i'_1, \alpha'_1] \times v[j'_1, \beta'_1] \times w[k'_1, \varepsilon'_1]$ distinct from λ' with $\lambda' \subset \lambda'_1$. Hence $j'_1 = J$. It is easy to see that λ'_1 is maximal in Λ''. So we have $\lambda' \subset \Lambda' \cap \Lambda'' = d^+_n \Lambda' = d^-_n \Lambda''$. Since $\dim(d^+_n \Lambda') \leq n$, we have $J \leq n$, as required.

Now since $g^v_J(\Lambda) = g^v_J(\Lambda') \cup g^v_J(\Lambda'')$, and

$$\begin{aligned} & d^+_{n-J} g^v_J(\Lambda') \\ = \; & g^v_J(d^+_n \Lambda') \\ = \; & g^v_J(d^-_n \Lambda'') \\ = \; & d^-_{n-J} g^v_J(\Lambda''), \end{aligned}$$

we can see that $g^v_J(\Lambda')\#_{n-J} g^v_J(\Lambda'')$ is defined, and $g^v_J(\Lambda) = g^v_J(\Lambda')\#_{n-J} g^v_J(\Lambda'')$. So $g^v_J(\Lambda)$ is a molecule, as required by the first condition. We now verify that Λ satisfies the second condition. If $p < J$, then $g^v_J(d^\gamma_p \Lambda) = \emptyset$, as required. If $p = n \geq J$, then

$$\begin{aligned} & g^v_J(d^-_p \Lambda) \\ = \; & g^v_J(d^-_p \Lambda') \\ = \; & d^-_{p-J} g^v_J(\Lambda') \\ = \; & d^-_{p-J} g^v_J(\Lambda); \end{aligned}$$

and similarly we have $g^v_J(d^+_p \Lambda) = d^+_{p-J} g^v_J(\Lambda)$. If $J \leq p < n$, then

$$\begin{aligned} & g^v_J(d^\gamma_p \Lambda) \\ = \; & g^v_J(d^\gamma_p \Lambda') \\ = \; & d^\gamma_{p-J} g^v_J(\Lambda') \\ = \; & d^\gamma_{p-J} g^v_J(\Lambda) \end{aligned}$$

If $p \geq J$ and $p > n$, then

$$\begin{aligned} & g^v_J(d^\gamma_p \Lambda) \\ = \; & g^v_J(d^\gamma_p \Lambda' \#_n d^\gamma_p \Lambda'') \\ = \; & g^v_J(d^\gamma_p \Lambda') \cup g^v_J(d^\gamma_p \Lambda'') \\ = \; & d^\gamma_{p-J} g^v_J(\Lambda') \cup d^\gamma_{p-J} g^v_J(\Lambda''). \end{aligned}$$

and

$$\begin{aligned} & d^+_{n-J} d^\gamma_{p-J} g^v_J(\Lambda') \\ = \; & d^+_{n-J} g^v_J(\Lambda') \\ = \; & d^-_{n-J} g^v_J(\Lambda'') \\ = \; & d^-_{n-J} d^\gamma_{p-J} g^v_J(\Lambda''), \end{aligned}$$

thus $d^{\gamma}_{p-J}g^{v}_{J}(\Lambda')\#_{n-J}d^{\gamma}_{p-J}g^{v}_{J}(\Lambda'')$ is defined and

$$\begin{aligned} & g^{v}_{J}(d^{\gamma}_{p}\Lambda) \\ = \; & d^{\gamma}_{p-J}g^{v}_{J}(\Lambda')\#_{n-J}d^{\gamma}_{p-J}g^{v}_{J}(\Lambda'') \\ = \; & d^{\gamma}_{p-J}[g^{v}_{J}(\Lambda')\#_{n-J}g^{v}_{J}(\Lambda'')] \\ = \; & d^{\gamma}_{p-J}g^{v}_{J}(\Lambda). \end{aligned}$$

Therefore Λ satisfies the second condition.

Finally, condition 3 can be easily verified by using condition 2 and the fact that g^{v}_{J} preserves unions.

This completes the proof.

□

Recall that there is a natural homomorphism $f^{v}_{J} : \mathcal{M}(u \times v \times w) \to \mathcal{M}(u \times v_J \times w)$ of ω-categories sending every atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ to $u[i,\alpha] \times v_J[j',\beta'] \times w[k,\varepsilon]$, where $v_J[j',\beta'] = v_J[j,\beta]$ whenever $j < J$, and $v_J[j',\beta'] = v_J$ whenever $j \geq J$. According to the definitions of g^{v}_{J} and f^{v}_{J}, it is easy to see that $F^{v}_{J} = g^{v}_{J} \circ f^{v}_{J}$. Thus F^{v}_{J} sends molecules in $u \times v \times w$ to molecules in $u \times w^{J}$ or the empty set.

We can similarly define maps $g^{u}_{I} : \mathcal{M}(u_I \times v \times w) \to v^{I} \times w^{I}$ and $g^{w}_{K} : \mathcal{M}(u \times v \times w_K) \to u \times v$ which send molecules to molecules or the empty set. Moreover, we have natural homomorphisms $f^{u}_{I} : \mathcal{M}(u_I \times v \times w)$ and $f^{w}_{K} : \mathcal{M}(u \times v \times w_K)$ of ω-categories and we can see that $F^{u}_{I} = g^{u}_{I} \circ f^{u}_{I}$ and $F^{w}_{K} = g^{w}_{K} \circ f^{w}_{K}$. Therefore F^{u}_{I} and F^{w}_{K} sends molecules to molecules or the empty set.

We have now proved the following theorem

Theorem 2.2.4. *Molecules in $u \times v \times w$ are pairwise molecular.*

2.3 Properties of Pairwise Molecular Subcomplexes

In this section, we prove some basic properties of pairwise molecular subcomplexes. In the next section, we are going to show that some of these properties characterise pairwise molecular subcomplexes in $u \times v \times w$.

Lemma 2.3.1. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$ and let $u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ be distinct maximal atoms in Λ.*

1. *If $i_1 = i_2$ and $\alpha_1 = -\alpha_2$, then Λ has a maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ with $i > i_1 = i_2$, $v[j, \beta] \supset v[j_1, \beta_1] \cap v[j_2, \beta_2]$ and $k \geq \min\{k_1, k_2\}$.*

2. *If $i_1 = i_2$ and $\alpha_1 = -\alpha_2$, then Λ has a maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ with $i > i_1 = i_2$, $j \geq \min\{j_1, j_2\}$ and $w[k, \varepsilon] \supset w[k_1, \varepsilon_1] \cap w[k_2, \varepsilon_2]$.*

3. *If $j_1 = j_2$ and $\beta_1 = -\beta_2$, then Λ has a maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ with $j > j_1 = j_2$, $u[i, \alpha] \supset u[i_1, \alpha_1] \cap u[i_2, \alpha_2]$ and $k \geq \min\{k_1, k_2\}$.*

4. *If $j_1 = j_2$ and $\beta_1 = -\beta_2$, then Λ has a maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ with $j > j_1 = j_2$, $i \geq \min\{i_1, i_2\}$ and $w[k, \varepsilon] \supset w[k_1, \varepsilon_1] \cap w[k_2, \varepsilon_2]$.*

5. *If $k_1 = k_2$ and $\varepsilon_1 = -\varepsilon_2$, then Λ has a maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ with $k > k_1 = k_2$, $u[i, \alpha] \supset u[i_1, \alpha_1] \cap u[i_2, \alpha_2]$ and $j \geq \min\{j_1, j_2\}$.*

6. *If $k_1 = k_2$ and $\varepsilon_1 = -\varepsilon_2$, then Λ has a maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ with $k > k_1 = k_2$, $i \geq \min\{i_1, i_2\}$ and $v[j, \beta] \supset v[j_1, \beta_1] \cap v[j_2, \beta_2]$.*

Proof. The proof of these conditions are similar, we only prove the second one. Suppose that $i_1 = i_2$ and $\alpha_1 = -\alpha_2$. Let $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$. Let $J = \min\{j_1, j_2\}$. It is evident that $F^v_J(\lambda_1) = u[i_1, \alpha_1] \times w^J[k_1, \varepsilon_1] \subset F^v_J(\Lambda)$ and $F^v_J(\lambda_2) = u[i_2, \alpha_2] \times w^J[k_2, \varepsilon_2] \subset F^v_J(\Lambda)$. Since $F^v_J(\Lambda)$ is a molecule in $u \times w^J$, it follows from the formation of maximal atoms in $F^v_J(\Lambda)$ that $F^v_J(\lambda_1)$ or $F^v_J(\lambda_2)$ is not maximal in $F^v_J(\Lambda)$, and $F^v_J(\Lambda)$ has a maximal atom $\mu = u[l, \sigma] \times w^J[n, \omega]$ with $l > i$ and $w[n, \omega] \supset w[k_1, \varepsilon_1]$ or $w[n, \omega] \supset w[k_2, \varepsilon_2]$. By the definition of F^v_J, it is easy to see that every maximal atom in $F^v_J(\Lambda)$ is an image of a maximal atom in Λ. Therefore Λ has a maximal atom $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ with $u[i, \alpha] = u[l, \sigma]$, $j \geq J$ and $w[k, \varepsilon] = w[n, \omega]$, as required.

This completes the proof. □

The next property says that certain signs in a pair of 'adjacent' maximal atoms of a pairwise molecular subcomplexes are related. Before we prove this property, we need to give the precise definition of adjacency of a pair of maximal atoms.

Definition 2.3.2. Let Λ be a subcomplex. A pair of distinct maximal atoms $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ in Λ is *adjacent* if, for every maximal atom $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ in Λ with $i \geq \min\{i_1, i_2\}$, $j \geq \min\{j_1, j_2\}$ and $k \geq \min\{k_1, k_2\}$, one has

$$\begin{aligned} & \min\{i_1, i\} + \min\{j_1, j\} + \min\{k_1, k\} \\ = \ & \min\{i_1, i_2\} + \min\{j_1, j_2\} + \min\{k_1, k_2\} \end{aligned}$$

or

$$\begin{aligned} & \min\{i_2, i\} + \min\{j_2, j\} + \min\{k_2, k\} \\ = \ & \min\{i_1, i_2\} + \min\{j_1, j_2\} + \min\{k_1, k_2\}. \end{aligned}$$

The following proposition may be helpful to understand the concept of adjacency.

Proposition 2.3.3. *Let Λ be a subcomplex satisfying condition 1 for pairwise molecular subcomplexes. A pair of distinct maximal atoms $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ in Λ is adjacent if and only if the following conditions hold.*

- *If $i_1 = i_2$, then there is no maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ such that $i \geq i_1 = i_2$, $j > \min\{j_2, j_2\}$ and $k > \min\{k_1, k_2\}$.*
- *If $j_1 = j_2$, then there is no maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ such that $j \geq j_1 = j_2$, $i > \min\{i_2, i_2\}$ and $k > \min\{k_1, k_2\}$.*
- *If $k_1 = k_2$, then there is no maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ such that $k \geq k_1 = k_2$, $i > \min\{i_2, i_2\}$ and $j > \min\{j_2, j_2\}$.*
- *If $i_1 > i_2$ and $j_1 > j_2$, then there is no maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ such that $i > i_2$, $j \geq j_2$ and $k > k_1$; and there is no maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ such that $i \geq i_2$, $j > j_2$ and $k > k_1$.*

- *If $i_1 > i_2$ and $k_1 > k_2$, then there is no maximal atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ such that $i > i_2$, $j > j_1$ and $k \geq k_2$; and there is no maximal atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ such that $i \geq i_2$, $j > j_1$ and $k > k_2$.*

- *If $j_1 > j_2$ and $k_1 > k_2$, then there is no maximal atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ such that $i > i_1$, $j > j_2$ and $k \geq k_2$; and there is no maximal atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ such that $i > i_1$, $j \geq j_2$ and $k > k_2$.*

Proof. The proof is a straightforward verification from the definition of adjacency and condition 1 for pairwise molecular subcomplexes. □

Example 2.3.4. For the subcomplex in Example 2.1.5, all the adjacent pairs of maximal atoms are

$$u_8^+ \times v_2^+ \times w_1^- \text{ and } u_5^- \times v_2^+ \times w_5^-;$$

$$u_8^+ \times v_2^+ \times w_1^- \text{ and } u_9^+ \times v_1^- \times w_2^+;$$

$$u_5^- \times v_2^+ \times w_5^- \text{ and } u_1^- \times v_2^+ \times w_8^+;$$

$$u_5^- \times v_2^+ \times w_5^- \text{ and } u_9^+ \times v_1^- \times w_2^+;$$

$$u_5^- \times v_2^+ \times w_5^- \text{ and } u_4^- \times v_1^- \times w_6^+;$$

$$u_5^- \times v_2^+ \times w_5^- \text{ and } u_8^- \times v_0^- \times w_5^-;$$

$$u_5^- \times v_2^+ \times w_5^- \text{ and } u_5^- \times v_0^+ \times w_6^+;$$

$$u_1^- \times v_2^+ \times w_8^+ \text{ and } u_9^+ \times v_1^- \times w_2^+;$$

$$u_1^- \times v_2^+ \times w_8^+ \text{ and } u_4^- \times v_1^- \times w_6^+;$$

$$u_1^- \times v_2^+ \times w_8^+ \text{ and } u_0^+ \times v_1^+ \times w_9^+;$$

$$u_1^- \times v_2^+ \times w_8^+ \text{ and } u_2^- \times v_0^- \times w_9^+;$$

$$u_9^+ \times v_1^- \times w_2^+ \text{ and } u_8^- \times v_0^- \times w_5^-;$$

$$u_4^- \times v_1^- \times w_6^+ \text{ and } u_4^- \times v_0^- \times w_7^+;$$

$$u_0^+ \times v_1^+ \times w_9^+ \text{ and } u_2^- \times v_0^- \times w_9^+;$$

$$u_8^- \times v_0^- \times w_5^- \text{ and } u_5^- \times v_0^+ \times w_6^+;$$

$$u_5^- \times v_0^+ \times w_6^+ \text{ and } u_4^- \times v_0^- \times w_7^+;$$

$$u_4^- \times v_0^- \times w_7^+ \text{ and } u_2^- \times v_0^- \times w_9^+.$$

Let Λ be a subcomplex of $u \times v \times w$ satisfying condition 1 for pairwise molecular subcomplexes. Let J be a fixed non-negative integer. A maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ in Λ is *(v, J)-projection maximal* if $j \geq J$ and there is no maximal atom $u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ with $i' \geq i$, $J \leq j' < j$ and $k' \geq k$.

Similarly, we can define a maximal atom to be *(u, I)-projection maximal* and *(w, K)-projection maximal.*

It is evident that a maximal atom λ in Λ is (v, J)-projection maximal implies that $F_J^v(\lambda)$ is maximal in $F_J^v(\Lambda)$. Conversely, for every maximal atom μ in $F_J^v(\Lambda)$, there is a maximal atom μ' in Λ such that $F_J^v(\mu') = \mu$. The following proposition implies that μ' is actually (v, J)-projection maximal.

Proposition 2.3.5. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$ and λ be a maximal atom in Λ. Then*

1. *λ is (u, I)-projection maximal if and only if $F_I^u(\lambda)$ is maximal in $F_J^v(\Lambda)$.*

2. *λ is (v, J)-projection maximal if and only if $F_J^v(\lambda)$ is maximal in $F_J^v(\Lambda)$.*

3. *λ is (w, K)-projection maximal if and only if $F_K^w(\lambda)$ is maximal in $F_K^w(\Lambda)$.*

Proof. The arguments for the three cases are similar. We only give the proof for the second one.

Suppose that λ is not (v, J)-projection maximal. Let $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$. Then there is a maximal atom $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ in Λ such that $J \leq j' < j$ $i' \geq i$ and $k' \geq k$. By condition 1 for pairwise molecular subcomplexes, we have $i' > i$

or $k' > k$. If $u[i', \alpha'] \supset u[i, \alpha]$ and $w[k', \varepsilon'] \supset w[k, \varepsilon]$, then it is evident that $F_J^v(\lambda) \subsetneqq F_J^v(\mu)$ so that $F_J^v(\lambda)$ is not maximal in $F_J^v(\Lambda)$. Now suppose that $u[i', \alpha'] \not\supset u[i, \alpha]$ or $w[k', \varepsilon'] \not\supset w[k, \varepsilon]$. $u[i', \alpha'] = u[i, -\alpha]$ or $w[k', \varepsilon'] = w[k, -\varepsilon]$. Thus we can get a maximal atom $\lambda'' = u[i'', \alpha''] \times v[j'', \beta''] \times w[k'', \varepsilon'']$ such that $J \leq j'' < j$ and $u[i'', \alpha''] \supset u[i, \alpha]$ and $w[k'', \varepsilon''] \supset w[k, \varepsilon]$ by applying Lemma 2.3.1. It follows that $F_J^v(\lambda)$ is not maximal in $F_J^v(\Lambda)$.

Conversely, suppose that $F_J^v(\lambda)$ is not maximal in $F_J^v(\lambda)$. It follows evidently from the definition that λ is not (v, J)-projection maximal.

This completes the proof.

$\square$

Example 2.3.6. For the subcomplex in Example 2.1.5, there is no (v, J)-projection maximal atoms for $J > 2$. The $(v, 2)$-projection maximal atoms are

$$u_8^+ \times v_2^+ \times w_1^-,$$

$$u_5^- \times v_2^+ \times w_5^-,$$

$$u_1^- \times v_2^+ \times w_8^+.$$

The $(v, 1)$-projection maximal atoms are

$$u_9^+ \times v_1^- \times w_2^+,$$

$$u_5^- \times v_2^+ \times w_5^-,$$

$$u_4^- \times v_1^- \times w_6^+,$$

$$u_1^- \times v_2^+ \times w_8^+,$$

$$u_0^+ \times v_1^+ \times w_9^+.$$

The $(v, 0)$-projection maximal atoms are

$$u_9^+ \times v_1^- \times w_2^+,$$

$$u_8^- \times v_0^- \times w_5^-,$$

$$u_5^- \times v_0^+ \times w_6^+,$$

$$u_4^- \times v_0^- \times w_7^+,$$

$$u_2^- \times v_0^- \times w_9^+.$$

One can similarly work out all the (u, I)-projection maximal atoms and all the (w, K)-projection maximal atoms.

Lemma 2.3.7. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$. Let $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ be a pair of adjacent maximal atoms in Λ.*

1. *If $i_1 > i_2$ and $j_1 < j_2$, then there is a pair of adjacent (w, K)-projection maximal atoms $\lambda_1' = u[i_1', \alpha_1'] \times v[j_1', \beta_1'] \times w[k_1', \varepsilon_1']$ and $\lambda_2' = u[i_2', \alpha_2'] \times v[j_2', \beta_2'] \times w[k_2', \varepsilon_2']$ with $K = \min\{k_1, k_2\}$ such that $u[i_2', \alpha_2'] = u[i_2, \alpha_2]$, $v[j_1', \beta_1'] = v[j_1, \beta_1]$ and $\min\{k_1', k_2'\} = K$.*

2. *If $i_1 > i_2$ and $k_1 < k_2$, then there is a pair of adjacent (v, J)-projection maximal atoms $\lambda_1' = u[i_1', \alpha_1'] \times v[j_1', \beta_1'] \times w[k_1', \varepsilon_1']$ and $\lambda_2' = u[i_2', \alpha_2'] \times v[j_2', \beta_2'] \times w[k_2', \varepsilon_2']$ with $J = \min\{j_1, j_2\}$ such that $u[i_2', \alpha_2'] = u[i_2, \alpha_2]$, $w[k_1', \varepsilon_1'] = w[k_1, \varepsilon_1]$ and $\min\{j_1', j_2'\} = J$.*

3. *If $j_1 > j_2$ and $k_1 < k_2$, then there is a pair of adjacent (u, I)-projection maximal atoms $\lambda_1' = u[i_1', \alpha_1'] \times v[j_1', \beta_1'] \times w[k_1', \varepsilon_1']$ and $\lambda_2' = u[i_2', \alpha_2'] \times v[j_2', \beta_2'] \times w[k_2', \varepsilon_2']$ with $I = \min\{i_1, i_2\}$ such that $v[j_2', \beta_2'] = v[j_2, \beta_2]$, $w[k_1', \varepsilon_1'] = w[k_1, \varepsilon_1]$ and $\min\{i_1', i_2'\} = I$.*

Proof. The arguments for these three cases are similar. We only give the proof for the second case.

Let $\lambda_1' = u[i_1', \alpha_1'] \times v[j_1', \beta_1'] \times w[k_1', \varepsilon_1']$ and $\lambda_2' = u[i_2', \alpha_2'] \times v[j_2', \beta_2'] \times w[k_2', \varepsilon_2']$ be the (v, J)-projection maximal atoms such that $i_t' \geq i_t$ and $k_t' \geq k_t$ for $t = 1, 2$. It follows from Lemma 2.3.1 and the adjacency of λ_1 and λ_2 that $u[i_2', \alpha_2'] = u[i_2, \alpha_2]$, $w[k_1', \varepsilon_1'] = w[k_1, \varepsilon_1]$ and $\min\{j_1', j_2'\} = \min\{j_1, j_2\}$, and λ_1' and λ_2' are adjacent, as required.

This completes the proof. $\square$

Now we can prove the sign conditions for pairwise molecular subcomplexes.

Proposition 2.3.8. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$. Then the following sign conditions hold.*

Sign conditions: for a pair of adjacent maximal atoms $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ in Λ, let $i = \min\{i_1, i_2\}$, $j = \min\{j_1, j_2\}$ and $k = \min\{k_1, k_2\}$.

1. *If $i = i_1 < i_2$ and $j = j_2 < j_1$, then $\beta_2 = -(-)^i \alpha_1$;*

2. *if $i = i_1 < i_2$ and $k = k_2 < k_1$, then $\varepsilon_2 = -(-)^{i+j} \alpha_1$;*

3. *if $j = j_1 < j_2$ and $k = k_2 < k_1$, then $\varepsilon_2 = -(-)^j \beta_1$.*

Proof. Suppose that $i_1 > i_2$ and $k_1 < k_2$. Let $J = \min\{j_1, j_1\}$. We must prove $\varepsilon_1 = -(-)^{i_2+J} \alpha_2$.

According to Lemma 2.3.7, we may assume that λ_1 and λ_2 are (v, J)-projection maximal. It is evident that $F_J^v(\lambda_1) = u[i_1, \alpha_1] \times w^J[k_1, \varepsilon_1]$ and $F_J^v(\lambda_2) = u[i_2, \alpha_2] \times w^J[k_2, \varepsilon_2]$, and they are maximal atoms in the molecule $F_J^v(\Lambda)$. Moreover, by the adjacency of λ_1 and λ_2, we can see that $F_J^v(\lambda_1)$ and $F_J^v(\lambda_2)$ are adjacent maximal atoms in $F_J^v(\Lambda)$. Since $F_J^v(\Lambda)$ is a molecule in $u \times w^J$, we have $\varepsilon_1 = -(-)^{i_2+J} \alpha_2$, as required.

The other cases can be proved similarly.

This completes the proof. □

Compared with the properties for molecules in the product of two globes, there is a new feature caused by the middle factors, as follows.

Proposition 2.3.9. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$. Let $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ be a pair of adjacent maximal atoms in Λ. If $i_1 > i_2$, $k_1 < k_2$ and $\min\{j_1, j_2\} > 0$, then there is a maximal atom $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ such that $j = \min\{j_1, j_2\} - 1$, $i > i_2$ and $k > k_1$.*

Proof. Let $J = \min\{j_1, j_2\}$. According to Lemma 2.3.7, we may assume that λ_1 and λ_2 are (v, J)-projection maximal. There are several cases, as follows.

1. Suppose that both λ_1 and λ_2 are $(v, J-1)$-projection maximal. Then $F_{J-1}(\lambda_1)$ and $F_{J-1}(\lambda_2)$ are maximal atoms in the molecule $F_{J-1}(\Lambda)$ of the ω-complex $u \times w^{J-1}$.

It is evident that $F_{J-1}(\lambda_t) = u[i_t, \alpha_t] \times w^{J-1}[k_t, \varepsilon_t]$ for $t = 1, 2$, and $\varepsilon_1 = -(-)^{i_2+J}\alpha_2$ by Proposition 2.3.8. Hence, according to the formation of molecules in $u \times w^{J-1}$, we can see that $F_{J-1}(\lambda_1)$ and $F_{J-1}(\lambda_2)$ are not adjacent in $u \times w^{J-1}$. So $F_{J-1}(\Lambda)$ has a maximal atom $\mu = u[i, \alpha] \times w^{J-1}[k, \varepsilon]$ with $i > i_2$ and $k > k_1$. It follows that there is a maximal atom $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ such that $F_{J-1}(\lambda) = \mu$ and hence $i > i_2$ and $k > k_1$. By the adjacency of λ_1 and λ_2, we must have $j = J - 1$. Therefore λ is as required.

2. Suppose that λ_1 is not $(v, J-1)$-projection maximal. Then there is a maximal atom $\lambda_1' = u[i_1', \alpha_1'] \times v[j_1', \beta_1'] \times w[k_1', \varepsilon_1']$ with $j_1' \geq J - 1$ such that $i_1' \geq i_1$, $j_1' < j_1$ and $k_1' \geq k_1$. It is evident that $j_1' = J - 1$. If $k_1' > k_1$, then λ_1' is as required. Suppose that $k_1' = k_1$ and $\varepsilon_1' = -\varepsilon_1$. By applying Lemma 2.3.1 to λ_1 and λ_1', one can get a maximal atom as required. The argument is similar if λ_2 is not $(v, J-1)$-projection maximal and $i_2' > i_2$, or if λ_2 is not $(v, J-1)$-projection maximal, $i_2' = i_2$ and $\alpha_2' = -\alpha_2$, where $\lambda_2' = u[i_2', \alpha_2'] \times v[j_2', \beta_2'] \times w[k_2', \varepsilon_2']$ is a maximal atom with $j_2' = J - 1$ such that $i_2' \geq i_2$, $j_2' < j_2$ and $k_2' \geq k_2$. There remains the case that $u[i_2', \alpha_2'] = u[i_2, \alpha_2]$ and $w[k_1', \varepsilon_1'] = w[k_1, \varepsilon_1]$. In this case, since λ_1' and λ_2' are maximal atoms with $j_1' = j_2' = J-1$ and $\varepsilon_1' = -(-)^{i_2'+J}\alpha_2'$, it follows from Proposition 2.3.8 that λ_1' and λ_2' are not adjacent in Λ. Therefore Λ has a maximal atom as required.

This completes the proof. □

Proposition 2.3.10. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$. If Λ has three pairwise adjacent maximal atoms $\lambda_1 = u[i_1, \alpha_1] \times v[j, \beta_1] \times w[k, \varepsilon_1]$ $\lambda_2 = u[i, \alpha_2] \times v[j_2, \beta_2] \times w[k, \varepsilon_2]$ and $\lambda_3 = u[i, \alpha_3] \times v[j, \beta_3] \times w[k_3, \varepsilon_2]$ with $i_1 > i$, $j_2 > j$ and $k_3 > k$, then $\alpha_2 = \alpha_3$ or $\beta_1 = \beta_3$ or $\varepsilon_1 = \varepsilon_2$.*

Proof. Suppose otherwise that $\alpha_2 = -\alpha_3$ and $\beta_1 = -\beta_3$ and $\varepsilon_1 = -\varepsilon_2$. Applying Lemma 2.3.1 to λ_1 and λ_2, one can get a maximal atom $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ with $k' > k$, $u[i', \alpha'] \supset u[i, \alpha_2]$ and $j' \geq j$. Since λ_1 and λ_3 are adjacent, we must have $i' = i$ and $\alpha' = \alpha_2 = -\alpha_3$. Since λ_2 and λ_3 are adjacent, we also have $j' = j$. Note that λ' and λ_3 are distinct, we get a contradiction to the first condition for pairwise molecular subcomplexes.

This completes the proof. □

Proposition 2.3.11. *Let Λ be a pairwise molecular subcomplex in $u \times v \times w$. Let $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ be maximal atoms in Λ.*

1. *If $i_1 < i_2$ and $j_1 > j_2$, and if there is no maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ such that $i > i_1$, $j > j_2$ and $k \geq \min\{k_1, k_2\}$, then $\beta_2 = -(-)^{i_1}\alpha_1$;*

2. *if $i_1 < i_2$ and $k_1 > k_2$, and if there is no maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ such that $i > i_1$, $j \geq \min\{j_1, j_2\}$ and $k > k_2$, then $\varepsilon_2 = -(-)^{i_1+\min\{j_1,j_2\}}\alpha_1$;*

3. *if $j_1 < j_2$ and $k_1 > k_2$, and if there is no maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ such that $i \geq \min\{i_1, i_2\}$, $j > j_1$ and $k > k_2$, then $\varepsilon_2 = -(-)^{j_1}\beta_1$.*

Note 2.3.12. We some times say a pair of maximal atoms as in condition 1 to be $(1, 2)$-*adjacent*; a pair of maximal atoms as in condition 2 to be $(1, 3)$-*adjacent*; and a pair of maximal atoms as in condition 3 to be $(2, 3)$-*adjacent*. It is evident that two maximal atoms are (r, s)-adjacent ($1 \leq r < s \leq 3$) if they are adjacent. However, in general, the reverse is not true. For example, in the pairwise molecular subcomplex in Example 2.1.5, the maximal atoms $u_5^- \times v_2^+ \times w_5^-$ and $u_4^- \times v_0^- \times w_7^+$ are $(1, 2)$-adjacent, but they are not adjacent.

Proof. The arguments for the first and the third cases are similar. We give the proof for the first and the second case.

In the first case, let $\lambda_1' = u[i_1', \alpha_1'] \times v[j_1', \beta_1'] \times w[k_1', \varepsilon_1']$ be the maximal atom in Λ with $i_1' \geq i_1$, $j_1' > j_2$ and $k_1' \geq \min\{k_1, k_2\}$ such that j_1' is minimal; let $\lambda_2' = u[i_2', \alpha_2'] \times v[j_2', \beta_2'] \times w[k_2', \varepsilon_2']$ be the maximal atom in Λ with $i_2' > i_1$, $j_2' \geq j_2$ and $k_2' \geq \min\{k_1, k_2\}$ such that i_2' is minimal. According to the assumption and Lemma 2.3.1, we have $u[i_1', \alpha_1'] = u[i_1, \alpha_1]$ and $v[j_2', \beta_2'] = v[j_2, \beta_2]$. It is evident that λ_1' and λ_2' are adjacent. It follows from the sign condition for λ_1' and λ_2' that $\beta_2 = -(-)^{i_1}\alpha_1$, as required.

In the second case, we claim that λ_1 is adjacent to λ_2 so that $\varepsilon_2 = -(-)^{i_1+\min\{j_1,j_2\}}\alpha_1$, as required. In fact, suppose otherwise that λ_1 and λ_2 are not adjacent. Then $j_1 \neq j_2$. We may assume that $j_1 < j_2$. In this case, there exists a maximal atom $\lambda_1' = u[i_1', \alpha_1'] \times v[j_1', \beta_1'] \times w[k_1', \varepsilon_1']$ such that $i_1' \geq i_1$, $j_1' \geq j_1$ and $k_1' > k_2$. By the assumption, we have $i'_1 = i_1$. By condition 1 for pairwise molecular subcomplexes, we have $j'_1 > j_1$ and $k_2 < k'_1 < k_1$. It follows from Lemma 2.3.9 that there is a maximal atom $\mu = u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ such that $l > i'_1 = i_1$, $m \geq \min\{j'_1, j_2\} - 1 \geq j_1$ and $n > k_2$. This contradicts the assumption.

This completes the proof.

□

2.4 An Alternative Description for Pairwise Molecular Subcomplexes

In this section, we give an alternative description for pairwise molecular subcomplexes of $u \times v \times w$, as follows.

Theorem 2.4.1. *Let Λ be a subcomplex of $u \times v \times w$. Then Λ is pairwise molecular if and only if the following conditions hold.*

1. *There are no distinct maximal atoms $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ and $u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ such that $i \leq i'$, $j \leq j'$ and $k \leq k'$.*

2. *Sign conditions: for a pair of adjacent maximal atoms $\lambda_1 = u[i_1,\alpha_1] \times v[j_1,\beta_1] \times w[k_1,\varepsilon_1]$ and $\lambda_2 = u[i_2,\alpha_2] \times v[j_2,\beta_2] \times w[k_2,\varepsilon_2]$ in Λ, let $i = \min\{i_1, i_2\}$, $j = \min\{j_1, j_2\}$ and $k = \min\{k_1, k_2\}$. If $i = i_1 < i_2$ and $j = j_2 < j_1$, then $\beta_2 = -(-)^i\alpha_1$; if $i = i_1 < i_2$ and $k = k_2 < k_1$, then $\varepsilon_2 = -(-)^{i+j}\alpha_1$; if $j = j_1 < j_2$ and $k = k_2 < k_1$, then $\varepsilon_2 = -(-)^j\beta_1$.*

3. *Let $u[i_1,\alpha_1] \times v[j_1,\beta_1] \times w[k_1,\varepsilon_1]$ and $u[i_2,\alpha_2] \times v[j_2,\beta_2] \times w[k_2,\varepsilon_2]$ be a pair of maximal atoms in Λ. If $i_1 = i_2$ and $\alpha_1 = -\alpha_2$, then Λ has a maximal atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ with $i > i_1 = i_2$, $j \geq \min\{j_1, j_2\}$ and $k \geq \min\{k_1, k_2\}$; if $j_1 = j_2$ and $\beta_1 = -\beta_2$, then Λ has a maximal atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ with $j > j_1 = j_2$, $i \geq \min\{i_1, i_2\}$ and $k \geq \min\{k_1, k_2\}$; if $k_1 = k_2$ and $\varepsilon_1 = -\varepsilon_2$, then Λ has a maximal atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ with $k > k_1 = k_2$, $i \geq \min\{i_1, i_2\}$ and $j \geq \min\{j_1, j_2\}$.*

4. *If Λ has a pair of adjacent maximal atoms $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ with $i_2 < i_1$, $k_1 < k_2$ and $\min\{j_1, j_2\} > 0$, then Λ has a maximal atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ with $i > i_2$, $j = \min\{j_1, j_2\} - 1$ and $k > k_1$.*

5. *If Λ has three pairwise adjacent maximal atoms $\lambda_1 = u[i_1, \alpha_1] \times v[j, \beta_1] \times w[k, \varepsilon_1]$ $\lambda_2 = u[i, \alpha_2] \times v[j_2, \beta_2] \times w[k, \varepsilon_2]$ and $\lambda_3 = u[i, \alpha_3] \times v[j, \beta_3] \times w[k_3, \varepsilon_3]$ with $i_1 > i$, $j_2 > j$ and $k_3 > k$, then $\alpha_2 = \alpha_3$ or $\beta_1 = \beta_3$ or $\varepsilon_1 = \varepsilon_2$.*

Note 2.4.2. In condition 4, it is easy to see that $\beta = -(-)^{i_2}\alpha_2$ and $\varepsilon_1 = -(-)^j\beta$ by sign conditions and condition 3.

In the last section, we have proved that the five conditions in the theorem are necessary for a pairwise molecular subcomplexes. The sufficiency is implied by the following Proposition 2.4.8 and the comments after the proposition.

Some of the following lemmas are preliminaries for the proof of Proposition 2.4.8, while some of them are designed for better understanding the five conditions in Theorem 2.4.1.

Lemma 2.4.3. *Let Λ be a subcomplex of $u \times v \times w$ satisfying the five conditions in Theorem 2.4.1. For a pair of adjacent maximal atoms $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ in Λ, let $i = \min\{i_1, i_2\}$, $j = \min\{j_1, j_2\}$ and $k = \min\{k_1, k_2\}$.*

1. *If $i = i_1 < i_2$ and $j = j_1 < j_2$, then $\beta_1 = (-)^i\alpha_1$;*

2. *if $i = i_1 < i_2$ and $k = k_1 < k_2$, then $\varepsilon_1 = (-)^{i+j}\alpha_1$;*

3. *if $j = j_1 < j_2$ and $k = k_1 < k_2$, then $\varepsilon_2 = (-)^j\beta_1$.*

Proof. Suppose that $i = i_1 < i_2$ and $j = j_1 < j_2$. By condition 1, we have $k_1 > k_2$. It follows from sign conditions that $\varepsilon_2 = -(-)^{i+j}\alpha_1$ and $\varepsilon_2 = -(-)^j\beta_1$. Thus $\beta_1 = (-)^i\alpha_1$, as required.

The other cases can be argued similarly. □

Lemma 2.4.4. *Let Λ be a subcomplex satisfying the five conditions in Theorem 2.4.1. Let $u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ be a pair of maximal atoms in Λ.*

1. *If $i_1 = i_2$, $\alpha_1 = -\alpha_2$, $j_1 < j_2$ and $k_1 > k_2$, then Λ has a maximal atom $u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ such that $i' > i_1 = i_2$, $v[j', \beta'] \supset v[j_1, \beta_1]$ and $w[k', \varepsilon'] \supset w[k_2, \varepsilon_2]$;*

2. *if $j_1 = j_2$, $\beta_1 = -\beta_2$, $i_1 < i_2$ and $k_1 > k_2$, then Λ has a maximal atom $u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ such that $j' > j_1 = j_2$, $u[i', \alpha'] \supset u[i_1, \alpha_1]$ and $w[k', \varepsilon'] \supset w[k_2, \varepsilon_2]$;*

3. *if $k_1 = k_2$, $\varepsilon_1 = -\varepsilon_2$, $i_1 < i_2$ and $j_1 > j_2$, then Λ has a maximal atom $u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ with $k' > k_1 = k_2$, $u[i', \alpha'] \supset u[i_1, \alpha_1]$ and $v[j', \beta'] \supset v[j_2, \beta_2]$.*

Proof. The arguments for the above three cases are similar. We prove only the first case.

Let $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$. Let $i = i_1 = i_2$. Suppose that λ_1 and λ_2 are not adjacent. Then, by the definition of adjacency, Λ has a maximal atom $\lambda_1' = u[i_1', \alpha_1'] \times v[j_1', \beta_1'] \times w[k_1', \varepsilon_1']$ with $i_1' \geq i$, $j_1 < j_1' < j_2$ $k_2 < k_1' < k_1$. If $i_1' > i$, then λ_1' is as required by the lemma. If $i_1' = i$, then $\alpha_1' = -\alpha_1$ or $\alpha_1' = -\alpha_2$. By repeating this process, we can get either a maximal atom as required or a pair of adjacent maximal atoms $\lambda_1'' = u[i_1'', \alpha_1''] \times v[j_1'', \beta_1''] \times w[k_1'', \varepsilon_1'']$ and $\lambda_2'' = u[i_2'', \alpha_2''] \times v[j_2'', \beta_2''] \times w[k_2'', \varepsilon_2'']$ with $i_1'' = i_2'' = i_1 = i_2$, $\alpha_1'' = -\alpha_2''$, $v[j_1, \beta_1] \subset v[j_1'', \beta_1''] \cap v[j_2'', \beta_2'']$ and $w[k_2, \varepsilon_2] \subset w[k_1'', \varepsilon_1''] \cap w[k_2'', \varepsilon_2'']$. In the following proof, we may assume that $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ are adjacent.

Let $\alpha_1 = -\gamma$, $j = j_1 < j_2$, $\beta = \beta_1$, $k = k_2$ and $\varepsilon = \varepsilon_2$. Thus $\alpha_2 = \gamma$, $k = k_2 < k_1$ and $\varepsilon = -(-)^j \beta$. By condition 3, Λ has a maximal atom $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ with $i' > i$, $j' \geq j$ and $k' \geq k$. We choose λ' such that i' is minimal. By condition 1, we have $j' < j_2$ and $k' < k_1$. Since λ_1 and λ_2 are adjacent, we have $j' = j$ or $k' = k$. Now there are two cases, as follows.

1. If $j' = j$ and $k' > k$, we claim that $\beta' = \beta$ which means that λ' is as required.

Indeed, suppose otherwise that $\beta' = -\beta$, then, by condition 3 in Theorem 2.4.1, there is a maximal atom $\lambda'' = u[i'', \alpha''] \times v[j'', \beta''] \times w[k'', \varepsilon'']$ in Λ with $j'' > j$, $i'' \geq i$ and

$k'' \geq k' > k$. This contradicts the adjacency of λ_1 and λ_2.

The argument for the case $j' > j$ and $k' = k$ is similar.

2. Suppose that $j' = j$ and $k' = k$. By the choice of λ', it is easy to see that λ' is adjacent to both λ_1 and λ_2. So $\beta' = -(-)^i\gamma$ and $\varepsilon' = (-)^{i+j}\gamma$. Thus $\varepsilon' = -(-)^j\beta'$. Since λ_1 is adjacent to λ_2, we can see that $\varepsilon_2 = -(-)^j\beta_1$. By condition 5, one has $\beta' = \beta_1$ or $\varepsilon' = \varepsilon_2$. Therefore $\beta' = \beta_1$ and $\varepsilon' = \varepsilon_2$ which means that λ' is as required.

This completes the proof of the lemma. □

Lemma 2.4.5. *Let Λ be a subcomplex of $u \times v \times w$ satisfying conditions 1 and 2 in Theorem 2.4.1 . Then Λ satisfies condition 5 if and only if for any triple of pairwise adjacent maximal atoms $\lambda_1 = u[i_1, \alpha_1] \times v[j, \beta_1] \times w[k, \varepsilon_1]$, $\lambda_2 = u[i, \alpha_2] \times v[j_2, \beta_2] \times w[k, \varepsilon_2]$ and $\lambda_3 = u[i, \alpha_3] \times v[j, \beta_3] \times w[k_3, \varepsilon_2]$ with $i_1 > i$, $j_2 > j$ and $k_3 > k$, there is a maximal atom in Λ containing $u[i, \gamma] \times v[j, (-)^i\gamma] \times w[k, (-)^{i+j}\gamma]$ for $\gamma = +$ or $\gamma = -$.*

Proof. Suppose that Λ satisfies condition 5 in Theorem 2.4.1. Then $\alpha_2 = \alpha_3$ or $\beta_1 = \beta_3$ or $\varepsilon_1 = \varepsilon_2$.

Suppose that $\alpha_2 = \alpha_3$ and let $\gamma = \alpha_1 = \alpha_2$. Then $\beta_1 = -(-)^i\gamma$ and $\varepsilon_1 = -(-)^{i+j}\gamma$ by the sign conditions. If $\beta_3 = (-)^i\gamma$, then $u[i, \gamma] \times v[j, (-)^i\gamma] \times w[k, (-)^{i+j}\gamma] \subset \lambda_3$ and $u[i, -\gamma] \times v[j, -(-)^i\gamma] \times w[k, -(-)^{i+j}\gamma] \subset \lambda_1$, as required. If $\beta_3 = -(-)^i\gamma$, then $\varepsilon_2 = (-)^{i+j}\gamma$ by the sign condition for λ_2 and λ_3. Therefore $u[i, \gamma] \times v[j, (-)^i\gamma] \times w[k, (-)^{i+j}\gamma] \subset \lambda_2$ and $u[i, -\gamma] \times v[j, -(-)^i\gamma] \times w[k, -(-)^{i+j}\gamma] \subset \lambda_1$, as required.

The other cases can be argued similarly.

Conversely, suppose that Λ has a maximal atom $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ containing $u[i, \gamma] \times v[j, (-)^i\gamma] \times w[k, (-)^{i+j}\gamma]$ for $\gamma = +$ or $\gamma = -$. By the pairwise adjacency of λ_1, λ_2 and λ_3, it is easy to see that λ' must be λ_1, λ_2 or λ_3. If $\lambda' = \lambda_1$, then $\beta_1 = (-)^i\gamma$ and $\varepsilon_1 = (-)^{i+j}\gamma$. It follows from the sign condition in Theorem 2.4.1 that $\alpha_2 = \alpha_3 = -\gamma$, as required by condition 5 in Theorem 2.4.1.

The other cases can be argued similarly.

This completes the proof. □

Lemma 2.4.6. *Let Λ be a subcomplex satisfying conditions 1, 2, 3 and 5 in Theorem 2.4.1. Then Λ satisfies condition 4 if and only if, for any pair of adjacent maximal atoms*

$\lambda_1 = u[i_1,\alpha_1] \times v[j_1,\beta_1] \times w[k,\varepsilon]$ and $\lambda_2 = u[i,\alpha] \times v[j_2,\beta_2] \times w[k_2,\varepsilon_2]$ with $i < i_1$, $k < k_2$ and $j = \min\{j_1, j_2\} > 0$, there is a maximal atom containing $u[i,-\alpha] \times v[j-1,-(-)^i\alpha] \times w[k,(-)^{i+j}\alpha]$.

Proof. The necessity is obvious. We now prove the sufficiency. We can assume that $j = j_1 \leq j_2$, and hence $\varepsilon = -(-)^{i+j}\alpha$.

By the assumption, there is a maximal atom $\lambda' = u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ containing $u[i,-\alpha] \times v[j-1,-(-)^i\alpha] \times w[k,(-)^{i+j}\alpha]$. Thus $i' \geq i$, $j' \geq j-1$ and $k' \geq k$. We claim that $j' = j-1$ and hence $\beta' = -(-)^i\alpha$.

Indeed, suppose otherwise that $j' > j-1$. Then $i' = i$ by the adjacency of λ_1 and λ_2. Hence $\alpha' = -\alpha$. Note that the proof of Lemma 2.4.4 does not use condition 4. So, by applying Lemma 2.4.4 to λ_2 and λ', one can get a maximal atom $\lambda_3 = u[i_3,\alpha_3] \times v[j_3,\beta_3] \times w[k_3,\varepsilon_3]$ with $i_3 > i$, $j_3 \geq j$ and $w[k_3,\varepsilon_3] \supset w[k_2,\varepsilon_2] \cap w[k',\varepsilon']$. Since $k_2 > k$, $w[k',\varepsilon'] \supset w[k,(-)^{i+j}\alpha] = w[k,-\varepsilon]$ and $k_2 \neq k'$, we can see that λ_3 is distinct from λ_1 and λ_2. This contradicts the adjacency of λ_1 and λ_2.

Now, if $i' > i$ and $k' > k$, then λ' is as required. Suppose that $i' = i$. Then $\alpha' = -\alpha$ and $k' > k$. Thus, by Lemma 2.4.4, Λ has a maximal atom $\lambda'' = u[i'',\alpha''] \times v[j'',\beta''] \times w[k'',\varepsilon'']$ with $i'' > i$, $j'' = j-1$ (by the adjacency of λ_1 and λ_2), $\beta'' = \beta' = -(-)^i\alpha$ and $k'' \geq k' > k$, with the required property. The argument for the case $k' = k$ is similar.

This completes the proof of the lemma. $\square$

Lemma 2.4.7. *Let Λ be a subcomplex of $u \times v \times w$ satisfying the five conditions in Theorem 2.4.1 . Then*

1. *Every maximal atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ with $j = J$ is (v,J)-projection maximal.*

2. *For every maximal atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ with $j \geq J$, there is a (v,J)-projection maximal atom $u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ such that $u[i,\alpha] \subset u[i',\alpha']$ and $w[k,\varepsilon] \subset w[k',\varepsilon']$.*

3. *All the (v,J)-projection maximal atoms, if exist, can be listed as $\lambda_1, \cdots, \lambda_S$ with $\lambda_s = u[i_s,\alpha_s] \times v[j_s,\beta_s] \times w[k_s,\varepsilon_s]$ such that $i_1 > \cdots > i_S$ and $k_1 < \cdots < k_S$ and $\varepsilon_{s-1} = (-)^{i_s+J}\alpha_s$ for $1 < s \leq S$.*

4. *For two consecutive (v, J)-projection maximal atoms λ_{s-1} and λ_s in the above list, either $j_{s-1} = J$ or $j_s = J$ for $1 < s \leq S$.*

Proof. In this proof, all the maximal atoms refer to maximal atoms with dimension of second factors not less than J.

Condition 1 follows from the definition of projection maximal.

To prove condition 2, suppose that $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ is not (v, J)-projection maximal. Then there is a (v, J)-projection maximal atom $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ such that $i_1 \geq i$, $j_1 < j$ and $k_1 \geq k$. If $u[i_1, \alpha_1] \supset u[i, \alpha]$ and $w[k_1, \varepsilon_1] \supset w[k, \varepsilon]$, then λ_1 is as required. Suppose that $u[i_1, \alpha_1] \not\supset u[i, \alpha]$. Then $i_1 = i$ and $\alpha_1 = -\alpha$. Moreover, we have $k_1 > k$ by condition 1 in Theorem 2.4.1. Hence, by Lemma 2.4.4, there is a maximal atom $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ in Λ such that $i_2 > i$, $j_2 < j$, $v[j_1, \beta_1] \subset v[j_2, \beta_2]$ and $w[k, \varepsilon] \subset w[k_2, \varepsilon_2]$. This shows that $u[i_2, \alpha_2] \supset u[i, \alpha]$ and $w[k_2, \varepsilon_2] \supset w[k, \varepsilon]$ and $j_2 < j$. Therefore condition 2 holds by induction. The argument for the case $w[k_1, \varepsilon_1] \not\supset w[k, \varepsilon]$ is similar.

Condition 4 follows easily from condition 4 in Theorem 2.4.1, while Condition 3 follows easily from the definition of projection maximal and condition 1 and condition 2 (sign conditions) in Theorem 2.4.1.

This completes the proof. □

Proposition 2.4.8. *Let Λ be a subcomplex of $u \times v \times w$ satisfying the five conditions in Theorem 2.4.1. Let the (v, J)-projection maximal atoms in Λ be listed as $\lambda_1, \cdots, \lambda_S$ with $\lambda_s = u[i_s, \alpha_s] \times v[j_s, \beta_s] \times w[k_s, \varepsilon_s]$ for $1 \leq s \leq S$ such that $i_1 > \cdots > i_S$ and $k_1 < \cdots < k_S$. Then $F^v_J(\Lambda) = u[i_1, \alpha_1] \times w^J[k_1, \varepsilon_1] \cup \cdots \cup u[i_S, \alpha_S] \times w^J[k_S, \varepsilon_S]$.*

Proof. This is a direct consequence of Proposition 2.1.3 for F^v_J and Lemma 2.4.7. □

Corollary 2.4.9. *Let Λ be a subcomplex of $u \times v \times w$ satisfying the five conditions in Theorem 2.4.1. Then $F^v_J(\Lambda)$ is a molecule in $u \times w^J$ or the empty set for every non-negative integer J.*

We can similarly show that $F^u_I(\Lambda)$ and $F^w_K(\Lambda)$ are molecules or the empty set for a subcomplex Λ of $u \times v \times w$ satisfying the five conditions in Theorem 2.4.1. This completes the proof of Theorem 2.4.1.

2.5 Sources and Targets of Pairwise Molecular Subcomplexes

In this section, we study source and target operators d_p^γ on pairwise molecular subcomplexes in $u \times v \times w$. The main result in this section is that $d_p^\gamma \Lambda$ is pairwise molecular for every pairwise molecular subcomplex Λ of $u \times v \times w$.

Recall that $d_p^\gamma \Lambda$ is a union of interiors of atoms. We first prove that $d_p^\gamma \Lambda$ is a subcomplex of Λ.

Lemma 2.5.1. *Let Λ be a subcomplex of $u \times v \times w$ and $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ be a p-dimensional atom in Λ with* $\operatorname{Int}\lambda \subset d_p^\gamma \Lambda$.

1. *If there is an atom $\lambda' = u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ in Λ such that $\lambda \subset \lambda'$ and $i' > i$, then $\alpha = \gamma$.*
2. *If there is an atom $\lambda' = u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ in Λ such that $\lambda \subset \lambda'$ and $j' > j$, then $\beta = (-)^i \gamma$.*
3. *If there is an atom $\lambda' = u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ in Λ such that $\lambda \subset \lambda'$ and $k' > k$, then $\varepsilon = (-)^{i+j} \gamma$.*

Proof. Suppose that there is an atom $\lambda' = u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ in Λ such that $\lambda \subset \lambda'$ and $i' > i$. Then $\lambda \subset u[i+1,\alpha'] \times v[j,\beta] \times w[k,\varepsilon] \subset \Lambda$ and $\dim(u[i+1,\alpha'] \times v[j,\beta] \times w[k,\varepsilon]) = p+1$. Since $\operatorname{Int}\lambda \subset d_p^\gamma \Lambda$, we have $\lambda \subset d_p^\gamma(u[i+1,\alpha'] \times v[j,\beta] \times w[k,\varepsilon])$. It follows easily from Lemma 2.1.2 that $\alpha = \gamma$, as required.

The arguments for other cases are similar. □

Proposition 2.5.2. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$. Let $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ be a p-dimensional atom such that* $\operatorname{Int}\lambda \subset d_p^\gamma \Lambda$.

1. *If there is a maximal atom λ' in Λ such that $i' > i$, $j' \geq j$ and $k' \geq k$, then $\alpha = \gamma$;*
2. *if there is a maximal atom λ' in Λ such that $i' \geq i$, $j' > j$ and $k' \geq k$, then $\beta = (-)^i \gamma$;*

3. *if there is a maximal atom λ' in Λ such that $i' \geq i$, $j' \geq j$ and $k' > k$, then $\varepsilon = (-)^{i+j}\gamma$.*

Proof. The arguments for the three cases are similar. We give the proof for the first case.

Since $\operatorname{Int}\lambda \subset \Lambda$, there is a maximal atom $\mu = u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ such that $\lambda \subset \mu$. If μ can be chosen such that $l > i$, then we have $\alpha = \gamma$ by Lemma 2.5.1 , as required. In the following proof, we may assume that μ cannot be chosen such that $l > i$ so that $u[l,\sigma] = u[i,\alpha]$.

Suppose that there is a maximal atom $\lambda' = u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ such that $i' > i$ $j' \geq j$ and $k' \geq k$. Then we have $v[j',\beta'] = v[j,-\beta]$ or $w[k',\varepsilon'] = w[k,-\varepsilon]$. By applying Lemma 2.4.4, we may assume that $v[j',\beta'] = v[j,-\beta]$ and $m > j$, or assume that $w[k',\varepsilon'] = w[k,-\varepsilon]$ and $n > k$.

Suppose that $w[k',\varepsilon'] = w[k,-\varepsilon]$ and $n > k$. Then $\varepsilon = (-)^{i+j}\gamma$ by Lemma 2.5.1. If $\min\{j',m\} = j$, and if λ' is $(1,3)$-adjacent to μ, then $\alpha = \sigma = -(-)^{i+j}\varepsilon' = \gamma$ by Proposition 2.3.11, as required. Otherwise, by the definition of adjacency or condition 4 in Theorem 2.4.1, we may chose λ' and μ such that $\min\{k',n\} > k$ so that $v[j',\beta'] = v[j,-\beta]$; according Lemma 2.4.4, we may also assume that $m > j$; thus $\beta = (-)^i\gamma$. In this case, according to the assumptions, λ' must be $(1,2)$-adjacent to μ. It follows from Proposition 2.3.11 that $\alpha = \sigma = -(-)^i\beta' = \gamma$, as required.

Suppose that $v[j',\beta'] = v[j,-\beta]$ and $m > j$. Then we can get $\alpha = \gamma$, as required, by a similar argument.

This completes the proof.

□

Lemma 2.5.3. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$. Let $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ be a $p-1$ dimensional atom such that* $\operatorname{Int}\lambda \subset d_p^\gamma\Lambda$.

1. *If there is a maximal atom λ' in Λ with $\lambda' \supset \lambda$ such that $i' > i$ and $j' > j$, then $\alpha = \gamma$ or $\beta = -(-)^i\gamma$;*

2. *if there is a maximal atom λ' in Λ with $\lambda' \supset \lambda$ such that $i' > i$ and $k' > k$, then $\alpha = \gamma$ or $\varepsilon = -(-)^{i+j}\gamma$;*

3. if there is a maximal atom λ' in Λ with $\lambda' \supset \lambda$ such that $j' > j$ and $k' > k$, then $\beta = (-)^i\gamma$ or $\varepsilon = -(-)^{i+j}\gamma$.

Proof. The arguments for the three cases are similar. We give the proof for the first one.

Suppose that there is a maximal atom λ' in Λ with $\lambda' \supset \lambda$ such that $i' > i$ and $j' > j$. Then $\lambda \subset u[i+1, \alpha'] \times v[j+1, \beta'] \times w[k, \varepsilon] \subset \Lambda$. Since $\operatorname{Int}\lambda \subset d_p^\gamma \Lambda$, we have $\lambda \subset d_p^\gamma(u[i+1, \alpha'] \times v[j+1, \beta'] \times w[k, \varepsilon])$. It follows easily from Lemma 2.1.2 that $\alpha = \gamma$ or $\beta = -(-)^i\gamma$, as required.

This completes the proof.

□

Proposition 2.5.4. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$. Let $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ be a $p-1$ dimensional atom such that* $\operatorname{Int}\lambda \subset d_p^\gamma \Lambda$.

1. If there is a maximal atom λ' in Λ such that $i' > i$, $j' > j$ and $k' \geq k$, then $\alpha = \gamma$ or $\beta = -(-)^i\gamma$;

2. if there is a maximal atom λ' in Λ such that $i' > i$, $j' \geq j$ and $k' > k$, then $\alpha = \gamma$ or $\varepsilon = -(-)^{i+j}\gamma$;

3. if there is a maximal atom λ' in Λ such that $i' \geq i$, $j' > j$ and $k' > k$, then $\beta = (-)^i\gamma$ or $\varepsilon = -(-)^{i+j}\gamma$.

Proof. The arguments for case 1 and case 3 are similar. We give the proofs for case 1 and case 2.

1. Suppose that there is a maximal atom $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ in Λ such that $i' > i$, $j' > j$ and $k' \geq k$. If λ' can be chosen such that $\lambda' \supset \lambda$, then we have $\alpha = \gamma$ or $\beta = -(-)^i\gamma$, as required, by Lemma 2.5.3. In the following, we assume that λ' cannot be chosen such that $\lambda' \supset \lambda$ so that $w[k', \varepsilon'] = w[k, -\varepsilon]$. Let $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ be a maximal atom in Λ such that $\lambda \subset \lambda_1$. Then $u[i_1, \alpha_1] = u[i, \alpha]$ or $v[j_1, \beta_1] = v[j, \beta]$ by the assumption. According to Lemma 2.4.4, we may assume that $k_1 > k$. Now there are several cases, as follows.

Suppose that λ_1 cannot be chosen such that $i_1 > i$ or $j_1 > j$. According to Lemma 2.4.4, it is easy to see that λ_1 and λ' are adjacent. Thus $\alpha = \gamma$ (when $\varepsilon = -(-)^{i+j}\gamma$) or $\beta = -(-)^i\gamma$ (when $\varepsilon = (-)^{i+j}\gamma$) by sign conditions, as required.

Suppose that λ_1 can be chosen such that $i_1 > i$. Suppose also that $\alpha = -\gamma$. Then $v[j_1, \beta_1] = v[j, \beta]$ by the assumptions. According to Lemma 2.5.3, it is easy to see that $\varepsilon = -(-)^{i+j}\gamma$, hence $\varepsilon' = (-)^{i+j}\gamma$. It is evident that λ_1 and λ' are $(2,3)$-adjacent. It follows from Proposition 2.3.11 that $\beta = \beta_1 = -(-)^i\gamma$, as required.

Suppose that λ_1 can be chosen such that $j_1 > j$ and that λ_1 cannot be chosen such that $i_1 > i$. Suppose also that $\beta = (-)^i\gamma$. According to Lemma 2.4.4, condition 4 in Theorem 2.4.1 and the assumptions, it is easy to see that λ_1 and λ' are adjacent and $\min\{j', j_1\} = j + 1$. It follows from condition 4 in Theorem 2.4.1 that there is a maximal atom $\lambda'' = u[i'', \alpha''] \times v[j'', \beta''] \times w[k'', \varepsilon'']$ such that $i'' > i$, $j'' = j$ and $k'' > k$. Moreover, we have $\beta'' = -(-)^i\alpha$ by Note 2.4.2. By the assumptions, we have $\beta'' = -\beta = -(-)^i\gamma$. It follows that $\alpha = \gamma$, as required.

This completes the proof for case 1.

2. Suppose that there is a maximal atom λ' in Λ such that $i' > i$, $j' \geq j$ and $k' > k$. If λ' can be chosen such that $\lambda' \supset \lambda$, then we have $\alpha = \gamma$ or $\varepsilon = -(-)^i\gamma$, as required, by Lemma 2.5.3. In the following, we assume that λ' cannot be chosen such that $\lambda' \supset \lambda$ so that $v[j', \beta'] = v[j, -\beta]$. Let $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ be a maximal atom in Λ such that $\lambda \subset \lambda_1$. Then $u[i_1, \alpha_1] = u[i, \alpha]$ or $w[k_1, \varepsilon_1] = w[k, \varepsilon]$ by the assumption. According to Lemma 2.4.4, we may assume that $j_1 > j$. Now there are several cases, as follows.

Suppose that λ_1 cannot be chosen such that $i_1 > i$ or $k_1 > k$. Then it is easy to see that λ_1 adjacent to λ'. It follows from sign conditions that $\alpha = \gamma$ (when $\beta' = -(-)^i\gamma$) or $\varepsilon = -(-)^{i+j}\gamma$ (when $\beta' = (-)^i\gamma$), as required.

Suppose that λ_1 can be chosen such that $i_1 > i$. Suppose also that $\alpha = -\gamma$. Then $\beta = -(-)^i\gamma$ by Lemma 2.5.3, and hence $\beta' = -\beta = (-)^i\gamma$. Moreover, we can see that λ' and λ_1 are $(2,3)$-adjacent. It follows from Lemma 2.3.11 that $\varepsilon = -(-)^{i+j}\gamma$, as required.

Suppose that λ_1 can be chosen such that $k_1 > k$. By a similar argument as in the above case, we can get $\alpha = \gamma$ or $\varepsilon = -(-)^{i+j}\gamma$, as required.

This completes the proof.

$\square$

Lemma 2.5.5. *Let x be a union of interiors of atoms in an ω-complex. Then x is a subcomplex if and only if for every atom a in x with* $\operatorname{Int} a \subset x$ *and every atom b with $b \subset a$, one has* $\operatorname{Int} b \subset x$.

Proof. The necessity is evident. To prove the sufficiency, it suffices to prove that for every atom a with $\operatorname{Int} a \subset x$ we have $a \subset x$. Note that a can be written as a union of interiors of atoms b with $b \subset a$. The sufficiency follows. $\square$

Proposition 2.5.6. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$. Then $d_p^\gamma \Lambda$ is a subcomplex.*

Proof. From Lemma 1.2.10, we have already known that $d_p^\gamma \Lambda$ is a union of interiors of atoms. By Lemma 2.5.5, it suffices to prove that for every atom λ with $\operatorname{Int} \lambda \subset d_p^\gamma \Lambda$ and every atom λ_1 with $\lambda_1 \subset \lambda$, one has $\operatorname{Int} \lambda_1 \subset d_p^\gamma \Lambda$. It is evident that there is a sequence $\lambda \supset \lambda_1^1 \supset \lambda_1^2 \cdots \supset \lambda_1$ such that the difference of the dimensions of any pair of consecutive atoms is 1. We may assume that $\dim \lambda_1 = \dim \lambda - 1$.

Let $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$. Since $\operatorname{Int} \lambda \subset d_p^\gamma \Lambda \subset \Lambda$ and Λ is a subcomplex, we have $\lambda_1 \subset \lambda \subset \Lambda$ and $\dim \lambda_1 \leq \dim \lambda \leq p$. Suppose that $\mu = u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ is an atom with $\dim \mu = p + 1$ and $\lambda_1 \subset \mu \subset \Lambda$. We must prove $\lambda_1 \subset d_p^\gamma \mu$.

If $\lambda \subset \mu$, then $\lambda_1 \subset \lambda \subset d_p^\gamma \mu$ since $\lambda \subset d_p^\gamma \Lambda$. If $l > i+1$ or $m > j+1$ or $n > k+1$, then we evidently have $\lambda_1 \subset d_p^\gamma \mu$ by Lemma 2.1.2. In the following, we may further assume that $\lambda \not\subset \mu$ and that $l \leq i+1$ and $m \leq j+1$ and $n \leq k+1$. Thus $u[i,\alpha] \not\subset u[l,\sigma]$ or $v[j,\beta] \not\subset v[m,\tau]$ or $w[k,\varepsilon] \not\subset w[n,\omega]$; moreover, if $u[i,\alpha] \not\subset u[l,\sigma]$, then we have $u[l,\sigma] = u[i,-\alpha]$ or $u[l,\sigma] = u[i-1,\sigma]$, we also have $v[j,\beta] \subset v[m,\tau]$ and $w[k,\varepsilon] \subset w[n,\omega]$; if $v[j,\beta] \not\subset v[m,\tau]$, then we have $v[m,\tau] = v[j,-\beta]$ or $v[m,\tau] = v[j-1,\tau]$, we also have $u[i,\alpha] \subset u[l,\sigma]$ and $w[k,\varepsilon] \subset w[n,\omega]$; if $w[k,\varepsilon] \not\subset w[n,\omega]$, then we have $w[n,\omega] = w[k,-\varepsilon]$ or $w[n,\omega] = u[k-1,\omega]$, we also have $u[i,\alpha] \subset u[l,\sigma]$ and $v[j,\beta] \subset v[m,\tau]$; Note that $\dim \mu = p+1$ and $\dim \lambda \leq p$, we now have 3 cases, as follows.

1. Suppose that $u[l,\sigma]=u[i,-\alpha]$ or $v[m,\tau]=v[j,-\beta]$ or $w[n,\omega]=w[k,-\varepsilon]$; suppose also that $\dim\lambda=p$. Then only one of the equations $l=i+1$, $m=j+1$ and $n=k+1$ holds. The arguments for the three cases are similar, we only give the proof for the case $v[m,\tau]=v[j,-\beta]$ and $\dim\lambda=p$. In this case, we must have $\mu=u[i+1,\sigma]\times v[j,-\beta]\times w[k,\varepsilon]$ or $\mu=u[i,\alpha]\times v[j,-\beta]\times w[k+1,\omega]$. Hence λ_1 is of the form $\lambda_1=u[i,\alpha]\times v[j-1,\tilde{\beta}]\times w[k,\varepsilon]$.

Suppose that $\mu=u[i+1,\sigma]\times v[j,-\beta]\times w[k,\varepsilon]$. Then there is a maximal atom $\mu'=u[l',\sigma']\times v[m',\tau']\times w[n',\omega']$ such that $l'>i$, $m'\geq j$ and $n'\geq k$. It follows from Proposition 2.5.2 that $\alpha=\gamma$. This implies $\lambda_1\subset d_p^{\gamma}\mu$, as required.

Suppose that $\mu=u[i,\alpha]\times v[j,-\beta]\times w[k+1,\omega]$. Then there is a maximal atom $\mu'=u[l',\sigma']\times v[m',\tau']\times w[n',\omega']$ such that $l'\geq i$, $m'\geq j$ and $n'>k$. It follows from Proposition 2.5.2 that $\varepsilon=(-)^{i+j}\gamma$. This implies $\lambda_1\subset d_p^{\gamma}\mu$, as required.

2. Suppose that $l=i-1$ or $m=j-1$ or $n=k-1$; suppose also that $\dim\lambda=p$. The arguments for these three cases are similar. We only give the proof for the case $m=j-1$ and $\dim\lambda=p$. In this case, we have $l=i+1$ and $n=k+1$ because $\dim\mu=p+1$; we also have $\lambda_1=u[i,\alpha]\times v[j-1,\tau]\times w[k,\varepsilon]$. To get $\lambda_1\subset d_p^{\gamma}\mu$, by Lemma 2.1.2, it suffices to prove that $\alpha=\gamma$ or $\varepsilon=(-)^{i+j}\gamma$.

Let $\lambda'=u[i',\alpha']\times v[j',\beta']\times w[k',\varepsilon']$ be a maximal atom in Λ such that $\lambda\subset\lambda'$. Let $\mu'=u[l',\sigma']\times v[m',\tau']\times w[n',\omega']$ be a maximal atom in Λ such that $\mu\subset\mu'$. If λ' can be chosen such that $i'>i$ or $k'>k$, then we have $\alpha=\gamma$ or $\varepsilon=(-)^{i+j}\gamma$ which implies that $\lambda_1\subset d_p^{\gamma}\mu$, as required. If there is a maximal atom $\mu''=u[l'',\sigma'']\times v[m'',\tau'']\times w[n'',\omega'']$ with $\mu''\supset\lambda_1$ and $m''>m$ such that $l''>i$ or $n''>k$, then, by Proposition 2.5.2, we have $\alpha=\gamma$ or $\varepsilon=(-)^{i+j}\gamma$ which implies that $\lambda_1\subset d_p^{\gamma}\mu$, as required. Now suppose that λ' cannot be chosen such that $i'>i$ or $k'>k$. Suppose also that there is no maximal atom $\mu''=u[l'',\sigma'']\times v[m'',\tau'']\times w[n'',\omega'']$ with $\mu''\supset\lambda_1$ and $m''>m$ such that $l''>i$ or $n''>k$. Then $u[i',\alpha']=u[i,\alpha]$, $w[k',\varepsilon']=w[k,\varepsilon]$ and $v[m',\tau']=v[m,\tau]$. Moreover, it is easy to see that λ' and μ' are adjacent. It follows from the sign condition for λ' and μ' that $\alpha=\gamma$ (when $\tau=-(-)^{i}\gamma$) or $\varepsilon=(-)^{i+j}\gamma$ (when $\tau=-(-)^{i}\gamma$). This implies that $\lambda_1\subset d_p^{\gamma}\mu$, as required.

3. Suppose that $u[l,\sigma]=u[i,-\alpha]$ or $v[m,\tau]=v[j,-\beta]$ or $w[n,\omega]=w[k,-\varepsilon]$; suppose also that $\dim\lambda=p-1$. The arguments for these three cases are similar. We only give the proof for the case $v[m,\tau]=v[j,-\beta]$ and $\dim\lambda=p-1$. In this case, we have $l=i+1$ and $n=k+1$. Moreover, we can see that λ_1 is of the form $\lambda_1=u[i,\alpha]\times v[j-1,\tilde{\beta}]\times w[k,\varepsilon]$. According to Lemma 2.5.4, we have $\alpha=\gamma$ or $\varepsilon=-(-)^{i+j}\gamma$. This implies that $\lambda_1\subset d_p^\gamma\mu$, as required.

This completes the proof. $\square$

We can now start to prove that $d_p^\gamma\Lambda$ is pairwise molecular for a molecular subcomplex Λ in $u\times v\times w$ by verifying conditions in Definition 2.1.4.

By Lemma 1.2.10, the maps F_I^u, F_J^v and F_K^w are defined on $d_p^\gamma\Lambda$ for every subcomplex Λ of $u\times v\times w$.

Proposition 2.5.7. *Let Λ be a pairwise molecular subcomplex of $u\times v\times w$. If $p\geq J$ and $F_J^v(\Lambda)\neq\emptyset$, then $F_J^v(d_p^\gamma\Lambda)=d_{p-J}^\gamma F_J^v(\Lambda)$; therefore $F_J^v(d_p^\gamma\Lambda)$ is a molecule in $u\times w^J$.*

Proof. Firstly, we prove that $d_{p-J}^\gamma F_J^v(\Lambda)\subset F_J^v(d_p^\gamma\Lambda)$.

Let $u[i,\alpha]\times w^J[k,\varepsilon]$ be an atom in $u\times w^J$ such that $\operatorname{Int}(u[i,\alpha]\times w^J[k,\varepsilon])\subset d_{p-J}^\gamma F_J^v(\Lambda)$. We must show that $\operatorname{Int}(u[i,\alpha]\times w^J[k,\varepsilon])\subset F_J^v(d_p^\gamma\Lambda)$. Clearly, we have $u[i,\alpha]\times w^J[k,\varepsilon]\subset F_J^v(\Lambda)$. So it is easy to see that $u[i,\alpha]\times v[J,\beta]\times w[k,\varepsilon]\subset\Lambda$ for some sign β. We are going to prove $\operatorname{Int}(u[i,\alpha]\times v[J,\beta]\times w[k,\varepsilon])\subset d_p^\gamma\Lambda$ by verifying conditions in Lemma 1.2.11. It is evident that $\dim(u[i,\alpha]\times v[J,\beta]\times w[k,\varepsilon])\leq p$. To verify the other conditions, we consider two cases, as follows.

1. Suppose that β can be chosen such that $\beta=(-)^i\gamma$. Suppose also that there is an atom $u[i',\alpha']\times v[j',\beta']\times w[k',\varepsilon']\subset\Lambda$ such that $u[i,\alpha]\times v[j,\beta]\times w[k,\varepsilon]\subset u[i',\alpha']\times v[j',\beta']\times w[k',\varepsilon']$. Then $u[i,\alpha]\times w^J[k,\varepsilon]\subset u[i',\alpha']\times w^J[k',\varepsilon']$ in $u\times w^J$. Therefore $u[i,\alpha]\times w^J[k,\varepsilon]\subset d_{p-J}^\gamma(u[i',\alpha']\times w^J[k',\varepsilon'])$. It follows easily that $u[i,\alpha]\times v[J,\beta]\times w[k,\varepsilon]\subset d_p^\gamma(u[i',\alpha']\times v[j',\beta']\times w[k',\varepsilon'])$, as required by the second condition of Lemma 1.2.11.

2. Suppose that β cannot be chosen such that $\beta=(-)^i\gamma$. Suppose also that there is an atom $u[i',\alpha']\times v[j',\beta']\times w[k',\varepsilon']\subset\Lambda$ such that $u[i,\alpha]\times v[j,\beta]\times w[k,\varepsilon]\subset u[i',\alpha']\times v[j',\beta']\times w[k',\varepsilon']$. Then $j'=J$ and $\beta'=\beta=-(-)^i\gamma$ from Lemma 2.5.1. By an

argument similar to the above case, it is easy to see that $u[i,\alpha] \times v[J,\beta] \times w[k,\varepsilon] \subset d_p^\gamma(u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon'])$, as required by the second condition of Lemma 1.2.11.

We have now shown that $\mathrm{Int}(u[i,\alpha] \times v[J,\beta] \times w[k,\varepsilon]) \subset d_p^\gamma\Lambda$. It follows that $\mathrm{Int}(u[i,\alpha] \times w^J[k,\varepsilon]) = F_J^v(\mathrm{Int}(u[i,\alpha] \times v[J,\beta] \times w[k,\varepsilon])) \subset F_J^v(d_p^\gamma\Lambda)$. This completes the proof that $d_{p-J}^\gamma F_J^v(\Lambda) \subset F_J^v(d_p^\gamma\Lambda)$.

Conversely, let $\lambda = u[i,\alpha] \times w^J[k,\varepsilon]$ be an atom such that $\mathrm{Int}\,\lambda \subset F_J^v(d_p^\gamma\Lambda)$. We must show that $\mathrm{Int}\,\lambda \subset d_{p-J}^\gamma F_J^v(\Lambda)$. It is easy to see that there is an atom $u[i,\alpha]\times v[j,\beta]\times w[k,\varepsilon]$ in Λ such that $\mathrm{Int}(u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]) \subset d_p^\gamma\Lambda$ and $j \geq J$. Since $d_p^\gamma\Lambda$ is a subcomplex of $u\times v\times w$, we can see that $u[i,\alpha]\times v[J,\beta']\times w[k,\varepsilon] \subset d_p^\gamma\Lambda$ for some sign β'. It follows that $\dim\lambda \leq p-J$. Clearly, we have $\lambda \subset F_J^v(\Lambda)$. This shows that the first condition of Lemma 1.2.11 is satisfied. To verify the other condition of 1.2.11, let $\mu = u[l,\sigma] \times w^J[n,\omega]$ be an atom in $F_J^v(\Lambda)$ such that $\lambda \subset \mu$ and $\dim\mu = p-J+1$. We must prove that $\lambda \subset d_{p-J}^\gamma\mu$. It is evident that there is an atom $u[l,\sigma]\times v[J,\tau']\times w[n,\omega]$ in Λ for some sign τ'. If $l > i+1$ or $n > k+1$, then it is evident that $\lambda \subset d_p^\gamma\mu$, as required. In the following proof, we may assume that $l \leq i+1$ and $n \leq k+1$ so that $\dim\lambda = p-J$ or $\dim\lambda = p-J-1$. Now there are various cases, as follows.

Suppose that β' and τ' can be chosen such that $\beta' = \tau'$. Then $u[i,\alpha] \times v[J,\beta'] \times w[k,\varepsilon] \subset (d_p^\gamma\Lambda \cap (u[l,\sigma]\times v[J,\tau']\times w[n,\omega]) \subset d_p^\gamma(u[l,\sigma]\times v[J,\tau']\times w[n,\omega])$ by Proposition 1.2.6. It follows easily that $\lambda \subset d_{p-J}^\gamma\mu$, as required.

Suppose that β' and τ' cannot be chosen such that $\beta' = \tau'$. Suppose also that $J > 0$. Since $d_p^\gamma\Lambda$ is a subcomplex, we know that $u[i,\alpha]\times v[J-1,\pm]\times w[k,\varepsilon] \subset d_p^\gamma\Lambda$. This implies that $u[i,\alpha] \times v[J-1,\pm] \times w[k,\varepsilon] \subset d_p^\gamma(u[l,\sigma] \times v[J,\tau'] \times w[n,\omega])$. It follows easily that $\lambda \subset d_{p-J}^\gamma\mu$, as required.

There remain the case that $J = 0$ and β' and τ' cannot be chosen such that $\beta' = \tau'$. If $\dim\lambda = p$, by Proposition 2.5.2, we can get $\alpha = \gamma$ when $l > i$, while $\varepsilon = (-)^i\gamma$ when $n > k$; thus $\lambda \subset d_p^\gamma\mu$, as required. If $\dim\lambda = p-1$, then $l = i+1$ and $n = k+1$; by Proposition 2.5.4, we can get $\alpha = \gamma$ or $\varepsilon = -(-)^i\gamma$; thus $\lambda \subset d_p^\gamma\mu$, as required.

This completes the proof.

□

We can prove the following two results by similar arguments.

Proposition 2.5.8. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$. If $p \geq I$ and $F_I^u(\Lambda) \neq \emptyset$, then $F_I^u(d_p^\gamma \Lambda) = d_{p-I}^\gamma F_I^u(\Lambda)$.*

Proposition 2.5.9. *Let Λ be a pairwise molecular subcomplex of $u \times v \times w$. If $p \geq K$ and $F_K^w(\Lambda) \neq \emptyset$, then $F_K^w(d_p^\gamma \Lambda) = d_{p-K}^\gamma F_K^w(\Lambda)$.*

We also need to show that $d_p^\gamma \Lambda$ satisfies condition 1 for pairwise molecular subcomplexes for a pairwise molecular subcomplex Λ.

Lemma 2.5.10. *Let Λ be a pairwise molecular subcomplex. Then there are no distinct maximal atoms $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ and $\lambda' = u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ in $d_p^\gamma \Lambda$ such that $i \leq i'$, $j \leq j'$ and $k \leq k'$.*

Proof. Let $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ and $\lambda' = u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ be a pair of maximal atom in $d_p^\gamma \Lambda$ such that $i \leq i'$, $j \leq j'$ and $k \leq k'$. We must prove that $\lambda = \lambda'$. Suppose that $\dim \lambda < p$ or $\dim \lambda' < p$. By Lemma 1.4.16, we can see that λ is a maximal atom in Λ when $\dim \lambda < p$ and λ' is a maximal atom in Λ when $\dim \lambda' < p$. According to condition 1 for pairwise molecular subcomplex Λ of $u \times v \times w$, it is evident that $\lambda = \lambda'$, as required. In the following argument, we may assume that $\dim \lambda = p$ and $\dim \lambda = p$ so that $i = i'$, $j = j'$ and $k = k'$.

Now suppose otherwise that $\lambda \neq \lambda'$. Then $\alpha' = -\alpha$ or $\beta' = -\beta$ or $\varepsilon' = -\varepsilon$. We may assume that $\alpha' = -\alpha$. In this case, we have $F_j^v(\lambda) \subset F_j^v(d_p^\gamma \Lambda) = d_{p-j}^\gamma F_j^v(\Lambda)$ and similarly $F_j^v(\lambda') \subset d_{p-j}^\gamma F_j^v(\Lambda)$ by Proposition 2.5.7. Since $\dim F_j^v(\lambda) = \dim F_j^v(\lambda') = p - j$ and $\dim(d_{p-j}^\gamma F_j^v(\Lambda)) \leq p - j$, we can see that $F_j^v(\lambda)$ and $F_j^v(\lambda')$ are maximal atoms in the molecule $d_{p-j}^\gamma F_j^v(\Lambda)$. Note that $F_j^v(\lambda) = u[i,\alpha] \times w^j[k,\varepsilon]$ and $F_j^v(\lambda') = u[i,-\alpha] \times w^j[k,\varepsilon']$. We get a contradiction to condition 1 in Theorem 1.3.7.

The arguments for the case $\beta' = -\beta$ or $\varepsilon' = -\varepsilon$ are similar.

This completes the proof. $\square$

Now we can prove the main result in this section.

Proposition 2.5.11. *Let Λ be a pairwise molecular subcomplex. Then so is $d_p^\gamma\Lambda$.*

Proof. We have shown that $d_p^\gamma\Lambda$ satisfies condition 1 for pairwise molecular subcomplexes. Moreover, by Proposition 2.5.7, 2.5.8 and 2.5.9, we have $F_I^u(d_p^\gamma\Lambda) = d_{p-I}^\gamma F_I^u(\Lambda)$, $F_J^v(d_p^\gamma\Lambda) = d_{p-J}^\gamma F_J^v(\Lambda)$ and $F_K^w(d_p^\gamma\Lambda) = d_{p-K}^\gamma F_K^w(\Lambda)$ for all $I \geq p$, $J \geq p$ and $K \geq p$. Since $F_I^u(\Lambda)$, $F_J^v(\Lambda)$ and $F_K^w(\Lambda)$ are molecules or the empty set for all I, J and K, we can see that $F_I^u(d_p^\gamma\Lambda)$, $F_J^v(d_p^\gamma\Lambda)$ and $F_K^w(d_p^\gamma\Lambda)$ are molecules or the empty set for all I, J and K. It follows that $d_p^\gamma\Lambda$ is pairwise molecular.

This completes the proof.

□

The following theorem gives the algorithm of constructing $d_p^\gamma\Lambda$ for a pairwise molecular subcomplex Λ in $u \times v \times w$.

Theorem 2.5.12. *Let Λ be a pairwise molecular subcomplex. Then the dimension of every maximal atom in $d_p^\gamma\Lambda$ is not greater than p. Moreover, an atom of dimension less than p is a maximal atom in $d_p^\gamma\Lambda$ if and only if it is a maximal atom in Λ; an atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ of dimension p is a maximal atom in $d_p^\gamma\Lambda$ if and only if there is a maximal atom $u[i'',\alpha''] \times v[j'',\beta''] \times w[k'',\varepsilon'']$ in Λ such that $i'' \geq i$, $j'' \geq j$ and $k'' \geq k$, and the signs α, β and γ satisfy the following conditions:*

1. *if $u[i'',\alpha''] \times v[j'',\beta''] \times w[k'',\varepsilon'']$ can be chosen such that $i'' > i$, then $\alpha = \gamma$; otherwise $\alpha = \alpha''$;*

2. *if $u[i'',\alpha''] \times v[j'',\beta''] \times w[k'',\varepsilon'']$ can be chosen such that $j'' > j$, then $\beta = (-)^i\gamma$; otherwise $\beta = \beta''$;*

3. *if $u[i'',\alpha''] \times v[j'',\beta''] \times w[k'',\varepsilon'']$ can be chosen such that $k'' > k$, then $\varepsilon = (-)^{i+j}\gamma$; otherwise $\varepsilon = \varepsilon''$.*

Note 2.5.13. It follows easily from condition 3 in Theorem 4.4.1 that α, β and γ are well defined.

Proof. Evidently, the dimension of every maximal atom in $d_p^\gamma\Lambda$ is not greater than p. Let Λ_1 be the union of the atoms as described in this theorem. It is easy to see that

Λ_1 satisfies condition 1 for pairwise molecular subcomplexes. To prove the theorem, by Proposition 2.1.7, it suffices to prove that $F^u_I(\Lambda_1) = F^u_I(d^\gamma_p\Lambda)$, $F^v_J(\Lambda_1) = F^v_J(d^\gamma_p\Lambda)$ and $F^w_K(\Lambda_1) = F^w_K(d^\gamma_p\Lambda)$ for all I, J and K. The arguments for the three equations are similar, we prove only the second one. If $J > p$, then it is easy to see that $F^v_J(\Lambda_1) = \emptyset = F^v_J(d^\gamma_p\Lambda)$, as required. In the remaining proof, we may assume that $J \leq p$. We have known that $F^v_J(d^\gamma_p\Lambda) = d^\gamma_{p-J}F^v_J(\Lambda)$. we need only to prove that $F^v_J(\Lambda_1) = d^\gamma_{p-J}F^v_J(\Lambda)$.

By the definition of F^v_J, it is easy to see that $F^v_J(\Lambda_1)$ and $d^\gamma_{p-J}F(\Lambda)$ are subcomplexes of $u \times w^J$. We are going to prove that $F^v_J(\Lambda_1)$ and $d^\gamma_{p-J}F^v_J(\Lambda)$ consist of the same maximal atoms so that they are equal.

Let $\mu = u[i,\alpha] \times w^J[k,\varepsilon]$ be a maximal atom in $F^v_J(\Lambda_1)$. Then Λ_1 has a (v,J)-projection maximal atom λ of the form $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$. Hence Λ has a maximal atom $\lambda' = u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ with $i \leq i'$, $j \leq j'$ and $k \leq k'$.

Suppose that $j = J$ and $i+j+k = p$. Since $u[i',\alpha'] \times w^J[k',\varepsilon']$ is an atom in $F^v_J(\Lambda)$ and $i+k = p-J$, we know that $d^\gamma_{p-J}F^v_J(\Lambda)$ has a maximal atom of the form $u[i,\alpha''] \times w^J[k,\varepsilon'']$. Moreover, we can see that there is a maximal atom $u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ in Λ such that $l > i$, $m \geq j$ and $n \geq k$ if and only if there is a maximal atom $u[l,\sigma] \times w^J[n,\omega]$ $F^v_J(\Lambda)$ such that $l > i$ and $n \geq k$; and we can also see that there is a maximal atom $u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ in Λ such that $l \geq i$, $m \geq j$ and $n > k$ if and only if there is a maximal atom $u[l,\sigma] \times w^J[n,\omega]$ in $F^v_J(\Lambda)$ such that $l \geq i$ and $n > k$. It follows from 1.3.7 that $\alpha = \alpha''$ and $\varepsilon = \varepsilon''$. This implies that μ is a maximal atom in $d^\gamma_{p-J}F(\Lambda)$.

Suppose that $j = J$ and $i+j+k < p$. Then λ is also a maximal atom in Λ. Therefore $\mu = F^v_J(\lambda)$ is a maximal atom in $F^v_J(\Lambda)$. Since $i+k < p-J$, we know that μ is a maximal atom in $d^\gamma_{p-J}F(\Lambda)$.

There remains the case that $j > J$. In this case, there are no maximal atom $u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ in Λ with $l \geq i$ and $m \geq J$ and $n \geq k$ such that $l > i$ or $n > k$. So $i = i'$, $\alpha = \alpha'$, $k = k'$ and $\varepsilon = \varepsilon'$. On the other hand, since $\mu = u[i,\alpha] \times w^J[k,\varepsilon] = u[i',\alpha'] \times w^J[k',\varepsilon'] = F^v_J(\lambda')$, we see that μ is a maximal atom in $F^v_J(\Lambda)$. Because $i'+k' = i+k \leq p-j < p-J$, we know that μ is a maximal atom in $d^\gamma_{p-J}F^v_J(\Lambda)$.

This shows that every maximal atom in $F^v_J(\Lambda_1)$ is a maximal atom in $d^\gamma_{p-J}F^v_J(\Lambda)$.

Conversely, let $\mu = u[i,\alpha] \times w^J[k,\varepsilon]$ be a maximal atom in $d^{\gamma}_{p-J}F^v_J(\Lambda)$. Then $F^v_J(\Lambda)$ has a maximal atom $\mu' = u[i',\alpha'] \times w^J[k',\varepsilon']$ with $i \leq i'$ and $k \leq k'$. Therefore Λ has a (v,J)-projection maximal atom of the form $\lambda' = u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$.

Suppose that $i+k = p-J$. Then Λ_1 has a (v,J)-projection maximal atom of the form $\lambda = u[i,\alpha''] \times v[J,\beta''] \times w[k,\varepsilon'']$. We can see that there is a maximal atom $u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ in Λ such that $l > i$, $m \geq J$ and $n \geq k$ if and only if there is a maximal atom $u[l,\sigma] \times w^J[n,\omega]$ $F^v_J(\Lambda)$ such that $l > i$ and $n \geq k$; and we can also see that there is a maximal atom $u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ in Λ such that $l \geq i$, $m \geq J$ and $n > k$ if and only if there is a maximal atom $u[l,\sigma] \times w^J[n,\omega]$ in $F^v_J(\Lambda)$ such that $l \geq i$ and $n > k$. So $\alpha'' = \alpha$ and $\varepsilon'' = \varepsilon$. Since $F^v_J(\lambda) = u[i,\alpha''] \times w^J[k,\varepsilon''] = \mu$, we can see that μ is a maximal atom in $F^v_J(\Lambda_1)$.

Suppose that $i+k < p-J$. Then $\mu = u[i,\alpha] \times w^J[k,\varepsilon]$ is also a maximal atom in $F^v_J(\Lambda)$. So Λ has a (v,J)-projection maximal atom $\lambda' = u[i,\alpha] \times v[j',\beta'] \times w[k,\varepsilon]$. Now, if $j' = J$, then $i+j'+k < p$; hence λ' is also a maximal atom in Λ_1 and $F^v_J(\lambda) = u[i,\alpha] \times w^J[k,\varepsilon]$ is a maximal atom in $F^v_J(\Lambda_1)$. Suppose that $j' > J$. Then it is easy to see that there is no maximal atom $u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ in Λ with $l \geq i$ and $m \geq J$ and $n \geq k$ such that $l > i$ or $n > k$. Hence Λ_1 has a (v,J)-projection maximal atom of the form $\lambda'' = u[i,\alpha] \times v[j'',\beta''] \times w[k,\varepsilon]$. Therefore we see that $F^v_J(\lambda'') = u[i,\alpha] \times w^J[k,\varepsilon] = \mu$ is a maximal atom in $F^v_J(\Lambda_1)$.

This shows that every maximal atom in $d^{\gamma}_{p-J}F^v_J(\Lambda)$ is a maximal atom in $F^v_J(\Lambda_1)$.

This completes the proof.

□

2.6 Composition of Pairwise Molecular Subcomplexes

In this section, we consider composition of pairwise molecular subcomplexes in $u \times v \times w$. We first give the construction of composites of pairwise molecular subcomplexes. Then we show that composites of pairwise molecular subcomplexes are pairwise molecular.

Lemma 2.6.1. *Let Λ^- and Λ^+ be pairwise molecular subcomplexes. If $d_p^+\Lambda^- = d_p^-\Lambda^+$, then for every maximal atom $\lambda^- = u[i^-,\alpha^-]\times v[j^-,\beta^-]\times w[k^-,\varepsilon^-]$ in Λ^- and every maximal atom $\lambda^+ = u[i^+,\alpha^+]\times v[j^+,\beta^+]\times w[k^+,\varepsilon^+]$ in Λ^+ one has $\min\{i^-,i^+\}+\min\{j^-,j^+\}+\min\{k^-,k^+\} \leq p$.*

Proof. Let $l = \min\{i^-,i^+\}$, $m = \min\{j^-,j^+\}$ and $n = \min\{k^-,k^+\}$. Suppose otherwise that $l+m+n > p$. Then there is an ordered triple $\{i,j,k\}$ with $i \leq l$, $j \leq m$, $k \leq n$ and $i+j+k = p$. Since $l+m+n > p$, we have $i < l$, $j < m$ or $k < m$. If $i < l$, then $d_p^+\Lambda^-$ has a maximal atom of the form $u[i,+]\times v[j,\beta]\times w[k,\varepsilon]$, while $d_p^-\Lambda^+$ has a maximal atom of the form $u[i,-]\times v[j,\beta']\times w[k,\varepsilon']$ by Theorem 2.5.12. This contradicts condition 1 for pairwise molecular subcomplex $d_p^+\Lambda^- = d_p^-\Lambda^+$. The arguments for the cases $j < m$ and $k < m$ are similar. □

Lemma 2.6.2. *Let Λ^- and Λ^+ be pairwise molecular subcomplexes in $u \times v \times w$. If $d_p^+\Lambda^- = d_p^-\Lambda^+$, then*

$$F_I^u(\Lambda^-)\cap F_I^u(\Lambda^+) = F_I^u(\Lambda^-\cap\Lambda^+) = F_I^u(d_p^+\Lambda^-) = F_I^u(d_p^-\Lambda^+),$$

$$F_J^v(\Lambda^-)\cap F_J^v(\Lambda^+) = F_J^v(\Lambda^-\cap\Lambda^+) = F_J^v(d_p^+\Lambda^-) = F_J^v(d_p^-\Lambda^+)$$

and

$$F_K^w(\Lambda^-)\cap F_K^w(\Lambda^+) = F_K^w(\Lambda^-\cap\Lambda^+) = F_K^w(d_p^+\Lambda^-) = F_K^w(d_p^-\Lambda^+)$$

for all I, J and K.

Proof. The arguments for the three formulae are similar. We give the proof for the second one. There are two cases, as follows.

1. Suppose that $J > p$. We claim that $F_J^v(\Lambda^-)\cap F_J^v(\Lambda^+) = \emptyset$.

Indeed, suppose otherwise that $F^v_J(\Lambda^-) \cap F^v_J(\Lambda^+) \neq \emptyset$. Then it is evident that there are atoms $\mu^- = u[l^-, \sigma^-] \times v[m^-, \tau^-] \times w[n^-, \omega^-]$ in Λ^- and $\mu^+ = u[l^+, \sigma^+] \times v[m^+, \tau^+] \times w[n^+, \omega^+]$ in Λ^+ such that $m^- \geq J > p$ and $m^+ \geq J > p$. According to Theorem 2.5.12, this implies that there are maximal atoms $u[0, \alpha_1'] \times v[p, +] \times w[0, \varepsilon']$ and $u[0, \alpha_1''] \times v[p, -] \times w[0, \varepsilon''] \times$ in $d_p^+\Lambda^-$ and $d_p^-\Lambda^+$ respectively. This contradicts the condition 1 for pairwise molecular subcomplex $d_p^+\Lambda^- = d_p^-\Lambda^+$.

Now we have $F^v_J(d_p^+\Lambda^-) \subset F^v_J(\Lambda^- \cap \Lambda^+) \subset F^v_J(\Lambda^-) \cap F^v_J(\Lambda^+) = \emptyset$. Therefore $F^v_J(d_p^+\Lambda^-) = F^v_J(\Lambda^- \cap \Lambda^+) = F^v_J(\Lambda^-) \cap F^v_J(\Lambda^+)$, as required.

2. Suppose that $J \leq p$. Since $d_p^+\Lambda^- = d_p^-\Lambda^+$, we have $d^+_{p-J}F^v_J(\Lambda^-) = F^v_J(d_p^+\Lambda^-) = F^v_J(d_p^-\Lambda^+) = d^-_{p-J}F^v_J(\Lambda^+)$. Because $F^v_J(\Lambda^-)$ and $F^v_J(\Lambda^+)$ are molecules, we can see that $F^v_J(\Lambda^-)\#_{p-J}F^v_J(\Lambda^+)$ is defined. Hence $F^v_J(\Lambda^-) \cap F^v_J(\Lambda^+) = d^+_{p-J}F^v_J(\Lambda^-) = F^v_J(d_p^+\Lambda^-) \subset F^v_J(\Lambda^- \cap \Lambda^+)$. Since we automatically have $F^v_J(\Lambda^- \cap \Lambda^+) \subset F^v_J(\Lambda^-) \cap F^v_J(\Lambda^+)$, we get $F^v_J(\Lambda^-) \cap F^v_J(\Lambda^+) = F^v_J(\Lambda^- \cap \Lambda^+) = F^v_J(d_p^+\Lambda^-)$, as required.

This completes the proof.

□

Proposition 2.6.3. *Let Λ^- and Λ^+ be pairwise molecular subcomplexes. If $d_p^+\Lambda^- = d_p^-\Lambda^+$, then $\Lambda^- \cap \Lambda^+ = d_p^+\Lambda^-(= d_p^-\Lambda^+)$; hence $\Lambda^-\#_p\Lambda^+$ is defined.*

Proof. Let $M = d_p^+\Lambda^- = d_p^-\Lambda^+$. It is evident that $M \subset \Lambda^- \cap \Lambda^+$. To prove the reverse inclusion, it suffices to prove that every maximal atom in $\Lambda^- \cap \Lambda^+$ is contained in M.

Suppose otherwise that there is a maximal atom $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ in $\Lambda^- \cap \Lambda^+$ such that $\lambda \not\subset M$. Since $u[i, \alpha] \times v[j, \beta] = F^w_k(\lambda) \subset F^w_k(\Lambda^- \cap \Lambda^+) = F^w_k(M)$, we can see that M has a maximal atom $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ such that $u[i, \alpha] \subset u[i', \alpha']$ and $v[j, \beta] \subset v[j', \beta']$ and $k' \geq k$. Because $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ is maximal in $\Lambda^- \cap \Lambda^+$ and $M \subset \Lambda^- \cap \Lambda^+$, we have $k' = k$ and $\varepsilon' = -\varepsilon$. Now we know that $\lambda \cup \lambda' \subset \Lambda^-$ and $\lambda \cup \lambda' \subset \Lambda^+$. By applying Lemma 2.4.4 to Λ^- and Λ^+, it is easy to see that there are maximal atoms $\lambda^- = u[i^-, \alpha^-] \times v[j^-, \beta^-] \times w[k^-, \varepsilon^-]$ in Λ^- and $\lambda^+ = u[i^+, \alpha^+] \times v[j^+, \beta^+] \times w[k^+, \varepsilon^+]$ in Λ^+ such that $u[i^-, \alpha^-] \cap u[i^+, \alpha^+] \supset u[i, \alpha]$ and $v[j^-, \beta^-] \cap v[j^+, \beta^+] \supset v[j, \beta]$ and $\min\{k^-, k^+\} > k$. Since λ is maximal atom in $\Lambda^- \cap \Lambda^+$, we have $k^- = k^+ = k + 1$ and $\varepsilon^- = -\varepsilon^+$.

Now, we have $u[i,\alpha] \times v[j,\beta] \subset F^w_{k+1}(\Lambda^-) \cap F^w_{k+1}(\Lambda^+) = F^w_{k+1}(\Lambda^- \cap \Lambda^+)$. Therefore $\Lambda^- \cap \Lambda^+$ has a maximal atom $\lambda'' = u[i'',\alpha''] \times v[j'',\beta''] \times w[k'',\varepsilon'']$ with $u[i'',\alpha''] \supset u[i,\alpha]$ and $v[j'',\beta''] \supset v[j,\beta]$ and $k'' > k$. This contradicts that λ is a maximal atom in $\Lambda^- \cap \Lambda^+$.

This completes the proof.

□

The following Proposition tells us how to construct the composite of a pair of pairwise molecular subcomplexes of $u \times v \times w$.

Proposition 2.6.4. *Let Λ^- and Λ^+ be pairwise molecular subcomplexes of $u \times v \times w$. If $d^+_p\Lambda^- = d^-_p\Lambda^+$, then the maximal atoms in the composite $\Lambda^- \#_p \Lambda^+$ are the q-dimensional common maximal atoms of Λ^- and Λ^+ with $q \le p$ and the r-dimensional atoms in either Λ^- and Λ^+ with $r > p$.*

Proof. Let Λ be the subcomplex of $u \times v \times w$ as described in the proposition. We must prove that $\Lambda = \Lambda^- \cup \Lambda^+$. Clearly, we have $\Lambda \subset \Lambda^- \cup \Lambda^+$; it suffices to prove that $\Lambda^- \cup \Lambda^+ \subset \Lambda$. By the formation of Λ, we must prove that, for each maximal atom $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ in either Λ^- or Λ^+ with $i+j+k \le p$ and such that λ is not a common maximal atom in Λ^- and Λ^+, $\lambda \subset \Lambda$. It is easy to see that this can only happen when $i+j+k = p$. Suppose that λ is a maximal atom in Λ^γ which is not a maximal atom in $\Lambda^{-\gamma}$. Then λ must be a maximal atom in $d^+_p\Lambda^- = d^-_p\Lambda^+$ which implies that $\lambda \subset \lambda^{-\gamma}$ for some maximal atom $\lambda^{-\gamma} = u[i^{-\gamma},\alpha^{-\gamma}] \times v[j^{-\gamma},\beta^{-\gamma}] \times w[k^{-\gamma},\varepsilon^{-\gamma}]$ with $i^{-\gamma} + j^{-\gamma} + k^{-\gamma} > p$. Thus $\lambda \subset \Lambda$. Therefore, we have $\Lambda^- \cup \Lambda^+ \subset \Lambda$.

This completes the proof.

□

Now we can show that the composites of pairwise molecular subcomplexes in $u \times v \times w$ are pairwise molecular.

Proposition 2.6.5. *Let Λ^- and Λ^+ be pairwise molecular subcomplexes. If $d^+_p\Lambda^- = d^-_p\Lambda^+$, then $\Lambda^- \#_p \Lambda^+$ is a pairwise molecular subcomplex of $u \times v \times w$.*

Proof. Let $\Lambda = \Lambda^- \#_p \Lambda^+$. According to Lemma 2.6.1, it is easy to see that Λ satisfies condition 1 for pairwise molecular subcomplexes. Moreover, we have $F_I^u(\Lambda^- \#_p \Lambda^+) = F_I^u(\Lambda^- \cup \Lambda^+) = F_I^u(\Lambda^-) \cup F_I^u(\Lambda^+)$.

Now suppose that $p \geq I$. We have $d_{p-I}^+ F_I^u(\Lambda^-) = F_I^u(d_p^+ \Lambda^-) = F_I^u(d_p^- \Lambda^+) = d_{p-I}^- F_I^u(\Lambda^+)$. Thus $F_I^u(\Lambda^- \#_p \Lambda^+) = F_I^u(\Lambda^-) \#_{p-I} F_I^u(\Lambda^+)$. Therefore $F_I^u(\Lambda^- \#_p \Lambda^+)$ is a molecule.

Suppose that $p < I$. Then it is easy to see that $F_I^u(\Lambda^-) = \emptyset$ or $F_I^u(\Lambda^+) = \emptyset$. (Otherwise, we have $F_I^u(\Lambda^- \cap \Lambda^+) \neq \emptyset$. This would lead to a contradiction to Lemma 2.6.1.) Therefore $F_I^u(\Lambda^- \#_p \Lambda^+)$ is a molecule or the empty set.

We have now proved that $F_I^u(\Lambda^- \#_p \Lambda^+)$ is a molecule or the empty set for all I.

Similarly, we can see that $F_J^v(\Lambda^- \#_p \Lambda^+)$ and $F_K^w(\Lambda^- \#_p \Lambda^+)$ are molecules or the empty set for all J and K.

It follows from Definition 2.1.4 that Λ is a pairwise molecular subcomplex of $u \times v \times w$.

□

2.7 Decomposition of Pairwise Molecular Subcomplexes

The aim of this section is to prove the main theorem in this chapter.

Theorem 2.7.1. *If Λ is a pairwise molecular subcomplex of $u \times v \times w$, then Λ is a molecule.*

It is trivial that the theorem holds when Λ is an atom. Thus we may assume that Λ is a pairwise molecular subcomplex in $u \times v \times w$ which is not an atom throughout this section. We are going to show that Λ is a molecule.

Let

$$p = \max\{\dim(\lambda \cap \mu) \colon \lambda \text{ and } \mu \text{ are distinct maximal atoms in } \Lambda\}.$$

Recall that p is called frame dimension of Λ. It is evident that there are at least two maximal atoms λ and μ in Λ with $\dim \lambda > p$ and $\dim \mu > p$. By Lemma 2.4.4, it is easy to see that p is the maximal number among the numbers $\min\{i_1, i_2\} + \min\{j_1, j_2\} + \min\{k_1, k_2\}$, where $\lambda_1 = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\lambda_2 = u[i_2, \alpha_2] \times v[j_2, \beta_2] \times w[k_2, \varepsilon_2]$ run over all pairs of distinct maximal atoms in Λ.

Lemma 2.7.2. *Let $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ and $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ are maximal atoms in Λ with $\min\{i, i'\} + \min\{j, j'\} + \min\{k, k'\} = p$.*

1. *If $i = i'$, $\alpha = -\alpha'$ and $j < j'$, then $\beta = (-)^i \alpha$;*

2. *If $j = j'$, $\beta = -\beta'$ and $k < k'$, then $\varepsilon = (-)^j \beta$;*

3. *If $k = k'$, $\varepsilon = -\varepsilon'$ and $j < j'$, then $\varepsilon = (-)^j \beta$;*

Proof. The arguments for the three cases are similar, we prove only for the first case.

Suppose that $i = i'$, $\alpha = -\alpha'$ and $j < j'$. According to Lemma 2.4.4, we can get a maximal atom $\lambda'' = u[i'', \alpha''] \times v[j'', \beta''] \times w[k'', \varepsilon'']$ with $i'' > i$, $v[j'', \beta''] \supset v[j, \beta]$ and $w[k'', \varepsilon''] \supset w[k', \varepsilon']$. Since $\min\{i, i'\} + \min\{j, j'\} + \min\{k, k'\} = p$, we have $j'' = j$ and $k'' = k'$. Hence $v[j'', \beta''] = v[j, \beta]$ and $w[k'', \varepsilon''] = w[k', \varepsilon']$. Moreover, it is easy to see that λ, λ' and λ'' are pairwise adjacent by the choice of p. It follows easily from the sign conditions that $\beta = (-)^i \alpha$, as required.

This completes the proof.

$\square$

We are going to prove that a pairwise molecular subcomplex Λ in $u \times v \times w$ is a molecule by showing that Λ can be properly decomposed into pairwise molecular subcomplexes. This decomposition depends essentially on the following total order on the set of maximal atoms in Λ.

For a pair of atoms $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ and $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ in Λ, we write $\lambda < \lambda'$ if one of the following holds:

- $\alpha = \alpha' = -$ and $i < i'$;

- $\alpha = \alpha' = +$ and $i > i'$;

- $\alpha = -$ and $\alpha' = +$;

- $i = i'$ are even, $\alpha = \alpha'$, $\beta = \beta' = -$ and $j < j'$;
- $i = i'$ are even, $\alpha = \alpha'$, $\beta = \beta' = +$ and $j' < j$;
- $i = i'$ are even, $\alpha = \alpha'$, $\beta = -$ and $\beta' = +$
- $i = i'$ are odd, $\alpha = \alpha'$, $\beta = \beta' = +$ and $j < j'$;
- $i = i'$ are odd, $\alpha = \alpha'$, $\beta = \beta' = -$ and $j' < j$;
- $i = i'$ are odd, $\alpha = \alpha'$, $\beta = +$ and $\beta' = -$.

It is evident that the relation $<$ is a total order on the set of maximal atoms in Λ.

Lemma 2.7.3. *For any pair of maximal atoms λ and λ' in Λ with $\dim \lambda > p$ and $\dim \lambda' > p$, if $\lambda < \lambda'$, then $\lambda \cap \lambda' \subset d_p^+ \lambda \cap d_p^- \lambda'$.*

Proof. Let $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ and $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$. According to the choice of p, it is evident that $\min\{i, i'\} + \min\{j, j'\} + \min\{k, k'\} \leq p$. We now consider five cases, as follows.

1. Suppose that $\min\{i, i'\} + \min\{j, j'\} + \min\{k, k'\} = p$. Then λ and λ' are adjacent by the choice of p. According to Lemma 2.7.2 and sign conditions for pairwise molecular subcomplexes, it is easy to see that $\lambda \cap \lambda' \subset d_p^+ \lambda \cap d_p^- \lambda'$, as required.

2. Suppose that $\min\{i, i'\} + \min\{j, j'\} + \min\{k, k'\} < p - 1$. Then it is easy to see that $\lambda \cap \lambda' \subset d_p^+ \lambda \cap d_p^- \lambda'$, as required.

3. Suppose that $\min\{i, i'\} + \min\{j, j'\} + \min\{k, k'\} = p - 1$ and that λ and λ' are adjacent. There are two case, as follows: (1) $i = i'$; (2) $i \neq i'$. In case (1), it is evident that $\lambda \cap \lambda' \subset d_p^+ \lambda \cap d_p^- \lambda'$, as required. In case (2), it follows easily from the sign conditions that $\lambda \cap \lambda' \subset d_{p-1}^+ \lambda \cap d_{p-1}^- \lambda'$; thus $\lambda \cap \lambda' \subset d_p^+ \lambda \cap d_p^- \lambda'$, as required.

4. Suppose that $\min\{i, i'\} + \min\{j, j'\} + \min\{k, k'\} = p - 1$ and that λ and λ' are not adjacent. Suppose also that $i = i'$ or $j = j'$ or $k = k'$. Then it is easy to see that $\lambda \cap \lambda' \subset d_p^+ \lambda \cap d_p^- \lambda'$, as required.

5. Suppose that $\min\{i, i'\} + \min\{j, j'\} + \min\{k, k'\} = p - 1$ and that λ and λ' are not adjacent. Suppose also that $i \neq i'$ and $j \neq j'$ and $k \neq k'$. Then there are several cases,

as follows. (1) $i < i'$ and $j < j'$, or $i < i'$ and $k < k'$; (2) $i < i'$ and $j > j'$ and $k > k'$; (3) $i > i'$ and $j > j'$, or $i > i'$ and $k > k'$; (4) $i > i'$ and $j < j'$ and $k < k'$. In case (1), we have $\alpha = -$; it follows easily that $\lambda \cap \lambda' \subset d_p^+\lambda \cap d_p^-\lambda'$, as required. Similarly, in case (3), we have $\alpha' = +$; this also implies that $\lambda \cap \lambda' \subset d_p^+\lambda \cap d_p^-\lambda'$, as required. There remain case (2) and case (4).

To give the proof for case (2), suppose that $\min\{i,i'\} + \min\{j,j'\} + \min\{k,k'\} = p - 1$ and that λ and λ' are not adjacent; suppose also that $i < i'$ and $j > j'$ and $k > k'$. Then $\alpha = -$ and there is a maximal atom $\lambda'' = u[i'',\alpha''] \times v[j'',\beta''] \times w[k'',\varepsilon'']$ in Λ distinct from λ' such that $i'' > i$, $j'' \geq j'$ and $k'' \geq k'$. By the choice of p, we can see that λ'' is adjacent to both λ and λ', and we have $i'' = i + 1$. According to condition 1 for pairwise molecular subcomplexes, we have $j'' > j'$ or $k'' > k'$.

In case (2), suppose that $j'' > j$. Then $\min\{j'',j\} = j' + 1$ and $k'' = k'$ by the choice of p. Hence $\varepsilon'' = -[-(-)^{i+j'+1}] = -(-)^{i+j'}$. If $\varepsilon' = \varepsilon'' = -(-)^{i+j'}$, then it is easy to see that $\lambda \cap \lambda' \subset d_p^+\lambda \cap d_p^-\lambda'$, as required. If $\varepsilon' = -\varepsilon'' = (-)^{i+j'}$, then we can get $\varepsilon' = (-)^{j'}\beta'$, i.e., $(-)^{j'}\beta' = (-)^{i+j'}$; thus $\beta' = (-)^i$; this implies that $\lambda \cap \lambda' \subset d_p^+\lambda \cap d_p^-\lambda'$, as required.

In case (2), suppose that $k'' > k$. Then $j'' = j'$ by the choice of p. We can also have $\beta'' = -(-)^i\alpha = (-)^i$ by the sign conditions. If $\beta' = \beta'' = (-)^i$, then it is easy to see that $\lambda \cap \lambda' \subset d_p^+\lambda \cap d_p^-\lambda'$, as required. If $\beta' = -\beta'' = -(-)^i$, then we can get $\varepsilon' = (-)^{j'}\beta' = -(-)^{i+j'}$; this implies that $\lambda \cap \lambda' \subset d_p^+\lambda \cap d_p^-\lambda'$, as required.

This completes the proof for case (2).

To give the proof for case (4), suppose that $\min\{i,i'\} + \min\{j,j'\} + \min\{k,k'\} = p - 1$ and that λ and λ' are not adjacent; suppose also that $i > i'$ and $j < j'$ and $k < k'$. Then $\alpha = \alpha' = +$ and there is a maximal atom $\lambda'' = u[i'',\alpha''] \times v[j'',\beta''] \times w[k'',\varepsilon'']$ in Λ distinct from λ' such that $i'' > i'$, $j'' \geq j$ and $k'' \geq k$. By the choice of p, we can see that λ'' is adjacent to both λ and λ', and we have $i'' = i' + 1$. According to condition 1 for pairwise molecular subcomplexes, we have $j'' > j$ or $k'' > k$.

In case (4), suppose that $j'' > j$. Then $\min\{j'',j'\} = j + 1$ and $k'' = k$ by the choice of p. Hence $\varepsilon'' = [-(-)^{i'+j+1}] = (-)^{i'+j}$. If $\varepsilon = \varepsilon'' = (-)^{i'+j}$, then it is easy to see that $\lambda \cap \lambda' \subset d_p^+\lambda \cap d_p^-\lambda'$, as required. If $\varepsilon = -\varepsilon'' = -(-)^{i'+j}$, then we can get $\varepsilon = (-)^j\beta$, i.e.,

$(-)^j\beta = -(-)^{i'+j}$; thus $\beta' = -(-)^{i'}$; this implies that $\lambda \cap \lambda' \subset d_p^+\lambda \cap d_p^-\lambda'$, as required.

In case (4), suppose that $k'' > k$. Then $j'' = j$ by the choice of p. We can also have $\beta'' = -(-)^{i'}\alpha = -(-)^{i'}$ by the sign conditions. If $\beta = \beta'' = -(-)^{i'}$, then it is easy to see that $\lambda \cap \lambda' \subset d_p^+\lambda \cap d_p^-\lambda'$, as required. If $\beta = -\beta'' = (-)^{i'}$, then we can get $\varepsilon = (-)^j\beta = (-)^{i'+j}$; this implies that $\lambda \cap \lambda' \subset d_p^+\lambda \cap d_p^-\lambda'$, as required.

This completes the proof for case (4), thus completes the proof of the lemma.

□

By this lemma, we can arrange all the maximal atoms in Λ with dimension greater than p as

$$\lambda_1, \lambda_2, \cdots, \lambda_n$$

such $\lambda_i \cap \lambda_j \subset d_p^+\lambda_i \cap d_p^-\lambda_j$ for $i < j$.

Let $\Lambda^- = d_p^-\Lambda \cup \lambda_1$ and $\Lambda^+ = d_p^+\Lambda \cup \lambda_2 \cdots \lambda_n$. We are going to prove that Λ^- and Λ^+ are pairwise molecular subcomplexes and Λ can be decomposed into Λ^- and Λ^+.

Lemma 2.7.4. *Λ^- satisfies condition 1 for pairwise molecular subcomplexes.*

Proof. We first prove that $d_p^-\lambda_1 \subset d_p^-\Lambda$. Suppose that $\xi \in d_p^-\lambda_1$. Then, for every maximal atom λ' in Λ with $\xi \in \lambda'$, if $\lambda' = \lambda_t$ for some $t > 1$, then $\xi \in \lambda_1 \cap \lambda_t \subset d_p^-\lambda_t = d_p^-\lambda'$; if $\dim \lambda' \leq p$, then we automatically have $\xi \in d_p^-\lambda'$. It follows from Lemma 1.4.17 that $d_p^-\lambda_1 \subset d_p^-\Lambda$, as required.

We now verify that Λ^- satisfies condition 1 for pairwise molecular subcomplexes. It suffices to prove that any maximal atom $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ in $d_p^-\Lambda$ with $i \leq i_1$, $j \leq j_1$ and $k \leq k_1$ is contained in λ_1. By the formation of $d_p^-\lambda_1$ and $d_p^-\Lambda$, it is easy to see that λ is a maximal atom in $d_p^-\lambda_1$, and hence $\lambda \subset \lambda_1$, as required. □

Lemma 2.7.5. *Λ^+ satisfies condition 1 for pairwise molecular subcomplexes.*

Proof. It suffices to prove that any maximal atom $\lambda = u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ in $d_p^+\Lambda$ with $i \leq i_t$, $j \leq j_t$ and $k \leq k_t$ for some $2 \leq t \leq n$ is contained in some λ_s for $2 \leq s \leq n$. It is evident that $i + j + k = p$.

Let r be the maximal integer between 2 and n such that $i \leq i_r$, $j \leq j_r$ and $k \leq k_r$. Then $d_p^+\lambda_r$ has a maximal atom $\lambda' = u[i,\alpha'] \times v[j,\beta'] \times w[k,\varepsilon']$. By the choice of r, it is evident that $\operatorname{Int}\lambda' \cap \lambda_t = \emptyset$ for any $t > r$. Moreover, for any $1 \leq s < r$, we have $\lambda' \cap \lambda_s \subset \lambda_r \cap \lambda_s \subset d_p^+\lambda_s$. By Lemma 1.4.17, it is easy to see that $\operatorname{Int}\lambda' \subset d_p^+\Lambda$ and hence $\lambda' \subset d_p^+\Lambda$. So, by condition 1 for the pairwise molecular subcomplex $d_p^+\Lambda$, we can see that $\lambda = \lambda' \subset \lambda_r$, as required.

This completes the proof. $\square$

Lemma 2.7.6. *Let $p \geq I$ and let λ_1 be (u, I)-projection maximal. Then*

1. *$F_I^u(\Lambda^-)$ and $F_I^u(\Lambda^+)$ are molecules in $v^I \times w^I$.*

2. *$d_{p-I}^+ F_I^u(\Lambda^-) = d_{p-I}^- F_I^u(\Lambda^+)$, hence $F_I^u(\Lambda^-)\#_{p-I}F_I^u(\Lambda^+)$ is defined.*

3. *$F_I^u(\Lambda) = F_I^u(\Lambda^-)\#_{p-I}F_I^u(\Lambda^+)$.*

Proof. Since F_I^u preserves unions, we have $F_I^u(\Lambda^-) = F_I^u(d_p^-\Lambda \cup \lambda_1) = F_I^u(d_p^-\Lambda) \cup F_I^u(\lambda_1)$ and $F_I^u(\Lambda^+) = F_I^u(d_p^+\Lambda \cup \lambda_2 \cup \cdots \cup \lambda_n) = F_I^u(d_p^+\Lambda) \cup F_I^u(\lambda_2) \cup \cdots \cup F_I^u(\lambda_n)$. If $\dim F_I^u(\lambda_1) = j_1 + k_1 \leq p - I$, then $F_I^u(\lambda_1)$ is a maximal atom in $d_{p-I}^- F_I^u(\Lambda)$ by Theorem 1.4.16; hence $F_I^u(\Lambda^-) = F_I^u(d_p^-\Lambda) = d_{p-I}^- F_I^u(\Lambda)$ and similarly $F_I^u(\Lambda^+) = F_I^u(\Lambda)$; it follows easily that $F_I^u(\Lambda^-)$ and $F_I^u(\Lambda^+)$ are molecules and $d_{p-I}^+ F_I^u(\Lambda^-) = d_{p-I}^- F_I^u(\Lambda^+)$, as required. If $F_I^u(\Lambda) = F_I^u(\lambda_1)$, then λ_s are not (u, I)-projection maximal for $s = 2, \ldots, n$; thus $F_I^u(\lambda_s) = F_I^u(\lambda_1 \cap \lambda_s) \subset F_I^u(d_p^+\lambda_1) = d_{p-I}^+ F_I^u(\lambda_1) = d_{p-I}^+ F_I^u(\Lambda)$; it follows easily that $F_I^u(\Lambda^+) = F_I^u(d_p^+\Lambda) = d_{p-I}^+ F_I^u(\Lambda)$; it is also evident that $F_I^u(\Lambda^-) = F_I^u(\lambda_1) = F_I^u(\Lambda)$; therefore $F_I^u(\Lambda^-)$ and $F_I^u(\Lambda^+)$ are molecules and $d_{p-I}^+ F_I^u(\Lambda^-) = d_{p-I}^- F_I^u(\Lambda^+)$, as required. In the following proof, we may assume that $\dim F_I^u(\lambda_1) > p - I$ and $F_I^u(\Lambda)$ has at least two distinct maximal atoms.

Let

$$q = \max\{\dim(\mu \cap \mu') : \mu \text{ and } \mu' \text{ are distinct maximal atoms in } F_I^u(\Lambda)\}.$$

It is clear that $q \leq p - I$ by the choice of p. Let $\mu = v^I[m, \tau] \times w^I[n, \omega]$ be a maximal atom in $F_I^u(\Lambda)$ distinct from $F_I^u(\lambda_1)$. If $\dim(F_I^u(\lambda_1) \cap \mu) < p - I$, then it is easy to see that $F_I^u(\lambda_1) \cap \mu \subset d_{p-I}^+ F_I^u(\lambda_1) \cap d_{p-I}^- \mu$ by the construction of molecule $F_I^u(\Lambda)$ in $v^I \times w^I$ (Theorem 1.3.7). Suppose that $\dim(F_I^u(\lambda_1) \cap \mu) = p - I$. Then there is a maximal atom $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ in Λ such that $F_I^u(\lambda') = \mu$. If $i_1 \leq i'$ and $j_1 < j'$, then

$i_1 = I$ by the choice of p, $\alpha_1 = -$ when $i_1 < i'$ by the definition of natural order and $k' < k_1$ by condition 1 for pairwise molecular subcomplexes; hence $\beta_1 = -(-)^I$ by the sign conditions for λ_1 and λ' or by the definition of natural order; thus $\omega = \varepsilon' = (-)^{I+j_1}$ by the sign condition for λ_1 and λ'; it follows easily that $F_I^u(\lambda_1) \cap \mu \subset d_{p-I}^+ F_I^u(\lambda_1) \cap d_{p-I}^- \mu$. If $i_1 \leq i'$ and $j_1 > j'$, then it is easy to see that $i_1 = I$, $\tau = \beta' = (-)^I$ and $\varepsilon_1 = -(-)^{I+m}$ by the sign condition for pairwise molecular subcomplexes or the definition of the natural order; it follows easily that $F_I^u(\lambda_1) \cap \mu \subset d_{p-I}^+ F_I^u(\lambda_1) \cap d_{p-I}^- \mu$. If $i_1 > i'$, then, by an similar argument, one can get $F_I^u(\lambda_1) \cap \mu \subset d_{p-I}^+ F_I^u(\lambda_1) \cap d_{p-I}^- \mu$. We have now shown that $F_I^u(\lambda_1) \cap \mu \subset d_{p-I}^+ F_I^u(\lambda_1) \cap d_{p-I}^- \mu$ for every maximal atom μ in $F_I^u(\Lambda)$.

Moreover, we have $F_I^u(\Lambda^-) = d_{p-I}^- F_I^u(\Lambda) \cup F_I^u(\lambda_1)$ and

$$F_I^u(\Lambda^+) = d_{p-I}^+ F_I^u(\Lambda) \cup \bigcup \{\mu : \mu \text{ is a maximal atom in } F_I^u(\Lambda) \text{ with } \mu \neq F_I^u(\lambda_1)\}$$

(Notice that it is possible that $F_I^u(\Lambda^+) = d_{p-I}^+ F_I^u(\Lambda)$). It follows from Theorem 1.4.13 that $F_I^u(\Lambda^-)$ and $F_I^u(\Lambda^+)$ are molecules in $v^I \times w^I$, $d_{p-I}^+ F_I^u(\Lambda^-) = d_{p-I}^- F_I^u(\Lambda^+)$ and $F_I^u(\Lambda) = F_I^u(\Lambda^-) \#_{p-I} F_I^u(\Lambda^+)$, as required.

This completes the proof.

□

Lemma 2.7.7. *Let $p \geq J$ and let λ_1 be a (v, J)-projection maximal atom. Then*

1. *$F_J^v(\Lambda^-)$ and $F_J^v(\Lambda^+)$ are molecules in $u \times w^J$.*

2. *$d_{p-J}^+ F_J^v(\Lambda^-) = d_{p-J}^- F_J^v(\Lambda^+)$, hence $F_J^v(\Lambda^-) \#_{p-J} F_J^v(\Lambda^+)$ is defined.*

3. *$F_J^v(\Lambda) = F_J^v(\Lambda^-) \#_{p-J} F_J^v(\Lambda^+)$.*

Proof. Since F_J^v preserves unions, we have $F_J^v(\Lambda^-) = F_J^v(d_p^- \Lambda \cup \lambda_1) = F_J^v(d_p^- \Lambda) \cup F_J^v(\lambda_1)$ and $F_J^v(\Lambda^+) = F_J^v(d_p^+ \Lambda \cup \lambda_2 \cup \cdots \cup \lambda_n) = F_J^v(d_p^+ \Lambda) \cup F_J^v(\lambda_2) \cup \cdots \cup F_J^v(\lambda_n)$. If $\dim F_J^v(\lambda_1) = i_1 + k_1 \leq p - J$, then it is evident that $F_J^v(\Lambda^-) = F_J^v(d_p^- \Lambda) = d_{p-J}^- F_J^v(\Lambda)$ and $F_J^v(\Lambda^+) = F_J^v(\Lambda)$; it follows easily that $F_J^v(\Lambda^-)$ and $F_J^v(\Lambda^+)$ are molecules and $d_{p-J}^+ F_J^v(\Lambda^-) = d_{p-J}^- F_J^v(\Lambda^+)$, as required. If $F_J^v(\Lambda) = F_J^v(\lambda_1)$, then λ_s are not (v, J)-projection maximal for $s \neq 1$; thus $F_J^v(\lambda_s) = F_J^v(\lambda_1 \cap \lambda_s) \subset F_J^v(d_p^+ \lambda_1) = d_{p-J}^+ F_J^v(\lambda_1) = d_{p-J}^+ F_J^v(\Lambda)$; it follows easily that $F_J^v(\Lambda^+) = F_J^v(d_p^+ \Lambda) = d_{p-J}^+ F_J^v(\Lambda)$; it is also evident that $F_J^v(\Lambda^-) = F_J^v(\lambda_1) = F_J^v(\Lambda)$; therefore $F_J^v(\Lambda^-)$ and $F_J^v(\Lambda^+)$ are molecules and $d_{p-J}^+ F_J^v(\Lambda^-) = d_{p-J}^- F_J^v(\Lambda^+)$, as required. In the following proof, we may assume that $\dim F_J^v(\lambda_1) > p - J$ and $F_J^v(\Lambda)$ has at least two distinct maximal atoms.

Let

$$q = \max\{\dim(\mu \cap \mu') : \mu \text{ and } \mu' \text{ are distinct maximal atoms in } F_J^v(\Lambda)\}.$$

It is clear that $q \leq p - J$ by the choice of p. Let $\mu = u[l, \sigma] \times w^J[n, \omega]$ be a maximal atom in $F_J^v(\Lambda)$ distinct from $F_J^v(\lambda_1)$. If $\dim(F_J^v(\lambda_1) \cap \mu) < p - J$, then it is easy to see that $F_J^v(\lambda_1) \cap \mu \subset d_{p-J}^+ F_J^v(\lambda_1) \cap d_{p-J}^- \mu$ by the construction of molecule $F_J^v(\Lambda)$ in $u \times w^J$ (Theorem 1.3.7). Now suppose that $\dim(F_J^v(\lambda_1) \cap \mu) = p - J$. Let $\lambda' = u[l, \sigma] \times v[j', \beta'] \times w[n, \omega]$ be the (v, J)-projection maximal atom in Λ such that $F_J^v(\lambda') = \mu$. Then $\dim \lambda' > p$. We can also see that $\min\{j_1, j'\} = J$ by the choice of p and λ is adjacent to λ'. Since $F_J^v(\Lambda)$ is a molecule in $u \times w^J$, we have $i_1 \neq l$ and $k_1 \neq n$. If $i_1 < l$, then $\alpha_1 = -$ and $k_1 > n$; it follows from the sign condition for λ_1 and λ' that $\omega = (-)^{i_1 + J}$ which implies that $F_J^v(\lambda_1) \cap \mu \subset d_{p-J}^+ F_J^v(\lambda_1) \cap d_{p-J}^- \mu$. Similarly, if $i_1 > l$, then $\sigma_1 = +$ and $k_1 < n$; it follows from the sign condition for λ_1 and λ' that $\varepsilon_1 = -(-)^{l+J}$ which implies that $F_J^v(\lambda_1) \cap \mu \subset d_{p-J}^+ F_J^v(\lambda_1) \cap d_{p-J}^- \mu$.

Moreover, we have $F_J^v(\Lambda^-) = d_{p-J}^- F_J^v(\Lambda) \cup F_J^v(\lambda_1)$ and

$$F_J^v(\Lambda^+) = d_{p-J}^+ F_J^v(\Lambda) \cup \bigcup\{\mu : \mu \text{ is a maximal atom in } F_J^v(\Lambda) \text{ with } \mu \neq F_J^v(\lambda_1)\}.$$

According to Proposition 1.4.13, we can see that $F_J^v(\Lambda^-)$ and $F_J^v(\Lambda^+)$ are molecules in $u \times w^J$, $d_{p-J}^+ F_J^v(\Lambda^-) = d_{p-J}^- F_J^v(\Lambda^+)$ and $F_J^v(\Lambda) = F_J^v(\Lambda^-) \#_{p-J} F_J^v(\Lambda^+)$, as required.

This completes the proof.

$\square$

Lemma 2.7.8. *Let $p \geq K$ and λ_1 be a (w, K)-projection maximal atom. Then*

1. *$F_K^w(\Lambda^-)$ and $F_K^w(\Lambda^+)$ are molecules in $u \times v$.*

2. *$d_{p-K}^+ F_K^w(\Lambda^-) = d_{p-K}^- F_K^w(\Lambda^+)$, hence $F_K^w(\Lambda^-) \#_{p-K} F_K^w(\Lambda^+)$ is defined.*

3. *$F_K^w(\Lambda) = F_K^w(\Lambda^-) \#_{p-K} F_K^w(\Lambda^+)$.*

Proof. The argument is similar to the proof of Lemma 2.7.7. $\square$

Proposition 2.7.9. *Let Λ be a pairwise molecular subcomplex. Then*

1. *Λ^- and Λ^+ are pairwise molecular subcomplexes.*

2. *$d_p^+\Lambda^- = d_p^-\Lambda^+$, hence the composite $\Lambda^-\#_p\Lambda^+$ is defined.*

3. *$\Lambda = \Lambda^-\#_p\Lambda_+$.*

Proof. If λ_1 is not a (v, J)-projection maximal atom in Λ, then it is easy to see that $F_J^v(\Lambda^-) = F_J^v(d_p^-\Lambda)$ and $F_J^v(\Lambda^+) = F_J^v(\Lambda)$ by the choice of p and Lemmas 2.7.4 and 2.7.5; hence $F_J^v(\Lambda^-)$ and $F_J^v(\Lambda^+)$ are the empty set or molecules in $u \times w^J$. Similarly, if λ_1 is not (u, I)-projection maximal atom in Λ, then $F_I^u(\Lambda^-)$ and $F_I^u(\Lambda^+)$ are molecules in $v^I \times w^I$ or the empty set; if λ_1 is not (w, K)-projection maximal atom in Λ, then $F_K^w(\Lambda^-)$ and $F_K^w(\Lambda^+)$ are molecules in $u \times v$ or the empty set.

According to the above argument and Lemmas 2.7.6 to 2.7.8, we can see that $F_I^u(\Lambda^-)$, $F_I^u(\Lambda^+)$, $F_J^v(\Lambda^-)$, $F_J^v(\Lambda^+)$, $F_K^w(\Lambda^-)$ and $F_K^w(\Lambda^+)$ are molecules in the corresponding ω-complexes or the empty set for all I, J and K. Thus Λ^- and Λ^+ are pairwise molecular.

Now, if $p \geq J$ and λ_1 is not (v, J)-projection maximal, then $F_J^v(d_p^+\Lambda^-) = d_{p-J}^+F_J^v(\Lambda^-) = d_{p-J}^+F_J^v(d_p^-\Lambda)) = d_{p-J}^-F_J^v(\Lambda) = d_{p-J}^-F_J^v(\Lambda^+) = F_J^v(d_p^-\Lambda^+)$; if $p < J$, then $F_J^v(d_p^+\Lambda^-) = \emptyset = F_J^v(d_p^-\Lambda^+)$. It follows from Lemmas 2.7.6 to 2.7.8 and Propositions 2.5.7 to 2.5.9 that $F_I^u(d_p^+\Lambda^-) = F_I^u(d_p^-\Lambda^+)$, $F_J^v(d_p^+\Lambda^-) = F_J^v(d_p^-\Lambda^+)$ and $F_K^w(d_p^+\Lambda^-) = F_K^w(d_p^-\Lambda^+)$ for all I, J and K. By Lemma 2.1.7, we can see that $d_p^+\Lambda^- = d_p^-\Lambda^+$. Hence $\Lambda^-\#_p\Lambda^+$ is defined. Clearly, we have $\Lambda = \Lambda^- \cup \Lambda^+$. Therefore $\Lambda = \Lambda^-\#_p\Lambda^+$.

This completes the proof. $\square$

We have now proved that a pairwise molecular subcomplex Λ in $u \times v \times w$ can be decomposed into pairwise molecular subcomplexes $\Lambda = \Lambda^-\#_p\Lambda^+$. It is evident that this is a proper decomposition. By induction, we can see that Λ can be eventually decomposed into atoms. Thus Λ is a molecule. So we get the proof for Theorem 2.7.1.

Chapter 3

Construction of Molecules in the Product of Three Infinite-Dimensional Globes

According to Proposition 2.2.1, the maximal atoms in a molecule of $u \times v \times w$ can be listed as $\lambda_1, \lambda_2, \ldots, \lambda_R$ with $\lambda_r = u_{i_r}^{\alpha_r} \times v_{j_r}^{\beta_r} \times w_{k_r}^{\varepsilon_r}$ such that $j_1 \geq \cdots \geq j_R$ and such that $i_r > i_{r+1}$ when $1 \leq r < R$ and $j_r = j_{r+1}$.

In this chapter, we aim to construct molecules by listing their maximal atoms as described above. The point in this chapter is that this is easily achieved inductively. In more detail, let maximal atoms $\lambda_1, \ldots, \lambda_r$ be an initial segment of the list. One can easily determine whether $\lambda_1 \cup \cdots \cup \lambda_r$ is already a molecule and determine the set of possible next maximal atoms λ_{r+1}.

Throughout this chapter, the $(v, J+1)$-projection maximal atoms in a subcomplex of $u \times v \times w$ are called the *lowest* maximal atoms above level J. An atom with dimension of second factor equal to J is said to be *at level* J, while an atom with dimension of second factor great than J is said to be *above level* J. For the convenience of the statement, we allow J to be -1.

3.1 Another Description of Molecules

In this section, we give another description of molecules in terms of the second factor on which the construction of the molecules is based.

Proposition 3.1.1. *Let Λ be a subcomplex. Suppose that all the maximal atoms above level J satisfy all the conditions in Theorem 2.4.1. Suppose also that all the maximal atoms at level J together with all the lowest maximal atoms above level J satisfy all the conditions in Theorem 2.4.1. Then all the maximal atoms above level $J-1$ satisfy all the conditions in Theorem 2.4.1.*

Proof. Let $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ be a maximal atom above level J. Suppose that λ is not lowest above level J. Then $j > J+1$ and there is a lowest maximal atom $\lambda' = u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ above level J such that $i' \geq i$, $J < j' < j$ and $k' \geq k$. Let $\mu = u[l, \sigma] \times v[J, \tau] \times w[n, \omega]$ be a maximal atom at level J. Note that there are no three pairwise adjacent maximal atoms as in the hypothesis of the condition 5 such that two of them are at level J and one of them is above level J and not lowest, hence the condition 5 is automatically satisfied by maximal atoms above level $J-1$. Now, it suffices to prove that λ and μ satisfies the conditions 1 to 4.

The condition 1 for λ and μ follows easily from the condition 1 for λ' and μ.

To verify the conditions 2, 3 and 4 for λ and μ, suppose that λ and μ are adjacent. Then $i \geq l$ or $k \geq n$. The arguments for these two cases are similar. We only give the proof for the case $i \geq l$.

Suppose that $i \geq l$. Then $k < n$ by condition 1 for λ and μ and $k' = k$ by the adjacency of λ and μ. Hence $\varepsilon' = \varepsilon$ by condition 3 in Theorem 2.4.1 for λ and λ' and the adjacency of λ and μ. Moreover, we can see that $i' > i$ by condition 1 for λ and λ'. Thus the condition 3 for λ and μ is automatically satisfied (whenever $i = l$ and $\alpha = -\sigma$). Finally, we can see that the conditions 2 and 4 for λ and μ follow from the corresponding conditions for λ' and μ.

This completes the proof. ∎

The following proposition also characterises molecules in $u \times v \times w$.

Proposition 3.1.2. *Let Λ be a subcomplex. Then Λ is a molecule if and only if the following conditions hold for every non-negative integer J:*

1. *For every non-negative integer J, all the maximal atoms $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ at level J, if there are any, can be listed by decreasing i and increasing k.*

2. *Suppose that $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ is a lowest maximal atom above level J and $u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ is a maximal atom at level J. If $l \leq i$, then $n > k$.*

3. *Let all the lowest maximal atoms $\lambda_s = u[i_s,\alpha_s] \times v[j_s,\beta_s] \times w[k_s,\varepsilon_s]$ above level J, if there are any, be listed as $\lambda_1, \cdots, \lambda_S$ by decreasing i_s and increasing k_s; let all the maximal atoms $\mu_t = u[l_t,\sigma_t] \times v[m_t,\tau_t] \times w[n_t,\omega_t]$ at level J, if there are any, be listed as $\mu_1, \cdots, \mu_T$ by decreasing l_t and increasing n_t.*

   (a) If $1 < s \leq S$, then there exists μ_t such that $l_t > i_s$ and $n_t > k_{s-1}$.

   (b) If $l_t > i_s$ and $n_t > k_{s-1}$ $(1 < s \leq S)$, then $\tau_t = -(-)^{i_s}\alpha_s = -(-)^J\varepsilon_{s-1}$;

   if $l_t > i_1$, then $\tau_t = -(-)^{i_1}\alpha_1$;

   if $n_t > k_S$, then $\tau_t = -(-)^J\varepsilon_S$;

   if J is the greatest dimension of second factors of maximal atoms in Λ, then $\tau_1 = \cdots = \tau_T$.

   (c) If $1 < t \leq T$ and if there is no λ_s such that $i_s > l_t$ and $k_s > n_{t-1}$, then $\omega_{t-1} = -(-)^{l_t+J}\sigma_t$.

   (d) Suppose that $n_t < k_s$. If $l_{t+1} \leq i_s$ $(1 \leq t < T)$, or if $s = S$ and $t = T$, then $\omega_t = -(-)^{i_s+J}\alpha_s$.

   (e) Suppose that $l_t < i_s$. If $n_{t-1} \leq k_s$ $(1 < t \leq T)$ or if $s = t = 1$, then $\sigma_t = -(-)^{l_t+J}\varepsilon_s$.

   (f) Suppose that $i_s = l_t$. If $k_s > n_{t-1}$ $(1 < t \leq T)$, or if $s = t = 1$, then $\alpha_s = \sigma_t$.

   (g) Suppose that $k_s = n_t$. If $i_s > l_{t+1}$ $(1 \leq t < T)$, or if $s = S$ and $t = T$, then $\varepsilon_s = \omega_t$.

(h) If $1 \le t < T$ and $i_s = l_{t+1}$ and $k_s = n_t$, then $\alpha_s = \sigma_{t+1}$ or $\varepsilon_s = \omega_t$.

Remark. 1. By induction, it follows easily from condition 1 and 2 in the proposition that, for every integer J less than the greatest dimension of second factors of maximal atoms of Λ, all the lowest maximal atoms $u[i_s, \alpha_s] \times v[j_s, \beta_s] \times w[k_s, \varepsilon_s]$ above level J can be listed by decreasing i_s and increasing k_s, as required by the assumption in Condition 3.

2. By condition 3b in the proposition, if $l_t > i_s$ and $n_t \ge k_s$, then $\tau_t = -(-)^{i_s}\alpha_s$; if $l_t \ge i_s$ and $n_t > k_s$, then $\tau_t = -(-)^J\varepsilon_s$. (Hence $l_t > i_s$ and $n_t > k_s$ cannot hold simultaneously unless $\varepsilon_s = (-)^{i_s+J}\alpha_s$.)

3. It follows from the first part of condition 3b that $\varepsilon_{s-1} = (-)^{i_s+J}\alpha_s$ which we have known from earlier part of construction.

Proof. Suppose that Λ is a molecule. Then Λ satisfies all the conditions in Theorem 2.4.1. We are going to verify all the conditions in this proposition.

Firstly, it follows easily from condition 1 in Theorem 2.4.1 that, for every integer J, all the maximal atoms $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ at level J, if there are any, can be listed by decreasing i and increasing k, as required.

Next, suppose that $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ is a lowest maximal atom above level J and $u[l, \sigma] \times v[m, \tau] \times w[n, \omega]$ is a maximal atom at level J. If $l \le i$, then it follows easily from condition 1 in Theorem 2.4.1 that $n > k$.

Finally, let all the lowest maximal atoms $\lambda_s = u[i_s, \alpha_s] \times v[j_s, \beta_s] \times w[k_s, \varepsilon_s]$, above level J, if there are any, be listed as $\lambda_1, \cdots, \lambda_S$ by decreasing i_s and increasing k_s; let all the maximal atoms $\mu_t = u[l_t, \sigma_t] \times v[m_t, \tau_t] \times w[n_t, \omega_t]$ at level J, if there are any, be listed as $\mu_1, \cdots, \mu_T$ by decreasing l_t and increasing n_t. (These can be done by condition 1 in Theorem 2.4.1.) We must verify conditions 3a to 3h. By the definition of lowest, it is easy to see that every pair of consecutive maximal atoms in the list $\lambda_1, \cdots, \lambda_S$ are adjacent.

3a Condition 3a follows from condition 4 in Theorem 2.4.1.

3b Condition 3b follows from conditions 2 and 3 in Theorem 2.4.1.

3c Condition 3c follows from condition 2 in Theorem 2.4.1 since μ_{t-1} and μ_t are adjacent under the hypothesis of condition 3c.

3d Condition 3d follows from condition 2 in Theorem 2.4.1 since λ_s and μ_t are adjacent under the hypothesis of condition 3d.

3e Condition 3e holds by an argument similar to the proof of condition 3d.

3f Condition 3f follows from condition 3 in Theorem 2.4.1.

3g Condition 3g also follows from condition 3 in Theorem 2.4.1.

3h Condition 3h follows from condition 5 in Theorem 2.4.1.

To prove the sufficiency, suppose that Λ satisfies all the conditions in the proposition. It is evident that the maximal atoms at the highest level satisfy conditions 1 to 5 in Theorem 2.4.1. Suppose that J less than the highest level and all maximal atoms above level J satisfy conditions 1 to 5 in Theorem 2.4.1. By induction and the proposition 3.1.1, it suffices to prove that all the maximal atoms at level J together with all the lowest maximal atoms above level J satisfy conditions 1 to 5 in Theorem 2.4.1.

Condition 1. By the conditions 1 and 2 in the proposition, condition 1 in Theorem 2.4.1 is satisfied by all the maximal atoms at level J together with all the lowest maximal atoms above level J.

Condition 2. By condition 3c in the proposition, a pair of adjacent maximal atoms at level J satisfies condition 2 in Theorem 2.4.1. Let λ_s be a lowest maximal atom above level J and let μ_t be a maximal atom at level J. Suppose that λ_s and μ_t are adjacent.

Case 1. If $l_t \geq i_s$ and $n_t \geq k_s$, then condition 2 for λ_s and μ_t is satisfied by remark 2 after the proposition.

Case 2. Suppose that $n_t < k_s$. Then $l_t > i_s$ and, by the adjacency of λ_s and μ_t, we have $l_t > i_s$ and $l_{t+1} \leq i_s$ whenever $t < T$. Hence λ_s and μ_t satisfy condition 2 in Theorem 2.4.1 by conditions 3a, 3b and 3d in this proposition.

Case 3. Suppose that $l_t < i_s$. The argument is similar to the above case.

This completes the proof that all the maximal atoms at level J together with all the lowest maximal atoms above level J satisfy conditions 2 in Theorem 2.4.1.

Condition 3. Suppose that μ_t and μ_{t+1} are a pair of adjacent maximal atoms at level J. Suppose also that there is no maximal atom $\lambda = u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ above level J with $i \geq l_{t+1}$ and $k \geq n_t$, then conditions 3b implies that $\tau_t = \tau_{t+1}$, as required by condition 3 in Theorem 2.4.1.

Indeed, if $l_{t+1} > i_1$ or $n_t > k_S$, then it follows easily from condition 3b in the proposition that $\tau_t = \tau_{t+1}$, as required. If $l_{t+1} \leq i_1$ and $n_t \leq k_S$, then $k_1 < n_t$ and $i_S < l_{t+1}$ by the hypothesis, i.e., $i_S < l_{t+1} \leq i_1$ and $k_1 < n_t \leq k_S$. Now let λ_s be such that $i_{s+1} < l_{t+1} \leq i_s$. Then $k_s < n_t$ by the hypothesis. So we have $l_t > l_{t+1} > i_{s+1}$ and $n_{t+1} > n_t > k_s$. So by condition 3b in the proposition, it is easy to see that $\tau_t = \tau_{t+1}$, as required by condition 3 in Theorem 2.4.1.

To finish the proof of condition 3, let λ_s be a lowest maximal atom above level J and μ_t be a maximal atom at level J. If $i_s = l_t$, or if $k_s = k_t$, then λ_s and μ_t are adjacent. Therefore, by conditions 3f and 3g in the proposition, it is evident that condition 3 in Theorem 2.4.1 hold for λ_s and μ_t.

Condition 4. By conditions 3a in the proposition, it is evident that condition 4 in Theorem 2.4.1 is satisfied by a pair of adjacent maximal atoms at level J since they are consecutive in the list of lowest maximal atoms above level $J - 1$. Now if λ_s is a lowest maximal atom above level $J - 1$, and if λ_s and μ_t are adjacent, then λ_s and μ_t are consecutive in the list of lowest maximal atoms above level $J - 1$. So, similar to the above case, the condition 4 in Theorem 2.4.1 holds for λ_s and μ_t. Suppose that λ_s is not the lowest maximal atom above level $J - 1$. Suppose also that $l_t < i_s$. Then $n_t > k_s$. In this case, there must be a maximal atom $\mu' = u[l', \sigma'] \times v[m', \tau'] \times w[n', \omega']$ at level J such that $n' = k_s$. Hence $l' > i_s$. It is evident that μ' and μ_t are adjacent. Since we have known that condition 4 in Theorem 2.4.1 holds for μ' and μ_t, we can see that condition 4 in Theorem 2.4.1 hold for λ_s and μ_t. If $n_t < k_s$, then we can see that condition 4 in Theorem 2.4.1 holds for λ_s and μ_t by a similar argument.

Condition 5. By condition 3h in the proposition, condition 5 in Theorem 2.4.1 is satisfied by all the maximal atoms at level J together with all the lowest maximal atoms above level J.

This completes the proof.

$\square$

We can now characterise the sets of maximal atoms in molecules of $u \times v \times w$.

Let $\mathcal{A}$ be a finite and non-empty set of atoms in $u \times v \times w$. For a fixed integer J, An atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ above level J in $\mathcal{A}$ is *lowest above level* J if there is no atom $u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ in $\mathcal{A}$ with $i' \geq i$, $J < j' < j$ and $k' \geq k$.

Suppose that there are no distinct atoms $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ and $u[i',\alpha'] \times v[j',\beta'] \times w[k',\varepsilon']$ in $\mathcal{A}$ such that $i \leq i'$, $j \leq j'$ and $k \leq k'$. Let Λ be the union of atoms in $\mathcal{A}$. Then it is evident that the maximal atoms in Λ are exactly the atoms in $\mathcal{A}$. Moreover, it is easy to see that, for every integer J with $J \geq -1$, an maximal atom in Λ is lowest above level J in Λ if and only if it is lowest above level J in $\mathcal{A}$.

Proposition 3.1.3. *Let $\mathcal{A}$ be a finite and non-empty set of atoms in $u \times v \times w$. Then $\mathcal{A}$ is the set of maximal atoms in a molecule if and only if the following conditions hold for every non-negative integer J:*

1. *For every non-negative integer J, all the atoms $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ at level J in $\mathcal{A}$, if there are any, can be listed by decreasing i and increasing k.*

2. *Suppose that $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ is a lowest atom above level J in $\mathcal{A}$ and $u[l,\sigma] \times v[m,\tau] \times w[n,\omega]$ is an atom at level J in $\mathcal{A}$. If $l \leq i$, then $n > k$.*

3. *Let all the lowest atoms $\lambda_s = u[i_s,\alpha_s] \times v[j_s,\beta_s] \times w[k_s,\varepsilon_s]$ above level J in $\mathcal{A}$, if there are any, be listed as $\lambda_1, \cdots, \lambda_S$ by decreasing i_s and increasing k_s; let all the atoms $\mu_t = u[l_t,\sigma_t] \times v[m_t,\tau_t] \times w[n_t,\omega_t]$ at level J in $\mathcal{A}$, if there are any, be listed as $\mu_1, \cdots, \mu_T$ by decreasing l_t and increasing n_t.*

   (a) For $1 < s \leq S$, there exists μ_t such that $l_t > i_s$ and $n_t > k_{s-1}$.

(b) *If* $l_t > i_s$ *and* $n_t > k_{s-1}$ $(1 < s \leq S)$, *then* $\tau_t = -(-)^{i_s}\alpha_s = -(-)^J\varepsilon_{s-1}$;

if $l_t > i_1$, *then* $\tau_t = -(-)^{i_1}\alpha_1$;

if $n_t > k_S$, *then* $\tau_t = -(-)^J\varepsilon_S$;

if J *is the greatest dimension of second factors of atoms in* $\mathcal{A}$, *then* $\tau_1 = \cdots = \tau_T$.

(c) *If* $1 < t \leq T$ *and if there is no* λ_s *such that* $i_s > l_t$ *and* $k_s > n_{t-1}$, *then* $\omega_{t-1} = -(-)^{l_t+J}\sigma_t$.

(d) *Suppose that* $n_t < k_s$. *If* $l_{t+1} \leq i_s$ $(1 \leq t < T)$, *or if* $s = S$ *and* $t = T$, *then* $\omega_t = -(-)^{i_s+J}\alpha_s$.

(e) *Suppose that* $l_t < i_s$. *If* $n_{t-1} \leq k_s$ $(1 < t \leq T)$ *or if* $s = t = 1$, *then* $\sigma_t = -(-)^{l_t+J}\varepsilon_s$.

(f) *Suppose that* $i_s = l_t$. *If* $k_s > n_{t-1}$ $(1 < t \leq T)$, *or if* $s = t = 1$, *then* $\alpha_s = \sigma_t$.

(g) *Suppose that* $k_s = n_t$. *If* $i_s > l_{t+1}$ $(1 \leq t < T)$, *or if* $s = S$ *and* $t = T$, *then* $\varepsilon_s = \omega_t$.

(h) *If* $1 \leq t < T$ *and* $i_s = l_{t+1}$ *and* $k_s = n_t$, *then* $\alpha_s = \sigma_{t+1}$ *or* $\varepsilon_s = \omega_t$.

Note: By induction, it follows easily from condition 1 and condition 2 in the proposition that, for every integer J less than the greatest dimension of second factors of atoms in $\mathcal{A}$, all the lowest atoms $u[i_s, \alpha_s] \times v[j_s, \beta_s] \times w[k_s, \varepsilon_s]$ above level J in $\mathcal{A}$ can be listed by decreasing i_s and increasing k_s, as required by the assumption in condition 3 of the proposition.

Proof. Suppose that $\mathcal{A}$ is the set of maximal atoms in a molecule Λ. Then an atom $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ in $\mathcal{A}$ is at level J in $\mathcal{A}$ if and only if it is at level J in Λ; $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ is above level J in $\mathcal{A}$ if and only if it is above level J in Λ; while $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ is lowest above level J in $\mathcal{A}$ if and only if it is lowest above level J in Λ. So the necessity follows from the necessity part of Proposition 3.1.2.

Conversely, suppose that a finite and non-empty set $\mathcal{A}$ satisfy conditions 1 to 3. It follows from condition 1 and 2 that $\mathcal{A}$ is the set of maximal atoms in a subcomplex Λ.

As in the proof of the necessity, an atom $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ in $\mathcal{A}$ is at level J in $\mathcal{A}$ if and only if it is at level J in Λ; $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ is above level J in $\mathcal{A}$ if and only if it is above level J in Λ; while $u[i,\alpha] \times v[j,\beta] \times w[k,\varepsilon]$ is lowest above level J in $\mathcal{A}$ if and only if it is lowest above level J in Λ. Therefore the sufficiency follows from the sufficiency part of Proposition 3.1.2.

This completes the proof □

3.2 Construction of Molecules

In this section, we propose an approach of constructing all the molecules in $u \times v \times w$ based on Proposition 3.1.3. The justification will be given in the next section.

We start at the top level and go down.

First choose top level $\bar{J}$ and a fixed sign $\bar{\beta}$ associated with the top level; then choose a list of atoms of the form $u[\bar{i}_s, \bar{\alpha}_s] \times v[\bar{J}, \bar{\beta}] \times w[\bar{k}_s, \bar{\varepsilon}_s]$ for $1 \leq s \leq \bar{S}$, where $\bar{S} \geq 1$, such that $\bar{i}_1 > \cdots > \bar{i}_{\bar{S}}$ and $\bar{k}_1 < \cdots < \bar{k}_{\bar{S}}$ and $\bar{\varepsilon}_{s-1} = -(-)^{\bar{i}_s + \bar{J}} \bar{\alpha}_s$ for $\bar{s} > 1$.

For an integer J with $0 \leq J < \bar{J}$, suppose that the atoms above level J are already constructed. Suppose also that the lowest atoms above level J are $u[i_s, \alpha_s] \times v[j_s, \beta_s] \times w[k_s, \varepsilon_s]$ with $1 \leq s \leq S$ such that $i_1 > \cdots > i_S$ and $k_1 < \cdots < k_S$. By condition 1 in Theorem 2.4.1, the atoms at level J, if there are any, can be listed as a sequence $u[l_t, \sigma_t] \times v[J, \tau_t] \times w[n_t, \omega_t]$ with $1 \leq t \leq T$, where $T \geq 1$, such that $l_1 > \cdots > l_T$ and $n_1 < \cdots < n_T$.

We are going to give all possibilities for the sequence of atoms at level J.

We first determine the possibilities for the sequence $(l_1, n_1, \cdots, l_T, n_T)$ working from left to right.

1. We now determine all the possibilities for l_1 and n_1.

   We determine l_1 as follows.

   (a) If $S = 1$, then there may or may not be atoms at level J; if there is at least one atom at level J, then $l_1 \geq 0$.

   (b) If $S > 1$, then there must be at least one atom at level J and $l_1 > i_2$.

For a fixed l_1, we determine n_1 as follows.

(a) If $l_1 > i_1$ and $\varepsilon_s = (-)^{i_s+J}\alpha_s$ for every s, then $n_1 \geq 0$.

(b) If $l_1 > i_1$ and if there exists s such that $\varepsilon_s = -(-)^{i_s+J}\alpha_s$, then $0 \leq n_1 \leq k_{s_1}$, where s_1 is the least s with $\varepsilon_s = -(-)^{i_s+J}\alpha_s$.

(c) If $l_1 \leq i_1$ and $\varepsilon_s = (-)^{i_s+J}\alpha_s$ for every s with $s > 1$, then $n_1 > k_1$.

(d) If $l_1 \leq i_1$ and there exists s with $s > 1$ such that $\varepsilon_s = -(-)^{i_s+J}\alpha_s$, then $k_1 < n_1 \leq k_{s_2}$, where s_2 is the least s with $s > 1$ and $\varepsilon_s = -(-)^{i_s+J}\alpha_s$.

2. Suppose that $t_0 > 1$ and that l_t and n_t for all $t < t_0$ are already constructed. We are going to determine all the possibilities for l_{t_0} and n_{t_0}.

We determine l_{t_0} as follows. There are various cases.

(a) If $n_{t_0-1} > k_S$ and $l_{t_0-1} = 0$, then there are no more atoms at level J.

(b) If $n_{t_0-1} > k_S$ and $l_{t_0-1} > 0$, then there may or may not be another atom at level J; if there is another atom at level J, then $0 \leq l_{t_0} < l_{t_0-1}$.

(c) Suppose that $n_{t_0-1} = k_S$. Then there may or may not be another atom at level J. Suppose also that there is another atom at level J. If $\varepsilon_S = (-)^{i_S+J}\alpha_S$, then $0 \leq l_{t_0} < l_{t_0-1}$; if $\varepsilon_S = -(-)^{i_S+J}\alpha_S$, then $0 \leq l_{t_0} \leq i_S$.

(d) If $S > 1$ and $k_{S-1} < n_{t_0-1} < k_S$, then there may or may not be another atom at level J; if there is another atom at level J, then $0 \leq l_{t_0} < l_{t_0-1}$.

(e) If $S = 1$ and $n_{t_0-1} < k_1$, then there may or may not be another atom at level J; if there is another atom at level J, then $0 \leq l_{t_0} < l_{t_0-1}$.

(f) Suppose that $1 \leq s < S$ and $n_{t_0-1} = k_s$. Then there must be another atom at level J. Moreover, if $\varepsilon_s = (-)^{i_s+J}\alpha_s$, then $i_{s+1} < l_{t_0} < l_{t_0-1}$; if $\varepsilon_s = -(-)^{i_s+J}\alpha_s$, then $i_{s+1} < l_{t_0} \leq i_s$.

(g) If $1 < s < S$ and $k_{s-1} < n_{t_0-1} < k_s$, then there must be another atom at level J and $i_{s+1} < l_{t_0} < l_{t_0-1}$.

(h) If $S > 1$ and $n_{t_0-1} < k_1$, then there must be another atom at level J and $i_2 < l_{t_0} < l_{t_0-1}$.

For a fixed l_{t_0}, we can determine n_{t_0} as follows.

(a) If $l_{t_0} > i_1$ and $\varepsilon_s = (-)^{i_s+J}\alpha_s$ for every s, then $n_{t_0} > n_{t_0-1}$.

(b) If $l_{t_0} > i_1$ and there is s such that $\varepsilon_s = -(-)^{i_s+J}\alpha_s$, then $n_{t_0-1} < n_{t_0} \leq k_{s_3}$, where s_3 be the least s with $\varepsilon_s = -(-)^{i_s+J}\alpha_s$.

(c) If $l_{t_0} \leq i_S$, then $n_{t_0} > \max\{n_{t_0-1}, k_S\}$.

(d) If $S > 1$ and $i_{s_4} < l_{t_0} \leq i_{s_4-1}$ for some s_4, and if $\varepsilon_s = (-)^{i_s+J}\alpha_s$ for every s with $s \geq s_4$, then $n_{t_0} > \max\{k_{s_4-1}, n_{t_0-1}\}$.

(e) If $S > 1$ and $i_{s_4} < l_{t_0} \leq i_{s_4-1}$ for some s_4, and if there is s such that $s \geq s_4$ and $\varepsilon_s = -(-)^{i_s+J}\alpha_s$, then $\max\{k_{s_4-1}, n_{t_0-1}\} < n_{t_0} \leq k_{s_5}$, where s_5 is the least s with $s \geq s_4$ and $\varepsilon_s = -(-)^{i_s+J}\alpha_s$.

This completes the construction of the sequence $(l_1, n_1, \cdots, l_T, n_T)$.

We now determine the signs σ_t, τ_t and ω_t for each t.

We can determine τ_t for each t, as follows.

1. If $l_t > i_1$, then $\tau_t = -(-)^{i_1}\alpha_1$.

2. If $s > 1$ and $i_s < l_t \leq i_{s-1}$, then $\tau_t = -(-)^{i_s}\alpha_s$.

3. If $l_t \leq i_S$ (in this case, we have $n_t > k_S$), then $\tau_t = -(-)^J\varepsilon_S$.

We now determine signs σ_t and ω_t for each t.

We first determine σ_1.

1. If $l_1 > i_1$, then σ_1 is arbitrary.

2. If $l_1 = i_1$, then $\sigma_1 = \alpha_1$.

3. If $l_1 < i_1$, then $\sigma_1 = -(-)^{l_1+J}\varepsilon_1$.

We next determine ω_{t-1} and σ_t for $1 < t \leq T$. Note that there can be at most one value of s such that $i_s \geq l_t$ and $k_s \geq n_{t-1}$ by the construction of l_t. There are various cases, as follows.

1. If there is no s such that $i_s \geq l_t$ and $k_s \geq n_{t-1}$, then ω_{t-1} is arbitrary and $\sigma_t = -(-)^{l_t+J}\omega_{t-1}$ for a fixed ω_{t-1}.

2. If there exists s such that $i_s > l_t$ and $k_s > n_{t-1}$, then $\omega_{t-1} = -(-)^{i_s+J}\alpha_s$ and $\sigma_t = -(-)^{l_t+J}\varepsilon_s$.

3. If there exists s such that $i_s = l_t$ and $k_s > n_{t-1}$, then $\omega_{t-1} = -(-)^{i_s+J}\alpha_s$ and $\sigma_t = \alpha_s$.

4. If there exists s such that $i_s > l_t$ and $k_s = n_{t-1}$, then $\omega_{t-1} = \varepsilon_s$ and $\sigma_t = -(-)^{l_t+J}\varepsilon_s$.

5. Suppose that there exists s such that $i_s = l_t$ and $k_s = n_{t-1}$. If $\varepsilon_s = (-)^{i_s+J}\alpha_s$, then ω_{t-1} is arbitrary and $\sigma_t = -(-)^{l_t+J}\omega_{t-1}$ for a fixed ω_{t-1}. If $\varepsilon_s = -(-)^{i_s+J}\alpha_s$, then $\omega_{t-1} = \varepsilon_s$ and $\sigma_t = \alpha_s$.

Finally, we determine ω_T.

1. If $n_T > k_S$, then ω_T is arbitrary.

2. If $n_T = k_S$, then $\omega_T = \varepsilon_T$.

3. If $n_T < k_S$, then $\omega_T = -(-)^{i_S+J}\alpha_S$.

This completes the construction of all the possibilities for the sequence of atoms at level J. Therefore, by induction, we can construct all the molecules in $u \times v \times w$.

Remark 3.2.1. In a subcomplex as constructed in the last section, we verify that the permitted value of l_t and n_t form non-empty intervals of integers for each t.

By the construction of atoms at level J, it is evident that a lowest atom above level J and an atom at level J satisfy condition 1 in Theorem 2.4.1.

1. It is evident that the permitted value of l_1 and n_1 form non-empty intervals of integers.

2. In the construction of l_{t_0}, it is evident that the permitted values of l_{t_0} form a non-empty interval of integers in (c) part two and (f) part two. If $n_{t_0-1} \leq k_S$, then we have $l_{t_0-1} > i_S \geq 0$. Therefore the permitted values of l_{t_0} in (b), (c) part one, (d) and (e) form a non-empty interval of integers. Finally, if $s < S$ and $n_{t_0-1} \leq k_s$, then we have $l_{t_0-1} > i_s \geq i_{s+1} + 1$ by condition 1 for $u[i_s, \alpha_s] \times v[j_s, \beta_s] \times w[k_s, \varepsilon_s]$ and $u[l_{t_0-1}, \sigma_{t_0-1}] \times v[J, \tau] \times w[n_{t_0-1}, \omega_{t_0-1}]$. This implies that the permitted values of l_{t_0} form a non-empty interval of integers in (f) part one, (g) and (h).

3. In the construction of n_{t_0}, it is evident that the permitted values of n_{t_0} forms a non-empty interval of integers in (a), (c) and (d).

   Suppose that $l_{t_0} > i_{s_4}$ for some s_4. Suppose also that there is s with $s \geq s_4$ such that $\varepsilon_s = -(-)^{i_s+J}\alpha_s$. Let s_5 be the least s with $s \geq s_4$ and $\varepsilon_s = -(-)^{i_s+J}\alpha_s$. We claim that $n_{t_0-1} < k_{s_5}$ which implies that the permitted values of n_{t_0} forms a non-empty interval of integers in (b) and (e).

   Indeed, since $l_{t_0-1} > l_{t_0} > i_{s_4} \geq i_{s_5}$, we have $n_{t_0-1} \leq k_{s_5}$ by the construction of l_{t_0-1} and n_{t_0-1}. If $n_{t_0-1} = k_{s_5}$, then $l_{t_0} \leq i_{s_5} \leq i_{s_4}$ by the construction of l_{t_0}; this contradicts the assumption on l_{t_0}. Therefore we have $n_{t_0-1} < k_{s_5}$, as required.

Therefore, the permitted value of l_t and n_t form non-empty intervals of integers for each t.

Example 3.2.2. The the molecule in Example 2.1.5 is really constructed by the approach in this section. The construction of the example involves most of the above cases.

3.3 Justification

In this section, we prove that the construction in the last section indeed gives molecules in $u \times v \times w$.

Lemma 3.3.1. *In a subcomplex as constructed in the last section, for every level J, the atoms $u[l_t, \sigma_t] \times v[J, \tau_t] \times w[n_t, \omega_t]$ with $1 \leq t \leq T$ at level J satisfy $l_1 > \cdots > l_T$ and $n_1 < \cdots < n_T$.*

Proof. By induction, it suffices to verify that $l_{t_0} < l_{t_0-1}$ and $n_{t_0} > n_{t_0-1}$ in the construction of l_{t_0} and n_{t_0}.

In the construction of l_{t_0}, we have already required that $l_{t_0} < l_{t_0-1}$ except in (c) part two and (f) part two. Now if $n_{t_0-1} = k_s$ for some s, then $i_s < l_{t_0-1}$ by the earlier part of construction (or, more precisely, by the induction hypothesis); hence $l_{t_0} < l_{t_0-1}$ in (c) part two and (f) part two, as required.

In the construction of n_{t_0}, we have already required that $n_{t_0} > n_{t_0-1}$ in all cases.

Therefore, the atoms $u[l_t, \sigma_t] \times v[J, \tau_t] \times w[n_t, \omega_t]$ at level J as constructed can be listed by decreasing l_t and increasing n_t for each level J, as required.

This completes the proof. □

Lemma 3.3.2. *In a subcomplex as constructed in the last section, all the atoms constructed satisfy condition 1 in Theorem 2.4.1. Hence all the lowest atoms $u[i_s, \alpha_s] \times v[j_s, \beta_s] \times w[k_s, \varepsilon_s]$ above level $J-1$ can be listed such that $i_1 > \cdots > i_S$ and $k_1 < \cdots < k_S$.*

Proof. We first show that all the atoms constructed satisfy condition 1 in Theorem 2.4.1.

It is evident that all the atoms at the top level $\bar{J}$ satisfy condition 1 in Theorem 2.4.1.

Suppose that $J < \bar{J}$ and that all the atoms above level J satisfy condition 1 in Theorem 2.4.1. We are going to show that all the atoms above level $J-1$ satisfy condition 1 in Theorem 2.4.1.

It follows from the Lemma 3.3.1 that a pair of atoms at level J satisfy condition 1 in Theorem 2.4.1. Let $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ be an atom above level J and $u[l, \sigma] \times v[J, \omega] \times w[n, \omega]$ be an atom at level J. If $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ is lowest above level J, then it is evident that $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ and $u[l, \sigma] \times v[J, \omega] \times w[n, \omega]$ satisfy condition 1 in Theorem 2.4.1 by the construction of atoms at level J. If $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ is not lowest above level J, then there is a lowest atom $u[i', \alpha'] \times v[j', \beta'] \times w[k', \varepsilon']$ above level J with $i' \geq i$, $j' < j$ and $k' \geq k$; hence condition 1 for $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$ and $u[l, \sigma] \times v[J, \omega] \times w[n, \omega]$ follows easily from condition 1 for $u[i, \alpha] \times v[j, \beta] \times w[k, \varepsilon]$

and $u[l,\sigma]\times v[J,\omega]\times w[n,\omega]$. Thus all the atoms above level $J-1$ satisfy condition 1 in Theorem 2.4.1.

Therefore, all the atoms satisfy condition 1 in Theorem 2.4.1.

Now let $u[i,\alpha]\times v[j,\beta]\times w[k,\varepsilon]$ and $u[i',\alpha']\times v[j',\beta']\times w[k',\varepsilon']$ be a pair of lowest atoms above level J. By the definition of lowest, we have $i\neq i'$ and $k\neq k'$. Moreover, it follows easily from condition 1 for pairwise molecular subcomplexes for $u[i,\alpha]\times v[j,\beta]\times w[k,\varepsilon]$ and $u[i',\alpha']\times v[j',\beta']\times w[k',\varepsilon']$ that $i>i'$ if and only if $k<k'$. Hence all the lowest atoms $u[i_s,\alpha_s]\times v[j_s,\beta_s]\times w[k_s,\varepsilon_s]$ above level J can be listed by decreasing i_s and increasing k_s, as required. $\square$

Lemma 3.3.3. *In a subcomplex as constructed in the last section, let all the lowest atoms* $u[\hat{i}_s,\hat{\alpha}_s]\times v[\hat{j}_s,\hat{\beta}_s]\times w[\hat{k}_s,\hat{\varepsilon}_s]$ *above level* $J-1$ *be listed such that* $\hat{i}_1>\cdots>\hat{i}_{\hat{S}}$ *and* $\hat{k}_1<\cdots<\hat{k}_{\hat{S}}$. *Then* $\min\{\hat{j}_{s-1},\hat{j}_s\}=J$ *and* $\hat{\varepsilon}_{s-1}=-(-)^{\hat{i}_s+J}\hat{\alpha}_s$ *for every* $1<s\leq\hat{S}$.

Proof. Let $u[\hat{i}_{s-1},\hat{\alpha}_{s-1}]\times v[\hat{j}_{s-1},\hat{\beta}_{s-1}]\times w[\hat{k}_{s-1},\hat{\varepsilon}_{s-1}]$ and $u[\hat{i}_s,\hat{\alpha}_s]\times v[\hat{j}_s,\hat{\beta}_s]\times w[\hat{k}_s,\hat{\varepsilon}_s]$ be a pair of consecutive lowest atoms above level $J-1$. We first show that $\min\{\hat{j}_{s-1},\hat{j}_s\}=J$.

Indeed, suppose otherwise that $\min\{\hat{j}_{s-1},\hat{j}_s\}>J$. Then we can see that $u[\hat{i}_{s-1},\hat{\alpha}_{s-1}]\times v[\hat{j}_{s-1},\hat{\beta}_{s-1}]\times w[\hat{k}_{s-1},\hat{\varepsilon}_{s-1}]$ and $u[\hat{i}_s,\hat{\alpha}_s]\times v[\hat{j}_s,\hat{\beta}_s]\times w[\hat{k}_s,\hat{\varepsilon}_s]$ are lowest atoms above level J and they are consecutive in the list for lowest atoms above level J. It follows from the construction that there is an atom $u[l,\sigma]\times v[J,\hat{\beta}]\times w[n,\omega]$ at level J with $l>\hat{i}_s$ and $n>\hat{k}_{s-1}$. Since $u[\hat{i}_{s-1},\hat{\alpha}_{s-1}]\times v[\hat{j}_{s-1},\hat{\beta}_{s-1}]\times w[\hat{k}_{s-1},\hat{\varepsilon}_{s-1}]$ and $u[\hat{i}_s,\hat{\alpha}_s]\times v[\hat{j}_s,\hat{\beta}_s]\times w[\hat{k}_s,\hat{\varepsilon}_s]$ are lowest atom above level $J-1$, we have $\hat{i}_{s-1}>l>\hat{i}_s$ and $\hat{k}_{s-1}<n<\hat{k}_s$. This contradicts the assumption that $u[\hat{i}_{s-1},\hat{\alpha}_{s-1}]\times v[\hat{j}_{s-1},\hat{\beta}_{s-1}]\times w[\hat{k}_{s-1},\hat{\varepsilon}_{s-1}]$ and $u[\hat{i}_s,\hat{\alpha}_s]\times v[\hat{j}_s,\hat{\beta}_s]\times w[\hat{k}_s,\hat{\varepsilon}_s]$ are a pair of consecutive lowest atoms above level $J-1$.

Now we are going to show that $\hat{\varepsilon}_{s-1}=-(-)^{\hat{i}_s+J}\hat{\alpha}_s$ for every $1<s\leq\hat{S}$. Note that either $u[\hat{i}_{s-1},\hat{\alpha}_{s-1}]\times v[\hat{j}_{s-1},\hat{\beta}_{s-1}]\times w[\hat{k}_{s-1},\hat{\varepsilon}_{s-1}]$ or $u[\hat{i}_s,\hat{\alpha}_s]\times v[\hat{j}_s,\hat{\beta}_s]\times w[\hat{k}_s,\hat{\varepsilon}_s]$ is an atom at level J by the first part of the lemma. Now there are several cases, as follows.

If both $u[\hat{i}_{s-1},\hat{\alpha}_{s-1}]\times v[\hat{j}_{s-1},\hat{\beta}_{s-1}]\times w[\hat{k}_{s-1},\hat{\varepsilon}_{s-1}]$ and $u[\hat{i}_s,\hat{\alpha}_s]\times v[\hat{j}_s,\hat{\beta}_s]\times w[\hat{k}_s,\hat{\varepsilon}_s]$ are atoms at level J, then, by the construction of the signs for atoms at level J, it is evident that $\hat{\varepsilon}_{s-1}=-(-)^{\hat{i}_s+J}\hat{\alpha}_s$ for every $1<s\leq\hat{S}$, as required.

Suppose that $u[\hat{i}_{s-1},\hat{\alpha}_{s-1}]\times v[\hat{j}_{s-1},\hat{\beta}_{s-1}]\times w[\hat{k}_{s-1},\hat{\varepsilon}_{s-1}]$ is an atom above level J and

$u[\hat{i}_s, \hat{\alpha}_s] \times v[\hat{j}_s, \hat{\beta}_s] \times w[\hat{k}_s, \hat{\varepsilon}_s]$ is an atom at level J. Then $u[\hat{i}_s, \hat{\alpha}_s] \times v[\hat{j}_s, \hat{\beta}_s] \times w[\hat{k}_s, \hat{\varepsilon}_s] = u[l_t, \sigma_t] \times v[J, \tau_t] \times w[n_t, \omega_t]$ for some t in the construction. If $t = 1$, then we have $u[\hat{i}_{s-1}, \hat{\alpha}_{s-1}] \times v[\hat{j}_{s-1}, \hat{\beta}_{s-1}] \times w[\hat{k}_{s-1}, \hat{\varepsilon}_{s-1}] = u[i_1, \alpha_1] \times v[j_1, \beta_1] \times w[k_1, \varepsilon_1]$ and $\hat{\varepsilon}_{s-1} = -(-)^{\hat{i}_s+J}\hat{\alpha}_s$ by the construction, as required. If $t > 1$, then it is easy to see that $s > 2$ and $u[\hat{i}_{s-2}, \hat{\alpha}_{s-2}] \times v[\hat{j}_{s-2}, \hat{\beta}_{s-2}] \times w[\hat{k}_{s-2}, \hat{\varepsilon}_{s-2}] = u[l_{t-1}, \sigma_{t-1}] \times v[J, \tau_{t-1}] \times w[n_{t-1}, \omega_{t-1}]$ by the first part of this lemma; thus we have $\hat{\varepsilon}_{s-1} = -(-)^{\hat{i}_s+J}\hat{\alpha}_s$ by the construction of signs, as required.

Suppose that $u[\hat{i}_{s-1}, \hat{\alpha}_{s-1}] \times v[\hat{j}_{s-1}, \hat{\beta}_{s-1}] \times w[\hat{k}_{s-1}, \hat{\varepsilon}_{s-1}]$ is an atom above level J and $u[\hat{i}_s, \hat{\alpha}_s] \times v[\hat{j}_s, \hat{\beta}_s] \times w[\hat{k}_s, \hat{\varepsilon}_s]$ is an atom at level J. By an argument similar to the above case, one can also get $\hat{\varepsilon}_{s-1} = -(-)^{\hat{i}_s+J}\hat{\alpha}_s$ for every $1 < s \leq \hat{S}$, as required.

This completes the proof. $\square$

By Proposition 3.1.3 and the remark after the statement of the proposition, it is easy to see that every molecule can be constructed as above. Now we are going to prove that every subcomplex of $u \times v \times w$ constructed as above is indeed a molecule.

Proposition 3.3.4. *Let Λ be a subcomplex whose maximal atoms are as constructed above. Then Λ is a molecule.*

Proof. Let $\mathcal{A}$ be a set of atoms as constructed above. It suffices to show that $\mathcal{A}$ satisfies all the conditions in Proposition 3.1.3.

By Lemmas 3.3.1 and 3.3.2, it is easy to see that conditions 1 and 2 hold.

Now let all the lowest atoms $\lambda_s = u[i_s, \alpha_s] \times v[j_s, \beta_s] \times w[k_s, \varepsilon_s]$ with dimension of second factors greater than J in $\mathcal{A}$, if there are any, be listed as $\lambda_1, \cdots, \lambda_S$ by decreasing i_s and increasing k_s; let all the atoms $\mu_t = u[l_t, \sigma_t] \times v[m_t, \tau_t] \times w[n_t, \omega_t]$ with dimension of second factors equal to J in $\mathcal{A}$, if there are any, be listed as $\mu_1, \cdots, \mu_T$ by decreasing l_t and increasing n_t. By the construction of l_t and n_t for all t, we can see that condition 3a hold. Moreover, by the construction of signs σ_t and ω_t, it is easy to see that conditions 3c to 3h hold. To complete the proof, we need only to verify condition 3b.

Suppose that $l_t > i_s$ and $n_t > k_{s-1}$ $(1 < s \leq S)$. Let $\hat{s}$ be such that $i_{\hat{s}} < l_t \leq i_{\hat{s}-1}$. By the construction, we have $\tau_t = -(-)^{i_{\hat{s}}}\alpha_{i_{\hat{s}}}$. If $s = \hat{s}$, then $\tau_t = -(-)^{i_s}\alpha_s = -(-)^{i_s}[-(-)^{i_s+J+1}]\varepsilon_{s-1} = -(-)^J\varepsilon_{s-1}$, as required, since $\varepsilon_{s-1} = -(-)^{i_s+J+1}\alpha_s$ by Lemma

3.3.3. Suppose that $\hat{s} < s$. Then $l_t > i_{\hat{s}} > \cdots > i_s$ and $n_t > k_{s-1} > \cdots > k_{\hat{s}}$. Hence $\varepsilon_{\hat{s}} = (-)^{i_{\hat{s}}+J}\alpha_{\hat{s}}, \cdots, \varepsilon_{s-1} = (-)^{i_{s-1}+J}\alpha_{s-1}$ by the construction of signs. It follows that

$$\begin{aligned}
\tau_t &= -(-)^{i_{\hat{s}}}\alpha_{i_{\hat{s}}} \\
&= -(-)^{i_{\hat{s}}}(-)^{i_{\hat{s}}+J}\varepsilon_{\hat{s}} \\
&= -(-)^{J}\varepsilon_{\hat{s}} \\
&= -(-)^{J}[-(-)^{i_{\hat{s}+1}+J+1}\alpha_{i_{\hat{s}+1}}] \\
&= -(-)^{i_{\hat{s}+1}}\alpha_{i_{\hat{s}+1}} \\
&= \cdots \\
&= -(-)^{J}\varepsilon_{s-1} \\
&= -(-)^{i_s}\alpha_s
\end{aligned}$$

as required by condition 3b.

The other parts of condition 3b can be seen easily from the construction of the sign τ_t for each t.

This completes the proof.

□

Chapter 4

Molecules in the Product of Four Infinite-Dimensional Globes

In this chapter, we study molecules in the product of four infinite dimensional globes. Similar to the results for molecules in the product of three infinite dimensional globes, we are going to give some equivalent descriptions for the molecules in the product of four infinite dimensional globes. The discussion is in parallel to that in chapter 2. There are some new features because of the two 'middle' factors.

In this chapter, all the subcomplexes refer to finite and non-empty subcomplexes in the ω-complex $u_1 \times u_2 \times u_3 \times u_4$, and all the integers refer to non-negative integers.

Recall that the ω-complex u^I is equivalent to the infinite dimensional globe u. It is easy to see that this equivalence induces an equivalence $u \times v \times w$ to $u^I \times v^J \times w^K$ of ω-complexes sending every atom $u_i^\alpha \times v_j^\beta \times w_k^\varepsilon$ to $u^I[i, (-)^I\alpha] \times v^J[j, (-)^J\beta] \times w^K[k, (-)^K\varepsilon]$. Thus all the results for the molecules in the product of three globes can be generalised to the molecules in the product of three 'twisted' infinite dimensional globes $u^I \times v^J \times w^K$. In particular, a pairwise molecular subcomplex in $u^I \times v^J \times w^K$ is defined as the image of a pairwise molecular subcomplex in $u \times v \times w$ under the above equivalence of ω-complexes and a subcomplex of $u^I \times v^J \times w^K$ is a molecule if and only if it is pairwise molecular. We are not going to make any more comments of this kind.

4.1 The Definition of Pairwise Molecular Subcomplexes

In this section, we define projection maps and give the definition of pairwise molecular subcomplexes of $u_1 \times u_2 \times u_3 \times u_4$. Some proofs are omitted because the arguments are very similar to that in Chapter 3.

For an atom $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in $u_1 \times u_2 \times u_3 \times u_4$, let

$$F_{I_1}^{u_1}(\operatorname{Int}\lambda) = \begin{cases} \operatorname{Int}(u_2^{I_1}[i_2, \alpha_2] \times u_3^{I_1}[i_3, \alpha_3] \times u_4^{I_1}[i_4, \alpha_4]), & \text{when } i_1 \geq I_1; \\ \emptyset, & \text{when } j < J. \end{cases}$$

This gives a map sending interiors of atoms in $u_1 \times u_2 \times u_3 \times u_4$ to interiors of atoms in $u_2^{I_1} \times u_3^{I_1} \times u_4^{I_1}$ or the empty set.

Since interiors of atoms are disjoint, it is clear that the map $F_{I_1}^{u_1}$ can be extended uniquely to a map sending unions of interiors of atoms in $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ to unions of interiors of atoms in $u_2^{I_1} \times u_3^{I_1} \times u_4^{I_1}$ by requiring it union-preserving.

We can similarly define a map $F_{I_2}^{u_2}$ sending unions of interiors of atoms in $u_1 \times u_2 \times u_3 \times u_4$ to unions of interiors of atoms in $u_1 \times u_3^{I_2} \times u_4^{I_2}$, a map $F_{I_3}^{u_3}$ sending unions of interiors of atoms in $u_1 \times u_2 \times u_3 \times u_4$ to unions of interiors of atoms in $u_1 \times u_2 \times u_4^{I_3}$ and a map $F_{I_4}^{u_4}$ sending unions of interiors of atoms in $u_1 \times u_2 \times u_3 \times u_4$ to unions of interiors of atoms in $u_1 \times u_2 \times u_3$.

It is easy to see that every atom in $u_1 \times u_2 \times u_3 \times u_4$ can be written as a union of interiors of atoms. It follows that $F_{I_1}^{u_1}$, $F_{I_2}^{u_2}$, $F_{I_3}^{u_3}$ and $F_{I_4}^{u_4}$ are defined on subcomplexes of $u_1 \times u_2 \times u_3 \times u_4$ and preserve unions.

We shall prove that $F_{I_s}^{u_s}$ sends atoms to atoms or the empty set so that it sends subcomplexes to subcomplexes for every s. We need a preliminary result.

Lemma 4.1.1. *Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ be an atom in $u_1 \times u_2 \times u_3 \times u_4$.*

1. *If $i_1 + i_2 + i_3 + i_4 \leq p$, then $d_p^\gamma \lambda = \lambda$.*

2. *If $i_1+i_2+i_3+i_4>p$, then the set of maximal atoms in $d_p^\gamma\lambda$ consists of all the atoms $u_1[l_1,\sigma_1]\times u_2[l_2,\sigma_2]\times u_3[l_3,\sigma_3]\times u_4[l_4,\sigma_4]$ such that $l_1\le i_1$, $l_2\le i_2$, $l_3\le i_3$, $l_4\le i_4$, where the signs σ_1, σ_2, σ_3 and σ_4 are determined as follows:*

   (a) *if $l_1=i_1$, then $\sigma_1=\alpha_1$; If $l_1<i_1$, then $\sigma_1=\gamma$;*

   (b) *if $l_2=i_2$, then $\sigma_2=\alpha_2$; If $l_2<i_2$, then $\sigma_2=(-)^{l_1}\gamma$;*

   (c) *if $l_3=i_3$, then $\sigma_3=\alpha_3$; If $l_3<i_3$, then $\sigma_3=(-)^{l_1+l_2}\gamma$;*

   (d) *if $l_4=i_4$, then $\sigma_4=\alpha_4$; If $l_4<i_4$, then $\sigma_4=(-)^{l_1+l_2+l_3}\gamma$.*

Proposition 4.1.2. *Let $\lambda=u_1[i_1,\alpha_1]\times u_2[i_2,\alpha_2]\times u_3[i_3,\alpha_3]\times u_4[i_4,\alpha_4]$ be an atom in $u_1\times u_2\times u_3\times u_4$. Then*

1. $$F_{I_1}^{u_1}(\lambda)=\begin{cases} u_2^{I_1}[i_2,\alpha_2]\times u_3^{I_1}[i_3,\alpha_3]\times u_4^{I_1}[i_4,\alpha_4], & \text{when } i_1\ge I_1;\\ \emptyset, & \text{when } i_1<I_1;\end{cases}$$

2. $$F_{I_2}^{u_2}(\lambda)=\begin{cases} u_1[i_1,\alpha_1]\times u_3^{I_2}[i_3,\alpha_3]\times u_4^{I_2}[i_4,\alpha_4], & \text{when } i_2\ge I_2;\\ \emptyset, & \text{when } i_2<I_2;\end{cases}$$

3. $$F_{I_3}^{u_3}(\lambda)=\begin{cases} u_1[i_1,\alpha_1]\times u_2[i_2,\alpha_2]\times u_4^{I_3}[i_4,\alpha_4], & \text{when } i_3\ge I_3;\\ \emptyset, & \text{when } i_3<I_3;\end{cases}$$

4. $$F_{I_4}^{u_4}(\lambda)=\begin{cases} u_1[i_1,\alpha_1]\times u_2[i_2,\alpha_2]\times u_3[i_3,\alpha_3], & \text{when } i_4\ge I_4;\\ \emptyset, & \text{when } i_4<I_4.\end{cases}$$

In particular, the maps $F_{I_1}^{u_1}$, $F_{I_2}^{u_2}$, $F_{I_3}^{u_3}$ and $F_{I_4}^{u_4}$ send atoms to atoms or the empty set.

Proof. The arguments for the four cases are similar. We only prove the second one. The proof is given by induction on dimension of atoms.

For an atom $\lambda=u_1[i_1,\alpha_1]\times u_2[i_2,\alpha_2]\times u_3[i_3,\alpha_3]\times u_4[i_4,\alpha_4]$ in $u_1\times u_2\times u_3\times u_4$, if

$\dim\lambda = 0$, then $i_1 = i_2 = i_3 = i_4 = 0$; hence

$$\begin{aligned}
& F^{u_2}_{I_2}(\lambda) \\
= \ & F^{u_2}_{I_2}(\operatorname{Int}\lambda) \\
= \ & \begin{cases} \operatorname{Int}(u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4]), & \text{when } I_2 = 0; \\ \emptyset, & \text{when } I_2 > 0 \end{cases} \\
= \ & \begin{cases} u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4], & \text{when } I_2 = 0; \\ \emptyset, & \text{when } I_2 > 0 \end{cases}
\end{aligned}$$

as required.

Suppose that $p > 0$ and that the proposition holds for every atom of dimension less then p. Suppose also that $\lambda = u_1[i_1,\alpha_1]\times u_2[i_2,\alpha_2]\times u_3[i_3,\alpha_3]\times u_4[i_4,\alpha_4]$ is a p-dimensional atom. If $i_2 < I_2$, then it is easy to see that $F^{u_2}_{I_2}(\lambda) = \emptyset$, as required. If $i_2 > I_2$, then we have

$$\begin{aligned}
& F^{u_2}_{I_2}(\lambda) \\
= \ & F^{u_2}_{I_2}(\operatorname{Int}\lambda \cup \partial^-\lambda \cup \partial^+\lambda) \\
\supset \ & F^{u_2}_{I_2}(\partial^+\lambda) \\
\supset \ & F^{u_2}_{I_2}(u_1[i_1,\alpha_1] \times u_2[i_2-1,(-)^{i_1}] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4]) \\
= \ & u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4]
\end{aligned}$$

since $u[i,\alpha] \times v[j-1,(-)^i] \times w[k,\varepsilon]$ is an atom of dimension $p-1$; the reverse inclusion holds automatically; so $F^{u_2}_{I_2}(\lambda) = u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4]$, as required. Suppose that $i_2 = I_2$. Then $\partial^\gamma\lambda$ is the union of atoms $u_1[l_1,\sigma_1] \times u_2[l_2,\sigma_2] \times u_3[l_3,\sigma_3] \times u_4[l_4,\sigma_4]$ with $l_1 + l_2 + l_3 + l_4 = p - 1$ such that

1. if $l_1 = i_1$, then $\sigma_1 = \alpha_1$; if $l_1 = i_1 - 1$, then $\sigma_1 = \gamma$;
2. if $l_2 = i_2$, then $\sigma_2 = \alpha_2$; if $l_2 = i_2 - 1$, then $\sigma_2 = (-)^{i_1}\gamma$;
3. if $l_3 = i_3$, then $\sigma_3 = \alpha_3$; if $l_3 = i_3 - 1$, then $\sigma_3 = (-)^{i_1+I_2}\gamma$.
4. if $l_4 = i_4$, then $\sigma_4 = \alpha_4$; if $l_4 = i_4 - 1$, then $\sigma_4 = (-)^{i_1+I_2+i_3}\gamma$.

It follows easily from the inductive hypothesis and Theorem 2.5.12 that $F^{u_2}_{I_2}(\partial^\gamma\lambda) = \partial^\gamma(u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4])$ for every sign γ. Therefore

$$\begin{aligned}
& F^{u_2}_{I_2}(\lambda) \\
= \;& F^{u_2}_{I_2}(\operatorname{Int}\lambda) \cup F^{u_2}_{I_2}(\partial^-\lambda) \cup F^{u_2}_{I_2}(\partial^+\lambda) \\
= \;& \operatorname{Int}(u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4]) \cup \partial^-(u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4]) \cup \\
& \partial^+(u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4]) \\
= \;& u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4],
\end{aligned}$$

as required.

This completes the proof of the proposition. □

We now define the concept of pairwise molecular subcomplexes as follows.

Definition 4.1.3. Let Λ be a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Then Λ is *pairwise molecular* if

1. There are no distinct maximal atoms $u_1[i_1,\alpha_1] \times u[i_2,\alpha_2] \times u[i_3,\alpha_3] \times u[i_4,\alpha_4]$ and $u_1[i'_1,\alpha'_1] \times u[i'_2,\alpha'_2] \times u[i'_3,\alpha'_3] \times u[i'_4,\alpha'_4]$ in Λ such that $i_1 \le i'_1$, $i_2 \le i'_2$, $i_3 \le i'_3$ and $i_4 \le i'_4$.

2. $F^{u_1}_{I_1}(\Lambda)$ is a molecule in $u_2^{I_1} \times u_3^{I_1} \times u_4^{I_1}$ or the empty set for every integer I_1.

3. $F^{u_2}_{I_2}(\Lambda)$ is a molecule in $u_1 \times u_3^{I_2} \times u_4^{I_2}$ or the empty set for every integer I_2.

4. $F^{u_3}_{I_3}(\Lambda)$ is a molecule in $u_1 \times u_2 \times u_4^{I_3}$ or the empty set for every integer I_3.

5. $F^{u_4}_{I_4}(\Lambda)$ is a molecule in $u_1 \times u_2 \times u_3$ or the empty set for every integer I_4.

Note. The reason that a subcomplex satisfying the above conditions is said to be pairwise molecular is made clear in the following Proposition 4.1.6.

One of the main result in this chapter is as follows.

Theorem 4.1.4. *A subcomplex in $u_1 \times u_2 \times u_3 \times u_4$ is a molecule if and only if it is pairwise molecular.*

Proposition 4.1.5. *Let Λ be a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Then $F^{u_s}_{I_s}[F^{u_t}_{I_t}(\Lambda)] = F^{u_t}_{I_t}[F^{u_s}_{I_s}(\Lambda)]$ for all s and t with $1 \le s,t \le 4$ and $s \ne t$.*

Proof. Let λ be an atom in $u_1 \times u_2 \times u_3 \times u_4$. It is evident from the definition that $F^{u_s}_{I_s}[F^{u_t}_{I_t}(\lambda)] = F^{u_t}_{I_t}[F^{u_s}_{I_s}(\lambda)]$ for all s and t with $1 \leq s, t \leq 4$ and $s \neq t$. Since $F^{u_s}_{I_s}$ and $F^{u_t}_{I_t}$ preserve unions, we can see that $F^{u_s}_{I_s}[F^{u_t}_{I_t}(\Lambda)] = F^{u_t}_{I_t}[F^{u_s}_{I_s}(\Lambda)]$, as required.

□

For every finite non-empty subcomplex Λ of $u_1 \times u_2 \times u_3 \times u_4$, the subcomplex $F^{u_s}_{I_s}[F^{u_t}_{I_t}(\Lambda)] = F^{u_t}_{I_t}[F^{u_s}_{I_s}(\Lambda)]$ is denoted by $F^{u_s,u_t}_{I_s,I_t}(\Lambda)$.

Proposition 4.1.6. *Let Λ be a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Then Λ is pairwise molecular if and only if the following conditions hold.*

1. *There are no distinct maximal atoms $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ in Λ such that $i_1 \leq i'_1$, $i_2 \leq i'_2$, $i_3 \leq i'_3$ and $i_4 \leq i'_4$.*

2. *If $F^{u_1}_{I_1}(\Lambda) \neq \emptyset$, then $F^{u_1}_{I_1}(\Lambda)$ satisfies condition 1 for pairwise molecular subcomplexes in $u_2^{I_1} \times u_3^{I_1} \times u_4^{I_1}$.*

3. *If $F^{u_2}_{I_2}(\Lambda) \neq \emptyset$, then $F^{u_2}_{I_2}(\Lambda)$ satisfies condition 1 for pairwise molecular subcomplexes in $u_1 \times u_3^{I_2} \times u_4^{I_2}$.*

4. *If $F^{u}_{I_3}(\Lambda) \neq \emptyset$, then $F^{u_3}_{I_3}(\Lambda)$ satisfies condition 1 for pairwise molecular subcomplexes in $u_1 \times u_2 \times u_4^{I_3}$.*

5. *If $F^{u_4}_{I_4}(\Lambda) \neq \emptyset$, then $F^{u_4}_{I_4}(\Lambda)$ satisfies condition 1 for pairwise molecular subcomplexes in $u_1 \times u_2 \times u_3$.*

6. *$F^{u_1,u_2}_{I_1,I_2}(\Lambda)$ is a molecule in $u_3^{I_1+I_2} \times u_4^{I_1+I_2}$ or the empty set for every pair of integers I_1 and I_2.*

7. *$F^{u_1,u_3}_{I_1,I_3}(\Lambda)$ is a molecule in $u_2^{I_1} \times u_4^{I_1+I_3}$ or the empty set for every pair of integers I_1 and I_3.*

8. *$F^{u_1,u_4}_{I_1,I_4}(\Lambda)$ is a molecule in $u_2^{I_1} \times u_3^{I_1}$ or the empty set for every pair of integers I_1 and I_4.*

9. $F^{u_2,u_3}_{I_2,I_3}(\Lambda)$ *is a molecule in* $u_1 \times u_4^{I_2+I_3}$ *or the empty set for every pair of integers* I_2 *and* I_3.

10. $F^{u_2,u_4}_{I_2,I_4}(\Lambda)$ *is a molecule in* $u_1 \times u_3^{I_2}$ *or the empty set for every pair of integers* I_2 *and* I_4.

11. $F^{u_3,u_4}_{I_3,I_4}(\Lambda)$ *is a molecule in* $u_1 \times u_2$ *or the empty set for every pair of integers* I_3 *and* I_4.

Proof. Suppose that Λ is pairwise molecular. Then $F^{u_s}_{I_s}(\Lambda)$ is a molecule or the empty set for every s. It follows from definition of $F^{u_s,u_t}_{I_s,I_t}$ and Theorem 2.1.6 that conditions 1 to 11 hold.

Conversely, suppose that Λ satisfies condition 1 to 11. By the definition of $F^{u_s,u_t}_{I_s,I_t}$, we can see that $F^{u_1}_{I_1}[F^{u_4}_{I_4}(\Lambda)]$, $F^{u_2}_{I_2}[F^{u_4}_{I_4}(\Lambda)]$ and $F^{u_3}_{I_3}[F^{u_4}_{I_4}(\Lambda)]$ are molecules in the corresponding (twisted) products of two globes or the empty set. Since $F^{u_4}_{I_4}(\Lambda)$ satisfies condition 1 for pairwise molecular subcomplexes, it follows from Theorem 2.1.6 that $F^{u_4}_{I_4}(\Lambda)$ is a molecule in $u_1 \times u_2 \times u_3$ or the empty set. Similarly, we can prove that $F^{u_4}_{I_4}(\Lambda)$, $F^{u_4}_{I_4}(\Lambda)$ and $F^{u_4}_{I_4}(\Lambda)$ are molecules in the corresponding (twisted) product of three globes or the empty set. This shows that Λ is pairwise molecular, as required.

This completes the proof. $\square$

We end this section by a proposition which is used later in this chapter.

Proposition 4.1.7. *Let* Λ *and* Λ' *be subcomplexes of* $u_1 \times u_2 \times u_3 \times u_4$ *satisfying condition 1 for pairwise molecular subcomplexes. If* $F^{u_s}_{I_s}(\Lambda) = F^{u_s}_{I_s}(\Lambda')$ *for every* s *and every* I_s *with* $1 \leq s \leq 4$, *then* $\Lambda = \Lambda'$.

Proof. It suffices to prove that Λ and Λ' consists of the same maximal atoms.

Let $u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4]$ be a maximal atom in Λ. It is easy to see that $u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha_3]$ is a maximal atom in $F^{u_4}_{i_4}(\Lambda) = F^{u_4}_{i_4}(\Lambda')$. Thus Λ' has a maximal atom $u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i'_4,\alpha'_4]$ with $i'_4 \geq i_4$. Since $u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha_3] \not\subset F^{u_4}_{i_4+1}(\Lambda) = F^{u_4}_{i_4+1}(\Lambda')$, we have $i'_4 = i_4$. One can similarly get a maximal atom $u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha'_3] \times u_4[i_4,\alpha_4]$ of Λ'. It follows from condition 1 for pairwise molecular subcomplexes that $\alpha'_4 = \alpha_4$ and $\alpha'_3 = \alpha_3$. This shows that $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ is a maximal atom in Λ'.

Symmetrically, we can see that every maximal atom in Λ' is a maximal atom in Λ.

This completes the proof that $\Lambda = \Lambda'$. □

4.2 Molecules Are Pairwise Molecular

In this section, we prove that molecules in $u_1 \times u_2 \times u_3 \times u_4$ are pairwise molecular. The argument is different from that in section 2 of chapter 2. We show that $F_{I_s}^{u_s}$ sends molecules to molecules or the empty set for every value of s without introducing $g_{I_s}^{u_s}$ (see section 2 of chapter 2).

We first show that molecules satisfy condition 1 for pairwise molecular subcomplexes.

Proposition 4.2.1. *In a molecule of $u_1 \times u_2 \times u_3 \times u_4$, there are no distinct maximal atoms $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ such that $i_s \leq i'_s$ for all $1 \leq s \leq 4$.*

Proof. Suppose otherwise that there are maximal atoms $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ in the molecule with $\lambda \neq \lambda'$ such that $i_s \leq i'_s$ for all $1 \leq s \leq 4$. By decomposing the given molecule, one can get composite of molecules $\Lambda \#_n \Lambda'$ or $\Lambda' \#_n \Lambda$ such that $\lambda \subset \Lambda$, $\lambda \not\subset \Lambda'$, $\lambda' \subset \Lambda'$ and $\lambda' \not\subset \Lambda$. We may assume that the given molecule is decomposed into $\Lambda \#_n \Lambda'$. We now consider two cases, as follows.

1. Suppose that $n \geq i_1 + i_2 + i_3 + i_4$. Then, by Lemma 1.4.16, we have $\lambda \subset d_n^+ \Lambda = d_n^- \Lambda' \subset \Lambda'$. This is a contradiction.

2. Suppose that $n < i_1 + i_2 + i_3 + i_4$. Consider the (natural) homomorphism $F : u_1 \times u_2 \times u_3 \times u_4 \to u_{i_1} \times u_{i_2} \times u_{i_3} \times u_{i_4}$. Since $\Lambda \#_n \Lambda'$ exists, we know that $F(\Lambda \#_n \Lambda') = F(\Lambda) \#_n F(\Lambda')$ exists. On the other hand, one can see that $u_{i_1} \times u_{i_2} \times u_{i_3} \times u_{i_4} \subset F(\lambda) \cap F(\lambda') \subset F(\Lambda) \cap F(\Lambda')$. Therefore $\dim[F(\Lambda) \cap F(\Lambda')] \geq i_1 + i_2 + i_3 + i_4 > n$. Since $\dim[d_n^+ F(\Lambda)] \leq n$, one gets $d_n^+ F(\Lambda) \neq F(\Lambda) \cap F(\Lambda')$. This contradicts that $F(\Lambda \#_n \Lambda') = F(\Lambda) \#_n F(\Lambda')$ exists.

This completes the proof.

$\square$

We next show that $F_{I_s}^{u_s}$ sends molecules to molecules or the empty set for every value of s. The arguments for different values of s are similar. We only give the proof for $s = 2$.

Lemma 4.2.2. *Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ be an atom in the ω-complex $u_1 \times u_2 \times u_3 \times u_4$ and $\Lambda, \Lambda' \in \mathcal{M}(u_1 \times u_2 \times u_3 \times u_4)$. Then*

1. *$F_{I_2}^{u_2}(\lambda) \in \mathcal{A}(u_1 \times u_3^{I_2} \times u_4^{I_2}) \cup \{\emptyset\}$;*

2. *If $\Lambda \#_n \Lambda'$ is defined, then $F_{I_2}^{u_2}(\Lambda \#_n \Lambda') = F_{I_2}^{u_2}(\Lambda) \cup F_{I_2}^{u_2}(\Lambda')$;*

3. *$F_{I_2}^{u_2}(\Lambda) \neq \emptyset$ if and only if there is a maximal atom $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in Λ such that $i_2 \geq I_2$;*

4. $$F_{I_2}^{u_2}(d_p^\gamma \lambda) = \begin{cases} d_{p-I_2}^\gamma F_{I_2}^{u_2}(\lambda) & \textit{when } p \geq I_2 \textit{ and } i_2 \geq I_2, \\ \emptyset & \textit{when } p < I_2 \textit{ or } i_2 < I_2; \end{cases}$$

Proof. The proof of the first three conditions is a trivial verification from the definition of $F_{I_2}^{u_2}$. We now verify condition 4.

If $p < I_2$ or $i_2 < I_2$, then it is evident that $F_{I_2}^{u_2}(d_p^\gamma \lambda) = \emptyset$ by the definition of $F_{I_2}^{u_2}$.

Now, suppose that $p \geq I_2$ and $i_2 \geq I_2$. Then $F_{I_2}^{u_2}(\lambda) = u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4]$. By proposition 4.1.1, the set of all maximal atoms in $d_p^\gamma \lambda$ consists of all $u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4]$ with $l_s \leq i_s$ for all $1 \leq s \leq 4$ such that $l_1 + l_2 + l_3 + l_4 = p$, where the signs σ_s $(s = 1, 2, 3, 4)$ are determined as follows:

1. If $l_1 = i_1$, then $\sigma_1 = \alpha_1$; if $l_1 < i_1$, then $\sigma_1 = \gamma$.

2. If $l_2 = i_2$, then $\sigma_2 = \alpha_2$; if $l_2 < i_2$, then $\sigma_2 = (-)^{l_1}\gamma$.

3. If $l_3 = i_3$, then $\sigma_3 = \alpha_3$; if $l_3 < i_3$, then $\sigma_3 = (-)^{l_1+l_2}\gamma$.

4. If $l_4 = i_4$, then $\sigma_4 = \alpha_4$; if $l_4 < i_4$, then $\sigma_4 = (-)^{l_1+l_2+l_3}\gamma$.

From this description and the formation of $d_{p-I_2}^\gamma(u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4])$ in $u_1 \times u_3^{I_2} \times u_4^{I_2}$ (Theorem 2.5.12), it is easy to see that $F_{I_2}^{u_2}(d_p^\gamma \lambda) = d_{p-I_2}^\gamma F_{I_2}^{u_2}(\lambda)$, as required. $\square$

Lemma 4.2.3. *Let Λ be a molecule in $u_1 \times u_2 \times u_3 \times u_4$. If Λ is decomposed into $\Lambda = \Lambda' \#_n \Lambda''$ and if $F^{u_2}_{I_2}(\Lambda') \neq \emptyset$ and $F^{u_2}_{I_2}(\Lambda') \neq \emptyset$, then $n \geq I_2$.*

Proof. Let $f^{u_2}_{I_2} : \mathcal{M}(u_1 \times u_2 \times u_3 \times u_4) \to \mathcal{M}(u_1 \times u_{I_2} \times u_3 \times u_4)$ be the natural homomorphism of ω-categories sending every maximal atom $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in $u_1 \times u_2 \times u_3 \times u_4$ with $i_2 < I_2$ to $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$, and sending every maximal atom $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in $u_1 \times u_2 \times u_3 \times u_4$ with $i_2 \geq I_2$ to $u_1[i_1, \alpha_1] \times u_{I_2} \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$. Then $f^{u_2}_{I_2}(\Lambda) = f^{u_2}_{I_2}(\Lambda') \#_n f^{u_2}_{I_2}(\Lambda'')$ is defined. Thus $d^+_n f^{u_2}_{I_2}(\Lambda') = d^-_n f^{u_2}_{I_2}(\Lambda'') = f^{u_2}_{I_2}(\Lambda') \cap f^{u_2}_{I_2}(\Lambda'')$. Since $F^{u_2}_{I_2}(\Lambda') \neq \emptyset$ and $F^{u_2}_{I_2}(\Lambda'') \neq \emptyset$, we know that there are maximal atoms $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ and $\lambda'' = u_1[i''_1, \alpha''_1] \times u_2[i''_2, \alpha''_2] \times u_3[i''_3, \alpha''_3] \times u_4[i''_4, \alpha''_4]$ in Λ' and Λ'' respectively with $i'_2 \geq I_2$ and $i''_2 \geq I_2$ such that $f^{u_2}_{I_2}(\lambda')$ and $f^{u_2}_{I_2}(\lambda'')$ are maximal atoms in $f^{u_2}_{I_2}(\Lambda')$ and $f^{u_2}_{I_2}(\Lambda'')$ respectively. If $f^{u_2}_{I_2}(\lambda')$ is not maximal in $f^{u_2}_{I_2}(\Lambda)$, then it is easy to see that there is a maximal atom in $f^{u_2}_{I_2}(\Lambda'')$ containing $f^{u_2}_{I_2}(\lambda')$; it follows that $n \geq \dim d^-_n f^{u_2}_{I_2}(\Lambda'') = \dim(f^{u_2}_{I_2}(\Lambda') \cap f^{u_2}_{I_2}(\Lambda'')) \geq \dim f^{u_2}_{I_2}(\lambda') \geq I_2$. Similarly, if $f^{u_2}_{I_2}(\lambda'')$ is not maximal in $f^{u_2}_{I_2}(\Lambda)$, then $n \geq I_2$, as required. In the following proof, we may assume that both $f^{u_2}_{I_2}(\lambda')$ and $f^{u_2}_{I_2}(\lambda'')$ are maximal in $f^{u_2}_{I_2}(\Lambda)$. Now there are two cases as follows.

1. Suppose that $f^{u_2}_{I_2}(\lambda') \cap f^{u_2}_{I_2}(\lambda'') \neq \emptyset$. Then it is easy to see that $n \geq \dim d^+_n f^{u_2}_{I_2}(\Lambda') = \dim(f^{u_2}_{I_2}(\Lambda') \cap f^{u_2}_{I_2}(\Lambda'')) \geq I_2$, as required.

2. Suppose that $f^{u_2}_{I_2}(\lambda') \cap f^{u_2}_{I_2}(\lambda'') = \emptyset$, then we must have $i'_1 = i''_1 = 0$, $i'_3 = i''_3 = 0$ or $i'_4 = i''_4 = 0$. We may assume that $i'_1 = i''_1 = 0$. Thus $\alpha'_1 = -\alpha''_1$. In this case, consider the natural homomorphism $f^{u_1,u_2}_{0,I_2} : \mathcal{M}(u_1 \times u_2 \times u_3 \times u_4) \to \mathcal{M}(u_0 \times u_{I_2} \times u_3 \times u_4)$. It is easy to see that $f^{u_1,u_2}_{0,I_2}(\lambda')$ and $f^{u_1,u_2}_{0,I_2}(\lambda'')$ are maximal in $f^{u_1,u_2}_{0,I_2}(\Lambda)$. It follows that $i'_3 \neq i''_3$ and $i'_4 \neq i''_4$ and hence $f^{u_1,u_2}_{0,I_2}(\Lambda') \cap f^{u_1,u_2}_{0,I_2}(\Lambda'') \neq \emptyset$. Since $f^{u_1,u_2}_{0,I_2}(\Lambda) = f^{u_1,u_2}_{0,I_2}(\Lambda') \#_n f^{u_1,u_2}_{0,I_2}(\Lambda'')$ we can see that $n \geq \dim d^+_n f^{u_1,u_2}_{0,I_2}(\Lambda') = \dim(f^{u_1,u_2}_{0,I_2}(\Lambda') \cap f^{u_1,u_2}_{0,I_2}(\Lambda'')) \geq I_2$, as required.

This completes the proof. □

Proposition 4.2.4. *Let $F^{u_2}_{I_2} : \mathcal{M}(u_1 \times u_2 \times u_3 \times u_4) \to \mathcal{C}(u_1 \times u_3^{I_2} \times u_4^{I_2})$ be the map as above. Then*

1. $F^{u_2}_{I_2}(\mathcal{M}(u_1 \times u_2 \times u_3 \times u_4)) \subset \mathcal{M}(u_1 \times u_3^{I_2} \times u_4^{I_2}) \cup \{\emptyset\}$;

2. For every molecule Λ in $u_1 \times u_2 \times u_3 \times u_4$, we have

$$F_{I_2}^{u_2}(d_p^\gamma \Lambda) = \begin{cases} d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda) & \textit{when } p \geq I_2 \textit{ and } F_{I_2}^{u_2}(\Lambda) \neq \emptyset, \\ \emptyset & \textit{when } p < I_2 \textit{ or } F_{I_2}^{u_2}(\Lambda) = \emptyset. \end{cases}$$

3. If $\Lambda \#_n \Lambda'$ is defined, then

$$F_{I_2}^{u_2}(\Lambda \#_n \Lambda') = \begin{cases} F_{I_2}^{u_2}(\Lambda) \#_{n-I_2} F_{I_2}^{u_2}(\Lambda') & \textit{when } F_{I_2}^{u_2}(\Lambda) \neq \emptyset \textit{ and } F_{I_2}^{u_2}(\Lambda') \neq \emptyset, \\ F_{I_2}^{u_2}(\Lambda') & \textit{when } F_{I_2}^{u_2}(\Lambda) = \emptyset, \\ F_{I_2}^{u_2}(\Lambda) & \textit{when } F_{I_2}^{u_2}(\Lambda') = \emptyset. \end{cases}$$

Proof. We are going to prove the first two conditions by induction and then prove the third condition.

By Lemma 4.2.2, it is evident that the first two conditions hold when Λ is an atom.

Now suppose that $q > 1$ and the first two conditions hold for molecules which can be written as a composite of less than q atoms. Suppose also that Λ is a molecule which can be written as a composite of q atoms. Since $q > 1$, we have a proper decomposition $\Lambda = \Lambda' \#_n \Lambda''$ such that Λ' and Λ'' are molecules which can be written as composites of less than q atoms. According to the induction hypothesis, we know that the first two conditions hold for Λ' and Λ''. We must show that the first two conditions in the proposition hold for Λ. There are two cases, as follows.

1. Suppose that $F_{I_2}^{u_2}(\Lambda') = \emptyset$ or $F_{I_2}^{u_2}(\Lambda'') = \emptyset$. We may assume that $F_{I_2}^{u_2}(\Lambda') = \emptyset$. In this case, we have $F_{I_2}^{u_2}(\Lambda) = F_{I_2}^{u_2}(\Lambda'')$. Thus $F_{I_2}^{u_2}(\Lambda) \in \mathcal{M}(u_1 \times u_3^{I_2} \times u_4^{I_2}) \cup \{\emptyset\}$ as required by the first condition. Moreover, if $p \neq n$, then

$$\begin{aligned} & F_{I_2}^{u_2}(d_p^\gamma \Lambda) \\ = \; & F_{I_2}^{u_2}(d_p^\gamma \Lambda' \#_n d_p^\gamma \Lambda'') \\ = \; & F_{I_2}^{u_2}(d_p^\gamma \Lambda') \cup F_{I_2}^{u_2}(d_p^\gamma \Lambda'') \\ = \; & \left\{ \begin{cases} d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda'') & \text{when } p \geq I_2 \text{ and } F_{I_2}^{u_2}(\Lambda'') \neq \emptyset, \\ \emptyset & \text{when } F_{I_2}^{u_2}(\Lambda'') = \emptyset \text{ or } p < I_2, \end{cases} \right. \\ = \; & \left\{ \begin{cases} d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda) & \text{when } p \geq I_2 \text{ and } F_{I_2}^{u_2}(\Lambda) \neq \emptyset, \\ \emptyset & \text{when } F_{I_2}^{u_2}(\Lambda) = \emptyset \text{ or } p < I_2, \end{cases} \right. \end{aligned}$$

as required by the second condition. Suppose that $p = n \geq I_2$. Then $F_{I_2}^{u_2}(d_p^+\Lambda') = \emptyset$. So $F_{I_2}^{u_2}(d_p^-\Lambda'') = \emptyset$. Hence, by the hypothesis, one gets $F_{I_2}^{u_2}(\Lambda'') = \emptyset$. Therefore $F_{I_2}^{u_2}(\Lambda) = \emptyset$ and $F_{I_2}^{u_2}(d_p^\gamma\Lambda) = \emptyset$, as required by the second condition.

2. Suppose that $F_{I_2}^{u_2}(\Lambda') \neq \emptyset$ and $F_{I_2}^{u_2}(\Lambda'') \neq \emptyset$. By Lemma 4.2.3, we have $n \geq I_2$. Since $F_{I_2}^{u_2}(\Lambda) = F_{I_2}^{u_2}(\Lambda') \cup F_{I_2}^{u_2}(\Lambda'')$ and

$$\begin{aligned} & d_{n-I_2}^+ F_{I_2}^{u_2}(\Lambda') \\ = \; & F_{I_2}^{u_2}(d_n^+\Lambda') \\ = \; & F_{I_2}^{u_2}(d_n^-\Lambda'') \\ = \; & d_{n-I_2}^- F_{I_2}^{u_2}(\Lambda''), \end{aligned}$$

we can see that $F_{I_2}^{u_2}(\Lambda')\#_{n-I_2}F_{I_2}^{u_2}(\Lambda'')$ is defined, and $F_{I_2}^{u_2}(\Lambda) = F_{I_2}^{u_2}(\Lambda')\#_{n-I_2}F_{I_2}^{u_2}(\Lambda'')$. So $F_{I_2}^{u_2}(\Lambda)$ is a molecule, as required by the first condition. We now verify that Λ satisfy the second condition. If $p < I_2$, then $F_{I_2}^{u_2}(d_p^\gamma\Lambda) = \emptyset$, as required. If $p = n \geq I_2$, then

$$\begin{aligned} & F_{I_2}^{u_2}(d_p^-\Lambda) \\ = \; & F_{I_2}^{u_2}(d_p^-\Lambda') \\ = \; & d_{p-I_2}^- F_{I_2}^{u_2}(\Lambda') \\ = \; & d_{p-I_2}^- F_{I_2}^{u_2}(\Lambda); \end{aligned}$$

and similarly we have $F_{I_2}^{u_2}(d_p^+\Lambda) = d_{p-I_2}^+ F_{I_2}^{u_2}(\Lambda)$. If $I_2 \leq p < n$, then

$$\begin{aligned} & F_{I_2}^{u_2}(d_p^\gamma\Lambda) \\ = \; & F_{I_2}^{u_2}(d_p^\gamma\Lambda') \\ = \; & d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda') \\ = \; & d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda). \end{aligned}$$

If $p \geq I_2$ and $p > n$, then

$$\begin{aligned} & F_{I_2}^{u_2}(d_p^\gamma\Lambda) \\ = \; & F_{I_2}^{u_2}(d_p^\gamma\Lambda'\#_n d_p^\gamma\Lambda'') \\ = \; & F_{I_2}^{u_2}(d_p^\gamma\Lambda') \cup F_{I_2}^{u_2}(d_p^\gamma\Lambda'') \\ = \; & d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda') \cup d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda'') \end{aligned}$$

and

$$\begin{aligned} & d^{+}_{n-I_2} d^{\gamma}_{p-I_2} F^{u_2}_{I_2}(\Lambda') \\ = \; & d^{+}_{n-I_2} F^{u_2}_{I_2}(\Lambda') \\ = \; & d^{-}_{n-I_2} F^{u_2}_{I_2}(\Lambda'') \\ = \; & d^{-}_{n-I_2} d^{\gamma}_{p-I_2} F^{u_2}_{I_2}(\Lambda''), \end{aligned}$$

thus $d^{\gamma}_{p-I_2} F^{u_2}_{I_2}(\Lambda') \#_{n-I_2} d^{\gamma}_{p-I_2} F^{u_2}_{I_2}(\Lambda'')$ is defined and

$$\begin{aligned} & F^{u_2}_{I_2}(d^{\gamma}_{p}\Lambda) \\ = \; & d^{\gamma}_{p-I_2} F^{u_2}_{I_2}(\Lambda') \#_{n-I_2} d^{\gamma}_{p-I_2} F^{u_2}_{I_2}(\Lambda'') \\ = \; & d^{\gamma}_{p-I_2} [F^{u_2}_{I_2}(\Lambda') \#_{n-I_2} F^{u_2}_{I_2}(\Lambda'')] \\ = \; & d^{\gamma}_{p-I_2} F^{u_2}_{I_2}(\Lambda). \end{aligned}$$

Therefore Λ satisfies the second condition.

Finally, condition 3 can be easily verified by using condition 2 and the fact that $F^{u_2}_{I_2}$ preserves unions.

This complete the proof.

$\square$

In particular, we have that $F^{u_2}_{I_2}$ sends molecules to molecules in $u_1 \times u_3^{I_2} \times u_4^{I_2}$ or the empty set.

We can similarly prove that $F^{u_s}_{I_s}$ sends molecules to molecules in the corresponding (twisted) product of three infinite dimensional globes or the empty set for every value of s. Thus we have proved the main theorem in this section.

Theorem 4.2.5. *Molecules in $u_1 \times u_2 \times u_3 \times u_4$ are pairwise molecular.*

We finish this section by a property of molecules in $u_1 \times u_2 \times u_3 \times u_4$. It can be proved from the results later in this chapter. But the proof here is also interesting.

Proposition 4.2.6. *Let Λ be a molecule in $u_1 \times u_2 \times u_3 \times u_4$. Let $\lambda_1 = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\lambda_2 = u_1[i_1, -\alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$. If $\lambda_1 \subset \Lambda$ and $\lambda_2 \subset \Lambda$, then there is an atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ in Λ with $i' > i_1 = i_2$ such that $\lambda' \supset \lambda_1$ and $\lambda' \supset \lambda_2$.*

Proof. If Λ is an atom, then the required property holds automatically.

Suppose that the required property hold for every molecule which can be written as a composite of less than q atoms. Suppose also that Λ can be written as a composite of q atoms.

It is easy to see that there is a composite of molecules $\Lambda_1 \#_n \Lambda_2$ with $\lambda_1 \subset \Lambda_1$ and $\lambda_2 \subset \Lambda_2$ such that both Λ_1 and Λ_2 can be written as composites of less than q atoms. By applying the natural homomorphism $f_{i_1}^{u_1} : \mathcal{M}(u_1 \times u_2 \times u_3 \times u_4) \to \mathcal{M}((u_{i_1} \times u_2 \times u_3 \times u_4)$ of ω-categories, one gets

$$f_{i_1}^{u_1}(\Lambda_1 \#_n \Lambda_2) = f_{i_1}^{u_1}(\Lambda_1) \#_n f_{i_1}^{u_1}(\Lambda_2)).$$

This implies that

$$f_{i_1}^{u_1}(\Lambda_1 \cap \Lambda_2) = f_{i_1}^{u_1}(\Lambda_1) \cap f_{i_1}^{u_1}(\Lambda_2)).$$

Since $f_{i_1}^{u_1}(\lambda_1) = f_{i_1}^{u_1}(\lambda_2) \subset f_{i_1}^{u_1}(\Lambda_1) \cap f_{i_1}^{u_1}(\Lambda_2))$, we have $f_{i_1}^{u_1}(\lambda_1) = f_{i_1}^{u_1}(\lambda_2) \subset f_{i_1}^{u_1}(\Lambda_1 \cap \Lambda_2)$. It follows that $\lambda_3 = u_1[i_1, \beta_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4] \subset \Lambda_1 \cap \Lambda_2$ for some sign β. Now if $\beta_1 = -\alpha$, then one can get an atom in Λ_1 as required by applying the induction hypothesis on λ_1 and λ_3 in Λ_1; if $\beta_1 = \alpha$, then one can get an atom in Λ_2 as required by applying the induction hypothesis on λ_2 and λ_3 in Λ_2.

This completes the proof. $\square$

4.3 Properties of Pairwise Molecular Subcomplexes

In this section, we study some properties of pairwise molecular subcomplexes in $u_1 \times u_2 \times u_3 \times u_4$. In the next section, we are going to prove that some of these conditions are sufficient for a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ to be pairwise molecular.

Lemma 4.3.1. *Let Λ be a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Let $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ be a pair of distinct maximal atoms in Λ. If $i_s = i'_s$ and $\alpha_s = -\alpha'_s$ for some $1 \leq s \leq 4$, then for every t with $t \neq s$, there is a maximal atom $u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4]$ in Λ such that $l_s > i_s = i'_s$, $l_t \geq \min\{i_t, i'_t\}$ and $u_r[l_r, \sigma_r] \supset u_r[i_r, \alpha_r] \cap u_r[i'_r, \alpha'_r]$ for all $r \in \{1, 2, 3, 4\} \setminus \{s, t\}$.*

Proof. Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$. The arguments for various cases are similar, we give the proof for $s = 1$ and $t = 4$. Let $I_4 = \min\{i_4, i'_4\}$. Since Λ is pairwise molecular, we can see that $F^{u_4}_{I_4}(\Lambda)$ is a molecule in $u_1 \times u_2 \times u_3$. Since $F^{u_4}_{I_4}(\lambda) = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3]$ and $F^{u_4}_{I_4}(\lambda) = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3]$, it follows easily from Lemma 2.4.4 that there is a maximal atom $u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3]$ such that $l_1 > i_1 = i'_1$, $u_2[l_2, \sigma_2] \supset u_2[i_2, \alpha_2] \cap u_2[i'_2, \alpha'_2]$ and $u_3[l_3, \sigma_3] \supset u_3[i_3, \alpha_3] \cap u_3[i'_3, \alpha'_3]$. Therefore Λ has a maximal atom $u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4]$ such that $l_1 > i_1 = i'_1$, $u_2[l_2, \sigma_2] \supset u_2[i_2, \alpha_2] \cap u_2[i'_2, \alpha'_2]$, $u_3[l_3, \sigma_3] \supset u_3[i_3, \alpha_3] \cap u_3[i'_3, \alpha'_3]$ and $l_4 \geq I_4$, as required.

This completes the proof. □

The following definition of *adjacency* for a pair of maximal atoms in a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ is inspired by Propositions 2.3.3 and 2.3.11.

Definition 4.3.2. Let Λ be a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Let $1 \leq s < t \leq 4$. A pair of maximal atoms $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ are (s,t)-*adjacent* if $\max\{i_s, i'_s\} + \max\{i_t, i'_t\} > \max\{i_s + i_t, i'_s + i'_t\}$ and if there is no maximal atom $u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ with $j_r \geq \min\{i_r, i'_r\}$ for all $1 \leq r \leq 4$ such that $\min\{i_s, j_s\} + \min\{i_t, j_t\} > \min\{i_s, i'_s\} + \min\{i_t, i'_t\}$ and $\min\{i'_s, j_s\} + \min\{i'_t, j_t\} > \min\{i_s, i'_s\} + \min\{i_t, i'_t\}$. A pair of maximal atoms $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ are *adjacent* if they are (s,t)-adjacent for all $1 \leq s < t \leq 4$ such that $\max\{i_s, i'_s\} + \max\{i_t, i'_t\} > \max\{i_s + i_t, i'_s + i'_t\}$.

Example 4.3.3. Suppose that $\lambda = u_1[5, \alpha_1] \times u_2[0, \alpha_2] \times u_3[1, \alpha_3] \times u_4[1, \alpha_4]$ and $\mu = u_1[0, \beta_1] \times u_2[5, \beta_2] \times u_3[1, \beta_3] \times u_4[2, \beta_4]$ are a pair of maximal atoms in a subcomplex. If Λ has a maximal atom $\nu = u_1[1, \varepsilon_1] \times u_2[1, \varepsilon_2] \times u_3[2, \varepsilon_3] \times u_4[1, \varepsilon_4]$, then λ and μ are not $(1,2)$-adjacent.

The following proposition shows that the definition of adjacency is in consistent with that in Chapter 2.

Proposition 4.3.4. *Let Λ be a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ satisfying condition 1 for pairwise molecular subcomplexes. Then a pair of distinct maximal atoms $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $u_1[i_1', \alpha_1'] \times u_2[i_2', \alpha_2'] \times u_3[i_3', \alpha_3'] \times u_4[i_4', \alpha_4']$ are adjacent if and only if there is no maximal atom $u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ with $j_r \geq \min\{i_r, i_r'\}$ for all $1 \leq r \leq 4$ such that*

$$\sum_{r=1}^{4} \min\{i_r, j_r\} > \sum_{r=1}^{4} \min\{i_r, i_r'\}$$

and

$$\sum_{r=1}^{4} \min\{i_r', j_r\} > \sum_{r=1}^{4} \min\{i_r, i_r'\}.$$

Proof. The proof is a straightforward verification from the definition of adjacency. $\square$

The concept of projection maximal can be defined in the similar way as that in chapter 2.

Let Λ be a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ satisfying condition 1 for pairwise molecular subcomplex. Let I_r be a fixed non-negative integer. A maximal atom $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in Λ is (u_r, I_r)-*projection maximal* if $i_r \geq I_r$ and there is no maximal atom $u_1[i_1', \alpha_1'] \times u_2[i_2', \alpha_2'] \times u_3[i_3', \alpha_3'] \times u_4[i_4', \alpha_4']$ such that $I_r \leq i_r' < i_r$ and $i_s' \geq i_s$ for all $s \in \{1, 2, 3, 4\} \setminus \{r\}$.

Evidently, if a maximal atom λ in Λ is (u_r, I_r)-projection maximal, then $F_{I_r}^{u_r}(\lambda)$ is maximal in $F_{I_r}^{u_r}(\Lambda)$. Conversely, for every maximal atom μ in $F_{I_r}^{u_r}(\Lambda)$, there is a maximal atom μ' in Λ such that $F_{I_r}^{u_r}(\mu') = \mu$. The following proposition implies that μ' is actually (u_r, I_r)-projection maximal.

Proposition 4.3.5. *Let Λ be a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ and λ be a maximal atom in Λ. Let $1 \leq r \leq 4$. Then λ is (u_r, I_r)-projection maximal if and only if $F_{I_r}^{u_r}(\lambda)$ is maximal in $F_{I_r}^{u_r}(\Lambda)$.*

Proof. The proof is similar to that in Proposition 2.3.5. $\square$

Lemma 4.3.6. *Let Λ be a pairwise molecular subcomplex in $u_1 \times u_2 \times u_3 \times u_4$. Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\mu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ be a pair of (s, t)-adjacent maximal atoms in Λ for some $1 \leq s < t \leq 4$. If $i_s > j_s$*

and $i_t < j_t$, then for $r \in \{1,2,3,4\} \setminus \{s,t\}$ there is a pair of (s,t)-adjacent and (u_r, I_r)-projection maximal atoms $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ and $u_1[j'_1, \beta'_1] \times u_2[j'_2, \beta'_2] \times u_3[j'_3, \beta'_3] \times u_4[j'_4, \beta'_4]$ such that $u_s[j'_s, \beta'_s] = u_s[j_s, \beta_s]$, $u_t[i'_t, \alpha'_t] = u_t[i_t, \alpha_t]$ and $\min\{i'_r, j'_r\} = \min\{i_r, j_r\}$, where $I_r = \min\{i_r, j_r\}$. Moreover, for $\bar{r} \in \{1,2,3,4\} \setminus \{r,s,t\}$, if $s < \bar{r} < t$, then $\min\{i'_{\bar{r}}, j'_{\bar{r}}\} = \min\{i_{\bar{r}}, j_{\bar{r}}\}$.

Proof. Let $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ be the (u_r, I_r)-projection maximal atom such that $i'_s \geq i_s$, $i'_t \geq i_t$ and $i'_{\bar{r}} \geq i_{\bar{r}}$. Let $\mu' = u_1[j'_1, \beta'_1] \times u_2[j'_2, \beta'_2] \times u_3[j'_3, \beta'_3] \times u_4[j'_4, \beta'_4]$ be the (u_r, I_r)-projection maximal atom such that $j'_s \geq j_s$, $j'_t \geq j_t$ and $j'_{\bar{r}} \geq j'_{\bar{r}}$. It follows easily from Lemma 4.3.1 and the adjacency of λ and μ that λ' and μ' are (s,t)-adjacent and that $u_s[j'_s, \beta'_s] = u_s[j_s, \beta_s]$, $u_t[i'_t, \alpha'_t] = u_t[i_t, \alpha_t]$, $\min\{i'_r, j'_r\} = I_r$ and $\min\{i'_{\bar{r}}, j'_{\bar{r}}\} \geq \min\{i_{\bar{r}}, j_{\bar{r}}\}$.

Moreover, if $s \leq \bar{r} \leq t$, we show that $\min\{i'_{\bar{r}}, j'_{\bar{r}}\} = \min\{i_{\bar{r}}, j_{\bar{r}}\}$ by contradiction. The arguments for various choices of r, s and t are similar. We give the proof for $s = 1$, $t = 4$ and $r = 3$. In this case, we have $\bar{r} = 2$. Suppose otherwise that $\min\{i'_2, j'_2\} > \min\{i_2, j_2\}$. Then $F^{u_3}_{I_3}(\lambda')$ and $F^{u_3}_{I_3}(\mu')$ are maximal atoms in $F^{u_3}_{I_3}(\Lambda)$. Note that $F^{u_3}_{I_3}(\lambda') = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_4^{I_3}[i_4, \alpha_4]$ and $F^{u_3}_{I_3}(\mu') = u_1[j_1, \beta_1] \times u_2[j'_2, \beta'_2] \times u_4^{I_3}[j'_4, \beta'_4]$. Since $F^{u_3}_{I_3}(\Lambda)$ is a molecule in $u_1 \times u_2 \times u_4^{I_3}$, it follows from condition 4 in Theorem 2.4.1 that there is a maximal atom $u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_4^{I_3}[l_4, \sigma_4]$ in $F^{u_3}_{I_3}(\Lambda)$ such that $l_1 > j_1$, $l_2 = \min\{i'_2, j'_2\} - 1 \geq \min\{i_2, j_2\}$ and $l_4 > i_4$. Thus there is a maximal atom $\nu = u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4]$ in Λ such that $l_3 \geq I_3$. This contradicts the assumption that λ and μ are $(1,4)$-adjacent.

This completes the proof. □

Proposition 4.3.7. *Let Λ be a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ be a pair of distinct maximal atoms in Λ. If $1 \leq s < r < t \leq 4$ and λ and λ' are (s,t)-adjacent, and if $i_s > i'_s$, $\min\{i_r, i'_r\} > 0$ and $i'_t > i_t$, then Λ has a maximal atom $u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4]$ such that $l_s > i'_1$, $l_r = \min\{i_r, i'_r\} - 1$, $l_t > i_t$ and $l_{\bar{r}} \geq \min\{i_{\bar{r}}, i'_{\bar{r}}\}$, where $\bar{r} \in \{1,2,3,4\} \setminus \{r,s,t\}$.*

Proof. The arguments for various cases are similar. We only give the proof for the case

$s = 1$, $r = 3$ and $t = 4$.

Let $I_3 = \min\{i_3, i'_3\}$. According to Lemma 4.3.6, we may assume that λ and λ' are (u_3, I_3)-projection maximal so that $F^{u_3}_{I_3}(\lambda)$ and $F^{u_3}_{I_3}(\lambda')$ are maximal atoms in $F^{u_3}_{I_3}(\Lambda)$.

Now $F^{u_3}_{I_3}(\lambda) = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_4^{I_3}[i_4, \alpha_4]$ and $F^{u_3}_{I_3}(\lambda') = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_4^{I_3}[i'_4, \alpha'_4]$. It is easy to see that $F^{u_3}_{I_3}(\lambda)$ and $F^{u_3}_{I_3}(\lambda')$ are adjacent in $F^{u_3}_{I_3}(\Lambda)$. According to condition 4 in Theorem 2.4.1, there is a maximal atom $u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_4^{I_3}[l_4, \sigma_4]$ such that $l_1 > i'_1$, $l_2 = \min\{i_2, i'_2\} - 1$ and $l_4 > i_4$. This implies that there is a maximal atom $u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4]$ in Λ as required.

This completes the proof.

$\square$

We also need to extend the concept of projection maximal to maximal atoms in $u_1 \times u_2 \times u_3 \times u_4$ with respect to two factors, as follows.

Let $1 \leq s < t \leq 4$. Let I_s and I_t be fixed non-negative integers. A maximal atom $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in Λ is $(u_s, u_t; I_s, I_t)$-*projection maximal* if $i_s \geq I_s$ and $i_t \geq I_t$, and if there is no maximal atom $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ such that $i'_s \geq I_s$ and $i'_t \geq I_t$ and such that $i'_r \geq i_r$ for all $r \in \{1, 2, 3, 4\} \setminus \{s, t\}$ and $i'_r > i_r$ for some $r \in \{1, 2, 3, 4\} \setminus \{s, t\}$.

Evidently, if a maximal atom λ in Λ is $(u_s, u_t; I_s, I_t)$-projection maximal, then $F^{u_s, u_t}_{I_s, I_t}(\lambda)$ is maximal in $F^{u_s, u_t}_{I_s, I_t}(\Lambda)$. Conversely, for every maximal atom μ in $F^{u_s, u_t}_{I_s, I_t}(\Lambda)$, there is a maximal atom μ' in Λ such that $F^{u_s, u_t}_{I_s, I_t}(\mu') = \mu$. The following Proposition implies that μ' is actually $(u_s, u_t; I_s, I_t)$-projection maximal.

Proposition 4.3.8. *Let Λ be a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ and λ be a maximal atom in Λ. Let $1 \leq s < t \leq 4$. Then λ is $(u_s, u_t; I_s, I_t)$-projection maximal if and only if $F^{u_s, u_t}_{I_s, I_t}(\lambda)$ is maximal in $F^{u_s, u_t}_{I_s, I_t}(\Lambda)$.*

Proof. The argument is similar to that in Proposition 4.3.5. $\square$

Lemma 4.3.9. *Let Λ be a pairwise molecular subcomplex in $u_1 \times u_2 \times u_3 \times u_4$. Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\mu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ be a pair of (s, t)-adjacent maximal atoms in Λ for some $1 \leq s < t \leq 4$. If*

$i_{\bar{s}} > j_s$ and $i_t < j_t$, then there is a pair of (s,t)-adjacent and $(u_{\bar{s}}, u_{\bar{t}}; I_{\bar{s}}, I_{\bar{t}})$-projection maximal atoms $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ and $u_1[j'_1, \beta'_1] \times u_2[j'_2, \beta'_2] \times u_3[j'_3, \beta'_3] \times u_4[j'_4, \beta'_4]$ such that $u_s[j'_s, \beta'_s] = u_s[j_s, \beta_s]$, $u_t[i'_t, \alpha'_t] = u_t[i_t, \alpha_t]$ and $\min\{i'_{\bar{s}}, j'_{\bar{s}}\} \geq I_{\bar{s}}$ and $\min\{i'_{\bar{t}}, j'_{\bar{t}}\} \geq I_{\bar{t}}$, where $\bar{s}$ and $\bar{t}$ are distinct elements in $\{1,2,3,4\} \setminus \{s,t\}$ and $I_{\bar{s}} = \min\{i_{\bar{s}}, j_{\bar{s}}\}$ and $I_{\bar{t}} = \min\{i_{\bar{t}}, j_{\bar{t}}\}$. Moreover, if $s < \bar{s} < t$, then $\min\{i'_{\bar{s}}, j'_{\bar{s}}\} = I_{\bar{s}}$; if $s < \bar{t} < t$, then $\min\{i'_{\bar{t}}, j'_{\bar{t}}\} = I_{\bar{t}}$.

Proof. Let $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ be the $(u_{\bar{s}}, u_{\bar{t}}; I_{\bar{s}}, I_{\bar{t}})$-projection maximal atom such that $i'_s \geq i_s$ and $i'_t \geq i_t$. Let $\mu' = u_1[j'_1, \beta'_1] \times u_2[j'_2, \beta'_2] \times u_3[j'_3, \beta'_3] \times u_4[j'_4, \beta'_4]$ be the $(u_{\bar{s}}, u_{\bar{t}}; I_{\bar{s}}, I_{\bar{t}})$-projection maximal atom such that $j'_s \geq j_s$ and $j'_t \geq j_t$. It follows easily from Lemma 4.3.1 and the adjacency of λ and μ that $u_s[j'_s, \beta'_s] = u_s[j_s, \beta_s]$, $u_t[i'_t, \alpha'_t] = u_t[i_t, \alpha_t]$, $\min\{i'_{\bar{s}}, j'_{\bar{s}}\} \geq I_{\bar{s}}$, $\min\{i'_{\bar{t}}, j'_{\bar{t}}\} \geq I_{\bar{t}}$ and λ' is adjacent to μ', as required.

The second part follows easily from the adjacency of λ and μ and Lemma 4.3.7.

This completes the proof. □

Proposition 4.3.10. *Let Λ be a pairwise molecular subcomplex in $u_1 \times u_2 \times u_3 \times u_4$. Then the following sign conditions hold.*

Sign conditions: for a pair of adjacent maximal atoms $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ in Λ, let $l_r = \min\{i_r, i'_r\}$ for $1 \leq r \leq 4$.

1. *If λ and μ are $(1,2)$-adjacent, and if $l_1 = i_1 < i'_1$ and $l_2 = i'_2 < i_2$, then $\alpha'_2 = -(-)^{l_1}\alpha_1$;*
2. *If λ and μ are $(1,3)$-adjacent, and if $l_1 = i_1 < i'_1$ and $l_3 = i'_3 < i_3$, then $\alpha'_3 = -(-)^{l_1+l_2}\alpha_1$;*
3. *If λ and μ are $(1,4)$-adjacent, and if $l_1 = i_1 < i'_1$ and $l_4 = i'_4 < i_4$, then $\alpha'_4 = -(-)^{l_1+l_2+l_3}\alpha_1$;*
4. *If λ and μ are $(2,3)$-adjacent, and if $l_2 = i_2 < i'_2$ and $l_3 = i'_3 < i_3$, then $\alpha'_3 = -(-)^{l_2}\alpha_2$;*

5. *If* λ *and* μ *are* $(2,4)$*-adjacent, and if* $l_2 = i_2 < i_2'$ *and* $l_4 = i_4' < i_4$*, then* $\alpha_4' = -(-)^{l_2+l_3}\alpha_2$*;*

6. *If* λ *and* μ *are* $(3,4)$*-adjacent, and if* $l_3 = i_3 < i_3'$ *and* $l_4 = i_4' < i_4$*, then* $\alpha_4' = -(-)^{l_3}\alpha_3$*.*

Proof. The arguments for the above cases are similar. We only give the proof for case 3.

Let $\lambda = u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4]$ and $\lambda' = u_1[i_1',\alpha_1'] \times u_2[i_2',\alpha_2'] \times u_3[i_3',\alpha_3'] \times u_4[i_4',\alpha_4']$ be a pair of $(1,4)$-adjacent maximal atoms in Λ. Suppose that $i_1' > i_1$ and $i_4' < i_4$. Let $\min\{i_2,i_2'\} = l_2$ $\min\{i_3,i_3'\} = l_3$. We must prove $\alpha_4' = -(-)^{l_2+l_3}\alpha_1$. According to Lemma 4.3.9, we may assume that λ and λ' are $(u_2,u_3;I_2,I_3)$-projection maximal atoms.

It is evident that $F^{u_2,u_3}_{l_2,l_3}(\lambda) = u_1[i_1,\alpha_1] \times u_4^{l_2+l_3}[i_4,\alpha_4]$ and $F^{u_2,u_3}_{l_2,l_3}(\lambda') = u_1[i_1',\alpha_1'] \times u_4^{l_2+l_3}[i_4',\alpha_4']$, and they are maximal atoms in the molecule $F^{u_2,u_3}_{l_2,l_3}(\Lambda)$. Moreover, by the adjacency of λ and λ', we can see that $F^{u_2,u_3}_{l_2,l_3}(\lambda)$ and $F^{u_2,u_3}_{l_2,l_3}(\lambda')$ are adjacent maximal atoms in $F^{u_2,u_3}_{l_2,l_3}(\Lambda)$. According to the formation of molecules in $u \times w^{l_2+l_3}$, we get $\alpha_4' = -(-)^{l_2+l_3}\alpha_1$, as required.

This completes the proof. □

4.4 An Alternative Description for Pairwise Molecular Subcomplexes

In this section, we give an alternative description for pairwise molecular subcomplexes of $u_1 \times u_2 \times u_3 \times u_4$, as follows.

Theorem 4.4.1. *Let* Λ *be a subcomplex of* $u_1 \times u_2 \times u_3 \times u_4$*. Then* Λ *is pairwise molecular if and only if*

1. *There are no distinct maximal atoms* $u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4]$ *and* $u_1[i_1',\alpha_1'] \times u_2[i_2',\alpha_2'] \times u_3[i_3',\alpha_3'] \times u_4[i_4',\alpha_4']$ *in* Λ *such that* $i_1 \le i_1'$*,* $i_2 \le i_2'$*,* $i_3 \le i_3'$ *and* $i_4 \le i_4'$*.*

2. *Sign conditions: for a pair of maximal atoms* $u_1[i_1,\alpha_1]\times u_2[i_2,\alpha_2]\times u_3[i_3,\alpha_3]\times u_4[i_4,\alpha_4]$ *and* $u_1[i_1',\alpha_1']\times u_2[i_2',\alpha_2']\times u_3[i_3',\alpha_3']\times u_4[i_4',\alpha_4']$ *in* Λ*, let* $l_r=\min\{i_r,i_r'\}$ *for* $1\le r\le 4$.

   (a) *If* λ *and* μ *are* $(1,2)$*-adjacent, and if* $l_1=i_1<i_1'$ *and* $l_2=i_2'<i_2$*, then* $\alpha_2'=-(-)^{l_1}\alpha_1$*;*

   (b) *If* λ *and* μ *are* $(1,3)$*-adjacent, and if* $l_1=i_1<i_1'$ *and* $l_3=i_3'<i_3$*, then* $\alpha_3'=-(-)^{l_1+l_2}\alpha_1$*;*

   (c) *If* λ *and* μ *are* $(1,4)$*-adjacent, and if* $l_1=i_1<i_1'$ *and* $l_4=i_4'<i_4$*, then* $\alpha_4'=-(-)^{l_1+l_2+l_3}\alpha_1$*;*

   (d) *If* λ *and* μ *are* $(2,3)$*-adjacent, and if* $l_2=i_2<i_2'$ *and* $l_3=i_3'<i_3$*, then* $\alpha_3'=-(-)^{l_2}\alpha_2$*;*

   (e) *If* λ *and* μ *are* $(2,4)$*-adjacent, and if* $l_2=i_2<i_2'$ *and* $l_4=i_4'<i_4$*, then* $\alpha_4'=-(-)^{l_2+l_3}\alpha_2$*;*

   (f) *If* λ *and* μ *are* $(3,4)$*-adjacent, and if* $l_3=i_3<i_3'$ *and* $l_4=i_4'<i_4$*, then* $\alpha_4'=-(-)^{l_3}\alpha_3$.

3. *Let* $\lambda=u_1[i_1,\alpha_1]\times u_2[i_2,\alpha_2]\times u_3[i_3,\alpha_3]\times u_4[i_4,\alpha_4]$ *and* $\lambda'=u_1[i_1',\alpha_1']\times u_2[i_2',\alpha_2']\times u_3[i_3',\alpha_3']\times u_4[i_4',\alpha_4']$ *be a pair of distinct maximal atoms in* Λ*. If* $i_s=i_s'$ *and* $\alpha_s=-\alpha_s'$ *for some* $1\le s\le 4$*, then for every* $t\in\{1,2,3,4\}\setminus\{s\}$*, there is a maximal atom* $u_1[l_1,\sigma_1]\times u_2[l_2,\sigma_2]\times u_3[l_3,\sigma_3]\times u_4[l_4,\sigma_4]$ *in* Λ *such that* $l_s>i_s=i_s'$*,* $l_t\ge\min\{i_t,i_t'\}$ *and* $u_r[l_r,\sigma_r]\supset u_r[i_r,\alpha_r]\cap u_r[i_r',\alpha_r']$ *for all* $r\in\{1,2,3,4\}\setminus\{s,t\}$.

4. *Let* $\lambda=u_1[i_1,\alpha_1]\times u_2[i_2,\alpha_2]\times u_3[i_3,\alpha_3]\times u_4[i_4,\alpha_4]$ *and* $\lambda'=u_1[i_1',\alpha_1']\times u_2[i_2',\alpha_2']\times u_3[i_3',\alpha_3']\times u_4[i_4',\alpha_4']$ *be a pair of distinct maximal atoms in* Λ*. For* $1\le s<r<t\le 4$*, if* λ *and* λ' *are* (s,t)*-adjacent, and if* $i_s>i_s'$*,* $\min\{i_r,i_r'\}>0$ *and* $i_t'>i_t$*, then* Λ *has a maximal atom* $u_1[l_1,\sigma_1]\times u_2[l_2,\sigma_2]\times u_3[l_3,\sigma_3]\times u_4[l_4,\sigma_4]$ *such that* $l_s>i_1'$*,* $l_r=\min\{i_r,i_r'\}-1$*,* $l_t>i_t$ *and* $l_{\bar r}\ge\min\{i_{\bar r},i_{\bar r}'\}$ *for* $\bar r\in\{1,2,3,4\}\setminus\{r,s,t\}$.

Note 4.4.2. In condition 4, we have a similar relations for the signs σ_r, α_s' and α_t as that in Note 2.4.2. For instance, if $s=1$, $r=2$ and $t=3$, then we have $\sigma_2=-(-)^{i_1'}\alpha_1'$ and $\alpha_t=-(-)^{l_2}\sigma_2$.

In the last section, we have proved that the four conditions in this theorem are necessary for pairwise molecular subcomplexes. We now prove the sufficiency. The proof is separated into several lemmas.

Lemma 4.4.3. *Let Λ be a subcomplex satisfying the four conditions in Theorem 4.4.1. Let $\lambda = u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4]$ and $\mu = u_1[j_1,\beta_1] \times u_2[j_2,\beta_2] \times u_3[j_3,\beta_3] \times u_4[j_4,\beta_4]$ be maximal atoms in Λ. If $i_s = j_s$ and $\alpha_s = -\beta_s$ for some $1 \le s \le 4$, then there is a maximal atom $\nu = u_1[l_1,\sigma_1] \times u_2[l_2,\sigma_2] \times u_3[l_3,\sigma_3] \times u_4[l_4,\sigma_4]$ in Λ with $l_s > i_s = j_s$ such that $l_t \ge \min\{i_t, j_t\}$ and $u_t[l_t,\sigma_t] \supset u_t[i_t,\alpha_t] \cap u_t[j_t,\beta_t]$ for all $t \in \{1,2,3,4\} \setminus \{s\}$.*

Proof. We first prove the lemma when λ and μ are adjacent. The arguments for various cases are similar. We only give the proof for $s = 1$, $i_2 > j_2$ and $i_3 < j_3$.

If $i_4 = j_4$ and $\alpha_4 = -\beta_4$, then we can get the required μ simply by applying condition 3 in Theorem 4.4.1. We now suppose that $i_4 \ne j_4$ or that $i_4 = j_4$ and $\alpha_4 = \beta_4$. In this case, we have $u_4[i_4,\alpha_4] \cap u_4[j_4,\beta_4] = u_4[i_4,\alpha_4]$ or $u_4[i_4,\alpha_4] \cap u_4[j_4,\beta_4] = u_4[j_4,\beta_4]$. We may assume that $u_4[i_4,\alpha_4] \cap u_4[j_4,\beta_4] = u_4[i_4,\alpha_4]$. According to condition 3 in Theorem 4.4.1, we can get maximal atoms $\lambda' = u_1[i'_1,\alpha'_1] \times u_2[i'_2,\alpha'_2] \times u_3[i'_3,\alpha'_3] \times u_4[i'_4,\alpha'_4]$ such that $i'_1 > i_1 = j_1$, $i'_2 \ge j_2$, $u_3[i_3,\alpha_3] \subset u_3[i'_3,\alpha'_3]$ and $u_4[i_4,\alpha_4] \subset u_4[i'_4,\alpha'_4]$. Similarly, we can get maximal atoms $\mu' = u_1[j'_1,\beta'_1] \times u_2[j'_2,\beta'_2] \times u_3[j'_3,\beta'_3] \times u_4[j'_4,\beta'_4]$ such that $j'_1 > i_1 = j_1$, $j'_3 \ge i_3$, $u_2[j_2,\beta_2] \subset u_2[j'_2,\beta'_2]$ and $u_4[i_4,\alpha_4] \subset u_4[j'_4,\beta'_4]$.

Now, suppose otherwise that Λ does not have a maximal atom $\nu = u_1[l_1,\sigma_1] \times u_2[l_2,\sigma_2] \times u_3[l_3,\sigma_3] \times u_4[l_4,\sigma_4]$ as required. Then $i'_2 = j_2$ and $\alpha'_2 = -\beta_2$, and $j'_3 = i_3$ and $\beta'_3 = -\alpha_3$. By applying condition 3 in Theorem 4.4.1 to μ and λ', it is easy to see that $i'_3 = i_3$; thus $\alpha'_3 = \alpha_3$. Similarly, we have $j'_2 = j_2$ and $\beta'_2 = \beta_2$. By applying condition 3 in Theorem 4.4.1 to λ' and μ', we get a maximal atom $\nu'' = u_1[l''_1,\sigma''_1] \times u_2[l''_2,\sigma''_2] \times u_3[l''_3,\sigma''_3] \times u_4[l''_4,\sigma''_4]$ such that $l''_1 \ge \min\{i'_1, j'_1\} > i_1 = j_1$, $l''_2 > i'_2 = j'_2 = j_2$, $l''_3 \ge i'_3 = j'_3 = i_3$, $u_4[l''_4,\sigma''_4] \supset u_4[i_4,\alpha_4]$. By the hypothesis, we have $l''_3 = i_3$ and $\sigma''_3 = -\alpha_3$. By applying condition 3 in Theorem 4.4.1 to λ and ν'', we can get a maximal atom $\lambda'' = u_1[i''_1,\alpha''_1] \times u_2[i''_2,\alpha''_2] \times u_3[i''_3,\alpha''_3] \times u_4[i''_4,\alpha''_4]$ such that $i''_1 \ge i_1 = j_1$, $i''_2 \ge \min\{i_2, l''_2\} > j_2$, $i''_3 > i_3$ and $i''_4 \ge i_4$. This contradicts the adjacency of λ and μ.

The other cases can be proved similarly.

Now we give the proof for the general case by induction.

1. Suppose that $\sum_{r=1}^{4} \min\{i_r, j_r\}$ is maximal among the non-negative integers $\sum_{r=1}^{4} \min\{i'_r, j'_r\}$ with $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ and $u_1[j'_1, \beta'_1] \times u_2[j'_2, \beta'_2] \times u_3[j'_3, \beta'_3] \times u_4[j'_4, \beta'_4]$ running over all pairs of distinct maximal atoms in Λ. It is evident that λ is adjacent to μ, hence the lemma holds for λ and μ.

2. Suppose that $q \geq 0$ and the lemma holds for every pair of distinct maximal atoms $u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ and $u_1[j'_1, \beta'_1] \times u_2[j'_2, \beta'_2] \times u_3[j'_3, \beta'_3] \times u_4[j'_4, \beta'_4]$ with $\sum_{r=1}^{4} \min\{i'_r, j'_r\} > q$. Suppose also that $\sum_{r=1}^{4} \min\{i_r, j_r\} = q$.

If λ and μ are adjacent, then the lemma holds by the first part of the proof.

Suppose that λ and μ are not adjacent. According to Theorem 4.3.4, there is a maximal atom $\nu' = u_1[l'_1, \sigma'_1] \times u_2[l'_2, \sigma'_2] \times u_3[l'_3, \sigma'_3] \times u_4[l'_4, \sigma'_4]$ with $l'_r \geq \min\{i_r, j_r\}$ for $1 \leq r \leq 4$ such that

$$\sum_{r=1}^{4} \min\{i_r, l_r\} > \sum_{r=1}^{4} \min\{i_r, j_r\}$$

and

$$\sum_{r=1}^{4} \min\{j_r, l_r\} > \sum_{r=1}^{4} \min\{i_r, j_r\}.$$

By possibly multiple applying the above argument, condition 3 in Theorem 4.4.1, and the induction hypothesis, we can get either a maximal atom $\mu = u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4]$ as required, or a pair of maximal atoms $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ and $\mu' = u_1[j'_1, \beta'_1] \times u_2[j'_2, \beta'_2] \times u_3[j'_3, \beta'_3] \times u_4[j'_4, \beta'_4]$ with $i'_s = i_s = j_s = j'_s$ and $\alpha'_s = -\beta'_s$ such that $\min\{i_t, j_t\} \leq \min\{i'_t, j'_t\}$ and $u_t[i_t, \alpha_t] \cap u_t[j_t, \beta_t] \subset u_t[i'_t, \alpha'_t] \cap u_t[j'_t, \beta'_t]$ for all $t \in \{1, 2, 3, 4\} \setminus \{s\}$, and such that $\sum_{r=1}^{4} \min\{i_r, j_r\} < \sum_{r=1}^{4} \min\{i'_r, j'_r\}$. It follows from the induction hypothesis that the lemma holds for λ and μ.

This completes the proof.

□

Note that the proof in Proposition 4.3.5 uses only the definition for projection maximal, condition 1 for pairwise molecular subcomplexes and Lemma 4.3.1. By condition 1 and condition 3 in Theorem 4.4.1, we have the following two propositions.

Proposition 4.4.4. *Let Λ be subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ satisfying the four conditions* *in Theorem 4.4.1 and λ be a maximal atom in Λ. Let $1 \leq r \leq 4$. Then λ is (u_r, I_r)-projection maximal if and only if $F^{u_r}_{I_r}(\lambda)$ is maximal in $F^{u_r}_{I_r}(\Lambda)$.*

Proposition 4.4.5. *Let Λ be a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ satisfying the four conditions in Theorem 4.4.1 and λ be a maximal atom in Λ. Let $1 \leq s < t \leq 4$. Then λ is $(u_s, u_t; I_s, I_t)$-projection maximal if and only if $F^{u_s,u_t}_{I_s,I_t}(\lambda)$ is maximal in $F^{u_s,u_t}_{I_s,I_t}(\Lambda)$.*

Proposition 4.4.6. *Let Λ be a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ satisfying the four conditions in Theorem 4.4.1. Let I_2 and I_3 be fixed non-negative integers. Then*

1. *Every maximal atom $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in Λ with $i_2 = I_2$ and $i_3 = I_3$ is $(u_2, u_3; I_2, I_3)$-projection maximal.*

2. *For every maximal atom $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ with $i_2 \geq I_2$ and $i_3 \geq I_3$, there is a $(u_2, u_3; I_2, I_3)$-projection maximal atom $u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ such that $u_1[i_1, \alpha_1] \subset u_1[j_1, \beta_1]$ and $u_4[i_4, \alpha_4] \subset u_4[j_4, \beta_4]$.*

3. *All $(u_2, u_3; I_2, I_3)$-projection maximal atoms , if exist, can be listed as $\lambda^{(1)}, \cdots, \lambda^{(S)}$ with $\lambda^{(s)} = u_1[i_1^{(s)}, \alpha_1^{(s)}] \times u_2[i_2^{(s)}, \alpha_2^{(s)}] \times u_3[i_3^{(s)}, \alpha_3^{(s)}] \times u_4[i_4^{(s)}, \alpha_4^{(s)}]$ such that $i_1^{(1)} \geq \cdots \geq i_1^{(S)}$ and $i_4^{(1)} \leq \cdots \leq i_4^{(S)}$. Moreover, in this list, if $1 \leq s < S$, then $i_1^{(s)} = i_1^{(s+1)}$ if and only if $i_4^{(s)} = i_4^{(s+1)}$; in this case, one must also have $\alpha_1^{(s)} = \alpha_1^{(s+1)}$ and $\alpha_4^{(s)} = \alpha_4^{(s+1)}$.*

4. *For two consecutive $(u_2, u_3; I_2, I_3)$-projection maximal atoms $\lambda^{(s)}$ and $\lambda^{(s+1)}$ in the above list with $i_1^{(s)} > i_1^{(s+1)}$, one has $\min\{i_2^{(s)}, i_2^{(s+1)}\} = I_2$ and $\min\{i_3^{(s)}, i_3^{(s+1)}\} = I_3$.*

Proof. Condition 1 follows from the definition of projection maximal and condition 1 in Theorem 4.4.1.

To prove condition 2, suppose that $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ is not $(u_2, u_3; I_2, I_3)$-projection maximal. Then there is a $(u_2, u_3; I_2, I_3)$-projection maximal atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ such that $i'_1 \geq i_1$ and $i'_4 \geq i_4$. If $u_1[i'_1, \alpha'_1] \supset u_1[i_1, \alpha_1]$ and $u_4[i'_4, \alpha'_4] \supset u_4[i_4, \alpha_4]$, then λ' is as required. Suppose that $u_1[i'_1, \alpha'_1] \not\supset u_1[i_1, \alpha_1]$. Then $i'_1 = i_1$ and $\alpha'_1 = -\alpha_1$. Moreover, we have $i'_4 > i_4$ by the definition of $(u_2, u_3; I_2, I_3)$-projection maximal. Hence, by condition 3 in Theorem 4.4.1, there is a maximal atom $\mu = u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4]$ in Λ such that $l_1 > i_1 = i'_1$, $l_2 \geq I_2$, $l_3 \geq I_3$ and $u_4[l_4, \sigma_4] \supset u_4[i'_4, \alpha'_4]$. If μ is $(u_2, u_3; I_2, I_3)$-projection maximal, then μ can be taken as $u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$, as required. If μ is not $(u_2, u_3; I_2, I_3)$-projection maximal, then one can get a $(u_2, u_3; I_2, I_3)$-projection maximal atom μ as required by repeating the above argument for λ.

To prove condition 3, by the definition of $(u_2, u_3; I_2, I_3)$-projection maximal, it suffices to show that Λ does not have $(u_2, u_3; I_2, I_3)$-projection maximal atoms $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\mu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ such that $i_1 = j_1$ and $\alpha_1 = -\beta_1$ or such that $i_4 = j_4$ and $\alpha_4 = -\beta_4$. This follows easily from condition 3 in Theorem 4.4.1.

Finally, condition 4 follows easily from condition 4 in Theorem 4.4.1.

This completes the proof. □

Proposition 4.4.7. *In a subcomplex Λ of $u_1 \times u_2 \times u_3 \times u_4$ satisfying the four conditions in Theorem 4.4.1, let the $(u_2, u_3; I_2, I_3)$-projection maximal atoms be listed as $\lambda^{(1)}, \cdots, \lambda^{(S)}$ with $\lambda^{(s)} = u_1[i_1^{(s)}, \alpha_1^{(s)}] \times u_2[i_2^{(s)}, \alpha_2^{(s)}] \times u_3[i_3^{(s)}, \alpha_3^{(s)}] \times u_4[i_4^{(s)}, \alpha_4^{(s)}]$ such that $i_1^{(1)} \geq \cdots \geq i_1^{(S)}$ and $i_4^{(1)} \leq \cdots \leq i_4^{(S)}$, taking $S = 0$ if there are none. Then*

1. $F^{u_2,u_3}_{I_2,I_3}(\Lambda) = u_1[i_1^{(1)}, \alpha_1^{(1)}] \times u_4^{I_2+I_3}[i_4^{(1)}, \alpha_4^{(1)}] \cup \cdots \cup u_1[i_1^{(S)}, \alpha_1^{(S)}] \times u_4^{I_2+I_3}[i_4^{(S)}, \alpha_4^{(S)}]$.

2. *$F^{u_2,u_3}_{I_2,I_3}(\Lambda)$ is a molecule in $u_1 \times u_4^{I_2+I_3}$ or the empty set.*

Proof. The first part is a direct consequence of the definition for $F^{u_2,u_3}_{I_2,I_3}$ and conditions 2 and 3 in the above lemma. Note that a pair of consecutive $(u_2, u_3; I_2, I_3)$-projection maximal atoms $\lambda^{(s)}$ and $\lambda^{(s+1)}$ with $i_1^s > i_1^{s+1}$ in the above lemma are (1,4)-adjacent and $\min\{i_2^{(r)}, i_2^{(r+1)}\} = I_2$ and $\min\{i_3^{(r)}, i_3^{(r+1)}\} = I_3$. It follows from the sign conditions that $\alpha_4^{(r)} = -(-)^{i_1^{(r+1)}+I_2+I_3}\alpha_1^{(r+1)}$. This implies that $F^{u_2,u_3}_{I_2,I_3}(\Lambda)$ is a molecule in $u_1 \times u_4^{I_2+I_3}$ or the empty set, as required.

□

We can similarly prove that $F^{u_s,u_t}_{I_s,I_t}(\Lambda)$ is a molecule or the empty set for every pair of s and t with $1 \leq s < t \leq 4$ in the corresponding products of three (twisted) infinite dimensional globes.

Lemma 4.4.8. *Let Λ be a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ satisfying the four conditions in Theorem 4.4.1. Then $F_{I_r}^{u_r}(\Lambda)$ satisfies condition 1 for pairwise molecular subcomplexes.*

Proof. The argument for various choices of r are similar. We only give the proof for $r = 4$.

Suppose otherwise that $F_{I_4}^{u_4}(\Lambda)$ does not satisfy condition 1 for pairwise molecular subcomplexes. The there is a pair of maximal atoms $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3]$ and $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3]$ such that $i_s \leq i'_s$ for all $1 \leq s \leq 3$. Thus we have $i_t = i'_t$ and $\alpha_t = -\alpha'_t$ for some $1 \leq t \leq 3$. Hence there is a pair of (u_4, I_4)-projection maximal atoms of the form $\mu = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\mu' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$. By condition 1 in Theorem 4.4.1, we evidently have $i_4 > i'_4 \geq I_4$. It follows from condition 3 in Theorem 4.4.1 that there is a maximal atom $u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4]$ such that $l_s \geq i_s$ for all $1 \leq s \leq 3$, $l_4 \geq i'_4 \geq I_4$ and $l_t > i_t$. This contradicts that μ is (u_4, I_4)-projection maximal.

This completes the proof.

□

By the comment after Proposition 4.4.7 and Lemma 4.4.8, we have now completed the proof for Theorem 4.4.1.

4.5 Sources and Targets of Pairwise Molecular Subcomplexes

In this section, we study source and target operators d_p^γ on pairwise molecular subcomplexes in $u_1 \times u_2 \times u_3 \times u_4$. We shall prove that $d_p^\gamma \Lambda$ is pairwise molecular for every pairwise molecular subcomplex Λ of $u_1 \times u_2 \times u_3 \times u_4$.

Recall that $d_p^\gamma \Lambda$ is a union of interiors of atoms for every subcomplex Λ of $u_1 \times u_2 \times u_3 \times u_4$. Hence the maps $F_{I_r}^{u_r}$ and $F_{I_s, I_t}^{u_s, u_t}$ are defined on $d_p^\gamma \Lambda$.

We first show that $d_p^\gamma \Lambda$ is a subcomplex for a pairwise molecular subcomplex Λ. The proof is separated into several Lemmas.

Lemma 4.5.1. *Let Λ be a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ and $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ be a p-dimensional atom in Λ with $\operatorname{Int} \lambda \subset d_p^\gamma \Lambda$. If there is an atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ in Λ with $\lambda' \supset \lambda$ such that $i'_s > i_s$ for some s, then $\alpha_s = (-)^{i_1 + \cdots + i_{s-1}} \gamma$.*

Proof. The proof is similar to that in 2.5.1. □

Lemma 4.5.2. *Let Λ be a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ be a p-dimensional atom with $\operatorname{Int} \lambda \subset d_p^\gamma \Lambda$. For $1 \leq s \leq 4$, if there is a maximal atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ in Λ with $i'_s > i_s$ such that $i'_r \geq i_r$ for all $r \in \{1, 2, 3, 4\}$, then $\alpha_s = (-)^{i_1 + \cdots + i_{s-1}} \gamma$.*

Proof. The arguments for various choices of s are similar. We only give the proof for $s = 1$.

Suppose that there is a maximal atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ such that $i'_1 > i_1$ and $i'_r \geq i_r$ for all $r \in \{2, 3, 4\}$. If λ' can be chosen such that $\lambda \subset \lambda'$, then we have $\alpha_1 = \gamma$ by Lemma 4.5.1 , as required. In the following proof, we may assume that λ' cannot be chosen such that $\lambda \subset \lambda'$ so that $u_t[i'_t, \alpha'_t] = u_t[i_t, -\alpha_t]$ for some $t \in \{2, 3, 4\}$. Since $\operatorname{Int} \lambda \subset \Lambda$, there is a maximal atom $\mu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ such that $\lambda \subset \mu$. By the assumption on the choice of λ', we have $u_1[j_1, \beta_1] = u_1[i_1, \alpha_1]$. By applying Lemma 4.4.3, we may further assume that $j_t > i_t$.

1. Suppose that λ' and μ can be chosen such that $\min\{i'_r, j_r\} > i_r$ for two value of $r \in \{2, 3, 4\}$. According to condition 4 in Theorem 4.4.1, we may suppose that $\min\{i'_r, j_r\} > i_r$ for $r = 3$ and $r = 4$, and that $j_2 > i_2$. Thus $i'_2 = i_2$ and $\alpha' = -\alpha_2$. By Lemma 4.5.1, we have $\alpha_2 = (-)^i \gamma$, and hence $\alpha'_2 = -(-)^i \gamma$. According to the assumption, it is easy to see that λ' is $(1, 2)$-adjacent to μ. It follows easily that $\alpha_1 = \gamma$, as required.

2. Suppose that λ' and μ cannot be chosen as in case 1. Suppose also that λ' and μ can be chosen such that $\min\{i'_r, j_r\} > i_r$ for only one value of $r \in \{2, 3, 4\}$. According to condition 4 in Theorem 4.4.1, we may suppose that $\min\{i'_4, j_4\} > i_4$, and that $j_2 > i_2$ or $j_3 > i_3$. We may further suppose that $j_2 > i_2$ and $i'_2 = i_2$ and $\alpha'_2 = -\alpha_2$. Note that, in this case, λ' is $(1, 2)$-adjacent to μ. By an argument similar to that in case 1, we have $\alpha_1 = \gamma$, as required.

3. Suppose that λ' and μ cannot be chosen as in case 1 and case 2. Then λ' is adjacent to μ. By an argument similar to that in case 1, we have $\alpha_1 = \gamma$, as required.

This completes the proof.

□

Lemma 4.5.3. *Let Λ be a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ be a $p-1$ dimensional atom such that $\operatorname{Int} \lambda \subset d_p^\gamma \Lambda$. If there is a maximal atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ in Λ with $\lambda' \supset \lambda$ such that $i'_s > i_s$ and $i'_t > i_t$ for some $1 \leq s < t \leq 4$, then $\alpha_s = (-)^{i_1+\cdots+i_{s-1}}\gamma$ or $\alpha_t = -(-)^{i_1+\cdots+i_{t-1}}\gamma$.*

Proof. The argument is similar to that in Lemma 2.5.3. □

Proposition 4.5.4. *Let Λ be a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ be a $p-1$ dimensional atom such that $\operatorname{Int} \lambda \subset d_p^\gamma \Lambda$. If there is a maximal atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ in Λ with $i'_s > i_s$ and $i'_t > i_t$ for some $1 \leq s < t \leq 4$ such that $u_r[i'_r, \alpha'_r] \supset u_r[i_r, \alpha_r]$ for at least three values of $r \in \{1, 2, 3, 4\}$, then $\alpha_s = (-)^{i_1+\cdots+i_{s-1}}\gamma$ or $\alpha_t = -(-)^{i_1+\cdots+i_{t-1}}\gamma$.*

Proof. The arguments for various cases are similar. We give the proof for the following case.

Suppose that there is a maximal atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ in Λ such that $i'_1 > i_1$, $i'_2 > i_2$, $i'_3 \geq i_3$ and $u_4[i'_4, \alpha'_4] \supset u_4[i_4, \alpha_4]$. If λ' can be chosen such that $\lambda' \supset \lambda$, then we have $\alpha_1 = \gamma$ or $\alpha_2 = -(-)^{i_1}\gamma$, as required, by Lemma 4.5.3. In the following, we assume that λ' cannot be chosen such that $\lambda' \supset \lambda$ so that $u_3[i'_3, \alpha'_3] = u_3[i_3, -\alpha_3]$. Let $\mu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ be a maximal atom in Λ such that $\lambda \subset \mu$. Then $u_1[j_1, \beta_1] = u_1[i_1, \alpha_1]$ or $u_2[j_2, \beta_2] = u_2[i_2, \alpha_2]$. According to Lemma 4.4.3, we may assume that $j_3 > i_3$. Now we consider three cases, as follows.

1. Suppose that μ cannot be chosen such that $j_1 > i_1$ or $j_2 > i_2$. We claim that $\alpha'_3 = -(-)^{i_1+i_2}\alpha_1$ and $\alpha'_3 = -(-)^{i_2}\alpha_2$.

Indeed, if λ' and μ are $(1,3)$-adjacent, then we have $\alpha'_3 = -(-)^{i_1+i_2}\alpha_1$ by sign conditions. Suppose that λ' and μ are not $(1,3)$-adjacent. Then there is a maximal atom

$\lambda'' = u_1[i''_1, \alpha''_1] \times u_2[i''_2, \alpha''_2] \times u_3[i''_3, \alpha''_3] \times u_4[i''_4, \alpha''_4]$ such that $i''_1 > i_1$, $i''_2 \geq i_2$, $i''_3 > i_3$ and $i''_4 \geq \min\{i'_4, j_4\}$. According to the assumptions and Lemma 4.4.3, we can see that $u_4[i''_4, \alpha''_4] = u_4[i_4, -\alpha_4]$. It follows easily from Lemma 4.4.3 that there is a maximal atom $\mu' = u_1[j'_1, \beta'_1] \times u_2[j'_2, \beta'_2] \times u_3[j'_3, \beta'_3] \times u_4[j'_4, \beta'_4]$ such that $u_1[j'_1, \beta'_1] = u_1[i_1, \alpha_1]$, $u_2[j'_2, \beta'_2] = u_2[i_2, \alpha_2]$, $j'_3 > i_3$ and $j'_4 > i_4$. By applying Lemma 4.4.3 and the assumptions, we can also get a maximal atom $\nu = u_1[k_1, \varepsilon_1] \times u_2[k_2, \varepsilon_2] \times u_3[k_3, \varepsilon_3] \times u_4[k_4, \varepsilon_4]$ in Λ such that $k_1 > i_1$, $k_2 \geq i_2$, $u_3[k_3, \varepsilon_3] = u_3[i'_3, \alpha'_3]$ and $k_4 > i_4$. It is evident that ν and μ' are $(1,3)$-adjacent. It follows from sign conditions that $\alpha'_3 = -(-)^{i_1+i_2}\alpha_1$, as required. We can similarly show that $\alpha'_3 = -(-)^{i_2}\alpha_2$.

Now, if $\alpha'_3 = -(-)^{i_1+i_2}\gamma$, then $\alpha_1 = \gamma$, as required; if $\alpha_3 = (-)^{i_1+i_2}\gamma$, then $\alpha_2 = -(-)^{i_1}\gamma$, as required.

2. Suppose that μ can be chosen such that $j_1 > i_1$. Suppose also that $\alpha_1 = -\gamma$. Then $u_2[j_2, \beta_2] = u_2[i_2, \alpha_2]$ by the assumptions. According to Lemma 4.5.3, we have $\alpha_3 = -(-)^{i_1+i_2}\gamma$, and hence $\alpha'_3 = (-)^{i_1+i_2}\gamma$. By Lemma 4.4.3, it is easy to see that λ' and μ can be chosen such that they are $(2,3)$-adjacent. It follows from sign conditions that $\alpha_2 = \beta_2 = -(-)^{i_1}\gamma$, as required.

3. Suppose that μ can be chosen such that $j_2 > i_2$ and that $\alpha_2 = (-)^{i_1}\gamma$. Suppose also that μ cannot be chosen such that $j_1 > i_1$. According to condition 4 in Theorem 4.4.1, Lemma 4.4.3 and the assumptions, it is easy to see that λ' and μ can be chosen such that they are $(1,3)$-adjacent and $\min\{j_2, i'_2\} = i+1$. According to condition 4 in Theorem 4.4.1, there is a maximal atom $\lambda'' = u_1[i''_1, \alpha''_1] \times u_2[i''_2, \alpha''_2] \times u_3[i''_3, \alpha''_3] \times u_4[i''_4, \alpha''_4]$ such that $i''_1 > i_1$, $j''_2 = j_2$, $i''_3 > i_3$ and $i''_4 \geq \min\{i'_4, j_4\} \geq i_4$. Moreover, we have $\alpha'_2 = -\alpha_2 = -(-)^{i_1}\gamma$ by the assumptions. According to Note 4.4.2, we have $\alpha''_2 = -(-)^{j_1}\beta_1 = -(-)^{i_1}\alpha_1$. This implies that $\alpha_1 = \gamma$, as required.

This completes the proof.

$\square$

Lemma 4.5.5. *Let Λ be a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ be a $p-2$ dimensional atom such that $\operatorname{Int}\lambda \subset d_p^\gamma\Lambda$. If there is a maximal atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$*

in Λ *with* $\lambda' \supset \lambda$ *such that* $i'_r > i_r$, $i'_s > i_s$ *and* $i'_t > i_t$ *for some* $1 \leq r < s < t \leq 4$, *then* $\alpha_r = (-)^{i_1+\cdots+i_{r-1}}\gamma$ *or* $\alpha_s = -(-)^{i_1+\cdots+i_{s-1}}\gamma$ *or* $\alpha_t = (-)^{i_1+\cdots+i_{t-1}}\gamma$.

Proof. The argument is similar to that in Lemma 2.5.3. □

Lemma 4.5.6. *Let* Λ *be a pairwise molecular subcomplex of* $u_1 \times u_2 \times u_3 \times u_4$. *Let* $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ *be a* $p-2$ *dimensional atom such that* $\operatorname{Int}\lambda \subset d_p^\gamma\Lambda$. *If there is a maximal atom* $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ *in* Λ *such that* $i'_r > i_r$, $i'_s > i_s$ *and* $i'_t > i_t$ *for some* $1 \leq r < s < t \leq 4$, *and such that* $i'_{\bar{r}} \geq i_{\bar{r}}$ *for* $\bar{r} \in \{1,2,3,4\} \setminus \{r,s,t\}$, *then* $\alpha_r = (-)^{i_1+\cdots+i_{r-1}}\gamma$ *or* $\alpha_s = -(-)^{i_1+\cdots+i_{s-1}}\gamma$ *or* $\alpha_t = (-)^{i_1+\cdots+i_{t-1}}\gamma$.

Proof. The arguments for various choices of r, s and t are similar. We give the proof for $r = 1$, $s = 3$ and $t = 4$.

Suppose that there is a maximal atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ such that $i'_1 > i_1$, $i'_2 \geq i_2$, $i'_3 > i_3$ and $i'_4 > i_4$. If λ' can be chosen such that $u_2[i'_2, \alpha'_2] \supset u_2[i_2, \alpha_2]$, then we have $\alpha_1 = \gamma$, $\alpha_3 = -(-)^{i_1+i_2}\gamma$ or $\alpha_4 = (-)^{i_1+i_2+i_3}\gamma$, as required, by Lemma 4.5.5. In the following, we assume that λ' cannot be chosen such that $u_2[i'_2, \alpha'_2] \supset u_2[i_2, \alpha_2]$ so that $u_2[i'_2, \alpha'_2] = u_2[i_2, -\alpha_2]$. Let $\mu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ be a maximal atom in Λ such that $\mu \supset \lambda$. According to Lemma 4.4.3, we can assume that $j_2 > i_2$. We consider three cases, as follows.

1. Suppose that μ can be chosen such that there are exactly three value of $r \in \{1, 2, 3, 4\}$ such that $j_r > i_r$.

a. Suppose that μ can be chosen such that $j_2 > i_2$, $j_3 > i_3$ and $j_4 > i_4$. Then $u_1[j_1, \beta_1] = u_1[i_1, \alpha_1]$ by the assumptions. We also have $\alpha_2 = (-)^{i_1}\gamma$, $\alpha_3 = -(-)^{i_1+i_2}\gamma$ or $\alpha_4 = (-)^{i_1+i_2+i_4}\gamma$ by Lemma 4.5.5. If $\alpha_3 = -(-)^{i_1+i_2}\gamma$ or $\alpha_4 = (-)^{i_1+i_2+i_4}\gamma$, then α_3 or α_4 is as required. If $\alpha_2 = (-)^{i_1}\gamma$, then $\alpha'_2 = -(-)^{i_1}\gamma$; moreover, it is easy to see that λ' is $(1,2)$-adjacent to μ; it follows from sign conditions that $\alpha_1 = \beta_1 = \gamma$, as required.

b. Suppose that μ can be chosen such that $j_1 > i_1$, $j_2 > i_2$ and $j_4 > i_4$. Then we can get $\alpha_1 = \gamma$, $\alpha_3 = -(-)^{i_1+i_2}\gamma$ or $\alpha_4 = (-)^{i_1+i_2+i_4}\gamma$, as required, by similar arguments as in case a.

c. Suppose that μ can be chosen such that $j_1 > i_1$, $j_2 > i_2$ and $j_3 > i_3$. Suppose also that μ cannot be chosen such that $j_1 > i_1$, $j_2 > i_2$ and $j_4 > i_4$. According to the assumptions, it is easy to see that λ' and μ are $(2,4)$-adjacent and $\min\{i'_3, j_3\} = i_3 + 1$. By condition 4 in Theorem 4.4.1, there is a maximal atom $\lambda'' = u_1[i''_1, \alpha''_1] \times u_2[i''_2, \alpha''_2] \times u_3[i''_3, \alpha''_3] \times u_4[i''_4, \alpha''_4]$ in Λ such that $i''_1 > i$, $i''_2 > i_2$, $i''_3 = i_3$ and $i''_4 > i_4$. According to the assumptions, we have $\alpha''_3 = -\alpha_3$. Now, if $\alpha_3 = -(-)^{i_1+i_2}\gamma$, then it is as required. If $\alpha_3 = (-)^{i_1+i_2}\gamma$, then $\alpha''_3 = -(-)^{i_1+i_2}\gamma$; therefore we have $\alpha_4 = (-)^{i_1+i_2+i_3}\gamma$ by Note 4.4.2, as required.

2. Suppose that μ cannot be chosen such that there are three value of $r \in \{1,2,3,4\}$ such that $j_r > i_r$. Suppose also that μ can be chosen such that there are two value of $r \in \{1,2,3,4\}$ such that $j_r > i_r$.

a. Suppose that μ can be chosen such that $j_1 > i_1$ and $j_2 > i_2$. Then we have $u_3[j_3, \beta_3] = u_3[i_3, \alpha_3]$ and $u_4[j_4, \beta_4] = u_4[i_4, \alpha_4]$ by the assumptions. Moreover, by Lemma 4.4.3 and the assumptions, we can see that λ' is both $(2,3)$-adjacent and $(2,4)$-adjacent to μ. It follows easily that $\alpha_3 = -(-)^{i_1+i_2}\gamma$ or $\alpha_4 = (-)^{i_1+i_2+i_3}\gamma$, as required.

b. Suppose that μ can be chosen such that $j_2 > i_2$ and $j_4 > i_4$. We can get $\alpha_1 = \gamma$ or $\alpha_3 = -(-)^{i_1+i_2}\gamma$ by similar arguments as that in case a.

c. Suppose that μ can be chosen such that $j_2 > i_2$ and $j_3 > i_3$. Then we have $u_1[j_1, \beta_1] = u_1[i_1, \alpha_1]$ and $u_4[j_4, \beta_4] = u_4[i_4, \alpha_4]$ by the assumptions. According to condition 4 in Theorem 4.4.1, Lemma 4.4.3 and the assumptions, it is easy to see that λ' is both $(1,2)$-adjacent and $(2,4)$-adjacent to μ, and that $\min\{i'_3, j_3\} = i_3 + 1$. It follows easily from sign conditions that $\alpha_1 = \gamma$ or $\alpha_4 = (-)^{i_1+i_2+i_3}\gamma$, as required.

3. There remains the case that μ cannot be chosen such that $j_1 > i_1$ or $j_3 > i_3$ or $j_4 > i_4$. In this case, we have $u_r[j_r, \beta_r] = u_r[i_r, \alpha_r]$ for all $r \in \{1,3,4\}$. By Lemma 4.4.3, it is easy to see that λ' and μ are adjacent. It follows from sign conditions that $\alpha_1 = \gamma$ or $\alpha_3 = -(-)^{i_1+i_2}\gamma$ or $\alpha_4 = (-)^{i_1+i_2+i_3}\gamma$, as required.

This completes the proof.

□

Proposition 4.5.7. *Let Λ be a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. Then*

$d_p^\gamma \Lambda$ *is a subcomplex.*

Proof. We have already seen that $d_p^\gamma \Lambda$ is a union of interiors of atoms. By Lemma 2.5.5, it suffices to prove that for every atom λ with $\operatorname{Int} \lambda \subset d_p^\gamma \Lambda$ and every atom λ_1 with $\lambda_1 \subset \lambda$, one has $\operatorname{Int} \lambda_1 \subset d_p^\gamma \Lambda$. It is evident that there is a sequence $\lambda \supset \lambda_1^1 \supset \lambda_1^2 \supset \cdots \supset \lambda_1$ such that the difference of the dimensions of any pair of consecutive atoms is 1. We may assume that $\dim \lambda_1 = \dim \lambda - 1$.

Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$. Since $\operatorname{Int} \lambda \subset d_p^\gamma \Lambda \subset \Lambda$ and Λ is a subcomplex, we have $\lambda_1 \subset \lambda \subset \Lambda$ and $\dim \lambda_1 \leq \dim \lambda \leq p$. Suppose that $\mu = u_1[l_1, \sigma_1] \times u_2[l_2, \sigma_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4]$ is an atom with $\dim \mu = p+1$ and $\lambda_1 \subset \mu \subset \Lambda$. We must prove $\lambda_1 \subset d_p^\gamma \mu$.

If $\lambda \subset \mu$, then $\lambda_1 \subset \lambda \subset d_p^\gamma \mu$ since $\lambda \subset d_p^\gamma \Lambda$. If $l_s > i_s + 1$ for some s, then $l_s > i_s^1 + 1$ for some s; hence we also have $\lambda_1 \subset d_p^\gamma \mu$ by the formation of $d_p^\gamma \mu$. In the following, we may further assume that $\lambda \not\subset \mu$ and that $l_s \leq i_s + 1$ for every s. Thus $u_t[i_t, \alpha_t] \not\subset u_t[l_t, \sigma_t]$ for some t, and hence $u_t[l_t, \sigma_t] = u_t[i_t, -\alpha_t]$ or $u_t[l_t, \sigma_t] = u_t[i_t - 1, \tilde{\alpha}_t]$; moreover, we can see that $u_s[i_s, \alpha_s] \subset u_s[l_s, \sigma_s]$ for every s with $s \neq t$. Note that $\dim \mu = p+1$ and $\dim \lambda \leq p$, we now have 5 cases, as follows.

1. Suppose that $u_t[l_t, \sigma_t] = u_t[i_t, -\alpha_t]$ for some t and $l_s = i_s + 1$ for only one value of s. The proofs for different choices of t and the one value for s are similar. We give the proof only for the case $l_1 = i_1 + 1$ and $u_2[l_2, \sigma_2] = u_2[i_2, -\alpha_2]$. In this case, we can see that λ_1 is of the form $\lambda_1 = u_1[i_1, \alpha_1] \times u_2[i_2 - 1, \tilde{\alpha}_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$; we also have $\dim \lambda = p$. According to Proposition 4.5.2, we have $\alpha_1 = \gamma$. This implies that $\lambda_1 \subset d_p^\gamma \mu$, as required, by Lemma 2.1.2.

2. Suppose that $l_t = i_t - 1$ and $l_s = i_s + 1$ for only two values of s. The proofs for different choices of t and the two values for s are similar. We give the proof only for the case $l_2 = i_2 - 1$, $l_1 = i_1 + 1$ and $l_3 = i_3 + 1$. In this case, we can see that λ_1 is of the form $\lambda_1 = u_1[i_1, \alpha_1] \times u_2[i_2 - 1, \sigma_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$; we also have $\dim \lambda = p$ and $u_4[l_4, \sigma_4] = u_4[i_4, \alpha_4]$. To get $\lambda_1 \subset d_p^\gamma \mu$, it suffices to prove that $\alpha_1 = \gamma$ or $\alpha_3 = (-)^{i_1+i_2}\gamma$ by Lemma 2.1.2.

Let $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ be a maximal atom in Λ such that $\lambda \subset \lambda'$. Let $\mu' = u_1[l'_1, \sigma'_1] \times u_2[l'_2, \sigma'_2] \times u_3[l'_3, \sigma'_3] \times u_4[l'_4, \sigma'_4]$ be a maximal atom in Λ such that $\mu \subset \mu'$. If λ' can be chosen such that $i'_1 > i_1$ or $i'_3 > i_3$, then we have $\alpha_1 = \gamma$ or $\alpha_3 = (-)^{i_1+i_2}\gamma$ by Lemma 4.5.1 which implies that $\lambda_1 \subset d_p^\gamma \mu$, as required. If there is a maximal atom $\mu'' = u_1[l''_1, \sigma''_1] \times u_2[l''_2, \sigma''_2] \times u_3[l''_3, \sigma''_3] \times u_4[l''_4, \sigma''_4]$ with $l''_r \geq i_r$ for all $r \in \{1, 2, 3, 4\}$ such that $l''_1 > i_1$ or $l''_3 > i_3$, then, by Proposition 4.5.2, we have $\alpha_1 = \gamma$ or $\alpha_3 = (-)^{i_1+i_2}\gamma$ which implies that $\lambda_1 \subset d_p^\gamma \mu$, as required.

Now suppose that λ' cannot be chosen such that $i'_1 > i_1$ or $i'_3 > i_3$. Suppose also that there is no maximal atom $\mu'' = u_1[l''_1, \sigma''_1] \times u_2[l''_2, \sigma''_2] \times u_3[l''_3, \sigma''_3] \times u_4[l''_4, \sigma''_4]$ with $l''_r \geq i_r$ for all $r \in \{1, 2, 3, 4\}$ such that $l''_1 > i_1$ or $l''_3 > i_3$. Then $u_1[i'_1, \alpha'_1] = u_1[i_1, \alpha_1]$, $u_3[i'_3, \alpha'_3] = u_3[i_3, \alpha_3]$ and $u_2[l'_2, \sigma'_2] = u_2[l_2, \sigma_2] = u_2[i_2 - 1, \tilde{\alpha}_2]$. It is easy to see that λ' is both $(1, 2)$-adjacent to μ' and $(2, 3)$-adjacent to μ'. If $\sigma_2 = -(-)^{i_1}\gamma$, then we can see that $\alpha_1 = \gamma$ by sign conditions; if $\sigma_2 = (-)^{i_1}\gamma$, then we can see that $\alpha_3 = (-)^{i_1+i_2}\gamma$ by sign conditions. These imply that $\lambda_1 \subset d_p^\gamma \mu$, as required.

3. Suppose that $u_t[l_t, \sigma_t] = u_t[i_t, -\alpha_t]$ for some t and $l_s = i_s + 1$ for only 2 value of s. The proofs for different choices of t and the 2 values for s are similar. We give the proof only for the case $u_2[l_2, \sigma_2] = u_2[i_2, -\alpha_2]$, $l_1 = i_1 + 1$ and $l_3 = i_3 + 1$. In this case, we can see that λ_1 is of the form $\lambda_1 = u_1[i_1, \alpha_1] \times u_2[i_2 - 1, \tilde{\alpha}_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$; we also have $\dim \lambda = p - 1$. According to Proposition 4.5.4, we have $\alpha_1 = \gamma$ or $\alpha_3 = -(-)^{i_1+i_2}\gamma$. This implies that $\lambda_1 \subset d_p^\gamma \mu$, as required, by Lemma 4.1.1.

4. Suppose that $l_t = i_t - 1$ and $l_s = i_s + 1$ for the other three values of s. The proofs for different choices of t are similar. We give the proof only for the case $u_2[l_2, \sigma_2] = u_2[i_2 - 1, \tilde{\alpha}_2]$, $l_1 = i_1 + 1$, $l_3 = i_3 + 1$ and $l_4 = l_4 + 1$. In this case, we can see that λ_1 is of the form $\lambda_1 = u_1[i_1, \alpha_1] \times u_2[i_2 - 1, \sigma_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$; we also have $\dim \lambda = p - 1$. To get $\lambda_1 \subset d_p^\gamma \mu$, it suffices to prove that $\alpha_1 = \gamma$ or $\alpha_3 = (-)^{i_1+i_2}\gamma$ or $\alpha_4 = -(-)^{i_1+i_2+i_3}\gamma$ by Lemma 4.1.1.

Let $\mu' = u_1[l'_1, \sigma'_1] \times u_2[l'_2, \sigma'_2] \times u_3[l'_3, \sigma'_3] \times u_4[l'_4, \sigma'_4]$ be a maximal atom in Λ such that $\mu \subset \mu'$. Let $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ be a maximal atom in Λ such that $\lambda \subset \lambda'$. If μ' can be chosen such that $l'_2 \geq i_2$, then we have $\alpha_1 = \gamma$ or $\alpha_4 = -(-)^{i_1+i_2+i_3}\gamma$, as required, by Lemma 4.5.4. In the following proof, we may assume that μ' cannot be chosen such that $l_2' \geq i_2$. Now there are two cases, as follows.

a. Suppose that λ' can be chosen such that there are exactly two values of $s \in \{1,3,4\}$ such that $i_s' > i_s$. If λ' can be chosen such that $i_1' > i_1$ and $i_4' > i_4$, or such that $i_3' > i_3$ and $i_4' > i_4$. Then it is easy to see that $\alpha_1 = \gamma$ or $\alpha_3 = (-)^{i_1+i_2}\gamma$ or $\alpha_4 = -(-)^{i_1+i_2+i_3}\gamma$, as required, by Proposition 4.5.4. Suppose that λ' can be chosen such that $i_1' > i_1$ and $i_3' > i_3$. Suppose also that $\alpha_1 = -\gamma$. By the assumptions, we have $u_4[i_4', \alpha_4'] = u_4[i_4, \alpha_4]$. According to Proposition 4.5.4, we also have $\alpha_3 = -(-)^{i_1+i_2}\gamma$. By the assumptions and condition 4 in Theorem 4.4.1, it is easy to see that λ' is $(2,4)$-adjacent to μ' and $\min\{i_3', l_3'\} = i_3 + 1$. According to condition 4 in Theorem 4.4.1, there exists a maximal atom $\nu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ such that $j_1 > i_1$, $j_2 \geq i_2$, $j_3 \geq i_3$ and $j_4 > i_4$. If $u_3[j_3, \beta_3] \supset u_3[i_3, \alpha_3]$, then it is easy to see that $\alpha_4 = -(-)^{i_1+i_2+i_3}\gamma$, as required, by Proposition 4.5.4. If $j_3 = i_3$ and $\beta_3 = -\alpha_3 = (-)^{i_1+i_2}\gamma$, then, by Note 4.4.2, we also have $\alpha_4 = \alpha_4' = -(-)^{j_3}\beta_3 = -(-)^{i_1+i_2+i_3}\gamma$, as required.

b. Suppose that λ' cannot be chosen such that there are two values of $s \in \{1,3,4\}$ such that $i_s' > i_s$. By our assumptions, it is easy to see that $i_s' > i_s$ for at most one value of $s \in \{1,3,4\}$.

Suppose that λ' can be chosen such that $i_1' > i_1$. Then $u_3[i_3', \alpha_3'] = u_3[i_3, \alpha_3]$ and $u_4[i_4', \alpha_4'] = u_4[i_4, \alpha_4]$ by the assumptions. It is also easy to see that μ' is both $(2,3)$-adjacent and $(2,4)$-adjacent to λ'. It follows easily that $\alpha_3 = (-)^{i_1+i_2}\gamma$ (when $\sigma_2 = (-)^{i_1+i_2}\gamma$) or $\alpha_4 = -(-)^{i_1+i_2+i_3}\gamma$ (when $\sigma_2 = -(-)^{i_1+i_2}\gamma$), as required.

Suppose that λ' can be chosen such that $i_4' > i_4$ or $\lambda' = \lambda$. By arguments similar to that in the last paragraph, we can get $\alpha_1 = \gamma$ or $\alpha_3 = (-)^{i_1+i_2}\gamma$ or $\alpha_4 = -(-)^{i_1+i_2+i_3}\gamma$, as required.

Suppose that λ' can be chosen such that $i_3' > i_3$. According to the assumptions and condition 3 in Theorem 4.4.1, it is easy to see that λ' and μ' are both $(1,2)$-adjacent and $(2,4)$-adjacent. By condition 3 in Theorem 4.4.1 again, we can also get $\min\{i_3', l_3'\} = i_3+1$. It follows easily that $\alpha_1 = \gamma$ (when $\sigma_2 = -(-)^{i_1+i_2}\gamma$) or $\alpha_4 = -(-)^{i_1+i_2+i_3}\gamma$ (when $\sigma_2 = (-)^{i_1+i_2}\gamma$), as required.

5. Suppose that $u_t[l_t, \sigma_t] = u_t[i_t, -\alpha_t]$ for some t and $l_s = i_s + 1$ for 3 values of s. The

proofs for different choices of t and the 3 values for s are similar. We give the proof only for the case $u_2[l_2, \sigma_2] = u_2[i_2, -\alpha_2]$, $l_1 = i_1 + 1$ and $l_3 = i_3 + 1$ and $l_4 = i_4 + 1$. In this case, we can see that λ_1 is of the form $\lambda_1 = u_1[i_1, \alpha_1] \times u_2[i_2 - 1, \tilde{\alpha}_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$; we also have $\dim \lambda = p - 2$. According to Proposition 4.5.4, we have $\alpha_1 = \gamma$ or $\alpha_3 = -(-)^{i_1+i_2}\gamma$ or $\alpha_4 = (-)^{i_1+i_2+i_3}\gamma$. This implies that $\lambda_1 \subset d_p^\gamma \mu$, as required, by Lemma 2.1.2.

This completes the proof. $\square$

Proposition 4.5.8. *Let Λ be a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$. For $1 \leq s \leq 4$, if $p \geq I_s$ and $F_{I_s}^{u_s}(\Lambda) \neq \emptyset$, then $F_{I_s}^{u_s}(d_p^\gamma \Lambda) = d_{p-I_s}^\gamma F_{I_s}^{u_s}(\Lambda)$.*

Proof. The arguments for various choices of s are similar. We give the proof for $s = 2$.

We first prove that $d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda) \subset F_{I_2}^{u_2}(d_p^\gamma \Lambda)$.

Let $u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4]$ be a maximal atom in $d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda)$. We must show that $u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4] \subset F_{I_2}^{u_2}(d_p^\gamma \Lambda)$. Since $u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4] \subset d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda) \subset F_{I_2}^{u_2}(\Lambda)$, we can see that $u_1[i_1, \alpha_1] \times u_2[I_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4] \subset \Lambda$ for some sign α_2. We now consider three cases, as follows.

1. Suppose that $\dim(u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4]) < p - I_2$, then $u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4]$ is a maximal atom in $F_{I_2}^{u_2}(\Lambda)$. We claim that $u_1[i_1, \alpha_1] \times u_2[I_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4] \subset d_p^\gamma \Lambda$. Firstly, it is evident that $\dim(u_1[i_1, \alpha_1] \times u_2[I_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]) < p$. Moreover, suppose that there is a maximal atom $u_1[i_1', \alpha_1'] \times u_2[i_2', \alpha_2'] \times u_3[i_3', \alpha_3'] \times u_4[i_4', \alpha_4']$ in Λ such that $u_1[i_1', \alpha_1'] \times u_2[i_2', \alpha_2'] \times u_3[i_3', \alpha_3'] \times u_4[i_4', \alpha_4'] \supset u_1[i_1, \alpha_1] \times u_2[I_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$. Then it is evident that $u_r[i_r', \alpha_r'] = u_r[i_r, \alpha_r]$ for all $r \in \{1, 3, 4\}$. According to Lemma 4.1.1, we have $u_1[i_1, \alpha_1] \times u_2[I_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4] \subset d_p^\gamma(u_1[i_1', \alpha_1'] \times u_2[i_2', \alpha_2'] \times u_3[i_3', \alpha_3'] \times u_4[i_4', \alpha_4'])$. It follows easily from Lemma 1.2.11 that $u_1[i_1, \alpha_1] \times u_2[I_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4] \subset d_p^\gamma \Lambda$. Hence $u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4] \subset F_{I_2}^{u_2}(d_p^\gamma \Lambda)$.

2. Suppose that $\dim(u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4]) = p - I_2$. Suppose also that α_2 can be chosen such that $\alpha_2 = (-)^{i_1}\gamma$. We claim that $u_1[i_1, \alpha_1] \times u_2[I_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4] \subset d_p^\gamma \Lambda$. Indeed, it is evident that $\dim(u_1[i_1, \alpha_1] \times u_2[I_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]) = p$. Moreover, suppose that there is a maximal atom $u_1[i_1', \alpha_1'] \times u_2[i_2', \alpha_2'] \times u_3[i_3', \alpha_3'] \times u_4[i_4', \alpha_4']$ in Λ such that $u_1[i_1', \alpha_1'] \times u_2[i_2', \alpha_2'] \times u_3[i_3', \alpha_3'] \times u_4[i_4', \alpha_4'] \supset u_1[i_1, \alpha_1] \times$

$u_2[I_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4]$. Then $u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4] \subset u_1[i_1',\alpha_1'] \times u_3^{I_2}[i_3',\alpha_3'] \times u_4^{I_2}[i_4',\alpha_4'] \subset F_{I_2}^{u_2}(\Lambda)$. Thus $u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4] \subset d_{p-I_2}^{\gamma}(u_1[i_1',\alpha_1'] \times u_3^{I_2}[i_3',\alpha_3'] \times u_4^{I_2}[i_4',\alpha_4'])$ by Lemma 1.2.11. It follows easily from Lemma 2.1.2 and 4.1.1 that $u_1[i_1,\alpha_1] \times u_2[I_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4] \subset d_p^{\gamma}(u_1[i_1',\alpha_1'] \times u_2[i_2',\alpha_2'] \times u_3[i_3',\alpha_3'] \times u_4[i_4',\alpha_4'])$. Therefore $u_1[i_1,\alpha_1] \times u_2[I_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4] \subset d_p^{\gamma}\Lambda$, and hence $u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4] \subset F_{I_2}^{u_2}(d_p^{\gamma}\Lambda)$, as required.

3. Suppose that $\dim(u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4]) = p - I_2$. Suppose also that α_2 cannot be chosen such that $\alpha_2 = (-)^{i_1}\gamma$. Then $i_2' = I_2$ and $\alpha_2' = \alpha_2 = -(-)^{i_1}\gamma$. By arguments similar to those in case 2, it is easy to see that $u_1[i_1,\alpha_1] \times u_2[I_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4] \subset d_p^{\gamma}\Lambda$. Therefore $u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4] \subset F_{I_2}^{u_2}(d_p^{\gamma}\Lambda)$, as required.

This completes the proof that $d_{p-I_2}^{\gamma}F_{I_2}^{u_2}(\Lambda) \subset F_{I_2}^{u_2}(d_p^{\gamma}\Lambda)$.

Conversely, let $\lambda = u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4]$ be an atom in $u_1 \times u_3^{I_2} \times u_4^{I_2}$ such that $\operatorname{Int}\lambda \subset F_{I_2}^{u_2}(d_p^{\gamma}\Lambda)$. We must show that $\operatorname{Int}\lambda \subset d_{p-I_2}^{\gamma}F_{I_2}^{u_2}(\Lambda)$.

It is easy to see that there is an atom $u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4]$ in Λ such that $\operatorname{Int}(u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4]) \subset d_p^{\gamma}\Lambda$ and $i_2 \geq I_2$. Since $d_p^{\gamma}\Lambda$ is a subcomplex of $u_1 \times u_2 \times u_3 \times u_4$, we can see that $u_1[i_1,\alpha_1] \times u_2[I_2,\alpha_2'] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4] \subset d_p^{\gamma}\Lambda$ for some sign α_2'. It follows that $\dim(u_1[i_1,\alpha_1] \times u_3^{I_2}[i_3,\alpha_3] \times u_4^{I_2}[i_4,\alpha_4]) \leq p - I_2$. Clearly, we have $\lambda \subset F_{I_2}^{u_2}(\Lambda)$. To prove that $\operatorname{Int}\lambda \subset d_{p-I_2}^{\gamma}F_{I_2}^{u_2}(\Lambda)$, it suffices to verify the second condition in the Lemma 1.2.11. Let $\mu = u_1[l_1,\sigma_1] \times u_3^{I_2}[l_3,\sigma_3] \times u_4^{I_2}[i_4,\sigma_4]$ be an atom in $F_{I_2}^{u_2}(\Lambda)$ such that $\lambda \subset \mu$ and $\dim\mu = p - I_2 + 1$. We must prove that $\lambda \subset d_{p-I_2}^{\gamma}\mu$. It is evident that $u_1[l_1,\sigma_1] \times u_2[I_2,\sigma_2'] \times u_3[l_3,\sigma_3] \times u_4[l_4,\sigma_4] \subset \Lambda$ for some sign σ_2'. If $l_1 > i_1 + 1$, $l_3 > i_3 + 1$ or $l_4 > i_4 + 1$, then it is evident that $\lambda \subset d_p^{\gamma}\mu$, as required. In the following prove, we may assume that $l_1 \leq i_1 + 1$, $l_3 \leq i_3 + 1$ and $l_4 \leq i_4 + 1$ so that $\dim\lambda = p - I_2$ or $\dim\lambda = p - I_2 - 1$ or $\dim\lambda = p - I_2 - 2$. Now there are various cases, as follows.

Suppose that α_2' and σ_2' can be chosen such that $\alpha_2' = \sigma_2'$. Then $u_1[i_1,\alpha_1] \times v[I_2,\alpha_2'] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4] \subset d_p^{\gamma}\Lambda \cap (u_1[l_1,\sigma_1] \times u_2[I_2,\sigma_2'] \times u_3[l_3,\sigma_3] \times u_4[l_4,\sigma_4]) \subset d_p^{\gamma}(u_1[l_1,\sigma_1] \times u_2[I_2,\sigma_2'] \times u_3[l_3,\sigma_3] \times u_4[l_4,\sigma_4])$. It follows easily from Lemma 4.1.1 and Lemma 2.1.2

that $\lambda \subset d^{\gamma}_{p-I_2}\mu$, as required.

Suppose that α'_2 and σ'_2 cannot be chosen such that $\alpha'_2 = \sigma'_2$. Suppose also that $I_2 > 0$. Since $d^{\gamma}_p\Lambda$ is a subcomplex, we know that $u_1[i_1, \alpha_1] \times u_2[I_2 - 1, \pm] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4] \subset d^{\gamma}_p\Lambda$. This implies that $u_1[i_1, \alpha_1] \times u_2[I_2 - 1, \pm] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4] \subset d^{\gamma}_p(u_1[l_1, \sigma_1] \times u_2[I_2, \sigma'_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4])$. Hence $u_1[i_1, \alpha_1] \times u_2[I_2, \sigma'_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4] \subset d^{\gamma}_p(u_1[l_1, \sigma_1] \times u_2[I_2, \sigma'_2] \times u_3[l_3, \sigma_3] \times u_4[l_4, \sigma_4])$. It follows easily from Lemma 2.1.2 and Lemma 4.1.1 that $\lambda \subset d^{\gamma}_{p-I_2}\mu$, as required.

There remains the case that $J = 0$ and α'_2 and τ' cannot be chosen such that $\alpha'_2 = \sigma'_2$. By arguments similar to those in cases 1, 3 and 5 in the proof of Proposition 2.5.6, we can get $\lambda \subset d^{\gamma}_{p-I_2}\mu$, as required.

This completes the proof.

□

Lemma 4.5.9. *Let Λ be a pairwise molecular subcomplex. Then $d^{\gamma}_p\Lambda$ satisfies condition 1 for pairwise molecular subcomplexes.*

Proof. Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ be a pair of maximal atom in $d^{\gamma}_p\Lambda$ with $i_s \leq i'_s$ for every value of s. We must prove that $\lambda = \lambda'$. Suppose that $\dim \lambda < p$ or $\dim \lambda' < p$. Then it is easy to see that λ is a maximal atom in Λ when $\dim \lambda < p$ and λ' is a maximal atom in Λ when $\dim \lambda' < p$. According to condition 1 for Λ, we can see that $\lambda = \lambda'$, as required. In the following argument, we may assume that $\dim \lambda = p$ and $\dim \lambda = p$ so that $i_s = i'_s$ for every value of s.

Now suppose otherwise that $\lambda \neq \lambda'$. Then $u_t[i'_t, \alpha'_t] = u_t[i_t, -\alpha_t]$ for some t. Let r be such that $r \neq t$. By Proposition 4.5.8, we have $F^{u_r}_{i_r}(\lambda) \subset F^{u_r}_{i_r}(d^{\gamma}_p\Lambda) = d^{\gamma}_{p-i_r}F^{u_r}_{i_r}(\Lambda)$ and similarly $F^{u_r}_{i_r}(\lambda') \subset d^{\gamma}_{p-i_r}F^{u_r}_{i_r}(\Lambda)$. Since $\dim F^{u_r}_{i_r}(\lambda) = \dim F^{u_r}_{i_r}(\lambda') = p - i_r$ and $\dim d^{\gamma}_{p-i_r}F^{u_r}_{i_r}(\Lambda) \leq p - i_r$, we can see that $F^{u_r}_{i_r}(\lambda)$ and $F^{u_r}_{i_r}(\lambda')$ are maximal atoms in the molecule $d^{\gamma}_{p-i_r}F^{u_r}_{i_r}(\Lambda)$. It follows from condition 1 for $d^{\gamma}_{p-i_r}F^{u_r}_{i_r}(\Lambda)$ that $F^{u_r}_{i_r}(\lambda) = F^{u_r}_{i_r}(\lambda')$. This contradicts the hypothesis that $u_t[i'_t, \alpha'_t] = u_t[i_t, -\alpha_t]$.

This completes the proof.

□

Proposition 4.5.10. *Let Λ be a pairwise molecular subcomplex. Then so is $d_p^\gamma\Lambda$.*

Proof. We have shown in the last Lemma that $d_p^\gamma\Lambda$ satisfies condition 1 for pairwise molecular subcomplexes. Moreover, by Proposition 4.5.8, we have $F_{I_s}^{u_s}(d_p^\gamma\Lambda) = d_{p-I_s}^\gamma F_{I_s}^{u_s}(\Lambda)$ for all values of $s \in \{1,2,3,4\}$ and all I_s with $I_s \le p$. Since $F_{I_s}^{u_s}(\Lambda)$ is a molecule or empty set for every value of s and every I_s, we can see that $F_{I_s}^{u_s}(d_p^\gamma\Lambda)$ is a molecule or empty set for every value of s and every I_s. It follows from the definition that $d_p^\gamma\Lambda$ is pairwise molecular.

This completes the proof. □

Theorem 4.5.11. *Let Λ be a pairwise molecular subcomplex. Then the dimension of every maximal atom in $d_p^\gamma\Lambda$ is not greater than p. Moreover, an atom of dimension less than p is a maximal atom in $d_p^\gamma\Lambda$ if and only if it is a maximal atom in Λ; an atom $u_1[i_1,\alpha_1] \times u_2[i_2,\alpha_2] \times u_3[i_3,\alpha_3] \times u_4[i_4,\alpha_4]$ of dimension p is a maximal atom in $d_p^\gamma\Lambda$ if and only if there is a maximal atom $u_1[k_1,\varepsilon_1] \times u_2[k_2,\varepsilon_2] \times u_3[k_3,\varepsilon_3] \times u_4[k_4,\varepsilon_4]$ in Λ such that $k_s \ge i_s$ for $1 \le s \le 4$, and the signs α_s $(1 \le s \le 4)$ satisfy the following conditions: if $u_1[k_1,\varepsilon_1] \times u_2[k_2,\varepsilon_2] \times u_3[k_3,\varepsilon_3] \times u_4[k_4,\varepsilon_4]$ can be chosen such that $k_s > i_s$ and $k_r \ge i_r$ for $1 \le r \le 4$, then $\alpha_s = (-)^{i_1+\cdots+i_{s-1}}\gamma$; otherwies $\alpha_s = \varepsilon_s$.*

Note 4.5.12. It follows easily from condition 3 in Theorem 4.4.1 that α_s are well defined for all $1 \le s \le 4$.

Proof. By the definition of $d_p^\gamma\Lambda$, it is evident that the dimension of every maximal atom in $d_p^\gamma\Lambda$ is not greater than p.

Let Λ_1 be the subcomplex of $u_1 \times u_2 \times u_3 \times u_4$ as described in this theorem. It is easy to see that Λ_1 satisfies condition 1 for pairwise molecular subcomplexes. By Lemma 4.1.7, it suffices to prove that $F_{I_s}^{u_s}(\Lambda_1) = F_{I_s}^{u_s}(d_p^\gamma\Lambda)$ for all I_s and all $1 \le s \le 4$. The arguments for different choices of s are similar. We now give the proof for $F_{I_2}^{u_2}(\Lambda_1) = F_{I_2}^{u_2}(d_p^\gamma\Lambda)$. If $I_2 > p$, then it is easy to see that $F_{I_2}^{u_2}(\Lambda_1) = \emptyset = F_{I_2}^{u_2}(d_p^\gamma\Lambda)$, as required. In the remaining proof, we may assume that $I_2 \le p$. We have known that $F_{I_2}^{u_2}(d_p^\gamma\Lambda) = d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda)$. We need only to prove that $F_{I_2}^{u_2}(\Lambda_1) = d_{p-I_2}^\gamma F_{I_2}^{u_2}(\Lambda)$.

By the definition of $F_{I_2}^{u_2}$, it is easy to see that $F_{I_2}^{u_2}(\Lambda_1)$ and $d_{p-I_2}^\gamma F(\Lambda)$ are subcomplexes of $u_1 \times u_3^{I_2} \times u_4^{I_2}$. We are going to prove that $F_{I_2}^{u_2}(\Lambda_1)$ and $d_{p-I_2}^{\gamma} F_{I_2}^{u_2}(\Lambda)$ consists of the same maximal atoms so that they are equal.

We first prove every maximal atom in $F_{I_2}^{u_2}(\Lambda_1)$ is a maximal atom in $d_{p-I_2}^{\gamma} F_{I_2}^{u_2}(\Lambda)$.

Let $\mu = u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4]$ be a maximal atom in $F_{I_2}^{u_2}(\Lambda_1)$. Then Λ_1 has a (u_2, I_2)-projection maximal atom λ of the form $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ with $i_2 \geq I_2$. Hence Λ has a maximal atom $\lambda' = u_1[i'_1, \alpha'_1] \times u_2[i'_2, \alpha'_2] \times u_3[i'_3, \alpha'_3] \times u_4[i'_4, \alpha'_4]$ with $i_s \leq i'_s$ for all $1 \leq s \leq 4$.

Suppose that $i_2 = I_2$ and $\dim \lambda = p$. Since $u_1[i'_1, \alpha'_1] \times u_3^{I_2}[i'_3, \alpha'_3] \times u_4^{I_2}[i'_4, \alpha'_4]$ is a atom in $F_{I_2}^{u_2}(\Lambda)$ and $i_1 + i_3 + i_4 = p - I_2$, we know that $d_{p-I_2}^{\gamma} F_{I_2}^{u_2}(\Lambda)$ has a maximal atom of the form $u_1[i_1, \alpha''_1] \times u_3^{I_2}[i_3, \alpha''_3] \times u_4^{I_2}[i_4, \alpha''_4]$. Moreover, we can see that for a fixed $t \in \{1, 3, 4\}$ there is a maximal atom $u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ in Λ with $i_s \leq j_s$ for all $s \in \{1, 3, 4\}$ such that $i_t < j_t$ if and only if there is a maximal atom $u_1[j_1, \beta_1] \times u_3^{I_2}[j_2, \beta_3] \times u_4^{I_2}[j_4, \beta_4]$ in $F_{I_2}^{u_2}(\Lambda)$ with $i_s \leq j_s$ for all $s \in \{1, 3, 4\}$ such that $i_t < j_t$. It follows that from Theorem 2.5.12 that $\alpha_s = \alpha''_s$ for all $s \in \{1, 3, 4\}$, thus μ is a maximal atom in $d_{p-I_2}^{\gamma} F(\Lambda)$.

Suppose that $i_2 = I_2$ and $\dim \lambda < p$. Then λ is also a maximal atom in Λ. Therefore $\mu = F_{I_2}^{u_2}(\lambda)$ is a maximal atom in $F_{I_2}^{u_2}(\Lambda)$. Since $i_1 + i_3 + i_4 < p - I_2$, we know that μ is a maximal atom in $d_{p-I_2}^{\gamma} F(\Lambda)$.

There remains the case that $i_2 > I_2$. In this case, there is no maximal atom $u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ in Λ with $j_s \geq i_s$ for all $s \in \{1, 3, 4\}$ such that $j_s > i_s$ for some $s \in \{1, 3, 4\}$. So $i_s = i'_s$ and $\alpha_s = \alpha'_s$ for all $s \in \{1, 3, 4\}$. On the other hand, since $\mu = u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4] = u_1[i'_1, \alpha'_1] \times u_3^{I_2}[i'_3, \alpha'_3] \times u_4^{I_2}[i'_4, \alpha'_4] = F_{I_2}^{u_2}(\lambda')$, we can see that μ is a maximal atom in $F_{I_2}^{u_2}(\Lambda)$. Because $\dim \mu = i'_1 + i'_3 + i'_4 < p - I_2$, it follows from Theorem 2.5.12 that μ is a maximal atom in $d_{p-I_2}^{\gamma} F_{I_2}^{u_2}(\Lambda)$.

This shows that every maximal atom in $F_{I_2}^{u_2}(\Lambda_1)$ is a maximal atom in $d_{p-I_2}^{\gamma} F_{I_2}^{u_2}(\Lambda)$.

We next prove that every maximal atom in $d_{p-I_2}^{\gamma} F_{I_2}^{u_2}(\Lambda)$ is a maximal atom in $F_{I_2}^{u_2}(\Lambda_1)$.

Let $\mu = u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4]$ be a maximal atom in $d_{p-I_2}^{\gamma} F_{I_2}^{u_2}(\Lambda)$. Then $F_{I_2}^{u_2}(\Lambda)$ has a maximal atom $\mu' = u_1[i'_1, \alpha'_1] \times u_3^{I_2}[i'_3, \alpha'_3] \times u_4^{I_2}[i'_4, \alpha'_4]$ with $i_s \leq i'_s$ for all $s \in \{1, 2, 3\}$. Therefore Λ has a (u_2, I_2)-projection maximal atom λ' of the form

$\lambda' = u_1[i_1', \alpha_1'] \times u_2[i_2', \alpha_2'] \times u_3[i_3', \alpha_3'] \times u_4[i_4', \alpha_4']$.

Suppose that $i_1 + i_3 + i_4 = p - I_2$. Then Λ_1 has a (u_2, I_2)-projection maximal atom of the form $\lambda = u_1[i_1, \alpha_1''] \times u_2[i_2, \alpha_2''] \times u_3[i_3, \alpha_3''] \times u_4[i_4, \alpha_4'']$. It is easy to see that for a fixed $t \in \{1,3,4\}$ there is a maximal atom $u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ in Λ with $i_s \leq j_s$ for all $s \in \{1,3,4\}$ such that $i_t < j_t$ if and only if there is a maximal atom $u_1[j_1, \beta_1] \times u_3^{I_2}[j_2, \beta_3] \times u_4^{I_2}[j_4, \beta_4]$ in $F_{I_2}^{u_2}(\Lambda)$ with $i_s \leq j_s$ for all $s \in \{1,3,4\}$ such that $i_t < j_t$. It follows that $\alpha_s'' = \alpha_s$ for $s = 1,3,4$. Therefore we have $F_{I_2}^{u_2}(\lambda) = u_1[i_1, \alpha_1''] \times u_2[i_2, \alpha_2''] \times u_3[i_3, \alpha_3''] \times u_4[i_4, \alpha_4''] = \mu$. This implies that μ is a maximal atom in $F_{I_2}^{u_2}(\Lambda_1)$.

Suppose that $i_1 + i_3 + i_4 < p - I_2$. Then $\mu = u_1[i_1, \alpha_1] \times u_3^{I_2}[i_3, \alpha_3] \times u_4^{I_2}[i_4, \alpha_4]$ is also a maximal atom in $F_{I_2}^{u_2}(\Lambda)$. So Λ has a (u_2, I_2)-projection maximal atom $\lambda' = u_1[i_1, \alpha_1] \times u_2[i_2', \alpha_2'] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ with $i_2' \geq I_2$. Now, if $i_2' = I_2$, then $\dim \lambda' < p$; hence λ' is also a maximal atom in Λ_1; it follows that $\mu = F_{I_2}^{u_2}(\lambda')$, and hence μ is a maximal atom in $F_{I_2}^{u_2}(\Lambda_1)$. Suppose that $i_2' > I_2$. Then it is easy to see that there is no maximal atom $u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ in Λ with $j_s \geq i_s$ for all $s \in \{1,3,4\}$ such that $j_s > i_s$ for some $s \in \{1,3,4\}$. Hence Λ_1 has a (u_2, I_2)-projection maximal atom of the form $\lambda' = u_1[i_1, \alpha_1] \times u_2[i_2', \alpha_2'] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$. Hence we see that $\mu = F_{I_2}^{u_2}(\lambda')$ and μ is a maximal atom in $F_{I_2}^{u_2}(\Lambda_1)$.

This shows that every maximal atom in $d_{p-I_2}^{\gamma} F_{I_2}^{u_2}(\Lambda)$ is a maximal atom in $F_{I_2}^{u_2}(\Lambda_1)$.

This completes the proof. $\square$

4.6 Composition of pairwise molecular subcomplexes

In this section, we study composition of pairwise molecular subcomplexes in $u_1 \times u_2 \times u_3 \times u_4$.

Lemma 4.6.1. *Let Λ^- and Λ^+ be pairwise molecular subcomplexes. If $d_p^+\Lambda^- = d_p^-\Lambda^+$, then for every pair of maximal atoms $\lambda^- = u_1[i_1^-, \alpha_1^-] \times u_2[i_2^-, \alpha_2^-] \times u_3[i_3^-, \alpha_3^-] \times u_4[i_4^-, \alpha_4^-]$ in Λ^- and $\lambda^+ = u_1[i_1^+, \alpha_1^+] \times u_2[i_2^+, \alpha_2^+] \times u_3[i_3^+, \alpha_3^+] \times u_4[i_4^+, \alpha_4^+]$ in Λ^+ one has*

$$\sum_{s=1}^{4} \min\{i_s^-, i_s^+\} \leq p.$$

Proof. Let $l_s = \min\{i_s^-, i_s^+\}$. Suppose otherwise that $\sum_{s=1}^4 l_s > p$. Then there is an ordered triple $\{i_1, i_2, i_3, i_4\}$ with $i_s \leq l_s$ for every value of s such that $\sum_s^4 i_s = p$. Since $\sum_{s=1}^4 l_s > p$, we have $i_t < l_t$ for some t. If $i_1 < l_1$, then, by Theorem 4.5.11, we have $d_p^+\Lambda^-$ has a maximal atom of the form $u_1[i_1, +] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$, while $d_p^-\Lambda^+$ has a maximal atom of the form $u_1[i_1, -] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$. This contradicts condition 1 for the pairwise molecular subcomplex $d_p^+\Lambda^- = d_p^-\Lambda^+$. The argument for the cases $i_2 < l_2$, $i_3 < l_3$ and $i_4 < l_4$ are similar.

This completes the proof. □

Lemma 4.6.2. *Let Λ^- and Λ^+ be pairwise molecular subcomplexes in $u_1 \times u_2 \times u_3 \times u_4$. If $d_p^+\Lambda^- = d_p^-\Lambda^+$, then $F_{I_s}^{u_s}(\Lambda^-) \cap F_{I_s}^{u_s}(\Lambda^+) = F_{I_s}^{u_s}(\Lambda^- \cap \Lambda^+) = F_{I_s}^{u_s}(d_p^+\Lambda^-) = F_{I_s}^{u_s}(d_p^-\Lambda^+)$ for every value of s and every integer I_s.*

Proof. The proofs for different values of s are similar. We give the proof for $s = 2$. There are two cases, as follows.

1. Suppose that $I_2 > p$. We first claim that $F_{I_2}^{u_2}(\Lambda^-) \cap F_{I_2}^{u_2}(\Lambda^+) = \emptyset$.

Indeed, suppose otherwise that $F_{I_2}^{u_2}(\Lambda^-) \cap F_{I_2}^{u_2}(\Lambda^+) \neq \emptyset$. Then it is evident that there are atoms $\mu^- = u_1[j_1^-, \beta_1^-] \times u_2[j_2^-, \beta_2^-] \times u_3[j_3^-, \beta_3^-] \times u_4[j_4^-, \beta_4^-]$ in Λ^- and $\mu^+ = u_1[j_1^+, \beta_1^+] \times u_2[j_2^+, \beta_2^+] \times u_3[j_3^+, \beta_3^+] \times u_4[j_4^+, \beta_4^+]$ in Λ^+ such that $j_2^- \geq I_2 > p$ and $j_2^+ \geq J > p$. This implies that there are maximal atoms $u_1[0, \alpha_1'] \times u_2[p, +] \times u_3[0, \alpha_3'] \times u_4[0, \alpha_4']$ and $u_1[0, \alpha_1''] \times u_2[p, -] \times u_3[0, \alpha_3''] \times u_4[0, \alpha_4'']$ in $d_p^+\Lambda^-$ and $d_p^-\Lambda^+$ respectively. This contradicts the condition 1 for pairwise molecular subcomplex $d_p^+\Lambda^- = d_p^-\Lambda^+$.

Now we have $F_{I_2}^{u_2}(d_p^+\Lambda^-) \subset F_{I_2}^{u_2}(\Lambda^- \cap \Lambda^+) \subset F_{I_2}^{u_2}(\Lambda^-) \cap F_{I_2}^{u_2}(\Lambda^+) = \emptyset$. Therefore $F_{I_2}^{u_2}(d_p^+\Lambda^-) = F_{I_2}^{u_2}(\Lambda^- \cap \Lambda^+) = F_{I_2}^{u_2}(\Lambda^-) \cap F_{I_2}^{u_2}(\Lambda^+)$, as required.

2. Suppose that $I_2 \leq p$. Since $d_p^+\Lambda^- = d_p^-\Lambda^+$, we have $d_{p-I_2}^+F_{I_2}^{u_2}(\Lambda^-) = F_{I_2}^{u_2}(d_p^+\Lambda^-) = F_{I_2}^{u_2}(d_p^-\Lambda^+) = d_{p-I_2}^-F_{I_2}^{u_2}(\Lambda^+)$. Because $F_{I_2}^{u_2}(\Lambda^-)$ and $F_{I_2}^{u_2}(\Lambda^+)$ are molecules, we can see that $F_{I_2}^{u_2}(\Lambda^-)\#_{p-J}F_{I_2}^{u_2}(\Lambda^+)$ is defined by Proposition 2.6.3. Hence $F_{I_2}^{u_2}(\Lambda^-) \cap F_{I_2}^{u_2}(\Lambda^+) = d_{p-I_2}^+F_{I_2}^{u_2}(\Lambda^-) = F_{I_2}^{u_2}(d_p^+\Lambda^-) \subset F_{I_2}^{u_2}(\Lambda^- \cap \Lambda^+)$. Since we automatically have $F_{I_2}^{u_2}(\Lambda^- \cap \Lambda^+) \subset F_{I_2}^{u_2}(\Lambda^-) \cap F_{I_2}^{u_2}(\Lambda^+)$, we get $F_{I_2}^{u_2}(\Lambda^-) \cap F_{I_2}^{u_2}(\Lambda^+) = F_{I_2}^{u_2}(\Lambda^- \cap \Lambda^+) = F_{I_2}^{u_2}(d_p^+\Lambda^-)$, as required.

This completes the proof

□

Proposition 4.6.3. *Let Λ^- and Λ^+ be pairwise molecular subcomplexes. If $d_p^+\Lambda^- = d_p^-\Lambda^+$, then $\Lambda^- \cap \Lambda^+ = d_p^+\Lambda^- (= d_p^-\Lambda^+)$; hence $\Lambda^- \#_p \Lambda^+$ is defined.*

Proof. Let $M = d_p^+\Lambda^- = d_p^-\Lambda^+$. It is evident that $M \subset \Lambda^- \cap \Lambda^+$. To prove the reverse inclusion, it suffices to prove that every maximal atom in $\Lambda^- \cap \Lambda^+$ is contained in M.

Suppose otherwise that there is a maximal atom $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in $\Lambda^- \cap \Lambda^+$ such that $\lambda \not\subset M$. Since $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] = F_{i_4}^{u_4}(\lambda) \subset F_{i_4}^{u_4}(\Lambda^- \cap \Lambda^+) = F_{i_4}^{u_4}(M)$, we can see that M has a maximal atom $\lambda' = u_1[i_1', \alpha_1'] \times u_2[i_2', \alpha_2'] \times u_3[i_3', \alpha_3'] \times u_4[i_4', \alpha_4']$ such that $u_s[i_s, \alpha_s] \subset u_s[i_s', \alpha_s']$ for $s = 1, 2, 3$. Because $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ is maximal in $\Lambda^- \cap \Lambda^+$ and $M \subset \Lambda^- \cap \Lambda^+$, we have $i_4' = i_4$ and $\alpha_4' = -\alpha_4$. Now we know that $\lambda \cup \lambda' \subset \Lambda^-$ and $\lambda \cup \lambda' \subset \Lambda^+$. By Lemma 4.4.3, it is easy to see that there are maximal atoms $\lambda^- = u_1[i_1^-, \alpha_1^-] \times u_2[i_2^-, \alpha_2^-] \times u_3[i_3^-, \alpha_3^-] \times u_4[i_4^-, \alpha_4^-]$ in Λ^- and $\lambda^+ = u_1[i_1^+, \alpha_1^+] \times u_2[i_2^+, \alpha_2^+] \times u_3[i_3^+, \alpha_3^+] \times u_4[i_4^+, \alpha_4^+]$ in Λ^+ such that $u_s[i_s^-, \alpha_s^-] \cap u_s[i_s^+, \alpha_s^+] \supset u_s[i_s, \alpha_s]$ for $s = 1, 2, 3$ and $\min\{i_4^-, i_4^+\} > i_4$. Since λ is a maximal atom in $\Lambda^- \cap \Lambda^+$, we have $i_4^- = i_4^+ = i_4 + 1$ and $\alpha_4^- = -\alpha_4^+$.

Now, we have $u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \subset F_{i_4+1}^{u_4}(\Lambda^-) \cap F_{i_4+1}^{u_4}(\Lambda^+) = F_{i_4+1}^{u_4}(\Lambda^- \cap \Lambda^+)$. Therefore $\Lambda^- \cap \Lambda^+$ has a maximal atom $\lambda'' = u_1[i_1'', \alpha_1''] \times u_2[i_2'', \alpha_2''] \times u_3[i_3'', \alpha_3''] \times u_4[i_4'', \alpha_4'']$ with $u_s[i_s'', \alpha_s''] \supset u_s[i_s', \alpha_s']$ for $s = 1, 2, 3$ and $i_4'' > i_4$. This contradicts the assumption that λ is a maximal atom in $\Lambda^- \cap \Lambda^+$.

This completes the proof.

□

Proposition 4.6.4. *Let Λ^- and Λ^+ be pairwise molecular subcomplexes of $u_1 \times u_2 \times u_3 \times u_4$. If $d_p^+\Lambda^- = d_p^-\Lambda^+$, then the maximal atoms in the composite $\Lambda^- \#_p \Lambda^+$ are the common q-dimensional maximal atoms of Λ^- and Λ^+ for $q \leq p$ together with the r-dimensional maximal atoms in either Λ^- or Λ^+ for $r > p$.*

Proof. Let Λ be the union of the maximal atoms described in the proposition. We must prove that $\Lambda = \Lambda^- \cup \Lambda^+$. Clearly, we have $\Lambda \subset \Lambda^- \cup \Lambda^+$; it suffices to prove that $\Lambda^- \cup \Lambda^+ \subset \Lambda$. By the formation of Λ, we must prove that $\lambda \subset \Lambda$ for every maximal atom

$\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in either Λ^- or Λ^+ with $\dim \lambda \leq p$ and such that λ is not a common maximal atom in Λ^- and Λ^+. In this case, it is easy to see that $\dim \lambda = p$. Suppose that λ is a maximal atom in Λ^γ which is not a maximal atom in $\Lambda^{-\gamma}$. Then λ must be a maximal atom in $d_p^+\Lambda^- = d_p^-\Lambda^+$ which implies that $\lambda \subset \lambda^{-\gamma}$ for some maximal atom $\lambda^{-\gamma} = u_1[i_1^{-\gamma}, \alpha_1^{-\gamma}] \times u_2[i_2^{-\gamma}, \alpha_2^{-\gamma}] \times u_3[i_3^{-\gamma}, \alpha_3^{-\gamma}] \times u_4[i_4^{-\gamma}, \alpha_4^{-\gamma}]$ with $\dim \lambda^{-\gamma} > p$. Thus $\lambda \subset \Lambda$. Therefore, we have $\Lambda^- \cup \Lambda^+ \subset \Lambda$.

This completes the proof.

□

Proposition 4.6.5. *Let Λ^- and Λ^+ be pairwise molecular subcomplexes. If $d_p^+\Lambda^- = d_p^-\Lambda^+$, then $\Lambda^-\#_p\Lambda^+$ is a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$.*

Proof. Let $\Lambda = \Lambda^-\#_p\Lambda^+$. According to Lemma 4.6.1 and Proposition 4.6.4, it is easy to see that Λ satisfies condition 1 for pairwise molecular subcomplexes. Moreover, we have $F_{I_s}^{u_s}(\Lambda^-\#_p\Lambda^+) = F_{I_s}^{u_s}(\Lambda^- \cup \Lambda^+) = F_{I_s}^{u_s}(\Lambda^-) \cup F_{I_s}^{u_s}(\Lambda^+)$ for every value of s.

Now suppose that $p \geq I_s$. We have $d_{p-I_s}^+ F_{I_s}^{u_s}(\Lambda^-) = F_{I_s}^{u_s}(d_p^+\Lambda^-) = F_{I_s}^{u_s}(d_p^-\Lambda^+) = d_{p-I_s}^- F_{I_s}^{u_s}(\Lambda^+)$. Thus $F_{I_s}^{u_s}(\Lambda^-\#_p\Lambda^+) = F_{I_s}^{u_s}(\Lambda^-)\#_{p-I_s} F_{I_s}^{u_s}(\Lambda^+)$. Therefore $F_{I_s}^{u_s}(\Lambda^-\#_p\Lambda^+)$ is a molecule.

Suppose that $p < I_s$. Then it is easy to see that $F_{I_s}^{u_s}(\Lambda^-) = \emptyset$ or $F_{I_s}^{u_s}(\Lambda^+) = \emptyset$. (Otherwise, we have $F_{I_s}^{u_s}(\Lambda^- \cap \Lambda^+) \neq \emptyset$. This would lead to a contradiction to Lemma 4.6.1.) Therefore $F_{I_s}^{u_s}(\Lambda^-\#_p\Lambda^+)$ is a molecule or empty set.

We have now proved that $F_{I_s}^{u_s}(\Lambda^-\#_p\Lambda^+)$ is a molecule or empty set for every value of s and every I_s. Evidently, Λ satisfies condition 1 for pairwise molecular subcomplexes. It follows from the definition that Λ is a pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$.

□

4.7 Decomposition of Pairwise Molecular subcomplexes

The aim of this section is to prove the following theorem.

Theorem 4.7.1. *If Λ is a pairwise molecular subcomplex, then Λ is a molecule.*

It is trivial that the theorem holds when Λ is an atom. Thus we may assume that Λ is a pairwise molecular subcomplex in $u_1 \times u_2 \times u_3 \times u_4$ which is not an atom throughout this section. We are going to show that Λ is a molecule.

Let

$$p = \max\{\dim(\lambda \cap \mu) \colon \lambda \text{ and } \mu \text{ are distinct maximal atoms in } \Lambda\}.$$

It is evident that there are at least two maximal atoms λ and μ in Λ with $\dim \lambda > p$ and $\dim \mu > p$. By Theorem 4.4.3 for pairwise molecular subcomplex Λ, it is easy to see that p is the maximal number among the numbers $\sum_{s=1}^{4} \min\{i_s, j_s\}$, where $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\mu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ run over all pairs of distinct maximal atoms in Λ.

Lemma 4.7.2. *Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\mu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ be maximal atoms in Λ with $\sum_{s=1}^{4} \min\{i_s, j_s\} = p$.*

1. *Let $i_1 = j_1$ and $\alpha_1 = -\beta_1$. If $i_2 < j_2$, then $\alpha_2 = (-)^{i_1}\alpha_1$; if $i_3 < j_3$, then $\alpha_3 = (-)^{i_1+\min\{i_2,j_2\}}\alpha_1$; if $i_4 < j_4$, then $\alpha_4 = (-)^{i_1+\min\{i_2,j_2\}+\min\{i_3,j_3\}}\alpha_1$.*

2. *Let $i_2 = j_2$ and $\alpha_2 = -\beta_2$. If $i_1 < j_1$, then $\alpha_2 = (-)^{i_1}\alpha_1$; if $i_3 < j_3$, then $\alpha_3 = (-)^{i_2}\alpha_2$; if $i_4 < j_4$, then $\alpha_4 = (-)^{i_2+\min\{i_3,j_3\}}\alpha_2$.*

3. *Let $i_3 = j_3$ and $\alpha_3 = -\beta_3$. If $i_1 < j_1$, then $\alpha_3 = (-)^{i_1+\min\{i_2,j_2\}}\alpha_1$; if $i_2 < j_2$, then $\alpha_3 = (-)^{i_2}\alpha_2$; if $i_4 < j_4$, then $\alpha_4 = (-)^{i_3}\alpha_3$.*

4. *Let $i_4 = j_4$ and $\alpha_4 = -\beta_4$. If $i_1 < j_1$, then $\alpha_4 = (-)^{i_1+\min\{i_2,j_2\}+\min\{i_3,j_3\}}\alpha_1$; if $i_2 < j_2$, then $\alpha_4 = (-)^{i_2+\min\{i_3,j_3\}}\alpha_1$; if $i_3 < j_3$, then $\alpha_4 = (-)^{i_3}\alpha_1$.*

Proof. The arguments for various cases are similar. We give the proof only for the first case.

Suppose that $i_1 = j_1$, $\alpha_1 = -\beta_1$ and $i_2 < j_2$. According to Theorem 4.4.3 for pairwise molecular subcomplexes, we can get a maximal atom $\nu = u_1[k_1, \varepsilon_1] \times u_2[k_2, \varepsilon_2] \times u_3[k_3, \varepsilon_3] \times u_4[k_4, \varepsilon_4]$ with $k_1 > i_1 = j_1$, $u_2[k_2, \varepsilon_2] \supset u_2[i_2, \alpha_2]$, $k_3 \geq \min\{i_3, j_3\}$ and

$k_4 \geq \min\{i_4, j_4\}$. Since $\sum_{s=1}^{4} \min\{i_s, j_s\} = p$, we have $k_s = \min\{i_s, j_s\}$ for $s = 2, 3, 4$. Hence $u_2[k_2, \varepsilon_2] = u_2[i_2, \alpha_2]$. Moreover, it is easy to see that λ, μ and ν are pairwise adjacent by the choice of p. It follows easily from the sign conditions that $\alpha_2 = (-)^{i_1}\alpha_1$, as required. The arguments for other cases are similar.

This completes the proof.

$\square$

To decompose Λ into atoms, we need a total order $<$ on the atoms in the product of four globes analogous to that on the atoms in the product of three globes. For a pair of atom atoms $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\mu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$ in $u_1 \times u_2 \times u_3 \times u_4$, we write $\lambda < \mu$ if one of the following holds:

- $\alpha_1 = \beta_1 = -$ and $i_1 < j_1$;
- $\alpha_1 = \beta_1 = +$ and $i_1 > j_1$;
- $\alpha_1 = -$ and $\beta_1 = +$;
- $i_1 = j_1$ are even, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2 = -$ and $i_2 < j_2$;
- $i_1 = j_1$ are even, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2 = +$ and $i_2 > j_2$;
- $i_1 = j_1$ are even, $\alpha_1 = \beta_1$, $\alpha_2 = -$ and $\beta_2 = +$.
- $i_1 = j_1$ are odd, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2 = +$ and $i_2 < j_2$;
- $i_1 = j_1$ are odd, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2 = -$ and $i_2 > j_2$;
- $i_1 = j_1$ are odd, $\alpha_1 = \beta_1$, $\alpha_2 = +$ and $\beta_2 = -$.
- $i_1 = j_1$, $\alpha_1 = \beta_1$, $j_1 = j_2$, $\beta_1 = \beta_2$, $i_1 + i_2$ is even, $\alpha_3 = \beta_3 = -$ and $i_3 < j_3$;
- $i_1 = j_1$, $\alpha_1 = \beta_1$, $j_1 = j_2$, $\beta_1 = \beta_2$, $i_1 + i_2$ is even, $\alpha_3 = \beta_3 = +$ and $i_3 > j_3$;
- $i_1 = j_1$, $\alpha_1 = \beta_1$, $j_1 = j_2$, $\beta_1 = \beta_2$, $i_1 + i_2$ is even, $\alpha_3 = -$ and $\beta_3 = +$;
- $i_1 = j_1$, $\alpha_1 = \beta_1$, $j_1 = j_2$, $\beta_1 = \beta_2$, $i_1 + i_2$ is odd, $\alpha_3 = \beta_3 = +$ and $i_3 < j_3$;

- $i_1 = j_1$, $\alpha_1 = \beta_1$, $j_1 = j_2$, $\beta_1 = \beta_2$, $i_1 + i_2$ is odd, $\alpha_3 = \beta_3 = -$ and $i_3 > j_3$;
- $i_1 = j_1$, $\alpha_1 = \beta_1$, $j_1 = j_2$, $\beta_1 = \beta_2$, $i_1 + i_2$ is even, $\alpha_3 = +$ and $\beta_3 = -$.

It is evident that the relation $<$ is a total order on the set of atoms in $u_1 \times u_2 \times u_3 \times u_4$.

Lemma 4.7.3. *For any pair of maximal atoms λ and μ in Λ with $\dim \lambda > p$ and $\dim \mu > p$, if $\lambda < \mu$, then $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$.*

Proof. In the proof of this lemma, we use Lemma 4.1.1 without comments.

Let $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ and $\mu = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$. We consider several cases, as follows.

1. Suppose that $\sum_{s=1}^4 \min\{i_s, j_s\} = p$. Then λ and μ are adjacent by the choice of p. According to Lemma 4.7.2 and sign conditions for pairwise molecular subcomplexes, it is easy to see that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required.

2. Suppose that $\sum_{s=1}^4 \min\{i_s, j_s\} < p - 2$. According to condition 1 for pairwise molecular subcomplexes, it is evident that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required.

3. Suppose that $\sum_{s=1}^4 \min\{i_s, j_s\} = p - 1$ and that λ and μ are adjacent. There are several case, as follows: (1) $i_1 = j_1$ and $\alpha_1 = \beta_1$; (2) $i_1 = j_1$, $\alpha_1 = -\beta_1$, $i_2 < j_2$, $\alpha_2 = (-)^{i_1}\alpha_1$; (3) $i_1 = j_1$, $\alpha_1 = -\beta_1$, $i_2 > j_2$ and $\beta_2 = (-)^{i_1}\beta_1$; (4) $i_1 \neq j_1$; (5) $i_1 = j_1$, $\alpha_1 = -\beta_1$, $i_2 < j_2$, $\alpha_2 = -(-)^{i_1}\alpha_1$, $i_3 > j_3$ and $i_4 < j_4$; (6) $i_1 = j_1$, $\alpha_1 = -\beta_1$, $i_2 < j_2$ $\alpha_2 = -(-)^{i_1}\alpha_1$, $i_3 > j_3$ and $i_4 < j_4$; (7) $i_1 = j_1$, $\alpha_1 = -\beta_1$, $i_2 > j_2$ $\beta_2 = -(-)^{i_1}\beta_1$, $i_3 > i_3$ and $i_4 < j_4$; (8) $i_1 = j_1$, $\alpha_1 = -\beta_1$, $i_2 > j_2$ $\beta_2 = -(-)^{i_1}\beta_1$, $i_3 < i_3$ and $i_4 > j_4$. In the first 4 cases, it follows from the sign conditions that $\lambda \cap \mu \subset d_{p-1}^+\lambda \cap d_{p-1}^-\mu$, thus $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required. The arguments for cases (5) to (8) are similar, we give the proof for only case (5). In this case, we have $\alpha_1 = -$ and hence $\beta_1 = +$, $\alpha_2 = (-)^{i_1}$, $\beta_3 = -(-)^{i_1+i_2}$ and $\alpha_4 = (-)^{i_1+i_2+j_3}$, thus $\lambda \cap \mu \subset (u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[j_3 + 1, \tilde{\alpha_3}] \times u_4[i_4, \alpha_4]) \cap (u_1[j_1, \beta_1] \times u_2[i_2 + 1, \tilde{\beta_2}] \times u_3[j_3, \beta_3] \times u_4[i_4, (-)^{i_1+i_2+j_3}]) \subset d_p^+\lambda \cap d_p^-\mu$, as required.

4. Suppose that $\sum_{s=1}^4 \min\{i_s, j_s\} = p - 1$ and that λ and μ are not adjacent. Suppose also that $i_s = j_s$ for two values of s. Then it is easy to see that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required.

5. Suppose that $\sum_{s=1}^{4} \min\{i_s, j_s\} = p-1$ and that λ and μ are not adjacent. Suppose also that $i_1 = j_1$, $i_2 \neq j_2$, $i_3 \neq j_3$ and $i_4 \neq j_4$. There are various cases: (1) $i_2 < j_2$, $i_3 < j_3$ and $i_4 > j_4$; (2) $i_2 < j_2$, $i_3 > j_3$ and $i_4 < j_4$; (3) $i_2 < j_2$, $i_3 > j_3$ and $i_4 > j_4$; (4) $i_2 > j_2$, $i_3 > j_3$ and $i_4 < j_4$; (5) $i_2 > j_2$, $i_3 < j_3$ and $i_4 > j_4$; (3) $i_2 > j_2$, $i_3 < j_3$ and $i_4 < j_4$. The arguments for cases (1), (2), (4) and (5) are similar, and the arguments for cases (3) and (6) are similar. We give the proof for cases (1) and (3).

In case (1), we have $i_1 = j_1$, $i_2 < j_2$, $i_3 < j_3$ and $i_4 > j_4$. We claim that $\alpha_2 = -(-)^{i_1}$ or $\alpha_3 = (-)^{i_1+i_2}$ which implies that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required. Indeed, suppose otherwise that $\alpha_2 = (-)^{i_1}$ and $\alpha_3 = -(-)^{i_1+i_2}$. By the definition of $<$, we must have $\alpha_1 = -$ and $\beta_1 = +$. If $\beta_4 = -(-)^{i_1+i_2+i_3}$, then Λ has a maximal atom $\nu = u_1[k_1, \varepsilon_1] \times u_2[k_2, \varepsilon_2] \times u_3[k_3, \varepsilon_3] \times u_4[k_4, \varepsilon_4]$ with $k_1 \geq i_1 = i_2$, $k_2 \geq i_2$, $k_3 > i_3$ and $k_4 > j_4$; moreover, we can see that $k_2 = i_2$, $\min\{k_3, j_3\} = i_3 + 1$ and $\min\{k_4, i_4\} = j_4 + 1$ by the definition of p; furthermore, we can see that ν is adjacent to both λ and μ; it follows that $\varepsilon_2 = -(-)^{i_1}$; this contradicts the sign condition for α_3 and ε_2. If $\beta_4 = (-)^{i_1+i_2+i_3}$, then Λ has a maximal atom $\nu' = u_1[k'_1, \varepsilon'_1] \times u_2[k'_2, \varepsilon'_2] \times u_3[k'_3, \varepsilon'_3] \times u_4[k'_4, \varepsilon'_4]$ with $k'_1 \geq i_1 = i_2$, $k'_2 > i_2$, $k'_3 \geq i_3$ and $k'_4 > j_4$; moreover, we can see that $\min\{k_2, j_2\} = i_2 + 1$, $k_3 = i_3$ and $\min\{k_4, i_4\} = j_4 + 1$ by the definition of p; furthermore, we can see that ν is adjacent to both λ and μ; it follows that $\varepsilon_1 = + = -\alpha_1$ when $k_1 = i_1$; this contradicts the sign condition for α_1 and α_2.

In case (3), we have $i_1 = j_1$, $i_2 < j_2$, $i_3 > j_3$ and $i_4 > j_4$. We claim that $\beta_3 = (-)^{i_1+i_2}$ or $\beta_4 = -(-)^{i_1+i_2+j_3}$ which implies that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required. Indeed, suppose otherwise that $\beta_3 = -(-)^{i_1+i_2}$ and $\beta_4 = (-)^{i_1+i_2+i_3}$. If $\alpha_2 = -(-)^{i_1}$, then Λ has a maximal atom $\nu = u_1[k_1, \varepsilon_1] \times u_2[k_2, \varepsilon_2] \times u_3[k_3, \varepsilon_3] \times u_4[k_4, \varepsilon_4]$ adjacent to both λ and μ with $k_1 = i_1$, $\min\{k_2, j_2\} = i_2+1$, $\min\{k_3, i_3\} = j_3+1$ and $k_4 = j_4$. By the sign condition for μ and ν, we have $\varepsilon_4 = (-)^{i_1+i_2+i_3}$. This contradicts the sign condition for α_2 and ε_4. If $\alpha_2 = (-)^{i_1}$, then $\alpha_1 = -$ and $\beta_1 = +$ by the definition of $<$. According to the sign conditions for μ and ν, we get $\varepsilon_1 = +$. This contradicts the sign condition for β_1 and α_2.

6. Suppose that $\sum_{s=1}^{4} \min\{i_s, j_s\} = p-1$ and that λ and μ are not adjacent. Suppose also that $i_1 \neq j_1$, $i_2 = j_2$, $i_3 \neq j_3$ and $i_4 \neq j_4$. There are various cases: (1) $i_1 < j_1$, $i_3 < j_3$

and $i_4 > j_4$; (2) $i_1 < j_1$, $i_3 > j_3$ and $i_4 < j_4$; (3) $i_1 < j_1$, $i_3 > j_3$ and $i_4 > j_4$; (4) $i_1 > j_1$, $i_3 > j_3$ and $i_4 < j_4$; (5) $i_1 > j_1$, $i_3 < j_3$ and $i_4 > j_4$; (6) $i_1 > j_1$, $i_3 < j_3$ and $i_4 < j_4$. The arguments for cases (1), (2), (4) and (5) are similar, and the arguments for cases (3) and (6) are similar. We give the proof for cases (1) and (3).

In case (1), we have $i_1 < j_1$, $i_2 = j_2$, $i_3 < j_3$ and $i_4 > j_4$. According to the definition of $<$, we get $\alpha_1 = -$. It follows easily that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required.

In case (3), we have $i_1 < j_1$, $i_2 = j_2$, $i_3 > j_3$ and $i_4 > j_4$. According to the definition of $<$, we get $\alpha_1 = -$. We claim that $\beta_3 = (-)^{i_1+i_2}$ or $\beta_4 = -(-)^{i_1+i_2+j_3}$ which implies that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required. Indeed, suppose otherwise that $\beta_3 = -(-)^{i_1+i_2}$ and $\beta_4 = (-)^{i_1+i_2+j_3}$. Then Λ has a maximal atom $\nu = u_1[k_1, \varepsilon_1] \times u_2[k_2, \varepsilon_2] \times u_3[k_3, \varepsilon_3] \times u_4[k_4, \varepsilon_4]$ adjacent to both λ and μ with $\min\{k_1, j_1\} = i_1 + 1$, $k_2 \geq i_2$, $\min\{k_3, i_3\} = j_3 + 1$ and $k_4 = j_4$. According to the sign conditions for λ and ν, we have $\varepsilon_4 = -(-)^{i_1+i_2+j_3}$ which contradicts the sign condition for β_3 and ε_4.

7. Suppose that $\sum_{s=1}^{4} \min\{i_s, j_s\} = p - 1$ and that λ and μ are not adjacent. Suppose also that $i_1 \neq j_1$, $i_2 \neq j_2$, $i_3 = j_3$ and $i_4 \neq j_4$. By similar arguments as in case 6, we can get $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required.

8. Suppose that $\sum_{s=1}^{4} \min\{i_s, j_s\} = p - 1$ and that λ and μ are not adjacent. Suppose also that $i_1 \neq j_1$, $i_2 \neq j_2$, $i_3 \neq j_3$ and $i_4 = j_4$. By similar arguments as in case 6, we can get $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required.

9. Suppose that λ and μ are not adjacent. Suppose also that $\sum_{s=1}^{4} \min\{i_s, j_s\} = p - 1$ and $i_s \neq j_s$ for all values of s. There are various cases. The arguments for these cases are similar. We give the proof for two cases.

Suppose that $i_1 < j_1$, $i_2 < j_2$, $i_3 > j_3$ and $i_4 > j_4$. Then $\alpha_1 = -$ by the definition of $<$. We claim that $\beta_3 = (-)^{i_1+i_2}$ or $\beta_4 = -(-)^{i_1+i_2+i_3}$ which implies that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required. Indeed, suppose otherwise that $\beta_3 = -(-)^{i_1+i_2}$ and $\beta_4 = (-)^{i_1+i_2+i_3}$. Then Λ has a maximal atom $\nu = u_1[k_1, \varepsilon_1] \times u_2[k_2, \varepsilon_2] \times u_3[k_3, \varepsilon_3] \times u_4[k_4, \varepsilon_4]$ adjacent to both λ and μ with $k_1 = i_1 + 1$, $k_2 = i_2$, $k_3 = j_3 + 1$ and $k_4 = j_4$; it follow from the sign conditions for ν and λ that $\varepsilon_4 = -(-)^{i_1+i_2+j_3}$ which contradicts the sign condition for β_3 and ε_4.

Suppose that $i_1 < j_1$, $i_2 > j_2$, $i_3 > j_3$ and $i_4 > j_4$. By similar arguments as that in the above case, we can prove that $\beta_2 = (-)^{i_1}$ and $\beta_3 = (-)^{i_1+i_2}$, or $\beta_2 = (-)^{i_1}$ and $\beta_4 = -(-)^{i_1+i_2+i_3}$, or $\beta_3 = -(-)^{i_1+i_2}$ and $\beta_4 = -(-)^{i_1+i_2+i_3}$, which implies that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required.

10. Suppose that $\sum_{s=1}^4 \min\{i_s, j_s\} = p-2$. If $i_s = j_s$ for some value of s, then it is evident that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required.

Now suppose that $i_s \neq j_s$ for every value of s. If λ and μ are adjacent, then we have $\lambda \cap \mu \subset d_{p-2}^+\lambda \cap d_{p-2}^-\mu \subset d_p^+\lambda \cap d_p^-\mu$, as required.

If $i_1 < j_1$ and if $i_s < j_s$ for some value of s with $s = 2,3,4$, then we have $\alpha_1 = -$; it follows easily that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required. If $i_1 > j_1$ and if $i_s > j_s$ for some value of s with $s = 2,3,4$, then we have $\beta_1 = +$; it follows easily that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required.

There remain two cases: (1) λ and μ are not adjacent and $i_1 < j_1$ and $i_s > j_s$ for $s = 2,3,4$; (2) λ and μ are not adjacent and $i_1 > j_1$ and $i_s < j_s$ for $s = 2,3,4$. The arguments for the two cases are similar. We give the proof for the first case.

In the first case, we have $\alpha_1 = -$ by the definition of $<$. We claim that $\beta_2 = (-)^{i_1}$ or $\beta_3 = -(-)^{i_1+j_2}$ or $\beta_4 = (-)^{i_1+j_2+j_3}$ which implies that $\lambda \cap \mu \subset d_p^+\lambda \cap d_p^-\mu$, as required. Indeed, suppose otherwise that $\beta_2 = -(-)^{i_1}$ and $\beta_3 = (-)^{i_1+j_2}$ and $\beta_4 = -(-)^{i_1+j_2+j_3}$. Then there is a maximal atom $\nu = u_1[k_1,\varepsilon_1] \times u_2[k_2,\varepsilon_2] \times u_3[k_3,\varepsilon_3] \times u_4[k_4,\varepsilon_4]$ such that $k_1 > i_1$, $k_2 \geq j_2$, $k_3 \geq j_3$ and $k_4 \geq j_4$. According to sign conditions and condition 4 in Theorem 4.4.1, we can see that, for each fixed value of s with $s = 2,3,4$, ν can be chosen such that $k_s > j_s$. Moreover, by the choice of p, there are at most two values of s with $s = 2,3,4$ such that $k_s > j_s$. Now there are several cases, as follows.

(a). Suppose that ν can be chosen such that there are two values of s with $s = 2,3,4$ such that $k_s > j_s$. The arguments for various choices of the two values are similar. We give the proof for $k_2 > j_2$ and $k_3 > j_3$. In this case, we have $k_2 = j_2 + 1$ and $k_3 = j_3 + 1$ and $k_4 = j_4$ by the choice of p, and λ and ν are adjacent. It follows from sign conditions that $\varepsilon_4 = (-)^{i_1+j_2+j_3} = -\beta_4$. According to Lemma 4.4.3, we can get a maximal atom $\mu' = u_1[j'_1,\beta'_1] \times u_2[j'_2,\beta'_2] \times u_3[j'_3,\beta'_3] \times u_4[j'_4,\beta'_4]$ such that $j'_1 \geq \min\{j_1, k_1\} > i_1$, $u_2[j'_2,\beta'_2] \supset u_2[j_2,\beta_2]$, $u_3[j'_3,\beta'_3] \supset u_3[j_3,\beta_3]$ and $j'_4 > j_4$. If $j'_2 > j_2$, then it is evident that $j_2' = j_2 + 1$ and $u_3[j_3', \beta_3'] = u_3[j_3, \beta_3]$ by the choice of p, and μ' is adjacent to λ; this gives a contradiction to the sign condition for β_3' and α_1. Suppose that $j_2' = j_2$. Then $\beta_2' = \beta_2 = -(-)^{i_1}$. Thus λ and μ' are not $(1,2)$-adjacent. It follows that there is a maximal atom $\lambda' = u_1[i_1', \alpha_1'] \times u_2[i_2', \alpha_2'] \times u_3[i_3', \alpha_3'] \times u_4[i_4', \alpha_4']$ in Λ such that $i_1' > i_1$, $i_2' > j_2' = j_2$, $i_3' \geq j_3$ and $i_4' \geq \min\{j_4', i_4\} > j_4$. By the choice of p, it is easy to see that $i_2' = j_2 + 1$, $i_3' = j_3$ and $i_4' = j_4 + 1$, and that λ' is adjacent to both λ and μ'. This leads to a contradiction to the sign conditions for α_1, α_3' and β_2'.

(b). Suppose that ν cannot be chosen such that there are two values of s with $s = 2,3,4$ such that $k_s > j_s$. Then, for each value of s with $s = 2,3,4$, ν can be chosen such that $k_s > j_s$. In particular, ν can be chosen such that $k_1 > i_1$ and $k_2 > j_2$. Moreover, we have $k_2 = j_2 + 1$ or $k_2 = j_2 + 2$ by the choice of p. By the assumption, we can see that λ is both $(1,3)$-adjacent and $(1,4)$-adjacent to ν.

Suppose that $k_2 = j_2 + 1$. It follows from sign conditions that $\varepsilon_3 = -(-)^{i_1+j_2} = -\beta_3$ and $\varepsilon_4 = -(-)^{i_1+j_2+j_3} = \beta_4$. According to Lemma 4.4.3 and the assumptions, there is a maximal atom $\nu' = u_1[k_1', \varepsilon_1'] \times u_2[k_2', \varepsilon_2'] \times u_3[k_3', \varepsilon_3'] \times u_4[k_4', \varepsilon_4']$ such that $k_1' \geq \min\{j_1, k_1\} > i_1$, $u_2[k_2', \varepsilon_2'] = u_2[j_2', \beta_2']$, $k_3' > j_3$ and $u_4[k_4', \varepsilon_4'] = u_4[j_4, \beta_4]$. It follows that λ and ν' are not $(1,2)$-adjacent. Thus Λ has a maximal atom $\nu'' = u_1[k_1'', \varepsilon_1''] \times u_2[k_2'', \varepsilon_2''] \times u_3[k_3'', \varepsilon_3''] \times u_4[k_4'', \varepsilon_4'']$ such that $k_1'' > i_1$, $k_2'' > k_2' = j_2$, $k_3'' \geq \min\{k_3', i_3\} > j_3$ and $k_4'' \geq k_4' = j_4$. This contradicts to the assumption on the choice of ν.

Suppose that $k_2 = j_2 + 2$. Then one can get a contradiction by a similar argument.

This completes the proof.

$\square$

By this lemma, we can arrange all the maximal atoms in Λ with dimension greater than p as

$$\lambda_1, \lambda_2, \cdots, \lambda_n$$

such $\lambda_i \cap \lambda_j \subset d_p^+ \lambda_i \cap d_p^- \lambda_j$ for $i < j$. We denote λ_k in the list by

$$\lambda_k = u_1[i_1^{(k)}, \alpha_1^{(k)}] \times u_2[i_2^{(k)}, \alpha_2^{(k)}] \times u_3[i_3^{(k)}, \alpha_3^{(k)}] \times u_4[i_4^{(k)}, \alpha_4^{(k)}].$$

Let $\Lambda^- = d_p^- \Lambda \cup \lambda_1$ and $\Lambda^+ = d_p^+ \Lambda \cup \lambda_2 \cdots \cup \lambda_n$. We are going to prove that Λ^- and Λ^+ are pairwise molecular subcomplexes and Λ can be decomposed into Λ^- and Λ^+.

Lemma 4.7.4. Λ^- *satisfies condition 1 for pairwise molecular subcomplexes.*

Proof. We first prove that $d_p^-\lambda_1 \subset d_p^-\Lambda$. Suppose that $\xi \in d_p^-\lambda_1$. Then, for every maximal atom λ' in Λ with $\xi \in \lambda'$, if $\lambda' = \lambda_t$ for some $t > 1$, then $\xi \in \lambda_1 \cap \lambda_t \subset d_p^-\lambda_t = d_p^-\lambda'$; if $\dim \lambda' \leq p$, then we automatically have $\xi \in d_p^-\lambda'$. It follows from Lemma 1.4.17 that $d_p^-\lambda_1 \subset d_p^-\Lambda$, as required.

We now verify that Λ^- satisfies condition 1 for pairwise molecular subcomplexes. It suffices to prove that any maximal atom $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in $d_p^-\Lambda$ with $i_s \leq i_s^{(1)}$ for $s = 1, 2, 3, 4$ is contained in λ_1. By the formation of $d_p^-\lambda_1$ and $d_p^-\Lambda$, it is easy to see that λ is a maximal atom in $d_p^-\lambda_1$, and hence $\lambda \subset \lambda_1$, as required. □

Lemma 4.7.5. Λ^+ *satisfies condition 1 for pairwise molecular subcomplexes.*

Proof. It suffices to prove that any maximal atom $\lambda = u_1[i_1, \alpha_1] \times u_2[i_2, \alpha_2] \times u_3[i_3, \alpha_3] \times u_4[i_4, \alpha_4]$ in $d_p^+\Lambda$ with $i_s \leq i_s^{(t)}$ for $s = 1, 2, 3, 4$ and some $2 \leq t \leq n$ is contained in some λ_r with $2 \leq r \leq n$. It is evident that $\dim \lambda = p$.

Let r be the maximal integer t between 2 and n such that with $i_s \leq i_s^{(t)}$ for $s = 1, 2, 3, 4$. Then $d_p^+\lambda_r$ has a maximal atom of the form $\lambda' = u_1[i_1, \alpha_1'] \times u_2[i_2, \alpha_2'] \times u_3[i_3, \alpha_3'] \times u_4[i_4, \alpha_4']$. By the choice of r, it is evident that $\operatorname{Int} \lambda' \cap \lambda_t = \emptyset$ for any $t > r$. Moreover, for any $1 \leq s < r$, we have $\lambda' \cap \lambda_s \subset \lambda_r \cap \lambda_s \subset d_p^+\lambda_s$. By Lemma 1.4.17, we can see that $\operatorname{Int} \lambda' \subset d_p^+\Lambda$ and hence $\lambda' \subset d_p^+\Lambda$. So, by condition 1 for the pairwise molecular subcomplex $d_p^+\Lambda$, we can see that $\lambda = \lambda' \subset \lambda_r$, as required.

This completes the proof. □

Lemma 4.7.6. *Let* $1 \leq r \leq 4$. *If* $p \geq I_r$ *and* λ_1 *is a* (u_r, I_r)*-projection maximal atom, then*

1. $F_{I_r}^{u_r}(\Lambda^-)$ *and* $F_{I_r}^{u_r}(\Lambda^+)$ *are molecules;*

2. $d_{p-I_r}^+ F_{I_r}^{u_r}(\Lambda^-) = d_{p-I_r}^- F_{I_r}^{u_r}(\Lambda^+)$, *hence* $F_{I_r}^{u_r}(\Lambda^-) \#_{p-I_r} F_{I_r}^{u_r}(\Lambda^+)$ *is defined;*

3. $F_{I_r}^{u_r}(\Lambda) = F_{I_r}^{u_r}(\Lambda^-) \#_{p-I_r} F_{I_r}^{u_r}(\Lambda^+)$.

Proof. The arguments for various choices of r are similar. We prove only for $r = 1$.

Since $F^{u_1}_{I_1}$ preserves unions, we have $F^{u_1}_{I_1}(\Lambda^-) = F^{u_1}_{I_1}(d^-_p\Lambda \cup \lambda_1) = F^{u_1}_{I_1}(d^-_p\Lambda) \cup F^{u_1}_{I_1}(\lambda_1)$ and $F^{u_1}_{I_1}(\Lambda^+) = F^{u_1}_{I_1}(d^+_p\Lambda \cup \lambda_2 \cup \cdots \cup \lambda_n) = F^{u_1}_{I_1}(d^+_p\Lambda) \cup F^{u_1}_{I_1}(\lambda_2) \cup \cdots \cup F^{u_1}_{I_1}(\lambda_n)$. If $\dim F^{u_1}_{I_1}(\lambda_1) \leq p - I_1$, then it is evident that $F^{u_1}_{I_1}(\Lambda^-) = F^{u_1}_{I_1}(d^-_p\Lambda) = d^-_{p-I_1}F^{u_1}_{I_1}(\Lambda)$ and $F^{u_1}_{I_1}(\Lambda^+) = F^{u_1}_{I_1}(\Lambda)$; it follows easily that $F^{u_1}_{I_1}(\Lambda^-)$ and $F^{u_1}_{I_1}(\Lambda^+)$ are molecules and $d^+_{p-I_1}F^{u_1}_{I_1}(\Lambda^-) = d^-_{p-I_1}F^{u_1}_{I_1}(\Lambda^+)$, as required. If $F^{u_1}_{I_1}(\Lambda)$ consists of only one maximal atom, then $F^{u_1}_{I_1}(\Lambda) = F^{u_1}_{I_1}(\lambda_1)$; it follows that $F^{u_1}_{I_1}(\Lambda^-) = F^{u_1}_{I_1}(\lambda_1) = F^{u_1}_{I_1}(\Lambda)$ and $F^{u_1}_{I_1}(\Lambda^+) = F^{u_1}_{I_1}(d^+_p\Lambda) = d^+_{p-I_1}F^{u_1}_{I_1}(\Lambda)$; hence $F^{u_1}_{I_1}(\Lambda^-)$ and $F^{u_1}_{I_1}(\Lambda^+)$ are molecules and $d^+_{p-I_1}F^{u_1}_{I_1}(\Lambda^-) = d^-_{p-I_1}F^{u_1}_{I_1}(\Lambda^+)$, as required. In the following proof, we may assume that $\dim F^{u_1}_{I_1}(\lambda_1) > p - I_1$ and $F^{u_1}_{I_1}(\Lambda)$ consists of at least two distinct maximal atoms.

Let

$$q = \max\{\dim(\mu \cap \mu') : \mu \text{ and } \mu' \text{ are distinct maximal atoms in } F^{u_1}_{I_1}(\Lambda)\}.$$

It is clear that $q \leq p - I_1$ by the choice of p. Let $\mu = u^{I_1}_2[j_2, \beta_2] \times u^{I_1}_3[j_3, \beta_3] \times u^{I_1}_4[j_4, \beta_4]$ be a maximal atom in $F^{u_1}_{I_1}(\Lambda)$ distinct from $F^{u_1}_{I_1}(\lambda_1)$ such that $\dim \mu > p - I_1$. We first prove that $F^{u_1}_{I_1}(\lambda_1) \cap \mu \subset d^+_{p-I_1}F^{u_1}_{I_1}(\lambda_1) \cap d^-_{p-I_1}\mu$.

Since μ is a maximal atom in $F^{u_1}_{I_1}(\Lambda)$, there is a (u_1, I_1)-projection maximal atom $\tilde{\mu}$ of the form $\tilde{\mu} = u_1[j_1, \beta_1] \times u_2[j_2, \beta_2] \times u_3[j_3, \beta_3] \times u_4[j_4, \beta_4]$. We consider several cases, as follows.

1. Suppose that $\min\{i_1, j_1\} = I_1$. Since $\lambda_1 \cap \tilde{\mu} \subset d^+_p\lambda_1 \cap d^-_p\tilde{\mu}$, it is easy to see that $F^{u_1}_{I_1}(\lambda_1) \cap \mu \subset d^+_{p-I_1}F^{u_1}_{I_1}(\lambda_1) \cap d^-_{p-I_1}\mu$, as required.

2. Suppose that $\min\{i_1, j_1\} > I_1 + 1$. Then $\min\{i_2, j_2\} + \min\{i_3, j_3\} + \min\{i_4, j_4\} \leq p - I_1 - 2$. It follows easily that $F^{u_1}_{I_1}(\lambda_1) \cap \mu \subset d^+_{p-I_1}F^{u_1}_{I_1}(\lambda_1) \cap d^-_{p-I_1}\mu$, as required.

3. Suppose that $\min\{i_1, j_1\} = I_1 + 1$. Then $\min\{i_2, j_2\} + \min\{i_3, j_3\} + \min\{i_4, j_4\} \leq p - I_1 - 1$. If $\min\{i_2, j_2\} + \min\{i_3, j_3\} + \min\{i_4, j_4\} < p - I_1 - 1$, then it is evident that $F^{u_1}_{I_1}(\lambda_1) \cap \mu \subset d^+_{p-I_1}F^{u_1}_{I_1}(\lambda_1) \cap d^-_{p-I_1}\mu$, as required. If $\min\{i_2, j_2\} + \min\{i_3, j_3\} + \min\{i_4, j_4\} = p - I_1 - 1$, and if $i_s = j_s$ for some value of s with $s = 2, 3, 4$, then it is evident that $F^{u_1}_{I_1}(\lambda_1) \cap \mu \subset d^+_{p-I_1}F^{u_1}_{I_1}(\lambda_1) \cap d^-_{p-I_1}\mu$, as required. If $\min\{i_2, j_2\} + \min\{i_3, j_3\} + \min\{i_4, j_4\} = p - I_1 - 1$, and if $i_s \neq j_s$ for $s = 2, 3, 4$, then $\min\{i_1, j_1\} + \min\{i_2, j_2\} + \min\{i_3, j_3\} + \min\{i_4, j_4\} = p$; thus λ_1 and $\tilde{\mu}$ are adjacent; it follows easily from the sign condition for λ_1 and $\tilde{\mu}$ that $F^{u_1}_{I_1}(\lambda_1) \cap \mu \subset d^+_{p-I_1}F^{u_1}_{I_1}(\lambda_1) \cap d^-_{p-I_1}\mu$, as required.

Now, we have $F_{I_1}^{u_1}(\Lambda^-) = d_{p-I_1}^- F_{I_1}^{u_1}(\Lambda) \cup F_{I_1}^{u_1}(\lambda_1)$ and

$$F_{I_1}^{u_1}(\Lambda^+) = d_{p-I_1}^+ F_{I_1}^{u_1}(\Lambda) \cup \bigcup\{\mu : \mu \text{ is a maximal atom in } F_{I_1}^{u_1}(\Lambda) \text{ with } \mu \neq F_{I_1}^{u_1}(\lambda_1)\}$$

(Note that it is possible that $F_{I_1}^{u_1}(\Lambda^+) = d_{p-I_1}^+ F_{I_1}^{u_1}(\Lambda)$). It follows from Theorem 1.4.13 that $F_{I_1}^{u_1}(\Lambda^-)$ and $F_{I_1}^{u_1}(\Lambda^+)$ are molecules in $u_2^{I_1} \times u_3^{I_1} \times u_4^{I_1}$, $d_{p-I_1}^+ F_{I_1}^{u_1}(\Lambda^-) = d_{p-I_1}^- F_{I_1}^{u_1}(\Lambda^+)$ and $F_{I_1}^{u_1}(\Lambda) = F_{I_1}^{u_1}(\Lambda^-)\#_{p-I_1} F_{I_1}^{u_1}(\Lambda^+)$, as required.

This completes the proof.

□

Proposition 4.7.7. *Let Λ be a pairwise molecular subcomplex. Then*

1. *Λ^- and Λ^+ are pairwise molecular subcomplexes.*

2. *$d_p^+\Lambda^- = d_p^-\Lambda^+$, hence the composite $\Lambda^-\#_p\Lambda^+$ is defined.*

3. *$\Lambda = \Lambda^-\#_p\Lambda_+$.*

Proof. We first prove that Λ^- and Λ^+ are pairwise molecular subcomplexes. If λ_1 is not a (u_1, I_1)-projection maximal atom in Λ, then it is easy to see that $F_{I_1}^{u_1}(\Lambda^-) = F_{I_1}^{u_1}(d_p^-\Lambda)$ and $F_{I_1}^{u_1}(\Lambda^+) = F_{I_1}^{u_1}(\Lambda)$ by the choice of p and Lemmas 4.7.4 and 4.7.5; hence $F_{I_1}^{u_1}(\Lambda^-)$ and $F_{I_1}^{u_1}(\Lambda^+)$ are the empty set or molecules in $u_2^{I_1} \times u_3^{I_1} \times u_4^{I_1}$. If λ_1 is a (u_1, I_1)-projection maximal atom in Λ, then we have already seen that $F_{I_1}^{u_1}(\Lambda^-)$ and $F_{I_1}^{u_1}(\Lambda^+)$ are molecules in $u_2^{I_1} \times u_3^{I_1} \times u_4^{I_1}$ from Lemma 4.7.6. Consequently, $F_{I_1}^{u_1}(\Lambda^-)$ and $F_{I_1}^{u_1}(\Lambda^+)$ are the empty set or molecules in $u_2^{I_1} \times u_3^{I_1} \times u_4^{I_1}$ for every integer I_1. Similarly, $F_{I_s}^{u_s}(\Lambda^-)$ and $F_{I_s}^{u_s}(\Lambda^+)$ are the empty set or molecules in the corresponding ω-complex for every value of s and every integer I_s. It follows that Λ^- and Λ^+ are pairwise molecular subcomplex of $u_1 \times u_2 \times u_3 \times u_4$.

Now, if $p \geq I_1$ and λ_1 is not (u_1, I_1)-projection maximal, then we can see that

$$\begin{aligned}
& F_{I_1}^{u_1}(d_p^+\Lambda^-) \\
= \; & d_{p-I_1}^+ F_{I_1}^{u_1}(\Lambda^-) \\
= \; & d_{p-I_1}^+ F_{I_1}^{u_1}(d_p^-\Lambda)) \\
= \; & d_{p-I_1}^- F_{I_1}^{u_1}(\Lambda) \\
= \; & d_{p-I_1}^- F_{I_1}^{u_1}(\Lambda^+) \\
= \; & F_{I_1}^{u_1}(d_p^-\Lambda^+);
\end{aligned}$$

if $p < I_1$, then $F^{u_1}_{I_1}(d^+_p\Lambda^-) = \emptyset = F^{u_1}_{I_1}(d^+_p\Lambda^-)$; if $p \geq I_1$ and λ is (u_1, I_1)-projection maximal, then $F^{u_1}_{I_1}(d^+_p\Lambda^-) = F^{u_1}_{I_1}(d^-_p\Lambda^+)$ by Proposition 4.5.8. Consequently, we have $F^{u_1}_{I_1}(d^+_p\Lambda^-) = F^{u_1}_{I_1}(d^+_p\Lambda^-)$ for every value of I_1. Similarly, we can see that we have $F^{u_s}_{I_s}(d^+_p\Lambda^-) = F^{u_s}_{I_s}(d^-_p\Lambda^+)$ for every value of s and every value of I_s. If follows from Proposition 4.1.7 that $d^+_p\Lambda^- = d^-_p\Lambda^+$. Clearly, we have $\Lambda = \Lambda^- \cup \Lambda^+$. Therefore $\Lambda = \Lambda^- \#_p \Lambda^+$, as required. This completes the proof. □

We have now proved that a pairwise molecular subcomplex Λ in $u_1 \times u_2 \times u_3 \times u_4$ can be decomposed into pairwise molecular subcomplexes $\Lambda = \Lambda^- \#_p \Lambda^+$. It is evident that this is a proper decomposition. By induction, we can see that Λ can be eventually decomposed into atoms. Thus Λ is a molecule. So we get the proof for Theorem 4.7.1.

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