SEMILINEAR PROBLEMS AND SPECTRAL THEORY

Wenying Feng

A thesis presented to the

University of Glasgow

Faculty of Science

for the degree of

Doctor of Philosophy

MAY 1997

Contents

STATEMENT

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

Chapter 1 covers basic definitions, notations and some known results which will be used in this thesis.

Much of the work in chapter 4 was done in collaboration with Professor J.R.L. Webb and has been published in [21].

Much of Chapter 5 is joint work with Professor J.R.L. Webb and will be published in [18], [22], [23].

The results in chapter 2, 3 and 6 are the original work of the author alone and some of them have been submitted for publication in [19].

ACKNOWLEDGEMENT

I wish to give my deepest gratitude to my supervisor Prof. J.R.L. Webb, FRSE. It is he, who guided me into the field of nonlinear analysis when I had no background in this field. His encouragement and patient help motivated me during the period of the research. Without these, this thesis would not be possible.

I would like to express my thanks to Prof. R. W. Ogden and to Prof. K. A. Brown for their every possible help within the department.

I would also like to give my thanks to the Faculty of Science, University of Glasgow and to the Committee of Vice-Chancellors and Principals for their financial support by awarding me a University Scholarship and an ORS.

Finally, I am grateful to my parents, my husband and my daughter for their love, support, and understanding.

SUMMARY

The subject of this thesis is that part of nonlinear functional analysis which deals with the solvability of semilinear differential equations and the study of spectral theory for nonlinear operators.

Chapter one is an introduction to the concepts used through the thesis, including measures of noncompactness, (p, k)-epi mappings and related properties, Fredholm operators of index zero, coincidence-degree theory for semilinear operators, L-k-set contractions, A-proper operators and so on.

The work in chapter two is based on the study of [16]. In [16], a spectrum for nonlinear operators was introduced by Furi, Martelli and Vignoli. Their spectrum need not contain the eigenvalues [9]. We establish a new spectral theory for nonlinear operators which contains all eigenvalues as in the linear case. We compare the new spectrum with that of [16] and the one of [48] and prove that all three spectra may be empty, which answers one of the open questions in [48]. Some applications of the new theory, including the generalization of three well known theorems, the study of the solvability of a Cauchy problem and a Hammerstein integral equation, are obtained in the last section of this chapter.

In chapter three, by generalizing the concept of $(0, k)$-epi mappings to that of $(0, L, k)$-epi mappings, we introduce the definition of spectrum for semilinear operators (L, N), where L is a Fredholm operator of index zero, N is a nonlinear operator. When L is the identity map, this spectrum reduces to the spectrum defined in Chapter 2. We prove that it has similar properties with the spectrum of nonlinear operators. Also in the last section, by using this theory, we discuss the solvability of semilinear operator equations and extend some existence results.

In chapter four, we obtain some surjectivity results on the mapping $\lambda T - S$, where T is a homeomorphism and S is a nonlinear map. We generalize one of the results of [12] in finite dimensional space to infinite dimensional space, which solves the open question

of [12]. We also apply our theorems to the study of a nonlinear Sturm-Liouville problem on the half line following the work by Toland [66] and to prove the existence of a solution for a second order differential equations which was studied in [29].

Much of the work in this Chapter is joint work with J.R.L. Webb and has been published in [21].

Chapter five is related to some recent work by Gupta, Ntouyas, Tsamatos and Lakshmikantham [24]-[30]. They proved existence results for m-point boundary value problems for second order ordinary differential equations under nonresonance assumptions and they also assume that the nonlinear part has a linear growth. We obtain results for these boundary value problems in the resonance case. Moreover, our assumptions allow the nonlinear part to have nonlinear growth. Some examples show that there exist equations to which our theorems can be used but the previous results do not apply.

Much of the work in this Chapter is joint work with J.R.L. Webb and part of this chapter will be published in [18], [22], [23].

In chapter six, we study second order ordinary differential equations subject to Dirichlet, Neumann, periodic and antiperiodic boundary conditions. We make use of an abstract continuation type theorem [56], [57] for semilinear equations involving A-proper mappings to obtain approximation solvability results for these boundary value problems. The results in this chapter generalize the results of [60], [61]. Also we give examples to show that our theorems permit the treatment of equations to which the results of [4], [32], [57] can not be used.

Part of this chapter has been submitted for publication, [19].

Introduction

In view of the importance of the spectral theory for linear operators, it is not surprising that various attempts have been made to define and study the spectrum also for nonlinear operators. Clearly, a good definition should preserve as many properties of the spectrum for classical bounded linear operators as possible and reduce to the familiar spectrum in the case of linear operators. Spectra of nonlinear operators have been defined and studied by many authors, in particular, [1], [12], [16]. The spectrum introduced by Furi, Martelli and Vignoli has found many interesting applications (see [16]). This spectrum is defined by using three extended real numbers $\alpha(f), \omega(f), d(f)$, and the concept of *stably-solvable* operators, (the detailed definitions will be given later). In [16], it was proved that this spectrum preserve many properties of the spectrum of linear operators. For example, it is closed; the boundary $\partial\sigma(f) \subset \sigma_\pi(f)$ ($\lambda \in \sigma_\pi(f)$ if and only if $d(\lambda - f) = 0$ or $\omega(\lambda - f) = 0$, when f is a bounded linear operator, $\sigma_\pi(f)$ is the approximate point spectrum of f); it is bounded when f is quasibounded and α-Lipschitz; it is upper semicontinuous and so on. However, in [9], it was indicated that this spectrum does not contain the eigenvalues in some cases. In fact, it may be disjoint from the eigenvalues, which is an important part of the spectrum in the linear case.

In chapter two, a new spectrum for nonlinear operators, which contains all the eigenvalues, as in the linear case, will be introduced. We shall do this by using the three real numbers, $\omega(f), m(f)$ and $\nu(f)$. We shall prove that this spectrum is compact, and upper semicontinuous. It is also contains all bifurcation points and asymptotic bifurcation points. Moreover, the nonlinear resolvent also has properties similar to the linear resolvent.

In section 2.2, we obtain some properties for the eigenvalues in the spectrum of a positively homogeneous operator. We shall prove that if f is a positively homogeneous operator and $\lambda \in \sigma(f)$ (the spectrum of f) with $|\lambda| > \alpha(f)$, then there exists $t_0 \in (0, 1]$ such that λ / t_0 is an eigenvalue of f [Theorem 2.2.8]. This result can be used to discuss the existence of solutions for nonlinear operator equations (see section 2.5). Furthermore.

we obtain the result that when f is odd and positively homogeneous, $\lambda \in \sigma(f)$ with $|\lambda| > \alpha(f)$, then λ is an eigenvalue of f [Theorem 2.2.9]. This theorem is a generalization of the results of [48], [70]. In this section, we also obtain the result about the estimation of the radius of the spectrum for a positively homogeneous operator. We will give an example to show that our estimate is best possible.

In section 2.3, we shall compare our new spectrum with the spectrum introduced by Furi, Martelli and Vignoli and the Lipschitz spectrum introduced in [39]. We shall prove that our spectrum lies properly between the other two spectra. Then, in section 2.4, a counterexample will show that all spectra may be empty, which answers one of the open questions of [48].

In the last section of chapter two, we shall discuss applications of the new theory. By using this theory, we shall study the solvability of some nonlinear operator equations, including a global Cauchy problem, a Hammerstein integral equation and Urysohn operators. A result on conditions for a compact, positive operator to have a positive eigenvalue and eigenvector will also be obtained. In section 10 of [16], three well known theorems: the Birkoff-kellogg theorem, the Hopf theorem on spheres and the Borsuk-Ulam theorem, were proved by applying their spectral theory. We shall show that our new theory enables us not only to prove, but also to generalize these theorems. We shall also obtain a generalization of Theorem 10.1.2 of [16].

The aim of chapter three is to extend the theory in chapter two to semilinear operators, (L, N), where L is a linear operator that is Fredholm of index zero, and N is a nonlinear operator. To do this, firstly we introduce the L-stably solvable mappings, which is a generalization of the stably-solvable operators defined in [16]. We shall show that some properties of stably-solvable operators hold true for L-stably solvable mappings, for example, the Continuation Principle. Then, in section 3.2, we shall extend the notion of $(0, k)$-epi mappings, which were defined in [65], to $(0, L, k)$-epi mappings for semilinear operators. Some properties of the $(0, L, k)$-epi mappings, such as existence results, normalization property, localization property, homotopy property, will be proved. These results generalize the results of [17], [43] and [65].

In section 3.3, we define $\sigma(L, N)$, the spectrum of semilinear operators (L, N). We shall prove that this spectrum contains all eigenvalues of (L, N) and it is closed. When L is the identity map, this spectrum reduces to the spectrum defined in Chapter 2.

Section 3.4 is dedicated to the study of the decomposition of the spectrum of (L, N). According to the decomposition of the spectrum $\sigma_{fmv}(f)$, we can decompose $\sigma(L, N)$ into $\sigma_\delta(L, N), \sigma_m(L, N), \sigma_\omega(L, N)$ and $\sigma_\pi(L, N)$. We shall prove that if N is a continuous L-k-set contraction and an odd map, $\lambda \in \sigma(L, N)$ with $|\lambda| > k$, then $\lambda \in \sigma_m(L, N)$ [Theorem 3.4.1]. We shall also study the boundary of the spectrum and show that if λ is in the boundary of the spectrum, then either $\lambda L - N$ is a surjective map or $\lambda \in \sigma_\pi(L, N)$. At the end of this section, we shall prove a theorem which gives information about the structure of $\sigma(L, N)$ when N is a continuous L-compact map defined on an infinite dimensional Banach space [Theorem 3.4.6]. The results will be used in section 3.6.

In section 3.5, we shall study eigenvalues of (L, N) when N is an asymptotically linear operator or a positively homogeneous operator.

In the last section of this chapter, we obtain some applications of this theory. By applying this theory, we can extend some existence results for semilinear operator equations. A different condition for the existence of a solution of the equation $\lambda L - T = 0$ can be obtained from that given in [31]. Also, Theorem 2.2 of [46] on the existence results of Leray-Schauder type can be generalized by using this theory.

The authors of [12] gave theorems for operators of the form $\lambda T - S$ of Fredholm alternative type under the assumptions that T is an odd (K, L, a)-homeomorphism and $S : X \to Y$ is an odd compact (completely continuous) operator. Furthermore, they showed the existence of a solution of the nonlinear operator equation

$$\lambda T(x) - S(x) = f \tag{0.1}$$

for each $f \in Y$ provided $\lambda \neq 0$ if T is an odd a-homogeneous and S is an odd b-strongly quasihomogeneous with $a > b$. In the case $a < b$ they proved the same assertion in finite dimensional spaces but said it was unsolved in the infinite-dimensional case.

In chapter 4, we shall obtain some surjectivity results on the mapping $\lambda T - S$ under weaker conditions. Our assumptions do not assume that S is an odd map. We employ different methods which allow us to answer some of their open questions. By introducing the concept of a-stably solvable operator and proving the Continuation Principle for this kind of map, we can obtain a result which generalizes the result of existence of a solution of (0.1) in case $a < b$ to the infinite-dimensional case. These results seem not to be able to be proven by their methods.

In section 4.2, we give some examples of ordinary differential equations for which the existence of a solution can be obtained by applying the theorems. It is possible to give simple examples that show that our results are real extensions of the earlier ones, but we prefer to give more substantial applications. We shall discuss a nonlinear Sturm-Liouville problem on the half line following the work by Toland [66]. He studied eigenvalues and asymptotic bifurcation points whereas we obtain surjectivity when λ is not one of these eigenvalues.

We also discuss existence of solutions to a three point boundary value problem recently studied by Gupta, Ntouyas and Tsamatos [29]. The boundary conditions are of the type $x(0) = 0, x(1) = \alpha x(\eta)$. Those authors assume that $\alpha < 1/\eta$ but we suppose only that $\alpha \neq 1/\eta$. We obtain a different criterion for existence which improves on Theorem 4 of [29] in some cases but is less good in others.

Chapter 5 follows the recent work done by Gupta, Ntouyas, Tsamatos and Lakshmikantham [24]-[30]. They studied the so-called nonlocal boundary value problems, which were studied also by Il'in and Moiseev [37] and S.A. Marano [47].

Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Carathéodory's conditions and $e : [0,1] \to \mathbb{R}$ be a function in $L^1(0,1)$, $a_i \in \mathbb{R}$ with all of the $a_i's$ having the same sign, $\xi_i \in (0,1), i = 1, 2, ..., m-2$, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$. Consider the following second order ordinary differential equation:

$$x''(t) = f(t, x(t), x'(t)) + e(t) \qquad t \in (0,1), \tag{0.2}$$

with one of the following boundary value conditions:

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{0.3}$$

$$x(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \tag{0.4}$$

It is known that the problem of the existence of a solution for these boundary-value problems can be studied respectively via the existence of a solution for equation (0.2) subject to one of the following three-point boundary-value conditions (see [29], [30]):

$$x'(0) = 0, \qquad x(1) = \alpha x(\eta), \tag{0.5}$$

$$x(0) = 0, \qquad x(1) = \alpha x(\eta), \tag{0.6}$$

where $\alpha \in \mathbb{R}$ and $\eta \in (0,1)$ are given.

In [30], the existence results for the BVP (0.2), (0.5) with the condition $\alpha \neq 1$ were proved and in [29], results for the BVP (0.2), (0.6) were obtained when $\alpha\eta < 1$. These assumptions ensure that the linear part L is invertible. They assume also that the non-linear part f has a linear growth. The method they used is Leray-Schauder Continuation theorem and Wirtinger type inequalities.

In section 5.1, we also assume that f has a linear growth. We shall prove the existence results for BVP (0.2), (0.5) with the condition $\alpha = 1$ and (0.2), (0.6) with the condition $\alpha = 1/\eta$. In these cases, L is noninvertible, the so-called resonance case. The Leray-Schauder degree theory can not be used. Our results make use of the coincidence degree theory of Mawhin [46].

In section 5.2, we shall obtain two uniqueness results for these kind of boundary value problems.

In section 5.3, we shall prove existence results for BVP (0.2), (0.5) and BVP (0.2), (0.6) which allow f to have nonlinear growth. We do this by imposing a decomposition condition for f and by showing that the growth of certain nonlinear terms is not restricted provided they satisfy a sign condition. We obtain appropriate *a priori* bounds and apply degree theory. Moreover, by using the coincidence degree theory of Mawhin, we also able to give existence results when the linear operator L is non-invertible and f has nonlinear growth. This allows us to treat the BVP (0.2), (0.5) with $\alpha = 1$, and the BVP (0.2), (0.6) with $\alpha = \frac{1}{\eta}$ without the restriction that f has a linear growth.

We shall give examples of equations which can be treated by our results but the results of [24], [25], [29], [30], [61] cannot be applied.

In section 5.4, we shall prove existence results for BVP (0.2), (0.5) with $|\alpha| \leq 1$ and f has a different nonlinear growth with that in section 5.3. As a special case, we allow f to have quadratic growth. Moreover, as a corollary of our theorem, we obtain a result on the Neumann boundary value problem which generalizes one of the results of [57]. We also prove a similar result for the m-point BVP (0.2), (0.3) when $|\sum_{i=1}^{m-2} a_i| \leq 1$.

For our results in this chapter, it is important that all the a_i's have the same sign and our result for the m-point BVP makes use of the estimates obtained in the proof for the three point BVP. Gupta [27] has considered a different m-point boundary value problem where the a_i's do not have the same sign and this technique cannot be used.

Finally in chapter 6, we shall establish some new existence results on the solvability of the following second order ODE's of the form

$$x'' = f(t, x, x') \tag{0.7}$$

subject to one of the following boundary conditions:

$$x(0) = x(1) = 0, \tag{0.8}$$

$$x'(0) = x'(1) = 0, \tag{0.9}$$

$$x(0) = x(1), \quad x'(0) = x'(1), \tag{0.10}$$

$$x(0) = -x(1), \quad x'(0) = -x'(1). \tag{0.11}$$

The solvability of (0.7) subject to various boundary conditions has been extensively studied by many authors([4], [32], [54]-[62], [64]). In a recent paper [4], a decomposition condition for f is imposed to ensure the solvability of (0.7) with the boundary condition (0.8). The theorems of [4] were proved by using the transversality theorem.

In section 6.1, by applying the abstract continuation type theorem of W.V. Petryshyn on A-proper mappings, we obtain approximation solvability results for BVP (0.7), (0.8).

These results include the result of [4]. Then, in section 6.2, under the assumption that f can be suitably decomposed, some feebly a-solvability results for (0.7) with the boundary conditions (0.9-0.11) are obtained.

Applying our theorems to the following BVPs, which were studied respectively in [60] and [61]:

$$x'' + \overline{g}(x)x' + \overline{f}(t, x, x', x'') = y(t), \quad x(0) = x(1),\ x'(0) = x'(1), \tag{0.12}$$

and

$$(p(t)x') + \overline{f}(t, x, x', x'') = y(t), \quad x'(0) = x'(T) = 0, \tag{0.13}$$

we can show that certain assumptions made in [60] and [61] are redundant. Our results are therefore substantial generalizations of the results in [60], [61]. Some examples will show that our theorems permit the treatment of equations to which the results of [4], [32], [57] do not apply.

Chapter 1

Preliminaries

In this chapter, we collect some of the notions and definitions that we shall often use in this thesis. We shall recall some known results without proof, but the references are given.

1.1 Generalities

The symbol $\mathbb{K}$ will stand either for the field of complex numbers $\mathbb{C}$ or for the field of real numbers $\mathbb{R}$.

The capital letters E, F, X and Y, unless otherwise stated, will be used to denote Banach spaces over $\mathbb{C}$ or $\mathbb{R}$. Spheres, open and closed balls centered at the origin and with radius $r > 0$ are denoted by

$$S_r = \{x \in E : \|x\| = r\},\ O_r = \{x \in E : \|x\| < r\},\ B_r = \{x \in E : \|x\| \leq r\},$$

respectively.

Given a map $f : E \to F$ we denote the image of f by $\mathrm{im}(f)$ and the kernel of f by $\ker(f)$ which is the set of all $x \in E$ such that $f(x) = 0$. A continuous map $f : E \to F$ is said to be *quasibounded* if there exist two constants $A, B \geq 0$ such that $\|f(x)\| \leq A\|x\|+B$ for all $x \in E$. If f is quasibounded, then it sends bounded sets into bounded sets and

$$|f| = \limsup_{\|x\|\to\infty} \frac{\|f(x)\|}{\|x\|} < +\infty.$$

$|f|$ is called the *quasinorm* of f. A continuous map $f : E \to F$ is called *asymptotically linear* if there exists a bounded linear operator $T : E \to F$ such that $|f - T| = 0$. T is called the *asymptotic derivative* of f. f is said to be *positively homogeneous* if for any $x \in E$ and $t \geq 0$,

$$f(tx) = tf(x).$$

Following [16], we call f *stably-solvable* if and only if given any compact map $h : E \to F$ (that is, a continuous map such that $h(\Omega)$ is relatively compact whenever $\Omega \subset E$ is bounded) with zero quasinorm, there is at least one element x of E such that $f(x) = h(x)$. Note that if f is stably-solvable, it is clearly surjective. If f is linear, it is stably-solvable if and only if it is surjective.

The following Continuation Principle for stably-solvable maps was proved in [16].

Theorem 1.1.1. *Let $f : E \to F$ be stably-solvable and $h : E \times [0,1] \to F$ be compact and such that $h(x, 0) = 0$ for all $x \in E$. Let*

$$S = \{x \in E : f(x) = h(x,t) \textit{ for some } \ t \in [0,1]\}.$$

If $f(S)$ is bounded then the equation

$$f(x) = h(x, 1)$$

has a solution.

In chapter 4, we shall generalize the concept of stably-solvable map to that of *a-stably-solvable* map and prove that for a-stably-solvable maps, the Continuation Principle holds true.

We also recall that f is called a *strong surjection* if the equation $f(x) = h(x)$ has a solution for every continuous map $h : E \to F$ with $\overline{h(E)}$ compact.

Let $f : E \to F$ be a continuous map. $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of f if there exists $x \in E$, $x \neq 0$ such that $f(x) = \lambda x$. Suppose that $f(0) = 0$, λ is said to be a *bifurcation point* of f if there exist sequences λ_n, $x_n \neq 0$ such that $f(x_n) = \lambda_n x_n$,

$\lambda_n \to \lambda$, $x_n \to 0$ as $n \to \infty$. λ is called an *asymptotic bifurcation point* of f if there exist sequences λ_n and $x_n \neq 0$ such that $f(x_n) = \lambda_n x_n$, $\lambda_n \to \lambda$, $\|x_n\| \to \infty$ as $n \to \infty$.

1.2 The measure of noncompactness and definitions of $\alpha(f), \omega(f), d(f)$

.

Definition 1.2.1. Let X be a metric space and $\Omega \subset X$ a bounded subset. The *measure of noncompactness* of Ω, $\alpha(\Omega)$, is defined by (see [7])

$$\alpha(\Omega) = \inf\{d > 0 : \ \Omega \text{ can be covered by a finite number of subsets of } X \text{ of diameter at most } d\}.$$

This notion of measure of noncompactness was introduced by Kuratowski [40].

Let A and B be bounded subsets of a metric space X. Then

1. $\alpha(A) = 0$ if and only if $\overline{A}$ is compact, where $\overline{A}$ denotes the closure of A;
2. $A \subset B$ implies $\alpha(A) \leq \alpha(B)$;
3. $\alpha(A) = \alpha(\overline{A})$;
4. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

   Furthermore, if X is a normed space, then

5. $\alpha(\lambda A) = |\lambda|\alpha(A)$, $\lambda \in \mathbb{K}$;
6. $\alpha(\text{co}(A)) = \alpha(A)$, where $\text{co}(A)$ denotes the convex hull of A;
7. $|\alpha(A) - \alpha(B)| \leq \alpha(A + B) \leq \alpha(A) + \alpha(B)$.

For the proof of these facts we refer to [7] , [42] or [48].

The concept of an α-Lipschitz mapping is of importance in this thesis. It is defined by the following (see [7]).

Definition 1.2.2. Let $\Omega \subset E$ and $f : \Omega \to E$ continuous. f will be called *α-Lipschitz* if $\alpha(f(B)) \leq k\alpha(B)$ for some $k \geq 0$ and all bounded $B \subset \Omega$. If $k < 1$, f is called a *strict α-contraction* or *k-set contraction.* f is said to be a *condensing* mapping if for each noncompact bounded subset B of Ω, $\alpha(f(B)) < \alpha(B)$.

In the sequel, we shall use the following lemma for the measure of noncompactness.

Lemma 1.2.3. ([66], lemma 4.8) *Let f be a real-valued function defined on the Banach space E, which is bounded on bounded subsets of E. For $u \in E$, define $F : E \to E$ by $Fu = f(u)u$. If Ω is a bounded subset of E, then*

$$\alpha(F(\Omega)) \leq l\alpha(\Omega)$$

where

$$l = \sup_{u \in \Omega} |f(u)|.$$

We recall that a continuous mapping $f : E \to F$ is called *proper* if for every compact subset K of F, $f^{-1}(K)$ is compact. The concept of *k-proper* mappings was defined in [65]. Let $k \geq 0$. A mapping f from a subset $\overline{\Omega}$ of E to F, written $f : \overline{\Omega} \to F$, is said to be k-proper if f is continuous and

$$\alpha(f^{-1}(S)) \leq k\alpha(S)$$

for each bounded set $S \subset F$. By the property of the measure of noncompactness, if f is k-proper for some $k > 0$, then f is proper.

Given a continuous map f from a subset $D(f)$ of E to F. The three real numbers $\alpha(f), \omega(f)$ and $d(f)$, which were used in [16] to define the spectrum, are defined by the following:

$$\begin{aligned}
\alpha(f) &= \inf\{k \geq 0 : \alpha(f(\Omega)) \leq k\alpha(\Omega) \text{ for every bounded } \Omega \subset D(f)\}, \\
\omega(f) &= \sup\{k \geq 0 : \alpha(f(\Omega)) \geq k\alpha(\Omega) \text{ for every bounded } \Omega \subset D(f)\}, \\
d(f) &= \liminf_{\|x\| \to \infty, x \in D(f)} \frac{\|f(x)\|}{\|x\|}.
\end{aligned}$$

Let $f, g : E \to E$ be continuous. The main properties of $\alpha(f), \omega(f)$ and $d(f)$, which were proved in [16], are contained in the following.

Proposition 1.2.4. *1.* $\alpha(\lambda f) = |\lambda|\alpha(f)$, $\lambda \in \mathbb{K}$.

2. $|\alpha(f) - \alpha(g)| \leq \alpha(f+g) \leq \alpha(f) + \alpha(g)$.

3. $\alpha(f) = 0$ *if and only if* f *is compact.*

4. If $\dim(E) = +\infty$ *and* f *is compact, then* $\alpha(\lambda - f) = |\lambda|$.

Proposition 1.2.5. *1.* $\omega(\lambda f) = |\lambda|\omega(f)$, $\lambda \in \mathbb{K}$.

2. $\omega(f)\omega(g) \leq \omega(fg) \leq \alpha(f)\omega(g)$.

3. If $\omega(f) > 0$, *then* f *is proper on bounded closed sets. If moreover,* $d(f) > 0$, *then* f *is proper.*

4. If $\dim(E) = +\infty$ *then* $\omega(f) \leq \alpha(f)$, *and* $\omega(f) = +\infty$ *if* $\dim(E) < +\infty$.

5. $\omega(f) - \omega(g) \leq \omega(f+g) \leq \omega(f) + \alpha(g)$.

6. $|\omega(f) - \omega(g)| \leq \alpha(f-g)$.

7. If f *is a homeomorphism and* $\omega(f) > 0$, *then* $\alpha(f^{-1})\omega(f) = 1$.

8. If $\dim E = \infty$ *and* f *is compact, then* $\omega(\lambda - f) = |\lambda|$.

Proposition 1.2.6. *1.* $0 \leq d(f) \leq |f|$.

2. $d(\lambda f) = |\lambda| d(f)$, $\lambda \in \mathbb{K}$.

3. $d(f) - |g| \leq d(f+g) \leq d(f) + |g|$.

4. $|d(f) - d(g)| \leq |f - g|$.

5. If f *is a homeomorphism with quasibounded inverse, then* $d(f) = |f^{-1}|^{-1}$.

1.3 Spectra for nonlinear operators

In this section, we recall the different spectra for nonlinear operators. In chapter 2, we shall compare the spectrum we introduce in section 2.1 with these. In the following, E is a Banach space and $f : E \to E$ is a continuous (nonlinear) operator. $\alpha(f), \omega(f)$ and $d(f)$ are as in section 1.2.

The following definition was given in [16].

Definition 1.3.1. f is said to be *regular* if it is stably-solvable and $d(f)$ and $\omega(f)$ are both positive. Let

$$\rho_{fmv}(f) = \{\lambda \in \mathbb{C}, \lambda - f \text{ is regular}\},$$

be the *resolvent set* of f and let $\sigma_{fmv}(f) = \mathbb{C} \backslash \rho_{fmv}(f)$ be the FMV-spectrum of f.

The following result was proved in [16].

Theorem 1.3.2. *Let* $r(f) = \sup\{|\lambda| : \lambda \in \sigma_{fmv}(f)\}$ *and* $q(f) = \max\{\alpha(f), |f|\}$. *Then* $r(f) \leq q(f)$.

According to [16], $\sigma_{fmv}(f)$ can be decomposed into the following two parts.

$$\begin{aligned} \sigma_\pi(f) &= \{\lambda \in \mathbb{K} : d(\lambda - f) = 0 \text{ or } \omega(\lambda - f) = 0\}, \\ \sigma_\delta(f) &= \{\lambda \in \mathbb{K} : \lambda - f \text{ is not stably-solvable}\}. \end{aligned}$$

They proved that $\sigma_\pi(f)$ is closed and the boundary of the spectrum $\partial\sigma_{fmv}(f)$ is contained in $\sigma_\pi(f)$. More precisely, $\sigma_\pi(f)$ can be regarded as the union of the following two sets

$$\sigma_\omega(f) = \{\lambda \in \mathbb{K} : \omega(\lambda - f) = 0\} \text{ and } \Sigma(f) = \{\lambda \in \mathbb{K} : d(\lambda - f) = 0\}.$$

In Chapter 3, we will discuss the corresponding decomposition of the spectrum for semi-linear operators.

The following proposition gives information about the structure of $\sigma_{fmv}(f)$ when f is a compact map.

Proposition 1.3.3. ([16], p.272) *Let $f : E \to E$ be a compact map defined on an infinite dimensional Banach space E. Then*

1. $\sigma_\omega(f) = 0$, *therefore* $\sigma_\pi(f) = \{0\} \cup \Sigma(f)$;

2. $f(E) \neq E$. *In particular,* $0 \in \sigma_\delta(f)$;

3. $0 \notin \Sigma(f)$ *implies that the connected component of* $\mathbb{K} \setminus \Sigma(f)$ *containing zero lies entirely in* $\sigma_\delta(f)$. *In particular,* 0 *is an interior point of* $\sigma_\delta(f)$;

4. *If moreover f is positively homogeneous, then*

$$\Sigma(f) \setminus \{0\} = \{\lambda \in \mathbb{K} : \lambda x = f(x) \textit{ for some } x \neq 0\}.$$

In [39], a spectrum for Lipschitz continuous operators was introduced by Kachurovskij.

Definition 1.3.4. Let $Lip(E)$ be the space of all Lipschitz mappings. For $A \in Lip(E)$, the Lipschitz constant

$$\|A\|_{lip} := \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}$$

is finite. Then the Lip-spectrum for A is defined by

$$\sigma_{lip}(A) = \{\lambda : (\lambda - A)^{-1} \text{ does not exists or } (\lambda - A)^{-1} \notin Lip(E)\}.$$

Definition 1.3.4 implies that $\lambda \notin \sigma_{lip}(A)$ only in case $(\lambda - A)$ is bijective.

Many properties for $\sigma_{lip}(A)$ can be found in [48]. An open question in [48] is whether $\sigma_{lip}(A)$ is nonempty. In section 2.4, we shall give an answer to this question by giving an example where $\sigma_{lip}(A)$ is empty.

For a continuous linear operator, the two spectra above coincide with the usual definition of the spectrum. If f is nonlinear, $\sigma_{fmv}(f)$ may be disjoint from eigenvalues of f, which can not happen in the linear case. The spectrum $\sigma_{lip}(A)$ contains all the eigenvalues of A and is compact. However, it may be empty (see the example in section 2.4). The spectrum we will introduce in Chapter 2, which applies to all continuous nonlinear operators, is compact, upper semicontinuous and contains all eigenvalues. However, it too may be empty.

1.4 (p, k)-epi mappings

The notion of p-epi *mappings* was introduced by Furi, Martelli and Vignoli [17] as follows:

Definition 1.4.1. If E and F are normed linear spaces, $\Omega \subset E$ is a bounded open set and $p \in F$ then a continuous mapping $f : \overline{\Omega} \to F$ with $f(x) \neq p$ for any $x \in \partial\Omega$ is called p-epi if for each compact mapping $h : \overline{\Omega} \to F$ with $h \equiv 0$ on $\partial\Omega$, the equation

$$f(x) = h(x) + p$$

has a solution in Ω.

Then, in [65], this concept was generalized to (p, k)-epi *mapping* by allowing the mapping h to be a k-set contraction rather then just a compact mapping and requiring E and F to be Banach spaces. Thus the class of (p, k)-epi mappings is smaller than that of p-epi mappings.

In the following, Ω is an open bounded subset of E.

Definition 1.4.2. A continuous mapping $f : \overline{\Omega} \to F$ is said to be *p-admissible* $(p \in F)$ if $f(x) \neq p$ for $x \in \partial\Omega$.

Definition 1.4.3. A 0-admissible mapping $f : \overline{\Omega} \to F$ is said to be $(0, k)$-epi if for each k-set contraction $h : \overline{\Omega} \to F$ with $h(x) \equiv 0$ on $\partial\Omega$ the equation $f(x) = h(x)$ has a solution in Ω. Similarly, a p-admissible mapping $f : \overline{\Omega} \to F$ is said to be (p, k)-epi if the mapping $f - p$ defined by

$$(f - p)(x) = f(x) - p, \quad x \in \overline{\Omega},$$

is $(0, k)$-epi.

(p, k)-epi mappings also were defined in the whole space in [65]. Let $f : E \to F$ be a continuous mapping of E into F. For $p \in F$, f is said to be p-admissible if $f^{-1}(p)$ is a bounded subset of E. f is said to be (p, k)-epi if f is (p, k)-epi on the closure of every bounded open set $\Omega \supset f^{-1}(p)$, that is, $f_{\overline{\Omega}}$, the restriction of f to $\overline{\Omega}$, is (p,k)-epi for each bounded open subset Ω containing $f^{-1}(p)$.

It was proved in [65] that the (p,k)-epi mappings, similarly to the p-epi mappings, have 'existence', 'boundary dependence', 'normalization', 'localization' and 'homotopy' properties similar to those of topological degree theory.

Property 1.4.4. *(Existence property)*

If $f : \overline{\Omega} \to F$ is a (p,k)-epi mapping, then the equation $f(x) = p$ has a solution in Ω.

Property 1.4.5. *(Normalization property)*

The inclusion mapping $i : \overline{\Omega} \to E$ is (p,k)-epi for $k \in [0,1)$ if and only if $p \in \Omega$.

Property 1.4.6. *(Localization property)*

If $f : \overline{\Omega} \to F$ is $(0,k)$-epi and $f^{-1}(0)$ is contained in an open set $\Omega_1 \subset \Omega$, then f restricted to Ω_1 is also $(0,k)$-epi.

Property 1.4.7. *(Homotopy property)*

Let $f : \overline{\Omega} \to F$ be $(0,k)$-epi and $h : [0,1] \times \overline{\Omega} \to F$ be an α-set contraction with $0 \le \alpha \le k < 1$ such that $h(0,x) = 0$ for all $x \in \overline{\Omega}$. Further let

$$f(x) + h(t,x) \neq 0$$

for all $x \in \partial\Omega$ and for all $t \in [0,1]$. Then $f(x) + h(1,x) : \overline{\Omega} \to F$ is $(0, k-\alpha)$-epi.

Property 1.4.8. *(Boundary dependence property)*

Let $f : \overline{\Omega} \to F$ be $(0,k)$-epi and $g : \overline{\Omega} \to F$ be an α-set contraction with $0 \le \alpha \le k < 1$ and $g \equiv 0$ on $\partial\Omega$. Then $(f+g) : \overline{\Omega} \to F$ is $(0, k-\alpha)$-epi.

The following theorem will be used in the sequel.

Theorem 1.4.9. *[65] Let $f : \overline{\Omega} \to F$ be continuous, injective and k_1-proper. Then $f(\Omega)$ is open if and only if f is (p,k)-epi for each $p \in f(\Omega)$ and each nonnegative k satisfying the condition $k_1 k < 1$.*

The theory of (p, k)-epi mappings is based on elementary tools such as the Schauder fixed point theorem, Urysohn's Lemma, etc. (p, k)-epi mappings may act between different spaces. Degree theories are often used to establish the existence of solutions of nonlinear problems. However in applications, such as to differential and functional differential equations, to apply degree theory it is often necessary to reformulate the problems as nonlinear self mappings acting on some space, whereas the theory of (p, k)-epi maps may applied directly.

In chapter 2, we shall use $(0, k)$-epi mapping theory to establish some spectral theory for nonlinear operators. In chapter 3, to study the spectral theory for semilinear operators, we shall generalize the $(0, k)$-epi mappings to $(0, L, k)$-epi mappings for semilinear operators, where L is Fredholm of index zero.

1.5 Coincidence degree theory

Let $L : \mathrm{dom}(L) \subset E \to F$ be a linear operator. Recall that L is called *Fredholm of index zero* if the following conditions hold:

1. $\mathrm{im}(L)$ is closed in F.

2. $\ker(L)$ and the cokernel of L, $F/\mathrm{im}(L)$, are finite dimensional and with equal dimension.

Now suppose that $L : \mathrm{dom}(L) \subset E \to F$ is a closed Fredholm operator of index zero, assume that $\ker(L) \neq \{0\}$ and $\mathrm{dom}(L)$ is dense in E. Let $E = \ker(L) \bigoplus E_1$, $F = F_0 \bigoplus \mathrm{im}(L)$ and $P : E \to \ker(L)$, $Q : F \to F_0$ be the respective projections. Also let L_P denote the invertible operator L restricted to $\mathrm{dom}(L) \cap E_1$ into $\mathrm{im}(L)$, write $K_P = {L_P}^{-1}$, $K_{PQ} = K_P(I - Q)$, and let $\Pi : F \to F/\mathrm{im}(L)$ be the quotient map, and let $\Lambda : F/\mathrm{im}(L) \to \ker(L)$ be the linear isomorphism (see [31]).

Let Ω be an open bounded subset of E such that $\text{dom}(L) \cap \Omega \neq \emptyset$ and $N : \overline{\Omega} \to F$ be a nonlinear mapping.

Definition 1.5.1. N will be said to be *L-compact* if

1. $\Pi N : \overline{\Omega} \to \text{coker}(L)$ is continuous and $\Pi N(\overline{\Omega})$ bounded.

2. $K_{PQ}N : \overline{\Omega} \to E$ is compact.

N is said to be a L-k-set contraction if

1. $\Pi N : \overline{\Omega} \to \text{coker}(L)$ is continuous and $\Pi N(\overline{\Omega})$ bounded.

2. $K_{PQ}N : \overline{\Omega} \to E$ is a k-set contraction.

Let L and N be as above and assume that $k < 1$ and that

$$0 \notin (L - N)(\text{dom}(L) \cap \partial\Omega).$$

Then $M_\Lambda = P + (\Lambda\Pi + K_{PQ})N$ is a k-set contraction and hence the degree

$$d[I - M_\Lambda, \Omega, 0]$$

is defined. The following definition can be found in [31].

Definition 1.5.2. The coincidence degree $d[(L, N), \Omega]$ of L and N in Ω is the integer

$$d[(L, N), \Omega] = d[I - M_\Lambda, \Omega, 0],$$

where the right hand number is the degree for k-set contractive perturbations of the identity [52].

The coincidence degree for (L, N) when N is L-compact was introduced by J. Mawhin in 1972 [44] and systematic expositions were given in [31] and [45].

The results for the existence of a solution for the second order ordinary differential equations in chapter 5 will be obtained by applying the following useful continuation theorem of coincidence degree theory, which was first proved in [44].

Theorem 1.5.3. (see [46], p.84)

Let L be Fredholm of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied.

1. $Lx + \lambda Nx \neq 0$ *for each* $(x, \lambda) \in [(D(L) \setminus \ker L) \cap \partial\Omega] \times (0, 1)$.

2. $Nx \notin \mathrm{im}(L)$ *for each* $x \in \ker(L) \cap \partial\Omega$.

3. $\deg(QN|_{\ker L}, \Omega \cap \ker(L), 0) \neq 0$, *where* $Q : F \to F_0$ *is a continuous projection as above.*

Then the equation $Lx + Nx = 0$ has at least one solution in $D(L) \cap \overline{\Omega}$.

We shall also, by applying the spectral theory for semilinear operators in chapter 3, obtain a generalization of the following existence theorem of Leray-Schauder type [46].

Theorem 1.5.4. *Let $F = L + N$ with $N : \overline{\Omega} \to F$ L-compact, let $A : E \to F$ be a linear L-compact mapping and $z \in (L + A)(\mathrm{dom}(L) \cap \Omega)$ satisfy the following conditions:*

1. $\ker(L + A) = \{0\}$.

2. $Lx + (1 - \lambda)(Ax - z) + \lambda Nx \neq 0$ *for each* $x \in \mathrm{dom}(L) \cap \partial\Omega$ *and each* $\lambda \in (0, 1)$.

Then equation $Lx + Nx = 0$ has at least one solution in $\mathrm{dom}(L) \cap \overline{\Omega}$.

1.6 A-proper maps

The notion of an *A-proper* mapping was introduced by W.V. Petryshyn in 1967. The basic theory of A-proper maps has been given in [58] and in [59]. One of the main purposes of the book [59] is to use the topological degree for densely defined A-proper operators in the systematic study of the solvability or approximation solvability of the semilinear equation

$$Lx - Nx = y, \quad x \in \overline{\Omega},\ y \in F,$$

where $L : \text{dom}(L) \subset E \to F$ is a Fredholm mapping of index $i(L) \geq 0$, N is a nonlinear mapping such that $L - N$ or $T_\lambda \equiv L - \lambda N : \overline{\Omega} \subset E \to F$ is A-proper for each $\lambda \in (0, 1]$ with respect to a suitable approximation scheme.

In chapter 5, we shall use the following abstract continuation type theorem [56], [57] for semilinear equations involving A-proper mappings, to obtain the approximation solvability results for the second order ODEs subject to Dirichlet, Neumann, periodic and antiperiodic boundary conditions.

Firstly we recall the definition of the A-proper mappings [57].

Definition 1.6.1. If $\{E_n\} \subset E$ and $\{F_n\} \subset F$ are sequences of finite dimensional oriented spaces and $Q_n : F \to F_n$ is a linear projection for each $n \in \mathbb{R}^+$, then the scheme $\Gamma = \{E_n, F_n, Q_n\}$ is said to be *admissible* for maps from E to F provided that $\dim E_n = \dim F_n$ for each n, $\text{dist}(x, E_n) \equiv \inf\{\|x - v\|_E : v \in E_n\} \to 0$ as $n \to \infty$ for each x in E, and $Q_n y \to y$ for each y in F. For a given map $T : D \subset E \to F$ the equation

$$Tx = y \tag{1.1}$$

is said to be *feebly approximation-solvable* (*a-solvable*) relative to Γ if there exists $N_y \in \mathbb{R}^+$ such that the finite dimensional equation

$$T_n(x) = Q_n y, \ (x \in D_n \equiv D \cap E_n, Q_n y \in F_n, T_n = Q_n T|_{D_n}), \tag{1.2}$$

has a solution $x_n \in D_n$ for each $n \geq N_y$ such that $x_{n_j} \to x \in D$ in E and $Tx = y$.

Definition 1.6.2. T is said to be A-proper relative to Γ if $T_n : D_n \subset E_n \to F_n$ is continuous for each $n \in \mathbb{R}^+$ and if $\{x_{n_j} | x_{n_j} \in D_{n_j}\}$ is any bounded sequence in E such that $T_{n_j}(x_{n_j}) \to g$ for some g in F, then there is a subsequence $\{x_{n_k}\}$ of $\{x_{n_j}\}$ and $x \in D$ such that $x_{n_k} \to x$ in E and $Tx = g$.

It is known that for (1.1) to be a-solvable relative to a given Γ the operator T has essentially to be A-proper relative to Γ [54].

Now let $L : E \to F$ be a Fredholm operator of index zero. It was shown [55] that an admissible scheme Γ_L (depending on L) can be constructed such that L is A-proper relative to Γ_L. Indeed, suppose that $E = \ker(L) \bigoplus E_1$, $F = F_0 \bigoplus \mathrm{im}(L)$. Let Q be the projection of F onto F_0. There is a compact map $C : E \to F_0$ such that $K = L - C$ is a homeomorphism of E onto F and choosing $\{\tilde{E_n}\} \subset E$ such that $F_n = K(\tilde{E_n})$ for $n \in \mathbb{Z}^+$, then, it can be shown that $\Gamma_L = \{\tilde{E}_n, F_n, Q_n\}$ is admissible and L is A-proper relative to Γ_L. In the following, we shall assume that there exists a continuous bilinear form $[\cdot, \cdot]$ on $F \times E$ mapping (y, x) into $[y, x]$ such that $y \in \mathrm{im}(L)$ if and only if $[y, x] = 0$ for every $x \in \ker(L)$.

Theorem 1.6.3. [56][57] *Let L be a Fredholm operator of index zero and $N : E \to F$ be a nonlinear map. Suppose there exists a bounded open set $G \subset E$ with $0 \in G$ such that*

1. *$L - \lambda N : \overline{G} \to F$ is A-proper relative to Γ_L for each $\lambda \in [0, 1]$ with $N(\overline{G})$ bounded;*
2. *$Lx \neq \lambda Nx - \lambda y$ for $x \in \partial G$ and $\lambda \in (0, 1]$.*
3. *$QNx - Qy \neq 0$ for $x \in \partial G \cap \ker(L)$.*
4. *Either $[QNx - Qy, x] \geq 0$ or $[QNx - Qy, x] \leq 0$ for $x \in \partial G \cap \ker(L)$.*

Then the equation

$$Lx - Nx = y$$

is feebly a-solvable relative to Γ_L and in particular it has a solution $x \in G$.

Chapter 2

A new spectral theory for nonlinear operators and its applications

The spectral theory for nonlinear operators has been extensively studied, see for example [1], [12], [16]. Different attempts have been made to define the spectra for nonlinear operators. In particular, the spectrum introduced by Furi, Martelli and Vignoli has found many interesting applications, see [16]. However, it was indicated in [9] that this spectrum may be disjoint from the eigenvalues in some cases. The main aim of this chapter is to give a new definition for the spectrum of nonlinear operators, which is closed and contains all the eigenvalues as in the case of linear operators. A counterexample proves that the spectrum may be empty, which answers one of the open questions in [48].

As the last part of this chapter, the applications of the new theory will be discussed. This theory enables us to generalize three well known theorems: the Birkoff-Kellogg theorem, the Hopf theorem on spheres and the Borsuk-Ulam theorem. Existence of non-trivial solutions for a global Cauchy problem, Hammerstein integral equations and a Urysohn operator are obtained by using this theory. Also, we shall apply this theory to obtain a generalization of a theorem in [16] and then discuss bifurcation points and asymptotic bifurcation points for a Urysohn operator.

2.1 A new definition of the spectrum for continuous operators

Throughout the following, E is a Banach space, $f : E \to E$ is a continuous nonlinear operator. $\alpha(f), \omega(f)$ and $d(f)$ are as in section 1.2. Let

$$m(f) = \sup\{k \geq 0 : \|f(x)\| \geq k\|x\| \text{ for all } x \in E\}.$$

If for every $x \neq 0, f(x) \neq 0$, let

$$\begin{aligned}\nu_r(f,0) \quad = \quad & \inf\{k \geq 0, \text{ there exists a } k\text{-set contraction } g : B_r \to E, \text{ with} \\ & g \equiv 0 \text{ on } \partial B_r \text{ s.t. } f(x) = g(x) \text{ has no solutions in } B_r\},\end{aligned}$$

where $B_r = \{x \in E : \|x\| \leq r\}$ and ∂B_r denotes the boundary of B_r. Let $\nu(f) = \inf\{\nu_r(f,0), r > 0\}$. We will call $\nu(f)$ the *measure of solvability* of f at 0 [65]. Notice that, $\nu(f) > 0$ if and only if there exists $\varepsilon > 0$, such that $f(x)$ is $(0,\varepsilon)$-epi on every B_r with $r > 0$. We begin with the following definition.

Definition 2.1.1. Suppose that $f : E \to E$ is a continuous map, then f is said to be *regular* if

$$\omega(f) > 0, \quad m(f) > 0, \quad \nu(f) > 0.$$

For each $\lambda \in \mathbb{C}$, if $\lambda I - f$ is *regular*, λ is said to be in the resolvent set of f. Let $\rho(f)$ denote the resolvent set of f, then the spectrum of f is defined as follows:

$$\sigma(f) = \{\lambda \in \mathbb{C} : \ \lambda I - f \text{ is not regular }\} = \mathbb{C} \backslash \rho(f).$$

We will see that all regular maps are onto.

Proposition 2.1.2. *If f is a regular map, then f is surjective.*

Proof: Since f is regular, $m(f) > 0$ and $\|f(x)\| \geq m(f)\|x\|$. Thus $\|f(x)\| \to \infty$ as $\|x\| \to \infty$. Also we have $\nu(f) > 0$, so there exists $\varepsilon > 0$ such that $f(x)$ is $(0,\varepsilon)$-epi on every B_r with $r > 0$. By Corollary 3.2 [65], f is a surjective map. $\square$

The following theorem characterizes regular maps among continuous linear operators.

Theorem 2.1.3. *Suppose that E is a normed linear space, $f : E \to E$ is a continuous linear operator. Then f is regular if and only if f is a linear homeomorphism.*

Proof: Assume that f is regular. By Proposition 2.1.2, f is surjective. Also, we have that $\|f(x)\| \geq m(f)\|x\|$ with $m(f) > 0$, so f is one to one and $\|f^{-1}(x)\| \leq (1/m(f))\|x\|$. Thus f^{-1} is a continuous operator, f is a linear homeomorphism.

Conversely, suppose that f is a linear homeomorphism. Then f^{-1} is a bounded linear operator and for every $x \in E$,

$$\|f(x)\| \geq \frac{1}{\|f^{-1}\|}\|x\|.$$

This ensures that

$$m(f) \geq \frac{1}{\|f^{-1}\|}, \quad \omega(f) \geq \frac{1}{\|f^{-1}\|}.$$

Let $0 < \varepsilon < 1/\|f^{-1}\|$, then f is $(0, \varepsilon)$-epi on every B_r with $r > 0$ [65]. Hence $\nu(f) > \varepsilon > 0$ and f is regular. □

Remark 2.1.4. By Theorem 2.1.3, we know that for a linear operator f, the spectrum of f in Definition 2.1.1 is the same as the usual spectrum of f.

It is well known in linear spectral theory that $\sigma(f)$ is a closed set and $\rho(f)$ is an open set. The following theorem shows that this property holds true for the spectrum of nonlinear maps given by Definition 2.1.1.

Theorem 2.1.5. *For every continuous map f, $\rho(f)$ is an open set and $\sigma(f)$ is closed.*

Proof: Suppose that $\lambda \in \rho(f)$, then

$$\omega(\lambda I - f) > 0, \ m(\lambda I - f) > 0.$$

$\lambda I - f$ is $(0, \varepsilon)$-epi on every B_r with $r > 0$ for some $\varepsilon > 0$. Now let

$$\delta_1 = \omega(\lambda I - f)/2,\ \delta_2 = m(\lambda I - f)/2,\ \delta_3 = \varepsilon/2$$

and $\delta = \min\{\delta_1,\ \delta_2,\ \delta_3\}$. Assume that $\mu \in \mathbb{C}$, $|\mu - \lambda| < \delta$. We shall prove that $\mu \in \rho(f)$. By Proposition 1.2.5, we have

$$|\omega(\mu I - f) - \omega(\lambda I - f)| \leq \alpha(\mu I - \lambda I) = |\mu - \lambda| < \frac{\omega(\lambda I - f)}{2},$$

so that

$$\omega(\mu I - f) > \frac{\omega(\lambda I - f)}{2} > 0.$$

For every $x \in E$,

$$\|\mu x - f(x)\| \geq \|\lambda x - f(x)\| - |\mu - \lambda|\|x\| \geq \frac{m(\lambda I - f)}{2}\|x\|,$$

so

$$m(\mu I - f) \geq \frac{m(\lambda I - f)}{2}.$$

Furthermore, let $h : [0, 1] \times E \to E$ be defined by $h(t, x) = t(\mu - \lambda)I$. Then h is a $(\mu - \lambda)$-set contraction. Let

$$S = \{x \in E : \lambda x - f(x) + t(\mu - \lambda)x = 0 \text{ for some } t \in (0, 1]\}.$$

Then for every $x \in S$,

$$\|\lambda x - f(x)\| = \|t(\mu - \lambda)x\| \geq m(\lambda I - f)\|x\|.$$

Hence

$$|\mu - \lambda|\|x\| \geq m(\lambda I - f)\|x\|.$$

Thus $x = 0$ and $S = \{0\}$. By Property 1.4.7, $\mu I - f$ is $(0, \varepsilon - |\mu - \lambda|)$-epi on B_r. Then

$$\nu(\mu I - f) > \varepsilon - |\mu - \lambda| > \frac{\varepsilon}{2} > 0.$$

Therefore, $\mu \in \rho(f)$. □

We recall that for a bounded linear operator, its spectrum is always bounded. The following theorem generalizes this result to the nonlinear case.

Theorem 2.1.6. *Let $f : E \to E$ be a continuous map. Assume that $\alpha(f) < \infty$ and there exists a constant $M > 0$ such that $\|f(x)\| \le M\|x\|$ for every $x \in E$. Then $\sigma(f)$ is bounded.*

Proof: Let $\lambda \in \mathbb{C}$ with $|\lambda| > \max\{M, \alpha(f)\}$, we shall prove that $\lambda \in \rho(f)$.

Firstly, by Proposition 1.2.5 we have

$$\omega(\lambda I - f) \ge |\lambda| - \alpha(f) > 0.$$

Also, for every $x \in E$, the inequality

$$\|(\lambda I - f)(x)\| \ge (|\lambda| - M)\|x\|$$

implies that $m(\lambda I - f) > 0$. Now let $\varepsilon > 0$ be such that $\alpha(f) + \varepsilon < |\lambda|$, we shall show that $\lambda I - f$ is $(0, \varepsilon)$-epi on every B_r with $r > 0$.

Suppose h is an α-lipschitz map with constant ε, and $h(x) = 0$ for $x \in \partial B_r$. Let

$$h_1(x) = \begin{cases} h(x) & \text{for } \|x\| \le r, \\ 0 & \text{for } \|x\| > r. \end{cases}$$

h_1 is continuous on E. For any bounded subset $A \subset E$,

$$\begin{aligned} \alpha(h_1(A)) &= \alpha(h_1(A \cap B_r)) \\ &= \alpha(h(A \cap B_r)) \\ &\le \varepsilon\alpha(A \cap B_r) \le \varepsilon\alpha(A). \end{aligned}$$

Hence h_1 is also an α-Lipschitz map with constant ε. Let

$$S = \left\{x : \; x - t\frac{f(x)}{\lambda} = \frac{h_1(x)}{\lambda}, \; t \in [0, 1]\right\}.$$

For every x with $\|x\| \ge r$ we have $h_1(x) = 0$ and

$$\|\lambda\left(x - t\frac{f(x)}{\lambda}\right)\| \ge |\lambda|\|x\| - \|f(x)\| \ge (|\lambda| - M)\|x\| > 0.$$

This implies that $S \subset B_r$. Since h_1/λ is an $\varepsilon/|\lambda|$-set contraction, $\varepsilon/|\lambda| < 1$, and $h_1(x)/\lambda \equiv 0$ on ∂B_r, the fact that I is $(0, \varepsilon)$-epi ensures that the equation

$$x = \frac{h_1(x)}{\lambda}$$

has a solution in B_r. Thus $S \neq \emptyset$, $S \cap \partial B_r = \emptyset$. Next we have

$$S \subset [0,1]\frac{f(S)}{\lambda} + \frac{h_1(S)}{\lambda}.$$

Therefore,

$$\alpha(S) \leq \frac{1}{|\lambda|}\alpha(f(S)) + \frac{1}{|\lambda|}\alpha(h_1(S)) \leq \frac{\alpha(f) + \varepsilon}{|\lambda|}\alpha(S).$$

Hence S is compact because $\alpha(f) + \varepsilon < |\lambda|$ and S is closed. Let $\phi(x)$ be the Urysohn's function such that

$$\phi(x) = \begin{cases} 1 & \text{for } x \in S, \\ 0 & \text{for } \|x\| \geq r, \end{cases}$$

and let

$$g(x) = \phi(x)\frac{f(x)}{\lambda} + \frac{h_1(x)}{\lambda}.$$

Then g is an $(\alpha(f) + \varepsilon)/|\lambda|$-set contraction, $g(x) \equiv 0$ on the boundary of B_r. Hence $x = g(x)$ has a solution $x_0 \in B_r$. Then $x_0 \in S$ so $\phi(x_0) = 1$ and $h_1(x_0) = h(x_0)$. Thus x_0 is a solution of the equation

$$x - \frac{f(x)}{\lambda} = \frac{h(x)}{\lambda}.$$

This ensures that $\lambda I - f$ is $(0, \varepsilon)$-epi on B_r, so we have $\nu(f) \geq \varepsilon > 0$, $\lambda I - f$ is regular, and λ is in the resolvent set of f. $\square$

Remark 2.1.7. For nonlinear map f with $f(0) = 0$, we define the norm of f by

$$\|f\| = \inf\{k > 0 : \ \|f(x)\| \leq k\|x\|\}.$$

Then the radius of the spectrum of f

$$r_\sigma(f) = \sup\{|\lambda| : \lambda \in \sigma(f)\} \leq \max\{\alpha(f), \|f\|\}.$$

If $f(0) \neq 0$, then for any $\lambda \in \mathbb{C}$, either $\lambda I - f$ is not surjective, or there exists $x \in E$, $x \neq 0$, such that $\lambda x - f(x) = 0$, and λ is an eigenvalue of f. By the following theorem, in both cases, $\lambda \in \sigma(f)$. Thus $\sigma(f) = \mathbb{C}$. Hence in what follows, unless otherwise stated, we shall assume that $f(0) = 0$.

Theorem 2.1.8. *All eigenvalues of f are in the spectrum of f.*

Proof: If there exists $x \in E$ $x \neq 0$ such that $f(x) = \lambda x$, then $m(\lambda I - f) = 0$, so $\lambda \in \sigma(f)$. $\square$

Notice that, the above simple theorem represents the big difference between $\sigma_{fmv}(f)$ and Definition 2.1.1. According to their definition, the spectrum may be disjoint with its eigenvalues [9], but it is well known that for a linear operator, one of the important parts of its spectrum is the point spectrum, the eigenvalues.

The following Lemma enables us to prove the upper semicontinuity of the spectrum.

Lemma 2.1.9. *Let $A \subset \mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) be compact with $A \cap \sigma(f) = \emptyset$. Then there exists $\varepsilon > 0$ such that for $\mu \in A$ and $g : E \to E$, a continuous mapping with $\|f - g\| < \varepsilon$, $\alpha(f - g) < \varepsilon$, it follows that $\mu \notin \sigma(g)$, where*

$$\|f - g\| = \inf\{k \geq 0 : \ \|f(x) - g(x)\| \leq k\|x\|, \ x \in E\}.$$

Proof: For every $\lambda \in A$, we have

$$\omega(\lambda I - f) > 0, \quad m(\lambda I - f) > 0, \quad \text{and} \quad \nu(\lambda I - f) > 0.$$

Thus $\lambda I - f$ is $(0, \varepsilon_0)$-epi for some $\varepsilon_0 > 0$ on every B_r. By the proof of Theorem 2.1.5, there exists $\delta_\lambda > 0$ such that for every λ' with $|\lambda' - \lambda| < \delta_\lambda$,

$$\omega(\lambda' I - f) > \frac{\omega(\lambda I - f)}{2}, \quad m(\lambda' I - f) > \frac{m(\lambda I - f)}{2},$$

and $\lambda' I - f$ is $(0, \varepsilon_0/2)$-epi on B_r. Let

$$0 < \varepsilon_\lambda < \min\{\omega(\lambda I - f)/2, \ m(\lambda I - f)/2, \ \varepsilon_0/2\},$$

and assume that $\|g - f\| < \varepsilon_\lambda$, $\alpha(g - f) < \varepsilon_\lambda$. Then

$$\omega(\lambda' I - g) \geq \omega(\lambda' I - f) - \alpha(f - g) > 0,$$

and

$$\|(\lambda' I - g)(x)\| \geq \|\lambda' x - f(x)\| - \|f(x) - g(x)\| > \left(\frac{m(\lambda I - f)}{2} - \varepsilon_\lambda\right)\|x\|.$$

Hence $m(\lambda' I - g) > 0$. Furthermore, for every $t \in (0, 1]$, $x \neq 0$,

$$\|\lambda' x - f(x) + t(f(x) - g(x))\| \geq \|\lambda' x - f(x)\| - \|f(x) - g(x)\| > 0.$$

By the Continuation Principle for $(0, k)$-epi mappings [43], $\lambda' I - g$ is $(0, r_0)$-epi for any $r_0 > 0$ with

$$r_0 < \min\left\{\frac{\omega(\lambda I - f)}{2} - \alpha(f - g),\ \frac{\varepsilon_0}{2} - \alpha(f - g)\right\}.$$

This implies that $\nu(\lambda' I - g) > 0$, so $\lambda' \in \rho(g)$.

The above discussion implies that $\cup_{\lambda \in A} O(\lambda, \delta_\lambda) \supset A$, where $O(\lambda, \delta_\lambda) = \{\lambda' \in \mathbb{K} : |\lambda' - \lambda| < \delta_\lambda\}$. Since A is a compact set, there exist a finite collection such that $\cup_{i=1}^{n} O(\lambda_i, \delta_{\lambda_i}) \supset A$. Let $\varepsilon = \min\{\varepsilon_{\lambda_1}, \varepsilon_{\lambda_2}, \cdots, \varepsilon_{\lambda_n}\}$, and suppose that

$$\|g - f\| < \varepsilon,\ \alpha(g - f) < \varepsilon.$$

Then for $\mu \in A$, if $\mu \in O(\lambda_i, \delta_{\lambda_i})$, $i \in \{1, \cdots, n\}$, we have

$$\|g - f\| < \varepsilon_{\lambda_i},\ \alpha(g - f) < \varepsilon_{\lambda_i}.$$

so $\mu \notin \sigma(g)$. $\square$

The following theorem whose proof follows that of Theorem 8.3.2 [16] concerns the upper semicontinuity of the spectrum. For the convenience of the reader, we give its proof here.

Theorem 2.1.10. *Let $p(E) = \{f : \alpha(f) < +\infty,$ there exists $M > 0,$ such that $\|f(x)\| \leq M\|x\|$ for all $x \in E\}$. The multivalued map $\sigma : p(E) \to \mathbb{K}$ which assigns each f to its spectrum $\sigma(f)$, is upper semicontinuous (with compact values).*

Proof: Let $U \supset \sigma(f)$ be an open set. Take $r > \max\{\alpha(f), \|f\|\}+1$, $B_r = \{\mu \in \mathbb{K}, |\mu| \leq r\}$ and let $M = B_r \backslash U$, then M is compact and $M \cap \sigma(f) = \emptyset$. By Lemma 2.1.9, there exists $0 < \varepsilon < 1$, such that for $g : E \to E$ with $\|f - g\| < \varepsilon$, $\alpha(f-g) < \varepsilon$, one has $M \cap \sigma(g) = \emptyset$. Moreover,

$$\alpha(g) \leq \alpha(f) + \alpha(g - f) \leq \alpha(f) + \varepsilon < \alpha(f) + 1 < r,$$

$$\|g\| \leq \|f\| + \|g - f\| \leq \|f\| + \varepsilon < \|f\| + 1 < r.$$

By Remark 2.1.7, $\sigma(g) \subset B_r$, hence $\sigma(g) \subset B_r \backslash M \subset U$. □

Proposition 2.1.11. *All bifurcation points and asymptotic bifurcation points of f are in the spectrum of f.*

Proof: Assume that λ is a bifurcation point of f, then $m(\lambda I - f) = 0$. For otherwise there would be sequences $\lambda_n \in \mathbb{K}$ and $x_n \in E$ such that $\lambda_n \to \lambda$, $x_n \neq 0$, $\|x_n\| \to 0$ and $f(x_n) = \lambda_n x_n$. Then we would have

$$\|\lambda x_n - \lambda_n x_n\| = \|\lambda x_n - f(x_n)\| \geq m(\lambda I - f)\|x_n\|.$$

This gives $|\lambda - \lambda_n| \geq m(\lambda I - f) > 0$, a contradiction. Hence $\lambda \in \sigma(f)$.

Similarly, if λ is an asymptotic bifurcation point of f, we obtain $\lambda \in \sigma(f)$. □

The following properties of the spectrum are easily checked.

Proposition 2.1.12. *Let $f : E \to E$ be a continuous operator. Then for $\lambda \in \mathbb{K}$,*

(1) $\sigma(\lambda f) \equiv \lambda\sigma(f), \quad \sigma(0) = 0, \quad \sigma(I) = 1, \quad \sigma(\lambda I) = \lambda.$

(2) $\sigma(\lambda I + f) \equiv \lambda + \sigma(f).$

We close this section with the following proposition devoted to the the study of the nonlinear resolvent.

Proposition 2.1.13. *Assume that $A : E \to E$ is a continuous mapping and $\lambda, \mu \in \rho(A)$. Let*

$$R_A(\lambda) = (A - \lambda I)^{-1}, \quad R_A(\mu) = (A - \mu I)^{-1}$$

be the multivalued maps. Then

$$R_A(\lambda)x \subset R_A(\mu)(I + (\lambda - \mu)R_A(\lambda))x, \quad x \in E.$$

If $\mu I - A$ *is injective, then*

$$R_A(\lambda)x = R_A(\mu)(I + (\lambda - \mu)R_A(\lambda))x, \quad x \in E.$$

Proof: Let $y \in R_A(\lambda)x$, so that $(A - \lambda I)y = x$. Then we can write $Ay - \mu y = x + (\lambda - \mu)y$, so that

$$y \in (A - \mu I)^{-1}(x + (\lambda - \mu)y) \subset (A - \mu I)^{-1}(x + (\lambda - \mu)R_A(\lambda)x).$$

This implies that

$$R_A(\lambda)x \subset R_A(\mu)(I + (\lambda - \mu)R_A(\lambda))x.$$

Now, suppose that $\mu I - A$ is injective. For $y \in R_A(\lambda)x$, we have

$$\begin{aligned} x + (\lambda - \mu)y &= (A - \lambda I)y + (\lambda - \mu)y \\ &= (A - \mu I)y \\ &\in (A - \mu I)R_A(\lambda)x. \end{aligned}$$

Hence

$$x + (\lambda - \mu)R_A(\lambda)x \subset (A - \mu I)R_A(\lambda)x.$$

This ensures that

$$(A - \mu I)^{-1}(I + (\lambda - \mu)R_A(\lambda))x \subset R_A(\lambda)x.$$

Then

$$R_A(\mu)(I + (\lambda - \mu)R_A(\lambda))x \subset R_A(\lambda)x.$$

We have completed the proof. $\square$

2.2 Positively homogeneous operators

Firstly, in this section, we shall prove a result on the spectrum of a positively homogeneous operator according to the definition of $\sigma_{fmv}(f)$. Then, according to our new definition, some special properties of the spectrum of a positively homogeneous operator will be obtained.

To prove Theorem 2.2.3, we need the following lemma.

Lemma 2.2.1. *Let E, F be Banach spaces and $T : E \to F$ be a $(0, k_1)$-epi, $S : E \to F$ be a k_2-set contraction. Suppose that $\lambda \neq 0$, $|\lambda| k_1 \geq k_2$. Let*

$$V = \{x \in E : \ \lambda T(x) = tS(x), \ \textit{for some}\ t \in (0, 1]\}.$$

Then the following alternative holds:

Either $\lambda T - S$ is $(0, |\lambda| k_1 - k_2)$-epi or V is unbounded.

Proof: Let $h(t, x) : [0, 1] \times E \to F$ be defined by $h(t, x) = -tS(x)$. For every subset I_1 of $[0, 1]$ and bounded subset Ω of E, we have

$$\begin{aligned} \alpha(h(I_1 \times \Omega)) &\leq \alpha(\mathrm{co}(S(\Omega) \cup \{0\})) = \alpha(S(\Omega)) \\ &\leq k_2 \alpha(\Omega) \leq k_2 \alpha(I_1 \times \Omega). \end{aligned}$$

Thus $\alpha(h) \leq k_2$. Also, $h(0, x) = 0$ for all $x \in E$. So, if V is bounded, by Property 1.4.7, $\lambda T + h(1, x) = \lambda T - S$ is $(0, |\lambda| k_1 - k_2)$-epi. □

Corollary 2.2.2. *If f is a k-set contraction and $|\lambda| > k$. Then for any $1 > k_1 \geq k/|\lambda|$, either $V = \{x : \ \lambda x = tf(x)\ \textit{for some}\ t \in (0, 1]\}$ is un bounded or $I - f/\lambda$ is $(0, k_1 - k/|\lambda|)$-epi.*

Proof: By Lemma 2.2.1 and the fact that for any $k/|\lambda| \leq k_1 < 1$, $I : E \to E$ is $(0, k_1)$-epi. □

In [16], the authors gave a decomposition for the spectrum $\sigma_{fmv}(f)$. They defined the sets

$$\begin{aligned}\sigma_d(f) &= \left\{\lambda \in \mathbb{K} : d(\lambda I - f) = \liminf_{\|x\|\to\infty} \frac{\|\lambda x - f(x)\|}{\|x\|} = 0\right\},\\ \sigma_\omega(f) &= \{\lambda \in \mathbb{K} : \omega(\lambda I - f) = 0\},\\ \sigma_\delta(f) &= \{\lambda \in \mathbb{K} : \lambda I - f \text{ is not stably-solvable }\}.\end{aligned}$$

The following theorem enables us to determine the relation between the eigenvalues and $\sigma_{fmv}(f)$ for positively homogeneous operators. This result can be used to discuss the existence of a solution for some nonlinear operator equations (see section 2.5).

Theorem 2.2.3. *Let $f : E \to E$ be a positively homogeneous operator.*

1. *If λ is an eigenvalue of f, then $\lambda \in \sigma_d(f) \subset \sigma_{fmv}(f)$;*
2. *If f is α-Lipschitz with constant k, $|\lambda| > k$ and also $\lambda \in \sigma_d(f)$, then λ is an eigenvalue of f;*
3. *If f is α-Lipschitz with constant k, $|\lambda| > k$ and $\lambda \in \sigma_{fmv}(f)$, then there exists $t_0 \in (0, 1]$ such that λ/t_0 is an eigenvalue of f.*

Proof: (1) Since f is positively homogeneous, for every eigenvalue λ of f there exists $x \in E$, $\|x\| = 1$ such that $f(x) = \lambda x$. Let $y_n = nx$, then

$$\lim_{n\to\infty} \frac{\|(\lambda I - f)y_n\|}{\|y_n\|} = 0.$$

Hence $\lambda \in \sigma_d(f)$.

(2) The condition $\lambda \in \sigma_d(f)$ ensures that there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ with $\|x_n\| \to \infty$, such that

$$\frac{\|(\lambda I - f)(x_n)\|}{\|x_n\|} \to 0, \quad (n \to \infty).$$

So,

$$\begin{aligned}|\lambda|\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right) &\le \alpha\left(\bigcup_{n=1}^{\infty}\left(\lambda\frac{x_n}{\|x_n\|} - f\left(\frac{x_n}{\|x_n\|}\right)\right)\right) + \alpha\left(f\left(\bigcup_{n=1}^{\infty}\frac{x_n}{\|x_n\|}\right)\right)\\ &\le k \quad \alpha\left(\bigcup_{n=1}^{\infty}\frac{x_n}{\|x_n\|}\right).\end{aligned}$$

Since $|\lambda| > k$, we obtain that $\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right) = 0$. This implies there exists a subsequence $\frac{x_{n_k}}{\|x_{n_k}\|} \to x_0$ $(k \to \infty)$ and $\lambda x_0 = f(x_0)$. Since $x_0 \neq 0$, λ is an eigenvalue of f.

(3) The assumption that f is α-Lipschitz implies $\alpha(f) \leq k$. If $|\lambda| > k$, then

$$\omega(\lambda I - f) \geq \omega(\lambda I) - \alpha(f) > 0.$$

So, $\lambda \notin \sigma_\omega(f)$. In the case that $\lambda \in \sigma_d(f)$, by (2) λ is an eigenvalue of f, so $t_0 = 1$. Now assume that $\lambda \notin \sigma_d(f)$, thus $d(\lambda I - f) > 0$. Let

$$V = \{x :\ \lambda x = tf(x) \text{ for some } t \in (0,1]\}.$$

If V is unbounded let $\{x_n\} \in V$ with $\|x_n\| \to \infty$, and $t_n \in (0,1]$ be such that

$$\lambda x_n = t_n f(x_n).$$

Then

$$|\lambda|\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right) \leq \alpha\left([0,1] \times f\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right)\right) \leq k\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right).$$

Hence $\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right) = 0$. So there exists a subsequence $\frac{x_{n_k}}{\|x_{n_k}\|} \to x_0$ as $k \to \infty$ and $\|x_0\| = 1$. It follows that $\lambda x_0 = t_0 f(x_0)$ for some $t_0 \in (0,1]$, so λ/t_0 is an eigenvalue of f.

In the case V is bounded, by Corollary 2.2.2, $\lambda I - f$ is $(0, |\lambda|k_1 - k)$-epi for every $1 > k_1 \geq k/|\lambda|$. $d(\lambda I - f) > 0$ ensures that $\|(\lambda I - f)(x)\| \to \infty$ as $\|x\| \to \infty$. By Corollary 3.2 of [65], for a compact operator $h : E \to E$ with bounded support, the equation $\lambda I - f = h$ has a solution. Then by Proposition 5.1.1 of [16], $\lambda I - f$ is stably-solvable. Thus $\lambda \notin \sigma_{fmv}(f)$. This contradiction completes the proof. $\square$

In the following, we shall prove some properties of positively homogeneous $(0,k)$-epi mappings from a Banach space E to a Banach space F, which will be used later.

Proposition 2.2.4. *Suppose that $f : E \to F$ is a positively homogeneous mapping and f is $(0,\varepsilon)$-epi on B_r for some $\varepsilon > 0, r > 0$. Then f is $(0,\varepsilon)$-epi on every B_R with $R > 0$.*

Proof: Since f is positively homogeneous and f is $(0,\varepsilon)$-epi on B_r, $f(x) \neq 0$ for all $x \neq 0$. Thus f is 0-admissible on B_R. Assume that $h : E \to F$ is an ε-set contraction with

$h(x) = 0$ for $x \in \partial B_R$. Let

$$h_1(x) = \frac{r}{R} h\left(\frac{R}{r}x\right).$$

Then for every bounded set $A \subset E$,

$$\begin{aligned} \alpha(h_1(A)) &= \frac{r}{R}\alpha\left(h\left(\frac{R}{r}A\right)\right) \\ &\leq \frac{r}{R}\varepsilon\alpha\left(\frac{R}{r}A\right) \\ &= \varepsilon\alpha(A). \end{aligned}$$

Hence h_1 is an ε-set contraction too. Furthermore, $h_1(x) = 0$ for $x \in \partial B_r$. Thus the equation

$$f(x) = \frac{r}{R} h\left(\frac{R}{r}x\right)$$

has at least one solution $x_0 \in E$ and $\|x_0\| < r$. Then $(R/r)x_0 \in B_R$ is a solution of the equation $f(x) = h(x)$. $\square$

Remark 2.2.5. Proposition 2.2.4 and Property 1.4.6 show that a positively homogeneous mapping f is $(0, \varepsilon)$-epi on $\overline{\Omega_1}$, where $\Omega_1 \supset f^{-1}(0)$ and Ω_1 is a bounded set of E, if and only if f is $(0, \varepsilon)$-epi on the closure of all bounded open sets $\Omega \supset f^{-1}(0)$. This is not true if f is not positively homogeneous as the following example shows. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 - 1$, and let $\Omega_1 = (-2, -1/2) \bigcup (1/2, 2)$. Then f is $(0, k)$-epi for every $k \geq 0$ on $\overline{\Omega_1}$. But f is not 0-epi on $(-n, n)$ for $n > 2$.

Proposition 2.2.6. *Suppose $f : E \to F$ is positively homogeneous, $\omega(f) > 0$ and f is $(0, \varepsilon)$-epi on B_r with $r > 0$. Then for every $p \in F$, there exists $R > 0$ such that f is (p, ε_1)-epi on B_R for some $\varepsilon_1 > 0$.*

Proof: For $p \in F$, let $g(x) = f(x) - p$. Then $\alpha(g - f) = 0$. Let

$$S = \{x : \ f(x) + t(g(x) - f(x)) = f(x) - tp = 0 \text{ for some } t \in (0, 1]\}.$$

We shall show that S is bounded. Assume there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset S$ with $\|x_n\| \to \infty$ as $n \to \infty$. Let $t_n \in (0,1]$ such that $f(x_n) = t_n p$. Then

$$f\left(\frac{rx_n}{\|x_n\|}\right) = \frac{rt_n p}{\|x_n\|} \to 0 \text{ as } n \to \infty.$$

So we have the following:

$$\omega(f)\alpha\left(\bigcup_{n=1}^{\infty} \frac{rx_n}{\|x_n\|}\right) \le \alpha\left(f\left(\bigcup_{n=1}^{\infty} \frac{rx_n}{\|x_n\|}\right)\right) \le \alpha\left(\bigcup_{n=1}^{\infty} f\left(\frac{rx_n}{\|x_n\|}\right)\right) = 0.$$

Since $\omega(f) > 0$, we have $\alpha\left(\bigcup_{n=1}^{\infty} \frac{rx_n}{\|x_n\|}\right) = 0$. This implies that that there exists a subsequence $\frac{rx_{n_k}}{\|x_{n_k}\|} \to x_0$ and $\|x_0\| = r$. Thus

$$\lim_{k\to\infty} f\left(\frac{rx_{n_k}}{\|x_{n_k}\|}\right) = f(x_0) = 0.$$

This contradicts the fact f is 0-admissible on B_r.

Now, let $R > 0$ be such that $S \subset B_R$. Then $\partial B_R \cap S = \emptyset$. By Proposition 2.2.4, f is $(0,\varepsilon)$-epi on B_R. So, the Continuation Principle of $(0,k)$-epi maps [43] ensures that $g(x)$ is at least $(0,\varepsilon_1)$-epi for $0 < \varepsilon_1 < \omega(f)$ and $\varepsilon_1 \le \varepsilon$. Thus f is (p,ε_1)-epi on B_R. □

The following proposition characterizes regular maps among positively homogeneous operators.

Proposition 2.2.7. *Suppose that f is positively homogeneous. Then f is regular if and only if*

1. $\omega(f) > 0$.

2. *There exists $\varepsilon > 0$ such that f is $(0,\varepsilon)$-epi on B_1.*

Proof: Clearly, we only need to prove that if f satisfies 1 and 2, then f is regular.

Suppose $\omega(f) > 0$ and there exists $\varepsilon > 0$ such that f is $(0,\varepsilon)$-epi on B_1. By Proposition 2.2.4, f is $(0,\varepsilon)$-epi on every B_r with $r > 0$. So $\nu(f) > 0$. Now assume that $m(f) = 0$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty} \in E$, $x_n \neq 0$ such that $\|f(x_n)\| < \frac{1}{n}\|x_n\|$. This implies that $f\left(\frac{x_n}{\|x_n\|}\right) \to 0$ as $n \to \infty$. Moreover,

$$\omega(f)\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right) \le \alpha\left(\bigcup_{n=1}^{\infty} f\left(\frac{x_n}{\|x_n\|}\right)\right) = 0.$$

Hence $\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right) = 0$. This implies $\left\{\frac{x_n}{\|x_n\|}\right\}$ has a subsequence $\frac{x_{n_k}}{\|x_{n_k}\|} \to x_0, \quad \|x_0\| = 1$, and $f(x_0) = 0$. This is a contradiction with f is $(0, \varepsilon)$-epi on B_1. So $m(f) > 0$, hence f is regular. □

It is known that for a linear operator f, if $\lambda \in \sigma(f)$ and $|\lambda| > \alpha(f)$, then λ is an eigenvalue of f [70]. We shall give an example later to show that for nonlinear operators, this property is not true. But if f is positively homogeneous, we have the following result on eigenvalues in the spectrum. This theorem can be used to obtain existence results for some nonlinear operator equations as in examples which will be given later.

Theorem 2.2.8. *Let $f : E \to E$ be a positively homogeneous operator and $\lambda \in \sigma(f)$ with $|\lambda| > \alpha(f)$. Then there exists $t_0 \in (0,1]$ such that λ/t_0 is an eigenvalue of f.*

Proof: By the assumption $|\lambda| > \alpha(f)$, we have $\omega(\lambda I - f) \geq |\lambda| - \alpha(f) > 0$. Let

$$S = \{x \in E : \ \|x\| = 1, \lambda x - tf(x) = 0 \text{ for some } t \in (0,1]\}.$$

If $S = \emptyset$, then by Property 1.4.7, $I - f/\lambda$ is $(0, r - \alpha(f)/|\lambda|)$-epi on B_1 for every $1 > r > \alpha/|\lambda|$, since f/λ is a $\alpha(f)/|\lambda|$-set contraction. It follows that $\lambda I - f$ is $(0, r|\lambda| - \alpha(f))$-epi on B_1. By Proposition 2.2.7, we know that $\lambda I - f$ is regular, so $\lambda \in \rho(f)$. This contradiction ensures that $S \neq \emptyset$. Thus there exists $t_0 \in (0,1]$ and $x_0 \in E$ with $\|x_0\| = 1$, such that $\lambda x_0 - t_0 f(x_0) = 0$, so λ/t_0 is an eigenvalue of f. □

The following result, which generalizes theorems of [48], [70], shows that for an odd and positively homogeneous mapping, the result on eigenvalues of a linear operator remains valid.

Theorem 2.2.9. *Suppose $f : E \to E$ is odd and positively homogeneous, $\lambda \in \sigma(f)$ with $|\lambda| > \alpha(f)$. Then λ is an eigenvalue of f.*

Proof: Assume that $m(\lambda I - f) > 0$. Then there exists $m > 0$ such that

$$\|(\lambda I - f)x\| \geq m\|x\| \text{ for all } x \in E.$$

Since f is odd, by Theorem 9.4 of [7],

$$\deg(I - \frac{f}{\lambda},\, O_1,\, 0) \neq 0.$$

This ensures that for every k_1 satisfying $0 \leq \alpha(f)/|\lambda| < k_1 < 1$, $(I - f/\lambda)$ is $(0, k_1 - \alpha(f)/|\lambda|)$-epi ([65] Theorem 2.8). So, $\lambda I - f$ is $(0, |\lambda| k_1 - \alpha(f))$-epi on B_1. Also we know that $\omega(\lambda I - f) \geq |\lambda| - \alpha(f) > 0$. Thus, $\lambda \in \rho(f)$. This contradiction shows that $m(\lambda I - f) = 0$. Therefore there exists a sequence $\{x_n\}_{n=1}^{\infty} \in E$ such that

$$\|\lambda x_n - f(x_n)\| < \frac{1}{n}\|x_n\|.$$

Hence,

$$\|\lambda \frac{x_n}{\|x_n\|} - f\left(\frac{x_n}{\|x_n\|}\right)\| < \frac{1}{n} \to 0, \quad \text{as } n \to \infty.$$

Moreover, we have

$$\omega(\lambda I - f)\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right) \leq \alpha\left(\bigcup_{n=1}^{\infty} (\lambda I - f)\frac{x_n}{\|x_n\|}\right) = 0.$$

This implies $\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right) = 0$. So $\left\{\frac{x_n}{\|x_n\|}\right\}$ has a convergent subsequence. Suppose that $\frac{x_{n_k}}{\|x_{n_k}\|} \to x_0$. Then $f(x_0) = \lambda x_0$ and $\|x_0\| = 1$, so λ is an eigenvalue of f. □

The following result follows directly from Theorem 2.2.9, which generalizes the result in the spectral theory for linear compact operators.

Corollary 2.2.10. *Suppose that f is a compact, odd and homogeneous operator. Then for $\lambda \in \sigma(f)$, if $\lambda \neq 0$, λ is an eigenvalue of f.*

In the following , $r_\sigma(f)$ denotes the radius of its spectrum. It is known that for a continuous linear operator f, $r_\sigma(f) = \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}}$. The following theorem gives an estimate for the radius of the spectrum of positively homogeneous maps.

Theorem 2.2.11. *Let E be a Banach space over $\mathbb{R}$ and $f : E \to E$ be a positively homogeneous operator with*

$$\alpha(f) < \infty,\ \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}} < \infty.$$

If

$$\lambda > \max\left\{\alpha(f), \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}}\right\},$$

then $\lambda \in \rho(f)$. *If also* $\|x_1\| = \|x_2\|$ *implies* $\|f(x_1)\| = \|f(x_2)\|$, *then*

$$r_\sigma(f) \le \max\left\{\alpha(f), \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}}\right\}. \tag{2.1}$$

Proof: Suppose

$$\lambda > \max\left\{\alpha(f), \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}}\right\}.$$

Let

$$V = \{x : \lambda x - tf(x) = 0 \text{ for some } t \in (0,1]\}.$$

We claim that $V = \{0\}$. Indeed, otherwise assume $x_0 \in V$ and $x_0 \neq 0$. Let $t_0 \in (0,1]$ be such that

$$\lambda x_0 - t_0 f(x_0) = 0.$$

Then

$$\|f\| = \sup_{\|x\|=1} \|f(x)\| \ge \|f\left(\frac{x_0}{\|x_0\|}\right)\| = \frac{|\lambda|}{t_0} \ge |\lambda|.$$

Also

$$\|f^2\left(\frac{x_0}{\|x_0\|}\right)\| = \|f\left(\frac{\lambda}{t_0}\frac{x_0}{\|x_0\|}\right)\| \ge \frac{\lambda^2}{{t_0}^2}. \tag{2.2}$$

So we have

$$\|f^2\| \ge \frac{|\lambda|^2}{{t_0}^2} \ge |\lambda|^2.$$

By induction, we obtain $\|f^n\|^{\frac{1}{n}} \ge |\lambda|$. This contradicts the assumption that $\lambda > \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}}$. Now, $\lambda I - f = \lambda(I - f/\lambda)$ and f/λ is an $\alpha(f)/|\lambda|$-set contraction. So, by Property 1.4.7, $I - f/\lambda$ is $(0, r - \alpha(f)/|\lambda|)$-epi on B_1 for every r satisfying $\alpha(f)/|\lambda| < r < 1$. Thus $\lambda I - f$ is $(0, r|\lambda| - \alpha(f))$-epi on B_1. Furthermore, $\omega(\lambda I - f) > 0$. Proposition 2.2.7 implies that $\lambda \in \rho(f)$.

In the case that $\|x_1\| = \|x_2\|$ implies $\|f(x_1)\| = \|f(x_2)\|$, (2.2) is also true for $\lambda < 0$. So by the same proof as above, we obtain that if

$$|\lambda| > \max\left\{\alpha(f), \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}}\right\},$$

then $\lambda \in \rho(f)$. Therefore we have (2.1). $\square$

The following example shows that the estimate in Theorem 2.2.11 is best possible.

Example 2.2.12. Let $f : E \to E$ be defined by $f(x) = x + a\|x\|e$, where $e \in E$ and $\|e\| = 1$, a is a positive number. Then $1 + a$ is an eigenvalue of f, hence, $1 + a \in \sigma(f)$. Also, $\|f^n\| = (1 + a)^n$, so we have $\liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}} = 1 + a$. If let $f(x) = a\|x\|e$, then f is positively homogeneous and even. a is an eigenvalue of f and $\|f^n\| = a^n$. Hence $\liminf \|f^n\|^{\frac{1}{n}} = a$ and $r_\sigma(f) = a$.

It was proved in [9] that if $f, g : E \to E$ are continuous map and $f - g = h$, with h compact and quasinorm $|h| = 0$, then $\sigma_{fmv}(f) = \sigma_{fmv}(g)$. With the new definition of the spectrum, we have the following properties. Firstly, we prove a lemma, which is essential for the proof of Theorem 2.2.14.

Lemma 2.2.13. *Suppose $f, g : E \to E$ are positively homogeneous. Suppose also that $\omega(f) > 0$, $f - g = h$, and h is compact, $h(E)$ is bounded. Then $\nu(f) > 0$ if and only if $\nu(g) > 0$.*

Proof: Assume that $\nu(f) > 0$. For $0 < \varepsilon < \nu(f)$, f is $(0, \varepsilon)$-epi on B_1. Let $h_1(t, x) = -th(x)$. Then $h_1 : [0, 1] \times E \to E$ is a compact map and $h_1(0, x) \equiv 0$. Let

$$S = \{x \in E,\ f(x) - th(x) = 0, \text{ for some } t \in (0, 1]\}.$$

Assume that there exists $x_n \in S$ with $\|x_n\| \to \infty$ as $n \to \infty$. Then

$$f\left(\frac{x_n}{\|x_n\|}\right) = t_n \frac{h(x_n)}{\|x_n\|} \to 0 \quad \text{as} \quad n \to \infty,$$

letting $u_n = \frac{x_n}{\|x_n\|}$, then

$$\omega(f)\alpha(\cup_{n=1}^{\infty} u_n) \leq \alpha \cup_{n=1}^{\infty} f(u_n).$$

Since $\omega(f) > 0$, we have $\alpha \cup_{n=1}^{\infty} (u_n) = 0$. Hence, there exists $u_{n_k} \to x_0$, $f(x_0) = 0$. This contradicts f is $(0, \varepsilon)$-epi on B_1. So, S is bounded. By Property 1.4.7, g is $(0, \varepsilon_1)$-epi on

B_1 for every $0 < \varepsilon_1 < \varepsilon$. Hence, $\nu(g) > 0$. Since

$$\omega(g) \geq \omega(f) - \alpha(h) = \omega(f) > 0,$$

by the same argument it can be proved that if $\nu(g) > 0$, then $\nu(f) > 0$ $\square$

The following Theorem follows easily from Proposition 2.2.7 and Lemma 2.2.13.

Theorem 2.2.14. *Suppose f and g are positively homogeneous operators and $f - g = h$, h is a compact map with $h(E)$ is bounded. Then $\sigma(f) = \sigma(g)$.*

We close this section with the following proposition.

Proposition 2.2.15. *Let $f : E \to E$ be positively homogeneous and $g : E \to E$ be a continuous map. Assume that $f - g = h$ and h is compact with $|h| = 0$. Then λ is an eigenvalue of f if $\lambda \in \sigma(f) \backslash \sigma(g)$.*

Proof: Firstly, we have that $\omega(\lambda I - g) > 0$ since $\lambda \in \rho(g)$. Therefore

$$\omega(\lambda I - f) = \omega(\lambda I - f) + \alpha(h) \geq \omega(\lambda I - f + h) = \omega(\lambda I - g) > 0.$$

Now, assume that $m(\lambda I - f) > 0$. Then there exists $m > 0$, such that

$$\|(\lambda I - f)x\| \geq m\|x\| \text{ for all } x \in E.$$

Let

$$S = \{x \in E : \ \lambda x - f(x) + h(x) - th(x) = 0, \ t \in (0, 1]\}.$$

Then S is bounded. Otherwise there would exist $x_n \in S$, with $\|x_n\| \to \infty$. Let $t_n \in (0, 1]$ be such that

$$\lambda x_n - f(x_n) + (1 - t_n)h(x_n) = 0.$$

Then

$$\frac{\lambda x_n - f(x_n)}{\|x_n\|} \to 0 \quad \text{as} \quad n \to \infty.$$

This contradicts the assumption $m(\lambda I - f) > 0$. By Property 1.4.7, $\lambda I - f$ is $(0,\varepsilon)$-epi on every B_r with $r > 0$ since $\lambda I - f + h = \lambda I - g$ is $(0,\varepsilon)$-epi on it. So we obtain $\lambda \in \rho(f)$. This contradiction implies that $m(\lambda I - f) = 0$. Therefore, there exists $x_n \in E$ such that

$$\|\lambda x_n - f(x_n)\| < \frac{1}{n}\|x_n\|, \quad x_n \neq 0.$$

Also we have

$$\omega(\lambda I - f)\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right) \leq \alpha\left(\bigcup_{n=1}^{\infty}(\lambda I - f)\frac{x_n}{\|x_n\|}\right).$$

So $\alpha\left(\bigcup_{n=1}^{\infty} \frac{x_n}{\|x_n\|}\right) = 0$. Let $\frac{x_{n_k}}{\|x_{n_k}\|} \to x_0$, then $\|x_0\| = 1$ and $\lambda x_0 - f(x_0) = 0$. □

2.3 Comparison of spectra

In this section, we shall compare definitions for the spectrum. Some examples will show differences.

We begin with an example concerning eigenvalues and the spectrum $\sigma_{fmv}(f)$.

Example 2.3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^3$. Then $\sigma_{fmv}(f) = \emptyset$ (see [16]) and we will show that $\sigma(f) = \{0\} \cup \{\text{eigenvalues of } f\}$. In fact, for every $\lambda \in (0,\infty)$, we have $f(\lambda^{\frac{1}{2}}) = \lambda\lambda^{\frac{1}{2}}$. Thus λ is an eigenvalue of f, so $(0,\infty) \subset \sigma(f)$. Next, $0 \in \sigma(f)$ since $m(f) = 0$. Furthermore, let $\lambda \in (-\infty, 0)$, then

$$|\lambda x - f(x)| = |\lambda x - x^3| \geq -\lambda|x| \text{ for } x \in \mathbb{R}.$$

Hence $m(\lambda I - f) \geq -\lambda > 0$. Also, $\omega(\lambda I - f) > 0$ and $(\lambda I - f)x = \lambda x - x^3$ is $(0,\varepsilon)$-epi for $\varepsilon > 0$. This implies that $\nu(\lambda I - f) > 0$. Therefore, $(-\infty, 0) = \rho(f)$ and $\sigma(f) = \{0\} \cup \{\text{eigenvalues of } f\}$.

Theorem 2.3.2. *Suppose $f : E \to E$ is continuous and $f(0) \neq 0$. Let $E(f)$ be the set of all eigenvalues of f. Then $\rho_{fmv}(f) \subset E(f)$ and $\rho_{lip}(f) \subset E(f)$.*

Proof: Assume that $\lambda \in \rho_{fmv}(f)$ or $\lambda \in \rho_{lip}(f)$, then $\lambda I - f$ is surjective. Suppose $(\lambda I - f)x = 0$, then $x \neq 0$, λ is an eigenvalue of f. □

Remark 2.3.3. Let $E(f)$ denote the set of all eigenvalues of f . Theorem 2.3.2 implies that for a continuous map f with $f(0) \neq 0$, we have the following

$$\sigma(f)\backslash\sigma_{fmv}(f) \subset E(f), \quad \sigma(f)\backslash\sigma_{lip}(f) \subset E(f).$$

Example 2.3.4. Let γ be the α-Lipschitz retraction of the closed unit ball B_1 of a Banach space E onto its boundary [13]. Define $f : E \to E$ by

$$f(x) = \begin{cases} \gamma(x) & \text{for } x \in B_1 \\ x & \text{for } x \notin B_1 \end{cases}$$

Then $|f| = 1$. So $\sigma_{fmv}(f)$ is bounded [16]. But $\sigma(f) = \mathbb{C}$ since $f(0) \neq 0$.

The following simple example shows the difference between the spectrum for linear operators and that for nonlinear maps.

Example 2.3.5. Let $f : E \to E$ be defined by $f(x) = \|x\|^2 e$, where $e \in E$ and $\|e\| = 1$. Then f is obviously a compact map and $\sigma(f) = \mathbb{C}$. In fact, for any $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, λ is an eigenvalue of f, since

$$f(\overline{\lambda}e) = \|\overline{\lambda}e\|^2 e = \lambda\overline{\lambda}e.$$

The following is an interesting result of the theory.

Theorem 2.3.6. *Suppose that $f : E \to E$ is a continuous operator and $f(0) = 0$. Then*

$$\sigma_{lip}(f) \supset \sigma(f) \supset \sigma_{fmv}(f).$$

Proof: We shall prove that $\rho_{lip}(f) \subset \rho(f) \subset \rho_{fmv}(f)$.

(a) Assume that $\lambda \in \rho(f)$. Then $\omega(\lambda I - f) > 0$ and $m(\lambda I - f) > 0$. Hence there exists $m > 0$ such that

$$\|(\lambda I - f)(x)\| \geq m\|x\| \text{ for all } x \in E.$$

This ensures that

$$d(\lambda I - f) = \liminf_{\|x\|\to\infty} \frac{\|(\lambda I - f)(x)\|}{\|x\|} \geq m > 0.$$

Moreover, $\nu(\lambda I - f) > 0$ implies that there exists $\varepsilon > 0$ such that $\lambda I - f$ is $(0, \varepsilon)$-epi on B_r with $r > 0$. So $\lambda I - f$ is stably-solvable [16]. Hence $\lambda \in \rho_{fmv}(f)$ and therefore $\rho(f) \subset \rho_{fmv}(f)$.

(b) Suppose that $\lambda \in \rho_{lip}(f)$, then $\lambda I - f$ is one to one, onto, and $(\lambda I - f)^{-1}$ is a Lipschitz map. Let $L > 0$ be the Lipschitz constant, thus for $y_1, y_2 \in E$,

$$\|(\lambda I - f)^{-1} y_1 - (\lambda I - f)^{-1} y_2\| \leq L \|y_1 - y_2\|.$$

This implies that

$$\|(\lambda I - f)x_1 - (\lambda I - f)x_2\| \geq \frac{1}{L}\|x_1 - x_2\| \text{ for } x_1, x_2 \in E. \tag{2.3}$$

Let $x_2 = 0$, we have

$$\|(\lambda I - f)x_1\| \geq \frac{1}{L}\|x_1\| \quad \text{for all } x_1 \in E.$$

Hence $m(\lambda I - f) > 0$. Also, by (2.3), $\omega(\lambda I - f) \geq 1/L > 0$.

Let $r > 0$ and $O_r = \{x : \ \|x\| < r\}$. $\lambda I - f : \ O_r \to E$ is continuous, injective and $1/L$-proper [65]. Furthermore $(\lambda I - f)O_r$ is open because $(\lambda I - f)^{-1}$ is continuous. By our assumption $(\lambda I - f)(0) = 0$. By theorem 2.3 of [65], $\lambda I - f$ is $(0, k)$-epi on B_r for each nonnegative k satisfying the condition $k < L$. Hence $\nu(\lambda I - f) > 0$, $\lambda \in \rho(f)$ and $\rho_{lip}(f) \subset \rho(f)$. $\square$

The following shows that $\sigma_{lip}(f) \subsetneqq \sigma(f) \subsetneqq \sigma_{fmv}(f)$.

Example 2.3.7. Let $\psi : \ \mathbb{R} \to \mathbb{R}$ be defined by

$$\psi(x) = \begin{cases} x & \text{for} \quad x \leq 1, \\ 1 & \text{for} \quad 1 < x < 2, \\ x - 1 & \text{for} \quad x \geq 2. \end{cases}$$

Let $f = I - \psi$. Then for $x \in \mathbb{R}$ we have

$$(1/2)\|x\| \leq \|\psi(x)\| \leq 2\|x\|.$$

Thus $\|f(x)\| \leq 3\|x\|$. Also, $\omega(I-f) > 0$ for f is a compact map. $I - f$ is $(0, \varepsilon)$-epi for $\varepsilon > 0$ on every $[-n, n]$. Hence $\lambda = 1 \in \rho(f)$.

Obviously, $I - f = \psi$ is not one to one, so $1 \in \sigma_{lip}(f)$. Thus $\sigma(f) \subsetneqq \sigma_{lip}(f)$.

Example 2.3.1 shows that $\sigma(f) \stackrel{-}{\subsetneqq} \sigma_{fmv}(f)$.

The following two results characterize the spectrum for positively homogeneous maps and maps that are derivable at 0 respectively.

Theorem 2.3.8. *Suppose that f is positively homogeneous and $\lambda \in \sigma(f) \backslash \sigma_{fmv}(f)$. Then one of the following cases occur:*

1. *$\lambda I - f$ is not injective.*
2. *$(\lambda I - f)^{-1}$ is not continuous.*

Proof: Suppose that $\lambda \in \sigma(f) \setminus \sigma_{fmv}(f)$ and $\lambda I - f$ is injective. We shall show that $(\lambda I - f)^{-1}$ is not continuous. Firstly $\lambda \notin \sigma_{fmv}(f)$ ensures that $\omega(\lambda I - f) > 0$ and $\lambda I - f$ is surjective. Also $\lambda I - f$ is injective implies that $\lambda x - f(x) \neq 0$ for $x \neq 0$ since $f(0) = 0$. Hence $\lambda I - f$ is $\omega(\lambda I - f)$-proper [65]. Assume $(\lambda I - f)^{-1}$ is continuous, then $\lambda I - f$ maps every open ball O_r to an open set. It follows that $\lambda I - f$ is $(0, k)$-epi for each nonnegative k satisfying $k < 1/\omega(\lambda I - f)$. By Proposition 2.2.7, we have $\lambda \in \sigma(f)$. This contradiction shows that $(\lambda I - f)^{-1}$ is not continuous. □

As the last result in this section, we have the following:

Theorem 2.3.9. *Let $f : E \to E$ be derivable at 0 with derivative T and $\lambda \in \sigma(f) \setminus \sigma_{fmv}(f)$. Then one of the following cases occur:*

1. *λ is an eigenvalue of f.*
2. *$\lambda I - f$ is not injective.*
3. *$(\lambda I - f)^{-1}$ is not continuous.*

4. $\lambda \in \sigma(T)$.

Proof: Let $\lambda \in \sigma(f) \setminus \sigma_{fmv}(f)$, then $\lambda I - f$ is onto and $\omega(\lambda I - f) > 0$. Now suppose that $m(\lambda I - f) = 0$. It follows that for each $n \in \mathbb{N}$, there exists $x_n \in E$ satisfying

$$\|\lambda x_n - f(x_n)\| < (1/n)\|x_n\|.$$

Assume that there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ with $\|x_{n_k}\| \to \infty$ as $k \to \infty$. Then

$$d(\lambda I - f) = \liminf_{\|x\| \to \infty} \frac{\|\lambda x - f(x)\|}{\|x\|} = 0.$$

This contradicts $\lambda \in \rho(f)$. So, $\{\|x_n\|\}_{n=1}^{\infty}$ is bounded and

$$\omega(\lambda I - f)\alpha\left(\cup_{n=1}^{\infty} x_n\right) \leq \alpha\left(\cup_{n=1}^{\infty}(\lambda I - f)x_n\right) = 0.$$

This implies $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence. Suppose $x_n \to x_0$ as $(n \to \infty)$. If $x_0 \neq 0$, λ is an eigenvalue of f. In the case $x_0 = 0$, we have

$$\frac{\|\lambda x_n - Tx_n - Rx_n\|}{\|x_n\|} < \frac{1}{n} \to 0.$$

Thus $\lambda \in \sigma(T)$ since $\frac{\|Rx_n\|}{\|x_n\|} \to 0$.

In the case $m(\lambda I - f) > 0$, assume that $\lambda I - f$ is injective , by the same argument as that in the proof of Theorem 2.3.8, $(\lambda I - f)^{-1}$ is not continuous. □

2.4 Nonemptiness of Spectra

A well known result in the spectral theory for linear operators is that the spectrum of a continuous linear operator, which is defined on a complex Banach space, is not empty. In the nonlinear case, for the spectrum $\sigma_{fmv}(f)$, this property does not hold (see the counterexample in [16]). An open question in [48] is the following: does this nonemptiness property hold for the spectrum $\sigma_{lip}(f)$? In this section, we shall give an example which answers this question in the negative.

Firstly, we have the following simple theorem.

Theorem 2.4.1. *Let $f : E \to E$ be a continuous map. $\sigma(f) \neq \emptyset$ provided that f satisfies one of the following conditions:*

1. *$f(0) \neq 0$.*

2. *f is compact and E is an infinite dimensional Banach space.*

3. *$\alpha(f) < d(f)$.*

Proof: It is easily seen that 1 follows from Remark 2.1.7 and the other two cases are direct corollaries of Theorem 2.3.6 and Theorem 8.2.1, Proposition 8.2.2 of [16]. □

Question 4.1 [48]: Suppose that E is a Banach space over the complex field $\mathbb{C}$ and that $A \in Lip(E)$. Is the spectrum of A, $\sigma_{lip}(A)$, nonempty?

In the following, an operator f is given which satisfies

$$\sigma(f) = \sigma_{lip}(f) = \sigma_{fmv}(f) = \emptyset.$$

Example 2.4.2. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by

$$f(x, y) = (\overline{y}, i\overline{x}), \quad (x, y) \in \mathbb{C}^2.$$

Then f is a continuous map and $f \in Lip(\mathbb{C}^2)$ since

$$\|f(x, y) - f(u, v)\| = \|(x, y) - (u, v)\|, \quad (x, y), (u, v) \in \mathbb{C}^2.$$

For every $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, we consider the map $\lambda I - f : \mathbb{C}^2 \to \mathbb{C}^2$.

(1) $\lambda I - f$ is one to one.

Suppose that $(\lambda I - f)(x_1, y_1) = (\lambda I - f)(x_2, y_2)$, then

$$|\lambda|^2(x_1 - x_2) = \overline{\lambda}(\overline{y_1} - \overline{y_2}). \tag{2.4}$$

Also $\lambda y_1 - i\overline{x_1} = \lambda y_2 - i\overline{x_2}$, so

$$i(x_2 - x_1) = \overline{\lambda}(\overline{y_1} - \overline{y_2}). \tag{2.5}$$

From (2.4) and (2.5) we get that $|\lambda|^2(x_1 - x_2) = -i(x_1 - x_2)$, thus $x_1 = x_2, y_1 = y_2$.

(2) $\lambda I - f$ is surjective.

For every $(x, y) \in \mathbb{C}^2$, let

$$u = \frac{\overline{\lambda}x + \overline{y}}{|\lambda|^2 + i}, \quad v = \frac{i\overline{\lambda}y - \overline{x}}{|\lambda|^2 i + 1}.$$

Then by calculation, we see that $(\lambda I - f)(u, v) = (x, y)$. Hence $\lambda I - f$ is onto.

(3)$(\lambda I - f)^{-1} \in Lip(\mathbb{C}^2)$.

Let $g = (\lambda I - f)^{-1}$, then

$$g(x, y) = \left(\frac{\overline{\lambda}x + \overline{y}}{|\lambda|^2 + i}, \frac{i\overline{\lambda}y - \overline{x}}{|\lambda|^2 i + 1}\right).$$

Suppose $(x_1, y_1) \in \mathbb{C}^2$, $(x_2, y_2) \in \mathbb{C}^2$, let $x = x_1 - x_2,\ y = y_1 - y_2$, then

$$|g(x_1, y_1) - g(x_2, y_2)|^2 = |\frac{1}{|\lambda|^2 + i}|^2 |\overline{\lambda}x + \overline{y}|^2 + |\frac{1}{i - |\lambda|^2}|^2 |i\overline{x} + \overline{\lambda}y|^2.$$

Let $r = |\frac{1}{|\lambda|^2 + i}| > 0$. Since

$$|\overline{\lambda}xy + \lambda\overline{yx} + \lambda i\overline{xy} - i\overline{\lambda}xy| \leq 2|\lambda|(|x|^2 + |y|^2),$$

we have

$$|g(x_1, y_1) - g(x_2, y_2)|^2 \leq r^2(|\lambda| + 1)^2(|x|^2 + |y|^2).$$

So,

$$|g(x_1, y_1) - g(x_2, y_2)| \leq r(|\lambda| + 1)|(x_1, y_1) - (x_2, y_2)|.$$

$g = (\lambda I - f)^{-1}$ is a Lipschitz map with constant $r(|\lambda| + 1)$.

In the case $\lambda = 0$, $|f(x)| = |x|$. Also, f is one to one, onto with $|f^{-1}(x)| = |x|$. By the argument above, for every $\lambda \in \mathbb{C}$, λ is in $\rho_{lip}(f)$. Thus, $\sigma_{lip}(f) = \emptyset$. By Theorem 2.3.6, $\sigma(f) = \sigma_{lip}(f) = \sigma_{fmv}(f) = \emptyset$.

Remark 2.4.3. In [49], the authors showed that $\sigma_{lip}(f)$ is always nonempty in the one-dimensional case and asked whether this is also true in higher dimensions. In [1] (which was seen after this part of the work had been completed), the authors gave a negative answer to this question by using Example 2.4.2. We found example 2.4.2 in [33] where it was used to show another fact.

We close this section with the following result regarding operators which are asymptotically linear or derivable at 0.

Proposition 2.4.4. *Let $f : E \to E$ be continuous and $f = T + R$, where T is a linear operator and R satisfies one of the following conditions:*

1. $\frac{\|R(x)\|}{\|x\|} \to 0$ *as* $\|x\| \to 0$.

2. $\frac{\|R(x)\|}{\|x\|} \to 0$ *as* $\|x\| \to \infty$.

Then $\lambda \in \sigma(f)$ provided λ is an eigenvalue of T.

Proof: Let $x_0 \in E$ and $x_0 \neq 0$ be such that $T(x_0) = \lambda x_0$. For $r \in \mathbb{R}$ with $r \geq 0$ we have

$$\|\lambda r x_0 - f(r x_0)\| = \|R(r x_0)\|.$$

So, in case (1), letting $r \to 0$ and in case (2) letting $r \to \infty$, we have

$$\frac{\|\lambda r x_0 - f(r x_0)\|}{r\|x_0\|} \to 0.$$

This implies that $m(\lambda I - f) = 0$. Hence $\lambda \in \sigma(f)$. □

2.5 Applications

In this section, firstly by applying the spectral theory, we shall study the solvability of some nonlinear operator equations. Some existence results will be obtained. Then three well known theorems will be generalized by using the theory. At the end of this section, we shall study bifurcation points and asymptotic bifurcation points of a Urysohn Operator.

We shall use the classical space $C[0,1]$ with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$. We recall that a cone K in a Banach space E is a closed subset of E such that

(1) $x, y \in K$, $a, b \geq 0$ imply $ax + by \in K$;

(2) $x \in K$ and $-x \in K$ imply $x = 0$.

A cone K is said to be normal if there exists a constant $\gamma > 0$ such that

$$\|x + y\| \geq \gamma \|x\|$$

for every $x, y \in K$. A cone defines a relation $\leq$ by means of $x \leq y$ if and only if $y - x \in K$ and an order preserving operator $T : E \to E$ is then defined by the condition $0 \leq x$ implies $0 \leq Tx$.

Example 2.5.1. We look for a non-trivial solution of the following global Cauchy problem depending on a parameter

$$x'(t) = \lambda\sqrt{x^2(t) + x^2(1-t)}, \quad x(0) = 0, t \in [0,1]. \tag{2.6}$$

Changing the problem into an integral equation we study the existence of an eigenvalue and eigenvector of the operator

$$Tx(t) = \int_0^t \sqrt{x^2(s) + x^2(1-s)}\, ds. \tag{2.7}$$

It is easily verified that $T : C[0,1] \to C[0,1]$ is positively homogeneous, order preserving, and $\|Tx\| \leq \sqrt{2}\|x\|$.

Now we shall prove that $\mu I - T$ is not surjective for every $0 < \mu < 1/\sqrt{2}$ and so $[0, 1/\sqrt{2}] \subset \sigma(T)$. Assume it is surjective, then there exists $x_0 \in C[0,1]$ such that $\mu x_0 - Tx_0 = \mu$. For every $t \in [0,1]$,

$$\mu(x_0(t) - 1) = Tx_0(t) \geq 0 \Longrightarrow x_0(t) \geq 1.$$

So for each natural number n, $T^n x_0 \geq T^n 1$. On the other hand,

$$Tx_0 \leq \mu x_0, \text{ so } T^n x_0 \leq \mu^n x_0.$$

Hence

$$\mu^n x_0 \geq T^n x_0 \geq T^n(1).$$

This implies that

$$(\sqrt{2})^n \mu^n x_0 \geq (\sqrt{2})^n T^n(1) \geq 0.$$

Also we know that $K = \{x(t) \in C[0,1] : \ x(t) \geq 0\}$ is a normal cone in $C[0,1]$ and $(\sqrt{2})^n\mu^n \to 0$ as $n \to \infty$. Thus we obtain that $(\sqrt{2})^n T^n(1) \to 0$ $(n \to \infty)$. This contradicts $T^n(1) \geq (1/\sqrt{2})^n t$. So, $\mu I - T$ is not onto.

Assume that for some $-1/\sqrt{2} < \mu < 0$, $\mu I - T$ is onto. Then for each $y \in C[0,1]$, there exists $x \in C[0,1]$, such that $\mu x - Tx = y$. Hence $(-\mu)(-x) - T(-x) = y$. So, $-\mu - T$ is onto. This is a contradiction since $1/\sqrt{2} > -\mu > 0$.

By the above argument, $[-1/\sqrt{2}, 1/\sqrt{2}] \subset \sigma(T)$ since the spectrum of T is closed. Also, we know that T is a compact map [43], so $\alpha(T) = 0$. By Theorem 2.2.8, there exist $\mu_1 \geq 1/\sqrt{2}$ and $\mu_2 \leq -1/\sqrt{2}$ such that μ_1 and μ_2 are eigenvalues of T. Moreover, by Theorem 2.2.11, we obtain that $1/\sqrt{2} \leq |\mu_i| \leq \liminf_{n\to\infty} \|T^n\|^{1/n}$. Therefore, problem (2.6) has at least two non-trivial solutions.

Remark 2.5.2. It was known that for the above operator T, $(1/\sqrt{2})\ln(1+\sqrt{2})$ is the only positive eigenvalue of T (see [43]). So, for each $\lambda \in [-1/\sqrt{2}, 1/\sqrt{2}]$, λ is not an eigenvalue of T. This shows that Theorem 2.2.9 is not true for positively homogeneous maps. It is well known that it is true for linear operators.

Example 2.5.3. The following Hammerstein integral equation

$$Ku(s) = \int_0^1 k(s,t) f(t, u(t))\, dt, \tag{2.8}$$

where $k(s,t)$ is continuous in the closed square $\{0 \leq s \leq 1, 0 \leq t \leq 1\}$ and f is a continuous map of $[0,1] \times \mathbb{R} \to \mathbb{R}$ which satisfies the condition $|f(t,u)| \leq a + b|u|$ with $a \geq 0, b \geq 0$, has been studied in [48]. Some existence results on the the equation

$$\lambda u(s) = Ku(s) = \int_0^1 k(s,t) f(t, u(t))\, dt \tag{2.9}$$

were obtained there. Let $M = \max_{(s,t)} |k(s,t)|$. We shall show (2.9) has at least one solution $u(t) \in C[0,1]$ provided that $|\lambda| > Mb$.

Proof: It is easy to see that $K : C[0,1] \to C[0,1]$ is a compact map, since for every bounded set $A \subset C[0,1]$, KA is equicontinuous and for every $\xi \in [0,1]$, $(KA)(\xi)$ is relatively compact. Now, let $\lambda \in \mathbb{K}$ with $|\lambda| > Mb$. We know that $\omega(\lambda I - K) > 0$. Assume that $m(\lambda I - K) > 0$, then there exists $m > 0$ such that

$$\|\lambda u - Ku\| \geq m\|u\| \text{ for } u \in C[0,1].$$

Let

$$S = \{u \in C[0,1] : \ \lambda u - tKu = 0, \ t \in [0,1]\}.$$

Then for $u \in S$ we have $\|u\| \leq Ma/(|\lambda| - Mb)$, so, S is bounded. This implies that $\lambda I - K$ is $(0,\varepsilon)$-epi on every B_r with $r > 0$ and $0 < \varepsilon < |\lambda|$, so $\nu(\lambda I - K) > 0$. Hence, $\lambda \in \rho(K)$. It follows that $\lambda u = Ku$ has a solution.

In the case that $m(\lambda I - K) = 0$, there exist $u_n \in C[0,1]$ such that

$$\|\lambda u_n - Ku_n\| < (1/n)\|u_n\|.$$

So

$$|\lambda|\|u_n\| - Ma - Mb\|u_n\| \leq (1/n)\|u_n\|.$$

This ensures that $\{\|u_n\|_{n=1}^{\infty}\}$ is bounded since $|\lambda| > Mb$. Thus

$$\|\lambda u_n - Ku_n\| \to 0, \ (n \to \infty).$$

Furthermore

$$\omega(\lambda I - K)\alpha(\cup_{n=1}^{\infty} u_n) \leq \alpha((\lambda I - K) \cup_{n=1}^{\infty} u_n).$$

So, $\{u_n\}_{n=1}^{\infty}$ has a convergent subsequence. Let $u_{n_k} \to u_0$ $(k \to \infty)$, then $\lambda u_0 = Ku_0$, thus u_0 is a solution of (2.9).

In the case $a = 0$, $|f(t,u)| \leq b|u|$, we have $\|Ku\| \leq Mb\|u\|$ for each $u \in C[0,1]$. For $\lambda \in \mathbb{C}$ with $|\lambda| > Mb$, by Remark 2.1.7, $\lambda \in \rho(f)$, so, $\lambda I - K$ is surjective. Thus, for each $v \in C[0,1]$, there exists $u \in C[0,1]$ satisfying

$$\lambda u(s) - (Ku)(s) = v(s).$$

□

Example 2.5.4. Suppose the Urysohn operator A is defined by

$$(A\varphi)(t) = \int_\Omega g(t, s, \varphi(s))\, ds \quad \text{for all } t \in \Omega \text{ and } \varphi \in L^2, \tag{2.10}$$

where $\Omega \subset \mathbb{R}$ is bounded. The following conditions on the kernel g was assumed in [48]:

1. $|g(t, s, x)| \le \beta_r(t, s)$ for $(t, s, x) \in \Omega^2 \times \mathbb{K}$ with $|x| \le r$ and there is $M_r > 0$ such that

$$\int_\Omega \beta_r(t, s)\, ds \le M_r \quad \text{for all } t \in \Omega;$$

2. $|g(t, s, x) - g(\tau, s, x)| \le \gamma_r(t, \tau, s)$ for (τ, s, x) in $\Omega^2 \times \mathbb{K}$ with $|x| \le r$ and

$$\lim_{t \to \tau} \int_\Omega \gamma_r(t, \tau, s)\, ds = 0 \ \text{ uniformly for } \quad \tau \in \Omega;$$

3. There exists a measurable function $\psi : \Omega^2 \to [0, \infty)$ such that

$$|g(t, s, x)| \le \psi(t, s)(1 + |x|) \text{ for all } (t, s, x) \in \Omega^2 \times \mathbb{K}$$

and

$$M = \int_\Omega \int_\Omega (\psi(t, s))^2\, ds\, dt < \infty.$$

We shall prove that $(A\varphi)(t) = \varphi(t)$ has a solution if $2M^{1/2} < 1$.

Proof: It was proved in [48] that, under the above assumptions, A is a compact operator from L^2 into L^2 and $\|A\varphi\|_2 \le 2M^{1/2}(1 + \|\varphi\|_2^2)^{1/2}$.

(a) Suppose that for every $n \in \mathbb{N}^+$, there exist $\varphi_n \in L^2$ such that $\|\varphi_n - A\varphi_n\| < (1/n)\|\varphi_n\|$. Then $\|\varphi_n\|_{n=1}^\infty$ is a bounded set. Otherwise

$$(1/n)\|\varphi_n\| > \|\varphi_n\| - \|A\varphi_n\| > \|\varphi_n\| - 2M^{1/2}(1 + \|\varphi_n\|^2)^{1/2}$$

implies that

$$1 - 2M^{1/2}\left(\frac{1}{\|\varphi_n\|^2} + 1\right)^{1/2} < \frac{1}{n}.$$

Letting $n \to \infty$, we would have $2M^{1/2} \ge 1$, a contradiction. Thus $\|\varphi_n - A\varphi_n\| \to 0$. So $\{\varphi_n\}_{n=1}^\infty$ has a convergent subsequence since A is a compact map. Let $\varphi_{n_k} \to \varphi_0 \in L^2$ $(k \to \infty)$, then $\varphi_0 = A\varphi_0$ and φ_0 is a solution of the equation.

(b) Suppose there exists $m > 0$ such that $\|(I - A)\varphi\| \geq m\|\varphi\|$. Let

$$S = \{\varphi \in L^2 : \varphi - tA\varphi = 0, \ t \in [0,1]\}.$$

For each $\varphi \in S$, we have

$$\|\varphi\| \leq \|A\varphi\| \leq 2M^{1/2}(1 + \|\varphi\|^2)^{1/2},$$

so,

$$\|\varphi\|^2 \leq \frac{4M}{1 - 4M}.$$

Thus S is bounded. It follows that $I - A$ is $(0,\varepsilon)$-epi for $0 < \varepsilon < 1$. Hence $I - A$ is regular. $1 \in \rho(A)$ and $I - A$ is surjective. Hence, the equation $A\varphi = \varphi$ has a solution in L^2. $\square$

The following theorem gives a condition for a compact, positive operator to have a positive eigenvalue and eigenvector.

Theorem 2.5.5. *Suppose E is a Banach space and K is a cone of E. For $r > 0$, let $K_r = K \cap B_r$, $\partial K_r = \{x \in K : \|x\| = r\}$.*

1. *Assume that $f : K \to K$ is a positively homogeneous, compact operator and $\inf_{\{\|x\|=1, x \in K\}} \|f(x)\| > 0$. Then there exist $r > 0$ and $x_0 \in K$, $\|x_0\| = 1$ such that $f(x_0) = rx_0$.*

2. *Assume that $F : K_r \to K$ is compact and $\inf\{\|F(x)\| : \|x\| = r\} > 0$. Then F has a positive eigenvalue with the eigenvector $x_0 \in \partial K_r$.*

Proof: (1) Since K is a cone of E, K is a closed, convex subset of E. By the Dugundj extension theorem, there exists an extension f_1 with $f_1(E) \subset K$ and with f_1 compact map [7]. Let $F : E \to K$ be defined by

$$F(x) = \begin{cases} \|x\| f_1\left(\frac{x}{\|x\|}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $F : E \to K$ is a positively homogeneous and compact operator. Also, $F|_K = f$. Suppose $\sigma(F) \cap \mathbb{R}^+ = \emptyset$ and $u \in K$, $u \neq 0$. Since $(1/n) \in \rho(F)$, $(1/n)I - F$ is onto, and there exist $\{x_n\}_{n=1}^{\infty} \in K$ satisfying the following:

$$(1/n)x_n - F(x_n) = u.$$

Then, $F(x_n) \in K$ implies that $x_n \in K$. So $F(x_n) = f(x_n)$. Thus

$$(1/n)x_n - f(x_n) = u. \tag{2.11}$$

Assume that $\{\|x_n\|_{n=1}^{\infty}\}$ is unbounded and $\|x_{n_k}\| \to \infty$ $(k \to \infty)$. Then,

$$\frac{1}{n}\frac{x_{n_k}}{\|x_{n_k}\|} - f\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right) = \frac{u}{\|x_{n_k}\|} \to 0 \ (k \to \infty).$$

Thus $f\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right) \to 0$. This contradicts $\inf_{\{x \in K, \|x\|=1\}} \|f(x)\| > 0$. So we obtain that $\{\|x_n\|_{n=1}^{\infty}\}$ is bounded. Next, (2.11) implies that

$$0 \leq u \leq (1/n)x_n \to 0 \ (n \to \infty).$$

Thus $u = 0$. This contradiction shows that $\sigma(F) \bigcap \mathbb{R}^+ \neq \emptyset$. Let $r_1 > 0$ and $r_1 \in \sigma(F)$. By Theorem 2.2.8, there exists $r \geq r_1$ such that r is an eigenvalue of F. Let $x_0 \in E$ with $\|x_0\| = 1$ be such that $F(x_0) = rx_0$. Then $x_0 \in K$ since $r > 0$ and $F(x_0) \in K$. Thus $f(x_0) = F(x_0) = rx_0$.

(2) Let $\overline{F} : K \to K$ be defined as follows:

$$\overline{F}(x) = \begin{cases} \|x\| F\left(\frac{rx}{\|x\|}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

$\overline{F}$ is a compact and positively homogeneous map since F is a compact map [43]. Furthermore,

$$\inf_{\|x\|=1} \|(\overline{F})(x)\| = \inf_{\|x\|=r} \|F(x)\| > 0.$$

So, by (1), there exists $x' \in K$ with $\|x'\| = 1$ and $\lambda > 0$ such that $\overline{F}(x') = \lambda x'$. Thus

$$F\left(\frac{rx'}{\|x'\|}\right) = \lambda x'.$$

Let $x_0 = rx'$ then $x_0 \in K$ and $\|x_0\| = r$. Also $F(x_0) = (\lambda/r)x_0$. $\square$

The following well known Birkoff-Kellogg theorem was proved in [16] by applying their spectral theory.

Theorem 2.5.6. (Birkoff-Kellogg Theorem) Let E be an infinite dimensional Banach space and $S = \{x \in E : \|x\| = 1\}$. Let $f : S \to E$ be continuous and compact such that $f(S)$ is bounded away from zero ($\inf_{x \in S} \|f(x)\| > 0$). Then f has a positive eigenvalue.

The following theorem is a generalization of Theorem 2.5.6.

Theorem 2.5.7. *Let E be an infinite dimensional Banach space and let $f : S \to E$ be continuous and bounded. Assume that $\inf_{x\in S} \|f(x)\| > \alpha(f)$. Then for every complex number λ with $\lambda \neq 0$ there exists $r > 0$ such that $r\lambda$ is an eigenvalue of f. In particular, f has a positive and a negative eigenvalue.*

Proof: Let $\tilde{f} : E \to E$ be the positively homogeneous operator which is defined as follows:

$$\tilde{f}(x) = \begin{cases} \|x\| f(x/\|x\|) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then we have

$$d(\tilde{f}) = \liminf_{\|x\|\to\infty} \frac{\|\tilde{f}(x)\|}{\|x\|} = \inf_{x\in S} \|f(x)\|, \quad |\tilde{f}| = \limsup_{\|x\|\to\infty} \frac{\|\tilde{f}(x)\|}{\|x\|} = \sup_{x\in S} \|f(x)\|,$$

and $\alpha(\tilde{f}) = \alpha(f)$ (see [43]). Let $B(\tilde{f})$ be the set of all asymptotic bifurcation points of $\tilde{f}$. By Theorem 11.1.1 of [16], there exists $\mu > 0$ such that $\mu \in B(\tilde{f})$. Let $\{\mu_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty} \in E$ be sequences such that $\mu_n \to \mu$, $\|x_n\| \to \infty$ as $n \to \infty$ and $\tilde{f}(x_n) = \mu_n x_n$. Then we have

$$\frac{\|(\mu I - \tilde{f})x_n\|}{\|x_n\|} = \frac{\|\mu x_n - \mu_n x_n\|}{\|x_n\|} \to 0, \quad \text{as } n \to \infty.$$

This implies that $d(\mu I - \tilde{f}) = 0$. So, by Theorem 2.3.6, $\mu \in \sigma_{fmv}(\tilde{f}) \subset \sigma(\tilde{f})$. Assume that $\mu \leq \alpha(\tilde{f})$, then $\mu < d(\tilde{f})$, since by our assumption, $\alpha(\tilde{f}) = \alpha(f) < d(\tilde{f})$. So,

$$d(\mu I - \tilde{f}) \geq d(\tilde{f}) - \mu > 0.$$

This contradicts $d(\mu I - \tilde{f}) = 0$. Thus we have $\mu > \alpha(\tilde{f})$. By Theorem 2.2.8, there exists $t_0 \in (0, 1]$ such that μ/t_0 is an eigenvalue of $\tilde{f}$. Let $x_0 \in E$ with $\|x_0\| = 1$ be such that $\tilde{f}(x_0) = (\mu/t_0)x_0$. Then $f(x_0) = rx_0$, where $r = \mu/t_0$.

For every complex number λ with $|\lambda| = 1$, writing $\overline{\lambda} = e^{i\theta}$, we have

$$\inf_{x\in S} \|\overline{\lambda} f(x)\| > \alpha(f) = \alpha(\overline{\lambda} f).$$

By the above argument, there exists $r > \alpha(f)$ and $x_0 \in E$ with $x_0 \neq 0$ such that $\overline{\lambda} f(x_0) = rx_0$. Thus $f(x_0) = (r\lambda)x_0$. In the case $|\lambda| \neq 1$, let $\lambda_1 = \lambda/|\lambda|$. By the same argument, there exists $r > 0$ such that $r\lambda$ is an eigenvalue of f. □

The following example shows that there exist mappings f to which Theorem 2.5.6 does not apply but Theorem 2.5.7 can be used.

Example 2.5.8. Let $B_1 = \{x \in E : \|x\| \leq 1\}$ and $g : E \to B_1$ be the radial retraction of E onto the unit ball, that is

$$g(x) = \begin{cases} x/\|x\| & \text{if } \|x\| > 1, \\ x & \text{if } 0 \leq \|x\| \leq 1. \end{cases}$$

Since $g(\Omega) \subset \text{co}(\Omega \cup 0)$, g is a 1-set contraction [7]. Let $y \in E$ with $\|y\| > 2$ and $f : S \to E$ be defined by

$$f(x) = y + g(x), \ x \in E.$$

Then,

$$\inf_{x\in S} \|f(x)\| = \inf_{x\in S} \|y + g(x)\| \geq \|y\| - \sup_{x\in S} \|g(x)\| = \|y\| - 1 > 1.$$

Next,

$$\alpha(f) = \alpha(y + g) = \alpha(g) = 1.$$

So, $\inf_{x\in S} \|f(x)\| > \alpha(f)$. Furthermore, we have

$$\sup_{\|x\|=1} \|f(x)\| = \sup_{\|x\|=1} \|y + g(x)\| \leq \|y\| + 1.$$

Hence, f satisfies the conditions of Theorem 2.5.7. So, for every $z \in \mathbb{C}$, there exists $\lambda > 1$ such that λz is an eigenvalue of f.

Theorem 2.5.7 enables us to give the following generalization of theorem 10.1.2 of [16].

Theorem 2.5.9. *Let S be the unit sphere in an infinite dimensional Banach space E and let $f : S \to S$ be a continuous strict set contraction. Then every $\lambda \in \mathbb{K}$, $|\lambda| = 1$, is an eigenvalue of f. In particular, f has a fixed point and an antipodal point.*

Proof: Since $\alpha(f) < 1$, we have

$$\inf_{x\in S} \|f(x)\| = 1 > \alpha(f), \qquad \sup_{x\in S} \|f(S)\| < +\infty.$$

By Theorem 2.5.7, for every complex number $|\lambda| = 1$, there exists $\alpha > 0$, such that $\alpha\lambda$ is an eigenvalue of f. Thus there exists $x_\lambda \in S$ such that $f(x_\lambda) = \alpha\lambda x_\lambda$. Since $\|f(x_\lambda)\| = 1$, $|\alpha\lambda| = \alpha = 1$, so λ is an eigenvalue of f.

In particular, for $\lambda = 1$, there exists $x \in S$, such that $f(x) = x$. For $\lambda = -1$, there exists $x \in S$, such that $f(x) = -x$. $\square$

In the following, let $\mathbb{R}^n$ be the n-dimensional space and $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. The Hopf theorem on spheres is as follows [16]:

Theorem 2.5.10. *Let $f : S^{2n} \to \mathbb{R}^{2n+1}$ be continuous. Assume $\langle f(x), x\rangle = 0$ for all $x \in S^{2n}$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^{2n+1}$. Then, f vanishes at some point $x \in S^{2n}$.*

The new theory enables us to give a generalization of Theorem 2.5.10. Firstly, we need the following lemma. We shall let E^* denote the dual space of E and $J : E \to E^*$ the duality mapping, that is, for $x \in E$,

$$J(x) = \{x^* \in E^* : x^*(x) = \|x\|^2 = \|x^*\|^2\}.$$

We recall that for a linear operator A in a Hilbert space, the *numerical range* of A is the set of values of (Ax, x) for all x with $\|x\| = 1$ [67]. It is known that the numerical range of A is a convex set. If we let $V(A)$ be the numerical range of A, then $\sigma(A) \subset \overline{V(A)}$.

For a nonlinear operator f, we have the following.

Lemma 2.5.11. *Suppose $f : E \to E$ is a continuous operator. Let*

$$V(f) = \left\{ \frac{(f(x), x^*)}{\|x\|^2} \right\} \cup \{0\}, \ x \in E, x^* \in J(x).$$

Then $\lambda \in \overline{\text{co}}V(f)$ provided that $\lambda \in \sigma(f)$ and $|\lambda| > \alpha(f)$.

(Where $\overline{\text{co}}$ denote the closed convex hull.)

Proof: We shall prove that $|\lambda| > \alpha(f)$ and $\text{dist}(\lambda, \overline{\text{co}}V(f)) > 0$ implies that $\lambda \in \rho(f)$.

If $|\lambda| > \alpha(f)$ and $\text{dist}(\lambda, \overline{\text{co}}V(f)) > 0$, then

$$\begin{aligned} 0 < d &= \text{dist}(\lambda, \overline{\text{co}}V(f)) \\ &\leq |\lambda - \frac{(f(x), x^*)}{\|x\|^2}| \\ &= |\frac{\lambda(x, x^*) - (f(x), x^*)}{\|x\|^2}| \\ &= \frac{|(\lambda x - f(x), x^*)|}{\|x\|^2} \\ &\leq \frac{\|\lambda x - f(x)\|}{\|x\|}. \end{aligned}$$

So, $\|\lambda x - f(x)\| \geq d\|x\|$. This implies that $m(\lambda I - f) > 0$. Also, $\omega(\lambda I - f) > 0$ since $|\lambda| > \alpha(f)$. Let

$$M = \{x \in E : \ \lambda x - tf(x) = 0, \ t \in [0, 1]\}.$$

For every $x \in M$, suppose there exists $t \in [0, 1]$ with $t \neq 0$, such that $\lambda x = tf(x)$. Then

$$\frac{(f(x), x^*)}{\|x\|^2} = \frac{((\lambda/t)x, x^*)}{\|x\|^2} = \frac{\lambda}{t}.$$

Hence,

$$\lambda = t\frac{(f(x), x^*)}{\|x\|^2} \in \overline{\text{co}}V(f).$$

This contradicts our assumption $d = \text{dist}(\lambda, \overline{\text{co}}V(f)) > 0$. Thus, $M = \{0\}$. By Corollary 2.2.2, $\lambda I - f$ is $(0, \varepsilon)$-epi for some $\varepsilon > 0$, thus $\nu(\lambda I - f) > 0$, hence $\lambda I - f$ is regular. This shows that

$$\{\lambda : |\lambda| > \alpha(f)\} \cap \sigma(f) \subset \overline{\text{co}}V(f).$$

□

The following theorem is a generalization of Theorem 2.5.10.

Theorem 2.5.12. *1. Let $f : S^n \to \mathbb{R}^{n+1}$ be continuous. Assume $\langle f(x), x\rangle = 0$ for all $x \in S^n$, where $\langle \cdot, \cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^{n+1}$. Then, f vanishes at some point $x \in S^n$ provided either n is an even number or $f(S^n)$ is contained in a proper subspace of $\mathbb{R}^{n+1}$.*

2. Let E be an infinite dimensional Banach space and $f : S \to E$ be a continuous compact mapping. Assume that $(f(x), x^*) = 0$ for all $x^* \in Jx$, $x \in S$. Then $\inf_{x\in S} \|f(x)\| = 0$.

Proof: (1) Let

$$\tilde{f}(x) = \begin{cases} \|x\| f(x/\|x\|) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then for every $x \in \mathbb{R}^{n+1}$ with $x \neq 0$,

$$\langle \tilde{f}(x), x\rangle = \|x\| \langle f(x/\|x\|), x/\|x\|\rangle = 0.$$

So, $V(\tilde{f}) = \{0\}$. Also, $\alpha(\tilde{f}) = 0$ since $\mathbb{R}^{n+1}$ is finite dimensional. By Lemma 2.5.11, we have $\sigma(\tilde{f}) \subset \{0\}$.

(a) Assume that n is even. Let $B(\tilde{f})$ denote the set of all asymptotic bifurcation points of $\tilde{f}$, then by theorem 11.1.3 of [16], $B(\tilde{f}) \neq \emptyset$. By Proposition 2.1.11, $B(\tilde{f}) \subset \sigma(\tilde{f})$. Hence, $0 \in B(\tilde{f})$. Suppose that $\{x_n\}_{n=1}^\infty \in \mathbb{R}^{n+1}$ and $\lambda_n \in \mathbb{K}$ satisfy $\|x_n\| \to \infty$, $\lambda_n \to 0$ as $n \to \infty$ and $\lambda_n x_n = \tilde{f}(x_n)$. Then

$$\tilde{f}\left(\frac{x_n}{\|x_n\|}\right) = \lambda_n \frac{x_n}{\|x_n\|} \to 0.$$

Moreover, $\left\{\frac{x_n}{\|x_n\|}\right\}_{n=1}^\infty$ has a convergent subsequence $\frac{x_{n_k}}{\|x_{n_k}\|} \to x_0$ $(k \to \infty)$. Thus $\|x_0\| = 1$ and $\tilde{f}(x_0) = f(x_0) = 0$.

(b) In the case that $f(S^n)$ is contained in a proper subspace of $\mathbb{R}^{n+1}$, $\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ cannot be surjective. So, $0 \in \sigma(\tilde{f})$. Also, we know that $\omega(\tilde{f}) = \infty$ since $\mathbb{R}^{n+1}$ is finite dimensional. Firstly, if $d(\tilde{f}) = 0$, it follows that there exists $\{x_n\}_{n=1}^\infty \in \mathbb{R}^{n+1}$ with $\|x_n\| = 1$ such that

$$\tilde{f}(x_n) = f(x_n) \to 0 \ (n \to \infty).$$

So, $f(x_0) = 0$ for some $x_0 \in \mathbb{R}^{n+1}$ and $\|x_0\| = 1$. Secondly, if $d(\tilde{f}) > 0$, then $0 \in \mathbb{K}\backslash\sigma_\pi(\tilde{f})$ and $1 \notin \sigma(\tilde{f})$. By Theorem 11.1.2 of [16], we have $B(\tilde{f}) \bigcup \sigma_\omega(\tilde{f})$ separates 1 from 0. So, $B(\tilde{f}) \bigcup \sigma_\omega(\tilde{f}) \neq \emptyset$. This implies that $B(\tilde{f}) \neq \emptyset$ since $\sigma_\omega(\tilde{f}) = \emptyset$. By the same argument with that in (a), there exists $x_0 \in S^n$ such that $f(x_0) = 0$.

(2) Suppose E is an infinite dimensional Banach space. Let $\tilde{f}$ be as in (1), then $V(\tilde{f}) = \{0\}$. By Theorem 8.2.1 of [16], $\Sigma(\tilde{f}) \neq \emptyset$ (see section 1.3 for the definition of $\Sigma(\tilde{f})$). Again applying Lemma 2.5.11, we have

$$\Sigma(\tilde{f}) \subset \sigma(\tilde{f}) \subset V(\tilde{f}).$$

So, $0 \in \Sigma(\tilde{f})$. Thus there exist $x_n \in E$ with $\|x_n\| = 1$ such that $\tilde{f}(x_n) = f(x_n) \to 0$. Hence $\inf_{x \in E} \|f(x)\| = 0$. □

In the following example, we shall use the spectral theory to study bifurcation points and asymptotic bifurcation points for a Urysohn Operator.

Example 2.5.13. Let Ω denote a closed bounded set in a finite-dimensional space and for simplicity, we shall assume that $\operatorname{meas}\Omega = 1$. Let $k : \Omega \to \mathbb{R}$ be continuous. We consider the compact operator $A : \mathbb{R} \times C(\Omega) \to C(\Omega)$ defined by

$$A(\mu, x)(t) = \mu \int_\Omega k(s,t) f(x(s))\, ds, \tag{2.12}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous.

We make the following hypotheses:

1. $k(s,t) \geq 0$ for $(s,t) \in \Omega \times \Omega$ and $M = \max_{t\in\Omega} \int_\Omega k(s,t)\, ds > 0$;

2. $f(x) \geq 0$ and $f(0) > 1/M$;

3. $f(x) \geq 0$ and $\inf_{x \geq 0} f(x) = d > 0$;

4. $f(x) \geq 0$, $\inf_{x\geq 0} f(x) = d > 0$ and $\sup_{x \geq 0} |f(x)| = D < \infty$.

Suppose (1) and (2) are satisfied or (1) and (3) are satisfied, then, 0 is a bifurcation point for the equation $A(\mu, x) = x$. If (1) and (4) are satisfied, then, ∞ is an asymptotic bifurcation point of $A(\mu, x) = x$.

(Recall that ∞ is called an asymptotic bifurcation point of $A(\mu, x) = x$ if there exist μ_n and $x_n \in E, x_n \neq 0$ such that $A(\mu_n, x_n) = x_n$, $\mu_n \to \infty$ and $\|x_n\| \to \infty$ as $n \to \infty$).

Proof: (a) Suppose that $k(s,t)$ and f satisfy (1) and (2). For every $r > 0$, define $A_r : C(\Omega) \to C(\Omega)$ by

$$(A_r x)(t) = \begin{cases} \|x\| \int_\Omega k(s,t) f\left(r\frac{x(s)}{\|x\|}\right) ds & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.13}$$

A_r is a positively homogeneous, compact operator. Suppose $1 \in \rho(A_r)$. Then, $I - A_r$ is surjective, so there exists $x \in C(\Omega)$ such that $x - A_r x = 1$. Obviously, $\|x\| \neq 0$. Also, since $f(x) \geq 0$, $x(t) \geq 1$ and

$$0 \leq \|x\| \int_\Omega k(s,t) f\left(r\frac{x(s)}{\|x\|}\right) ds < x(t).$$

Hence

$$\int_\Omega k(s,t) f\left(r\frac{x(s)}{\|x\|}\right) ds < \frac{x(t)}{\|x\|}. \tag{2.14}$$

Since $f(0) > 1/M$, there exists $\delta > 0$ such that $|x| < \delta$ implies $f(x) > 1/M$. Let $r_n > 0$, $r_n \to 0$, and assume $r_n < \delta$. Then $f\left(r_n\frac{x_n(s)}{\|x_n\|}\right) > \frac{1}{M}$, so that

$$\max_{t\in\Omega} \int_\Omega k(s,t) f\left(r_n\frac{x_n(s)}{\|x_n\|}\right) ds \geq \frac{1}{M} \max_{t\in\Omega} \int_\Omega k(s,t) ds = 1.$$

This contradicts (2.14). Hence $1 \in \sigma(A_{r_n})$. By Theorem 2.2.8, there exists $\lambda_n \geq 1$ such that λ_n an eigenvalue of A_{r_n}. Suppose that $A_{r_n} x_n = \lambda_n x_n$ and $x_n \neq 0$. We have

$$\frac{r_n}{\lambda_n} \int_\Omega k(s,t) f\left(r_n\frac{x_n(s)}{\|x_n\|}\right) ds = r_n\frac{x_n(t)}{\|x_n\|}.$$

Let $y_n = r_n\frac{x_n}{\|x_n\|}$, then

$$A\left(\frac{r_n}{\lambda_n}, y_n\right) = y_n,$$

also, $\|y_n\| \to 0$ and $r_n/\lambda_n \to 0$ as $n \to \infty$. Thus 0 is a bifurcation point for the equation $A(\mu, x) = x$.

(b) Suppose $k(s,t)$ and f satisfy conditions (1) and (3). Let $K = C_+(\Omega)$ be the cone of non-negative functions. For $r > 0$, let $A_r : C_+(\Omega) \to C_+(\Omega)$ be defined as in (a). Assume $x \in C_+(\Omega)$, and $\|x\| = 1$. Then

$$\|A_r x\| = \max_{t\in\Omega} \int_\Omega k(s,t) f(rx(s))\, ds \geq d \max_{t\in\Omega} \int_\Omega k(s,t)\, ds = dM > 0.$$

So, $\inf_{\{x\in K, \|x\|=1\}} \|A_r x\| > 0$. By Theorem 2.5.5, there exist $\lambda_r > 0$ and $x_r \in C_+(\Omega)$ with $\|x_r\| = 1$ such that $(A_r x)(t) = \lambda_r x_r(t)$. Therefore,

$$\frac{1}{\lambda_r} \int_\Omega k(s,t) f(rx_r(s))\, ds = x_r(t).$$

Moveover

$$\lambda_r = \max_{t\in\Omega} \int_\Omega k(s,t) f(rx_r(s))\, ds \geq dM.$$

Let $y_r = rx_r$, then $\|y_r\| = r$ and

$$\frac{r}{\lambda_r} \int_\Omega k(s,t) f(rx_r(s))\, ds = y_r(t). \tag{2.15}$$

Let $r \to 0$, we obtain 0 is a bifurcation point of $A(\mu, x) = x$.

(c) Assume conditions (1) and (4) are satisfied. Let λ_r and $y_r(t)$ be as in the proof of (b). Then from (2.15),

$$\lambda_r = \max_{t\in\Omega} \int_\Omega k(s,t) f(rx_r(s))\, ds \leq MD.$$

Let $r \to \infty$, it follows $r/\lambda_r \to \infty$. Hence, $\|y_r\| = r \to \infty$. Thus ∞ is an asymptotic bifurcation point of $A(\mu, x) = x$. $\square$

The following theorem was given in [16] (see Theorem 10.1.4). Here, we shall give a different proof by using our theory.

Theorem 2.5.14. *Let E be an infinite dimensional Banach space and let $f : S \to S$ be continuous and compact. Then f cannot be odd.*

Proof: Assume f is odd. Let

$$\tilde{f}(x) = \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Then $\tilde{f}$ is an odd and positively homogeneous compact operator [43]. So, by Theorem 2.2.9, $\lambda \in \sigma(\tilde{f})\backslash\{0\}$ implies that λ is an eigenvalue of $\tilde{f}$. On the other hand, 0 is an interior point of $\sigma(\tilde{f})$ [16]. So, for $\delta > 0$, there exists $0 < \lambda < \delta$, such that λ is an eigenvalue of $\tilde{f}$. Thus there exists $x \neq 0$ such that

$$f\left(\frac{x}{\|x\|}\right) = \lambda\frac{x}{\|x\|}.$$

This contradicts $f : S \to S$. Hence, f cannot be odd. $\square$

The following Borsuk-Ulam Theorem was proved in [16] by applying their theory.

Theorem 2.5.15. *Let $\phi : S \to E$ be a compact vector field such that $\phi(S)$ is contained in a proper closed subspace F of E. Then there exists $x \in S$ such that $\phi(x) = \phi(-x)$.*

We conclude this chapter with a generalization of the above theorem, which also was studied by Edmunds and Webb [8] and others, see [8] for references.

Theorem 2.5.16. *Let $g : S \to E$ be condensing and suppose $(I-g)(S)$ is contained in a proper subspace F of E. Then there exists a point $x \in S$ such that $(I-g)(x) = (I-g)(-x)$.*

Proof: (1) First suppose g is an odd k-set contraction, so that $\alpha(g) < 1$.

Let $\overline{\psi} : E \to E$ be defined by $\overline{\psi}(x) = x - \overline{g}(x)$, where

$$\overline{g}(x) = \begin{cases} \|x\|g(\frac{x}{\|x\|}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We claim that $\overline{\psi}$ is not surjective. For suppose $y_0 \in E \setminus F$ and there exists $x_0 \in E$ such that

$$x_0 - \|x_0\|g(\frac{x_0}{\|x_0\|}) = y_0,$$

that is

$$\|x_0\|(I-g)(\frac{x_0}{\|x_0\|}) = y_0.$$

Since $(I-g)(x_0/\|x_0\|) \in F$, we get $y_0 \in F$, a contradiction. Thus $1 \in \sigma(\overline{g})$. Also $\alpha(\overline{g}) = \alpha(g) < 1$ and $\overline{g}$ is a positively homogeneous odd operator. By Theorem 2.2.9, 1 is an eigenvalue of $\overline{g}$, so there exists $x_0 \in E,\ x_0 \neq 0$, such that

$$x_0 - \|x_0\| g(\frac{x_0}{\|x_0\|}) = 0.$$

Since g is odd,

$$(I-g)(-\frac{x_0}{\|x_0\|}) = (I-g)(\frac{x_0}{\|x_0\|}) = 0.$$

(2) Next suppose g is an odd condensing map. Let $0 < k_n < 1$, $k_n \to 1$ $(n \to \infty)$ and define $h_n : S \to E$ by $h_n(x) = k_n g(x)$, so that $\alpha(h_n) < 1$. We show that for all large enough n, $(I-h_n)(S)$ is contained in F. Let $y_0 \in E \setminus F$. For any k, there exist $n(k) > k$ and $x_n \in S$ such that

$$(I-h_n)(x_n) = \lambda_n y_0.$$

Since $\{|\lambda_n|\}$ is bounded, it has a convergent subsequence. For simplicity we write $\alpha(x_n)$ in place of $\alpha\left(\bigcup_{n=1}^{\infty} x_n\right)$. Then, $\alpha(\lambda_n y_0) = 0$. From

$$x_n - g(x_n) = -(1-k_n)g(x_n) + \lambda_n y_0$$

we obtain $\alpha(x_n - g(x_n)) = 0$. If $\alpha(x_n) \neq 0$, then

$$\alpha(x_n) \leq \alpha(x_n - g(x_n)) + \alpha(g(x_n)) < \alpha(x_n),$$

a contradiction. Hence $\alpha(x_n) = 0$, so $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \to x_0,\ x_0 \in S$. By the above we have

$$x_0 - g(x_0) = \lambda_0 y_0.$$

If $\lambda_0 = 0,\ x_0 - g(x_0) = 0, (-x_0) - g(-x_0) = -x_0 + g(x_0) = 0$, so that

$$(I-g)(x_0) = (I-g)(-x_0)$$

If $\lambda_0 \neq 0$, $y_0 = (x_0 - g(x_0))/\lambda_0 \in F$, a contradiction with $y_0 \in E \setminus F$.

So for all large enough n, $(I-h_n)(S)$ is contained in a proper subspace of E. By (1), there exists $x_n \in S$ such that

$$(I-h_n)(x_n) = (I-h_n)(-x_n), \text{ where } n > N.$$

Therefore,

$$2x_n = k_n g(x_n) - k_n g(-x_n).$$

If $x_n \neq 0$,

$$\begin{aligned} 2\alpha(x_n) &\leq \alpha((k_n - 1)g(x_n) + (1 - k_n)g(-x_n) + 2g(x_n)) \\ &\leq 2\alpha(g(x_n)) \\ &< 2\alpha(x_n), \end{aligned}$$

a contradiction. Hence $\alpha(x_n) = 0$ and so x_n has a convergent subsequence, say $x_{n_k} \to x_0$ with $\|x_0\| = 1$. Thus $2x_0 = g(x_0) - g(-x_0)$, that is $(I - g)(x_0) = (I - g)(-x_0)$.

(3) Finally, if g is not odd, let $g_1 = (g(x) - g(-x))/2$, g_1 is odd and condensing. Also $(I - g_1)(S)$ is contained in F. By (2), there exist $x_0 \in S$ such that

$$x_0 - g_1(x_0) = -x_0 - g_1(-x_0).$$

Hence

$$(I - g)(x_0) = (I - g)(-x_0).$$

$\square$

Chapter 3

Spectral theory for semilinear operators and its applications

Let E and F be two Banach spaces. In this chapter, $L : \mathrm{dom}(L) \subset E \to F$ will denote a closed Fredholm operator of index zero with $\ker(L) \neq \{0\}$ and $\mathrm{dom}(L)$ dense in E. $N : \overline{\Omega} \to F$ will be a continuous nonlinear map, where $\Omega \subset E$ is an open bounded set and $\mathrm{dom}(L) \cap \Omega \neq \emptyset$. We shall introduce the spectrum for the semilinear operator $L - N$, and prove that it has the similar properties with that of the spectrum of nonlinear operators. Also, we shall give some applications of the theory.

3.1 L-stably solvable mappings

In the following, subspaces E_1, F_0 and maps $P, Q, L_P, K_P, K_{PQ}, \Pi, \Lambda$ will be as defined in section 1.5. We will generalize the stably-solvable mapping for nonlinear operators to the L-stably solvable mapping for semilinear operators. Firstly, we give a definition and then we will prove two lemmas which we will make use of in the sequel. We also suppose that $N : E \to F$.

Definition 3.1.1. For $\lambda \in \mathbb{C}$, let $f_\lambda(L, N) : E \to E$ be defined by

$$f_\lambda(L, N)(x) = \lambda(I - P)x - (\Lambda\Pi + K_{PQ})Nx,$$

and let $\Sigma(L, N)$ denote the set

$$\left\{\lambda \in \mathbb{C} : \liminf_{\|x\| \to \infty} \frac{\|f_\lambda(L, N)(x)\|}{\|x\|} = 0\right\}.$$

In the following, we will use f_λ for $f_\lambda(L, N)$ when there is no confusion.

Lemma 3.1.2. *Let $h : F/\operatorname{im}(L) \to F_0$ be the natural linear isomorphism and let $J = h\Lambda^{-1}$. Then $\Lambda\Pi + K_{PQ} : F \to \operatorname{dom}(L)$ is a linear isomorphism and $L + JP : \operatorname{dom}(L) \to F$ is invertible with $(L + JP)^{-1} = \Lambda\Pi + K_{PQ}$.*

Proof: Obviously, $J : \ker(L) \to F_0$ is an isomorphism.

For $x \in \operatorname{dom}(L)$, suppose that $(L + JP)x = 0$. Then $Lx = -JPx \in F_0$, so $JPx = Lx = 0$, $x \in \ker(L)$ and then $JPx = Jx = 0$. This implies that $x = 0$. Hence $L + JP$ is invertible on $\operatorname{dom}(L)$.

Now, let $y \in F$ and suppose $(\Lambda\Pi + K_{PQ})y = 0$. Then

$$\Lambda\Pi y = -K_P(I - Q)y \in \operatorname{dom}(L) \cap E_1.$$

This implies that $\Lambda\Pi y = 0$ and $(I - Q)y = 0$. Thus $y \in \operatorname{im}(L) \cap F_0$, so that $y = 0$. Therefore, $\Lambda\Pi + K_{PQ}$ is one to one. For every $y \in F$, we have

$$(L + JP)(\Lambda\Pi + K_{PQ})y = JP\Lambda\Pi y + (I - Q)y = h\Pi y - Qy + y = y.$$

Hence $L + JP$ is onto. Also, for every $x \in \operatorname{dom}(L)$, we have

$$(\Lambda\Pi + K_{PQ})(L + JP)x = K_P(I - Q)Lx + \Lambda\Pi JPx = (I - P)x + J^{-1}h\Pi JPx = x.$$

Hence, $\Lambda\Pi + K_{PQ}$ is the (bounded) inverse of $L + JP$. □

Lemma 3.1.3. *Let $y \in F$ and $\lambda \in \mathbb{C}$. Then*

1. *$\lambda Lx - Nx = y$ if and only if $f_\lambda(x) = (\Lambda\Pi + K_{PQ})y$.*

2. *$\lambda L - N : \operatorname{dom}(L) \to F$ is onto if and only if $f_\lambda : E \to \operatorname{dom}(L)$ is onto.*

Proof: Let $y \in F$, then

$$\begin{aligned}
\lambda Lx - Nx = y &\Longleftrightarrow \lambda Lx = Q(Nx+y) + (I-Q)(Nx+y) \\
&\Longleftrightarrow \lambda Lx = (I-Q)(Nx+y),\ Q(Nx+y) = 0 \\
&\Longleftrightarrow \lambda K_P Lx = K_P(I-Q)(Nx+y),\ \Pi(Nx+y) = 0 \\
&\Longleftrightarrow \lambda(I-P)x = K_{PQ}(Nx+y),\ \Lambda\Pi(Nx+y) = 0 \\
&\Longleftrightarrow \lambda(I-P)x = (\Lambda\Pi + K_{PQ})(Nx+y) \\
&\Longleftrightarrow f_\lambda(x) = (\Lambda\Pi + K_{PQ})y.
\end{aligned}$$

Suppose $\lambda L - N$ is onto. For every $x \in \operatorname{dom}(L)$, by Lemma 3.1.2, $x = (\Lambda\Pi + K_{PQ})y$ for some $y \in F$. Let $x_0 \in \operatorname{dom}(L)$ be such that $\lambda Lx_0 - Nx_0 = y$. Then by 1,

$$f_\lambda(x_0) = (\Lambda\Pi + K_{PQ})y = x.$$

Hence, $f_\lambda : E \to \operatorname{dom}(L)$ is onto.

On the other hand, suppose $f_\lambda : E \to \operatorname{dom}(L)$ is onto. Then for every $y \in F$, $(\Lambda\Pi + K_{PQ})y \in \operatorname{dom}(L)$. There exists $x_0 \in E$ such that $f_\lambda(x_0) = (\Lambda\Pi + K_{PQ})y$. Thus $x_0 \in \operatorname{dom}(L)$ and $\lambda Lx_0 - Nx_0 = y$. Hence $\lambda L - N$ is onto. □

The following concept is a generalization of stably-solvable operators which were introduced in [16].

Definition 3.1.4. $\lambda L - N$ is said to be L-stably solvable if the equation

$$\lambda Lx - Nx = h(x) \tag{3.1}$$

has a solution $x \in \operatorname{dom}(L)$ for every continuous bounded L-compact map (see Definition 1.5.1) $h : E \to F$ with quasinorm $|h| = 0$.

$\lambda L - N$ is called a L-strong surjection if equation (3.1) has a solution $x \in \operatorname{dom}(L)$ for every continuous map $h : E \to F$ with $h(E)$ bounded and $\overline{K_{PQ}h(E)}$ compact.

It is clear that when L is the identity, L-stably solvable operators coincides with the usual stably solvable operators of [16] and L-strong surjection is the same as strong surjection.

In the following, we will let $H = L + JP$. Then $\Lambda\Pi + K_{PQ} = H^{-1}$. The following theorem is a generalization of Proposition 5.1.1 of [16].

Theorem 3.1.5. *Suppose that $\lambda \notin \Sigma(L, N)$. The following conditions are equivalent.*

1. *$\lambda Lx - Nx = h(x)$ has a solution $x \in \operatorname{dom}(L)$ for every bounded continuous L-compact map $h : E \to F$ with bounded support.*

2. *$\lambda L - N$ is an L-strong surjection.*

3. *$\lambda L - N$ is L-stably solvable.*

Proof: (3)$\Rightarrow$ (2) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (2) Suppose $h : E \to F$ is a continuous map with $h(E)$ bounded and $\overline{K_{PQ}h(E)}$ compact. Then h is L-compact. Let $0 \leq \sigma_n \leq 1$ be continuous with

$$\sigma_n(x) = \begin{cases} 1 & \text{if } \|x\| \leq n, \\ 0 & \text{if } \|x\| > 2n. \end{cases}$$

There exist $x_n \in E$ such that

$$\lambda L(x_n) - N(x_n) = \sigma_n(x_n)h(x_n).$$

If $\{x_n\}$ is unbounded, $\lambda \notin \Sigma(L, N)$ implies that $\lim_{n\to\infty} \|f_\lambda(x_n)\| = \infty$. By Lemma 3.1.3,

$$\lim_{n\to\infty} |\sigma_n(x_n)| \|H^{-1}h(x_n)\| \to \infty.$$

This contradicts $h(E)$ is bounded. So, there exists $M > 0$ such that $\|x_n\| < M$ for all n. For $n > M$, $\sigma_n(x_n) = 1$, $\lambda L(x_n) - N(x_n) = h(x_n)$. Thus $\lambda L - N$ is an L-strong surjection.

(1) $\Rightarrow$ (3) Suppose $h : E \to F$ is a continuous, bounded L-compact map with $|h| = 0$. Let $\sigma_n(x)$ be defined as above. Then $\sigma_n(x)h(x)$ is a bounded L-compact map, also with bounded support. The equation

$$\lambda Lx - Nx = \sigma_n(x)h(x)$$

has a solution $x_n \in \operatorname{dom}(L)$. Assume that $\{x_n\}$ is unbounded. Again by Lemma 3.1.3,

$$\frac{\|f_\lambda(x_n)\|}{\|x_n\|} = \|\sigma_n(x_n)H^{-1}\frac{h(x_n)}{\|x_n\|}\| \leq \|H^{-1}\|\frac{\|h(x_n)\|}{\|x_n\|} \to 0.$$

This contradicts $\lambda \notin \Sigma(L,N)$. As in the proof of (1) $\Rightarrow$ (2), for n sufficiently large, $\sigma_n(x_n) = 1$. Thus $(\lambda L - N)(x_n) = h(x_n)$. Hence, $\lambda L - N$ is L-stably solvable. □

We need the following Lemma to prove the Continuation Principle for L-stably solvable maps.

Lemma 3.1.6. *Let $B_r = \{x \in F : \|x\| \leq r\}$, π be the radial retraction, $\pi : F \to B_r$. Let $h : E \to F$ be a continuous L-compact map. Then, $\pi h : E \to F$ is L-compact.*

Proof: Clearly, $\Pi \pi h$ is bounded and continuous. Assume that Ω is a bounded subset of E. Then

$$\pi h(\Omega) \subset \operatorname{co}(0 \cup h(\Omega)).$$

So $K_{PQ}\pi h(\Omega) \subset \operatorname{co} K_{PQ}(0 \cup h(\Omega))$ since K_{PQ} is linear. Then

$$\alpha(K_{PQ}\pi h(\Omega)) \leq \alpha(K_{PQ}h(\Omega)) = 0.$$

Thus, $K_{PQ}\pi h$ is a compact map. So, πh is L-compact. □

Theorem 3.1.7. *(*Continuation Principle for L-stably solvable maps*)*.

Let $\lambda L - N : E \to F$ be a L-stably solvable map and $h : E \times [0,1] \to F$ be a continuous L-compact map such that $h(x,0) = 0$ for every $x \in E$. Let

$$S = \{x \in E : \lambda Lx - Nx = h(x,t) \mathit{\ for\ some\ } t \in [0,1]\}.$$

If $(\lambda L - N)(S)$ is bounded, then the equation $\lambda Lx - Nx = h(x,1)$ has a solution.

Proof: Suppose $r > 0$ is such that $(\lambda L - N)(S)$ is contained in the interior of B_r. Let $\varphi : F \to [0,1]$ be continuous and such that $\varphi(y) = 1$ if $y \in \overline{(\lambda L - N)(S)}$, $\varphi(y) = 0$ for all

$\|y\| \geq r$. Let $\pi : F \to B_r$ be the radial retraction. Then we have

$$\lim_{\|x\|\to\infty} \frac{\pi h(x, \varphi(\lambda L - N)(x))}{\|x\|} = 0.$$

Also, by Lemma 3.1.6, πh is a L-compact map. So, the equation

$$\lambda Lx - Nx = \pi h(x, \varphi(\lambda L - N)(x)) \tag{3.2}$$

has a solution x_0. If $\|\lambda L(x_0) - N(x_0)\| \geq r$, then by (3.2), we obtain $\lambda L(x_0) - N(x_0) = 0$. So $\|\lambda L(x_0) - N(x_0)\| < r$. This implies that $\|\pi h(x_0, \varphi(\lambda L - N)(x_0))\| < r$. Hence,

$$\pi h(x_0, \varphi(\lambda L - N)(x_0)) = h(x_0, \varphi(\lambda L - N)(x_0)),$$

so that

$$\lambda Lx_0 - Nx_0 = h(x_0, \varphi(\lambda L - N)(x_0)).$$

Thus $x_0 \in S$ and $\varphi(\lambda L - N)(x_0) = 1$, and then $\lambda Lx_0 - Nx_0 = h(x_0, 1)$. □

3.2 $(0, L, k)$-epi mappings

In this section, we will generalize $(0, k)$-epi mappings for nonlinear operators to $(0, L, k)$-epi mapping for semilinear operators. We will show that properties of $(0, k)$-epi mappings hold true for $(0, L, k)$-epi mappings. Later, the results will be used in the study of the spectrum for semilinear operators.

Definition 3.2.1. Let $L : E \supset \mathrm{dom}(L) \to F$ be a closed Fredholm operator of index zero. Let $\Omega \subset E$ be a bounded open set and $N : \overline{\Omega} \to F$ be a continuous map. Suppose $0 \notin (L - N)(\mathrm{dom}(L) \cap \partial\Omega)$. We say that $L - N$ is a $(0, L, k)$-epi mapping in $\mathrm{dom}(L) \cap \overline{\Omega}$ if for every continuous bounded map $h : \overline{\Omega} \to F$ such that

1. h is a L-k-set contraction;

2. $h(x) \equiv 0$ for $x \in \partial\Omega$,

the equation $Lx - Nx = h(x)$ has at least one solution in $\mathrm{dom}(L) \cap \Omega$.

The following property is clear.

Property 3.2.2. (Existence result)

Suppose $L - N$ *is* $(0, L, k)$-epi *for some* $k \geq 0$. *Then the equation* $Lx - Nx = 0$ *has a solution in* $\mathrm{dom}(L) \cap \Omega$.

Property 3.2.3. (Normalization property)

Suppose that L *is invertible and* N *is a continuous* L-k-*set contraction with* $k < 1$, $(L - N)(\mathrm{dom}(L) \cap \partial\Omega) \neq 0$. *If* $0 \in \Omega$, *then* $L - N$ *is* $(0, L, k_1)$-epi *for every* k_1 *with* $k + k_1 < 1$.

Proof: Let $h : \overline{\Omega} \to F$ be a L-k_1-set contraction with $k_1 + k < 1$. Suppose that h is continuous and bounded with $h(x) \equiv 0$ on $\partial\Omega$. $\ker(L) = \{0\}$ implies that $\Lambda\Pi = 0$, $Q = 0$ and $P = 0$. Hence $L^{-1} : F \to \mathrm{dom}(L)$ is a bounded linear operator and $L^{-1}(N + h) : \overline{\Omega} \to E$ is a $(k_1 + k)$-set contraction. Define $h_1 : E \to E$ by

$$h_1(x) = \begin{cases} L^{-1}(Nx + h(x)) & x \in \mathrm{dom}(L) \cap \overline{\Omega}, \\ 0 & x \notin \mathrm{dom}(L) \cap \overline{\Omega}. \end{cases}$$

Then h_1 is a $(k_1 + k)$-set contraction and $h_1(\mathrm{dom}(L) \cap \overline{\Omega})$ is bounded. Now, let

$$M = \sup\{\|h_1(x)\| : x \in \mathrm{dom}(L) \cap \overline{\Omega}\} \text{ and } B = \{x \in E : \|x\| \leq M\}.$$

Then $h_1 : B \to B$ is a $(k_1 + k)$-set contraction and $k_1 + k < 1$. So h_1 has a fixed point $x_0 \in B$. Assume that $x_0 \notin \mathrm{dom}(L) \cap \overline{\Omega}$. Then $h_1(x_0) = 0$, thus $x_0 = 0$. This contradicts $0 \in \Omega$. So

$$x_0 \in \mathrm{dom}(L) \cap \overline{\Omega} \text{ and } h_1(x_0) = L^{-1}(Nx_0 + h(x_0)) = x_0.$$

Hence

$$Lx_0 - Nx_0 = h(x_0).$$

If $x_0 \in \mathrm{dom}(L) \cap \partial\Omega$, then $h(x_0) = 0$ and $Lx_0 - Nx_0 \neq 0$, this is a contradiction. Hence $x_0 \in \mathrm{dom}(L) \cap \Omega$ and $L - N$ is $(0, L, k_1)$-epi on $\mathrm{dom}(L) \cap \overline{\Omega}$. $\square$

Property 3.2.4. (Localization property)

If $L - N : \mathrm{dom}(L) \cap \overline{\Omega} \to F$ *is* $(0, L, k)$-epi *and* $(L - N)^{-1}(0) \subset \mathrm{dom}(L) \cap \Omega_1$, *where* $\Omega_1 \subset \Omega$ *is an open set. Then* $L - N$ *restricted to* $\mathrm{dom}(L) \cap \overline{\Omega_1}$ *is a* $(0, L, k)$-epi *map.*

Proof: Let $h : \overline{\Omega_1} \to F$ be a L-k-set contraction with $h(x) \equiv 0$ on $\partial\Omega_1$. Define $h_1 : E \to F$ to be the following L-k-set contraction

$$h_1(x) = \begin{cases} h(x) & x \in \overline{\Omega_1}, \\ 0 & x \notin \overline{\Omega_1}. \end{cases}$$

Let $h_2 = h_1|_{\overline{\Omega}}$. Then h_2 is a L-k-set contraction with $h_2(x) \equiv 0$ on $\partial\Omega$. So

$$Lx - Nx = h_2(x)$$

has a solution $x_0 \in \mathrm{dom}(L) \cap \Omega$. Since $(L - N)^{-1}(0) \subset \mathrm{dom}(L) \cap \Omega_1$, $x_0 \in \mathrm{dom}(L) \cap \Omega_1$. Hence $L - N$ is a $(0, L, k)$-epi map on $\mathrm{dom}(L) \cap \overline{\Omega_1}$. □

Property 3.2.5. (Homotopy property)

Let $L - N : \mathrm{dom}(L) \cap \overline{\Omega} \to F$ *be* $(0, L, k)$-epi *and* $h : [0, 1] \times \overline{\Omega} \to F$ *be a continuous* L-k_1*-set contraction,* $0 \le k_1 \le k < 1$ *and* $h(0, x) = 0$ *for all* $x \in \overline{\Omega}$. *Furthermore, let* $Lx - Nx + h(t, x) \neq 0$ *for all* $x \in \mathrm{dom}(L) \cap \partial\Omega$ *and all* $t \in [0, 1]$. *Then* $Lx - Nx + h(1, x) : \mathrm{dom}(L) \cap \overline{\Omega} \to F$ *is* $(0, L, k - k_1)$-epi.

Proof: Let $g : \overline{\Omega} \to F$ be a continuous L-$(k - k_1)$-set contraction with $g(x) \equiv 0$ on $\partial\Omega$. Let

$$S = \{x \in \mathrm{dom}(L) \cap \overline{\Omega}, Lx - Nx = g(x) - h(t, x), \text{ for some } t \in [0, 1]\}.$$

Then S is bounded. We will prove that S is closed. Suppose that $x_n \in S$, $x_n \to x_0$ $(n \to \infty)$. Let $t_n \in [0, 1]$ satisfy

$$Lx_n - Nx_n = g(x_n) - h(t_n, x_n).$$

$\{t_n\}$ has a convergent subsequence $t_{n_k} \to t_0$. Since N, g, h are continuous operators, we have

$$L(x_{n_k}) = N(x_{n_k}) + g(x_{n_k}) - h(t_{n_k}, x_{n_k}) \to N(x_0) + g(x_0) - h(t_0, x_0), (n_k \to \infty).$$

L is a closed operator implies that $x_0 \in \text{dom}(L)$ and

$$N(x_0) + g(x_0) - h(t_0, x_0) = L(x_0).$$

Hence $x_0 \in S$ and S is closed. Since g is a L-k-set contraction, there exists $x' \in \text{dom}(L) \cap \Omega$ such that

$$Lx' - Nx' = g(x'),$$

so $x' \in S$ and S is not empty. Moreover $S \cap \text{dom}(L) \cap \partial\Omega = \emptyset$ implies that $S \cap \partial\Omega = \emptyset$. By Urysohn's Lemma, there exists continuous function $0 \leq \phi \leq 1$, such that

$$\phi(x) = \begin{cases} 1 & x \in S, \\ 0 & x \in \partial\Omega. \end{cases}$$

Let

$$\overline{h}(x) = g(x) - h(\phi(x), x), \; x \in \overline{\Omega},$$

then $\overline{h}$ is a L-k-set contraction, also $\overline{h}(x) \equiv 0$ on $\partial\Omega$. Thus the equation

$$Lx - Nx = \overline{h}(x)$$

has a solution $x'' \in \text{dom}(L) \cap \Omega$. However, $x'' \in S$, $\phi(x'') = 1$, so

$$L(x'') - N(x'') + h(1, x'') = g(x''), \; x'' \in \text{dom}(L) \cap \Omega.$$

Hence $Lx - Nx + h(1, x)$ is $(0, L, k - k_1)$-epi. □

3.3 Regular maps and the spectrum of semilinear operators

In this section, we will introduce the spectrum for semilinear operators $L - N$ by first defining regular maps. In the following, ω, m and ν are as defined in Chapters 1 and 2.

Definition 3.3.1. For $\lambda \in \mathbb{C}$, the operator $\lambda L - N$ is said to be regular if $\omega(f_\lambda) > 0$, $m(f_\lambda) > 0$ and $\nu(f_\lambda) > 0$. The resolvent set of (L, N) is defined by

$$\rho(L, N) = \{\lambda \in \mathbb{C} : \lambda L - N \text{ is regular}\}.$$

The spectrum is the set $\sigma(L, N) = \mathbb{C} \setminus \rho(L, N)$.

Relative to the measure of solvability of f at 0, $\nu(f)$, which was defined in Chapter 2, we have the following definition.

Definition 3.3.2. Let $r > 0$ and

$$\nu_{Lr}(\lambda L - N, 0) = \inf\{k \geq 0 : \text{ there exists a } L\text{-}k\text{-set contraction } g : B_r \to F, \text{ with}$$

$$g \equiv 0 \text{ on } \partial B_r \text{ s.t. } f(x) = g(x) \text{ has no solutions in } B_r \cap \mathrm{dom}(L)\}.$$

Let

$$\nu_L(\lambda L - N) = \inf\{\nu_{Lr}(\lambda L - N, 0), r > 0\}.$$

We will call $\nu_L(\lambda L - N)$ the measure of solvability of $\lambda L - N$ at 0.

It is clear that $\nu_L(\lambda L - N) > 0$ if and only if there exists $\varepsilon > 0$, such that $\lambda L - N$ is $(0, L, \varepsilon)$-epi on $\mathrm{dom}(L) \cap B_r$ for every $r > 0$. Now, we establish some properties of regular mappings.

Proposition 3.3.3. *If $\lambda L - N$ is regular, then $\omega(\lambda L - N) > 0$, $m(\lambda L - N) > 0$ and $\nu_L(\lambda L - N) > 0$. Suppose that L is a bounded linear operator, then $\lambda L - N$ is regular if and only if the above conditions are satisfied.*

Proof: Firstly, we have

$$\omega(f_\lambda) = \omega(K_P(\lambda L) - K_P(I - Q)N).$$

By Proposition 1.2.5,

$$\omega(K_P)\omega(\lambda L - N) \leq \omega(f_\lambda) \leq \alpha(K_P)\omega(\lambda L - N).$$

Since $\alpha(K_P) \leq \|K_P\| < +\infty$, if $\omega(f_\lambda) > 0$, then $\omega(\lambda L - N) > 0$. If L is bounded, then $\mathrm{dom}(L) = E$, $\omega(K_P) > 0$. Hence $\omega(\lambda L - N) > 0$ implies $\omega(f_\lambda) > 0$.

Next, $f_\lambda = H^{-1}(\lambda H(I-P) - N)$ and $H(I-P) = L$, (recall $H = L + JP$). So,

$$\|H^{-1}(\lambda L - N)x\| = \|f_\lambda(x)\| \geq m(f_\lambda)\|x\|.$$

Since H^{-1} is bounded,

$$\|H^{-1}(\lambda L - N)x\| \leq \|H^{-1}\|\|(\lambda L - N)x\|.$$

Hence

$$\|(\lambda L - N)x\| \geq \frac{m(f_\lambda)}{\|H^{-1}\|}\|x\|.$$

Thus $m(f_\lambda) > 0$ implies $m(\lambda L - N) > 0$. When L is bounded, we have

$$\|H\|\|f_\lambda(x)\| \geq \|(\lambda L - N)x\| \geq m(\lambda L - N)\|x\|.$$

So, $m(\lambda L - N) > 0$ implies that $m(f_\lambda) > 0$.

Now suppose $\nu(f_\lambda) > 0$. Thus there exists $\varepsilon > 0$ such that f_λ is $(0, \varepsilon)$-epi on every B_r with $r > 0$. Let $h : B_r \to F$ be a bounded continuous L-ε-set contraction with $h(x) \equiv 0$ on ∂B_r. Then $(\Lambda\Pi + K_{PQ})h : B_r \to E$ is a continuous operator.

$$\alpha((\Lambda\Pi + K_{PQ})h) = \alpha(K_{PQ}h) \leq \varepsilon,$$

$(\Lambda\Pi + K_{PQ})h(x) \equiv 0$ on ∂B_r. By the definition of ε-epi map, the equation

$$f_\lambda(x) = (\Lambda\Pi + K_{PQ})h(x)$$

has a solution $x_0 \in B_r$, $\|x_0\| < r$. So

$$\lambda(I-P)x_0 = (\Lambda\Pi + K_{PQ})(h(x_0) + N(x_0)) = H^{-1}(h(x_0) + N(x_0)).$$

Thus $\lambda Lx_0 - Nx_0 = h(x_0)$. Since $H^{-1} : F \to \mathrm{dom}(L)$, we have

$$\lambda(I-P)x_0 \in H^{-1}F \subset \mathrm{dom}(L),$$

so $x_0 \in \mathrm{dom}(L) \cap \{x : \|x\| < r\}$. We obtain that $\lambda L - N$ is $(0, L, \varepsilon)$-epi on $\mathrm{dom}(L) \cap B_r$, thus $\nu_L(\lambda L - N) > 0$. Suppose that L is bounded and there exists $\varepsilon > 0$ such that $\lambda L - N$ is $(0, L, \varepsilon)$-epi on $\mathrm{dom}(L) \cap B_r$. Let $h : B_r \to F$ be a continuous map with $\alpha(h) \leq \varepsilon$ and $h(x) \equiv 0$ on ∂B_r. Then $\alpha H^{-1} H h \leq \varepsilon$. So Hh is a L-ε-set contraction. By the definition, the equation

$$\lambda L x - N x = H h(x)$$

has a solution $x_0 \in \mathrm{dom}(L) \cap O_r = O_r$. By Lemma 3.1.3, $f_\lambda(x_0) = h(x_0)$. Hence f_λ is $(0, \varepsilon)$-epi on B_r. We have completed the proof of the Proposition. □

Proposition 3.3.4. *Let $\lambda \in \mathbb{C}$, $\lambda \notin \Sigma(L, N)$ and $\nu_L(\lambda L - N) > 0$. Then $\lambda L - N$ is onto.*

Proof: Let $y \in F$, $h(x, t) = -ty$ and

$$S = \{x \in \mathrm{dom}(L) \cap E, \lambda L x - N x = ty, \text{ for some } t \in [0, 1]\}.$$

Then S is bounded. Otherwise there exist $x_n \in S$, $\|x_n\| \to \infty$. Let $t_n \in [0, 1]$ be such that

$$\lambda L x_n - N x_n = t_n y.$$

By Lemma 3.1.3, we have

$$\lambda (I - P) x_n - (\Lambda \Pi + K_{PQ}) N(x_n) = t_n (\Lambda \Pi + K_{PQ}) y.$$

Then

$$\frac{\|\lambda (I - P) x_n - (\Lambda \Pi + K_{PQ}) N(x_n)\|}{\|x_n\|} = \frac{\|t_n (\Lambda \Pi + K_{PQ}) y\|}{\|x_n\|} \to 0 \ (n \to \infty).$$

This contradicts $\lambda \notin \Sigma(L, N)$. Let $R > 0$ be such that $S \subset B_R$. Then

$$\lambda L x - N x - t y \neq 0 \text{ for } x \in \mathrm{dom}(L) \cap \partial B_R.$$

By Property 3.2.5, $\lambda L - N - y$ is $(0, L, \varepsilon)$-epi for some $\varepsilon > 0$ on $\mathrm{dom}(L) \cap B_R$. By the existence result, there exist $x_0 \in \mathrm{dom}(L) \cap B_R$ such that $\lambda L x_0 - N x_0 = y$. Thus $\lambda L - N$ is onto. □

By Proposition 3.3.4 and Proposition 3.3.3, we obtain the following result.

Theorem 3.3.5. *Assume that $\lambda \in \rho(L, N)$, then $\lambda L - N$ is onto.*

Theorem 3.3.6. *All eigenvalues of (L, N) are in the spectrum of (L, N).*

Proof: Suppose there exists $0 \neq x_0 \in \text{dom}(L)$ such that $\lambda L x_0 - N x_0 = 0$. Then by Lemma 3.1.3, $f_\lambda(x_0) = 0$. Hence $m(f_\lambda) = 0$ and $\lambda \in \sigma(L, N)$. $\square$

Now, we will prove that the spectrum of semilinear operators is closed.

Theorem 3.3.7. *$\rho(L, N)$ is an open set and $\sigma(L, N)$ is closed.*

Proof: Suppose $\lambda_1 \in \rho(L, N)$. Let λ_2 be such that

$$|\lambda_2 - \lambda_1| < \min\{\omega(f_{\lambda_1}), \nu(f_{\lambda_1}), m(f_{\lambda_1})/\|I - P\|\}.$$

Then $f_{\lambda_2} = f_{\lambda_1} + (\lambda_2 - \lambda_1)(I - P)$. So,

$$\omega(f_{\lambda_2}) \geq \omega(f_{\lambda_1}) - \alpha((\lambda_2 - \lambda_1)(I - P)) = \omega(f_{\lambda_1}) - |\lambda_2 - \lambda_1| > 0.$$

Also, we have

$$\|f_{\lambda_2}(x)\| \geq \|f_{\lambda_1}(x)\| - |\lambda_2 - \lambda_1| \|(I - P)x\| \geq (m(f_{\lambda_1}) - |\lambda_2 - \lambda_1| \|(I - P)\|)\|x\|.$$

Thus $m(f_{\lambda_2}) > 0$. Let $h(t, x) = t(\lambda_2 - \lambda_1)(I - P)x$, $t \in [0, 1]$. Then

$$\alpha(h) \leq |\lambda_2 - \lambda_1| < \nu(f_{\lambda_1})$$

and $h(0, x) = 0$ for all $x \in E$. If $f_{\lambda_1}(x) + t(\lambda_2 - \lambda_1)(I - P)x = 0$, then

$$m(f_{\lambda_1})\|x\| \leq \|f_{\lambda_1}(x)\| \leq |\lambda_2 - \lambda_1| \|I - P\| \|x\|.$$

Hence $x = 0$. Since f_{λ_1} is $(0, \varepsilon)$-epi on every ball for some $\varepsilon > 0$, by Property 1.4.7, f_{λ_2} is $(0, \varepsilon - \alpha(h))$-epi on it. Thus $\nu(f_{\lambda_2}) > 0$, $\lambda_2 L - N$ is regular, so, $\lambda_2 \in \rho(L, N)$. $\square$

In the following, we shall study some properties of regular mappings.

Proposition 3.3.8. *Let $\lambda L - N : \mathrm{dom}(L) \to F$ be a regular map, $g : E \to F$ be a L-k-set contraction with $k < \min\{\omega(f_\lambda), \nu(f_\lambda)\}$. Assume that there exists constant $0 < l < m(f_\lambda)$ such that*

$$\|(\Lambda\Pi + K_{PQ})g(x)\| \leq l\|x\| \textit{ for all } x \in E.$$

Then $\lambda L - N + g$ is regular.

Proof: It is easy to see that $f_\lambda(L, N - g) = f_\lambda + H^{-1}g$. So,

$$\omega(f_\lambda(L, N - g)) \geq \omega(f_\lambda) - \alpha(H^{-1}g) \geq \omega(f_\lambda) - k > 0.$$

Next, we have

$$\|f_\lambda(L, N - g)(x)\| \geq (m(f_\lambda) - l)\|x\|,$$

thus $m(f_\lambda(L, N - g)) > 0$. Let $h(t, x) = t(\Lambda\Pi + K_{PQ})g(x)$. Then

$$\alpha(h) \leq \alpha(K_{PQ}g) < \nu(f_\lambda).$$

If $f_\lambda(x) + h(t, x) = 0$, we have

$$m(f_\lambda)\|x\| \leq \|f_\lambda(x)\| = \|t(\Lambda\Pi + K_{PQ})g(x)\| \leq l\|x\|.$$

By our assumption, $x = 0$. Thus

$$S = \{x \in E : f_\lambda(x) + t(\Lambda\Pi + K_{PQ})g(x) = 0 \text{ for some } t \in [0, 1]\}$$

is bounded. Let $k < \varepsilon < \nu(f_\lambda)$, by Property 1.4.7, $f_\lambda(L, N - g)$ is $(0, (\varepsilon - \alpha(K_{PQ}g)))$-epi on every ball. Thus, $\nu(f_\lambda(L, N - g)) > 0$. We have proved that $\lambda L - N + g$ is regular. $\square$

In [31], the concept of regular map for $L - N$, when N is also linear, was defined. The following theorem shows that regular maps according to Definition 3.3.1 for $L - N$ when N is linear is exactly same as the definition given in [31].

Theorem 3.3.9. *Suppose N is a linear operator. Then $\lambda L - N$ is regular if and only if $\lambda L - N$ has a continuous inverse.*

Proof: If $\lambda L - N$ is regular, it is one to one, onto and there exists $m > 0$ such that $\|(\lambda L - N)(x)\| \geq m\|x\|$. So $(\lambda L - N)^{-1}$ is a continuous operator.

On the other hand, assume that $\lambda L - N$ has a continuous inverse. If there exists $x_0 \in E$ such that $f_\lambda(x_0) = 0$, then by Lemma 3.1.3, $\lambda Lx_0 - Nx_0 = 0$. Thus $x_0 = 0$, so that f_λ is one to one.

Let $x' \in E$ and $y_0 = Hx'$. Since $\lambda L - N$ is onto, there exists $x_0 \in \mathrm{dom}(L)$ such that $\lambda Lx_0 - Nx_0 = y_0$. This implies that $f_\lambda(x_0) = H^{-1}y_0 = x'$. Hence, $f_\lambda : E \to E$ is onto. Since f_λ is continuous, f_λ^{-1} is also continuous. By Proposition 3.2.1 of [16], $\omega(f_\lambda) > 0$. Let $m' > 0$ be such that $\|f_\lambda^{-1}(x)\| \leq m'\|x\|$ for every $x \in E$. Then $\|f_\lambda(x)\| \geq (1/m')\|x\|$. Thus $m(f_\lambda) > 0$.

Furthermore, for every bounded subset $S \subset F$, let $A = f_\lambda^{-1}(S)$, then A is a bounded subset of E.

$$\omega(f_\lambda)\alpha(A) \leq \alpha(f_\lambda(A)) \Longrightarrow \alpha(f_\lambda^{-1}(S)) \leq (1/\omega(f_\lambda))\alpha(S).$$

Hence $f_\lambda : O_r \to F$ is continuous, injective and $1/\omega(f_\lambda)$-proper (where $O_r = \{x \in E : \|x\| < r\}$). Since f_λ^{-1} is continuous, $f_\lambda(O_r)$ is an open set. By Theorem 2.3 of [65], f_λ is $(0, k)$-epi for $0 < k < \omega(f)$. Hence $\nu(f_\lambda) > 0$. By the above arguments, $\lambda L - N$ is regular. □

Corollary 3.3.10. *If N is a linear operator and $L+N$ is regular, then $L+N$ is $(0, L, \varepsilon)$-epi for all $\varepsilon < \omega(f)$.*

Remark 3.3.11. From the above theorem, we obtain that if N is a linear operator, $\lambda \in \rho(L, N)$ if and only if λ is a regular value for (L, N), as defined in [31].

Proposition 3.3.12. *Let $\mu L - N : E \to F$ be regular and $h : E \to F$ be a mapping such that there exists $r > 0$, h is a L-k-set contraction on B_r and $\|h(x)\| \leq l\|x\|$ $(l > 0)$ for all $x \in B_r$. Then there exists $\varepsilon > 0$ such that the equation*

$$\mu Lx - Nx = \lambda h(x)$$

has a solution $x_0 \in E$ for all $\lambda \in \mathbb{C}$ with $|\lambda| < \varepsilon$.

Proof: Let $\pi : E \to B_r$ be the radial retraction of E onto B_r. Let $g(x) = h\pi(x)$. Then for $\lambda \in \mathbb{C}$, we have

$$\alpha((\Lambda\Pi + K_{PQ})\lambda g) = |\lambda|\alpha((\Lambda\Pi + K_{PQ})h\pi) \leq |\lambda|k\alpha(\pi) \leq |\lambda|k.$$

For $x \in E$,

$$\begin{aligned} \|(\Lambda\Pi + K_{PQ})\lambda g(x)\| &= \|H^{-1}\lambda h\pi(x)\| \\ &\leq |\lambda|\|H^{-1}\|\|h\pi(x)\| \\ &\leq \begin{cases} l|\lambda|\|H^{-1}\|\|x\| & \|x\| \leq r \\ l|\lambda|\|H^{-1}\|r & \|x\| > r \end{cases} \\ &\leq l|\lambda|\|H^{-1}\|\|x\|. \end{aligned}$$

Let $\varepsilon > 0$ be such that

$$\varepsilon k < \min\{\omega(f_\mu), \nu(f_\mu)\}, \ \varepsilon l\|H^{-1}\| < m(f_\mu), \ \varepsilon l < m(\mu L - N).$$

Then for $\lambda \in \mathbb{C}$ with $|\lambda| < \varepsilon$,

$$\alpha(H^{-1}\lambda g) < \min\{\omega(f_\mu), \nu(f_\mu)\},$$

and

$$\|H^{-1}\lambda g(x)\| \leq \varepsilon l\|H^{-1}\|\|x\| < m(f_\mu)\|x\|.$$

By proposition 3.3.8, $\mu L - N - \lambda g$ is regular. So, there exists $x_0 \in E$ such that

$$\mu L x_0 - N x_0 = \lambda g(x_0) = \lambda h\pi(x_0).$$

Moreover we have

$$m(\mu L - N)\|x_0\| \leq \|\mu L x_0 - N x_0\| \leq |\lambda| l r.$$

Thus,

$$\|x_0\| \leq \frac{lr}{m(\mu L - N)}|\lambda| < \frac{\varepsilon l r}{m(\mu L - N)} < r.$$

This implies $\pi(x_0) = x_0$, so that $\mu L x_0 - N x_0 = \lambda h(x_0)$. $\square$

3.4 The decomposition of the spectrum

In this section, we shall discuss the decomposition of the spectrum. As in the earlier case (see section 1.3), we shall use following symbols:

$$\sigma_\delta(L,N) = \{\lambda : \nu(f_\lambda) = 0\}, \tag{3.3}$$

$$\sigma_m(L,N) = \{\lambda : m(f_\lambda) = 0\}, \tag{3.4}$$

$$\sigma_\omega(L,N) = \{\lambda : \omega(f_\lambda) = 0\}, \tag{3.5}$$

$$\sigma_\pi(L,N) = \sigma_m(L,N) \cup \sigma_\omega(L,N). \tag{3.6}$$

Theorem 3.4.1. *Suppose that N is a continuous L-k-set contraction, N is an odd mapping, $\lambda \in \sigma(L,N)$ with $|\lambda| > k$. Then $\lambda \in \sigma_m(L,N)$.*

Proof: Firstly we have that

$$\omega(f_\lambda) \geq |\lambda| - \alpha(H^{-1}N) = |\lambda| - k > 0.$$

Assume that $m(f_\lambda) > 0$. Then for every $x \in E$, $\|f_\lambda(x)\| \geq m(f_\lambda)\|x\|$. Let $r > 0$ and $B_r = \{x : \|x\| \leq r\}$. Suppose $g : B_r \to E$ is an ε-set contraction with $\varepsilon < |\lambda| - k$ and $g(x) \equiv 0$ on ∂B_r. We shall prove that $f_\lambda(x) = g(x)$ has a solution $x_0 \in B_r \setminus \partial B_r$. Let

$$h(x) = Px + (1/\lambda)(\Lambda\Pi + K_{PQ})Nx + g(x)/\lambda.$$

Then

$$\alpha(h) \leq (1/|\lambda|)(k + \alpha(g)) \leq (1/|\lambda|)(k + \varepsilon) < 1.$$

Hence h is a k_1-set contraction with $k_1 < 1$. For $x \in \partial B_r$, $(I-h)(x) = (1/\lambda)f_\lambda(x) \neq 0$. Also $h|_{\partial B_r}$ is odd. So, $d(I-h, O_r, 0) \neq 0$. Then there exists $x_0 \in B_r \setminus \partial B_r$ such that

$$x_0 = Px_0 + (1/\lambda)(\Lambda\Pi + K_{PQ})Nx_0 + g(x_0)/\lambda.$$

Therefore f_λ is $(0,\varepsilon)$-epi on B_r. By the definition, $\lambda \in \rho(L,N)$. This contradicts $\lambda \in \sigma(L,N)$. So, we must have $m(f_\lambda) = 0$. $\square$

Proposition 3.4.2. *Bifurcation points and asymptotic bifurcation points of* (L, N) *are in* $\sigma_m(L, N)$.

Proof: Suppose λ is a bifurcation point or an asymptotic bifurcation point of (L, N). Then there exists $(\lambda_n, x_n) \in \mathbb{R} \times E$ such that $\lambda_n \to \lambda$, $\|x_n\| \to 0$ or $\|x_n\| \to \infty$ and $\lambda_n Lx_n - Nx_n = 0$. Then

$$\lambda_n(I - P)x_n - (\Lambda\Pi + K_{PQ})Nx_n = 0.$$

Hence

$$\frac{\lambda(I - P)x_n - (\Lambda\Pi + K_{PQ})Nx_n}{\|x_n\|} = \frac{(\lambda - \lambda_n)(I - P)x_n}{\|x_n\|} \to 0,$$

thus $\lambda \in \sigma_m(L, N)$. □

In [31], it was proved that if N is linear and L-compact, $\mu \in \mathbb{C}$, μ is not a regular value of (L, N), then μ is an eigenvalue of (L, N). The following result generalizes this result.

Theorem 3.4.3. *Assume that* N *is a* L-k-*set contraction,* N *is odd and positively homogeneous. If* $\lambda \in \sigma(L, N)$ *and* $|\lambda| > k$, *then* λ *is an eigenvalue of* (L, N).

Proof: By Theorem 3.4.1, we know that $\lambda \in \sigma_m(L, N)$. So, for $n \in \mathbb{N}$, there exists $x_n \in E$ such that

$$\|\lambda(I - P)x_n - H^{-1}N(x_n)\| < (1/n)\|x_n\|.$$

Since N is positively homogeneous, we have

$$\|\lambda(I - P)\frac{x_n}{\|x_n\|} - H^{-1}N\left(\frac{x_n}{\|x_n\|}\right)\| \to 0, \ (n \to \infty).$$

Let $A = \cup_{n=1}^{\infty}\{x_n/\|x_n\|\}$. Then A is bounded and

$$\omega(f_\lambda)\alpha(A) \leq \alpha(f_\lambda(A)) = 0.$$

Since N is a L-k-set contraction and $|\lambda| > k$, $\omega(f_\lambda) > 0$. This implies that $\alpha(A) = 0$. Thus A has a convergent subsequence. Assume that $x_n/\|x_n\| \to x_0$, then $\|x_0\| = 1$ and

$$\lambda(I - P)x_0 - H^{-1}N(x_0) = 0.$$

By Lemma 3.1.3, we obtain $\lambda L x_0 - N x_0 = 0$. Thus λ is an eigenvalue of (L, N). $\square$

It is known that the boundary of the spectrum of a linear operator is contained in its approximate point spectrum. In the nonlinear case, the boundary of $\sigma_{fmv}(f)$ is contained in $\sigma_\pi(f)$ (see section 1.3). Thus if $\lambda \in \partial\sigma_{fmv}(f)$, then $d(\lambda - f) = 0$ or $\omega(\lambda - f) = 0$. To prove a similar result in the semilinear case for the boundary of the spectrum, we first prove the following result.

Proposition 3.4.4. *$\sigma_m(L, N)$ and $\sigma_\omega(L, N)$ are closed sets. Let*

$$U = \{\lambda : \lambda \in (\sigma_\delta(L, N) \setminus \sigma_\pi(L, N)) \textit{ and } (\lambda L - N \textit{ is not onto })\},$$

then U is an open set.

Proof: (1) Suppose that $\lambda_n \in \sigma_m(L, N)$ and $\lambda_n \to \lambda$. We shall show that $\lambda \in \sigma_m(L, N)$. Otherwise, there exists $m > 0$ such that $\|f_\lambda(x)\| \geq m\|x\|$ for every $x \in E$. Since

$$f_{\lambda_n} = f_\lambda + (\lambda_n - \lambda)(I - P),$$

we have

$$\|f_{\lambda_n}(x)\| \geq (m - |\lambda - \lambda_n|\|I - P\|)\|x\|.$$

If $|\lambda_n - \lambda| < m/\|I - P\|$, then $\lambda_n \notin \sigma_m(L, N)$. We reach a contradiction.

(2) Suppose $\lambda_n \in \sigma_\omega(L, N)$ and $\lambda_n \to \lambda$. If $\omega(f_\lambda) > 0$, then by the following

$$\omega(f_{\lambda_n}) = \omega(f_\lambda + (\lambda_n - \lambda)(I - P)x) \geq \omega(f_\lambda) - |\lambda - \lambda_n|,$$

we obtain that $\lambda_n \notin \sigma_\omega(L, N)$ if $|\lambda - \lambda_n| < \omega(f_\lambda)$. Thus $\omega(f_\lambda) = 0$ and $\lambda \in \sigma_\omega(L, N)$.

(3) Suppose that $\lambda \in U$, then $\lambda \in (\sigma_\pi(L, N))^c$. By (1) and (2), $\sigma_\pi(L, N)$ is closed. So there exists $\delta > 0$ such that for $\mu \in \mathbb{C}$ with $|\mu - \lambda| < \delta$, $\mu \notin \sigma_\pi(L, N)$. Now assume

that there exist $\mu_n \notin U$, $\mu_n \to \lambda$. Suppose that $|\mu_n - \lambda| < \delta$, then $\mu_n \notin \sigma_\pi(L, N)$. So $\mu_n \notin \sigma_\delta(L, N) \setminus \sigma_\pi(L, N)$ implies that $\mu_n \notin \sigma_\delta(L, N)$. Then $\mu_n L - N$ is regular, hence is onto. By Lemma 3.1.3, f_{μ_n} is onto. If $\mu_n \in \sigma_\delta(L, N) \setminus \sigma_\pi(L, N)$, then f_{μ_n} is also onto because $\mu_n \notin U$.

Therefore, for $y \in E$, there exist $x_n \in E$ such that $f_{\mu_n}(x_n) = y$. Then

$$\begin{aligned}
\|y\| &= \|\mu_n(I-P)x_n - (\Lambda\Pi + K_{PQ})Nx_n\| \\
&\geq \|f_\lambda(x_n)\| - |\mu_n - \lambda|\|I-P\|\|x_n\| \\
&\geq m\|x_n\| - |\mu_n - \lambda|\|I-P\|\|x_n\|.
\end{aligned}$$

Suppose that $|\mu_n - \lambda| < m/2(\|I-P\|)$, then $\|x_n\| \leq 2\|y\|/m$. Thus $\{x_n\}_{n=1}^\infty$ is bounded. Since $\|(I-P)x_n\| \leq \|I-P\|\|x_n\|$, we have

$$f_\lambda(x_n) = y + (\lambda - \mu_n)(I-P)(x_n) \to y \ (n \to \infty).$$

Then

$$\omega(f_\lambda)\alpha\left(\cup_{n=1}^\infty\{x_n\}\right) \leq \alpha\left(f_\lambda\left(\cup_{n=1}^\infty\{x_n\}\right)\right) \leq \alpha\left(\cup_{n=1}^\infty\{f_\lambda(x_n)\}\right) = 0.$$

Since $\omega(f_\lambda) > 0$, we obtain $\alpha\left(\cup_{n=1}^\infty x_n\right) = 0$. Thus $\{x_n\}_{n=1}^\infty$ has a convergent subsequence. Let $x_{n_k} \to x_0$, then $f_\lambda(x_{n_k}) \to f_\lambda(x_0) = y$. We have shown that f_λ is onto, so $\lambda L - N$ is onto. This contradicts $\lambda \in U$. By the above argument, there exists $\delta_1 > 0$ such that $|\mu - \lambda| < \delta_1$ implies $\mu \in U$, thus U is an open set. □

The result for the boundary of the spectrum in the semilinear case is the following one.

Theorem 3.4.5. *$\partial\sigma(L, N) \subset \{\sigma_\pi(L, N)\} \cup \{\lambda : \lambda L - N$ is onto $\}$.*

Proof: Suppose that $\lambda \in \partial\sigma(L, N)$. Since $\sigma(L, N)$ is closed, we have $\lambda \in \sigma(L, N)$. Assume that $\lambda \notin \sigma_\pi(L, N)$ and $\lambda L - N$ is not onto. Then $\lambda \in U$. By Proposition 3.4.4, there exists $\delta > 0$ such that for every $\mu \in \mathbb{C}$ with $|\mu - \lambda| < \delta$, $\mu \in U$. On the other hand, there exist $\mu_n \notin \sigma(L, N)$ with $\mu_n \to \lambda$. When $|\mu_n - \lambda| < \delta$, we have $\mu_n \in U$. This contradiction proves that $\lambda \in \sigma_\pi(L, N)$ or $\lambda L - N$ is onto. □

Comparing with Proposition 1.3.3, we can prove the following theorem which gives information about the structure of $\sigma(L, N)$ when N is a L-compact map.

Theorem 3.4.6. *Let $N : E \to F$ be a continuous L-compact map defined on an infinite dimensional Banach space E. Then*

1. $\sigma_\omega(L, N) = \{0\}$ *therefore* $\sigma_\pi(L, N) = \{0\} \cup \sigma_m(L, N)$.
2. $0 \in \sigma_\delta(L, N) \cup \Sigma(L, N)$.
3. *If* $0 \notin \sigma_m(L, N)$, *then* 0 *is an interior point of* $\sigma_\delta(L, N)$.

Proof: (1) For $\lambda \in \mathbb{C}$, we have

$$\omega(f_\lambda) = \omega(\lambda(I - P) - (\Lambda\Pi + K_{PQ})N) = |\lambda|.$$

So, if $\lambda \neq 0$, we have $\lambda \notin \sigma_\omega(L, N)$.

(2) Since N is L-compact, $(\Lambda\Pi + K_{PQ})N$ is compact. Let $B_n = \{x \in E, \|x\| \leq n\}$. Then

$$f_0(E) = \cup_{n=1}^{\infty}(\Lambda\Pi + K_{PQ})N(B_n).$$

Since $\dim(E) = \infty$, $(\Lambda\Pi + K_{PQ})N(B_n)$ is compact, so $(\Lambda\Pi + K_{PQ})N(B_n)$ are nowhere dense subsets of E. E is of second category, hence $f_0(E) \neq E$. If $0 \notin \Sigma(L, N)$ and $0 \notin \sigma_\delta(L, N)$, by Proposition 3.3.4, f_0 would be onto. This proves (2).

(3) Suppose $0 \notin \sigma_m(L, N)$, then 0 is an isolated point of $\sigma_\pi(L, N)$. Hence to prove (3), it is sufficient to show that for λ sufficiently small, f_λ is not onto. Assume to the contrary that $\lambda_n \in \mathbb{C}$, $\lambda_n \to 0$, and f_{λ_n} is onto. Then for each $y \in E$, there exist $x_n \in E$ such that

$$\lambda_n(I - P)x_n - (\Lambda\Pi + K_{PQ})Nx_n = -y.$$

Hence,

$$\|y\| \geq \|(\Lambda\Pi + K_{PQ})Nx_n\| - |\lambda_n|\|I - P\|\|x_n\| \geq m(f_0)\|x_n\| - |\lambda_n|\|I - P\|\|x_n\|.$$

For $|\lambda_n|\|I-P\| < m(f_0)/2$, we have $\|x_n\| \leq 2\|y\|/m(f_0)$, so $\{x_n\}_{n=1}^{\infty}$ is bounded. It follows that

$$(\Lambda\Pi + K_{PQ})N(x_n) = \lambda_n(I-P)x_n + y \to y, (n \to \infty).$$

This shows that $y \in \bigcup_{n=1}^{\infty} \overline{(\Lambda\Pi + K_{PQ})N(B_n)}$. So $E = \bigcup_{n=1}^{\infty} \overline{(\Lambda\Pi + K_{PQ})N(B_n)}$. By Baire's theorem, $E_n = \overline{(\Lambda\Pi + K_{PQ})N(B_n)}$ has an interior point for some n. But E_n is compact, this contradicts $\dim(E) = \infty$. Thus for λ sufficiently small, $\lambda \in \sigma(L,N)$, $\lambda \notin \sigma_\pi(L,N)$. So, we have $\lambda \in \sigma_\delta(L,N)$. □

Remark 3.4.7. If N is an odd L-compact mapping, we claim that $0 \in \sigma_m(L,N)$. Otherwise by Theorem 3.4.6, 0 would be in interior point of $\sigma_\delta(L,N)$. By Theorem 3.4.1, each $0 \neq \lambda \in \sigma(L,N)$ is in $\sigma_m(L,N)$. Since $\sigma_m(L,N)$ is a closed set, we would have $0 \in \sigma_m(L,N)$. This contradicts $0 \notin \sigma_m(L,N)$.

Proposition 3.4.8. *Suppose N is positively homogeneous. Then*

$$\sigma_m(L,N) \setminus \sigma_\omega(L,N) \subset \{\lambda : \lambda \textit{ is an eigenvalue of} (L,N)\} \subset \sigma_m(L,N).$$

Proof: Let $\lambda \in \sigma_m(L,N) \setminus \sigma_\omega(L,N)$. Then there exist $\{x_n\} \subset E$, $\|x_n\| = 1$ such that

$$\|\lambda(I-P)x_n - (\Lambda\Pi + K_{PQ})N(x_n)\| < \frac{1}{n} \to 0 \ (n \to \infty).$$

So, $\omega(f_\lambda)\alpha(\cup_{n=1}^{\infty} x_n) \leq \alpha\left(\cup_{n=1}^{\infty} f_\lambda(x_n)\right) = 0$. Now $\lambda \notin \sigma_\omega(L,N)$ implies that $\omega(f_\lambda) > 0$. Hence $\alpha\left(\cup_{n=1}^{\infty} x_n\right) = 0$ and $\{x_n\}$ has a convergent subsequence. Suppose that $x_{n_k} \to x_0$, then $\|x_0\| = 1$ and $f_\lambda(x_0) = 0$. By Lemma 3.1.3, $\lambda L(x_0) - N(x_0) = 0$. Hence λ is an eigenvalue of (L,N). The other part of the proposition is clear. □

Proposition 3.4.9. $\mathbb{C} \setminus (\sigma_\pi(L,N) \cup \{\lambda : \lambda L - N \textit{ is onto}\}) \subset \sigma_\delta(L,N) \setminus \partial\sigma(L,N)$. *Therefore,* $\sigma(L,N) \setminus \partial\sigma(L,N) \subset \sigma_\delta(L,N)$.

Proof: Let $\lambda \in \mathbb{C} \backslash (\sigma_\pi(L,N) \cup \{\lambda : \lambda L - N \text{ is onto}\})$, then $\lambda \notin \sigma_\pi(L,N)$ and $\lambda L - N$ is not onto. So $\lambda \in \sigma_\delta(L,N)$. By Theorem 3.4.5, $\lambda \notin \partial\sigma(L,N)$. Thus $\lambda \in \sigma_\delta(L,N) \setminus \partial\sigma(L,N)$.
□

3.5 The spectrum of (L, N) when N is asymptotically linear or positively homogeneous

In this section, we will study the spectrum of (L, N) where N is an asymptotically linear operator or a positively homogeneous operator. We will use the following symbols. Let

$\Pi(L,N) = \{\lambda \in \mathbb{C} : \|\lambda Lx - Nx\| \geq m\|x\| \text{ for some } m > 0 \text{ and all } x \in \operatorname{dom} L\}$.

$\Phi_+(L,N) = \{\lambda \in \mathbb{C} :$ there exists a linear operator $T : E \to F$ such that

$$\lambda L - N = T \dot{+} R, \tag{3.7}$$

where $\ker(T) = \{0\}$ and $\operatorname{im}(T)$ is closed, $\|R(x)\|/\|x\| \to 0$ as $\|x\| \to \infty\}$.

$\Phi_0(L,N) = \{\lambda \in \mathbb{C}$: there exists a linear operator $T : E \to F$ such that equation (3.7) is satisfied, where $\ker(T) = \{0\}$ and $\operatorname{im}(T)$ is closed, $\|R(x)\|/\|x\| \to 0$ as $\|x\| \to 0\}$.

Remark 3.5.1. When $T : D(T) \subset H \to H$ is a linear operator, in a Hilbert space H, the field of regularity $\Pi(T)$ of T is defined (see [10]) to be the set of values $\lambda \in \mathbb{C}$, for which there exists a positive constant k such that

$$\|(\lambda I - T)(u)\| \geq k\|u\| \text{ for all } u \in \operatorname{dom}(T).$$

According to this definition, $\Pi(L,N)$ can be called the field of regularity of (L,N).

Lemma 3.5.2. *1. If N is an asymptotically linear operator, then $\Pi(L,N) \subset \Phi_+(L,N)$.*

2. If N is differentiable at 0, then $\Pi(L,N) \subset \Phi_0(L,N)$.

In the proof of the above lemma, we will use the concept of reduced minimum modulus of a linear operator. Let $T : D(T) \subset E \to F$ be a bounded linear operator. $\gamma(T)$ is the greatest number γ such that

$$||Tu|| \geq \gamma \operatorname{dist}(u, \ker(T)) \text{ for all } u \in D(T).$$

$\gamma(T)$ is called the reduced minimum modulus of T (see [41] p.231). If $\ker(T) = 0$, $\gamma(T)$ is equal to the minimum modulus of T, which is defined as $\inf ||Tu||/||u||$ for $0 \neq u \in D(T)$. The reduced minimum modulus is useful in proving that T has a closed range by the following theorem.

Theorem 3.5.3. ([41] p.231) *T has closed range if and only if $\gamma(T) > 0$.*

Proof of lemma 3.5.2:

(1) Suppose $\lambda \in \Pi(L, N)$ and $m > 0$ with $||(\lambda L - N)x|| \geq m||x||$ for all $x \in E$. Since N is asymptotically linear, we can write

$$\lambda L - N = T + R,$$

where T is linear and $||R(x)||/||x|| \to 0$ as $||x|| \to \infty$. Assume that $x_0 \in E$ and $Tx_0 = 0$. If $x_0 \neq 0$, then

$$\frac{||R(nx_0)||}{||nx_0||} \geq m.$$

This contradicts $||R(x)||/||x|| \to 0$ as $||x|| \to \infty$. Hence $\ker(T) = 0$. For every $y_0 \in E$ with $||y_0|| = 1$, we have

$$||Tny_0 + R(ny_0)|| \geq mn, \text{ so } ||Ty_0 + R(ny_0)/n|| \geq m.$$

Let $n \to \infty$, we obtain $||T(y_0)|| \geq m$. Thus the reduced minimum modulus of T is positive, so by Theorem 3.5.3, $\operatorname{im}(T)$ is closed. Hence $\lambda \in \Phi_+(L, N)$.

(2)Follows by exactly similar arguments to case (1). □

By using the above lemma, we can prove the following result.

Proposition 3.5.4. *Let $L : E \to F$ be a Fredholm operator of index 0. Suppose N is an asymptotically linear operator and is a L-k-set contraction. Assume that*

$$N = l - R,$$

where l is linear, R is compact and $\|R(x)\|/\|x\| \to 0$ as $\|x\| \to \infty$. If $\lambda \in \sigma(L, N)$ and $|\lambda| > k$, then $\lambda \in \sigma_m(L, N)$.

Proof: Since $|\lambda| > k$, $\omega(\lambda L - N) > 0$, so $\lambda \notin \sigma_\omega(L, N)$. Assume that $\lambda \notin \sigma_m(L, N)$, then $\lambda \in \Pi(L, N)$. By the proof of (1) of Lemma 3.5.2, $\lambda \in \Phi_+(L, N)$ and

$$\lambda L - N = T + R,$$

where $T = \lambda L - l : E \to F$ is a linear operator, $\ker(T) = \{0\}$ and $\mathrm{im}(T)$ is closed. By Lemma 3.1.3,

$$\lambda(I - P) - H^{-1}N = H^{-1}T + H^{-1}R.$$

$H^{-1}T : E \to E$ is a linear isomorphism, so $H^{-1}T$ is $(0, \varepsilon)$-epi for some $0 < \varepsilon < \frac{1}{\|H^{-1}T\|}$ (see [65]). Let

$$U = \{x : H^{-1}Tx + tH^{-1}Rx = 0, t \in [0, 1]\}.$$

Then U is bounded. Otherwise there would exist $x_n \in E$ with $\|x_n\| \to \infty$, such that

$$H^{-1}T(x_n/\|x_n\|) = -t_n H^{-1}(R(x_n)/\|x_n\|) \to 0 \text{ as } (n \to \infty).$$

Thus $T(x_n/\|x_n\|) \to 0$ as $n \to \infty$. This contradicts the fact that $\mathrm{im}(T)$ is closed. Hence for r_1 sufficiently large, if $x \in E$ with $\|x\| = r_1$, we have

$$H^{-1}Tx + tH^{-1}Rx \neq 0.$$

This ensures that $H^{-1}T + H^{-1}R$ is $(0, \varepsilon)$-epi on B_{r_1}. Since

$$(H^{-1}T + H^{-1}R)^{-1}\{0\} = f_\lambda^{-1}\{0\} = \{0\},$$

we have $H^{-1}T + H^{-1}R$ is $(0, \varepsilon)$-epi on every ball B_r with $r > 0$. By the definition, $\lambda \in \rho(L, N)$. This contradiction shows that $\lambda \in \sigma_m(L, N)$. □

Corollary 3.5.5. *Suppose that N is linear and is a L-k-set contraction. If $\lambda \in \sigma(L,N)$ and $|\lambda| > k$, then λ is an eigenvalue of (L,N).*

Remark 3.5.6. The above corollary extends Theorem 5.3.3 of [70], where L is the identity map.

Proposition 3.5.7. *Suppose that T is a positively homogeneous L-k-set contraction. $\lambda \in \sigma(L,T)$ with $|\lambda| > k$. Then there exists $t \in (0,1]$, such that λ/t is an eigenvalue of $(L,(I-Q)T)$.*

Proof: Assume that $m(f_\lambda) = 0$. Then there exist $\{x_n\} \in E$, such that

$$\|\lambda(I-P)\left(\frac{x_n}{\|x_n\|}\right) - (\Lambda\Pi + K_{PQ})T\left(\frac{x_n}{\|x_n\|}\right)\| < \frac{1}{n} \to 0,\ (n \to \infty).$$

Since $|\lambda| > k$, $\omega(f_\lambda) > 0$. So $\{x_n/\|x_n\|\}$ has a convergent subsequence. Assume that $\{x_n/\|x_n\|\} \to x_0$, then $f_\lambda(x_0) = 0$. Hence

$$\lambda L x_0 = T x_0 = QTx_0 + (I-Q)Tx_0.$$

This implies that $\lambda L x_0 = (I-Q)Tx_0$. Thus λ is an eigenvalue of $(L,(I-Q)T)$. In this case $t = 1$.

Now suppose that $m(f_\lambda) > 0$. Let

$S = \{x \in E : \lambda x - t\lambda Px - t(\Lambda\Pi + K_{PQ})Tx = 0,\ t \in [0,1]\}. We have the following two cases.$

(1)There exist $x_n \in S$ with $\|x_n\| \to \infty$. Let $y_n = x_n/\|x_n\|$, then

$$\lambda(y_n) - t_n\lambda P(y_n) - t_n(\Lambda\Pi + K_{PQ})T(y_n) = 0,$$

where $t_n \in [0,1]$. Suppose that $t_n \to t_0 \in [0,1]$, then

$$\begin{aligned} &\lambda(y_n) - t_0\lambda P(y_n) - t_0(\Lambda\Pi + K_{PQ})T(y_n) \\ = \;& (t_n - t_0)\lambda P(y_n) + (t_n - t_0)(\Lambda\Pi + K_{PQ})T(y_n). \end{aligned}$$

So

$$\begin{aligned}
&\alpha(\{y_n\}_{n=1}^{\infty})\omega(\lambda I - t_0\lambda P - t_0(\Lambda\Pi + K_{PQ})T) \\
\leq\ & \alpha y_n - t_0\lambda P y_n - t_0(\Lambda\Pi + K_{PQ})Ty_n = 0.
\end{aligned}$$

Since $|\lambda| > k$, $\omega(\lambda I - t_0\lambda P - t_0(\Lambda\Pi + K_{PQ})T) > 0$. Thus $\alpha(\{y_n\}_{n=1}^{\infty}) = 0$. So $\{y_n\}_{n=1}^{\infty}$ has a convergent subsequence. Suppose that $y_n \to x_0$. Then we have

$$\lambda x_0 - t_0\lambda P x_0 - t_0(\Lambda\Pi + K_{PQ})Tx_0 = 0.$$

Since $\|x_0\| = 1$, $t_0 \neq 0$. Then

$$\lambda(I - P)x_0 - t_0 K_{PQ}Tx_0 = t_0\lambda P x_0 - \lambda P x_0 + t_0\Lambda\Pi T x_0 \in \ker(L).$$

But $\lambda(I - P)x_0 - t_0 K_{PQ} T x_0 \in E_1 \cap \mathrm{dom}(L)$, so we have

$$\lambda(I - P)x_0 = t_0 L_P^{-1}(I - Q)Tx_0.$$

Thus

$$(\lambda/t_0)Lx_0 - (I - Q)Tx_0 = 0,$$

and λ/t_0 is an eigenvalue of $(L, (I - Q)T)$.

Now assume that there exists $R > 0$ such that $S \subset B_R \backslash \partial B_R$. Then for every $x \in \partial B_R$ we have

$$\lambda x - t\lambda P(x) - t(\Lambda\Pi + K_{PQ})T(x) \neq 0, \ t \in [0, 1].$$

By Property 1.4.7, $\lambda I - \lambda P - (\Lambda\Pi + K_{PQ})T$ is $(0, \varepsilon)$-epi for some $\varepsilon > 0$ on B_R. By our assumption, $m(f_\lambda) > 0$. Thus f_λ is $(0, \varepsilon)$-epi on every B_r with $r > 0$. Hence $\nu(f_\lambda) > 0$. This shows that f_λ is regular and $\lambda \in \rho(L, T)$, which contradicts $\lambda \in \sigma(L, T)$. The above arguments complete the proof. □

The following corollary is obvious.

Corollary 3.5.8. *Suppose T is a positively homogeneous L-k-set contraction. If the eigenvalues of $(L, (I - Q)T)$ are bounded, then $\sigma(L, T)$ is bounded.*

Proposition 3.5.9. *Suppose $T = B + R$, where B is a linear operator, either $\frac{\|R(x)\|}{\|x\|} \to 0$ as $\|x\| \to \infty$ or as $\|x\| \to 0$. If λ is an eigenvalue of (L, B), then $\lambda \in \sigma(L, T)$.*

Proof: Let $x_0 \in E$, $x_0 \neq 0$ be such that $\lambda L x_0 - B x_0 = 0$. If $\|R(x)\|/\|x\| \to 0$ as $\|x\| \to \infty$, then

$$\frac{\lambda L(nx_0) - T(nx_0)}{n} \to 0.$$

Thus $m(\lambda L, T) = 0$, and $\lambda \in \sigma(L, T)$. If $\|R(x)\|/\|x\| \to 0$ as $\|x\| \to 0$, then

$$\frac{\lambda L(x_0/n) - T(x_0/n)}{1/n} \to 0, \ (n \to \infty).$$

Hence $\lambda \in \sigma(L, T)$. □

3.6 Applications of the spectral theory

By applying the spectral theory for semilinear operators, we can extend some existence results for semilinear operator equations. The first theorem in this section generalizes Corollary 1 in [45], which has been widely applied to the study of differential equations. In [45], the mapping A was assumed to be linear and L-compact, and N was L-compact.

Theorem 3.6.1. *Let $A : E \to F$ be an odd and positively homogeneous operator, which is a L-k_1-set contraction on the unit closed ball of E and such that $\ker(L - A) = \{0\}$. If $0 \in \Omega$ and $N : \Omega \to F$ is a L-k_2-set contraction with*

$$Lx \neq (1-t)Ax + tNx \text{ for all } t \in [0,1], \ x \in \mathrm{dom}(L) \cap \partial\Omega,$$

then the equation

$$Lx = Nx$$

has at least one solution in $\mathrm{dom}(L) \cap \overline{\Omega}$ provided that $2k_1 + k_2 < 1$.

Proof: Since $\ker(L - A) = \{0\}$, 1 is not an eigenvalue of (L, A). By Theorem 3.4.3, $1 \in \rho(L, A)$. Theorem 3.4.1 and Remark 2.2.5 imply that $f_1 = (I - P) - (\Lambda\Pi + K_{PQ})A$

is $(0,\varepsilon)$ -epi on $\overline{\Omega}$ for every $0 \le \varepsilon < 1-k_1$. So by Proposition 3.3.3, $L-A$ is $(0,L,\varepsilon)$-epi on $\overline{\Omega}$ for every $0 \le \varepsilon < 1-k_1$. Let $h:[0,1]\times\overline{\Omega}\to F$ be defined by

$$h(t,x) = tAx - tNx,\ t\in[0,1].$$

Then $h(0,x)=0$, h is a L-(k_1+k_2)-set contraction and for all $x\in \mathrm{dom}(L)\cap\partial\Omega$, $t\in[0,1]$,

$$Lx - Ax + h(t,x) = Lx-(1-t)Ax - tNx \neq 0.$$

By the Property 3.2.5, $L-N$ is $(0,L,\varepsilon)$-epi for $0<\varepsilon<1-2k_1-k_2$. So there exists $x_0\in\Omega$ such that $Lx_0 = Nx_0$. □

Suppose that $T:E\to F$ is a k-set contraction and is asymptotically linear. It was proved in Lemma XI.3 of [31] that if $0\le k\le\omega(L)$ and B is the asymptotic derivative of T, and $\mathrm{im}(T-B)\subset\mathrm{im}(L-B)$, then $Lx-Tx=0$ has a solution. In the following, by using a different method, we obtain a different condition for the existence of a solution of equation $\lambda Lx - Tx = 0$.

Theorem 3.6.2. *Suppose that T is an asymptotically linear operator and a k-set contraction with constant k, B is the asymptotic derivative of T. Let $\lambda\in\mathbb{C}$ with $|\lambda| > 3k/\omega(L)$. Then $\lambda Lx - Tx = 0$ has a solution provided that λ is not an eigenvalue of (L,B).*

Proof: Suppose that λ is not an eigenvalue of (L,B). Since T is a k-set contraction, B is also a k-set contraction with $\alpha(B)=k<|\lambda|\omega(L)$ [66]. So $\lambda L - B$ is a Fredholm operator of index zero [31]. Let $P:E\to\ker(\lambda L-B)$ be the projector, $J:F_1\to\ker(\lambda L - B)$ be the linear isomorphism, where F_1 is a subspace of F with $F_1\oplus\mathrm{im}(\lambda L-B)=F$. Suppose that $T=B+R$ and let

$$S=\{x:\lambda Lx - Bx - J^{-1}Px + t[J^{-1}Px - R(x)]=0, t\in[0,1]\}.$$

Case (1). Assume that there exist $x_n\in S$ with $\|x_n\|\to\infty$. Let $y_n = x_n/\|x_n\|$ and $t_n\in[0,1]$ be such that

$$\lambda Ly_n - By_n - J^{-1}Py_n + t_nJ^{-1}Py_n \to 0,\ (n\to\infty).$$

Since $J^{-1}P$ is compact, we can assume that

$$J^{-1}Py_n - t_nJ^{-1}Py_n \to y_0 \in F_1, \ (n \to \infty).$$

Then

$$\lambda Ly_n - By_n \to y_0, \ (n \to \infty).$$

Since $|\lambda| > \alpha(B)/\omega(L)$, we obtain $\omega(\lambda L - B) > 0$ and $\alpha(\{y_n\}_{n=1}^{\infty}) = 0$. Suppose that $y_{n_k} \to x_0$. Then $\|x_0\| = 1$ and

$$\lambda Ly_{n_k} \to y_0 + B(x_0).$$

Since L is a closed operator, $\lambda Lx_0 - Bx_0 = y_0 \in F_1$. This ensures that $\lambda Lx_0 - Bx_0 = 0$. Hence λ is an eigenvalue of (L, B). This contradicts our assumption.

Case (2). Suppose that there exists $r > 0$ such that for all $x \in \partial B_r$ we have

$$\lambda Lx - Bx - J^{-1}Px + t[J^{-1}Px - Rx] \neq 0, \ t \in [0,1].$$

Since $\lambda L - B - J^{-1}P$ is one to one and $J^{-1}P$ is a linear compact operator, by Theorem 3.4.3, $1 \in \rho(\lambda L - B, J^{-1}P)$. Thus $\lambda L - B - J^{-1}P$ is regular. Let $L_1 = \lambda L - B$, and

$$f_1(x) = (I - P)x - (\Lambda\Pi + K_{PQ})J^{-1}Px.$$

By Theorem 3.3.9, f_1 is $(0, \varepsilon)$-epi for any $\varepsilon < \omega(f_1) = 1$. Also by Proposition 3.3.3, $L_1 - J^{-1}P$ is $(0, L_1, \varepsilon)$-epi for every $\varepsilon < 1$. Let

$$H(x,t) = t[J^{-1}Px - R(x)],$$

and note that $\alpha(H) \leq \alpha(R)$. Since $|\lambda| > 3k/\omega(L)$, we have

$$\alpha(R) = \alpha(T - B) \leq 2k < |\lambda|\omega(L) - k \leq |\lambda|\omega(L) - \alpha(B) \leq \omega(\lambda L - B) = \omega(L_1).$$

Hence H is a L_1-k-set contraction with $k = \alpha(R)/\omega(L_1) < 1$ [31]. Now suppose that $\alpha(R)/\omega(L_1) < \varepsilon_1 < 1$, then $L_1 - J^{-1}P$ is $(0, L_1, \varepsilon_1)$-epi . Again by Property 3.2.5, we obtain that

$$L_1x - J^{-1}Px + H(x,1) = \lambda L - T$$

is $(0, L_1, \varepsilon)$-epi for some $\varepsilon > 0$ on B_r. Hence there exists $x_0 \in E$ such that $\lambda Lx_0 - Tx_0 = 0$. We are done. □

Corollary 3.6.3. *In Theorem 3.6.2, if* $T(0) \neq 0$*, then either* λ *is an eigenvalue of* (L, B) *or* λ *is an eigenvalue of* (L, T).

Theorem 3.6.4. *Let* $N : E \to F$ *be L-compact and* $A : E \to F$ *be an odd, positively homogeneous operator which is L-compact. Suppose that*

1. $\ker(L + A) = \{0\}$.

2. $S = \{Ax - Nx, x \in E\}$ *is bounded.*

Then $L + N$ *maps* $\mathrm{dom}(L)$ *onto* F.

Proof: Since $\ker(L + A) = \{0\}$, 1 is not an eigenvalue of (L, A). By Theorem 3.4.3, $1 \in \rho(L, A)$. Thus there exists $\varepsilon > 0$ such that $L + A$ is $(0, L, \varepsilon)$-epi on every ball B_r and $m(f_1(L, A)) > 0$. Let $h : E \to F$ be a bounded, continuous L-compact map, and assume that the support of h is bounded. Then there exists a ball $B_r \supset \mathrm{supp}(h)$. So $h(x) = 0$ for $x \in \partial B_r$. By the definition of $(0, L, \varepsilon)$-epi mappings, the equation $Lx + Ax = h(x)$ has a solution $x \in \mathrm{dom}(L) \cap B_r$. By Theorem 3.1.5, $L + A$ is L-stably solvable. For any $y \in F$, let

$$h_1(t, x) = tAx - tNx + ty.$$

Then h_1 is a L-compact map. Let

$$S_0 = \{x \in E : Lx + Ax = h_1(t, x) \text{ for some } t \in [0, 1]\}.$$

Since $\|h_1(t, x)\| \leq \|Ax - Nx\| + \|y\|$, we obtain that $(L + A)(S_0)$ is bounded. Theorem 3.1.7 ensures that there exists $x_0 \in \mathrm{dom}(L)$ such that

$$Lx_0 + Ax_0 = Ax_0 - Nx_0 + y,$$

thus $L + N$ is onto. □

In [46], Mawhin gave some existence theorems of Leray-Schauder type. The following theorem generalizes Theorem 2.2 of that paper.

Theorem 3.6.5. *Let Ω be an open bounded connected and convex subset of E and $0 \in \Omega$. Let $N : \overline{\Omega} \to F$ be a L-k_1-set contraction. Let $A : E \to F$ be a linear L-k_2-set contraction and $h : \overline{\Omega} \to F$ be a L-compact map such that*

(1) $h(\partial\Omega) \subset (L + A)(\mathrm{dom}(L) \cap \Omega)$.

(2) $\ker(L + A) = \{0\}$.

(3) $Lx + (1 - t)(Ax - hx) + tNx \neq 0$ *for* $x \in \partial\Omega \cap \mathrm{dom}(L)$ *and* $t \in (0, 1)$.

(4) $k_1 + 2k_2 < 1$.

Then the equation $Lx + Nx = 0$ has at least one solution in $\mathrm{dom}(L) \cap \overline{\Omega}$.

Proof: The condition $\ker(L+A) = \{0\}$ implies that $1 \in \rho(L, A)$ since A is a linear L-k_2-set contraction with $k_2 < 1$. So by Corollary 3.3.10, $L + A$ is $(0, L, k)$-epi for every $k < \omega(f)$, where $f = (I - P) - (\Lambda\Pi + K_{PQ})A$ and $\omega(f) > 1 - k_2$. Let $h_1(t, x) = th(x)$. Then $h_1(t, x) : [0, 1] \times \overline{\Omega} \to F$ is a L-compact map. Assume that there exists $x_0 \in \partial\Omega \cap \mathrm{dom}(L)$ such that

$$Lx_0 + Ax_0 - t_0h(x_0) = 0.$$

Then

$$Lx_0 + Ax_0 = t_0h(x_0) = t_0(L + A)x_1,$$

where $x_1 \in \mathrm{dom}(L) \cap \Omega$. Thus $x_0 = t_0x_1$, $t_0 \in [0, 1]$. This contradicts the connectedness and convexity of Ω. Hence for $x \in \partial\Omega \cap \mathrm{dom}(L)$, we have

$$Lx + Ax \neq th(x).$$

By Property 3.2.5, we obtain that $L + A - h$ is $(0, L, k)$-epi for every $k < \omega(f)$ on $\mathrm{dom}(L) \cap \overline{\Omega}$. Let $h_2 : [0, 1] \times \overline{\Omega} \to F$ be defined by

$$h_2(t, x) = t(Ax - h(x)) + tNx.$$

Then h_2 is a L-$(k_2 + k_1)$-set contraction and $k_2 + k_1 < 1 - k_2 < \omega(f)$. The assumption (3) and Property 3.2.5 imply that $L + N : \mathrm{dom}(L) \cap \overline{\Omega} \to F$ is a $(0, L, \varepsilon)$-epi for some $\varepsilon > 0$. Hence there exists $x_0 \in \mathrm{dom}(L) \cap \Omega$ which is a solution of the equation $Lx + Nx = 0$. $\square$

Remark 3.6.6. In Theorem 3.6.5, let $k_1 = k_2 = 0$, $h : \overline{\Omega} \to F$ be defined by $h(x) = z$ with $z \in (L + A)(\mathrm{dom}(L) \cap \Omega)$. Then this Theorem reduces to Theorem 2.2 of [46].

Chapter 4

Surjectivity results for nonlinear mappings without oddness conditions

The results in this Chapter follow the work of Fučik, Nečas, Souček and Souček in [12]. Much of this is joint work with J.R.L. Webb and has been published in [21].

The authors of [12] gave theorems for operators of the form $\lambda T - S$ of Fredholm alternative type under the assumptions that T is an odd (K, L, a)-homeomorphism and $S : X \to Y$ is an odd compact operator. Furthermore, they showed that the existence of a solution of the nonlinear operator equation

$$\lambda T(x) - S(x) = f \tag{4.1}$$

for each $f \in Y$ provided $\lambda \neq 0$ if T is an odd a-homogeneous and S is an odd b-strongly quasihomogeneous with $a > b$ (the definitions will be given later). In the case $a < b$ they proved the same assertion in finite dimensional spaces but said it was unsolved in the infinite-dimensional case.

In this chapter, we shall obtain some surjectivity results on the mapping $\lambda T - S$ under weaker conditions. One of the theorems generalizes the result of existence of a solution of (4.1) in case $a < b$ to the infinite-dimensional case. These results seem not to be able to be proven by their methods. We conclude with some examples of ordinary differential equations to which by applying the theorems some conditions for the existence of a solution can be obtained.

4.1 Surjectivity theorems

Given any continuous map f from a complex Banach space X into a Banach space Y, let $\Omega \subset X$ be a bounded set and let $\alpha(\Omega)$, $\alpha(f)$, $\omega(f)$, $d(f)$ and $|f|$ be defined as in Chapter 1. We will also make use the spectrum $\sigma_{fmv}(f)$, which was introduced in 1.3 of Chapter 1. The authors of [12] studied operators T that are (K, L, a)-homeomorphisms, where a (not necessarily linear) map $T : X \to Y$ is said to be a (K, L, a)-homeomorphism if

(a) T is a homeomorphism of X onto Y, and

(b) there exists real numbers $K > 0, a > 0, L > 0$ such that

$$L\|x\|^a \leq \|T(x)\| \leq K\|x\|^a \quad \text{for each } x \in X.$$

Theorem 1.2 of [12] and its generalization Theorem 1.2′ of [6] contain an error which makes part of their results false. The following is a correct version of their results.

Theorem 4.1.1. *Let T be an odd (K, L, a)-homeomorphism of X onto Y and $S : X \to Y$ be an odd compact operator. Let*

$$A := \limsup_{\|x\| \to \infty} \frac{\|Sx\|}{\|x\|^a} \quad \text{and} \quad B := \liminf_{\|x\| \to \infty} \frac{\|Sx\|}{\|x\|^a}.$$

(A and B can be $+\infty$). Then for $|\lambda| \notin [B/K, A/L] \cup \{0\}$ the operator $\lambda T - S$ maps X onto Y.

Proof: By Theorem 1.1 of [12], to prove the assertion it is sufficient to show that

$$\|\lambda Tx - Sx\| \to \infty \text{ as } \|x\| \to \infty \text{ if } |\lambda| > A/L \text{ or } |\lambda| < B/K.$$

Suppose that there exist $\|x_n\| \to \infty$ with $\|\lambda T x_n - S x_n\|$ bounded. Then

$$\frac{|\lambda| \|T x_n\|}{\|x_n\|^a} - \frac{\|S x_n\|}{\|x_n\|^a} \to 0. \tag{4.2}$$

Now

$$L \leq \frac{\|T x_n\|}{\|x_n\|^a} \leq K, \text{ for every } n \in \mathbb{N},$$

so when $|\lambda| > A/L$,

$$\limsup_{n \to \infty} \frac{\|S x_n\|}{\|x_n\|^a} = \limsup_{n \to \infty} \frac{|\lambda| \|T x_n\|}{\|x_n\|^a} \geq |\lambda| L > A,$$

a contradiction. Similarly when $|\lambda| < B/K$,

$$\liminf_{n \to \infty} \frac{\|S x_n\|}{\|x_n\|^a} \leq |\lambda| K < B,$$

another contradiction. When $B = \infty$, $\|S x_n\| / \|x_n\|^a \to \infty$ and (4.2) is contradicted for every λ. $\square$

Remark 4.1.2. Theorem 1.2 of [12] claims the same result if $|\lambda| \notin [A/K, A/L] \cup \{0\}$. The following simple examples show that this is incorrect and that the estimates $|\lambda| < B/K$ and $|\lambda| > A/L$ are sharp.

Example 4.1.3. Let T and $S : \mathbb{C} \to \mathbb{C}$ be defined by

$$T(z) = z, \; S(z) = z(1/2 + (1/2)|\sin x|), \text{ where } z = x + iy.$$

Then $a = 1$, $K = L = 1$, T and S are both odd. Also $A = 1$, $B = 1/2$, S, T are compact maps. Let $\lambda = 1/2 = B/K < A/K$. Then $\lambda T - S$ is not onto since $z/2 - S(z) = i$ has no solution. Otherwise there would exist z such that $z|\sin x| = -2i$, so z must be totally imaginary. But then $\sin x = 0$ so we have the impossible equation $0 = 2i$.

Example 4.1.4. Let T be as in example 4.1.3 and let $S(z) = z(1/2 + (1/2)|\cos x|)$. Then a, K, L, A, B are as in example 4.1.3. Let $\lambda = 1 = A/L$. Then $\lambda T - S$ is not onto since $z(1 - |\cos x|) = 2i$ has no solution.

Remark 4.1.5. Theorem 1.2′ of [6] claims that if $A = \infty$, then $\lambda T - S$ is surjective for every $\lambda \neq 0$. The next example shows that this is not true and that the estimate $|\lambda| < B/K$ is also sharp in the case $A = \infty$.

Example 4.1.6. Let T be as in example 4.1.3 and let $S(z) = z + |z|z(1 - \cos x)$. Then $A = \infty$ and $B = 1$. Let $\lambda = 1 = B/K$. Then $\lambda T - S$ is not onto since $|z|z(1 - \cos x) = i$ has no solution.

The next result generalizes Theorem 4.1.1 by allowing more general operators.

Theorem 4.1.7. *Let $T : D(T) \subseteq X \to Y$ be an operator satisfying the following conditions:*

1. *T is one to one, onto and $T^{-1} : Y \to D(T)$ is continuous;*

2. *There exist real numbers $L > 0, a > 0$ and $b > 0$ such that*

$$\|T(x)\| \geq L\|x\|^a - b \quad \text{for every } x \in D(T);$$

3. *T is bounded, that is, maps bounded sets into bounded sets.*

Let $S : X \to Y$ be bounded, continuous and suppose that

$$\limsup_{x \in D(T), \|x\| \to \infty} \frac{\|S(x)\|}{\|x\|^a} = A.$$

Then $\lambda T - S$ maps $D(T)$ onto Y under any one of the following conditions.

1. *$|\lambda| > \max\{\frac{A}{L}, \frac{\alpha(S)}{\omega(T)}\}$.*

2. *S is compact, and $|\lambda| > \frac{A}{L}$.*

3. *Y is a finite dimensional space, and $|\lambda| > \frac{A}{L}$.*

4. *S is compact, $A = 0$, and $\lambda \neq 0$.*

Proof: Clearly it suffices to prove case 1. Also it is clear that $\lambda T - S$ maps $D(T)$ onto Y if $I - F$ maps Y onto Y where $F : Y \to Y$ is defined by $F(y) = ST^{-1}(y/\lambda)$.

For every bounded set $\Omega \subset Y$, we have

$$\begin{aligned} \alpha(F(\Omega)) &= \alpha(ST^{-1}(\Omega/\lambda)) \\ &\leq \alpha(ST^{-1})\alpha(\Omega/\lambda) \\ &\leq \frac{1}{|\lambda|}\frac{\alpha(S)}{\omega(T)}\alpha(\Omega). \end{aligned}$$

Therefore,

$$\alpha(F) \leq \frac{1}{|\lambda|}\frac{\alpha(S)}{\omega(T)} < 1.$$

[If S is compact or F is finite dimensional, then $\alpha(F) = 0$.]

Also we have,

$$\begin{aligned} |F| &= \limsup_{\|y\|\to\infty} \|F(y)\|/\|y\| \\ &= \limsup_{\|y\|\to\infty} \|ST^{-1}(y/\lambda)\|/\|y\|. \end{aligned}$$

Writing $x = T^{-1}(y/\lambda)$, we have $Tx = y/\lambda$, and we obtain

$$\begin{aligned} |F| &= \limsup_{\|Tx\|\to\infty} \frac{\|S(x)\|}{|\lambda|\|Tx\|} \\ &= \limsup_{x\in D(T), \|x\|\to\infty} \frac{\|S(x)\|}{|\lambda|\|Tx\|} \\ &\leq \limsup_{x\in D(T), \|x\|\to\infty} \frac{\|S(x)\|}{|\lambda|(L\|x\|^a - b)} \\ &= \frac{A}{|\lambda|L} < 1. \end{aligned}$$

Hence, by theorem 1.3.2, $1 \in \rho(F)$, in particular $I - F$ maps Y onto Y. □

Remark 4.1.8. A result similar to Theorem 4.1.7 was obtained in [69], where a different method was used.

We recall the following concepts from [12].

Definition 4.1.9. Suppose that $a > 0$.

(a) A map $F_0 : X \to Y$ is called *a-homogeneous* if $F_0(tu) = t^a F_0(u)$ for every $t \geq 0$ and $u \in X$.

(b) $F : X \to Y$ is said to be *a-quasihomogeneous* relative to F_0 if $F_0 : X \to Y$ is a-homogeneous and

$$t_n \searrow 0, \; u_n \rightharpoonup u_0, \; t_n^a F(u_n/t_n) \to g \in Y$$

together imply that $g = F_0(u_0)$. [Here $u_n \rightharpoonup u_0$ denotes weak convergence.]

(c) $F : X \to Y$ is said to be *a-strongly quasihomogeneous* relative to F_0 if

$$t_n \searrow 0, \; u_n \rightharpoonup u_0 \; \text{ imply that } \; t_n^a F(u_n/t_n) \to F_0(u_0) \in Y.$$

It is known [12] that in case (c) F_0 is a-homogeneous and also must be strongly continuous, that is $u_n \rightharpoonup u_0$ implies $F_0 u_n \to F_0 u_0$

By applying Theorem 4.1.7 instead of Corollary 1.1 of [12] , we obtain the following generalization of Theorem 4.1 of [12], where we can dispense with the assumption that T, S are odd maps.

Theorem 4.1.10. *Let X be reflexive and let T satisfy the conditions of Theorem 4.1.7. Let $S : X \to Y$ be a compact b-strongly quasihomogeneous operator relative to S_0 and suppose that $a > b$. Then for $\lambda \neq 0$, $\lambda T - S$ maps $D(T)$ onto Y.*

Proof: By Theorem 4.1.7 part 4, it suffices to show that

$$\limsup_{\|x\| \to \infty, \, x \in D(T)} \frac{\|S(x)\|}{\|x\|^a} = 0.$$

This was proved in Theorem 4.1 of [12] but we include the proof for completeness. If this is false, there is a sequence $\{x_n\}$ with $\|x_n\| \to \infty$ and $\varepsilon > 0$ such that $\|Sx_n\|/\|x_n\|^a \geq \varepsilon$, for all sufficiently large n. Letting $u_n = x_n/\|x_n\|$ and $t_n = 1/\|x_n\|$ we have, for a subsequence, that

$$S(x_n)/\|x_n\|^b \to S_0(u_0)$$

Since $a > b$ this gives $S(x_n)/\|x_n\|^a \to 0$, a contradiction. $\square$

Remark 4.1.11. The authors of [12] say that the case $a < b$ seems to be unsolved in the infinite dimensional case. We shall give an answer below, see Theorem 4.1.16.

We introduce the following extension of the concept of stably solvable maps (see 1.1 of Chapter 1) which is appropriate to our needs.

Definition 4.1.12. A continuous map $f : D(f) \subseteq X \to Y$ is said to be *a-stably-solvable* for some $a > 0$ if the equation

$$f(x) = h(x)$$

has a solution $x \in D(f)$ for any continuous compact map $h : X \to Y$ with

$$|h|_a := \limsup_{\|x\|\to\infty} \frac{\|h(x)\|}{\|x\|^a} = 0.$$

Lemma 4.1.13. *Suppose $T : D(T) \subseteq X \to Y$ is as in Theorem 4.1.7. Then T is a-stably-solvable.*

Proof: Let $h : X \to Y$ be a compact map with $|h|_a = 0$. Then $\alpha(T^{-1}h) = 0$, and

$$\begin{aligned}\limsup_{\|x\|\to\infty} \frac{\|T^{-1}h(x)\|}{\|x\|} &= \limsup_{\|x\|\to\infty} \frac{\|T^{-1}h(x)\|}{\|h(x)\|^{\frac{1}{a}}} \left(\frac{\|h(x)\|}{\|x\|^a}\right)^{\frac{1}{a}} \\ &\leq \limsup_{\|x\|\to\infty} \left(\frac{1}{L}\right)^{\frac{1}{a}} \left(\frac{\|h(x)\|}{\|x\|^a}\right)^{\frac{1}{a}} \to 0.\end{aligned}$$

Therefore, $|T^{-1}h| = 0$. This implies that $1 \in \rho(T^{-1}h)$, so that $I - T^{-1}h$ is onto, that is, there exists $x \in D(T)$ such that $x = T^{-1}h(x)$, that is, $Tx = hx$. □

Lemma 4.1.14. (The Continuation Principle for a-stably-solvable maps)

Let $f : D(f) \subseteq X \to Y$ be a-stably-solvable , $h : X \times [0,1] \to Y$ be continuous, compact and such that $h(x,0) = 0$ for all $x \in D(f)$. Let

$$U = \{x \in D(f),\ f(x) = h(x,t) \ \textit{for some} \ \ t \in [0,1]\}.$$

Then, if $f(U)$ is bounded, the equation

$$f(x) = h(x, 1)$$

has a solution.

Proof: Let $O_r = \{y \in Y,\ \|y\| < r\}$, and let $r > 0$ be chosen so that $\overline{f(U)} \subset O_r$. Let $\varphi : Y \to [0, 1]$ be continuous and such that

$$\varphi(y) = \begin{cases} 1, & \text{for } y \in \overline{f(U)}, \\ 0, & \text{for } \|y\| \geq r, \end{cases}$$

and let π be the radial retraction of Y onto $B_r = \overline{O_r}$. Then the equation

$$f(x) = \pi h(x, \varphi(f(x)))$$

has a solution $x_0 \in D(f)$ since πh is compact and

$$|\pi h|_a = \lim_{\|x\| \to \infty} \frac{\|(\pi h)(x)\|}{\|x\|^a} = 0.$$

If $\|f(x_0)\| \geq r$, then $\varphi(f(x_0)) = 0$, and $f(x_0) = \pi h(x_0, 0) = 0$, a contradiction. Thus $\|f(x_0)\| < r$, and $f(x_0) = h(x_0, \varphi(f(x_0)))$, which shows that $x_0 \in U$ and therefore $f(x_0) = h(x_0, 1)$. □

Theorem 3.1 of [12] gave theorems of Fredholm alternative type for the couple (T, S) when T, S were both odd. Recall that λ is said to be an eigenvalue for the couple (T_0, S_0) if there is $x_0 \neq 0$ such that $\lambda T_0 x_0 - S_0 x_0 = 0$. Using Lemmas 4.1.13 and 4.1.14 we can give the following result when neither T nor S is odd.

Theorem 4.1.15. *Let X be a reflexive Banach space, and let T be as in Theorem 4.1.7 with $D(T) = X$ and also a-quasihomogeneous relative to T_0. Let $S : X \to Y$ be a compact a-strongly-quasihomogeneous operator relative to S_0. If $\lambda \neq 0$, and for every $t \in (0, 1]$, λ / t is not an eigenvalue for the couple (T_0, S_0), then $\lambda T - S$ maps X onto Y.*

Proof: For arbitrary $y \in Y$, let

$$U = \{x \in X,\ \lambda T(x) = h(x,t) = t[S(x) + y],\ t \in [0,1]\},$$

We show that U is bounded. If not, there exists $x_n \subset U$, $\|x_n\| \to \infty$, such that

$$\lambda T(x_n) = t_n[S(x_n) + y],\ \ t_n \in [0,1],$$

so that

$$\begin{aligned}
\frac{\lambda T(x_n)}{\|x_n\|^a} &= t_n \left(\frac{S(x_n)}{\|x_n\|^a} + \frac{y}{\|x_n\|^a} \right) \\
&= t_n \frac{1}{\|x_n\|^a} S\left(\frac{x_n/\|x_n\|}{1/\|x_n\|} \right) + t_n \frac{y}{\|x_n\|^a}.
\end{aligned}$$

Without loss of generality we assume that $x_n/\|x_n\| \rightharpoonup x_0$, $t_n \to t_0 \in [0,1]$. Then there exists a subsequence $\{x_{n_k}\}$ such that

$$t_{n_k} \frac{1}{\|x_{n_k}\|^a} S\left(\frac{x_{n_k}/\|x_{n_k}\|}{1/\|x_{n_k}\|} \right) \to t_0 S_0(x_0),$$

$$\lim_{n\to\infty} \frac{\lambda T(x_{n_k})}{\|x_{n_k}\|^a} = t_0 S_0(x_0).$$

Since T is a-quasihomogeneous relative to T_0, we obtain

$$\lambda T_0(x_0) = t_0 S_0(x_0).$$

However,

$$\frac{\|\lambda T(x_{n_k})\|}{\|x_{n_k}\|^a} \geq |\lambda| L - \frac{|\lambda| b}{\|x_{n_k}\|^a} > 0,$$

for n_k sufficiently large so that $\|t_0 S_0(x_0)\| > 0$. Hence $t_0 \neq 0$, and $S_0(x_0) \neq 0$.

From the definition of a-strongly-quasihomogeneous operator it is easy to show that $S_0(0) = 0$. Thus $x_0 \neq 0$, and λ/t_0 is an eigenvalue of (T_0, S_0), a contradiction. Thus U is bounded. By Lemma 4.1.13, $\lambda T : X \to Y$ is a-stably-solvable. So by Lemma 4.1.14, the equation $\lambda T(x) = S(x) + y$ has a solution $x \in X$, that is $\lambda T - S$ is onto. □

The next two results extend Theorem 4.2 of [12] to the infinite dimensional case.

Theorem 4.1.16. *Let X be a reflexive Banach space. Let T be a bounded, odd mapping satisfying the following conditions.*

1. $T : D(T) \subseteq X \to Y$ *is one to one, onto and* $T^{-1} : Y \to D(T)$ *is continuous.*

2. There exist real numbers $K > 0, a > 0$ *and* q *such that*

$$\|T(x)\| \leq K\|x\|^a + q \text{ for every } x \in D(T).$$

Suppose that S *is odd, continuous and b-strongly quasihomogeneous relative to* S_0*, and that* $\inf_{\{\|x\|=1\}}\|S_0(x)\| > 0$*. If* $a < b$*, then for every* λ *with* $|\lambda| > \alpha(S)/\omega(T)$*,* $\lambda T - S$ *is a-stably-solvable.*

Proof: First we show that there exists $R > 0$ such that $\lambda x - T^{-1}Sx \neq 0$ whenever $\|x\| \geq R$. If there exists $\{x_n\} \subset X$, $\|x_n\| \to \infty$ such that

$$\lambda x_n - T^{-1}S(x_n) = 0$$

we may assume that $\frac{x_n}{\|x_n\|} \rightharpoonup x_0$. Then we have

$$\frac{\|S(x_n)\|}{\|x_n\|^b} = \frac{\|T(\lambda x_n)\|}{\|x_n\|^b} \leq \frac{|\lambda|K\|x_n\|^a + q}{\|x_n\|^b} \to 0.$$

Since S is b-strongly quasihomogeneous relative to S_0, we have

$$\frac{1}{\|x_n\|^b}S(x_n) = \frac{1}{\|x_n\|^b}S\left(\frac{x_n/\|x_n\|}{1/\|x_n\|}\right) \to S_0(x_0).$$

As S_0 is strongly continuous we also have $S_0\left(\frac{x_n}{\|x_n\|}\right) \to S_0(x_0)$. Since $\inf_{\|x\|=1}\|S_0(x)\| > 0$ it follows that $S_0(x_0) \neq 0$, this contradicts the above. Let $O_r = \{x \in X, \|x\| < r\}$, where $r > R$. Then $\alpha(T^{-1}S) < |\lambda|$ and the topological degree $d(I - T^{-1}S/\lambda,\ O_r,\ 0)$ is odd, hence nonzero (see, for example, [7]). For a compact operator $h : X \to Y$ with $h = 0$ for $\|x\| = r$,

$$d\left(I - T^{-1}S/\lambda - T^{-1}h/\lambda,\ O_r,\ 0\right) \neq 0$$

because of boundary value dependence of degree.

For each $n \in \mathbb{N}$ let σ_n be continuous, $0 \leq \sigma_n \leq 1$ and such that

$$\sigma_n(x) = \begin{cases} 1 & \text{for } \|x\| \leq n, \\ 0 & \text{for } \|x\| \geq 2n. \end{cases}$$

Then, if $h : X \to Y$ is a compact operator with $|h|_a = 0$, for every $n > R/2$, the equation

$$\lambda T(x) - S(x) = \sigma_n(x)h(x)$$

has a solution $x_n \in D(T)$. If for all n, we have $\|x_n\| > n$, then

$$\frac{\lambda T(x_n) - S(x_n)}{\|x_n\|^b} = \frac{\sigma_n(x_n)h(x_n)}{\|x_n\|^b}.$$

Assume that $x_n/\|x_n\| \rightharpoonup x_0$. Then from

$$\frac{\lambda T(x_n) - S(x_n)}{\|x_n\|^b} \to -S_0(x_0) \neq 0 \quad (n \to \infty),$$

and

$$\frac{\sigma_n(x_n)h(x_n)}{\|x_n\|^b} = \sigma_n(x_n)\frac{h(x_n)}{\|x_n\|^a}\frac{\|x_n\|^a}{\|x_n\|^b} \to 0 \quad (n \to \infty),$$

we reach a contradiction. Hence there exists n, such that $\|x_n\| \leq n$, and then

$$\lambda T(x_n) - S(x_n) = h(x_n),$$

and we are done. □

Theorem 4.1.17. *Let X be a reflexive Banach space, $T, T_1 : D(T) \to Y$ and $S, S_1 : X \to Y$ be of the form $T = T_1 + R$, $S = S_1 + R'$, where T_1 satisfies the same conditions as T in Theorem 4.1.16, S_1 is odd, continuous and b-strongly quasihomogeneous relative to S_0, and $R, R' : X \to Y$ are compact operators with $|R|_a = |R'|_a = 0$. Suppose that $a < b$, and that $\inf_{\{\|x\|=1\}}\|S_0(x)\| > 0$. Then $\lambda T - S$ maps $D(T)$ onto Y for every λ with $|\lambda| > \alpha(S)/\omega(T)$.*

Proof: For $y \in Y$, let $h(x) = -\lambda R(x) + R'(x) + y$, so that h is compact and $|h|_a = 0$. By Theorem 4.1.16, the equation

$$\lambda T_1(x) - S_1(x) = h(x)$$

has a solution $x_0 \in D(T)$. Hence

$$\lambda T(x_0) - S(x_0) = y,$$

that is $\lambda T - S$ is onto. □

Remark 4.1.18. The author of [6] claims that if T is an odd (K, L, a)-homeomorphism, S is an odd, compact and b-strongly quasihomogeneous relative to S_0, $S_0 \not\equiv 0$ and $a < b$, then $\lambda T - S$ is surjective for all $\lambda \neq 0$. His proof contains an error. In our theorem 4.1.16, we are only able to prove the result under the stronger condition $\inf_{\{||x||=1\}} ||S_0(x)|| > 0$.

4.2 Applications of the theorems

The following applications are examples of situations that can be settled by the theorems in section 4.1, but apparently cannot be handled by the results in [12].

Example 4.2.1. We consider a nonlinear Sturm-Liouville problem on an unbounded domain, namely the following nonlinear differential equation:

$$\begin{gathered} -(p(x)u'(x))' + q(x)u(x) = \lambda\{u(x) + g(x)f(u(x))\}, \\ \text{for} \quad x \in (0, \infty), \text{ and } \quad u(0) = 0. \end{gathered} \tag{4.3}$$

In [66] it was shown that certain eigenvalues λ are asymptotic bifurcation points. Under the same assumptions we will show that if v is continuous, the equation

$$\begin{gathered} -(p(x)u'(x))' + q(x)u(x) = \lambda\{u(x) + g(x)f(u(x))\} + v(x) \\ \text{for} \quad x \in (0, \infty), \quad \text{and} \quad u(0) = 0. \end{gathered} \tag{4.4}$$

has a solution when λ is not one of these eigenvalues.

We recall the assumptions made in [66].

1. $p : [0, \infty) \to \mathbb{R}$ is continuous and continuously differentiable on $(0, \infty)$, with p' bounded and $0 < P_1 \leq p(x) \leq P_2 < \infty$ for all $x \in [0, \infty)$.

2. $q : [0, \infty) \to \mathbb{R}$ is continuous with
$0 < Q_1 \leq q(x) \leq Q_2 < \infty$ for all $x \in [0, \infty)$.

3. f is a continuously differentiable function from $\mathbb{R}$ into itself, and there exist positive real numbers P and K such that $|f(p)| \leq K|p|^r$ for all $p \geq P$, for some $r < 1$.

4. $g \in H_0^1(0,\infty)$.

For $u : [0,\infty) \to \mathbb{R}$ and $x \geq 0$ let H be defined by $(Hu)(x) = g(x)f(u(x))$. Let $A : H_0^1 \cap W^{2,2} \to L^2$ be the self-adjoint extension of the operator A_0 defined by

$$A_0 u = -(p(x)u'(x))' + q(x)u(x)$$

with domain the set of twice continuously differentiable functions with compact support in $(0,\infty)$. Then A is a positive self-adjoint operator in L^2 and its positive square root $A^{\frac{1}{2}}$ is a linear homeomorphism of H_0^1 onto L^2, where H_0^1 is the closure of C_0^∞ in $W^{1,2}$ and C_0^∞ is the linear space of all infinitely differentiable, real-valued functions with compact support in $(0,\infty)$ (see [66]).

We claim (and will show below) that for $0 < |\lambda| < Q := \liminf_{x\to\infty} q(x)$, and λ not an eigenvalue of A, the operator

$$u \mapsto u - \lambda A^{-1}u + \lambda A^{-1/2}HA^{-1/2}u$$

from $L^2 \to L^2$ is onto. Assuming this, it follows that the equation

$$Au = \lambda u + \lambda Hu + v$$

has a solution $u \in H_0^1 \cap W^{2,2}$ for any $v \in L^2$ ([66], Lemma 4.18). Hence if v is continuous, using the same arguments as in Lemma 4.20 of [66], it follows that the equation (4.4) has a solution.

We now establish the claim made above. Let $\mu = 1/\lambda$, and let $T, S : L^2 \to L^2$ be defined by

$$Tu = \mu u - A^{-1}u, \quad Su = A^{-1/2}HA^{-1/2}u.$$

Suppose that $|\mu| > \alpha(A^{-1}) = 1/Q$ ([66], Theorem 4.23), and that μ is not an eigenvalue of A^{-1}. Then T is a bounded linear operator, which is one to one, onto, and has a continuous inverse. So it is a $(K, L, 1)$-homeomorphism of L^2 onto L^2. Furthermore, T is 1-quasihomogeneous relative to T since it has continuous inverse. It has been shown that S is a compact operator and the quasinorm $|S| = 0$ in the space L^2 ([66], Lemma 4.17). Assume that there exist $u_n \in L^2$ with $u_n \rightharpoonup u_0$, $t_n \searrow 0$ such that

$$t_n \| S(u_n/t_n) \| > \varepsilon_0 > 0.$$

Then $\{\|u_n/t_n\|_2\}$ is unbounded. If $\|u_{n_k}/t_{n_k}\|_2 \to \infty$, $(n_k \to \infty)$, then we have

$$\|t_{n_k} S\left(u_{n_k}/t_{n_k}\right)\|_2 = \frac{\|S\left(u_{n_k}/t_{n_k}\right)\|_2}{\|u_{n_k}\|_2/t_{n_k}}\|u_{n_k}\|_2 \to 0,$$

a contradiction. Thus we have shown that S is a 1-strongly quasihomogeneous operator relative to $S_0 = 0$ in the space L^2. For any $t \in (0, 1]$,

$$(1/t)(\mu I - A^{-1})(u) = 0 \Longrightarrow u = 0,$$

so $1/t$ is not an eigenvalue of the couple $(T, 0)$. By Theorem 4.1.15, $T - S$ maps L^2 onto L^2. Thus we have reached the conclusion.

The following second-order m-point nonlinear boundary value problem has been studied recently by Gupta, Ntouyas and Tsamatos ([25], [29], [30]):

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)) + e(t) \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = \textstyle\sum_{i=1}^{m-2} a_i x(\xi_i). \end{gathered} \tag{4.5}$$

It was shown that the problem of existence of a solution for the BVP (4.5) can be studied via the three-point boundary value problem

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)) + e(t) \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = \alpha x(\eta), \end{gathered} \tag{4.6}$$

where $\eta \in (0, 1)$ and $\alpha \in \mathbb{R}$.

Some conditions for the existence of a solution for the BVP (4.6) were obtained in [29] using the Leray-Schauder continuation theorem. Their results suppose that $\alpha < 1/\eta$. By using Theorem 4.1.7, we obtain the following result which gives a different condition for the existence of a solution for (4.6) under the more general hypothesis $\alpha \neq 1/\eta$.

We shall use the classical spaces $C[0, 1]$, $C^1[0, 1]$, $C^2[0, 1]$ and $L^p[0, 1]$. We denote the norm in $L^p[0, 1]$ by $\|\cdot\|_p$. We also use the Sobolev space $W^{2,1}(0, 1)$ (see [34]) which may be defined by

$$W^{2,1}(0, 1) = \{x : \ [0, 1] \to \mathbb{R} : \ x, x' \text{ are absolutely continuous on } [0, 1] \text{ with } x'' \in L^1[0, 1]\}$$

with its usual norm.

We also recall the following definition.

Definition 4.2.2. A function $f : [0,1] \times \mathbb{R}^2 \to R$ satisfies Carathéodory's conditions if

1. For each $(x, y) \in \mathbb{R}^2$, the function $t \in [0,1] \to f(t, x, y) \in \mathbb{R}$ is measurable on $[0,1]$;

2. for a.e. $t \in [0,1]$, the function $(x, y) \in \mathbb{R}^2 \to f(t, x, y) \in \mathbb{R}$ is continuous on $\mathbb{R}^2$, and for each $r > 0$, there exists $g_r \in L^1[0,1]$ such that

$$|f(t, x, y)| \leq |g_r(t)|$$

for a.e. $t \in [0,1]$ and for every $(x, y) \in \mathbb{R}^2$ with $\sqrt{x^2 + y^2} \leq r$.

Theorem 4.2.3. *Let f: $[0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function which satisfies Carathéodory's conditions. Assume that there exist functions p, q, r in $L^1(0,1)$ such that*

$$|f(t, x_1, x_2)| \leq p(t)|x_1| + q(t)|x_2| + r(t)$$

for a.e. $t \in [0,1]$ and all $(x_1, x_2) \in \mathbb{R}^2$. Also let $\eta \in (0,1), \alpha \geq 0, \alpha \neq 1/\eta$ be given. Then for any given $e \in L^1(0,1)$ the boundary value problem (4.6) has at least one solution in $C^1[0,1]$ provided that

$$\|p\|_1 + \|q\|_1 < \begin{cases} (1 - \alpha\eta)/2, & \text{if} \quad \alpha\eta < 1, \\ (\alpha\eta - 1)/2\alpha\eta, & \text{if} \quad \alpha\eta > 1. \end{cases} \tag{4.7}$$

Proof: Let X denote the Banach space $C^1[0,1]$ with the norm

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}.$$

Let Y denote the Banach space $L^1(0,1)$ with its usual norm.

The linear operator $L : D(L) \subset X \to Y$ is defined by setting

$$D(L) = \{x \in W^{2,1}(0,1) : \ x(0) = 0, \ x(1) = \alpha x(\eta)\},$$

and for $x \in D(L)$,

$$Lx = x''.$$

For $x \in X$, let

$$(Nx)(t) = f(t, x(t), x'(t)), \quad t \in [0,1].$$

Then N is a bounded map from X into Y. It can be shown that $L : D(L) \subset X \to Y$ is one to one and onto when $\alpha \neq 1/\eta$. In fact, $L^{-1} = K$, where $K : Y \to D(L) \subset X$ is the linear operator defined by

$$(Ky)(t) = \int_0^t (t-s)y(s)\,ds + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)y(s)\,ds - \frac{t}{1-\alpha\eta}\int_0^1 (1-s)y(s)\,ds.$$

For $y \in Y$, we have

$$\|Ky\|_\infty \leq \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right)\|y\|_1,$$

where $\|y\|_1$ is the norm of y in the space $L^1(0,1)$. Also

$$\|(Ky)'\|_\infty \leq \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right)\|y\|_1.$$

Thus we have

$$\|Ky\| \leq \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right)\|y\|_1.$$

Let $T = I$ and $S = KN$. Then $\alpha(S) = 0$ by the Arzéla-Ascoli theorem. Also we have

$$\begin{aligned}
A &= \limsup_{\|x\|\to\infty} \frac{\|S(x)\|}{\|x\|} \\
&= \limsup_{\|x\|\to\infty} \frac{\|KN(x)\|}{\|x\|} \\
&\leq \limsup_{x\to\infty} \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right)\frac{\|N(x)\|_1}{\|x\|} \\
&\leq \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right)\limsup_{\|x\|\to\infty} \frac{\|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty + \|r\|_1}{\|x\|} \\
&\leq \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right)\limsup_{\|x\|\to\infty} \frac{(\|p\|_1 + \|q\|_1)\|x\| + \|r\|_1}{\|x\|} \\
&= \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right)(\|p\|_1 + \|q\|_1) \\
&= \begin{cases} \frac{2}{1-\alpha\eta}(\|p\|_1 + \|q\|_1) & \text{for} \quad \alpha\eta < 1 \\ \frac{2\alpha\eta}{\alpha\eta - 1}(\|p\|_1 + \|q\|_1) & \text{for} \quad \alpha\eta > 1 \end{cases}
\end{aligned}$$

By the assumption (4.7) we see that $A < 1$. Hence, from Theorem 4.1.7, the operator $T - S = I - KN$ maps X onto X.

Hence, given any $e \in L^1(0,1)$, there exits $x \in C^1[0,1]$ such that

$$x(t) - (KNx)(t) = Ke(t).$$

Thus $x = KNx + Ke \in D(L)$ and

$$Lx - Nx = e.$$

This proves that the BVP (4.6) has at least one solution in $C^1[0,1]$. $\square$

Remark 4.2.4. When $\alpha\eta < 1$, the condition (4.7) gives a better result than Theorem 4 of [29] in case $\alpha(1-\eta) > 2$ since their condition demands $\|p\|_1 + \|q\|_1 < \frac{1-\alpha\eta}{\alpha(1-\eta)}$, but is worse in the case $\alpha(1-\eta) < 2$. Also our result can be applied when $\alpha\eta > 1$.

Chapter 5

Solvability of m-point boundary value problems for second order ordinary differential equations

Much of this chapter is joint work with J.R.L. Webb and will be published in [18], [22] and [23].

In this chapter, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ will be a function satisfying Carathéodory's conditions (see definition 4.2.2) and $e : [0,1] \to \mathbb{R}$ be a function in $L^1(0,1)$, $a_i \in \mathbb{R}$ with all of the $a_i s$ having the same sign, $\xi_i \in (0,1), i = 1, 2, ..., m-2$, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$. We consider the following second order ordinary differential equation:

$$x''(t) = f(t, x(t), x'(t)) + e(t) \qquad t \in (0,1), \tag{5.1}$$

with one of the following boundary value conditions:

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{5.2}$$

$$x(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \tag{5.3}$$

It is known that the problem of the existence of a solution for the boundary-value problem (5.1), (5.2) and (5.1), (5.3) can be studied respectively via the existence of a solution for equation (5.1) subject to one of the following three-point boundary-value conditions (see [29] [30]):

$$x'(0) = 0, \qquad x(1) = \alpha x(\eta), \tag{5.4}$$

$$x(0) = 0, \qquad x(1) = \alpha x(\eta), \tag{5.5}$$

where $\alpha \in \mathbb{R}$ and $\eta \in (0,1)$ be given.

In this chapter, we will study the existence and uniqueness results for the above problems. The work in this chapter is related to the recent work of Gupta, Ntouyas, Tsamatos and Lakshmikantham [24]-[30].

5.1 Three-point boundary value problems at resonance

In [30] the authors studied the problem (5.1), (5.4) in the case that $\alpha \neq 1$ and in [29], under the assumption that $\alpha < \frac{1}{\eta}$, they obtained some results for the existence of a solution of (5.1), (5.5). In both cases above, the linear operator L defined in a suitable Banach space is invertible. Also, they always assume that f has a linear growth. In this section, we shall prove the existence results for problem (5.1), (5.4) with the condition $\alpha = 1$ and (5.1), (5.5) with the condition $\alpha = \frac{1}{\eta}$. In these cases, L is non-invertible, so the Leray-Schauder continuation theory can not be used. Our results makes use of the following coincidence degree continuation theorem of Mawhin [46].

Theorem 5.1.1. (see [46], p.84)

Let $L : E \to F$ be Fredholm of index zero and let N be L-compact on $\overline{\Omega}$. Assume that $F = \operatorname{im}(L) \oplus F_0$ and the following conditions are satisfied.

1. $Lx + \lambda Nx \neq 0$ *for each* $(x, \lambda) \in [(D(L) \setminus \ker L) \cap \partial\Omega] \times (0,1)$.

2. $Nx \notin \operatorname{im}(L)$ *for each* $x \in \ker(L) \cap \partial\Omega$.

3. $\deg(QN|_{\ker L}, \Omega \cap \ker(L), 0) \neq 0$, *where* $Q : F \to F_0$ *is a continuous projection.*

Then the equation $Lx + Nx = 0$ has at least one solution in $D(L) \cap \overline{\Omega}$.

We also, in this section, assume that f has a linear growth. Later we will weaker this considerably, see sections 5.3 and 5.4.

Theorem 5.1.2. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying the following conditions:*

1. *There exist functions p, q, r in $L^1[0,1]$ such that*

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t) \quad \textit{for a.e.} \quad t \in [0,1] \quad (u,v) \in \mathbb{R}^2;$$

2. *There exists $N_0 \in \mathbb{R}$, $N_0 > 0$ such that for all $u \in \mathbb{R}$ with $|u| > N_0$, one has*

$$|f(t,u,v)| \geq l|u| - n|v| - M \quad \textit{for all} \quad t \in [0,1] \quad (u,v) \in \mathbb{R}^2,$$

   where $l > 0, M > 0, n \geq 0$;

3. *There exists $R > 0$ such that for all $|u| > R$ either*

$$uf(t,u,0) \geq 0 \quad \textit{for a.e.} \quad t \in [0,1]$$

   or

$$uf(t,u,0) \leq 0 \quad \textit{for a.e.} \quad t \in [0,1].$$

Then the BVP(5.1), (5.4) with $\alpha = 1$ has at least one solution in $C^1[0,1]$ provided

$$\left(2 + \frac{n}{l}\right) \|p\|_1 + \|q\|_1 < 1. \tag{5.6}$$

To prove the theorem above, we need the following lemmas. We shall denote by X the Banach space $C^1[0,1]$ with the norm $\|x\| = \max\{\|x\|_\infty, \ \|x'\|_\infty\}$, and Y denotes the Banach space $L^1[0,1]$ with its usual norm. Define L to be the linear operator from $D(L) \subset X$ to Y with

$$D(L) = \{x \in W^{2,1}(0,1) : x'(0) = 0, x(1) = x(\eta)\}$$

and for $x \in D(L)$,

$$Lx = x''.$$

Also let $N : X \to Y$ be the nonlinear operator defined by setting

$$N(x)(t) = -f(t, x(t), x'(t)) - e(t), \quad t \in [0,1],$$

so that N is a bounded map under our hypotheses.

Lemma 5.1.3. *Suppose L is as above, then $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero. Furthermore, let $X = \ker(L) \oplus X_1$, the linear operator $K : \mathrm{im}(L) \to D(L) \cap X_1$ defined by*

$$(Ky)(t) = \int_0^t (t-s)y(s)\,ds \quad \textit{for } y \in \mathrm{im}(L)$$

is such that

$$K = L_1^{-1},$$

where $L_1 = L|_{D(L)\cap X_1}$. Also we have that $\|Ky\| \le \|y\|_1$ for $y \in \mathrm{im}(L)$.

Proof: It is clear that $\ker(L) = \{x(t) \equiv c,\ c \in \mathbb{R}\}$. Also we have that

$$\begin{aligned} \mathrm{im}(L) &= \left\{ y \in L^1[0,1] : \int_\eta^1 Y(\tau)\,d\tau = 0, \text{ where } Y(\tau) = \int_0^\tau y(s)\,ds. \right\} \\ &= \left\{ y \in L^1[0,1] : \quad \int_\eta^1 (1-s)y(s)\,ds + (1-\eta)\int_0^\eta y(s)\,ds = 0 \right\}. \end{aligned} \tag{5.7}$$

(5.7) can be shown as follows. For $y \in \mathrm{im}(L)$, there exists $x \in D(L)$ such that $x'' = y$. Then

$$Y(\tau) = \int_0^\tau x''(s)\,ds = x'(\tau) - x'(0) = x'(\tau),$$

$$\int_\eta^1 Y(\tau)\,d\tau = \int_\eta^1 x'(\tau)\,d\tau = x(1) - x(\eta) = 0.$$

On the other hand, for $y \in L^1[0,1]$ with $\int_\eta^1 Y(\tau)\,d\tau = 0$, letting

$$x(t) = \int_0^t \int_0^\tau y(s)\,ds\,d\tau,$$

we have $x \in D(L)$ and $Lx = y$. Thus we have shown that (5.7) holds.

Suppose that $y \in L^1[0,1]$, let

$$Qy = \frac{2}{1-\eta^2}\int_\eta^1 Y(\tau)\,d\tau,$$

and let $y_1 = y - Qy$. It can be proved that $y_1 \in \mathrm{im}(L)$, so $Y = \mathrm{im}(L) + \mathbb{R}$. Also $\mathbb{R} \cap \mathrm{im}(L) = \{0\}$, hence $Y = \mathrm{im}(L) \oplus \mathbb{R}$ and $\dim(\ker(L)) = \dim(\mathbb{R}) = 1$, and L is a Fredholm operator of index zero. Now we define the projection $P : X \to \ker(L)$ by setting $(Px)(t) = x(0)$. Let $X_1 = \{x \in X, x(0) = 0\}$. Then for $x \in D(L) \cap X_1$,

$$(KL_1x)(t) = Kx''(t) = \int_0^t\int_0^\tau x''(s)\,ds\,d\tau = \int_0^t x'(\tau)\,d\tau = x(t),$$

and for $y \in \mathrm{im}(L)$,

$$(L_1Ky)(t) = \left(\int_0^t Y(\tau)\,d\tau\right)'' = y(t),$$

thus $K = {L_1}^{-1}$. We also have

$\|Ky\| \le \|y\|_1$ since

$$\|Ky\|_\infty \le \int_0^1\int_0^1 |y(s)|\,ds\,d\tau \le \|y\|_1,$$

and

$$(Ky)'(t) = \int_0^t y(s)\,ds, \quad \text{so} \quad \|(Ky)'(t)\|_\infty \le \|y(t)\|_1.$$

This completes the proof of Lemma 5.1.3. □

Lemma 5.1.4. *Let*

$$U_1 = \{x \in D(L)\backslash \ker(L), Lx + \lambda Nx = 0 \quad \textit{for some } \lambda \in [0,1]\},$$

then U_1 is a bounded subset of X.

Proof: Suppose that $x \in U_1$, and $Lx = -\lambda Nx$. Then $\lambda \neq 0$ and $QNx = 0$, so that

$$\int_\eta^1\int_0^\tau f(t, x(t), x'(t))\,dt\,d\tau = -\int_\eta^1\int_0^\tau e(t)\,dt\,d\tau.$$

Hence there exist $\xi \in (\eta, 1)$ and $\zeta \in (0, \xi)$ such that

$$f(\zeta, x(\zeta), x'(\zeta)) = -\frac{1}{(1-\eta)\xi}\int_\eta^1\int_0^\tau e(t)\,dt\,d\tau,$$

so we obtain

$$|f(\zeta, x(\zeta), x'(\zeta))| \leq \frac{\|e\|_1}{\eta}. \tag{5.8}$$

Also for $x \in D(L) \setminus \ker(L)$, by Lemma 5.1.3 and condition (1),

$$\begin{aligned}\|(I-P)x\| &= \|KL(I-P)x\| \leq \|L(I-P)x\|_1 = \|Lx\|_1 \\ &\leq \|Nx\|_1 \leq \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty + \|r\|_1 + \|e\|_1. \end{aligned} \tag{5.9}$$

If for some $t_0 \in [0,1]$, $|x(t_0)| \leq N_0$ then writing

$$x(0) = x(t_0) - \int_0^{t_0} x'(t)\, dt,$$

we have

$$|x(0)| \leq N_0 + \|x'\|_1 \leq N_0 + \|x'\|_\infty. \tag{5.10}$$

On the other hand, if $|x(t)| > N_0$ for all $t \in [0,1]$, then by condition (2),

$$|f(t, x(t), x'(t))| \geq l|x(t)| - n|x'(t)| - M. \tag{5.11}$$

In this case from (5.8) and (5.11), we obtain

$$|x(\zeta)| \leq \frac{M}{l} + \frac{\|e\|_1}{l\eta} + \frac{n}{l}|x'(\zeta)|.$$

Since $x(0) = x(\zeta) - \int_0^\zeta x'(t)\, dt$, we have

$$|x(0)| \leq \frac{M}{l} + \frac{\|e\|_1}{l\eta} + \left(\frac{n}{l} + 1\right)\|x'\|_\infty. \tag{5.12}$$

From (5.10) and (5.12), we see that in every case we have the inequality

$$\|Px\| = |x(0)| \leq \max\left\{\frac{M}{l} + \frac{\|e\|_1}{l\eta}, N_0\right\} + \left(\frac{n}{l} + 1\right)\|x'\|_\infty. \tag{5.13}$$

From (5.9) and (5.13), we obtain

$$\begin{aligned}\|x\|_\infty &\leq \|x\| \\ &\leq \|(I-P)(x)\| + \|Px\| \\ &\leq \|p\|_1\|x\|_\infty + \left(\|q\|_1 + 1 + \frac{n}{l}\right)\|x'\|_\infty + C,\end{aligned}$$

where $C = \|r\|_1 + \|e\|_1 + \max\left\{\frac{M}{l} + \frac{\|e\|_1}{l\eta}, N_0\right\}$. Thus

$$\|x\|_\infty \le \frac{\|q\|_1 + 1 + \frac{n}{l}}{1 - \|p\|_1}\|x'\|_\infty + \frac{C}{1 - \|p\|_1}. \tag{5.14}$$

From (5.9) we know that

$$\|x''\|_1 = \|L(x)\|_1 \le \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty + \|r\|_1 + \|e\|_1.$$

By (5.14),

$$\|x''\|_1 \le \frac{\|p\|_1 + \|q\|_1 + \frac{n}{l}\|p\|_1}{1 - \|p\|_1}\|x'\|_\infty + C_1,$$

where $C_1 = \|r\|_1 + \|e\|_1 + \frac{C\|p\|_1}{1-\|p\|_1}$. For each $t \in [0,1]$, $x'(t) = \int_0^t x''(s)\,ds$ and hence

$$\|x'\|_\infty \le \|x''\|_1. \tag{5.15}$$

Therefore

$$\|x''\|_1 \le \frac{\|p\|_1 + \|q\|_1 + n/l\|p\|_1}{1 - \|p\|_1}\|x''\|_1 + C_1.$$

Let $C_{pq} = \frac{\|p\|_1+\|q\|_1+\frac{n}{l}\|p\|_1}{1-\|p\|_1}$, so that $C_{pq} < 1$ by our assumption. Then we have that $\|x''\|_1 \le \frac{C_1}{1-C_{pq}}$ and so $\|x'\|_\infty \le \frac{C_1}{1-C_{pq}}$. Hence using (5.14) we obtain

$$\|x\|_\infty \le \frac{\|q\|_1 + \frac{n}{l} + 1}{1 - \|p\|_1}\frac{C_1}{1 - C_{pq}} + \frac{C}{1 - \|p\|_1} := N_1,$$

so that $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\} \le N_1$. We have shown that U_1 is bounded. □

Lemma 5.1.5. *The set $U_2 = \{x \in \ker(L) : Nx \in \operatorname{im}(L)\}$ is bounded.*

Proof: Let $x \in U_2$ so that $x(t) \equiv C$ and $QNx = 0$ and therefore

$$\int_\eta^1 \int_0^\tau f(s, C, 0)\,ds\,d\tau = -\int_\eta^1 \int_0^\tau e(s)\,ds\,d\tau.$$

Hence there exist $\xi_1 \in (\eta, 1)$ and $\zeta_1 \in (0, \xi_1)$ such that

$$f(\zeta_1, C, 0) = -\frac{1}{(1-\eta)\xi_1}\int_\eta^1 \int_0^\tau e(s)\,ds\,d\tau.$$

Therefore

$$|f(\zeta_1, C, 0)| \le \frac{\|e\|_1}{\eta}.$$

It follows that

$$|C| \leq \max\left\{N_0, \frac{M}{l} + \frac{\|e\|_1}{l\eta}\right\}.$$

For otherwise, we have $|C| > N_0$, by condition (2),

$$\frac{\|e\|_1}{\eta} \geq |f(\xi_1, C_1, 0)| \geq l|C| - M,$$

hence $|C| \leq \frac{M}{l} + \frac{\|e\|_1}{l\eta}$, a contradiction. This gives us

$$U_2 \subset \left\{x \in D(L) : \|x\| \leq N_2 = \max\left\{N_0, \frac{M}{l} + \frac{\|e\|_1}{l\eta}\right\}\right\}.$$

$\square$

Lemma 5.1.6. *If in the condition (3), we assume that there exists $R > 0$ such that for all $|u| > R$, $uf(t, u, 0) \leq 0$, for a.e. $t \in [0, 1]$, and let*

$$U_3 = \{x \in \ker(L) : H(x, \lambda) = \lambda Jx + (1 - \lambda)QNx = 0,\ \lambda \in [0, 1]\},$$

where $Y = \operatorname{im}(L) \oplus Y_1$ and $J : \ker(L) \to Y_1$ is the identity isomorphism, then U_3 is bounded.

Proof: Assume that $C_n \in U_3$ and $\|C_n\| \to \infty$ as $n \to \infty$. There exist $\lambda_n \in [0, 1]$ such that

$$\lambda_n C_n + (1 - \lambda_n)(QN)(C_n) = 0.$$

$\{\lambda_n\}$ has a convergent subsequence, for simplicity of notation we suppose that $\lambda_n \to \lambda_0$. We also can get that $\lambda_0 \neq 1$ since otherwise we would have

$$\lambda_n = -(1 - \lambda_n)\frac{(QN)C_n}{C_n},$$

and from

$$\begin{aligned}\frac{\|(QN)C_n\|}{\|C_n\|} &\leq \frac{\|Q\| \cdot \|N(C_n)\|_1}{\|C_n\|} \\ &\leq \frac{\|Q\|\{\|p\|_1|C_n| + \|r\|_1 + \|e\|_1\}}{|C_n|} \\ &\leq \|Q\| \cdot \|p\|_1 + \frac{\|Q\|\|r\|_1 + \|Q\|\|e\|_1}{|C_n|},\end{aligned}$$

where $\|Q\|$ denote the norm of the 1-dimensional linear projection Q, we obtain

$$(1-\lambda_n)\frac{\|(QN)C_n\|}{\|C_n\|} \to 0 \ (n \to \infty),$$

this contradicts $\lambda_n \to 1$. So for n sufficiently large, $1-\lambda_n \neq 0$, and therefore

$$\frac{\lambda_n}{1-\lambda_n} C_n = Q(f(t, C_n, 0) + e(t)),$$

that is,

$$\frac{\lambda_n}{1-\lambda_n} = \frac{2}{1-\eta^2}\int_\eta^1\int_0^\tau \frac{f(s, C_n, 0)}{C_n}\, ds\, dt + \frac{2}{C_n(1-\eta^2)}\int_\eta^1\int_0^\tau e(s)\, ds\, d\tau.$$

Since $|C_n| \to \infty$, we may assume that $|C_n| > \max\{N_0, R\}$. Then for n sufficiently large we obtain

$$\left|\frac{f(t, C_n, 0)}{C_n}\right| \geq l - \frac{M}{|C_n|} \geq \frac{l}{2}.$$

We are assuming that $C_n f(t, C_n, 0) \leq 0$ for a.e. $t \in [0,1]$, therefore

$$\frac{f(t, C_n, 0)}{C_n} \leq -\frac{l}{2}.$$

Hence, using Fatou's Lemma we obtain

$$\begin{aligned}
\overline{\lim}_{n\to\infty} & \left\{\int_\eta^1\int_0^\tau \frac{f(s, C_n, 0)}{C_n}\, ds\, d\tau + \frac{1}{C_n}\int_\eta^1\int_0^\tau e(s)\, ds\, d\tau\right\} \\
&\leq \overline{\lim}_{n\to\infty}\int_\eta^1\int_0^\tau \frac{f(s, C_n, 0)}{C_n}\, ds\, d\tau \\
&\leq \int_\eta^1\int_0^\tau \overline{\lim}_{n\to\infty}\frac{f(s, C_n, 0)}{C_n}\, ds\, d\tau \leq -\frac{l}{4}(1-\eta^2).
\end{aligned}$$

This contradicts $\lambda_n/1-\lambda_n \geq 0$. Hence U_3 is bounded. □

The proof of Theorem 5.1.2 is now an easy consequence of the above lemmas and Theorem 5.1.1.

Proof: Firstly, by the Arzela-Ascoli Theorem, it can be shown that the linear operator $K : \operatorname{im}(L) \to D(L)\cap X_1$ in Lemma 5.1.3 is a compact operator [45], so N is an L-compact mapping. Let Ω be a bounded open set containing $\overline{\bigcup_{i=1}^3 U_i}$, then by the above lemmas, we have verified the hypotheses 1 and 2 of Theorem 5.1.1. Let $H(x, \lambda) = \lambda Jx + (1-\lambda)QNx$, with J as in lemma 5.1.6. By the homotopy property of degree,

$$\deg(QN|_{\ker L}, \Omega\cap\ker(L), 0) = \deg(J, \Omega\cap\ker(L), 0) \neq 0.$$

Thus by Theorem 5.1.1, $Lx + Nx = 0$ has at least one solution in $D(L) \cap \overline{\Omega}$ so that BVP(5.1),(5.4) with $\alpha = 1$ has a solution. $\square$

Remark 5.1.7. In the condition (3) of Theorem 5.1.2, if we assume that there exists a $R > 0$ such that for all $|u| > R$,

$$uf(t, u, 0) \geq 0 \quad \text{for a.e.} \quad t \in [0, 1],$$

then in Lemma 5.1.6, we let

$$U_3 = \{x \in \ker(L), H(x, \lambda) = -\lambda Jx + (1 - \lambda)QNx = 0, \lambda \in [0, 1]\}.$$

Exactly as in Lemma 5.1.6, we can prove that U_3 is bounded, so in the proof of Theorem 5.1.2, we have that

$$\deg(QN|_{\ker(L)}, \Omega \cap \ker(L), 0) = \deg(-J, \Omega \cap \ker(L), 0) \neq 0.$$

The other part of the proof is the same.

The next theorem deals with the BVP (5.1), (5.5).

Theorem 5.1.8. *Let $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying the following conditions:*

1. *There exist functions p, q, r in $L^1[0, 1]$ such that*

$$|f(t, u, v)| \leq p(t)|u| + q(t)|v| + r(t) \quad \textit{for a.e.} \quad t \in [0, 1] \quad (u, v) \in \mathbb{R}^2;$$

2. *There exists $N_0 \in \mathbb{R}$, $N_0 > 0$ such that for all $v \in \mathbb{R}$ with $|v| > N_0$, one has*

$$|f(t, u, v)| \geq -l|u| + n|v| - M \quad \textit{for all} \quad t \in [0, 1] \quad (u, v) \in \mathbb{R}^2,$$

where $l \geq 0, M > 0, n > 0$;

3. There exists $R > 0$ such that for all $|v| > R$ either

$$vf(t, vt, v) \geq 0 \quad \textit{for a.e.} \quad t \in [0, 1]$$

or

$$vf(t, vt, v) \leq 0 \quad \textit{for a.e.} \quad t \in [0, 1].$$

Then the BVP(5.1), (5.5) with $\alpha = \frac{1}{\eta}$ and with e continuous on $[0, 1]$ has at least one solution in $C^1[0, 1]$ provided

$$2(\|p\|_1 + \|q\|_1) + \frac{l}{n} < 1. \tag{5.16}$$

In the following proof, we denote by L the linear operator from $D(L) \subset X \to Y$ with

$$D(L) = \left\{ x \in W^{2,1}(0, 1) : \quad x(0) = 0, \quad x(1) = \frac{1}{\eta} x(\eta) \right\}$$

and for $x \in D(L)$,

$$Lx = x''.$$

Lemma 5.1.9. *Suppose L is as above, then $L : D(L) \subset X \to Y$ is Fredholm operator of index zero. Furthermore, the linear operator $K : \mathrm{im}(L) \to D(L) \cap X_1$ which is as follows:*

$$(Ky)(t) = \int_0^t \int_0^\tau y(s)\, ds\, d\tau \quad \textit{for} \quad y \in \mathrm{im}(L)$$

is such that

$$K = L_1^{-1},$$

where $L_1 = L|_{D(L) \cap X_1}$. Also we have that $\|Ky\| \leq \|y\|_1$ for $y \in \mathrm{im}(L)$.

Proof: It is easy to show that $\ker(L) = \{ct : c \in \mathbb{R}\}$. We shall show that

$$\mathrm{im}(L) = \left\{ y \in L^1[0, 1] : \int_0^1 Y(t)\, dt = \int_0^1 Y(\eta t)\, dt, \text{ where } Y(t) = \int_0^t y(s) ds \right\}.$$

For $y \in \mathrm{im}(L)$, $y = x''$ with $x \in D(L)$, therefore we have

$$\int_0^1 Y(t)\, dt = \int_0^1 (x'(t) - x'(0))\, dt = x(1) - x'(0),$$

$$\int_0^1 Y(\eta t)\,dt = \int_0^1 (x'(\eta t) - x'(0))\,dt = \frac{1}{\eta}x(\eta) - x'(0),$$

since $x(1) = \frac{1}{\eta}x(\eta)$, thus $\int_0^1 Y(t)\,dt = \int_0^1 Y(\eta t)\,dt$. On the other hand, suppose $y \in L^1[0,1]$ is such that $\int_0^1 Y(t)\,dt = \int_0^1 Y(\eta t)\,dt$. Let $x(t) = \int_0^t Y(s)\,ds$, then $x \in D(L)$ and $x'' = y$, thus $y \in \mathrm{im}(L)$.

For $y \in L^1[0,1]$, let

$$Qy = \frac{2}{1-\eta}\int_0^1 \int_{\eta t}^t y(s)\,ds\,dt,$$

and let $y_1 = y - Qy$. Then $Y_1(t) = \int_0^t y(s)\,ds - (Qy)t$, where Qy can be rewritten as

$$Qy = \frac{2}{1-\eta}\left\{\int_0^1 \int_0^t y(s)\,ds\,dt - \int_0^1 \int_0^{\eta t} y(s)\,ds\,dt\right\},$$

$$\int_0^1 \int_0^t y(s)\,ds\,dt - \frac{Qy}{2} = \int_0^1 \int_0^{\eta t} y(s)\,ds\,dt - \eta\frac{Qy}{2},$$

so that

$$\int_0^1 Y_1(t)dt = \int_0^1 Y_1(\eta t)dt.$$

This shows that $y_1 \in \mathrm{im}(L)$, and therefore $Y = \mathrm{im}(L) + \mathbb{R}$. Also $\mathbb{R} \cap \mathrm{im}(L) = \{0\}$, hence $Y = \mathrm{im}(L) \oplus \mathbb{R}$ and $\dim(\ker(L)) = \dim(\mathbb{R}) = 1$, and L is a Fredholm operator of index zero. Now we define $P : X \to \ker(L)$ by setting $(Px)(t) = x'(0)t$. Let $X_1 = \{x \in X, x'(0) = 0\}$. For $x \in D(L) \cap X_1$,

$$(KL_1x)(t) = Kx''(t) = \int_0^t \int_0^\tau x''(s)\,ds\,d\tau = \int_0^t (x'(\tau) - x'(0))\,d\tau = x(t) - x(0) = x(t).$$

And for $y \in \mathrm{im}(L)$,

$$(L_1Ky)(t) = \left(\int_0^t \int_0^\tau y(s)\,ds\,d\tau\right)'' = y(t),$$

thus $K = {L_1}^{-1}$. Since

$$\|Ky\|_\infty \le \|y\|_1,$$

and

$$|(Ky)'(t)| = |\int_0^t y(s)\,ds| \le \|y\|_1,$$

we obtain $\|Ky\| \le \|y\|_1$.

$\square$

In the following, the mapping $N : X \to Y$ is defined as in the proof of Theorem 5.1.2.

Lemma 5.1.10. *Let*

$$U_1 = \{x \in D(L)\backslash \ker(L),\ Lx + \lambda Nx = 0 \textit{ for some } \lambda \in [0,1]\},$$

then U_1 is a bounded subset of X.

Proof: Suppose that $x \in U_1$, and $Lx = -\lambda Nx$, then $\lambda \neq 0$ and $QNx = 0$. Therefore

$$\int_0^1 \int_{\eta\tau}^{\tau} (f(t, x(t), x'(t)) + e(t))\, dt\, d\tau = 0,$$

and hence there exists $\gamma \in (0,1)$ such that

$$|f(\gamma, x(\gamma), x'(\gamma))| = |e(\gamma)| \leq \|e\|_\infty. \tag{5.17}$$

Also for $x \in D(L) \setminus \ker(L)$, by Lemma 5.1.9 and condition (1) of Theorem 5.1.8,

$$\begin{aligned}\|(I-P)x\| &= \|KL(I-P)x\| \leq \|L(I-P)x\|_1 = \|L(x)\|_1 \\ &\leq \|N(x)\|_1 \leq \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty + \|r\|_1 + \|e\|_1. \end{aligned}\tag{5.18}$$

If for some $t_0 \in [0,1]$, $|x'(t_0)| \leq N_0$ then

$$|x'(0)| = |x'(t_0) - \int_0^{t_0} x''(t)\, dt| \leq N_0 + \|x''\|_1; \tag{5.19}$$

If for all $t \in [0,1]$, $|x'(t)| > N_0$, by (5.17) and condition (2) of Theorem 5.1.8, we get that

$$|x'(\gamma)| \leq \frac{\|e\|_\infty + M}{n} + \frac{l}{n}\|x\|_\infty,$$

so

$$|x'(0)| = |x'(\gamma) - \int_0^{\gamma} x''(t)\, dt| \leq \frac{\|e\|_\infty + M}{n} + \frac{l}{n}\|x\|_\infty + \|x''\|_1. \tag{5.20}$$

By (5.19) and (5.20),

$$\|Px\| = |x'(0)| \leq \max\left\{\frac{\|e\|_\infty + M}{n}, N_0\right\} + \frac{l}{n}\|x\|_\infty + \|x''\|_1. \tag{5.21}$$

Since $x(t) = \int_0^t x'(s)\, ds$, we obtain

$$\|x\|_\infty \leq \|x'\|_1 \leq \|x'\|_\infty.$$

Thus from (5.18) and (5.21), we have

$$\begin{aligned}\|x'\|_\infty &\leq \|x\| \\ &\leq \|(I-P)(x)\| + \|Px\| \\ &\leq \left(\|p\|_1 + \|q\|_1 + \frac{l}{n}\right)\|x'\|_\infty + \|x''\|_1 + C,\end{aligned}$$

where $C = \|r\|_1 + \|e\|_1 + \max\left\{\frac{\|e\|_\infty + M}{n}, N_0\right\}$. By our assumption, $\|p\|_1 + \|q\|_1 + \frac{l}{n} < 1$. Let

$$C_1 = 1 - \|p\|_1 - \|q\|_1 - \frac{l}{n}.$$

Then

$$\|x'\|_\infty \leq \frac{1}{C_1}\|x''\|_1 + \frac{C}{C_1}. \tag{5.22}$$

We have the following inequalities:

$$\begin{aligned}\|x''\|_1 = \|Lx\|_1 \leq \|Nx\|_1 &\leq \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty + \|r\|_1 + \|e\|_1 \\ &\leq (\|p\|_1 + \|q\|_1)\|x'\|_\infty + \|r\|_1 + \|e\|_1 \\ &\leq \frac{\|p\|_1 + \|q\|_1}{C_1}\|x''\|_1 + C_2,\end{aligned}$$

where $C_2 = \frac{(C\|p\|_1 + \|q\|_1)}{C_1} + \|r\|_1 + \|e\|_1$. By condition (5.16), $\|p\|_1 + \|q\|_1 < C_1$. So

$$C_3 = \frac{\|p\|_1 + \|q\|_1}{C_1} < 1.$$

Thus $\|x''\|_1 \leq \frac{C_2}{1-C_3}$. By (5.22),

$$\|x\|_\infty \leq \|x'\|_\infty \leq \frac{C_2}{C_1(1-C_3)} + \frac{C}{C_1},$$

and so

$$U_1 \subset \{x \in D(L), \|x\| \leq N_1\}.$$

where $N_1 = \frac{C_2}{C_1(1-C_3)} + \frac{C}{C_1}$. □

Lemma 5.1.11. *Let $U_2 = \{x \in \ker(L) : Nx \in \operatorname{im}(L)\}$, then U_2 is bounded.*

Proof: Suppose that $x \in U_2$ and therefore $x(t) = Ct$, where C is a constant, and $QNx = 0$. Thus

$$\int_0^1 \int_{\eta\tau}^{\tau} f(t, Ct, C)\, dt\, d\tau = -\int_0^1 \int_{\eta\tau}^{\tau} e(t)\, dt\, d\tau.$$

So there exists $\xi \in (0,1)$ such that

$$|f(\xi, C\xi, C)| = |e(\xi)| \leq \|e\|_\infty.$$

It follows that

$$|C| \leq \max\left\{N_0, \frac{M + \|e\|_\infty}{n - l}\right\}.$$

For otherwise, we have $|C| > N_0$. By condition (2),

$$\|e\|_\infty \geq -l|C\xi| + n|C| - M \geq -l|C| + n|C| - M.$$

Hence $|C| \leq \frac{M+\|e\|_\infty}{n-l}$, a contradiction. This gives us

$$U_2 \subset \left\{x \in D(L) : \|x\| \leq N_2 = \max\left\{N_0, \frac{M + \|e\|_\infty}{n - l}\right\}\right\}.$$

□

Lemma 5.1.12. *If in the condition (3) of Theorem 5.1.8, we assume that there exists $R > 0$ such that for all $|v| > R$, $vf(t, vt, v) \leq 0$, for a.e. $t \in [0,1]$, letting*

$$U_3 = \{x \in \ker(L) : H(x, \lambda) = \lambda Jx + (1-\lambda)QNx = 0, \lambda \in [0,1]\},$$

where $J : \ker(L) \to Y_1$, defined by $J(Ct) = C$ for $Ct \in \ker(L)$, is the linear isomorphism, then U_3 is bounded.

Proof: Assume that $x_n(t) = C_n t \in U_3$ and $\|C_n t\| = |C_n| \to \infty$ as $n \to \infty$. There exist $\lambda_n \in [0,1]$ such that

$$\lambda_n C_n + (1 - \lambda_n)(QN)(C_n t) = 0.$$

$\{\lambda_n\}$ has a convergent subsequence, we just suppose that $\lambda_n \to \lambda_0$. We also have $\lambda_0 \neq 1$ since otherwise we have

$$\lambda_n = -(1 - \lambda_n)\frac{(QN)C_n t}{C_n}.$$

From

$$\frac{\|(QN)C_nt\|}{|C_n|} \leq \frac{\|Q\| \cdot \|N(C_nt)\|_1}{|C_n|} \leq \frac{\|Q\|\{\|p\|_1|C_n| + \|q\|_1|C_n| + \|r\|_1 + \|e\|_1\}}{|C_n|}$$
$$\leq \|Q\|(\|p\|_1 + \|q\|_1) + \frac{\|Q\|\|r\|_1 + \|Q\|\|e\|_1}{|C_n|},$$

where $\|Q\|$ denote the norm of the 1-dimensional linear projection Q, we obtain

$$(1-\lambda_n)\frac{\|(QN)C_nt\|}{|C_n|} \to 0 \ (n \to \infty).$$

This contradicts $\lambda_n \to 1$. So for n sufficiently large, $1 - \lambda_n \neq 0$, and thus

$$\frac{\lambda_n}{1-\lambda_n}C_n = Q(f(t, C_nt, C_n) + e(t)),$$

$$\frac{\lambda_n}{1-\lambda_n} = \frac{2}{1-\eta}\int_0^1 \int_{\eta\tau}^{\tau} \frac{f(s, C_ns, C_n)}{C_n}\, ds\, dt + \frac{2}{C_n(1-\eta)}\int_0^1 \int_{\eta\tau}^{\tau} e(s)\, ds\, d\tau.$$

Since $|C_n| \to \infty$, we may assume that $|C_n| > \max\{N_0, R\}$. Then for n sufficiently large we have that

$$\left|\frac{f(t, C_nt, C_n)}{C_n}\right| \geq n - l - \frac{M}{|C_n|} \geq \frac{n-l}{2}.$$

By our assumption, $C_n f(t, C_nt, C_n) \leq 0$ for a.e. $t \in [0,1]$, so that

$$\frac{f(t, C_nt, C_n)}{C_n} \leq -\frac{n-l}{2}.$$

Also by our condition (1) we know that

$$\left|\frac{f(t, C_nt, C_n)}{C_n}\right| \leq p(t) + q(t) + \frac{r(t)}{R} \in L^1[0,1].$$

Hence, by Fatou's Lemma,

$$\overline{\lim}_{n\to\infty}\left\{\int_0^1 \int_{\eta\tau}^{\tau} \frac{f(s, C_ns, C_n)}{C_n}\, ds\, d\tau + \frac{1}{C_n}\int_0^1 \int_{\eta\tau}^{\tau} e(s)\, ds\, d\tau\right\}$$
$$\leq \overline{\lim}_{n\to\infty}\int_0^1 \int_{\eta\tau}^{\tau} \frac{f(s, C_ns, C_n)}{C_n}\, ds\, d\tau$$
$$\leq \int_0^1 \int_{\eta\tau}^{\tau} \overline{\lim}_{n\to\infty}\frac{f(s, C_ns, C_n)}{C_n}\, ds\, d\tau \leq -\frac{(1-\eta)(n-l)}{4}.$$

This contradicts $\frac{\lambda_n}{1-\lambda_n} \geq 0$. Thus U_3 is bounded. □

By using the above lemmas and the method in the proof of Theorem 5.1.2, the proof of Theorem 5.1.8 now follows easily. We therefore omit it.

5.2 Uniqueness results

In this section, we shall prove uniqueness of solutions to the BVP (5.1), (5.4) and the BVP (5.1), (5.5) under stronger hypotheses than previously.

Theorem 5.2.1. *Suppose that the conditions (1) and (2) in Theorem 5.1.2 are replaced by the following conditions respectively:*

1. *There exist functions p, q in $L^1[0,1]$ such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq p(t)|u_1 - u_2| + q(t)|v_1 - v_2|$$

for $t \in [0,1], (u_1,v_1), (u_2,v_2) \in \mathbb{R}^2$;

2. *There exist $l > 0, n \geq 0$ such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \geq l|u_1 - u_2| - n|v_1 - v_2|$$

for $t \in [0,1]$ and $(u_1,v_1), (u_2,v_2) \in \mathbb{R}^2$.

Then the BVP (5.1), (5.4) with $\alpha = 1$ has exactly one solution in $C^1[0,1]$ provided

$$\left(2 + \frac{n}{l}\right)\|p\|_1 + \|q\|_1 < 1.$$

Proof: Let X, Y, L, Q, P be as in the proof of Theorem 5.1.2. The existence of a solution for the boundary-value problem (5.1), (5.4) with $\alpha = 1$ follows from Theorem 5.1.2.

Now suppose that $x_1, x_2 \in C^1[0,1]$ are two solutions of (5.1), (5.4) with $\alpha = 1$. Then

$$x_i''(t) = f(t, x_i(t), x_i'(t)) + e(t),$$

and $x_i'(0) = 0, x_i(1) = x_i(\eta), i = 1, 2.$

Write $x = x_1 - x_2$, so that x satisfies the equation

$$x''(t) = f(t, x_1(t), x_1'(t)) - f(t, x_2(t), x_2'(t)). \tag{5.23}$$

Hence

$$Q(f(t, x_1(t), x_1'(t)) - f(t, x_2(t), x_2'(t))) = 0,$$

and therefore

$$\int_\eta^1 \int_0^\tau (f(t, x_1(t), x_1'(t)) - f(t, x_2(t), x_2'(t)))\, dt\, d\tau = 0.$$

It follows that there exists $\xi \in (0, 1)$ such that

$$f(\xi, x_1(\xi), x_1'(\xi)) - f(\xi, x_2(\xi), x_2'(\xi)) = 0.$$

By our hypotheses (2), we have

$$0 = |f(\xi, x_1(\xi), x_1'(\xi)) - f(\xi, x_2(\xi), x_2'(\xi))| \geq l|x(\xi)| - n|x'(\xi)|,$$

from which we obtain

$$|x(\xi)| \leq \frac{n}{l}|x'(\xi)| \leq \frac{n}{l}\|x'\|_\infty.$$

Hence we have

$$\|Px\| = |x(0)| \leq |x(\xi)| + |\int_0^\xi x'(t)\, dt| \leq \left\{\frac{n}{l} + 1\right\} \|x'\|_\infty.$$

Next by Lemma 5.1.3 and (5.23) we have

$$\|(I - P)x\| \leq \|Lx\|_1 = \|x''\|_1 \leq \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty.$$

This gives

$$\|x\|_\infty \leq \|x\| \leq \|Px\| + \|(I - P)x\| \leq \left\{\frac{n}{l} + 1 + \|q\|_1\right\} \|x'\|_\infty + \|p\|_1\|x\|_\infty,$$

so that

$$\|x\|_\infty \leq \frac{\|q\|_1 + 1 + \frac{n}{l}}{1 - \|p\|_1}\|x'\|_\infty. \tag{5.24}$$

Again from (5.23) we obtain

$$\|x''\|_1 \leq \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty,$$

and using (5.24) this gives

$$\|x''\|_1 \leq \frac{\|p\|_1 + \|q\|_1 + \frac{n}{l}\|p\|_1}{1 - \|p\|_1}\|x''\|_1.$$

By our assumption the coefficient on the right is less than 1 so we have $\|x''\|_1 = 0$. Hence by (5.24)

$$\|x\|_\infty = \|x'\|_\infty = \|x''\|_1 = 0,$$

and since x is continuous, $x(t) = 0$ for all $t \in [0,1]$, that is $x_1 = x_2$. □

We also have a uniqueness result for the second set of boundary conditions.

Theorem 5.2.2. *Suppose that the conditions (1) and (2) in Theorem 5.1.8 are replaced by the following conditions respectively:*

1. *There exist functions p, q in $L^1[0,1]$ such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le p(t)|u_1 - u_2| + q(t)|v_1 - v_2|$$

for $t \in [0,1], (u_1,v_1), (u_2,v_2) \in \mathbb{R}^2$;

2. *There exist $l \ge 0, n > 0$ such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \ge -l|u_1 - u_2| + n|v_1 - v_2|$$

for $t \in [0,1]$ and $(u_1,v_1),(u_2,v_2) \in \mathbb{R}^2$.

Then the BVP (5.1), (5.5) with $\alpha = \frac{1}{\eta}$ has exactly one solution in $C^1[0,1]$ provided

$$2(\|p\|_1 + \|q\|_1) + \frac{l}{n} < 1.$$

Proof: Let X, Y, L, Q, P be as in the proof of Theorem 5.1.8. The existence of a solution for the boundary-value problem (5.1), (5.5) with $\alpha = \frac{1}{\eta}$ follows from Theorem 5.1.8.

Now suppose that $x_1, x_2 \in C^1[0,1]$ are two solutions of (5.1), (5.5) with $\alpha = \frac{1}{\eta}$, and write $x = x_1 - x_2$. Then

$$x''(t) = f(t, x_1(t), x_1'(t)) - f(t, x_2(t), x_2'(t)), \tag{5.25}$$

and $x(0) = 0, x(1) = \frac{1}{\eta}x(\eta)$. Also

$$Q(f(t, x_1(t), x_1'(t)) - f(t, x_2(t), x_2'(t))) = 0,$$

therefore

$$\int_0^1 \int_{\eta\tau}^{\tau} (f(t, x_1(t), x_1'(t)) - f(t, x_2(t), x_2'(t)))\, dt, d\tau = 0.$$

Hence there exists $\zeta \in (0, 1)$ such that

$$f(\zeta, x_1(\zeta), x_1'(\zeta)) - f(\zeta, x_2(\zeta), x_2'(\zeta)) = 0.$$

By our hypotheses (2), we have

$$0 \geq -l|x(\zeta)| + n|x'(\zeta)|,$$

and so

$$|x'(\zeta)| \leq \frac{l}{n}\|x\|_\infty.$$

Hence

$$\|Px\| = |x'(0)| \leq |x'(\zeta)| + |\int_0^\zeta x''(t)\, dt| \leq \frac{l}{n}\|x\|_\infty + \|x''\|_1.$$

Next by Lemma 5.1.9 we obtain

$$\|(I - P)x\| \leq \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty.$$

This gives

$$\|x'\|_\infty \leq \|x\| \leq \|Px\| + \|(I - P)x\| \leq \left\{\frac{l}{n} + \|p\|_1 + \|q\|_1\right\}\|x'\|_\infty + \|x''\|_1,$$

so that

$$\|x'\|_\infty \leq \frac{1}{C}\|x''\|_1,$$

where $C = 1 - \frac{l}{n} - \|p\|_1 - \|q\|_1$. Since $x(0) = 0$, we have $\|x\|_\infty \leq \|x'\|_\infty$, and therefore

$$\|x''\|_1 \leq \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty \leq (\|p\|_1 + \|q\|_1)\|x'\|_\infty.$$

This yields

$$\|x''\|_1 \leq \frac{\|p\|_1 + \|q\|_1}{C}\|x''\|_1.$$

$\|x''\|_1 = 0$, by our assumption. Hence we also have

$$\|x\|_\infty = \|x'\|_\infty = \|x''\|_1 = 0$$

and uniqueness is shown. □

Remark 5.2.3. The uniqueness of solutions to the BVP (5.1), (5.5) with $\alpha < 1/\eta$, was proved in [29] under weaker conditions,. Theorems 5.2.1 and 5.2.2 treat the resonance cases for BVPs (5.1), (5.4) and (5.1), (5.5), but the assumptions are stronger than that assumed in [29].

5.3 Three-point boundary value problems with nonlinear growth

In this section, we shall study the BVP (5.1), (5.4) and BVP (5.1), (5.5) under the assumption that f has a nonlinear growth. We shall treat both the non-resonance case and the resonance case. We do this by imposing a decomposition condition for f and by showing that the growth of certain nonlinear terms is not restricted provided they satisfy a sign condition. Some examples will be given to show that there exist equations which can be treated by our results but the results of [24], [25], [29], [30], [61] cannot be applied. This section is joint work with J.R.L. Webb and to appear in [23].

5.3.1 Non-resonance results

In our first result we show that the growth of certain nonlinear terms is not restricted provided they satisfy a sign condition. This idea is similar to, but different from, one used in [4].

Theorem 5.3.1. *Assume that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and has the decomposition*

$$f(t,x,p) = g(t,x,p) + h(t,x,p)$$

such that

1. $pg(t,x,p) \leq 0$ *for all* $(t,x,p) \in [0,1] \times \mathbb{R}^2$*;*
2. $|h(t,x,p)| \leq a(t)|x| + b(t)|p| + u(t)|x|^r + v(t)|p|^k + c(t)$ *for all* $(t,x,p) \in [0,1] \times \mathbb{R}^2$*, where* a, b, u, v, c *are in* $L^1[0,1]$ *and* $0 \leq r, k < 1$.

Then, for $\alpha \neq 1$, there exists a solution $x \in C^1[0,1]$ to BVP(5.1), (5.4) provided that

$$\begin{cases} \|a\|_1 + \|b\|_1 < 1/2 & \text{if } \alpha \leq 0, \\ \frac{\alpha-\eta}{\alpha-1}\|a\|_1 + \|b\|_1 < 1/2 & \text{if } \alpha > 1, \\ \frac{2-\alpha-\eta}{1-\alpha}\|a\|_1 + \|b\|_1 < 1/2 & \text{if } 0 < \alpha < 1. \end{cases} \tag{5.26}$$

Proof: Let X denote the Banach space $C^1[0,1]$ and Z denote the Banach space $L^1[0,1]$. We define a linear mapping L: $D(L) \subset X \to Z$ by setting

$$D(L) = \{x \in W^{2,1}(0,1) : \; x'(0) = 0, \; x(1) = \alpha x(\eta)\},$$

and for $x \in D(L)$, $\quad L(x) = x''$.

Let $N : X \to Z$ be the nonlinear mapping defined by

$$(Nx)(t) = f(t, x, x'), \quad t \in [0,1].$$

Since $\alpha \neq 1$, L is a one-to-one linear mapping. Let $K := L^{-1}$ so that $KN : X \to X$ is compact by the Arzéla-Ascoli theorem. By the Leray-Schauder degree theory, to obtain the existence of a solution for BVP (5.1), (5.4) in $C^1[0,1]$ it suffices to prove the set of all possible solutions of the following family of equations

$$x''(t) = \lambda f(t, x, x') + \lambda e(t), \quad t \in (0,1), \tag{5.27}$$

$$x'(0) = 0, \quad x(1) = \alpha x(\eta) \tag{5.28}$$

is bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

To do this, suppose x is a solution of (5.27), (5.28), so that $x \in D(L)$ and

$$x'x'' = \lambda x' f(t, x, x') + \lambda x' e.$$

An integration shows that:

$$\begin{aligned} \frac{1}{2}x'^2(t) &= \lambda \int_0^t x' g(s, x, x')\, ds + \lambda \int_0^t x' h(s, x, x')\, ds + \lambda \int_0^t x' e(s)\, ds \\ &\leq \int_0^1 |x' h(t, x, x')|\, dt + \int_0^1 |x'(t)||e(t)|\, dt \\ &\leq \|x'\|_\infty \left\{ \|a\|_1 \|x\|_\infty + \|b\|_1 \|x'\|_\infty + \|u\|_1 \|x\|_\infty^r + \|v\|_1 \|x'\|_\infty^k + \|c\|_1 + \|e\|_1 \right\}. \end{aligned}$$

Suppose $\|x'\|_\infty \neq 0$, otherwise we are done since $\alpha \neq 1$ implies that $x \equiv 0$. Then we have the inequality

$$(\frac{1}{2} - \|b\|_1)\|x'\|_\infty \leq \|a\|_1\|x\|_\infty + \|u\|_1\|x\|_\infty^r + \|v\|_1\|x'\|_\infty^k + \|c\|_1 + \|e\|_1. \tag{5.29}$$

Case 1. Assume that $\alpha \leq 0$. Since $x(1)$ and $x(\eta)$ have opposite signs in this case, there exists $\xi \in (\eta, 1]$ such that $x(\xi) = 0$. Therefore for each $t \in [0, 1]$, we have

$$|x(t)| = |\int_\xi^t x'(s)\, ds| \leq \|x'\|_\infty,$$

and so $\|x\|_\infty \leq \|x'\|_\infty$. Since $\|a\|_1 + \|b\|_1 < 1/2$, from (5.29), we obtain

$$(\frac{1}{2} - \|a\|_1 - \|b\|_1)\|x'\|_\infty \leq \|u\|_1\|x'\|_\infty^r + \|v\|_1\|x'\|_\infty^k + \|c\|_1 + \|e\|_1.$$

This implies that there exists $M_1 > 0$ such that $\|x'\|_\infty \leq M_1$, hence also $\|x\|_\infty \leq M_1$.

Case 2. Assume that $1 \neq \alpha > 0$. By (5.4) and the mean value theorem there exists $\xi \in (\eta, 1)$ such that

$$x(\eta) = \frac{1-\eta}{\alpha - 1}x'(\xi),$$

([30], Lemma 2.2). Thus for every $t \in [0, 1]$,

$$x(t) = \int_\eta^t x'(s)\, ds + \frac{1-\eta}{\alpha-1}x'(\xi),$$

and hence

$$\|x\|_\infty \leq \frac{|\alpha - 1| + 1 - \eta}{|\alpha - 1|}\|x'\|_\infty. \tag{5.30}$$

By our assumption (5.26),

$$C := \|a\|_1\frac{|\alpha - 1| + 1 - \eta}{|\alpha - 1|} + \|b\|_1 < \frac{1}{2}.$$

From (5.29), we obtain

$$(\frac{1}{2} - C)\|x'\|_\infty \leq \|u\|_1\left(\frac{|\alpha - 1| + 1 - \eta}{|\alpha - 1|}\right)^r \|x'\|_\infty^r + \|v\|_1\|x'\|_\infty^k + \|c\|_1 + \|e\|_1.$$

Hence $\max\{\|x\|_\infty, \|x'\|_\infty\} \leq M_2$. □

We will need the following simple lemma to deal with the second set of boundary conditions.

Lemma 5.3.2. *Let $x \in C^1[0,1]$ satisfy $x(0) = 0$, $x(1) = \alpha x(\eta)$, where $\eta \in (0,1)$, and $\alpha > 1$, $\alpha \neq 1/\eta$. Then there exist ζ and $C_0 \in (0,1)$ such that $x'(\zeta) = C_0 x(1)$.*

Proof: If $\alpha\eta > 1$, let $C_0 = 1/(\alpha\eta)$. There exists $\zeta \in (0,\eta)$ with

$$x'(\zeta) = \frac{x(\eta) - x(0)}{\eta} = C_0 x(1).$$

If $\alpha\eta < 1$, let $C_0 = (\alpha - 1)/(\alpha(1-\eta))$. There exists $\zeta \in (\eta, 1)$ with

$$x'(\zeta) = \frac{x(1) - x(\eta)}{1-\eta} = \frac{\alpha - 1}{1-\eta} x(\eta) = C_0 x(1).$$

□

Remark 5.3.3. If $\alpha \leq 1$ the lemma holds with $C_0 = 0$ (see [29], Lemma 2). For, if $\alpha < 0$, $x(1)$ and $x(\eta)$ have opposite signs so there are at least two zeros of x. The case $\alpha = 0$ is simple, the case $\alpha = 1$ follows from the mean value theorem and for $0 < \alpha < 1$, x has either a positive maximum or a negative minimum in $(0,1)$.

Theorem 5.3.4. *Assume that the mapping f satisfies the condition:*

$$|f(t,x,p)| \leq a(t)|x| + b(t)|p| + u(t)|x|^r + v(t)|p|^k + c(t)$$

for all $(t,x,p) \in [0,1] \times \mathbb{R}^2$, where a, b, u, v, c are in $L^1[0,1]$ and $0 \leq r, k < 1$. Then there exists a solution $x \in C^1[0,1]$ to BVP (5.1), (5.5) with $\alpha \neq 1/\eta$ provided that

$$\begin{cases} \|a\|_1 + \|b\|_1 < 1/2 & \text{if } \alpha \leq 1, \\ \|a\|_1 + \|b\|_1 < \frac{1}{2}\left(1 - \frac{(\alpha-1)^2}{\alpha^2(1-\eta)^2}\right) & \text{if } 1 < \alpha < \frac{1}{\eta}, \\ \|a\|_1 + \|b\|_1 < \frac{1}{2}(1 - 1/(\alpha^2\eta^2)) & \text{if } \frac{1}{\eta} < \alpha. \end{cases} \tag{5.31}$$

Proof: By the same argument as in the proof of Theorem 5.3.1, it suffices to show that all possible solutions of the following family of equations

$$x''(t) = \lambda f(t, x, x') + \lambda e(t), \quad t \in (0,1), \tag{5.32}$$

$$x(0) = 0, \quad x(1) = \alpha x(\eta) \tag{5.33}$$

are bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

Suppose that x is a solution of (5.32) and let ζ be as in lemma 5.3.2, or remark 5.3.3 and write $C = \|c\|_1 + \|e\|_1$. Multiplying both sides of equation (5.32) by x' and integrating, we obtain

$$\begin{aligned}\frac{1}{2}x'^2(t) &\leq \frac{1}{2}x'^2(\zeta) + \|x'\|_\infty(\|a\|_1\|x\|_\infty + \|b\|_1\|x'\|_\infty + \|u\|_1\|x\|_\infty^r + \|v\|_1\|x'\|_\infty^k + C) \\ &= \frac{1}{2}C_0^2 x(1)^2 + \|x'\|_\infty(\|a\|_1\|x\|_\infty + \|b\|_1\|x'\|_\infty + \|u\|_1\|x\|_\infty^r + \|v\|_1\|x'\|_\infty^k + C).\end{aligned}$$

By (5.31),

$$\|a\|_1 + \|b\|_1 < \frac{1}{2}(1 - C_0^2).$$

Since $\|x\|_\infty \leq \|x'\|_\infty$, if $\|x'\|_\infty \neq 0$, we have

$$\left(\frac{1}{2}(1 - C_0^2) - (\|a\|_1 + \|b\|_1)\right)\|x'\|_\infty \leq \|u\|_1\|x'\|_\infty^r + \|v\|_1\|x'\|_\infty^k + \|c\|_1 + \|e\|_1.$$

This implies that $\|x'\|_\infty$ is bounded since $0 \leq r, k < 1$, that is, there is $M > 0$ such that $\|x\|_\infty \leq \|x'\|_\infty \leq M$. This completes the proof of the Theorem. □

Remark 5.3.5. The assumptions of Theorem 2.3, Theorem 2.4 of [30] and Theorem 3, Theorem 4 of [29] are special cases of our Theorem 5.3.1 and Theorem 5.3.4 when $g(t,x,p) \equiv 0$, $u(t) \equiv 0$ and $v(t) \equiv 0$. But their results allow $\|a\|_1 + \|b\|_1 < 1$ and our results above, in these special cases, need $\|a\|_1 + \|b\|_1 < 1/2$.

Remark 5.3.6. Equation (5.1) subject to the boundary condition

$$x(0) = x'(1) = 0 \tag{5.34}$$

can be considered as a limiting case of the boundary conditions $x(0) = 0, x(\eta) = x(1)$ when $\eta \to 1$. Hence we have the following result.

Corollary 5.3.7. *Assume that the mapping f is as in Theorem 5.3.4. Then there exists a solution $x \in C^1[0,1]$ to BVP (5.1), (5.34) provided that*

$$\|a\|_1 + \|b\|_1 < \frac{1}{2}.$$

In order to compare the above results with the results obtained in [24], [25], [29], [30], we consider the following simple examples.

Example 5.3.8. The equation

$$x'' = -x'^{2n+1} - x'^{2m+1}x^2\ln(1+x'^2) + \sin(t)$$

subject to the boundary condition (5.4) ($\alpha \neq 1$) has a solution by Theorem 5.3.1. Since we cannot find $p, q, r \in L^1[0,1]$ such that

$$\begin{aligned} |f(t,x,p)| &= |-p^{2n+1} - p^{2m+1}x^2\ln(1+p^2) + \sin(tp)| \\ &\leq p(t)|x| + q(t)|p| + r(t), \text{ for all } (t,x,p) \in [0,1] \times \mathbb{R}^2, \end{aligned}$$

(take $t_0 \in [0,1], x = 0$, and let $p \to +\infty$) we cannot apply the results of [30].

Example 5.3.9. Consider the equation

$$x'' = \frac{\pi^2}{4}x^{\frac{1}{2}} + \frac{\pi^2}{4}x'^{\frac{1}{2}} + \sin(t)$$

subject to the boundary condition

$$x(0) = 0,\ x(1) = x(1/2) \quad \text{or} \quad x(0) = 0,\ x'(1) = 0.$$

The existence of solutions followings Theorem 5.3.4 or Corollary 5.3.7. But it is obvious that the results of [24], [25] and [29] cannot be applied.

5.3.2 Results at resonance

In the following, we shall prove existence results for BVP (5.1), (5.4) with the condition $\alpha = 1$ and BVP (5.1), (5.5) with $\alpha = 1/\eta$ under the assumption that f has a nonlinear growth. In these cases, BVPs are the following:

$$x''(t) = f(t, x(t), x'(t)) + e(t) \qquad t \in (0,1), \tag{5.35}$$

subject to one of the following conditions:

$$x'(0) = 0, \qquad x(1) = x(\eta), \tag{5.36}$$

$$x(0) = 0, \qquad \eta x(1) = x(\eta). \tag{5.37}$$

Theorem 5.3.10. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and suppose f has the decomposition*

$$f(t,x,p) = g(t,x,p) + h(t,x,p).$$

Assume that

1. *There exists a constant $M \geq 0$ such that*

$$x[f(t,x,0) + e(t)] > 0, \text{ for } |x| > M, t \in [0,1];$$

2. *$pg(t,x,p) \leq 0$ for all $(t,x,p) \in [0,1] \times [-M,M] \times \mathbb{R}$;*

3. *$|h(t,x,p)| \leq a(t)|x| + b(t)|p| + u(t)|x|^r + v(t)|p|^k + c(t)$ for $(t,x,p) \in [0,1] \times [-M,M] \times \mathbb{R}$, where $0 \leq r, k < 1$ and $a, b, u, v, c \in L^1[0,1]$.*

Then the BVP (5.35), (5.36) has at least one solution in $C^1[0,1]$ provided that $\|b\|_1 < \frac{1}{2}$.

Proof: Let X and Z be the Banach spaces as in the proof of Theorem 5.3.1. Define L to be the linear operator from $D(L) \subset X$ to Z with

$$D(L) = \{x \in W^{2,1}(0,1) : \ x'(0) = 0, x(1) = x(\eta)\}$$

and $Lx = x''$, $x \in D(L)$. We define $N : X \to Z$ by setting

$$N(x)(t) = -f(t, x(t), x'(t)) - e(t), \quad t \in [0,1].$$

Then $L : D(L) \subset X \to Z$ is Fredholm of index zero (see [22]) and the continuous projections $P : X \to \ker(L)$ and $Q : Z \to Z_1$ can be defined by

$$(Px)(t) = x(0)$$

and

$$Qy = \frac{2}{1-\eta^2}\int_\eta^1\int_0^\tau y(s)\,ds\,d\tau,$$

where $\ker(L)$ and $\mathrm{im}(L)$ are described in section 5.1. We shall prove that the conditions of Theorem 5.1.1 are satisfied. Let

$$U_1 = \{x \in D(L) : Lx + \lambda N(x) = 0,\ \lambda \in (0,1)\}.$$

Suppose $x \in U_1$, and let $t_0 \in [0,1)$ be such that $|x(t_0)| = \max_{t\in[0,1]}|x(t)|$. Assume that $|x(t_0)| > M$. Then we have the following two cases,

Case 1: $t_0 \neq 0$.

If $x(t_0) > M$, then $x'(t_0) = 0, x''(t_0) \leq 0$, so we have

$$0 \geq x(t_0)x''(t_0) = \lambda x(t_0)[f(t_0, x(t_0), 0,) + e(t_0)] > 0,$$

a contradiction. If $x(t_0) < -M$, then $x'(t_0) = 0, x''(t_0) \geq 0$, we have

$$0 \geq x(t_0)x''(t_0) = \lambda x(t_0)[f(t_0, x(t_0), 0) + e(t_0)] > 0,$$

a contradiction again.

Case 2: $t_0 = 0$. If $x(0) > M$, then by condition (1),

$$x''(0) = \lambda[f(0, x(0), 0,) + e(0)] > 0.$$

This implies that $x'(t)$ is increasing for sufficiently small t. Since $x'(0) = 0$, $x'(t) > 0$ for t small enough. Thus $x(t)$ is increasing, contradicting $x(0) = \max_{t\in[0,1]}|x(t)|$.

If $x(0) < -M$, then a similar argument shows that x is decreasing and a contradiction is obtained. Thus we have shown

$$\|x\|_\infty \leq M,\ x \in U_1. \tag{5.38}$$

Next, for $x \in U_1$,

$$x'x'' = \lambda x'g(t,x,x') + \lambda x'h(t,x,x') + \lambda x'e(t).$$

This implies that, for every $t \in [0,1]$,

$$\begin{aligned}
\frac{1}{2}x'^2(t) &= \lambda\int_0^t x'g(s,x,x')\,ds + \lambda\int_0^t x'h(s,x,x')\,ds + \lambda\int_0^t x'e(s)\,ds \\
&\leq \int_0^1 |x'h(t,x,x')|\,dt + \|x'\|_\infty\|e(t)\|_1 \\
&\leq \|x'\|_\infty\left(\int_0^1 |h(t,x,x')|\,dt + \|e(t)\|_1\right).
\end{aligned}$$

Assume that $\|x'\|_\infty \neq 0$, then (5.38) ensures that

$$\frac{1}{2}\|x'\|_\infty \leq \|b\|_1\|x'\|_\infty + \|v\|_1\|x'\|_\infty{}^k + C_1,$$

where $C_1 = \|a\|_1 M + \|u\|_1 M^r + \|c\|_1 + \|e\|_1$. By our assumption $\|b\|_1 < \frac{1}{2}$, so

$$\left(\frac{1}{2} - \|b\|_1\right)\|x'\|_\infty \leq \|v\|_1\|x'\|_\infty{}^k + C_1.$$

As $0 \leq k < 1$, there exists $M_1 > 0$ such that $\|x'\|_\infty \leq M_1$. Hence

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\} \leq \max\{M, M_1\},$$

and we have proved that U_1 is bounded.

Let $U_2 = \{x \in \ker L : Nx \in \operatorname{im}(L)\}$. Suppose that $x \in U_2$ and $x \equiv C_0$ for $t \in [0,1]$. Then $C_0 > M$ implies that

$$\int_\eta^1 \int_0^\tau (f(t, C_0, 0) + e(t))\, dt\, d\tau > 0$$

and $C_0 < -M$ implies that

$$\int_\eta^1 \int_0^\tau (f(t, C_0, 0) + e(t))\, dt\, d\tau < 0.$$

In both cases, $N(C_0) = f(t, C_0, 0) + e(t) \notin \operatorname{im}(L)$. Therefore $\|x\|_\infty = |C_0| \leq M$.

Next let $U_3 = \{x \in \ker(L) : H(x,\mu) = \mu QNx + (1-\mu)x = 0,\ \mu \in [0,1]\}$. For $x \equiv C_0 \in U_3$, we have

$$\frac{2\mu}{1-\eta^2}\int_\eta^1 \int_0^\tau (f(s, C_0, 0) + e(s))\, ds\, d\tau = -(1-\mu)C_0.$$

If $\mu = 0$, then $C_0 = 0$. If $\mu > 0$, suppose $C_0 > M$, then

$$f(s, C_0, 0) + e(s) > 0 \quad \text{for all } s \in [0,1],$$

contradicting $-(1-\mu)C_0 \leq 0$. On the other hand, if $C_0 < -M$, then

$$f(s, C_0, 0) + e(s) < 0 \quad \text{for all } s \in [0,1],$$

contradicting $-(1-\mu)C_0 \geq 0$. Thus,

$$U_3 \subset \{x \in \ker(L) : \|x\| \leq M\}.$$

Now, writing $X = \ker(L) \oplus X_1$, $Z = \mathrm{im}(L) \oplus Z_1$ and $L_1 = L|_{D(L)\cap X_1}$, the operator $K = {L_1}^{-1} : \mathrm{im}(L) \to D(L) \cap X_1$ is the linear operator defined by (see section 5.1)

$$(Ky)(t) = \int_0^t (t-s)y(s)\,ds \quad \text{for } y \in \mathrm{im}(L).$$

By the Arzéla-Ascoli Theorem, it can be shown that K is compact [45], so N is L-compact. Let Ω be a bounded open subset of X such that $\overline{\bigcup_{i=1}^{3} U_i} \subset \Omega$. The above bounds show that the hypotheses of Theorem 5.1.1 are satisfied. Hence $Lx + Nx = 0$ has at least one solution in $D(L) \cap \overline{\Omega}$, and the BVP (5.35), (5.36) has a solution. □

If we let $\eta \to 1$ in Theorem 5.3.10, as in Corollary 5.3.7, we obtain the following result:

Corollary 5.3.11. *Let f, g, h be mappings as in Theorem 5.3.10. Then for every $e \in L^1[0,1]$, the following Neumann boundary value problem*

$$x''(t) = f(t, x(t), x'(t)) + e(t), \tag{5.39}$$

$$x'(0) = x'(1) = 0, \tag{5.40}$$

has at least one solution in $C^1[0,1]$ provided $\|b\|_1 < 1/2$.

Remark 5.3.12. Comparing Corollary 5.3.11 with Theorem 2.1 of [61], we can see that these two results concerning Neumann BV problems are very different. Neither contains the other. The following example shows that there exist equations which can be treated by our Corollary 5.3.11 but their Theorem 2.1 cannot be used.

Example 5.3.13. Consider the boundary value problem

$$x'' = -x'^{2n+1}x^{2m} + x + \sin(t) + \cos(t), \tag{5.41}$$

$$x'(0) = x'(1) = 0, \tag{5.42}$$

Let $g(t,x,p) = -x'^{2n+1}x^{2m}$, $h(t,x,p) = x + \sin(t) + \cos(t)$, then corollary 5.3.11 ensures that there exists a solution $x \in C^1[0,1]$ to (5.41), (5.42). But, it is easily seen that

Theorem 2.1 of [61] does not apply to the above problem. Moreover, by Theorem 5.3.10, for $\eta \in (0,1)$, equation (5.41) subject to the following boundary conditions:

$$x'(0) = 0, \quad x(1) = x(\eta),$$

has at least one solution in $C^1[0,1]$.

We now treat the boundary condition (5.37). In the following, we assume that the mapping N and the linear operator L are the same as in the proof of Theorem 5.3.10 and let

$$D(L) = \{x \in W^{2,1}(0,1) : \ x(0) = 0, \ \eta x(1) = x(\eta)\}.$$

Theorem 5.3.14. *Let f satisfy the following conditions:*

(1) There exists $M_1 > 0$ such that, for $x \in D(L)$, if $|x'(t)| > M_1$ for all $t \in [0,1]$, then

$$\int_0^1 \int_{\eta s}^{s} (f(t, x(t), x'(t)) + e(t)) \, dt \, ds \neq 0;$$

(2) There exists $M_2 > 0$, such that for all $v \in \mathbb{R}$ with $|v| > M_2$ one has either

$$(2a) \quad v(f(t, vt, v) + e(t)) \geq 0 \quad \text{for a.e.} \quad t \in [0,1],$$

or

$$(2b) \quad v(f(t, vt, v) + e(t)) \leq 0 \quad \text{for a.e.} \quad t \in [0,1];$$

(3) $|f(t,x,p)| \leq a(t)|x| + b(t)|p| + u(t)|x|^r + v(t)|p|^k + c(t)$ for $(t,x,p) \in [0,1] \times [-M_1, M_1] \times \mathbb{R}$, where $0 \leq r, k < 1$ and $a, b, u, v, c \in L^1[0,1]$.

Then the BVP (5.35), (5.37) has at least one solution in $C^1[0,1]$ provided that

$$\|a\|_1 + \|b\|_1 < \frac{1}{2}.$$

Proof: Let

$$U_1 = \{x \in D(L) : Lx + \lambda Nx = 0, \ \lambda \in (0,1)\}.$$

Then for $x \in U_1$, $Nx \in \text{im}(L)$. By the characterisation of $\text{im}(L)$ (see section 5.1),

$$\int_0^1 \int_{\eta s}^s (f(t, x(t), x'(t)) + e(t))\, dt\, ds = 0.$$

So, by our assumption (1), there exists $\xi \in [0,1]$ such that $|x'(\xi)| \le M_1$. By a similar argument to that in the proof of Theorem 5.3.10 we have

$$\begin{aligned}
\frac{1}{2}x'^2(t) &\le \frac{1}{2}x'^2(\xi) + \|x'\|_\infty \int_0^1 |f(t,x,x')|\, dt + \|x'\|_\infty \|e\|_1 \\
&\le \frac{1}{2}{M_1}^2 + \|x'\|_\infty(\|a\|_1\|x\|_\infty + \|b\|_1\|x'\|_\infty + \|u\|_1\|x\|_\infty^k + \|v\|_1\|x'\|_\infty^r + \|e\|_1).
\end{aligned}$$

Since $x(0) = 0$, $\|x\|_\infty \le \|x'\|_\infty$, we obtain

$$\left(\frac{1}{2} - \|a\|_1 - \|b\|_1\right) {\|x'\|_\infty}^2 \le \frac{1}{2}{M_1}^2 + \|x'\|_\infty^{k+1} + \|v\|_1\|x'\|_\infty^{r+1} + \|x'\|_\infty\|e\|_1.$$

This implies that there exists $M_0 > 0$ such that $\|x\|_\infty \le \|x'\|_\infty \le M_0$.

Now, let

$$U_2 = \{x \in \ker(L) : \quad Nx \in \text{im}(L)\}.$$

For $x \in U_2$, it is easy to see that $x = C_0 t$ for some $C_0 \in \mathbb{R}$. $Nx \in \text{im}(L)$ implies that

$$\int_0^1 \int_{\eta s}^s (f(t, C_0 t, C_0) + e(t))\, dt\, ds = 0.$$

By our assumption (1), $\|x\|_\infty = |C_0| \le M_1$. Under the assumption (2a), we let

$$U_3 = \{x = C_0 t \in \ker(L) : \ \mu QNC_0 + (1-\mu)C_0 = 0\},$$

and under the assumption (2b), we let

$$U_3 = \{x = C_0 t \in \ker(L) : \ \mu QNC_0 - (1-\mu)C_0 = 0\}.$$

Using (2), it is easily proved that U_3 is bounded. So, following the same argument as in the proof of Theorem 5.3.10, the proof of Theorem 5.3.14 is completed. □

5.4 Quadratic growth

In this section, we shall prove an existence result for the BVP (5.1), (5.4), thus the following problem

$$x''(t) = f(t, x(t), x'(t)) + e(t) \qquad t \in (0,1), \tag{5.43}$$

$$x'(0) = 0, \quad x(1) = \alpha x(\eta), \tag{5.44}$$

with $|\alpha| \le 1$ and f has a different nonlinear growth from that in section 5.3. As a special case, we allow f to have quadratic growth. Moreover, as a corollary of our theorem, we obtain a result on the Neumann BVP which generalizes one of the results of [57]. We also prove a similar result for the BVP (5.1), (5.2) when $|\sum_{i=1}^{m-2} a_i| \le 1$.

5.4.1 Boundary value problem (5.43), (5.44)

Theorem 5.4.1. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and have the decomposition*

$$f(t,x,p) = g(t,x,p) + h(t,x,p).$$

Suppose the following conditions hold:

1. *There exists a constant $M \ge 0$ such that*

$$x[f(t,x,0) + e(t)] > 0 \quad \text{for} \quad |x| > M, \quad t \in [0,1];$$

2. *$pg(t,x,p) \le 0$ for all $(t,x,p) \in [0,1] \times [-M,M] \times \mathbb{R}$;*

3. *There are continuous functions $A, B : [0,1] \times [-M,M] \to \mathbb{R}^+$ such that*

$$|h(t,x,p)| \le A(t,x)p^2 + B(t,x) \quad \text{for} \quad p \in \mathbb{R}.$$

Then the BVP (5.43), (5.44) with $|\alpha| \le 1$ has at least one solution in $C^1[0,1]$.

Proof: Let X denote the Banach space $C^1[0,1]$ with the norm

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}.$$

Let L, N, P and Q be the operators defined in section 5.1. Again, we shall prove that the conditions of Theorem 5.1.1 are satisfied. Let

$$U_1 = \{x \in D(L) : Lx + \lambda N(x) = 0, \lambda \in (0,1)\}.$$

Suppose $x \in U_1$, then $|\alpha| \leq 1$ implies that there exists $t_0 \in [0,1)$ such that $|x(t_0)| = \max_{t\in[0,1]} |x(t)|$. As in the proof of theorem 5.3.10, we obtain that

$$\|x(t)\|_\infty \leq M \quad \text{for all} \quad x \in U_1. \tag{5.45}$$

Since $x'(0) = 0$, each $t \in [0,1]$ for which $x'(t) \neq 0$ belongs to $[\mu,\gamma]$ such that $x'(t)$ maintains a fixed sign on $[\mu,\gamma]$ and $x'(\mu) = 0$. Assume that $x'(t) \geq 0$ on $[\mu,\gamma]$, let $x \in U_1$, then

$$\begin{aligned} x'x'' &= \lambda x' g(t,x,x') + \lambda x' h(t,x,x') + \lambda x' e(t) \\ &\leq \lambda x' h(t,x,x') + \lambda x' e(t) \\ &\leq x'(Ax'^2 + B), \end{aligned} \tag{5.46}$$

where

$$A = \max\{|A(t,x)| : t \in [0,1], |x| \leq M\}$$

and

$$B = \max\{|B(t,x)| + \|e\|_\infty : t \in [0,1], |x| \leq M\}.$$

Therefore,

$$\int_\mu^t \frac{2Ax''x'}{Ax'^2 + B}\, ds \leq 2A \int_\mu^t x'\, ds.$$

Hence we have

$$\ln\left(\frac{Ax'^2 + B}{B}\right) \leq 4AM,$$

and so

$$|x'(t)| \leq M_1 \quad \text{for some } M_1 > 0.$$

If, on the other hand, $x'(\mu) = 0$ and $x'(t) \leq 0$ on $[\mu,\gamma]$, similarly we get

$$x'x'' \leq -x'(Ax'^2 + B),$$

from which, as before, we obtain the bound M_1 on x'. Hence for each $x \in U_1, \|x\| \leq \max\{M, M_1\}$.

If $\alpha \neq 1$, $P = 0$, $Q = 0$, and also N is a L-compact operator [22]. Condition (2) and (3) of Theorem 5.1.1 are trivial. Hence the existence of a solution for BVP (5.43), (5.44) follows from the boundedness of U_1. Now, suppose that $\alpha = 1$, as in the proof of theorem 5.3.10, let

$$U_2 = \{x \in \ker L : Nx \in \operatorname{im}(L)\}$$

and

$$U_3 = \{x \in \ker(L) : H(x, \mu) = \mu QNx + (1-\mu)x = 0,\ \mu \in [0,1]\}.$$

By the same arguments as that in the proof of that theorem, we obtain that U_2 and U_3 are bounded and then again, the existence of a solution follows from Theorem 5.1.1. □

Remark 5.4.2. In Theorem 5.4.1, taking $g(t, x, p) = 0$, gives a result which allows f to have quadratic growth.

In Theorem 5.4.1, let $\alpha = 1$, then there exists $\xi \in (\eta, 1)$ such that $x'(\xi) = 0$. So, accordingly, we can consider equation (5.43) subject to the following Neumann boundary value condition

$$x'(0) = x'(1) = 0. \tag{5.47}$$

The following corollary follows immediately from Theorem 5.4.1 when $\eta \to 1$.

Corollary 5.4.3. *Assume that the mappings f, g, h are as in Theorem 5.4.1. Then the BVP (5.43), (5.47) has at least one solution in $C^1[0,1]$.*

Remark 5.4.4. In Corollary 5.4.3, letting $g(t, x, p) = 0$, we obtain Corollary 3.1 of [57] for the boundary condition $x'(0) = x'(1) = 0$ (condition II of [57]). By the same method used in the proof of Theorem 5.4.1, we can prove that the result of Theorem 5.4.1 is also true for the other boundary conditions studied in [57].

Example 5.4.5. Consider the following equation:

$$x'' = -x'^{2n+1}x^{2m} + x'^2 + x + \beta\sin(t), \quad t \in [0,1], \tag{5.48}$$

where n, m are natural numbers. The existence of a solution for the BVP (5.48), (5.44) (respectively BVP (5.48), (5.47)) can be obtained by Theorem 5.4.1 (respectively Corollary 5.4.3). But, it is easy to see that the results of [30] and [57] can not be applied to the above problems.

5.4.2 M-point boundary value problem

Now, we prove an existence result for the following m-point boundary value problem:

$$x''(t) = f(t, x(t), x'(t)) + e(t) \qquad t \in (0, 1), \tag{5.49}$$

$$x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{5.50}$$

with nonlinear growth term.

Theorem 5.4.6. *Let f, g, h satisfy the assumptions of Theorem 5.4.1 and suppose that $|\sum_{i=1}^{m-2} a_i| \leq 1$. Then for each bounded function $e : [0, 1] \to \mathbb{R}$, the BVP (5.49), (5.50) has at least one solution in $C^1[0, 1]$.*

Proof: Let Banach spaces X, Y and the bounded nonlinear map N be as in the proof of Theorem 5.4.1. Let $L_m : D(L_m) \subset X \to Y$ be the linear operator with

$$D(L_m) = \{x \in W^{2,1}(0, 1) : \ x'(0) = 0, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)\},$$

and for $x \in D(L_m), L_m x = x''$. We note that $x \in C^1[0, 1]$ is a solution of (5.49), (5.50) if and only if $x \in D(L_m)$ and

$$L_m x + N x = 0.$$

Let $U_{m1} = \{x \in D(L_m) : L_m(x) + \lambda N(x) = 0, \lambda \in (0, 1)\}$. Suppose that $x \in U_{m1}$, then $x \in W^{2,1}(0, 1)$ with $x'(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$. This implies that there exists $\eta \in [\xi_1, \xi_{m-2}]$, depending on x, such that x is a solution of the boundary value problem

$$x''(t) - \lambda f(t, x(t), x'(t)) - \lambda e(t) = 0, \tag{5.51}$$

$$x'(0) = 0, \qquad x(1) = \alpha x(\eta), \tag{5.52}$$

where $\alpha = \sum_{i=1}^{m-2} a_i$, and so $|\alpha| \leq 1$. Note that in the proof of the boundedness of U_1 in the proof of Theorem 5.4.1, we obtained that $U_1 \subset \{x \in X : \|x\| \leq \max\{M, M_1\}\}$ and M_1 only depends on M, A and B, so the bound of U_1 does not depend on η. Thus by the proof given there, if x is a solution of (5.51), (5.52) then $\|x\| \leq \max\{M, M_1\}$. Hence U_{m1} is bounded.

First, suppose that $\sum_{i=1}^{m-2} a_i \neq 1$, then L_m is an invertible linear operator. The linear operator $K_m : Y \to D(L_m)$ defined by

$$K_m y(t) = \int_0^t (t-s)y(s)\, ds + C, \quad y \in Y,$$

where $C\left(1 - \sum_{i=1}^{m-2} a_i\right) = \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)\, ds - \int_0^1 (1-s)y(s)\, ds$, is such that $K_m L_m = I$ and $L_m K_m = I$. Also, by Arzéla-Ascoli theorem, $K_m N : X \to X$ is compact, thus N is a L_m-compact map. By Theorem 5.1.1, the existence of the solution follows from the boundedness of U_{m1}.

Now suppose that $\sum_{i=1}^{m-2} a_i = 1$. Then $\ker(L_m) = \mathbb{R} \subset X$ and we can prove that $Y = \mathbb{R} \oplus Y_1$, where

$$Y_1 = \operatorname{im}(L_m) = \left\{ y \in L^1[0,1] : \quad \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \int_0^\tau y(s)\, ds\, d\tau = 0 \right\},$$

hence L_m is Fredholm of index zero. Let $P : X \to \ker(L_m)$ and $Q : Y \to \mathbb{R}$ be the projections defined by

$$Px = x(0), \quad \text{and} \quad Qy = \frac{2}{\sum_{i=1}^{m-2} a_i(1-\xi_i{}^2)} \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \int_0^\tau y(s)\, ds\, d\tau.$$

Let $X_1 = (I-P)X$ and let $K_m : Y_1 \to D(L_m) \cap X_1$ be the linear operator defined by

$$(K_m y)(t) = \int_0^t \int_0^\tau y(s)\, ds\, d\tau.$$

It is easy to check that $K_m = (L_m|_{D(L_m)\cap X_1})^{-1}$. Also, by the Arzéla-Ascoli theorem,

$$K_m(I-Q)N : X \to D(L_m) \cap X_1$$

is compact. So, N is L_m-compact. Let U_{m2} and U_{m3} denote the following subsets of X:

$$U_{m2} = \{x \in \ker(L_m) : \; Nx \in \operatorname{im}(L_m)\}, \quad U_{m3} = \{\mu QNx + (1-\mu)x = 0, \; \mu \in [0,1]\}.$$

Then by Theorem 5.1.1, to prove there exists a solution for BVP (5.49), (5.50), we only need to prove U_{m2} and U_{m3} are bounded. By the assumption that all a_i have the same sign and by similar arguments to those given in the proof of the boundedness of U_2 and U_3 in the proof of Theorem 5.4.1, we can obtain that for all $x \in U_{m1}$ and all $x \in U_{m2}$, we have $\|x\| \leq M$. This completes the proof. $\square$

Remark 5.4.7. For our results it is important that all the a_i's have the same sign and our result for the m-point BVP makes use of the estimates obtained in the proof for the three point BVP. Gupta [27] has considered a different m-point boundary value problem where the a_i's do not have the same sign and the above technique cannot be used.

Chapter 6

Two-point boundary value problems

Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. The purpose of this chapter is to establish some new existence results on the solvability of second order ODE's of the form

$$x'' = f(t, x, x') \tag{6.1}$$

subject to one of the following boundary conditions:

$$x(0) = x(1) = 0, \tag{6.2}$$

$$x'(0) = x'(1) = 0, \tag{6.3}$$

$$x(0) = x(1), \quad x'(0) = x'(1), \tag{6.4}$$

$$x(0) = -x(1), \quad x'(0) = -x'(1). \tag{6.5}$$

The solvability of (6.1) subject to the above Dirichlet, Neumann, periodic and antiperiodic boundary conditions has been extensively studied by many authors (see[3], [4], [32], [54]-[63]). In a recent paper [4], a decomposition condition for f is imposed to ensure the solvability of (6.1) with the boundary condition (6.2). In this chapter, under the assumption that f can be suitably decomposed, we shall apply the abstract continuation type theorem of W.V.Petryshyn on A-proper mappings to prove approximation solvability results for (6.1) with the boundary conditions (6.2)-(6.5). Approximation solvability includes the classical Galerkin method. One of our theorems includes the result of [4]. When f is independent of x'', our results generalize the results of [60], [61] and show that certain restrictions imposed in [60], [61] are not needed in this case. Some examples show that our theorems permit the treatment of equations to which the results of [4], [32], [57] do not apply. The work in this chapter has been submitted in [19].

6.1 Dirichlet boundary value problem

The proof of the following theorems make use of the continuation theorem on A-proper mappings (see theorem 1.6.3 and section 1.6 for definitions and references).

We shall call equation (6.1) subject to the boundary conditions (6.2)-(6.5) (P1)-(P4) respectively. Our first three theorems deal with the simple case (P1).

Theorem 6.1.1. *Let $f : [0,1] \times \mathbb{R}^2 \to R$ be continuous. Consider the following BVP:*

$$x'' = f(t, x, x'), \quad x(0) = x(1) = 0. \tag{P1}$$

Assume that f has the decomposition

$$f(t, x, p) = g(t, x, p) + h(t, x, p)$$

such that

1. $\int_0^1 xg(t, x, x')\,dt \geq 0$ *for all $x \in C^2[0,1]$ with $x(0) = x(1) = 0$,*
2. $|h(t, x, p)| \leq a|x| + b|p|$, *where $a > 0, b > 0$ and $a + b\pi < \pi^2$.*

Then (P1) is feebly a-solvable in $C^2[0,1]$.

Proof: Let $X = C_0^2 = \{x \in C^2[0,1], \quad x(0) = x(1) = 0\}$ endowed with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}$, where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Let $\|\cdot\|_2$ be the usual norm of $L^2(0,1)$ and let $L : X \to C[0,1]$ be the linear operator defined by

$$Lx = x'' \qquad \text{for } x \in X.$$

Define $N : C^1[0,1] \to C[0,1]$ to be the nonlinear mapping

$$Nx(t) = f(t, x(t), x'(t)).$$

Let $J : C_0^2 \to C^1[0,1]$ denote the compact natural embedding. Since NJ is compact, $L - \lambda NJ : C_0^2 \to C[0,1]$ is *A-proper* for each $\lambda \in [0,1]$, [54]. Also, L is invertible, so by Theorem 1.6.3, the *a-solvability* of (P1) follows provided there exists an open bounded set $G \subset C_0^2$ such that

$$Lx - \lambda NJx \neq 0 \quad \text{for} \quad (x, \lambda) \in (C_0^2 \cap \partial G) \times (0, 1].$$

This is equivalent to proving the following subset of C_0^2 is bounded:

$$U = \{x \in C_0^2, \quad Lx - \lambda NJx = 0, \ \lambda \in (0,1]\}.$$

Let $x \in U$, then

$$x'' = \lambda(g(t, x, x') + h(t, x, x')).$$

Applying Wirtinger's inequality [35]: $\|x\|_2 \leq (1/\pi)\|x'\|_2$ we obtain

$$\begin{aligned}
\|x'\|_2^2 &= -\int_0^1 xx''\, dt \\
&= -\lambda \int_0^1 xg(t, x, x')\, dt - \lambda \int_0^1 xh(t, x, x')\, dt \\
&\leq -\lambda \int_0^1 xh(t, x, x')\, dt \\
&\leq a \int_0^1 |x|^2\, dt + b \left(\int_0^1 |x|^2\, dt\right)^{\frac{1}{2}} \left(\int_0^1 |x'|^2\, dt\right)^{\frac{1}{2}} \\
&\leq \frac{a + b\pi}{\pi^2} \|x'\|_2^2.
\end{aligned}$$

By our assumption, $a + b\pi < \pi^2$, so $x' = 0$. Since $x \in C_0^2$, $x = 0$. This completes the proof. $\square$

Remark 6.1.2. In the case $g(t, x, x') = r(x)x'$, where r is continuous and $r(x) \in C^1[0,1]$, the condition 1 of Theorem 6.1.1 is always satisfied, since $\int_0^1 xr(x)x'\, dt = 0$ for all $x \in C_0^2$.

We will use the following condition A (see [4]) and condition B for a continuous function $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$.

Condition A: $|g(t, x, p)| \leq A(t, x)\omega(p^2)$ for all $(t, x, p) \in [0,1] \times \mathbb{R}^2$, where $A(t, x)$ is bounded on each compact subset of $[0,1]\times\mathbb{R}$, $\omega\in C(\mathbb{R},(0,+\infty))$ is nondecreasing and satisfies

$$\int_0^{+\infty}\frac{ds}{\omega(s)}=\infty.$$

Condition B: $|g(t,x,p)|\leq\sum_{i=1}^{r}B_i(t,x)\omega_i(p)$ for all $(t,x,p)\in[0,1]\times\mathbb{R}^2$, where $B_i(t,x)$ is bounded on compact subsets of $[0,1]\times\mathbb{R}$ and $\omega_i(p)$ are functions such that

$$\int_0^1|x'(t)|^2\,dt\leq M\quad\text{implies}\quad\int_0^1|\omega_i(x'(t))|\,dt\leq M_0,$$

where M, M_0 are constants, M_0 may depend on M.

The following theorem is a generalization of Theorem 1 of [4].

Theorem 6.1.3. *Let f have the decomposition*

$$f(t,x,p)=g(t,x,p)+h(t,x,p).$$

Assume that

1. $\int_0^1 xg(t,x,x')\,dt\geq 0$ *for all* $x\in C_0^2$;
2. $|h(t,x,p)|\leq a|x|+b|p|+\sum_{i=1}^{n}c_i|x|^{\alpha_i}+\sum_{j=1}^{m}d_j|p|^{\beta_j}$,
   where $a\geq 0, b\geq 0,\quad 0\leq\alpha_i,\beta_j<1$;
3. $g(t,x,p)$ *satisfies condition A or condition B.*

Then (P1) is feebly a-solvable in $C^2[0,1]$ provided that $a+b\pi<\pi^2$.

Proof: By the same argument as in the proof of Theorem 6.1.1, we only need to prove that the set

$$U=\{x\in C_0^2,\quad Lx-\lambda NJx=0,\ \lambda\in(0,1)\}$$

is bounded. As in the proof of Theorem 6.1.1, for $x \in U$,

$$\begin{aligned}
\|x'\|_2^2 &\le \int_0^1 |xh(t,x,x')|\,dt \\
&\le \int_0^1 |x| \left(a|x| + b|x'| + \sum_{i=1}^n c_i |x|^{\alpha_i} + \sum_{j=1}^m d_j |x'|^{\beta_j} \right) dt \\
&\le a\|x\|_2^2 + b\|x\|_2\|x'\|_2 + \sum_{i=1}^n c_i \|x\|_2 \left(\int_0^1 |x|^{2\alpha_i} \right)^{\frac{1}{2}} + \sum_{j=1}^m d_j \|x\|_2 \left(\int_0^1 |x'|^{2\beta_j} \right)^{\frac{1}{2}} \\
&\le \left(\frac{a}{\pi^2} + \frac{b}{\pi} \right) \|x'\|_2^2 + \sum_{i=1}^n \frac{c_i}{\pi} \|x'\|_2 \|x\|_2^{\alpha_i} + \sum_{i=j}^m \frac{d_j}{\pi} \|x'\|_2 \|x'\|_2^{\beta_j} \\
&\le \left(\frac{a}{\pi^2} + \frac{b}{\pi} \right) \|x'\|_2^2 + \frac{1}{\pi} \sum_{i=1}^n c_i \|x'\|_2^{1+\alpha_i} + \frac{1}{\pi} \sum_{j=1}^m d_j \|x'\|_2^{1+\beta_j}.
\end{aligned}$$

Suppose that $\|x'\|_2 \neq 0$, since otherwise $x = 0$. By our assumption $(a + b\pi)/\pi^2 < 1$, we have

$$\left(1 - \frac{a + b\pi}{\pi^2} \right) \|x'\|_2 \le \frac{1}{\pi} \sum_{i=1}^n c_i \|x'\|_2^{\alpha_i} + \frac{1}{\pi} \sum_{j=1}^m d_j \|x'\|_2^{\beta_j}.$$

If $\|x'\|_2 \to \infty$, we will have a contradiction since $0 \le \alpha_i, \beta_i < 1$. So there exists a constant $M > 0$ such that $\|x'\|_2 \le M$. This implies

$$\|x\|_\infty \le \int_0^1 |x'|\,dt \le \|x'\|_2 \le M.$$

Suppose that g satisfies condition A, then

$$|x''| \le A_1 \omega(x'^2) + C + b|x'| + \sum_{j=1}^m d_j |x'|^{\beta_j},$$

where A_1, C are positive constants. Since

$$|x'|^{\beta_j} \le \frac{1}{2}(1 + |x'|^{2\beta_j}) \le 1 + |x'|^2,$$

we have

$$\begin{aligned}
|x''| &\le A_1 \omega(x'^2) + C + d(1 + |x'|^2) \\
&\le A(\omega(x'^2) + 2 + |x'|^2), \qquad (6.6)
\end{aligned}$$

where $A = \max\{A_1, C, d\}$. As in the proof of Theorem 1 of [4], (6.6) implies that $|x'|$ is bounded (for completeness, we give the proof here). Each $t \in [0,1]$ for which $x'(t) \neq 0$ belongs to some interval $[s_1, s_2] \subset [0,1]$ with $x'(t) \neq 0$ on (s_1, s_2) and $x'(s_1) = 0$ or

$x'(s_2) = 0$. Suppose that $x'(s_1) = 0$ and $x'(t) > 0$ on (s_1, s_2). Define $z(t) = x'(t), t \in [s_1, s_2]$. Then (6.6) implies that

$$\frac{2z(t)z'(t)}{\omega(z^2(t)) + z^2(t) + 2} \leq 2Ax'(t),\ t \in [s_1, s_2].$$

By integrating this inequality we obtain

$$\int_0^{z^2(t)} \frac{ds}{\omega(s) + s + 2} \leq 4AM,\ t \in (s_1, s_2).$$

The assumption $\omega \in C(\mathbb{R}, (0, +\infty))$ is nondecreasing and satisfies

$$\int_0^{+\infty} \frac{ds}{\omega(s)} = \infty,$$

implies (see [3]) that

$$\int_0^{\infty} \frac{ds}{\omega(s) + s + 2} = \infty.$$

This ensures there exists a constant $M_1 > 0$ such that $|x'(t)| \leq M_1,\ t \in [s_1, s_2]$. Considering all the possible cases, we obtain there exists a constant M_1 such that $\|x'\|_\infty \leq M_1$. Let

$$M_2 = \sup_{t\in[0,1],|x|\leq M,|p|\leq M_1} |f(t, x, p)|,$$

then $\|x\| \leq \max\{M, M_1 M_2\}$. Hence, U is bounded.

If g satisfies condition B, then there exists $A_2 > 0$ such that

$$|x''| \leq A_2 \left(\sum_{i=1}^{r} |\omega_i(x')| + |x'|^2 + 1 \right).$$

Hence

$$\int_0^1 |x''|\, dt \leq A_2 \left(\sum_{i=1}^{r} \int_0^1 |\omega(x')|\, dt + \int |x'|^2\, dt + 1 \right) \leq A_2(rM_0 + M + 1) = M_3.$$

Suppose that $\xi \in [0, 1]$ is such that $x'(\xi) = 0$. Then $x'(t) = \int_\xi^t x''(s)\, ds$, and hence

$$\|x'\|_\infty \leq \|x''\|_1 \leq M_3.$$

It follows that U is bounded. □

Remark 6.1.4. Theorem 1 of [4] is the special case of our Theorem 6.1.3 when $a = 0$, $b = 0$ and $n = m = 1$.

Example 6.1.5. Consider the following boundary value problem:

$$x'' = x^{2n+1}x'^2 + x' - x^{\frac{1}{3}} + 1 + \sin(t), \quad x(0) = x(1) = 0,$$

where n is a natural number. Let

$$g(t, x, p) = x^{2n+1}p^2 \quad \text{and} \quad h(t, x, p) = p - x^{\frac{1}{3}} + 1 + \sin(t).$$

Then by Theorem 6.1.3, this BVP is feebly a-solvable in $C^2[0, 1]$ and in particular it has a solution in $C^2[0, 1]$.

Obviously, Theorem 1 of [4] can not be applied to it. Also, we can not find constants $M > 0$ and $a, b \in \mathbb{R}$ such that

$$x \geq M \Longrightarrow f(t, x, 0) > a \quad \text{while} \quad x \leq -M \Longrightarrow f(t, x, 0) < b$$

since $f(t, x, 0) \to -\infty$ as $x \to \infty$ and $f(t, x, 0) \to \infty$ as $x \to -\infty$. Hence Theorem 4.1 of [32] and Theorem 2.1 of [57] can not be applied.

6.2 Neumann, periodic and antiperiodic boundary value problems

Now, we consider (P2), (P3) and (P4). These are resonance cases, since the linear part is noninvertible. In the following, let

$$X_i = \{x \in C^2[0, 1] : \ x \text{ satisfies the boundary condition } (1.i), i = 2, 3 \text{ or } 4\}.$$

and

$$U_i = \{x \in X_i : x'' = \lambda f(t, x, x'), \lambda \in (0, 1]\}.$$

Theorem 6.2.1. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that*

$$f(t, x, p) = g(t, x, p) + h(t, x, p),$$

and f, g and h satisfy the following conditions:

1. *There exists a constant $M_0 > 0$ such that $xf(t,x,0) > 0$ for $|x| > M_0$;*

2. *(a): g satisfies Condition A or*

   (b): g satisfies Condition B and $\int_0^1 xg(t,x,x')\,dt \geq 0$ for all $x \in X_i$;

3. $|h(t,x,p)| \leq C(t,x) + D(t,x)|p|^2 + \sum_{j=1}^n d_j(t,x)|p|^{\beta_j}$,

   where $C(t,x), D(t,x)$, and $d_j(t,x)$ are bounded on compact subsets of $[0,1]\times\mathbb{R}$ and

   $0 \leq \beta_j < 2$.

Let $M = \max_{t\in[0,1],|x|\leq M_0} |D(t,x)|$, then (Pi) is feebly a-solvable relative to Γ provided that $M_0 M < 1$.

Proof: Let $L : X_i \to C[0,1]$ be the linear operator defined by $Lx = x''$. Then it is easily seen that L is a Fredholm operator of index zero and $\ker(L) = \mathbb{R}$. Let $Nx = f(t,x,x')$ be the nonlinear map from $C^1[0,1]$ to $C[0,1]$ and $J_i : X_i \to C^1[0,1]$ be the compact continuous embedding. Then $L - \lambda NJ_i$ is A-proper for each $\lambda \in [0,1]$. Moreover, let $Qy = \int_0^1 y\,dt$ be the projection and

$$[y,x] = \int_0^1 y(t)x(t)\,dt$$

be the bilinear form on $C[0,1]\times X_i$. For any $x \equiv c \in \ker(L)$, if $c > M_0$, then by assumption 1, $f(t,c,0) > 0$ and if $c < -M_0$, then $f(t,c,0) < 0$. Hence $\|x\| = |c| > M_0$ implies $QNJ_ic \neq 0$. Assumption 1 also ensures that $[QNJ_ic, c] \geq 0$ for any $c \in \ker(L)$ with $|c| > M_0$. So, by Theorem 1.6.3, to prove (Pi) is feebly a-solvable, we only need to prove U_i is bounded.

Suppose that $x \in U_i$, the lemma 2.2 of [57] implies that $\|x\|_\infty \leq M_0$. Suppose g satisfies (2a), then by assumption 3, we obtain

$$\begin{aligned}
|x''(t)| &\leq A(t,x)\omega((x'(t))^2) + C(t,x) + D(t,x)|x'(t)|^2 + \sum_{j=1}^n d_j(t,x)|x'(t)|_{\beta_j} \\
&\leq A_1\omega((x'(t))^2) + C_1 + M|x'(t)|^2 + \sum_{j=1}^n d_{j1}(|x'(t)|_2 + 1) \\
&\leq A_2(\omega(x'(t))^2 + 2 + |x'(t)|^2),
\end{aligned}$$

where $A_1 = \max_{t\in[0,1],|x|\le M_0} |A(t,x)|$, C_1, d_{j1} are defined similarly and A_2 is a constant. As above, there exists $M_1 > 0$, such that $\|x'\|_\infty \le M_1$. This implies that U_i is bounded.

Suppose that g satisfies (2b), then

$$
\begin{aligned}
\|x'\|_2^2 &= -\int_0^1 xx''\,dt \\
&= -\lambda\int_0^1 xg(t,x,x')\,dt - \lambda\int_0^1 xh(t,x,x')\,dt \\
&\le \int |x||h(t,x,x')|\,dt \\
&\le M_0\int_0^1 \left(|C(t,x)| + D(t,x)|x'|^2 + \sum_{j=1}^n d_j(t,x)|x'|^{\beta_j}\right) dt \\
&\le M_0C' + M_0M\int_0^1 |x'|^2\,dt + \sum_{j=1}^n d_j{}'\int_0^1 |x'|^{\beta_j}\,dt.
\end{aligned}
$$

Since $M_0M < 1$, and by Holder's inequality,

$$\int_0^1 |x'|^{\beta_j}\,dt \le \left(\int_0^1 |x'|^2\,dt\right)^{\frac{1}{2}\beta_j} = \|x'\|_2^{\beta_j},$$

so

$$(1 - M_0M)\|x'\|_2^2 \le M_0C' + \sum_{j=1}^n d_j{}'\|x'\|_2^{\beta_j}.$$

This implies that there exists $M_2 > 0$ such that $\|x'\|_2 \le M_2$ for $0 \le \beta_j < 2$. Since g satisfies condition B, we obtain

$$\int_0^1 |x''(t)|\,dt \le A\int_0^1 |\omega(x')|\,dt + C' + M\int_0^1 |x'|^2\,dt + \sum_{j=1}^n d_j{}'\int_0^1 (|x'(t)|^2 + 1) \le M_3.$$

$x \in X_i$ implies that there exists $\xi \in [0,1]$ such that $x'(\xi) = 0$, hence

$$\|x'\|_\infty = \|\int_\xi^t x''(s)\,ds\|_\infty \le \|x''\|_1 \le M_3.$$

Thus, we have proved that U_i is bounded, which completes the proof. □

Remark 6.2.2. In assumption 3 of Theorem 6.2.1, since $|p|^\beta \le 1 + |p|^2$, the third term is included in the first two terms, but it is convenient to make this split since the bound on the $|p|^2$ term only is important.

Remark 6.2.3. In [61] the authors obtained the results on the existence of a solution to the following boundary value problem:

$$(a(t)x')' + \overline{f}(t, x, x', x'') = y(t), \quad x'(0) = x'(T) = 0, \tag{6.7}$$

and in [60] they studied the BVP

$$x'' + g_1(x)x' + \overline{f}(t, x, x', x'') = y(t), \quad x(0) = x(1),\ x'(0) = x'(1). \tag{6.8}$$

In (6.7), $a \in C^1[0, T]$, $a_0 = \min\{a(t) : 0 \le t \le T\} > 0$, $a_1 = \max\{|a'(t)| : 0 \le t \le T\}$. When $\overline{f}$ is independent of x'', let

$$\overline{h}(t, x, x') = \overline{f}(t, x, x') - y(t).$$

Taking $T = 1$ (for simplicity), (6.7) can be rewritten in the following form:

$$x'' = -\frac{a'(t)}{a(t)}x' - \frac{\overline{h}(t, x, x')}{a(t)}, \quad x'(0) = x'(1) = 0. \tag{6.9}$$

To apply Theorem 6.2.1 to BVP(6.9), let

$$g(t, x, p) = -\frac{a'(t)}{a(t)}p \quad \text{and} \quad h(t, x, p) = -\frac{\overline{h}(t, x, p)}{a(t)}.$$

Then g satisfies Condition A with $\omega(p) = p^{\frac{1}{2}}$. Assume that $|\overline{f}(t, x, p)| \le A + B|x| + C|p|$, since the condition (H4i) or (H4ii) of [61] implies assumption 1 of Theorem 6.2.1, we obtain BVP(6.9) is feebly a-solvable provided (H4i) or (H4ii) of [61] holds. Thus when f does not depend on x'', in Theorem 2.1 of [61], the conditions $BT^2 + \pi T(C + p_1) \le \pi^2 p_0$ of (H1) and (H2), (H3) are not necessary.

Similarly when $\overline{f}$ is independent of x'', (6.8) can be rewritten as

$$x'' = -g_1(x)x' - \overline{h}(t, x, x'), \quad x(0) = x(1),\ x'(0) = x'(1). \tag{6.10}$$

Let

$$g(t, x, p) = -g_1(x)p \quad \text{and} \quad h(t, x, p) = -\overline{h}(t, x, p).$$

Then g satisfies condition B since $\int_0^1 xg_1(x)x'\,dt = 0$ for any $x \in X_3$. Assume that $|\overline{f}(t, x, p)| \le A + B|x| + C|p|$, then condition (H4) of [60] ensures assumption 1 of Theorem

6.2.1. Applying Theorem 6.2.1, we obtain that BVP(6.10) is feebly a-solvable provided (H4) of [60] holds. Hence in this case, in Theorem 2.1 of [60], the conditions $B+\pi C < 2\pi^2$ of (H1) and (H2),(H3) are not needed.

Theorem 6.2.4. *Let $f(t,x,p) = g(t,x,p) + h(t,x,p)$. Assume that*

1. *There exists $M_0 > 0$ such that $xf(t,x,0) > 0$ for $|x| > M_0$;*
2. *$pg(t,x,p) \geq 0$ or $pg(t,x,p) \leq 0$ for $(t,x,p) \in [0,1] \times \mathbb{R}^2$;*
3. *$|h(t,x,p)| \leq C(t,x) + D(t,x)|x'| + \sum_{j=1}^{n} d_j(t,x)|x'|^{\alpha_j}$, where $C(t,x), D(t,x)$, and $d_j(t,x)$ are bounded on compact subsets of $[0,1] \times \mathbb{R}$ and $0 \leq \alpha_j < 1$.*

Let $M = \max_{t\in[0,1],|x|\leq M_0} |D(t,x)|$, then (P3), (P4) are feebly a-solvable relative to Γ provided that $M < 1/4$. (P2) is feebly a-solvable relative to Γ provided that $M < 1/2$ if $pg(t,x,p) \leq 0$ and $M < 1/4$ if $pg(t,x,p) > 0$.

Proof: By the same argument as that in the proof of Theorem 6.2.1, we only need to prove U_i is bounded. Let $x \in U_i$, then $\|x\|_\infty \leq M_0$ by Lemma 2.3 of [57]. Let $\xi \in [0,1]$ be such that $x'(\xi) = 0$, and assume that $M < 1/4$. Then

$$\begin{aligned} \frac{1}{2}(x'(t))^2 &= \lambda \int_\xi^t x'g(s,x,x')\,ds + \lambda \int_\xi^t x'h(s,x,x')\,ds \\ &\leq |\int_0^1 x'g(s,x,x')\,ds| + \int_0^1 |x'h(s,x,x')|\,ds. \end{aligned} \tag{6.11}$$

Since $x \in X_i$, so

$$\int_0^1 x'x''\,dt = \lambda \int_0^1 (x'g(t,x,x') + x'h(t,x,x'))\,dt = 0.$$

Hence,

$$\frac{1}{2}(x'(t))^2 \leq 2\int_0^1 |x'h(s,x,x')|\,ds.$$

Thus

$$\begin{aligned} \frac{1}{4}(x'(t))^2 &\leq \|x'\|_\infty \int_0^1 \left(C(t,x) + D(t,x)|x'| + \sum_{j=1}^{n} d_j(t,x)|x'|^{\alpha_j} \right) dt \\ &\leq \|x'\|_\infty \left(C' + M\|x'\|_\infty + \sum_{j=1}^{n} d'_j \|x'\|_\infty^{\alpha_j} \right). \end{aligned}$$

Assume that $\|x'\|_\infty \neq 0$, then

$$\left(\frac{1}{4} - M\right) \|x'\|_\infty \leq C' + \sum_{j=1}^{n} d'_j \|x'\|_\infty^{\alpha_j}.$$

Since $\alpha_j < 1$, we obtain there exists $M_1 > 0$ such that $\|x'\|_\infty \leq M_1$. In the (p2) case, if $pg(t,x,p) \leq 0$ and $M < 1/2$, instead of (6.11), we will have

$$\begin{aligned} \frac{1}{2}(x'(t))^2 &= \lambda \int_0^t x' g(s,x,x')\,ds + \lambda \int_0^t x' h(s,x,x')\,ds \\ &\leq \int_0^1 |x' h(s,x,x')|\,ds. \end{aligned} \tag{6.12}$$

So, by the same proof as above, there exists $M_2 > 0$ such that $\|x'\|_\infty \leq M_2$. Thus in every case, U_i is bounded. □

Example 6.2.5. We study the following equation

$$x'' = \pm x'^{2n+1} + Q(t,x) + |x'|^{\frac{1}{2}} \tag{6.13}$$

subject to the boundary conditions (6.3)-(6.5), where n is a natural number and $Q(t,x)$ is a continuous function. Assume that there exists $M_0 > 0$, for which $xQ(t,x) > 0$ for $|x| > M_0$. By Theorem 6.2.4, the above BVP is feebly a-solvable since $D(t,x) = 0$. Since we can not find $A(t,x)$ such that

$$| \pm p^{2n+1} + Q(t,x) + |p|^{\frac{1}{2}} | \leq A(t,x)p^2 + C(t,x),$$

Theorem 2.1 of [57] and Theorem 4.1 of [32] can not be used.

In our last theorem, we impose a condition which is similar to the condition H3 of [61].

Theorem 6.2.6. *Let $f(t,x,p) = g(t,x,p) + h(t,x,p)$. Assume that*

1. *There exists $M_1 > 0$ such that either $cf(t,c,0) \geq 0$ for all $|c| \geq M_1$ or $cf(t,c,0) \leq 0$ for all $|c| \geq M_1$;*

2. There exists $M_2 > 0$ such that $\int_0^1 f(t, x, x')\, dt \neq 0$ for $x \in X_i$ with $|x(t)| > M_2$ for $t \in [0, 1]$;

3. $pg(t, x, p) \geq 0$ or $pg(t, x, p) \leq 0$ for $(t, x, p) \in [0, 1] \times \mathbb{R}^2$;

4. $|h(t, x, p)| \leq a|x| + b|p| + c|x|^\alpha + d|p|^\beta + e$, where $0 \leq \alpha, \beta < 1$, and a, b, c, d, e are constants.

Then (P3), (P4) are feebly a-solvable relative to Γ provided that $a + b < 1/4$. (P2) is feebly a-solvable relative to Γ provided that $a + b < 1/2$ if $pg(t, x, p) \leq 0$ and $a + b < 1/4$ if $pg(t, x, p) > 0$.

Proof: Let L, N, J_i, Q and the bilinear form $[y, x]$ be as in the proof of Theorem 6.2.1. For $c \in \ker(L)$, by assumption 2, $QNc \neq 0$ if $|c| \geq M_2$. Moreover, according to assumption 1, $[QNc, c] \geq 0$ for all $|c| \geq M_1$ or $[QNc, c] \leq 0$ for all $|c| \geq M_1$. Hence, by Theorem 1.6.3, (Pi) is feebly a-solvable if U_i is bounded.

Let $x \in U_i$ and $\xi \in [0, 1]$ be such that $x'(\xi) = 0$. By assumption 3 and 4, using the same calculation with that in (6.11) and (6.12), we obtain that

$$\frac{1}{4}\|x'\|_\infty^2 \leq \|x'\|_\infty \left(a\|x\|_\infty + b\|x'\|_\infty + c\|x\|_\infty^\alpha + d\|x'\|_\infty^\beta + e\right)$$

and in the case (P2), if $pg(t, x, p) \leq 0$,

$$\frac{1}{2}\|x'\|_\infty^2 \leq \|x'\|_\infty \left(a\|x\|_\infty + b\|x'\|_\infty + c\|x\|_\infty^\alpha + d\|x'\|_\infty^\beta + e\right).$$

Assume that $\|x'\|_\infty \neq 0$. Since $x \in X_i$, $Nx \in \operatorname{im}(L)$, so $QNx = 0$. Assumption 2 ensures that there exists $\zeta \in [0, 1]$ such that $|x(\zeta)| \leq M_2$. Writing $x(t) = \int_\zeta^t x'(s)\, ds + x(\zeta)$ gives

$$\|x\|_\infty \leq \|x'\|_1 + M_2 \leq \|x'\|_\infty + M_2. \tag{6.14}$$

From the above discussion, we obtain

$$\left(\frac{1}{4} - a - b\right) \|x'\|_\infty \leq M + c(\|x'\|_\infty + M_2)^\alpha + d\|x'\|_\infty^\beta + e.$$

In the case (P2) with $pg(t, x, p) \leq 0$, a similar inequality is obtained. These imply that there exists $M_3 > 0$ such that in both cases, $\|x'\|_\infty \leq M_3$. By (6.14), $\|x\|_\infty \leq M_3$. Thus, we have proved that U_i is bounded. $\square$

Remark 6.2.7. It is easy to see that in condition 4 of Theorem 6.2.6, $c|x|^\alpha$ and $d|p|^\beta$ can respectively be replaced by $\sum_{i=1}^n c_i|x|^{\alpha_i}$ and $\sum_{j=1}^m d_j|p|^{\beta_j}$, where $0 \le \alpha_i, \beta_j \le 1$.

Remark 6.2.8. The results can also be proved by applying the coincidence degree theory of Mawhin. The difference is that we only can obtain the existence of a solution but not the approximation solvability. The latter gives constructive results for the solutions.

Bibliography

[1] J. Appell and M. Dörfner, Some spectral theory for nonlinear operators, Nonlinear Anal. TMA. **28** (1997), 1955–1976.

[2] S. R. Bhatt and S. J. Bhatt, On the nonlinear spectrum of Furi and Vignoli, Math. Today **2** (1984), 49–64.

[3] A. Constantin, Global existence of solutions for perturbed differential equations, Ann. Mat. Pura Appl. **168** (1995), 237–299.

[4] A. Constantin, On a two-point boundary value problem, J. Math. Anal. Appl. **193** (1995), 318–328.

[5] A. Canada and R. Ortega, Existence theorems for equations in normed spaces and boundary value problems for nonlinear vector ordinary differential equations, Procceedings of the Royal Society of Edinburgh, **98A** (1984), 1–11.

[6] A. Canfora, Existence and application of a surjectivity theorem of Fučik, (Italian), Ricerche Mat. **27** (1978), 283–299.

[7] K. Deimling, Nonlinear functional analysis, Springer-Verlag, Berlin, Heidelberg, New York 1985.

[8] D. E. Edmunds and J. R. L. Webb, Some generalizations of the Borsuk-Ulam theorem, Math. Proc. Camb. Phil. Soc. **82** (1977), 119–125.

[9] D. E. Edmunds and J. R. L. Webb, Remarks on nonlinear spectral theory, Boll. Un. Mat. Ital. **B(2)** (1983), 377–390.

[10] D. E. Edmunds and W. D. Evans, Spectral theory and differential operators, Oxford Science Publications, Oxford University Press, New York (1987).

[11] S. Fučik, Solvability of nonlinear equations and boundary value problems, D. Reidel Publishing Company, Holland, (1980).

[12] S. Fučik, J. Nečas, J. Souček, V. Souček, Spectral analysis of nonlinear operators, Lecture Notes in Mathematics **346**, Springer-Verlag, 1973.

[13] M. Furi and M. Martelli, On α-Lipschitz retractions of the unit closed ball onto its boundary, Accad. Naz. Lincei, **57** (1974), 61–65.

[14] M. Furi and A. Vignoli, A nonlinear spectral approach to surjectivity in Banach spaces, J. Funct. Anal. **20** (1975), 304–318.

[15] M. Furi and A. Vignoli, Spectrum for nonlinear maps and bifurcation in the nondifferentiable case, Ann. Mat. Pura Appl. **113** (1977), 265–285.

[16] M. Furi, M. Martelli, A. Vignoli, Contributions to the spectral theory for nonlinear operators in Banach spaces, Ann. Mat. Pura Appl. **118** (1978), 229–294.

[17] M. Furi, M. Martelli, A. Vignoli, On the solvability of nonlinear operator equations in normed spaces, Ann. Mat. Pura Appl. **124** (1980), 321–343.

[18] W. Feng, On an m-point boundary value problem, Proceedings, WCNA-96, to appear.

[19] W. Feng, Decomposition conditions for two-point boundary value problems, submitted to Nonlinear Anal. TMA.

[20] W. Feng, A new spectral theory for nonlinear operators and its applications, submitted to Ann. Mat. Pura Appl.

[21] W. Feng and J. R. L. Webb, Surjectivity results for nonlinear mappings without oddness conditions, Comment. Math. Univ. Carolinae, **38** (1997), 15–28 .

[22] W. Feng and J. R. L. Webb, Solvability of three-point boundary value problems at resonance, Proceedings, WCNA-96, to appear.

[23] W. Feng and J. R. L. Webb, Solvability of m-point boundary value problems with nonlinear growth, to appear in J. Math. Anal. Appl. (1997).

[24] C. P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation, J. Math. Anal. Appl. **168** (1992), 540–551.

[25] C. P. Gupta, A note on a second order three-point boundary value problem, J. Math. Anal. Appl. **186** (1994), 277–281.

[26] C. P. Gupta, Existence theorems for a second order m-point boundary value problem at resonance, Internat J. Math. Math. Sci. **18** (1995), 705–710.

[27] C. P. Gupta, A Dirichlet type multi-point boundary value problem for second order ordinary differential equations, Nonlinear Anal. TMA **26** (1996), 925–931.

[28] C. P. Gupta and V. Lakshmikantham, Existence and uniqueness theorems for a third-order three-point boundary value problem, Nonlinear Anal. TMA **16** (1991), 949–957.

[29] C. P. Gupta, S. K. Ntouyas and P. Ch. Tsamatos, On an m-point boundary-value problem for second-order ordinary differential equations, Nonlinear Anal. TMA **23** (1994), 1427–1436.

[30] C. P. Gupta, S. K. Ntouyas and P. Ch. Tsamatos, Solvability of an m-point boundary value problem for second order ordinary differential equations, J. Math. Anal. Appl. **189** (1995), 575–584.

[31] R. E. Gaines and J. Mawhin, Coincidence degree and nonlinear differential equations, Lecture Notes in Mathematics, **568**, Springer Verlag, 1977.

[32] A. Granas, R. B. Guenther and J. W. Lee, On a theorem of S. Bernstein, Pacific J. Math.**74** (1978), 67–82.

[33] K. Georg and M. Martelli, On spectral theory for nonlinear operators, J. Funct. Anal. **24** (1977), 140–147.

[34] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, 1977.

[35] G. H. Hardy, J. E. Littlewood, and G. Polya, Inequalities, Cambridge Univ. Press, London/New York, 1967.

[36] E. Hewitt and K. Stromberg, Real and abstract analysis, Springer-Verlag, Berlin-Heideberg, New York 1969.

[37] V. Il'in and E. Moiseev, Nonlocal boundary value problems of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, Differential Equations, **23** (1987), 803–810.

[38] J. Ize, I. Massabo, J. Pejsachowicz, and A. Vignoli, Nonlinear multiparametric equations: structure and topological dimension of global branches of solutions, Nonlinear Func. Anal. Appl. (Berkeley, Calif, 1983) 529-540. Proc. Sympos. Pure. Math. **45** Part 1, Amer. Math. Soc. Providence, R.I. (1986), 529–540.

[39] R. I. Kachurovskij, Regular points, spectrum and eigenfunctions of nonlinear operators (in Russian), Dokl. Akad. Nauk SSSR **188** (1969), 274–277; Engl. transl.:Soviet Math. Dokl. **10** (1969), 1101–1105.

[40] C. Kuratowski, Sur les éspaces complètes, Fund. Math. **15** (1930), 301–335.

[41] T. Kato, Perturbation theory of linear operators, Springer-Verlag, Berlin Heidelberg, New York (1976).

[42] N. G. Lloyd, Degree theory, Cambridge University Press, London 1978.

[43] M. Martelli, Positive eigenvectors of wedge maps, Ann. Mat. Pura Appl.**145** (1986), 1–32.

[44] J. Mawhin, Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, J. Differential Equations **12** (1972), 610–636.

[45] J. Mawhin, Topological degree methods in nonlinear boundary value problems, in NSF-CBMS Regional Conference Series in Math. No.**40**, American Mathematical Society, Providence, R.I. (1979).

[46] J. Mawhin, Topological degree and boundary value problems for nonlinear differential equations, in Topological methods for ordinary differential equations (Eds. P. M. Fitzpatrick, M. Martelli, J. Mawhin, R. Nussbaum), Lecture Notes in Mathematics, **1537**, Springer Verlag (1991), 74–142.

[47] S. A. Marano, A Remark on a third-order three-point boundary value problem, Bull. Austral. Math. Soc. **49** (1994), 1–5.

[48] R. H. Martin, Nonlinear operators and differential equations in Banach spaces, John Wiley and Sons, Inc., 1976.

[49] I. J. Maddox and A. W. Wickstead, The spectrum of uniformly Lipschitz mappings, Proc. Royal Ir. Acad. **89 A** (1989), 101–114.

[50] M. Mininni, Coincidence degree and solvability of some nonlinear functional equations in normed spaces: a spectral approach, Nonlinear Anal. TMA **1** (1976), 105–122.

[51] J. W. Neuberger, Existence of a spectrum for nonlinear transformations, Pacific J. Math. **31** (1969), 157–159.

[52] R. D. Nussbaum, The fixed point index for local condensing maps, Ann. Mat. Pura Appl. **89** (1971), 217–258.

[53] R. D. Nussbaum, Degree theory for local condensing maps, J. Math. Anal. Appl. **37** (1972), 741–766.

[54] W. V. Petryshyn, On the approximation-solvability of nonlinear equations, Math. Ann. **177** (1968), 156–164.

[55] W. V. Petryshyn, Existence theorems for semilimear abstract and differential equations with noninvertible linear parts and noncompact perturbations, in "Nonlinear Equations in Abstract Spaces," (V. Lakshmikantham, Ed.), Academic Press, New York, (1978), 275–316.

[56] W. V. Petryshyn, Some further results on periodic solutions of certain higher order nonlinear differential equations, Nonlinear Anal. TMA **8** (1984), 1055–1069.

[57] W. V. Petryshyn, Solvability of various boundary value problems for the equation $x'' = f(t, x, x', x'') - y$, Pacific J. Math. **122** (1986), 169–195.

[58] W. V. Petryshyn, Approximation solvability of nonlinear functional and differential equations, Marcel Dekker, inc., New York, Basel, Hong Kong, (1993).

[59] W. V. Petryshyn, Generalized topological degree and semilinear equations, Cambridge University Press, New York, (1995).

[60] W. V. Petryshyn and Z. S. Yu, Periodic solutions of nonlinear second-order differential equations which are not solvable for the highest derivative, J. Math. Anal. Appl. **89** (1982), 462–488.

[61] W. V. Petryshyn and Z. S. Yu, Solvability of Neumann BV problems for nonlinear second-order ODE's which need not be solvable for highest-order derivative, J. Math. Anal. Appl. **91** (1983), 244–253.

[62] W. V. Petryshyn and Z. S. Yu, On the solvability of an equation describing the periodic motions of a satellite in its elliptic orbit, Nonlinear Anal. TMA **9** (1985), 969–975.

[63] J. Pejsachowicz and A. Vignoli, On differentiability and surjectivity of α-Lipschitz mappings, Ann. Mat. Pura Appl. **101** (1974), 49–63.

[64] D. O'Regan, Singular Sturm-Liouville problems and existence of solutions to singular nonlinear boundary value problems, Nonlinear Anal. TMA **20** (1993), 767–779.

[65] E. U. Tarafdar and H. B. Thompson, On the solvability of nonlinear noncompact operator equations, J. Austral. Math. Soc. (Series A) **43** (1987), 103–126.

[66] J. F. Toland, Topological methods for nonlinear eigenvalue problems, Battelle, Advanced studies center, Geneva, Switzerland, Mathematics Peport No **77**, 1973.

[67] A. E. Taylor and D. C. Lay, Introduction to functional analysis, second edition, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1980.

[68] J. R. L. Webb, A fixed point theorem and applications to functional equations in Banach spaces, Boll. Un. Mat. Ital. **4** (1971), 775–788.

[69] J. R. L. Webb, On degree theory for multivalued mappings and applications, Boll. Un. Mat. Ital. **9** (1974), 137–158.

[70] W. Zhao, Geometrical coefficients and measures of noncompactness, Ph. D thesis, (1992).