A SPECTRAL EIGENVALUE METHOD FOR MULTILAYRED CONTINUUA

by

ABDUL-FATTAH KARI A. BUKHARI

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To my mother and my family

Preface

This thesis is submitted to the Mathematics Department in the University of Glasgow in accordance with the requirements of the degree of Doctor of Philosophy.

I would like to take this opportunity to express my best wishes and sincere gratitude to my supervisor, Dr. K.A. Lindsay, for his advice, help and encouragement throughout my research time in the Mathematics Department. I would also like to extend my best wishes and thanks to the Head of Department, Professor Brown, and to the many other members of staff and research students at the Mathematics Department for their support and kindness during my stay at Glasgow.

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Finally, I offer my best wishes and my thanks to my mother and my family for their patience with me and their continuous support.

Contents

Chapter 1

Introduction

1.1 Introduction

This thesis explores the use of spectral methods in the stability analysis of a number of problems arising in Applied Mathematics and Physics. On some occasions new results will be obtained; on other occasions existing results will be confirmed or extended by a more accurate technique. The application of spectral methods to multi-layered regions is novel.

The stability analysis of a linear system of partial differential equations is connected in a natural way with the computation of the eigenvalues of a boundary value problem. In this work, time is removed from the evolution equations of the original problem in such a way that *instability* ensues whenever the resulting linearized boundary value problem has eigenvalues with positive real part. Two distinct modes of instability are possible depending on whether an eigenvalue is real (stationary instability) or complex (overstability); in either case, the magnitude of the related eigenfunction grows exponentially in time. As a rough rule of thumb, stationary instability tends to occur in systems in which a single driving mechanism, say thermally induced buoyancy, overpowers a single damping mechanism such as viscosity. When additional mechanisms are present, say the influence of magnetic fields, overstability is often a possibility. In this case, instability ensues through "hunting", that is, oscillations of increasing magnitude. In practice, the mathematical problem reduces to one of finding the *critical eigenvalue* or eigenvalue with largest real part. If stationary instability is the preferred mechanism then the critical eigenvalue is zero, otherwise it is purely complex and, for example, will be one of a complex conjugate pair if the eigenvalue problem happens to be real.

There are essentially two distinct strategies for determining eigenvalues depending on whether or not the location of the competitive (most unstable) eigenvalue is known for a particular set of system parameters. If a good estimate of a particular eigenvalue is available then methods such as *Inverse Iteration* or *Compound Matrices* can often be used effectively to confirm the existence of this eigenvalue and further improve its value. The Inverse Iteration method relies upon the power method for eigenvalue calculation. Its principal drawback lies in the treatment of complex boundary conditions and the need to compute matrix inner products in higher precision. The Compound Matrices technique is a clever adaptation of the shooting method for linear eigenvalue problems. Instead of computing the target function for a shooting method by the evaluation of a determinant, the target is computed as a linear combination of the solutions of a system of ordinary differential equations. This avoids the inescapable numerical inaccuracies which are inherent in the evaluation of determinants. Compound Matrices are particularly accurate but usually at the expense of solving a large system of linear differential equations.

Often it is possible to establish the "principle" of exchange of stabilities, that is, the eigenvalues of the system are either always real or, if complex eigenvalues are possible, then these can never be destabilising (for example, always have negative real parts if the eigenfunction dependence on time is $e^{\sigma t}$). Of course, this is a matter for mathematical proof and not a principle of nature as the title might suggest. When exchange of stabilities is operative then Inverse Iteration and Compound Matrices have a role to play in eigenvalue determination. In the work explored here, no such principle exists generally and so methods which rely on an initial guess for the competitive eigenvalue are at best dangerous. Also, they will often produce wrong estimates of critical eigenvalues since they are insensitive to the discontinuous dependence of critical eigenvalues on parametric variables. Spectral methods, because they estimate the spectrum of the linear operator, are extremely flexible to the rapid changes which occur for small changes in the system parameters. They are responsive to the "jostling for leading position" typically displayed by eigenvalues as system parameters such as wavenumbers change even by small amounts.

The use of spectral series in the analysis of eigenvalue problems can probably be attributed to Lanczos [26] but it was Orszag's [37] computational treatment of the Orr-Sommerfeld (OS) equation using Chebyshev polynomials that established the prominence and viability of this method. Orszag's original calculations were done on a CDC Cyber with a rounding error of 10/11 decimal places using a fourth order spectral (tau) representation of the OS equation with 25 polynomials. Under these circumstances, his calculations were at the very limits of feasibility since the matrix entries in the 4th order spectral representation grow like M^7 where M is the number of polynomials in use. When $M > 30$, rounding errors dominate if not before. Orszag circumvented this difficulty by recognising two important properties of the OS equation: firstly, its eigenfunctions are either odd or even functions (and so do not require a fully Chebyshev expansion) and secondly, its coefficients are polynomials (in fact, at worst quadratic) and so enjoy an exact representation in the Chebyshev basis. Although polynomial coefficients appear often in applications, spectral methods work effectively for non-polynomial coefficients.

Implementations of spectral methods are commonly classified as Galerkin, Collocation or Tau depending on the nature of the eigenfunction and the way in which the original eigenvalue problem is approximated within a function space. In a Galerkin [10] method, the eigenfunction is expressed as a truncated sum of basis functions which individually satisfy the boundary conditions. The coefficients of the eigenfunction are chosen so that the residual (the remainder or measure of the extent to which the spectral solution fails to satisfy the eigenvalue problem) is orthogonal to each basis function with respect to a suitable norm. The first attempt to use spectral Galerkin methods in the numerical solution of partial differential equations (meteorological modelling) has been attributed to Silberman [41]. Orszag [37], [38] and Eliasen, Machenhauer & Rasmussen [7] have shown that spectral Galerkin methods (based on transforms) are practical for high resolution calculations for differential systems involving quadratic nonlinearities but are impractical for more complicated nonlinearities.

The Tau approach is a variant of the Galerkin method in which the basis functions do not individually satisfy the boundary conditions. These enter the problem as a restriction on the spectral coefficients of the eigenvector. Experience reveals that Tau methods are normally superior to Galerkin methods and will be the preferred method in this thesis.

Finally, Collocation methods differ from both Galerkin and Tau methods in the respect that the entries of eigenvectors now represent values of the independent variables at predetermined points (usually the zeros of some basis function) rather than spectral coefficients; otherwise the method is ostensibly the same. The Collocation approach looks attractive at first sight because it deals directly with values instead of coefficients but it is conjectured that this advantage may be illusory for two reasons.

(a) For a given mathematical representation of the eigenvalue problem, the Collocation matrices contain entries which are larger than their corresponding Galerkin or Tau counterparts. Hence they are more susceptible to instabilities induced by rounding error.

(b) A diverging series (perhaps corresponding to a solution which is blowing up) could be represented by its spectral coefficients whereas a Collocation representation is doomed.

The application of spectral methods to a variety of subject areas is described by Haltiner and Williams [16], Mercier [30], Gottlieb and Hussaini [13], Deville [6], Jarraud and Baede [21], Hussaini, Salas and Zang [19], Zang and Hussaini [46], Gottlieb [14], Hussaini and Zang [20] and Canuto, Hussaini, Quarteoni and Zang [3]. For finite intervals, Chebyshev and Legendre polynomials have a similar quality of spectral performance although Chebyshev polynomials are much easier to use in practice, not to mention the obvious connection between Chebyshev spectral series and the Fast Fourier Transform (FFT). Similarly, infinite and semi-infinite regions can be treated using Hermite and Laguerre polynomials respectively but it is usually preferable to map such regions into $[-1, 1]$ using, for example, $y = \tanh(\alpha x)$ or $y = -1 + 2e^{\alpha x}$ and then employ Chebyshev expansions.

In practice, spectral methods convert the boundary value problem into the generalised eigenvalue problem $AY = \sigma BY$ where A and B are square matrices and Y is an eigenvector to be assimilated with the coefficients of spectral series in the Galerkin and Tau approaches and function values in the Collocation approach. In particular, B is typically singular. In this work, the generalised eigenvalue problem is treated using NAG routines F02GJX and F02BJX for complex and real matrices respectively. The mathematical construction of the generalised eigenvalue problem from the boundary value problem and the treatment of boundary conditions is described in Chapter 2 of this thesis.

As has already been mentioned, Orszag treated the OS equation as a single 4th order eigenvalue problem. Of course, the mathematical problem could equivalently be represented as a pair of second order equations or a quartet of first order equations. Mathematically these are all equivalent but numerically the approaches are subtley different in that lowering the order of the component equations endows more derivatives with spectral expansions, effectively treating them as independent variables. In the first order system, each derivative enjoys its own expansion. The potential benefits of this approach are, hopefully, increased accuracy and ease of coding but these must be counterbalanced by the significant increase in the size of the matrices A and B. This issue is addressed in Chapter 3 in which the OS equation is investigated using a pair of second order equations (the D^2 approach) and a quartet of first order equations (the D approach). Briefly, computations suggest that the D method is more accurate than the D^2 method but not overpoweringly so. When memory is scarce such as in eigenvalue calculations in 2D problems, it is clear that the D^2 method is the way forward.

The Orr-Sommerfeld eigenvalue problem, previously solved in Chapter 3 by Chebyshev Tau methods, is now solved in Chapter 4 using Legendre spectral methods.

In Chapter 5 of this thesis, the Chebyshev Tau method is applied to the stability of Benard-Marangoni convection in a horizontal layer of viscous, electrically conducting fluid with an imposed axial magnetic field of constant magnitude. Wilson [45] develops a comprehensive analysis of stationary stability for this problem in a variety of circumstances including situations in which the free surface is flat or wrinkled. His results are checked and, where necessary, extended to include overstable regimes of parameter space. Roughly speaking, overstability is preferred increasingly as the ratio of the magnetic to viscous Prandtl numbers exceeds unity.

Chapter 6 of this thesis extends the spectral methodology of chapter 2 into multi-layered regions.

Chapter 7 of this thesis deals with the investigation of linear stability analysis for a layer of porous medium permeated and superposed by a layer of incompressible viscous fluid. Chen & Chen [5] have addressed this problem when heating is applied at the bottom boundary. Their investigation assumed stationary instability from the outset and used a shooting technique based on 4th order Runge-Kutta approximations for integration of all differential equations. To check Chen & Chen's results, Chebyshev spectral methods are applied to this two-layer problem. Their results for the behaviour of the Rayleigh number appear qualitatively accurate for small wavenumbers only. As the wavenumber increases, the discrepancy between their results and these generated by spectral methods grows. The trouble is that the already poor characteristics of a determinant based method are further eroded by the presence of the second layer and the need to reorganise boundary conditions for crossing the inner boundary.

Chapter 8 of this thesis explores the Chen & Chen [5] problem when salting effects are included in addition to heating. They considered the solution in which *heating and salting are applied at the top boundary* (in connection with the solidification of a liquid melt). Again, stationary instability is assumed from the outset. It is expected that overstability may be the preferred mechanism in certain regions of parameter space but computation reveals that the eigenvalues are always real in the applications of Chen & Chen. This contrasts sharply with the situation in which heating and salting are applied at the lower boundary; here stationary and overstable instability are both possible! By taking suitable inner products of the governing differential equations, it can be discerned that analytical explanations for these observations are not possible. Again the presence of a second layer visibly accelerates the deterioration in the numerical accuracy achievable by an unsophisticated shooting method.

In Chapter 9 of the thesis, spectral methods are used to investigate an eigenvalue problem arising in MHD and originally treated by Lamb [25] using Inverse Iteration. In this model, the Earth is described as a solid cylinder (inner core) of finite electrical conductivity surrounded by a cylindrical annulus (outer core) of incompressible viscous fluid with finite conductivity, both electrically and thermally, and this in turn is surrounded by a non-conducting region of infinite extent (mantle). The finite conductivity of the solid inner core and the non-conducting nature of the mantle essentially guarantees an active interaction with the fluid outer core leading to a genuine three-layer eigenvalue problem: the inner core and mantle cannot be replaced by simple boundary conditions. Moreover, it is a three-layer problem with a subtle twist arising from the fact that Laplace's equation has divergent solutions in the inner core and mantle. Lamb deals with these difficulties analytically and, in so doing, reduces the full problem to an eigenvalue problem in the fluid outer core. In the absence of analytical solutions in the inner core and mantle, Lamb's methodology would likely fail. Methods such as Inverse Iteration and Compound Matrices experience severe difficulties whenever governing equations have potentially singular solutions simply because they proceed by constructing eigenfunctions numerically whereas a spectral method constructs the coefficients of a spectral expansion and is more able to suppress singular behaviour in favour of regularity. This example from MHD provides a good illustration of this point.

Chapter 2

Eigenvalue Determination using Spectral Methods

2.1 Introduction to Orthogonal Polynomials

Orthogonal polynomials play an important role in many areas of mathematics. The following sections review properties that are relevant to eigenvalue analysis.

2.1.1 Some General Aspects of Orthogonal Polynomials

Let f and g be two functions defined over $[a, b]$ $(-\infty \leq a < b \leq \infty)$ then the inner product of f and g over the interval $[a, b]$ with respect to the weight function $w(x)$ (> 0) is denoted by the symbolism $< f, g >$ and defined by (see [43])

$$< f, g >= \int_a^b w(x) f(x) g(x)\, dx \tag{2.1.1}$$

whenever this integral exists.

If $< f, g >= 0$ for non-trivial f, g then the functions f and g are said to be mutually orthogonal. In this work, the class of functions for consideration is restricted to the space of real polynomials. Let $\pi_0 = 1$, the canonical polynomial of degree one. Given a real interval $[a, b]$ and a suitable weight function $w(x)$, a family of mutually orthogonal real polynomials $\pi_0(x), \ldots, \pi_n(x) \ldots$ can be constructed by a Gram-Schmidt procedure in which $\pi_n(x)$ has leading term x^n. In particular, each family is uniquely determined by this condition, the choice of interval and the weight function. Hence

$$< \pi_n, \pi_m >= \int_a^b w(x)\pi_n(x)\pi_m(x)\,dx = \begin{cases} 0 & \text{for } n \neq m \\ \mu_n & \text{for } n = m. \end{cases} \tag{2.1.2}$$

There is a close connection between the interval $[a, b]$, the associated weight function $w(x)$ and an associated family of second order ordinary differential equations which possess the set of orthogonal polynomials as a family of solutions. However, in this work, the family of π's is regarded as the solution of the second order difference equation

$$\pi_{n+1}(x) = (x - \delta_{n+1})\pi_n(x) - \gamma_{n+1}^2\pi_{n-1}(x)\,, \quad (n \geq 0) \quad \pi_{-1}(x) = 0\,, \pi_0(x) = 1. \tag{2.1.3}$$

where δ_{n+1} and γ_{n+1}^2 are defined by

$$\delta_{n+1} = \frac{< x\pi_n, \pi_n >}{\mu_n} \quad \text{for } n \geq 0\,, \qquad \gamma_{n+1}^2 = \begin{cases} 0 & \text{for } n = 0 \\ \dfrac{\mu_n}{\mu_{n-1}} & \text{for } n \geq 1\,. \end{cases} \tag{2.1.4}$$

These definitions of δ_{n+1} and γ_{n+1}^2 essentially ensure that the π's form a mutually orthogonal set of polynomials.

Spectral analysis with polynomials depends critically on the fact that the space of polynomials is closed under differentiation and multiplication; that is, the derivative of a polynomial is a polynomial and the product of two polynomials is a polynomial. Hence

(i) the derivative of $\pi_k(x)$ can be represented by a linear combination of the polynomials $\pi_0(x), \ldots, \pi_{k-1}(x)$,

(ii) the product of $\pi_n(x)$ and $\pi_k(x)$ can be expressed as a linear combination of the polynomials $\pi_0(x), \ldots, \pi_{n+k}(x)$.

In fact, the special nature of the π's allow a stronger claim for the second property. Suppose that π_n and π_k are two members of the family defined by the recurence relation (2.1.3) then $\pi_n(x)\pi_k(x)$ can be expressed as a linear combination of $\pi_{|n-k|}(x), \ldots, \pi_{n+k}(x)$. That is, the terms $\pi_0(x) \ldots \pi_{|n-k-1|}(x)$ are absent unexpectedly from the product.

This result is proved by induction. If either of n or k is zero then the result is trivial since one of the polynomials is unity. Also if $n = k$ then the product $\pi_n\pi_k$ is just a polynomial of degree $2n$ and the result is again unremarkable. Without any loss of generality, assume now that $n > k > 0$. From (2.1.3), $\pi_1(x) = x - \delta_1$ and the definition of the polynomial family can be recast in the form

$$\pi_{n+1}(x) = (\pi_1(x) + \delta_1 - \delta_{n+1})\pi_n(x) - \gamma_{n+1}^2 \pi_{n-1}(x) \tag{2.1.5}$$

and this in turn can be rewritten as

$$\pi_1(x)\pi_n(x) = \pi_{n+1}(x) + (\delta_{n+1} - \delta_1)\pi_n(x) + \gamma_{n+1}^2 \pi_{n-1}(x) \ .$$

Hence the result is true when $n > k = 1$.

Now assume that the result is true for all $n > k \geq 1$. In view of the alternative definition (2.1.5) for the π's relating $\pi_{k+1}(x)$, $\pi_k(x)$ and $\pi_{k-1}(x)$,

$$\begin{aligned}
\pi_n(x)\pi_{k+1}(x) &= (\pi_1(x) + \delta_1 - \delta_{k+1})\pi_n(x)\pi_k(x) - \gamma_{k+1}^2 \pi_{k-1}(x)\pi_n(x) \\
&= (\pi_1(x) + \delta_1 - \delta_{k+1}) \sum_{r=n-k}^{n+k} a_r \pi_r(x) - \gamma_{k+1}^2 \sum_{r=n-k+1}^{n+k-1} b_r \pi_r(x) \\
&= \pi_1(x) \sum_{r=n-k}^{n+k} a_r \pi_r(x) + \sum_{r=n-k}^{n+k} c_r \pi_r(x) \\
&= \sum_{r=n-k}^{n+k} a_r \pi_r(x)\pi_1(x) + \sum_{r=n-k}^{n+k} c_r \pi_r(x) \\
&= \sum_{r=n-k-1}^{n+k+1} d_r \pi_r(x) + \sum_{r=n-k}^{n+k} c_r \pi_r(x) \\
&= \sum_{r=n-k-1}^{n+k+1} e_r \pi_r(x).
\end{aligned}$$

Hence the claim is justified. To sum up, it is possible to determine coefficients D_{nm} and β_{nmk} for $n \geq m$ so that

$$\left.\begin{aligned}
\frac{d}{dx}\pi_n(x) &= \sum_{r=0}^{n-1} D_{rn} \pi_r(x) \ , && (n \geq 1) \\
\pi_n(x)\pi_m(x) &= \sum_{r=0}^{2m} \beta_{nmr} \pi_{n+m-r}(x) \ , && (n \geq m) \ .
\end{aligned}\right\} \tag{2.1.6}$$

2.1.2 Function Approximation

Let $f(x)$ be a continuous function satisfying the property

$$< f, f >= \int_a^b w(x) f^2(x)\, dx < \infty$$

then $f(x)$ has a spectral representation with respect to the polynomial family $\pi_0(x), \ldots$. Suppose that the representation is

$$f(x) = \sum_{r=0}^{\infty} f_r \pi_r(x) \tag{2.1.7}$$

then it is clear that the coefficients f_k can be evaluated by first multiplying the spectral series (2.1.7) with $w(x)\pi_k(x)$ and then integrating the result over the interval $[a, b]$. Thus

$$\begin{aligned}
\int_a^b w(x)f(x)\pi_k(x)\,dx &= \int_a^b \left(\sum_{r=0}^{\infty} f_r w(x)\pi_r(x)\pi_k(x)\,dx\right) \\
&= \sum_{r=0}^{\infty} f_r \int_a^b w(x)\pi_r(x)\pi_k(x)\,dx \\
&= f_k\mu_k
\end{aligned}$$

leading to the final conclusion that the coefficients f_k are eventually determined by the evaluation of the integral

$$f_k = \frac{1}{\mu_k}\int_a^b w(x)f(x)\pi_k(x)\,dx\ . \tag{2.1.8}$$

These coefficients must now be calculated for a general family of polynomials $\pi_n(x)$ and any function f whose value can be determined at any point x. The answer to this question lies in an understanding of the analysis and methods of Gaussian quadrature.

2.1.3 Gaussian Quadrature

Suppose $\pi_0(x), \ldots, \pi_n(x) \ldots$ is a family of orthogonal polynomials in the sense of (2.1.2) where $\pi_k(x)$ has degree k, then $\pi_k(x)$ has k distinct zeros in $[a, b]$. Since $\pi_0(x) = 1$ then the claim is true for $k = 0$. Now let $k \geq 1$ and observe that

$$\int_a^b w(x)\pi_0(x)\pi_k(x)\,dx = \int_a^b w(x)\pi_k(x)\,dx = 0$$

so that $\pi_k(x)$ certainly changes sign at least once in $[a, b]$, that is, $\pi_k(x)$ has at least one zero in $[a, b]$. Suppose that $z_1 \ldots z_m$ $(m < k)$ are the m zeros of $\pi_k(x)$ in $[a, b]$ occuring to an *odd* power. Hence $\pi_k(x)$ changes sign at $z_1 \ldots z_m$ in $[a, b]$. Consider the polynomial

$$P(x) = (z_1 - x)(z_2 - x)\ldots(z_m - x) = \prod_{r=1}^{r=m}(z_r - x)\ .$$

Since $\pi_k(x)$ and $P(x)$ change sign at the same places then $\pi_k(x)P(x)$ has fixed sign in $[a, b]$ and so

$$\int_a^b w(x)\pi_k(x)P(x)\,dx$$

is single signed. However $P(x)$ can be represented by the expansion

$$P(x) = \sum_{r=0}^{r=m} \alpha_r\pi_r(x) \qquad (m < k)$$

so that

$$\int_a^b w(x)\pi_k(x)P(x)\,dx = \sum_{r=0}^{r=m} \alpha_r \int_a^b w(x)\pi_k(x)\pi_r(x)\,dx = \sum_{r=0}^{r=m} \alpha_r \times 0 = 0.$$

This contradicts the single signed nature of the previous integral. Hence $\pi_k(x)$ must have k odd zeros in $[a,b]$, that is, $\pi_k(x)$ has k distinct zeros in $[a,b]$.

Let $x_0 \ldots x_n$ be the $(n+1)$ zeros of $\pi_{n+1}(x)$ in (a,b) then these zeros, supplemented by the points $x_{-1} = a$ and $x_{n+1} = b$, form a dissection of $[a,b]$ with respect to which the Gaussian quadrature

$$\int_a^b w(x)f(x)\,dx = \sum_{r=0}^{n} a_r f(x_r) \tag{2.1.9}$$

has maximum precision $(2n+1)$. Here the a_r's are determined by the weighted integration of the rth Lagrange interpolating polynomial over $[a,b]$.

2.1.4 The Zeros of $\pi_n(x)$

The proof that $\pi_n(x)$ has n distinct zeros in $[a,b]$ unfortunately offers no clues as to how these zeros might be determined. The answer to this question involves an investigation into the eigenvalues of the symmetric tridiagonal matrix T_{n+1} given by

$$T_{n+1} = \begin{bmatrix} \delta_1 & \gamma_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \gamma_3 & \delta_3 & \gamma_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \gamma_n & \delta_n & \gamma_{n+1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{n+1} & \delta_{n+1} \end{bmatrix}.$$

Define $\chi_{n+1}(\lambda) = \det(T_{n+1} - \lambda I)$. Then $\chi_{n+1}(\lambda)$ is the characteristic polynomial of T_{n+1} and therefore has $(n+1)$ real zeros which can be determined to a high degree of accuracy using the QR algorithm; an iterative scheme for upper triangularising general upper Hessenberg matrices of which a tridiagonal matrix is a simple example. Indeed $\chi_{n+1}(x)$

is just the value of the determinant

$$\begin{vmatrix} \delta_1 - x & \gamma_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \gamma_2 & \delta_2 - x & \gamma_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \gamma_3 & \delta_3 - x & \gamma_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \gamma_n & \delta_n - x & \gamma_{n+1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{n+1} & \delta_{n+1} - x \end{vmatrix} .$$

When this determinant is evaluated about the last row, it is obvious that

$$\chi_{n+1}(x) = (\delta_{n+1} - x)\chi_n(x) - \gamma_{n+1}^2 \chi_{n-1}(x)$$

and this is algebraically equivalent to the conclusion $\pi_{n+1}(x) = (-1)^{n+1}\chi_{n+1}(x)$. Hence $\pi_{n+1}(x)$ and $\chi_{n+1}(x)$ have the same zeros, namely, the eigenvalues of T_{n+1}. Moreover suppose that $v_j \mathbf{e}_j$ is the unit eigenvector corresponding to eigenvalue x_j then the Gaussian quadrature weight a_j is given by

$$a_j = (v_1)^2 \mu_0 = v_1^2 \int_a^b w(x)\, dx \,. \qquad (2.1.10)$$

Details of this proof are available in Bulirsch and Stoer [43].

2.1.5 Application to Chebyshev Polynomials

Chebyshev polynomials are commonly defined over $[-1, 1]$ by the relation

$$T_n(z) = \cos(n\theta), \qquad z = \cos\theta \qquad (2.1.11)$$

from which it is immediately clear that $T_n(z)$ has n zeros in $(-1, 1)$ located at

$$z_k = \cos\left(\frac{(1+2k)\pi}{2n}\right), \qquad k = 0, \ldots, n-1 \,.$$

This is the first useful property of Chebyshev polynomials. Moreover it can be shown that the appropriate form of (2.1.9) is

$$\int_{-1}^{1} \frac{f(x)\, dx}{\sqrt{1-x^2}} = \frac{\pi}{n} \sum_{r=0}^{n-1} f(x_r) \,. \qquad (2.1.12)$$

The spectral coefficients of f are now determined from (2.1.8) by

$$f_k = \frac{1}{\mu_k} \int_{-1}^{1} \frac{f(x)\pi_k(x)\, dx}{\sqrt{1-x^2}} = \frac{\pi}{n\mu_k} \sum_{r=0}^{n-1} f(x_r) T_k(x_r) \,.$$

After some algebra, it can be verified that

$$f_1 = \frac{1}{n}\sum_{r=0}^{n-1} f(x_r)\ , \quad f_k = \frac{\pi}{n}\sum_{r=0}^{n-1} f(x_r)\cos\left(\frac{k(1+2r)\pi}{2n}\right)\ . \tag{2.1.13}$$

The sequence of spectral coefficients determined by (2.1.13) ensures that all polynomials up to order $(n-1)$ are exactly represented, that is, it is an expansion based on interpolation.

However, there is a more elegant association of a function f with a Chebyshev expansion but it is not interpolating. The method is based on Gauss-Chebyshev-Lobatto quadratures nodes with

$$x_k = a\cos^2(k\pi/2n) + b\sin^2(k\pi/2n), \qquad k = 0\ldots n \tag{2.1.14}$$

the optimal weighted quadrature when the endpoints of the interval are nodes of the dissection. Let $f(x)$, $x \in [a, b]$, be a continuous function then

$$x = a + \frac{1}{2}(z+1)(b-a), \qquad x \in [a,b], \quad z \in [-1,1] \tag{2.1.15}$$

maps $x \in [a, b]$ into $z \in [-1, 1]$. Assume that $f(x) = F(z)$ is approximated by an expansion in Chebyshev polynomials up to order N, that is

$$f(x) = F(z) = \sum_{r=0}^{N} f_r T_r(z), \tag{2.1.16}$$

then it can be shown that $F_k = f(x_k)$ $0 \leq k \leq N$, the value of f at the Chebyshev nodes (2.1.14) and $f_0, \ldots, f_N$, the coefficients of the Chebyshev spectral series (2.1.16) are connected by (the detail in Appendix A)

$$\begin{aligned}
f_k &= \frac{\pi}{N c_k}\sum_{j=0}^{N}\frac{1}{c_j}F_j\cos(kj\pi/N)\ , \quad && k = 0, \ldots, N\ ,\\
F_j &= \sum_{k=0}^{N} f_k\cos(kj\pi/N)\ , && j = 0, \ldots, N\ ,
\end{aligned}$$

where

$$c_k = \begin{cases} 2 & k = 0 \text{ or } N, \\ 1 & 0 < r < N. \end{cases}$$

2.2 Common Families of Polynomials

There are four families of orthogonal polynomials that are particularly familiar in applied mathematics. These are the Chebyshev polynomials $T_n(x)$, the Legendre polynomials

$P_n(x)$, the Hermite polynomials $H_n(x)$ and the Laguerre polynomials $L_n(x)$, (see Jean [22]). Hildebrand [17] provides a comprehensive description of many of their properties. Hermite and Laguerre polynomials fit the defining property (2.1.3) and are suitable for infinite and semi-infinite regions respectively. It is possible to use them in eigenvalue expansions but they are difficult to handle and it is almost invariably preferable to map infinite or semi-infinite regions into $[-1, 1]$ and use Chebyshev or Legendre polynomials. The relevant properties of these polynomials are now reviewed.

2.2.1 Chebyshev Polynomials

The Chebyshev Polynomial $T_n(x)$	
Range $[a, b]$	$[-1, 1]$
Weight function	$w(x) = (1 - x^2)^{-1/2}$
μ_{n+1}	$\begin{cases} \pi & \text{for } n = 0 \\ \frac{\pi}{2} & \text{for } n \geq 1 \end{cases}$
δ_{n+1}	0
γ_{n+1}^2	$\begin{cases} 0 & \text{for } n = 0 \\ 1 & \text{for} n \geq 1 \end{cases}$
Recursion relation	$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

The differentiation of Chebyshev spectral series relies on the trigonometric identities

$$\frac{\sin(2k\theta)}{\sin\theta} = 2\sum_{r=1}^{k} \cos(2r-1)\theta, \qquad \frac{\sin(2k+1)\theta}{\sin\theta} = 1 + 2\sum_{r=1}^{k} \cos(2r\theta).$$

In combination with the definition (2.1.11) of $T_n(z)$, these identities lead immediately to the results

$$\begin{aligned} \frac{dT_{2k}(z)}{dz} &= 4k\sum_{r=1}^{k} T_{2r-1}(z) , \\ \frac{dT_{2k+1}(z)}{dz} &= 2(2k+1)\sum_{r=1}^{k} T_{2r}(z) + (2k+1)T_0(z) . \end{aligned} \qquad (2.2.17)$$

Suppose that $f(x)$ is a differentiable function in the interval $[a, b]$ then

$$\frac{df}{dx} = c\frac{dF}{dz} = c\sum_{k=1}^{N} f_k \frac{dT_k(z)}{dz} = c(\sum_{k=1}^{N/2} f_{2k}\frac{dT_{2k}(z)}{dz} + \sum_{k=1}^{(N-1)/2} f_{2k+1}\frac{dT_{2k+1}(z)}{dz}) \qquad (2.2.18)$$

where $c = 2/(b-a)$ and the integer part only of $N/2$ is considered. Results (2.2.17) are now substituted into expression (2.2.18) to obtain

$$\frac{df}{dx} = c(\sum_{k=1}^{N/2} 4kf_{2k} \sum_{r=1}^{k} T_{2r-1}(z) + \sum_{k=1}^{(N-1)/2} (2k+1)f_{2k+1}(T_0(z) + 2\sum_{k=1}^{k} T_{2r}(z)). \qquad (2.2.19)$$

The order of summation on each double sum in the expression (2.2.19) is reversed to get

$$\begin{aligned} \frac{df}{dx} &= c\sum_{r=1}^{N/2}\sum_{k=r}^{N/2} 4kf_{2k}T_{2r-1}(z) + c\sum_{k=0}^{(N-1)/2} (2k+1)f_{2k}T_0(z) \\ &\quad + c\sum_{r=1}^{(N-1)/2}\sum_{k=r}^{(N-1)/2} (4k+2)f_{2k+1}T_{2r}(z). \end{aligned} \qquad (2.2.20)$$

Let D be the $(N+1) \times (N+1)$ upper triangular matrix whose non-zero entries are

$$\left.\begin{aligned} D_{0,2k+1} &= (2k+1)\ , \\ D_{r,r+2k+1} &= 2(r+2k+1)\ , \end{aligned}\right\} \quad k \geq 0\ . \qquad (2.2.21)$$

then it follows directily from (2.2.20) that

$$\frac{df}{dx} = \sum_{r=0}^{N}(\sum_{k=0}^{N} cD_{rk}f_k)T_r(z). \qquad (2.2.22)$$

Thus spectral differentiation of the function f is effected by multiplying the $(N+1)$ dimensional column vector $(f_0, f_1, \ldots, f_N)^T$ of Chebyshev coefficients of f by the differentiation matrix cD. Appendix 1 provides a FORTRAN77 subroutines.

2.2.2 Legendre Polynomials

The Legendre Polynomial $P_n(x)$	
Range $[a,b]$	$[-1,1]$
Weight function	$w(x) = 1$
μ_{n+1}	$\dfrac{2}{2n+1}$
δ_{n+1}	0
γ_{n+1}^2	$\begin{cases} 0 & \text{for } n=0 \\ \dfrac{n}{n+1} & \text{for } n \geq 1 \end{cases}$
Recursion relation	$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

The derivative of $P_n(x)$ is related to $P_0(x)\dots P_{n-1}(x)$ by

$$\begin{aligned} \frac{dP_{2k+2}}{dx} &= \sum_{r=0}^{k}(4r+3)P_{2r+1}(x)\ , \\ \frac{dP_{2k+1}}{dx} &= \sum_{r=0}^{k}(4r+1)P_{2r}(x) \end{aligned} \tag{2.2.23}$$

and the product $P_n(x)P_m(x)$ has Legendre polynomial form

$$P_n(x)P_m(x) = \sum_{r=0}^{m} \frac{A_{m-r}A_rA_{n-r}}{A_{n+m-r}} \frac{2n+2m-4r+1}{2n+2m-2r+1} P_{n+m-2r}(x)\ , \quad n \geq m, \tag{2.2.24}$$

where $A_k = \dfrac{1}{2^k}\left(\begin{array}{c} 2k \\ k \end{array}\right)$. The expressions (2.2.23) must be recast in terms of the matrix D introduced in (2.1.6). It is obvious that the non-zero entries of D for Legendre polynomials are

$$D_{i,2j+i+1} = 2i+1\ , \quad i,j \geq 0\ . \tag{2.2.25}$$

In fact, Legendre polynomials have similar qualities as Chebyshev polynomials except that the formula for their product is clearly more complex than the comparable Chebyshev formula. The similarity is more obvious when the recurrence relation is expressed in the form

$$P_{n+1}(x) = \left(2 - \frac{1}{n+1}\right) xP_n(x) - \left(1 - \frac{1}{n+1}\right) P_{n-1}(x)\ .$$

For any sizeable n, this is effectively the same recurrence relation as applies to Chebyshev polynomials. It is only for small values of n that Chebyshev and Legendre polynomials differ markedly. In particular, there is usually nothing to be gained by using Legendre polynomials over a finite interval if Chebyshev polynomials are equally appropriate.

2.3 Stability Analysis of One Layer

Let $\mathcal{L}$ be a layer containing a continuum which interact thermally, mechanically and magnetically with the world outside via its boundaries. Suppose that the equations describing the physical problem are non-dimensionalised so that $z = 1$ is its upper boundary of the layer and $z = -1$ is its lower boundary. The standard linear stability problem for this configuration can be systematically reduced to the eigenvalue problem

$$\frac{dY}{dz} = AY + \sigma BY\ , \tag{2.3.26}$$

where Y is an n vector with components $y_1, \ldots, y_n$, A and B are complex $n \times n$ matrices and σ is the eigenvalue to be determined. The equation (2.3.26) is to be supplemented by n boundary conditions. These describe the interaction of $\mathcal{L}$ with its environment and specify the behaviour of thermal, mechanical and magnetic effects etc, on the boundary. They are linear in nature, involve only the components of Y and can be expressed in matrix form

$$\begin{aligned} U_k^T Y &= 0\ , \qquad 1 \le k \le m < n \quad (z = 1) \\ L_k^T Y &= 0\ , \qquad 1 \le k \le n - m \quad (z = -1) \end{aligned} \tag{2.3.27}$$

where U_k and L_k are families of n-vectors with constant entries.

Some Remarks

(a) Boundary conditions may contain the eigenvalue σ. Indeed this happens in the Calculus of Variations when transversality conditions are in operation.

(b) From a purely mathematical point of view, the boundary conditions can be distributed arbitrarily between the upper and lower boundaries of the layer. However, in practice, boundary conditions relate to the physical properties of macroscopic quantities such as stress, velocity, temperature etc. and these conditions appear in pairs — one for the upper boundary and one for the lower. Hence n, the order of the systems describing layer $\mathcal{L}$, is almost invariably even and $m = n/2$.

2.3.1 The Eigenvalue Problem

Let complex $n \times n$ matrices A and B and $n \times 1$ vector Y be defined by

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, \tag{2.3.28}$$

where the matrices A and B and the vector Y are represented by the structure of the eigenvalue problem (2.3.26) as

$$\frac{dY}{dz} = AY + \sigma BY\ , \qquad z \in [-1, 1]\ . \tag{2.3.29}$$

Similarly, the boundary conditions (2.3.27), can be recast in the simpler form

$$C_k^T Y = 0 \quad 1 \le k \le n \tag{2.3.30}$$

where the interpretation of C_k is

$$C_k^T = \begin{cases} U_k^T & 1 \le k \le m\ , \\ L_k^T & m < k \le n\ . \end{cases}$$

Further progress is achieved by approximating each component of Y with an expansion in terms of some family of orthogonal polynomials. Since the interval is $[-1, 1]$ then the previous theory suggests that Legendre or Chebyshev polynomials will be most suitable. As has been previously mentioned, the mathematical quality of the representation is effectively equivalent for both sets of polynomials. However, the especially simple form for the product $T_n(x)T_m(x)$ favours Chebyshev polynomials, particularly if the entries of A and B are non-constant. Henceforth Chebyshev polynomials will be employed.

2.3.2 Representation of the Eigenvalue Problem

Suppose that each component of Y is approximated by a series of $(M + 1)$ Chebyshev polynomials so that

$$y_r(z) = \sum_{k=0}^{M} \alpha_{kr} T_k(z)\ , \qquad \text{for } 1 \le r \le N\ . \tag{2.3.31}$$

Of course, (2.3.31) is not an exact solution of (2.3.29) and actually satisfies the modified differential equation

$$\frac{dY}{dz} = A(z)Y + \sigma B(z)Y + R_M(z)$$

where $R_M(z)$ is the *residual*; in this case, an n dimensional vector representing the remainder term. Indeed, (2.3.31) is a solution of (2.3.29) in the sense that the residual is orthogonal to $T_0, \ldots, T_{M-1}$, that is,

$$\int_{-1}^{1} \frac{R_M(z)T_k(z)\, dz}{\sqrt{1-z^2}} = 0, \quad 0 \le k < M.$$

Notionally $R_M \to 0$ as $M \to \infty$. This criterion effectively means that the coefficients of the first M Chebyshev polynomials in the representation of R_M must be zero. It now remains to construct R_M. This process involves three steps.

step 1 Compute the Chebyshev coefficients of the derivative term dY/dz from the Chebyshev coefficients of Y.

step 2 Compute the Chebyshev coefficients of the terms AY and BY from the coefficients of Y.

step 3 Convert the boundary conditions into relationships between the Chebyshev coefficients of Y.

Each of these procedures is now discussed in detail.

2.3.3 Treatment of Derivative

Consider first dy_r/dz. From (2.3.31)

$$\begin{aligned}\frac{dy_r}{dz} &= \sum_{k=0}^{M}\alpha_{kr}\frac{dT_k(z)}{dz}\\ &= \sum_{k=0}^{M}\alpha_{kr}\sum_{j=0}^{M}D_{jk}T_j(z)\\ &= \sum_{j=0}^{M}\left(\sum_{k=0}^{M}D_{jk}\alpha_{kr}\right)T_j(z)\,.\end{aligned} \tag{2.3.32}$$

Thus the coefficients of the Chebyshev expansion of dy_r/dz are computed from the coefficients of the Chebyshev expansion of y_r by premultiplying the vector

$$\boldsymbol{\alpha}_r = [\alpha_{0r},\ldots,\alpha_{Mr}]^T \tag{2.3.33}$$

by the $(M+1)\times(M+1)$ matrix D. Now redefine Y to be the vector of length $N(M+1)$ formed from $[y_1,\ldots,y_r,\ldots,y_N]$ by expanding y_r to the vector $\boldsymbol{\alpha}_r$, that is, Y has block matrix form

$$Y^T = [\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_N]\,. \tag{2.3.34}$$

The Chebyshev coefficients of dY/dz are now obtained from Y by matrix multiplication. In block matrix notation

$$\frac{dY}{dz} = \begin{bmatrix} D & 0 & 0 & 0 & \cdots & 0\\ 0 & D & 0 & 0 & \cdots & 0\\ 0 & 0 & D & 0 & \cdots & 0\\ 0 & 0 & 0 & D & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & D\end{bmatrix}\begin{bmatrix}\boldsymbol{\alpha}_1\\ \boldsymbol{\alpha}_2\\ \boldsymbol{\alpha}_3\\ \boldsymbol{\alpha}_4\\ \vdots\\ \boldsymbol{\alpha}_N\end{bmatrix}. \tag{2.3.35}$$

2.3.4 Treatment of Matrix Product

The contribution made to the derivative dy_r/dz by the matrix product AY has form $\sum_{i=1}^{N} A_{ri}(z)y_i(z)$, that is, it is a sum of terms of the form $f(z)g(z)$ where $f(z) = A_{ri}(z)$

and $g(z) = y_i(z)$. Suppose that

$$f(z) = \sum_{i=0}^{\infty} f_i T_i(z) , \qquad g(z) = \sum_{j=0}^{M} g_j T_j(z)$$

then the product $h(z) = f(z)g(z)$ is

$$\begin{aligned} h(z) &= \sum_{i=0}^{\infty}\sum_{j=0}^{M} f_i g_j T_i(z) T_j(z) \\ &= \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{M} f_i g_j (T_{i+j}(z) + T_{|i-j|}(z)) \\ &= \frac{1}{2}\sum_{k=0}^{M}\sum_{j=0}^{k} f_{k-j} g_j T_k(z) + \frac{1}{2}\sum_{k=M}^{\infty}\sum_{j=0}^{M} f_{k-j} g_j T_k(z) \\ &\quad + \frac{1}{2}\sum_{k=0}^{M}\sum_{j=k}^{M} f_{j-k} g_j T_k(z) + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{j=0}^{M} f_{k+j} g_j T_k(z). \end{aligned} \tag{2.3.36}$$

From the final expression for $h(z)$ in equations (2.3.36), the first $(M+1)$ Chebyshev coefficients of h can be determined. Unless f is constant, Chebyshev coefficients of order higher than $(M+1)$ also appear in the expression. However, these are of no consequence since they contribute only to the part of R_M which vanishes under the inner product with $T_0(z), \ldots T_{M-1}(z)$. It follows almost immediately from (2.3.36) that

$$h_k = \begin{cases} f_0 g_0 + \dfrac{1}{2}\displaystyle\sum_{j=1}^{M} f_j g_j , & k = 0 \\ \dfrac{1}{2}\displaystyle\sum_{j=0}^{k} f_{k-j} g_j + \dfrac{1}{2}\displaystyle\sum_{j=k}^{M} f_{j-k} g_j + \dfrac{1}{2}\displaystyle\sum_{j=0}^{M} f_{k+j} g_j , & k \geq 1 \end{cases} \tag{2.3.37}$$

As in the derivation of the Chebyshev coefficients of the derivative terms, it is self evident that result (2.3.37) can be re-expressed in the form of the matrix multiplication

$$\frac{1}{2}\begin{bmatrix} 2f_0 & f_1 & f_2 & f_3 & \cdots & f_M \\ 2f_1 & 2f_0 + f_2 & f_1 + f_3 & f_2 + f_4 & \cdots & f_{M-1} + f_{M+1} \\ 2f_2 & f_1 + f_3 & 2f_0 + f_4 & f_1 + f_5 & \cdots & f_{M-2} + f_{M+2} \\ 2f_3 & f_2 + f_4 & f_1 + f_5 & 2f_0 + f_6 & \cdots & f_{M-3} + f_{M+3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2f_M & f_{M-1} + f_{M+1} & f_{M-2} + f_{M+2} & f_{M-3} + f_{M+3} & \cdots & 2f_0 + f_{2M} \end{bmatrix}\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_M \end{bmatrix} .$$

Thus each term $A_{ri}(z)y_i(z)$ in the computation of AY can be expanded into a matrix product of F_{ri} and $\boldsymbol{\alpha}_i$ where F_{ri} is the $(M+1) \times (M+1)$ matrix associated with $A_{ri}(z)$ as illustrated in the previous expression and $\boldsymbol{\alpha}_i$ has its usual meaning from (2.3.33). In particular, if A_{ri} is constant then, trivially, $F_{ri} = A_{ri}I_M$ (since $f_k = 0,\ k > 0$) where I_M is the $(M+1) \times (M+1)$ identity matrix. Hence AY has matrix form

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & \cdots & F_{1N} \\ F_{21} & F_{22} & F_{23} & F_{24} & \cdots & F_{2N} \\ F_{31} & F_{32} & F_{33} & F_{34} & \cdots & F_{3N} \\ F_{41} & F_{42} & F_{43} & F_{44} & \cdots & F_{4N} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{N1} & F_{N2} & F_{N3} & F_{N4} & \cdots & F_{NN} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \boldsymbol{\alpha}_3 \\ \boldsymbol{\alpha}_4 \\ \vdots \\ \boldsymbol{\alpha}_N \end{bmatrix} . \tag{2.3.38}$$

Clearly an identical analysis applies to the term BY. Of course, in practical applications of these ideas, many of the F matrices are either zero or multiples of the identity and the detail of building them is in Appendix B .

2.3.5 Boundary Conditions

Recall that Chebyshev polynomials can be defined by the property $T_n(\cos\theta) = \cos(n\theta)$ so that

$$T_n(1) = 1 \ , \qquad\qquad T_n(-1) = (-1)^n \ .$$

Thus the eigenfunction expansion (2.3.31) for y_k leads to the obvious conclusions

$$\begin{aligned} y_k(1) &= \sum_{i=0}^{M} \alpha_{ik} T_i(1) = \sum_{i=0}^{M} \alpha_{ik} \\ y_k(-1) &= \sum_{i=0}^{M} \alpha_{ik} T_i(-1) = \sum_{i=0}^{M} \alpha_{ik}(-1)^i \end{aligned} \tag{2.3.39}$$

In terms of the constant vectors $\mathbf{p}$ and $\mathbf{q}$ of length L given by

$$\mathbf{p} = (1, 1, \ldots, 1, \ldots, 1) \ , \qquad \mathbf{q} = (1, -1, 1, -1, \ldots (-1)^r \ldots (-1)^M) \ ,$$

the boundary conditions (2.3.39) become

$$y_k(1) = \boldsymbol{p}.\boldsymbol{\alpha}_k \ , \qquad y_k(-1) = \boldsymbol{q}.\boldsymbol{\alpha}_k \ . \tag{2.3.40}$$

Each boundary condition (2.3.30) is converted into a linear relationship among the entries α_{ij} of Y by expanding each component part y_k $(1 \leq k \leq n)$ into a multiple of the vector $\mathbf{p}$ if the boundary condition is applied at $z = 1$ or a multiple of $\mathbf{q}$ if it is applied at $z = -1$.

2.4 The Linear Eigenvalue Problem

The final outcome of this analysis is that the derivative dY/dz and the matrix products AY and BY in (2.3.26) can all be processed so that equation (2.3.26) reduces to a generalized eigenvalue problem of the type $EY = \sigma FY$ where E and F are complex square $N(M+1) \times N(M+1)$ matrices. The N boundary condition are now used to replace the Nth, $2N$th, $3N$th ... , $N(M+1)$th rows of $EY = \sigma FY$ and the final eigenvalue problem is produced. At this stage, a numerical eigenvalue routine is called and the complex eigenvalues computed along with the corresponding eigenvectors, if required. In all subsequent work, NAG routines FO2BJF and F02GJF are called for real E and F and complex E and F respectively. This technique is exemplified for the convection problem arising when a viscous fluid overburdens a porous layer.

Chapter 3

Introductory Applications Using Spectral Methods

3.1 Introduction

This section introduces the Chebyshev Tau method via the eigenvalue problems associated with the shear flow of a viscous fluid (Orr-Sommerfeld problem) and the convection of a layer of electrically conducting fluid in the presence of an axial magnetic field (Magnetic Benard problem). In both of these problems, the nature of the spectrum depends critically on the choice of parameters. In the former, all eigenvalues are essentially complex but different eigenvalues are critical for different values of the Reynolds number. In the latter, the spectrum contains real and complex conjugate pairs of eigenvalues so that in certain parameter regions, real eigenvalues are critical whereas in others, it is the complex eigenvalues that dominate. This chapter applies Chebyshev spectral methods to these problems, initially by way of illustration but more significantly because tracking techniques such as Inverse Iteration and Compound Matrices are unable to handle the subtle and rapid changes undergone by the spectrum in response to "small" changes in problem parameters.

Finally, an opportunity is taken to compare the relative accuracy of two popular implementations of the Chebyshev-Tau method. One treats the eigenvalue problem as a system of first order differential equations whereas the other expresses the eigenvalue problem in terms of systems of second order differential equations. The former strategy needs larger spectral matrices than the latter but this apparent disadvantage is counterbalanced by technically simpler boundary conditions and the need for less polynomials.

3.2 Orr-Sommerfeld Problem

The Orr-Sommerfeld equation [24] arises in the stability analysis of the laminar flow of a viscous fluid down a cylindrical pipe under a pressure gradient (Poiseuille flow) or between two parallel rigid boundaries, one of which is induced to move at constant speed (Couette flow). The equation has non-dimensional form

$$(D^2 - a^2)^2 w = iaR[(\bar{u} - \sigma)(D^2 - a^2) - D^2\bar{u}]w \,, \qquad x \in (-1, 1) \,, \tag{3.2.1}$$

where $D\phi = d\phi/dx$, R is the Reynolds No., a is a wavenumber, σ is the eigenvalue and $\bar{u}(x)$ is the laminar solution. In Poiseuille flow, $\bar{u}(x) = 1 - x^2$ and in Couette flow, $\bar{u} = x$. In both cases, equation (3.2.1) must be supplemented by the boundary conditions

$$w(1) = w(-1) = 0 \,, \qquad Dw(1) = Dw(-1) = 0 \,. \tag{3.2.2}$$

The critical Reynold number R_{crit} is determined by the criterion $\text{Re}(\sigma) = 0$, that is, the real part of σ is zero. For Poiseuille flow, it can be shown that the critical Reynolds No. is $R_{\text{crit}} \approx 5751.9$ occurring at wavenumber $a_{\text{crit}} = 1.0215$ while in the case of Couette flow, the critical Reynolds number is $R_{\text{crit}} = 45310.9$ occurring at wavenumber $a_{\text{crit}} = 1.0207$. However, the traditional eigenvalue problem in this context occurs when the wavenumber a is fixed at unity and R is allowed to vary.

3.2.1 System Formulation

In terms of the variables

$$z_1 = w \,, \qquad z_2 = D^2 w \,, \tag{3.2.3}$$

the Orr-Sommerfeld equation (3.2.1) may be represented by the two differential equations

$$\begin{aligned} z_2 &= D^2 z_1 \,, \\ (D^2 - 2a^2)z_2 + a^4 z_1 &= iaR(\bar{u} - \sigma)(z_2 - a^2 z_1) - iaRD^2\bar{u}\, z_1 \,, \end{aligned} \tag{3.2.4}$$

with boundary conditions $z_1 = Dz_1 = 0$ on $x = \pm 1$. Equations (3.2.4) may be re-expressed in the matrix format

$$\frac{d^2 Z}{dx^2} = UZ + \sigma VZ \tag{3.2.5}$$

where U and V are the 2×2 matrices

$$U = \begin{bmatrix} 0 & 1 \\ -iaR(D^2\bar{u} + a^2\bar{u}) - a^4 & iaR\bar{u} + 2a^2 \end{bmatrix} , \qquad V = \begin{bmatrix} 0 & 0 \\ ia^3R & -iaR \end{bmatrix} . \tag{3.2.6}$$

and $Z = (z_1, z_2)^T$. Equations (3.2.4) may be reduced further to a system of four first order differential equations by introducing variables y_1, y_2, y_3 and y_4 by the definitions

$$y_1 = z_1 , \qquad y_2 = Dz_1 , \qquad y_3 = D^2 z_1 , \qquad y_4 = D^3 z_1 .$$

Thereafter it is verified easily that equations (3.2.4), when expressed in terms of $y_1 \ldots y_4$, become the first order system

$$\begin{gathered} Dy_1 = y_2 , \qquad Dy_2 = y_3 , \qquad Dy_3 = y_4 , \\ Dy_4 = 2a^2 y_3 - a^4 y_1 + iaR(\bar{u} - \sigma)(y_3 - a^2 y_1) - iaRD^2\bar{u}y_1 , \end{gathered} \tag{3.2.7}$$

with the boundary conditions $y_1 = y_2 = 0$ on $x = \pm 1$. As before, equations (3.2.7) can be reformulated in the matrix format

$$\frac{dY}{dx} = AY + \sigma BY \tag{3.2.8}$$

where A and B are the 4×4 matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -iaR(\bar{u} + D^2\bar{u}) - a^4 & 0 & iaR\bar{u} + 2a^2 & 0 \end{bmatrix} , \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ia^3R & 0 & -iaR & 0 \end{bmatrix} .$$

and $Y = (y_1, y_2, y_3, y_4)^T$. Following the methods described in chapter 2, eigenvalue problems (3.2.5) and (3.2.8) can be converted into the generalised form $EV = \sigma FV$ where E and F are block matrices of suitable dimension.

3.2.2 Second Order System

Here z_1 and z_2 are represented by series involving Chebyshev polynomials $T_0(x)$ to $T_{M-1}(x)$. Equations (3.2.4) assume the generalised eigenvalue form $EV = \sigma FV$ in which E and F have 2×2 block matrix form

$$\begin{gathered} E = \begin{bmatrix} D^2 & -I \\ iaR(P + a^2Q) + a^4 I & D^2 - iaRQ - 2a^2 I \end{bmatrix} , \\ F = \begin{bmatrix} 0 & 0 \\ ia^3RI & -iaRI \end{bmatrix} . \end{gathered} \tag{3.2.9}$$

In (3.2.9), the forms of P and Q depend on the problem under consideration and are specified by

Poiseuille Flow	Couette Flow
$P_{ij} = 2\delta_{ij}$	$P_{ij} = 0$
$Q_{i,(i+2)} = -1/4 \quad Q_{i,(i-2)} = -1/4$	$Q_{i,(i+1)} = 1/2 \quad Q_{i,(i-1)} = 1/2$
$Q_{i,i} = 1/2$	$Q_{2,1} = 1$
$Q_{2,2} = 1/4 \quad Q_{3,1} = -1/2$	Rest zero
Rest zero	

(3.2.10)

whenever the appropriate matrix entries exist. The formulation of the eigenvalue problem is now completed by replacing the $(M-1)th$, Mth, $(2M-1)th$ and $2Mth$ rows of E and F with the boundary information. It does not matter how the four boundary conditions are ordered but numerical performance is usually enhanced if the boundary data is inserted so that the largest entries occupy the top right of E and F. Let $\boldsymbol{p} = p_k\boldsymbol{e}_k$, $\boldsymbol{q} = q_k\boldsymbol{e}_k$, $\boldsymbol{r} = r_k\boldsymbol{e}_k$ and $\boldsymbol{s} = s_k\boldsymbol{e}_k$ be M dimensional vectors whose $k\,th$ entries are respectively

$$p_k = 1\ , \quad q_k = (-1)^k\ , \quad r_k = k^2\ , \quad s_k = k^2(-1)^k \qquad k = 0 \ldots M-1\ . \tag{3.2.11}$$

In terms of these vectors, the boundary condition rows and their location are, in block matrix notation,

Condition	Row	E	F
$z_1 = 0$ on $x = -1$	$M-1$	$(\mathbf{q}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0})$
$z_1 = 0$ on $x = 1$	M	$(\mathbf{p}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0})$
$Dz_1 = 0$ on $x = -1$	$2M-1$	$(\mathbf{r}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0})$
$Dz_1 = 0$ on $x = 1$	$2M$	$(\mathbf{s}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0})$

. (3.2.12)

3.2.3 First Order System

Here y_1, y_2, y_3 and y_4 are represented by series involving Chebyshev polynomials $T_0(x)$ to $T_{M-1}(x)$. Equations (3.2.8) are converted into the generalised eigenvalue form $EV = \sigma FV$

in which E and F have 4×4 block matrix form

$$E = \begin{bmatrix} D & -I & 0 & 0 \\ 0 & D & -I & 0 \\ 0 & 0 & D & -I \\ iaR(P + a^2Q) + a^4 & 0 & -iaRQ - 2a^2 I & D \end{bmatrix},$$
$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ia^3 R & 0 & -iaRI & 0 \end{bmatrix}. \tag{3.2.13}$$

The matrices P and Q appearing in (3.2.13) are specified in (3.2.10) whenever the appropriate matrix entries exist. The formulation of the eigenvalue problem is now completed by replacing the Mth, $2Mth$, $3Mth$ and $4Mth$ rows of E and F with the boundary information. In terms of the M dimensional vectors described in (3.2.11), the boundary condition rows and their location are, in block matrix notation,

Condition	Row	E	F
$y_1 = 0$ on $x = -1$	M	$(\mathbf{q}, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$
$y_1 = 0$ on $x = 1$	$2M$	$(\mathbf{p}, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$
$y_2 = 0$ on $x = -1$	$3M$	$(\mathbf{0}, \mathbf{q}, \mathbf{0}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$
$y_2 = 0$ on $x = 1$	$4M$	$(\mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$

. (3.2.14)

3.2.4 Results

Both techniques extracted capably the competitive eigenvalues in the spectrum of the Orr-Sommerfeld equation over a range of problem parameters. For example, for the matrix representation of the Poiseuille flow problem in terms of a first order system[1], 50 polynomials resolved the leading eigenvalues of the spectrum when $R = 10,000$ and $a = 1$, in agreement with Lindsay [28]. The corresponding eigenvectors can be used to determine the related eigenfunctions although, as expected, eigenfunction determination requires more polynomials, relatively speaking.

The comparative performance of first order and second order differential equation rep-

[1] Appendix 2 gives the appropriate Fortran77 program. Eigenvalues were extracted using routine F02GJF, NAG's implementation of the QZ algorithm due to Moler and Stewart [31].

resentations of eigenvalue problems is currently of great interest. Higher order representations are of limited interest due to the growth of numerical errors arising from powers of D, the differentiation matrix, even although Orszag's [38] original work on the Orr-Sommerfeld equation treated it as a single fourth order equation. The leading eigenvalue for Poiseuille and Couette flow was calculated for various orders of polynomial approximation using both the D and D^2 representations. The results are recorded in table 3.1 and displayed in figure 3.1. Since the eigenvalue is naturally a complex number, accuracy was measured as the modulus of the difference between σ_M, the eigenvalue estimate using M polynomials and σ_∞, an estimate based on a very large number of polynomials.

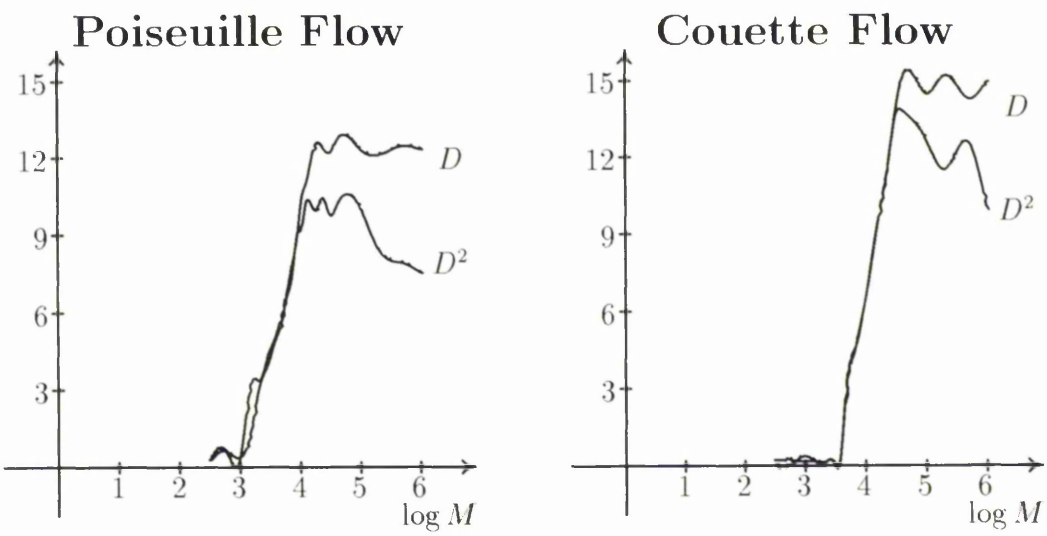


Figure 3.1: Graph of $-\log_{10}|\sigma_M - \sigma_\infty|$ versus $\log M$.

Some remarks are appropriate.

(a) The behaviour of the D^2 curves suggests that, from a practical point of view, there is an optimal number of polynomials to use in an eigenfunction expansion. Thus there is a compromise between the inaccuracy due to a truncated mathematical description of the eigenfunctions and the accumulation of rounding error due to finite precision arithmetic.

(b) For expansions using less than this optimal number of polynomials, both the D and D^2 methods return similar levels of resolution although the D matrices are four times the size of D^2 matrices.

(c) The resolution of the D method is significantly better than D^2 in this instance. This is more a feature of the Orr-Sommerfeld problem. For less punishing applications (e.g. the magnetic Benard problem), the difference in performance is less marked.

	Poiseuille Flow		Couette Flow	
Value of M	Accuracy D method	Accuracy D^2 method	Accuracy D method	Accuracy D^2 method
12	5.489×10^{-1}	5.489×10^{-1}	9.605×10^{-1}	5.957×10^{-1}
16	2.432×10^{-1}	2.432×10^{-1}	8.530×10^{-1}	6.278×10^{-1}
20	4.113×10^{-1}	4.113×10^{-1}	4.243×10^{-1}	6.701×10^{-1}
24	5.858×10^{-04}	3.897×10^{-03}	7.032×10^{-1}	6.947×10^{-1}
28	4.134×10^{-04}	6.229×10^{-04}	7.173×10^{-1}	7.115×10^{-1}
32	4.237×10^{-05}	9.444×10^{-05}	7.301×10^{-1}	7.243×10^{-1}
36	9.761×10^{-06}	1.365×10^{-05}	5.464×10^{-1}	5.120×10^{-1}
40	1.298×10^{-06}	2.420×10^{-06}	4.683×10^{-4}	6.951×10^{-4}
44	1.521×10^{-07}	2.382×10^{-07}	8.253×10^{-5}	4.910×10^{-5}
48	1.361×10^{-08}	3.385×10^{-08}	1.222×10^{-5}	1.592×10^{-5}
52	1.328×10^{-09}	1.056×10^{-09}	1.529×10^{-6}	1.740×10^{-6}
56	3.479×10^{-11}	5.409×10^{-10}	1.525×10^{-7}	1.622×10^{-7}
60	1.239×10^{-11}	7.451×10^{-11}	1.701×10^{-8}	2.275×10^{-8}
70	3.322×10^{-13}	1.059×10^{-10}	9.263×10^{-11}	8.812×10^{-11}
80	4.084×10^{-13}	3.682×10^{-11}	1.894×10^{-12}	1.671×10^{-12}
90	6.204×10^{-13}	1.683×10^{-10}	1.859×10^{-14}	2.533×10^{-14}
100	2.103×10^{-13}	7.327×10^{-11}	9.020×10^{-16}	1.565×10^{-14}
150	5.068×10^{-13}	1.056×10^{-10}	3.360×10^{-15}	2.530×10^{-13}
200	7.813×10^{-13}	3.757×10^{-09}	6.753×10^{-16}	3.208×10^{-12}
300	3.480×10^{-13}	1.198×10^{-08}	5.226×10^{-15}	2.744×10^{-13}
400	4.679×10^{-13}	2.764×10^{-08}	1.226×10^{-15}	8.165×10^{-11}

Table 3.1: Decimal accuracy in leading eigenvalue versus number of polynomials deployed.

3.2.5 Eigenvalue Distribution

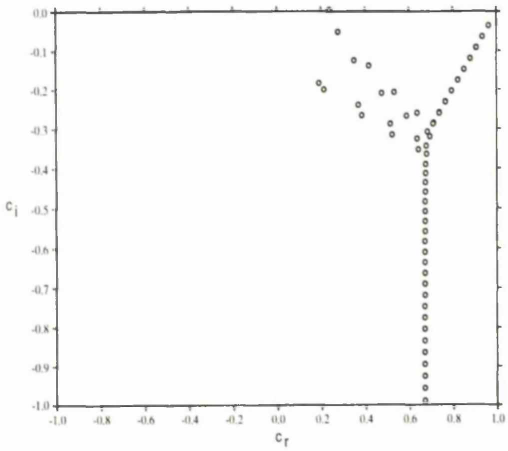
The previous observations concentrated on the resolution of the competitive eigenvalue in the Orr-Sommerfeld equation. It's also interesting to probe the structure of its spectrum using the D and D^2 strategies. The distribution of the first 30 or so eigenvalues of the OS equation were calculated using the D and D^2 methods with 200 polynomials and displayed for various values of R, the Reynolds number. Figure 3.2 deals with Poiseuille flow for some low Reynolds numbers (≤ 10000) whereas figure 3.4 deals with selected Reynolds numbers up to 50000.

The distribution of eigenvalues for plane Couette flow, using both the D and D^2 methods, is displayed in figure 3.3 for selected Reynolds numbers up to 13000.

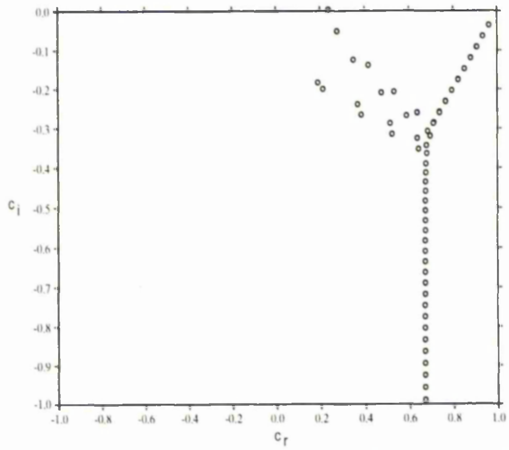
3.3 Modifying Boundary Conditions

The formulation of eigenvalue problems into pairs of second order differential equations followed by a spectral analysis based on the D^2 method is often at ease with the natural specification of boundary conditions in the sense that these are frequently paired. For example, in the Orr-Sommerfeld problem, the boundary conditions $w(1) = w(-1) = 0$ and $Dw(1) = Dw(-1) = 0$ translate into $z_1 = 0$ on $x = \pm 1$ and $Dz_1 = 0$ on $x = \pm 1$ respectively. An obvious disadvantage of this approach is that the second pair of boundary conditions also relate to z_1. Thus, although the reformulation of the original differential equations gives z_1 and z_2 equal status, the boundary conditions inherently prefer z_1 to z_2.

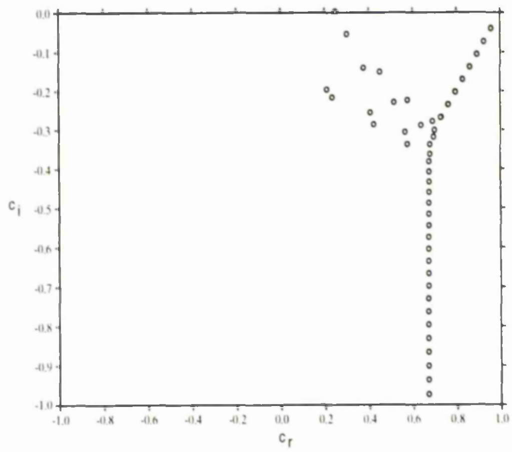
More generally, the governing differential equations of an eigenvalue problem can always be rewritten as a system in which each variable is independent and enjoys its own spectral expansion. However, this impartiality may be undermined by boundary conditions in the sense that particular variables dominate. For example, z_1 is preferred to z_2 in the Orr-Sommerfeld problem. Ideally, the boundary conditions $Dz_1 = 0$ should be transferred onto the variable z_2 without reference to z_1. More generally, in block format, the boundary conditions should assume an upper triangular structure whose main diagonal is non-zero. The following analysis describes a mechanism with the potential to achieve this aim, and exemplifies it for the Orr-Sommerfeld equation.



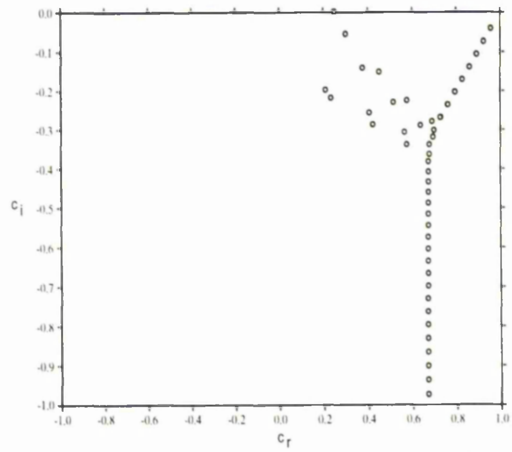
Poiseuille spectrum using D method when $a = 1$ and $R = 10000$



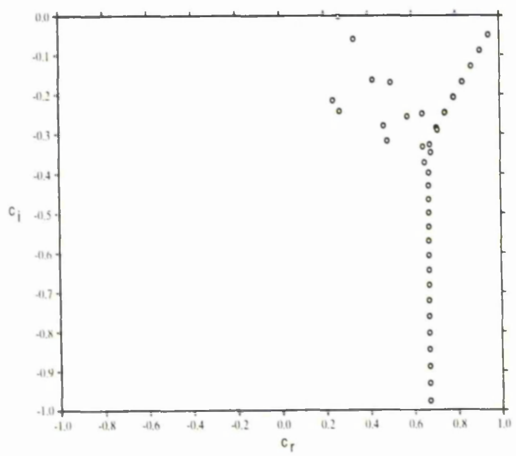
Poiseuille spectrum using D^2 method when $a = 1$ and $R = 10000$



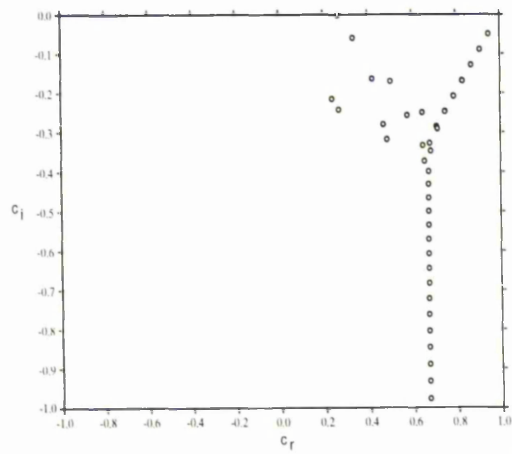
Poiseuille spectrum using D method when $a = 1$ and $R = 7500$



Poiseuille spectrum using D^2 method when $a = 1$ and $R = 7500$

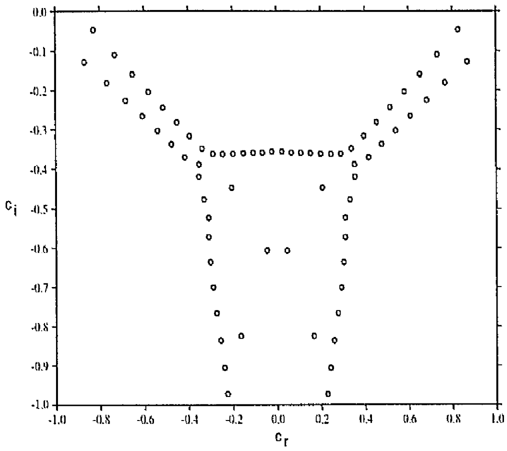


Poiseuille spectrum using D method when $a = 1$ and $R = 5000$

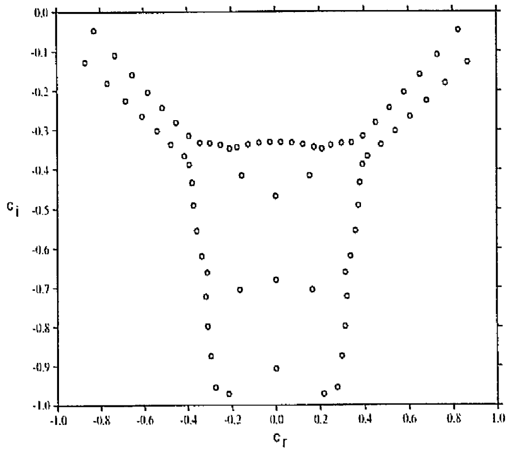


Poiseuille spectrum using D^2 method when $a = 1$ and $R = 5000$

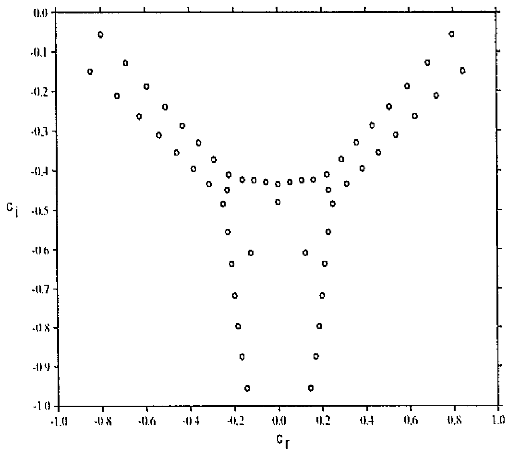
Figure 3.2: Eigenvalue distribution for D and D^2 methods



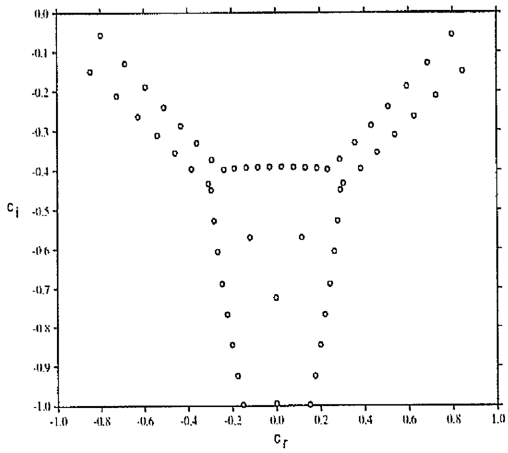
Couette spectrum using D method when $a = 1$ and $R = 13000$



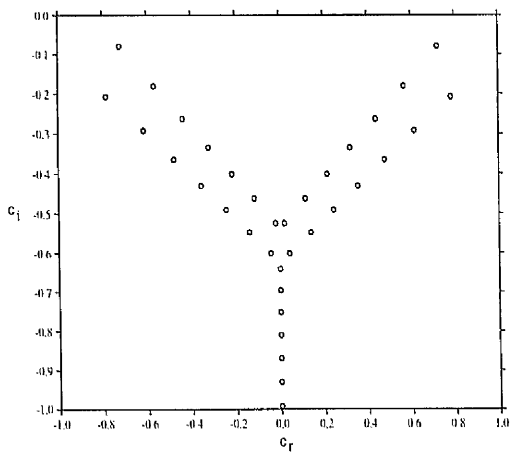
Couette spectrum using D^2 method when $a = 1$ and $R = 13000$



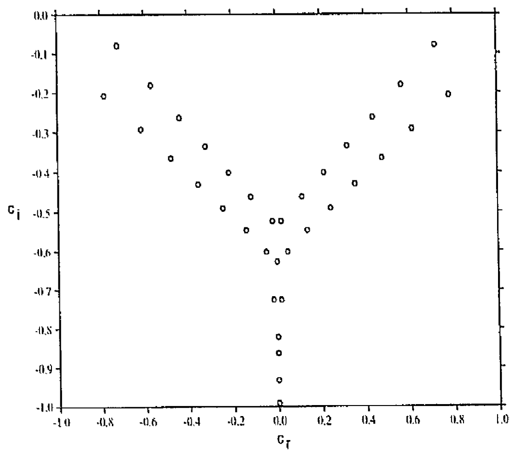
Couette spectrum using D method when $a = 1$ and $R = 8000$



Couette spectrum using D^2 method when $a = 1$ and $R = 8000$

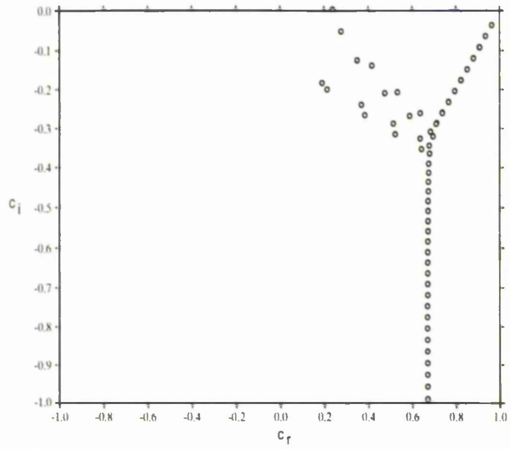


Couette spectrum using D method when $a = 1$ and $R = 3000$

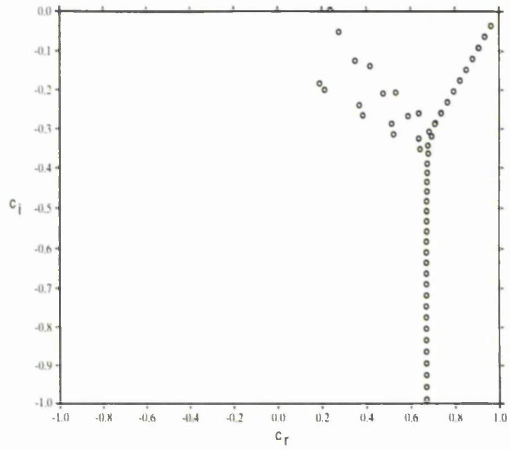


Couette spectrum using D^2 method when $a = 1$ and $R = 3000$

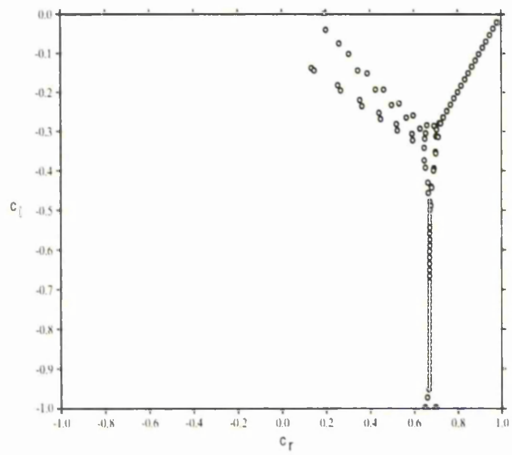
Figure 3.3: Eigenvalue distribution for D and D^2 methods



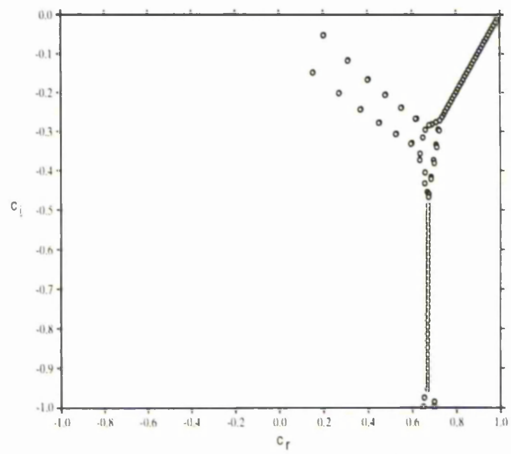
Poiseuille spectrum using D method when $a = 1$ and $R = 10000$



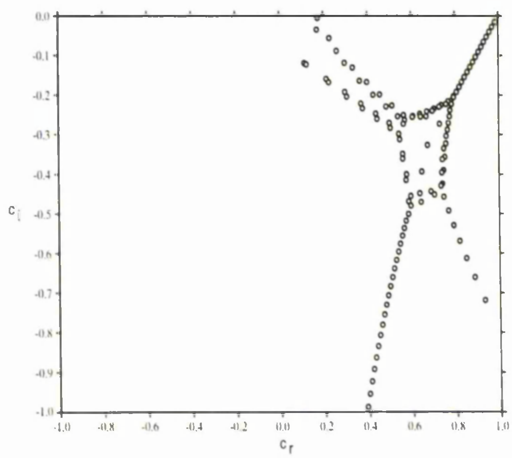
Poiseuille spectrum using D^2 method when $a = 1$ and $R = 10000$



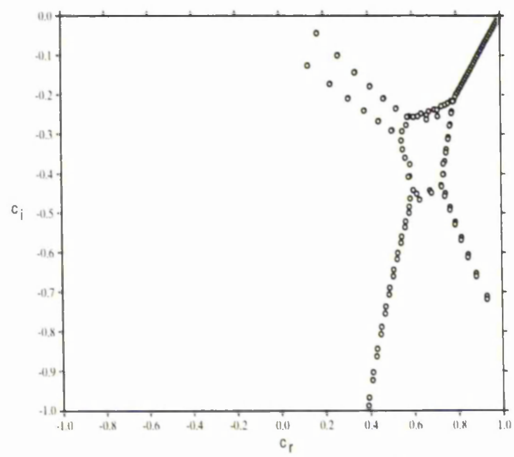
Poiseuille spectrum using D method when $a = 1$ and $R = 30000$



Poiseuille spectrum using D^2 method when $a = 1$ and $R = 30000$



Poiseuille spectrum using D method when $a = 1$ and $R = 50000$



Poiseuille spectrum using D^2 method when $a = 1$ and $R = 50000$

Figure 3.4: Eigenvalue distribution for D and D^2 methods

Results from the modified boundary value problem are then compared against those derived using the formulation described in the previous section.

3.4 The Procedure

Let $\phi = D^2 w$ where $w = 0$ on $x = \pm 1$. The idea is to construct a representation of w in terms of ϕ, the arbitrary constants arising in this computation being used to satisfy two boundary conditions, in this case $w = 0$ on $x = \pm 1$. Clearly

$$Dw = A + \int_{-1}^{x} \phi(u)\, du$$

which on further integration yields

$$w = A(x+1) + \int_{-1}^{x} \left(\int_{-1}^{t} \phi(u)\, du \right) dt = A(x+1) + \int_{-1}^{x} (x-t)\phi(t)\, dt\,. \tag{3.4.15}$$

By construction, the formula for w in (3.4.15) automatically satisfies $w(-1) = 0$ and can be made to satisfy $w(1) = 0$ by choosing

$$A = -\frac{1}{2} \int_{-1}^{1} (1-t)\phi(t)\, dt\,. \tag{3.4.16}$$

Hence

$$Dw(-1) = -\frac{1}{2} \int_{-1}^{1} (1-t)\phi(t)\, dt\,, \qquad Dw(1) = \frac{1}{2} \int_{-1}^{1} (1+t)\phi(t)\, dt\,. \tag{3.4.17}$$

For later convenience let the sequence $f_1, f_2, \ldots$ be defined by

$$f_n = \int_{-1}^{1} T_{2n-2}(t)\, dt = \int_{0}^{\pi} \sin\theta \cos(2n-2)\theta\, d\theta\,, \qquad n \geq 1 \tag{3.4.18}$$

and let $\phi(t)$ have spectral representation

$$\phi(t) = \sum_{n=1}^{\infty} \phi_n T_{n-1}(t)\,. \tag{3.4.19}$$

In view of the sequential odd and even nature of Chebyshev polynomials, it follows that

$$\begin{aligned}
\int_{-1}^{1} t\phi(t)\, dt &= \sum_{n=1}^{\infty} \phi_n \int_{-1}^{1} T_1(t) T_{n-1}(t)\, dt \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \phi_n \int_{-1}^{1} (T_n(t) + T_{n-2}(t))\, dt \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \phi_{2n} \int_{-1}^{1} (T_{2n}(t) + T_{2n-2}(t))\, dt \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \phi_{2n} (f_{n+1} + f_n)\,.
\end{aligned}$$

In conclusion,

$$\int_{-1}^{1} \phi(t)\,dt = \sum_{n=1}^{\infty} \phi_{2n-1} f_n \ , \qquad \int_{-1}^{1} t\phi(t)\,dt = \sum_{n=1}^{\infty} \phi_{2n}(f_{n+1} + f_n) \ . \tag{3.4.20}$$

By elementary calculus, it is verify easily that $f_n = -2/[(2n-1)(2n-3)]$ and so it now follows from (3.4.20) that

$$\int_{-1}^{1} \phi(t)\,dt = -\sum_{n=1}^{\infty} \frac{2\phi_{2n-1}}{(2n-1)(2n-3)} \ , \quad \int_{-1}^{1} t\phi(t)\,dt = -\sum_{n=1}^{\infty} \frac{4\phi_{2n}}{(2n-3)(2n+1)} \ . \tag{3.4.21}$$

In this particular example, $Dw = 0$ on $x = \pm 1$ and so in view of (3.4.17), ϕ is required to satisfy the conditions

$$\int_{-1}^{1} \phi(t)\,dt = \int_{-1}^{1} t\phi(t)\,dt = 0$$

and these in turn lead to the boundary conditions

$$\sum_{n=1}^{\infty} \frac{\phi_{2n-1}}{(2n-1)(2n-3)} = \sum_{n=1}^{\infty} \frac{\phi_{2n}}{(2n-3)(2n+1)} = 0 \ . \tag{3.4.22}$$

These are implemented in the same fashion as in the previous section.

3.5 Results

These ideas successfully extract the eigenvalues of the Orr-Sommerfeld equation over a range of problem parameters. The results presented in table 3.2 compare the accuracy of this new technique with that of the conventional D^2 method.

In conclusion, this procedure compares favourably with the conventional D^2 approach, being rarely inferior and often almost an order of magnitude better.

3.6 Benard Convection of a Conducting Fluid

Suppose that an incompressible, thermally and electrically conducting Navier-Stokes fluid occupies the horizontal layer $0 \le z \le 1$ and is subject to constant gravitational acceleration in the negative z direction and imposed magnetic field in the positive z direction. It is possible to find an equilibrium configuration for this layer in which the fluid is stationary, the magnetic field is constant at its imposed value and heat is conducted across the layer so that the thermal boundary conditions are satisfied. After a non-dimensionalisation

	Accuracy in Poiseuille Flow		Accuracy in Couette Flow	
Value of M	Conventional D^2 method	Modified D^2 method	Conventional D^2 method	Modified D^2 method
12	5.489×10^{-1}	5.489×10^{-1}	5.957×10^{-1}	5.957×10^{-1}
16	2.432×10^{-1}	2.432×10^{-1}	6.278×10^{-1}	6.278×10^{-1}
20	4.113×10^{-1}	4.113×10^{-1}	6.701×10^{-1}	6.701×10^{-1}
24	3.897×10^{-03}	3.897×10^{-03}	6.947×10^{-1}	6.947×10^{-1}
28	6.229×10^{-04}	6.229×10^{-04}	7.115×10^{-1}	7.115×10^{-1}
32	9.444×10^{-05}	9.444×10^{-05}	7.243×10^{-1}	7.243×10^{-1}
36	1.365×10^{-05}	1.365×10^{-05}	5.120×10^{-1}	5.120×10^{-1}
40	2.420×10^{-06}	2.420×10^{-06}	6.951×10^{-4}	6.951×10^{-1}
44	2.382×10^{-07}	2.382×10^{-07}	4.910×10^{-5}	4.910×10^{-5}
48	3.385×10^{-08}	3.385×10^{-08}	1.592×10^{-5}	1.592×10^{-5}
52	1.056×10^{-09}	1.037×10^{-09}	1.740×10^{-6}	1.740×10^{-6}
56	5.409×10^{-10}	5.080×10^{-10}	1.622×10^{-7}	1.622×10^{-7}
60	7.451×10^{-11}	6.678×10^{-11}	2.275×10^{-8}	2.275×10^{-8}
70	1.059×10^{-10}	2.496×10^{-11}	8.812×10^{-11}	8.823×10^{-11}
80	3.682×10^{-11}	2.702×10^{-11}	1.671×10^{-12}	1.589×10^{-12}
90	1.683×10^{-10}	3.789×10^{-11}	2.533×10^{-14}	1.385×10^{-13}
100	7.327×10^{-11}	3.745×10^{-11}	1.565×10^{-14}	1.613×10^{-14}
150	1.056×10^{-10}	1.403×10^{-10}	2.530×10^{-13}	2.783×10^{-13}
200	3.757×10^{-09}	1.839×10^{-10}	3.208×10^{-12}	1.380×10^{-12}
300	1.198×10^{-08}	1.471×10^{-09}	2.744×10^{-13}	1.508×10^{-11}
400	2.764×10^{-08}	1.950×10^{-09}	8.165×10^{-11}	1.587×10^{-11}

Table 3.2: Decimal accuracy in leading eigenvalue versus number of polynomials.

and normal modes procedure, it can be shown that the linear stability analysis of this state is controlled by the eigenvalues, σ, of the system of differential equations

$$\begin{aligned} (D^2-a^2)^2w - QD^2w - \sqrt{R}a^2\theta &= \sigma\Big((D^2-a^2)w - \sqrt{Q}P_m^{-1}Db\Big) , \\ (D^2-a^2)b + \sqrt{Q}Dw &= \sigma P_m b , \\ \sqrt{R}w + (D^2-a^2)\theta &= \sigma P_r \theta , \end{aligned} \tag{3.6.23}$$

where w is the axial component of velocity, a is the wavenumber, R is the Rayleigh number, θ is the temperature, b is the axial component of magnetic induction, Q is the Chandrasekhar number, P_r and P_m are the viscous and magnetic Prandtl numbers respectively and D is the differential operator d/dz. Chandrasekhar [4] provides further details of this procedure.

3.6.1 First Order Formulation

Let variables $y_1, \ldots, y_8$ be defined by

$$\begin{aligned} y_1 &= w\,, & y_2 &= Dw\,, & y_3 &= D^2 w\,, & y_4 &= D^3 w\,, \\ y_5 &= \theta\,, & y_6 &= D\theta\,, & y_7 &= b\,, & y_8 &= Db\,, \end{aligned} \tag{3.6.24}$$

then it is verified easily that equations (3.6.23) can be rewritten as the 8th order system

$$\begin{gathered} Dy_1 = y_2\,, \qquad Dy_2 = y_3\,, \qquad Dy_3 = y_4\,, \\ Dy_4 = -a^4 y_1 + (2a^2 + Q) y_3 + \sqrt{R} a^2 y_5 + \sigma[(y_3 - a^2 y_1) - \sqrt{Q} P_m y_8]\,, \\ Dy_5 = y_6\,, \qquad Dy_6 = -\sqrt{R} y_1 + (a^2 + \sigma P_r) y_5\,, \\ Dy_7 = y_8\,, \qquad Dy_8 = -\sqrt{Q} y_2 + (a^2 + \sigma P_m) y_7\,. \end{gathered} \tag{3.6.25}$$

Since all the coefficients in these equations are constant, no auxiliary matrices are required. It follows almost immediately that σ satisfies the generalised eigenvalue problem $EV = \sigma F V$ where E and F are respectively the 8×8 block matrices

$$E = \begin{bmatrix} D & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D & -I & 0 & 0 & 0 & 0 \\ a^4 I & 0 & -(Q + 2a^2) I & D & -\sqrt{R} a^2 I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D & -I & 0 & 0 \\ \sqrt{R} I & 0 & 0 & 0 & -a^2 I & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D & -I \\ 0 & \sqrt{Q} & 0 & 0 & 0 & 0 & -a^2 I & D \end{bmatrix} \tag{3.6.26}$$

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a^2 I & 0 & I & 0 & 0 & 0 & 0 & -\sqrt{Q} P_m I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_r I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P_m I & 0 \end{bmatrix} . \tag{3.6.27}$$

It only remains to replace the Mth, $2Mth$, ... , $8Mth$ rows of E and F with the appropriate boundary information. For illustrative purposes, suppose that the layer of fluid is contained within two rigid boundaries that are electrically and thermally perfectly conducting. The appropriate boundary conditions are then

$$w = Dw = \theta = b = 0 \qquad \text{on } z = 0 \text{ and } z = 1 , \tag{3.6.28}$$

that is,

$$y_1 = 0 , \quad y_2 = 0 , \quad y_5 = 0 , \quad y_7 = 0 \qquad \text{on } z = 0 \text{ and } z = 1 .$$

In terms of the M dimensional vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ defined in (3.2.11), the boundary conditions and their location are, in block matrix notation,

Condition	Row	E	F
$y_1 = 0$ on $x = -1$	M	$(\mathbf{q}, 0, 0, 0, 0, 0, 0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0)$
$y_1 = 0$ on $x = 1$	$2M$	$(\mathbf{p}, 0, 0, 0, 0, 0, 0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0)$
$y_2 = 0$ on $x = -1$	$3M$	$(0, \mathbf{q}, 0, 0, 0, 0, 0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0)$
$y_2 = 0$ on $x = 1$	$4M$	$(0, \mathbf{p}, 0, 0, 0, 0, 0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0)$
$y_5 = 0$ on $x = -1$	$5M$	$(0, 0, 0, 0, \mathbf{q}, 0, 0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0)$
$y_5 = 0$ on $x = 1$	$6M$	$(0, 0, 0, 0, \mathbf{p}, 0, 0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0)$
$y_7 = 0$ on $x = -1$	$7M$	$(0, 0, 0, 0, 0, 0, \mathbf{q}, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0)$
$y_7 = 0$ on $x = 1$	$8M$	$(0, 0, 0, 0, 0, 0, \mathbf{p}, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0)$

3.6.2 Second Order Formulation

Let z_1, z_2, z_3 and z_4 be defined in terms of w, θ and b by

$$z_1 = w , \quad z_2 = D^2 w , \quad z_3 = \theta , \quad z_4 = b , \tag{3.6.29}$$

then in terms of these variables, the eigenvalue problem (3.6.23) becomes

$$\begin{aligned}
D^2 z_1 &= z_2 \,, \\
D^2 z_2 &= (2a^2+Q)z_2 - a^4 z_1 + \sqrt{R}a^2 z_3 + \sigma(z_2 - a^2 z_1 - \sqrt{Q}P_m^{-1} D z_4) \,, \\
D^2 z_3 &= -\sqrt{R} z_1 + a^2 z_3 + \sigma P_r z_3 \,, \\
D^2 z_4 &= -\sqrt{Q} D z_1 + a^2 z_4 + \sigma P_m z_4 \,,
\end{aligned} \tag{3.6.30}$$

with boundary conditions

$$z_1 = 0 \,, \quad D z_1 = 0 \,, \quad z_3 = 0 \,, \quad z_4 = 0 \,, \qquad \text{on } z = 0 \text{ and } z = 1 \,. \tag{3.6.31}$$

By a routine calculation, it follows from (3.6.30) that σ satisfies the generalised eigenvalue problem $EV = \sigma FV$ where E and F are respectively the 4×4 block matrices

$$\begin{aligned}
E &= \begin{bmatrix} D^2 & -I & 0 & 0 \\ a^4 I & D^2 - (Q+2a^2)I & -\sqrt{R}a^2 I & 0 \\ \sqrt{R} I & 0 & D^2 - a^2 I & 0 \\ \sqrt{Q} D & 0 & 0 & D^2 - a^2 I \end{bmatrix} , \\
F &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -a^2 I & I & 0 & -\sqrt{Q}P_m^{-1} D \\ 0 & 0 & P_r I & 0 \\ 0 & 0 & 0 & P_m I \end{bmatrix} .
\end{aligned} \tag{3.6.32}$$

It only remains to replace the $(M-1)th$, Mth, ... , $(4M-1)th$ and $4Mth$ rows of E and F with the appropriate boundary information. In terms of the M dimensional vectors $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ and $\boldsymbol{s}$ defined in (3.2.11), the boundary conditions and their location are, in block matrix notation,

Condition	Row	E	F
$z_1 = 0$ on $x = -1$	$M-1$	$(\mathbf{q},0,0,0)$	$(0,0,0,0)$
$z_1 = 0$ on $x = 1$	M	$(\mathbf{p},0,0,0)$	$(0,0,0,0)$
$Dz_1 = 0$ on $x = -1$	$2M-1$	$(\mathbf{r},0,0,0)$	$(0,0,0,0)$
$Dz_1 = 0$ on $x = 1$	$2M$	$(\mathbf{s},0,0,0)$	$(0,0,0,0)$
$z_3 = 0$ on $x = -1$	$3M-1$	$(0,0,\mathbf{q},0)$	$(0,0,0,0)$
$z_3 = 0$ on $x = 1$	$3M$	$(0,0,\mathbf{p},0)$	$(0,0,0,0)$
$z_4 = 0$ on $x = -1$	$4M-1$	$(0,0,0,\mathbf{q})$	$(0,0,0,0)$
$z_4 = 0$ on $x = 1$	$4M$	$(0,0,0,\mathbf{p})$	$(0,0,0,0)$

Rayleigh No. R^2	Largest Eigenvalue σ Real	Imaginary
3731	-1.4802	± 0.3716
	-5.5340	0.0000
3732	-1.4769	± 0.2507
	-5.5349	0.0000
3733	-1.3448	0.0000
	-1.6023	0.0000
	-5.5349	0.0000
	-36.3329	± 21.2906
3734	-1.1604	0.0000
	-1.7801	0.0000
	-5.5349	0.0000
	-36.3306	± 21.2412

Table 3.3: Benard spectrum around $R = 3732$.

Appendix 3 gives a Fortran77 program based on routine F02BJF to solve these two eigenvalue problems.

3.7 Results

For given values of the nondimensional parameters and a fixed wavenumber a, R is adjusted so that all eigenvalues have negative real part except the leading eigenvalue which has zero real part. This procedure defines $R = R(a)$. The critical Rayleigh number R_{crit} and critical wavenumber a_{crit} are determined so that $R(a) \geq R_{\text{crit}} = R(a_{\text{crit}})$, that is, $R(a)$ has a minimum value R_{crit} at $a = a_{\text{crit}}$. The key step in this procedure is the identification of the leading eigenvalue for all values of R. Table 3.3 displays the top of the spectrum as R varies between 3731 and 3734. The dynamic nature of these excerpts gives a clear indication as to the nature of eigenvalue problems and provides an unambiguous warning that non-spectral methods should be used with extreme caution unless supported by corroborative mathematics such as a "principle of exchange of stabilities".

Benard Convection for $a = 5.5576$, $R = 15500$ and $Q = 1000$			
	Regular form of boundary conditions	Regular form of boundary conditions	Modified form of boundary conditions
Value of M	Accuracy for D $\lvert\sigma - \sigma_c\rvert$	Accuracy for D^2 $\lvert\sigma - \sigma_c\rvert$	Accuracy for D^2 $\lvert\sigma - \sigma_c\rvert$
10	1.837×10^{-01}	3.251×10^{-01}	3.251×10^{-01}
15	2.123×10^{-03}	2.176×10^{-03}	2.176×10^{-03}
20	8.735×10^{-06}	2.802×10^{-05}	2.802×10^{-05}
25	1.854×10^{-08}	1.923×10^{-08}	1.921×10^{-08}
30	1.058×10^{-10}	1.121×10^{-10}	1.268×10^{-10}
35	8.804×10^{-11}	2.840×10^{-10}	2.090×10^{-10}
40	1.239×10^{-10}	2.501×10^{-10}	4.769×10^{-10}
50	1.925×10^{-10}	1.043×10^{-09}	4.869×10^{-10}
60	1.181×10^{-10}	1.808×10^{-09}	1.485×10^{-09}
70	1.064×10^{-10}	1.620×10^{-09}	2.031×10^{-09}
80	1.729×10^{-10}	3.272×10^{-09}	1.980×10^{-09}
90	1.214×10^{-10}	8.093×10^{-09}	7.450×10^{-09}
120	1.781×10^{-10}	4.821×10^{-09}	1.966×10^{-09}
150	6.205×10^{-10}	1.370×10^{-08}	8.489×10^{-09}
190	6.495×10^{-10}	6.938×10^{-08}	2.570×10^{-08}

Table 3.4: Comparison of D and D^2 methods in Benard convection

3.8 Modified Boundary Conditions

As with the Orr-Sommerfeld equation, the accuracy of the eigenvalue determination was estimated for the D method, the D^2 method and the D^2 method with modified boundary conditions. The results are displayed in table 3.4. The D method was superior to the D^2 methods at each level of polynomial approximation although the difference was not so marked as with the OS equation. Within the D^2 formulation of the problem, the format of the boundary conditions seemed to make little difference although the modified version is marginally superior.

Chapter 4

Eigenvalue Calculations using Legendre Polynomials

4.1 Introduction

This chapter is intended to provide some results comparing the performance of Legendre and Chebyshev polynomial series in eigenvalue calculations in addition to illustrating the details of implementation for a Legendre spectral series. It is convenient to repeat the treatment of the eigenvalue problem for the Orr-Sommerfeld equation in the case of Poiseuille and Couette flow and compare results with those established previously in chapter 3 using the Chebyshev Tau method.

4.2 Changes for Legendre Polynomials

The Legendre treatment of the Orr-Sommerfeld problem differs overtly from the Chebyshev calculation in the respect that the matrix Q (describing $\bar{u}$) and the differentiation matrix D need to be replaced by their Legendre equivalent form. Note that P is unchanged since it represents $D^2\bar{u}$ - a constant in the OS equation. In fact, minor alterations are also required in the D^2 implementation since the derivative of Legendre and Chebyshev polynomials at $x = \pm 1$ are slightly different. No adjustment to the boundary rows is required in the D method since $T_n(1) = P_n(1) = 1$, $T_n(-1) = P_n(-1) = (-1)^n$. Recall from (2.2.25) that

$$D_{i,i+2j+1} = 2i + 1 \ , \qquad i, j \geq 0 \ .$$

Now suppose that

$$f(x) = \sum_{k=0}^{\infty} f_k P_k(x) \tag{4.2.1}$$

then, in view of the product property for Legendre polynomials,

$$\begin{aligned}
xf(x) &= \sum_{k=0}^{\infty} f_k x P_k(x) \\
&= f_0 P_1(x) + \sum_{k=1}^{\infty} f_k \left(\frac{k+1}{2k+1} P_{k+1}(x) + \frac{k}{2k+1} P_{k-1}(x) \right) \\
&= f_0 P_1(x) + \sum_{k=2}^{\infty} \frac{k}{2k-1} f_{k-1} P_k(x) + \sum_{k=0}^{\infty} \frac{k+1}{2k+3} f_{k+1} P_k(x) \\
&= \sum_{k=1}^{\infty} \frac{k}{2k-1} f_{k-1} P_k(x) + \sum_{k=0}^{\infty} \frac{k+1}{2k+3} f_{k+1} P_k(x) \, .
\end{aligned}$$

A similar calculation, when applied to $x(xf(x))$, yields

$$\begin{aligned}
x^2 f(x) &= \sum_{k=1}^{\infty} \frac{k}{2k-1} f_{k-1} x P_k(x) + \sum_{k=0}^{\infty} \frac{k+1}{2k+3} f_{k+1} x P_k(x) \\
&= \sum_{k=1}^{\infty} \frac{k}{2k-1} f_{k-1} \left(\frac{k+1}{2k+1} P_{k+1}(x) + \frac{k}{2k+1} P_{k-1}(x) \right) + \frac{f_1}{3} x \\
&\quad + \sum_{k=1}^{\infty} \frac{k+1}{2k+3} f_{k+1} \left(\frac{k+1}{2k+1} P_{k+1}(x) + \frac{k}{2k+1} P_{k-1}(x) \right) \\
&= \sum_{k=2}^{\infty} \frac{k(k-1)}{(2k-3)(2k-1)} f_{k-2} P_k(x) + \sum_{k=0}^{\infty} \frac{(k+1)^2}{(2k+1)(2k+3)} f_k P_k(x) + \frac{f_1}{3} x \\
&\quad + \sum_{k=2}^{\infty} \frac{k^2}{(2k-1)(2k+1)} f_k P_k(x) + \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{(2k+3)(2k+5)} f_{k+2} P_k(x) \\
&= \sum_{k=2}^{\infty} \frac{k(k-1)}{(2k-3)(2k-1)} f_{k-2} P_k(x) + \sum_{k=0}^{\infty} \frac{(k+1)^2}{(2k+1)(2k+3)} f_k P_k(x) \\
&\quad + \sum_{k=1}^{\infty} \frac{k^2}{(2k-1)(2k+1)} f_k P_k(x) + \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{(2k+3)(2k+5)} f_{k+2} P_k(x) \\
&= \sum_{k=2}^{\infty} \frac{k(k-1)}{(2k-3)(2k-1)} f_{k-2} P_k(x) + \sum_{k=0}^{\infty} \frac{2k^2+2k-1}{(2k-1)(2k+3)} f_k P_k(x) \\
&\quad + \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{(2k+3)(2k+5)} f_{k+2} P_k(x)
\end{aligned}$$

In conclusion,

$$\begin{aligned}
(1-x^2) f(x) &= -\sum_{k=2}^{\infty} \frac{k(k-1)}{(2k-3)(2k-1)} f_{k-2} P_k(x) \\
&\qquad + \sum_{k=0}^{\infty} \frac{2(k^2+k-1)}{(2k-1)(2k+3)} f_k P_k(x) \\
&\qquad\quad - \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{(2k+3)(2k+5)} f_{k+2} P_k(x) \qquad (4.2.2) \\
xf(x) &= \sum_{k=1}^{\infty} \frac{k}{2k-1} f_{k-1} P_k(x) + \sum_{k=0}^{\infty} \frac{k+1}{2k+3} f_{k+1} P_k(x)
\end{aligned}$$

Since $\bar{u}(x) = 1 - x^2$ in Poiseuille flow and $\bar{u}(x) = x$ in Couette flow then equations (4.2.2) form the basis for the determination of Q in Poiseuille and Couette flow respectively. For each flow, the non-zero entries of Q are

Poiseuille Flow

$$Q_{k,k-2} = -\frac{k(k-1)}{(2k-3)(2k-1)}\,, \qquad Q_{k,k+2} = \frac{(k+1)(k+2)}{(2k+3)(2k+5)}\,,$$
$$Q_{k,k} = \frac{2(k^2+k-1)}{(2k-1)(2k+3)}\,, \tag{4.2.3}$$

Couette Flow

$$Q_{k,k-1} = \frac{k}{2k-1}\,, \qquad Q_{k,k+1} = \frac{k+1}{2k+3}\,. \tag{4.2.4}$$

4.3 Boundary Conditions

Since $T_n(1) = P_n(1) = 1$ and $T_n(-1) = P_n(-1) = (-1)^n$ then boundary conditions involving function values are treated identically for Chebyshev and Legendre matrices. In terms of the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ defined in (3.2.11),

$$y_r(1) = \sum_{k=0}^{M} \alpha_{rk} P_k(1) = \boldsymbol{\alpha}_r.\boldsymbol{p}\,, \qquad y_r(-1) = \sum_{k=0}^{M} \alpha_{rk} P_k(-1) = \boldsymbol{\alpha}_r.\boldsymbol{q}\,. \tag{4.3.5}$$

Hence boundary conditions for Chebyshev and Legendre spectral series based on the D method are always identical.

However, the D^2 technique requires derivatives of the spectral polynomials at $z = \pm 1$. For Legendre polynomials, it is easily established that

$$\frac{dP_{2k}(1)}{dx} = \sum_{r=0}^{k-1} 4r+3 = k(2k+1)\,, \qquad \frac{dP_{2k-1}(1)}{dx} = \sum_{r=0}^{k-1} 4r+1 = k(2k-1)\,. \tag{4.3.6}$$

Moreover $P_{2k}(x)$ is an even function of x so that $P'_{2k}(x)$ is an odd function whereas $P_{2k-1}(x)$ is an odd function of x so that $P'_{2k-1}(x)$ is an even function. Hence it follows from (4.3.6) that

$$\frac{dP_k(1)}{dx} = \frac{k(k+1)}{2} \qquad \frac{dP_k(-1)}{dx} = \frac{k(k+1)}{2}(-1)^k\,. \tag{4.3.7}$$

4.4 Results

Legendre polynomial series were employed to compute the competitive eigenvalue of the Orr-Sommerfeld equation for Poiseuille and Couette flow using the D and D^2 methodologies. Computations were done with wavenumber $a = 1$ and Reynolds number $R = 10000$

for Poiseuille flow and $R = 13000$ for Couette flow. Comparative results are presented in tables 4.1 and 4.2.

Spectral Accuracy of Poiseuille Flow				
Value of	D Method		D^2 Method	
M	Chebyshev	Legendre	Chebyshev	Legendre
20	4.113×10^{-1}	4.085×10^{-1}	4.113×10^{-1}	4.085×10^{-1}
32	4.237×10^{-5}	3.381×10^{-5}	9.444×10^{-5}	6.059×10^{-5}
50	1.328×10^{-9}	2.996×10^{-9}	1.056×10^{-9}	5.685×10^{-9}
100	2.103×10^{-13}	7.822×10^{-13}	7.327×10^{-11}	5.658×10^{-11}
200	7.813×10^{-13}	4.362×10^{-12}	3.757×10^{-9}	4.279×10^{-10}
400	4.679×10^{-13}	2.563×10^{-12}	2.764×10^{-8}	8.977×10^{-9}

Table 4.1: Decimal accuracy versus number of polynomials used.

Spectral Accuracy of Couette Flow				
Value of	D Method		D^2 Method	
M	Chebyshev	Legendre	Chebyshev	Legendre
20	4.113×10^{-1}	8.279×10^{-1}	6.701×10^{-1}	6.647×10^{-1}
32	4.237×10^{-5}	7.339×10^{-1}	7.243×10^{-1}	7.324×10^{-1}
50	1.328×10^{-9}	2.139×10^{-6}	1.740×10^{-6}	3.401×10^{-6}
100	2.103×10^{-13}	1.295×10^{-15}	1.565×10^{-14}	5.979×10^{-15}
200	7.813×10^{-13}	1.898×10^{-15}	3.208×10^{-12}	3.326×10^{-14}
400	4.679×10^{-13}	1.295×10^{-15}	8.165×10^{-11}	2.189×10^{-14}

Table 4.2: Decimal accuracy versus number of polynomials used.

It is clear from these calculations that both families of polynomials perform equivalently. Indeed, the results on Couette flow suggest that Legendre polynomials may have a slight edge over Chebyshev polynomials for this particular problem. We believe that this is probably an anomaly generated by the way in which the errors were estimated. As has already been stated, analysis suggests that both sets of polynomials are effectively equivalent.

Chapter 5

Benard-Marangoni Convection in a Layer of Conducting Fluid with Imposed Magnetic Field

5.1 Introduction

Let x_i be a set of Cartesian coordinates with associated base unit vectors $\mathbf{e}_i$ where it will be understood in all subsequent analysis that roman indices take values 1, 2 and 3 whereas greek indices take values 1 and 2 only. Suppose that an incompressible, thermally and electrically conducting Navier-Stokes fluid occupies the horizontal layer $0 \leq x_3 \leq d$ and is subject to constant gravitational acceleration $-g\mathbf{e}_3$ and imposed magnetic field $H\mathbf{e}_3$. The fluid motion is constrained by a rigid lower boundary maintained at constant temperature T_{L} and an upper free boundary whose temperature T_{U} is maintained by the radiative transfer of heat into an impinging passive inviscid fluid at constant temperature T_∞ and constant pressure P_∞. This configuration possesses a steady state solution in which the fluid is stationary, the magnetic field remains at its imposed value but heat is conducted across the layer at a constant rate determined by the thermal boundary conditions. This work aims to explore the stability of this "conduction solution" and is novel in the respect that it presents a comprehensive treatment of the linearised problem, that is, one which can distinguish between stationary and overstable modes. Previous analysis of this problem either considered restricted situations in which the eigenvalues of the linear problem could be determined explicitly, or alternatively, investigated circumstances in which zero was an eigenvalue of the linearized problem. This latter possibility is only sensible provided a "principle of exchange of stabilities" can be established.

Chandrasekhar's [4] work on the convection of conducting fluids established that overstability can be the preferred mechanism in particular parameter regions. For example, in the magnetic Benard problem for a layer of fluid in the absence of surface tension effects, overstability can be the preferred mechanism when the magnetic Prandtl number exceeds the viscous Prandtl number. Hence it seems plausible that overstable convection features strongly here.

5.2 Basic Equations

Let V_i, H_i, B_i, J_i and E_i be respectively the components of the fluid velocity, magnetic field, magnetic induction, current density and electric field with respect to the base vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. In Cartesian tensor notation, the equations expressing conservation of momentum in the fluid layer have component form

$$\frac{\partial V_i}{\partial t} + V_j V_{i,j} = -\frac{1}{\rho_0} P_{,i} + \nu V_{i,jj} - \frac{\rho}{\rho_0} g(1 - \alpha(T - T_0))\delta_{i3} + \frac{1}{\rho_0}(\mathbf{J} \times \mathbf{B})_i \tag{5.2.1}$$

where T is the Kelvin temperature of the fluid, ρ is the fluid density at temperature T, ρ_0 is the fluid density at a reference Kelvin temperature T_0 (taken to be T_U in this work), P is the hydrostatic pressure and ν (constant and independent of temperature) is the kinematic viscosity of the fluid. In keeping with the classical Boussinesq[1] approximation, it will henceforth be assumed that

$$\rho(T) = \rho_0(1 - \alpha(T - T_0)) = \rho_0(1 - \alpha(T - T_U)) \tag{5.2.2}$$

in which α is the coefficient of volume expansion of the fluid and is assumed to be constant. Furthermore, incompressibility of the fluid and the non-existence of magnetic monopoles require that $\mathbf{V}$ and $\mathbf{B}$ are both solenoidal vectors. Hence

$$\operatorname{div} \mathbf{V} = V_{i,i} = 0\ , \qquad \operatorname{div} \mathbf{B} = B_{i,i} = 0\ . \tag{5.2.3}$$

Suppose also that the magnetization in the fluid is directly proportional to the applied field and that the fluid behaves like an Ohmic conductor so that the magnetic field

[1]The Boussinesq approximation asserts that variations in density due to temperature manifest themselves only through buoyancy. The classical view supposes that density is a linear function of temperature although it is well known that this is not a good approximation for water [30].

$\mathbf{H}$, magnetic induction $\mathbf{B}$, current density $\mathbf{J}$ and electric field $\mathbf{E}$ are connected by the constitutive relations

$$\mathbf{B} = \mu \mathbf{H} \, , \qquad \mathbf{J} = \bar{\sigma}(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \, , \tag{5.2.4}$$

and the Maxwell equations

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \, , \qquad \mathbf{J} = \frac{1}{4\pi} \operatorname{curl} \mathbf{H} \, , \tag{5.2.5}$$

where μ (constant) is the magnetic permeability, $\bar{\sigma}$ is the electrical conductivity and the displacement current has been neglected in the second of these Maxwell equations as is customary in situations when free charge is instantaneously dispersed. On taking the curl of equation $(5.2.4)_2$ and replacing the electric field by the Maxwell relation $(5.2.5)_1$, the magnetic field $\mathbf{H}$ is now readily seen to satisfy the partial differential equation

$$\eta \operatorname{curl} \operatorname{curl} \mathbf{H} = -\frac{\partial \mathbf{H}}{\partial t} + \operatorname{curl} (\mathbf{V} \times \mathbf{H}) \tag{5.2.6}$$

where $\eta = (4\pi\mu\bar{\sigma})^{-1}$ is the electrical resistivity. In addition, the constant nature of μ makes the magnetic field $\mathbf{H}$ a solenoidal vector. Equation (5.2.6) is now reworked using standard vector identities to yield in sequence

$$\begin{aligned}
\frac{\partial \mathbf{H}}{\partial t} &= \operatorname{curl} (\mathbf{V} \times \mathbf{H}) - \eta \operatorname{curl} \operatorname{curl} \mathbf{H} \\
&= \mathbf{V} (\operatorname{div} \mathbf{H}) - \mathbf{H} (\operatorname{div} \mathbf{V}) + (\mathbf{H}.\operatorname{grad})\mathbf{V} - (\mathbf{V}.\operatorname{grad})\mathbf{H} - \eta \operatorname{curl} \operatorname{curl} \mathbf{H} \\
&= (\mathbf{H}.\operatorname{grad})\mathbf{V} - (\mathbf{V}.\operatorname{grad})\mathbf{H} - \eta \operatorname{curl} \operatorname{curl} \mathbf{H} \\
&= (\mathbf{H}.\operatorname{grad})\mathbf{V} - (\mathbf{V}.\operatorname{grad})\mathbf{H} - \eta \operatorname{grad}(\operatorname{div} \mathbf{H}) + \eta \triangle \mathbf{H} \\
&= (\mathbf{H}.\operatorname{grad})\mathbf{V} - (\mathbf{V}.\operatorname{grad})\mathbf{H} + \eta \triangle \mathbf{H}
\end{aligned}$$

with component form

$$\frac{\partial H_i}{\partial t} + V_j H_{i,j} = H_j V_{i,j} + \eta H_{i,jj} \, . \tag{5.2.7}$$

Equation (5.2.7) describes the temporal evolution of the magnetic field. Moreover, the relations (5.2.4) and (5.2.5) can be used to recast the Lorentz force $\mathbf{J} \times \mathbf{B}$ into

$$\mathbf{J} \times \mathbf{B} = \frac{\mu}{4\pi} (\operatorname{curl} \mathbf{H}) \times \mathbf{H} = \frac{\mu}{4\pi} (\mathbf{H}.(\operatorname{grad} \mathbf{H}) - \operatorname{grad} (H^2/2))$$

leading to the notion that the Lorentz force is derived from a magnetic stress tensor $\sigma_{ij}^{(\mathrm{m})}$. In view of the fact that

$$(\mathbf{J} \times \mathbf{B})_i = \sigma_{ji,j}^{(\mathrm{m})} = \frac{\mu}{4\pi} (H_j H_i - \frac{1}{2} H^2 \delta_{ij})_{,j} \, . \tag{5.2.8}$$

the momentum equation (5.2.1) can now be manipulated into the format

$$\frac{\partial V_i}{\partial t} + V_j V_{i,j} = -\frac{1}{\rho_0}\Pi_{,i} + \nu V_{i,jj} - g(1-\alpha(T-T_0))\delta_{i3} + \frac{\mu}{4\pi\rho_0} H_j H_{i,j} \ , \qquad (5.2.9)$$

where $\Pi = P + \mu H^2/8\pi$ is pressure. Assuming that energy losses due to viscous dissipation can be neglected, conservation of energy contributes the field equation

$$\frac{\partial T}{\partial t} + V_j T_{,j} = \kappa T_{,jj} \qquad (5.2.10)$$

where κ (constant) is the thermal diffusivity of the fluid. To summarise, the convection problem is described by the differential equations

$$\begin{aligned} \frac{\partial V_i}{\partial t} + V_j V_{i,j} &= -\frac{1}{\rho_0}\Pi_{,i} + \nu V_{i,jj} - g(1-\alpha(T-T_0))\delta_{i3} + \frac{\mu}{4\pi\rho_0} H_j H_{i,j} \ , \\ \frac{\partial T}{\partial t} + V_j T_{,j} &= \kappa T_{,jj} \ , \\ \frac{\partial H_i}{\partial t} + V_j H_{i,j} &= H_j V_{i,j} + \eta H_{i,jj} \ . \end{aligned} \qquad (5.2.11)$$

where $\mathbf{V}$ and $\mathbf{H}$ are solenoidal vector fields. Equations (5.2.11) need boundary conditions on $x_3 = 0$ and $x_3 = d$.

5.3 Boundary Conditions

Suppose from the outset that the region exterior to the fluid layer is filled with non-conducting material so that no currents can flow there. At the boundaries between conducting and non-conducting materials, the component of the current density normal to the interface is zero. At any interface, normal components of magnetic induction are always continuous and so the natural way to guarantee the current condition is to extend continuity to all components of the magnetic induction. Within an insulating material, the first Maxwell equation in (5.2.5) indicates that $\mathbf{H}$ is irrotational so that $\mathbf{H}$ is the gradient of $\phi(t, x_i)$ where ϕ is a solution of the Laplace equation since $\operatorname{div}\mathbf{H} = 0$. That is,

$$H_i = H\delta_{i3} + \phi_{,i} \ , \qquad \phi_{,jj} = 0 \ .$$

Since $x_3 = 0$ is a rigid boundary at fixed temperature T_{L} then the appropriate boundary conditions there are

$$V_i = 0 \ , \qquad T = T_{\mathrm{L}} \ , \qquad B_i = \mu H_i \quad \text{continuous} \qquad (5.3.12)$$

with irrotational magnetic field in $x_3 < 0$. The treatment of the upper boundary $x_3 = d$ is more involved since it can move. Suppose that it has equation $x_3 = d + F(t, x_\alpha)$ at time t with unit normal $\mathbf{n} = n_i \mathbf{e}_i$ directed from the viscous fluid into the passive inviscid fluid. The boundary conditions come from four sources.

Heat Transfer The heat flux passing from the viscous to inviscid fluid is $-kn_iT_{,i}$ and this is equal to $h(T - T_\infty)$, the heat loss due to radiation (Newtonian cooling). Here k (constant) is the thermal conductivity of the fluid and h (constant) is the heat transfer coefficient. Hence the thermal boundary condition is

$$kT_{,i}n_i + h(T - T_\infty) = 0 \, . \tag{5.3.13}$$

Material Surface Fluid particles on the surface $x_3 = d + F(t, x_\alpha)$ remain there and so

$$\frac{dx_3}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_\alpha}\frac{dx_\alpha}{dt}$$

and this leads to the condition

$$V_3 - \frac{\partial F}{\partial x_\alpha} V_\alpha = \frac{\partial F}{\partial t} \, . \tag{5.3.14}$$

Magnetic Condition Since the region $x_3 > d$ is electrically insulating then $B_i = \mu H_i$ is continuous across $x_3 = d$ and the magnetic field in $x_3 > d$ is irrotational, that is, derived from a potential function.

Stress Conditions Stress conditions at the interface between the fluid layer and the passive inviscid gas are based on the assumption that the discontinuity experienced by the stress vector in crossing the interface is balanced by the divergence of the surface stress tensor, assumed here to be due entirely to a temperature dependent surface tension in the absence of interfacial mechanical shear stress. Specifically the surface stress tensor is

$$S^{\alpha\beta} = \sigma(T) a^{\alpha\beta} \tag{5.3.15}$$

where $a^{\alpha\beta}$ is the surface metric tensor and σ is surface tension. Although simplistic, this view of an interface is ubiquitous in the literature. Of course, in reality the interfacial region has a finite dimension (of the order of microns) and is perhaps more accurately modelled by mixture theory in the respect that molecules of each bulk fluid can coexist at each interfacial point. Clearly this is an area for future research.

The stress vector for the passive gas impinging on the top boundary is $\mathbf{t} = -P_\infty n_i \mathbf{e}_i$ and, in view of observation (5.2.8), the stress vector for the fluid is

$$\mathbf{t} = n_i \left[-P\delta_{ij} + \rho_0 \nu (V_{i,j} + V_{j,i}) + \frac{\mu}{4\pi}(H_j H_i - \frac{1}{2}H^2\delta_{ij}) \right] \mathbf{e}_j \ .$$

The divergence of the surface stress tensor is

$$\left(S^{\alpha\beta} x_{i,\beta} \right)_{,\alpha} \mathbf{e}_i = \left(\sigma a^{\alpha\beta} x_{i,\beta} \right)_{,\alpha} \mathbf{e}_i = \left(\sigma_{,\alpha} a^{\alpha\beta} x_{i,\beta} + \sigma b^\alpha_\alpha n_i \right) \mathbf{e}_i$$

where b^α_α is the mean curvature of the interface. Thus the stress boundary condition has component form

$$\begin{aligned} \sigma_{,\alpha} a^{\alpha\beta} x_{i,\beta} + \sigma b^\alpha_\alpha n_i &= -P_\infty n_i + (P + \frac{\mu}{8\pi}H^2) n_i \\ &\quad - \rho_0 \nu (V_{i,j} + V_{j,i}) n_j - \frac{\mu}{4\pi}(H_j n_j) H_i \ , \end{aligned} \tag{5.3.16}$$

which can be decomposed further into the tangential and normal components

$$\begin{aligned} \sigma(T) b^\alpha_\alpha &= -P_\infty + \left(P + \frac{\mu}{8\pi}H^2 \right) - 2\rho_0 \nu V_{i,j} n_i n_j - \frac{\mu}{4\pi}(H_j n_j)^2 \ , \\ \sigma_{,\alpha} &= -\rho_0 \nu (V_{i,j} + V_{j,i}) n_j x_{i,\alpha} - \frac{\mu}{4\pi}(H_j n_j)(H_i x_{i,\alpha}) \ . \end{aligned} \tag{5.3.17}$$

5.4 The Base Solution

It is easily verified that equations (5.2.11) have a steady state conduction solution in which the viscous fluid is stationary, the top surface is flat, the magnetic field is constant at the imposed value and the fluid interior is permeated by temperature and pressure fields which are functions of x_3 only. The actual solution satisfying all the boundary conditions on $x_3 = 0$ and $x_3 = d$ is

$$\begin{gathered} V_i = 0 \ , \qquad H_i = H\delta_{i3} \ , \qquad F(x_\alpha, t) = 0 \ , \qquad T_E(x_3) = T_{\mathrm{L}} + \beta x_3 \ , \\ \Pi_E = P_\infty + \frac{\mu}{8\pi}H^2 + \rho_0 g \int_{x_3}^{d} [1 - \alpha(T_{\mathrm{L}} + \beta z - T_{\mathrm{U}})] \, dz \end{gathered} \tag{5.4.18}$$

where β denotes the temperature gradient and is determined from the thermal boundary condition at $x_3 = d$, the thermal condition at $x_3 = 0$ being satisfied trivially. In terms of the Nusselt number

$$N_u = \frac{hd}{k} \ , \qquad \text{(Nusselt number)} \tag{5.4.19}$$

in which k and h are respectively the thermal conductivity of the fluid and the heat transfer coefficient of the radiant boundary, it follows directly from the equilibrium temperature profile in (5.4.18) and the Robin condition (5.3.13) that $T_{\rm U}$ satisfies

$$T_{\rm U} = \frac{T_{\rm L} + N_u T_\infty}{1+N_u} \,, \qquad \beta d = \frac{N_u}{1+N_u}(T_\infty - T_{\rm L}) \,. \tag{5.4.20}$$

5.5 The Perturbed Equations

Let $\mathbf{h} = h_i \mathbf{e}_i$, θ and p be perturbations of the magnetic field, temperature and pressure respectively about their equilibrium values $H\mathbf{e}_3$, $T_E(x_3)$ and $\Pi_E(x_3)$ so that

$$V_i = v_i \,, \qquad H_i = H\delta_{i3} + h_i \,, \qquad T = T_E(x_3) + \theta \,, \qquad \Pi = \Pi_E + p \,. \tag{5.5.21}$$

It can be established easily from (5.2.11) that h_i, θ and p satisfy the field equations

$$\begin{aligned} \frac{\partial v_i}{\partial t} + v_j v_{i,j} &= -\frac{1}{\rho_0} p_{,i} + \nu v_{i,jj} + g\alpha\theta\delta_{i3} + \frac{\mu}{4\pi\rho_0}(Hh_{i,3} + h_j h_{i,j}) \,, \\ \frac{\partial \theta}{\partial t} + \beta v_3 + v_j \theta_{,j} &= \kappa \theta_{,jj} \,, \\ \frac{\partial h_i}{\partial t} + v_j h_{i,j} &= h_j v_{i,j} + H v_{i,3} + \eta h_{i,jj} \,, \end{aligned} \tag{5.5.22}$$

where $\mathbf{v}$ and $\mathbf{h}$ are solenoidal vector fields. The boundary conditions on the lower boundary $x_3 = 0$ become

$$\theta = 0 \,, \qquad v_i = 0 \,, \qquad \mu h_i \quad \text{continuous} \tag{5.5.23}$$

and the conditions on the upper boundary $x_3 = d + F(t, x_\alpha)$ corresponding to (5.3.13) and (5.3.14) are modified respectively to

$$\begin{aligned} (n_3 - 1)(T_{\rm U} - T_{\rm L}) + dn_i\theta_{,i} + N_u\theta + N_u(T_{\rm U} - T_{\rm L})\frac{F}{d} &= 0 \,, \\ v_3 - \frac{\partial F}{\partial x_\alpha} v_\alpha &= \frac{\partial F}{\partial t} \,. \end{aligned} \tag{5.5.24}$$

The surface stress conditions (5.3.16) and (5.3.17) require significantly more effort. The modified form of the normal component of (5.3.16) is

$$\begin{aligned} \sigma(T_{\rm U} + (T_{\rm U} - T_{\rm L})Fd^{-1} + \theta)b^\alpha_\alpha \;=\; & p - \frac{\mu}{4\pi}\Big[(h_j n_j)^2 + 2Hn_3 h_j n_j\Big] - \rho_0 g F \\ & + \frac{\mu}{4\pi}H^2(1 - n_3^2) - 2\rho_0 \nu v_{i,j} n_i n_j \\ & + \frac{\rho_0 g \alpha}{2d}(T_{\rm U} - T_{\rm L})F^2 \,, \end{aligned} \tag{5.5.25}$$

whereas the tangential surface stress condition (5.3.17) yields

$$\begin{aligned} \frac{d\sigma}{dT}\left[(T_\mathrm{U}-T_\mathrm{L})d^{-1}F_{,\alpha}+\theta_{,\alpha}\right] &= -\rho_0\nu(v_{i,j}+v_{j,i})n_j x_{i,\alpha} \\ &\quad -\frac{\mu}{4\pi}(Hn_3+h_jn_j)(h_ix_{i,\alpha}) \,. \end{aligned} \tag{5.5.26}$$

In (5.5.26) it is assumed that the derivative of the surface tension with respect to temperature is evaluated at $T = T_\mathrm{U} + (T_\mathrm{U} - T_\mathrm{L})d^{-1}F + \theta$.

5.6 The Non-dimensional Equations

Equations (5.5.22) and boundary conditions (5.5.24), (5.5.25) and (5.5.26) are now non-dimensionalised in the customary manner. Spatial coordinates x_i are scaled with respect to d, time t with respect to d^2/κ so that the non-dimensional form of the upper surface becomes $x_3 = 1 + f(t, x_\alpha)$. Similarly perturbed velocity components are scaled with respect to κ/d, magnetic field components with respect to $H\kappa/\eta$, pressures with respect to $\rho_0\nu\kappa/d^2$ and temperatures with respect to $|T_\mathrm{L} - T_\mathrm{U}|$. The corresponding non-dimensional form of (5.5.22) is

$$\begin{aligned} P_r^{-1}\left(\frac{\partial v_i}{\partial t}+v_jv_{i,j}\right) &= -p_{,i}+v_{i,jj}+R\theta\delta_{i3}+Q(h_{i,3}+P_m^{-1}h_jh_{i,j})\,, \\ \frac{\partial\theta}{\partial t}+v_j\theta_{,j} &= \gamma v_3+\theta_{,jj}\,, \\ P_m^{-1}\left(\frac{\partial h_i}{\partial t}+v_jh_{i,j}-h_jv_{i,j}\right) &= v_{i,3}+h_{i,jj}\,, \end{aligned} \tag{5.6.27}$$

where the Viscous Prandtl No. P_r, the Magnetic Prandtl No. P_m, the Chandrasekhar No. Q and the Rayleigh No. R are defined by

$$P_r = \frac{\nu}{\kappa}\,, \qquad P_m = \frac{\eta}{\kappa}\,, \qquad Q = \frac{\mu H^2 d^2}{4\pi\rho_0\nu\eta}\,, \qquad R = \frac{\alpha g d^3|T_\mathrm{L}-T_\mathrm{U}|}{\kappa\nu}\,. \tag{5.6.28}$$

Furthermore, $\gamma = \mathrm{sign}(T_\mathrm{L} - T_\mathrm{U})$ indicates the boundary at which heat is supplied. When $\gamma = +1$, the fluid layer is heated on its lower boundary whereas when $\gamma = -1$, the upper boundary is heated. Equations (5.6.27) are to be supplemented with rescaled boundary conditions on the upper and lower boundaries. The non-dimensional boundary conditions on $x_3 = 0$ are derived from (5.5.23) and are

$$v_i = 0\,, \qquad \theta = 0\,, \qquad \lim_{x_3\to 0^+} h_i = \frac{\mu_0}{\mu}\lim_{x_3\to 0^-}\frac{\partial\phi_\mathrm{L}}{\partial x_i}\,. \tag{5.6.29}$$

in which ϕ_{L} is the non-dimensional magnetic potential function[2] in the region $x_3 < 0$. The rescaled equation of the upper boundary is $x_3 = 1 + f(t, x_\alpha)$. Here the heat transfer condition, free surface condition and continuity of magnetic induction require that

$$\begin{aligned} \gamma(1-n_3) + n_i\theta_{,i} + N_u(\theta - \gamma f) = 0 \,, \\ v_3 - \frac{\partial f}{\partial x_\alpha} v_\alpha = \frac{\partial f}{\partial t} \,, \\ \lim_{x_3 \to 1^-} h_i = \frac{\mu_0}{\mu} \lim_{x_3 \to 1^+} \frac{\partial \phi_{\mathrm{U}}}{\partial x_i} \,, \end{aligned} \tag{5.6.30}$$

where ϕ_{U} is the non-dimensional magnetic potential function in the region $x_3 > 1 + f$. Again, the surface stress conditions require more effort. The rescaled form of (5.5.26) arising from the normal component of the surface stress is

$$\begin{aligned} C_r^{-1} \frac{\sigma(T_{\mathrm{U}} + (\theta - \gamma f)|T_{\mathrm{L}} - T_{\mathrm{U}}|)}{\sigma(T_{\mathrm{U}})} b_\alpha^\alpha &= p - Q[P_m^{-1}(h_j n_j)^2 + 2n_3 h_j n_j] - B_o C_r^{-1} f \\ &\quad + Q P_m (1 - n_3^2) - 2v_{i,j} n_i n_j - \frac{\gamma R}{2} f^2 \,, \end{aligned} \tag{5.6.31}$$

in which C_r, the Crispation No., and B_o, the Bond No., are defined by

$$B_o = \frac{\rho_0 g d^2}{\sigma(T_{\mathrm{U}})} \,, \qquad C_r = \frac{\rho_0 \nu \kappa}{d \sigma(T_{\mathrm{U}})} \,. \tag{5.6.32}$$

The rescaled version of the tangential surface stress condition (5.5.26) is

$$\begin{aligned} M \frac{\sigma'(T_{\mathrm{U}} + |T_{\mathrm{L}} - T_{\mathrm{U}}|)(\theta - \gamma f)}{\sigma'(T_{\mathrm{U}})} (\gamma f_{,\alpha} - \theta_{,\alpha}) &= (v_{i,j} + v_{j,i}) n_j x_{i,\alpha} \\ &\quad + Q h_i x_{i,\alpha} (n_3 + P_m^{-1} h_j n_j) \end{aligned} \tag{5.6.33}$$

where M is the Marangoni No. defined by

$$M = \frac{d|T_{\mathrm{L}} - T_{\mathrm{U}}|\sigma'(T_{\mathrm{U}})}{\rho_0 \nu \kappa} \,. \tag{5.6.34}$$

5.7 The Linearised Problem

Until now the analysis has been exact. Henceforth suppose that perturbations in $\mathbf{v}$, $\mathbf{h}$, θ and p are so small that their products can be ignored whenever they occur — this is the linear approximation. In this scenario, the approximated form for equations (5.6.27) is

$$\begin{aligned} P_r^{-1} \frac{\partial v_i}{\partial t} &= -p_{,i} + v_{i,jj} + R\theta\delta_{i3} + Q h_{i,3} \,, \\ \frac{\partial \theta}{\partial t} &= \gamma v_3 + \theta_{,jj} \,, \\ P_m^{-1} \frac{\partial h_i}{\partial t} &= v_{i,3} + h_{i,jj} \,. \end{aligned} \tag{5.7.35}$$

[2]Both ϕ_{L} and ϕ_{U} have been rescaled with $Hd\kappa/\eta$.

The boundary conditions on the lower boundary $x_3 = 0$ are unchanged from those described in (5.6.29). On the upper boundary $x_3 = 1 + f(t, x_\alpha)$, the outward unit normal has components

$$n_1 = \frac{-f_{,1}}{\sqrt{1+f_{,1}^2+f_{,2}^2}} , \qquad n_2 = \frac{-f_{,2}}{\sqrt{1+f_{,1}^2+f_{,2}^2}} , \qquad n_3 = \frac{1}{\sqrt{1+f_{,1}^2+f_{,2}^2}} \tag{5.7.36}$$

so that the linearised upper boundary conditions described in (5.6.30), (5.6.31) and (5.6.33) are

$$\begin{aligned} \frac{\partial\theta}{\partial x_3} + N_u(\theta - \gamma f) = 0 \, , \\ v_3 = \frac{\partial f}{\partial t} \, , \\ \lim_{x_3\to 1^-} h_i = \frac{\mu_0}{\mu} \lim_{x_3\to 1^+} \frac{\partial \phi_{\rm U}}{\partial x_i} \, , \\ C_r^{-1} b_\alpha^\alpha = p - 2Qh_3 - B_o C_r^{-1} f - 2\frac{\partial v_3}{\partial x_3} \, , \\ M(\gamma f_{,\alpha} - \theta_{,\alpha}) = v_{3,\alpha} + \frac{\partial v_\alpha}{\partial x_3} + Qh_\alpha \, . \end{aligned} \tag{5.7.37}$$

5.8 Magnetic Boundary Conditions

Recall that the magnetic field in an insulating material is irrotational and is derived from a potential function $\phi(t, x_i)$ which is the solution of Laplace's equation.

Let $\phi = \psi(t, x_3)e^{i(px_1+qx_2)}$ then

$$\frac{\partial^2\psi}{\partial x_3^2} - a^2\psi = 0 \, , \quad a^2 = p^2 + q^2 \, , \qquad \frac{\partial\psi}{\partial x_3} \to 0 \text{ as } |x_3| \to \infty \, .$$

Trivially $\phi_{\rm U}$ and $\phi_{\rm L}$ have functional form

$$\phi_{\rm L} = C_{\rm L}(t)e^{ax_3}e^{i(px_1+qx_2)} \, , \qquad \phi_{\rm U} = C_{\rm U}(t)e^{-ax_3}e^{i(px_1+qx_2)} \, . \tag{5.8.38}$$

When $x_3 < 0$ then $\mathbf{h} = C_{\rm L}(t)(ip, iq, a)$ and continuity of the magnetic induction across $x_3 = 0$ requires that

$$h_{3,3} = \frac{1}{\mu} b_{3,3} = -\frac{1}{\mu} b_{\alpha,\alpha} = \frac{\mu_0}{\mu} a^2 C_{\rm L}(t) = \frac{a}{\mu} b_3 = a h_3$$

with a similar argument on $x_3 = 1 + f$. Hence the magnetic boundary conditions are

$$\begin{aligned} \frac{\partial h_3}{\partial x_3} - a h_3 = 0 \, , \qquad x_3 = 0 \, . \\ \frac{\partial h_3}{\partial x_3} + a h_3 = 0 \, , \qquad x_3 = 1 \, . \end{aligned} \tag{5.8.39}$$

In fact, the description of the convection problem is best expressed in terms of the behaviour of the third components of the velocity and magnetic field. It is convenient to write $w = v_3$, $h = h_3$ and represent partial differentiation of an arbitrary function ψ with respect to x_3 by $D\psi$, the 3-D Laplacian of ψ by $\triangle\psi$ and the 2-D Laplacian of ψ by $\triangle_2\psi$. On taking the double curl of the first of equations (5.7.35) and then using the second of these equations to replace the Laplacian of h_3, it follows that

$$\begin{aligned} \frac{\partial}{\partial t}(P_r^{-1}\triangle w - QP_m^{-1}Dh) &= \triangle^2 w - QD^2 w + R\triangle_2\theta \, , \\ P_m^{-1}\frac{\partial h}{\partial t} &= \triangle h + Dw \, , \\ \frac{\partial \theta}{\partial t} &= \gamma w + \triangle\theta \, . \end{aligned} \tag{5.8.40}$$

In this new notation, the boundary conditions on $x_3 = 0$ finally become

$$w = 0 \, , \quad Dw = 0 \, , \quad \theta = 0 \, , \quad Dh - ah = 0 \, , \tag{5.8.41}$$

and on $x_3 = 1$ the boundary conditions are

$$\begin{aligned} & D\theta + N_u(\theta - \gamma f) = 0 \, , \\ & w - \frac{\partial f}{\partial t} = 0 \, , \\ & Dh + ah = 0 \, , \\ & \triangle_2 f + fB_o - C_r(p - 2Dw - 2Qh) = 0 \, , \\ & M(\theta - \gamma f)_{,\alpha} + (Dv_\alpha + w_{,\alpha} + Qh_\alpha) = 0 \end{aligned} \tag{5.8.42}$$

in which the linearised form of the Gauss-Weingarten relations have been used to replace b^α_α by $\triangle_2 f$. The treatment of the boundary conditions on $x_3 = 1$ is completed by computing the surface divergence of the last condition in (5.8.42). The result is

$$\triangle_2 w + M\triangle_2(\theta - f) - D^2 w - QDh = 0 \, . \tag{5.8.43}$$

5.9 Normal Modes Analysis

When solutions to equations (5.7.35) are sought in the form

$$\begin{aligned} \phi(t, x_i) &= \phi(x_3)e^{\sigma t}e^{i(px_1+qx_2)} \qquad \phi = \{w, h, p, \theta\} \, , \\ f(t, x_\alpha) &= f_0 e^{\sigma t}e^{i(px_1+qx_2)} \qquad f_0 \text{ constant} \, , \end{aligned} \tag{5.9.44}$$

it is relatively straightforward to establish that σ is an eigenvalue of the system

$$\begin{aligned} \sigma P_r^{-1}(D^2-a^2)w - \sigma Q P_m^{-1} Dh &= (D^2-a^2)^2 w - QD^2 w - Ra^2\theta \,, \\ \sigma P_m^{-1} h &= (D^2-a^2)h + Dw \,, \\ \sigma\theta &= \gamma w + (D^2-a^2)\theta \,. \end{aligned} \tag{5.9.45}$$

The boundary conditions on $x_3 = 0$ are

$$w = 0\,, \qquad Dw = 0\,, \qquad \theta = 0\,, \qquad Dh - ah = 0\,. \tag{5.9.46}$$

As a preamble to the formulation of the final boundary conditions on $x_3 = 1$, it follows from (5.2.9) that

$$\triangle_2 p = -\triangle(Dw) - QD^2 h + P_r^{-1}\frac{\partial}{\partial t}(Dw)$$

so that the pressure everywhere is given by the equation

$$p = \frac{1}{a^2}(D^3 w - a^2 Dw + QD^2 h - \sigma P_r^{-1} Dw)\,. \tag{5.9.47}$$

Hence the boundary conditions on $x_3 = 1$ are

$$\begin{gathered} D\theta + N_u(\theta - \gamma f_0) = 0\,, \\ w = \sigma f_0\,, \\ Dh + ah = 0\,, \\ a^2(B_o - a^2)f_0 + C_r(QDw - D^3 w + 3a^2 Dw + Qa^2 h) = \sigma C_r(QP_m^{-1}h - P_r^{-1}Dw)\,, \\ (D^2+a^2)w + M(\theta - \gamma f_0)a^2 + QDh = 0\,. \end{gathered} \tag{5.9.48}$$

Let variables $y_1, \ldots, y_8$ be defined by

$$\begin{aligned} y_1 &= w\,, & y_2 &= Dw\,, & y_3 &= D^2 w\,, & y_4 &= D^3 w\,, \\ y_5 &= \theta\,, & y_6 &= D\theta\,, & y_7 &= h\,, & y_8 &= Dh\,, \end{aligned} \tag{5.9.49}$$

then it is straightforward to verify that equations (5.8.40) can be rewritten as the 8th order system

$$
\begin{aligned}
Dy_1 &= y_2 \\
Dy_2 &= y_3 \\
Dy_3 &= y_4 \\
Dy_4 &= -a^4 y_1 + (2a^2 + Q)y_3 + R_a a^2 y_5 \\
&\qquad + \sigma[P_r^{-1}(y_3 - a^2 y_1) - QP_m^{-1} y_8] \qquad (5.9.50) \\
Dy_5 &= y_6 \\
Dy_6 &= -y_1 + (a^2 + \sigma)y_5 \\
Dy_7 &= y_8 \\
Dy_8 &= -y_2 + (a^2 + \sigma P_m^{-1})y_7 \,.
\end{aligned}
$$

The boundary conditions on $x_3 = 0$ are now

$$y_1 = 0\,, \qquad y_2 = 0\,, \qquad y_5 = 0\,, \qquad y_8 - ay_7 = 0\,. \qquad (5.9.51)$$

The boundary conditions on $x_3 = 1$ are more involved and comprise 4 conditions to complete this eigenvalue problem plus a further condition to establish the height of the free surface. It is convenient to use the relation

$$(D^2 + a^2)w + M_a(\theta - f)a^2 = 0$$

to eliminate occurrences of f everywhere in the equations (5.8.42). This is always possible provided $M_a > 0$. If $M_a = 0$ then the condition $D\theta + N_u(\theta - f) = 0$ plays a similar role etc. In terms of $y_1, \ldots, y_8$, the appropriate conditions are

$$
\begin{aligned}
N_u(y_3 + a^2 y_1) - a^2 M_a y_6 &= 0\,, \\
\sigma(y_3 + a^2 y_1) + \sigma a^2 M_a y_5 - a^2 M_a y_1 &= 0\,, \\
(B_o - a^2)\,[y_3 + a^2 y_1 + a^2 M_a y_5] + M_a C_r(Qy_2 - y_4 + 3a^2 y_2) & \qquad (5.9.52) \\
+\sigma M_a C_r(P_r^{-1} y_2 - QP_m^{-1} y_7) &= 0\,, \\
y_8 + ay_7 &= 0\,.
\end{aligned}
$$

5.9.1 Benard-Marangoni Convection

The equations (5.8.40) can be reformulated in the form of

$$\frac{dY}{dx_3} = AY + \sigma BY$$

where A and B are the real 8×8 matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -a^4 & 0 & (2a^2+Q) & 0 & R_a a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & a^2 & 0 \end{bmatrix} \qquad (5.9.53)$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a^2 P_r^{-1} & 0 & P_r^{-1} & 0 & 0 & 0 & 0 & -QP_m^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P_m^{-1} & 0 \end{bmatrix}. \qquad (5.9.54)$$

Since equations (5.8.40) have constant coefficients then they can be converted into the spectral representation $EV = \sigma FV$ as indicated previously where E and F have block matrix form

$$E = \begin{bmatrix} D & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D & -I & 0 & 0 & 0 & 0 \\ a^4 I & 0 & -(Q+2a^2)I & D & -R_a a^2 I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D & -I & 0 & 0 \\ I & 0 & 0 & 0 & -a^2 I & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D & -I \\ 0 & I & 0 & 0 & 0 & 0 & -a^2 I & D \end{bmatrix} \qquad (5.9.55)$$

and

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a^2P_r^{-1}I & 0 & P_r^{-1}I & 0 & 0 & 0 & 0 & -QP_m^{-1}I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P_m^{-1}I & 0 \end{bmatrix} . \tag{5.9.56}$$

It only remains to replace the Mth, $2Mth$, ... , $8Mth$ rows of E and F with the boundary information. From a mathematical standpoint, it does not matter how the eight boundary conditions are ordered but numerical performance is usually enhanced if the boundary data is inserted so that it favours the Upper Hessenberg format which is generated by Householder operations on E. We deal with each boundary condition in turn:

Mth row This comes from the boundary condition $(5.9.51)_1$. The rows of E and F are replaced respectively by the block forms

$$(\mathbf{q}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) , \qquad (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) .$$

2Mth row This comes from the boundary condition $(5.9.52)_1$. The rows of E and F are replaced respectively by the block forms

$$(N_u a^2\mathbf{p}, \mathbf{0}, N_u\mathbf{p}, \mathbf{0}, \mathbf{0}, -M_a a^2\mathbf{p}, \mathbf{0}, \mathbf{0}) , \qquad (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) .$$

3Mth row This comes from the boundary condition $(5.9.52)_2$. The rows of E and F are replaced respectively by the block forms

$$(M_a a^2\mathbf{p}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) , \qquad (a^2\mathbf{p}, \mathbf{0}, \mathbf{p}, \mathbf{0}, M_a a^2\mathbf{p}, \mathbf{0}, \mathbf{0}, \mathbf{0}) .$$

4Mth row This comes from the boundary condition $(5.9.52)_3$. The rows of E and F are replaced respectively by the block forms

$$(((B_o a^2 - a^4) + 3a^2 M_a C_r)\mathbf{p}, QM_aC_r\mathbf{p}, (B_o - a^2)\mathbf{p},$$
$$-M_aC_r, (B_o - a^2)M_a a^2\mathbf{p}, \mathbf{0}, \mathbf{0}, \mathbf{0}) ,$$

$$(\mathbf{0}, -C_rM_aP_r^{-1}\mathbf{p}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, QC_rM_aP_m^{-1}\mathbf{p}, \mathbf{0}) .$$

5M*th* row This comes from the boundary condition $(5.9.51)_2$. The rows of E and F are replaced respectively by the block forms

$$(\mathbf{0}, \mathbf{q}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})\,, \qquad (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})\,.$$

6M*th* row This comes from the boundary condition $(5.9.51)_2$. The rows of E and F are replaced respectively by the block forms

$$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{q}, \mathbf{0}, \mathbf{0}, \mathbf{0})\,, \qquad (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})\,.$$

7M*th* row This comes from the boundary condition $(5.9.51)_4$. The rows of E and F are replaced respectively by the block forms

$$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, -a\mathbf{q}, \mathbf{q})\,, \qquad (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})\,.$$

8M*th* row This comes from the boundary condition $(5.9.52)_4$. The rows of E and F are replaced respectively by the block forms

$$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, a\mathbf{p}, \mathbf{p})\,, \qquad (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})\,.$$

Appendix 4 shows the appropriate Fortran77 programs using NAG routine F02BJF to solve this problem using both first and second order systems.

5.10 Results

In his work on the influence of a uniform magnetic field on the onset of instability conducting fluid, Wilson [45] obtained a series of results using constant viscous and Magnetic Prandtl numbers $P_r = 1$ and $P_m = 1$ respectively.

The aim of this chapter is firstly, to investigate the effect of a vertical magnetic field on a layer of conducting fluid with $P_r = P_m = 1$; secondly, to compare the results obtained with those reported by Wilson, and finally to study the variations on the stability mode using various values of the Magnetic Prandtl number.

5.10.1 Non-Deformable Free Surface $C_r = 0$

Results and figures that are found when $C_r = 0$, $P_r = P_m = 1$ and for different values of parameters R_a, Q, M_a and B_o for purely buoyancy-driven and purely thermocapillary-driven convection in the case $N_u = 0$ and $N_u \longrightarrow \infty$ respectively are identcal to those reported by Wilson. Because of similarity, they are not mentioned in this subsection with some exceptions due to their relative results. By reducing the value of P_m from unity the results are found to be different from those reported by Wilson.

The curve representing the marginal stability in a plane $U = U(R_a, a)$, with the parameters P_r, P_m, Q, M_a, B_o and N_u, separates the plane into two parts. The part which is above the curve represents overstable modes whereas that below the curve represents stable modes. It should be noted that the simplest result is obtained with the parameters $P_r = 1$, $P_m = 1$, $Q = 0$, $M_a = 0$ and $B_o = 0$, but for the number N_u there are two cases, $N_u = 0$ or $N_u \longrightarrow \infty$.

The critical value of the Rayleigh number is the minimum of the corresponding marginal stability curve and is denoted by

$$R_c = R_c(P_r, P_m, M_a, Bo, N_u)$$

and a corresponding critical wavenumber is denoted by

$$a_c = a_c(P_r, P_m, M_a, Bo, N_u).$$

For example, the critical value of Rayleigh numbers are respectively $R_c = 669.00040$ and $R_c \longrightarrow 1100.6520$ at the corresponding wavenumbers $a_c = 2.08560$ and $a_c = 2.6820$ when $N_u = 0$ and $N_u \longrightarrow \infty$. The instability case generates real eigenvalues as long as $P_m{}^3 is \geq P_r$ as shown in second column of tables 5.1 and 5.2, while if $P_m d < P_r$, then the overstability case starts to generate the complex eigenvalues as shown in third and fourth columns of tables 5.1 and 5.2 when $P_m = 0.5$ and 5.1 and 5.2 when $P_m = 0.1$ for $N_u = 0$ and $N_u \longrightarrow \infty$ respectively. Briefly, the overstability case starts to appear in the third and fourth columns of the table 5.1 at $Q = 1438.450$, $R_c = 12494.474$ and $a_c = 5.156$ and at $Q = 29.769$, $R_c = 1248.138$ and $a_c = 2.567$ when $P_m = 0.5$ and $P_m = 0.1$ respectively for $N_u = 0$, and it is shown that in the third and fourth columns of the table 5.2 at $Q = 885.867$, $R_c \longrightarrow 13413.185$ and $a_c = 4.671$ and $Q = 18.330$, $R_c \longrightarrow 1471.840$ and $a_c = 2.732$ when $P_m = 0.5$ and $P_m = 0.1$ respectively for $N_u \longrightarrow \infty$.

It can be concluded that the complex eigenvalues begin to appear early as the value of P_m decreases. This is evident from figure 5.1 in which the curve begin to diverge from its path when the value of P_m is reduced.

[3]The Magnetic Prandtl numl:er in this prol lem is the reverse of the same numl:er in Benard Magnetic prol lem.

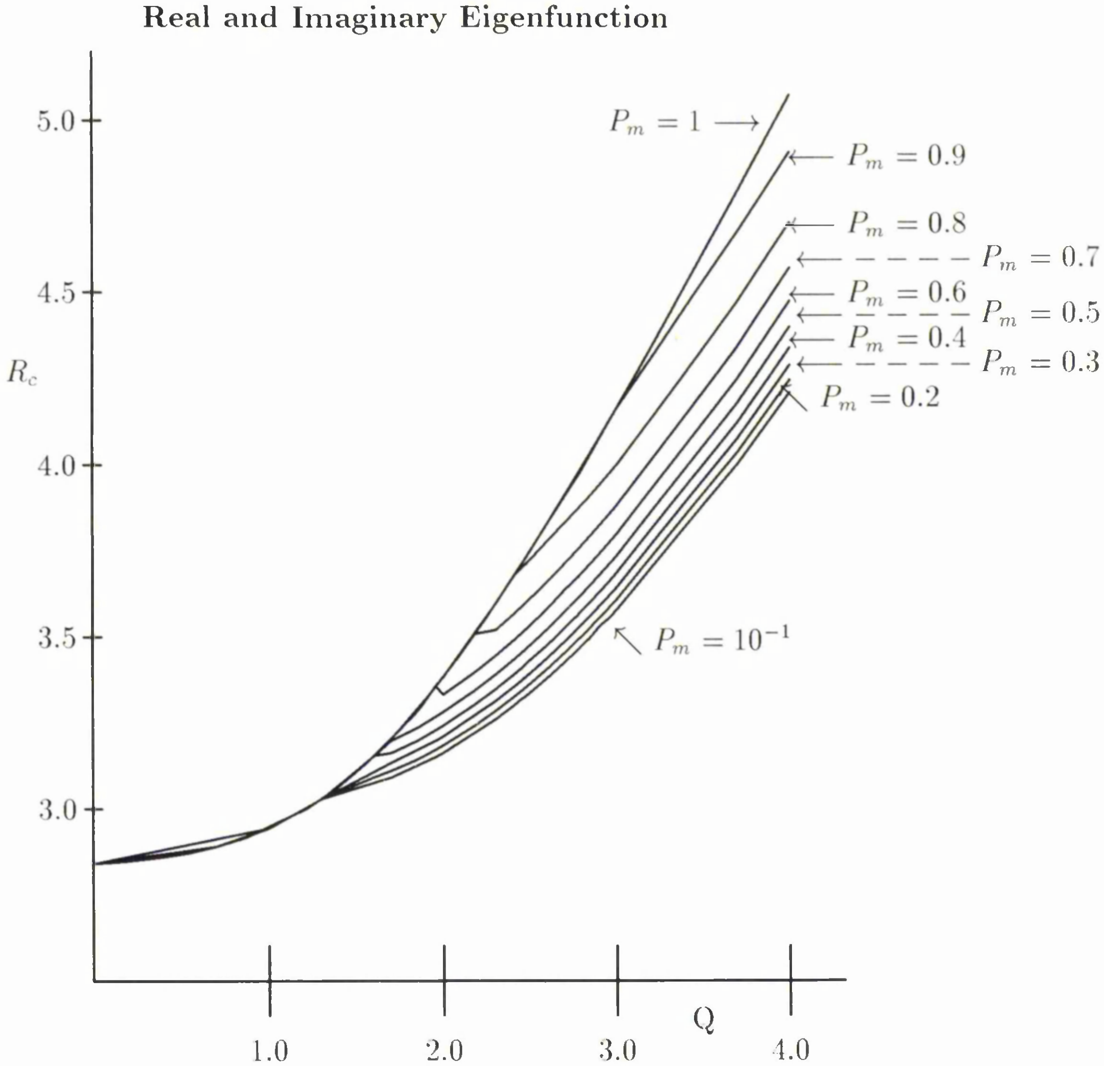


Figure 5.1: $Log_{10}R_a$ versus $Log_{10}Q$ with $N_u = B_o = 0$ for a range of $P_m = 0.1, 0.2, \ldots, 1$.

Also, the marginal stability curve in a plane $U = U(M_a, a)$, with the parameters P_r, P_m, Q, M_a, B_o and N_u, divides the plane into two parts. The unstable modes are represented above the curve, whereas the stable modes are represented below it. The critical value of the Marangoni number in this problem is the minimum of the corresponding marginal stability curve which is denoted by

$$M_c = M_c(P_r, P_m, R_a, Cr, Bo, N_u)$$

and a corresponding critical wavenumber is denoted by

$$a_c = a_c(P_r, P_m, R_a, Cr, Bo, N_u).$$

The result, which is very simple, occurs when the parameters take their values as $P_r = 1$, $P_m = 1$, $Q = 0$, $R_a = 0$ and $B_o = 0$, but for the number N_u there are two cases, $N_u = 0$

Critical Rayleigh No. R_c and Wavenumber a_c for various P_r and P_m but $C_r = 0$, $N_u = 0$						
	$P_r = 1.0$, $P_m = 1.0$		$P_r = 1.0$ $P_m = 0.5$		$P_r = 1.0$ $P_m = 0.1$	
Q	R_c	a_c	R_c	a_c	R_c	a_c
0.000	668.998	2.086	669.00	2.086	669.00	2.086
1.000	690.373	2.109	690.37	2.109	690.37	2.109
1.624	703.602	2.123	703.60	2.123	703.60	2.123
2.637	724.922	2.146	724.92	2.146	724.92	2.146
4.281	759.143	2.180	759.14	2.180	759.14	2.180
6.952	813.768	2.233	813.77	2.233	813.77	2.233
11.288	900.328	2.310	900.33	2.310	900.33	2.310
18.330	1036.32	2.420	1036.3	2.420	1036.3	2.420
29.764	1248.14	2.567	1248.1	2.567	1248.1*	2.567
48.329	1575.59	2.758	1575.6	2.758	1438.7*	2.402
78.476	2079.38	2.993	2079.4	2.993	1560.8*	2.531
127.43	2853.12	3.276	2853.1	3.275	1733.0*	2.697
206.91	4042.82	3.602	4042.8	3.602	1973.1*	2.901
335.98	5877.87	3.977	5438.0	3.982	2303.3*	3.141
545.56	8720.24	4.398	7016.5	4.340	2754.8*	3.416
885.87	13144.5	4.866	9260.4	4.729	3371.0*	3.723
1438.5	20062.7	5.383	12494.*	5.156	4215.2*	4.061
2335.7	30930.5	5.948	17216.*	5.628	5380.9*	4.433
3792.7	48074.6	6.552	24188.*	6.150	7007.1*	4.839
6158.5	75223.0	7.229	34589.*	6.727	9302.0*	5.285
10000.	118360.	7.949	50249.*	7.360	12580.*	5.773

Table 5.1: Critical Rayleigh and Wave numbers for various P_r, P_m when $C_r = N_u = 0$.

or $N_u \longrightarrow \infty$, for example, the critical Marangoni numbers as recovered by Pearson [39], are $M_c = 79.60669$ at $a_c = 1.99291$ when $Nu = 0$ and $\dfrac{M_c}{Nu} \longrightarrow 32.073$ at $a_c = 3.0141$ as $Nu \longrightarrow \infty$ as shown in a second column of a table 5.3. In this case, the value of P_m should be sufficiently smaller than the value of P_r so that the overstablitiy case begins to generate the complex eigenvalues. For example, when $P_m = 0.1$, the complex eigenvalues

Critical Rayleigh No. R_c and Wavenumber a_c for various P_r and P_m but $C_r = 0$, $N_u = 0$						
	$P_r = 1.0$, $P_m = 1.0$		$P_r = 1.0$ $P_m = 0.5$		$P_r = 1.0$ $P_m = 0.1$	
Q	R_c	a_c	R_c	a_c	R_c	a_c
0.000	1100.65	2.682	1100.65	2.682	1100.65	2.682
1.000	1127.50	2.710	1127.49	2.710	1127.50	2.710
1.624	1144.06	2.727	1144.06	2.727	1144.06	2.727
2.637	1170.68	2.753	1170.68	2.753	1170.68	2.753
4.281	1213.22	2.793	1213.22	2.793	1213.22	2.793
6.952	1280.66	2.855	1280.66	2.855	1280.66	2.855
11.288	1386.51	2.944	1386.51	2.944	1386.51	2.944
18.330	1550.70	3.069	1550.70	3.069	1471.84*	2.732
29.764	1802.36	3.236	1802.36	3.236	1516.39*	2.774
48.329	2184.26	3.447	2184.26	3.447	2184.26*	3.447
78.476	2760.23	3.705	2760.23	3.705	2760.23*	3.705
127.43	3628.19	4.006	3628.19	4.006	3544.79*	3.246
206.91	4835.73	4.350	4835.73	4.350	4234.13*	3.490
335.98	6919.70	4.735	6919.70	4.735	5242.26*	3.788
545.56	9947.77	5.160	9947.77	5.160	6712.45*	4.138
885.87	14602.0	5.626	13413.2*	4.671	8858.81*	4.539
1438.5	21805.8	6.134	18827.2*	5.175	12005.6*	4.988
2335.7	33027.9	6.686	25884.9*	5.665	16748.0*	5.487
3792.7	50612.5	7.285	37010.5*	6.238	23547.1*	6.035
6158.5	78310.2	7.933	53756.3*	6.865	33879.4*	6.634
10000.	122136.	8.636	79152.4*	7.548	49473.2*	7.288

Table 5.2: Critical Rayleigh and Wave numbers for various P_r, P_m when $C_r = 0$ and $N_u \longrightarrow \infty$.

appear at $Q = 48.329$, $M_c = 170.240$ and $a_c = 2.458$ as shown in a third column of table 5.3 while the eigenvalues are real when $P_m = 0.5$ for $N_u = 0$, and they are real for any value of P_m when $N_u \longrightarrow \infty$ as shown in a table 5.4.

The conditions of the onset of steady convection in the case $P_m = 1$ for $M_c{}^*$ and a_c

Critical Marangoni No. R_c and Wavenumber a_c for various P_r and P_m but $C_r = 0$, $N_u = 0$				
	$P_r = 1.0$, $P_m = 1.0$		$P_r = 1.0$ $P_m = 0.1$	
Q	M_c	a_c	M_c	a_c
0.000	79.607	1.993	79.607	1.993
1.000	82.172	2.015	82.172	2.015
1.624	83.759	2.028	83.759	2.028
2.637	86.315	2.049	86.315	2.049
4.281	90.412	2.081	90.412	2.081
6.952	96.940	2.130	96.940	2.130
11.288	107.25	2.202	107.25	2.202
18.330	123.38	2.303	123.38	2.303
29.764	148.33	2.440	148.33	2.440
48.329	186.54	2.616	170.24*	2.458
78.476	224.68	2.834	198.28*	2.612
127.43	332.94	3.095	239.87*	2.811
206.91	467.12	3.400	301.91*	3.055
335.98	672.08	3.753	395.05*	3.344
545.56	986.91	4.157	535.97*	3.679
885.87	1473.5	4.619	750.92*	4.061
1438.5	2230.1	5.147	1080.5*	4.489
2335.7	3412.4	5.748	1595.1*	4.971
3792.7	5269.2	6.432	2398.1*	5.580
6158.5	8196.8	7.210	3698.6*	5.831
10000.	12831.	8.091	5100.6*	6.127

Table 5.3: Critical Marangoni and Wave numbers for various P_r, P_m when $C_r = N_u = 0$.

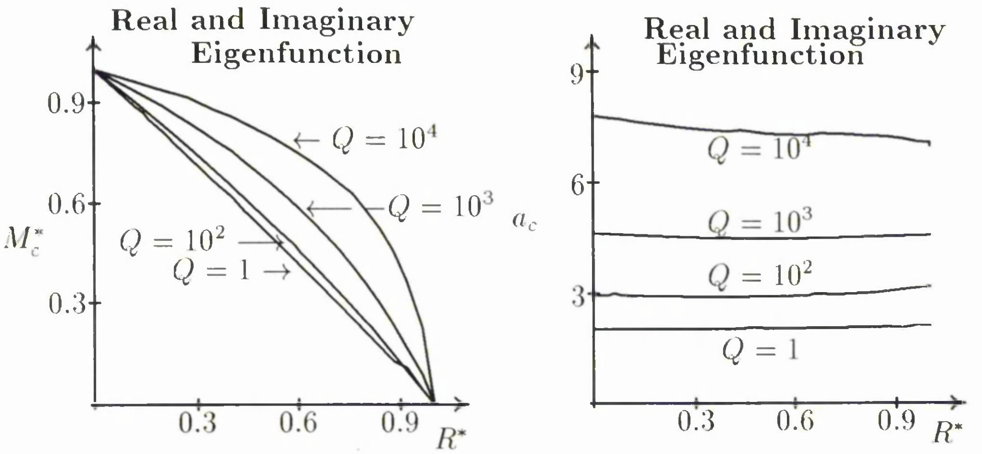
when $Nu = 0$ together with $M_c{}^*/N_u$ and a_c as $Nu \longrightarrow \infty$ plotted as function of R^* where M^* is the value M_a divided by the value of M_c when $R_a = 0$ and R^* is the value of R_a divided by R_c at $M_a = 0$. The resulting figures for $P_m < 1$ represent several relations. A relation between $M_c{}^*$ and R^*, as well as the corresponding a_c and R^* and the relation between $M_a{}^*/N_u$ and R^*, as well as the corresponding a_c and $R_a{}^*$ when $Q = 1$, $Q = 10^2$,

Critical Marangoni No. R_c and Wavenumber a_c for various P_r and P_m but $C_r = 0$, $N_u = 0$					
$P_r = 1.0$, $P_m = 1.0$					
Q	M_c	a_c	Q	M_c	a_c
0.000	32.073	3.014	127.43	79.553	5.5782.086
1.000	32.726	3.055	206.91	98.410	6.5172.109
1.624	33.127	3.080	335.98	123.37	7.7692.123
2.637	33.766	3.120	545.56	156.05	9.4692.146
4.281	34.776	3.182	885.87	198.35	11.802.180
6.952	36.353	3.278	1438.5	252.64	14.752.233
11.288	38.768	3.422	2335.7	322.02	18.432.310
18.330	42.381	3.631	3792.7	410.60	23.042.420
29.764	47.648	3.925	6158.5	523.73	28.802.567
48.329	55.121	4.328	10000.	668.22	36.002.402
78.476	65.476	4.867			

Table 5.4: Critical Marangoni and Wave numbers for various P_r, P_m when $C_r = 0$ and $N_u \longrightarrow \infty$.

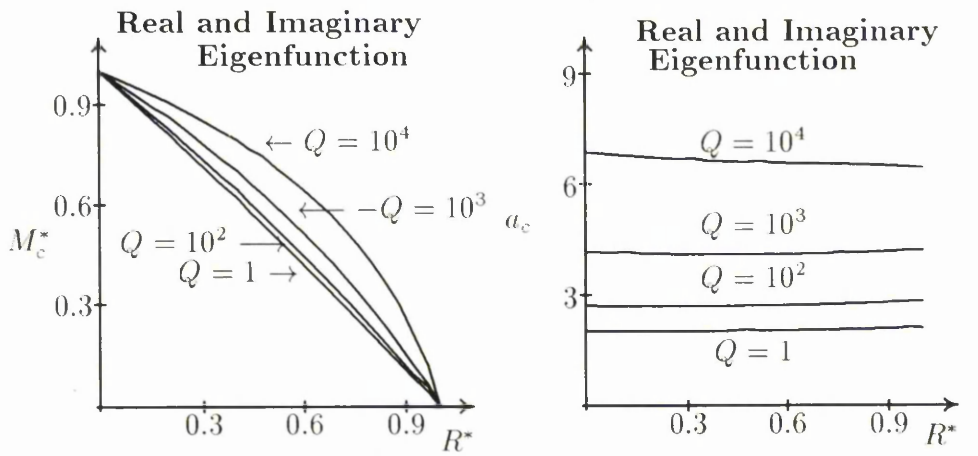
$Q = 10^3$ and $Q = 10^4$, the graphs in both cases $N_u = 0$ and $N_u \longrightarrow \infty$ are different from those obtained by Wilson for $P_m = 1$. This is mainly due to the overstability case, which generates complex eigenvalues. In the case $N_u = 0 when P_m = 0.5$ and 0.1 in figures 5.2 a and 5.3 a, the convexity of the curves representing the relation of M_c^* and R^* is more pronounced than that obtained by Wilson for $P_m = 1$ when $Q = 10^4$, while in figures 5.2 b and 5.3 b concavity of Wilson's curves representing the relation between a_c and R^* is more pronounced than those obtained by this thesis when $Q = 10^4$. In case $N_u \longrightarrow \infty$ when $P_m = 0.5$ and 0.1 the figures 5.4 a and 5.5 a representing the relation M_c^*/N_u versus R^*, are different from that obtained by Wilson's. The figures 5.4 b and 5.5 b which illustrates the relation between a_c and R^* show a value of a_c around 25 and 19 for $P_m = 0.5$ and $P_m = 0.1$ respectively while it showed around 40 in wilson's result when $Q = 10^4$.

There are many relations of Chandrasekhar [4] number Q with Rayleigh number R_a and wavenumber a. These relations take the forms as R_a/Q, $\dfrac{R_a - \pi^2 Q}{Q^{2/3}}$ and $\dfrac{a}{Q^{1/6}}$ with



a: $\mathrm{Log}_{10} M_c{}^*$ versus $\mathrm{Log}_{10} R^*$ b: $\mathrm{Log}_{10} a_c$ versus $\mathrm{Log}_{10} R^*$

Figure 5.2: $M_c{}^*$ and a_c are plotted as a function of $R_a{}^*$ for $Q = 1$, $Q = 10^2$, $Q = 10^3$ and $Q = 10^4$ when $N_u = 0$ and $P_m = 0.5$

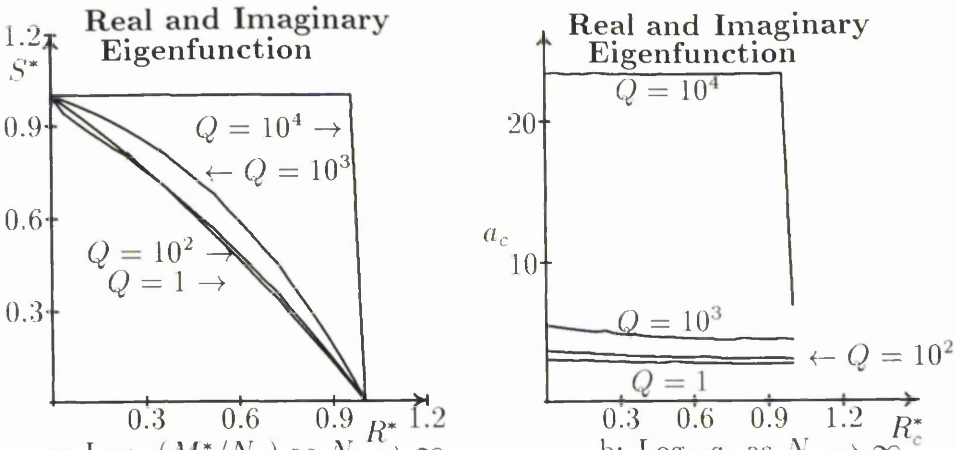


a: $\mathrm{Log}_{10} M_c{}^*$ versus $\mathrm{Log}_{10} R^*$ b: $\mathrm{Log}_{10} a_c$ versus $\mathrm{Log}_{10} R^*$

Figure 5.3: $M_c{}^*$ and a_c are plotted as a function of $R_a{}^*$ for $Q = 1$, $Q = 10^2$, $Q = 10^3$ and $Q = 10^4$ when $N_u = 0$ and $P_m = 0.1$.

Q. Graphs of these three relations for $P_m = 1$, $N_u = 1$, 10^2 and 10^4 are the same as those of Wilson's, but for $P_m = 0.5$ the lower part of curves are lower than those of Wilson's as illustrated in figures 5.6 *a* and 5.6 *b* and for $P_m = 0.1$ they are more lower as illustrated in figures 5.8 *a* 5.8 *b*. The lower part of the curves in figures 5.7 and 5.9 disappeared because the critical Rayleigh number in the overstablity case is less than the critical Raleigh number in the instability case beyond the values $Q = 127.428$ for $P_m = 0.5$ and $Q = 48.329$ for $P_m = 0.1$, therefore the value

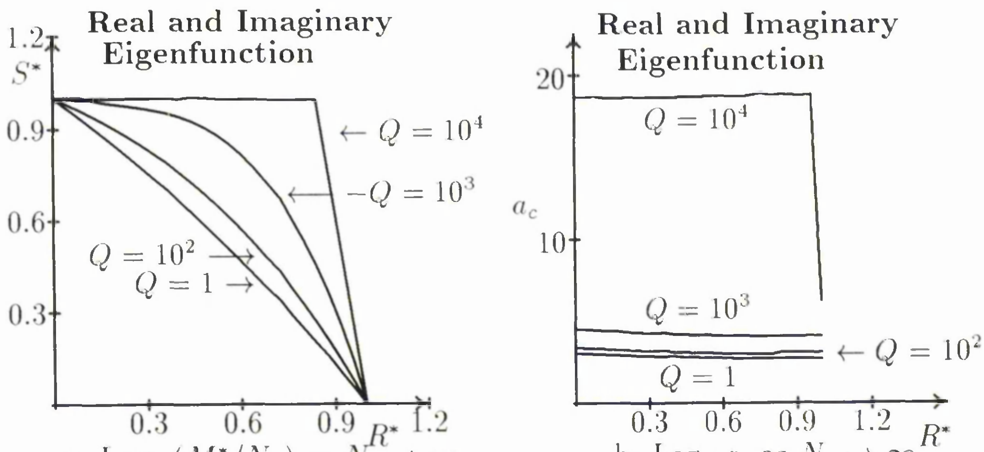
$$\frac{R_c - \pi^2 Q}{Q^{2/3}}$$



a: $\text{Log}_{10}(M_c^*/N_u)$ as $N_u \rightarrow \infty$ versus $\text{Log}_{10}R^*$

b: $\text{Log}_{10}a_c$ as $N_u \rightarrow \infty$ versus $\text{Log}_{10}R^*$

Figure 5.4: M_c^*/N_u and a_c are plotted as a function of R^* for $Q = 1$, $Q = 10^2$, $Q = 10^3$ and $Q = 10^4$ when $N_u \longrightarrow \infty$ and $P_m = 0.5$ and $S^* = \lim_{N_u \rightarrow \infty} M_c^*/N_u$.



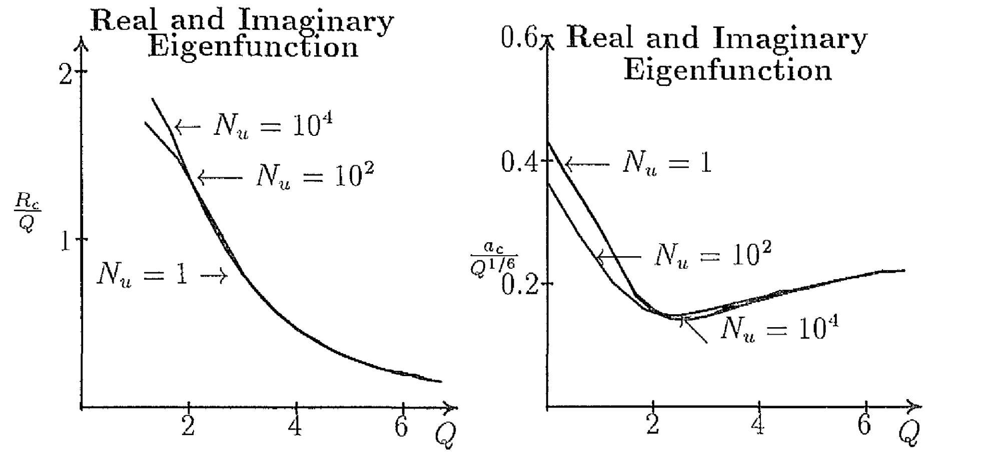
a: $\text{Log}_{10}(M_c^*/N_u)$ as $N_u \rightarrow \infty$ versus $\text{Log}_{10}R^*$

b: $\text{Log}_{10}a_c$ as $N_u \rightarrow \infty$ versus $\text{Log}_{10}R^*$

Figure 5.5: M_c^*/N_u and a_c are plotted as a function of R^* for $Q = 1$, $Q = 10^2$, $Q = 10^3$ and $Q = 10^4$ when $N_u \rightarrow \infty$ and $P_m = 0.1$ and $S^* = \lim_{N_u \rightarrow \infty} M_c^*/N_u$.

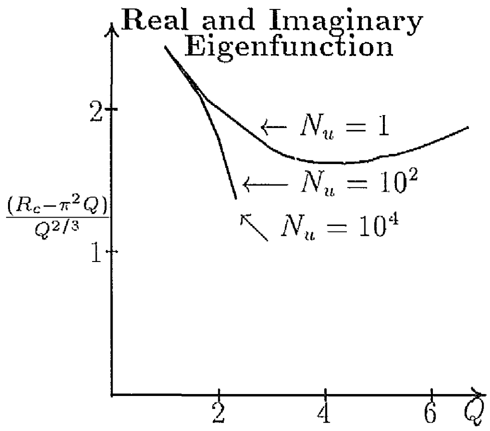
is less than zero for the overstability case while it is greater than zero for the instability case.

Also, there are many relations between Marangoni number M_a and the Chandrasekhar [4] number Q and there are relations between the wave number a and Chandrasekhar number Q. These relations take the forms as M_a/Q, $(M_a - Q)/Q^{3/4}$ and $a/Q^{1/4}$ with Q respectively. Graphs of these three relations for $P_m = 1$ and $N_u = 0$, 1, 5 and 10 are the same as those of Wilson's, while the lower part of the curves $P_m = 0.5$ are lower than those of Wilson's and for $P_m = 0.1$ are more lower as shown in figures 5.10 a, 5.10 b, 5.11,



a: $\text{Log}_{10}(R_c/Q)$ versus $\text{Log}_{10}Q$ b: $\text{Log}_{10}(a_c/Q^{1/6})$ versus $\text{Log}_{10}Q$.

Figure 5.6: R_c/Q and $a_c/Q^{1/6}$ are plotted as function of Q for $N_u = 1$, 10^2 and 10^4 when $P_m = 0.5$.



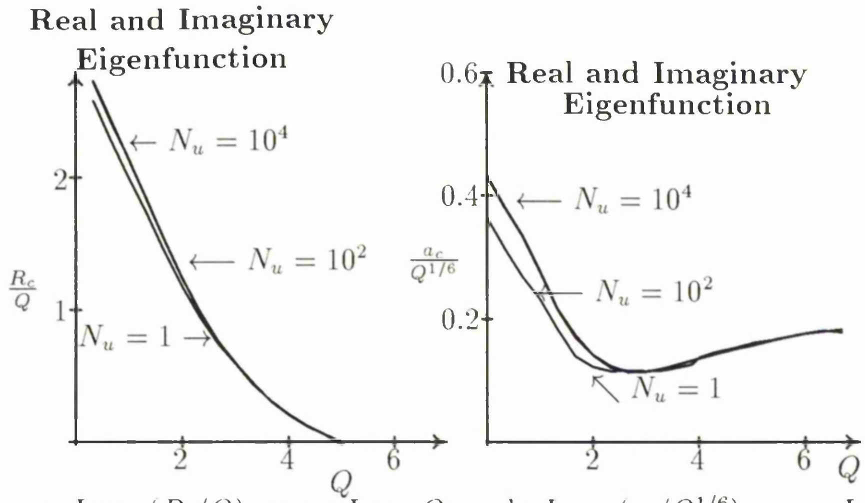
$\text{Log}_{10}((R_c - \pi^2 Q)/Q^{2/3})$ versus $\text{Log}_{10}Q$.

Figure 5.7: $(R_c - \pi^2 Q)/Q^{2/3}$ is plotted as function of Q for $N_u = 1$, 10^2 and 10^4 when $P_m = 0.5$.

5.12 a, 5.12 b and 5.13.

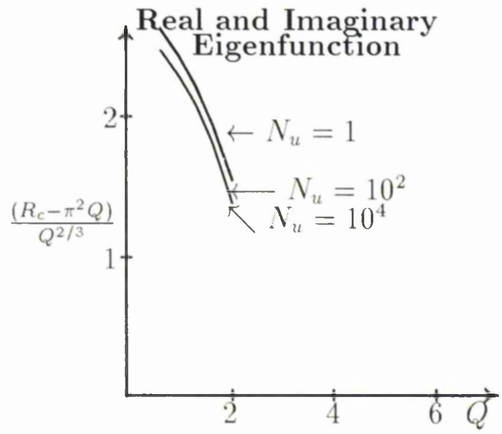
5.10.2 Deformable Free Surface

In the case of a deformable free surface with $C_r \neq 0$, the marginal stability curves are slightly different from those of Wilson for any value of P_m less than unity, even when $P_m = 1$ for most cases. Consider $C_r = 0$, 0.005, 0.01, 0.011 and 0.012 when $P_r = 1$, $P_m = 1$, $Q = 0$, B=1 and $N_u = 0$, as well as the convexity of curves of critical Marangoni number M_c and a corresponding wavenumber a that are plotted as having Rayleigh number R_a in figures 5.14 a and 5.14 b. These have the reverse convexity of those of



a: $\mathrm{Log}_{10}(R_c/Q)$ versus $\mathrm{Log}_{10}Q$ b: $\mathrm{Log}_{10}(a_c/Q^{1/6})$ versus $\mathrm{Log}_{10}Q$.

Figure 5.8: R_c/Q and $(a_c/Q^{1/6})$ are plotted as function of Q for $N_u = 1$, 10^2 and 10^4 when $P_m = 0.1$.

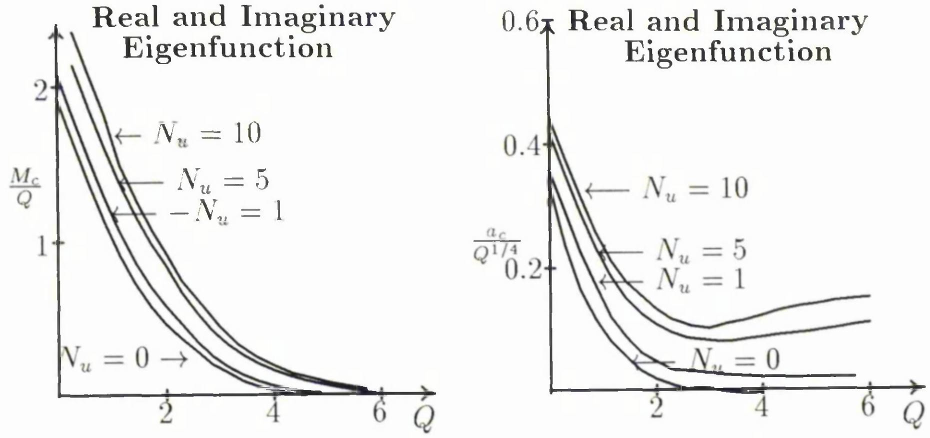


$\mathrm{Log}_{10}((R_c - \pi^2 Q)/Q^{2/3})$ versus $\mathrm{Log}_{10}Q$.

Figure 5.9: $(R_c - \pi^2 Q)/Q^{2/3}$ is plotted as function of Q for $N_u = 1$, 10^2 and 10^4 when $P_m = 0.1$.

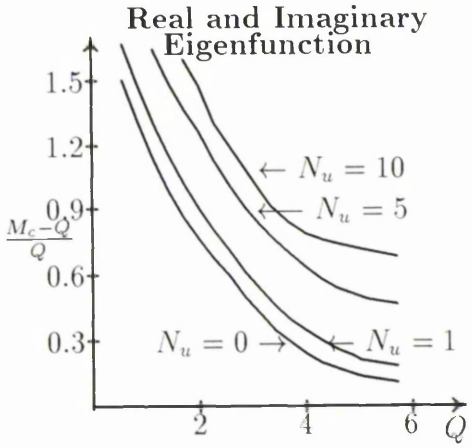
Wilson except for $C_r = 0$. Moreover, the curves for $C_r = 0.011$ and $C_r = 0.012$ change their direction at $M_c = 20$ (see figures 5.14 *a* and 5.14 *b*). However, when $P_m = 0.1$, figures 5.15 *a* and 5.15 *b* are the same as figures 5.14 *a* and 5.14 *b*, except that the curve for $C_r = 0.011$ does not change its direction. The effect of the Magnetic Prandtl number P_m is not remarkable in these cases because the value of Q is not large enough to cause the overstability case.

Figures 5.16 *a*, 5.16 *b*, 5.17 *a* and 5.17 *b* show the values of the critical Marangoni number M_c and the corresponding wavenumber a_c and are plotted as functions of Q when $R_a = 300$, $B_o = 1$ and $N_u = 0$ for the values $C_r = 0$, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3} and 10^{-2} when $P_m = 1$ and $P_m = 0.1$. For each of the figures 5.16 *a* and 5.17 *a*, there is a certain point



a: $\mathrm{Log}_{10}(M_c/Q)$ versus $\mathrm{Log}_{10}Q$ b: $\mathrm{Log}_{10}(a_c/Q^{1/4})$ versus $\mathrm{Log}_{10}Q$

Figure 5.10: M_c/Q and $a_c/Q^{1/4}$ are plotted as function of Q for $N_u = 0$, 1, 5 and 10 when $B_o = 0$ and $P_m = 0.5$.

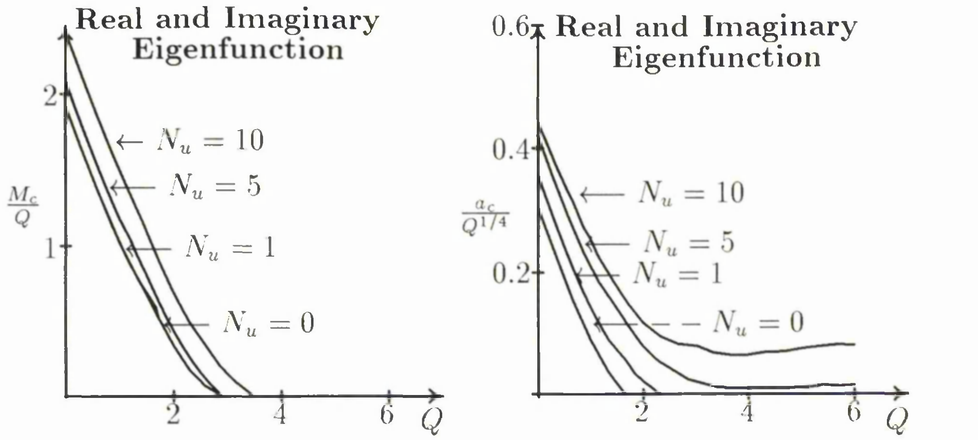


$\mathrm{Log}_{10}((M_c - Q)/Q^{3/4})$ versus $\mathrm{Log}_{10}Q$

Figure 5.11: $M_c/Q^{3/4}$ plotted as function of Q for $N_u = 0$, 1, 5 and 10 when $B_o = 0$ and $P_m = 0.5$. .

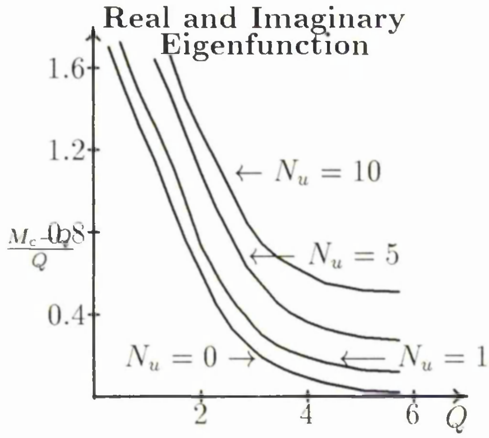
at which the curve changes direction right and then upward. These curves are different from Wilson's whereas in figures 5.16 b and 5.17 b they are the same.

The critical Marangoni number M_c and the wavenumber a_c are plotted as functions of C_r when $R_a = 300$, $B_o = 1$ and $N_u = 0$ for the values $Q = 1$, 10^2, and 10^4 when $P_r = 1$ and $P_m = 1$. The results in figures 5.18 a and 5.18 b are the same as those of Wilson, while the results are different from those of Wilson for $P_m = 0.1$, as C_r increases. The value of M_c increases when $Q = 10^4$ only; otherwise it decreases when $Q = 1$ and 10^2. Moreover, the parts of curves parallel to the C_r-axes in 5.19 a and 5.19 b. are in the opposite direction to those in figures 5.18 a and 5.18 b when $P_m = 1$



a: $\text{Log}_{10}(M_c/Q)$ versus $\text{Log}_{10}Q$ b: $\text{Log}_{10}(a_c/Q^{1/4})$ versus $\text{Log}_{10}Q$

Figure 5.12: M_c/Q and $a_c/Q^{1/4}$ are plotted as function of Q for $N_u = 0, 1, 5$ and 10 when $B_o = 0$ and $P_m = 0.1$.

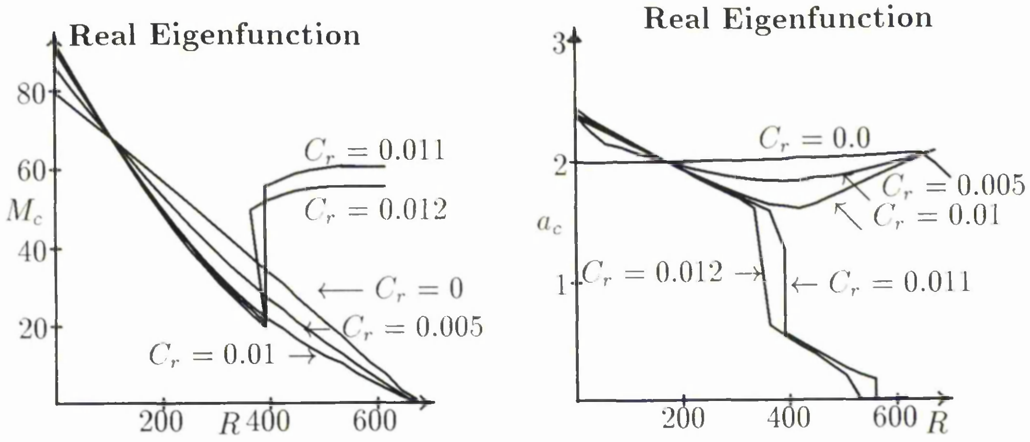


$\text{Log}_{10}((M_c - Q)/Q^{3/4})$ versus $\text{Log}_{10}Q$

Figure 5.13: $M_c/Q^{3/4}$ plotted as function of Q for $N_u = 0, 1, 5$ and 10 when $B_o = 0$ and $P_m = 0.1$. .

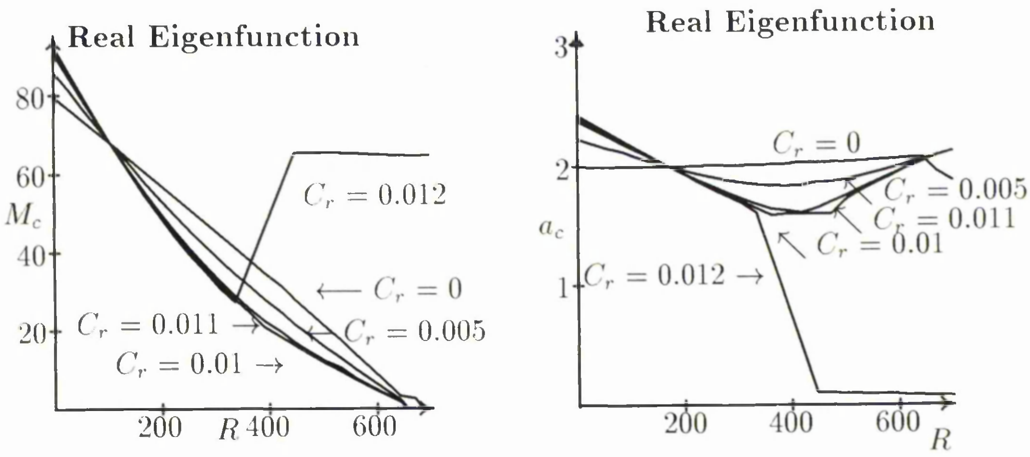
Figurs 5.20 a, 5.20 b, 5.21 a and 5.21 b show that the critical values of R_c and corresponding wavenumber a_c are plotted as functions of Q when $M = 25$, $B_o = 1$ and $N_u = 0$ at the values $C_r = 1, 10^{-1}, 10^{-2}$ and 10^{-3} and for $P_r = 1$ and $P_m = 1$. As Q increases, the values of R_c and a_c remain constant up to certain points depending on the value of C_r then they start to increase as shown in figures 5.20 a and 5.20 b and they are different from Wilson's, i.e. the curve for $C_R = 1$ is lower than the curve for $C_r = 10^{-3}$ which is the opposite to those of Wilson's. Now for $P_m = 0.1$, curves of figures 5.21 a and 5.21 b are totally different from those in 5.20 a and 5.20 b.

Figures 5.22 a, 5.22 b, 5.23 a and 5.23 b display critical the Rayleigh number R_c and a corresponding wavenumber a_c which are plotted as functions of C_r when $M = 25$, $B_o = 1$



a: M_c versus R_a b: a_c versus R_a.

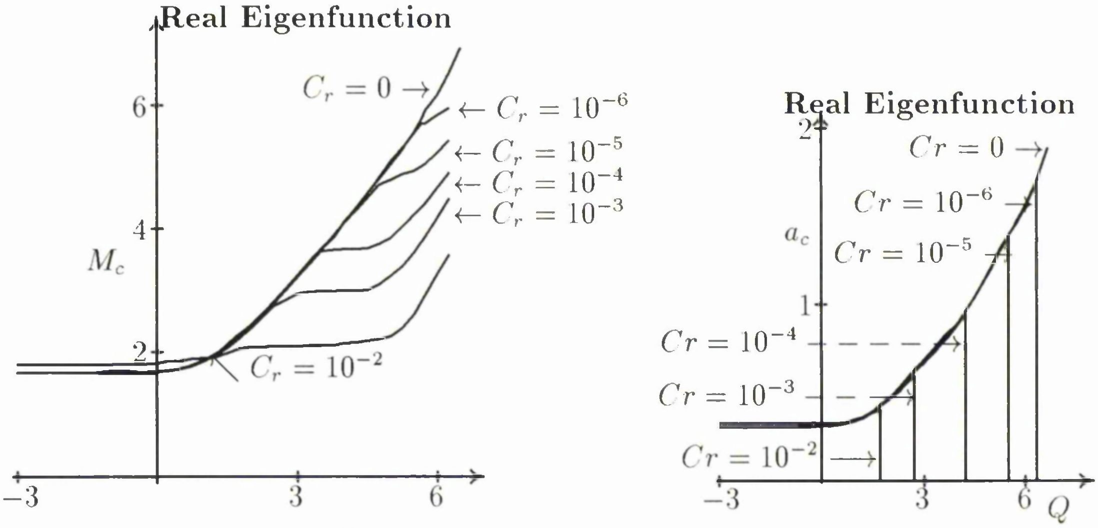
Figure 5.14: M_c and a_c are plotted as function of R_a for C_r = 0, 0.005, 0.1, 0.011 and 0.012 when $Q = 0$, $B_o = 1$, $N_u = 0$ and $P_m = 1$.



a: M_c versus R_a b: a_c versus R_a.

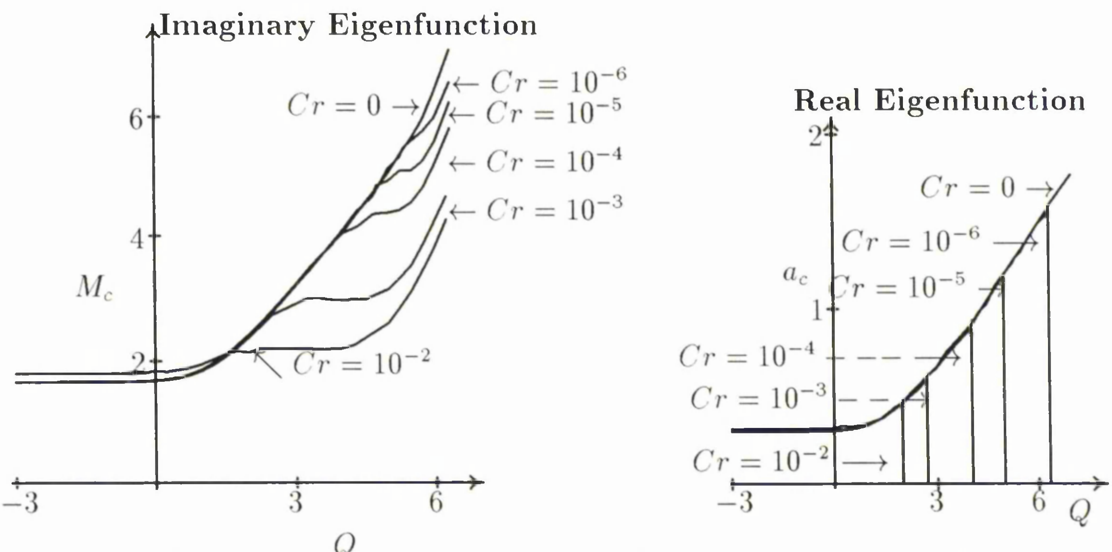
Figure 5.15: M_c and a_c are plotted as function of R_a for C_r = 0, 0.005, 0.1, 0.011 and 0.012 when $Q = 0$, $B_o = 1$, $N_u = 0$ and $P_m = 0.1$.

and $N_u = 0$ with the values $Q = 1$, 10^2, 10^4 and $Q = 10^6$ for $P_r = 1$ and $P_m = 0.1$ and $P_m = 1$. These figures are totally different from those of Wilson. At certain points in these figures, the values of R_c and a_c decrease while, in Wilson's figures they increase as the value of c_r increases.



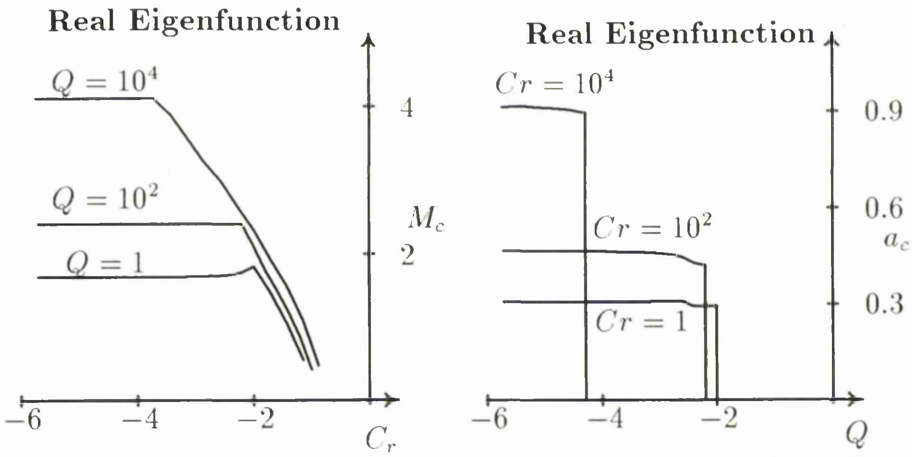
a: $\text{Log}_{10} M_c$ versus $\text{Log}_{10} Q$ b: $\text{Log}_{10} a_c$ versus $\text{Log}_{10} Q$.

Figure 5.16: M_c and a_c are plotted as function of Q for $Cr = 0$, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3} and 10^{-2} when $R_a = 300$, $B_o = 1$, $N_u = 0$ and $P_m = 1$.



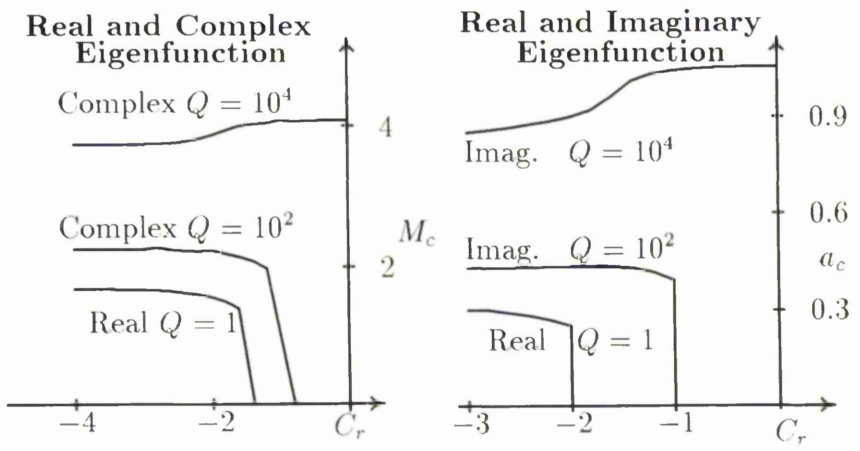
a: $\text{Log}_{10} M_c$ versus $\text{Log}_{10} Q$ b: $\text{Log}_{10} a_c$ versus $\text{Log}_{10} Q$

Figure 5.17: M_c and a_c are plotted as function of Q for $Cr = 0$, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3} and 10^{-2} when $R_a = 300$, $B_o = 1$, $N_u = 0$ and $P_m = 0.1$.



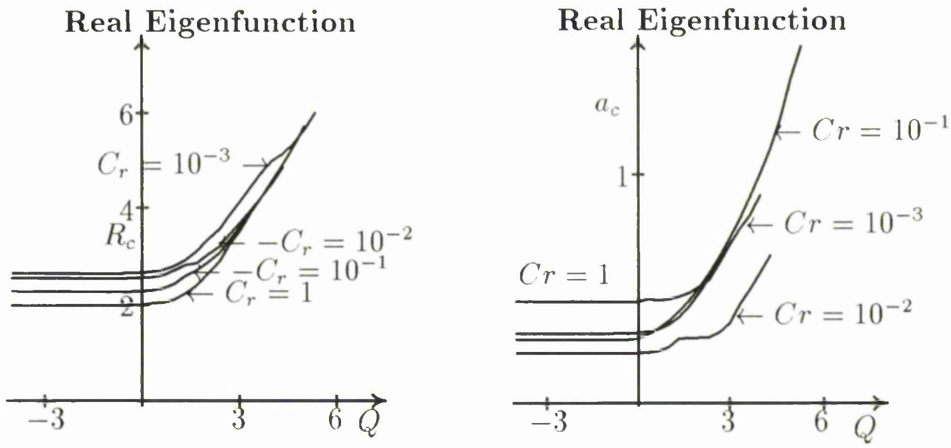
a: $\text{Log}_{10}M_c$ versus $\text{Log}_{10}C_r$ b: $\text{Log}_{10}a_c$ versus $\text{Log}_{10}C_r$

Figure 5.18: M_c and a_c are plotted as function of C_r for $Q = 1$, 10^2 and 10^4 when $R_a = 300$, $B_o = 1$, $N_u = 0$ and $P_m = 1$.



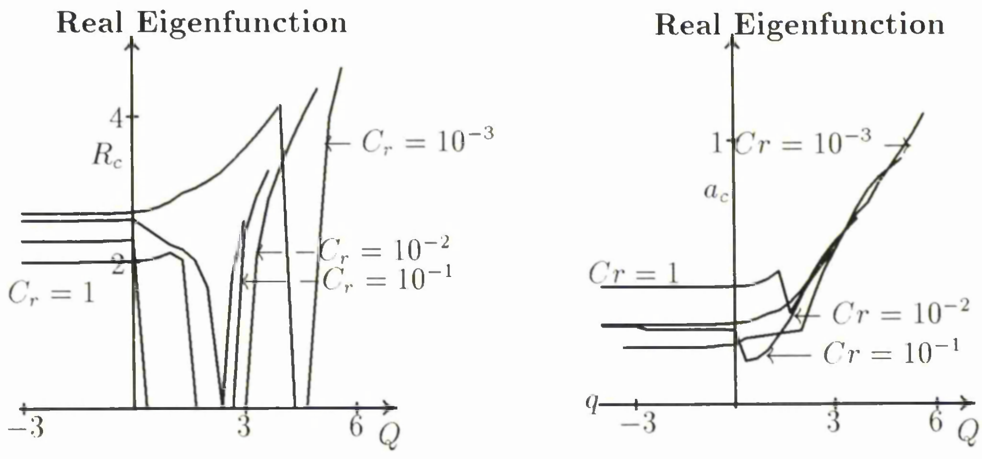
a: $\text{Log}_{10}M_c$ versus $\text{Log}_{10}C_r$ b: $\text{Log}_{10}a_c$ versus $\text{Log}_{10}C_r$

Figure 5.19: M_c and a_c are plotted as function of C_r for $Q = 1$, 10^2 and 10^4 when $R_a = 300$, $B_o = 1$, $N_u = 0$ and $P_m = 0.1$.



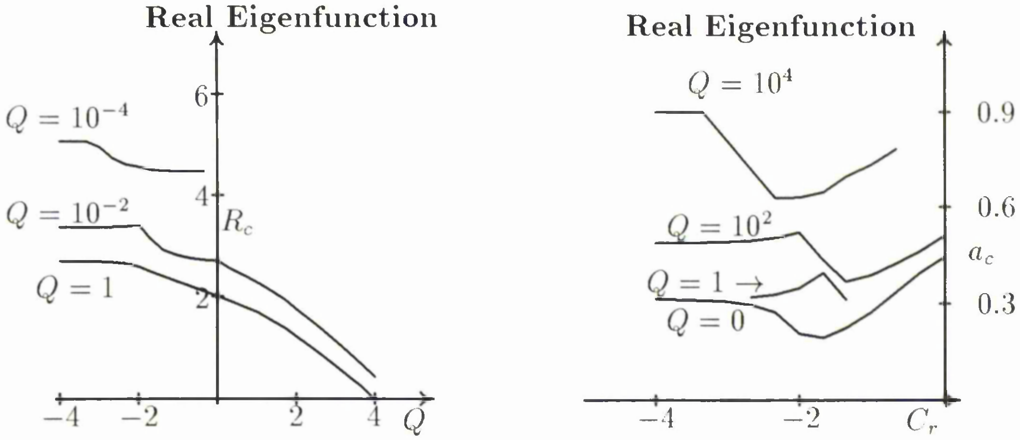
a: $\mathrm{Log}_{10} R_c$ versus $\mathrm{Log}_{10} Q$ b: $\mathrm{Log}_{10} a_c$ versus $\mathrm{Log}_{10} Q$

Figure 5.20: R_c and a_c are plotted as function of Q for $C_r = 10^{-2}$, 10^{-3}, 10^{-4}, and 0 when $M_a = 25$, $B_o = 1$, $N_u = 0$ and $P_m = 1$.



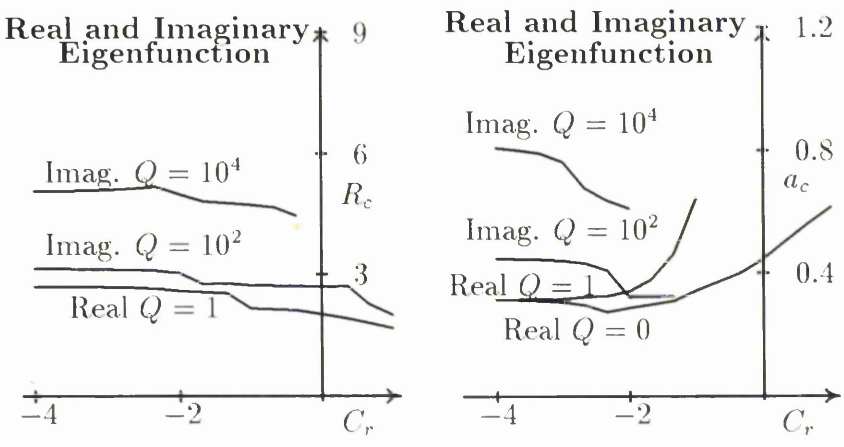
a: $\mathrm{Log}_{10} R_c$ versus $\mathrm{Log}_{10} Q$ b: $\mathrm{Log}_{10} R_c$ versus $\mathrm{Log}_{10} Q$

Figure 5.21: R_c and a_c are plotted as function of Q for $C_r = 10^{-2}$, 10^{-3}, 10^{-4}, and 0 when $M_a = 25$, $B_o = 1$, $N_u = 0$ and $P_m = 0.1$.



a: $\text{Log}_{10}R_c$ versus $\text{Log}_{10}C_r$ b: $\text{Log}_{10}R_c$ versus $\text{Log}_{10}Q$

Figure 5.22: R_c and a_c are plotted as function of Q for $C_r = 10^{-2}, 10^{-3}, 10^{-4}$, and 0 when $M_a = 25$, $B_o = 1$. $N_u = 0$ and $P_m = 1$.



a: $\text{Log}_{10}R_c$ versus $\text{Log}_{10}Cr$ b: $\text{Log}_{10}R_c$ versus $\text{Log}_{10}Q$

Figure 5.23: R_c and a_c are plotted as function of Cr for $Q = 1, 10^2, 10^4$ when $M_a = 25$, $B_o = 1$, $N_u = 0$ and $P_m = 0.1$.

Chapter 6

Eigenvalue Determination using Spectral Methods for Multi-Layers

6.1 Introduction

This chapter extends spectral methods for single layers into those for multiple layers. The analysis deals with two layers only but it is clear that the methodology expands to many layers. A three layer problem in Magnetohydrodynamics is illustrated later.

6.2 Stability Analysis of Two Layers

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two layers, the first one stacked on the second, so that the bottom of $\mathcal{L}_1$ and the top of $\mathcal{L}_2$ form a common interface, say $\mathcal{L}$. Each layer contains a different continuum but they interact thermally, mechanically and magnetically with each other across $\mathcal{L}$ and with the world outside across their outer boundaries. The equations describing the physical problem are now non-dimensionalised so that the upper and lower layers are mapped into $-1 \leq z_1 \leq 1$ and $-1 \leq z_2 \leq 1$ respectively. Thus $z_1 = 1$ is the upper boundary of the top layer, $z_2 = -1$ is the lower boundary of the bottom layer and $z_1 = -1$, $z_2 = 1$ both denote the interface between the two layers. The standard linear stability problem for this configuration can be systematically reduced to the eigenvalue problem

$$\beta_1 \frac{dY_1}{dz_1} = \frac{1}{\alpha_1} A_1 Y_1 + \frac{\sigma}{\alpha_1} B_1 Y_1 \, , \qquad \beta_2 \frac{dY_2}{dz_2} = \frac{1}{\alpha_2} A_2 Y_2 + \frac{\sigma}{\alpha_2} B_2 Y_2 \tag{6.2.1}$$

where β_1 and β_2 are time scales, α_1 and α_2 are length scales for the $\mathcal{L}_1$ and $\mathcal{L}_2$ layers respectively, Y_1 is an m vector with components $y_{u_1}, \ldots, y_{u_m}$, Y_2 is an n vector with components $y_{l_1}, \ldots, y_{l_n}$, A_1 and B_1 are complex $m \times m$ matrices, A_2 and B_2 are complex $n \times n$ matrices and σ is the eigenvalue to be determined. In effect, the governing equations in layers $\mathcal{L}_1$ and $\mathcal{L}_2$ have order m and n respectively and equations (6.2.1) are the representation of these equations as a first order system. To complete the eigenvalue problem, equations (6.2.1) must be supplemented by $m - s$ boundary conditions on $z_1 = 1$, $n - r$ boundary conditions on $z_2 = -1$ and $r + s$ boundary conditions on $z_1 = -1$, $z_2 = 1$. In practice, these boundary conditions relate to the physical properties of macroscopic quantities such as stress, velocity, temperature etc. These conditions appear in pairs; one for the upper boundary and one for the lower boundary. The order of the systems describing layers $\mathcal{L}_1$ and $\mathcal{L}_2$ is almost invariably even with $s = m/2$, $r = n/2$. Notice also that the boundary conditions can contain the eigenvalue σ, for example. This happens in the Calculus of Variations when transversality conditions are operative. Boundary conditions are normally based on thermal, mechanical, magnetic and other properties of each boundary and have the general form:

Upper boundary ($z_1 = 1$) These conditions describe the interaction of $\mathcal{L}_1$ with the exterior region $z_1 > 1$ and are linear, involving only combinations of the components of Y_1. In matrix notation, they can be expressed as:

$$U_k^T Y_1 = 0 \,, \qquad 1 \leq k \leq m - s \tag{6.2.2}$$

where U_k are a family of m-vectors with constant entries.

Interface boundary ($z_1 = -1$, $z_2 = 1$) These conditions describe the interaction between $\mathcal{L}_1$ and $\mathcal{L}_2$ and are also linear in nature but now connect the components of Y_1 and Y_2. In matrix notation, they can be expressed as:

$$P_k^T Y_1 + Q_k^T Y_2 = 0 \,, \qquad 1 \leq k \leq r + s \tag{6.2.3}$$

where P_k are a family of m-vectors, Q_k are a family of n-vectors, both with constant entries.

Lower boundary ($z_2 = -1$) These conditions describe the interaction of $\mathcal{L}_2$ with the exterior region $z_2 < -1$ and are linear in nature, involving only combinations of the components of Y_2. In matrix notation, they can be expressed as:

$$L_k^T Y_2 = 0 \;, \qquad 1 \leq k \leq n - r \tag{6.2.4}$$

where L_k are a family of n-vectors with constant entries.

6.2.1 The Extended Eigenvalue Problem

It is self evident that the structure of the eigenvalue problem (6.2.1) is qualitatively the same for each layer and so it is sensible to incorporate them both into a single problem. Let complex $N \times N$ matrices A and B and $N \times 1$ vector Y be defined by

$$\begin{gathered} Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} , \qquad A = \begin{bmatrix} \dfrac{1}{\alpha_1} A_1 & 0 \\ 0 & \dfrac{1}{\alpha_2} A_2 \end{bmatrix} , \quad B = \begin{bmatrix} \dfrac{1}{\alpha_1} B_1 & 0 \\ 0 & \dfrac{1}{\alpha_2} B_2 \end{bmatrix} , \\ S = \begin{bmatrix} \beta_1 I & 0 \\ 0 & \beta_2 I \end{bmatrix} , \end{gathered} \tag{6.2.5}$$

where $N = n + m$ and I is an identity matrix.

$$S \frac{dY}{dz} = AY + \sigma BY \;, \qquad z \in [-1, 1] \,. \tag{6.2.6}$$

Similarly, the boundary conditions (6.2.2), (6.2.4) and (6.2.3) can be recast in the simpler form

$$C_k^T Y = 0 \quad 1 \leq k \leq N \tag{6.2.7}$$

where the interpretation of C_k is

$$C_k = \begin{cases} [U_k^T, 0] & 1 \leq k \leq m - s \;, \\ [P_j^T, Q_j^T] & j = k - m + s \;, \quad m - s + 1 \leq k \leq m + r \;, \\ [0, L_j^T] & j = k - m - r \;, \quad m + r + 1 \leq k \leq N \,. \end{cases}$$

The treatment of the boundary value problem now proceeds similarly to that in the single layer situation for N independent variables with the appropriate interpretation of Y.

Chapter 7

Convection in a Horizontal Porous Layer Superposed by a Fluid Layer

7.1 Introduction

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two horizontal layers such that the bottom of the layer $\mathcal{L}_1$ touches the top of the layer $\mathcal{L}_2$. A right handed system of Cartesian coordinates (x_i $i = 1, 2, 3$) is chosen so that the interface is the plane $x_3 = 0$, the top boundary of $\mathcal{L}_1$ is $x_3 = d_f$ and the lower boundary of $\mathcal{L}_2$ is $x_3 = -d_m$. Suppose that the upper layer $\mathcal{L}_1$ is filled with an incompressible viscous fluid whereas the lower layer $\mathcal{L}_2$ is occupied by a porous medium permeated by the fluid. Gravity acts in the negative x_3 direction and the porous medium is heated at its lower boundary. Convection takes place in which temperature driven buoyancy effects are damped by viscous effects. A stationary fluid with a thermal gradient in the x_3 direction (the so-called "conduction solution") is one possible solution to this problem and so it is natural to investigate its stability. This question has recently been addressed by Chen [5] who derived the appropriate equations.

Briefly, the fluid flow in the porous layer, with thickness d_m, is governed by Darcy's law, whereas the fluid flow in the upper layer $\mathcal{L}_2$,with thickness d_f, is governed by the Navier-Stokes equations. Convection is driven by the temperature dependence of the fluid density. Typically, the Oberbeck-Boussinesq approximation is made in which concepts like local thermal equilibrium, heating from viscous dissipation, radiative effects etc. are ignored as are variations in fluid density except where they occur in the momentum equation. Let T denote the Kelvin temperature of the fluid and T_0 be a constant reference Kelvin temperature. Then for the purpose of this work, the fluid density ρ_f is related to T by

$$\rho_f = \rho_0[1 - \alpha(T - T_0)] , \tag{7.1.1}$$

where ρ_0 is the density of the fluid at T_0 and α (supposed constant) is the coefficient of volume expansion of the fluid. In many situations (7.1.1) is inadequate. For example, the description of water[1] around 4°K. However, the objective here is to emulate the work of Chen [5].

7.2 The Governing Equations of Natural Convection

The field equations for this problem are written separately for the porous medium and overlying fluid layer. The governing equations for porous medium are represented by

$$\begin{aligned} \frac{\rho_0}{\phi}\frac{\partial \mathbf{V}_m}{\partial t} &= -\nabla P_m - \frac{\mu}{\kappa}\mathbf{V}_m + \rho_f \mathrm{g} , \\ (\rho c)_m \frac{\partial T_m}{\partial t} + (\rho c_p)_f \mathbf{V}_m . \nabla T_m &= k_m \nabla^2 T_m \end{aligned} \tag{7.2.2}$$

where T_m is the Kelvin temperature of the porous medium, $\mathbf{V}_m$ is the solenoidal seepage velocity, P_m is the hydrostatic pressure, μ is the dynamic viscosity of the fluid, K is the permeability of the porous substrate, ϕ is its porosity, k_m is the overall thermal conductivity of the porous medium, $(\rho c_p)_f$ is the heat capacity per unit volume of the fluid at constant pressure and $(\rho c)_m$ is the overall heat capacity per unit volume of the porous medium at constant pressure. In fact,

$$(\rho c)_m = \phi(\rho c_p)_f + (1 - \phi)(\rho c_p)_m$$

where $(\rho c_p)_m$ is the heat capacity per unit volume of the porous substrate. The governing equations for the fluid layer are

$$\begin{aligned} \rho_0\Big(\frac{\partial \mathbf{V}_f}{\partial t} + \mathbf{V}_f . \nabla \mathbf{V}_f\Big) &= -\nabla P_f + \mu \nabla^2 \mathbf{V}_f + \rho_f \mathrm{g} \\ (\rho c_p)_f \Big(\frac{\partial T_f}{\partial t} + \mathbf{V}_f . \nabla T_f\Big) &= k_f \nabla^2 T_f \end{aligned} \tag{7.2.3}$$

where T_f is the Kelvin temperature of the fluid layer, $\mathbf{V}_f$ is solenoidal fluid velocity, P_f is the hydrostatic pressure and k_f is the thermal conductivity of the fluid.

[1] George et. al. [11] describes convection in lakes in which the bottom can be represented by a porous layer which is under-pinned by an impermeable permafrost boundary.

7.3 Boundary Conditions

The convection problem is completed by the specification of boundary conditions at the upper surface of the viscous fluid layer, at the interface between the fluid and porous medium layers and at the lower boundary of the porous medium layer. Many combinations of boundary conditions are possible but for comparison with Chen [5], $x_3 = d_f$ is assumed to be rigid and held at constant temperature T_u, whereas $x_3 = -d_m$ is assumed to be impenetrable and at constant temperature T_l. In terms of w_f and w_m, the axial velocity components of the fluid in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, these requirements lead to the three conditions:

$$T_f(d_f) = T_u \ , \qquad w_f(d_f) = 0 \ , \qquad \frac{\partial w_f(d_f)}{\partial x_3} = 0 \ , \tag{7.3.4}$$

on the top boundary $\mathcal{L}_1$ and the two conditions

$$T_m(-d_m) = T_l \ , \qquad w_m(-d_m) = 0 \ , \tag{7.3.5}$$

on the lower boundary of $\mathcal{L}_2$. Strictly speaking, the rigid boundary condition on $x_3 = d_f$ is $\boldsymbol{v}_f = \mathbf{0}$; the format (7.3.4) specifically uses the fact that $\boldsymbol{v}_f$ is solenoidal (incompressiblity constraint). The fluid/porous-medium interface boundary conditions are based on the assumption that temperature, heat flux, normal fluid velocity and normal stress are continuous so that

$$\begin{aligned} T_m(0) &= T_f(0) \ , & k_m \frac{\partial T_m(0)}{\partial x_3} &= k_f \frac{\partial T_f(0)}{\partial x_3} \ , \\ w_m(0) &= w_f(0) \ , & -P_f(0) + 2\mu \frac{\partial w_f(0)}{\partial x_3} &= -P_m(0) \end{aligned} \tag{7.3.6}$$

respectively. This leaves one final condition to be specified on the interface. Several possible forms[2] have been proposed for the missing condition but the most popular of these is undoubtedly due to Beavers and Joseph [2] who suggest that

$$\frac{\partial u_f}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}}(u_f - u_m), \qquad \frac{\partial v_f}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}}(v_f - v_m) \ , \tag{7.3.7}$$

where u_f, v_f are the limiting tangential components of the fluid velocity as the interface is approached from the fluid layer $\mathcal{L}_1$, whereas u_m, v_m are the same limiting components of

[2] Jones [23] proposes continuity of shear stress at the interface . In truth, the nature of this boundary condition has little impact on results under most circumstances.

tangential fluid velocity as the interface is approached from the porous layer $\mathcal{L}_2$. Clearly, discontinuities in shear velocity across the interface are inherent in this specification of the last boundary condition. Equations (7.2.2), (7.2.3) together with boundary conditions (7.3.4), (7.3.5), (7.3.6) and (7.3.7) possess a static (equilibrium) solution in which the fluid is stationary everywhere and heat is conducted across the layers in accordance with the thermal boundary conditions. Specifically,

$$\boldsymbol{V}_f = \boldsymbol{0} \,, \qquad \boldsymbol{V}_m = \boldsymbol{0} \,,$$

and the static temperature and hydrostatic pressure fields satisfy the equations

$$-\nabla P_m + \rho_f \mathrm{g} = \boldsymbol{0} \,, \qquad -\nabla P_f + \rho_f \mathrm{g} = \boldsymbol{0} \,, \qquad \nabla^2 T_m = \nabla^2 T_f = 0 \,, \tag{7.3.8}$$

together with the exterior boundary conditions

$$T_f(d_f) = T_u \,, \qquad T_m(-d_m) = T_l \,, \tag{7.3.9}$$

and the interfacial conditions

$$T_m(0) = T_f(0) \,, \qquad k_m \frac{\partial T_m(0)}{\partial x_3} = k_f \frac{\partial T_f(0)}{\partial x_3} \,, \qquad P_f(0) = P_m(0) \,. \tag{7.3.10}$$

In conclusion, it follows almost immediately that the equilibrium temperature fields in the fluid and porous medium are respectively

$$\begin{aligned} T_f &= T_0 - (T_0 - T_u)\frac{x_3}{d_f} & 0 \le x_3 \le d_f \,, \\ T_m &= T_0 - (T_l - T_0)\frac{x_3}{d_m} & -d_m \le x_3 \le 0 \,, \end{aligned} \tag{7.3.11}$$

where T_0 is the temperature on $\mathcal{L}$ and is determined by the continuity of heat flux across $x_3 = 0$.

7.4 Perturbed Equations

Suppose that the static equilibrium solution is now perturbed so that the velocity, pressure and temperature fields in the fluid and porous layers are respectively

$$\boldsymbol{v}_f \,, \qquad P_f + p_f \,, \qquad T_0 - (T_0 - T_u)\frac{x_3}{d_f} + \theta_f \,, \tag{7.4.12}$$

and

$$\boldsymbol{v}_m \,, \qquad P_m + p_m \,, \qquad T_0 - (T_l - T_0)\frac{x_3}{d_m} + \theta_m \,. \tag{7.4.13}$$

Taking account of the properties of the equilibrium solution, it follows from the general field equations (7.2.3) and (7.2.2) that $\boldsymbol{v}_f$, p_f and θ_f satisfy

$$\begin{aligned} \rho_0\Big(\frac{\partial \boldsymbol{v}_f}{\partial t} + \boldsymbol{v}_f.\boldsymbol{\nabla}\boldsymbol{v}_f\Big) &= -\nabla p_f + \mu\nabla^2\boldsymbol{v}_f - \rho_0\alpha\theta_f\boldsymbol{g} \\ (\rho c_p)_f\left[\frac{\partial \theta_f}{\partial t} + \boldsymbol{v}_f.\Big(\nabla\theta_f - \frac{(T_0-T_u)}{d_f}\boldsymbol{e}_3\Big)\right] &= k_f\nabla^2\theta_f \ , \end{aligned} \tag{7.4.14}$$

where $\boldsymbol{v}_f$ is solenoidal whereas $\boldsymbol{v}_m$, p_m and θ_m satisfy

$$\begin{aligned} \frac{\rho_0}{\phi}\frac{\partial \boldsymbol{v}_m}{\partial t} &= -\nabla p_m - \frac{\mu}{\kappa}\boldsymbol{v}_m - \rho_0\alpha\theta_m\boldsymbol{g} \ , \\ (\rho c)_m\frac{\partial \theta_m}{\partial t} + (\rho c_p)_f\boldsymbol{v}_m.\Big(\nabla\theta_m - \frac{(T_l-T_0)}{d_m}\boldsymbol{e}_3\Big) &= k_m\nabla^2\theta_m \ . \end{aligned} \tag{7.4.15}$$

with $\boldsymbol{v}_m$ solenoidal. The modified boundary conditions on the upper boundary of the fluid layer ($x_3 = d_f$), the fluid/porous interface ($x_3 = 0$) and the lower boundary of the porous layer ($x_3 = -d_m$) are respectively

$$\theta_f(d_f) = 0 \ , \qquad w_f(d_f) = 0 \ , \qquad \frac{\partial w_f(d_f)}{\partial x_3} = 0 \ ,$$

$$\begin{aligned} &\theta_f(0) = \theta_m(0) \ , \qquad k_f\frac{\partial \theta_f(0)}{\partial x_3} = k_m\frac{\partial \theta_m(0)}{\partial x_3} \ , \\ &-p_f(0) + 2\mu\frac{\partial w_f(0)}{\partial x_3} = -p_m(0) \ , \qquad w_f(0) = w_m(0) \ , \\ &\frac{\partial u_f(0)}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}}(u_f(0) - u_m(0)) \ , \qquad \frac{\partial v_f(0)}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}}(v_f(0) - v_m(0)) \ , \end{aligned} \tag{7.4.16}$$

$$\theta_m(-d_m) = 0 \ , \qquad w_m(-d_m) = 0 \ .$$

7.5 Non-dimensionalisation

The non-dimensionalisation of (7.4.14), (7.4.15) and the boundary conditions (7.4.16) is technical but routine. Nield [32] presents a detailed description of the procedure. Most importantly, each layer has a different length and time scale. Using the scaling suggested by Chen & Chen [5], non dimensional spatial coordinates $\hat{\boldsymbol{x}}_f$, time $\hat{t}_f$, perturbed velocity $\hat{\boldsymbol{v}}_f$, pressure $\hat{p}_f$ and temperature $\hat{\theta}_f$ in the upper (fluid) layer are introduced by the definitions

$$\begin{aligned} &\boldsymbol{x} = d_f\hat{\boldsymbol{x}}_f \ , \qquad t_f = \frac{d_f^2}{\lambda_f}\hat{t}_f \ , \qquad \boldsymbol{v}_f = \frac{\lambda_f}{d_f}\hat{\boldsymbol{v}}_f \ , \\ &p_f = \frac{\mu\lambda_f}{d_f^2}\hat{p}_f \ , \qquad \theta_f = |T_0 - T_u|\hat{\theta}_f \ . \end{aligned} \tag{7.5.17}$$

Here λ_f is the thermal diffusivity of the fluid phase and is defined by $\lambda_f = k_f/(\rho c_p)_f$. With this change of variables, the equations (7.4.14) describing the motion of the fluid layer now assume the non-dimensional form

$$
\begin{aligned}
\frac{\partial \hat{\boldsymbol{v}}_f}{\partial \hat{t}_f} + \hat{\boldsymbol{v}}_f . \boldsymbol{\nabla}_f \hat{\boldsymbol{v}}_f &= \mathrm{Pr}_f \left[-\nabla_f \hat{p}_f + \nabla_f^2 \hat{\boldsymbol{v}}_f + \mathrm{Ra}_f \hat{\theta}_f \boldsymbol{e}_3 \right] \\
\frac{\partial \hat{\theta}_f}{\partial \hat{t}_f} + \hat{\boldsymbol{v}}_f . \left(\nabla_f \hat{\theta}_f - \mathrm{sign}(T_0 - T_u) \boldsymbol{e}_3 \right) &= \nabla_f^2 \hat{\theta}_f
\end{aligned}
\qquad (7.5.18)
$$

where Pr_f and Ra_f denote respectively the Prandtl number and Rayleigh number of the fluid layer and are defined by

$$
\mathrm{Pr}_f = \frac{\mu}{\rho_0 \lambda_f}\,, \qquad \mathrm{Ra}_f = \frac{g\alpha d_f^3 |T_0 - T_u|}{\mu \lambda_f}\,. \qquad (7.5.19)
$$

A similar procedure is applied to the porous medium in which non- dimensional spatial coordinates $\hat{\boldsymbol{x}}_m$, time $\hat{t}_m$, perturbed velocity $\hat{\boldsymbol{v}}_m$, pressure $\hat{p}_m$ and temperature $\hat{\theta}_m$ are introduced by the definitions

$$
\begin{gathered}
\boldsymbol{x} = d_m \hat{\boldsymbol{x}}_m\,, \qquad t_m = \frac{d_m^2}{\lambda_m} \hat{t}_m\,, \qquad \boldsymbol{v}_m = \frac{\lambda_m}{d_m} \hat{\boldsymbol{v}}_m\,, \\
p_m = \frac{\mu \lambda_m}{K} \hat{p}_m\,, \qquad \theta_m = |T_l - T_0| \hat{\theta}_m\,.
\end{gathered}
\qquad (7.5.20)
$$

Here λ_m is the thermal diffusivity of the porous medium and is defined by $\lambda_m = k_m/(\rho c_p)_f$. With this change of variables, the equations (7.4.15) governing the motion of the fluid in the porous layer now become

$$
\begin{aligned}
\frac{\mathrm{Da}}{\phi} \frac{\partial \hat{\boldsymbol{v}}_m}{\partial \hat{t}_m} &= \mathrm{Pr}_m \left[-\nabla_m \hat{p}_m - \hat{\boldsymbol{v}}_m + \mathrm{Ra}_m \hat{\theta}_m \boldsymbol{e}_3 \right], \\
G_m \frac{\partial \hat{\theta}_m}{\partial \hat{t}_m} + \hat{\boldsymbol{v}}_m . \left(\nabla_m \hat{\theta}_m - \mathrm{sign}(T_l - T_0) \boldsymbol{e}_3 \right) &= \nabla_m^2 \hat{\theta}_m\,,
\end{aligned}
\qquad (7.5.21)
$$

where $G_m = (\rho c)_m/(\rho c_p)_f$ and Pr_m, Da and Ra_m denote respectively the Prandtl number, Darcy number and Rayleigh number of the porous layer and are defined by

$$
\mathrm{Pr}_m = \frac{\mu}{\rho_0 \lambda_m}\,, \qquad \mathrm{Da} = \frac{K}{d_m^2}\,, \qquad \mathrm{Ra}_m = \frac{g\rho_0 \alpha K d_m |T_l - T_0|}{\mu \lambda_m}\,. \qquad (7.5.22)
$$

The scalings (7.5.17) and (7.5.20) are now used to non-dimensionalise the boundary conditions (7.4.16). The procedure is straightforward and yields

$$
\hat{\theta}_f(1) = 0\,, \qquad \hat{w}_f(1) = 0\,, \qquad \frac{\partial \hat{w}_f(1)}{\partial x_3} = 0\,,
$$

$$
\begin{gathered}
\hat{\theta}_f(0) = \epsilon_T \hat{\theta}_m(0) \ , \qquad \frac{\partial \hat{\theta}_f(0)}{\partial x_3} = \frac{\partial \hat{\theta}_m(0)}{\partial x_3} \ , \\
\epsilon_T \hat{d} \mathrm{Da}\Big(\hat{p_f} - 2\frac{\partial \hat{w}_f}{\partial x_3}\Big) = \hat{p}_m \ , \qquad \epsilon_T \hat{w}_f(0) = \hat{w}_m(0) \ , \\
\epsilon_T \frac{\partial \hat{u}_f}{\partial x_3} = \frac{\alpha_{\mathrm{BJ}}}{\hat{d}\sqrt{\mathrm{Da}}}(\epsilon_T \hat{u}_f - \hat{u}_m) \ , \qquad \epsilon_T \frac{\partial \hat{v}_f}{\partial x_3} = \frac{\alpha_{\mathrm{BJ}}}{\hat{d}\sqrt{\mathrm{Da}}}(\epsilon_T \hat{v}_f - \hat{v}_m) \ , \\
\hat{\theta}_m(-1) = 0 \ , \qquad \hat{w}_m(-1) = 0 \ .
\end{gathered}
\tag{7.5.23}
$$

where the parameters ϵ_T, $\hat{d}$ and $\hat{k}$ are defined by

$$
\epsilon_T = \frac{\hat{d}}{\hat{k}} \ , \qquad \hat{d} = \frac{d_m}{d_f} \ , \qquad \hat{k} = \frac{k_m}{k_f} \ .
$$

7.6 Linearisation of Problem

Until this point, no approximations have been made in the derivation of the perturbation equations. All subsequent analyses in this chapter are based on the linearised version of equations (7.5.18) and (7.5.21), obtained from them by ignoring all "product terms". For the fluid layer, w_f and θ_f satisfy

$$
\begin{aligned}
\frac{\partial \boldsymbol{v}_f}{\partial_f} &= \mathrm{Pr}_f \left[-\nabla_f p_f + \nabla_f^2 \boldsymbol{v}_f + \mathrm{Ra}_f \theta_f \boldsymbol{e}_3 \right] \\
\frac{\partial \theta_f}{\partial t_f} - H w_f &= \nabla_f^2 \theta_f
\end{aligned}
\tag{7.6.24}
$$

and for the porous layer, w_m and θ_m satisfy

$$
\begin{aligned}
\frac{\mathrm{Da}}{\phi} \frac{\partial \boldsymbol{v}_m}{\partial t_m} &= \mathrm{Pr}_m \Big[-\nabla_m \hat{p}_m - \boldsymbol{v}_m + \mathrm{Ra}_m \theta_m \boldsymbol{e}_3 \Big] \ , \\
G_m \frac{\partial \theta_m}{\partial t_m} - H w_m &= \nabla_m^2 \theta_m \ .
\end{aligned}
\tag{7.6.25}
$$

where $H = \mathrm{sign}(T_0 - T_u) = \mathrm{sign}(T_l - T_0)$ and the "hat" superscript has been dropped although all variables are non-dimensional. Since the boundary conditions (7.5.23) are already linear, no further action is required here except to remove superscripts.

By taking the double curl of the momentum equation in each layer, the hydrostatic pressures are suppressed. The specification of the final problem is completed by taking the third component of the reworked momentum equation in each layer together with the appropriate energy equation. In the fluid layer

$$
\begin{aligned}
\frac{1}{\mathrm{Pr}_f} \frac{\partial}{\partial t} \nabla^2 w_f &= \nabla^4 w_f + \mathrm{Ra}_f \Delta_2 \theta_f \\
\frac{\partial \theta_f}{\partial t} - H w_f &= \nabla^2 \theta_f \ ,
\end{aligned}
\tag{7.6.26}
$$

and in the porous layer

$$
\begin{aligned}
\frac{1}{\mathrm{Pr}_m}\frac{\mathrm{Da}}{\phi}\frac{\partial}{\partial t}\nabla^2 w_m &= -\nabla^2 w_m + \mathrm{Ra}_m \Delta_2 \theta_m \, , \\
G_m \frac{\partial \theta_m}{\partial t} - H w_m &= \nabla^2 \theta_m \, .
\end{aligned}
\tag{7.6.27}
$$

The Beaver-Joseph and normal stress interfacial boundary conditions must be reworked to eliminate pressure and horizontal components of velocity. Hydrostatic pressure terms are removed by computing the two-dimensional Laplacian of the boundary condition and by using the divergence of the respective momentum equation to eliminate the Laplacian of pressure. Similarily, both Beaver-Joseph conditions can be combined together by constructing the two-dimensional divergence of the tangential components of fluid velocity. The upshot of these considerations is that these boundary conditions are transformed to

$$
\hat{d}^3 \epsilon_T \mathrm{Da} \frac{\partial}{\partial x_3}\left(\nabla^2 w_f - \frac{1}{\mathrm{Pr}_f}\frac{\partial w_f}{\partial t} + 2\Delta_2 w_f\right) = -\left(\frac{\mathrm{Da}}{\mathrm{Pr}_m \phi}\frac{\partial}{\partial t} + 1\right)\frac{\partial w_m}{\partial x_3} \, . \tag{7.6.28}
$$

$$
\epsilon_T \hat{d} \frac{\partial}{\partial x_3}\left(w_f - \frac{\hat{d}\sqrt{\mathrm{Da}}}{\alpha_{\mathrm{BJ}}}\frac{\partial w_f}{\partial x_3}\right) = \frac{\partial w_m}{\partial x_3} \, . \tag{7.6.29}
$$

7.7 Linearisation of Equations

The linearisation of the equations (7.6.27) and (7.6.26) and the related boundary conditions is the resultant vector obtained by applying combining the relationships:

$$
\begin{aligned}
w_m(t, \mathrm{x}) &= w_m(x_3)\exp\left[i(p_m x + q_m y) + \sigma_m t\right] \, , \\
\theta_m(t, \mathrm{x}) &= \theta_m(x_3)\exp\left[i(p_m x + q_m y) + \sigma_m t\right] \, , \\
w_f(t, \mathrm{x}) &= w_f(x_3)\exp\left[i(p_f x + q_f y) + \sigma_f t\right] \, , \\
\theta_f(t, \mathrm{x}) &= \theta_f(x_3)\exp\left[i(p_f x + q_f y) + \sigma_f t\right] \, .
\end{aligned}
\tag{7.7.30}
$$

The governing equation of two layers can be represented as a system of equations, called the basic equations. Expressions (7.7.30) are substituted into equations (7.6.27) and (7.6.26) to obtain

$$
\begin{aligned}
\frac{\sigma_f}{\mathrm{Pr}_f}(D_f^2 - a_f^2) w_f &= (D_f^2 - a_f^2)^2 w_f - \mathrm{Ra}_f a_f^2 \theta_f \, , \\
\sigma_f \theta_f &= w_f + (D_f^2 - a_f^2)\theta_f \, , \\
-\frac{\mathrm{Da}}{\phi}\frac{\sigma_m}{\mathrm{Pr}_m}(D_m^2 - a_m^2) w_m &= (D_m^2 - a_m^2) w_m + \mathrm{Ra}_m a_m^2 \theta_m \, , \\
G_m \sigma_m \theta_m &= w_m + (D_m^2 - a_m^2)\theta_m \, ,
\end{aligned}
\tag{7.7.31}
$$

where $a_m^2 = p_m^2 + q_m^2$, $a_f^2 = p_f^2 + q_f^2$ are non-dimensionalised wave numbers in the porous medium and fluid respectively. For a given set of physical parameters and given a_m, Ra_m is determined by the condition that the real part of σ_f and σ_m are zero. However, in this particular problem it is a non-trivial fact that σ_f and σ_m are always real; in fact, there is a principle of exchange of stabilities[3]. Hence for a given a_m, Ra_m is computed when $\sigma_f = \sigma_m = 0$. The eigenvalue problem for σ_m and σ_f is completed by the specification of boundary conditions at $x_3 = 1$, $x_3 = 0$ and $x_3 = -1$. Pressures are computed from the two-dimensional divergence of the momentum equations whereas non-axial velocity components are eliminated by judicious use of the incompressibility constraints. Using these ideas, it can be verified from (7.5.23), (7.6.28) and (7.6.29) that the final boundary conditions are:-

Upper boundary $x_3 = 1$

$$w_f = 0 \,, \qquad D_f w_f = 0 \,, \qquad \theta_f = 0 \,, \tag{7.7.32}$$

Middle boundary $x_3 = 0$

$$\begin{aligned}
&\theta_f = \epsilon_T \theta_m \,, \qquad D_f \theta_f = D_m \theta_m \,, \qquad w_m = \epsilon_T w_f \,, \\
&\epsilon_T \hat{d} \left(D_f w_f - \frac{\hat{d}\sqrt{\mathrm{Da}}}{\alpha_{\mathrm{BJ}}} D_f^2 w_f \right) = D_m w_m \,, \\
&\hat{d}^3 \epsilon_T \mathrm{Da} \left(D_f^3 w_f - 3a_f^2 D_f w_f - \frac{\sigma_f}{\mathrm{Pr}_f} D_f w_f \right) = -\left(\frac{\mathrm{Da}}{\phi} \frac{\sigma_m}{\mathrm{Pr}_m} + 1 \right) D_m w_m \,,
\end{aligned} \tag{7.7.33}$$

Lower boundary $x_3 = -1$

$$w_m = 0 \,, \qquad \theta_m = 0 \,. \tag{7.7.34}$$

$$\begin{aligned}
D_m \psi &= \frac{d\psi_m}{dx_3} \quad (-1 < x_3 < 0) \,, & a_f &= \hat{d} a_m \,, \\
D_f \psi &= \frac{d\psi_f}{dx_3} \quad (0 < x_3 < 1) \,, & \sigma_f &= \frac{\hat{d}^2}{\hat{k}} \sigma_m \,.
\end{aligned} \tag{7.7.35}$$

[3]As a working rule, stationary convection is usually the only destabilising mechanism when two effects are competing (viscosity and thermal here) but once another stabilising effect such as a magnetic field comes into play, overstability now becomes possible, that is, stationary eigenvalues are fully complex.

7.8 First Order Formulation

Let variables $y_1, \ldots, y_{10}$ be defined by

$$
\begin{aligned}
y_1 &= w_f\,, & y_2 &= D_f w_f\,, & y_3 &= D_f^2 w_f\,,\\
y_4 &= D_f^3 w_f\,, & y_5 &= \theta_f\,, & y_6 &= D_f\theta_f\,,
\end{aligned}
\tag{7.8.36}
$$

$$
y_7 = w_m\,, \quad y_8 = D_m w_m \quad y_9 = \theta_m\,, \quad y_{10} = D_m\theta_m\,.
$$

The basic equations (7.7.31) can now be represented by the system of 10 first order differential equations

$$
\begin{aligned}
D_f y_1 &= y_2\,,\\
D_f y_2 &= y_3\,,\\
D_f y_3 &= -y_4\,,\\
D_f y_4 &= 2a_f^2 y_3 - a_f^4 y_1 + \mathrm{Ra}_f a_f^2 y_5 + \frac{\sigma_f}{\mathrm{Pr}_f}(y_3 - a_f^2 y_1)\,,\\
D_f y_5 &= y_6\,,\\
D_f y_6 &= a_f^2 y_5 - y_1 + \sigma_f y_5\,,\\
D_m y_7 &= y_8\,,\\
D_m y_8 &= a_m^2 y_7 - \mathrm{Ra}_m a_m^2 y_9 - \frac{\mathrm{Da}}{\phi}\frac{\sigma_m}{\mathrm{Pr}_m}(D_m y_8 - y_7)\,,\\
D_m y_9 &= y_{10}\,,\\
D_m y_{10} &= a_m^2 y_9 - y_7 + G_m\sigma_m y_9\,,
\end{aligned}
\tag{7.8.37}
$$

where

$$
\sigma_f = \frac{\hat{d}^2}{\hat{k}}\sigma_m. \tag{7.8.38}
$$

In terms of $y_1, \ldots, y_{10}$, the boundary conditions (7.7.32), (7.7.33) and (7.7.34) are re-expressed in the format

Upper boundary $x_3 = -1$

$$
y_1 = 0\,, \quad y_2 = 0\,, \quad y_5 = 0\,, \tag{7.8.39}
$$

Interface boundary $x_3 = 0$

$$
\begin{aligned}
& y_1 - \epsilon_T y_7 = 0\,, \qquad y_6 - y_{10} = 0\,,\\
& y_5 - \epsilon_T y_{11} = 0\,, \qquad y_8 = \hat{d}\epsilon_T(y_2 - \Delta y_3)\,,\\
\epsilon_T\hat{d}^3\mathrm{Da}(y_4 - 3a_f^2 y_2) + y_8 &= \epsilon_T\hat{d}^3\frac{\hat{d}^2\mathrm{Da}}{\hat{k}}\frac{\sigma}{\mathrm{Pr}_f}y_2 - \frac{\mathrm{Da}}{\phi}\frac{\sigma}{\mathrm{Pr}_m}y_8\,,
\end{aligned}
\tag{7.8.40}
$$

Lower boundary $x_3 = -1$

$$y_7 = 0 \, , \qquad\qquad y_9 = 0 \, . \tag{7.8.41}$$

where $\Delta = \hat{d}\sqrt{\mathrm{Da}}/\alpha_{BJ}$.

Choose one of σ_m and σ_f, say for example σ_m, and replace the value of σ_f from the relation (7.8.38) and nominate the chosen one to be σ. The eigenvalue problem for equations (7.7.31) can be reformulated in the form

$$AY = \sigma BY,$$

where A and B are real 10×10 matrices. These matrices are expressed by

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_f^4 & 0 & 2a_f^2 & 0 & a_f^2\mathrm{Ra}_f & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & a_f^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_m^2 & 0 & -a_m^2\mathrm{Ra}_m & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & a_m^2 & 0
\end{bmatrix}, \tag{7.8.42}$$

$$B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\dfrac{a_f^2\hat{d}^2}{\hat{k}\mathrm{Pr}_f} & 0 & \dfrac{\hat{d}^2}{\hat{k}\mathrm{Pr}_f} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dfrac{\mathrm{Da}}{\phi\mathrm{Pr}_m}a_m^2 & -\dfrac{\mathrm{Da}}{\phi\mathrm{Pr}_m}D_m & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_m & 0
\end{bmatrix}. \tag{7.8.43}$$

Each variable of $y_1, \ldots, y_{10}$ are assigned Chebyshev spectral expansion of order N and the coefficients of the expansion of these variables are replaced into a column vector Y of dimension $14(N+1)$. The eigenvalue problem then assumes the format $EY = \sigma FY$, where matrices E and F have block form

$$E = \begin{bmatrix} D & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_f & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ a^4{}_f I & 0 & -2a_f^2 I & D & -a_f^2 \mathrm{Ra}_f I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D & -I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & -a_f^2 I & D & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D_m & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_m^2 I & D & \mathrm{Ra}_m a_m^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D & -I \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -a_m^2 I & D \end{bmatrix}, \quad (7.8.44)$$

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{a_f^2 \hat{d}^2}{\hat{k}\mathrm{Pr}_f} I & 0 & \dfrac{\hat{d}^2}{\hat{k}\mathrm{Pr}_f} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{\mathrm{Da}}{\phi \mathrm{Pr}_m} a_m^2 I & -\dfrac{\mathrm{Da}}{\phi \mathrm{Pr}_m} D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_m I & 0 \end{bmatrix}, \quad (7.8.45)$$

with $D = D_f = D_m$ since the two layers have width one. Finally, it remains to incorporate the boundary conditions (7.8.39) into matrices E and F. Table (7.8.46) shows the forms of the boundary conditions (7.7.32), (7.7.33) and (7.7.34) expressed in terms of the variables $y_1, \ldots, y_{10}$ in (7.8.39), (7.8.40) and (7.8.41) and the equivalent spectral representations in terms of rows of E and F respectively.

These conditions replace the $(N+1)th, 2(N+1)th, \ldots, 10(N+1)th$ rows of E and F to produce the final forms for these matrices prior to the eigenvalue calculation.

<table>
<tr><th colspan="2">Manifestation in Matrix</th></tr>
<tr><th>E</th><th>F</th></tr>
<tr><td colspan="2">$y_1 = 0$</td></tr>
<tr><td>$[\mathrm{p}, 0, 0, 0, 0, 0, 0, 0, 0, 0]$</td><td>$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$</td></tr>
<tr><td colspan="2">$y_2 = 0$</td></tr>
<tr><td>$[0, \mathrm{p}, 0, 0, 0, 0, 0, 0, 0, 0]$</td><td>$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$</td></tr>
<tr><td colspan="2">$y_5 = 0$</td></tr>
<tr><td>$[0, 0, 0, 0, \mathrm{p}, 0, 0, 0, 0, 0]$</td><td>$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$</td></tr>
<tr><td colspan="2">$y_7 = \epsilon_T y_1$</td></tr>
<tr><td>$[\epsilon_T \mathrm{q}, 0, 0, 0, 0, 0, \mathrm{p}, 0, 0, 0]$</td><td>$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$</td></tr>
<tr><td colspan="2">$y_5 = \epsilon_T y_9$</td></tr>
<tr><td>$[0, 0, 0, 0, \mathrm{q}, 0, 0, 0, \epsilon_T \mathrm{p}, 0]$</td><td>$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$</td></tr>
<tr><td colspan="2">$y_6 = y_{10}$</td></tr>
<tr><td>$[0, 0, 0, 0, 0, \mathrm{q}, 0, 0, 0, \mathrm{p}]$</td><td>$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$</td></tr>
<tr><td colspan="2">$-y_4 + 3a_f^2 y_2 - \frac{1}{\epsilon_T \hat{d}^3 \mathrm{Da}} y_8 = -\frac{\hat{d}^2}{\hat{k}} \frac{\sigma}{\mathrm{Pr}_f} y_2 + \frac{1}{\epsilon_T \hat{d}^3 \phi} \frac{\sigma}{\mathrm{Pr}_m} y_8$</td></tr>
<tr><td>$[0, 3a_f^2 \mathrm{q}, 0, -\mathrm{q}, 0, 0, 0, -\frac{\mathrm{p}}{\mathrm{Da} \epsilon_T \hat{d}^3}, 0, 0]$</td><td>$[0, -\frac{\hat{d}^2}{\hat{k}} \frac{\sigma}{\mathrm{Pr}_f} \mathrm{q}, 0, 0, 0, 0, 0, \frac{1}{\hat{d}^3 \epsilon_T \phi} \frac{\sigma}{\mathrm{Pr}_m} \mathrm{p}, 0, 0]$</td></tr>
<tr><td colspan="2">$\epsilon_T \hat{d}(y_2 - \frac{\hat{d}\sqrt{\mathrm{Da}}}{\alpha_{BJ}} y_3) = y_8$</td></tr>
<tr><td>$[0, \hat{d}\epsilon_T \mathrm{q}, -\hat{d}^2 \epsilon_T \frac{\sqrt{\mathrm{Da}}}{\alpha_{BJ}} \mathrm{q}, 0, 0, 0, 0, -\mathrm{p}, 0, 0]$</td><td>$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$</td></tr>
<tr><td colspan="2">$y_7 = 0$</td></tr>
<tr><td>$[0, 0, 0, 0, 0, 0, \mathrm{q}, 0, 0, 0]$</td><td>$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$</td></tr>
<tr><td colspan="2">$y_9 = 0$</td></tr>
<tr><td>$[0, 0, 0, 0, 0, 0, 0, 0, \mathrm{q}, 0]$</td><td>$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$</td></tr>
</table>

The matrices F and E are loaded with a Fortran77 program using the routine F02BJF of the NAG libraries in appendix 5 to find the results of this problem for the first order system.

7.9 Results and Remarks

Chen [5] computes the stability curves for thermally-driven convection of a fluid layer with superposed porous layer heated from below for isothermal rigid boundaries, with thermal conductivity ratio $\hat{k} = 1.43$, Darcy number $\delta = 4 \times 10^{-6}$, Beavers-Joseph constant $\alpha_{BJ} = 0.1$ and for a variety of reciprocal depth ratios ranging from 0.33 to 0.1. The results of this thhesis are illustrated in table 7.1 and figure 7.1. They are qualitatively similar to those of Chen, but quantitatively dissimilar. For example, Chen quotes 40 (approximately) as the maximum peak of the stability curve $\hat{d}^{-1} = 0.12$, whereas the calculations here suggest something nearer 27. The differences are too large to be dismissed. Of course, one obvious explanation for this discrepancy is that the spectral method has failed. To check this possibility, the analysis was extended to the case of a porous medium layer sandwiched symmetrically between two layers of viscous fluid, a problem already treated by Pillatsis et al [40]. A significant number of Pillastsis results were tested here without deviation. Such high quality results are very much to their credit but they also validate the methodology. The three layer problem has not been presented here because it is essentially similar to the two layer problem but more technical.

A closer examination of Chen's method reveals that a 4th order Runge-Kutta integrator is used as the core of a shooting method geared to the calculation of a 7×7 determinant. Of course, this technique is intrinsically unsound, both numerically and logistically. The evaluation of high order determinants, and 7 is high, is prone to serious rounding errors not to mention the numerical errors involved in estimating the entries of the determinant. If a determinating method is to be used at all then ideally it should be implemented using the compound matrix methodology. A possible (unsophisticated) application of compound matrices determines 20 variables in the fluid region, 6 in the porous medium region and computes target functions using Laplace's expansion of a determinant. The conclusion is clear. This implementation of spectral method offers a straightforward and powerful way to determine critical eigenvalues irrespective of their type.

Wave No. a	Critical Rayleigh No. R_c					
	$\hat{d} = 0.1$	$\hat{d} = 0.12$	$\hat{d} = 0.13$	$\hat{d} = 0.14$	$\hat{d} = 0.2$	$\hat{d} = 0.33$
1.0	31.23	28.67	27.46	26.27	18.05	2.689
2.0	19.20	17.77	17.02	16.20	8.878	0.863
3.0	21.14	19.48	18.42	17.11	5.929	0.486
4.0	26.51	23.68	21.39	18.34	4.176	0.343
5.0	33.62	27.21	22.00	16.79	3.162	0.274
6.0	41.03	26.99	19.71	14.21	2.551	0.239
7.0	45.79	24.28	17.00	12.04	2.163	0.221
8.0	45.58	21.40	14.77	10.40	1.908	0.215
9.0	42.60	18.98	13.04	9.182	1.736	0.216
10.	39.12	17.05	11.73	8.273	1.622	0.224
11.	35.90	15.53	10.70	7.587	1.551	0.238
12.	33.12	14.32	9.902	7.068	1.506	0.257
13.	30.78	13.37	9.302	7.774	1.488	0.282
14.	28.83	12.62	8.831	6.379	1.491	0.314
15.	27.21	12.03	8.472	6.164	1.511	0.351
16.	25.87	11.57	8.203	6.780	1.360	0.396
17.	24.77	11.21	8.010	5.916	1.601	0.440
18.	23.86	10.95	7.880	5.867	1.668	0.508
19.	23.12	10.76	7.806	6.454	1.835	0.577
20.	22.53	10.64	7.800	5.890	2.018	0.656
21.	19.80	10.58	7.802	5.954	1.959	0.746
22.	21.78	10.58	7.864	6.051	2.087	0.847
23.	21.46	10.62	7.963	6.179	2.231	0.960
24.	21.30	9.560	8.100	6.350	2.393	1.088
25.	21.22	10.52	8.271	6.524	2.572	1.230

Table 7.1: Rayleigh number R_c for a given wavenumber a.

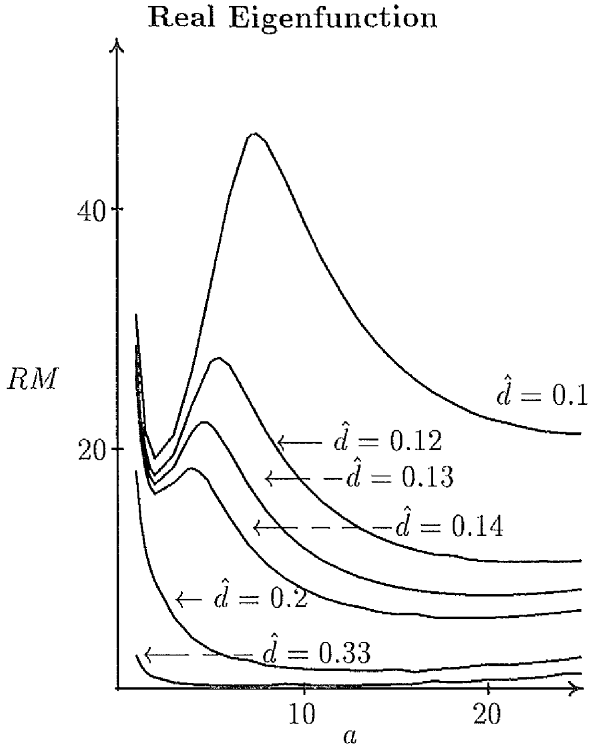


Figure 7.1: R_{a_m} versus a_m

7.10 Second Order Formulation

Let z_1, z_2, z_3 z_4 and z_5 be defined in terms of w_f, θ_f, w_m and θ_m by

$$z_1 = w_f \ , \quad z_2 = D_f^2 w_f \ , \quad z_3 = \theta_f \ , \quad z_4 = w_m \ , \quad z_5 = \theta_m \ . \tag{7.10.46}$$

Then in terms of these variables, the eigenvalue problem (7.10.46) becomes

$$\begin{aligned} D_f^2 z_1 &= z_2 \ , \\ D_f^2 z_2 &= 2a_f^2 z_2 - a_f^4 z_1 + \mathrm{Ra}_f a_f^2 z_3 + \frac{\sigma_f}{\mathrm{Pr}_f}(z_2 - a_f^2 z_1) \ , \\ D_f^2 z_3 &= a_f^2 z_3 - z_1 + \sigma_f z_3 \ , \\ D_m^2 z_4 &= a_m^2 z_4 - \mathrm{Ra}_m a_m^2 z_5 - \frac{\mathrm{Da}}{\phi}\frac{\sigma_m}{\mathrm{Pr}_m}(D_m^2 - a_m^2) z_4 \ , \\ D_m^2 z_5 &= a^2 z_5 - z_4 + \sigma_m G m z_5 \ , \end{aligned} \tag{7.10.47}$$

with boundary conditions Upper boundary $x_3 = 1$

$$z_1 = 0 \ , \quad D_f z_1 = 0 \ , \quad z_3 = 0 \ , \tag{7.10.48}$$

Middle boundary $x_3 = 0$

$$
\begin{aligned}
&z_3 = \epsilon_T z_5 \;, \qquad D_f z_3 = D_m z_5 \;, \qquad z_4 = \epsilon_T z_1 \;, \\
&\epsilon_T \hat{d} \left(D_f z_1 - \frac{\hat{d}\sqrt{\mathrm{Da}}}{\alpha_{\mathrm{BJ}}} z_2 \right) = D_m z_4 \;, \\
&\epsilon_T \hat{d}^3 \mathrm{Da} \left(D_f z_2 - 3a_f^2 D_f z_1 - \frac{\sigma_f}{\mathrm{Pr}_f} D_f z_1 \right) = -\left(\frac{\mathrm{Da}}{\phi} \frac{\sigma_m}{\mathrm{Pr}_m} + 1 \right) D_m z_4 \;,
\end{aligned}
\tag{7.10.49}
$$

Lower boundary $x_3 = -1$

$$
z_4 = 0 \;, \qquad z_5 = 0 \;. \tag{7.10.50}
$$

Recall that $D = D_f = D_m$ and $\sigma_f = \dfrac{\hat{d}^2}{\hat{k}} \sigma_m = \dfrac{\hat{d}^2}{\hat{k}} \sigma$, as mentioned before. It follows routinely from (7.10.47) that σ satisfies the generalised eigenvalue problem $EV = \sigma FV$, where E and F are respectively the 5×5 block matrices

$$
E = \begin{bmatrix}
D^2 I & -I & 0 & 0 & 0 \\
a_f^4 I & D^2 - 2a_f^2 I & -\mathrm{Ra}_f a_f^2 I & 0 & 0 \\
I & 0 & D^2 - a_f^2 I & 0 & 0 \\
0 & 0 & 0 & D^2 - a_m^2 I & \mathrm{Ra}_m a_m^2 I \\
0 & 0 & 0 & I & D^2 - a_f^2 I
\end{bmatrix} \tag{7.10.51}
$$

and

$$
F = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
\dfrac{\hat{d}^2}{\hat{k}} a_f^2 I & \dfrac{\hat{d}^2}{\hat{k}\mathrm{Pr}_f} I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & -\dfrac{\mathrm{Da}}{\phi} \dfrac{1}{\mathrm{Pr}_m} (D^2 - a_m^2) I & 0 \\
0 & 0 & 0 & 0 & GmI
\end{bmatrix} . \tag{7.10.52}
$$

The formulation of the eigenvalue problem is now completed by replacing the $(M-1)th$, Mth, $(2M-1)th$ and $2Mth$ rows of E and F with terms obtained using boundary information. From a mathematical standpoint, it does not matter how the two boundary conditions are ordered but numerical performance is usually enhanced if boundary data is inserted so that the largest entries occupy the top right of E and F. In terms of the M dimensional vectors

$$
\begin{aligned}
\mathrm{p} &= (p_1, p_2, \ldots, p_M) \;, \qquad & \mathrm{q} &= (q_1, q_2, \ldots, q_M) \;, \\
\mathrm{r} &= (r_1, r_2, \ldots, r_M) \;, \qquad & \mathrm{s} &= (s_1, s_2, \ldots, s_M) \;,
\end{aligned}
\tag{7.10.53}
$$

where $p_k = 1$, $q_k = (-1)^k$, $r_k = (k-1)^2$ and $s_k = (k-1)^2(-1)^k$ for $1 \leq k \leq M$, The boundary conditions and their locations are sequentially replaced for the M dimensional vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ defined in (7.10.53). It then remains only to replace the $(M-1)th$, Mth, ... , $(5M-1)th$ and $5Mth$ rows of E and F with the appropriate boundary information.

Row	E	F
	$z_4 = 0$ on $x_3 = -1$	
$M-1$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{q},\mathbf{0})$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$
	$z_1 = 0$ on $x_3 = 1$	
M	$(\mathbf{p},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$
	$z_5 = 0$ on $x_3 = -1$	
$2M-1$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{r})$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$
	$Dz_1 = 0$ on $x_3 = 1$	
$2M$	$(\mathbf{0},\mathbf{s},\mathbf{0},\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$
	$z_3 = 0$ on $x_3 = 1$	
$3M-1$	$(\mathbf{0},\mathbf{0},\mathbf{q},\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$
	$z_4 - \epsilon_T z_1 = 0$ on $x_3 = 0$	
$3M$	$(-\epsilon_T\mathbf{q},\mathbf{0},\mathbf{0},\mathbf{p},\mathbf{0})$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$
	$z_3 - \epsilon_T z_5 = 0$ on $x_3 = 0$	
$4M-1$	$(\mathbf{0},\mathbf{0},\mathbf{q},\mathbf{0},-\epsilon_T\mathbf{p})$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$
	$D_f z_3 - D_m z_5 = 0$ on $x_3 = 0$	
$4M$	$(\mathbf{0},\mathbf{0},\mathbf{s},\mathbf{0},-\mathbf{r})$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$
	$\epsilon_T\hat{d}^3\mathrm{Da}(Dz_2 - 3a_f^2 Dz_1 - \frac{\hat{d}^2}{\hat{k}}\frac{\sigma}{\mathrm{Pr}_f}Dz_1) = -Dz_4 - \frac{\mathrm{Da}}{\phi}\frac{\sigma}{\mathrm{Pr}_m}Dz_4$ on $x_3 = 0$	
$5M-1$	$(-3a_f^2\mathbf{s},\mathbf{s},\mathbf{0},\frac{1}{\epsilon_T\hat{d}^3\mathrm{Da}}\mathbf{r},\mathbf{0})$	$(\frac{\hat{d}^2}{\hat{k}}\frac{\sigma}{\mathrm{Pr}_f}\mathbf{s},\mathbf{0},\mathbf{0},\frac{\mathrm{Da}}{\phi\epsilon_T\hat{d}^3\mathrm{Da}}\frac{\sigma}{\mathbf{Pr}_m}\mathbf{r},\mathbf{0})$
	$\epsilon_T\hat{d}(D_f z_1 - \Delta z_2) = D_m z_4$ on $x_3 = 0$	
$5M$	$(\mathbf{s},\frac{\hat{d}\sqrt{\mathrm{Da}}}{\alpha_{BJ}}\mathbf{q},\mathbf{0},-\frac{1}{\epsilon_T\hat{d}}\mathbf{r},\mathbf{0})$	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$

The matrices F and E are loaded with the Fortran77 program using routine F02BJF of the NAG libraries, shown in Appendix 5 to find the results of this problem for the second

Value of wavenumber a	Accuracy for $\hat{d} = 0.1$ $\|\mathrm{Ra}_m - \mathrm{Ra}_{m2}\|$	Accuracy for $\hat{d} = 0.12$ $\|\mathrm{Ra}_m - \mathrm{Ra}_{m2}\|$	Accuracy for $\hat{d} = 0.13$ $\|\mathrm{Ra}_m - \mathrm{Ra}_{m2}\|$	Accuracy for $\hat{d} = 0.14$ $\|\mathrm{Ra}_m - \mathrm{Ra}_{m2}\|$
1	1.400×10^{-11}	2.250×10^{-10}	2.400×10^{-11}	1.150×10^{-9}
2	6.001×10^{-12}	5.230×10^{-11}	1.700×10^{-11}	3.700×10^{-11}
3	1.060×10^{-10}	1.120×10^{-10}	1.070×10^{-10}	1.470×10^{-10}
4	1.580×10^{-10}	2.880×10^{-10}	6.070×10^{-10}	1.309×10^{-9}
5	5.470×10^{-10}	1.481×10^{-10}	2.496×10^{-10}	3.168×10^{-9}

Table 7.2: Comparison of D and D^2 methods in Chen problem

order system.

7.10.1 Results and Remarks

The first order system (tenth order system problem) computes the marginal stability curves for a thermally-driven convection of a fluid layer with superposed porous layer heated from below for isothermal rigid boundaries, with reciprocal thermal conductivity ratio $\hat{k}^{-1} = 0.7$, Darcy number $\delta = 4 \times 10^{-6}$, Beavers-Joseph constant $\alpha_{BJ} = 0.1$ and for a variety of reciprocal depth ratios ranging from 0.1 to 0.33. The results in this section are illustrated in figure (7.1). They are qualitatively and quantitatively similar to those of first order system solutions, illustrated in this chapter. The difference between the Rayleigh numbers of a porous medium Ra_m for the first order system D and the Rayleigh numbers of a porous medium Ra_{m2} for the second order system D^2 is comparatively very small. Table (7.2) shows some examples of the accuracy of the second order system D^2 compared to that of the first order system D.

7.11 Conclusion

The present results are different from those of Chen regarding the locations of the curves of the Rayleigh numbers R_{a_m}, which are plotted as functions of wavenumbers a_m. The spectral methods have a strong ability to solve the multi-layered problem using both first order systems and second order systems. The results of this problem, using both first and second order systems, are identical.

Chapter 8

Finger Convection in a Horizontal Porous Layer Superposed by a Fluid Layer

8.1 Introduction

This chapter describes the onset of finger convection in a horizontal layer of porous medium of thickness d_m. This medium is covered and permeated by a horizontal layer of incompressible viscous fluid of thickness d_f in which a solute is dissolved. A right handed system of Cartesian coordinates x_i $i = 1 \ldots 3$ are chosen such that gravity acts in the negative x_3 direction and the origin of coordinates is arranged so that the fluid and porous media occupy respectively the layers $0 < x_3 < d_f$ and $-d_m < x_3 < 0$. Convection takes place in which temperature driven buoyancy effects and solute effects are damped by viscous effects. This problem has an equilibrium solution in which the fluid is stationary but there are thermal and salinity gradients in the x_3 direction in order to satisfy the field equations and boundary conditions. The stability of this equilibrium solution is of practical and theoretical importance. For example, Hills et al [18] and Maples & Poirier [29] model the directional solidification of molten alloys as a layer of porous material of variable permeability is separated from its melt by a mushy zone of dendrites. Glicksman et al [12] describe the interaction between the solidifying alloy and its melt by a doubly diffusive model.

8.2 The Governing Equations

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two horizontal layers such that the bottom of $\mathcal{L}_1$ interfaces with the top of $\mathcal{L}_2$. A right handed system of Cartesian coordinates x_i, $(i = 1, 2, 3)$ is chosen such that gravity acts in the negative x_3 direction and the interface is the plane $x_3 = 0$. With respect to these coordinates, the top boundary of $\mathcal{L}_1$ is $x_3 = d_f$ and the lower boundary of $\mathcal{L}_2$ is $x_3 = -d_m$. Suppose that the upper layer $\mathcal{L}_1$ is filled with an incompressible viscous fluid containing a dissolved solute (or salt) whereas the lower layer $\mathcal{L}_2$ is occupied by a porous medium permeated by the fluid. Heat is now applied to this configuration so that convection takes place in which temperature driven buoyancy and salting effects are damped by viscosity. One obvious solution to this problem occurs when the fluid is at rest and both layers are spanned by temperature and salinity gradients in the x_3 direction. This is the so-called "conduction solution" whose stability has been investigated recently by Chen [5].

Briefly, the fluid flow in the porous layer, thickness d_m, is governed by Darcy's law whereas the fluid flow in the upper layer $\mathcal{L}_1$, thickness d_f, is governed by the Navier-Stokes equations. Convection is driven by the dependence of the fluid density on temperature and salinity. Typically, the Oberbeck-Boussinesq approximation is made where concepts like local thermal equilibrium, heating from viscous dissipation, radiative effects etc. are ignored as are variations in fluid density except where they occur in the momentum equation. The fluid density ρ_f is related to the Kelvin temperature T and salinity S by

$$\rho_f = \rho_0[1 - \alpha(T - T_0) + \beta(S - S_0)] \tag{8.2.1}$$

where ρ_0 is the density at temperature T_0 and salinity S_0, and α, β (both assumed constant) are respectively the thermal and salting coefficients of volume expansion for the fluid. It is well known that these can be strongly temperature dependent so that (8.2.1) may be inappropriate[1] for large temperature and salinity variations.

Following the approach of Nield [33] and Chen & Chen [5], the momentum, energy and salting equations for the flow of an incompressible viscous fluid through a porous medium

[1] George et. al. [11] represent ρ_f by a polynomial of order three in their description of convection in lakes in which the bottom can be represented by a porous layer which is under-pinned by an impermeable permafrost boundary.

are

$$
\begin{aligned}
\frac{\rho_0}{\phi}\frac{\partial \mathbf{v}_m}{\partial t} &= -\nabla P_m - \frac{\mu}{K}\mathbf{v}_m + \rho_f \mathbf{g} \\
(\rho c)_m \frac{\partial T_m}{\partial t} + (\rho c_p)_f \mathbf{v}_m . \nabla T_m &= k_m \nabla^2 T_m \\
\phi \frac{\partial S_m}{\partial t} + \mathbf{v}_m . \nabla S_m &= \mathrm{D}_m \nabla^2 S_m
\end{aligned}
\tag{8.2.2}
$$

where the solenoidal vector $\mathbf{v}_m$ denotes fluid seepage velocity, P_m denotes pressure, μ denotes the dynamic viscosity (assumed constant) of the fluid, K and ϕ denote respectively the permeability and porosity of the porous substrate. ρ_f denotes the fluid density and is given by the formula (8.2.1), k_m and D_m are respectively the overall thermal conductivity and mass diffusivity of the porous layer, $(\rho c_p)_f$ is the heat capacity per unit volume of the fluid at constant pressure and $(\rho c)_m$ is the overall heat capacity per unit volume of the porous medium at constant pressure. In fact,

$$
(\rho c)_m = \phi(\rho c_p)_f + (1-\phi)(\rho c_p)_m
$$

where $(\rho c_p)_m$ is the heat capacity per unit volume of the porous substrate.

The top layer $\mathcal{L}_1$ is filled with incompressible Navier-Stokes fluid in which the conservation of momentum, energy and salting are expressed through the equations

$$
\begin{aligned}
\rho_0\left(\frac{\partial \mathbf{v}_f}{\partial t} + \mathbf{v}_f . \nabla \mathbf{v}_f\right) &= -\nabla P_f + \mu \nabla^2 \mathbf{v}_f + \rho_f \mathbf{g} \\
(\rho c_p)_f \left(\frac{\partial T_f}{\partial t} + \mathbf{v}_f . \nabla T_f\right) &= k_f \nabla^2 T_f \\
\frac{\partial S_f}{\partial t} + \mathbf{v}_f . \nabla S_f &= \mathrm{D}_f \nabla^2 S_f
\end{aligned}
\tag{8.2.3}
$$

where the solenoidal vector $\mathbf{v}_f$ denotes fluid velocity and k_f and D_f are respectively the thermal conductivity and mass diffusivity of the fluid. The Boussinesq approximation has been used in equations (8.2.2) and (8.2.3) and the convected terms in the fluid acceleration have been ignored and Darcy's law has been employed as is customary in the modelling of porous media.

The convection problem is completed by the specification of boundary conditions on the upper surface of the viscous fluid layer, at the interface between the fluid and porous layers and at the lower boundary of the porous layer. Many combinations of boundary conditions are possible but for comparison with Chen [5], isothermal rigid exterior boundaries are considered giving three conditions on each exterior boundary. Thus

$$\begin{aligned} T_f &= T_u\,, & S_f &= S_u\,, & \mathbf{v}_f &= \mathbf{0}\,, & x_3 &= d_f\,, \\ T_m &= T_l\,, & S_m &= S_l\,, & \mathbf{v}_m.\mathbf{e}_3 &= \mathbf{0}\,, & x_3 &= -d_m\,. \end{aligned} \tag{8.2.4}$$

where T_u and T_l are respectively the temperatures at the upper and lower exterior boundaries. At the fluid/porous-medium interface, temperature, heat flux, salinity, salt flux, normal fluid velocity and normal stress are assumed to be continuous. This leaves one final condition to be specified on the interface. Jones [23] advocates continuity of shear stress although this is perhaps an incongruous condition bearing in mind that in the formulation of the momentum equation for a porous medium, Darcy's law, replaces conventional viscous stress. The most commonly used boundary condition is due to Beavers and Joseph [2] who suggest that

$$\frac{\partial u_f}{\partial x_3} = \frac{\alpha_{\mathrm{BJ}}}{\sqrt{K}}(u_f - u_m)\,, \qquad \frac{\partial v_f}{\partial x_3} = \frac{\alpha_{\mathrm{BJ}}}{\sqrt{K}}(v_f - v_m)\,, \tag{8.2.5}$$

where u_f, v_f are the limiting tangential components of the fluid velocity as the interface is approached from the fluid layer $\mathcal{L}_1$ whereas u_m, v_m are the same limiting components of tangential fluid velocity as the interface is approached from the porous layer $\mathcal{L}_2$. Self-evidently, the Beavers-Joseph[2] this condition permits discontinuities in shear velocity across the interface. It is verified easily that the field equations (8.2.2) and (8.2.3) and all boundary conditions (8.2.4) are satisfied by the conduction solution

$$\begin{aligned} &\mathbf{v}_f = \mathbf{0}\,, \quad T_f|_E = T_0 + (T_u - T_0)\frac{x_3}{d_f}\,, \quad S_f|_E = S_0 + (S_u - S_0)\frac{x_3}{d_f}\,, \\ &\mathbf{v}_m = \mathbf{0}\,, \quad T_m|_E = T_0 + (T_0 - T_l)\frac{x_3}{d_m}\,, \quad S_m|_E = S_0 + (S_0 - S_l)\frac{x_3}{d_m}\,, \end{aligned} \tag{8.2.6}$$

where the interfacial temperature T_0 and salt concentration S_0 are determined by the continuity of heat flux and salt flux respectively and take the values

$$T_0 = \frac{k_m d_f T_l + k_f d_m T_u}{k_m d_f + k_f d_m}\,, \qquad S_0 = \frac{\mathrm{D}_m d_f S_l + \mathrm{D}_f d_m S_u}{\mathrm{D}_m d_f + \mathrm{D}_f d_m}\,. \tag{8.2.7}$$

This solution is accompanied by a hydrostatic pressure which is a function of x_3 only.

8.3 Perturbed Equations

Following the policy of Neild [34] and Chen [5], displacement and time are rescaled respectively by d_m and d_m^2/λ_m in the porous medium and by d_f and d_f^2/λ_f in the fluid

[2]Often the numerical results are insensitive to the choice of Beaver-Joseph condition or continuity of shear stress calculated in the porous medium in the conventional way.

where

$$\lambda_f = \frac{k_f}{(\rho c_p)_f}\,, \qquad \lambda_m = \frac{k_m}{(\rho c_p)_f}\,. \tag{8.3.8}$$

When non-dimensional velocity $\mathbf{u}_f$, temperature θ_f, salinity s_f and hydrostatic pressure p_f are introduced into the fluid equations (8.2.3) by the definitions

$$\begin{aligned} \mathbf{v}_f &= \frac{\nu}{d_f}\mathbf{u}_f\,, \qquad & T_f &= T_f|_E + \frac{|T_0 - T_u|\nu}{\lambda_f}\theta_f\,, \\ S_f &= S_f|_E + \frac{|S_u - S_0|\nu}{\mathrm{D}_f}s_f \qquad & P_f &= P_f|_E + \frac{\rho_0\nu^2}{d_f^2}p_f\,. \end{aligned} \tag{8.3.9}$$

The resulting non-dimensionalised fluid equations are

$$\begin{aligned} \frac{1}{\mathrm{Pr}_f}\frac{\partial \mathbf{u}_f}{\partial t_f} + \mathbf{u}_f.\nabla\mathbf{u}_f &= -\nabla p_f + \nabla^2\mathbf{u}_f + \mathrm{Ra}_f\theta_f\mathbf{e}_3 - \mathrm{Ra}_f^{(s)}s_f\mathbf{e}_3\,, \\ \frac{\partial\theta_f}{\partial t_f} + \mathrm{Pr}_f\mathbf{u}_f.\nabla\theta_f &= \mathrm{sign}(T_0 - T_u)w_f + \nabla^2\theta_f\,, \\ \frac{1}{\mathrm{Le}_f}\Big(\frac{\partial s_f}{\partial t_f} + \mathrm{Pr}_f\mathbf{u}_f.\nabla s_f\Big) &= \nabla^2 s_f + \mathrm{sign}(S_0 - S_u)w_f\,. \end{aligned} \tag{8.3.10}$$

Similarly, when non-dimensional velocity $\mathbf{u}_m$, temperature θ_m, salinity s_m and hydrostatic pressure p_m are introduced into the porous medium equations (8.2.2) by the definitions

$$\begin{aligned} \mathbf{v}_m &= \frac{\nu}{d_m}\mathbf{u}_m\,, \qquad & T_m &= T_m|_E + \frac{|T_0 - T_l|\nu}{\lambda_m}\theta_m\,, \\ S_m &= S_m|_E + \frac{|S_0 - S_l|\nu}{\mathrm{D}_m}s_m\,, \qquad & P_m &= P_m|_E + \frac{\rho_0\nu^2}{K}p_m\,, \end{aligned} \tag{8.3.11}$$

the porous medium equations assume the non-dimensional form

$$\begin{aligned} \frac{\mathrm{Da}}{\mathrm{Pr}_m}\frac{1}{\phi}\frac{\partial\mathbf{u}_m}{\partial t_m} &= -\nabla p_m - \mathbf{u}_m + \mathrm{Ra}_m\theta_m\mathbf{e}_3 - \mathrm{Ra}_m^{(s)}s_m\mathbf{e}_3\,, \\ G_m\frac{\partial\theta_m}{\partial t_m} + \mathrm{Pr}_m\mathbf{u}_m.\nabla\theta_m &= \nabla^2\theta_m + \mathrm{sign}(T_l - T_0)w_m\,, \\ \phi\frac{\partial s_m}{\partial t_m} + \mathbf{u}_m.\nabla s_m &= \mathrm{Le}_m\nabla^2 s_m + \mathrm{Le}_m\mathrm{sign}(S_l - S_0)w_m\,. \end{aligned} \tag{8.3.12}$$

where Da (Darcy number) and G_m are non-dimensional numbers defined by

$$\mathrm{Da} = \frac{K}{d_m^2}\,, \qquad G_m = \frac{(\rho c)_m}{(\rho c_p)_f}\,. \tag{8.3.13}$$

In (8.3.10) and (8.3.12), $\mathbf{u}_m$ and $\mathbf{u}_f$ are solenoidal vectors and the nondimensional Prandtl numbers Pr_m and Pr_f, Lewis numbers Le_m and Le_f, Rayleigh numbers Ra_m, Ra_f, $\mathrm{Ra}_m^{(s)}$

and $\mathrm{Ra}_f^{(s)}$ are defined by

$$
\begin{aligned}
\Pr_m = \frac{\nu}{\lambda_m}\,, \qquad \Pr_f = \frac{\nu}{\lambda_f}\,, \qquad \mathrm{Le}_m = \frac{\mathrm{D}_m}{\lambda_m}\,, \qquad \mathrm{Le}_f = \frac{\mathrm{D}_f}{\lambda_f}\,, \\
\mathrm{Ra}_m = \frac{g\alpha|T_0 - T_l|d_m K}{\nu\lambda_m}\,, \qquad \mathrm{Ra}_f = \frac{g\alpha|T_u - T_0|d_f^3}{\nu\lambda_f}\,, \\
\mathrm{Ra}_m^{(s)} = \frac{g\beta|S_0 - S_l|d_m K}{\nu \mathrm{D}_m}\,, \qquad \mathrm{Ra}_f^{(s)} = \frac{g\beta|S_u - S_0|d_f^3}{\nu \mathrm{D}_f}\,.
\end{aligned}
\tag{8.3.14}
$$

8.4 Linearised Problem

The linearearised approximation of equations (8.3.10) and (8.3.12) (currently exact) is constructed by ignoring all terms involving products of the unknown functions. Self-evidently, the linearised equations in the fluid layer are given by

$$
\begin{aligned}
\frac{1}{\Pr_f}\frac{\partial \mathbf{u}_f}{\partial t_f} &= -\nabla p_f + \nabla^2 \mathbf{u}_f + \mathrm{Ra}_f \theta_f \mathbf{e}_3 - \mathrm{Ra}_f^{(s)} s_f \mathbf{e}_3\,, \\
\frac{\partial \theta_f}{\partial t_f} &= \nabla^2 \theta_f + H_T w_f\,, \\
\frac{1}{\mathrm{Le}_f}\frac{\partial s_f}{\partial t_f} &= \nabla^2 s_f + H_S w_f\,,
\end{aligned}
\tag{8.4.15}
$$

and in the porous medium are given by

$$
\begin{aligned}
\frac{\mathrm{Da}}{\Pr_m}\frac{1}{\phi}\frac{\partial \mathbf{u}_m}{\partial t_m} &= -\nabla p_m - \mathbf{u}_m + \mathrm{Ra}_m \theta_m \mathbf{e}_3 - \mathrm{Ra}_m^{(s)} s_m \mathbf{e}_3\,, \\
G_m \frac{\partial \theta_m}{\partial t_m} &= \nabla^2 \theta_m + H_T w_m\,, \\
\frac{\phi}{\mathrm{Le}_m}\frac{\partial s_m}{\partial t_m} &= \nabla^2 s_m + H_S w_m\,,
\end{aligned}
\tag{8.4.16}
$$

where $H_T = \mathrm{sign}(T_l - T_0) = \mathrm{sign}(T_0 - T_u)$ and $H_S = \mathrm{sign}(S_l - S_0) = \mathrm{sign}(S_0 - S_u)$. From condition (8.2.4), the unknowns θ_m, θ_f, s_m, s_f, w_m and $\mathbf{v}_f$ satisfy the boundary conditions

$$
\begin{aligned}
\theta_f = 0\,, \qquad s_f = 0\,, \qquad \mathbf{u}_f = \mathbf{0}\,, \quad x_3 = 1\,, \\
\theta_m = 0\,, \qquad s_m = 0\,, \qquad w_m = 0\,, \quad x_3 = -1\,.
\end{aligned}
\tag{8.4.17}
$$

The formulation of the interfacial boundary conditions is technical but straightforward. Continuity of temperature, salinity, normal velocity, normal heat flux and normal salt flux yield sequentially

$$
\begin{aligned}
\gamma_T \theta_f = \epsilon_T \theta_m\,, \qquad \gamma_S s_f = \epsilon_S s_m\,, \qquad w_f = \hat{d} w_m\,, \\
\frac{\partial \theta_f}{\partial x_3} = \epsilon_T \frac{\partial \theta_m}{\partial x_3}\,, \qquad \frac{\partial s_f}{\partial x_3} = \epsilon_S \frac{\partial s_m}{\partial x_3}\,,
\end{aligned}
\tag{8.4.18}
$$

where $\hat{d} = d_f/d_m$. Continuity of normal stress and the Beavers-Joseph slip condition (see (8.2.5)) are respectively

$$p_f - 2\frac{\partial w_f}{\partial x_3} = \frac{\hat{d}^2}{\mathrm{Da}} p_m \Longrightarrow \Delta_2 p_f - 2\frac{\partial}{\partial x_3}\Delta_2 w_f = \frac{\hat{d}^2}{\mathrm{Da}}\Delta_2 p_m \,. \tag{8.4.19}$$

The substitution can be made from (8.2.2) and (8.2.3) into (8.4.19) to yield

$$\frac{\partial}{\partial x_3}(\nabla^2 \mathbf{u}_f) - \frac{1}{\mathrm{Pr}_f}\frac{\partial}{\partial t_f}\Big(\frac{\partial \mathbf{u}_f}{\partial x_3}\Big) + 2\frac{\partial}{\partial x_3}(\Delta_2 \mathbf{u}_f) = -\frac{\hat{d}^4}{\mathrm{Da}}\frac{\partial \mathbf{u}_m}{\partial x_3} - \frac{\hat{d}^4}{\phi \mathrm{Pr}_m}\frac{\partial}{\partial t_m}\Big(\frac{\partial}{\partial x_3}\mathbf{u}_m\Big) \tag{8.4.20}$$

and

$$\frac{\partial u_f}{\partial x_3} = \frac{\hat{d}\alpha_{\mathrm{BJ}}}{\sqrt{\mathrm{Da}}}(u_f - \hat{d}u_m)\,, \qquad \frac{\partial v_f}{\partial x_3} = \frac{\hat{d}\alpha_{\mathrm{BJ}}}{\sqrt{\mathrm{Da}}}(v_f - \hat{d}v_m)\,. \tag{8.4.21}$$

The derivative of the first equation with respect to x_1 and the second equation with respect to x_2 in (8.4.21) can then be added together to obtain

$$\frac{\hat{d}\alpha_{\mathrm{BJ}}}{\sqrt{\mathrm{Da}}}\Big[\frac{\partial w_f}{\partial x_3} - \hat{d}^2\frac{\partial w_m}{\partial x_3}\Big] = \frac{\partial^2 w_f}{\partial x_3^2}\,, \tag{8.4.22}$$

where the parameters ϵ_T, ϵ_S, γ_T and γ_S are defined by

$$\epsilon_T = \frac{\lambda_f}{\lambda_m}\,, \qquad \epsilon_S = \frac{\mathrm{D}_f}{\mathrm{D}_m}\,, \qquad \gamma_T = \frac{T_u - T_0}{T_0 - T_l}\,, \qquad \gamma_S = \frac{S_u - S_0}{S_0 - S_l}\,. \tag{8.4.23}$$

8.4.1 The Linearised Equations

A normal modes solution is sought for equations (8.4.16) and (8.4.15) in which all variables ψ_m in the porous medium and ψ_f in the fluid have respective representations

$$\psi_f = \psi_f(x_3)e^{\sigma_f t_f}e^{i(p_f x_1 + q_f x_2)}\,, \qquad \psi_m = \psi_m(x_3)e^{\sigma_m t_m}e^{i(p_m x_1 + q_m x_2)}\,.$$

When the fluid and porous momentum equations are treated twice by the curl operator to remove the pressure terms, it follows easily that the fluid layer equations can be recast in the form

$$\begin{aligned}
\frac{\sigma_f}{\mathrm{Pr}_f}(D_f^2 - a_f^2)w_f &= (D_f^2 - a_f^2)^2 w_f - a_f^2\mathrm{Ra}_f\theta_f + a_f^2\mathrm{Ra}_f^{(s)}s_f\,,\\
\sigma_f\theta_f - H_T w_f &= (D_f^2 - a_f^2)\theta_f\,, \qquad 0 \le x_3 1 \le 1\\
\frac{\sigma_f}{\mathrm{Le}_f}s_f - H_S w_f &= (D_f^2 - a_f^2)s_f\,,
\end{aligned} \tag{8.4.24}$$

the porous medium equations then become

$$\begin{aligned}
\left(\frac{\mathrm{Da}}{\mathrm{Pr}_m}\frac{\sigma_m}{\phi} + 1\right)(D_m^2 - a_m^2)w_m &= -a_m^2\mathrm{Ra}_m\theta_m + a_m^2\mathrm{Ra}_m^{(s)}s_m\,,\\
G_m\sigma_m\theta_m - H_T w_m &= (D_m^2 - a_m^2)\theta_m\,, \qquad -1 \le x_3 \le 0\,,\\
\frac{\sigma_m\phi}{\mathrm{Le}_m}s_m - H_S w_m &= (D_m^2 - a_m^2)s_m\,.
\end{aligned} \tag{8.4.25}$$

From these equations,

$$\begin{aligned} D_f\psi &= \frac{d\psi}{dx_3} \quad (0 < x_3 < 1)\,, & a_f &= \hat{d}a_m\,, \\ D_m\psi &= \frac{d\psi}{dx_3} \quad (-1 < x_3 < 0)\,, & \sigma_f &= \frac{\hat{d}^2}{\epsilon_T}\sigma_m\,, \end{aligned} \tag{8.4.26}$$

$$w_f = D_f w_f = \theta_f = s_f = 0 \qquad \text{on} \quad x_3 = 1\,, \tag{8.4.27}$$

$$\left.\begin{aligned} w_f = \hat{d}w_m\,, \qquad \gamma_T\theta_f =\in_T \theta_m\,, \qquad \gamma_S s_f = \in_S s_m\,, \\ D_f\theta_f =\in_T D_m\theta_m\,, \qquad D_f s_f &= \in_S D_m s_m\,, \\ D_f^3 w_f - 3a_f^2 D_f w_f + \frac{\hat{d}^4}{\mathrm{Da}} D_m w_m &= \frac{\sigma_f}{\mathrm{Pr}_f} D_f w_f - \frac{\hat{d}^4\sigma_m}{\phi \mathrm{Pr}_m} D_m w_m\,, \\ \frac{\hat{\alpha}\hat{d}}{\sqrt{\mathrm{Da}}}[D_f w_f - \hat{d}^2 D_m w_m] &= D_f^2 w_f \end{aligned}\right\} x_3 = 0. \tag{8.4.28}$$

$$w_m = \theta_m = s_m = 0 \qquad \text{on} \quad x_3 = -1\,. \tag{8.4.29}$$

8.5 Method of Solution

Let the variables $y_1, \ldots, y_{14}$ be defined by

$$\begin{aligned} y_1 &= w_f\,, & y_2 &= D_f w_f\,, & y_3 &= D_f^2 w_f\,, \\ y_4 &= D_f^3 w_f\,, & y_5 &= \theta_f\,, & y_6 &= D_f\theta_f\,, \\ y_7 &= s_f\,, & y_8 &= D_f s_f\,, & y_9 &= w_m\,, \\ y_{10} &= D_m w_m\,, & y_{11} &= \theta_m\,, & y_{12} &= D_m\theta_m\,, \\ & & y_{13} = s_m\,, & & y_{14} = D_m s_m\,. \end{aligned} \tag{8.5.30}$$

Then it is straightforward to verify that equations (8.4.24) and (8.4.25) can be rewritten as the 14th order system:

$$
\begin{aligned}
&D_f y_1 - y_2 = 0 \\
&D_f y_2 - y_3 = 0 \\
&D_f y_3 - y_4 = 0 \\
&D_f y_4 + a^4 y_1 - 2a_f^2 y_3 - \mathrm{Ra}_f a_f^2 y_5 + \mathrm{Ra}_f^{(s)} y_7 = \frac{\sigma_f}{\mathrm{Pr}_f}(y_3 - a_f^2 y_1) \, , \\
&D_f y_5 - y_6 = 0 \\
&D_f y_6 - y_1 - a_f^2 y_5 = \sigma_f y_5 \\
&D_f y_7 - y_8 = 0 \\
&D_f y_8 - y_1 - a_f^2 y_7 = \frac{\sigma_f}{\mathrm{Le}_f} y_7 \\
&D_m y_9 - y_{10} = 0 \\
&D_m y_{10} - a_m^2 y_9 + a_m^2 \mathrm{Ra}_m y_{11} - a_m^2 \mathrm{Ra}_m^{(s)} y_{13} = -\frac{\mathrm{Da}}{\phi} \frac{\sigma_m}{\mathrm{Pr}_m}(D_m y_{10} - a_m^2 y_9) \, , \\
&D_m y_{11} - y_{12} = 0 \, , \\
&D_m y_{12} - y_9 - a_m^2 y_{11} = G_m \sigma_m y_{11} \, , \\
&D_m y_{13} - y_{14} = 0 \, , \\
&D_m y_{14} - y_9 - a_m^2 y_{13} = \frac{\sigma_m \phi}{\mathrm{Le}_m} y_{13} \, .
\end{aligned}
\tag{8.5.31}
$$

To solve this problem, $\sigma = \sigma_f$ is chosen then from the relation $(8.4.26)_4$ $\sigma = \dfrac{\hat{d}^2}{\epsilon_T}\sigma_m$. Each variable of $y_1, \ldots, y_{14}$ is assigned a Chebyshev spectral expansion of order N and the coefficients of the expansion of these variables are replaced into a column vector Y of dimension $14(N+1)$. The eigenvalue problem then assumes the format $EY = \sigma FY$, where matrices E and F have respectively block form

$$E = \begin{bmatrix} D & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a^4 I & 0 & -2a_f^2 I & D & -a_f^2 \mathrm{Ra}_f I & 0 & a_f^2 \mathrm{Ra}_f^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & -a_f^2 I & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & -a_f^2 I & D & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_m^2 I & D & a_m^2 \mathrm{Ra}_m I & 0 & -a_m^2 \mathrm{Ra}_m^{(s)} I & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & -a_m^2 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D & -I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & a_m^2 & D \end{bmatrix} \tag{8.5.32}$$

$$
F = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\hat{d}^2}{\epsilon_T}\frac{a_f^2}{\mathrm{Pr}_f}I & 0 & \frac{\hat{d}^2}{\epsilon_T}\frac{1}{\mathrm{Pr}_f}I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\hat{d}^2}{\epsilon_T}I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\hat{d}^2}{\epsilon_T}\frac{1}{\mathrm{Le}_f}I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_m^2\mathrm{Da}}{\phi\mathrm{Pr}_m}I & 0 & -\frac{\mathrm{Da}}{\phi\mathrm{Pr}_m}I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_m I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\phi}{\mathrm{Le}_m}I & 0
\end{bmatrix}
\qquad (8.5.33)
$$

where D_f and D_m have the same values since each layer has a width from 0 to 1. Finally, it only remains to incorporate the boundary conditions (8.4.27), (8.4.28) and (8.4.29) into matrices E and F. These boundary conditions in (8.4.27), (8.4.28) and (8.4.29) can be expressed in terms of the variables $y_1, \ldots, y_{14}$ and the equivalent spectral representation in terms of rows of E and F are respectively:

$$\begin{gathered}
y_1 = 0\ , y_2 = 0\ , \qquad y_5 = 0\ , \qquad y_7 = 0\ , \qquad y_9 = 0\ , \qquad y_{11} = 0\ , \\
y_{13} = 0\ , \qquad y_1 - \hat{d} y_9 = 0\ , \qquad \gamma_T y_5 - \epsilon_T y_{11} = 0\ , \\
\gamma_S y_7 - \epsilon_S y_{13} = 0\ , \qquad y_6 - \epsilon_T y_{12} = 0\ , \qquad y_8 - \epsilon_S y_{14} = 0\ , \\
y_4 - 3a_f^2 y_2 + \frac{\hat{d}^4}{\mathrm{Da}} y_{10} = \frac{\hat{d}^2}{\epsilon_T} \frac{\sigma}{\mathrm{Pr}_f} y_2 - \frac{\hat{d}^4 \sigma}{\phi \mathrm{Pr}_m} y_{10}\ , \quad y_3 = \frac{\hat{d} \alpha_{\mathrm{BJ}}}{\sqrt{\mathrm{Da}}} (y_2 - \hat{d}^2 y_{10})\ .
\end{gathered} \tag{8.5.34}$$

Each boundary condition (8.5.34) is incorporated sequentially into the $(N+1)th, 2(N+1)th, \ldots 14(N+1)th$ rows of the matrices E and F. This completes the specification of the eigenvalue problem $EY = \sigma FY$. The details of each row (in matix format) are given in table 8.1. The calculations were done using the Fortran program listed in appendix 6. In fact,the eigenvalues all appear to be real so that $\sigma = 0$ is the critical eigenvalue. In this case, the eigenvalue problem can be recast in the format

$$E^* Y = \mathrm{Ra}_m^{(s)} F^* Y$$

where $E = E^* - \mathrm{Ra}_m^{(s)} F^*$ and E^* and F^* are respectively represented in block form

$$\hat{E} = \begin{bmatrix} D & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a^4 I & 0 & -2a_f^2 I & D & -a_f^2 \mathrm{Ra}_f I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & -a_f^2 I & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & -a_f^2 I & D & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_m^2 I & D & a_m^2 \mathrm{Ra}_m I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & -a_m^2 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D & -I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & a_m^2 & D \end{bmatrix} \tag{8.5.35}$$

$$\hat{F} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&a_f^2\dfrac{\hat{d}^4}{\epsilon_S^2\mathrm{Da}}&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&a_m^2&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \qquad (8.5.36)$$

8.6 Results

Chen [5] computes the critical salt Rayleigh numbers of a porous medium for a thermally-driven convection and a salinated convection of a fluid layer with superposed porous layer heated from above for isothermal rigid boundaries. Therefore it is necessary to stabilise the initial temperature distribution. The value of the initial Rayleigh number of a porous medium is 50 and the other values and numbers which are respectively required to solve this problem are Beavers-Joseph constants $\alpha_{\mathrm{BJ}} = 0.1$, $\sqrt{\mathrm{Da}} = 0.003$, $\epsilon_T = 0.7$, $\epsilon_S = 3.75$,

$$\mathrm{Le}_f = \frac{D_f}{\lambda_f}\,, \qquad \mathrm{Le}_m = \frac{D_m}{\lambda_m}\,, \qquad \mathrm{Pr}_{fs} = \frac{\mathrm{Pr}_f}{\mathrm{Le}_f}\,,$$

$$\mathrm{Pr}_{ms} = \frac{\mathrm{Pr}_m}{\mathrm{Le}_m}\,, \qquad \gamma_T = \frac{\hat{d}}{\epsilon_T}\,, \qquad \gamma_S = \frac{\hat{d}}{\epsilon_S}\,,$$

and these are determined by Chen [5]. The values of the depth ratios $\hat{d}$ vary between $10^{-4} - 1.5$. The eigenvalues σ are real for all values of $\hat{d}$. To determine whether or not the eigenvalues are real, it is known that the critical eigenvalues occur when $\sigma = 0$ and so Y satisfies the condition $EY = 0$. However, $E = E^* - \mathrm{Ra}_m^{(s)} F^*$ and so the possible values of $\mathrm{Ra}_m^{(s)}$ are now eigenvalues of the modified and generalised eigenvalue problem. In this case, the results show that the eigenvalues are real. Now returning back to the original problem, the values of the critical salt Rayleigh number $\mathrm{Ra}_m^{(s)}$ and the corresponding wavenumbers a_m for a porous layer considered here are different from those of Chen [5] for the values of depth ratios $\hat{d} = 0.1, \ldots, 1.5$ as illustrated in a table 8.2. The values of salt Rayleigh number Ra_{ms} are plotted as a function of the corresponding wavenumber a_m to produce the marginal stability curves for a range of depth ratios $\hat{d}$ as illustrated in figure 8.1. It is observed that the curves for $\hat{d} = 10^{-4}$, 10^{-3} and 10^{-1} are indistinguishable along with the curves for $\hat{d} = 0.5$, 1 and 1.5. This is also true in Chen's result with respect to the two curves for $\hat{d} = 10^{-4}$ and 10^{-2}. In addition, the curves for $\hat{d} = 0.5$, 1 and $= 1.5$ are lower than those of Chen. For the possible bimodal nature of marginal stability curves, the calculations have been extended to include $a_m = 20$ for $\hat{d} = 0.2$ and 0.1. As shown in figure 8.2, the curves are qualitatively and quantitatively different from those of Chen. For example, Chen quotes 102 (approximately) as the minimum value of the stability curve for $\hat{d} = 1.0$ whereas the result here is nearer 84. The differences are too large to be ignored. Of course, one obvious explanation for this discrepancy is that the spectral method has failed. To check this possibility, the spectral method is applied to the case of the two-layered problem (porous layer superposed by fluid layer) which is treated in chapter 7.

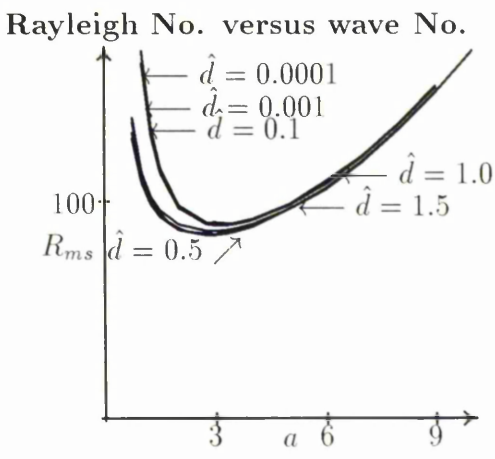


Figure 8.1: Ra_{ms} versus a_m for $\hat{d} = 0.0001$, 0.001, 0.1, 0.5, 1 and 1.5.

8.7 Conclusions

For the stabilised value of Rayleigh number $Ra_m = 50$ of a porous layer combined with other values, it is found that the critical salt Rayleigh numbers Ra_{ms}, here compared

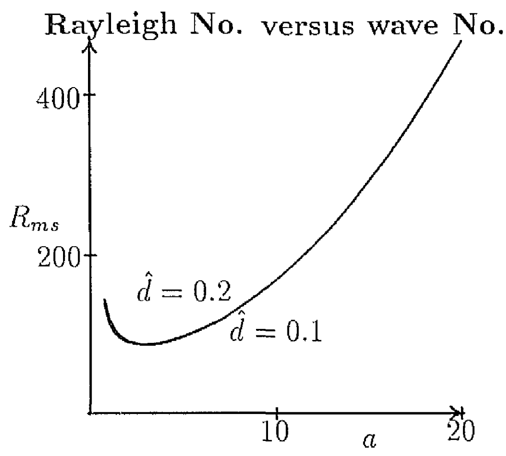


Figure 8.2: Ra_{ms} versus a_m for $\hat{d} = 0.1$ and 0.2.

with those of Chen, are almost identical for some $\hat{d} =$ values of depth ratios and lower than those of Chen for other $\hat{d}$ ratios as shown in the table (8.2). We believe that these differences are due in large measure to the inadequacy of Chen's scheme, as a numerical methodolgy particularly when an interior boundary has to be negotiated.

Manifestation in Matrix	
E	F
$y_1 = 0$	
$[\mathrm{p}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_2 = 0$	
$[0, \mathrm{p}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_5 = 0$	
$[0, 0, 0, 0, \mathrm{p}, 0, 0, 0, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_7 = 0$	
$[0, 0, 0, 0, 0, 0, \mathrm{p}, 0, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_1 = \hat{d} y_9$	
$[\mathrm{q}, 0, 0, 0, 0, 0, 0, 0, -\hat{d}\mathrm{p}, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$\gamma_T y_5 = \epsilon_T y_{11}$	
$[0, 0, 0, 0, \gamma_T \mathrm{q}, 0, 0, 00, 0, \epsilon_T \mathrm{p}, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_6 = \epsilon_T y_{12}$	
$[0, 0, 0, 0, 0, \mathrm{q}, 0, 0, 00, 0, \epsilon_T \mathrm{p}, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$\gamma_S y_7 = \epsilon_S y_{13}$	
$[0, 0, 0, 0, 0, 0, \gamma_S \mathrm{q}, 0, 0, 0, 0, 0, \epsilon_S \mathrm{p}, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_8 = \epsilon_S y_{14}$	
$[0, 0, 0, 0, 0, 0, 0, \mathrm{q}, 0, 0, 0, 0, 0, \epsilon_S \mathrm{p}]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$-y_4 + 3a_f^2 y_2 - (\hat{d}^4/\mathrm{Da}) y_{10} = -(\hat{d}^2/\epsilon_T)(\sigma/\mathrm{Pr}_f) y_2 + (\sigma \hat{d}^4/\phi \mathrm{Pr}_m) y_{10}$	
$[0, 3a_f^2 \mathrm{q}, 0, -\mathrm{q}, 0, 0, 0, 0, 0, -\frac{\hat{d}^4}{\mathrm{Da}}\mathrm{p}, 0, 0, 0, 0]$	$[0, -\frac{\hat{d}^2}{\epsilon_T}\frac{\sigma \mathrm{q}}{\mathrm{Pr}_f}, 0, 0, 0, 0, 0, 0, 0, \frac{\sigma \hat{d}^4 \mathrm{p}}{\phi \mathrm{Pr}_m}, 0, 0, 0, 0]$
$y_3 = (\alpha_{BJ} \hat{d}/\sqrt{\mathrm{Da}})(y_2 - \hat{d}^2 y_{10})$	
$[0, \frac{\mathrm{q}}{\sqrt{\mathrm{Da}}}, \frac{-\mathrm{q}}{\alpha_{\mathrm{BJ}} \hat{d}}, 0, 0, 0, 0, 0, 0, \frac{-\mathrm{p}}{\sqrt{\mathrm{Da}}}, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_9 = 0$	
$[0, 0, 0, 0, 0, 0, 0, 0, \mathrm{q}, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_{11} = 0$	
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathrm{q}, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_{13} = 0$	
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathrm{q}, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

Table 8.1: The form of the boundary rows

$\hat{d}$	$\mathrm{Ra}_m^{(s)}$	a_m	$\mathrm{Ra}_m^{(s)}$(Chen)	a_m(Chen)
0.01	88.8734	3.110	88.91	3.1
0.042	83.6874	2.700	83.43	2.7
0.044	83.7098	2.695	83.42	2.7
0.05	83.8767	2.690	83.48	2.7
0.1	86.2986	2.876	85.59	2.8
0.2	85.6721	2.951	97.23	3.6
0.4	84.7741	2.844	103.37	4.3
0.5	84.6067	2.822	103.96	4.3
1.0	84.3762	2.775		
1.5	84.3607	2.769		
2.0	84.3606	2.769		
5.0	84.6129	2.743		

Table 8.2: For $\hat{d}$, critical value of $\mathrm{Ra}_m^{(s)}$ and a_c.

Chapter 9

Magnetic Instability with a Finitely Conducting Inner Rigid Core

9.1 Introduction

The model of this chapter is an interior of the Earth divided into three layers; the first layer is the cylindrical rigid core of the Earth, the second layer is a cylindrical annulus of electrically conducting fluid rotating around its axis with angular velocity $\mathbf{\Omega}_0 = \Omega_0 \boldsymbol{e}_z$, and the third layer is a rigid outer region which can be called a mantle which is either a perfect electrical conductor or insulator. This chapter reviews Lamb's [25] investigation of the case that the mantle is a perfect insulator.

The magnetic field and velocity of the fluid core of the Earth in its basic state can be represented in cylindrical polar coordinates (r, θ, z) by

$$\mathbf{B}_0 = B(r)\boldsymbol{e}_\theta, \qquad \mathbf{U}_0 = \mathrm{U}(r)\boldsymbol{e}_\theta.$$

The ratio of the magnetic diffusivity in the inner core, denoted η_i, to that in the fluid, denoted η_0, is given by $\eta = \eta_i/\eta_0$ and is an important parameters of the problem. For the inner core, perfect conduction corresponds to $\eta = 0$ and perfect insulation corresponds as $\eta = \infty$. Significant results are established when η_i is finite, that is, η is finite.

9.2 The Governing Equations of Layers

The Earth is often modelled as an active spherical core enclosed by a solid annular region or mantle whose electrical properties resemble those of either a perfect insulator or perfect conductor. The core region is subdivided into an electrically conducting solid inner core surrounded by a spherical shell of electrically conducting incompressible viscous fluid rotating rapidly at constant angular velocity $\mathbf{\Omega} = \Omega_0$ about a north-south axis with an inner radius r_i and an outer radius r_0 respectively. As a precursor to the full problem, the most recent research models the Earth by three interacting cylindrical regions and it is this approach that will be emulated here. Lamb [25] discussed the effect of a finitely conducting inner core on magnetically driven instability in the absence of thermal buoyancy effects. The radius of the inner region varies between 0 and r_i and the outer region consists of everything beyond $r = r_0$. The fluid flow is related to the velocity mode $\mathbf{U}$ and it is permeated by a toroidal magnetic field $\mathbf{B}$. Here the aim is to reproduce a selection of these results. Lamb quotes the governing differential equations in the form

Inner Core

$$\frac{\partial \mathbf{B}}{\partial t} = \eta_i \Delta \mathbf{B} , \qquad \nabla . \boldsymbol{B} = 0 , \qquad r \in (0, r_i) , \tag{9.2.1}$$

Fluid Region

$$\begin{aligned}
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}.\nabla\mathbf{U} + 2\Omega_0 \times \mathbf{U} &= -\frac{1}{\rho_0}\nabla P + \frac{1}{\mu\rho_0}(\nabla \times \mathbf{B}) \times \mathbf{B} + \nu\Delta\mathbf{U} , \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{U} \times \mathbf{B}) + \eta_0 \Delta \mathbf{B} , \qquad r \in (r_i, r_0), \\
\nabla . \mathbf{B} &= \nabla . \mathbf{U} = \mathbf{0} ,
\end{aligned} \tag{9.2.2}$$

where P is the hydrostatic pressure, ν is the kinematic viscosity of the fluid, μ is the magnetic permeability of the fluid, η_0 , η_i are the magnetic diffusivity of the fluid and inner core respectively and ρ_0 is the fluid density. In cylindrical polar coordinates (r, θ, z) with unit base vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$, the underlying terrestrial magnetic field and flow velocity are azimuthal with form $\mathbf{B}_0 = B(r)\boldsymbol{e}_\theta$, $\mathbf{U}_0 = U(r)\boldsymbol{e}_\theta$ respectively and $\mathbf{\Omega} = \Omega_0 \boldsymbol{e}_z$. If $\mathbf{u}$ and $\mathbf{b}$ are respectively the perturbations in the fluid velocity and magnetic induction about the basic state by

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \qquad \mathbf{U} = \mathbf{U}_0 + \mathbf{u}.$$

where

$$\boldsymbol{b} = (b_r, b_\theta, b_z), \qquad \boldsymbol{u} = (u, v, w)$$

then the non-dimensional field equations will be in the form as quoted by Lamb.

Inner Core

$$\frac{\partial \boldsymbol{b}}{\partial t} = \eta \Lambda^{-1} \Delta \boldsymbol{b} \, . \qquad \nabla . \boldsymbol{b} = 0 \, , \qquad r \in (0, r_i) \, , \tag{9.2.3}$$

Fluid Region

$$\begin{aligned} \Lambda E_\eta \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{e}_z \times \boldsymbol{u} &= -\nabla p + (\nabla \times \boldsymbol{B}_0) \times \boldsymbol{b} + (\nabla \times \boldsymbol{b}) \times \boldsymbol{B}_0 + E \Delta \boldsymbol{u} \, , \\ \frac{\partial \boldsymbol{b}}{\partial t} &= \nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0) + \Lambda^{-1} \Delta \boldsymbol{b} \, , \quad r \in (r_i, r_0) \, , \\ \nabla . \boldsymbol{b} &= \nabla . \boldsymbol{u} = 0 \, , \end{aligned} \tag{9.2.4}$$

where p is the hydrostatic pressure and

$$E = \nu / 2\Omega_0 {r_0}^2, \qquad E_\eta = \eta_0 / 2\Omega_0 {r_0}^2 \, , \qquad \Lambda = {B_M}^2 / 2\eta_0 \Omega_0 \mu \rho_0$$

denote the viscous Ekman number, the magnetic Ekman number and the Elasser number respectively. It is common practice to write

$$B(r) = B_M r F(r) \, , \tag{9.2.5}$$

where B_M is the maximum value of $B(r)$ and $F(r)$ is a function to be specified later. The curl of each term of the momentum equation (9.2.4) is taken after replacing $\Delta \boldsymbol{u}$ by ($-$curl curl $\boldsymbol{u}$), and then (curl $\boldsymbol{b}$) and (curl $\boldsymbol{u}$) are replaced by $\boldsymbol{J}$ and $\boldsymbol{\xi}$ respectively. The field equations now become.

Inner Core

$$\frac{\partial \boldsymbol{b}}{\partial t} = \eta \Lambda^{-1} \Delta \boldsymbol{b} \, . \qquad \nabla . \boldsymbol{b} = 0 \, , \qquad r \in (0, r_i) \, , \tag{9.2.6}$$

Fluid region

$$\begin{aligned} \Lambda E_\eta \frac{\partial \xi}{\partial t} &+ \text{curl } (\boldsymbol{e}_z \times \boldsymbol{u}) = \text{curl } \left[\left(B' + \frac{B}{r} \right) \boldsymbol{e}_z \times \boldsymbol{b} \right] + \text{curl } (\boldsymbol{J} \times \boldsymbol{B}_0) \\ & \qquad\qquad - E \text{curl curl } \boldsymbol{\xi} \, , \\ \frac{\partial \boldsymbol{b}}{\partial t} &= \text{curl } (\boldsymbol{u} \times \boldsymbol{B}_0) - \Lambda^{-1} \text{curl curl } \boldsymbol{b} \, , \qquad r \in (r_i, r_0) \, , \\ \nabla . \boldsymbol{b} &= \nabla . \boldsymbol{u} = \boldsymbol{0} \, . \end{aligned} \tag{9.2.7}$$

The calculations on (9.2.6) and (9.2.7) are involved. The final system of equations is constructed from the components.

$$
\begin{aligned}
\text{curl }(\boldsymbol{u}\times\boldsymbol{B}_0) &= \frac{B(r)}{r}\frac{\partial u}{\partial\theta}\boldsymbol{e}_r \quad - \quad \Big(B(r)\frac{\partial w}{\partial z}+B(r)\frac{\partial u}{\partial r}+B'(r)u\Big)\boldsymbol{e}_\theta \\
&\qquad +\frac{B(r)}{r}\frac{\partial w}{\partial\theta}\boldsymbol{e}_z\ , \\
\text{curl }(\boldsymbol{e}_z\times\boldsymbol{u}) &= -\frac{\partial\boldsymbol{u}}{\partial z}\ , \\
\text{curl }\Big[\Big(B'(r)+\frac{B(r)}{r}\Big)\boldsymbol{e}_z\times\boldsymbol{b}\Big] &= -\Big(B'(r)+\frac{B(r)}{r}\Big)\frac{\partial\boldsymbol{b}}{\partial z} \\
&\qquad +\Big(B'(r)+\frac{B(r)}{r}\Big)'b_r\boldsymbol{e}_z\ , \\
\text{curl }(\boldsymbol{J}\times\boldsymbol{B}_0) &= \Big(\frac{B(r)}{r}\frac{\partial J_r}{\partial\theta},-B(r)\frac{\partial J_z}{\partial z}-B(r)\frac{\partial J_r}{\partial r} \\
&\qquad -B'(r)J_r,\frac{B(r)}{r}\frac{\partial J_z}{\partial\theta}\Big)\ .
\end{aligned}
\tag{9.2.8}
$$

Define

$$
\mathcal{J}(\psi)=\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}.
$$

$$
\begin{aligned}
(\text{curl curl }\boldsymbol{\varrho})_r &= -\mathcal{J}(\varrho_r)-\frac{2}{r}\frac{\partial\varrho_r}{\partial r}-\frac{\varrho_r}{r^2}-\frac{2}{r}\frac{\partial\varrho_z}{\partial z}\ , \\
(\text{curl curl }\boldsymbol{\varrho})_\theta &= -\mathcal{J}(\varrho_\theta)-\frac{2}{r^2}\frac{\partial\varrho_r}{\partial\theta}+\frac{\varrho_\theta}{r^2}\ , \\
(\text{curl curl }\boldsymbol{\varrho})_z &= -\mathcal{J}(\varrho_z)\ ,
\end{aligned}
\tag{9.2.9}
$$

where $\boldsymbol{\varrho}$ represents the vector variables $\boldsymbol{b}$, $\boldsymbol{u}$ and $\boldsymbol{\xi}$ by

$$
(\varrho_r,\varrho_\theta,\varrho_z)=\Big((b_r,b_\theta,b_z),(u,v,w),(\xi_r,\xi_\theta,\xi_z)\Big).
$$

A substitution is made using (9.2.9) in (9.2.6) and using (9.2.8) and (9.2.9) in (9.2.7) to obtain the necessary r and z components of magnetic equations in the inner rigid core and the outer fluid layer and r and θ components of the momentum equations of the outer fluid layer respectively as:

Inner Core

$$
\begin{aligned}
\frac{\partial b_r}{\partial t} &= \eta\Lambda^{-1}\Big(\mathcal{J}(b_r)+\frac{2}{r}\frac{\partial b_r}{\partial r}+\frac{b_r}{r^2}+\frac{2}{r}\frac{\partial b_z}{\partial z}\Big)\ , \\
\frac{\partial b_z}{\partial t} &= \eta\Lambda^{-1}\mathcal{J}(b_z)\ ,
\end{aligned}
\tag{9.2.10}
$$

Fluid Layer

$$
\begin{aligned}
\Lambda E_\eta \frac{\partial \xi_r}{\partial t} - \frac{\partial u}{\partial z} &= -\Big(B'(r) + \frac{B(r)}{r}\Big)\frac{\partial b_r}{\partial z} + \frac{B(r)}{r}\frac{\partial J_r}{\partial \theta} \\
&\quad + E\Big(\mathcal{J}(\xi_r) + \frac{2}{r}\frac{\partial \xi_r}{\partial r} + \frac{\xi_r}{r^2} + \frac{2}{r}\frac{\partial \xi_z}{\partial z}\Big)\,, \\
\Lambda E_\eta \frac{\partial \xi_\theta}{\partial t} - \frac{\partial v}{\partial z} &= -\Big(B'(r) + \frac{B(r)}{r}\Big)\frac{\partial b_\theta}{\partial z} - B(r)\Big(\frac{\partial J_z}{\partial z} + \frac{\partial J_r}{\partial r}\Big) - B'(r)J_r \\
&\quad + E\Big(\mathcal{J}(\xi_\theta) + \frac{2}{r^2}\frac{\partial \xi_r}{\partial \theta} - \frac{\xi_\theta}{r}\Big)\,, \\
\frac{\partial b_r}{\partial t} &= \frac{B(r)}{r}\frac{\partial u}{\partial \theta} + \Lambda^{-1}\Big(\mathcal{J}(b_r) + \frac{2}{r}\frac{\partial b_r}{\partial r} + \frac{b_r}{r^2} + \frac{2}{r}\frac{\partial b_z}{\partial z}\Big)\,, \\
\frac{\partial b_z}{\partial t} &= \frac{B(r)}{r}\frac{\partial w}{\partial \theta} + \Lambda^{-1}\mathcal{J}(b_z)\,.
\end{aligned}
\tag{9.2.11}
$$

Now the equations (9.2.10) and (9.2.11) have normal mode solutions

$$
\begin{aligned}
\boldsymbol{u} &= (u, v, w)e^{\sigma t}e^{i(m\theta+nz)}\,, \\
\boldsymbol{b} &= (b_r, b_\theta, b_z)e^{\sigma t}e^{i(m\theta+nz)}\,.
\end{aligned}
\tag{9.2.12}
$$

in which σ term represents the eigenvalues to be determined and m and n are wavenumbers.

The non-constant nature of the basic magnetic field B_0 ensures that all calculations will be algebraically complex but after a laborious calculation it can be shown that b_r, b_z, u_r and u_z satisfy the field equations

Inner Core

r, z-component of magnetic induction equations

$$
\begin{aligned}
\frac{\sigma\Lambda}{\eta} r^2 b_r &= r^2 D_{ic}^2 b_r + 3rD_{ic}b_r - (r^2G - 1)b_r + 2inrb_z\,, \\
\frac{\sigma\Lambda}{\eta} r^2 b_z &= r^2 D_{ic}^2 b_z + rD_{ic}b_z - r^2 G b_z\,,
\end{aligned}
\tag{9.2.13}
$$

Fluid Region

r, z-component of momentum and magnetic induction equations

$$
\begin{aligned}
\sigma\frac{\Lambda E_\eta}{E}[in(D_{oc}u+\frac{u}{r})-Gw] &= \frac{mn}{E}[-FD_{oc}b_r+F'b_r-\frac{u}{r}]-\frac{im}{E}FGb_z \\
+in[D_{oc}^3u+\frac{4}{r}D_{oc}^2u- & G \quad D_{oc}u+\frac{1}{r^2}D_{oc}u+\frac{G}{r}u-\frac{2n^2}{r}u-\frac{1}{r^3}u] \\
- & [\quad GD_{oc}^2w-G^2w+\frac{G}{r}D_{oc}w+\frac{2n^2}{r}D_{oc}w]\ , \\
\sigma\frac{\Lambda E_\eta}{E}[D_{oc}w-inu] &= -in[D_{oc}^2u+\frac{3}{r}D_{oc}u-Gu+\frac{1}{r^2}u] \\
+[D_{oc}^3w &+ \frac{1}{r}D_{oc}^2w-\frac{1}{r^2}D_{oc}w-GD_{oc}w+\frac{2G}{r}w] \\
&+\frac{im}{E}(FD_{oc}b_z+F'b_z-\frac{F}{r}b_z-inFb_r) \\
-\frac{2n}{mE}rF[D_{oc}b_r &+ inb_z+\frac{1}{r}b_r]+\frac{n}{mE}(u+rD_{oc}u+inrw)\ , \\
\sigma\Lambda b_r &= \Lambda imFu+D_{oc}^2b_r+\frac{3}{r}D_{oc}b_r-Gb_r+\frac{2in}{r}b_z \\
&+\frac{b_r}{r^2}\ , \\
\sigma\Lambda b_z &= \Lambda imFw+D_{oc}^2b_z+\frac{1}{r}D_{oc}b_z-Gb_z\ .
\end{aligned}
\tag{9.2.14}
$$

Mantle

$$
D_m^2\psi+\frac{1}{r}D_m\psi-G\psi=0\ , \tag{9.2.15}
$$

where $F=F(r)$ is defined in (9.2.5), $G=G(r)=\frac{m^2}{r^2}+n^2$, $D_{ic}=\frac{\partial}{\partial r_{ic}}$ is the partial derivative in inner core region, $D_{oc}=\frac{\partial}{\partial r_{oc}}$ is the partial derivative in outer core region (fluid layer) and $D_m=\frac{\partial}{\partial r_m}$ is the partial derivative in mantle region.

9.3 Boundary Conditions

The no slip boundary conditions are applied at the inner core bounding surface $r=r_i$ or at the core-mantle bounding surface $r=r_0$. The no slip condition

$$
\boldsymbol{u}=\boldsymbol{0}, \qquad r=r_i, r_0
$$

can be written in terms of its components

$$
u=0, \qquad v=0, \qquad w=0 \qquad r=r_i, r_0\ . \tag{9.3.16}
$$

since

$$\nabla . \boldsymbol{u} = 0 \Longrightarrow D_{oc}u + \frac{u}{r} + \frac{im}{r}v + inw = 0$$

then

$$v = 0 \quad \longrightarrow D_{oc}u = 0 \qquad\qquad r = r_i, r_0 \ . \tag{9.3.17}$$

On the other hand, magnetic conditions require the magnetic induction must be continuous everywhere. On the axis of rotation this implies

$$\left.\begin{array}{ll} D_{ic}b_r = b_z = 0 & \text{when} \quad m = 1 \\ b_r = b_z = 0 & \text{when} \quad m > 1 \end{array}\right\} \quad \text{at } r = 0 \ . \tag{9.3.18}$$

If the mantle is a perfectly conducting region then the tangential electric field at the core-mantle boundary surface is zero and this in turn infers that $b_r = 0$ and $D_{ic}b_z = 0$. In effect, this is a two- layered problem since the mantle region is disjoint from the inner and outer core (fluid layer). The conclusion of this work is that the mantle is taken to be a perfect insulator so that no currents cross the boundary but the region is permeated by a magnetic field which is derived from a potential function $\Phi = \phi(r)e^{\sigma t}e^{i(m\theta+nz)}$. It is trivial that $\phi(r)$ satisfies the differential equation

$$D_m^2\phi + \frac{1}{r}D_m\phi - G\phi = 0 \qquad \text{on} \quad r = r_0 \tag{9.3.19}$$

where

$$\boldsymbol{b} = -\nabla\Phi = -(\frac{d\phi}{dr}, \frac{im}{r}\phi, in\phi)e^{\sigma t}e^{i(m\theta+nz)}.$$

The current normal to the core-mantle boundary is zero in this case leading to the boundary condition

$$inD_{oc}b_r + \frac{in}{r}b_r + Gb_z = 0, \qquad\qquad r = r_0 \ . \tag{9.3.20}$$

The continuity of magnetic field induction is enforced elsewhere. The spectral approach to this boundary value problem is distinctively different from that of Lamb [25]. For example. Lamb's technique enforces boundary conditions at $r = 0$ and $r = r_0$ by determining suitable analytic solutions (Bessel functions with complex arguments in this instance) and then matching these solutions to the fluid boundary conditions at $r = r_i$ and $r = r_0$. Such a technique is messy and limiting. On the other hand, spectral methods succeed effortlessly without the need for an analytical solution. The explanation lies in the fact that methods based on computation of the solution interval inevitably experience severe difficulties when the underlying equations possess bounded and unbounded solutions in the interval. However, the spectral methods operate in frequency and not physical space and now it is easy to suppress the unbounded solution. Hence the continuous magnetic field conditions on the boundaries $r = r_i$ and $r = r_0$ are

$$\begin{array}{lll} b_r = -2D_m\psi, & b_z = -in\psi & r = r_0\ , \\ b_r = b_r, & b_z = b_z & r = r_i\ , \\ b_\theta = b_\theta \longrightarrow & D_{ic}b_r = D_{oc}b_r & r = r_i. \end{array} \tag{9.3.21}$$

The last continuous magnetic field condition on the boundary between the inner core and the outer core can be derived from

$$\nabla \times \boldsymbol{b} = \mathbf{0}$$

by taking its θ-component. Hence the boundary condition is

$$\frac{1}{\eta}(inb_r - D_{ic}b_z) - \frac{1}{\eta_i}(inb_r - D_{oc}b_z) = 0 \tag{9.3.22}$$

where η_i and $\eta = 1$ are magnetic diffusivity for inner core and outer core respectively and b_r represents in the first bracket, the inner core and in the second bracket, fluid layer. The last boundary condition of this problem is on the infinite boundary of the mantle and is represented by

$$\psi \to 0 \tag{9.3.23}$$

9.4 The Method of Solution

Let variables $y_1, \ldots, y_{16}$ be defined by

$$\begin{array}{llll} y_1 = b_r\ , & y_2 = D_{ic}b_r\ , & y_3 = b_z\ , & y_4 = D_{ic}b_z\ , \\ y_5 = u\ , & y_6 = D_{oc}u\ , & y_7 = D_{oc}^2 u\ , & \\ y_8 = w\ , & y_9 = D_{oc}w & y_{10} = D_{oc}^2 w\ , & \\ y_{11} = b_r\ , & y_{12} = D_{oc}b_r\ , & y_{13} = b_z\ , & y_{14} = D_{oc}b_z\ . \\ y_{15} = \psi\ , & y_{16} = D_m\psi\ . & & \end{array} \tag{9.4.24}$$

The spectral method needs to transfer a system of basic equations (9.2.13), (9.2.14) and (9.2.15) into a system of a linear ordinary differential equations of first order which can be described in terms of the variables $y_1, \ldots, y_{16}$ by

Inner Core

$$\begin{aligned} D_{ic}y_1 &= y_2 \ , \\ r^2 D_{ic}y_2 &= -3ry_2 + (r^2G - 1)y_1 - 2iny_3 + r^2\frac{\sigma\Lambda}{\eta}y_1 \ , \\ D_{ic}y_3 &= y_4 \ , \\ r^2 D_{ic}y_4 &= -ry_4 + r^2Gy_3 + r^2\frac{\sigma\Lambda}{\eta}y_3 \ , \end{aligned} \tag{9.4.25}$$

Outer Core (Fluid Layer)

$$\begin{aligned} D_{oc}y_5 &= y_6 \ , \\ D_{oc}y_6 &= y_7 \ ., \\ D_{oc}y_7 &= -\frac{4}{r}y_7 + (G - \frac{1}{r^2})y_6 - (\frac{G}{r} - \frac{2n^2}{r} - \frac{1}{r^3} - \frac{im}{rE})y_5 \ , \\ &\quad -\frac{iG}{n}y_{10} - (\frac{iG}{nr} + \frac{2in}{r})y_9 + \frac{iG^2}{n}y_8 + \frac{mG}{En}Fy_{13} \ , \\ &\quad -[\frac{im}{E}y_{12} + \frac{im}{E}F'y_{11}] + \frac{\Lambda E_\eta \sigma}{E}(y_6 + \frac{1}{r}y_5 + \frac{iG}{n}y_8) \ , \\ D_{oc}y_8 &= y_9 \ , \\ D_{oc}y_9 &= y_{10} \ , \\ D_{oc}y_{10} &= -\frac{1}{r}y_{10} + (G + \frac{1}{r^2})y_9 - \frac{2G}{r}y_8 + iny_7 + (\frac{3in}{r} - \frac{nr}{mE})y_6 \ , \\ &\quad -(inG - \frac{in}{r^2} + \frac{n}{mE})y_5 - (\frac{in^2r}{mE} + \frac{2G}{r})y_8 - \frac{im}{E}Fy_{14} \ , \\ &\quad -(\frac{im}{E}F' - \frac{im}{E}\frac{F}{r} - \frac{2in^2}{mE})y_{13} + \frac{2nF}{mE}y_{12} \ , \\ &\quad +(\frac{2n}{mE}\frac{F}{r} - \frac{mn}{E}F)y_{11} + \frac{\sigma\Lambda E_\eta}{E}(y_9 - iny_5) \ , \\ D_{oc}y_{11} &= y_{12} \ , \\ D_{oc}y_{12} &= -im\Lambda Fy_5 - \frac{3}{r}y_{12} + (G - \frac{1}{r})y_{11} - \frac{2in}{r}y_{13} + \Lambda\sigma y_{11} \ , \\ D_{oc}y_{13} &= y_{14} \\ D_{oc}y_{14} &= -im\Lambda Fy_8 - \frac{1}{r}y_{14} + Gy_{13} + \Lambda\sigma y_{13} \ , \end{aligned} \tag{9.4.26}$$

Mantle

$$\begin{aligned} D_m y_{15} &= y_{16} \ , \\ (1-r)^4 D_m y_{16} &= -(1-r)^3 y_{16} + [4n^2 + m^2(1-r)^2]y_{15} \ . \end{aligned} \tag{9.4.27}$$

The equations (9.4.25), (9.4.26) and (9.4.27), can be reformulated in the form of

$$\frac{dY}{dx_3} = AY + \sigma BY$$

where A and B are the complex 16×16 matrices. Also the spectral methods needs to express the boundary conditions (9.3.16), (9.3.17), (9.3.18), (9.3.20), (9.3.21), (9.3.22) and (9.3.23) in terms of the variables $y_1, \ldots y_{16}$. These are

$$\left.\begin{array}{lll} y_2 = y_3 = 0 & \text{when} & m = 1 \\ y_1 = y_3 = 0 & \text{when} & m > 1 \end{array}\right\} \quad \text{on} \quad r = 0\,,$$

$$\left.\begin{array}{r} y_1 - y_{11} = 0 \\ y_3 - y_{13} = 0 \\ y_2 - y_{12} = 0 \\ \eta(iny_1 - y_4) = iny_{11} - y_{14} \end{array}\right\} \quad \text{on} \quad r = r_i\,, \tag{9.4.28}$$

$$y_5 = y_6 = y_8 = 0 \qquad \text{on} \quad r = r_i \text{ and } r = 1\,,$$

$$\left.\begin{array}{r} y_{12} + \frac{1}{r}y_{11} + \frac{i}{n}Gy_{13} = 0 \\ y_{11} + 2y_{16} = 0 \\ y_{13} + iny_{15} = 0 \end{array}\right\} \quad \text{on} \quad r = 1\,,$$

$$y_{15} = 0 \qquad \text{on} \quad r = r_\infty\,.$$

Each variable of $y_1, ..., y_{16}$ is assigned a Chebyshev spectral expansion of order N and the coefficients of expansion of these variables are replaced in a column vector Y of dimension $16(N+1)$. The eigenvalue problem then assumes the format $\hat{E}Y = \sigma\hat{F}Y$, where matrices E and F have row and element representations. Since the Chebyshev polynomial is used, it is necessary to transfer the boundaries of the layers:

For the inner core an interval $[0, r_i]$ to an interval $[-1, 1]$ by a relation

$$r = -1 + 2\frac{2s}{r_i} \qquad \text{for} \quad s \in (0.r_i)$$

For the outer core an interval $[r_i, r_0]$ to an interval $[-1, 1]$ by a relation

$$r = 1 - 2\frac{1-s}{1-r_i} \qquad \text{for} \quad s \in (r_i, 1)$$

For the mantle an interval $[r_0, \infty]$ to an interval $[-1, 1]$ by a relation

$$r = 1 - 2/s \qquad \text{for} \quad s \in (1, \infty)$$

. The complex matrix $\hat{E}$ and $\hat{F}$ are expressed respectively in the forms

$$\hat{E} = \begin{bmatrix}
D_{ic} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_1 & A_2 & \frac{2in}{r^2}I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & D_{ic} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -GI & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_4 & A_1 & A_5 & -\frac{iG^2}{n}I & A_6 & \frac{i}{n}GI & -\frac{im}{E}F'I & \frac{im}{E}I & -\frac{mF}{nE}GI & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_7 & A_8 & -inI & A_9 & A_{10} & A_{11} & A_{12} & -\frac{2n}{mE}FI & A_{13} & \frac{im}{E}FI & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & im\Lambda FI & 0 & 0 & -GI & A_{11} & 2inI & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 \\
0 & 0 & 0 & 0 & im\Lambda FI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -GI & A_{14} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_m & -I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{15} & A_{16}
\end{bmatrix} \qquad (9.4.29)$$

where $A_1 = (\frac{1}{r^2} - G)I$, $A_2 = D_{ic} + \frac{3}{r}I$, $A_3 = D_{ic} + \frac{1}{r}I$, $A_4 = (\frac{G}{r} - \frac{2n^2}{r} - \frac{1}{r^3} + \frac{im}{rE})I$ $A_5 = (D_{oc} + \frac{4}{r}I)$, $A_6 = (\frac{iG}{nr} + \frac{2in}{r})I$, $A_7 = (inG - \frac{in}{r^2} + \frac{n}{mE})I$, $A_8 = (\frac{nr}{mE} + \frac{3in}{r})I$, $A_9 = (\frac{2G}{r} + \frac{(in^2r)}{mE})I$, $A_{10} = -(G + \frac{1}{r^2})I$, $A_{11} = D_{oc} + \frac{1}{r}I$, $A_{12} = (\frac{mnr}{E} - \frac{2n}{mE})\frac{F}{r}I$, $A_{13} = (\frac{im}{E}F' - \frac{im}{E}\frac{F}{r} - \frac{2in^2}{mE}F)I$, $A_{14} = D_{oc} + \frac{1}{r^2}I$, $A_{15} = [\frac{4n^2}{(1-r)^4} + \frac{m^2}{(1-r)^2}]I$ and $A_{16} = D_m + \frac{1}{(1-r)}I$

$$
\hat{F} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Lambda I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\Lambda E_\eta}{E}\frac{1}{r}I & \frac{\Lambda E_\eta}{E}D_{oc} & 0 & \frac{\Lambda E_\eta}{E}\frac{iG}{n}I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -in\frac{\Lambda E_\eta}{E}I & 0 & 0 & 0 & 0 & \frac{\Lambda E_\eta}{E}D_{oc} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{9.4.30}
$$

Now to avoid the non-singular terms of the equations in both inner core and Mantle layers, they are multiplied by r^2 and $(1-r)^4$ respectively, and then the matrices $\hat{E}$ and $\hat{F}$ become respectively

$$E_M = \begin{bmatrix}
D_{ic} & -I & 0 & .0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_1 & B_2 & 2in & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & D_{ic} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -r^2GI & B_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & B_4 & B_5 & B_6 & -\frac{iG^2}{n}I & B_7 & \frac{i}{n}GI & -\frac{im}{E}F'I & \frac{im}{E}I & -\frac{mF}{nE}GI & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & B_8 & B_9 & -inI & B_{10} & B_{11} & B_{12} & B_{13} & -\frac{2nF}{mE}I & B_{14} & \frac{im}{E}FI & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & im\Lambda FI & 0 & 0 & -GI & B_{12} & 2inI & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{oc} & -I & 0 & 0 \\
0 & 0 & 0 & 0 & im\Lambda FI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -GI & B_{15} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_m & -I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{16} & B_{17}
\end{bmatrix} \qquad (9.4.31)$$

where $B_1 = (1 - r^2G)I$, $B_2 = r^2D_{ic} + 3rI$, $B_3 = r^2D_{ic} + rI$, $B_4 = (\frac{G}{r} - \frac{2n^2}{r} - \frac{1}{r^3} + \frac{im}{rE})I$, $B_5 = (\frac{1}{r^2} - G)I$, $B_6 = (D_{oc} + \frac{4}{r}I)$, $B_7 = (\frac{iG}{nr} + \frac{2in}{r})I$, $B_8 = (inG - \frac{in}{r^2} + \frac{n}{mE})I$, $B_9 = (\frac{nr}{mE} + \frac{3in}{r})I$, $B_{10} = (\frac{2G}{r} + \frac{(in^2r)}{mE})I$, $B_{11} = -(G + \frac{1}{r^2})I$, $B_{12} = D_{oc} + \frac{1}{r}I$, $B_{13} = (\frac{mnr}{E} - \frac{2n}{mE})\frac{F}{r}I$, $B_{14} = (\frac{im}{E}F' - \frac{im}{E}\frac{F}{r} - \frac{2in^2}{mE}F)I$, $B_{15} = D_{oc} + \frac{1}{r^2}I$, $B_{16} = [4n^2 + m^2(1-r)^2]I$ and $B_{17} = (1-r)^4D_m + (1-r)^3I$

$$
F_M = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda r^2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Lambda r^2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\Lambda E_\eta}{E}\frac{1}{r}I & \frac{\Lambda E_\eta}{E}D_{oc} & 0 & \frac{\Lambda E_\eta}{E}\frac{iG}{n}I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -in\frac{\Lambda E_\eta}{E}I & 0 & 0 & 0 & 0 & \frac{\Lambda E_\eta}{E}D_{oc} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \qquad (9.4.32)
$$

The complex matrices E_M and F_M are now used in a QZ algorithem (F02GJF of a NAG routine) to compute their eigenvalues. Some coefficients of the system of ordinary differential equations (9.4.25), (9.4.26) and (9.4.27) are functions which should be treated by a Chebyshev polynomial expansion and these functions are:

For Rigid Inner Core

$$f_{+2}(r) = r^2, \qquad f_{+1}(r) = r$$

For Outer Core (Fluid Layer)

$$f_1(r) = \frac{1}{r}, \qquad f_2(r) = \frac{1}{r^2}, \qquad f_3(r) = \frac{1}{r^3}, \qquad f_4(r) = \frac{1}{r^4}$$

$$f_5 = B(r), \qquad f_6(r) = \frac{B(r)}{r}, \qquad f_7(r) = \frac{B(r)}{r^2}, \qquad f_8 = \frac{B(r)}{r^3}, \qquad f_9(r) = B'(r)/r$$

where $B(r) = rF(r)$ and $F(r)$ is the basic field generated by one of the two forms

$$\begin{aligned} F(r) &= r^\alpha && \text{for} \quad m = 1\,, \\ F(r) &= \frac{1}{1+\alpha}\left(\frac{4(1-r^\beta)(r^\beta - r_i^\beta)}{(1-r_i^\beta)^2} + \alpha\right) && \text{for} \quad m > 1\,. \end{aligned} \tag{9.4.33}$$

where m is azimuthal wavenumber. Here α and β are arbitrary parameters whose values are chosen to mimic the behavior of the terrestrial magnetic field. The first field is $(9.4.33)_1$ is monotone increasing for $\alpha > 0$ whereas the second field attains a maximum value in $(r_i, 1)$. The latter is a more realistic form for the magnetic field strength since this expression vanishes at $r = 1$ and $r = r_i$ when $\alpha = 0$. Both are normalised so that the maximum value of $F(r)$ in $(r_i, 1)$ is unity, hence the reason for the multiplier $(1+\alpha)^{-1}$. Here $\alpha = \beta = 1$. For rigid mantle

$$f_{m4} = (1-r)^4, \qquad f_{m3} = (1-r)^3, \qquad f_{m2} = (1-r)^2\,.$$

A matrix for each of the above functions can be calculated element-wise by using a Chebyshev series of the form

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)\,.$$

Each of the boundary conditions (9.4.28) is incorporated into the complex matrices E_M and F_M. The last rows $(N+1)th, 2(N+1)th, ..., 16(N+1)th$ of the matrices E_M and F_M contain the boundary conditions (9.4.28) as

Manifestation in Matrix	
E_M	F_M
$y_1 = 0$ when $m > 1$	$r = 0$,
$[\mathrm{p}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_2 = 0$ when $m = 1$	$r = 0$,
$[0, \mathrm{p}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_3 = 0$	$r = 0$,
$[0, 0, \mathrm{p}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_1 - y_{11} = 0$	$r = r_i$,
$[\mathrm{q}, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\mathrm{p}, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_2 - y_{12} = 0$	$r = r_i$,
$[0, \mathrm{q}, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\mathrm{p}, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_3 - y_{13} = 0$	$r = r_i$,
$[0, 0, \mathrm{q}, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\mathrm{p}, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$\eta(iny_1 - y_4) = iny_{11} - y_{14}$	$r = r_i$,
$[\eta\mathrm{q}, 0, 0, \frac{-\eta}{in}\mathrm{q}, 0, 0, 0, 0, 0, 0, \mathrm{p}, 0, 0, \frac{-1}{in}\mathrm{p}, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_5 = 0$	$r = r_i, 1$,
$[0, 0, 0, 0, \mathrm{p}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_6 = 0$	$r = r_i, 1$,
$[0, 0, 0, 0, 0, \mathrm{p}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_8 = 0$	$r = r_i, 1$,
$[0, 0, 0, 0, 0, 0, 0, \mathrm{p}, 0, 0, 0, 0, 0, 0, , 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_{12} + y_{11} + (i/n)(m^2 + n^2)y_{13} = 0$	$r = 1$,
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathrm{q}, \mathrm{q}, \frac{ik}{n}\mathrm{q}, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_{11} + 2y_{12} = 0$	$r = 1$,
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathrm{q}, 2\mathrm{q}, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_{13} + iny_{15} = 0$	$r = 1$,
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathrm{q}, 0, in\mathrm{p}, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
$y_{15} = 0$	$r = \infty$,
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathrm{q}, 0]$	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$.

where $k = m^2 + n^2$. Now the system $E_M Y = \sigma F_M Y$ is prepared to calculate eigenvalue when $\sigma = 0$ as illustrated in Appendix 7.

9.5 Results

The results can be represented as eigenvalues (values of σ) of together with plots of the real and imaginary parts of the corresponding eigenfunctions. Lamb investigates field gradient (ideal) instability, resistive instability and exceptional instability for $m = 1$ and $m > 1$ and the appropriate expression for $F(r)$ in (9.4.33). A selection of Lamb's results are reproduced here. Good agreement is found for some cases and in others it would appear that Lamb's analysis has missed the critical eigenvalue.

Numerical Procedure

For each set of numbers Λ, n_c, η and with the other parameters of this problem, the complex eigenvalues in this chapter are listed in a way such that the critical complex eigenvalue is the top value in the list and the other eigenvalues are listed in order. If there are two eigenvalues in table (9.2) for the same numbers Λ_c, n_c and η_i , then the first one is the first critical complex eigenvalue and the second one is the second complex eigenvalue in the list. The eigenfunctions of velocity components u_r and u_z and magnetic flux components b_r and b_z can be calculated directly but the second component of both velocity u_θ and magnetic flux b_θ can be calculated in terms of the other components u_r, u_z, b_r and b_z respectively using $\nabla.\mathbf{u} = 0 \quad and \quad \nabla.\mathbf{b} = 0$. These eigenfunctions have been normalised as $max|b_z| = 1$ and they are plotted as a real part and as an imaginary part.

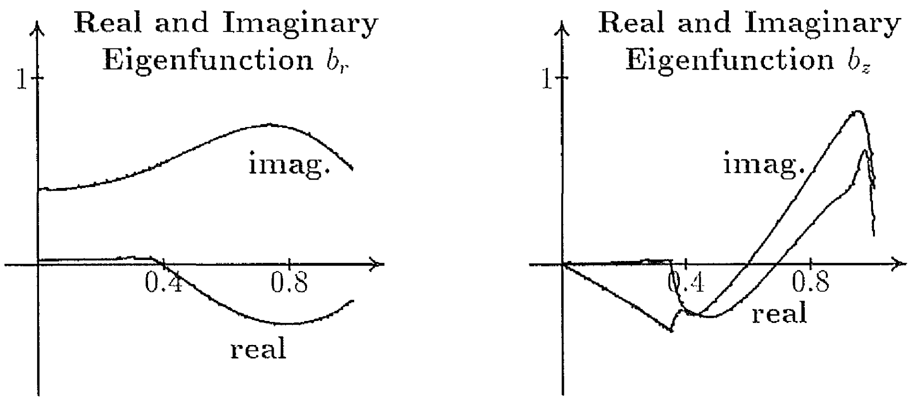
Comparing the results of C.J. Lamb's thesis with the results of this thesis for the complex eigenvalues and the corresponding eigenfunctions b_r, b_z, u_r and u_z, when the corresponding real eigenvalues are zero, some results are the same while the others are different. The results, both here and in Lamb's thesis, are evaluated for a chosen value of Λ and a chosen value of n with $E = E_\eta = 10^5$ and for different value of η_i by using Chebyshev spectral methods and LR algorithm respectively.

The results for field $(9.4.33)_1$ with a wavenumber $m = 1$ and $\alpha = 1$ are expressed in table

(9.1). The eigenfunctions b_r and b_z are plotted respectively in figures (9.1): a and b for $\Lambda = 192.42$, wavenumber $n_c = 2.370$ and the corresponding eigenvalue $\omega_c = -0.5053.$, and they are plotted respectively in figures (9.2): a and b for $\Lambda = 210.43$, $n_c = 4.508$ and the corresponding eigenvalue $\omega_c = -0.9799$. Table (9.1) shows the differences between the eigenvalues here and the eigenvalues in Lamb's [25]) thesis. Figures (9.1): a and b and (9.2): a and b for b_r and b_z show the differences between the eigenfunctions b_r and b_z here and those in Lamb's thesis.

Λ_c	n_c	w_c	η_i	Lamb's result	Discerption
192.42	2.370	-0.5053	10^3	-0.5052	the same
210.43	4.508	-0.9799	10^3	-0.9800	the same

Table 9.1: This table shows the value of Λ_c, n_c, w_c and η_i for first field $(9.4.33)_1$.

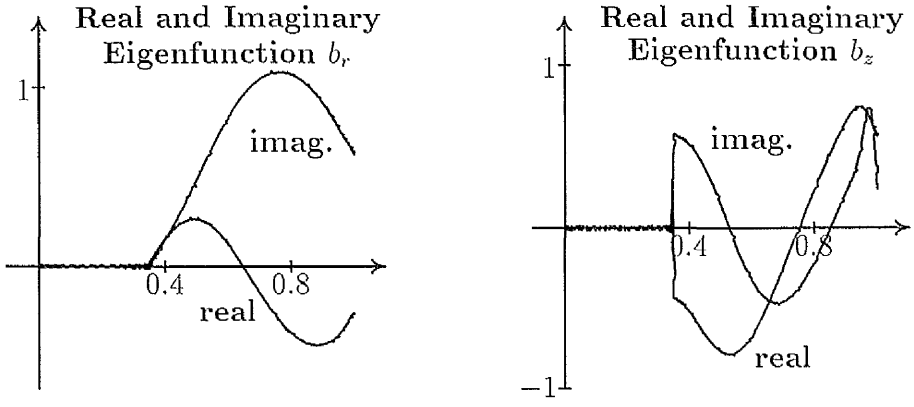


a: Eigenfunction b_r

b: Eigenfunction b_z

Figure 9.1: Critical eigenfunctions b_r and b_z when $\Lambda = 192.42$, $n_c = 2.370$ and $\eta_i = 10^3$.

The results in field $(9.4.33)_2$ with wavenumber $m = 2$, $\alpha = 0$ and $\beta = 1$ are expressed in table 9.2 which shows identical and different eigenvalues here and in Lamb's thesis. The first critical eigenvalues (the first values in both lists) are identical, the second critical eigenvalues (the first values in the list) are different while the third values (the second values in the list) are the same. The fourth critical eigenvalues (the first values in the list) is different, the fifth critical eigenvalues (the first values in the list) are different, while the sixth critical eigenvalues (the second values in the list) are the same. The eigenfunctions b_r, b_z, u_r and u_z are plotted respectively in figures 9.3 a and 9.3 b and 9.4 a and 9.4 b when $\Lambda = 508$, the wavenumber $n_c = 9.61$, $\eta_i = 10^3$ and the corresponding first critical



a: Eigenfunction b_r b: Eigenfunction b_z .

Figure 9.2: Critical eigenfunctions b_r and b_z when $\Lambda = 210.43$, $n_c = 4.508$ and $\eta_i = 10^3$.

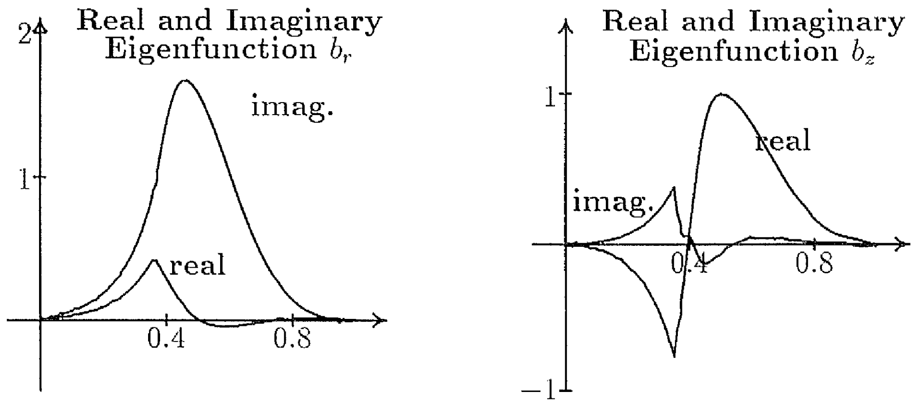
complex eigenvalue $\omega_c = -0.934$ in the table 9.2 are the same as those of Lamb. The eigenfunctions b_r, b_z, u_r and u_z are plotted respectively in figures 9.5 a and 9.5 b and 9.6 a and 9.6 b for $\Lambda = 623$, $n_c = 8.25$, $\eta_i = 1$.The corresponding second critical complex eigenvalues $\omega_c = 0.6024$ in the table (9.2) are different from those of Lamb, while the eigenfunctions are plotted respectively in figures 9.7 a and 9.7 b and 9.8 a and 9.8 b. The corresponding third complex eigenvalues (the second complex eigenvalues in the list) $\omega_c = -0.5677$ are the same as those of Lamb. The eigenfunctions b_r, b_z, u_r and u_z are plotted respectively in figures 9.9 a and 9.9 b and 9.10 a and 9.10 b when $\Lambda = 1213$, $n_c = 15.1$, $\eta_i = 10^{-3}$. The corresponding fourth critical complex eigenvalues $\omega_c = -0.971$ in the table (9.2) are different from those of Lamb. The eigenfunctions b_r, b_z, u_r and u_z are plotted respectivelyin figures (9.11): a and b and 9.12 a and 9.12 b when $\Lambda = 106$, $n_c = 4.3$, $\eta_i = 10^{-3}$ and the corresponding fifth critical complex eigenvalues $\omega_c = 0.377$ in the table 9.2 are different from those of Lamb, while the eigenfunctions which are plotted in figures (9.13): a and b and 9.14 a and 9.14 b and the corresponding sixth complex eigenvalues (the second complex eigenvalue in the list) $\omega_c = -0.001314$ are the same as those of Lamb.

Conclusion

The results obtained in this thesis indicate that the Chebyshev Tau method has the ability to handle eigenvalue problems easily and accurately. This method simplifies single and multi-layered problems, even with the boundary conditions that usually require much effort to be put into a useful form. The constant and variable coefficients of the basic

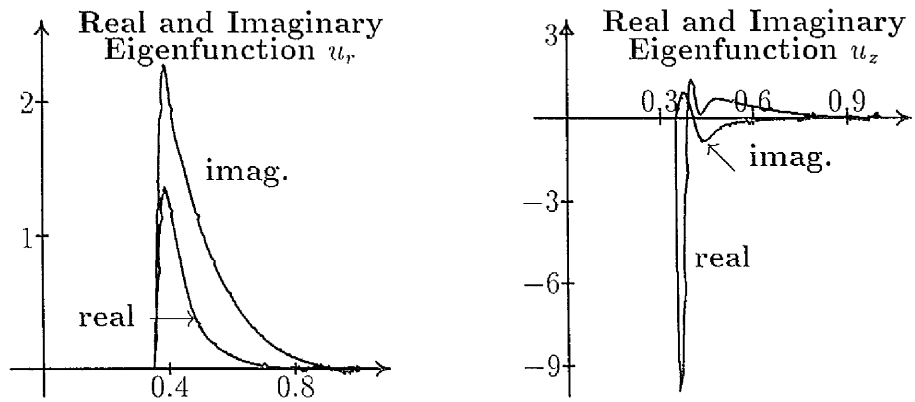
Λ_c	n_c	w_c	η_i	Lamb's result	Discerption
508	9.61	-0.9342	10^3	-0.934	identical
623	8.25	0.6024	1		different
623	8.25	-0.5677	1	-0.568	the same
1213	15.1	-0.971	10^{-3}	-0.969	different
106	4.3	0.377	10^{-3}		different
106	4.3	-0.01314	10^{-3}	-0.0131	different

Table 9.2: This table shows the value of Λ_c, n_c, w_c and η_i for the second field $(9.4.33)_2$.



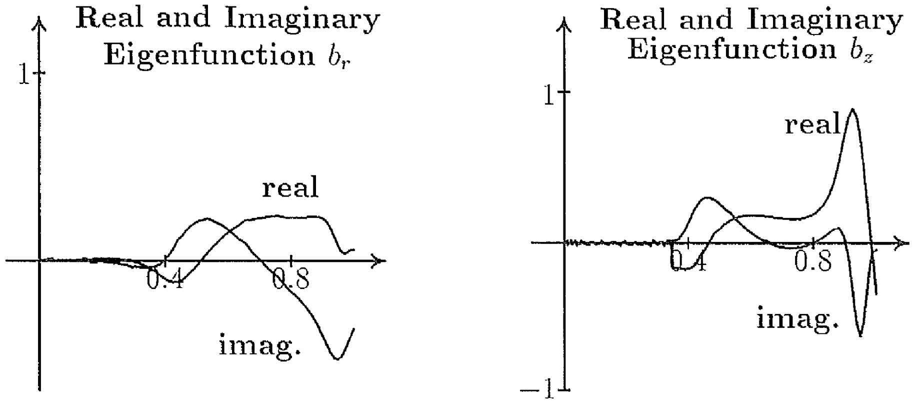
a: Eigenfunction b_r b: Eigenfunction b_z .

Figure 9.3: Critical eigenfunctions b_r and b_z when $\Lambda_c = 508$, $n_c = 9.61$, $\eta_i = 10^3$ and $\omega_c = -0.934$.



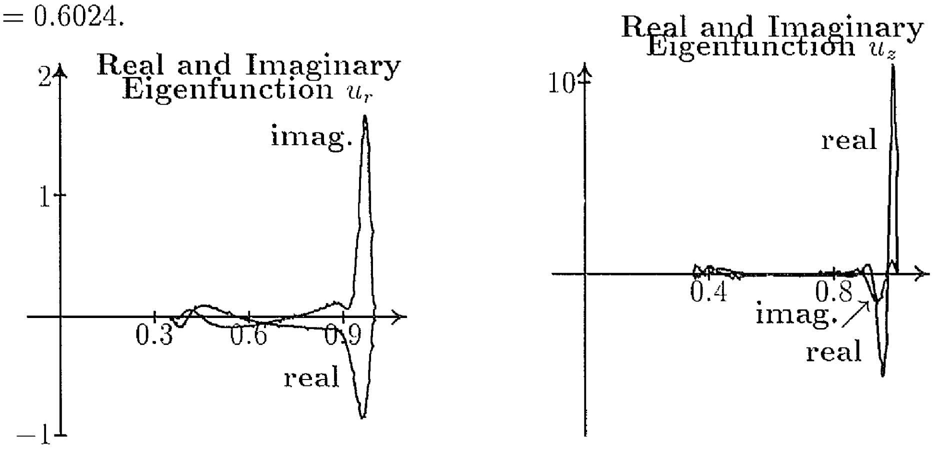
a: Eigenfunction u_r b: Eigenfunction u_z .

Figure 9.4: Critical eigenfunctions u_r and u_z $\Lambda_c = 508$, $n_c = 9.61$, $\eta_i = 10^3$ and $\omega_c = -0.934$.



a: Eigenfunction b_r b: Eigenfunction b_z .

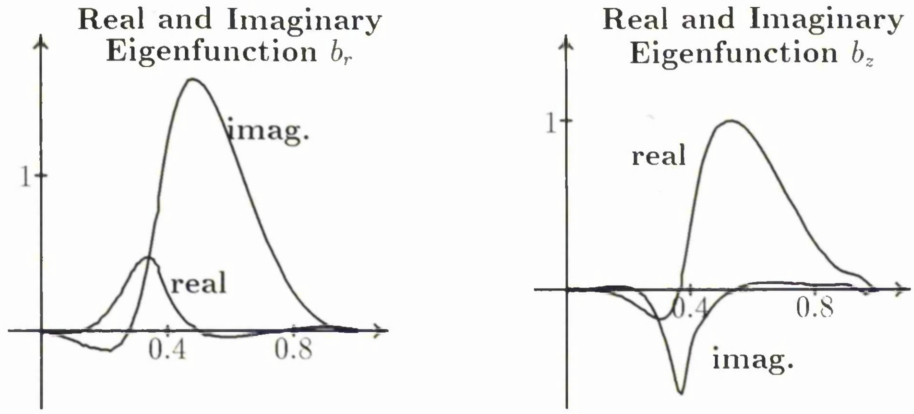
Figure 9.5: Critical eigenfunction b_r and b_z when $\Lambda_c = 623$,$n_c = 8.25$, $\eta_i = 1$ and $\omega_c = 0.6024$.



a: Eigenfunction u_r b: Eigenfunction u_z.

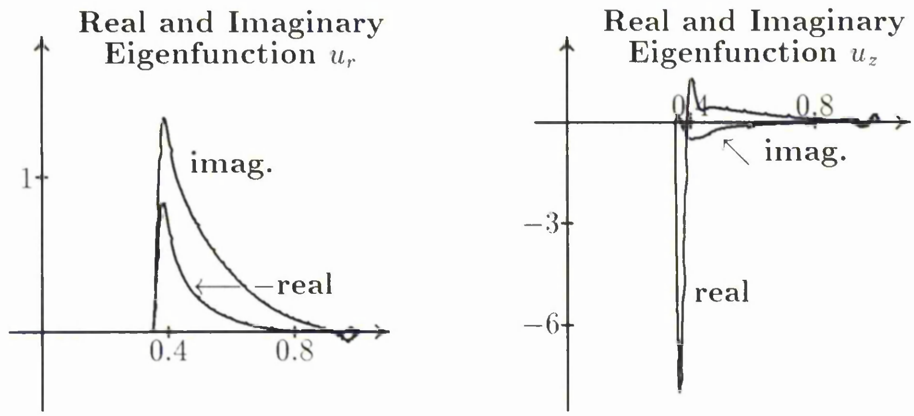
Figure 9.6: Critical eigenfunctions u_r and u_z when $\Lambda_c = 623$, $n_c = 8.25$, $\eta_i = 1$ and $\omega_c = 0.6024$.

equations in this problem do not cause any difficulty, as can be determined by this study. In addition, if there are some coefficients causing singularity in the equations, then it is required to multiply the equations by these coefficients so that they can be calculated directly using Chebyshev expansion. Moreover, the eigenvalue problems can be solved by using a second order system (D^2) as well as a first order system (D). In fact, the second order system gives a good opportunity to solve higher order differential equation problems by modifying some boundary conditions if necessary. Now Legendre spectral method has the same ability as the Chebyshev Tau method to treat eigenvalue problems, but the product variables using Legendre expansion is a difficult matter.



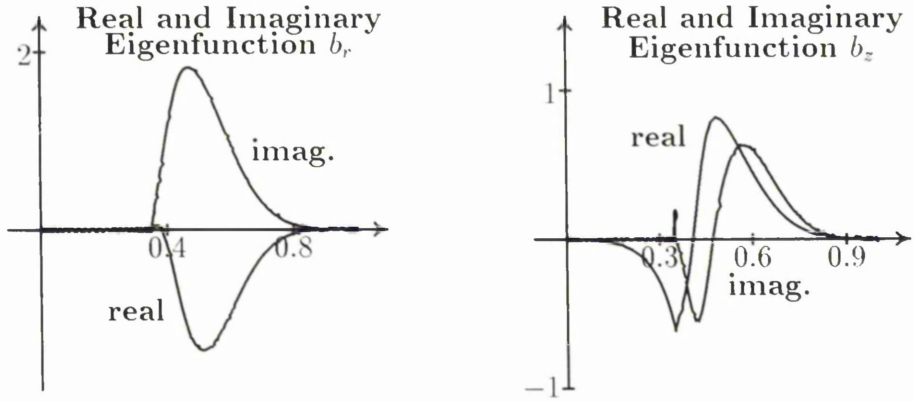
a: Eigenfunction b_r b: Eigenfunction b_z .

Figure 9.7: Eigenfunctions b_r and b_z when $\Lambda_c = 623$, $n_c = 8.25$, $\eta_i = 1$ and $\omega_c = -0.5677$.



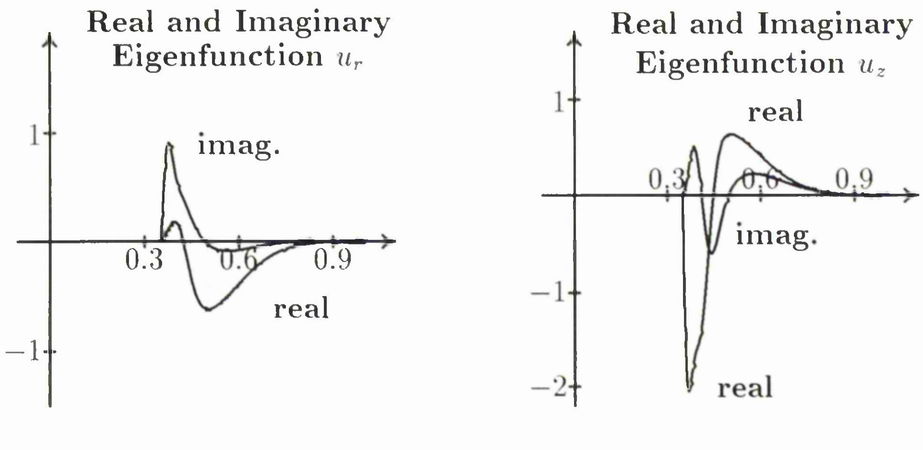
a: Eigenfunction u_r b: Eigenfunction u_z .

Figure 9.8: Eigenfunctions u_r and u_z when $\Lambda_c = 623$, $n_c = 8.25$, $\eta_i = 1$ and $\omega_c = -0.5677$.



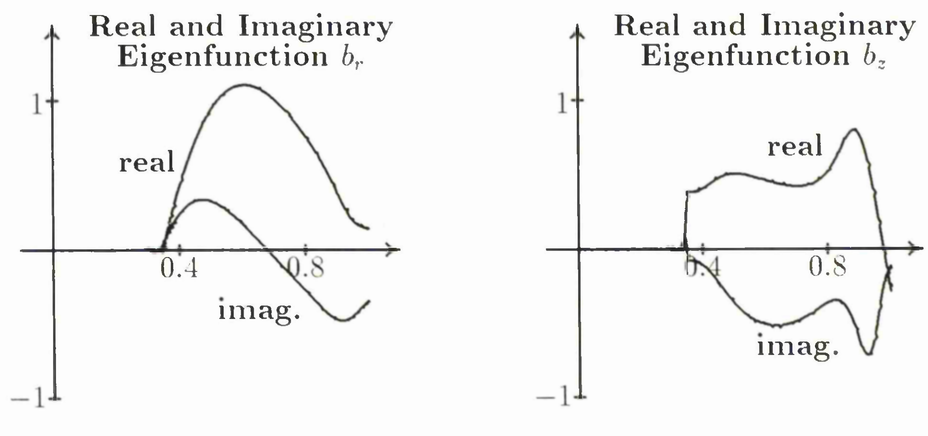
a: Eigenfunction b_r b: Eigenfunction b_z .

Figure 9.9: Critical eigenfunctions b_r and b_z when $\Lambda_c = 1213$, $n_c = 15.1$, $\eta_i = 10^{-3}$ and $\omega_c = -0.971$.



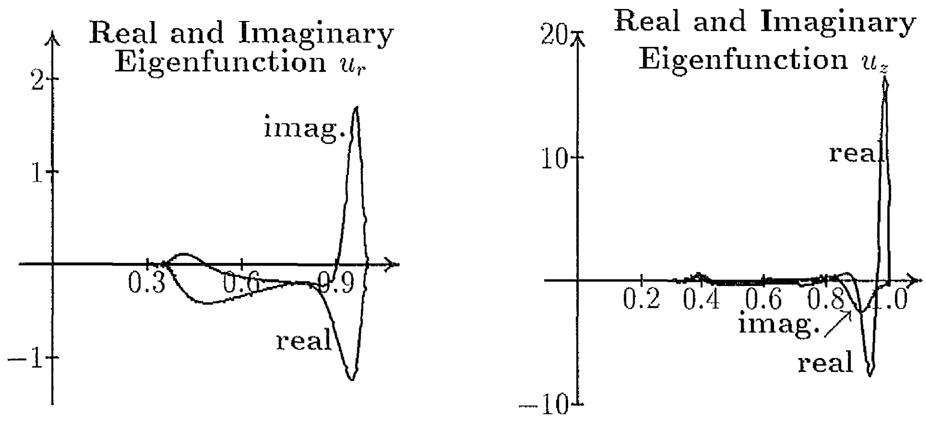
a: Eigenfunction u_r b: Eigenfunction u_z.

Figure 9.10: Critical eigenfunctions u_r and u_z when $\Lambda_c = 1213$, $n_c = 15.1$, $\eta_i = 10^{-3}$ and $\omega_c = -0.971$.



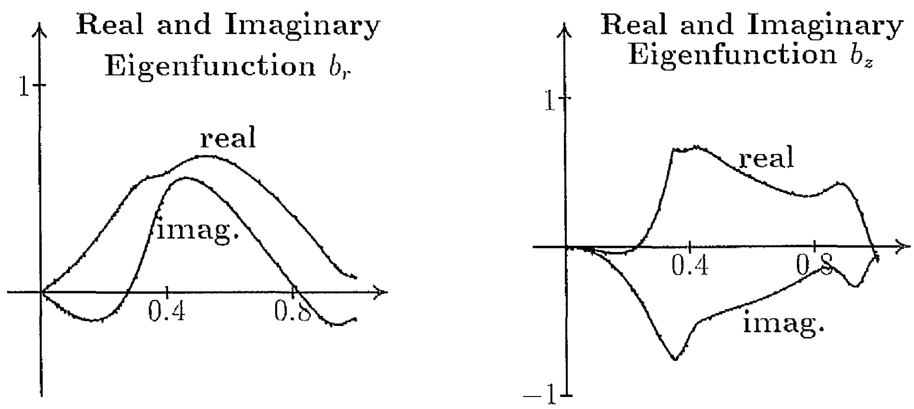
a: Eigenfunction b_r b: Eigenfunction b_z .

Figure 9.11: Critical eigenfunctions b_r and b_z when $\Lambda_c = 106$, $n_c = 4.30$, $\eta_i = 10^{-3}$ and $\omega_c = 0.377$.



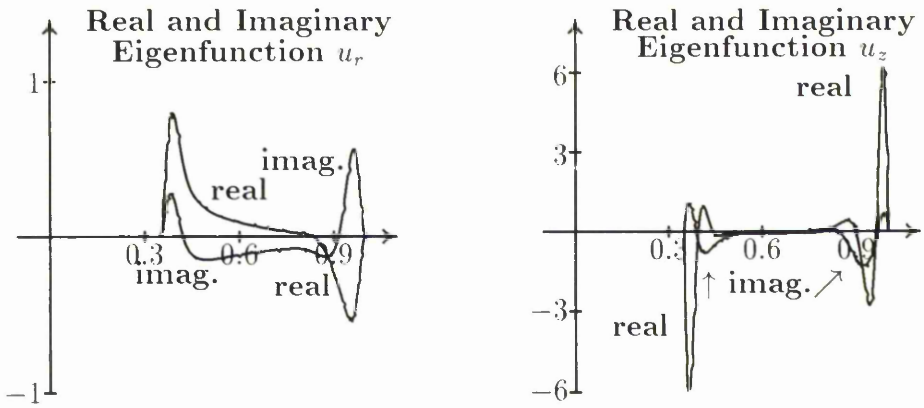
a: Eigenfunction u_r b: Eigenfunction u_z.

Figure 9.12: Critical eigenfunctions u_r and u_z when $\Lambda_c = 106$, $n_c = 4.30$, $\eta_i = 10^{-3}$ and $\omega_c = 0.377$.



a: Eigenfunction b_r b: Eigenfunction b_z.

Figure 9.13: Eigenfunctions b_r and b_z when $\Lambda_c = 106$, $n_c = 4.30$, $\eta_i = 10^{-3}$ and $\omega_c = -0.001314$.



a: Eigenfunction u_r

b: Eigenfunction u_z.

Figure 9.14: Eigenfunctions u_r and u_z when $\Lambda_c = 106$, $n_c = 4.30$, $\eta_i = 10^{-3}$ and $\omega_c = -0.001314$.

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Appendix A

Suppose that $f(x)$ $x \in [a, b]$ is a continuous function then

$$x = a + \frac{1}{2}(z+1)(b-a), \qquad x \in [a,b] \quad z \in [-1,1]\,, \tag{9.5.34}$$

maps $[a, b]$ into $[-1, 1]$. Let $F(z) = f(x)$ then by the orthogonal property of Chebyshev polynomials, the spectral series for F is

$$f(x) = F(z) = \sum_{r=0}^{\infty} f_r T_r(z), \qquad f_r = \frac{1}{c_r}\int_{-1}^{1} \frac{F(z)T_r(z)dz}{\sqrt{1-z^2}}, \tag{9.5.35}$$

with the assumption such a series exists. Of course, in a practical application, the series (9.5.35) for f is truncated to polynomials of degree N and the coefficients $f_0, \ldots, f_N$ are evaluated from (9.5.35) by Gaussian quadrature methods. Here I employ a Gauss-Chebyshev-Lobatto quadrature based on the nodes

$$\left.\begin{aligned} x_k &= a\cos^2(k\pi/2n) + b\sin^2(k\pi/2n) \\ z_k &= -\cos^2(k\pi/2n) + \sin^2(k\pi/2n) = -\cos(k\pi/n) \end{aligned}\right\} \quad k = 0, \ldots, n \tag{9.5.36}$$

From the first and second relations (9.5.35) at the nodes (9.5.36) yield

$$\begin{aligned} f(x_j) &= \sum_{r=0}^{N} f_r T_r(z_j)\,, \\ &= \sum_{r=0}^{N} T_r(z_j)\frac{1}{c_r}\int_{-1}^{1}\frac{F(z)T_r(z)dz}{\sqrt{1-z^2}} \\ &= \sum_{r=0}^{N}\frac{1}{c_r}\int_{-1}^{1}\frac{F(z)T_r(z)T_r(z_j)dz}{\sqrt{1-z^2}} \\ &= F_j \end{aligned} \tag{9.5.37}$$

after apply multiplying and adding processes to the function (9.5.37) yield

$$\frac{\pi}{Nc_k}\sum_{j=0}^{N}\frac{1}{c_j}F_jT_k(z_j) = \frac{\pi}{Nc_k}\sum_{j=0}^{N}\frac{1}{c_j}T_k(z_j)\sum_{r=0}^{N} f_r T_r(z_j)\,, \tag{9.5.38}$$

substitute from (9.5.37) into (9.5.38) to obtain

$$\begin{aligned} \frac{\pi}{Nc_k}\sum_{j=0}^{N}\frac{1}{c_j}F_jT_k(z_j) &= \frac{\pi}{Nc_k}\sum_{j=0}^{N}\frac{1}{c_j}T_k(z_j)f(x_j)\,, \\ &= \frac{\pi}{Nc_k}\sum_{j=0}^{N} F(z_j)T_k(z_j)\,. \end{aligned} \tag{9.5.39}$$

now substitute from the relation $(9.5.35)_2$ into (9.5.39) to get

$$\frac{\pi}{Nc_k}\sum_{j=0}^{N}\frac{1}{c_j}F_jT_k(z_j) = \frac{1}{c_r}\int_{-1}^{1}\frac{F(z)T_r(z)dz}{\sqrt{1-z^2}} = f_k\,. \tag{9.5.40}$$

Hence $F_k = f(x_k) \quad 0 \le k \le N$, the value of f at the nodes (9.5.36) and $f_0, \ldots, f_N$, the coefficients of the Chebyshev spectral series (9.5.40) are conected by

$$\begin{aligned} f_k &= \frac{\pi}{Nc_k}\sum_{j=0}^{N}\frac{1}{c_j}F_j\cos(kj\pi/N)\ , \quad k = 0,\ldots,N\ , \\ F_j &= \sum_{k=0}^{N} f_k\cos(kj\pi/N)\ , \qquad\qquad j = 0,\ldots,N\ , \end{aligned} \tag{9.5.41}$$

where

$$c_k = \begin{cases} 2 & k = 0 \text{ or } N, \\ 1 & 0 < r < N. \end{cases}$$

Appendix B

The sub-matrices F_{ri} are symmetric once the first row and column are disregarded. Here is a flow diagram for coding F_{ri} assuming that the starting matrix is an array of zeros.

- Disregard the first row and column of F_{ri}. Thereafter symmetry prevails and so fill in the (i, j)th entry for $j > i$ and use symmetry to also fill (j, i)th entry.
- Add diagonal entries to F_{ri}.
- Add extra entries to first row.

Appendix 1

This appendix contains the sample of using matrix in Chebyshev spectral method and subroutines of all programs that were used to get the results of the problems.

```
C   .. THIS SUBROUTINE IS USED TO GIVE THE VALUE OF FIRST DERIVATIVE FOR
C   .. CHEBYSHEV SPECTRAL METHODS OVER AN INTERVAL [-1,1] FOR M POLYNOMIAL ..
       SUBROUTINE DERIV1_C(M,D)
       DOUBLE PRECISION D(M,*)
       DO 100 I=1,M
          DO 200 J=1,M
             D(I,J) = 0.D0
200       CONTINUE
100    CONTINUE
       DO 300 I=1,M-1
          DO 400 J=I+1,M,2
             D(I,J) = DBLE(2*J-2)
400       CONTINUE
          D(1,I) = 0.5D0*D(1,I)
300    CONTINUE
       D(1,M) = 0.5D0*D(1,M)
       RETURN
       END

C   .. THIS SUBROUTINE USING TO GIVE THE VALUE OF SECOND DERIVATIVE OF
C   .. CHEBYSHEV SPECTRAL METHODS FOR AN INTERVAL [-1,1] FOR M POLYNOMIAL ..
       SUBROUTINE DERIV2_C(M,DD)
       DOUBLE PRECISION DD(M,*)
       DO 100 I=1,M
          DO 200 J=1,M
             DD(I,J) = 0.D0
200       CONTINUE
100    CONTINUE
       DO 300 I=1,M
          DO 400 J=I+2,M,2
             DD(I,J) = DBLE((J-1)*(J+I-2)*(J-I))
400       CONTINUE
          DD(1,I) = 0.5D0*DD(1,I)
300    CONTINUE
       DD(1,M) = 0.5D0*DD(1,M)
       RETURN
       END

C   .. THIS SUBROUTINE IS USED TO GIVE THE VALUE OF FIRST DERIVATIVE FOR
C   .. LEGENDRE SPECTRAL METHODS OVER AN INTERVAL [-1,1] FOR M POLYNOMIAL ..
       SUBROUTINE DERIV1_L(M,D)
       DOUBLE PRECISION D(0:M-1,0:M-1)
```

```
      DO 100 I=0,M-1
         DO 200 J=0,M-1
            D(I,J) = 0.D0
200      CONTINUE
100   CONTINUE
      DO 300 I=0,M-1
         DO 400 J=I+1,M-1,2
            D(I,J) = DBLE(2*I+1)
400      CONTINUE
300   CONTINUE
      RETURN
      END

C  .. THIS SUBROUTINE IS USED TO GIVE THE VALUE OF FIRST DERIVATIVE FOR
C  .. LEGENDRE SPECTRAL METHODS OVER AN INTERVAL [-1,1] FOR M POLYNOMIAL ..
      SUBROUTINE DERIV2_L(M,D)
      DOUBLE PRECISION D(0:M-1,0:M-1)
      DO 100 I=0,M-1
         DO 200 J=0,M-1
            D(I,J) = 0.D0
200      CONTINUE
100   CONTINUE
      DO 300 I=0,M-1
         DO 400 J=I+1,M-1,2
            D(I,J) = DBLE(2*I+1)
400      CONTINUE
300   CONTINUE
      DO 500 I=1,M-1
         DO 600 J=1,M
            TEMP = 0.D0
            DO 700 K=1,M
               TEMP = TEMP+D(I,K)*D(K,J)
700         CONTINUE
            DD(I,J) = TEMP
600      CONTINUE
500   CONTINUE
      RETURN
      END

C  .. THIS SUBROUTINE IS USED TO GIVE THE VALUE OF FIRST DERIVATIVE FOR
C  .. ALL PROBLEMS OVER AN INTERVAL [0,1] FOR M POLYNOMIAL ..
      SUBROUTINE DERIV_1(M,D)
      DOUBLE PRECISION D(M,*)
      DO 100 I=1,M
         DO 200 J=1,M
            D(I,J) = 0.D0
200      CONTINUE
100   CONTINUE
```

```
      DO 300 I=1,M-1
         DO 400 J=I+1,M,2
            D(I,J) = DBLE(4*J+4)
400      CONTINUE
         D(1,I) = 0.5D0*D(1,I)
300   CONTINUE
      D(1,M) = 0.5D0*D(1,M)
      RETURN
      END

C  .. THIS SUBROUTINE IS USED TO GIVE THE VALUE OF SECOND DERIVATIVE OF
C  .. ALL PROBLEMS OVER AN INTERVAL [0,1] FOR M POLYNOMIAL ..
      SUBROUTINE DERIV2_C(M,D)
      DOUBLE PRECISION D(M,*), DD(M,*)
      DO 100 I=1,M
         DO 200 J=1,M
            D(I,J) = 0.D0
            DD(I,J) = 0.D0
200      CONTINUE
100   CONTINUE
      DO 300 I=1,M-1
         DO 400 J=I+1,M,2
            D(I,J) = DBLE(4*J-4)
400      CONTINUE
         D(1,I) = 0.5D0*D(1,I)
300   CONTINUE
      D(1,M) = 0.5D0*D(1,M)
      DO 500 I=1,M-1
         DO 600 J=1,M
            TEMP = 0.D0
            DO 700 K=1,M
               TEMP = TEMP+D(I,K)*D(K,J)
700         CONTINUE
            DD(I,J) = TEMP
600      CONTINUE
500   CONTINUE
      RETURN
      END


C .. MINIMISATION SUBROUTINE USING FOR FINDING CRITICAL WAVENUMBER ..
C .. XLEFT AND XRIGHT ARE END VALUES OF AN INTERVAL IN WHICH THE VALUE
C .. OF WAVENUMBER EXISTS ..
C .. XSTAT IS THE WAVENUMBER ..
C .. TOL DETERMINES THE ACCURACY OF SEEKING VALUE ..
C .. VALUE IS THE SEEKING VALUE ..
C .. G IS THE FUNCTION ..
      MINMUM(XLEFT,XRIGHT,XSTAT,TOL,VALUE,G)
```

```
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      GRATIO = 0.618033988749890D0
      COEFF1 = -2.07808692123500D0
      D = ABS(COEFF1*LOG(TOL/(ABS(XRIGHT - XLEFT))))
      N = INT(D)
      A = XLEFT
      B = XRIGHT
      XLOWER = A + (B - A)*GRATIO**2
      XUPPER = A + (B - A)*GRATIO
      VLOWER = G(XLOWER)
      VUPPER = G(XUPPER)
      DO 5 J=1,N
         IF (VLOWER.GE.VUPPER) THEN
            A = XLOWER
            XLOWER = XUPPER
            VLOWER = VUPPER
            XUPPER = A + (B - A)*GRATIO
            VUPPER = G(XUPPER)
         ELSE
            B = XUPPER
            XUPPER = XLOWER
            VUPPER = VLOWER
            XLOWER = A + (B - A)*GRATIO**2
            VLOWER = G(XLOWER)
         ENDIF
         DIFF = ABS(VUPPER - VLOWER)
         IF (DIFF.LE.TOL) GOTO 102
5     CONTINUE
102   XSTAT = 0.5D0*(XUPPER + XLOWER)
      VALUE = G(XSTAT)
      RETURN
      END
```

Appendix 2

This appendix contains two FORTRAN77 programs to perform a stability analysis for the Orr-Sommerfeld eigenvalue problem using the Chebyshev spectral tau method. The first program treats the OS equation as a system of 4 first order differential equations whereas the second and the program treats the OS equation as a pair of second order equations for the conventional and modified boundary value technique. Poiseuille and Couette flow are implemented by calling the appropriate subroutine for the auxiliary matrices $PandQ$.

Fourth order system

```
      PROGRAM ORRSOM
**********************************************************************
*                                                                    *
*    USES D APPROACH ON THE ORR-SOMMERFELD EQUATION IN THE FORM      *
*                                                                    *
*        DY_1-Y_2=0,          DY_2-Y_3=0,           DY_3-Y_4=0,      *
*                                                                    *
*           DY_4-2a^2Y_3+a^4Y_1-iaRQ(Y_3-a^2Y_1)-2iaRY_1             *
*                                     = -iaR LAMBDA(Y_3-a^2Y_1)      *
*                                                                    *
*    WITH BOUNDARY CONDITIONS Y_1=0, Y_2=0  ON x=-1, 1               *
*                                                                    *
**********************************************************************

      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
C
C .. DECLARATIONS FOR QZ ALGORITHM ..
      LOGICAL MATV
      PARAMETER( M=50, N1=M, N2=2*M, N3=3*M, N4=4*M, MATV=.FALSE. )
      DIMENSION AR(N4,N4), AI(N4,N4), BR(N4,N4), BI(N4,N4), ALFR(N4),
     *          ALFI(N4), BETA(N4), ITER(N4)
C
C .. GENERAL DECLARATIONS ..
      CHARACTER*3 CODE
      PARAMETER( ZERO=0.D0, ONE=1.D0, WAVE_NO=ONE, REYNOLDS_NO=1.0D4 )
      DIMENSION D(M,M), P(M,M), Q(M,M)
C
C .. ZERO ALL MATRICES ..
      DO 100 I=1,N4
         DO 200 J=1,N4
            AR(I,J) = ZERO
            AI(I,J) = ZERO
            BR(I,J) = ZERO
            BI(I,J) = ZERO
```

```
200       CONTINUE
100    CONTINUE
C .. CHEBYSHEV TAU METHOS CALLS ..
C .. CALL FIRST DERIVATIVE MATRIX ..
       CALL DERIV1_C(M,D)
C .. CALL MATRIX CORRESPONDING TO FLOW (POISEUILLE or COUETTE) ..
       CALL POISEUILLE_C(M,P,Q)
C       CALL COUETTE_C(M,P,Q)
C
C .. LEGENDRE SPECTRAL METHOS CALLS ..
C .. CALL FIRST DERIVATIVE MATRIX ..
       CALL DERIV1_L(M,D)
C .. CALL MATRIX CORRESPONDING TO FLOW (POISEUILLE or COUETTE) ..
       CALL POISEUILLE_L(M,P,Q)
C       CALL COUETTE_L(M,P,Q)

C .. BUILD MATRIX A(N4,N4) AND B(N4,N4) IN EQUATIONS IN SEQUENCE..
C .. EQUATIONS 1 - 3 ..
       DO 1000 I=1,M-1
          DO 1100 K=0,2
             KK = K*M
             DO 1200 J=1,M
                AR(KK+I,KK+J) = D(I,J)
1200         CONTINUE
             AR(KK+I,KK+M+I) = -ONE
1100      CONTINUE
          AR(M+I,I) = -WAVE_NO**2
1000   CONTINUE
C
C .. EQUATION 4 ..
       DO 2000 I=1,M-1
          NV = N3+I
          DO 2100 J=1,M
             AR(NV,N3+J) = D(I,J)
             AI(NV,J) = P(I,J)*REYNOLDS_NO*WAVE_NO
             AI(NV,N2+J) = -Q(I,J)*REYNOLDS_NO*WAVE_NO
2100      CONTINUE
          AR(NV,N2+I) = -WAVE_NO**2
          BI(NV,N2+I) = -REYNOLDS_NO*WAVE_NO
2000   CONTINUE
C
C .. BOUNDARY CONDITIONS ..
       FAC = ONE
       DO 3000 I=1,M
          AR(N1,I) = ONE
          AR(N2,I) = FAC
          AR(N3,N1+I) = ONE
          AR(N4,N1+I) = FAC
```

```
         FAC = -FAC
3000  CONTINUE
C
C .. THE EIGENVALUE SOLVER ..
      EPS = -ONE
      IFAIL = 0
      CALL F02GJF(N4,AR,N4,AI,N4,BR,N4,BI,N4,EPS,ALFR,ALFI,BETA,MATV,
     *            VR,N4,VI,N4,ITER,IFAIL)

C .. CALL THE FOLLOWING STEPS  (10)

C ..  THE STEPS BELOW DETERMINE EIGENVALUES
      NL = 0
      DO 4000 K=1,N4
         IF ( BETA(K).NE.ZERO ) THEN
            NL = NL+1
            ALFR(NL) = ALFR(K)/BETA(K)
            ALFI(NL) = ALFI(K)/BETA(K)
            ITER(NL) = K
         ENDIF
4000  CONTINUE
C ..
C .. REORDER IMAGINARY PART OF EIGENVALUES. THE REQUIRED EIGENVALUE ..
C .. IS THE FIRST LARGEST IMAGINARY PART..
      DO 4100 I=1,NL-1
         RMAX = ALFI(I)
         INOW = I
         DO 4200 J=I+1,NL
            IF (RMAX.LT.ALFI(J)) THEN
               RMAX = ALFI(J)
               INOW = J
            ENDIF
4200     CONTINUE
         TEMP = ALFR(INOW)
         ALFR(INOW) = ALFR(I)
         ALFR(I) = TEMP
         TEMP = ALFI(INOW)
         ALFI(INOW) = ALFI(I)
         ALFI(I) = TEMP
         ITEMP = ITER(INOW)
         ITER(INOW) = ITER(I)
         ITER(I) = ITEMP
4100  CONTINUE
C ..
C .. OPEN FILE TO STORE EIGENVALUES ..
      NL = M/100
      CODE(1:1) = CHAR(48+NL)
      NL = M-100*NL
```

```
      CODE(2:2) = CHAR(48+NL/10)
      CODE(3:3) = CHAR(48+NL-10*(NL/10))
      OPEN(10,FILE='ORR4DATA.'//CODE)
      DO 4300 I=1,5
         WRITE(10,'(2F20.15)') ALFR(I),ALFI(I)
 4300 CONTINUE
      CLOSE(10)
      END
```

Second order system

```
      PROGRAM ORRSOM
*********************************************************************
*                                                                   *
*    Uses D^2 approach on the ORR-SOMMERFELD equation in the form   *
*                                                                   *
*                        Y_2=D^2 Y_1,                               *
*    (D^2-2a^2)Y_2+a^4Y_1=iaR(1-X^2-LAMBDA)(Y_2-a^2 Y_1)+2iaRY_1    *
*                                                                   *
*    with boundary conditions Y_1=0, DY_1=0 on x=-1,1               *
*                                                                   *
*********************************************************************

      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
C
C .. DECLARATIONS FOR QZ ALGORITHM ..
      LOGICAL MATV
      PARAMETER( M=40, N1=M, N2=2*M, MATV=.FALSE. )
      DIMENSION AR(N2,N2), AI(N2,N2), BR(N2,N2), BI(N2,N2), ALFR(N2),
     *          ALFI(N2), BETA(N2), ITER(N2)
C
C .. GENERAL DECLARATIONS ..
      CHARACTER*3 CODE
      PARAMETER( ZERO=0.D0, ONE=1.D0, WAVE_NO=1.D0, REYNOLDS_NO=1.D4)
      DIMENSION DD(M,M), Q(M,M), P(M,M), BCS(M)
C
C .. ZERO ALL MATRICES ..
      DO 100 I=1,N2
         DO 200 J=1,N2
            AR(I,J) = ZERO
            AI(I,J) = ZERO
            BR(I,J) = ZERO
            BI(I,J) = ZERO
200      CONTINUE
100   CONTINUE

C
C .. CHEBYSHEV TAU METHOS CALLS ..
C .. CALL SECOND DERIVATIVE MATRIX ..
      CALL DERIV2_C(M,DD)
C .. CALL MATRIX CORRESPONDING TO FLOW (POISEUILLE or COUETTE) ..
      CALL POISEUILLE_C(M,P,Q)
C       CALL COUETTE_C(M,P,Q)

C .. LEGENDRE SPECTRAL METHOS CALLS ..
C .. CALL SECOND DERIVATIVE MATRIX ..
      CALL DERIV2_L(M,D)
```

```
C .. CALL MATRIX CORRESPONDING TO FLOW (POISEUILLE or COUETTE) ..
      CALL POISEUILLE_L(M,P,Q)
C       CALL COUETTE_L(M,P,Q)

C .. SET OF CONSTANT PARAMETERS ..
      TMP1 = REYNOLDS_NO*WAVE_NO
      TMP2 = WAVE_NO**2
      TMP3 = REYNOLDS_NO*WAVE_NO**3
      TMP4 = WAVE_NO**4

C .. BUILD MATRIX A(N2,N2) AND B(N2,N2) IN EQUATIONS IN SEQUENCE..
C .. EQUATION 1 ..
      DO 1000 I=1,M-2
         DO 1100 J=1,M
            AR(I,J) = DD(I,J)
1100     CONTINUE
         AR(I,N1+I) = -ONE
1000  CONTINUE
C .. EQUATION 2 ..
      DO 2000 I=1,M-2
         NV = N1+I
         DO 2100 J=1,M
            AR(NV,N1+J) = DD(I,J)
            AI(NV,N1+J) = -TMP1*Q(I,J)
            AI(NV,J)    = TMP3*Q(I,J)+TMP1*P(I,J)
2100     CONTINUE
         AR(NV,I)  = TMP4
         AR(NV,NV) = AR(NV,NV)-2.D0*TMP2
         BI(NV,I)  = TMP3
         BI(NV,NV) = -TMP1
2000  CONTINUE
C
C .. ONE OF THE TWO BOUNDARY CONDITIONS WILL BE IN CHARGE ..
C
C (1).. CONVENTIONAL BOUNDARY CONDITIONS ..
      FAC = ONE
      DO 3000 I=1,M
         AR(N1-1,I) = ONE
         AR(N1,I) = FAC
         II = I-1
         AR(N2-1,I) = DBLE(II*II)
         AR(N2,I) = FAC*DBLE(II*II)
         FAC = -FAC
3000  CONTINUE

C (2).. MODIFIED BOUNDARY CONDITIONS ..
      DO 2500 I=1,M
         IF ( MOD(I,2).EQ.1 ) THEN
```

```
            BCS(I) = ONE/DBLE(I*(I-2))
         ELSE
            BCS(I) = ONE/DBLE((I+1)*(I-3))
         ENDIF
2500  CONTINUE

C .. BOUNDARY CONDITIONS ..
      FAC = ONE
      DO 3000 I=1,M
         AR(N1-1,I) = ONE
         AR(N1,I) = FAC
         FAC = -FAC
3000  CONTINUE
      DO 3100 I=1,M-2
         IF ( MOD(I,2).EQ.1 ) THEN
            AR(N2-1,N1+I) = BCS(I)
         ELSE
            AR(N2,N1+I) = BCS(I)
         ENDIF
3100  CONTINUE

C .. THE EIGENVALUE SOLVER ..
      EPS = -ONE
      IFAIL = 0
      CALL F02GJF(N2,AR,N2,AI,N2,BR,N2,BI,N2,EPS,ALFR,ALFI,BETA,MATV,
     *            VR,N2,VI,N2,ITER,IFAIL)
C ..
C .. THE STEPS IN (10) ARE NECESSARY TO DETERMINE THE EIGENVALUES IN
C .. THIS PROGRAM FOR K=1,N2 INSTEAD OF K=1,N4 ..
      END
```

Poiseuille and Couette Flow Subroutines

```
C .. CHEBYSHEV TAU METHOD SUBROUTINES  ..
C .. THIS SUBROUTINE GIVES PARAMETERS AND MATRICES RELATED TO
C .. POISEUILLE FLOW FOR ..
C .. M IS A POLYNOMIAL, P AND Q ARE MATRICES ..
      SUBROUTINE POISEUILLE_C(M,P,Q)
      DOUBLE PRECISION Q(M,*), P(M,*)
      DO 100 I=1,M
         DO 200 J=1,M
            Q(I,J) = 0.D0
            P(I,J) = 0.D0
200      CONTINUE
         Q(I,I) = 0.5D0
         P(I,I) = -2.D0
         IF ( I.LE.M-2 ) Q(I,I+2) = -0.25D0
```

```
         IF ( I.GE.3 ) Q(I,I-2) = -0.25D0
100   CONTINUE
      Q(3,1) = Q(3,1)-0.25D0
      Q(2,2) = Q(2,2)-0.25D0
      RETURN
      END

C .. THIS SUBROUTINE GIVES PARAMETERS AND MATRICES RELATED TO
C .. COUETTE FLOW ..
C .. M IS A POLYNOMIAL, P AND Q ARE MATRICES ..
      SUBROUTINE COUETTE_C(M,P,Q)
      DOUBLE PRECISION Q(M,*), P(M,*)
      DO 100 I=1,M
         DO 200 J=1,M
            Q(I,J) = 0.D0
            P(I,J) = 0.D0
200      CONTINUE
         IF ( I.LE.M-1 ) Q(I,I+1) = 0.5D0
         IF ( I.GE.3 ) Q(I,I-1) = 0.5D0
100   CONTINUE
      Q(2,1) = 1.0D0
      RETURN
      END


C .. LEGENDRE SPECTRAL METHOD SUBROUTINES ..
C .. THIS SUBROUTINE GIVES PARAMETERS AND MATRICES RELATED TO
C .. POISEUILLE FLOW FOR ..
C .. M IS A POLYNOMIAL, P AND Q ARE MATRICES ..
      SUBROUTINE POISEUILLE_L(M,P,Q)
      DOUBLE PRECISION Q(0:M-1,0:M-1), P(0:M-1,0:M-1)
      DO 100 I=0,M-1
         DO 200 J=0,M-1
            Q(I,J) = 0.D0
            P(I,J) = 0.D0
200      CONTINUE
100   CONTINUE
      DO 300 K=0,M-1
         Q(K,K) = 2.D0*DBLE(K*K+K-1)/DBLE(4*K*K+4*K-3)
         P(K,K) = -2.D0
300   CONTINUE
      DO 400 K=2,M-1
         Q(K,K-2) = -DBLE(K*K-K)/DBLE(4*K*K-8*K+3)
400   CONTINUE
      DO 500 K=0,M-3
         Q(K,K+2) = -DBLE(K*K+3*K+2)/DBLE(4*K*K+16*K+15)
500   CONTINUE
      RETURN
```

```
      END


C .. THIS SUBROUTINE GIVES PARAMETERS AND MATRICES RELATED TO
C .. COUETTE FLOW ..
C .. M IS A POLYNOMIAL, P AND Q ARE MATRICES ..
      SUBROUTINE COUETTE_L(M,P,Q)
      DOUBLE PRECISION Q(0:M-1,0:M-1), P(0:M-1,0:M-1)
      DO 100 I=0,M-1
         DO 200 J=0,M-1
            Q(I,J) = 0.D0
            P(I,J) = 0.D0
200      CONTINUE
         Q(I,I) = 0.D0
100   CONTINUE
      DO 300 K=1,M-1
         Q(K,K-1) = DBLE(K)/DBLE(2*K-1)
300   CONTINUE
      DO 400 K=0,M-2
         Q(K,K+1) = DBLE(K+1)/DBLE(2*K+3)
400   CONTINUE
      RETURN
      END
```

Appendix 3

This appendix contains two FORTRAN77 programs to perform a stability analysis for the Benard-convection eigenvalue problem using the Chebyshev spectral tau method. The first program treats the differential equation as a system of 8 first order differential equations whereas the second program treats the differential equation as a system of 4 second order differential equations for both the conventional and modified boundary value technique.

Eighth order system

```
      PROGRAM CONVEC
*************************************************************************
*                                                                       *
*    PROGRAM COMPUTES EIGENVALUES FOR BENARD CONVECTION IN A LAYER      *
*    OF CONDUCTING NAVIER-STOKES FLUID SUBJECT TO A CONSTANT AXIAL      *
*    MAGNETIC FIELD.                                                    *
*    EIGENFUNCTION ARE REPRESENTED BY CHEBYSHEV SPECTRAL SERIES AND     *
*    NAG ROUTINE F02BJF IS USED TO TREAT THE GENERALISED BOUNDARY       *
*    VALUE PROBLEM.                                                     *
*                                                                       *
*************************************************************************
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( M=20, N1=M, N2=2*M, N3=3*M, N4=4*M, N5=5*M, N6=6*M,
     *           N7=7*M, N8=8*M )
      DIMENSION A(N8,N8), B(N8,N8), ALFR(N8), ALFI(N8), BETA(N8),
     *          ITER(N8), D(M,M)
      CHARACTER*3 CODE
      CHARACTER*1 TYPE
      LOGICAL MATV
      PARAMETER( MATV=.FALSE., TOL=1.D-9, VMU=1.D0, PM=3.D0, PR=1.D0,
     *           BVAL=1.55D4, AVAL=5.5576D0, Q=1.D3 )
      RVAL = SQRT(BVAL)

c .. ZERO ALL ENTRIES OF A(N8,N8) AND B(N8,N8)
      DO 100 I=1,N8
      DO 100 J=1,N8
         A(J,I) = 0.D0
         B(J,I) = 0.D0
100   CONTINUE

C .. VARIABLES OF THE EQUATIONS ..
C       Y(1) ... W, Y(2) ... DW, Y(3) ... D^2W, Y(4) ... D^3W
C       Y(5) ... \THETA, Y(6) ... D\THETA, Y(7) ... b, Y(8) ... Db
```

```
C .. FILL A(N8,N8) WITH THE DIFFERENTIATION MATRIX
C .. CALL FIRST DERIVATIVE MATRIX ..
      CALL DERIV_1(M,D)

C .. FILL A(N8,N8) WITH THE DIFFERENTIATION MATRIX IN ORDER
      DO 1000 I=1,M
C .. EQUATION 1 ..
         DO 1100 J=1,M
            A(I,J) = D(I,J)
1100     CONTINUE
         A(I,N1+I) = -1.D0
C .. EQUATION 2 ..
         NV = N1+I
         DO 1200 J=1,M
            A(NV,N1+J) = D(I,J)
1200     CONTINUE
         A(NV,N2+I) = -1.D0
C .. EQUATION 3 ..
         NV = N2+I
         DO 1300 J=1,M
            A(NV,N2+J) = D(I,J)
1300     CONTINUE
         A(NV,N3+I) = -1.D0
C .. EQUATION 4 ..
         NV = N3+I
         DO 1400 J=1,M
            A(NV,N3+J) = D(I,J)
1400     CONTINUE
         A(NV,I)     = AVAL**4
         A(NV,N2+I) = -Q-2.D0*AVAL**2
         A(NV,N4+I) = -RVAL*AVAL**2
         B(NV,I)     = -AVAL**2
         B(NV,N2+I) = 1.D0
         B(NV,N7+I) = -SQRT(Q)*PM
C .. EQUATION 5 ..
         NV = N4+I
         DO 1500 J=1,M
            A(NV,N4+J) = D(I,J)
1500     CONTINUE
         A(NV,N5+I) = -1.D0
C .. EQUATION 6 ..
         NV = N5+I
         DO 1600 J=1,M
            A(NV,N5+J) = D(I,J)
1600     CONTINUE
         A(NV,N4+I) = -AVAL**2
         A(NV,I)     = RVAL
         B(NV,N4+I) = PR
```

```
C .. EQUATION 7 ..
          NV = N6+I
          DO 1900 J=1,M
             A(NV,N6+J) = D(I,J)
1900      CONTINUE
          A(NV,N7+I) = -1.D0
C .. EQUATION 8 ..
          NV = N7+I
          DO 2000 J=1,M
             A(NV,N7+J) = D(I,J)
2000      CONTINUE
          A(NV,N1+I) = SQRT(Q)
          A(NV,N6+I) = -AVAL**2
          B(NV,N6+I) = PM
1000   CONTINUE
       DO 3000 I=1,N8
          DO 3100 K=1,12
             A(K*M,I) = 0.D0
             B(K*M,I) = 0.D0
3100      CONTINUE
3000   CONTINUE
C
C .. BOUNDARY CONDITIONS - SET PARAMETER CONSTANTS FIRST
       FAC   = 1.D0
       DO 3200 I = 1,M
          A(N1,I)    = 1.D0
          A(N2,I)    = FAC
          A(N3,N1+I) = 1.D0
          A(N4,N1+I) = FAC
          A(N5,N4+I) = 1.D0
          A(N6,N4+I) = FAC
          A(N7,N6+I) = 1.D0
          A(N8,N6+I) = FAC
          FAC = -FAC
3200   CONTINUE

C .. THE NAG ROUTINE F02BJF IS CALLED AS THE EIGENVALUE SOLVER ..
       EPS = -1.D0
       IFAIL=0
       CALL F02BJF(N8,A,N8,B,N8,EPS,ALFR,ALFI,BETA,MATV,Z,N8,ITER,IFAIL)

C ..   CALL THE FOLLOWING STEPS  (20)
C ..   DETERMINE REAL PARTS AND IMAGINARY PARTS OF EIGENVALUES ..
       NL = 0
       DO 9000 K=1,N8
          IF (ABS(BETA(K)).GT.TOL) THEN
             NL=NL+1
             ALFR(NL) = ALFR(K)/BETA(K)
```

```
            ALFI(NL) = ALFI(K)/BETA(K)
            ITER(NL) = K
         ENDIF
9000  CONTINUE
C ..
C .. REORDER REAL PART OF EIGENVALUES. THE REQUIRED EIGENVALUE ..
C .. IS THE FIRST LARGEST REAL PART..
      DO 9100 I=1,NL-1
         RMAX = ALFR(I)
         INOW = I
         DO 9200 J=I+1,NL
            IF (RMAX.LT.ALFR(J)) THEN
               RMAX = ALFR(J)
               INOW = J
            ENDIF
9200     CONTINUE
         TEMP = ALFR(INOW)
         ALFR(INOW) = ALFR(I)
         ALFR(I) = TEMP
         TEMP = ALFI(INOW)
         ALFI(INOW) = ALFI(I)
         ALFI(I) = TEMP
         NTEMP = ITER(INOW)
         ITER(INOW) = ITER(I)
         ITER(I) = NTEMP
9100  CONTINUE
      END
```

Fourth order system

```
      PROGRAM BENARD
************************************************************************
*                                                                      *
*    PROGRAM COMPUTES EIGENVALUES FOR BENARD CONVECTION IN A LAYER     *
*    OF CONDUCTING NAVIER-STOKES FLUID SUBJECT TO A CONSTANT AXIAL     *
*    MAGNETIC FIELD.                                                   *
*    EIGENFUNCTION ARE REPRESENTED BY CHEBYSHEV SPECTRAL SERIES AND    *
*    NAG ROUTINE F02BJF IS USED TO TREAT THE GENERALISED BOUNDARY      *
*    VALUE PROBLEM.                                                    *
*                                                                      *
************************************************************************

      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      CHARACTER*3 CODE
      PARAMETER( M=20, N1=M, N2=2*M, N3=3*M, N4=4*M )
      DIMENSION A(N4,N4), B(N4,N4), ALFR(N4), ALFI(N4), BETA(N4),
     *          ITER(N4), D(M,M), DD(M,M), BCS(M)
      CHARACTER*1 TYPE
      LOGICAL MATV
      PARAMETER( MATV=.FALSE., TOL=1.D-9, VMU=1.D0, PM=3.D0, PR=1.D0,
     *           BVAL=1.55D4, AVAL=5.5576D0, Q=1.D3 )
      RVAL = SQRT(BVAL)
c .. ZERO ALL ENTRIES OF A(N4,N4) AND B(N4,N4)
      DO 100 I=1,N4
      DO 100 J=1,N4
         A(J,I) = 0.D0
         B(J,I) = 0.D0
100   CONTINUE
C .. VARIABLES OF THE EQUATIONS ..
C .. Y(1) ... W, Y(2) ... D^2W, Y(3) ... \THETA, Y(4) ... b
C
C .. FILL A(N4,N4) WITH THE FIRST DIFFERENTIATION MATRIX
C .. CALL FIRST DERIVATIVE MATRIX ..
      CALL DERIV_1(M,D)

C .. FILL A(N4,N4) WITH THE SECOND DIFFERENTIATION MATRIX
C .. CALL SECOND DERIVATIVE MATRIX ..
      CALL DERIV_2(M,D)
C ..
C .. FILL A(N4,N4) WITH THE DIFFERENTIATION MATRIX IN ORDER
      DO 1000 I=1,M
C .. EQUATION 1 ..
         DO 1100 J=1,M
            A(I,J) = DD(I,J)
1100     CONTINUE
         A(I,N1+I) = -1.D0
```

```
C .. EQUATION 2 ..
         NV = N1+I
         DO 1200 J=1,M
            A(NV,N1+J) = DD(I,J)
            B(NV,N3+J) = -SQRT(Q)*PM*D(I,J)
1200     CONTINUE
         A(NV,I)     = AVAL**4
         A(NV,NV)    = A(NV,NV)-Q-2.D0*AVAL**2
         A(NV,N2+I) = -RVAL*AVAL**2
         B(NV,I)     = -AVAL**2
         B(NV,NV)    = 1.D0
C .. EQUATION 3 ..
         NV = N2+I
         DO 1300 J=1,M
            A(NV,N2+J) = DD(I,J)
1300     CONTINUE
         A(NV,I) = RVAL
         A(NV,N2+I) = -AVAL**2
         B(NV,N2+I) = PR
C .. EQUATION 4 ..
         NV = N3+I
         DO 1400 J=1,M
            A(NV,N3+J) = DD(I,J)
            A(NV,J) = SQRT(Q)*D(I,J)
1400     CONTINUE
         A(NV,N3+I)  = -AVAL**2
         B(NV,N3+I) = PM
1000  CONTINUE

C .. THIS ROUTINE IS FOR MODIFIED BOUNDARY CONDITIONS ..
      DO 2000 I=1,M
         IF ( MOD(I,2).EQ.1 ) THEN
            BCS(I) = ONE/DBLE(I*(I-2))
         ELSE
            BCS(I) = ONE/DBLE((I+1)*(I-3))
         ENDIF
2000  CONTINUE
      DO 3000 I=1,N4
         DO 3100 K=1,4
            A(K*M,I)    = 0.D0
            A(K*M-1,I) = 0.D0
            B(K*M,I)    = 0.D0
            B(K*M-1,I) = 0.D0
3100     CONTINUE
3000  CONTINUE

C .. ONE OF THE TWO BOUNDARY CONDITIONS WILL BE USED ..
C
```

```
C (1).. BOUNDARY CONDITIONS AND THE CONVENTIONAL BOUNDARY CONDITIONS ..
      FAC   = 1.D0
      DO 3200 J = 1,M
         A(N1-1,J)  = 1.D0
         A(N1,J)    = FAC
C .. DW=0 ON UPPER BOUNDARY ..
         TEMP1 = 0.D0
         DO 3300 I=1,M
            TEMP1 = TEMP1+D(I,J)
3300     CONTINUE
         A(N2-1,J) = TEMP1
C .. DW=0 ON LOWER BOUNDARY ..
         TEMP2 = 0.D0
         FAC1  = 1.D0
         DO 3400 I=1,M
            TEMP2 = TEMP2+FAC1*D(I,J)
            FAC1 = -FAC1
3400     CONTINUE
         A(N2,J) = TEMP2
         A(N3-1,N2+J) = 1.D0
         A(N3,N2+J)   = FAC
         A(N4-1,N3+J) = 1.D0
         A(N4,N3+J)   = FAC
         FAC = -FAC
3200  CONTINUE
C
C (2).. THE BOUNDARY CONDITIONS AND THE MODIFIED BOUNDARY CONDITIONS ..
      FAC  = 1.D0
      DO 3200 I = 1,M
         A(N1-1,I)  = 1.D0
         A(N1,I)    = FAC
         A(N3-1,N2+I) = 1.D0
         A(N3,N2+I)   = FAC
         A(N4-1,N3+I) = 1.D0
         A(N4,N3+I)   = FAC
         FAC = -FAC
3200  CONTINUE
      DO 3300 I = 1,M-2
         IF ( MOD(I,2).EQ.1 ) THEN
            A(N2-1,N1+I) = BCS(I)
         ELSE
            A(N2,N1+I) = BCS(I)
         ENDIF
3300  CONTINUE

C .. THE NAG ROUTINE F02BJF IS CALLED AS THE EIGENVALUE SOLVER ..
      EPS = -1.D0
      IFAIL=0
```

```
      CALL F02BJF(N4,A,N4,B,N4,EPS,ALFR,ALFI,BETA,MATV,Z,N4,ITER,IFAIL)
C ..
C .. THE STEPS IN (20) ARE NECESSARY TO DETERMINE THE EIGENVALUES IN
C .. THIS PROGRAM FOR K=1,N4 INSTEAD OF K=1,N8 ..
      END
```

Appendix 4

```
      PROGRAM CONVEC
c
c ..  PARAMETERS FOR THE EIGENVALUE PROBLEM
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( TOL=5.D-9,PR=1.D0, PM=1.D0, NCRVL=20 )
      PARAMETER( RMAX=10.D0)
      DIMENSION  CR(0:NQVL)
C OR  DIMENSION  Q(0:NQVL)
      CHARACTER*1 TYPE
      CHARACTER*3 CODE
      COMMON / INFO1 / CRV, PMVAL, PRVAL, RM
c      COMMON / INFO1 / QV, PMVAL, PRVAL, RM
      COMMON / INFO2 / TYPE
      EXTERNAL EIGVAL
*************************************************************************
*                                                                       *
*     PROGRAM COMPUTES EIGENVALUES FOR BENARD AND BENARD/MARANGONI       *
*     CONVECTION IN A CONDUCTING MAGNETIC MEDIUM SUBJECT TO A           *
*     CONSTANT VERTICAL MAGNETIC FIELD.                                 *
*     A NAG ROUTINE F02BJF IS USED TO INTEGRATE THE SYSTEM OF FIRST     *
*     ORDER ORDINARY DIFFERENTIAL EQUATIONS USING CHEBYSHEV SPECTRAL    *
*     THE EIGENVALUE PROBLEM IS EIGHTH ORDER WITH FOUR BOUNDARY         *
*     CONDITIONS ON EACH BOUNDARY. THERE ARE SEVERAL CASES ACCORDING    *
*     TO THE BOUNDARY CONDITIONS.                                       *
*                                                                       *
*************************************************************************
C      FACTOR  = 2.D0*ATAN(1.D0)/DBLE(NCRVL)
      FACTOR  = 2.D0*ATAN(1.D0)/DBLE(NQVL)
      PMVAL = PM
      PRVAL = PR
      DO 100 I=0,NCRVL
C         CR(I) =  RMAX*(SIN(FACTOR*DBLE(I)))**2
        Q(I) =  RMAX*(SIN(FACTOR*DBLE(I)))**2
100   CONTINUE
      WRITE(6,*) 'ENTER VALUE FOR RVAL'
C      OPEN(16,FILE=FILE_IN,STATUS='OLD',ERR=888)
C      OPEN(13,FILE=FILE_OUT,STATUS='UNKNOWN')
      READ(5,*) RVAL
      RM = RVAL
      WRITE(6,*) 'ENTER VALUE FOR AVAL'
      READ(5,*) AVAL
C      CLOSE(16)
      OPEN(1,FILE=FNAME,STATUS='UNKNOWN')
      DO 200 I=0,NCRVL
C         CRV = CR(I)
        QV = Q(I)
```

```
C .. THIS SUBROUTINE IS USED TO MINIMISE THE REQUIRED WAVENUMBER ..
C .. ALEFT AND ARIGHT ARE THE END OF AN INTERVAL, ASTAT IS THE REQUIRED
C .. NUMBER, TOL IS FOR ACCURACY OF THE RESULT, R_VAL IS RAYLEIGH NUMBER
C .. AND EIGVAL IS A FUNCTION ..
         ALEFT = AVAL*0.8D0
         ARIGHT = AVAL*1.25D0
         CALL MINMUM(ALEFT,ARIGHT,ASTAT,TOL,R_VAL,EIGVAL)
      END


C .. THIS FUNCTION IS USED TO ITERATE THE RAYLEIGH NUMBER FOR THE THE ..
C .. GIVEN WAVENUMBER ..
      FUNCTION EIGVAL(AVAL)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION X(2), F(2)
      COMMON / INFO1 / CR, PM, PR, RM
C      COMMON / INFO1 / Q, PM, PR, RM
      X(1) = RM*0.8D0
      X(2) = RM*1.25D0
      F(1) = SIGMA(X(1),AVAL)
111   F(2) = SIGMA(X(2),AVAL)
      IF (ABS(F(1)-F(2)).LE.5.D-9) THEN
         EIGVAL = X(2)
         RM = EIGVAL
         WRITE(*,*) EIGVAL, AVAL
         RETURN
      ENDIF
      G = X(1)-F(1)*(X(2)-X(1))/(F(2)-F(1))
      X(1) = X(2)
      F(1) = F(2)
      X(2) = G
      GOTO 111
      END

C .. THIS FUNCTION IS USED TO TRANSFER THE PARAMETERS RMS AND AM ..
      FUNCTION SIGMA(RVAL,AVAL)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( M=20, N1=M, N2=2*M, N3=3*M, N4=4*M, N5=5*M, N6=6*M,
     *           N7=7*M, N8=8*M )
      DIMENSION A(N8,N8), B(N8,N8), D(M,M), ALFR(N8), ALFI(N8),
     *          BETA(N8), ITER(N8)
      CHARACTER*1 TYPE
      LOGICAL MATV
      PARAMETER( MATV=.FALSE., RVAL=300.D0, RNU=0.D0, B0=1.D0,
     *           CR=0.D0, ZERO=0.D0, ONE=1.D0, TOL=1.D-9 )
C      COMMON / INFO1 / CR, PM, PR, RM
      COMMON / INFO1 / Q, PM, PR, RM
      COMMON / INFO2 / TYPE
```

```
C .. VARIABLES OF THE EQUATIONS ..
C ..  Y(1) ... W, Y(2) ... DW, Y(3) ... D^2W, Y(4) ... D^3W,
C ..  Y(5) ... \THETA, Y(6) ... D\THETA, Y(7) ... h, Y(8) ... Dh

C .. ZERO ALL ENTRIES OF A(N8,N8), B(N8,N8) ..
      DO 300 I=1,N8
      DO 300 J=1,N8
            A(J,I) = 0.D0
            B(J,I) = 0.D0
300   CONTINUE

C .. BUILD THE CHEBYSHEV DERIVATIVE MATRIX D(N,N) ..
C .. CALL FIRST DERIVATIVE MATRIX ..
      CALL DERIV_1(M,D)

C .. FILL A(N8,N8) WITH THE DIFFERENTIATION MATRIX IN ORDER
      DO 1000 I=1,M
C .. EQUATION 1 ..
         DO 1100 J=1,M
            A(I,J) = D(I,J)
1100     CONTINUE
         A(I,N1+I) = -1.D0
C .. EQUATION 2 ..
         NV = N1+I
         DO 1200 J=1,M
            A(NV,N1+J) = D(I,J)
1200     CONTINUE
         A(NV,N2+I) = -1.D0
C .. EQUATION 3 ..
         NV = N2+I
         DO 1300 J=1,M
            A(NV,N2+J) = D(I,J)
1300     CONTINUE
         A(NV,N3+I) = -1.D0
C .. EQUATION 4 ..
         NV = N3+I
         DO 1400 J=1,M
            A(NV,N3+J) = D(I,J)
1400     CONTINUE
         A(NV,I)     = AVAL**4
         A(NV,N2+I) = -Q-2.D0*AVAL**2
         A(NV,N4+I) = -RVAL*AVAL**2
         B(NV,I)     = -AVAL**2/PR
         B(NV,N2+I) = 1.D0/PR
         B(NV,N7+I) = -Q/PM
C .. EQUATION 5 ..
         NV = N4+I
```

```
          DO 1500 J=1,M
             A(NV,N4+J) = D(I,J)
1500      CONTINUE
          A(NV,N5+I) = -1.D0
C .. EQUATION 6 ..
          NV = N5+I
          DO 1600 J=1,M
             A(NV,N5+J) = D(I,J)
1600      CONTINUE
          A(NV,I)    = 1.D0
          A(NV,N4+I) = -AVAL**2
          B(NV,N4+I) = 1.D0
C .. EQUATION 7 ..
          NV = N6+I
          DO 1700 J=1,M
             A(NV,N6+J) = D(I,J)
1700      CONTINUE
          A(NV,N7+I) = -1.D0
C .. EQUATION 8 ..
          NV = N7+I
          DO 1800 J=1,M
             A(NV,N7+J) = D(I,J)
1800      CONTINUE
          A(NV,N1+I) = 1.D0
          A(NV,N6+I) = -AVAL**2
          B(NV,N6+I) = 1.D0/PM
1000   CONTINUE
C
C      .. ZERO ALL ENTRIES OF 8 Mth ROWS..
       DO 2000 I=1,N8
          DO 2100 K=1,8
             A(K*M,I) = 0.D0
             B(K*M,I) = 0.D0
2100      CONTINUE
2000   CONTINUE
C
C .. BOUNDARY CONDITIONS - SET PARAMETER CONSTANTS FIRST
       C1 = AVAL**2
       C2 = RMVL
       C3 = RMVL*AVAL**2
       FAC = 1.D0
       DO 2200 I=1,M
          A(N1,I)    = -C3
          B(N1,I)    = C1
          B(N1,N2+I) = 1.D0
          B(N1,N4+I) = C3
          A(N2,I)    = FAC
          A(N3,I)    = (BO-C1)*C1
```

```
      A(N3,N1+I) = Q*C2*CR+3.D0*C3*CR
      A(N3,N2+I) = B0-C1
      A(N3,N3+I) = -C2*CR
      A(N3,N4+I) = (B0-C1)*C3
      B(N3,N1+I) = -C2*CR/PR
      B(N3,N6+I) = C2*CR*Q/PM
      A(N4,N1+I) = FAC
      A(N5,I) = RNU*C1
      A(N5,N2+I) = RNU
      A(N5,N5+I) = -C3
      A(N6,N4+I) = FAC
      A(N7,N6+I) = AVAL
      A(N7,N7+I) = 1.D0
      A(N8,N6+I) = -AVAL*FAC
      A(N8,N7+I) = FAC
      FAC = -FAC
2200  CONTINUE

C .. THE NAG ROUTINE F02BJF IS USED AS THE EIGENVALUE SOLVER ..
      EPS = -1.D0
      IFAIL=0
      CALL F02BJF(N8,A,N8,B,N8,EPS,ALFR,ALFI,BETA,MATV,Z,N8,ITER,IFAIL)

C .. THE REAL PARTS OF THE EIGENVALUES ARE ALFR(M) AND THE IMAGINARY ..
C .. PARTS ARE ALFI(M)
      NL = 0
      DO 60 I=1,N8
         IF (ABS(BETA(I)).GT.TOL) THEN
            NL=NL+1
            ALFR(NL) = ALFR(I)/BETA(I)
            ALFI(NL) = ALFI(I)/BETA(I)
         ENDIF
60    CONTINUE

C .. DETERMINE THE REQUIRED LARGEST REAL PART OF ALL EIGENVALUES ..
      ALARGE = ALFI(1)
      AREAL = ALFR(1)
      DO 65 I=2,NL
         IF (AREAL.GT.ALFR(I)) GOTO 65
         ALARGE = ALFI(I)
         AREAL = ALFR(I)
65    CONTINUE
      SIGMA = AREAL
      RETURN
      END
```

Appendix 5

This appendix contains two FORTRAN77 programs to perform a stability analysis for the Porous medium superposed fluid layer eigenvalue problem using the Chebyshev spectral tau method. The first program treats the governing equations as a system of 10 first order differential equations whereas the second program treats the governing equations as five second order differential equations.

Tenth order system

```
      PROGRAM BENARD
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( TOL=5.D-9, MVAL=25 )
      DIMENSION AVAL(0:MVAL), RVAL(0:MVAL)
      COMMON / INFO1 / RM
*************************************************************************
*                                                                       *
*     PROGRAM COMPUTES EIGENVALUES FOR BENARD AND BENARD/RAYLEIGH       *
*     CONVECTION IN A POROUS MEDIUM SUPERPOSED BY CLEAR FLUID HEATED    *
*     FROM BELOW.                                                       *
*     NAG ROUTINE F02BJF IS USED TO TREAT THE SYSTEM OF FIRST ORDER     *
*     ORDINARY DIFFERENTIAL EQUATIONS USING THE CHEBYSHEV TAU METH-     *
*     OD. THE EIGENVALUE PROBLEM IS 10TH ORDER WITH FIVE BOUNDARY       *
*     CONDITIONS ON EACH BOUNDARY.                                      *
*                                                                       *
*************************************************************************

      WRITE(6,*) 'ENTER VALUE FOR RM'
      READ(5,*) RM

C .. DETERMINE WAVENUMBERS ..
      DO 100 I=0,MVAL
         ANOW = DBLE(I)
         AVAL(I) = ANOW
         RVAL(I) = EIGVAL(ANOW)
         RM = RVAL(I)
         AM = AVAL(I)
100   CONTINUE
      END

C .. THIS FUNCTION IS USED TO ITERATE THE RAYLEIGH NUMBER FOR THE THE ..
C .. GIVEN WAVENUMBER ..
      FUNCTION EIGVAL(AM)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( EPS=5.D-8 )
```

```
      DIMENSION X(2), F(2)
      COMMON / INFO1 / RM
      X(1) = RM*0.9D0
      X(2) = RM*1.1D0
      F(1) = SIGMA(X(1),AM)
111   F(2) = SIGMA(X(2),AM)
      IF (ABS(F(1)-F(2)).LE.EPS) THEN
         EIGVAL = X(2)
         RETURN
      ENDIF
      XNOW = X(1)-F(1)*(X(2)-X(1))/(F(2)-F(1))
      X(1) = X(2)
      F(1) = F(2)
      X(2) = XNOW
      GOTO 111
      RETURN
      END

C .. THIS FUNCTION IS USED TO TRANSFER THE PARAMETERS RMS AND AM ..
      FUNCTION SIGMA(RM,AM)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( N=12, N2=2*N, N3=3*N, N4=4*N, N5=5*N, N6=6*N,
     *           N7=7*N, N8=8*N, N9=9*N, NX=10*N )
      PARAMETER( PR=1.D0, GM=1.D0, PHI=1.D0, DA=4.D-6, ALFA_BJ=0.1D0)
      DIMENSION A(NX,NX), B(NX,NX), D(N,N), ALFR(NX), ALFI(NX),
     *          BETA(NX), ITER(NX)
      CHARACTER*1 TYPE
      LOGICAL MATV
      PARAMETER( MATV=.FALSE., ZERO=0.D0, ONE=1.D0, TOL=5.D-9)
      HAT_D = ONE/0.12D0
      HAT_K = ONE/0.7D0
      PM    = PR/HAT_K
      DELTA = HAT_D*SQRT(DA)/ALFA_BJ
      AF    = AM/HAT_D
      EPS_T = HAT_D/HAT_K
      RF    = RM*HAT_K**2/(DA*HAT_D**4)

C .. VARIABLES OF THE EQUATIONS FOR FLUID ..
C .. Y(1) ... W, Y(2) ... DW,  Y(3) ... D^2W, Y(4) ... D^3W,
C .. Y(5) ... \THETA, Y(6) ... D\THETA
C .. VARIABLES OF THE EQUATIONS FOR POROUS MEDIUM ..
C .. Y(7) ... W, Y(8) ... DW, Y(9) ... \THETA, Y(10) ... D\THETA,
C
C .. FINDS REAL EIGENVALUES FOR BENARD CONVECTION IN TWO LAYERS
C        PROBLEM WITH BOUNDARIES USING CHEBYSHEV SPECTRAL METHODS ..
      DO 100 I=1,NX
      DO 100 J=1,NX
         A(J,I) = ZERO
```

```
         B(J,I) = ZERO
100   CONTINUE

C .. BUILD THE CHEBYSHEV DERIVATIVE MATRIX D(N,N) ..
C .. CALL FIRST DERIVATIVE MATRIX ..
      CALL DERIV_1(M,D)

C .. BUILD MATRIX A(NX,NX) AND B(NX,NX) IN EQUATIONS IN SEQUENCE..
      DO 1000 I=1,N-1
C .. EQUATION 1 (In fluid region) ..
         DO 1100 J=1,N
            A(I,J) = D(I,J)
1100     CONTINUE
         A(I,N1+I) = -ONE
C .. EQUATION 2 (In fluid region) ..
         DO 1200 J=1,N
            A(N1+I,N1+J) = D(I,J)
1200     CONTINUE
         A(N1+I,N2+I) = -ONE
C .. EQUATION 3 (In fluid region) ..
         DO 1300 J=1,N
            A(N2+I,N2+J) = D(I,J)
1300     CONTINUE
         A(N2+I,N3+I) = -ONE
C .. EQUATION 4 (In fluid region) ..
         DO 1400 J=1,N
            A(N3+I,N3+J) = D(I,J)
1400     CONTINUE
         A(N3+I,I) = AF**4
         A(N3+I,N2+I) = -2.D0*AF**2
         A(N3+I,N4+I) = -RF*AF**2
         B(N3+I,N2+I) = HAT_D**2/(PR*HAT_K)
         B(N3+I,I)    = -AF**2*HAT_D**2/(PR*HAT_K)
C .. EQUATION 5 (In fluid region) ..
         DO 1500 J=1,N
            A(N4+I,N4+J) = D(I,J)
1500     CONTINUE
         A(N4+I,N5+I) = -ONE
C .. EQUATION 6 (In fluid region) ..
         DO 1600 J=1,N
            A(N5+I,N5+J) = D(I,J)
1600     CONTINUE
         A(N5+I,I)    = ONE
         A(N5+I,N4+I) = -AF**2
         B(N5+I,N4+I) = HAT_D**2/HAT_K
C .. EQUATION 7 (In porous medium) ..
         DO 1900 J=1,N
            A(N6+I,N6+J) = D(I,J)
```

```
1900      CONTINUE
          A(N6+I,N7+I) = -ONE
C .. EQUATION 8 (In porous medium) ..
          DO 2000 J=1,N
             A(N7+I,N7+J) = D(I,J)
             B(N7+I,N7+J) = -DA*D(I,J)/(PHI*PM)
2000      CONTINUE
          A(N7+I,N6+I) = -AM**2
          A(N7+I,N8+I) = RM*AM**2
          B(N7+I,N6+I) = DA*AM**2/(PHI*PM)
C .. EQUATION 9 (In porous medium) ..
          DO 2100 J=1,N
             A(N8+I,N8+J) = D(I,J)
2100      CONTINUE
          A(N8+I,N9+I) = -ONE
C .. EQUATION 10 (In porous medium) ..
          DO 2200 J=1,N
             A(N9+I,N9+J) = D(I,J)
2200      CONTINUE
          A(N9+I,N6+I) = ONE
          A(N9+I,N8+I) = -AM**2
          B(N9+I,N8+I) = GM
1000   CONTINUE

C .. INTRODUCE BOUNDARY CONDITIONS ..
       FAC    = ONE
       FACTOR = DA*EPS_T*HAT_D**3
       DO 900 J=1,N
C .. 1st ROW ..
          A(N,J)      = ONE
C .. 2nd ROW ..
          A(N2,J+N)   = ONE
C .. 3rd ROW ..
          A(N3,J+N4)  = ONE
C .. 4th ROW ..
          A(N4,J)     = FAC*EPS_T
          A(N4,J+N6)  = -ONE
C .. 5th ROW ..
          A(N5,J+N4)  = FAC
          A(N5,J+N8)  = -EPS_T
C .. 6th ROW ..
          A(N6,J+N5)  = FAC
          A(N6,J+N9)  = -ONE
C .. 7th ROW ..
          A(N7,J+N)   = -3.D0*FAC*FACTOR*AF**2
          A(N7,J+N3)  = FAC*FACTOR
          A(N7,J+N7)  = ONE
          B(N7,J+N)   = FAC*FACTOR*HAT_D**2/(PR*HAT_K)
```

```
         B(N7,J+N7) = -DA/(PHI*PM)
C .. 8th ROW ..
         A(N8,J+N)  = FAC*HAT_D*EPS_T
         A(N8,J+N2) = -FAC*DELTA*HAT_D*EPS_T
         A(N8,J+N7) = -ONE
C .. 9th ROW ..
         A(N9,J+N6) = FAC
C .. 10th ROW ..
         A(NX,J+N8) = FAC
         FAC = -FAC
900   CONTINUE
      DO 100 I=1,N5
      DO 100 J=1,N5
         A(J,I) = ZERO
         B(J,I) = ZERO
100   CONTINUE

C .. THE NAG ROUTINE F02BJF IS USED AS THE EIGENVALUE SOLVER ..
      EPS = -ONE
      IFAIL = 0
      CALL F02BJF(NX,A,NX,B,NX,EPS,ALFR,ALFI,BETA,MATV,Z,NX,ITER,IFAIL)

C .. THE REAL PARTS OF THE EIGENVALUES ARE ALFR(M) AND THE IMAGINARY ..
C .. PARTS ARE ALFI(M)
      NL = 0
      DO 9000 K=1,NX
         IF (ABS(BETA(K)).GT.TOL) THEN
            NL       = NL+1
            ALFR(NL) = ALFR(K)/BETA(K)
            ALFI(NL) = ALFI(K)/BETA(K)
         ENDIF
9000  CONTINUE

C .. DETERMINE THE REQUIRED LARGEST REAL PART OF ALL EIGENVALUES ..
      ALARGE = ALFI(1)
      AREAL  = ALFR(1)
      DO 65 I=2,NL
         IF (AREAL.GT.ALFR(I)) GOTO 65
         ALARGE = ALFI(I)
         AREAL  = ALFR(I)
65    CONTINUE
      SIGMA   = AREAL
      RETURN
      END
```

Fifth order system

```
      PROGRAM BENARD
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( TOL=5.D-9, MVAL=25 )
      DIMENSION AVAL(0:MVAL), RVAL(0:MVAL)
      COMMON / INFO1 / RM
************************************************************************
*                                                                      *
*     PROGRAM COMPUTES EIGENVALUES FOR BENARD AND BENARD/RAYLEIGH      *
*     CONVECTION IN A POROUS MEDIUM SUPERPOSED BY CLEAR FLUID HEATED   *
*     FROM BELOW.                                                      *
*     NAG ROUTINE F02BJF IS USED TO TREAT THE SYSTEM OF FIRST ORDER    *
*     ORDINARY DIFFERENTIAL EQUATIONS USING THE CHEBYSHEV TAU METH-    *
*     OD. THE EIGENVALUE PROBLEM IS 5TH ORDER WITH FIVE BOUNDARY       *
*     CONDITIONS ON EACH BOUNDARY.                                     *
*                                                                      *
************************************************************************
      WRITE(6,*) 'ENTER VALUE FOR RM'
      READ(5,*) RM

C .. DETERMINE WAVENUMBERS ..
      DO 100 I=0,MVAL
         ANOW = DBLE(I)
         AVAL(I) = ANOW
         RVAL(I) = EIGVAL(ANOW)
         OPEN(1,FILE='BE_SEC2.DAT',STATUS='UNKNOWN')
         AM = AVAL(I)
         RM = RVAL(I)
100   CONTINUE
      END

C .. THIS FUNCTION IS USED TO ITERATE THE RAYLEIGH NUMBER FOR THE THE ..
C .. GIVEN WAVENUMBER ..
      FUNCTION EIGVAL(AM)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( EPS=5.D-8 )
      DIMENSION X(2), F(2)
      COMMON / INFO1 / RM
      X(1) = RM*0.9D0
      X(2) = RM*1.1D0
      F(1) = SIGMA(X(1),AM)
 111  F(2) = SIGMA(X(2),AM)
      IF (ABS(F(1)-F(2)).LE.EPS) THEN
         EIGVAL = X(2)
         RETURN
      ENDIF
      XNOW = X(1)-F(1)*(X(2)-X(1))/(F(2)-F(1))
```

```
      X(1) = X(2)
      F(1) = F(2)
      X(2) = XNOW
      GOTO 111
      RETURN
      END

C .. THIS FUNCTION IS USED TO TRANSFER THE PARAMETERS RMS AND AM ..
      FUNCTION SIGMA(RM,AM)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( N=25, N1=N, N2=2*N, N3=3*N, N4=4*N, N5=5*N)
      PARAMETER( PR=1.D0, PHI=1.D0, DA=4.D-6, ALFA_BJ=0.1D0,
     *           GM=1.D0)
      DIMENSION A(N5,N5), B(N5,N5), D(N,N), DD(N,N), ALFR(N5), ALFI(N5),
     *          BETA(N5), ITER(N5)
      CHARACTER*1 TYPE
      LOGICAL MATV
      PARAMETER( MATV=.FALSE., ZERO=0.D0, ONE=1.D0, TOL=5.D-9)
      HAT_D = ONE/0.1D0
      HAT_K = ONE/0.7D0
      PM    = (ONE/HAT_K)*PR
      DELTA = HAT_D*SQRT(DA)/ALFA_BJ
      AF    = AM/HAT_D
      EPS_T = HAT_D/HAT_K
      RF    = RM*HAT_K**2/(DA*HAT_D**4)

C .. FINDS REAL EIGENVALUES FOR BENARD CONVECTION FOR FREE
C        BOUNDARIES USING CHEBYSHEV SPECTRAL METHODS ..
      DO 100 I=1,N5
      DO 100 J=1,N5
         A(J,I) = ZERO
         B(J,I) = ZERO
100   CONTINUE

C .. VARIABLES OF THE EQUATIONS ..
C .. Y(1) ... W_f, Y(2) ... \xi, Y(3) ... \THETA_f, Y(4) ... W_m,
C       Y(5) ... \THETA_m

C .. BUILD THE CHEBYSHEV FIRST DERIVATIVE MATRIX D(N,N) ..
C .. CALL FIRST DERIVATIVE MATRIX ..
      CALL DERIV_1(M,D)

C .. BUILD THE CHEBYSHEV SECOND DERIVATIVE MATRIX DD(N,N) ..
C .. CALL SECOND DERIVATIVE MATRIX ..
      CALL DERIV_2(M,D)

C .. BUILD MATRIX A(N5,N5) AND B(N5,N5) IN EQUATIONS IN SEQUENCE..
      DO 1000 I=1,N-1
```

```
C .. EQUATION 1 (In fluid region) ..
        DO 1100 J=1,N
           A(I,J) = DD(I,J)
1100    CONTINUE
        A(I,N1+J) = -ONE
C .. EQUATION 2 (In fluid region) ..
         NI = N1+I
         DO 1200 J=1,N
            A(NI,N1+J) = DD(I,J)
1200     CONTINUE
         A(NI,N1+I) = AF**4
         A(NI,N1+I) = -2.D0*AF**2
         A(NI,N2+I) = -RF*AF**2
         B(NI,I)    = -HAD_D**2*AF**2/(PR*HAT_K)
         B(NI,N1+I) = HAD_D**2/(PR*HAT_K)
C .. EQUATION 3 (In fluid region) ..
         NI = N2+I
         DO 1300 J=1,N
            A(NI,N2+J) = DD(I,J)
1300     CONTINUE
         A(NI,I)    = ONE
         A(NI,N2+I) = -AF**2
         B(NI,N2+I) = HAT_D**2/HAT_K
C .. EQUATION 4 (In porous medium) ..
         NI = N3+I
         DO 1400 J=1,N
            A(NI,N3+J) = DD(I,J)
            B(NI,N3+J) = -DA*DD(I,J)/(PHI*PM)
1400     CONTINUE
         A(NI,N3+I) = -AM**2
         A(NI,N4+I) = RM*AM**2
         B(NI,N4+I) = DA*AM**2/(PHI*PM)
C .. EQUATION 5 (In porous medium) ..
         NI = N4+I
         DO 1500 J=1,N
            A(NI,N4+J) = DD(I,J)
1500     CONTINUE
         A(NI,N3+I) = ONE
         A(NI,N4+I) = -AM**2
         B(NI,N4+I) = GM
1000  CONTINUE
      DO 2000 K=1,5
         DO 2100 I=1,N5
            A(K*N,I)   = 0.D0
            A(K*N-1,I) = 0.D0
            B(K*N,I)   = 0.D0
            B(K*N-1,I) = 0.D0
2100     CONTINUE
```

```
2000  CONTINUE

C .. INTRODUCE BOUNDARY CONDITIONS ..
      FAC    = ONE
      FACTOR = DA*EPS_T*HAT_D**3
      DO 2200 J=1,N
C .. 1st ROW ..
         A(N1-1,J)     = ONE
C .. 2nd ROW ..
         A(N1,J+N2) = ONE
C .. 3rd ROW ..
         A(N2-1,J)    = FAC*EPS_T
         A(N2-1,J+N3) = -ONE
C .. 4th ROW ..
         A(N2,J+N2) = FAC
         A(N2,J+N4) = -EPS_T
C .. D VARIABLE BOUNDARY CONDITIONS ..
         TEMP1 = 0.D0
         TEMP2 = 0.D0
         FAC1  = ONE
         DO 2300 I=1,N
            TEMP1 = TEMP1+D(I,J)
            TEMP2 = TEMP2+FAC1*D(I,J)
            FAC1 = -FAC1
2300     CONTINUE
C .. 5th ROW ..
C .. DW=0 ON LOWER BOUNDARY ..
         A(N3-1,J)  = TEMP1
c .. 6th ROW
C .. D\THETA_F-D\THETA_M=0 ON INTERFACE BOUNDARY ..
         A(N3,J+N2) = TEMP2
         A(N3,J+N4) = -TEMP1
C .. 7th ROW ..
C .. NORMAL STRESS BOUNDARY CONDITIONS..
         A(N4-1,J)  = -3.D0*TEMP2*FACTOR*AF**2
         A(N4-1,J+N1) = TEMP2*FACTOR
         A(N4-1,J+N3) = TEMP1
         B(N4-1,J)  = TEMP2*FACTOR*HAT_D**2/(PR*HAT_K)
         B(N4-1,J+N3) = -TEMP1*DA/(PHI*PM)
C .. 8th ROW ..
C .. BEAVERS AND JOSEPH BOUNDARY CONDITIONS ..
         A(N4,J)  = TEMP1*HAT_D*EPS_T
         A(N4,J+N1) = -FAC*DELTA*HAT_D*EPS_T
         A(N4,J+N3) = -TEMP1
C .. 9th ROW ..
         A(N5-1,J+N3) = FAC
C .. 10th ROW ..
         A(N5,J+N4) = FAC
```

```
         FAC = -FAC
2200   CONTINUE

C .. THE NAG ROUTINE F02BJF IS USED AS THE EIGENVALUE SOLVER ..
      EPS = -ONE
      IFAIL = 0
      CALL F02BJF(N5,A,N5,B,N5,EPS,ALFR,ALFI,BETA,MATV,Z,N5,ITER,IFAIL)

C .. THE REAL PARTS OF THE EIGENVALUES ARE ALFR(M) AND THE IMAGINARY ..
C .. PARTS ARE ALFI(M)
      NL = 0
      DO 9000 K=1,N5
         IF (ABS(BETA(K)).GT.TOL) THEN
            NL       = NL+1
            ALFR(NL) = ALFR(K)/BETA(K)
            ALFI(NL) = ALFI(K)/BETA(K)
         ENDIF
9000  CONTINUE

C .. DETERMINE THE REQUIRED LARGEST REAL PART OF ALL EIGENVALUES ..
      ALARGE = ALFI(1)
      AREAL  = ALFR(1)
      DO 65 I=2,NL
         IF (AREAL.GT.ALFR(I)) GOTO 65
         ALARGE = ALFI(I)
         AREAL  = ALFR(I)
65    CONTINUE
      SIGMA   = AREAL
      RETURN
      END
```

Appendix 6

This appendix contains two FORTRAN77 programs to perform a stability analysis for the Porous medium superposed fluid layer eigenvalue problem using the Chebyshev spectral tau method. The first program treats the problem as the real eigenvalues exit ($\sigma \Longrightarrow$) whereas the second program treats the problem considering ($\sigma = 0$) and the eigenvalue is Ra_{ms}.

The system with the eigenvalue σ

```
      PROGRAM BENARD
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( TOL=5.D-9 )
      COMMON / INFO1 / RMS
      EXTERNAL EIGVAL
************************************************************************
*                                                                      *
*     PROGRAM COMPUTES EIGENVALUES FOR BENARD AND BENARD/RAYLEIGH      *
*     CONVECTION IN A POROUS MEDIUM SUPERPOSED BY CLEAR FLUID          *
*     HEATED AND SALTED FROM ABOVE.                                    *
*     NAG ROUTINE F02BJF IS USED TO TREAT THE SYSTEM OF FIRST ORDER    *
*     ORDINARY DIFFERENTIAL EQUATIONS USING THE CHEBYSHEV  TAU MET-    *
*     HOD. THE EIGENVALUE PROBLEM IS 14TH ORDER WITH SEVEN BOUNDARY    *
*     CONDITIONS ON EACH BOUNDARY.                                     *
*                                                                      *
************************************************************************
      WRITE(6,*) 'ENTER VALUE FOR RMSV'
      READ(5,*) RMSV
      RMS = RMSV

C     .. THERE IS TWO KINDS OF RESULTS ..

C (1) .. TO CALCULATE CRITICAL RAYLEIGH NUMBER RMS AND A CORRESPONDING ..
C     .. WAVENUMBER A_M ..
      WRITE(6,*) 'ENTER VALUE FOR AMVL'
      READ(5,*) AMVL
      AM = AMVL

C .. THIS SUBROUTINE IS USED TO MINIMISE THE REQUIRED WAVENUMBER ..
C .. ALEFT AND ARIGHT ARE THE END OF AN INTERVAL, ASTAT IS THE REQUIRED
C .. NUMBER, TOL IS FOR ACCURACY OF THE RESULT, RMSV IS RAYLEIGH NUMBER
C .. AND EIGVAL IS A FUNCTION ..
      ALEFT  = AM*0.8D0
      ARIGHT = AM*1.25D0
      CALL MINMUM(ALEFT,ARIGHT,ASTAT,TOL,RMSV,EIGVAL)
```

```
      WRITE(6,111)  ASTAT, RMSV, ALEFT, ARIGHT
111   FORMAT(5X,'    MINIMUM RAYLEIGH NUMBER IS',F10.4/
     *       5X,'MINIMUM ACHIEVED AT WAVENUMBER',F10.4/
     *       5X,'  INITIAL RAYLEIGH NUMBER RMSV',F10.4/
     *       5X,'             SEARCH INTERVAL (',F6.4,',',F6.4,')'//)
C         WRITE(1,'(5X,A11,F8.4,4X,A11,F25.10)') '  VENUMBER ',AVAL,
C     *        'EIGENVALUE ',RMSV
C      CLOSE(1)
      END

C (2) .. TO CALCULATE CRITICAL RAYLEIGH NUMBER RMS FOR A GIVEN ..
C     .. WAVENUMBER AM ..
      DO 100 I=0,MVAL
         ANOW = DBLE(I)
         AVAL(I) = ANOW
         RVAL(I) = EIGVAL(ANOW)
100   CONTINUE
      END

C .. THIS FUNCTION IS USED TO ITERATE THE RAYLEIGH NUMBER FOR THE THE ..
C .. GIVEN WAVENUMBER ..
      FUNCTION EIGVAL(AM)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( EPS=5.D-9 )
      DIMENSION X(2), F(2)
      COMMON / INFO1 / RMS
      X(1) = RMS*0.8D0
      X(2) = RMS*1.2D0
      F(1) = SIGMA(X(1),AM)
111   F(2) = SIGMA(X(2),AM)
      IF (ABS(F(1)-F(2)).LE.EPS) THEN
         EIGVAL = X(2)
         RMS = EIGVAL
         WRITE(6,*) EIGVAL, AM
         RETURN
      ENDIF
      XNOW = X(1)-F(1)*(X(2)-X(1))/(F(2)-F(1))
      X(1) = X(2)
      F(1) = F(2)
      X(2) = XNOW
      GOTO 111
      END

C .. THIS FUNCTION IS USED TO TRANSFER THE PARAMETERS RMS AND AM ..
      FUNCTION SIGMA(RMS,AM)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( N=20, N1=N, N2=2*N, N3=3*N, N4=4*N, N5=5*N, N6=6*N,
     *           N7=7*N, N8=8*N, N9=9*N, NA=10*N, NB=11*N, NC=12*N,
```

```
     *          ND=13*N, NE=14*N )
      PARAMETER( PR=1.D0,  ALPHA_H=0.1D0, DELTA=0.003D0, PHI=1.D0,
     *          VLEF=1.D0, EPS_T=0.7D0, EPS_S=3.75D0, GM=0.1D0 )
      DIMENSION A(NE,NE), B(NE,NE), D(N,N), ALFR(NE), ALFI(NE),
     *         BETA(NE), ITER(NE)
      CHARACTER*1 TYPE
      LOGICAL MATV
      PARAMETER( MATV=.FALSE., ZERO=0.D0, ONE=1.D0, TOL=5.D-12 )
      HAT_D   = 0.01D0
      RM      = 50.D0
      AF      = AM*HAT_D
      GAMMA_T = HAT_D/EPS_T
      GAMMA_S = HAT_D/EPS_S
      RF      = RM*HAT_D**4/((DELTA*EPS_T)**2)
      RFS     = RMS*HAT_D**4/((DELTA*EPS_S)**2)
      PM      = EPS_T*PR
      VLEM    = (EPS_T/EPS_S)*VLEF
      PRS     = PR/VLef
      PMS     = PM/VLem

C .. VARIABLES OF THE EQUATIONS FOR FLUID ..
C .. Y(1) ... W, Y(2) ... DW, Y(3) ... D^2W, Y(4) ... D^3W,
c .. Y(5) ... \THETA, Y(6) ... D\THETA, Y(7) ... \ZETA, Y(8) ... D\ZETA
C .. VARIABLES OF THE EQUATIONS FOR POROUS MEDIUM ..
C .. Y(9) ... W, Y(10) ... DW, Y(11) .... \THETA, Y(12) ... D\THETA,
C .. Y(13) ... \ZETA, Y(14) ... D\ZETA

C .. FINDS REAL EIGENVALUES FOR BENARD CONVECTION FOR FREE
C        BOUNDARIES USING CHEBYSHEV SPECTRAL METHODS ..
      DO 100 I=1,NE
         DO 150 J=1,NE
            A(J,I) = ZERO
            B(J,I) = ZERO
150      CONTINUE
100   CONTINUE

C      .. BUILD THE CHEBYSHEV DERIVATIVE MATRIX D(N,N) ..
C .. CALL FIRST DERIVATIVE MATRIX ..
      CALL DERIV_1(M,D)

C .. BUILD MATRIX A(NE,NE) AND B(NE,NE) IN EQUATIONS IN SEQUENCE ..
      DO 1000 I=1,N-1
C .. EQUATION 1 (In fluid region) ..
         DO 1100 J=1,N
            A(I,J) = D(I,J)
1100     CONTINUE
         A(I,N1+I) = -ONE
C .. EQUATION 2 (In fluid region) ..
```

```
      DO 1200 J=1,N
        A(N1+I,N1+J) = D(I,J)
1200  CONTINUE
      A(N1+I,N2+I) = -ONE
C .. EQUATION 3 (In fluid region) ..
      DO 1300 J=1,N
        A(N2+I,N2+J) = D(I,J)
1300  CONTINUE
      A(N2+I,N3+I) = -ONE
C .. EQUATION 4 (In fluid region) ..
      DO 1400 J=1,N
        A(N3+I,N3+J) = D(I,J)
1400  CONTINUE
      A(N3+I,I) = AF**4
      A(N3+I,N2+I) = -2.D0*AF**2
      A(N3+I,N4+I) = -RF*AF**2
      A(N3+I,N6+I) = RFS*AF**2
      B(N3+I,I)    = -AF**2*HAT_D**2/(PR*EPS_T)
      B(N3+I,N2+I) = HAT_D**2/(PR*EPS_T)
C .. EQUATION 5 (In fluid region) ..
      DO 1500 J=1,N
        A(N4+I,N4+J) = D(I,J)
1500  CONTINUE
      A(N4+I,N5+I) = -ONE
C .. EQUATION 6 (In fluid region) ..
      DO 1600 J=1,N
        A(N5+I,N5+J) = D(I,J)
1600  CONTINUE
      A(N5+I,I)    = -ONE
      A(N5+I,N4+I) = -AF**2
      B(N5+I,N4+I) = HAT_D**2/(EPS_T)
C .. EQUATION 7 (In fluid region) ..
      DO 1700 J=1,N
        A(N6+I,N6+J) = D(I,J)
1700  CONTINUE
      A(N6+I,N7+I) = -ONE
C .. EQUATION 8 (In fluid region) ..
      DO 1800 J=1,N
        A(N7+I,N7+J) = D(I,J)
1800  CONTINUE
      A(N7+I,I)    = -ONE
      A(N7+I,N6+I) = -AF**2
      B(N7+I,N6+I) = HAT_D**2/(VLEF*EPS_T)
C .. EQUATION 9 (In porous medium) ..
      DO 1900 J=1,N
        A(N8+I,N8+J) = D(I,J)
1900  CONTINUE
      A(N8+I,N9+I) = -ONE
```

```
C .. EQUATION 10 (In porous medium) ..
         DO 2000 J=1,N
            A(N9+I,N9+J) = D(I,J)
            B(N9+I,N9+J) = -(DELTA**2/(PHI*PM))*D(I,J)
2000     CONTINUE
         A(N9+I,N8+I) = -AM**2
         A(N9+I,NA+I) = RM*AM**2
         A(N9+I,NC+I) = -RMS*AM**2
         B(N9+I,N8+I) = (DELTA*AM)**2/(PHI*PM)
C .. EQUATION 11 (In porous medium) ..
         DO 2100 J=1,N
            A(NA+I,NA+J) = D(I,J)
2100     CONTINUE
         A(NA+I,NB+I) = -ONE
C .. EQUATION 12 (In porous medium) ..
         DO 2200 J=1,N
            A(NB+I,NB+J) = D(I,J)
2200     CONTINUE
         A(NB+I,N8+I) = -ONE
         A(NB+I,NA+I) = -AM**2
         B(NB+I,NA+I) = GM
C .. EQUATION 13 (In porous medium) ..
         DO 2300 J=1,N
            A(NC+I,NC+J) = D(I,J)
2300     CONTINUE
         A(NC+I,ND+I) = -ONE
C .. EQUATION 14 (In porous medium) ..
         DO 2400 J=1,N
            A(ND+I,ND+J) = D(I,J)
2400     CONTINUE
         A(ND+I,N8+I) = -ONE
         A(ND+I,NC+I) = -AM**2
         B(ND+I,NC+I) = PHI/VLEM
1000  CONTINUE

C .. INTRODUCE BOUNDARY CONDITIONS ..
      FAC = ONE
      DO 3000 J=1,N
C .. 1st ROW ..
         A(N1,J)     = ONE
C .. 2nd ROW ..
         A(N2,J)     = FAC
         A(N2,N8+J) = -HAT_D
C .. 3rd ROW ..
         A(N3,N1+J) = ONE
C .. 4rd ROW ..
         A(N4,N1+J) = FAC*ALPHA_H*HAT_D
         A(N4,N2+J) = -FAC*DELTA
```

```
      A(N4,N9+J) = -HAT_D**3*ALPHA_H
C .. 5th ROW ..
      A(N5,N1+J) = 3.D0*FAC*AF**2
      A(N5,N3+J) = -FAC
      A(N5,N9+J) = -HAT_D**4/DELTA**2
      B(N5,N1+J) = -FAC*HAT_D**2/(PR*EPS_T)
      B(N5,N9+J) = HAT_D**4/(PM*PHI)
C .. 6th ROW ..
      A(N6,N4+J) = ONE
C .. 7th ROW ..
      A(N7,N4+J) = FAC*GAMMA_T
      A(N7,NA+J) = -EPS_T
C .. 8th ROW ..
      A(N8,N5+J) = FAC
      A(N8,NB+J) = -EPS_T
C .. 9th ROW ..
      A(N9,N6+J) = ONE
C .. 10th ROW ..
      A(NA,N6+J) = FAC*GAMMA_S
      A(NA,NC+J) = -EPS_S
C .. 11th ROW ..
      A(NB,N7+J) = FAC
      A(NB,ND+J) = -EPS_S
C .. 12th ROW ..
      A(NC,N8+J)  = FAC
C .. 13th ROW ..
      A(ND,NA+J) = FAC
C .. 14th ROW ..
      A(NE,NC+J) = FAC
      FAC = -FAC
3000  CONTINUE

C .. THE NAG ROUTINE F02BJF IS USED AS THE EIGENVALUE SOLVER ..
      EPS = -ONE
      IFAIL = 0
      CALL F02BJF(NE,A,NE,B,NE,EPS,ALFR,ALFI,BETA,MATV,Z,NE,ITER,IFAIL)

C .. THE REAL PARTS OF THE EIGENVALUES ARE ALFR(M) AND THE IMAGINARY ..
C .. PARTS ARE ALFI(M)
      NL = 0
      DO 9000 K=1,NE
         IF (ABS(BETA(K)).GT.TOL) THEN
            NL       = NL+1
            ALFR(NL) = ALFR(K)/BETA(K)
            ALFI(NL) = ALFI(K)/BETA(K)
         ENDIF
9000  CONTINUE
```

```
C .. DETERMINE THE REQUIRED LARGEST REAL PART OF ALL EIGENVALUES ..
      ALARGE = ALFI(1)
      AREAL  = ALFR(1)
      DO 65 I=2,NL
         IF (AREAL.GT.ALFR(I)) GOTO 65
         ALARGE = ALFI(I)
         AREAL  = ALFR(I)
65    CONTINUE
      SIGMA = AREAL
      RETURN
      END
```

The system with the eigenvalue Ra_{ms} when $\sigma = 0$

```
      PROGRAM BENARD
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( TOL=5.D-9 )
      COMMON / INFO1 / RMS
      EXTERNAL EIGVAL
************************************************************************
*                                                                      *
*     PROGRAM COMPUTES EIGENVALUES FOR BENARD AND BENARD/RAYLEIGH      *
*     CONVECTION IN A POROUS MEDIUM SUPERPOSED BY CLEAR FLUID          *
*     HEATED AND SALTED FROM ABOVE.                                    *
*     NAG ROUTINE F02BJF IS USED TO TREAT THE SYSTEM OF FIRST ORDER    *
*     ORDINARY DIFFERENTIAL EQUATIONS USING THE CHEBYSHEV TAU METH-    *
*     OD. THE EIGENVALUE PROBLEM IS 14TH ORDER WITH SEVEN BOUNDARY     *
*     CONDITIONS ON EACH BOUNDARY.                                     *
*                                                                      *
************************************************************************

      WRITE(6,*) 'ENTER VALUE FOR RMSV'
      READ(5,*) RMS
      RMS = EIGVAL(3.11D0)
      END

C .. THIS FUNCTION IS USED TO ITERATE THE RAYLEIGH NUMBER FOR THE THE ..
C .. GIVEN WAVENUMBER ..
      FUNCTION EIGVAL(AM)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( EPS=5.D-9 )
      DIMENSION X(2), F(2)
      COMMON / INFO1 / RMS
      X(1) = RMS*0.9D0
      X(2) = RMS*1.1D0
      F(1) = SIGMA(X(1),AM)
111   F(2) = SIGMA(X(2),AM)
      IF (ABS(F(1)-F(2)).LE.EPS) THEN
         EIGVAL = X(2)
         RMS = EIGVAL
         WRITE(6,*) EIGVAL, AM
         RETURN
      ENDIF
      XNOW = X(1)-F(1)*(X(2)-X(1))/(F(2)-F(1))
      X(1) = X(2)
      F(1) = F(2)
      X(2) = XNOW
      GOTO 111
      END
```

```
C .. THIS FUNCTION IS USED TO TRANSFER THE PARAMETERS RMS AND AM ..
      FUNCTION SIGMA(RMS,AM)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( N=20, N1=N, N2=2*N, N3=3*N, N4=4*N, N5=5*N, N6=6*N,
     *           N7=7*N, N8=8*N, N9=9*N, NA=10*N, NB=11*N, NC=12*N,
     *           ND=13*N, NE=14*N )
      PARAMETER( PR=1.D0,  ALPHA_H=0.1D0, DELTA=0.003D0, PHI=1.D0,
     *           VLEF=1.D0, EPS_T=0.7D0, EPS_S=3.75D0, GM=0.1D0 )
      DIMENSION A(NE,NE), B(NE,NE), D(N,N), ALFR(NE), ALFI(NE),
     *          BETA(NE), ITER(NE)
      CHARACTER*1 TYPE
      LOGICAL MATV
      PARAMETER( MATV=.FALSE., ZERO=0.D0, ONE=1.D0, TOL=5.D-12 )
      HAT_D   = 0.01D0
      RM      = 50.D0
      AF      = AM*HAT_D
      GAMMA_T = HAT_D/EPS_T
      GAMMA_S = HAT_D/EPS_S
      RF      = RM*HAT_D**4/((DELTA*EPS_T)**2)
      RFS     = RMS*HAT_D**4/((DELTA*EPS_S)**2)
      PM      = EPS_T*PR
      VLEM    = (EPS_T/EPS_S)*VLEF
      PRS     = PR/VLef
      PMS     = PM/VLem
C
C .. FINDS REAL EIGENVALUES FOR BENARD CONVECTION FOR FREE
C       BOUNDARIES USING CHEBYSHEV SPECTRAL METHODS ..
      DO 100 I=1,NE
         DO 150 J=1,NE
            A(J,I) = ZERO
            B(J,I) = ZERO
150      CONTINUE
100   CONTINUE
C
C .. BUILD THE CHEBYSHEV DERIVATIVE MATRIX D(N,N) ..
C .. CALL FIRST DERIVATIVE MATRIX ..
      CALL DERIV_1(M,D)
C
C .. BUILD MATRIX A(NE,NE) AND B(NE,NE) IN EQUATIONS IN SEQUENCE..
      DO 1000 I=1,N-1

C .. EQUATION 1 (In fluid region) ..
         DO 1100 J=1,N
            A(I,J) = D(I,J)
1100     CONTINUE
         A(I,N1+I) = -ONE
C .. EQUATION 2 (In fluid region) ..
         DO 1200 J=1,N
```

```
         A(N1+I,N1+J) = D(I,J)
 1200 CONTINUE
      A(N1+I,N2+I) = -ONE
C .. EQUATION 3 (In fluid region) ..
      DO 1300 J=1,N
         A(N2+I,N2+J) = D(I,J)
 1300 CONTINUE
      A(N2+I,N3+I) = -ONE
C .. EQUATION 4 (In fluid region) ..
      DO 1400 J=1,N
         A(N3+I,N3+J) = D(I,J)
 1400 CONTINUE
      A(N3+I,I) = AF**4
      A(N3+I,N2+I) = -2.D0*AF**2
      A(N3+I,N4+I) = -RF*AF**2
      A(N3+I,N6+I) = RFS*AF**2
C .. EQUATION 5 (In fluid region) ..
      DO 1500 J=1,N
         A(N4+I,N4+J) = D(I,J)
 1500 CONTINUE
      A(N4+I,N5+I) = -ONE
C .. EQUATION 6 (In fluid region) ..
      DO 1600 J=1,N
         A(N5+I,N5+J) = D(I,J)
 1600 CONTINUE
      A(N5+I,I)    = -ONE
      A(N5+I,N4+I) = -AF**2
C .. EQUATION 7 (In fluid region) ..
      DO 1700 J=1,N
         A(N6+I,N6+J) = D(I,J)
 1700 CONTINUE
      A(N6+I,N7+I) = -ONE
C .. EQUATION 8 (In fluid region) ..
      DO 1800 J=1,N
         A(N7+I,N7+J) = D(I,J)
 1800 CONTINUE
      A(N7+I,I)    = -ONE
      A(N7+I,N6+I) = -AF**2
C .. EQUATION 9 (In porous medium) ..
      DO 1900 J=1,N
         A(N8+I,N8+J) = D(I,J)
 1900 CONTINUE
      A(N8+I,N9+I) = -ONE
C .. EQUATION 10 (In porous medium) ..
      DO 2000 J=1,N
         A(N9+I,N9+J) = D(I,J)
 2000 CONTINUE
      A(N9+I,N8+I) = -AM**2
```

```
      A(N9+I,NA+I) = RM*AM**2
      B(N9+I,NC+I) = -AM**2
C .. EQUATION 11 (In porous medium) ..
      DO 2100 J=1,N
         A(NA+I,NA+J) = D(I,J)
2100  CONTINUE
      A(NA+I,NB+I) = -ONE
C .. EQUATION 12 (In porous medium) ..
      DO 2200 J=1,N
         A(NB+I,NB+J) = D(I,J)
2200  CONTINUE
      A(NB+I,N8+I) = -ONE
      A(NB+I,NA+I) = -AM**2
C .. EQUATION 13 (In porous medium) ..
      DO 2300 J=1,N
         A(NC+I,NC+J) = D(I,J)
2300  CONTINUE
      A(NC+I,ND+I) = -ONE
C .. EQUATION 14 (In porous medium) ..
      DO 2400 J=1,N
         A(ND+I,ND+J) = D(I,J)
2400  CONTINUE
      A(ND+I,N8+I) = -ONE
      A(ND+I,NC+I) = -AM**2
1000 CONTINUE
C
C .. INTRODUCE BOUNDARY CONDITIONS ..
     FAC = ONE
     DO 3000 J=1,N
C .. 1st ROW ..
      A(N1,J)     = ONE
C .. 2nd ROW ..
      A(N2,J)     = FAC
      A(N2,N8+J) = -HAT_D
C .. 3rd ROW ..
      A(N3,N1+J) = ONE
C .. 4rd ROW ..
      A(N4,N1+J) = FAC*ALPHA_H*HAT_D
      A(N4,N2+J) = -FAC*DELTA
      A(N4,N9+J) = -HAT_D**3*ALPHA_H
C .. 5th ROW ..
      A(N5,N1+J) = 3.D0*FAC*AF**2
      A(N5,N3+J) = -FAC
      A(N5,N9+J) = -HAT_D**4/DELTA**2
C .. 6th ROW ..
      A(N6,N4+J) = ONE
C .. 7th ROW ..
      A(N7,N4+J) = FAC*GAMMA_T
```

```
         A(N7,NA+J) = -EPS_T
C .. 8th ROW ..
         A(N8,N5+J) = FAC
         A(N8,NB+J) = -EPS_T
C .. 9th ROW ..
         A(N9,N6+J) = ONE
C .. 10th ROW ..
         A(NA,N6+J) = FAC*GAMMA_S
         A(NA,NC+J) = -EPS_S
C .. 11th ROW ..
         A(NB,N7+J) = FAC
         A(NB,ND+J) = -EPS_S
C .. 12th ROW ..
         A(NC,N8+J)  = FAC
C .. 13th ROW ..
         A(ND,NA+J) = FAC
C .. 14th ROW ..
         A(NE,NC+J) = FAC
         FAC = -FAC
3000  CONTINUE
C .. THE NAG ROUTINE F02BJF IS USED AS THE EIGENVALUE SOLVER ..
      EPS = -ONE
      IFAIL = 0
      CALL F02BJF(NE,A,NE,B,NE,EPS,ALFR,ALFI,BETA,MATV,Z,NE,ITER,IFAIL)

C .. THE REAL PARTS OF THE EIGENVALUES ARE ALFR(M) AND THE IMAGINARY ..
C .. PARTS ARE ALFI(M)
      NL = 0
      DO 9000 K=1,NE
         IF (ABS(BETA(K)).GT.TOL) THEN
            NL       = NL+1
            ALFR(NL) = ALFR(K)/BETA(K)
            ALFI(NL) = ALFI(K)/BETA(K)
         ENDIF
9000  CONTINUE
C .. DETERMINE THE REQUIRED LARGEST REAL PART OF ALL EIGENVALUES ..
      ALARGE = ALFI(1)
      AREAL  = ALFR(1)
      DO 65 I=2,NL
         IF (AREAL.GT.ALFR(I)) GOTO 65
         ALARGE = ALFI(I)
         AREAL  = ALFR(I)
65    CONTINUE
      WRITE(*,222) AREAL,ALARGE
222   FORMAT(5X,'CURRENT EIGENVALUE IS (',F20.7','F20.7,')'/)
      IF (ABS(ALARGE).GE.TOL) THEN
         TYPE = 'C'
      ELSE
```

```
   TYPE = 'R'
ENDIF
SIGMA = AREAL
RETURN
END
```

Appendix 7

```
      PROGRAM CONVEC
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      IMPLICIT INTEGER*4(I-N)
************************************************************************
*                                                                      *
*     PROGRAM COMPUTES EIGENVALUES FOR MAGNETIC INSTABILITY WITH A     *
*     FLINTILY CONDUCTING INNER CORE SURROUNDED BY A CYLINDRICAL       *
*     LAYER OF CONDUCTING FLUID WHICH IS BOUNDED BY THE MANTLE.        *
*     NAG ROUTINE F02GJF IS USED TO TREAT THE SYSTEM OF FIRST ORDER    *
*     ORDINARY DIFFERENTIAL EQUATIONS USING CHEBYSHEV POLYNOMIALS.     *
*     THE EIGENVALUE PROBLEM IS 16 ORDER WITH EIGHT BOUNDARY           *
*     CONDITIONS ON EACH BOUNDARY                                      *
*                                                                      *
************************************************************************
       PARAMETER( M=32, N1=M, N2=2*M, N3=3*M, N4=4*M, N5=5*M, N6=6*M,
     *           N7=7*M, N8=8*M, N9=9*M, NA=10*M, NB=11*M, NC=12*M,
     *           ND=13*M, NE=14*M, NF=15*M, NG=16*M, NPTS=50 )
      DIMENSION AR(NG,NG), AI(NG,NG), BR(NG,NG), BI(NG,NG), DF(M,M),
     *          DC(M,M), DO(M,M), ALFR(NG), ALFI(NG), BETA(NG), W(2*M),
     *          ITER(NG), VR(NG,NG), VI(NG,NG), XP(0:2*NPTS),
     *          YREAL(0:2*NPTS), YIMAG(0:2*NPTS), XR(2), YR(2),
     *          WG(10*NPTS+5), MAG(2)
C     .. DECLARATION OF MATRICES USED IN FLUID REGION ..
      DIMENSION B(M,M), B1(M,M), B2(M,M), B3(M,M), BP1(M,M), RP1(M,M),
     *          R1(M,M), R2(M,M), R3(M,M), R4(M,M)
C
C     .. DECLARATION OF MATRICES USED IN INNER CORE ..
      DIMENSION RC1(M,M), RC2(M,M)
C
C     .. DECLARATION OF MATRICES USED IN OUTER CORE ..
      DIMENSION RO2(M,M), RO3(M,M), RO4(M,M)

      LOGICAL MATV, MORE, SPLINE, VIEW, BOX
      CHARACTER*60 TITLE
      PARAMETER(  ZERO=0.D0, ONE=1.D0, TOL=1.D-12 )
      PARAMETER( TITLE='Graph of Eigenfunctions' )
C
C     .. PARAMETERS OF THE PROBLEM ..
      PARAMETER( SIB=0.35D0, VLAM=508.D0, DNV=9.61D0, DMV=2.D0,
     *           E=1.D-5, E_ETA=1.D-5, ETA=1.D3 )
C
C     .. DECLARATION OF COEFFICIENT FUNCTIONS IN FLUID REGION ..
      EXTERNAL FB, FB1, FB2, FB3, FBP1, FRP1, FR1, FR2, FR3, FR4
C
C     .. DECLARATION OF COEFFICIENT FUNCTIONS IN INNER CORE ..
      EXTERNAL  FRC1, FRC2
C
```

```
C      .. DECLARATION OF COEFFICIENT FUNCTIONS IN OUTER CORE ..
       EXTERNAL  FRO2, FRO3, FRO4
C
C .. VARIABLES OF THE EQUATIONS FOR FLUID LAYER ..
C .. Y(1) = u,.. Y(2) = Du,..  Y(3) = D^2u,.. Y(4) = w,.. Y(5) = Dw,
C .. Y(6) = D^2w,.. Y(7) = b_r,.. Y(8) = Db_r,.. Y(9) = b_z,..
C .. Y(10=A) = D$b_z$,..
C .. VARIABLES OF THE EQUATIONS FOR INNER CORE ..
C .. Y(11=B) = b_r,.. Y(C=12) = Db_r,.. Y(D=13) = b_z,.. Y(E=14) = Db_z,
C .. VARIABLES OF THE EQUATIONS FOR MANTLE ..
C .. Y(D=15) = \phi,..  Y(E=16) = D\phi ..
C
C      .. ZERO ALL ENTRIES OF AR(NG,NG), AI(NG,NG), BR(NG,NG) AND BI(NG,NG)
       DO 100 I=1,NG
          DO 200 J=1,NG
             AR(J,I) = ZERO
             AI(J,I) = ZERO
             BR(J,I) = ZERO
             BI(J,I) = ZERO
200       CONTINUE
100    CONTINUE
C
C .. GET DIFFERENTIATION MATRICES IN [SIB,1], [0,SIB] AND [1,\INFTY] ..
C .. DF DIFFERENTIATION MATRIX IN FLUID LAYER ..
C .. DC DIFFERENTIATION MATRIX IN INNER CORE  ..
C .. DC DIFFERENTIATION MATRIX IN OUTER CORE  ..
       DO 300 I=1,M
          DO 400 J=1,M
             DF(I,J) = ZERO
             DC(I,J) = ZERO
             DO(I,J) = ZERO
400       CONTINUE
300    CONTINUE
       DO 500 I=1,M-1
          DO 600 J=I+1,M,2
             DF(I,J) = DBLE(4*J-4)/(ONE-SIB)
             DC(I,J) = DBLE(4*J-4)/SIB
             DO(I,J) = DBLE(2*J-2)
600       CONTINUE
500    CONTINUE
       DO 700 I=1,M
          DF(1,I) = 0.5D0*DF(1,I)
          DC(1,I) = 0.5D0*DC(1,I)
          DO(1,I) = 0.5D0*DO(1,I)
700    CONTINUE
C
C      .. GET ALL THE COEFFICIENT MATRICES ..
       CALL MATRIX(M,B,FB,W)
```

```
      CALL MATRIX(M,B1,FB1,W)
      CALL MATRIX(M,B2,FB2,W)
      CALL MATRIX(M,B3,FB3,W)
      CALL MATRIX(M,BP1,FBP1,W)
      CALL MATRIX(M,RP1,FRP1,W)
      CALL MATRIX(M,R1,FR1,W)
      CALL MATRIX(M,R2,FR2,W)
      CALL MATRIX(M,R3,FR3,W)
      CALL MATRIX(M,R4,FR4,W)
      CALL MATRIX(M,RC1,FRC1,W)
      CALL MATRIX(M,RC2,FRC2,W)
      CALL MATRIX(M,RO2,FRO2,W)
      CALL MATRIX(M,RO3,FRO3,W)
      CALL MATRIX(M,RO4,FRO4,W)
C
C     .. DEFINE CONSTANT MULTIPLYING FACTORS ..
      C1 = DMV**2
      C2 = DNV**2
      C3 = DMV*DNV/E
      C4 = DMV/E
      C5 = DNV/(E*DMV)
      C6 = VLAM*E_ETA/E
      C7 = DNV**2/(E*DMV)
      C8 = DMV**2/DNV
      C9 = VLAM/ETA
C
C   .. FILL THE MATRICES AR, AI, BR AND BI IN EQUATIONS IN SEQUENCE ..
C   .. EQUATION 1 ..
      DO 1000 I=1,M-1
         DO 1100 J=1,M
            AR(I,J) = DF(I,J)
1100     CONTINUE
         AR(I,N1+I) = -ONE
1000  CONTINUE
C   .. EQUATION 2 ..
      DO 1500 I=1,M-1
         NV = N1+I
         DO 1600 J=1,M
            AR(NV,N1+J) = DF(I,J)
1600     CONTINUE
         AR(NV,N2+I) = -ONE
1500  CONTINUE
C   .. EQUATION 3 ..
      DO 2000 I=1,M-1
         NV = N2+I
         DO 2100 J=1,M
            AR(NV,J)    = 2.D0*(C2*R1(I,J)+C1*R3(I,J))
            AI(NV,J)    = C7*RP1(I,J)
```

```
            AR(NV,N1+J) = -(ONE+C1)*R2(I,J)
            AR(NV,N2+J) = DF(I,J)+R1(I,J)
            AI(NV,N3+J) = (C1-ONE)*DNV*R2(I,J)
            AR(NV,N4+J) = C5*RP1(I,J)
            AI(NV,N4+J) = -3.D0*DNV*R1(I,J)
            AR(NV,N6+J) = (C3-2.D0*C5)*B1(I,J)
            AR(NV,N7+J) = -2.D0*C5*B(I,J)
            AI(NV,N8+J) = C4*BP1(I,J)-2.D0*C7*B(I,J)-C4*B2(I,J)
            AI(NV,N9+J) = C4*B1(I,J)
2100     CONTINUE
         AR(NV,N1+I) = AR(NV,N1+I)-C2
         BR(NV,N1+I) = C6
         AR(NV,N3+I) = C5
         AI(NV,N3+I) = AI(NV,N3+I)+DNV**3
         BI(NV,N3+I) = -DNV*C6
         AI(NV,N5+I) = -DNV
2000  CONTINUE
C  .. EQUATION 4 ..
      DO 2500 I=1,M-1
         NV = N3+I
         DO 2600 J=1,M
            AR(NV,N3+J) = DF(I,J)
2600     CONTINUE
         AR(NV,N4+I) = -ONE
2500  CONTINUE
C  .. EQUATION 5 ..
      DO 3000 I=1,M-1
         NV = N4+I
         DO 3100 J=1,M
            AR(NV,N4+J) = DF(I,J)
3100     CONTINUE
         AR(NV,N5+I) = -ONE
3000  CONTINUE
C  .. EQUATION 6 ..
      DO 3500 I=1,M-1
         NV = N5+I
         DO 3600 J=1,M
            AI(NV,J)    = -C8*(C1*R4(I,J)+2.D0*C2*R2(I,J))
            BI(NV,J)    = C8*C6*R2(I,J)
            AI(NV,N1+J) = 3.D0*DNV*R1(I,J)+C8*R3(I,J)
            AI(NV,N2+J) = C8*R2(I,J)
            AR(NV,N3+J) = (C1-ONE)*R3(I,J)-C2*R1(I,J)
            AI(NV,N3+J) = C4*R1(I,J)
            BR(NV,N3+J) = C6*R1(I,J)
            AR(NV,N4+J) = (ONE-C1)*R2(I,J)
            AR(NV,N5+J) = DF(I,J)+4.D0*R1(I,J)
            AI(NV,N6+J) = -C4*BP1(I,J)
            AI(NV,N7+J) = C4*B1(I,J)
```

```
            AR(NV,N8+J) = -C8*C4*B3(I,J)-C3*B1(I,J)
3600     CONTINUE
         AI(NV,I)     = AI(NV,I)-DNV**3
         BI(NV,I)     = BI(NV,I)+DNV*C6
         AI(NV,N2+I) = AI(NV,N2+I)+DNV
         AR(NV,N4+I) = AR(NV,N4+I)-C2
         BR(NV,N4+I) = C6
3500  CONTINUE
C  .. EQUATION 7 ..
      DO 4000 I=1,M-1
         NV = N6+I
         DO 4100 J=1,M
            AR(NV,N6+J) = DF(I,J)
4100     CONTINUE
         AR(NV,N7+I) = -ONE
4000  CONTINUE
C  .. EQUATION 8 ..
      DO 4500 I=1,M-1
         NV = N7+I
         DO 4600 J=1,M
            AI(NV,N3+J) = DMV*VLAM*B1(I,J)
            AR(NV,N6+J) = (ONE-C1)*R2(I,J)
            AR(NV,N7+J) = DF(I,J)+3.D0*R1(I,J)
            AI(NV,N8+J) = 2.D0*DNV*R1(I,J)
4600     CONTINUE
         AR(NV,N6+I) = AR(NV,N6+I)-C2
         BR(NV,N6+I) = VLAM
4500  CONTINUE
C  .. EQUATION 9 ..
      DO 5000 I=1,M-1
         NV = N8+I
         DO 5100 J=1,M
            AR(NV,N8+J) = DF(I,J)
5100     CONTINUE
         AR(NV,N9+I) = -ONE
5000  CONTINUE
C  .. EQUATION 10 ..
      DO 5500 I=1,M-1
         NV = N9+I
         DO 5600 J=1,M
            AI(NV,J) = DMV*VLAM*B1(I,J)
            AR(NV,N8+J) = -C1*R2(I,J)
            AR(NV,N9+J) = DF(I,J)+R1(I,J)
5600     CONTINUE
         AR(NV,N8+I) = AR(NV,N8+I)-C2
         BR(NV,N8+I) = VLAM
5500  CONTINUE
C
```

```
C   .. TREAT THE INNER CORE ..
C   .. EQUATION 11 ..
      DO 6000 I=1,M-1
         NV = NA+I
         DO 6100 J=1,M
            AR(NV,NA+J) = DC(I,J)
6100     CONTINUE
         AR(NV,NB+I) = -ONE
6000  CONTINUE
C   .. EQUATION 12 ..
      DO 6500 I=1,M-1
         NV = NB+I
         DO 6600 J=1,M
            TEMP = 3.D0*RC1(I,J)
            DO 6700 K=1,M
               TEMP = TEMP+RC2(I,K)*DC(K,J)
6700        CONTINUE
            AR(NV,NA+J) = -C2*RC2(I,J)
            BR(NV,NA+J) = C9*RC2(I,J)
            AR(NV,NB+J) = TEMP
            AI(NV,NC+J) = 2.D0*DNV*RC1(I,J)
6600     CONTINUE
         AR(NV,NA+I) = AR(NV,NA+I)+(ONE-C1)
6500  CONTINUE
C   .. EQUATION 13 ..
      DO 7000 I=1,M-1
         NV = NC+I
         DO 7100 J=1,M
            AR(NV,NC+J) = DC(I,J)
7100     CONTINUE
         AR(NV,ND+I) = -ONE
7000  CONTINUE
C   .. EQUATION 14 ..
      DO 7500 I=1,M-1
         NV = ND+I
         DO 7600 J=1,M
            TEMP = RC1(I,J)
            DO 7700 K=1,M
               TEMP = TEMP+RC2(I,K)*DC(K,J)
7700        CONTINUE
            AR(NV,NC+J) = -C2*RC2(I,J)
            BR(NV,NC+J) = C9*RC2(I,J)
            AR(NV,ND+J) = TEMP
7600     CONTINUE
         AR(NV,NC+I) = AR(NV,NC+I)-C1
7500  CONTINUE
C
C   .. TREATMENT OF THE OUTER CORE ..
```

```
C   .. EQUATION 15 ..
      DO 8000 I=1,M-1
         NV = NE+I
         DO 8100 J=1,M
            AR(NV,NE+J) = DO(I,J)
8100     CONTINUE
         AR(NV,NF+I) = -ONE
8000  CONTINUE
C   .. EQUATION 16 ..
      DO 8500 I=1,M-1
         NV = NF+I
         DO 8600 J=1,M
            TEMP = -RO3(I,J)
            DO 8700 K=1,M
               TEMP = TEMP+RO4(I,K)*DO(K,J)
8700        CONTINUE
            AR(NV,NE+J) = -C1*RO2(I,J)
            AR(NV,NF+J) = TEMP
8600     CONTINUE
         AR(NV,NE+I) = AR(NV,NE+I)-4.D0*C2
8500  CONTINUE
C
C   .. BOUNDARY CONDITIONS - SET PARAMETER CONSTANTS FIRST
      FAC = ONE
      DO 8800 J=1,M
C   .. W=0 ON R=SIB ..
         AR(N1,J) = FAC
C   .. W=0 ON R=ONE ..
         AR(N2,J) = ONE
C   .. U=0 ON R=SIB ..
         AR(N3,N3+J) = FAC
C   .. U=0 ON R=ONE ..
         AR(N4,N3+J) = ONE
C   .. V=0 ON R=SIB - I.E. D(U)=0 ..
         AR(N5,N4+J) = FAC
C   .. V=0 ON R=ONE - I.E. D(U)=0 ..
         AR(N6,N4+J) = ONE
C   .. D(B_r)+B_r+i(M**2+N**2)/N B_z=0 ON R=ONE ..
         AR(N7,N6+J) = ONE
         AR(N7,N7+J) = ONE
         AI(N7,N8+J) = C8+DNV
C   .. B_R=GAMMA*B_Z ON R=ONE ..
         AR(N8,N6+J) = ONE
         AR(N8,NF+J) = 2.DO*FAC
C   .. B_r CTS ON R=SIB ..
         AR(N9,N6+J) = FAC
         AR(N9,NA+J) = -ONE
C   .. B_z CTS ON R=SIB ..
```

```
         AR(NA,N8+J) = FAC
         AR(NA,NC+J) = -ONE
C  .. B_theta CTS ON R=SIB - DB_r IS CONTINUOUS ..
         AR(NB,N7+J) = FAC
         AR(NB,NB+J) = -ONE
C  .. ETA*(i*N*B_r-DB_z) CTS ON R=SIB ..
         AI(NC,N6+J) = FAC*DNV/ETA
         AR(NC,N9+J) = -FAC/ETA
         AI(NC,NA+J) = -DNV
         AR(NC,ND+J) = ONE
C  .. B_r=0 AT R=0 ..
         AR(ND,NA+J) = FAC
C  .. B_z=0 AT R=0 ..
         AR(NE,NC+J) = FAC
C  .. B_Z+iN*V=0 ON R=1 ..
         AR(NF,N8+J) = ONE
         AI(NF,NE+J) = DNV*FAC
C  .. V=0 ON R=INFINITY ..
         AR(NG,NE+J) = ONE
         FAC = -FAC
8800  CONTINUE
C
C .. THE NAG ROUTINE F02GJF IS USED AS THE EIGENVALUE SOLVER ..
      EPS = -ONE
      MATV = .TRUE.
      IFAIL = 0
      CALL F02GJF(NG,AR,NG,AI,NG,BR,NG,BI,NG,EPS,ALFR,ALFI,BETA,MATV,
     *            VR,NG,VI,NG,ITER,IFAIL)

C .. THE REAL PARTS OF THE EIGENVALUES ARE ALFR(M) AND THE IMAGINARY ..
C .. PARTS ARE ALFI(M)
      NL = 0
      DO 9000 K=1,NG
         IF (ABS(BETA(K)).GT.TOL) THEN
            NL=NL+1
            ALFR(NL) = ALFR(K)/BETA(K)
            ALFI(NL) = ALFI(K)/BETA(K)
            ITER(NL) = K
         ENDIF
9000  CONTINUE

C ..
C .. REORDER REAL PARTS OF EIGENVALUES. THE REQUIRED EIGENVALUE ..
C .. IS THE FIRST LARGEST REAL PART..
      DO 9100 I=1,NL-1
         RMAX = ALFR(I)
         INOW = I
         DO 9200 J=I+1,NL
```

```
            IF (RMAX.LT.ALFR(J)) THEN
               RMAX = ALFR(J)
               INOW = J
            ENDIF
9200     CONTINUE
         TEMP = ALFR(INOW)
         ALFR(INOW) = ALFR(I)
         ALFR(I) = TEMP
         TEMP = ALFI(INOW)
         ALFI(INOW) = ALFI(I)
         ALFI(I) = TEMP
         NTEMP = ITER(INOW)
         ITER(INOW) = ITER(I)
         ITER(I) = NTEMP
9100  CONTINUE
c9150  CONTINUE
C .. GET COEFFICIENTS OF EIGENVECTORS BR AND BZ FOR INNER CORE ..
C .. AND U, W, BR, AND BZ FOR FLUID REGION ..
      D_THETA = 4.D0*ATAN(ONE)/DBLE(NPTS)
      INOW = ITER(1)
      BMAX = ZERO
      DO 9300 I=0,NPTS
         THETA = D_THETA*DBLE(NPTS-I)
         TEMP = (COS(0.5D0*THETA))**2
         XP(I) = SIB*TEMP
         XP(I+NPTS) = SIB+(ONE-SIB)*TEMP
         SUMR_IN_BR = VR(NA+1,INOW)
         SUMI_IN_BR = VI(NA+1,INOW)
         SUMR_OUT_BR = VR(N6+1,INOW)
         SUMI_OUT_BR = VI(N6+1,INOW)
         SUMR_OUT_U = VR(N3+1,INOW)
         SUMI_OUT_U = VI(N3+1,INOW)
         SUMR_IN_BZ = VR(NC+1,INOW)
         SUMI_IN_BZ = VI(NC+1,INOW)
         SUMR_OUT_BZ = VR(N8+1,INOW)
         SUMI_OUT_BZ = VI(N8+1,INOW)
         SUMR_OUT_W = VR(1,INOW)
         SUMI_OUT_W = VI(1,INOW)
         DO 9400 K=2,M
            FAC = COS(THETA*DBLE(K-1))
            SUMR_IN_BR = SUMR_IN_BR+VR(NA+K,INOW)*FAC
            SUMI_IN_BR = SUMI_IN_BR+VI(NA+K,INOW)*FAC
            SUMR_OUT_BR = SUMR_OUT_BR+VR(N6+K,INOW)*FAC
            SUMI_OUT_BR = SUMI_OUT_BR+VI(N6+K,INOW)*FAC
            SUMR_OUT_U = SUMR_OUT_BR+VR(N3+K,INOW)*FAC
            SUMI_OUT_U = SUMI_OUT_BR+VI(N3+K,INOW)*FAC
            SUMR_IN_BZ = SUMR_IN_BZ+VR(NC+K,INOW)*FAC
            SUMI_IN_BZ = SUMI_IN_BZ+VI(NC+K,INOW)*FAC
```

```
            SUMR_OUT_BZ = SUMR_OUT_BZ+VR(N8+K,INOW)*FAC
            SUMI_OUT_BZ = SUMI_OUT_BZ+VI(N8+K,INOW)*FAC
            SUMR_OUT_W = SUMR_OUT_BZ+VR(K,INOW)*FAC
            SUMI_OUT_W=  SUMI_OUT_BZ+VI(K,INOW)*FAC
9400      CONTINUE
C
c          YREAL(I) = SUMR_IN_BR
c          YIMAG(I) = SUMI_IN_BR
c          YREAL(I+NPTS) = SUMR_OUT_BR
c          YIMAG(I+NPTS) = SUMI_OUT_BR
          YREAL(I+NPTS) = SUMR_OUT_U
          YIMAG(I+NPTS) = SUMI_OUT_U
c          YREAL(I) = SUMR_IN_BZ
c          YIMAG(I) = SUMI_IN_BZ
c          YREAL(I+NPTS) = SUMR_OUT_BZ
c          YIMAG(I+NPTS) = SUMI_OUT_BZ
c          YREAL(I+NPTS) = SUMR_OUT_W
c          YIMAG(I+NPTS) = SUMI_OUT_W
          TEMP = SQRT(SUMR_IN_BZ**2+SUMI_IN_BZ**2)
          IF (BMAX.LT.TEMP) THEN
             BMAX = TEMP
             BZ_R = SUMR_IN_BZ
             BZ_I = SUMI_IN_BZ
          ENDIF
          TEMP = SQRT(SUMR_OUT_BZ**2+SUMI_OUT_BZ**2)
          IF (BMAX.LT.TEMP) THEN
             BMAX = TEMP
             BZ_R = SUMR_OUT_BZ
             BZ_I = SUMI_OUT_BZ
          ENDIF
9300    CONTINUE
C
C .. NORMALISE FIELDS TO UNITY ..
        IF (BZ_R.LT.ZERO) BMAX = -BMAX
        IF ((BZ_R.EQ.ZERO).AND.(BZ_I.LT.ZERO)) BMAX = -BMAX
        DO 9500 I=0,2*NPTS
           YREAL(I) = YREAL(I)/BMAX
           YIMAG(I) = YIMAG(I)/BMAX
9500    CONTINUE
        END
C
C       .. THE ARE TWO FIELDS FOR THIS PROBLEM TO SOLVE ..
C       FIRST FIELD  WHEN F(R)=R^{ALPHA}
C
C                                (4(1-R^BETA)(R^BETA-{S^BETA}_IB)
C       SECOND FIELD  WHEN F(R)=----------------------------------+ALPHA
C                                   (1+ALPHA)(1-{S^BETA}_IB)^2
C
```

```
C (1) .. THE FUNCTIONS FOR FIRST FIELD ..
      FUNCTION FB(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL ALFA
      PARAMETER( ALFA=1.0, SIB=0.35D0, ONE=1.D0 )
      R = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      FB = R**(ALFA+1.0)
      RETURN
      END

      FUNCTION FB1(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL ALFA
      PARAMETER( ALFA=1.0, SIB=0.35D0, ONE=1.D0 )
      R = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      FB1 = R**ALFA
      RETURN
      END

      FUNCTION FB2(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL  ALFA
      PARAMETER( ALFA=1.0, SIB=0.35D0, ONE=1.D0 )
      R = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      FB2 = R**(ALFA-1.0)
      RETURN
      END

      FUNCTION FB3(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL ALFA
      PARAMETER( ALFA=1.0, SIB=0.35D0, ONE=1.D0 )
      R = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      FB3 = R**(ALFA-2.0)
      RETURN
      END

      FUNCTION FBP1(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL BETA, ALFA
      PARAMETER( ALFA=1.0, BETA=1.0, SIB=0.35D0, ONE=1.D0 )
      R = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      FBP1 = DBLE(ALFA+1.0)*R**(ALFA-1.0)
      RETURN
      END

      FUNCTION FR1(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
```

```
      PARAMETER( ONE=1.D0,SIB=0.35D0 )
      FR1 = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      FR1 = ONE/FR1
      RETURN
      END

      FUNCTION FR2(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( ONE=1.D0,SIB=0.35D0 )
      FR2 = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      FR2 = ONE/FR2**2
      RETURN
      END

      FUNCTION FR3(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( ONE=1.D0, SIB=0.35D0 )
      FR3 = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      FR3 = ONE/FR3**3
      RETURN
      END

      FUNCTION FR4(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( ONE=1.D0, SIB=0.35D0 )
      FR4 = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      FR4 = ONE/FR4**4
      RETURN
      END

      FUNCTION FRP1(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( ONE=1.D0, SIB=0.35D0 )
      FRP1 = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      RETURN
      END

      FUNCTION FRC1(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( SIB=0.35D0, HALF=0.5D0, ONE=1.D0 )
      FRC1 = HALF*SIB*(ONE+X)
      RETURN
      END

      FUNCTION FRC2(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( SIB=0.35D0, HALF=0.5D0, ONE=1.D0 )
      FRC2 = (HALF*SIB*(ONE+X))**2
```

```
      RETURN
      END

      FUNCTION FRO2(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( ONE=1.D0 )
      FRO2 = (ONE-X)**2
      RETURN
      END

      FUNCTION FRO3(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( ONE=1.D0 )
      FRO3 = (ONE-X)**3
      RETURN
      END

      FUNCTION FRO4(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( ONE=1.D0 )
      FRO4 = (ONE-X)**4
      RETURN
      END
C
C (2) .. THE FUNCTIONS FOR SECOND FIELD ..
      FUNCTION FB(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL BETA, ALFA
      PARAMETER( ALFA=0.0, BETA=1.0, SIB=0.35D0, ONE=1.D0 )
      R = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      TEMP = SIB**BETA
      FB = R**BETA
      FB = (4.D0*(ONE-FB)*(FB-TEMP)/(ONE-TEMP)**2)+ALFA
      FB = R*FB/(ONE+ALFA)
      RETURN
      END

      FUNCTION FB1(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL BETA, ALFA
      PARAMETER( ALFA=0.0, BETA=1.0, SIB=0.35D0, ONE=1.D0 )
      R = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      TEMP = SIB**BETA
      FB = R**BETA
      FB = (4.D0*(ONE-FB)*(FB-TEMP)/(ONE-TEMP)**2)+ALFA
      FB1 = FB/(ONE+ALFA)
      RETURN
      END
```

```
      FUNCTION FB2(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL BETA, ALFA
      PARAMETER( ALFA=0.0, BETA=1.0, SIB=0.35D0, ONE=1.D0 )
      R = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      TEMP = SIB**BETA
      FB = R**BETA
      FB = (4.D0*(ONE-FB)*(FB-TEMP)/(ONE-TEMP)**2)+ALFA
      FB2 = FB/((ONE+ALFA)*R)
      RETURN
      END

      FUNCTION FB3(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL BETA, ALFA
      PARAMETER( ALFA=0.0, BETA=1.0, SIB=0.35D0, ONE=1.D0 )
      R = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      TEMP = SIB**BETA
      FB = R**BETA
      FB = (4.D0*(ONE-FB)*(FB-TEMP)/(ONE-TEMP)**2)+ALFA
      FB3 = FB/((ONE+ALFA)*R**2)
      RETURN
      END

      FUNCTION FBP1(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL BETA, ALFA
      PARAMETER( ALFA=0.0, BETA=1.0, SIB=0.35D0, ONE=1.D0 )
      R = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      TEMP = SIB**BETA
      FB1 = R**BETA
      FB2 = R**2.D0*BETA
      FB = 4.D0*(((BETA+ONE)*FB1-(2.D0*BETA+ONE)*FB2+(BETA+ONE)*FB1*TEMP
     *     -TEMP)/(ONE-TEMP)**2)+ALFA
      FBP1 = FB/((ONE+ALFA)*R)
      RETURN
      END

      FUNCTION FR1(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( ONE=1.D0,SIB=0.35D0 )
      FR1 = SIB+0.5D0*(ONE-SIB)*(ONE+X)
      FR1 = ONE/FR1
      RETURN
      END

      FUNCTION FR2(X)
```

```
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
PARAMETER( ONE=1.D0,SIB=0.35D0 )
FR2 = SIB+0.5D0*(ONE-SIB)*(ONE+X)
FR2 = ONE/FR2**2
RETURN
END

FUNCTION FR3(X)
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
PARAMETER( ONE=1.D0, SIB=0.35D0 )
FR3 = SIB+0.5D0*(ONE-SIB)*(ONE+X)
FR3 = ONE/FR3**3
RETURN
END

FUNCTION FR4(X)
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
PARAMETER( ONE=1.D0, SIB=0.35D0 )
FR4 = SIB+0.5D0*(ONE-SIB)*(ONE+X)
FR4 = ONE/FR4**4
RETURN
END

FUNCTION FRP1(X)
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
PARAMETER( ONE=1.D0, SIB=0.35D0 )
FRP1 = SIB+0.5D0*(ONE-SIB)*(ONE+X)
RETURN
END

FUNCTION FRC1(X)
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
PARAMETER( SIB=0.35D0, HALF=0.5D0, ONE=1.D0 )
FRC1 = HALF*SIB*(ONE+X)
RETURN
END

FUNCTION FRC2(X)
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
PARAMETER( SIB=0.35D0, HALF=0.5D0, ONE=1.D0 )
FRC2 = (HALF*SIB*(ONE+X))**2
RETURN
END

FUNCTION FRO2(X)
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
PARAMETER( ONE=1.D0 )
FRO2 = (ONE-X)**2
```

```
      RETURN
      END

      FUNCTION FR03(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( ONE=1.D0 )
      FR03 = (ONE-X)**3
      RETURN
      END

      FUNCTION FR04(X)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      PARAMETER( ONE=1.D0 )
      FR04 = (ONE-X)**4
      RETURN
      END

C .. THIS SUBROUTINE COMPUTES THE CHEBYSHEV SPECTRAL MATRIX ..
C .. N IS NUMBER OF POLYNOMIAL, T IS THE MATRIX, FCN IS THE GIVEN ..
C .. FUNCTION AND W THE FUNCTION ..
      SUBROUTINE MATRIX(N,T,FCN,W)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
C
C .. PROGRAM COMPUTES THE CHEBYSHEV SPECTRAL MATRIX OF A ..
C .. SPECIFIED FUNCTION FCN(X) ..
      DIMENSION T(N,*), W(*)
C
C     .. GENERATE CHEBYSHEV NODES ..
      DX = 2.D0*ATAN(1.D0)/DBLE(2*N-1)
      DO 100 J=1,2*N-1
         SUM = 0.D0
         DO 200 I=1,2*N-1
            X = COS(DX*DBLE(2*I-1))
            SUM = SUM+FCN(X)*COS(DX*DBLE((J-1)*(2*I-1)))
200      CONTINUE
         W(J) = 2.D0*SUM/DBLE(2*N-1)
100   CONTINUE
      W(1) = 0.5D0*W(1)
C
C     .. BUILD CHEBYSHEV MATRIX ..
      T(1,1) = W(1)
      DO 300 I=2,N
         T(1,I) = 0.5D0*W(I)
         T(I,1) = W(I)
         T(I,I) = W(1)+0.5D0*W(2*I-1)
         DO 400 J=I+1,N
            T(I,J) = 0.5D0*(W(J-I+1)+W(J+I-1))
            T(J,I) = T(I,J)
```

```
400       CONTINUE
300    CONTINUE
       RETURN
       END
```