Some Special Classes of Modules

by

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A Thesis Presented To The
University Of Glasgow
Faculty Of Science
For The Degree Of
Doctor Of Philosophy

May, 1997

Statement

Chapter 1 consists of known results, stated for use in the rest of the thesis. Credit is given to the original authors where appropriate.

Chapter 2, Section 3 is a fully constructed version of an example whose outline appears in [34].

The remaining results in Chapter 2 and those in Chapters 3–5 are my own work, except where the text indicates otherwise.

Abstract

The aim of this report is to study rings whose module classes have special properties and modules which subgenerate module classes with special properties. Of the four main chapters, three concern classes where every member has a particular type of decomposition, while the other one concerns classes which are closed under carrying out particular operations. In each case, we will try and relate the property of the class to the submodule or ideal structure of the module or ring which induces it.

Chapter 2 is about a class of rings called the right co-H rings, which generalise the QF rings. Like QF rings, these can be classified in numerous different ways, the original four of which were shown to be equivalent by Oshiro in [34]. The original Theorem is reproduced here as Theorem 2.1.5. In this chapter, we show a new, shorter way to prove Oshiro's Theorem using a new result, Lemma 2.2.7. We also expand Oshiro's example from [34], and construct some concrete examples of rings which are right co-H but neither QF nor right H (the H property being a dual of the co-H property). At the end of the chapter there is a short discussion about whether it is possible to weaken the rather strong conditions required by Lemma 2.2.7.

Chapter 3 concerns the class $\sigma[M]$ of modules subgenerated by the module M. For an arbitrary M, we can only say that $\sigma[M]$ is closed under the operations of taking submodules, factor modules and direct sums. We will be asking which modules subgenerate classes which are closed under the operations of taking extensions and taking essential extensions, and trying to find out about their submodule structure. We are able to characterise these modules completely in two cases:

(1) Where the annihilator of the module is the annihilator of a finite subset of the

module (Corollary 3.2.16 and Corollary 3.2.17).

(2) Where the modules are indecomposable injectives over a commutative noetherian ring (Theorem 3.3.16).

We are also able to show a few other partial results.

In Chapter 4, we wonder whether we can classify rings whose right modules are direct sums of uniform modules. In the first part of the Chapter, we show some basic results on these rings. The main part of the Chapter is taken up by the construction of examples which suggest that a complete classification may be impossible.

Chapter 5 asks a generalisation of the central question of Chapter 4 - which modules M have the property that every module in $\sigma[M]$ is a direct sum of uniform modules? In the case where the annihilator of M is the annihilator of a finite subset of M, we are able to provide a complete answer or at least an answer in terms of the (unclassifiable?) rings of Chapter 4. In the case where the ring is commutative, we show that our property holds if and only if M is a pure-semisimple module in the sense of [52] (Theorem 5.2.4).

To illustrate the topics under discussion, lots of examples have been included, particularly in Chapter 4 where they are the inspiration for the chapter's conclusions. The aim is for this report to be as self-contained as possible and with this in mind proofs of some known results have been included. Also, for the sake of completeness, on occasion we will prove a result directly rather than deducing it from a known result not included in the report. In Chapters 4 and 5, however, we are forced to use some results which we will not prove, since some of these proofs are heavily dependent on theory which we do not have space to introduce.

Acknowledgements

I would like to thank all the people who have provided me with references and mathematical advice during my period of research, including Professor Ken Brown, Professor Nguyen Viet Dung, Professor Gerald Janusz, Dr. David Jordan, Professor Kiyoichi Oshiro, Professor Ana Viola-Prioli, Professor Jorge Viola-Prioli and Professor Robert Wisbauer. Also, I am grateful to all the staff and students of Glasgow University's department of mathematics who have helped me to make the most of my time here. Special thanks must go to Professor Jose Luis García, Professor Pedro Guil, Professor Alberto del Valle Robles, Professor Manuel Saorin and all the others at the University of Murcia who helped to make my short stay there both productive and enjoyable.

I should also thank EPSRC for providing funds to make these studies possible, and ERASMUS for helping out financially with my visit to Murcia.

Above all, I would like to thank Professor Patrick Smith, my supervisor, for his patience and help, and my parents, for all of their support over the years.

Contents

Chapter 1

Introduction

All of the results in this Introduction are well-established, and are stated for use in the later chapters. In the other Chapters, a result is original unless it is indicated otherwise.

Parts of Chapter 2 are to appear in the Rocky Mountain Journal of Mathematics under the title "A Simplified Proof of Oshiro's Theorem for co-Harada Rings".

1.1 Conventions and Basic Results

Throughout this report, all rings are assumed to have an identity and all modules are assumed to be unitary. No other properties except those normally ascribed to rings and modules will be assumed, unless stated otherwise. We will use the notation M_R to indicate that M is a right R-module, and correspondingly, ${}_RM$ to indicate that M is a left R-module.

$N \leq M$ will be used to indicate that N is a submodule of M and $N < M$ to indicate that N is a proper submodule of M. $N \subseteq M$ merely states that N is a subset of M, unless it is clear from the context that it is also a submodule. $N \leq_{ess} M$ indicates that N is an essential submodule of M, and $N \leq^{\oplus} M$ indicates that N is a direct summand of M.

If A and M are modules with a homomorphism $\theta : M \to A$ and N is a submodule of M, then the homomorphism $\theta|_N : N \to A$ is defined to be the restriction of θ to N.

For a module M and set Λ, M^{Λ} refers to a direct product of copies of M indexed by Λ, while $M^{(\Lambda)}$ refers to a corresponding direct sum. Of course, if the set Λ is finite, then these two objects are isomorphic. For a module M and a natural number n, we will use the notation M^n to refer to a direct sum of n copies of M, except in the case of a right (left) ideal I where the notation I^n refers to the right (left) ideal generated by the set of elements of the form $x_1 x_2 ... x_n$, where $x_j \in I$ for every j.

If S is a non-empty subset of a right R-module M, and m is an element of M, then $\mathbf{r}(S)$ and $\mathbf{r}(m)$ are the right ideals of R which annihilate S and m, respectively.

$Z(M)$ and $Soc(M)$ are used to indicate the singular submodule and the socle, respectively, of a module M.

We will sometimes use the shorthand $I \lhd R$ to indicate that I is a 2-sided ideal of R and $I \lhd_r R$ ($I \lhd_l R$) to indicate that I is a right (left) ideal of R.

$\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$ refer to the integers, the natural numbers (including zero) and the rational numbers, respectively. For any $n \in \mathbb{N}$, $\mathbb{Z}_n$ refers to the ring or $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$.

Definition 1.1.1 *Given two right R-modules $B \leq A$, a submodule C of A is said to be a* **complement** *of B in A if C is maximal with respect to the restriction $C \cap B = 0$.*

Lemma 1.1.2 *If A is a right R-module with submodules B and N such that $B \cap N = 0$, then B has a complement C in A such that $N \leq C$. Furthermore, $B \oplus C \leq_{ess} A$ and $(B \oplus C)/C \leq_{ess} A/C$.*

Proof Let $\mathcal{S} = \{Y : N \leq Y \leq A,\ Y \cap B = 0\}$. Clearly, $N \in \mathcal{S}$ and the union of any chain in $\mathcal{S}$ is in $\mathcal{S}$, so $\mathcal{S}$ has a maximal element C by Zorn's Lemma.

Suppose that $K \leq A$ and $K \cap (B \oplus C) = 0$. Then $B \cap (K \oplus C) = 0$, so by the maximality of C, $K = 0$. Thus $B \oplus C$ is an essential submodule of A.

Now take a module D such that $C < D \leq A$. Suppose that $D \cap (B \oplus C) \subseteq C$. Then $D \cap B \subseteq C \cap B = 0$, which is a contradiction, by the maximality of C. Hence $D \cap (B \oplus C) \not\subseteq C$ and so $(B \oplus C)/C \leq_{ess} A/C$. $\square$

Note that if we set $N = 0$, Lemma 1.1.2 can be used to show that every module has a complement in every module which contains it.

Example 1.1.3 Complements are not in general unique. For example, if we take a ring R, and let $M_R = R_R \oplus R_R$ and $M_R \geq N_R = R_R \oplus 0$, then clearly $A_R = 0 \oplus R_R$ and $B_R = (1,1)R_R$ are both complements of N in M since $M = N \oplus A = N \oplus B$.

Lemma 1.1.4 *Let R be a ring, U a uniform right R-module, and $M_1, M_2, ..., M_n$ a finite set of right R-modules such that $U \hookrightarrow M$ where $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. Then $U \hookrightarrow M_i$ for some $1 \leq i \leq n$.*

Proof Let θ be the embedding $U \hookrightarrow M$ and π_i the canonical surjection $M \twoheadrightarrow M_i$ for each $1 \leq i \leq n$. Then for every $u \in U$, $\theta(u) = \pi_1\theta(u) + \pi_2\theta(u) + ... + \pi_n\theta(u)$, so since θ is monomorphic, $\text{Ker}\,(\pi_1\theta) \cap \text{Ker}\,(\pi_2\theta) \cap ... \cap \text{Ker}\,(\pi_n\theta) = 0$. By the uniformity of U, there exists i such that $\text{Ker}\,(\pi_i\theta) = 0$, and so $\pi_i\theta$ is an embedding. □

Lemma 1.1.5 *Let R be a ring, U_R a uniform right R-module with non-zero socle, and $\{M_\lambda\}_{\lambda\in\Lambda}$ a set of modules with a monomorphism $U \hookrightarrow \Pi_{\lambda\in\Lambda}M_\lambda$. Then there exists $\alpha \in \Lambda$ such that $U \hookrightarrow M_\alpha$.*

Proof This can be shown by the same method that was used in the proof of Lemma 1.1.4, making use of the observation that the intersection of any set of non-zero submodules of a uniform module with non-zero socle must contain the socle and so is non-zero. □

The **modular law** used in the following Lemma states that if $A \leq B$ and C are right R-modules, then $B \cap (A \oplus C) = A \oplus (B \cap C)$. Its proof is elementary.

Lemma 1.1.6 *Let R be a ring, M_R a right R-module and N_R be the intersection of all of the essential submodules of M. Then $N = Soc(M)$.*

Proof Let S_R be a simple submodule of M and X_R an essential submodule of M. Then $S \cap X \neq 0$, so $S \cap X = S$, i.e. $S \subseteq X$. It follows that $Soc(M) \subseteq N$.

Now let A_R be a submodule of N_R, and let C_R be a complement of A_R in M_R. Then $N \subseteq A \oplus C$, so $N = N \cap (A \oplus C) = A \oplus (N \cap C)$ by the modular law, i.e. $A_R \leq^{\oplus} N_R$ for an arbitrary submodule A of N. Therefore N_R is semisimple and so $N \leq Soc(M)$. □

Definition 1.1.7 *A monomorphism $\theta : M \hookrightarrow N$ is said to be an* **essential embedding** *if $Im\theta$ is an essential submodule of N.*

1.2 Chain Conditions and the Jacobson Radical

For reasons of space, we will not prove all of the results in this section. Complete proofs, along with definitions of any unexplained terms can be found in [1].

Definition 1.2.1 *If R is a ring, then the* **Jacobson radical** *$J(R)$ of R is defined as the intersection of all of the maximal right ideals of R. By [1], Theorem 15.3, this is equal to the intersection of all of the maximal left ideals of R.*

Lemma 1.2.2 *If R is a ring with a Jacobson radical J and $j \in J$ then $1 - j$ is invertible.*

Proof Clearly, $R = (1-j)R+J$. Suppose that $(1-j)R \neq R$. Then, by Zorn's Lemma, there exists a maximal right ideal M of R which contains $(1-j)R$. It follows that $(1-j)R+J \leq M$, which is not true. Therefore $(1-j)R = R$ and similarly, $R(1-j) = R$. Hence $1-j$ is invertible. □

Lemma 1.2.3 *(Nakayama) Let R be a ring with a Jacobson radical J and let M be a non-zero finitely generated right R-module. Then $MJ \neq M$.*

Proof Let $\{m_1, ..., m_s\}$ be a generating set of M and suppose that $MJ = M$. Then there exist $j_i \in J$ for $1 \leq i \leq s$ such that $m_1 = m_1 j_1 + ... + m_s j_s$. Therefore, $m_1(1-j_1) = m_2 j_2 + ... + m_s j_s$ and so by Lemma 1.2.2, $m_1 \in m_2 R + ... + m_s R$. We can therefore remove m_1 from the generating set. Repeating the same process, we can remove $m_2, ..., m_s$ from the generating set, which is clearly absurd. Hence $MJ \neq M$. □

Corollary 1.2.4 *If R is a ring with Jacobson radical J and S is a semsimple right R-module, then $SJ = 0$.*

Proof Let T_R be a simple right R-module. Then by Lemma 1.2.3, $TJ = 0$. Since S is a sum of simple modules, it follows that $SJ = 0$. $\square$

Theorem 1.2.5 *(Hopkins-Levitzki) If R is a right artinian ring, then R is also right noetherian.*

Lemma 1.2.6 *If R is right noetherian, then every right R-module contains a uniform submodule.*

Proof Suppose that there exists a module M_R which does not contain a uniform submodule. Let N_R be a non-zero finitely generated submodule of M. Since N is not uniform, it contains two non-zero modules A_1 and B_1 such that $A_1 \cap B_1 = 0$. Since B_1 is not uniform, it in turn contains non-zero submodules A_2 and B_2 such that $A_2 \cap B_2 = 0$. Repeating, each B_j contains non-zero submodules A_{j+1} and B_{j+1} such that $A_{j+1} \cap B_{j+1} = 0$.

Since N is a finitely generated module over a right noetherian ring, it is noetherian. But N contains an infinite increasing chain:

$$A_1 \subsetneq A_1 \oplus A_2 \subsetneq A_1 \oplus A_2 \oplus A_3 \subsetneq \ldots$$

which is clearly a contradiction, so N and hence M must contain a uniform submodule. $\square$

By a similar proof, we can show the following:

Lemma 1.2.7 *Every module which is noetherian or artinian is a direct sum of indecomposable modules.*

Proof Suppose that M is either noetherian or artinian and is not a direct sum of indecomposable modules. Then there exists a non-trivial decomposition $M = M_1 \oplus A_1$. Furthermore either M_1 or A_1 must decompose non-trivially, so we can suppose that $A_1 = M_2 \oplus A_2$. Continuing in this way, we can form an infinite direct sum $M_1 \oplus M_2 \oplus \ldots$ contained in M, which is a contradiction, since M was asumed to be either noetherian or artinian. $\square$

Definition 1.2.8 *If M_R is artinian and noetherian, then there exists an $n \in \mathbb{N}$ and a chain:*

$$0 = M_0 \leq M_1 \leq M_2 \leq ... \leq M_n = M$$

such that for every $1 \leq i \leq n$, $(M_i/M_{i-1})_R$ is non-zero and simple. We call n the **length** *of M_R.*

Theorem 1.2.9 *(Jordan-Hölder) If M_R is an artinian and noetherian module, then the length of M_R is constant - that is to say it is independent of the chain of submodules chosen.*

Corollary 1.2.10 *If M_R is an artinian and noetherian module with a submodule N_R, then the length of M_R/N_R is equal to the length of M_R minus the length of N_R.*

Proof This is easily seen from Theorem 1.2.9 by considering the chain:

$$0 = M_0 \leq M_1 \leq ... \leq M_t = N \leq ... \leq M_n = M$$

where all of the factors M_{j+1}/M_j are simple. □

Definition 1.2.11 *A module M_R is said to be* **locally noetherian (artinian)** *if every finitely generated submodule of M_R is noetherian (artinian).*

It is easy to see that R_R is locally noetherian (artinian) $\Leftrightarrow$ R_R is noetherian (artinian).

Definition 1.2.12 *If R is a ring and $e = e^2 \in R$, then e is called an* **idempotent** *of R. If E is a set of idempotents of R where $ef = fe = 0$ for every $e \neq f \in E$, then E is said to be an* **orthogonal set of idempotents**. *If F is a finite set of idempotents such that $\sum_{f \in F} f = 1$, then F is said to be a* **complete set of idempotents**. *If e is an idempotent of R which cannot be written as the sum of two non-zero mutually orthogonal idempotents, then we say that e is a* **primitive idempotent** *of R.*

Definition 1.2.13 *A ring is said to be* **semiperfect** *if R/J is a semisimple ring and for every idempotent $e + J$ of the ring R/J, there exists an idempotent f of R such that $f + J = e + J$.*

Lemma 1.2.14 *([1], Theorem 15.20 and Proposition 27.1) If R is right artinian, then R is semiperfect and there exists $n \in \mathbb{N}$ such that $J^n = 0$.*

Lemma 1.2.15 *Let R be a semiperfect ring with a Jacobson radical J and let M be a right R-module. Then $Soc(M) = \{m \in M : mJ = 0\}$.*

Proof $Soc(M)J = 0$ by Corollary 1.2.4. Conversely, if $t \in M$ and $tJ = 0$, then $tR.J = 0$, so tR is a right R/J-module. By [1], Proposition 13.9, every module of a semisimple ring is semisimple and so tR is a semisimple right R/J-module. Since it has the same submodule structure as both a right R-module and a right R/J-module, it must also be semisimple as a right R-module. Therefore $t \in Soc(M)$. $\square$

Definition 1.2.16 *A module M is said to be* **uniserial** *if for every pair of submodules X and Y of M, either $X \subseteq Y$ or $Y \subseteq X$. A ring R is said to be* **right serial** *if it is a direct sum of right ideals, each of which is a uniserial right R-module. If R is left and right artinian and left and right serial, then R is called* **serial**.

Note that in the above definition a serial ring is taken to be artinian which means that, a little confusingly, a left serial and right serial ring is not necessarily serial. This is for the sake of brevity since all of the 2-sided serial rings we are interested in will be artinian. Most recent texts use the same convention, but some (e.g. [3]) use the term "serial ring" more generally, meaning a left and right serial ring which is not even necessarily noetherian. The reader should always check the authors' definition of a serial ring carefully when consulting any text.

1.3 Injectivity and Projectivity Conditions

For the definitions of projectivity and injectivity, the reader is referred to [1]. Recall that every module M has an **injective hull** denoted $E(M)$, in which M sits as an essential submodule.

Lemma 1.3.1 *([1], Proposition 18.13) Every direct sum of injective right modules of a right noetherian ring is injective.*

Lemma 1.3.2 *([1], Theorem 25.6) If R is right noetherian and E_R is injective, then E_R is a direct sum of indecomposable injective right R-modules.*

Definition 1.3.3 *A module M is said to be* **local** *if it has a maximal submodule which contains every proper submodule. A ring R is said to be* **local** *if $J(R)$ is a unique maximal left (or equivalently right - see [1] Proposition 15.15) ideal.*

Theorem 1.3.4 *([1], Theorem 27.11) Let R be a semiperfect ring. Then there exists a set of primitive orthogonal idempotents $\{e_1, e_2, ..., e_n\}$ in R such that every e_iR is indecomposable and local, every projective right R-module is isomorphic to $e_1R^{(A_1)} \oplus e_2R^{(A_2)} \oplus ... \oplus e_nR^{(A_n)}$ for suitable index sets $A_1, A_2, ..., A_n$ and every semisimple right R-module is isomorphic to $(e_1R/e_1J)^{(B_1)} \oplus (e_2R/e_2J)^{(B_2)} \oplus ... \oplus (e_nR/e_nJ)^{(B_n)}$ for suitable index sets $B_1, B_2, ..., B_n$.*

Theorem 1.3.5 *Let R be a right artinian ring with a set of primitive idempotents $\{e_1, e_2, ..., e_n\}$ of the type described in Theorem 1.3.4. Then there are exactly n isomorphism classes of indecomposable injective right R-modules which are given by $E(e_iR/e_iJ)$, and every injective right R-module is isomorphic to a direct sum of copies of these indecomposable injective right R-modules.*

Proof Every injective right R-module is a direct sum of indecomposable injective right R-modules by Theorem 1.2.5 and Lemma 1.3.2. Since R is right artinian, every indecomposable injective has non-zero socle, so must be isomorphic to some $E(e_iR/e_iJ)$ by Theorem 1.3.4. □

Definition 1.3.6 *If A_R and M_R are modules, then A is said to be* **M-injective** *if for any module N_R with a monomorphism $i : N_R \hookrightarrow M_R$ and homomorphism $\theta : N \to A$, there* *exists a homomorphism $\theta' : M \to A$ such that the diagram:*

$$\begin{tikzcd} N \arrow[r, hook, "i"] \arrow[d, "\theta"'] & M \arrow[dl, "\theta'"] \\ A & \end{tikzcd}$$

commutes. Equivalently, A is M-injective if for every submodule K_R of M and homomorphism $\phi : K_R \to A_R$, ϕ can be extended to a homomorphism $\phi' : M_R \to A_R$.

If the module M is M-injective, then we say that M is **quasi-injective**.

Lemma 1.3.7 *Let A_R and $N'_R \leq N_R$ be modules and $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of right R-modules such that A is N-injective and M_λ-injective for every $\lambda \in \Lambda$. Then:*

(i) A is N'-injective.

(ii) A is N/N'-injective.

(iii) A is $\bigoplus_{\lambda\in\Lambda} M_\lambda$-injective.

Proof (i) Let $N''_R \leq N'_R$ and $\theta : N'' \to A$. Then by assumption, θ can be extended to a homomorphism $\theta' : N \to A$ whose restriction $\theta'|_{N'} : N' \to A$ extends θ.

(ii) Let X_R be a module such that $N' \leq X \leq N$ and let $\theta : X/N' \to A$. We know that if $\pi : X \to X/N'$ is the canonical surjection, then there exists a homomorphism $\phi : N \to A$ which extends $\theta\pi$. Clearly, $\theta\pi(N') = 0$, so $\phi(N') = 0$ and hence we can define $\phi' : N/N' \to A : n + N' \mapsto \phi(n)$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\pi"'] & N \arrow[d, "\pi'"] \arrow[ddl, dotted, "\phi"] \\ X/N' \arrow[r, hook] \arrow[d, "\theta"'] & N/N' \\ A & \end{tikzcd}$$

Let $x \in X$. Then:

$$\phi'(x + N') = \phi(x) = \theta\pi(x) = \theta(x + N')$$

Therefore ϕ' extends θ.

(iii) Let $Y_R \leq M_R = \bigoplus_{\lambda \in \Lambda} M_\lambda$ and let $\theta : Y \to A$. Certainly, there exists a homomorphism $\phi : M \to E(A)$ which extends θ and we will show that for any such ϕ, $\phi(M) \subseteq A$.

Choose any $\lambda \in \Lambda$ and let $K = \{m \in M_\lambda : \phi(m) \in A\}$. By the M_λ-injectivity of A, the homomorphism $\phi|_K : K \to A$ extends to a homomorphism $\mu : M_\lambda \to A$. Now let $a \in A \cap (\phi - \mu)(M_\lambda)$. Then $a = \phi(m) - \mu(m)$ for some $m \in M_\lambda$ and so $\phi(m) = a + \mu(m) \in A$, i.e. $m \in K$. Thus, $\phi(m) = \mu(m)$ and so $a = 0$, hence $A \cap (\phi - \mu)(M_\lambda) = 0$. But we know that A is essential in $E(A)$ and so $(\phi - \mu)(M_\lambda) = 0$, which tells us that $\phi(M_\lambda) \subseteq A$.

Finally, $\phi(M) = \sum_{\lambda \in \Lambda} \phi(M_\lambda) \subseteq A$. $\square$

Lemma 1.3.8 *(Baer) Let M be a module over a ring R. Then M_R is injective $\Leftrightarrow$ M_R is R_R-injective.*

Proof $\Rightarrow$ This is clear.

$\Leftarrow$ Let I be any right ideal of R. Then by Lemma 1.3.7 (ii) M is R/I-injective. Let A_R be a module; then A is a homomorphic image of $\bigoplus_{a \in A} aR$. By Lemma 1.3.7 (iii), M is injective with respect to $\bigoplus_{a \in A} R/\mathbf{r}(a)$, which is isomorphic to $\bigoplus_{a \in A} aR$. It follows from Lemma 1.3.7 (ii) that M is injective with respect to A.

In other words, M_R is A_R-injective for any right R-module A_R which is just another way of saying that M_R is injective. $\square$

Corollary 1.3.9 *If R is a ring and M_R is a right R-module, then M_R is injective $\Leftrightarrow$ every homomorphism from an essential right ideal of R to M can be extended to a homomorphism from R to M.*

Proof $\Rightarrow$ This is obvious.

$\Leftarrow$ Let I be a right ideal of R and let $\theta : I \to M$ be an homomorphism. Let K_R be a complement of I_R in R_R. We can extend θ to a homomorphism $\phi : I \oplus K \to M : x + k \mapsto \theta(x)$, where $x \in I$ and $k \in K$. By Lemma 1.1.2, $I_R \oplus K_R$ is essential in R_R and so ϕ extends to a homomorphism from R_R to M_R. The result follows by Lemma 1.3.8. $\square$

Lemma 1.3.10 *If R is a right noetherian ring, then every right R-module is a direct sum of an injective module and a module with no non-zero injective submodules.*

Proof Let M_R be a right R-module. If M_R has no non-zero injective submodules, then the result follows trivially. Otherwise, M_R contains an injective E_R, and of course $M_R = E_R \oplus X_R$ for some submodule X_R of M_R.

Now consider the sets of non-zero injective submodules of M_R which are independent (that is to say that their sum is direct). Trivially, $\{E_R\}$ is a non-empty example of such a set. Furthermore the union of any chain of these sets is also such a set. Hence by Zorn's Lemma, there exists a maximal such set $\mathcal{S}$. Let T be the sum of the modules in $\mathcal{S}$. Then T is a direct sum of injective right R-modules, so is itself injective by Lemma 1.3.1, and hence $M_R = T_R \oplus Y_R$ for some submodule Y_R of M_R. Suppose that Y_R contains a non-zero injective submodule A_R. Then $M_R = T_R \oplus A_R \oplus B_R$ for some submodule B_R of Y_R. But this implies that $\mathcal{S} \cup \{A_R\}$ is a set of non-zero injective submodules of M_R which are independent, contradicting the maximality of $\mathcal{S}$. Therefore, Y_R contains no non-zero injective submodules. $\square$

Definition 1.3.11 *If R is a ring such that every submodule of a projective right R-module is projective, then we say that R is* **right hereditary**. *Right and left hereditary rings are called* **hereditary**.

Theorem 1.3.12 *(Cartan & Eilenberg) The following are equivalent, for a ring R:*

(i) R is right hereditary.

(ii) Every right ideal is projective as an R-module.

(iii) Every homomorphic image of an injective right R-module is injective.

Proof

(i) $\Rightarrow$ (ii) This is obvious, since every right ideal of R is a submodule of R_R.

(ii) $\Rightarrow$ (iii) Let I be a right ideal of R and let X_R be a submodule of an injective module E_R. Let $\theta : I \to E/X$ be a homomorphism. Since I_R is projective, there exists a homomorphism $\alpha : I_R \to E_R$ such that $\pi\alpha = \theta$, where $\pi : E \to E/X$ is the canonical projection. E is injective, so there exists $\beta : R_R \to E$, such that $\beta|_I = \alpha$.

$$\begin{tikzcd}
& I \arrow[r, hook] \arrow[d, "\theta"] \arrow[dl, dotted, "\alpha"'] & R \arrow[dll, dotted, "\beta"] \\
E \arrow[r, two heads, "\pi"'] & E/X &
\end{tikzcd}$$

Let $a \in I$. Then $\pi\beta(a) = \pi\alpha(a) = \theta(a)$ and so $\pi\beta$ extends θ. Hence by Lemma 1.3.8, E/X is injective.

(iii) $\Rightarrow$ (i) Let M_R be a module with a submodule K_R and P_R be a projective module with a submodule A_R. Let $\theta : A \to M/K$ be a homomorphism. By assumption, $E(M)/K$ is injective. Let i and j be the canonical injections of M and M/K into $E(M)$ and $E(M)/K$, respectively and let π and ρ be the canonical projections of M and $E(M)$ into M/K and $E(M)/K$, respectively. Consider the diagram:

$$\begin{tikzcd}
& A \arrow[r, hook] \arrow[d, "\theta"] & P \arrow[ddll, dotted, "\beta"] \arrow[ddl, dotted, "\alpha"] \\
M \arrow[r, two heads, "\pi"] \arrow[d, hook, "i"'] & M/K \arrow[d, hook, "j"] & \\
E(M) \arrow[r, two heads, "\rho"'] & E(M)/K &
\end{tikzcd}$$

α and β, which preserve the commutativity of the diagram, exist by the injectivity of $E(M)/K$ and projectivity of P, respectively. If $a \in A$, then $\rho\beta(a) = \alpha(a) = j\theta(a) \in M/K$, so $\rho\beta(a) = j\pi(m)$ for some $m \in M$. Using the diagram, $\rho\beta(a) = \rho i(m)$ hence $\beta(a) - i(m) \in \operatorname{Ker}\rho = K$. Now, $K \subseteq M$, so this implies that $\beta(a) \in M$, and hence $\rho\beta$ extends θ. $\square$

Finally, we will state the following result, which holds for projective modules over any ring.

Theorem 1.3.13 *(Kaplansky, [1] Corollary 26.2) Every projective module is a direct sum of countably generated modules.*

1.4 Small Modules and Projective Covers

Definition 1.4.1 *If* $A_R \leq B_R$ *are modules, we say that* **A is small in B** *(written* $A << B$*) if whenever there exists* $X_R \leq B_R$ *such that* $A + X = B$*, then* $X = B$*.* A_R *is said to be* **small** *if* $A << E(A)$*.*

Lemma 1.4.2 *Let* A *and* B *be modules such that* A *is a small submodule of* B*. Then:*

(i) A *is small in every module which contains* B*.*

and (ii) A *is small in every direct summand of* B *which contains* A*.*

Proof (i) Suppose that B is contained in a module C and that there exists a submodule X of C such that $C = A + X$. Then it follows that $B = A + (X \cap B)$, so $X \cap B = B$, i.e. $B \subseteq X$. Therefore, $A \subseteq X$ and so $C = X$. Hence A is small in C.

(ii) Assume that $B = D \oplus E$ and that $A \subseteq D$. Suppose that there exists $Y \subseteq D$ such that $A + Y = D$. Then $A + Y + E = B$, so by assumption, $Y + E = B$. But $Y \subseteq D$ and $D \cap E = 0$, so it follows that $Y = D$. Hence A is small in D. □

Lemma 1.4.3 *(W. W. Leonard) Let* M *be a right* R*-module. Then* M_R *is small* $\Leftrightarrow$ *there exists a right* R*-module* A *such that* M_R *is small in* A_R*.*

Proof $\Rightarrow$ This follows by the definition of a small module.

$\Leftarrow$ Suppose that M_R is small in some A_R. By Lemma 1.4.2 (i), M_R is small in $E(M)_R + A_R$. Being injective, $E(M)_R$ is a direct summand of $E(M)_R + A_R$, so M_R is small in $E(M)_R$ by Lemma 1.4.2 (ii). □

Lemma 1.4.4 *(i) Let* M *be a small module with a submodule* N*. Then* N *and* M/N *are small.*

(ii) Let A *and* B *be small submodules of* X *and* Y *respectively. Then* $A \oplus B$ *is a small submodule of* $X \oplus Y$*.*

Proof

(i) Suppose that $E(M)$ has a submodule X such that $N + X = E(M)$. Then $M + X = E(M)$, so $X = E(M)$. Hence N is small.

Suppose that $E(M)/N$ has a submodule Y/N such that $E(M)/N = M/N + Y/N$. Then $E(M) = M + Y$, so $Y = E(M)$. Hence M/N is small.

(ii) Let K be a submodule of $X \oplus Y$ such that $(A \oplus B) + K = X \oplus Y$. Clearly then $X = A + ((B+K) \cap X)$, so by the smallness of A in X, $X \subseteq B+K$. Therefore $B+K = X \oplus Y$ and so $Y = B + (K \cap Y)$ from which it follows that $Y \subseteq K$. Similarly, $X \subseteq K$ and it must be the case that $K = X \oplus Y$. Therefore, $A \oplus B$ is small in $X \oplus Y$. □

Corollary 1.4.5 *A direct sum of finitely many small modules is small.*

Proof This follows immediately from Lemma 1.4.4 (ii) and Lemma 1.4.3. □

Definition 1.4.6 *If $\pi : M_R \twoheadrightarrow N_R$ is an epimorphism such that $Ker\pi << M$, then we say that the pair (M, π) is a* **small cover** *of N. When M is projective and $Ker\pi$ is small in M, we call the pair a* **projective cover** *of N. In the latter case, we will sometimes refer to the module M without the corresponding surjection as the projective cover of N.*

Lemma 1.4.7 *([1], Lemma 17.17) Let M_R be a module with a projective cover (Q, π). Suppose that there is a projective module P with a surjection $\rho : P_R \twoheadrightarrow M_R$. Then there is a decomposition $P_R = P'_R \oplus P''_R$ such that:*

(i) There exists an isomorphism $\alpha : Q \cong P'$ and $\pi = (\rho|_{P'})\alpha$.

(ii) $P'' \subseteq Ker\rho$,

(iii) $(P', \rho|_{P'})$ is a projective cover of M_R.

Definition 1.4.8 *A ring R is said to be* **right perfect** *if every right R-module has a projective cover.*

Example 1.4.9 ([1], Corollary 28.8) Every right or left artinian ring is right perfect.

1.5 CS Modules and Lifting Modules

Definition 1.5.1 *If $N_R \leq M_R$, then N is said to be* **essentially closed** *in M if there does not exist a module $L_R > N_R$ such that $N \leq_{ess} L \leq M$.*

We will introduce the following together, as they are dual notions.

Definition 1.5.2 *A module M is said to be* **CS** *if every submodule of M is essential in a direct summand of M, or equivalently, if every essentially closed submodule of M is a direct summand of M. A module M is said to be* **$\sum$-CS** *if every direct sum of copies of M is CS. In some of the literature, CS modules are referred to as* **extending** *modules.*

A module M is said to be **lifting** *if for every submodule A of M, there exists a direct summand B of M contained in A such that $A/B << M/B$.*

Example 1.5.3 Let M_R be a uniform module with a non-zero submodule N_R. Then, trivially, $N \leq_{ess} M$ and so M is CS.

Example 1.5.4 Let L_R be a local module with maximum submodule M_R. Since L is local, every proper submodule of L is contained in M. It follows that the sum of any two proper submodules of L is contained in M, and so every submodule of L is small in L. Therefore L is lifting.

Lemma 1.5.5 *(i) Every injective module is CS.*

(ii) If R is a right perfect ring, then every projective right R-module is lifting.

Proof (i) Let E_R be an injective module with an essentially closed submodule C_R. By Lemma 1.1.2, C_R has a complement X_R in E_R such that $(C \oplus X)/X \leq_{ess} E/X$. So there exists an embedding $i : C \hookrightarrow E/X$ such that $\operatorname{Im} i \leq_{ess} E/X$. If we now consider the diagram:

$$\begin{tikzcd} C \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & E/X \arrow[dl, dotted, "\alpha"] \\ E & \end{tikzcd}$$

where j is the inclusion of C in E, then there exists a homomorphism $\alpha : E/X \to E$ which makes the diagram commute. Since j is a monomorphism and i is an essential embedding, it follows that α is monomorphic and furthermore that $\mathrm{Im}\, j \leq_{ess} \mathrm{Im}\, \alpha$. As C is essentially closed in E, this must mean that $\mathrm{Im}\, \alpha = \mathrm{Im}\, j = C$. If $e+X \in (E/X)\backslash \mathrm{Im}\, i$, then $\alpha(e+X) = c = \alpha i(c)$ for some $c \in C$. But then $0 \neq (e+X) - i(c) \in \mathrm{Ker}\, \alpha$ - a contradiction. Therefore $\mathrm{Im}\, i = E/X$, and since this was induced by the inclusion $(C \oplus X)/X \leq_{ess} E/X$, it follows that $(C \oplus X)/X = E/X$ and so $E = C \oplus X$. Hence C is a direct summand of E.

(ii) Let P_R be projective, $K_R \leq P_R$ and $\pi : P \twoheadrightarrow P/K$ be the canonical projection. Then by Lemma 1.4.7, there exists a decomposition $P = P' \oplus P''$ such that $(P', \pi|_{P'})$ is a projective cover of P/K and $P'' \leq K$. So $\mathrm{Ker}\, (\pi|_{P'}) = K \cap P' << P'$ and $K = (P' \cap K) \oplus P''$.

By the obvious isomorphism, it must be the case that $((K \cap P') \oplus P'')/P''$ is small in P/P'', i.e. $K/P'' << P/P''$. Hence P is lifting. □

The following class of rings were originally described in [17].

Definition 1.5.6 *Let R be a ring such that every singular right R-module is injective. Then we call R a* **right SI ring.**

Example 1.5.7 ([17], Ex. 3.2) Let k be a field and consider the ring $k^{\mathbb{N}}$ of countably infinite sequences of members of k, where addition and multiplication are defined componentwise. Let R be the subring of $k^{\mathbb{N}}$ consisting of all those members whose entries from k become constant after a finite portion of the string.

Clearly if we take $i \in \mathbb{N}$ and define S_i to be the set of members of R which are zero outside of the ith position, then S_i is an ideal and also a simple R-module. If we set $S = \bigoplus_{i \in \mathbb{N}} S_i$, then S is semisimple and it is easy to verify that S is essential and maximal in R. Since every essential ideal of R must contain the socle of R by Lemma 1.1.6, it follows that S is the only non-trivial essential ideal of R. Also, it is straightforward to show that each S_i is non-singular and so S is non-singular.

Let T be a singular R-module. By Corollary 1.3.9, in order to show that T_R is injective, it is enough to show that every homomorphism from S to T extends to one from R to T. Take a homomorphism $\theta : S \to T$. Since S is semisimple, every submodule of S is a direct summand of S, and therefore $\operatorname{Im}\theta \cong S/\operatorname{Ker}\theta$ must also be isomorphic to a direct summand of S. But since S is non-singular and T is singular, it must be the case that $\operatorname{Im}\theta = 0$. Hence θ is the trivial map and can be extended to the trivial map from R to T.

Therefore R is an SI ring.

Lemma 1.5.8 *(Goodearl, [17]) Let R be a right SI ring. Then R is right hereditary.*

Proof We will use Theorem 1.3.12. Let E_R be an injective module with a submodule X_R. Then, since E_R is injective, there exists a decomposition $E_R = E'_R \oplus E''_R$, where X_R is an essential submodule of E'_R by Lemma 1.5.5 (i). Clearly, $E/X \cong E'/X \oplus E''$ and E'/X is singular, hence injective. Thus E/X is injective. □

More information about small modules, CS modules and related topics can be found in [31].

Chapter 2

A Simplified Proof of Oshiro's Theorem for co-H Rings

2.1 The History

QF (Quasi-Frobenius) rings have been known about since Nakayama's papers in the early '40s. These have very pleasing properties - as well as being symmetric, the QF property itself has many equivalent characterisations.

Definition 2.1.1 *A ring R is said to be* **QF** *if R_R is injective and R is right noetherian.*

Theorem 2.1.2 *([26], Theorem 13.2.1, Theorem 13.6.1) For a ring R, the following are equivalent:*

(i) R is QF.

(ii) ${}_RR$ is injective and R is right noetherian.

(iii) R_R is injective and R is right artinian.

(iv) ${}_RR$ is injective and R is right artinian.

(v) R is right noetherian and if $A \lhd_r R$ and $B \lhd_l R$, then $\mathbf{rl}(A) = A$ and $\mathbf{lr}(B) = B$.

(vi) Every projective right R-module is injective.

(vii) Every injective right R-module is projective.

(viii) The left-right symmetric dual of any of the above.

Definition 2.1.3 *A ring R is said to be* **right QF-3** *if there exists a right R-module whose annihilator is zero and which is isomorphic to a direct summand of every right R-module whose annihilator is zero.*

There have been many generalised versions of the QF property - see for example Chapter 31 of [1] which describes QF-3 rings in more detail and also introduces QF-2 rings. In [34], Oshiro produced two particular generalisations of QF rings - right (and left) H rings and right (and left) co-H rings (the H standing for Harada whose earlier work had inspired these generalisations). As the names suggest, co-H rings are a kind of dual of H rings. Both H and co-H rings were shown to be QF-3 in [34]. Left H rings are not necessarily right H and left co-H rings are not necessarily right co-H, but Oshiro proved in [35] that a ring is left co-H if and only if it is right H. This was used in [37] to show that right co-H rings are right and left artinian.

Here, we will introduce the idea of H rings only in sketch form, as we are really interested in co-H rings.

Theorem 2.1.4 *([34], Theorem 2.11) For a ring R, the following are equivalent:*

(i) Every injective right R-module is lifting.

(ii) Every right R-module is the direct sum of an injective module and a small module.

(iii) (a) If E_R is an injective module with a small cover (X_R, π), then X_R is also injective,

(b) R is right perfect.

(iv) (a) Every non-small right R-module contains a non-zero injective submodule,

(b) R is right artinian.

Theorem 2.1.5 *([34], Theorem 3.18) For a ring R, the following are equivalent:*

(i) Every projective right R-module is CS.

(ii) Every right R-module is the direct sum of a projective module and a singular module.

(iii) The class of projective right R-modules is closed under taking essential extensions.

(iv) (a) Every right R-module which is not singular has a non-zero projective direct summand.

(b) R has ACC on the right annihilators of subsets of R.

Definition 2.1.6 *[34] A ring R is said to be* **right H** *if it satisfies the equivalent conditions of Theorem 2.1.4, and* **right co-H** *if it satisfies the equivalent conditions of Theorem 2.1.5. It is easy to see that R is right co-H if and only if R_R is a $\sum$-CS module.*

Lemma 2.1.7 *(Well-known) QF rings are left and right H and left and right co-H.*

Proof This follows by the equivalence of projective and injective modules over a QF ring and Lemma 1.5.5. □

There has been a great deal of interest in co-H rings and their generalisations, such as Σ-CS modules, in recent years - see for example [4], [5] and [6].

In the section which follows, we will prove Theorem 2.1.5. This was originally done in [34], but the proof there is rather difficult to follow. The tricky step is showing that condition (iv) is equivalent to the others. Note that all of the conditions, with the exception of (iv) (b) are conditions on the modules, rather than the ideals. The proof in [34] uses the ideal structure of the rings which makes it interesting, but somewhat indirect. A second proof of the result was obtained in [4], as a corollary to a larger theorem. Although shorter, this one was a little technical. It is a composite version we include here, some parts of which are new, while others come from the above sources. This proof has the advantages of being short and of using mainly well-known module theoretic ideas. Also, the methods we will use can be applied elsewhere. The key to the new proof is to convert condition (iv) (b) of the Theorem from an ideal theoretic to a module theoretic statement.

2.2 The New Proof

Before we can start, we will need a few lemmas.

Lemma 2.2.1 *(Well-known) Let $\{M_\lambda\}_\Lambda$ be a set of right R-modules with a set of submodules $\{N_\lambda\}_\Lambda$ such that $N_\lambda \leq M_\lambda$ for every $\lambda \in \Lambda$. Put $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ and $N = \bigoplus_{\lambda\in\Lambda} N_\lambda$. Then $N \leq_{ess} M \Leftrightarrow N_\lambda \leq_{ess} M_\lambda$ for every $\lambda \in \Lambda$.*

Proof $\Rightarrow$ Suppose that there is a $\lambda \in \Lambda$ such that N_λ is not essential in M_λ. Then there exists a submodule X of M_λ such that $X \cap N_\lambda = 0$. It follows that $X \cap N = 0$, and so N is not essential in M - a contradiction.

$\Leftarrow$ Suppose that $N_\lambda \leq_{ess} M_\lambda$ for every $\lambda \in \Lambda$, but N is not essential in M. Then there exists a non-zero $m \in M$ such that $mR \cap N = 0$. Now for each choice of m, the representation of m in the sum $\bigoplus_{\lambda \in \Lambda} M_\lambda$ has non-zero entries only in a finite subset F of Λ, so we can choose m such that $|F|$ is minimum and put $m = \sum_{f \in F} m_f$, where $0 \neq m_f \in M_f$. By assumption, $|F| > 1$.

If we fix $f \in F$, then there exists $r \in R$ such that $0 \neq m_f r \in N$. We know that $mr \neq 0$ and $mr \notin N$, so $mr - m_f r \notin N$, and in particular $mr - m_f r \neq 0$. If there then exists $s \in R$ such that $(mr - m_f r)s \in N$, then $mrs \in N$ and so $mrs = 0$. By the decomposition of M, it follows that $m_f rs = 0$ and hence $(mr - m_f r)R \cap N = 0$. But $mr - m_f r$ is non-zero in strictly less than $|F|$ entries of $\{M_\lambda\}_\Lambda$, which contradicts the minimality of $|F|$. $\square$

Lemma 2.2.2 *(Well-known) (i) If P is a projective right R-module with a submodule X, then P/X is singular $\Leftrightarrow$ $X \leq_{ess} P$.*

(ii) A projective singular module is zero.

Proof (i) $\Leftarrow$ is straightforward.

$\Rightarrow$ Let $I \lhd_r R$ such that R/I is singular. Then $I = \mathbf{r}(1 + I)$ must be an essential right ideal of R.

Now, say that the implication does not hold, and that there exists a projective right R-module P with a submodule X such that P/X is singular but X is not essential in P. Then there is a free right R-module F such that $F = P \oplus P'$ for some $P' \leq F$. By our earlier assumptions, $F/(X \oplus P') \cong (P \oplus P')/(X \oplus P') \cong P/X$ is singular and $X \oplus P'$ is not essential in F, so we can assume without loss of generality that P is free, i.e. $P = \bigoplus_{\lambda \in \Lambda} R_\lambda$, where each R_λ is a copy of R.

If we take R_λ to be one of the copies of R in the decomposition of P, then $R_\lambda/(R_\lambda \cap X) \cong (R_\lambda + X)/X \hookrightarrow P/X$ is singular and so by the first paragraph, $(R_\lambda \cap X) \leq_{ess} R_\lambda$. By Lemma

2.2.1, $\bigoplus_{\lambda\in\Lambda} R_\lambda \cap X \leq_{ess} \bigoplus_{\lambda\in\Lambda} R_\lambda = P$ and hence it follows that $X \leq_{ess} P$, contradicting our assumption.

(ii) Let A be a projective singular module. Then $A/0$ is singular, so by part (i), $0 \leq_{ess} A$. Hence $A = 0$. □

Lemma 2.2.3 *([18], Prop 1.4) Let A, B and C be right R-modules such that $C \leq B \leq_{ess} A$ and $C \leq_{cl} A$. Then $B/C \leq_{ess} A/C$.*

Proof Let $a \in A\backslash C$. Since C is not essential in $aR + C$, there exist $r \in R$ and $c \in C$ such that $ar + c \neq 0$ and $(ar + c)R \cap C = 0$. But since $B \leq_{ess} A$, there exists $s \in R$ such that $0 \neq ars + cs \in B$, and hence $ars \in B$. If $ars \in C$, then $ars + cs \in C$ - a contradiction.

Therefore, $0 \neq ars + C \in ((aR + C)/C) \cap (B/C)$, and the result follows. □

Lemma 2.2.4 *([14]) For any cardinal ξ, there exists a cardinal σ such that every ξ-generated module has at most σ submodules.*

Proof Every ξ-generated right R-module is an epimorphic image of $F = R_R^{(\xi)}$, so cannot have more submodules than F. □

Definition 2.2.5 *If $A \leq B$ and $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$, then A is said to be a* **local direct summand** *of B if for any finite subset F of Λ, $\bigoplus_{\lambda\in F} A_\lambda$ is a direct summand of B.*

Lemma 2.2.6 *(Adapted from [33].) If M_R is a module for which R has ACC on right ideals of the form $\mathbf{r}(m)$ where $m \in M$, then all local direct summands of M are essentially closed in M.*

Proof Let $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ be a local direct summand of M. Suppose that there is a submodule B of M such that $A \neq B$ and $A \leq_{ess} B$. Take $b \in B\backslash A$ such that $\mathbf{r}(b)$ is maximal in $\{\mathbf{r}(c) : c \in B\backslash A\}$. Then there exists $r \in R$ such that $0 \neq br \in A$.

Now, br is contained in $A_F = \bigoplus_{f\in F} A_f$ for some finite subset F of Λ and we know that there exists $K_F \leq M$ such that $M = A_F \oplus K_F$. If we put $b = a + k$, where $a \in A_F$ and $k \in K_F$, then $br = ar + kr$ and so $kr \in A_F \cap K_F = 0$.

Suppose that $x \in \mathbf{r}(b)$. Then $(a+k)x = 0$ and hence $kx = 0$. Thus $\mathbf{r}(b) \subseteq \mathbf{r}(k)$. But $r \in \mathbf{r}(k) \backslash \mathbf{r}(b)$, so that $\mathbf{r}(b) \neq \mathbf{r}(k)$. Furthermore, $k \in B \backslash A$, contradicting the maximality of $\mathbf{r}(b)$ for our choice of b. The result then follows. $\square$

The following Lemma is new and is primarily included to complete the proof of Theorem 2.1.5, but it can also be used in the proof of other results. We will use it here to prove Lemma 2.2.9.

Lemma 2.2.7 *If $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ is a local direct summand of B and R satisfies the ACC on right ideals of the form $\mathbf{r}(S)$, where S is a subset of A, then A is essentially closed in B.*

Proof Suppose that A is not essentially closed in B. Then there exists a submodule X of B such that A is an essential proper submodule of X. Clearly A must also be a local direct summand of X, and so without loss of generality, we can assume that A is essential in B.

By Lemma 2.2.6, it is enough to show that R satisfies ACC on right ideals of the form $\mathbf{r}(b)$, where $b \in B$. Let $\mathcal{F}$ be the set of all finite subsets of Λ. For each $F \in \mathcal{F}$ let $A_F = \bigoplus_{\lambda \in F} A_\lambda$, and fix K_F so that $B = A_F \oplus K_F$. Set $K = \bigcap_{F \in \mathcal{F}} K_F$. For each $k \in K \cap A$, there exists $F \in \mathcal{F}$ with $k \in A_F \cap K_F = 0$, so $K \cap A = 0$ and hence $K = 0$ since A is essential in B.

Let $b \in B$. Then for each $F \in \mathcal{F}$ let $a_F \in A_F$ and $k_F \in K_F$ with $b = a_F + k_F$. Let $\Omega_b = \{a_F : F \in \mathcal{F}\} \subseteq A$. Then $x \in \mathbf{r}(\Omega_b) \Leftrightarrow bx = k_F x$ $(\forall F \in \mathcal{F}) \Leftrightarrow bx \in K \Leftrightarrow x \in \mathbf{r}(b)$. Therefore, $\mathbf{r}(b) = \mathbf{r}(\Omega_b)$, so R satisfies the ACC on right ideals of the form $\mathbf{r}(b)$. $\square$

Definition 2.2.8 *An injective module E is said to be* **$\sum$-injective** *if every direct sum of copies of E is injective. E is said to be* **countably $\sum$-injective** *if every direct sum of countably many copies of E is injective.*

Lemma 2.2.9 *(Faith) The following are equivalent for an injective module E:*

(i) E is $\sum$-injective.

(ii) E is countably $\sum$-injective.

(iii) R satisfies ACC on annihilators of subsets of E.

Proof (i) $\Rightarrow$ (ii) This is obvious.

(ii) $\Rightarrow$ (iii) Let $S_1, S_2, \ldots$ be subsets of E such that:

$$\mathbf{r}(S_1) \subset \mathbf{r}(S_2) \subset \ldots$$

is an infinite increasing chain. Let $I = \bigcup_{j \in \mathbb{N}} \mathbf{r}(S_j)$, and for every $i \in \mathbb{N}$ fix $s_i \in S_i$ such that $s_i \mathbf{r}(S_{i+1}) \neq 0$. For every $x \in I$, there exists $j \in \mathbb{N}$ such that $S_k x = 0$ for all $k \geq j$, and so there exists a well-defined homomorphism:

$$\theta : I \to E^{(\mathbb{N})} : x \mapsto (s_1 x, s_2 x, \ldots)$$

Since E is countably $\sum$-injective, this extends to a homomorphism $\phi : R_R \to E^{(\mathbb{N})}$. Obviously, there exists $t \in \mathbb{N}$ and $e_i \in E$ for $1 \leq i \leq t$ such that $\phi(1) = (e_1, e_2, \ldots, e_t, 0, 0, \ldots)$. It follows that for any $x \in I$, $\phi(x) = (e_1 x, e_2 x, \ldots, e_t x, 0, 0, \ldots)$ and so $s_{t+1} x = 0$. This implies that $s_{t+1} I = 0$ and in particular $s_{t+1} \mathbf{r}(S_{t+2}) = 0$, which contradicts our choice of s_{t+1}. Therefore we cannot have an infinite increasing chain of annihilators of subsets of E and condition (iii) is proved.

(iii) $\Rightarrow$ (i) Let $X = \bigoplus_{\lambda \in \Lambda} E_\lambda$, where each E_λ is an isomorphic copy of E. Since any finite direct sum of copies of E is injective, X is a local direct summand of $E(X)$. The result then follows by Lemma 2.2.7. $\square$

We are now ready to proceed to the proof of Theorem 2.1.5. Note that in [34], the statement of condition (iv) (a) is slightly different - namely that all non-cosmall modules have a non-zero projective direct summand. In fact, a module is non-cosmall iff it is not singular. Oshiro was considering both co-H and, dually, H rings, so used the dual notation, but here we consider only co-H rings, so we will use the more usual notation.

Of the implications in the proof, (i)$\Rightarrow$(ii) is taken directly from [34] and is included for completeness, and (ii)$\Rightarrow$(iii) is a straightforward proof pointed out to me by Alberto del Valle Robles. (ii)$\Rightarrow$(iv) (b) is an application of ([14], Theorem 1.12) to the case where $M = R$, used in the same way as in [4]. Once again, the whole argument has been included here for completeness.

Proof of Theorem 2.1.5 (i)$\Rightarrow$(ii) Let M be a right R-module. There is a free module F, and an epimorphism $\theta : F \twoheadrightarrow M$. By the hypothesis, there exist P and Q such that $F = P \oplus Q$, with $\operatorname{Ker}\theta \leq_{ess} P$. We have $M = \theta(P) \oplus \theta(Q)$, where $\theta(P) \cong P/\operatorname{Ker}\theta$ is singular and $\theta(Q) \cong Q$ is projective. Hence condition (ii) holds.

(ii)$\Rightarrow$(iii) Let P be a projective module and M be an essential extension of P. By condition (ii), we have $M = Q \oplus S$, where Q is projective and S is singular. Now, $P/(P \cap Q) \cong (P+Q)/Q \hookrightarrow (Q \oplus S)/Q \cong S$. By Lemma 2.2.2, $P \cap Q \leq_{ess} P$, so $P \cap Q \leq_{ess} M$ and hence $Q \leq_{ess} M$. Thus $M = Q$, and the result is shown.

(iii)$\Rightarrow$(i) Let C be an essentially closed submodule of a projective module P. Now $C = E(C) \cap P$, so $P/C = P/(E(C) \cap P) \cong (E(C)+P)/E(C)$. By condition (iii), $E(C)+P$ is projective, and since $E(C) \leq^{\oplus} E(C) + P$, we have that P/C is isomorphic to a direct summand of $E(C) + P$, so P/C is projective and hence $C \leq^{\oplus} P$.

(ii)$\Rightarrow$(iv) (a) is trivial.

(iii)$\Rightarrow$(iv) (b) We use Lemma 2.2.9. Let ξ be an uncountably infinite cardinal such that $E(R)$ can be generated by ξ (or fewer) elements. By Lemma 2.2.4, there exists a cardinal σ, such that every ξ-generated module has no more than σ submodules. Take $\tau > \sigma$, and let $E = E(E(R)^{(\tau)})$. Then since E is projective, $E = \bigoplus_{\lambda \in \Lambda} E_\lambda$, where each E_λ is countably generated, by Theorem 1.3.13.

Take any copy of $E(R)$ in E, $E(R)_1'$, say. Then $E(R)_1' \leq \bigoplus_{\omega \in \Omega_1} E_\omega$, where $|\Omega_1| \leq \xi$. $\bigoplus_{\omega \in \Omega_1} E_\omega$ is ξ-generated, so has no more than σ submodules. Hence there is a copy of $E(R)$, $E(R)_2$ say, in E such that $E(R)_2 \cap \bigoplus_{\Omega_1} E_\omega = 0$, otherwise there would be at least τ non-zero submodules of $\bigoplus_{\Omega_1} E_\omega$, (formed by its intersections with the τ copies of $E(R)$ in E), contradicting the previous statement. Hence $\exists E(R)_2' \leq \bigoplus_{\Lambda \setminus \Omega_1} E_\lambda$, and so $E(R)_2' \leq \bigoplus_{\Omega_2} E_\omega$, where $|\Omega_2| \leq \xi$, $\Omega_2 \subseteq \Lambda \setminus \Omega_1$.

Repeating, we can find $E(R)_3' \leq \bigoplus_{\Lambda \setminus (\Omega_1 \cup \Omega_2)} E_\lambda$, etc. to obtain a countably infinite set of copies of $E(R)$ in E, each contained in its own independent subset of $\{E_\lambda\}_\Lambda$. Note that for every i, $E(R)_i'$ is injective so is a direct summand of $\bigoplus_{\Omega_i} E_\lambda$, and hence $\bigoplus_{i \in \mathbb{N}} E(R)_i'$ is a direct summand of E, so is injective. Thus by Lemma 2.2.9, we have that R has the ACC on right annihilators of subsets of $E(R)$ and hence on those of subsets of R.

(iv)$\Rightarrow$(i) Let P be a projective module, and C be an essentially closed submodule of P. We wish to prove that $C \leq^{\oplus} P$. If $C \leq_{ess} P$, then clearly $C = P$ and we are done. Otherwise, by Lemma 2.2.2 and condition (iv) (a), P/C has a non-zero projective direct summand, X/C.

Consider a set of non-zero independent projective submodules of P/C which form a local direct summand of P/C. The union of any chain of such sets is also a set of non-zero independent projectives which form a local direct summand, and $\{X/C\}$ is a nonempty example of such a set, so by Zorn's Lemma, we can find a maximal such set, $\{Q_\lambda/C\}_{\lambda\in\Lambda}$. Let $Q/C = \bigoplus_{\lambda\in\Lambda} Q_\lambda/C$.

Suppose that Q is not essential in P. Then by Lemma 2.2.2 and (iv) (a), we have $P/Q = A/Q \oplus B/Q$, where A/Q is a non-zero projective. Now, $P/B \cong (P/Q)/(B/Q) \cong A/Q$ is projective and so $P = B \oplus K$, where $K \neq 0$. But, since $C \subseteq B$, we have $P/C = B/C \oplus (K \oplus C)/C$, and moreover, it is easy to see that $\{Q_\lambda/C\}_{\lambda\in\Lambda}$ is a local direct summand of B/C, which implies that we can add $(K \oplus C)/C$ to our set, contradicting its maximality. Thus Q is essential in P, and so by Lemma 2.2.3, we have Q/C is essential in P/C.

The annihilator of an element of a projective module is the annihilator of a subset of R (since every projective module is contained in a free module) so by (iv) (b), R satisfies ACC on the right annihilators of the projective module Q/C. Hence by Lemma 2.2.7, we have $Q/C = P/C$, and thus P/C is projective, showing that C is a direct summand of P.
$\square$

2.3 An Example of a Family of co-H Rings

Our example is taken directly from [34]. The way we will proceed is to show how to construct a right co-H ring, given an arbitrary local QF ring. Having established that our ring is right co-H, we will then introduce some local QF rings which when used in our construction, give rise to rings which are not left co-H (and hence not QF). Before we can start, we will need some preliminary results.

Lemma 2.3.1 *(Harada) If R is a semiperfect ring such that every right R-module which is not singular has a non-zero projective direct summand, then every indecomposable projective right R-module is uniform.*

Proof Let P_R be an indecomposable projective. Then by Theorem 1.3.4, $P \cong eR$ for some primitive idempotent e of R and P is local. Suppose that P is not uniform and let $0 \neq M_R$ be a non-essential submodule of P. Then by Lemma 2.2.2 and the hypothesis, there exists a decomposition $P/M = A/M \oplus B/M$, where A/M is a non-zero projective. It is easy to see that $P = A + B$, and since P is local, this implies that $A = P$ or $B = P$. If $A = P$, then clearly $M \leq^{\oplus} P$ which is false. If $B = P$, then $A = M$, which is also false. Therefore every submodule of P is essential, i.e. P is uniform. $\square$

Lemma 2.3.2 *([21], Theorem 3.16) Let R be a semiperfect ring with Jacobson radical J and a complete set $\{e_i\} \cup \{g_j\}$ of primitive orthogonal idempotents such that each e_iR is non-small and each g_jR is small. Then every right R-module which is not singular has a projective direct summand if and only if the following hold:*

(i) e_iR is injective for every i,

(ii) for every j, there exists i such that $g_jR \hookrightarrow e_iR$,

(iii) for every i, there exists $n_i \geq 0$ such that e_iJ^t is projective for $0 \leq t \leq n_i$ and $e_iJ^{n_i+1}$ is singular.

Proof Firstly, suppose that every right R-module which is not singular contains a non-zero projective direct summand. Let f be a primitive idempotent of R. Then by Lemma 2.3.1 fR is uniform and so $E(fR)$ is uniform too. By Lemma 2.2.2, fR is not singular and hence $E(fR)$ is not singular. So $E(fR)$ contains a non-zero projective direct summand which can only be $E(fR)$ itself. Hence $E(fR)$ is an indecomposable projective, i.e. $E(fR) \cong f'R$ for some primitive idempotent f' of R.

Now, we know by Theorem 1.3.4 that $f'R$ is local, and so if fR is not small, $fR \cong f'R$, i.e. fR is injective. We have now proved conditions (i) and (ii). To prove (iii) it is enough to note that for every $k \in \mathbb{N}$, fJ^k is uniform, so must be either projective or singular. If it is singular, it is obvious that fJ^l is also singular for every $l > k$.

Conversely, suppose that the conditions (i), (ii) and (iii) hold and that M_R is not singular. Then there is a surjection $\pi : P \twoheadrightarrow M$, where P is projective. Since R is semiperfect, it follows by Theorem 1.3.4 that $P = \bigoplus_{\lambda\in\Lambda} P_\lambda$ where each P_λ is an indecomposable projective right R-module, so $M = \sum_{\lambda\in\Lambda} \pi(P_\lambda)$. By (i) and (ii), each P_λ embeds in an indecomposable injective and so must be uniform. If $\mu \in \Lambda$ and $\operatorname{Ker}\pi|_{P_\mu} \neq 0$ then $\pi(P_\mu)$ is singular, so by assumption there exists $\lambda \in \Lambda$ such that $\operatorname{Ker}\pi|_{P_\lambda} = 0$, i.e. $P_\lambda \hookrightarrow M$.

Using (i) and (ii) again with Theorem 1.3.4, there exists a primitive idempotent e of R such that eR is injective and $P_\lambda \hookrightarrow eR$. So if we consider:

$$\begin{tikzcd} P_\lambda \arrow[r, hook, "\pi|_{P_\lambda}"] \arrow[d, hook, "i"'] & M \arrow[dl, dotted, "\theta"] \\ eR & \end{tikzcd}$$

then there exists a homomorphism $\theta : M \to eR$ such that $\theta(M) \supseteq i(P_\lambda)$. By (iii), there exists $n \in \mathbb{N}$ such that eJ^t is projective for every $t \leq n$ and eJ^{t+1} is singular. Since every indecomposable projective right R-module is local, it is easy to see by an induction argument that eR/eJ^{t+1} is uniserial. Therefore a submodule of eR is projective if and only if it strictly contains eJ^{t+1}. Since $i(P_\lambda)$ is projective it must also be true that $\theta(M)$ is projective. This means that $M/\operatorname{Ker}\theta$ is projective and so M has a non-zero projective direct summand isomorphic to $M/\operatorname{Ker}\theta$. □

Lemma 2.3.3 *(Well-known) If Q is a QF ring, then $Soc(Q_Q) = Soc({}_QQ)$.*

Proof By Lemma 1.2.15, $\mathbf{l}(J) = Soc(Q_Q)$, and so using Theorem 2.1.2 (v), we can see that $J = \mathbf{r}(Soc(Q_Q))$. Let e be a non-zero primitive idempotent of Q and suppose that $Soc(Q_Q)e = 0$. Then $e \in J$. But it follows from Lemma 1.2.2 that $1 - e$ is invertible which cannot be true since $(1-e)e = 0$. Therefore $Soc(Q_Q)e \neq 0$.

By Lemma 2.1.7 and Lemma 2.3.1, Qe is a uniform left ideal. Since ${}_QQ$ is left artinian, $Soc({}_QQe)$ is an essential simple submodule of Qe. Clearly, $Soc(Q_Q)$ is a left ideal of Q, so $Soc(Q_Q)e$ is a submodule of ${}_QQe$ and hence $Soc({}_QQe) \subseteq Soc(Q_Q)e \subseteq Soc(Q_Q)$.

Since e was chosen arbitrarily, it must follow that $Soc({}_QQ) \subseteq Soc(Q_Q)$, and by similar reasoning, $Soc(Q_Q) \subseteq Soc({}_QQ)$. □

Lemma 2.3.4 *([34], Section 5) Let Q be a local QF ring with Jacobson radical J and right socle S_Q. Then there exists a well-defined ring:*

$$W = \begin{bmatrix} Q & Q/S \\ J & Q/S \end{bmatrix}.$$

Proof By Lemma 1.2.15, $SJ = 0$. Since the left and right socles of Q are equal by Lemma 2.3.3, we also have $JS = 0$.

Now consider the ring W. If $u, v, w, x, y, z \in Q$ and $j, k \in J$, then we define multiplication in W according to the following rule:

$$\begin{bmatrix} u & v+S \\ j & w+S \end{bmatrix} \begin{bmatrix} x & y+S \\ k & z+S \end{bmatrix} = \begin{bmatrix} ux+vk & uy+vz+S \\ jx+wk & jy+wz+S \end{bmatrix}$$

Clearly, $jx+wk \in J$. Since $SJ = 0$, the element $ux+vk$ is unaffected by the representative v of $v+S$. Each of the other three elements of the product is also unaffected by choosing the representatives of $v+S$, $w+S$, $y+S$ and $z+S$. □

In the proof of the following Theorem, we need to show that the right ideal generated by one of the primitive idempotents of the ring is injective. The shortest way to do this would be to use Theorem 3.1 of [12], but we can prove the result directly with only a little extra effort, which allows us the luxury of a self-contained proof.

Theorem 2.3.5 *([34], Section 5) W as defined in Lemma 2.3.4 is a right co-H ring.*

Proof W_Q and ${}_QW$ are finitely generated, so Q is left and right artinian and therefore noetherian and semiperfect. It follows that if conditions (i), (ii) and (iii) of Lemma 2.3.2 hold, then condition (iv) of Theorem 2.1.5 must hold.

Let $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. It is easy to see that eW_W is isomorphic to the right W-module $\begin{bmatrix} Q & Q/S \end{bmatrix}$ with the usual matrix multiplication. Consider the submodule $\begin{bmatrix} S & 0 \end{bmatrix}$ of $\begin{bmatrix} Q & Q/S \end{bmatrix}$. Since S_Q is simple, it follows that for any non-zero $s \in S$ we have $sQ = S$. Similarly, we have $\begin{bmatrix} s & 0 \end{bmatrix} \begin{bmatrix} Q & Q/S \\ J & Q/S \end{bmatrix} = \begin{bmatrix} S & 0 \end{bmatrix}$ for any non-zero $s \in S$ and so $\begin{bmatrix} S & 0 \end{bmatrix}$ is also simple.

Let $0 \neq q \in Q$ and $r \in Q$. Then there exists $a \in Q$ such that $0 \neq qa \in S$ and so $0 \neq \begin{bmatrix} q & r+S \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} S & 0 \end{bmatrix}$. Now let $q' \in Q$ and $r' \in Q \backslash S$. Since Q is right artinian and $r'J \neq 0$, there exists $j \in J$ such that $0 \neq r'j \in S$ and so $0 \neq \begin{bmatrix} q' & r'+S \end{bmatrix} \begin{bmatrix} 0 & 0 \\ j & 0 \end{bmatrix} \in \begin{bmatrix} S & 0 \end{bmatrix}$. It follows that $\begin{bmatrix} S & 0 \end{bmatrix}$ is essential in $\begin{bmatrix} Q & Q/S \end{bmatrix}$ and so $\begin{bmatrix} S & 0 \end{bmatrix}$ is the socle of $\begin{bmatrix} Q & Q/S \end{bmatrix}$.

It is easy to see that $\begin{bmatrix} J & Q/S \end{bmatrix}$ is a submodule of $\begin{bmatrix} Q & Q/S \end{bmatrix}$ and furthermore that $\begin{bmatrix} J & Q/S \end{bmatrix}$ is isomorphic to fW. Let $q \in Q \backslash J$ and let $r \in Q$. Since Q is local, q has an inverse q^{-1}, and $\begin{bmatrix} q & r+S \end{bmatrix} \begin{bmatrix} q^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, which is a generator of $\begin{bmatrix} Q & Q/S \end{bmatrix}$. It follows that $\begin{bmatrix} Q & Q/S \end{bmatrix}$ is local with maximal submodule $\begin{bmatrix} J & Q/S \end{bmatrix}$.

Now, $\begin{bmatrix} J & Q/S \end{bmatrix}$ has a submodule $\begin{bmatrix} J & J/S \end{bmatrix}$. If we take $j \in J$ and $q \in Q \backslash J$, then $\begin{bmatrix} j & q+S \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & q^{-1}+S \end{bmatrix} = \begin{bmatrix} 0 & 1+S \end{bmatrix}$, which is a generator of $\begin{bmatrix} J & Q/S \end{bmatrix}$. It follows that $\begin{bmatrix} J & Q/S \end{bmatrix}$ is also local with maximal submodule $\begin{bmatrix} J & J/S \end{bmatrix}$ and hence that $J(W) = eJ(W) \oplus fJ(W) = \begin{bmatrix} J & Q/S \\ J & J/S \end{bmatrix}$.

We know that $Soc(W_W) = \begin{bmatrix} S & 0 \\ S & 0 \end{bmatrix}$. Also $\begin{bmatrix} J & J/S \end{bmatrix} \begin{bmatrix} S & 0 \\ S & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and so $\begin{bmatrix} J & J/S \end{bmatrix}$ is singular.

By putting together what we have so far, we can see that $eJ(W) \cong fW$ and that $fJ(W)$ is singular. Hence W satisfies criteria (ii) and (iii) of Lemma 2.3.2 and we only need to prove criterion (i) - that eW is injective.

Since $W_W \cong eW \oplus eJ(W)$, by using Lemma 1.3.7 and Lemma 1.3.8, it is enough to show that $eW \cong \begin{bmatrix} Q & Q/S \end{bmatrix}$ is quasi-injective. Now, take a submodule N of $\begin{bmatrix} Q & Q/S \end{bmatrix}$. By considering the action of $\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & Q/S \end{bmatrix}$ on N it is apparent that $N = \begin{bmatrix} A & B/S \end{bmatrix}$, where A is a right ideal of Q and B is a right ideal of Q containing S. Also, if $a \in A$, then:

$$\begin{bmatrix} a & 0 \end{bmatrix} \begin{bmatrix} 0 & 1+S \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a+S \end{bmatrix}$$

and so $A \subseteq B$. Similarly, if $b \in B$ and $j \in J$,

$$\begin{bmatrix} 0 & b+S \end{bmatrix} \begin{bmatrix} 0 & 0 \\ j & 0 \end{bmatrix} = \begin{bmatrix} bj & 0 \end{bmatrix}$$

and so $BJ \subseteq A$.

Let $\theta : \begin{bmatrix} A & B/S \end{bmatrix} \to \begin{bmatrix} Q & Q/S \end{bmatrix}$. By considering the action of $\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$ on θ, we can see that the left-hand entries define a homomorphism $\phi : A_Q \to Q_Q$. Since Q_Q is injective, this extends to a homomorphism $\phi' : Q_Q \to Q_Q$. We can use ϕ' to define a homomorphism:

$$\theta' : \begin{bmatrix} Q & Q/S \end{bmatrix} \to \begin{bmatrix} Q & Q/S \end{bmatrix} : \begin{bmatrix} q & q'+S \end{bmatrix} \mapsto \begin{bmatrix} \phi'(q) & \phi'(q')+S \end{bmatrix}$$

It is easy to verify that θ' is a homomorphism which agrees with θ over $\begin{bmatrix} A & A/S \end{bmatrix}$.

Now suppose that $\psi : \begin{bmatrix} A & B/S \end{bmatrix} \to \begin{bmatrix} Q & Q/S \end{bmatrix}$ is a non-zero homomorphism such that $\psi\begin{bmatrix} A & A/S \end{bmatrix} = 0$. Then since $Soc(\operatorname{Im}\psi) \neq 0$, there exist $a \in A$ and $b \in B$ such that $\psi\begin{bmatrix} a & b+S \end{bmatrix} = \begin{bmatrix} t & 0 \end{bmatrix}$ where $0 \neq t \in S$. It follows that $\psi\begin{bmatrix} 0 & b+S \end{bmatrix} = \psi\begin{bmatrix} a & b+S \end{bmatrix} - \psi\begin{bmatrix} a & 0 \end{bmatrix} = \begin{bmatrix} t & 0 \end{bmatrix}$. So:

$$\begin{aligned} \psi\begin{bmatrix} 0 & b+S \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} t & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \psi\begin{bmatrix} 0 & 0 \end{bmatrix} &= \begin{bmatrix} t & 0 \end{bmatrix} \end{aligned}$$

so that $t = 0$, a contradiction. Therefore, $\psi\begin{bmatrix} A & B/S \end{bmatrix} = 0$.

Applying this last paragraph to $\theta'' - \theta$ where θ'' is the restriction of θ' to $\begin{bmatrix} A & B/S \end{bmatrix}$, we can see that θ' and θ agree over $\begin{bmatrix} A & B/S \end{bmatrix}$ and so θ' extends θ. □

Definition 2.3.6 *Let M be a module. For every $i \in \mathbb{N} \cup \{0\}$ we define the* **ith socle of M**, *$Soc_i(M)$ as follows:*

$$\begin{aligned} Soc_0(M) &= 0 \\ Soc_i(M)/Soc_{i-1}(M) &= Soc(M/Soc_{i-1}(M)) \quad \text{for } i \geq 1 \end{aligned}$$

Note that for any module M, $Soc_1(M)$ is equal to $Soc(M)$.

Lemma 2.3.7 *If Q is a local QF ring such that ${}_QQ/Soc({}_QQ)$ is uniform, then ${}_QQ$ is uniserial.*

Proof As Q is left artinian, we must have either $Soc_{i+1}({}_QQ) \neq Soc_i({}_QQ)$ or $Soc_i({}_QQ) = Q$, for every $i \in \mathbb{N}$.

Since Q is local, there is only one isomorphism class of simple left Q-modules. Also, ${}_QQ$ is indecomposable injective and hence uniform, so every indecomposable injective is isomorphic to ${}_QQ$. Therefore ${}_QQ/Soc({}_QQ) \hookrightarrow {}_QQ$, which means that $Soc_3({}_QQ)/Soc({}_QQ) \hookrightarrow Soc_2({}_QQ)$. This means that either $Soc_3({}_QQ) = Soc_2({}_QQ)$, in which case ${}_QQ$ is uniserial of length 2, or $Soc_3({}_QQ)/Soc_2({}_QQ)$ is simple.

Continuing, we see that for every i, $Soc_{i+1}({}_QQ)$ is either Q or the unique minimal submodule of ${}_QQ$ strictly containing $Soc_i({}_QQ)$. Since ${}_QQ$ is noetherian, the resulting chain must eventually stop. Therefore ${}_QQ$ is uniserial. $\square$

Lemma 2.3.8 *(Well-known) The following are local QF rings:*

(i) $Q = k[x,y]/(x^2, y^2)$, where k is a field and x and y commute.

(ii) The group ring $R = k[G]$, where k is a field of prime characteristic p and G is a finite p-group.

Proof (i) Q is a finite dimensional algebra and is therefore artinian. Furthermore, Q is commutative and the only ideals of Q are:

$$\begin{aligned} J &= Jac(Q) = xQ + yQ \\ S &= Soc(Q) = xyQ \\ X &= xQ \\ Y &= yQ \\ A_\alpha &= (x + \alpha y)Q \text{ where } 0 \neq \alpha \in k \end{aligned}$$

Thus J is the unique maximal ideal of Q, and hence Q is local.

It is easy to see that:

$$\begin{aligned} \mathbf{r}(J) &= S \\ \mathbf{r}(S) &= J \\ \mathbf{r}(X) &= X \\ \mathbf{r}(Y) &= Y \\ \mathbf{r}(A_\alpha) &= A_{-\alpha} \end{aligned}$$

So Q satisfies the double annihilator properties of Theorem 2.1.2 (v) and thus Q is QF.

(ii) R is a finite dimensional k-algebra and so is artinian. R_R is injective by [39] Theorem 3.2.8.

Let $r, s \in R$. Then, by considering the binomial expansion, along with the fact that k has characteristic p, it is easy to see that $(r+s)^p = r^p + s^p$. By induction, it follows that for $r_1, r_2, ..., r_m \in R$, $(r_1 + r_2 + ... + r_m)^p = r_1^p + r_2^p + ... + r_m^p$ and furthermore for any $n \in \mathbb{N}$, $(r_1 + r_2 + ... + r_m)^{p^n} = r_1^{p^n} + r_2^{p^n} + ... + r_m^{p^n}$.

We will describe those elements of R which are of the form ag, where $a \in k$ and $g \in G$, as **homogeneous**. Let ag be a homogeneous element of R. Then, since $g^{p^n} = 1_G$ for some $n \in \mathbb{N}$, it follows that $(ag)^{p^n} = a^{p^n} 1_G$. Now, if $r \in R$, by considering r as a sum of homogeneous elements, and using the working of the previous paragraph, it follows that $r^{p^t} = b.1_G$, for some $t \in \mathbb{N}$ and $b \in k$. If $b = 0$, then r is nilpotent, otherwise $r^{p^t}(b^{-1}.1_G) = 1_k.1_G$ so r is invertible. Therefore every element of R is either invertible or nilpotent.

Suppose that c and d are nilpotent elements of R and that r is an arbitrary element of R. Clearly cr and rc cannot be invertible, so they must be nilpotent. Also, if $c^{p^m} = d^{p^n} = 0$, then by the first paragraph, $(c+d)^{p^{m+n}} = 0$, so $c + d$ is nilpotent. Therefore, the non-invertible elements of R form an ideal and so R is local. □

Theorem 2.3.9 *If Q is a local QF ring which is not uniserial as a left module over itself, then $W(Q)$ as defined in Lemma 2.3.4 is not left co-H (and hence not QF).*

Proof Since f is a primitive idempotent of W, Wf is an indecomposable projective left W-module. But since ${}_QQ$ is not uniserial, it follows from Lemma 2.3.7 that $Soc_2({}_QQ)/Soc({}_QQ)$ is not simple. Hence there exist left ideals A and B of Q which are contained in $Soc_2({}_QQ)$ and which both strictly contain $Soc({}_QQ)$ such that $A \cap B = Soc({}_QQ)$. It follows that $\begin{bmatrix} 0 & A/S \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & B/S \\ 0 & 0 \end{bmatrix}$ are submodules of Wf whose intersection is zero. Hence by Lemma 2.3.1, W cannot be left co-H. $\square$

It is easy to see that Q as defined in Lemma 2.3.8 is not uniserial - we can take the right ideals X and Y and note that $X \not\subseteq Y$ and $Y \not\subseteq X$. Hence $W(Q)$ is not left co-H.

Let G be the group of quaternions (we will write these as $G = \{e, e^*, a, a^*, b, b^*, c, c^*\}$ instead of $G = \{1, -1, i, -i, j, -j, k, -k\}$ to avoid confusion). If k is the field $\mathbb{Z}_2$, then the group ring $R = k[G]$ is a local QF ring by Lemma 2.3.8.

Consider the following subsets of R:

$$\{0,\ e + e^* + a + a^*,\ b + b^* + c + c^*,\ e + e^* + a + a^* + b + b^* + c + c^*\}$$

$$\{0,\ e + e^* + b + b^*,\ a + a^* + c + c^*,\ e + e^* + a + a^* + b + b^* + c + c^*\}$$

It is not hard to see that each sum of two elements in one of the sets is also in that set. Furthermore, multiplying an element of one of the sets by a homogeneous element of R on the left gives an element of that same set. It follows that the two sets are unequal left ideals, each of four elements, and so ${}_RR$ is not uniserial. Therefore, $W(R)$ is not left co-H.

Note In fact, if Q is a QF ring which is left uniserial as a left Q-module but not semisimple, then $W(Q)$ is left and right co-H but not QF.

2.4 The New Lemma

Before we finish this chapter, we will take a further look at Lemmma 2.2.7. We already know that it has uses other than proving Theorem 2.1.5 - it was also used to prove Lemma 2.2.9. As an afterthought, we can use it to prove the following Corollary.

Corollary 2.4.1 *Let R be a ring which satisfies ACC on the right annihilators of subsets of R. Then every direct sum of projective injective modules is projective injective.*

Proof Let $X := \bigoplus_{\lambda\in\Lambda} X_\lambda$ be a direct sum of projective injective modules. We know that X is projective, and that our decomposition of X is a local direct summand of $E(X)$. As in the proof of (iv)$\Rightarrow$(i) in Theorem 2.1.5, R satisfies ACC on the annihilators of subsets of X, and so the result follows from Lemma 2.2.7. $\square$

Going back to Lemma 2.2.7, we might also wonder if we could weaken the conditions of the statement to produce something which can be used in a wider variety of situations. In particular, the condition that R must satisfy the ACC on right annihilators of *all* subsets of A is very strong. The perfect situation would be where it was enough to have the ACC on annihilators of subsets of each of the A_λ. Sadly, this is not sufficient, as the following example shows.

Example 2.4.2 Here we will produce a ring with a set of Σ-injective right modules whose direct sum is not injective. By Lemma 2.2.9, we know that R has the ACC on right annihilators of subsets of each of the injectives. If we could weaken the condition of Lemma 2.2.7 in the way suggested above, then this would force the sum of the injective modules in the set to be equal to its own injective hull, which we know to be false.

Let k be a field and let R be the ring formed by the direct product of countably infinitely many copies of k with multiplication defined componentwise. Label the copies of k with the natural numbers and let e_j be the element of R with 1 in the jth position and zeroes elsewhere. Let $I_j = e_jR$. It is easily seen that I_j is a simple R-module.

Fix $j \in \mathbb{N}$ and let M be a direct sum of copies of the R-module I_j and say that L is an ideal of R with an R-homomorphism $\theta : L \to M$. For every $m \in M$, $me_j = m$ and so for every $l \in L$, $\theta(l) = \theta(l)e_j = \theta(le_j)$. Also, $L = e_jL \oplus (1 - e_j)L$ and e_jL is clearly either e_jR or 0, so either $e_j \in L$ or $l = (1 - e_j)l$ for every $l \in L$.

If $e_j \notin L$, then $\forall l \in L$, $\theta(l) = \theta(le_j) = \theta((1 - e_j)le_j) = \theta(0) = 0$, i.e. θ is trivial and can be extended to the trivial homomorphism $0 : R \to M$.

If $e_j \in L$, then we can define a homomorphism $\theta' : R \to M : 1 \mapsto m$ where $m = \theta(e_j)$. Now, if $x \in L$, then $\theta'(x) = mx = \theta(e_j)x = \theta(e_jx) = \theta(x)e_j = \theta(x)$. So θ' extends θ.

Therefore, M is injective, i.e. I_j is Σ-injective.

Clearly, the I_js are linearly independent, and so we can form an ideal $I = \bigoplus_{j=1}^{\infty} I_j$. $1 \notin I$, i.e. I is a proper ideal. If $r \in R \backslash I$ then $r \neq 0$ and so r has a non-zero entry in the jth position, say. We have $0 \neq re_j \in rR \cap I$, so I is an essential ideal of R and hence cannot be injective.

Question In the statement of Lemma 2.2.7, can we alter "S is a subset" to read "S is a countable subset"?

This is not an unreasonable question to ask, being a generalisation of the equivalence of the countable and uncountable cases in Lemma 2.2.9. It is debatable however, whether such an alteration would increase the number of applications of the Lemma.

Chapter 3

Modules Which Subgenerate Classes With Extra Closure Properties

The first task of this chapter will be to define the notion of a "class" of modules. With our definition, we can then consider various different types of module classes. The particular classes we will be concentrating on each arise naturally from a single module. We will be trying to find links between the properties of a module and the properties of its class, asking which modules give us certain special types of class.

The classes arising from the single module are of great importance in much of modern module theory, for example in the work of Wisbauer (e.g. [52]) and the Viola-Priolis (e.g. [51]). In recent years, a lot of work has been published concerning modules whose class has certain extra properties. For example, in [8] a study is made of the classes arising from a single module which have the property that every member is CS.

A good reason for studying the class arising from M is that we can sometimes generalise results for, for example, the class of injective right R-modules to the class of M-injective modules in the class arising from M.

If a module gives us a class with additional properties, then we might be able to use results from other, more specialised types of class when studying this one. For example, Page and Zhou in [38] considered what they call a "natural class" where it is necessary that injective hulls of modules in the class are also in the class.

Since the classes of interest to us arise from the one single module, we might hope that it would be possible to find explicit "if and only if" relationships between the internal properties of the module and the properties of its class. However, if these sort of equivalences exist, for an arbitrary ring and module, it seems as though they would have to be impossibly complex.

The main questions of this chapter were also considered by Berning in [2] and Wisbauer in [53], although both of these authors take a more topological and categorical approach than we will use here, discussing equivalent properties on the category rather than the modules in the class. Later in the Chapter we will use some concepts which were introduced to algebra from topology, but we will try to avoid introducing too many non-module-theoretic ideas.

3.1 Classes of Modules

Definition 3.1.1 *For a ring R, a* **class** $\underline{X}$ *of right R-modules is a collection which includes the zero module and is closed under taking isomorphisms. We say that $\underline{X}$ is* **closed under extensions** *if whenever $A_R \leq B_R$ and A, $B/A \in \underline{X}$ then $B \in \underline{X}$, and that $\underline{X}$ is* **essentially closed** *if every module which has an essential submodule in $\underline{X}$ is also in $\underline{X}$.*

Example 3.1.2 (i) For any ring R, the class of finitely generated right R-modules is closed under extensions. In general however, the class of finitely generated right R-modules will not be essentially closed.

(ii) Let R be a right non-singular ring and let $N_R \leq M_R$ be modules such that N and M/N are singular. Let m be a non-singular element of M. There exists a non-zero right ideal I of R such that $\mathbf{r}(m) \cap I = 0$ and so $mI \cong I$ is a non-singular module. It follows that $mI \cap N = 0$, but this implies that $(mI \oplus N)/N \cong mI$, i.e. M/N has a non-zero non-singular submodule, which is false. Hence M is singular.

Now let A_R and B_R be modules such that A_R is singular and $A_R \leq_{ess} B_R$. Let b be a non-singular element of B. As before, there exists a non-zero right ideal K of R such that $\mathbf{r}(b) \cap K = 0$ and we know that $bK \cong K$ is non-singular. But it follows that $bK \cap A$ is non-zero, singular and non-singular, which is clearly false and so B is singular.

Hence, over a right non-singular ring, the class of singular right R-modules is essentially closed and closed under extensions.

Definition 3.1.3 *For a module M, we define the* **class $\sigma[M]$ of modules subgenerated by** M *to be the class of modules isomorphic to submodules of factor modules of direct sums of copies of M, i.e.:*

$$\sigma[M] = \{A_R : A \hookrightarrow \frac{M^{(\Lambda)}}{B} \text{ for some index set } \Lambda \text{ and } B \leq M^{(\Lambda)}\}$$

Example 3.1.4 (i) If S_R is a simple module, then $\sigma[S]$ consists of those modules which are isomorphic to direct sums of copies of S_R.

(ii) $\sigma[R_R]$ is the class of all right R-modules.

Lemma 3.1.5 *(Well-known) Let M_R be any right R-module. Then $\sigma[M]$ is closed under taking submodules, factor modules and direct sums.*

Proof These results are easily derived from the definition. □

Lemma 3.1.6 *(Well-known) If M is a right R-module, then for any right R-module X, $X \in \sigma[M] \Leftrightarrow xR \in \sigma[M]$ for every $x \in X$.*

Proof $\Rightarrow$ Immediate from Lemma 3.1.5.

$\Leftarrow$ X is a homomorphic image of $\bigoplus_{x \in X} xR$. □

Lemma 3.1.6 shows us that $\sigma[M]$ is defined completely by its cyclic modules, so we can associate $\sigma[M]$ with a set of right ideals:

$$\mathcal{T}(M) = \{I \lhd_r R : R/I \in \sigma[M]\}$$

This allows us to tie up the theory of classes of the form $\sigma[M]$ with the theory of kernel functors, as used in e.g. [48] and [50].

Definition 3.1.7 *If R is a ring, then a* **kernel functor** *τ is a map from the class of right R-modules to itself such that for all right R-modules M and M':*

(i) $\tau(M) \subseteq M$,

(ii) If $f : M \to M'$ is a homomorphism, then $f(\tau(M)) \subseteq \tau(M')$,

(iii) If $M' \leq M$, then $\tau(M') = M' \cap \tau(M)$.

The **torsion class** *of a kernel functor τ is the class of right R-modules M such that $\tau(M) = M$.*

Lemma 3.1.8 *(Well-known) The following are equivalent for a class $\underline{X}$ of right R-modules:*

(a) $\underline{X}$ is closed under taking submodules, factor modules and direct sums.

(b) $\underline{X} = \sigma[M]$, for some right R-module M.

(c) $\underline{X}$ is the torsion class of a kernel functor.

Proof (a) $\Rightarrow$ (b) Put $M = \bigoplus\{R/I : I \lhd_r R,\ R/I \in \underline{X}\}$. Since $\underline{X}$ has the closure properties of (a), it is obvious that $\sigma[M] \subseteq \underline{X}$. Now say that $N \in \underline{X}$. By Lemma 3.1.6, we only need to check that all the cyclic submodules of N are in $\sigma[M]$. Let $n \in N$, then $nR \cong R/\mathbf{r}(n)$ is isomorphic to a direct summand of M and so $nR \in \sigma[M]$.

(b) $\Rightarrow$ (c) We define a map from each right R-module to one of its subsets:

$$\tau : N \mapsto \{n \in N : nR \in \underline{X}\}$$

It is easy to see that for any module N, $\tau(N)$ is a submodule and that τ satisfies criterion (i) in the definition of a kernel functor.

Say that $f : N \to N'$ is a homomorphism and that $n \in \tau(N)$. Then $f(n)R = f(nR) \cong nR/(nR \cap \operatorname{Ker} f)$ which must be in $\sigma[M]$. So criterion (ii) is also satisfied.

Criterion (iii) is also satisfied by elementary considerations.

(c) $\Rightarrow$ (a) is straightforward. $\square$

Example 3.1.9 The following classes can all be defined in any of the ways described in Lemma 3.1.8.

(a) It is easy to see that the class of semisimple right R-modules is closed under taking submodules, factor modules and direct sums.

(b) Similarly, the class of singular right R-modules is closed under taking submodules, factor modules and direct sums.

(c) For any ring R and module M_R we can define the class:

$$\imath[M] = \{N_R : M \text{ is } N\text{-injective}\}$$

By using Lemma 1.3.7, we can see that $\imath[M]$ is closed under taking submodules, factor modules and direct sums. $\imath[M]$ must contain all of the semisimple right R-modules and so is non-empty.

(d) For a given ring R, the class of locally noetherian right R-modules and the class of locally artinian right R-modules are both closed under the three operations.

Example 3.1.10 Remember from the previous chapters that there exists a dual concept to that of a singular module - namely a small module. Since we know that for any ring R, the class of singular right R-modules is of the form $\sigma[M_R]$ for a suitable M_R, it seems reasonable to suppose that the class of small right R-modules is also of this type. This is not true in general. Recall Lemma 1.4.4 stated that the class of small right R-modules is closed under taking submodules, factor modules and finite direct sums. It was shown in [40] that this class is closed under arbitrary direct sums if and only if the Jacobson radical of every injective right R-module is small. The right artinian rings, for example, have this property.

$\mathbb{Z}$ is an example of a ring where the small modules do not form a class of the type $\sigma[M]$. If we take E to be $\mathbb{Z}_{2^\infty}$, the injective hull of $\mathbb{Z}_2$, then E is not finitely generated, but is uniserial and is the union of its submodules, which are all finite. Clearly each of these finite modules is small in E, so is small. It is easy to produce a projection from the direct sum of all of the submodules of E onto E. If this sum was small then Lemma 1.4.4 would force E to be small as well, which is clearly false since E is injective.

Lemma 3.1.8 states that for any module class with the appropriate closure properties, we can find a module M which subgenerates the class. Our choice of M is not unique of course - it is easy to see that $\sigma[M] = \sigma[M \oplus M] = \sigma[M^{(\mathbb{N})}]$. The proof above merely contains a way of constructing one suitable choice of M. This may not be the easiest one to work with, but there are certain properties all of the subgenerators of the class must share, as we will demonstrate in Corollary 3.1.15.

Even for some well-known, easily described module classes which are closed under taking submodules, factor modules and direct sums, there does not exist a "nice" subgenerator. We will produce an example of such a class, using the following Lemma.

Lemma 3.1.11 *If M and A are right R-modules, and G is a generating set of A, then the following are equivalent:*

(i) $A \in \sigma[M]$.

(ii) For every $a \in A$, there exists a finite set $S_a \subseteq M$ such that $\mathbf{r}(S_a) \subseteq \mathbf{r}(a)$.

(iii) For every $g \in G$, there exists a finite set $T_g \subseteq M$ such that $\mathbf{r}(T_g) \subseteq \mathbf{r}(g)$.

Proof (i) $\Rightarrow$ (ii) By Lemma 3.1.6, we know that $aR \in \sigma[M]$, so there exists a monomorphism $\theta : aR \hookrightarrow M^{(\Lambda)}/B$ for some index set Λ and submodule B of $M^{(\Lambda)}$. Since aR is cyclic, we can assume that Λ is finite, and the proof is completed by taking S_a to be a set of representatives of $\theta(a)$ in $M^{(\Lambda)}$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) If we take a particular $g \in G$, put $T_g = \{t_1, t_2, ..., t_n\}$ and take the cyclic submodule $C := (t_1, t_2, ..., t_n)R$ of M^n, then there is a surjection $C \twoheadrightarrow gR : (t_1, t_2, ..., t_n)r \mapsto gr$, and hence $gR \in \sigma[M]$. Furthermore, A is a homomorphic image of $\bigoplus_{g \in G} gR$ and so $A \in \sigma[M]$. $\square$

Example 3.1.12 It is easy to see that the class of torsion $\mathbb{Z}$-modules is closed under taking submodules, factor modules and direct sums. Therefore, we must be able to find a $\mathbb{Z}$-module M which subgenerates this class. Taking a prime p and a natural number n, then we know that there exists an index set Λ and a submodule N of $M^{(\Lambda)}$ such that $\mathbb{Z}_{p^n} \hookrightarrow M^{(\Lambda)}/N$. Since $\mathbb{Z}_{p^n}$ is finitely generated, $\mathbb{Z}_{p^n} \hookrightarrow (M^r + N)/N \cong M^r/(M^r \cap N)$ for some $r \in \mathbb{N}$, so without loss of generality, we can say that $\mathbb{Z}_{p^n} \hookrightarrow M^r/N$.

There must exist a finite set $m_1, m_2, ..., m_s \in M$ such that $\mathbf{r}(m_1 + N, m_2 + N, ..., m_s + N) = p^n\mathbb{Z}$, i.e. $\mathbf{r}(m_1 + N) \cap \mathbf{r}(m_2 + N) \cap ... \cap \mathbf{r}(m_s + N) = p^n\mathbb{Z}$. Since the only ideals of

$\mathbb{Z}$ which contain $p^n\mathbb{Z}$ are $p^k\mathbb{Z}$ where $k \leq n$, it follows that for every $1 \leq j \leq s$, there exists $k_j \leq n$ such that $\mathbf{r}(m_j + N) = p^{k_j}\mathbb{Z}$. Therefore there exists some $1 \leq i \leq s$, such that $\mathbf{r}(m_i + N) = p^n\mathbb{Z}$. So $(m_iR + N)/N \cong \mathbb{Z}_{p^n}$, i.e. $\mathbb{Z}_{p^n} \hookrightarrow M/N'$ for some submodule N' of M.

In other words, for any prime p and natural number n, $\mathbb{Z}_{p^n}$ is contained in a factor of M. M must therefore be very large, and in particular, not finitely generated. A suitable choice of M would be $\mathbb{Q}/\mathbb{Z}$.

Definition 3.1.13 *We will say that a module M_R is* **finitely annihilated (with respect to the set S)** *if there exists a finite subset $S \subseteq M$ such that $\mathbf{r}(M) = \mathbf{r}(S)$.*

Lemma 3.1.14 *Let R be a ring and let M_R and N_R be modules. Then the following hold:*

(a) $M \in \sigma[N] \Leftrightarrow \sigma[M] \subseteq \sigma[N]$.

(b) $\mathbf{r}(M) = \bigcap\{I \lhd_r R : R/I \in \sigma[M]\}$.

(c) M is finitely annihilated $\Leftrightarrow$ $\exists s \in \mathbb{N}$, $I_1, I_2, ..., I_s \lhd_r R$ such that for every $1 \leq j \leq s$, $R/I_j \in \sigma[M]$ and $\bigcap_{1 \leq j \leq s} I_j = \mathbf{r}(M)$.

Proof (a) This is straightforward.

(b) Let $K = \bigcap\{I \lhd_r R : R/I \in \sigma[M]\}$. Suppose that $I \lhd_r R$, $R/I \in \sigma[M]$. Then by Lemma 3.1.11, there exists a finite subset S of M such that $\mathbf{r}(S) \subseteq I$. Therefore, $\mathbf{r}(M) \subseteq I$ and hence $\mathbf{r}(M) \subseteq K$.

Now suppose that $r \in R \backslash \mathbf{r}(M)$. Then there exists $m \in M$ such that $mr \neq 0$. Clearly, $R/\mathbf{r}(m) \cong mR \in \sigma[M]$, but $r \notin \mathbf{r}(m)$ and so $r \notin K$. Hence $K \subseteq \mathbf{r}(M)$.

(c) $\Rightarrow$ Suppose that M is finitely annihilated with respect to the set $\{m_1, m_2, ..., m_t\}$. Then put $s = t$ and $I_j = \mathbf{r}(m_j)$ for $1 \leq j \leq t$.

$\Leftarrow$ By Lemma 3.1.11, for each I_j where $1 \leq j \leq s$, there exists a finite set $S_j = \{m_{j,1}, m_{j,2}, ..., m_{j,k_j}\} \subseteq M$ such that $\mathbf{r}(S_j) \subseteq I_j$. Then $\bigcap_{1 \leq j \leq s} \mathbf{r}(S_j) \subseteq \mathbf{r}(M)$ and so $\bigcap_{1 \leq j \leq s} \mathbf{r}(S_j) = \mathbf{r}(M)$. $\square$

Corollary 3.1.15 *Let M_R, N_R and X_R be modules such that $\sigma[M] = \sigma[N]$. Then:*

(a) $M \in \sigma[X]$ $\Leftrightarrow$ $N \in \sigma[X]$.

(b) $\mathbf{r}(M) = \mathbf{r}(N)$.

(c) M is finitely annihilated $\Leftrightarrow$ N is finitely annihilated.

(d) M_R is locally noetherian (artinian) $\Leftrightarrow$ N_R is locally noetherian (artinian).

Proof (a), (b) and (c) follow by (a), (b) and (c) respectively of Lemma 3.1.14. (d) follows by using part Lemma 3.1.14 (a) and Example 3.1.9 (d). $\square$

Corollary 3.1.15 (c) shows us that classes of the form $\sigma[M]$ can be divided into two different types - those which are subgenerated by a finitely annihilated module and those which are not. As we shall see the former type is far more easy to work with and in this case, we will be able to answer our questions precisely.

So now we know what the class $\sigma[M]$ is for a module M, and we have some idea of its properties. The questions we want to answer are:

- When is $\sigma[M]$ closed under extensions?
- When is $\sigma[M]$ essentially closed?
- When is $\sigma[M]$ closed under taking injective hulls?

In [50], a study was made of rings for which every right module satisfies the condition of the first question, using the kernel functor characterisation of the classes. The author completely characterised these rings in the commutative case and also proved some results in the general case.

A moment of consideration will show that the second and third questions are actually the same, and in fact any $\sigma[M]$ which satisfies the conditions of these two questions satisfies the condition of the first:

Lemma 3.1.16 *(Well-known) If $\underline{X}$ is a class of right R-modules which is closed under taking submodules, (finite) direct sums and essential extensions then $\underline{X}$ is closed under extensions.*

Proof Let $B_R \leq A_R$ with B, $A/B \in \underline{X}$. Let C be a complement of B in A. Then $C \cong (C \oplus B)/B \hookrightarrow A/B \in \underline{X}$, so we must have $B \oplus C \in \underline{X}$. $B \oplus C$ is an essential submodule of A and therefore $A \in \underline{X}$. $\square$

The following results will be required later.

Lemma 3.1.17 *Let R be a right noetherian ring and let M be a right R-module. Then $\sigma[M]$ is essentially closed $\Leftrightarrow$ $E(U) \in \sigma[M]$ for every cyclic uniform U in $\sigma[M]$.*

Proof $\Rightarrow$ This is trivial.

$\Leftarrow$ Let $X \in \sigma[M]$. By Lemma 1.2.6, X contains a uniform submodule, which we can assume is cyclic. It is easy to see that the union of any chain of sets of independent uniform cyclic submodules of X is a set of independent uniform cyclic submodules of X, so by Zorn's Lemma there exists a set $\{U_\lambda\}_{\lambda\in\Lambda}$ of uniform cyclic submodules of X such that $T = \bigoplus_{\lambda\in\Lambda} U_\lambda$ is essential in X. It follows that we can assume that $E(T) = E(X)$. Since R is right noetherian, $\bigoplus_{\lambda\in\Lambda} E(U_\lambda) = E(X)$ and so $E(X) \in \sigma[M]$, by our original assumption. $\square$

Corollary 3.1.18 *Let R be a right noetherian ring and $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of right R-modules such that $\sigma[M_\lambda]$ is essentially closed for every $\lambda \in \Lambda$. Then if $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, $\sigma[M]$ is also essentially closed.*

Proof By Lemma 3.1.17, it is enough to show that if U_R is a uniform cyclic module and $U_R \in \sigma[M]$, then $E(U_R) \in \sigma[M]$. Let $U_R \in \sigma[M]$ be a uniform cyclic module. Then there exists an index set Ω and a submodule K of $M^{(\Omega)}$ such that $U \hookrightarrow M^{(\Omega)}/K$. Since U is finitely generated, we can assume without loss of generality that Ω is finite, and better still, that $U \hookrightarrow (N_1 \oplus ... \oplus N_s)/K$ for some $s \in \mathbb{N}$ and $K \leq N_1 \oplus ... \oplus N_s$, where each N_i is a (not necessarily distinct) member of the set $\{M_\lambda\}_{\lambda\in\Lambda}$. Clearly for each uniform cyclic module U in $\sigma[M]$ we can find a set of N_i's of minimal size. We shall denote this size $s(U)$ and we will show that $E(U) \in \sigma[M]$ for every uniform cyclic $U \in \sigma[M]$ by induction on $s(U)$.

Clearly if $s(U) = 1$, then $U \in \sigma[N_1]$ and so $E(U) \in \sigma[N_1] \subseteq \sigma[M]$. Now assume that $s(U) > 1$ and that $E(V) \in \sigma[M]$ for every cyclic uniform member V of $\sigma[M]$ with $s(V) < s(U)$. Take $N = N_1 \oplus ... \oplus N_{s(U)}$ to be the direct sum of a set of minimal size with a submodule K such that $U \hookrightarrow N/K$. Let $A = N_1 + K$ and let $B = N_2 + ... + N_s + K$. Let i be the embedding $U \hookrightarrow (A/K) + (B/K)$.

If $i(U) \cap (B/K) \neq 0$, then there exists a uniform cyclic module W_R such that $W \hookrightarrow U$ and $W \hookrightarrow (B/K)$. It follows that $W \hookrightarrow (N_2 + ... + N_s + K)/K \cong (N_2 + ... + N_s)/((N_2 + ... + N_s) \cap K)$, so $s(W) < s$. Of course, $E(W) \cong E(U)$, so by assumption $E(U) \in \sigma[M]$.

If $i(U) \cap (B/K) = 0$, then:

$$i(U) \cong (i(U) + (B/K))/(B/K) \leq (N/K)/(B/K) \cong N/B = (A+B)/B \cong A/(A \cap B)$$

Now, $K \subseteq A \cap B$, so there exists a projection $A/K \twoheadrightarrow A/(A \cap B)$. So we have:

$$U \in \sigma[A/(A \cap B)] \subseteq \sigma[A/K] = \sigma[(N_1 + K)/K] = \sigma[N_1/(N_1 \cap K)] \subseteq \sigma[N_1]$$

Therefore $E(U) \in \sigma[N_1] \subseteq \sigma[M]$ and so $\sigma[M]$ is essentially closed. □

The implication in Corollary 3.1.18 cannot in general be reversed. Lemma 3.1.19, which illustrates a family of modules which subgenerate essentially closed classes, will allow us to construct Example 3.1.20 which contains modules A and B such that $\sigma[A \oplus B]$ is essentially closed but $\sigma[A]$ is not.

Lemma 3.1.19 *Let R be a right noetherian right hereditary ring and E_R be an injective module. Then $\sigma[E]$ is essentially closed.*

Proof Let $N \in \sigma[E]$. Then there exists an index set Λ and a submodule $B \leq E^{(\Lambda)}$ such that $N \hookrightarrow E^{(\Lambda)}/B$. Since R is right noetherian, $E^{(\Lambda)}$ is injective and since R is right hereditary, $E^{(\Lambda)}/B$ is also injective. Therefore, $E(N) \hookrightarrow E^{(\Lambda)}/B$ which implies that $E(N) \in \sigma[E]$. □

Example 3.1.20 Consider the ring $\mathbb{Z}$ and the $\mathbb{Z}$-module $\mathbb{Z}_2$. Clearly, $\sigma[\mathbb{Z}_2]$ consists of all those $\mathbb{Z}$ modules which are annihilated by the element 2 of $\mathbb{Z}$. It is easy to show that $\sigma[\mathbb{Z}_2]$ is not essentially closed. To show this we can take the module $\mathbb{Z}_4$ which is not annihilated by 2, but has an essential submodule $2\mathbb{Z}_4$ which is.

The injective hull of $\mathbb{Z}_2$ is isomorphic to the module $\mathbb{Z}_{2^\infty}$. By Lemma 3.1.19, $\sigma[\mathbb{Z}_{2^\infty}]$ is essentially closed, and since $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_{2^\infty}$, clearly $\sigma[\mathbb{Z}_2 \oplus \mathbb{Z}_{2^\infty}] = \sigma[\mathbb{Z}_{2^\infty}]$ is essentially closed.

To finish the section on classes of modules the table below summarises the closure properties of some of the most interesting types of class. The classes marked with a † are described in more detail in [44] and are completely defined by their closure properties in the table.

Given any class $\underline{X}$ of right R-modules, $\underline{X}^\perp$ is defined to be the class of right R-modules which do not have any submodules isomorphic to non-zero submodules of members of $\underline{X}$.

Class	$\subseteq$	/	$\oplus$	X	E	Π
$\sigma[M]$	$\bullet$	$\bullet$	$\bullet$			
Torsion†		$\bullet$	$\bullet$	$\bullet$		
Torsion-free†	$\bullet$		$\circ$	$\bullet$		$\bullet$
Hereditary Torsion†	$\bullet$	$\bullet$	$\bullet$	$\bullet$		
Hereditary Torsion-free†	$\bullet$		$\circ$	$\bullet$	$\bullet$	$\bullet$
Stable Hereditary Torsion†	$\bullet$	$\bullet$	$\bullet$	$\circ$	$\bullet$	
$\underline{X}^\perp$	$\bullet$		$\bullet$	$\circ$	$\bullet$	

$\oplus$ = arbitrary direct sums

X = extensions

E = essential extensions

Π = direct products

$\bullet$ = closed under the operation

$\circ$ = closed under the operation but this can be derived as a consequence of the other listed closure properties

The table shows us that, for example, if we have a module M such that $\sigma[M]$ is closed under extensions, then $\sigma[M]$ is a torsion class in the sense of [44]. This applies to the singular right modules of a right non-singular ring, as shown in Example 3.1.2 (ii).

[44] contains further reading about the general theory of classes.

3.2 Finitely Annihilated Modules

In this section we will completely answer our questions for finitely annihilated modules. Examples of such modules include finitely generated modules over commutative rings, finitely generated right modules over right FBN rings (which are defined shortly), and all right modules over right artinian rings.

Definition 3.2.1 *If R is a right noetherian ring and M is a right R-module, then an* **associated ideal** *of M is the annihilator of a non-zero submodule N of M which has the property that $\mathbf{r}(N) = \mathbf{r}(N')$ for every non-zero submodule N' of N. Without loss of generality, we can always assume that N is cyclic.*

The set of associated ideals of M is denoted by $\mathbf{Ass}(\mathbf{M})$.

Lemma 3.2.2 *(Well-known) Let R be a right noetherian ring with a right module M_R. Then we have the following:*

(i) $Ass(M)$ is a set of prime ideals of R.

(ii) If $M_R \neq 0$ then $Ass(M) \neq \varnothing$.

(iii) If M_R is uniform, then $|Ass(M)| = 1$.

(iv) If $N_R \leq_{ess} M_R$, then $Ass(N) = Ass(M)$.

Proof (i) Let $0 \neq N \leq M$ such that $\mathbf{r}(N) = \mathbf{r}(N')$ for every $0 \neq N' \leq N$. If we put $P = \mathbf{r}(N)$, it is clear that P is a 2-sided ideal. Suppose that there exist 2-sided ideals A and B of R such that $AB \subseteq P$, and $A \nsubseteq P$. Then $(NA)B = 0$, so $B \subseteq P$, since NA is a non-zero submodule of N. Hence P is prime.

(ii) Since R is right noetherian, we can choose $0 \neq N \leq M$ such that $\mathbf{r}(N)$ is maximal in the set of annihilators of non-zero submodules of M. If $0 \neq N' \leq N$, then clearly $\mathbf{r}(N') \supseteq \mathbf{r}(N)$, and so by maximality, $\mathbf{r}(N') = \mathbf{r}(N)$.

(iii) Suppose that M has two associated ideals, P and Q, and that V and V' are submodules of M such that $\mathbf{r}(X) = P$ for every non-zero submodule X of V and $\mathbf{r}(X') = Q$ for every non-zero submodule X' of V'. Since M is uniform, $V \cap V' \neq 0$. Then $P = \mathbf{r}(V \cap V') = Q$.

(iv) It is obvious that $Ass(N) \subseteq Ass(M)$. Suppose that $P \in Ass(M)$ and that $P = \mathbf{r}(A')$ for every non-zero submodule A' of some fixed submodule A of M. If we now consider $A \cap N$ (which is non-zero since N is essential in M), and let $0 \neq B \leq A \cap N$, then $\mathbf{r}(B) = P$, and so $P \in Ass(N)$. $\square$

In the light of Lemma 3.2.2 (iii), we will sometimes cheat a little with our notation when considering the associated ideal of a uniform module U and put $Ass(U) = P$ instead of the more formal $Ass(U) = \{P\}$ which is consistent with the non-uniform case.

Lemma 3.2.3 *(Well-known) Let R be a right noetherian ring and let P be a prime ideal of R. Then $Ass(R/P) = \{P\}$.*

Proof By Lemma 3.2.2 (ii), R/P has an associated prime ideal, Q. Now, there exists a right ideal A of R which strictly contains P such that $\mathbf{r}(A/P) = Q$. Obviously, $P \subseteq Q$. Furthermore, $AQ \subseteq P$ and so $(RA)Q \subseteq P$. Since RA is not contained in P, it follows that $Q \subseteq P$. Hence $P = Q$. □

Definition 3.2.4 *([30], 6.4.7) A ring is said to be* **right FBN** *if it is right noetherian and every essential right ideal of a prime factor ring of R contains a non-zero 2-sided ideal.*

Example 3.2.5 Every commutative noetherian ring is trivially FBN.

Definition 3.2.6 *A right R-module M is said to be* **faithful** *if* $\mathbf{r}(M) = 0$.

Theorem 3.2.7 *(Mewborn & Winton) Let R be a ring with the a.c.c. for right annihilators of subsets of R. Then $Z(R_R)$ is nilpotent.*

Proof Let $Z = Z(R_R)$. Then $\mathbf{r}(Z) \subseteq \mathbf{r}(Z^2) \subseteq ...$ is an ascending chain, and so there exists $n \in \mathbb{N}$ such that $\mathbf{r}(Z^n) = \mathbf{r}(Z^{n+1})$. Suppose that $Z^{n+1} \neq 0$. Then we can choose $a \in Z$ such that $Z^n a \neq 0$ and $\mathbf{r}(a)$ is maximal with a having this property. Let $b \in Z$, then $\mathbf{r}(b)$ is an essential right ideal of R and so $\mathbf{r}(b) \cap aR \neq 0$, i.e. there exists $r \in R$ such that $0 \neq ar \in \mathbf{r}(b)$. Obviously, $ba \in Z$, $\mathbf{r}(a) \subseteq \mathbf{r}(ba)$ and we know that $ar \neq 0$ and $bar = 0$. Therefore $\mathbf{r}(a)$ is strictly contained in $\mathbf{r}(ba)$, and so by the choice of a, $Z^n ba = 0$. Since b was chosen arbitrarily, we must have $Z^{n+1}a = 0$ and so $Z^n a = 0$ - a contradiction. □

The following Lemma is compiled from [44] Theorem 2.1 and Proposition 2.4, and [3] Theorem 7.8. I have included the proof in full here to avoid the reader having to look up multiple references.

Lemma 3.2.8 *(Krause, Gabriel, Cauchon) The following are equivalent for a right noetherian ring R.*

(i) R is right FBN.

(ii) Every finitely generated right R-module is finitely annihilated.

(iii) If M_R is finitely generated, then $R/\mathbf{r}(M) \hookrightarrow M^n$ for some $n \in \mathbb{N}$.

(iv) The mapping from the indecomposable injective right R-modules to the prime ideals of R given by $E \mapsto Ass(E)$ is bijective.

Proof (i) $\Rightarrow$ (ii) Suppose that M_R is a finitely generated right R-module which is not finitely annihilated. Then there is a submodule N of M maximal such that M/N is not finitely annihilated and so by shifting to M/N, we can assume that every homomorphic image of M satisfies the condition. Also, if we change the ring to $R/\mathbf{r}(M)$ then M is a faithful right $R/\mathbf{r}(M)$-module and $R/\mathbf{r}(M)$ is right FBN, so we can assume without loss of generality that M is faithful.

Now take $P \in Ass(M)$ and let U be a submodule of M such that $\mathbf{r}(V) = P$ for every $0 \neq V \leq U$. Clearly, U is a faithful right R/P-module. Suppose that $U_{R/P}$ has a non-zero singular element, u. Then in R/P, $I/P = \mathbf{r}(u)$ is essential as a right ideal, so by (i) contains a 2-sided ideal, K/P. In U_R, $uR.K = uK \subseteq uI = 0$, so uR is not faithful as an R/P-module - a contradiction. Hence U is non-singular as a right R/P-module.

Now by assumption $\mathbf{r}(M/U) = \mathbf{r}(m_1 + U, m_2 + U, ..., m_n + U)$ in R. If we put $A = \mathbf{r}(m_1, m_2, ..., m_n)$, then $MA \subseteq U$, so $MAP = 0$ and by faithfulness, $AP = 0$, i.e. A is a right R/P-module.

We know that $MA \neq 0$, so there exists $m_{n+1} \in M$ with $m_{n+1}A \neq 0$. If we now put $A_1 = \mathbf{r}(m_1, m_2, ..., m_{n+1})$, then $A/A_1 \cong m_{n+1}A \subseteq MA \subseteq U$, so A/A_1 is non-singular as an R/P-module. Again $MA_1 \neq 0$, so $\exists m_{n+2} \in M$ with $m_{n+2}A_1 \neq 0$, and we repeat to produce an infinite descending chain $A \supset A_1 \supset A_2 \supset ...$ such that A_i/A_{i+1} is non-zero and non-singular as a right R/P module.

Obviously, A_1 is not essential in A as a right R/P-module, so there exists $K_1 \leq A$ such that $K_1 \cap A_1 = 0$. Likewise, for every $i \geq 2$, there exists $K_i \leq A_{i-1}$ with $K_i \cap A_i = 0$. $K_1 \oplus K_2 \oplus ...$ is an infinite direct sum contained in R_R, which is clearly false, since R_R

is noetherian, and so the supposition must have been false, that is to say there is a finite subset of M whose annihilator is the annihilator of M.

(ii) $\Rightarrow$ (iii) Suppose that M_R is finitely annihilated with respect to the set $S = \{m_1, m_2, ..., m_n\}$. Now consider a mapping $\theta : R/\mathbf{r}(M) \to M^n : r + \mathbf{r}(M) \mapsto (m_1 r, m_2 r, ..., m_n r)$. It is easy to see that θ is well-defined and a monomorphism.

(iii) $\Rightarrow$ (iv) Let P be a prime ideal of R. By Lemma 3.2.3 and Lemma 3.2.2, $Ass(E(R/P)) = \{P\}$. It follows that for any uniform direct summand X of $E(R/P)$, $Ass(X) = P$ and so the mapping is surjective.

Let E and E' be indecomposable injective right R-modules such that $Ass(E) = Ass(E') = P$. To show that $E \cong E'$, it is enough to show that E and E' contain isomorphic submodules. Take $e \in E$ and $e' \in E'$ such that $P = \mathbf{r}(eR) = \mathbf{r}(e'R)$. By (iii),

$$R/P \hookrightarrow (eR)^n \quad R/P \hookrightarrow (e'R)^m$$

If we now consider a non-zero uniform submodule of R/P (which must exist by Lemma 1.2.6), then the result follows by Lemma 1.1.4.

(iv) $\Rightarrow$ (i) We firstly show that if (iv) is true for R, then it holds for all factor rings of R. Let $T = R/D$ be a factor ring of R, and let E_1 and E_2 be non-isomorphic indecomposable injective T-modules. Suppose that their injective hulls as R-modules are isomorphic. Then since these injective hulls are uniform, there must be an R-module M which embeds in E_1 and E_2. Since $MD = 0$, M is also a T-module, and so $E_1 \cong E(M) \cong E_2$ as T-modules, which we know to be false. Hence the injective hulls of E_1 and E_2 as R-modules are non-isomorphic, and so $Ass(E_1) \neq Ass(E_2)$ as R-modules. It is easy to show that $Ass_T(E_i) = (Ass_R(E_i))/D$ for $i = 1, 2$, so we have $Ass(E_1) \neq Ass(E_2)$ as T-modules, and the first part is proved.

Now, consider a prime factor ring of R, S, and say that S has an essential right ideal which does not contain a 2-sided ideal. We take a maximal such right ideal, I. Suppose that $I = A_1 \cap A_2$, where A_1 and A_2 are right ideals strictly containing I. By the maximality of I, $A_1 \geq B_1$ and $A_2 \geq B_2$, where B_1 and B_2 are non-zero 2-sided ideals of S. Now, $I = A_1 \cap A_2 \supseteq B_1 \cap B_2 \supseteq B_1 B_2$. Since S is prime, $B_1 B_2 \neq 0$, and so $B_1 \cap B_2$ is a non-zero

2-sided ideal contained in I. Thus it must be the case that I is irreducible, or to put it another way, S/I is a uniform S-module.

If $P = Ass(S/I)$, then $P = \mathbf{r}(K/I)$ for some right ideal K strictly containing I. Now K contains a non-zero 2-sided ideal L and $LP \subseteq KP \subseteq I$, so $LP = 0$, since I does not contain any non-zero 2-sided ideals. Since S is prime, it must be that $P = 0$ and so $Ass(S/I) = 0$.

Now let $Q \in Ass(S)$. Then, there exists a non-zero right ideal X of S such that $XQ = 0$. This means that $SXQ = 0$, so by the primeness of S, either $SX = 0$, which is false, or $Q = 0$, which must be the case. Hence $Ass(S/I) = Ass(S)$, so by (iv), S/I and S_S contain isomorphic submodules. However, S/I is clearly a singular right S-module and S_S is non-singular by Theorem 3.2.7, a contradiction. Therefore, (i) must be true. □

Definition 3.2.9 *An ideal I of R is said to satisfy the* **right Artin-Rees** *property (henceforth we will say that I is* **right AR***) if for any right ideal K of R, there exists $n \in \mathbb{N}$ such that $K \cap I^n \subseteq KI$.*

Lemma 3.2.10 *([19], Lemma 11.11) For an ideal I of a ring R, the following are equivalent:*

(i) I is right AR.

(ii) If $B_R \leq A_R$ where A is finitely generated, then $\exists n \in \mathbb{N}$ such that $B \cap AI^n \subseteq BI$.

(iii) If $B_R \leq_{ess} A_R$ where A is finitely generated and $BI = 0$, then $\exists n \in \mathbb{N}$ such that $AI^n = 0$.

Proof (i) $\Rightarrow$ (iii) Let $a \in A$, and let $K = \{r \in R : ar \in B\}$. Then $K \cap I^n \subseteq KI$ for some $n \in \mathbb{N}$, by condition (i).

Suppose that $x \in aI^n \cap B$. Then $x = as$, where $s \in K \cap I^n$. Hence $x \in aKI \subseteq BI = 0$, so $aI^n \cap B = 0$ and therefore $aI^n = 0$.

Since each of the generators of A is annihilated by a power of I, A itself is annihilated by some power of I.

(iii) $\Rightarrow$ (ii) Take C/BI to be a complement of B/BI in A/BI. By Lemma 1.1.2,

$$((C/BI) \oplus (B/BI))/(C/BI) \leq_{ess} (A/BI)/(C/BI)$$

and so

$$B/BI \hookrightarrow_{ess} A/C$$

So by condition (iii), $\exists n \in \mathbb{N}$ such that $(A/C)I^n = 0$, i.e. $AI^n \subseteq C$. Hence $B \cap AI^n \subseteq B \cap C \subseteq BI$.

(ii) $\Rightarrow$ (i) follows by putting $A = R$ and $B = K$. $\square$

Lemma 3.2.11 *(Well-known) Every ideal of a commutative noetherian ring is AR.*

Proof We will show condition (iii) of Lemma 3.2.10.

Let R be a commutative noetherian ring with a principal ideal $I = xR$ and let A be a finitely generated R-module. Let $B \leq_{ess} A$ be R-modules such that $BI = 0$. We now define $f : A \to A : a \mapsto ax$. It is easy to see that f is a homomorphism, and that $\operatorname{Ker} f \subseteq \operatorname{Ker} f^2 \subseteq \dots$. Since A is noetherian, there exists $s \in \mathbb{N}$ such that $\operatorname{Ker} f^t = \operatorname{Ker} f^s$ for every $t \geq s$. Suppose that $k \in \operatorname{Im} f^s \cap \operatorname{Ker} f^s$. Then $k = f^s(k')$ for some $k' \in A$ and $f^{2s}(k') = f^s(k) = 0$. But by the choice of s, this means that $f^s(k') = 0$ i.e. $k = 0$. So $\operatorname{Im} f^s \cap \operatorname{Ker} f^s = 0$, therefore $\operatorname{Im} f^s \cap B = 0$ and hence $\operatorname{Im} f^s = 0$. This means that $Ax^s = 0$, so $AI^s = 0$ and we have shown that every principal ideal is AR.

Now let $I = x_1R + x_2R + \dots + x_nR$ and put $I_i = x_iR$ for $1 \leq i \leq n$. Again, let $B \leq_{ess} A$ be R-modules with $BI = 0$. By the preceding paragraph, we know that for every $1 \leq i \leq n$, there exists s_i such that $AI_i^{s_i} = 0$. Put $t = max_{1 \leq i \leq n}\{s_i\}$ and consider $I^{nt} = (I_1 + I_2 + \dots + I_n)^{nt}$. It is clear that in the binomial expansion of this power of I, every component of the sum must contain some I_j to a power of at least t. Hence it follows that $AI^{nt} = 0$, and we are done. $\square$

We will now reduce our problem concerning modules to a problem concerning ideals.

Lemma 3.2.12 *If R is a ring with an ideal I, then $X \in \sigma[R_R/I_R] \Leftrightarrow XI = 0$.*

Proof $\Rightarrow$ $X \cong A/B$, where $B \leq A \leq (R/I)^{(\Lambda)}$ for some index set Λ, so $XI = 0$.

$\Leftarrow$ Let $x \in X$. Then $xI = 0$, so $\mathbf{r}(x) \supseteq \mathbf{r}(\bar{1})$, where $\bar{1}$ is the image of 1 in the canonical surjection of R onto R/I. Hence by Lemma 3.1.11, $X \in \sigma[R/I]$. $\square$

Note that in Lemma 3.2.12, the proof requires I to be a 2-sided ideal of R, even though the statement only seems to use the right ideal property of I. If I is a right ideal of R which is not a left ideal, then clearly $R/I \in \sigma[R/I]$, but there exists $r \in R$ such that $rI \nsubseteq I$ and so $(R/I)I \neq 0$.

Corollary 3.2.13 *If R is a ring and I is an ideal of R, then $\sigma[R_R/I_R]$ is closed under extensions $\Leftrightarrow$ $I = I^2$.*

Proof $\Rightarrow$ Consider the right R-module $X = R/I^2$. Now, $(XI)I = 0$ and $(X/XI)I = 0$, so by Lemma 3.2.12, XI and X/XI are both in $\sigma[R/I]$. Hence it follows that $X \in \sigma[R/I]$, and using Lemma 3.2.12 again, $XI = 0$. Therefore $I \subseteq I^2$ and so $I = I^2$.

$\Leftarrow$ Let $A_R \leq B_R$ such that A, $B/A \in \sigma[R/I]$. Then $AI = 0$ and $(B/A)I = 0$, i.e. $BI \subseteq A$, by Lemma 3.2.12. So $BI^2 = 0$, i.e. $BI = 0$, and the result follows by Lemma 3.2.12. □

Corollary 3.2.14 *Let R be a ring with an ideal I, and consider the following conditions:*

(i) $\sigma[R_R/I_R]$ is essentially closed.

(ii) $\sigma[R_R/I_R]$ is closed under extensions and I is right AR.

(iii) $I = I^2$ and I is right AR.

(iv) $I = Re$, where $e^2 = e \in R$.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (iv) $\Rightarrow$ (iii). Furthermore, if I is finitely annihilated as a right R-module (for example if R is commutative noetherian, right artinian or otherwise right FBN; or if I is finitely generated as a left R-module), then (iii) $\Rightarrow$ (iv).

Proof (i) $\Rightarrow$ (ii) Closure under extensions follows by Lemma 3.1.16. To show that I is right AR, consider a case where $B_R \leq_{ess} A_R$ and $BI = 0$. Then by Lemma 3.2.12, $B \in \sigma[R/I]$ and hence by assumption $A \in \sigma[R/I]$. By Lemma 3.2.12, $AI = 0$, and so I is right AR.

(ii)$\Rightarrow$(iii) This follows from Corollary 3.2.13.

(iii)$\Rightarrow$(i) Suppose that $B_R \leq_{ess} A_R$ and $B \in \sigma[R/I]$. Then by Lemma 3.1.6, we have to show that $aR \in \sigma[R/I]$ for every $a \in A$. It is easy to see that $(B \cap aR) \leq_{ess} aR$ and that

$B \cap aR \in \sigma[R/I]$. By Lemma 3.2.12, $(B \cap aR)I = 0$, so by the AR-property, there exists $n \in \mathrm{N}$ such that $aRI^n = 0$, i.e. $aRI = 0$. Hence by Lemma 3.2.12, $aR \in \sigma[R/I]$.

(iv) $\Rightarrow$ (iii) It is easy to see that $e = e^2 \in I^2$ and so $I = I^2$. Now let K be a right ideal of R and let $x \in K \cap I$. Then there exists $r \in R$ such that $x = re = re^2 = xe$. Therefore, $x \in KI$ and so I is right AR.

Now assume that I is finitely annihilated as a right R-module. It is well-known that in this case, (iii) $\Leftrightarrow$ (iv), but we include a proof here for completeness.

(iii) $\Rightarrow$ (iv) The first thing we must note is that by assumption, $K \cap I = KI$ for every right ideal K of R.

Since I_R is finitely generated, $\mathbf{r}(I) = \mathbf{r}(x_1, x_2, ..., x_n)$ for some $n \in \mathrm{N}$, $x_i \in I$. Now for each x_i, $x_iR = x_iR \cap I = x_iI$, so $x_i = x_iy_i$ for some $y_i \in I$ and hence $x_i(1 - y_i) = 0$.

Suppose that $n \geq 2$, and consider y_1 and y_2. Let $F = (1 - y_1)R + (1 - y_2)R$. Clearly:

$$(1 - y_2)R + I = R$$

so:

$$1 - y_1 = (1 - y_2)r + k$$

where $r \in R$ and $k \in I$. Now:

$$1 - y_1 - (1 - y_2)r = k \in F \cap I = FI$$

hence

$$1 - y_1 - (1 - y_2)r = \sum_{1 \leq j \leq m} ((1 - y_2)r_j + (1 - y_1)s_j)k_j$$

for some $m \in \mathrm{N}$, $r_j, s_j \in R$ and $k_j \in I$. Re-arranging this, we have:

$$(1 - y_1)(1 - \sum s_jk_j) = (1 - y_2)(r + \sum r_jk_j)$$

Put $a = \sum s_jk_j$. Then $(1 - y_1)(1 - a) = 1 - (y_1 + a - y_1a)$ and putting $b = y_1 + a - y_1a$, clearly $b \in I$ and $x_1(1 - b) = x_2(1 - b) = 0$. So without loss of generality, we can assume that $y_1 = y_2$, and by extending this argument that $y_1 = y_2 = ... = y_n$.

Therefore, there exists $e \in I$ such that $I(1 - e) = 0$. Obviously, $e(1 - e) = 0$ so $e = e^2$, and furthermore if $w \in I$, $w(1 - e) = 0$, i.e. $w = we$. Hence $I = Re$. $\square$

Lemma 3.2.15 *The following statements are equivalent for a module M_R:*

(i) M is finitely annihilated.

(ii) $\sigma[M] = \sigma[R_R/I_R]$ for some ideal I of R.

(iii) $\sigma[M] = \sigma[R/\mathbf{r}(M)]$.

Proof (i)$\Rightarrow$(ii) Let S be a finite subset of M such that $\mathbf{r}(S) = \mathbf{r}(M)$ and let $I = \mathbf{r}(S)$. Then $R/I \in \sigma[M]$ by Lemma 3.1.11 and $M \in \sigma[R/I]$ by Lemma 3.2.12. Hence the result follows.

(ii)$\Rightarrow$(i) & (iii) Let T be any finite subset of M. Then by Lemma 3.1.11, $R/\mathbf{r}(T) \in \sigma[M]$, and so by Lemma 3.2.12, $(R/\mathbf{r}(T))I = 0$, i.e. $\mathbf{r}(T) \supseteq I$. So $\mathbf{r}(M) \supseteq I$. But since $R/I \in \sigma[M]$, there exists a finite subset S of M, such that $\mathbf{r}(S) \subseteq I$. Hence $\mathbf{r}(M) \subseteq \mathbf{r}(S) \subseteq I \subseteq \mathbf{r}(M)$, i.e. $I = \mathbf{r}(S) = \mathbf{r}(M)$.

(iii)$\Rightarrow$(ii) is obvious. □

The corollaries which follow are immediate consequences of Corollary 3.2.13, Corollary 3.2.14 and Lemma 3.2.15. They can be applied to, for example, any right module over a right artinian ring, any finitely generated right module over a right FBN ring or any finitely generated module over a commutative ring.

Corollary 3.2.16 *Let R be a ring and M_R be finitely annihilated. Then the following are equivalent:*

(i) $\sigma[M]$ is closed under extensions.

(ii) $\mathbf{r}(M) = \mathbf{r}(M)^2$.

Corollary 3.2.17 *Let R be a ring and let M_R be a finitely annihilated module. Consider the following conditions:*

(i) $\sigma[M]$ is essentially closed.

(ii) $\sigma[M]$ is closed under extensions and $\mathbf{r}(M)$ is right AR.

(iii) $\mathbf{r}(M) = \mathbf{r}(M)^2$ and $\mathbf{r}(M)$ is right AR.

(iv) $\mathbf{r}(M) = Re$, where $e^2 = e \in R$.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (iv) $\Rightarrow$ (iii). Furthermore, if $\mathbf{r}(M)$ is finitely annihilated as *a right R-module (for example if R is commutative noetherian, right artinian or otherwise right FBN; or if* $\mathbf{r}(M)$ *is finitely generated as a left R-module), then (iii) $\Rightarrow$ (iv).*

The following example was suggested to me by Professor Ken Brown.

Example 3.2.18 Here we will show that we need M to be finitely annihilated for both Corollary 3.2.16 and Corollary 3.2.17 to work, even in the case where M is noetherian.

Let k be a field of characteristic 0 and let R be the first Weyl algebra over k with indeterminates x and y, i.e. $R = k[x, y]$ subject to the relation $yx = xy - 1$. It is well-known (see e.g. [30], 1.3.5) that R is simple.

Every $r \in R$ can be written uniquely in the form $r = \sum_{i\in\mathbb{N},j\in\mathbb{N}} a_{i,j}x^iy^j$ where every $a_{i,j} \in k$ and only finitely many of the $a_{i,j}$'s are non-zero. We can now define a right R-module $T = \{(b, c) : b, c \in k\}$ where right multiplication is defined by $(b, c)\sum_{i\in\mathbb{N},j\in\mathbb{N}} a_{i,j}x^iy^j = (ba_{0,0} + ca_{1,0}, ca_{0,0})$. Let M be the submodule $(1, 0)R$ of T.

It is easy to see that M is simple and that every element of $T\backslash M$ generates T. Therefore T/M is simple and T is uniserial. Since R is simple, $\mathbf{r}(M) = 0$, so conditions (ii) of Corollary 3.2.16 and (iii) of Corollary 3.2.17 are satisifed trivially. However, M is simple and T is not semisimple, so $T \notin \sigma[M]$. It can easily be seen that $T/M \cong M$, so $\sigma[M]$ is not closed under extensions and hence conditions (i) of Corollary 3.2.16 and (ii) of Corollary 3.2.17 do not hold.

3.3 Modules Over Commutative Noetherian Rings

In the previous section, we completely answered our questions for finitely annihilated modules, but we could not apply the results to arbitrary modules. It is easy to see that for any module M, $\sigma[M]$ is essentially closed if and only if $E(M) \in \sigma[M]$ and $\sigma[E(M)]$ is essentially closed. Therefore, it might be instructive if we could find out which injective modules have our properties. In this section we will answer our questions for the indecomposable injectives of a commutative noetherian ring.

Lemma 3.3.1 *(Well-known) Let R be a commutative noetherian ring. Then:*

(i) R is FBN.

(ii) R/P is a uniform R-module for every prime ideal P of R.

(iii) For an R-module M, $P \in Ass(M) \Leftrightarrow R/P \hookrightarrow M$.

Proof (i) This was established in Example 3.2.5.

(ii) Suppose that A/P and B/P are non-zero submodules of R/P. We know that $AB \not\subseteq P$, so $0 \neq (AB+P)/P \subseteq (A/P) \cap (B/P)$.

(iii) Suppose that $P \in Ass(M)$ and take a finitely generated non-zero submodule U of M such that $\mathbf{r}(V) = P$ for every $0 \neq V \leq U$. By Lemma 3.2.8, $R/P \hookrightarrow U^n$ for some $n \in \mathbb{N}$. Since R/P is uniform, we have by Lemma 1.1.4 that $R/P \hookrightarrow U \leq M$.

Conversely, suppose that $R/P \hookrightarrow M$. The primeness of P means that for every right ideal A which strictly contains P, we have that $\mathbf{r}(A/P) = P$, so $P \in Ass(M)$. □

Corollary 3.3.2 *Let E be an indecomposable injective module over a commutative noetherian ring R. Then $E \cong E(R/P)$ for some prime ideal P of R.*

Proof This follows immediately by Lemma 3.2.2 (ii) and Lemma 3.3.1 (iii). □

Definition 3.3.3 *If R is a ring with an ideal I, and M is an R-module, then we say that M is I-***torsion** *if for every $m \in M$ there exists $n_m \in \mathbb{N}$ such that $mI^{n_m} = 0$.*

Lemma 3.3.4 *(Well-known) Let R be a ring with an ideal I.*

(i) If M is a finitely generated I-torsion module, then there exists $n \in \mathbb{N}$ such that $MI^n = 0$.

(ii) If R is commutative and noetherian, and X is an I-torsion module with an essential extension Y, then Y is also I-torsion.

Proof (i) If we take $\{m_1, m_2, ..., m_s\}$ to be a set of generators of M then for each m_j there exists $n_{m_j} \in \mathbb{N}$ such that $m_j I^{n_{m_j}} = 0$. Now take $n = max_{1 \leq j \leq s}\{n_{m_j}\}$ and we are done.

(ii) Let $0 \neq y \in Y$. By (i), $\exists n \in \mathbb{N}$ such that $(yR \cap X)I^n = 0$. It is easy to see that $(yR \cap X) \leq_{ess} yR$. I is AR by Lemma 3.2.11, so $\exists m \in \mathbb{N}$ such that $yRI^m = 0$, i.e. $yI^m = 0$.
□

Corollary 3.3.5 *Let R be a commutative noetherian ring and I be an ideal of R. Then $E(R/I)$ is I-torsion.*

Proof This follows immediately from Lemma 3.3.4 (ii). □

We will derive our main theorem about indecomposable injectives over commutative noetherian rings from a theorem about general modules over commutative noetherian rings.

Theorem 3.3.6 *Let R be a commutative noetherian ring and M be an R-module. Then the following are equivalent:*

(i) $\sigma[M]$ is essentially closed.

(ii) $\sigma[M]$ is closed under extensions.

(iii) $E(R/P) \in \sigma[M]$ whenever P is a prime ideal of R with $R/P \in \sigma[M]$.

(iv) $R/P^n \in \sigma[M]$ for all $n \in \mathbb{N}$ and prime ideals P of R with $R/P \in \sigma[M]$.

(v) $R/P^n \in \sigma[M]$ for all $n \in \mathbb{N}$ and prime ideals P of R with $R/P \in \sigma[M]$ such that for all prime ideals $Q \subsetneq P$, $R/Q \notin \sigma[M]$.

Proof (i) $\Rightarrow$ (ii) follows from Lemma 3.1.16.

(ii) $\Rightarrow$ (iv) Suppose that $R/P^k \in \sigma[M]$ for some $k \in \mathbb{N}$. Then $(R/P^{k+1})/(P^k/P^{k+1}) \cong R/P^k \in \sigma[M]$. Also $(P^k/P^{k+1})P = 0$, so $P^k/P^{k+1} \in \sigma[R/P] \subseteq \sigma[M]$. Hence by assumption $R/P^{k+1} \in \sigma[M]$ and (iv) follows by induction.

(iv) $\Rightarrow$ (iii) If $e \in E(R/P)$, then $eP^n = 0$ for some $n \in \mathbb{N}$ by Corollary 3.3.5. By Lemma 3.2.12, $eR \in \sigma[R/P^n]$ and so $eR \in \sigma[M]$. Hence $E(R/P) \in \sigma[M]$ by Lemma 3.1.6.

(iii) $\Rightarrow$ (i) Suppose that $N \in \sigma[M]$. Since R is noetherian, it follows from Lemma 1.3.2 that $E(N) \cong \bigoplus_{\lambda\in\Lambda} E_\lambda$, where each E_λ is indecomposable. By Corollary 3.3.2, each E_λ is isomorphic to $E(R/P_\lambda)$ for some prime ideal P_λ.

For every λ, $X_\lambda = E_\lambda \cap N \neq 0$, so by the uniqueness of associated primes, $Ass(X_\lambda) = P_\lambda$. Hence $R/P_\lambda \hookrightarrow X_\lambda$. By (iii), $E_\lambda \in \sigma[M]$ for all $\lambda \in \Lambda$ and so $E(N) \in \sigma[M]$.

(iv) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (iv) Suppose that P is prime in R and that $R/P \in \sigma[M]$. It is well known (see e.g. [42] Corollary 15.5) that every chain of prime ideals of a commutative noetherian ring has finite length, so we can find an ideal P' which is minimal with respect to the conditions $P' \subseteq P$, P' is prime and $R/P' \in \sigma[M]$. By (v), $R/P'^n \in \sigma[M]$ for every $n \in \mathbb{N}$, and clearly $R/P'^n \twoheadrightarrow R/P^n$. It follows that $R/P^n \in \sigma[M]$ for every $n \in \mathbb{N}$. $\square$

Note that we cannot generalise Theorem 3.3.6 to all rings, as the following example will show. The proof of (iv) $\Rightarrow$ (iii) requires the prime P to be AR and the proof of (iii) $\Rightarrow$ (i) requires a strong relationship between the prime ideals and the indecomposable injectives. Neither of these conditions hold for an arbitrary ring but we do have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) for a right FBN ring where every prime ideal is AR. Non-commutative examples of such rings are so specialised that we will leave it to the concerned reader to check the proof in such a case. Theorem 3.3.16 can also be generalised a little.

Example 3.3.7 Let k be a field and consider the ring $R = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$. R is a finite dimensional algebra, so is left and right artinian. Every right R-module must be finitely annihilated, by the right artinian property of R, therefore R is right FBN by Lemma 3.2.8. If I is the ideal $\begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}$, then the only prime ideals (indeed, the only proper ideals) which contain I are $P = \begin{bmatrix} k & k \\ 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix}$. By Lemma 3.2.12, if A is a prime ideal such that $R_R/A_R \in \sigma[R_R/I_R]$ then $A \supseteq I$, so P and Q are the only such primes. Also, $P^2 = P$ and $Q^2 = Q$, so condition (iv) of Theorem 3.3.6 clearly holds for $M = R_R/I_R$.

However, $I^2 = 0 \neq I$, so $\sigma[R_R/I_R]$ is neither essentially closed nor closed under extensions by Corollaries 3.2.13 and 3.2.14.

It will shortly be revealed that over a commutative ring, the prime ideals P for which $\sigma[E(R/P)]$ is essentially closed are exactly the same ones which satisfy a different, already widely studied, condition. We will now introduce this other condition, and this requires a definition.

Definition 3.3.8 *Let R be a commutative noetherian ring. An ideal A of R is said to be* **primary** *if whenever $x, y \in R$ and $xy \in A$ then either $x \in A$ or $y^n \in A$ for some $n \in \mathbb{N}$.*

This is equivalent to saying that P is primary if whenever $x, y \in R\backslash A$ and $xy \in A$ then both $x^m \in A$ and $y^n \in A$ for some $m, n \in \mathbb{N}$.

Lemma 3.3.9 *(Well-known) Let R be a commutative noetherian ring. Every prime ideal of R is primary and any primary ideal A can be associated with a unique prime:*

$$P = \sqrt{A} = \{r \in R : r^n \in A \text{ for some } n \in \mathbb{N}\}.$$

In this situation we say that A is **P-primary.**

Proof It is a trivial matter to show that all prime ideals of R are primary.

For the second part, we must first of all show that P as defined is an ideal of R. Suppose that $r, s \in P$ and that $r^i, s^j \in A$. If we consider the binomial expansion of $(r-s)^{i+j-1}$, then we see that every term has either r raised to a power of at least i, or s raised to a power of at least j. Hence every term is in A, so $(r-s)^{i+j-1} \in A$, i.e. $r - s \in P$. It is straightforward to show that P is closed under multiplication by elements of R.

Now, suppose that $x,y \in R$ such that $xy \in P$. Then $(xy)^n \in A$ for some $n \in \mathbb{N}$, i.e. $x^n y^n \in A$. Hence either $x^n \in A$ or $y^{nm} \in A$ for some $m \in \mathbb{N}$, and so $x \in P$ or $y \in P$, which means that P is prime. $\square$

Lemma 3.3.10 *(Well-known) Given an ideal B and a prime ideal P of a commutative noetherian ring R, then B is P-primary if and only if the following hold:*

(i) $B \subseteq P$,

(ii) $P^n \subseteq B$ for some $n \in \mathbb{N}$,

(iii) $xy \in B \Rightarrow x \in B$ or $y \in P$.

Proof $\Rightarrow$ Trivially, we must have that $B \subseteq P$.

Suppose that $P = p_1R + p_2R + ... + p_kR$. Then for every $i \in \{1, 2, ..., k\}$, there exists $n_i \in \mathbb{N}$ such that $p_i^{n_i} \in B$. If we take n to be $k. \max n_i$, condition (ii) follows.

Now suppose that $xy \in B$. Then $x \in B$ or $y^t \in B$ for some $t \in \mathbb{N}$. In the latter case we must have $y \in P$.

$\Leftarrow$ Suppose that P and B satisfy conditions (i), (ii) and (iii) and that $xy \in B$ where $x \notin B$. Then $y \in P$, so $y^n \in B$ and therefore B is primary. Now let $r \in P$. Then $r^n \in B$, so $P \subseteq \sqrt{B}$. Let $s \in \sqrt{B}$ and let $m \in \mathbb{N}$ such that $s^m \in B$. Clearly then, $s^m \in P$, so $s \in P$ and hence $\sqrt{B} \subseteq P$. Therefore $P = \sqrt{B}$. $\square$

Lemma 3.3.11 *(Well-known) If R is a commutative noetherian ring with a prime ideal P, and B is a P-primary ideal of R, then $Ass(R/B) = \{P\}$.*

Proof By Lemma 3.2.2 (ii), $Ass(R/B) \neq \varnothing$. Suppose that $Q \in Ass(R/B)$. Then by Lemma 3.3.1 (iii), there exists a monomorphism $i : R/Q \hookrightarrow R/B$. Therefore $(R/Q)B = 0$ and so $B \subseteq Q$. By Lemma 3.3.10, $P^n \subseteq Q$ for some $n \in \mathbb{N}$, hence $P \subseteq Q$. Now if $r \in R$ such that $i(1+Q) = r+B$, then $rQ \subseteq B$. Since B is P-primary it follows by Lemma 3.3.10 (iii) that either $b \in B$, which is clearly false, or $Q \subseteq P$, which must therefore be true. Hence $Q = P$. $\square$

Definition 3.3.12 *If R is a commutative noetherian ring, P is a prime ideal and $n \in \mathbb{N}$, then the* **nth symbolic power** *of P is the ideal:*

$$P^{(n)} = \{p \in P : pr \in P^n \text{ for some } r \in R\backslash P\}$$

Lemma 3.3.13 *(Well-known) If R is a commutative noetherian ring, P is a prime ideal and $n \in \mathbb{N}$, then the nth symbolic power of P is P-primary and is contained in every P-primary ideal which contains P^n.*

Proof Suppose that $xy \in P^{(n)}$ and that $x \notin P^{(n)}$. We know that there exists $r \in R\backslash P$ such that $xyr \in P^n$, and this tells us that $yr \in P$, since $x \notin P^{(n)}$. Hence $y \in P$, so $y^n \in P^n \subseteq P^{(n)}$ and therefore $P^{(n)}$ is primary.

To show that $P^{(n)}$ is P-primary, we must show that $P = \sqrt{P^{(n)}}$. Obviously $P \subseteq \sqrt{P^n}$ and $\sqrt{P^n} \subseteq \sqrt{P^{(n)}}$, so it follows that $P \subseteq \sqrt{P^{(n)}}$. Let $q \in \sqrt{P^{(n)}}$. Then $q^m \in P^{(n)} \subseteq P$ for some $m \in \mathbb{N}$, hence $q \in P$.

Now suppose that A is P-primary and $A \supseteq P^n$. Let $p \in P^{(n)}$, then there exists $r \in R\backslash P$ such that $pr \in P^n \subseteq A$. By Lemma 3.3.10, it follows that $p \in A$. $\square$

Here we are only introducing primary ideals as a means of studying the central questions of this chapter, and consequently have been rather selective in the results we have reproduced. Further reading on the theory of primary ideals over commutative rings can be found in e.g. [42] and [43].

Definition 3.3.14 *If R is a commutative noetherian ring and P is a prime ideal of R, then we say that* **the P-adic topology is equivalent to the P-symbolic topology** *if for every $n \in \mathbb{N}$, there exists an $n' \in \mathbb{N}$ such that $P^{(n')} \subseteq P^n$. A prime which has this property is called a* **t-ideal** *(this notation is taken from [49]).*

The problem of classifying the t-ideals is discussed in [22], [41] and [49], mainly from a topological viewpoint.

Definition 3.3.15 *Let M_R and N_R be modules. Then the* **trace** *of M in N is defined as:*

$$Tr(M,N) = \sum_{\lambda \in Hom_R(M,N)} \lambda(M)$$

Theorem 3.3.16 *If R is a commutative noetherian ring and P is a prime ideal of R, then the following are equivalent:*

(i) $\sigma[E(R/P)]$ is essentially closed.

(ii) $\sigma[E(R/P)]$ is closed under extensions.

(iii) $R/P^n \in \sigma[E(R/P)]\ \forall n \in \mathbb{N}$.

(iv) $\sigma[E(R/P)]$ is the class of all P-torsion modules.

(v) $Tr(E(R/P), E(R/Q)) = E(R/Q)$ for every prime Q containing P.

(vi) P is a t-ideal.

Proof (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) This follows by Theorem 3.3.6 (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv).

(iii) $\Rightarrow$ (iv) Since $E(R/P)$ is P-torsion, it is clear that any member of $\sigma[E(R/P)]$ is also P-torsion. Conversely, if X is a P-torsion module, then every cyclic submodule of X is an epimorphic image of R/P^n for some $n \in \mathbb{N}$ and therefore X is in $\sigma[E(R/P)]$ by Lemma 3.1.6.

(iv) $\Rightarrow$ (v) Let $e \in E(R/Q)$. We know that $E(R/Q)$ is Q-torsion by Corollary 3.3.5, so there exists $n \in \mathbb{N}$ such that $eQ^n = 0$. Obviously $eP^n = 0$, hence $E(R/Q)$ is P-torsion and a member of $\sigma[E(R/P)]$. Therefore, there exists an index set Λ and a module $K \leq E(R/P)^{(\Lambda)}$ such that $E(R/Q) \hookrightarrow (E(R/P)^{(\Lambda)})/K$. Since $E(R/Q)$ is injective, it is isomorphic to a direct summand of $(E(R/P)^{(\Lambda)})/K$, so $(E(R/P)^{(\Lambda)}) \twoheadrightarrow E(R/Q)$ and (v) is proved.

(v) $\Rightarrow$ (i) Suppose that A is a prime ideal of R and $R/A \in \sigma[E(R/P)]$. Since $E(R/P)$ is P-torsion, it follows that R/A is also P-torsion, so there exists $n \in \mathbb{N}$ such that $P^n \subseteq A$ and thus $P \subseteq A$. By (v), $E(R/A) \in \sigma[E(R/P)]$ and (i) then follows by Theorem 3.3.6 (iii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (vi) Let $n \in \mathbb{N}$. Now $R/P^n \in \sigma[E(R/P)]$, so by Lemma 3.1.11, there is a finite subset $S = \{e_1, ..., e_m\}$ of $E(R/P)$ such that $\mathbf{r}(S) \subseteq P^n$. Since $E(R/P)$ is P-torsion, there exists $t \in \mathbb{N}$ such that $P^t \subseteq \mathbf{r}(S)$. We will now prove that $\mathbf{r}(S)$ is P-primary, and then it follows by Lemma 3.3.13 that $P^{(t)} \subseteq \mathbf{r}(S)$.

Suppose that $xy \in \mathbf{r}(S)$ but $x \notin \mathbf{r}(S)$. Then there is an e_i such that $e_i x \neq 0$. There exists $r \in R$ such that $0 \neq e_i x r \in R/P$, so since $e_i x r y = 0$, it follows that $y \in P$.

(vi) $\Rightarrow$ (iii) Let $n \in \mathbb{N}$. Then there is an integer $m \geq n$ such that $P^{(m)} \subseteq P^n$. $E(R/P^{(m)}) = \bigoplus_{\lambda \in \Lambda} E(R/Q_\lambda)$ where each Q_λ is a prime ideal of R, and so if we can show that $Q_\lambda = P$ for every $\lambda \in \Lambda$, this will mean that $E(R/P^{(m)}) \in \sigma[E(R/P)]$, so $R/P^{(m)} \in \sigma[E(R/P)]$, hence $R/P^n \in \sigma[E(R/P)]$ and we are done.

Let $Q = Q_\lambda$ for some $\lambda \in \Lambda$. $R/Q \hookrightarrow E(R/P^{(m)})$, so $Q \in Ass(E(R/P^{(m)})$ by Lemma 3.3.1 (iii). Now, since $P^{(m)}$ is P-primary, it follows by Lemma 3.3.11 and Lemma 3.2.2 (iv) that $Q = P$. $\square$

Lemma 3.3.17 *If P is a maximal ideal of a commutative noetherian ring R, then every power of P is P-primary, so in particular, $\sigma[E(R/P)]$ is essentially closed.*

Proof It is clear that P is P-primary. Now say that P^k is P-primary for some $k \in \mathbb{N}$. Suppose that $rx \in P^{k+1}$, where $r \in R \backslash P$, then by using Lemma 3.3.10 and the fact that P^k is P-primary, we get $x \in P^k$. Let $L = \{s \in R : xs \in P^{k+1}\}$. Clearly $P \subseteq L$ and $r \in L$, so $L = R$. Therefore $x \in P^{k+1}$ and so P^{k+1} is P-primary by Lemma 3.3.10. The result follows by induction. $\square$

Corollary 3.3.18 *If E_R is an injective module over a commutative artinian ring R, then $\sigma[E]$ is essentially closed.*

Proof Every injective R-module is a direct sum of indecomposable injectives, so by Corollary 3.1.18, it is enough to show that $\sigma[E]$ is essentially closed in the case where E is indecomposable. Therefore we will assume that E is indecomposable.

Clearly E has non-zero socle and so $E \cong E(R/P)$, where P is a maximal ideal of R. The result then follows by Lemma 3.3.17. $\square$

Corollary 3.3.19 *Let R be a commutative artinian ring and let M be an R-module. Then $\sigma[M]$ is essentially closed $\Leftrightarrow E(M) \in \sigma[M]$.*

Proof The result follows from Corollary 3.3.18. $\square$

In the following example, we will say that an element of a polynomial ring is **homogeneous** if it is an element of the base ring or a product of an element of the base ring and one or more of the indeterminates. More simply, an element is homogeneous if it is not built up using addition. A homogeneous element of a factor ring of a polynomial ring can be defined similarly.

Definition 3.3.20 *Let R be a commutative domain. An element x of R is called* **irreducible** *if whenever there exist a and b in R such that $ab = x$, then either a or b is a unit. R is called a* **unique factorization domain** *(or* **UFD** *for short) if the following conditions hold:*

(i) For every non-zero non-unit y of R there exist irreducible elements $c_1, c_2, ..., c_n$ of R such that $y = c_1c_2...c_n$,

(ii) If $d_1, d_2, ..., d_k$ and $g_1, g_2, ..., g_l$ are irreducible non-units of R such that $d_1d_2...d_k = g_1g_2...g_l$ then $k = l$ and there exists a permutation θ of the set $\{1, ..., k\}$ and a set of units $\{u_i\}_{1 \le i \le k}$ such that $d_i = g_{\theta(i)}u_i$ for every i.

Example 3.3.21 (Sharp, [42] Ex. 4.12) Let k be a field, and take $R = \frac{k[X,Y,Z]}{(XZ-Y^2)}$, i.e. R is a polynomial ring of x, y, z over k subject to the relationship $xz = y^2$. Let P be the ideal of R generated by x and y. It was shown by Sharp that R is a domain, that P is prime and that P^2 is not P-primary, i.e. $P^{(2)} \neq P^2$. In fact, it is true that for any $2 \leq n \in \mathbb{N}$, P^n is not P-primary. To see this, consider the element $x^{n-1}z = x^{n-2}y^2$ of P^n. Clearly $z \notin P$, so if P^n were P-primary, this would mean that $x^{n-1} \in P^n$. Suppose that this is the case. Then moving back up to $k[X, Y, Z]$, we can see that:

$$X^{n-1} = f_0(X,Y,Z)X^n + f_1(X,Y,Z)X^{n-1}Y + ... + f_n(X,Y,Z)Y^n + g(X,Y,Z)(XZ - Y^2)$$

for some $g(X,Y,Z), f_i(X,Y,Z) \in k[X,Y,Z]$ $(0 \leq i \leq n)$. If we put $Y = Z = 0$, then $X^{n-1} = f_0(X,Y,Z)X^n$ and so $X^{n-1}(1 - Xf_0(X,Y,Z)) = 0$. Since $k[X,Y,Z]$ is a domain, it follows that $Xf_0(X,Y,Z) = 1$, which is clearly impossible. Therefore P^n is not P-primary.

We will now show that for any $n \in N$, there exists $n' \in N$ such that $P^{(n')} \subseteq P^n$. It follows that P is an example of a t-ideal whose powers (apart from P^1, of course) are not primary.

First we will establish a system for uniquely describing each member of R. We can write $r \in R$ as:

$$a + \sum_{(0,0)\neq(i,j)\in\mathbb{N}^2} b_{i,j}x^iy^j + \sum_{l\in\mathbb{N},\ 1\leq m\in\mathbb{N}} c_{l,m}y^lz^m$$

To see that these descriptions are unique, suppose that the above is a description of 0, where not all of $a, b_{i,j}, c_{l,m}$ are 0. Going back up to $k[X,Y,Z]$, we see that:

$$a + \sum_{(i,j)\neq(0,0)} b_{i,j}X^iY^j + \sum_{m\geq 1} c_{l,m}Y^lZ^m \in (XZ - Y^2)$$

Since $k[X,Y,Z]$ is a UFD ([42] Theorem 1.42), $XZ - Y^2$ must be a factor of the left-hand side. Since XZ does not appear as part of any of the homogeneous terms there, this cannot happen and so this description of 0 in R does not exist.

Secondly we show that a member of R is in P^n if and only if each of the homogeneous elements in its unique description is in P^n. If $h_1, ..., h_t$ are homogeneous elements of R whose sum is in P^n, then

$$h_1 + ... + h_t = \quad x^n(h_{0,1} + ... + h_{0,a_0})$$

$$
\begin{aligned}
&+ \quad x^{n-1}y(h_{1,1}+\ldots+h_{1,a_1}) \\
&+ \quad \ldots \\
&+ \quad y^n(h_{n,1}+\ldots+h_{n,a_n})
\end{aligned}
$$

where the $h_{i,j}$s are homogeneous elements of R. If we multiply out the brackets and replace all occurrences of xz with y^2, then by the uniqueness of the descriptions, we can see that the right-hand side of the equation must be a rearrangement of the left-hand side. It is then easily seen that each h_i is a member of P^n.

Now, $P^{(n)} = \{p \in P : pr \in P^n \text{ for some } r \in R \backslash P\}$. So if $p \in P^{(n)}$, then $p(h_0+\ldots+h_n) \in P^n$ where at least one h_i is of the form c or cz^s, where $0 \neq c \in k$ and $s \geq 1$. By the preceding paragraph, we have that $ph_i \in P^n$ for this value of i, so on considering multiplication by c^{-1}, it is clear that either $p \in P^n$ or $pz^s \in P^n$. Furthermore p is a sum of homogeneous elements and if p' is one of these, $p' \in P^n$ or $p'z^s \in P^n$. So $P^{(n)}$ is generated by P^n and the homogeneous elements of P whose product with a power of z is in P^n. So to find $P^{(n)}$, we need only consider homogeneous elements of P.

Consider y^iz^j, where $i \geq 1$, and suppose that this is a member of P^n. Then as before:

$$
\begin{aligned}
y^iz^j = &\quad x^n(h_{0,1}+\ldots+h_{0,a_0}) \\
&+ \quad x^{n-1}y(h_{1,1}+\ldots+h_{1,a_1}) \\
&+ \quad \ldots \\
&+ \quad y^n(h_{n,1}+\ldots+h_{n,a_n})
\end{aligned}
$$

for suitable homogeneous terms $h_{d,e}$. Note that after multiplying out the right-hand side, the sum of the powers of x and y in each of the homogeneous terms is at least n, and after replacing all occurrences of xz with y^2, this is still the case. By the uniqueness of the descriptions the right-hand side now reduces to a single homogeneous term in P^n and so $i \geq n$. Therefore $y^iz^j \in P^n \Leftrightarrow i \geq n$. Furthermore, $y^iz^j \in P^{(n)} \Leftrightarrow y^iz^{j+s} \in P^n$ for some $s \geq 0 \Leftrightarrow i \geq n$. So $y^iz^j \in P^{(n)} \Leftrightarrow y^iz^j \in P^n$.

Now consider $x^iy^j \in P$. Then $x^iy^jz^s \in P^n$ for some s if and only if $x^iy^jz^s \in P^n$ for some $s \geq i$. If $s \geq i$, then $x^iy^jz^s = y^{j+2i}z^{s-i}$, which is in P^n if and only if $j+2i \geq n$. So

$x^iy^j \in P^{(n)}$ if and only if $2i + j \geq n$. If $x^iy^j \in P^{(2n)}$, then $2i + j \geq 2n$, so $i + j \geq n$ and hence $x^iy^j \in P^n$.

Therefore $P^{(2n)} \subseteq P^n$ and so the P-adic topology is equivalent to the P-symbolic topology, showing that $\sigma[E(R/P)]$ is essentially closed.

Example 3.3.22 Let k be a field and let $R = \frac{k[X,Y]}{(X^2,XY)}$, i.e. R is a polynomial ring in indeterminates x and y over k subject to the relations $x^2 = 0$ and $xy = 0$. Let P be the ideal of R generated by x. Then P is prime and $P^2 = 0$, so $\sigma[E(R/P)]$ is essentially closed if and only if 0 is P-primary. But $xy = 0$, $x \neq 0$ and $y \notin P$, so $\sigma[E(R/P)]$ cannot be essentially closed.

These examples show that establishing when $\sigma[E(R/P)]$ is essentially closed is a far from trivial problem. Obviously in a domain, the prime ideal 0 is a t-ideal, but [49] features an example of a prime ideal P of a domain such that P is not a t-ideal, so we cannot for example say that all prime ideals containing a prime t-ideal are t-ideals.

The situation becomes worse still when we start to consider non-indecomposable injective modules. We have the following Lemma, but as we shall see the implication cannot, in general, be reversed.

Lemma 3.3.23 *Let R be a commutative noetherian ring with a set of prime t-ideals $\{P_\lambda\}_{\lambda\in\Lambda}$. Then $\sigma[\bigoplus_{\lambda\in\Lambda} E(R/P_\lambda)]$ is essentially closed.*

Proof This follows by Theorem 3.3.16 and Corollary 3.1.18. □

Corollary 3.3.24 *Let R be a commutative noetherian domain whose only primes are* 0 *and the maximal ideals. Then if E is an injective R-module, $\sigma[E]$ is essentially closed.*

Proof Obviously, 0 is a t-ideal. Also, every other prime ideal of R is a t-ideal by Lemma 3.3.17. The result then follows by Lemma 3.3.23. □

We will now construct an example to disprove the converse of Lemma 3.3.23.

Example 3.3.25 Let R be the ring constructed in Example 3.3.22. It is easy to verify that $\mathbf{r}(x) = xR + yR$, that $\mathbf{r}(y) = xR$ and that $xR \cap yR = 0$. Also, $xR \oplus yR \leq_{ess} R$ and so $Ass(R) = \{xR + yR, xR\}$ by Lemma 3.2.2 (iv). Therefore, $E(R) \cong E(R/xR \oplus yR) \oplus E(R/xR)$, so $\sigma[E(R/xR + yR) \oplus E(R/xR)]$ contains every R-module and must be essentially closed. But, as we saw in Example 3.3.22, xR is not a t-ideal.

So, even in the case of a finite set of ideals $\{P_\lambda\}_{\lambda \in \Lambda}$, knowing that $\bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$ is essentially closed is not enough to tell us that each P_λ is a t-ideal. We can prove the following partial result:

Lemma 3.3.26 *Let R be a commutative noetherian ring with a set of prime ideals $\{P_\lambda\}_{\lambda \in \Lambda}$ such that $\sigma[E]$ is essentially closed, where $E = \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$. If $\alpha \in \Lambda$ is such that $P_\alpha + P_\beta = R$ for every $\alpha \neq \beta \in \Lambda$ then P_α is a t-ideal.*

Proof For any $n \in \mathbb{N}$, Theorem 3.3.6 tells us that $R/P_\alpha^n \in \sigma[E]$. Therefore there exist finite sets $\{e_1, ..., e_s\} \subseteq E(R/P_\alpha)$ and $\{f_1, ..., f_t\}$ where $f_i \in E(R/P_{\lambda_i})$ with $\lambda_i \neq \alpha$, such that $\mathbf{r}(e_1, ..., e_s, f_1, ..., f_t) \subseteq P_\alpha^n$. If we put $A = \mathbf{r}(e_1, ..., e_s)$, then $A.P_{\lambda_1}^{k_1}.\cdots.P_{\lambda_t}^{k_t} \subseteq P_\alpha^n$ for suitable $k_i \in \mathbb{N}$ by Corollary 3.3.5. Considering the binomial expansion of $(P_{\lambda_1} + P_\alpha)^{n+k_1}$, we see that every term must have either $P_{\lambda_1}^{k_1}$ or P_α^n as a factor. It follows that $A.(P_{\lambda_1} + P_\alpha)^{n+k_1}.\cdots.P_{\lambda_t}^{k_t} \subseteq P_\alpha^n$, i.e. $A.P_{\lambda_2}^{k_2}.\cdots.P_{\lambda_t}^{k_t} \subseteq P_\alpha^n$. Repeating this step for each P_{λ_i}, we deduce that $A \subseteq P_\alpha^n$. Therefore, $R/P_\alpha^n \in \sigma[E(R/P_\alpha)]$ and it follows that P_α is a t-ideal by Theorem 3.3.16. □

Finally, as an interesting by-product, we can combine results from the last two sections to produce some ideal theoretic results.

Corollary 3.3.27 *Let R be a commutative noetherian ring with an ideal I. Then the following are equivalent:*

(i) $I = I^2$.

(ii) $P^n \supseteq I$ for every prime P containing I and $n \in \mathbb{N}$.

(iii) $P^n \supseteq I$ for every $n \in \mathbb{N}$ and prime P containing I such that there are no prime ideals between P and I.

Proof (i) $\Rightarrow$ (ii) By condition (i) and Corollary 3.2.13, it follows that $\sigma[R_R/I_R]$ is closed under extensions. Obviously, $(R/P)I = 0$, so $R/P \in \sigma[R/I]$ by Lemma 3.2.12 and $R/P^n \in \sigma[R/I]$ by taking suitable extensions. Therefore, using Lemma 3.2.12, $(R/P^n)I = 0$, i.e. $I \subseteq P^n$.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) It is easy to see that condition (v) of Theorem 3.3.6 is satisfied, and we simply apply (v) $\Rightarrow$ (ii) of this Theorem, followed by Corollary 3.2.13. □

Corollary 3.3.28 *Let R be a commutative noetherian ring with a finitely generated uniform module U such that $\mathbf{r}(U)^2 = \mathbf{r}(U)$, and let $P = Ass(U)$. Then P is a t-ideal.*

Proof $\sigma[U]$ is essentially closed, by Corollary 3.2.17. $E(R/P) \cong E(U)$ by Lemma 3.3.1 (iii) and so $\sigma[E(R/P)]$ is also essentially closed. The result follows from Theorem 3.3.16. □

Corollary 3.3.29 *Let R be a commutative artinian ring with an injective module E. Then $\mathbf{r}(E) = \mathbf{r}(E)^2$.*

Proof This follows by Corollary 3.3.18 and Corollary 3.2.17. □

Chapter 4

SU Rings

Rings whose modules on one side are direct sums of uniserials have been shown to be serial (see, for example, [52] 55.16). Furthermore, it was shown in [9] that a left or right module over a serial ring is a direct sum of uniserial modules. Here we will consider a weaker condition on the module class - we want to find out about rings whose modules on one side are direct sums of uniform modules.

Artinian rings where every indecomposable right module is uniform (**SLCRT rings** as they were called) were considered in [46] and [47] by Tachikawa, where some necessary and sufficient conditions were discussed. Some of Tachikawa's findings were combined by Sumioka in [45] to give Theorem 4.0.1 below.

In the first section of this chapter, we will produce some necessary conditions for a ring to have the property that every right module is a direct sum of uniforms. Some of these, for example that such a ring is right serial, were known to Tachikawa, but modern techniques allow us to have simpler proofs. At the end of the section, we will introduce some necessary and sufficient conditions for a ring to have our property.

The second and third sections are devoted to constructions of families of examples and counterexamples which, in their complexity, illustrate the difficulty of dealing with our property, and perhaps show why there has been little written on the subject since Tachikawa's original papers.

Theorem 4.0.1 *([45]) Let R be an artinian ring with Jacobson radical J, and consider the following conditions:*

(a) Every indecomposable right R-module is local.

(b) Every indecomposable left R-module is uniform.

(c) (i) R is left serial,

(ii) For any uniserial left R-modules ${}_RU$ and ${}_RV$, every isomorphism $\theta : Soc({}_RU) \to Soc({}_RV)$ is either extendible to a homomorphism $\theta' : {}_RU \to {}_RV$ or cannot be extended to any homomorphism $\phi : {}_RW \to {}_RV$, where $Soc(U) < W \leq U$.

(iii) $length(eJ/eJ^2) \leq 2$, where e is a primitive idempotent of R.

(d) (i) R is left serial,

(ii) For every primitive idempotent e of R, $eJ_R = M_1 \oplus M_2$, where M_i is either zero or uniserial for $i = 1, 2$.

For any artinian ring R, (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d). If R is a finite dimensional algebra over a field, then all four conditions are equivalent.

In [46], Tachikawa claimed that conditions (b) and (c) of Theorem 4.0.1 were equivalent for any ring, but he later noticed that an extra, rather technical, condition on the ring, (D) as described in [45], was needed to make the proof of (c) $\Rightarrow$ (b) work. Every finite dimensional algebra over a field has condition (D).

4.1 The Theory

Throughout this chapter, we will freely use standard results on right artinian rings from Chapter 1. Recall that there exists $n \in \mathbb{N}$ such that R has a complete set of primitive orthogonal idempotents with n members. This contains a subset $\{e_1, ..., e_m\}$ such that if J is the Jacobson radical of R, then there are exactly (up to isomorphism) m simple right R-modules which are given by $\{e_iR/e_iJ\}$, m indecomposable right projectives which are given by $\{e_iR\}$ and m indecomposable right injectives which are given by $\{E(e_iR/e_iJ)\}$. Furthermore, any right projective is a direct sum of indecomposable right projectives and any right injective is a direct sum of indecomposable injectives.

Definition 4.1.1 *We say that R is a* **right pure-semisimple ring** *if every right R-module is a direct sum of indecomposable submodules. R is called a* **right SU ring** *if every right R-module is a direct sum of uniform submodules. If there are only finitely many isomorphism classes of finitely generated indecomposable right R-modules then R is said to be of* **finite (representation) type.**

Semisimple rings are trivially left and right SU. For a serial ring R, all left and right modules are direct sums of uniserial modules (see [9]), so R is left and right SU. These examples are all two-sided SU rings, which gives rise to the question of whether all right SU rings are also left SU. In fact the answer is "no", as we shall see later.

To complete some of the later proofs, we will need to state some results about pure-semisimple rings and rings of finite type. The proofs of these results make use of advanced category theory and the theory of rings without identity, so for reasons of space, we are unable to reproduce them here. [52] contains the proofs in full, as well as further reading on pure-semisimple rings.

Theorem 4.1.2 *(Zimmerman-Huisgen, see [52], 53.6) A ring R is right pure-semisimple $\Leftrightarrow$ every right R-module is the direct sum of finitely generated modules. When this happens, R is right artinian.*

Theorem 4.1.3 *(Well-known) Let R be a right pure-semisimple ring with a right module M such that $M = \bigoplus_{\lambda\in\Lambda} M_\lambda = \bigoplus_{\omega\in\Omega} N_\omega$ where each M_λ and each N_ω is indecomposable. Then $|\Lambda| = |\Omega|$ and there exists a bijection $\rho : \Lambda \to \Omega$ such that $M_\lambda \cong N_{\rho(\lambda)}$ for every $\lambda \in \Lambda$.*

Proof The main Theorem of [13] states that every decomposition of a module of a pure-semisimple ring into a direct sum of indecomposable submodules complements direct summands. In particular, every such decomposition complements maximal direct summands and therefore the decompositions are unique (up to isomorphism) by [1] Theorem 12.4. $\square$

The following Theorem is due to various people, including Auslander.

Theorem 4.1.4 *([52], 54.3) The following are equivalent for a ring R:*

(i) R is a ring of finite representation type.

(ii) R is left and right pure-semisimple (and consequently, left and right artinian by Theorem 4.1.2).

(iii) R is a ring of bounded representation type (i.e. $\exists n \in \mathbb{N}$ such that all finitely generated indecomposable modules have length n or less).

Corollary 4.1.5 *The finite type property is left-right symmetric - that is to say that R is a ring of finite representation type $\Leftrightarrow$ there are only finitely many isomorphism classes of finitely generated indecomposable left R-modules.*

Proof This follows from condition (ii) of Theorem 4.1.4. $\square$

Note It is an open question, whether or not all right pure-semisimple rings are also left pure-semisimple.

Definition 4.1.6 *If R is a ring and M_R is a right R-module, then $Z_2(M)$ is defined to be the submodule of M such that $Z_2(M)/Z(M) = Z(M/Z(M))$. If $Z_2(M) = M$, then we say that M is* **Goldie torsion**.

Lemma 4.1.7 *Let R and A be right SU rings. Then:*

(i) R is of finite representation type.

(ii) R is right serial.

(iii) Every quotient ring of R is right SU.

(iv) The ring direct sum $R \oplus A$ is right SU.

(v) If R is right non-singular, then R is (right and left) hereditary.

(vi) For every module M_R, $Z_2(M)$ is a direct summand of M.

(vii) If R is neither right non-singular nor right Goldie torsion, then there exists a decomposition:

$$R \cong \begin{bmatrix} S & M \\ 0 & T \end{bmatrix} \text{ where } Z_2\left(\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}\right) = \begin{bmatrix} S & M \\ 0 & 0 \end{bmatrix}$$

where T is a non-zero right non-singular right SU ring, S is a non-zero right SU ring, and ${}_SM_T$ is a bimodule, with M_T singular and serial.

Proof (i) R is a right pure-semisimple ring and hence R is right artinian. Every right R-module is a direct sum of finitely generated modules (by Theorem 4.1.2) and so for any simple T_R, $E(T)$ is finitely generated and thus has finite length. Since any indecomposable module X_R is uniform, $X \subseteq E(T)$ for some simple T_R, and so X has length less than or equal to $E(T)$. Furthermore, there are only finitely many simple right R-modules, and if we take one whose injective hull has maximal length, then this length is an upper bound for the lengths of indecomposable modules. Hence by Theorem 4.1.4, R is of finite type.

(ii) Since R is right artinian, $R_R = e_1R \oplus ... \oplus e_nR$, where each e_iR is local.

Take any e_i, and let $X, Y \subseteq e_iR$. Then $e_iR/(X \cap Y)$ is local and a direct sum of uniforms, so is uniform. But $X/(X \cap Y) \cap Y/(X \cap Y) = 0$, so $X/(X \cap Y) = 0$ or $Y/(X \cap Y) = 0$, i.e. $X \subseteq Y$ or $Y \subseteq X$. Hence e_iR is uniserial and so R is right serial.

(iii) If I is an ideal of R, every right R/I-module can be considered as a right R-module, and so as a direct sum of uniform right R-modules, which are also uniform R/I-modules. Hence R/I is also right SU.

(iv) Let $T = R \oplus A$ and M_T be a module. Then $M = MR \oplus MA$ is a decomposition as T-modules, and when considered as a right R module, MR must be a direct sum of uniforms. Subsets of MR are submodules with respect to R if and only if they are submodules with respect to T, since $MRA = 0$. Hence MR, and similarly MA, are also direct sums of uniform T-modules and we have the result.

(v) The first part of the proof, that R is right hereditary, is a modification of a proof of Warfield ([3], Theorem 8.15). It is clearly enough to show that every uniform right ideal is projective. If I is a uniform right ideal of R, then I embeds in eR for some primitive idempotent e of R by Lemma 1.1.4. Hence we just need to show that every right ideal contained in an indecomposable summand of R_R is projective.

Let K be a right ideal of R such that $K \subseteq eR$ for some primitive idempotent e of R. By (ii), eR is uniserial, and so $K = exR$ for some $x \in R$. Clearly, $exR = exe_1R + ... + exe_nR$, where $\{e_i\}_{1 \leq i \leq n}$ is a complete set of primitive orthogonal idempotents of R. Also, since eR is uniserial, it is easy to see that $exR = exe_jR$ for some $1 \leq j \leq n$. Therefore there is a surjection $e_jR \twoheadrightarrow exR$. Now, exR is non-singular and e_jR is uniserial, so this surjection must be a monomorphism. Hence $exR \cong e_jR$ is projective.

The second part of the proof, that R is left hereditary, follows by [3] Corollary 8.18.

(vi) Let U be a uniform direct summand of M. Then either $Z(U) = 0$ or $Z(U)$ is essential in U. In the former case, $Z_2(U) = 0$ and in the latter $Z_2(U) = U$, i.e. U is Goldie torsion. $Z_2(M)$ is the sum of those summands of M which are Goldie torsion.

(vii) By (vi), $Z_2(R_R)$ is a direct summand of R_R. Therefore, $Z_2(R_R) = eR$ for some idempotent e of R. Now, if x is a member of R, then $x \in Z_2(R_R)$ if and only if there exists an essential right ideal I of R such that xI is a singular right ideal. It is easy to see then that eR is a 2-sided ideal of R and so $(1-e)Re \subseteq (1-e)R \cap eR = 0$. So, using a standard type of decomposition of a ring, we can get:

$$R \cong \begin{bmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix} = \begin{bmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{bmatrix}$$

Let $S = eRe$, $T = (1-e)R(1-e)$ and $M = eR(1-e)$. Since eR is a 2-sided ideal, it follows that $T \cong R/eR$ is a right SU ring by part (iii).

Consider the left ideal $R(1-e)$ of R. We know that $R(1-e)Re = 0$, so $R(1-e)R \subseteq R(1-e)$. Therefore $R(1-e)$ is a 2-sided ideal of R and so $S \cong R/R(1-e)$ is also a right SU ring by part (iii).

For ease of notation, from now on we will assume that $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. Clearly, $\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ is an ideal. Let $f_1, \ldots, f_n$ be a set of primitive idempotents of R whose sum is e, then for each i, f_iR is uniserial and so $f_i \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ is uniserial as a right R-module. If we consider the action of a member $\begin{bmatrix} s & m \\ 0 & t \end{bmatrix}$ of R on the right R-module $f_i \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$, it is apparent that only the entry t has an effect. Therefore the structure of $f_i \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ as a right T-module is equivalent to its structure as a right R-module and hence right ideal of

R, so $f_i \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ is a uniserial right T-module. Since $\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ is equal to the sum of the $f_i \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$'s, it follows that $\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ is a direct sum of uniserial right R-modules and hence M_T is serial.

Since the structure of $\begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}$ as a right R-module is the same as its structure as a right T-module, it follows that $Soc\left(\begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & Soc(T_T) \end{bmatrix}$. Now we will find $Soc\left(\begin{bmatrix} S & M \\ 0 & 0 \end{bmatrix}\right)$. Consider the right ideals $\begin{bmatrix} Soc(\mathbf{l}_S(M))_S & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & Soc(M_T) \\ 0 & 0 \end{bmatrix}$, where $\mathbf{l}_S(M)$ is the ideal of S consisting of the elements which annihilate all of M. Since their right multiplication by a member of R is affected only by the S and T entries respectively, it follows that they are semisimple. Now let $s \in S$ and $m \in M$ such that at least one of them is non-zero. If m is non-zero, then there exists $t \in T$ such that $0 \neq mt \in Soc(M_T)$, so $\begin{bmatrix} s & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} 0 & mt \\ 0 & 0 \end{bmatrix}$. If s is non-zero and a member of $\mathbf{l}_S(M)$, then there exists $s' \in S$ such that $0 \neq ss' \in Soc(\mathbf{l}_S(M)_S)$, and so $\begin{bmatrix} s & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ss' & 0 \\ 0 & 0 \end{bmatrix}$. If s is non-zero and not in $\mathbf{l}_S(M)$, then there exists $m' \in M$ such that $sm' \neq 0$ and $t' \in T$ such that $0 \neq sm't' \in Soc(M_T)$, and so $\begin{bmatrix} s & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & m't' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & sm't' \\ 0 & 0 \end{bmatrix}$. It follows that $\begin{bmatrix} Soc(\mathbf{l}_S(M))_S & Soc(M_T) \\ 0 & 0 \end{bmatrix}$ is a semisimple essential submodule of $\begin{bmatrix} S & M \\ 0 & 0 \end{bmatrix}$ and hence $Soc(R_R) = \begin{bmatrix} Soc(\mathbf{l}_S(M))_S & Soc(M_T) \\ 0 & Soc(T_T) \end{bmatrix}$.

Let t be an arbitrary non-zero member of T. From our original decomposition, we know that $\begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix}$ is a non-singular element of R, so it follows that:

$$\begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} Soc(\mathrm{l}_S(M))_S & Soc(M_T) \\ 0 & Soc(T_T) \end{bmatrix} \neq 0$$

i.e. $t.Soc(T_T) \neq 0$. Therefore T is right non-singular and $(Soc(T_T))^2 = Soc(T_T)$.

Now, $(Soc(R_R))^2 = \begin{bmatrix} (Soc(\mathrm{l}_S(M))_S)^2 & Soc(M_T)Soc(T_T) \\ 0 & Soc(T_T) \end{bmatrix}$. Since $\begin{bmatrix} S & M \\ 0 & 0 \end{bmatrix}$ is Goldie torsion, it is annihilated by $(Soc(R_R))^2$, so

$$\begin{bmatrix} S & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (Soc(\mathrm{l}_S(M))_S)^2 & Soc(M_T)Soc(T_T) \\ 0 & Soc(T_T) \end{bmatrix} = 0$$

i.e.

$$\begin{bmatrix} (Soc(\mathrm{l}_S(M))_S)^2 & Soc(M_T)Soc(T_T) + M.Soc(T_T) \\ 0 & 0 \end{bmatrix} = 0$$

Therefore $M.Soc(T_T) = 0$, which is to say that M is a singular right T-module. □

Note that in Lemma 4.1.7, (ii) and (v) hold for any SLCRT ring - in fact, (ii) was shown to hold in such a case by Tachikawa in [46]. Also parts (iii), (iv) and (vii) are true if "right SU" is replaced throughout by "SLCRT".

Corollary 4.1.8 *A ring is left and right SU $\Leftrightarrow$ it is serial.*

Proof $\Rightarrow$ follows from Theorem 4.1.7 (i) and (ii).

$\Leftarrow$ is a consequence of [9], Theorem 1.2. □

Corollary 4.1.9 *A commutative ring is SU $\Leftrightarrow$ it is serial. When this happens, it is QF.*

Proof The first part follows directly from Corollary 4.1.8. For the second part, see [3] Theorem 6.17. □

Corollary 4.1.10 *Let R be a right non-singular right SU ring whose left and right Goldie dimensions are equal (i.e. the left and right socles have the same length). Then R is serial.*

Proof By Lemma 4.1.7 (i), (ii), (v) and Theorem 4.1.4, R is artinian, hereditary and right serial. Now, if e is a primitive idempotent of R, then Re is uniform, since otherwise the left Goldie dimension of R would exceed the right Goldie dimension. Furthermore, Je is the unique maximal submodule of Re and Je is projective since R is left hereditary, so must be isomorphic to Rf for some primitive idempotent f. Similarly, Rf in turn has its own unique maximal submodule which must be projective, and so on. Since R is left artinian, this process must stop after a finite number of steps and so R is left serial and hence generalised uniserial. □

We should note here that in the decomposition described in Lemma 4.1.7 (vii), it is not true in general that S_S is Goldie torsion, as the following example shows.

Example 4.1.11 Let k be any field and define the ring R' as follows:

$$R' := \begin{bmatrix} k & k & k & k \\ 0 & k & k & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{bmatrix}$$

It is easy to show that R' is a serial ring, and is therefore right SU. Now we define the ideal I of R' which consists of the elements of R which are zero everywhere except the top-right entry and the entry immediately below this one. We can then create a ring $R := R'/I$, which we will write as:

$$R := \begin{bmatrix} k & k & k & * \\ 0 & k & k & * \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{bmatrix}$$

R is a right SU ring by Lemma 4.1.7 (iii). It is not hard to show that:

$$Z_2(R_R) = \begin{bmatrix} k & k & k & * \\ 0 & k & k & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so by Lemma 4.1.7 (vii), there exists a decomposition, $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. If we go through the steps outlined in the proof of Lemma 4.1.7 (vii), we get that $S = T = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$ and $M = \begin{bmatrix} k & * \\ k & * \end{bmatrix}$. Since $S = T$, S must be right non-singular.

The following result is interesting but is not developed later, and consequently it is not necessary to define Morita invariance here. For an explanation the reader is referred to [1].

Theorem 4.1.12 *The right SU property of a ring is Morita invariant.*

Proof Let R be a right SU ring and S a ring Morita equivalent to R via an equivalence $F : Mod-R \to Mod-S$. By [1], Propositions 21.4, 21.5 and 21.6(5), F preserves injective mappings, direct sums and essential embeddings. It is easily seen that a right R-module M is uniform if and only if every monomorphism from a non-zero right R-module to M is an essential embedding. It follows that a direct sum of uniform right R-modules maps to a direct sum of uniform right S-modules via F. □

We will end this section with some different classifications of right SU rings.

Theorem 4.1.13 *For a ring R, the following are equivalent:*

(i) R is right SU.

(ii) R is right pure-semisimple and every (finitely generated) indecomposable right R-module is uniform.

(iii) R is right artinian, every indecomposable injective right R-module has finite length and every indecomposable right R-module is uniform.

(iv) R is right artinian, every indecomposable injective right R-module has finite length and every non-zero right R-module has a non-zero uniform direct summand.

(v) R is of finite representation type and every (finitely generated) indecomposable right R-module is uniform.

Proof (i) $\Leftrightarrow$ (ii) is straightforward from Theorem 4.1.2.

(ii) $\Rightarrow$ (iv) By Theorem 4.1.2.

(iv) $\Rightarrow$ (iii) Every non-zero indecomposable module must have a uniform direct summand, so must itself be uniform.

(iii) $\Rightarrow$ (ii) Every indecomposable module must be contained in an indecomposable injective, of which there are only finitely many, so there is a finite upper bound to the length of the indecomposable modules. Hence R is of finite representation type and is right pure-semisimple, by Theorem 4.1.4.

(i) $\Rightarrow$ (v) follows from Lemma 4.1.7.

(v) $\Rightarrow$ (ii) is a consequence of Theorem 4.1.4. $\square$

Note the similarity between condition (iii) of Theorem 4.1.13 and condition (b) of Theorem 4.0.1.

To summarise some of the main results of this section:

$$R \text{ is serial}$$
$$\Downarrow$$
$$R \text{ is right SU}$$
$$\Downarrow$$
$$R \text{ is artinian, right serial and of finite representation type}$$

Looking at this, the top and bottom conditions seem tantalisingly close, as though being right SU might turn out to be equivalent to one of these. In the next sections we will show that neither of these implications can be reversed.

Lemma 4.1.7 (vii) shows that any right SU ring which is not right singular and not right Goldie torsion can be decomposed to a matrix ring whose diagonal entries are SU rings which are factors of the original ring. This suggests that the non-singular cases and Goldie torsion cases might be important. In the following sections, we will provide counterexamples to the reverse of the above implications in both the non-singular (hereditary) case and the Goldie torsion case.

The examples we will construct are of left SU rings, rather than the right SU rings we have been discussing in this section. This is simply to allow for more readable typesetting.

4.2 The Hereditary Case

Throughout the rest of this section, we will take k to be an infinite field and R_n to be the ring of $n \times n$ matrices with entries from k on the diagonal and bottom row and zeroes elsewhere. [1]

In fact, all of the rings R_n are finite dimensional algebras and so we could use Theorem 4.0.1 to establish whether, in each case, the indecomposable modules are uniform. However, we will prove everything directly in short steps, which allows us to construct the module classes and thus illustrate how R_n becomes more complicated as n becomes larger.

One of the most useful properties of these rings is that we can always think of any left R_n-module as the direct sum of a semisimple module and a module which consists of a set of $n \times \Lambda$ matrices for some index set Λ, with left multiplication by members of R_n carried out by performing the usual matrix multiplication. This will be made clearer later on.

We will use the notation e_{ij} to refer to the matrix which has a 1 in the (i, j)th position (the i being the row number) and zeroes elsewhere.

Theorem 4.2.1 *The following hold:*

(i) R_n is hereditary, artinian, and left serial for every $n \in \mathbb{N}$.

[1] These rings come from [24] and I am grateful to the author for taking the trouble to explain some of their properties to me in private correspondence. Theorem 4.2.1 (iv) was stated in [24] and the proof of the second part of this result - namely that the ring R_n does not have finite type if $n > 4$ - is taken from the author's correspondence to me.

(ii) R_n is right serial if and only if $n \leq 2$.

(iii) R_n is left SU if and only if $n \leq 3$.

(iv) R_n is of finite type if and only if $n \leq 4$.

This Theorem, which we will prove in a series of steps, has two immediate corollaries:

Corollary 4.2.2 *R_3 is left SU but not right serial.*

Corollary 4.2.3 *R_4 is hereditary, artinian, left serial and of finite type, but not left SU.*

Proof of Theorem 4.2.1 (i) R_n is artinian, because it is a finite dimensional algebra.

It is easy to see that the left ideals of the form $R_n e_{ni}$ where $1 \leq i \leq n$ are simple, and that their sum is essential in ${}_{R_n}R_n$, so $Soc({}_{R_n}R_n)$ is the set of $n \times n$ matrices with entries from k on the bottom row and zeroes elsewhere. Similarly, $e_{ii}R_n$ where $1 \leq i \leq n-1$ and $e_{nj}R_n$ where $1 \leq j \leq n-1$ are simple right ideals of R_n whose sum is essential in R_{nR_n}, so $Soc(R_{nR_n})$ consists of the members of R_n whose bottom right entry is zero.

The elements e_{11}, e_{22}, ..., e_{nn} form an orthogonal set of idempotents of R_n and $R_n e_{nn}$ is simple as a left ideal, so e_{nn} is primitive. For $1 \leq i \leq n-1$, $R_n e_{ii}$ is the left ideal consisting of those members of R_n which are zero outside of the ith column. It is uniserial of length 2 since $Soc(R_n e_{ii}) = R_n e_{ni}$ which is simple and any element of $R_n e_{ii} \backslash R_n e_{ni}$ generates $R_n e_{ii}$. So the e_{ii} are primitive idempotents, and furthermore, R_n is left serial.

Since R_n is left artinian, the singular elements of a left R_n-module are precisely those which are annihilated on the left by $Soc({}_{R_n}R_n)$. It follows that R_n is left non-singular, and therefore hereditary by Lemma 4.1.7. □

Proof of Theorem 4.2.1 (ii) The case where $n = 1$ is trivial, since $R_1 \cong k$.

Suppose that $n \geq 2$. For $1 \leq i \leq n-1$, $e_{ii}R_n$ is simple, and therefore uniserial. $e_{nn}R_n$, however, has a socle of length $n-1$ which is also a maximal submodule, and hence is uniserial if and only if $n = 2$. □

Lemma 4.2.4 *(Probably well-known) Let R be a ring with an essential left socle (e.g. a left artinian ring). Then a non-singular left R-module M is injective $\Leftrightarrow$ every homomorphism $\theta : Soc({}_RR) \to M$ can be extended to a homomorphism $\theta' : {}_RR \to M$.*

Proof $\Rightarrow$ is straightforward.

$\Leftarrow$ Let K be a left ideal of R containing $Soc(_RR)$ and $\alpha : K \to M$ be a homomorphism such that $\alpha(Soc(_RR)) = 0$. Clearly, $\operatorname{Ker}\alpha$ is essential in K, so $\operatorname{Im}\alpha$ is singular. But M is non-singular and therefore $\alpha = 0$.

Let I be an essential left ideal of R and let $\theta : I \to M$ be a homomorphism. Of course, $Soc(_RR) \subseteq I$ and by assumption, we can extend $\theta|_{Soc(_RR)}$ to a homomorphism $\phi : R \to M$. Now consider the homomorphism $\mu = (\theta - \phi|_I) : I \to M$. Clearly, $\mu(Soc(_RR)) = 0$ and so by the previous paragraph, $\mu = 0$. Therefore ϕ extends θ and the result follows by Corollary 1.3.9. □

Lemma 4.2.5 *Suppose that $n \geq 2$. Then:*

(i) $E(R_n e_{nn})$ is isomorphic to the $n \times 1$ column matrix X with entries from k.

(ii) For $1 \leq i < n$, $R_n e_{ii}/J e_{ii}$ is an injective left R_n-module.

(iii) R_n is left SI.

Proof

(i) There is an embedding from $R_n e_{nn}$ to X defined simply by taking the bottom right entry of $R_n e_{nn}$ and putting it as the bottom entry of X with zeroes in the other positions. It is easy to show that this embedding is essential. No non-zero member of X is annihilated by $Soc(_{R_n}R_n)$, hence X is non-singular and so by Lemma 4.2.4 it is enough to show that any homomorphism from $Soc(_{R_n}R_n)$ to X extends to one from R_n.

Let $\theta : Soc(_{R_n}R_n) \to X$. Of course, $\operatorname{Im}(\theta) \subseteq Soc(X)$ and $Soc(X)$ consists of those elements of X with zeroes in all but the bottom position. So for $1 \leq i \leq n$, let $a_i \in k$ be the bottom entry of $\theta(e_{ni})$. Now define a homomorphism:

$$\theta' : R \to X : 1 \mapsto \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

For any $1 \leq j \leq n$, $\theta'(e_{nj}) = e_{nj}\theta'(1)$ which is clearly the element of X with a_j in the bottom entry and zeroes elsewhere. Hence for every $1 \leq j \leq n$, $\theta(e_{nj}) = \theta'(e_{nj})$ and so θ' extends θ.

(ii) It is clear that $R_n e_{ii}$ is isomorphic to the set of $n \times 1$ column matrices which are allowed non-zero entries only in the ith and nth positions. If we factor this by its maximal submodule $Je_{ii} = R_n e_{ni}$, this is the same as ignoring the nth position and so members are only differentiated by their entry in the ith position. Considering the left action of R_n on this factor module, only the (i,i)th entry of R_n affects the ith position of the module, and so $R_n e_{ii}/Je_{ii}$ can be thought of as a copy of k with left multiplication by R_n performed by multiplying by the $(i,i)th$ entry. Also, if we consider the submodule X_i of X consisting of those members of X with a zero in the ith position, then it is not hard to see that X/X_i can be thought of in exactly the same way. Hence $R_n e_{ii}/Je_{ii} \cong X/X_i$. Since R_n is left hereditary by Theorem 4.2.1 (i), this means that Re_{ii}/Je_{ii} is injective.

(iii) Part (ii) shows that every singular simple left R_n-module is injective and since R_n is left noetherian, every semisimple singular left R_n-module is also injective. Furthermore, since R_n is left artinian, every left R_n-module has an essential socle, so every singular left R_n-module contains an essential injective submodule and so must itself be injective. $\square$

Lemma 4.2.6 *([27]) Every left R_3-module is the direct sum of a projective module and an injective module.*

Proof Let ${}_{R_3}M$ be a module with no non-zero injective submodules. We will show that M is projective, and the result will follow by Lemma 1.3.10. Let (P, π) be a projective cover of M. Then by Theorem 1.3.4 $P = P_1 \oplus P_2 \oplus P_3$, where P_i is a direct sum of copies of $R_3 e_{ii}$.

Now suppose that $\pi(p_1 + p_2 + p_3) = 0$, where $p_i \in P_i$ and $p_3 \neq 0$. P_3 is semisimple, so there exists $T < P_3$ such that $R_3 p_3 \oplus T = P_3$. But this implies that $\operatorname{Ker}\pi + (P_1 \oplus P_2 \oplus T) = P$, contradicting the smallness of $\operatorname{Ker}\pi$ in P. Hence $\operatorname{Ker}\pi \leq P_1 \oplus P_2$, and $M = \pi(P_1 \oplus P_2) \oplus \pi(P_3)$ and $\pi(P_3) \cong P_3$ is projective. So without loss of generality, we can assume that $P = P_1 \oplus P_2$.

We can write $P = \bigoplus_{\lambda \in \Lambda} R_3 x_\lambda$, where each $R_3 x_\lambda$ is isomorphic to either $R_3 e_{11}$ or $R_3 e_{22}$. Now each $R_3 x_\lambda$ has length 2, so for any $r \in R_3$, $R_3 r x_\lambda$ has length at most 2. Suppose that

$\pi(r_1x_{\lambda_1}+r_2x_{\lambda_2}+...+r_nx_{\lambda_n})=0$ where $r_i\in R_3$ is such that $R_3r_1x_{\lambda_1}$ has length 2. Then $P=\operatorname{Ker}\pi+\bigoplus_{\lambda\in\Lambda\setminus\{\lambda_1\}}R_3x_\lambda$, which contradicts the smallness of $\operatorname{Ker}\pi$ in P. Hence for any choice of λ_1, $R_3r_1x_{\lambda_1}$ has length at most 1, i.e. is simple or zero. Therefore, $\operatorname{Ker}\pi\subseteq Soc(P)$.

If we now visualise P_1 as a set of 3-rowed matrices where each column represents a copy of R_3e_{11}, then we can have entries from k in the top and bottom rows, and the middle row must consist entirely of zeroes. Also, since each column represents a direct summand of P_1, we can only have finitely many non-zero columns in any given matrix. Let $0\neq p\in P_1$ such that $\pi(p)=0$. We know that R_3p is semisimple, so in our imaginary representation of P_1, p has non-zero entries only in the bottom row. There exists $r\in R_3$ such that $0\neq R_3rp$ is simple, so without loss of generality we can assume that R_3p is simple. If we now take the element p' of P_1 whose representation has the bottom row of p as its top row and two other rows of zeroes, then $Soc({}_{R_3}R_3)R_3p'\subseteq R_3p$ and since $Soc({}_{R_3}R_3)\leq_{ess}{}_{R_3}R_3$, this implies that $(R_3p')/(R_3p)$ is singular. Now $R_3p\subseteq\operatorname{Ker}\pi|_{R_3p'}$, so $\pi(R_3p')$ is singular and hence injective by Lemma 4.2.5 (iii). This, however, contradicts our first assumption, and so $\operatorname{Ker}\pi|_{P_1}=0$. Similarly, $\operatorname{Ker}\pi|_{P_2}=0$.

Now let $p_1\in P_1$ and $p_2\in P_2$ be such that p_1 and p_2 are not both zero and $\pi(p_1+p_2)=0$. By the last paragraph, we must have $p_1\neq0$ and $p_2\neq0$. There exists $r\in R_3$ such that $0\neq R_3rp_1$ is simple. Since $\pi(rp_1+rp_2)=0$, we must have $rp_2\neq0$, and so there exists $s\in R_3$ such that $0\neq R_3srp_2$ is simple and, as before, $srp_1\neq0$. So we can assume that R_3p_1 and R_3p_2 are non-zero and simple. Returning to our matrix representation of P_1, we can similarly represent P_2 as a set of 3-rowed matrices with entries from k, where each column represents a copy of R_3e_{22} (so must be zero in its top entry) and no matrix has infinitely many non-zero columns. Since R_3p_1 and R_3p_2 are semisimple (in fact, simple), it follows that p_1 and p_2 have non-zero entries only on the bottom row.

We now take q_1 to be the member of P_1 whose top row is the bottom row of p_1 and whose other rows are zero, and q_2 to be the member of P_2 whose middle row is the bottom row of p_2 and whose other rows are zero. On considering what happens to q_1 under left multiplication by members of R_3, it is clear that $\mathbf{l}(q_1)=R_3e_{22}\oplus R_3e_{33}$, and so $R_3q_1\cong R_3/(R_3e_{22}\oplus R_3e_{33})\cong R_3e_{11}$ which has length 2 by Corollary 1.2.10. Similarly, R_3q_2 has length 2. Now, $\pi(Soc(R_3q_1)) = \pi(R_3p_1) = R_3\pi(p_1) = R_3\pi(p_2) = \pi(R_3p_2) = \pi(Soc(R_3q_2))$ is non-zero simple and so $\pi(R_3q_1 \oplus R_3q_2)$ is uniform. Therefore, $\operatorname{Ker} \pi|_{R_3q_1 \oplus R_3q_2}$ is simple and the length of $\pi(R_3q_1 \oplus R_3q_2)$ is $4 - 1 = 3$ by Corollary 1.2.10.

Now, there is only one uniform module (up to isomorphism) of length 3 - namely $E(R_3e_{33})$ which is injective, in contradiction to our original assumption. It follows that $\operatorname{Ker} \pi = 0$ and therefore M is projective. □

Proof of Theorem 4.2.1 (iii) We already know that R_1 and R_2 are serial, so they are both left SU. Now we must show that R_3 is left SU and that R_n is not, for all other $n \in \mathbb{N}$.

Lemma 4.2.6 shows that each left R_3-module is the direct sum of a projective module and an injective module. We know by Lemma 1.3.2 that every injective left R_3-module is a direct sum of uniform modules, and by Theorem 1.3.4 that every projective left R_3-module is a direct sum of indecomposable projectives of the form R_3e_{ii}. Since each R_3e_{ii} is uniform, it follows that R_3 is a left SU ring.

To show that R_4 is not left SU, we will form a left R_4-module which is indecomposable but not uniform. We take the left R_4-module:

$$M := \left\{ \begin{bmatrix} a & 0 \\ b & b \\ 0 & c \\ d & e \end{bmatrix} : a, b, c, d, e \in k \right\}$$

with the normal left matrix action of R_4. Note that $Soc(M) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ k & k \end{bmatrix}$ which has length 2, so if M is decomposable, then $M = U \oplus V$, where U and V are uniform.

The problem with the decomposition is that one of the uniform submodules, U say, must have non-zero entries from 2 of the top 3 rows for the whole of M to be present in $U \oplus V$.

Suppose we have a pair of elements of U with a non-zero entry on the top row in one, and a non-zero entry on the second row in the other, i.e.

$$\begin{bmatrix} a & 0 \\ b & b \\ 0 & d \\ e & f \end{bmatrix}, \begin{bmatrix} g & 0 \\ h & h \\ 0 & j \\ l & m \end{bmatrix} \in U \quad \text{where} \quad a, h \neq 0$$

Then:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & b \\ 0 & d \\ e & f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \in U$$

and

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & h^{-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} g & 0 \\ h & h \\ 0 & j \\ l & m \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \in U$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \in U$$

i.e. $Soc(M) \subseteq U$, so that $U \leq_{ess} M$, which is clearly false. Similar arguments show that if U has a pair of elements with non-zero entries in the 1st and 3rd rows respectively, or with non-zero entries in the 2nd and 3rd respectively, then we arrive at the same contradiction. Furthermore, in the case where $n \geq 5$, we can construct a module which is indecomposable but not uniform by taking M and inserting suitably many rows of zeroes. □

Proof of Theorem 4.2.1 (iv) R_1, R_2 and R_3 are left SU and therefore of finite type by Lemma 4.1.7. Now we will show that R_4 is also of finite type. We will do this by showing that the indecomposable left R_4-modules have bounded length.

Let M be an indecomposable left R_4-module. R_4 is left SI by Lemma 4.2.5, so $Z(M)$ must be a direct summand of M and hence M is either singular or non-singular. If M

is singular, then M must be semisimple and therefore simple, so has length 1. We can therefore suppose that M is non-singular.

Now $Soc(M)$ is an essential submodule of M, so $Soc(M) \subseteq M \subseteq E(Soc(M))$. Since M is non-singular, $Soc(M) \cong Re_{44}^{(\Lambda)}$ for some index set Λ. This means that we can represent $E(Soc(M))$ as a direct sum of $|\Lambda|$ matrices of the form X as described in Lemma 4.2.5. This can be more simply thought of as the set of $4 \times |\Lambda|$ matrices over k which have only finitely many non-zero columns. $Soc(M)$ consists of those members of $E(Soc(M))$ which are zero outwith the bottom row.

If we now look at the rows of our matrix representation of M, and consider the left action of members of R_4 (in particular e_{ii} and e_{4i}) on them, it is clear that the different 4th rows we obtain form a $|\Lambda|$-dimensional space over k and that the top 3 rows give us subspaces of this. We will call the vector spaces we obtain from the 4 rows V_1, V_2, V_3 and V, respectively.

A subset of M written this way is a submodule if and only if the rows all form vector subspaces such that those of the top 3 are all contained in the bottom one. This means that M has a non-trivial module decomposition if and only if there exists a non-trivial vector space decomposition $V = V' \oplus V''$ such that $V_i = (V_i \cap V') \oplus (V_i \cap V'')$ for $i = 1, 2, 3$. Since we have already assumed that M is indecomposable, it follows that there cannot be such a vector space decomposition.

First of all, we will assume that V_1, V_2 and V_3 are not all 0 and not all V, since in these cases either V has dimension 1, in which case $M \hookrightarrow X$ and therefore M has length at most 4 or V has dimension at least 2, and any non-trivial decomposition $V = V' \oplus V''$ will produce a module decomposition.

Suppose that $V_1 + V_2 + V_3 \neq V$. Then put $V' = V_1 + V_2 + V_3$ and V'' as a complement of V' in V, and we have a decomposition. So we can assume that $V_1 + V_2 + V_3 = V$.

Suppose $V_1 \cap V_2 \cap V_3 \neq 0$. Then by putting $V' = V_1 \cap V_2 \cap V_3$ and setting V'' to be any vector space complement of V' in V, we form a decomposition. So we can suppose that $V_1 \cap V_2 \cap V_3 = 0$.

Suppose that $V_1 + V_2 = 0$. Then $V_1 = V_2 = 0$ and $V_3 = V$, so either V has dimension 1 or any non-trivial decomposition of V will suffice, as before. Hence we can assume $V_1+V_2 \neq 0$, $V_1 + V_3 \neq 0$ and $V_2 + V_3 \neq 0$.

Suppose that $V_1 + V_2 \neq V$. Then there exists a non-zero subspace U of V_3 such that $(V_1 + V_2) \oplus U = V$. This gives a decomposition $V' = V_1 + V_2$ and $V'' = U$. So we can assume that $V = V_1 + V_2 = V_1 + V_3 = V_2 + V_3$.

Suppose that $V_1 \cap V_2 = V$. Then $V_1 = V_2 = V$ and $V_3 = 0$, so again it follows that either V has dimension 1 or any non-trivial decomposition of V will work on M. Hence we can assume that $V_1 \cap V_2 \neq V$.

Suppose that $V_1 \cap V_2 \neq 0$. $V = (V_1 \cap V_2) \oplus V_3 \oplus T$ for some T, and we can make a decomposition using $V' = V_1 \cap V_2$ and $V'' = V_3 \oplus T$. So we can assume that $V_1 \cap V_2 = V_1 \cap V_3 = V_2 \cap V_3 = 0$.

Now, the only case remaining is when $V = V_1 \oplus V_2 = V_1 \oplus V_3 = V_2 \oplus V_3$. Take S to be a 1-dimensional subspace of V_1, and put $V_1 = S \oplus X$. Let $\pi_{V_2} : V \to V_2$ and $\pi_{V_3} : V \to V_3$ be the projections arising from the decomposition $V = V_2 \oplus V_3$. It is clear that π_{V_2} and π_{V_3} map V_1 isomorphically onto V_2 and V_3, respectively. So $V_2 = \pi_{V_2}(S) \oplus \pi_{V_2}(X)$ and $V_3 = \pi_{V_3}(S) \oplus \pi_{V_3}(X)$. Now we can put $V' = \pi_{V_2}(S) \oplus \pi_{V_3}(S)$ and $V'' = \pi_{V_2}(X) \oplus \pi_{V_3}(X)$. Since $\pi_{V_2}(S)$ is non-zero, for M to be indecomposable we must have that $\pi_{V_2}(X) = \pi_{V_3}(X) = 0$. But since the restriction of π_{V_2} to V_1 is an isomorphism, then this implies that $X = 0$. Hence $V_1 = S$ is simple, V_2 and V_3 are also simple, and V has length 2.

Thus, in any case, M has length at most 5 and so R_4 is of finite type by Theorem 4.1.4.

Now let $n > 4$. We will show that there are infinitely many isomorphism classes of indecomposable left R_n-modules of length 6 or less. Consider the projective left ideal $P = \bigoplus_{i=1}^{4} R_n e_{ii}$, which has length 8. We will define an equivalence relation $\sim$ on the class of length 2 semisimple submodules of P, by saying $S \sim T$ if and only if there is an automorphism of P which maps S onto T.

We know that for any $1 \leq i \leq 4$, $Soc(R_n e_{ii})$ is small in $R_n e_{ii}$, since $R_n e_{ii}$ is uniserial. Using Lemma 1.4.4 (ii), it follows that $Soc(P)$ is small in P.

Now suppose that S and T are length 2 semisimple submodules of P such that there is an isomorphism θ from P/S to P/T. If π_S and π_T are the canonical homomorphisms from

P to P/S and P/T respectively, then $\operatorname{Ker}\pi_T = T$ and $\operatorname{Ker}(\theta\pi_S) = S$ are both semisimple, so (P, π_T) and $(P, \theta\pi_S)$ are both projective covers of P/T. By Lemma 1.4.7, there exists an automorphism ϕ of P such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, hook, two heads, "\phi"] \arrow[d, "\pi_S"'] & P \arrow[d, "\pi_T"] \\ P/S \arrow[r, hook, two heads, "\theta"'] & P/T \end{tikzcd}$$

Clearly, $\pi_T\phi(S) = \theta\pi_S(S) = 0$, so $\phi(S) \subseteq T$. Since ϕ is an isomorphism, it follows that $\phi(S) = T$ and so $S \sim T$. Therefore, to prove the existence of infinitely many isomorphism classes of the form P/S where S is semisimple of length 2, it is enough to show that there are infinitely many equivalence classes produced by $\sim$.

If we let $p = e_{11} + e_{22} + e_{33} + e_{44}$, then it is not difficult to see that $P = R_n p$. It follows that every endomorphism of P is completely defined by its action on p. Recall that the elements of P only contain non-zero entries in their first 4 columns. We can therefore cut off the right-hand columns which are always zero and visualise P as the set of $n \times 4$ matrices with entries from k on the leading diagonal and bottom row, and zeroes elsewhere. The left action of R_n on P is still ordinary left multiplication. Now consider what happens if we multiply a member of P on the right by an element of D, the ring of 4×4 matrices with entries from k on the diagonal and zeroes elsewhere. The resultant matrix will be a member of P, so each element of D defines an endomorphism of P. Furthermore, it is easy to see that if $d \in D$, then $pd = 0$ if and only if $d = 0$. Therefore there is a ring monomorphism from D to the endomorphism ring of P.

Now let λ be an endomorphism of P. Suppose that:

$$\lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & y_4 \\ \vdots & \vdots & \vdots & \vdots \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$$

for some $y_i, z_i \in k$ $(1 \leq i \leq 4)$. Then on left multiplication by $e_{11} + e_{22} + e_{33} + e_{44}$, we see that:

$$\lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & y_4 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So in fact λ is equal to the endomorphism of P resulting from right multiplication by the element of D whose diagonal elements are y_1, y_2, y_3 and y_4 respectively. D is therefore isomorphic to the endomorphism ring of P and we can denote each endomorphism of P by $\lambda_{\alpha\beta\gamma\delta}$, where α, β, γ and δ are the diagonal elements of the appropriate member of D.

Let α, β, γ and δ be non-zero elements of k. Then $\lambda_{\alpha\beta\gamma\delta}\lambda_{\alpha^{-1}\beta^{-1}\gamma^{-1}\delta^{-1}} = \lambda_{\alpha^{-1}\beta^{-1}\gamma^{-1}\delta^{-1}}\lambda_{\alpha\beta\gamma\delta} = 1_D$, so $\lambda_{\alpha\beta\gamma\delta}$ is an isomorphism. Also, $\lambda_{0\beta\gamma\delta}$ sends the element of P with 1_k in the top-left entry and zeroes elsewhere to zero, so $\lambda_{0\beta\gamma\delta}$ is a non-isomorphism. Similarly, $\lambda_{\alpha\beta\gamma\delta}$ is a non-isomorphism if any of β, γ or δ are zero, so the automorphisms of P are precisely those members of D with 4 non-zero entries.

The length 2 semisimple submodules of P are precisely the 2-dimensional k-vector subspaces of $Soc(P)$, $Soc(P)$ being those members of P which are zero except in the 4th row. We will define a class of these two dimensional subspaces as follows:

$$S_\omega = \{(a, b, a+b, a+\omega b) : a, b \in k\}$$

where $\omega \in k$. Now $S_\chi \sim S_\psi$ if and only if there is an automorphism $\lambda_{\alpha\beta\gamma\delta}$ of P such that $S_\chi\lambda_{\alpha\beta\gamma\delta} = S_\psi$. Since $\lambda_{\alpha\beta\gamma\delta}$ has to be an isomorphism, we know that α, β, γ and δ must all be non-zero.

$$\begin{aligned} S_\chi\lambda_{\alpha\beta\gamma\delta} &= \{(a, b, a+b, a+\chi b) : a, b \in k\}\lambda_{\alpha\beta\gamma\delta} \\ &= \{(\alpha a, \beta b, \gamma a + \gamma b, \delta a + \delta\chi b) : a, b \in k\} \\ &= \{(\alpha a, \beta b, (\gamma/\alpha)\alpha a + (\gamma/\beta)\beta b, (\delta/\alpha)\alpha a + (\delta\chi/\beta)\beta b) : a, b \in k\} \end{aligned}$$

$$= \{(c, d, (\gamma/\alpha)c + (\gamma/\beta)d, (\delta/\alpha)c + (\delta\chi/\beta)d : c, d \in k\}$$

For this to be equal to S_ψ we must have $\gamma = \alpha = \beta = \delta$ and hence $\chi = \psi$. So, each S_χ is in a separate $\sim$-equivalence class and there must therefore be infinitely many of these classes. $\square$

4.3 The Goldie Torsion Case

In this section, we will produce more examples, this time of Goldie torsion rings, which show that the implications at the end of Section 4.1 cannot be reversed. We will be constructing rings similar to those in the last section, so we can omit some of the more detailed explanation.

Example 4.3.1 Take $R := \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{bmatrix}$ where multiplication of two elements of R is carried out by considering $2\mathbb{Z}_4$ as a $\mathbb{Z}_2$-module in the obvious way. It is easy to verify that R is left and right artinian (in fact, finite), left but not right serial, and that $Soc(_RR) = \begin{bmatrix} 0 & 0 \\ 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \end{bmatrix} = J(R)$. Now, $(Soc(_RR))^2 = 0$, so $Soc(_RR)$ is singular and hence $Z_2(_RR) = R$, i.e. R is left Goldie torsion.

Consider the left R-module Re_{11}/Je_{11}, which we will denote $X = \begin{bmatrix} \mathbb{Z}_2 \end{bmatrix}$. Let I be a left ideal of R and let $\theta : {}_RI \to X$. Suppose that $a, d \in \mathbb{Z}_2$, $b \in 2\mathbb{Z}_4$ and $c \in \mathbb{Z}_4$ such that $\theta\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right) = \begin{bmatrix} d \end{bmatrix}$. Then $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in I$, and:

$$\theta\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\theta\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} d \end{bmatrix} = \begin{bmatrix} d \end{bmatrix}$$

So:

$$\theta\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right) = \theta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)$$

This shows that the image of an element of I under the homomorphism θ depends only on its top-left matrix entry. Since this top-left entry, and similarly the top entry of X, can only be 0 or 1, it follows that either θ is the zero mapping or it maps elements with a 1 in the top-left to the element $\begin{bmatrix} 1 \end{bmatrix}$ of X. In the former case, θ can be extended to the zero mapping from ${}_RR$ to X and in the latter to the homomorphism which maps 1_R to the non-zero element of X. Therefore X is injective.

Let $Y = \begin{bmatrix} \mathbb{Z}_2 \\ \mathbb{Z}_4 \end{bmatrix}$. By Lemma 1.3.7, in order to show that ${}_RY$ is injective, it is enough to show that it is Re-injective and $R(1-e)$-injective. The left R-module Re has only one proper non-zero submodule, namely $\begin{bmatrix} 0 & 0 \\ 2\mathbb{Z}_4 & 0 \end{bmatrix}$. This has only one non-zero member, and the sum of this member with itself is zero. Therefore, there is only one non-zero homomorphism ϕ from this module to Y, for which $\phi\left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. We can extend ϕ to a homomorphism $\phi' : Re \to Y : \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Similarly, the left R-module $R(1-e)$ has only one non-trivial submodule, namely $\begin{bmatrix} 0 & 0 \\ 0 & 2\mathbb{Z}_4 \end{bmatrix}$, whose only homomorphism to Y is $\theta : \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. This can be extended to a homomorphism $\theta' : R(1-e) \to Y : \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore ${}_RY$ is injective.

${}_RY$ is artinian and $Soc({}_RY) = \begin{bmatrix} 0 \\ 2\mathbb{Z}_4 \end{bmatrix}$ is simple, so Y is uniform. Furthermore, $Soc({}_RY) \cong Re_{22}/Je_{22}$, so ${}_RY \cong E(Je_{22}/Re_{22})$. Also,

$$\begin{bmatrix} \mathbb{Z}_2 \\ \mathbb{Z}_4 \end{bmatrix} \supset \begin{bmatrix} 0 \\ \mathbb{Z}_4 \end{bmatrix} \supset \begin{bmatrix} 0 \\ 2\mathbb{Z}_4 \end{bmatrix} \supset \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is a chain with simple factors, so the length of ${}_RY$ is 3. Since every non-zero left R-module has non-zero socle and the two simple left R-modules have injective hulls of lengths 1 and 3, it follows that every uniform left R-module of length 3 is injective.

Let M be a left R-module - we will show that M is a direct sum of uniforms. By Lemma 1.3.10, M is the direct sum of an injective module and a module with no non-zero injective submodule. The injective summand is of course a direct sum of uniform modules, so we can assume without loss of generality that M contains no non-zero injective submodules.

Let (P, π) be a projective cover for M. Then $P = P_1 \oplus P_2$, where P_i is a direct sum of copies of Re_{ii}. As in the case of R_3 in the previous section, $\operatorname{Ker} \pi$ must be semisimple, in order for it to be small in P. Suppose that there exist $p_1 \in P_1$ and $p_2 \in P_2$ such that $\pi(p_1) = \pi(p_2) \neq 0$. We know that Rp_1 and Rp_2 are semisimple, and we can assume without loss of generality that $R(\pi(p_1))$ is simple.

As with the rings in the previous example, we can represent a direct sum of $|\Lambda|$ copies of P_1 as the set of $2 \times |\Lambda|$ matrices whose top rows contain entries from $\mathbb{Z}_2$ and whose bottom rows contain entries from $2\mathbb{Z}_4$, such that overall there are only finitely many non-zero entries in a given element. Similarly a direct sum of copies of P_2 can be represented as the set of $2 \times |\Lambda|$ matrices with a top row of zeroes and a bottom row of entries from $\mathbb{Z}_4$, only finitely many of which can be non-zero.

Now, we take p_1' to be the element of P_1 with a bottom row of zeroes and a top row which is the bottom row of p_1 with 2's replaced by 1's. In the same fashion, we take p_2' to be the element of P_2 whose bottom row is the bottom row of p_2 where 2's have been replaced by 1's (the top row, of course, must be zero by definition). Then $\pi(Rp_1') \cong Rp_1'$ and $\pi(Rp_2') \cong Rp_2'$ are isomorphic to Re_{11} and Re_{22} respectively, so are both uniform modules of length 2. $\operatorname{Ker} \pi|_{Rp_1'+Rp_2'}$ is simple, so $\pi(Rp_1'+Rp_2')$ is a uniform module of length 4-1=3, and so must be injective, contradicting our original assumption. Hence $\operatorname{Ker} \pi = (\operatorname{Ker} \pi \cap P_1) \oplus (\operatorname{Ker} \pi \cap P_2)$, so $M \cong (P_1/(\operatorname{Ker} \pi \cap P_1)) \oplus (P_2/(\operatorname{Ker} \pi \cap P_2))$ and we can consider the two summands separately.

Firstly, the submodules of P_1 are the subsets whose top and bottom rows are $\mathbb{Z}_2$ vector spaces such that the vector space of the top row is a subspace of the bottom row (if we equate the 1's of the top row with the 2's of the bottom row). Hence we can form two submodules of P_1:

(i) K, whose top and bottom rows are arbitrarily chosen bottom rows of $\operatorname{Ker}\pi \cap P_1$ (1's being replaced with 2's for the top row),

(ii) L, whose top and bottom rows are arbitrarily chosen members of a vector space complement of the bottom row of $\operatorname{Ker}\pi \cap P_1$ in the vector space formed by the bottom row of P_1 (1's being replaced with 2's for the top row of L).

Now, $\operatorname{Ker}\pi|_{P_1} = Soc(K)$, and $\pi(P_1) = \pi(K) \oplus \pi(L)$ which is the direct sum of a semisimple module and a projective module. Hence $\pi(P_1)$ is a direct sum of uniforms.

Secondly, the submodules of P_2 are simply the subsets which are $\mathbb{Z}_4$-modules. $\mathbb{Z}_4$ is a uniserial ring, and so SU, and hence any $\pi(P_2)$ is a direct sum of uniform modules.

Example 4.3.2 Let $S = \begin{bmatrix} \mathbb{Z}_2 & 0 & 0 \\ 0 & \mathbb{Z}_2 & 0 \\ 2\mathbb{Z}_4 & 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{bmatrix}$. As in Example 4.3.1, S is Goldie torsion.

Define a left S-module:

$$M := \left\{ \begin{bmatrix} a & 0 \\ 0 & b \\ c & d \end{bmatrix} : a, b \in \mathbb{Z}_2,\ c, d \in \mathbb{Z}_4,\ c, d \text{ are both odd or both even} \right\}$$

Then, $Soc(M)$ has length 2, and as in the case of R_3, if we try to find a decomposition of M, then one of the summands must turn out to be non-uniform - a contradiction.

Hence S is not left SU.

4.4 Summary

Theorem 4.0.1 allows us to identify finite dimensional algebras whose indecomposable left modules are uniform. We could have used this result on R_3, for example, and saved ourselves some work. Doing things longhand, however, tells us not only that R_3 is an SU ring, but also shows what the uniform left R_3-modules are. This extra information allows us to look at some of the similarities between the rings R_n for different values of n. Notice that for every value of n there is one simple indecomposable projective left R_n-module and $n-1$

uniserial indecomposable projective left R-modules of length 2; and $n-1$ simple projective right R_n-modules and one projective right R_n-module of length n. Since all of the left indecomposable projectives of any R_n have length 1 or 2, we must have $J(R_n)^2 = 0$ for every n (it is also easy to see this directly). The indecomposable injectives (in fact all the uniform left R_n-modules) are cyclic for every n. R_n is always left hereditary and left SI.

The family $\{R_n\}$ of rings illustrates how, even in the case of rings which can be simply described, we need to know the left *and* right ideal structures before we can begin to construct its classes of modules. In fact, even though we are armed with all the information about the left and right ideals, and this seems to be relatively straightforward, we only need to look as far as R_4 before we start to find rings with infinitely many isomorphism classes of indecomposable modules.

4.5 Questions

Some of the conditions required in Theorem 4.1.13 are very strong. Maybe in some of the cases where multiple conditions are required, one or more of these may be implied be the others. For instance we might ask the following:

- If every (finitely generated) indecomposable right R-module is uniform, then does it follow that R is right pure-semisimple?

- If R is right artinian and every indecomposable right R-module is uniform, then does it follow that every indecomposable injective right R-module has finite length? (Note that it is quite hard to construct artinian rings whose indecomposable injectives do not all have finite length - see [25] Example 6.3.15 for one example.)

- If R is right artinian and every non-zero right R-module has a non-zero uniform direct summand, then does it follow that every indecomposable injective right R-module has finite length?

Also there are other questions arising from the theory of this chapter:

- Can Corollary 4.1.10 be extended to the case where the left and right Goldie dimensions are equal, but the ring is not right non-singular?

- Lemma 4.1.7 (vii) shows a way of decomposing a right SU ring into a matrix ring. It might be interesting to try and find when we can work in the opposite direction, for example, given a pair of right SU rings S and T, for which bimodules ${}_SM_T$ is the ring given by $\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ a right SU ring? We know that if $M = 0$, then the matrix ring is right SU by Lemma 4.1.7 (iv).

Chapter 5

SU Modules

In this chapter, we will consider the structure of a module M which has the property that every module in $\sigma[M]$ is a direct sum of uniforms.

We have seen in earlier chapters that the class of singular right modules of a ring can be described in the form $\sigma[M]$ for some module M_R. Therefore, we will also be addressing the problem of when every singular right module of a ring is a direct sum of uniform modules in a series of corollaries and extra results.

We could, of course, attempt to make similar classifications of rings for any of the standard torsion functors of a ring - for example : for which right artinian rings R is every small right R-module a direct sum of uniforms (see Example 3.1.10)? In the case of semisimple modules, these are obviously always direct sums of uniforms, so the question does not arise.

Recall from Chapter 3 that it was easier to decide whether $\sigma[M]$ was essentially closed or not in the case where M was finitely annihilated. The same is true when deciding whether a module has the SU property or not, as we shall see.

In the second section, we shall see that a module over a commutative ring has the SU property if and only if it satisfies some other, already widely studied conditions.

5.1 SU modules over general rings

Definition 5.1.1 *A right R-module M_R is said to be* **finitely presented** *if:*

(i) M_R is finitely generated,

and (ii) whenever $K_R \leq N_R$ are modules such that N_R is finitely generated and $N/K \cong M$, then K_R is finitely generated.

Definition 5.1.2 *Let M_R be a module such that every module in $\sigma[M]$ is a direct sum of uniform modules. Then we will say that M_R is an* **SU module.** *Let S be a ring whose singular right modules are all direct sums of uniform modules. Then we will call S a* **right SSU ring.**

Definition 5.1.3 *A module M_R is said to be* **pure-semisimple** *if every module in $\sigma[M]$ is a direct sum of finitely presented modules.*

Lemma 5.1.4 *Submodules and factor modules of $\left\{ \begin{array}{c} \textit{pure-semisimple} \\ SU \end{array} \right\}$ modules are also $\left\{ \begin{array}{c} \textit{pure-semisimple} \\ SU \end{array} \right\}$.*

Proof This is trivial. □

Lemma 5.1.5 *([15] Corollary 4.7) A module M is pure-semisimple $\Leftrightarrow$ every module in $\sigma[M]$ is a direct sum of indecomposable modules.*

Corollary 5.1.6 *(Well-known) R is a right pure-semisimple ring $\Leftrightarrow$ R_R is a pure-semisimple module.*

Proof This follows by the definitions and Lemma 5.1.5. □

Lemma 5.1.7 *(Well-known) Let M_R be a pure-semisimple module. Then M_R is locally noetherian and every module in $\sigma[M]$ is locally noetherian.*

Proof Let $E_R \in \sigma[M]$ be an M-injective module. Then by Lemma 5.1.5, E is a direct sum of indecomposable modules, so M is locally noetherian by [52], 27.5. It follows by [52], 27.3 that every module in $\sigma[M]$ is locally noetherian. $\square$

Corollary 5.1.8 *A module M is pure-semisimple $\Leftrightarrow$ every module in $\sigma[M]$ is a direct sum of noetherian modules.*

Proof $\Rightarrow$ This follows by the definition of pure-semsimple and by Lemma 5.1.7.

$\Leftarrow$ This follows by Lemma 1.2.7 and Lemma 5.1.5. $\square$

Corollary 5.1.9 *A module M is SU $\Leftrightarrow$ M is pure-semisimple and every indecomposable noetherian module in $\sigma[M]$ is uniform.*

Proof $\Rightarrow$ This is trivial.

$\Leftarrow$ This follows by Corollary 5.1.8 and Lemma 1.2.7. $\square$

Lemma 5.1.10 *Let R be a ring and M be a right R-module such that $R/\mathbf{r}(M)$ is a right $\left\{\begin{array}{c}\textit{pure-semisimple}\\ SU\end{array}\right\}$ ring. Then M_R is $\left\{\begin{array}{c}\textit{a pure-semisimple}\\ \textit{an } SU\end{array}\right\}$ module.*

Proof Let $N \in \sigma[M]$. Then by Lemma 3.2.12 and Lemma 3.2.15, $N\mathbf{r}(M) = 0$. Thus N is a right $R/\mathbf{r}(M)$-module and hence is a direct sum of $\left\{\begin{array}{c}\text{indecomposables}\\ \text{uniforms}\end{array}\right\}$ as both an $R/\mathbf{r}(M)$-module and an R-module. $\square$

Lemma 5.1.11 *Let R be a ring and M_R be $\left\{\begin{array}{c}\textit{a pure-semisimple}\\ \textit{an } SU\end{array}\right\}$ module. If I is an ideal of R such that $R/I \in \sigma[M]$, then R/I is a right $\left\{\begin{array}{c}\textit{pure-semisimple}\\ SU\end{array}\right\}$ ring.*

Proof Every R/I-module N is also an R-module with $NI = 0$. Therefore by Lemma 3.2.12, $N_R \in \sigma[M]$ and both N_R and $N_{R/I}$ are direct sums of $\left\{\begin{array}{c}\text{indecomposables}\\ \text{uniforms}\end{array}\right\}$. Since N was chosen arbitrarily, R/I is a right $\left\{\begin{array}{c}\text{pure-semisimple}\\ \text{SU}\end{array}\right\}$ ring. □

Theorem 5.1.12 *Let R be a ring and M be a finitely annihilated module. Then M is* $\left\{\begin{array}{c}\textit{a pure-semisimple}\\ \textit{an SU}\end{array}\right\}$ *module* $\Leftrightarrow$ $R/\mathbf{r}(M)$ *is a right* $\left\{\begin{array}{c}\textit{pure-semisimple}\\ \textit{SU}\end{array}\right\}$ *ring.*

Proof $\Rightarrow$ Since M_R is finitely annihilated, it follows by Lemma 3.2.15 that $R/\mathbf{r}(M) \in \sigma[M]$. Hence by Lemma 5.1.11, $R/\mathbf{r}(M)$ is a right $\left\{\begin{array}{c}\text{pure-semisimple}\\ \text{SU}\end{array}\right\}$ ring.

$\Leftarrow$ This follows by Lemma 5.1.10. □

Note that we need the finite annihilator property for Theorem 5.1.12, as the following example shows.

Example 5.1.13 Taking $\mathbb{Z}$ as our ring and $\mathcal{P}$ to be the set of prime natural numbers, if we put $M = \bigoplus_{p\in\mathcal{P}} \mathbb{Z}_p$, then $\sigma[M] = \{\text{semisimple } \mathbb{Z}\text{-modules}\}$. We know that every semisimple module is a direct sum of uniform modules and so M is an SU module. However, $\mathbf{r}(M) = 0$ and $\mathbb{Z}/\mathbf{r}(M) \cong \mathbb{Z}$ is not an SU ring since it is not artinian.

Lemma 5.1.14 *Let R be a ring and M_R a module such that $\sigma[M]$ is the class of singular right R-modules (such an M exists by Example 3.1.9 (b)). Then the following are equivalent:*

(i) M is finitely annihilated.

(ii) $Soc(R_R) \leq_{ess} R_R$.

(iii) $\sigma[M] = \sigma[R/Soc(R_R)]$.

Proof (i) $\Rightarrow$ (ii) By Lemma 3.1.14 (c), there exists a finite set $I_1, I_2, ..., I_n$ of right ideals of R such that $R/I_j \in \sigma[M]$ for every $1 \leq j \leq n$ and $\bigcap_{1\leq j\leq n} I_j = \mathbf{r}(M)$. Since each R/I_j is singular, it follows by Lemma 2.2.2 that each I_j is an essential right ideal of R. Therefore, their intersection is an essential right ideal of R and so $\mathbf{r}(M)$ is essential in R_R.

Now, by Lemma 3.1.14 (b), $\mathbf{r}(M) = \bigcap\{K \lhd_r R : K_R \leq_{ess} R_R\}$. It follows from Lemma 1.1.6 that $\mathbf{r}(M) = Soc(R_R)$.

(ii) $\Rightarrow$ (iii) Clearly, $R_R/Soc(R_R)$ is singular and so $\sigma[R_R/Soc(R_R)] \subseteq \sigma[M]$. Conversely, if X_R is singular, then $X.Soc(R_R) = 0$ and so $X \in \sigma[R_R/Soc(R_R)]$ by Lemma 3.2.12.

(iii) $\Rightarrow$ (i) $R/Soc(R_R)$ is clearly finitely annihilated with respect to the set $\{1+Soc(R_R)\}$. It follows from Corollary 3.1.15 that M is finitely annihilated. $\square$

Corollary 5.1.15 *Let R be a ring such that $Soc(R_R) \leq_{ess} R_R$. Then R is a right SSU ring $\Leftrightarrow$ $R/Soc(R_R)$ is a right SU ring.*

Proof Since $Soc(R_R)$ is essential in R_R, the class of singular modules is equal to the class $\sigma[R/Soc(R_R)]$, by Lemma 5.1.14. Clearly then, R is right SSU if and only if $R/Soc(R_R)$ is an SU module. $R/Soc(R_R)$ is finitely annihilated (by $\{1 + Soc(R_R)\}$) and so we can apply Theorem 5.1.12 for the result. $\square$

To end this section, we have some extra results concerning Morita invariance. As with Theorem 4.1.12, these stand alone and can be ignored without detracting from the reader's understanding of later results.

Lemma 5.1.16 *Let R and S be Morita equivalent rings with an equivalence $F : Mod-R \to Mod-S$. Then $N \in \sigma[M_R] \Leftrightarrow F(N) \in \sigma[F(M)_S]$.*

Proof By [1] Propositions 21.4 and 21.5, F preserves direct sums, injections and surjections.

$\Rightarrow$ Suppose that $N \in \sigma[M_R]$. Then there exists an index set Λ and a module $K_R \leq M_R^{(\Lambda)}$ such that $N_R \hookrightarrow M_R^{(\Lambda)}/K_R$. Hence $F(N)_S \hookrightarrow F(M)_S^{(\Lambda)}/F(K)_S$ and the result follows.

$\Leftarrow$ This can be shown be repeating the steps in "$\Rightarrow$" using the corresponding inverse equivalence $G : Mod - S \to Mod - R$. $\square$

Lemma 5.1.17 *Let R and S be Morita equivalent rings with an equivalence $F : Mod-R \to Mod - S$. Then M_R is an SU module $\Leftrightarrow$ $F(M)_S$ is an SU module.*

Proof $\Rightarrow$ Let $G : Mod-S \to Mod-R$ be the corresponding inverse equivalence. Suppose that $N_S \in \sigma[F(M)_S]$. Then by Lemma 5.1.16, $G(N)_R \in \sigma[M_R]$, and so $G(N)_R$ is a direct sum of uniform modules. Using the same reasoning as in Theorem 4.1.12, $FG(N)_S \cong N_S$ is also a direct sum of uniform modules.

$\Leftarrow$ This is proved by carrying out "$\Rightarrow$" in reverse. □

Lemma 5.1.18 *Let R and S be Morita equivalent rings with an equivalence $F : Mod-R \to Mod-S$ and let X_R and M_R be right R-modules. Then:*

(i) X_R is singular $\Leftrightarrow$ $F(X)_S$ is singular.

(ii) $\sigma[M_R]$ is the class of singular right R-modules $\Leftrightarrow$ $\sigma[F(M)_S]$ is the class of singular right S-modules.

Proof (i) It is clearly enough to prove the implication in one direction. Suppose that X_R is singular. Then there exists an exact sequence:

$$0 \to K_R \to P_R \to X_R \to 0$$

where P_R is projective and the embedding $K_R \hookrightarrow P_R$ is essential. By [1] Propositions 21.4 and 21.6 (5), the sequence:

$$0 \to F(K)_S \to F(P)_S \to F(X)_S \to 0$$

is also exact with an essential embedding $F(K)_S \hookrightarrow F(P)_S$, so $F(X)_S$ must be singular.

(ii)Again, it is enough to prove the implication in one direction. Suppose that $\sigma[M_R]$ is the class of singular right R-modules. By part (i), $F(M)_S$ must be a singular module and so $\sigma[F(M)_S]$ is contained in the class of singular right S-modules. Also, if Y_S is a module which is singular, then $G(Y)_R \in \sigma[M_R]$, where G is the corresponding inverse equivalence, so $Y_S \in \sigma[F(M)_S]$ by Lemma 5.1.16. Therefore $\sigma[F(M)_S]$ is exactly the class of singular right S-modules. □

Corollary 5.1.19 *The SSU property of a ring is Morita invariant.*

Proof This follows by Lemma 5.1.17 and Lemma 5.1.18 (ii). □

5.2 SU modules over commutative rings

Recall Corollary 4.1.9 stated that a commutative ring is SU if and only if it is serial. In fact this result can be generalised as follows:

Theorem 5.2.1 *(Corollary 4.1.9 & [20] Theorem 4.3) Let R be a commutative ring. Then R is pure-semisimple $\Leftrightarrow$ R is serial $\Leftrightarrow$ R is SU.*

Definition 5.2.2 *A uniserial module M is said to be* **homo-uniserial** *if whenever A, B, C and D are submodules of M such that A and C are maximal submodules of B and D respectively, then $B/A \cong D/C$.*

Lemma 5.2.3 *(Well-known) Let R be a commutative serial ring. Then every R-module is a direct sum of homo-uniserial modules of finite length.*

Proof By [52] 55.16, every R-module is a direct sum of uniserial modules, so R is of finite representation type and hence every indecomposable R-module has finite length. Let X be a uniserial R-module. Then, since $R = e_1R \oplus ... \oplus e_nR$, where each e_i is a primitive idempotent of R, we can create a module decomposition $X = Xe_1R \oplus ... \oplus Xe_nR$. Since X is uniserial, it must be the case that $X = Xe_iR$ for some $1 \leq i \leq n$. Now, X_R has the same submodule structure as X_{e_iR} and e_iR is a local serial ring, so X_{e_iR} is homo-uniserial. □

Theorem 5.2.4 *Let M_R be a module over a commutative ring R. Then the following are equivalent:*

(i) M_R is SU.

(ii) M_R is pure-semisimple.

(iii) Every module in $\sigma[M]$ is a direct sum of homo-uniserial modules of finite length.

(iv) Every module in $\sigma[M]$ is a direct sum of homo-uniserial modules.

Proof (i) $\Rightarrow$ (ii) This follows by Lemma 5.1.5.

(ii) $\Rightarrow$ (iii) Let X be an indecomposable module in $\sigma[M]$. Then X is finitely generated and so X must be finitely annihilated. It follows that $R/\mathrm{r}(X)$ is an SU ring by Theorem

5.1.12, hence $R/\mathbf{r}(X)$ is serial by Theorem 5.2.1. By Lemma 5.2.3, $X_{R/\mathbf{r}(X)}$ and therefore X_R is homo-uniserial of finite length.

(iii) $\Rightarrow$ (iv) This is trivial.

(iv) $\Rightarrow$ (i) This follows by Lemma 5.1.5. $\square$

Lemma 5.2.5 *Let M_R be an SU module over a commutative ring R. Then M_R is locally artinian.*

Proof If $m \in M$, then $mR \cong R/\mathbf{r}(m)$ is SU and hence artinian. Therefore M is locally artinian. $\square$

Theorem 5.2.6 *Let M be a $\mathbb{Z}$-module. Then $M_{\mathbb{Z}}$ is an SU module $\Leftrightarrow$ $M \cong \bigoplus_{\lambda \in \Lambda} \mathbb{Z}_{p_\lambda^{n_\lambda}}$ where for each prime q, $\max\{n_\alpha : \alpha \in \Lambda, p_\alpha = q\}$ exists and is finite.*

Proof $\Rightarrow$ Theorem 5.2.4 (i) $\Rightarrow$ (iii) tells us that M is a direct sum of uniserial modules of finite length and it is easy to see that any such $\mathbb{Z}$-module is of the form $\mathbb{Z}_{q^a}$, where q is prime and $a \in \mathbb{N}$. Suppose that there exists a prime q such that for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $m > n$ and M has a summand of the form $\mathbb{Z}_{q^m}$. It is not hard to see then that $\mathbb{Z}_{q^\infty}$, the injective hull of $\mathbb{Z}_q$, is a homomorphic image of M and so $\mathbb{Z}_{q^\infty} \in \sigma[M]$, which must be false, since $\mathbb{Z}_{q^\infty}$ has infinite length. Therefore the powers q^b of q such that $\mathbb{Z}_{q^b}$ occurs as a summand of M must be bounded.

$\Leftarrow$ It is well-known (see e.g. [11] Theorem 8.4) that a torsion $\mathbb{Z}$-module is the direct sum of its q-torsion submodules, for each prime q. It is therefore enough to show that for a fixed prime q, a q-torsion module in $\sigma[M]$ is a direct sum of uniform submodules.

Suppose that A is a q-torsion module in $\sigma[M]$. Let t be the maximum number such that $\mathbb{Z}_{q^t}$ occurs as a direct summand of M. Let a be a member of A and let s be the least natural number such that $aq^s = 0$. Suppose that $s > t$. Since $A \in \sigma[M]$, there exists a natural number d such that q is not a factor of d and $q^t\mathbb{Z} \cap d\mathbb{Z} \subseteq \mathbf{r}(a)$, i.e. $q^t d\mathbb{Z} \subseteq \mathbf{r}(a)$, i.e. $aq^t d = 0$. Since the greatest common divisor of q^s and $q^t d$ is q^t, it follows that $aq^t = 0$ - a contradiction. Hence $aq^t = 0$ and therefore $Aq^t = 0$, which means that A is a Z_{q^t} module. Z_{q^t} is a serial ring and therefore A is a direct sum of uniserial modules of finite length. $\square$

5.3 Summary

It is very difficult to construct non-trivial examples of SU modules over an arbitrary ring, although Theorem 5.2.4 provides fairly strong equivalent conditions in the commutative case and Theorem 5.2.6 categorises them exactly in the case where the ring is $\mathbb{Z}$.

5.4 Question

- If M_R is an SU module over a general (i.e. non-commutative) ring R, is M locally artinian? We see no reason why there should not be a non-locally-artinian pure-semisimple module X (see [52]), but in the case where we demand that all of the indecomposables in $\sigma[X]$ are uniform, we may force extra conditions on these indecomposable modules.

  It is not hard to see that this question is equivalent to asking whether every uniform in $\sigma[M]$ must have non-zero socle and to asking whether every noetherian SU module is artinian.

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Index