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PROBLEMS IN THE ANALYSIS OF BINARY MIXTURE DISTRIBUTIONS

BY

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To Hina and Madeeha;

and Maria and my mother.

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CONTENTS

SUMMARY

In Chapter 1, a finite mixture distribution is defined and background literature is briefly mentioned.

A theorem is proved in Chapter 2 which states that if three density functions are proper mixtures of the same two components then one of them is a proper mixture of the other two. It is shown, with the help of a counter example, that the converse of this theorem is not necessarily true.

In Chapter 3, two critical papers are reviewed.

In Chapter 4, a distance function, based on the necessary condition of this theorem,is defined which 'measures' the deviation from the hypothesis that one of the densities is a proper mixture of the other two. The expressions for the mean and variance of this distance function are calculated for continuous data. Simulations are carried out to generate values of the distance function for a number of cases (null as well as non-null). For the several values of the distance function thus obtained the sample mean and sample variance were calculated and compared with the theoretical values obtained from the above mentioned expressions. The distribution of the distance function is considered and it is shown, diagrammatically, that the logarithm of this distance function is approximately normally distributed. Finally, in this chapter, a test of the null hypothesis is suggested. In Chapter 5, expressions for the mean and variance of the distance function are discussed for discrete data. A simulation study is carried out and a 'parametric' test of the null hypothesis is suggested and carried out on several data sets. Also, a non-parametric test of the null hypothesis is suggested and carried out on the same data sets. Finally, in this chapter, the various tests suggested are also applied to some fish data.

In Chapter 6, a Monte Carlo test based on the above-mentioned distance function is introduced.

In Chapter 7, a theorem is proved which determines another necessary condition for three densities to be proper mixtures of the same two components.

In Chapter 8, the theory and methods developed in this thesis (which are applicable only to the case of three density functions) are generalised to the case where we have more than three density functions. Finally, finite mixtures having more than two components are discussed in some detail.

CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 Definition of a finite mixture distribution

Suppose that a random variable or vector, X, takes values in a sample space, $\mathcal{X}$, and that its distribution can be represented by a probability density function (or mass function in the case of discrete $\mathcal{X}$) of the form

$$p(x) = \sum_{j=1}^{k} \Pi_i f_i(x) \qquad (x \varepsilon \mathcal{X}), \qquad (1)$$

where $\sum_{j=1}^{k} \Pi_j = 1$ and, for $j \varepsilon [1,2,\ldots.k]$,

(i) $\Pi_j > 0$

(ii) $f_j(.) \geq 0$

and (iii) $\int_{\mathcal{X}} f_j(x)dx = 1.$

In such a case, X is said to have a <u>finite mixture distribution</u> and p(.) is called a finite mixture density function.

The parameters Π_j and the functions $f_j(.)$, for $j \varepsilon [1,2,\ldots.k]$ will be called the mixing weights and the component densities, respectively, of the mixture.p(.) will be called a proper mixture of the $f_i(.)$'s if the parameters π_j (j= 1,2,3,....k) lie between 0 and 1.

1.2 Two types of applications of finite mixture models

Finite mixture models are applied in two different ways. One is called a direct application and the other an indirect one.

If we believe in the existence of k (refer to equation (1)) underlying categories such that the experimental unit on which the observation X is made belongs to one of these categories (we do not, however, observe directly the source of X) then in this form of application, $f_j(.)$ is the probablity distribution of X given that the observation actually derives from category j, and Π_j denotes the probability that the observation comes from this source. This is called a direct application of a finite mixture model.

By an indirect application we mean a situation where the finite mixture model is simply being used as a mathematical device in order to provide an indirect means of obtaining a simpler form of analysis.

1.3 Examples of applications of finite mixture models

We now mention, briefly, a few areas of application of finite mixture models where the model is intended as a direct representation of the underlying physical phenomenon.

For the continuous case, finite mixture distributions with normal components, and, for the discrete case, finite mixture distributions with multinomial components are most widely used.

Example 1 Fisheries research

Cassie (1954) gives frequency data for the lengths of 256 snappers. The underlying categories are the possible age groups to which an individual fish might belong. Thus the component densities describe the length distributions for fish of different ages and the mixing weights indicate the age distribution of snappers in the total population.

Example 2 Grain sizes in samples of particles

Brazier et al., (1983) give grain size distributions at various distances downwind from Mount St. Helens, deposited from a certain eruption. The samples are analysed by sieving and we get a distribution measured by percentages of total weight in various ranges of size.

At three different distances, recorded, there is evidence of a mixture of two components, although the mixing weights change from place to place.The physical explanation for the mixture offered by Brazier et al. is that some of the fine ash aggregates to form the larger particles that contribute to one of the components.

Example 3 Sex distribution of twin pairs

Blischke (1978, Vol 1,p.175) gives an example of a mixture of two trinomial distributions, i.e. ,the sex distribution of twin pairs.A twin pair must be from one of three classes: male/male, male/female,female/female. Thus we are dealing with a trinomial distribution.However, twin pairs come in two types,dizygotic and monozygotic, so that we have, in fact, a mixture of two trinomials. The mixing proportions are the relative magnitudes of the two types of twin pairs.

Finally we give a few examples of indirect applications of finite mixture models.

Example 4

To approximate an intractable heavy-tailed distribution we may use a two-component normal mixture where one component has an inflated variance.

Example 5

The kernel methods of density estimation employ a finite mixture as a smooth curve-fitting device; see Section 1.6.

1.4 Estimating the parameters of mixture distributions

If, in equation (1), $f_1(.),\ldots\ldots\ldots,f_k(.)$ have specified parametric forms then the right hand side of (1) will have the more explicit representation

$$\sum_{j=1}^{k} \pi_j f_j(x|\underline{\theta}_j)$$

where $\underline{\theta}_j$ denotes the parameters occurring in $f_j(.)$.

Many methods have been devised and used for estimating the parameters of mixture distrbutions. Some methods are of general applicability while there are methods which are restricted to particular types of components; see Everitt and Hand (1981) and Titterington, Smith and Makov (1985) for detailed accounts.

1.5 Binary mixtures and the associated invariant

In this thesis we will be interested in binary mixtures. A binary mixture is a mixture having two components. It will be shown that there is an invariant quantity (constant) associated with any three binary mixtures which are expressible as mixtures of the same two components.

If in a binary mixture we regard the two components as binary mixtures themselves, in the sense that the other component has zero weight, then it follows that there is an invariant quantity associated with each binary mixture.

Some statistical tests will be developed which are based upon the above-mentioned invariant.

In our work, parametric models will not be assumed for the component densities. As a consequence, kernel-based density estimates will form an important tool in our analysis and we provide a brief introduction to them now.

1.6 Kernel-based density estimation

A kernel based density estimate is defined as follows.

Let $[x_1,....x_n]$ be a sample of size n drawn from the true density f (x), say.

As mentioned in example 5 of section 1.3, we write down the estimate $\hat{f}(x)$ of $f(x)$ as

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} k\left[\frac{(x - x_i)}{h}\right]$$

where k is a density function which we will always take to be normal. Thus, f(x) is being approximated by a finite mixture of n components. Each component has, as its mean, one of the data points. However, each component has the same variance h^2 and all the components have the same mixing weight $\frac{1}{n}$. The function k is called the kernel of the density estimate.

Thus if k is taken to be normal then k(v) is a standard normal density.

The mean and variance of $\hat{f}(x)$ are given by

$$E\hat{f}(x) = f(x) + \tfrac{1}{2}h^2 I_1 f''(x) + o(h^2)$$

where $I_1 = \int v^2 k(v)\,dv$

and

$$\text{var } \hat{f}(x) = \frac{1}{nh} I_2 f(x) + O(1/n) + o(nh)^{-1}$$

where

$I_2 = \int k^2(v)\,dv$ and, in practice, $h=o(1)$

Thus, approximately,

$$E\hat{f}(x) = f(x) + \tfrac{1}{2}h^2 I_1 f''(x)$$

$$\text{var } \hat{f}(x) = (nh)^{-1} I_2 f(x).$$

Therefore the mean squared error of $\hat{f}(x)$ is given by

$$\text{MSE } \hat{f}(x) = \tfrac{1}{4}h^4 I_1^2 (f''(x))^2 + (nh)^{-1} I_2 f(x).$$

Differentiating MSE $\hat{f}(x)$ with respect to h and equating it to zero, gives , for fixed x,

$$h^5 = \frac{I_2 f(x)}{I_1^2 [f''(x)]^2} \cdot \frac{1}{n} .$$

Since the second derivative of MSE $\hat{f}(x)$ is positive, therefore the optimal h (asymptotically) to minimise MSE $\hat{f}(x)$ is proportional to $n^{-1/5}$. Also the integrated MSE of $\hat{f}(x)$ gives the optimal h (asymptotically) to be proportional to $n^{-1/5}$. We shall use this asymptotically optimal choice for the smoothing parameter in what follows.

CHAPTER 2

SOME MATHEMATICAL RESULTS ABOUT BINARY MIXTURES.

2.1 Introduction

Suppose $f_1(x)$, $f_2(x)$ and $f_3(x)$ are three distinct densities. For brevity we will write them as f_1, f_2 and f_3 respectively. In this chapter we determine the conditions under which one of the f_i's (i=1,2,3) is a proper mixture of the other two. Further, we determine the additional condition under which the f_i's are expressible as proper mixtures of the same two components.

2.2 A linear dependence condition

We prove a necessary and sufficient condition for f_1, f_2 and f_3 to be linearly dependent.

By definition, f_1, f_2 and f_3 are linearly dependent iff

$$Af_1+Bf_2+Cf_3=0 \text{ (i.e. } Af_1(x)+Bf_2(x)+Cf_3(x)=0 \ \forall x) \quad (1)$$

where the constants A, B and C are non-zero and $A+B+C=0$. (2)

Obviously, A,B and C cannot all be positive. Thus it follows that at least one of them is negative. We assume, without loss of generality that $A<0$.

Using (2), (1) can be written as

$$-(B+C)f_1+Bf_2+Cf_3=0. \quad (3)$$

Also, obviously, out of B and C one (at least) must be positive. Again, without loss of generality,we assume that B is positive. Thus (3) can be written as

$$f_1-\frac{B}{B+C}f_2-\frac{C}{B+C}f_3=0. \quad (4)$$

If C is positive, we write (4) as

$$f_1=\frac{B}{B+C}f_2+\frac{C}{B+C}f_3.$$

If C is negative we write (4) as

$$f_2 = \frac{B+C}{B} f_1 - \frac{C}{B} f_3 .$$

Thus, in either case, one of the f_i's is a proper mixture of the other two. This proves the necessary condition for the f_i's to be linearly dependent. If we assume that one of the f_i's is a proper mixture of the other two then it is obvious that the three f_i's are linearly dependent. Hence we have proved the following theorem.

Theorem 1

Three densities are linearly dependent iff one of them is a proper mixture of the other two.

The next theorem establishes the necessary condition (but not sufficient, as we will show later) that f_1, f_2 and f_3 are proper mixtures of the same two components.

Theorem 2

If f_1, f_2 and f_3 are proper mixtures of the same two components then $f_i - f_j = t_{ijik}(f_i - f_k)$, where t_{ijik} is constant, $i \neq j \neq k$ and $i,j,k \varepsilon [1,2,3]$.

Proof

Let g and h be the common components of the f_i's.

$$\text{Let } f_i = p_i g + (1 - p_i)h \quad (i=1,2,3). \tag{5}$$

Thus $$f_i - f_j = (p_i - p_j)(g - h)$$

and $$f_i - f_k = (p_i - p_k)(g - h).$$

Therefore

$$f_i - f_j = \frac{p_i - p_j}{p_i - p_k}(f_i - f_k)$$

where

$$\frac{p_i - p_j}{p_i - p_k} = \text{constant} = t_{ijik}$$. This proves Theorem 2.

Now, if f_1, f_2 and f_3 are related as

$$f_i - f_j = t_{ijik}(f_i - f_k)$$

then, rewriting this as

$$(1 - t_{ijik})f_i - f_j + t_{ijik} f_k = 0,$$

we note that the sum of the coefficients of f_i, f_j and f_k is zero. Thus by Theorem 1, one of the f_i's is a proper mixture of the other two. This proves the following theorem.

Theorem 3

If f_1, f_2 and f_3 are related as

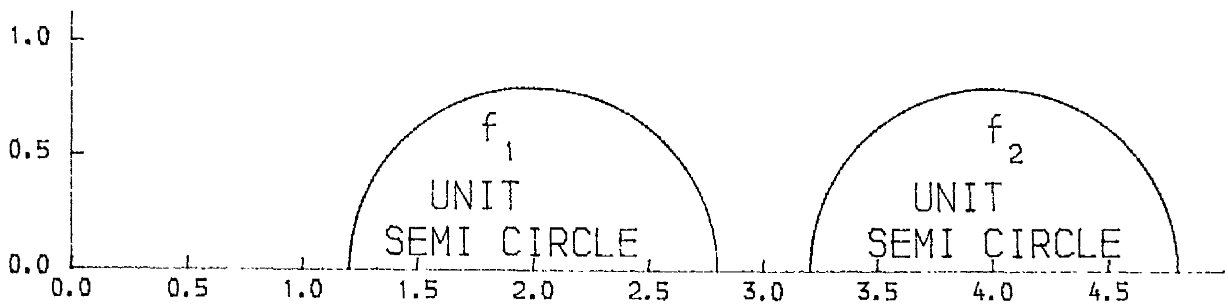
$$f_i - f_j = t_{ijik}(f_i - f_k) \qquad i \neq j \neq k$$

where t_{ijik} is constant, $i,j,k \epsilon [1,2,3]$,

then one of the f_i's is a proper mixture of the other two , and conversely.

The following graphical example demonstrates that the converse of theorem 2 is not necessarily true.

Example



Let $f_3 = af_1 + (1-a)f_2$, where $0 < a < 1$. Then, obviously, f_1 and f_2 are not proper mixtures of the same two components.

In the next section we determine the additional condition under which the converse of theorem 2 is true.

2.3 An additional theorem

Consider the identities

$$f_1 \equiv p_1 \frac{(1-p_2)f_1 - (1-p_1)f_2}{p_1 - p_2} + (1-p_1) \frac{p_1 f_2 - p_2 f_1}{p_1 - p_2} \tag{6}$$

$$f_2 \equiv p_2 \frac{(1-p_2)f_1 - (1-p_1)f_2}{p_1 - p_2} + (1-p_2) \frac{p_1 f_2 - p_2 f_1}{p_1 - p_2} \tag{7}$$

where $0 < p_1, p_2 < 1$ and without loss of generality, we choose $p_1 > p_2$.

Let

$$g \overset{\text{defn.}}{=} \frac{(1-p_2)f_1 - (1-p_1)f_2}{p_1 - p_2} \tag{8}$$

and

$$h \overset{\text{defn.}}{=} \frac{p_1 f_2 - p_2 f_1}{p_1 - p_2} . \tag{9}$$

Now g and h are densities iff

$$\text{(i)} \int g\,dx = 1 \text{ and } \int h\,dx = 1$$

$$\text{(ii)} \; g \geq 0 \text{ and } h \geq 0 \quad \forall x \; .$$

Condition (i) is obvious from (8) and (9). For condition (ii) to hold it is necessary (and sufficient) that

$$(1-p_2)f_1 \geq (1-p_1)f_2 \tag{10}$$

and

$$p_1 f_2 \geq p_2 f_1 . \tag{11}$$

(10) and (11) together imply that

$$\frac{1-p_1}{1-p_2} \leq \frac{f_1}{f_2} \leq \frac{p_1}{p_2} \; . \tag{12}$$

This means that f_1 and f_2 are proper mixtures of the same two components if (12) holds. In other words, if f_1 and f_2 are proper mixtures of the same two components then $\frac{f_1}{f_2}$ lies between finite and non-zero limits. Conversely, let $d_1 \leq \frac{f_1}{f_2} \leq d_2$ where d_1 is the infimum and d_2 the supremum. We take d_1 to be non-zero. Obviously d_2 is positive. Since f_1 and f_2 are densities therefore it is not possible that $f_1 \leq f_2$ or $f_1 \geq f_2$ for all x. This implies that $0<d_1<1$ and $d_2>1$.

Put $$d_1 = \frac{1-\bar{p}_1}{1-\bar{p}_2} \quad \text{and} \quad d_2 = \frac{\bar{p}_1}{\bar{p}_2} \; .$$

Thus $$\bar{p}_1 = \frac{d_2(1-d_1)}{d_2-d_1} \quad \text{and} \quad \bar{p}_2 = \frac{1-d_1}{d_2-d_1} \; .$$

Note that $\bar{p}_1 > \bar{p}_2$ and $0<\bar{p}_1, \bar{p}_2 < 1$. Thus we have proved the following theorem.

<u>Theorem 4</u>

Two densities f_1 and f_2 are proper mixtures of the same two components iff $\frac{f_1}{f_2}$ lies between finite non-zero limits at all points (except the points where $\frac{f_1}{f_2}$ takes the indeterminate form $\frac{0}{0}$).

2.4 Looking ahead

In Chapters 4 and 5, Theorem 3 is used as the basis for forming a distance function which measures the deviation from the hypothesis that one density is a proper mixture of the other two. In other words, if we decide that the condition in Theorem 3 is not satisfied then we can state that none of the f_i's is a proper mixture of the other two.

CHAPTER 3

A REVIEW OF TWO CRITICAL PAPERS

3.1 Introduction

The hypothesis that the distribution function of one of three populations is a proper mixture of the distribution functions of the other two populations may be tested either by a parametric method or by a non-parametric one. In this thesis the problem has been attempted by several non-parametric methods.

A review of the literature shows that this problem was attempted by Falmagne (1968) and Thomas (1969). The tests proposed by both were non-parametric. Furthermore, the test proposed by Falmagne was a non-statistical one .

This problem, for the continuous densities case, has not been attempted, to the best of our knowledge, by any parametric method.

If we are testing the hypothesis that the density functions, or, equivalently, distribution functions, of three populations are proper mixtures of the same two components, then we are considering a related problem as explained in what follows.

According to Theorem 2 of chapter two it is clear that if we decide that the above mentioned hypothesis is true then it follows that the null hypothesis that the distribution function of one of three populations is a proper mixture of the distribution functions of the other two populations is also true.

Thus if we consider the null hypothesis in its stronger form then the parametric approach involves fitting mixtures of the same two component densities and then deciding whether the parameters of the component densities are the same for each of the three given densities.

Kornbrot(1983) introduces a method by which the parameters of the fitted mixtures are estimated simultaneously as opposed to the other methods which estimate the parameters of the densities individually. But this paper does not suggest (or attempt) ways of deciding whether the component densities are the same for the given densities.

In the next section we elaborate on the papers by Falmagne (1968) and Thomas (1969).

3.2 Two non-parametric tests

In this section we first discuss the paper by Falmagne (1968). He proves a theorem which states that if F_1 and F_2 are proper mixtures of the same two component distribution functions which intersect at a point, say, $x=x_0$, then any mixture of the same two distribution functions also intersects at $x=x_0$. Note that the property expressed by this theorem also holds for density functions. Falmagne applies the theorem to six empirical distributions and concludes that a model involving two basic distributions cannot be fitted to the six empirical distributions for any reasonable hypothesis about the basic distributions. Observe that the test here is a non-statistical one.

Thomas (1969) discusses distribution-free methods for testing the hypothesis that the distribution function, say F_3, of one of the populations is a proper mixture of the distribution functions, say, F_1 and F_2, of the other two populations.

He introduces two statistics τ_n^2 and $\tau_n'^2$ which are defined as

$$\tau_n^2 = n \int_{-\infty}^{\infty} [\hat{p}\hat{F}_1(x)+\hat{q}\hat{F}_2(x)-\hat{F}_3(x)]^2 \, d\hat{F}_3(x)$$

and $$\tau_n'^2 = n \int_{-\infty}^{\infty} [\hat{p}\hat{F}_1(x)+\hat{q}\hat{F}_2(x)-\hat{F}_3(x)]^2 \, dF_3(x)$$

respectively.

$\hat{F}_1(x)$, $\hat{F}_2(x)$ and $\hat{F}_3(x)$ are the estimates of the distribution functions $F_1(x)$, $F_2(x)$ and $F_3(x)$ respectively.

$F_1(x)$, $F_2(x)$ and $F_3(x)$ are related as

$F_3(x)=pF_1(x)+qF_2(x)$, where $0<p<1$ and $q=1-p$.

$\hat{p}$ is the estimate of the mixing weight p, and similarly $\hat{q}$ the estimate of q.

Each of the three samples, for simplicity, is assumed to be of size 2n. Two sets are constructed. The first set consists of a random choice of n readings from each of the three samples. The second set consists of the remaining 3n readings. The first set is used to estimate p. The second set is used to construct $\hat{F}_1(x)$, $\hat{F}_2(x)$ and $\hat{F}_3(x)$. It is assumed that the estimate $\hat{p}$ of p constructed from set one is such that

$E(\hat{p}-p)=0(n^{-1})$

and $E(\hat{p}-p)^2=v^2 n^{-1}+0(n^{-2})$.

If $y_1,\ldots\ldots,y_n$ is the ordered sample from F_3 then the statistic τ_n^2 is given by

$$\tau_n^2 = \sum_{k=1}^{n} [\hat{p}\hat{F}_1(y_k)+\hat{q}\hat{F}_2(y_k)-\hat{F}_3(y_k)]^2 \equiv \sum_{k=1}^{n} [V(y_k)]^2$$

$$= n^{-2} \sum_{k=1}^{n} [\hat{p}s_k(1)+\hat{q}s_k(2)-k]^2. \qquad (1)$$

Here $s_k(h)$ is the number of x_{hi}'s less than or equal to x_{3k}, where the three samples are given by $(x_{11},\ldots\ldots,x_{1n}),(x_{21},\ldots\ldots x_{2n})$ and $(x_{31},\ldots x_{3n})$ for $\hat{F}_1(x)$, $\hat{F}_2(x)$ and $\hat{F}_3(x)$ respectively. Note that τ_n^2 does not involve the assumption that $F_3(x)$ is given by

$h=1,2 \qquad i=1,..,n \qquad k=1,..,n$

$$F_3(x)=pF_1(x)+qF_2(x). \qquad (2)$$

Thus equation (1) can be used to obtain $E\tau_n^2$ without the assumption given by equation (2).

The function $V(x)$, with the condition given by (2) incorporated in it, can be written as

$$V=p(\hat{F}_1-F_1)+q(\hat{F}_2-F_2)+(\hat{p}-p)(F_1-F_2)+F_3-\hat{F}_3. \qquad (3)$$

Thus

$$E\tau_n'^2=n\int_{-\infty}^{\infty} E[V^2(x)]\, dF_3(x) . \qquad (4)$$

Using (3) we have

$$nE(V^2)=p^2F_1(1-F_1)+q^2F_2(1-F_2)+v^2(F_1-F_2)^2+F_3(1-F_3).$$

Substituting in (4) we have, after performing the integrations

$$E\tau_n'^2=\frac{1}{3}+\frac{4}{3}v^2-\frac{1}{2}pq(1+2p)-pq(1-2p)\alpha$$

$$-2(v^2-p^2q)\beta_1-2(v^2-pq^2)\beta_2. \qquad (5)$$

In equation (5) $\beta_h=\int_{\infty}^{\infty} F_1(x)F_2(x)dF_h(x)$ (h=1,2)

and $\alpha=\int_{\infty}^{\infty}F_1(x)dF_2(x)$.

A theorem is proved which states that if (2) is true then for arbitrary $\delta>0$,

$$\lim_{n\to\infty} P[\,|\tau_n^2-\tau_n'^2|<\delta\,]=1 .$$

This theorem ensures that the moments of τ_n^2 are approximately equal to those of $\tau_n'^2$.

Consider (5). v^2 depends on how p is estimated.If the relation

$$\hat{p}=\frac{n_1\Sigma r_j(3)-0.5n_1n_2n_3}{n_2\Sigma s_k(1)+n_1\Sigma r_j(3)-n_1n_2n_3} \qquad (6)$$

is used, where

$r_j(3)$ is the number of x_{3k}'s less than or equal to x_{2j} and $s_k(1)$ is the number of x_{1i}'s less than or equal to x_{3k} and $n_1=n_2=n_3$, then v^2 can be expressed in terms of p and α where α can be estimated by

$$\hat{\alpha} = \frac{1}{n_1 n_2} \Sigma r_j(1) \ ;$$

$r_j(1)$ is the number of x_{1i}'s less than or equal to x_{2j}.

Thus, since p,α and similarly β_1 and β_2 can be estimated, from (5) it follows that $E\tau_n'^2$ can be estimated. Also $E\tau_n^2$ can be estimated, the difference being that $E\tau_n'^2$ contains the null assumption while $E\tau_n^2$ does not. Thus, when the null hypothesis is true, $E\tau_n'^2$ and $E\tau_n^2$, according to the theorem, should be approximately equal.

Thomas proposes the following test

Suppose $F_3(x)= p(x)F_1(x)+(1-p(x))\ F_2(x)$. (7)

H_0: $p(x)= p(0<p<1)$

H_1: $p(x)\neq$ const.

Let the quantity, under the null hypothesis, estimated by $\hat{p}$ in (6) be p_0. Thus (7) may be written as

$F_3(x)= p_0F_1(x)+q_0F_2(x)+e(x)$

where $e(x)= (p(x)-p_0)[F_1(x)-F_2(x)]$.

Thus if $\hat{p}$ is given by (6) and V is as defined earlier then

$$E(\tau_n'^2)=n \int_{-\infty}^{\infty} E(V^2(x))dF_3(x)+n \int_{-\infty}^{\infty} e^2(x)dF_3(x).$$

var $\tau_n'^2$ is seen to be of order O(1).

Under H_0, $E\tau_n'^2$ is given by (5) with $p=p_0$.

Under H_1, since $e(x)\neq 0$ in a non-degenerate interval, it follows that therefore $\int_{-\infty}^{\infty} e^2(x)dF_3(x)>0$, and $\lim_{n\to\infty} E\tau_n'^2=\infty$.

Thus, since under H_1, since var $\tau_n'^2 = O(1)$ for $c>0$,

we have $\lim_{n\to\infty} p\ (|D_n|>c)=1$.

Here D_n is the difference between the estimates of the two sides of equation (5).

A test with rejection region of the form $|D_n|>c$ would be consistent and asymptotically unbiased. However, Thomas (1969) does not describe how critical values for c might be obtained.

CHAPTER 4

A TEST STATISTIC: CONTINUOUS DATA

4.1 Introduction

According to Theorem 3 of Chapter 2, if f_1, f_2 and f_3 are three densities which are related as

$$f_i - f_j = t_{ijik}(f_i - f_k) \qquad i \neq j \neq k \tag{1}$$

where t_{ijik} is constant and i,j and $k \in [1,2,3]$, then one of the f_i's is a proper mixture of the other two,and conversely. This means that if (1) is not true then it is not possible that any of the f_i's is a proper mixture of the other two.This suggests that if we consider the square of

$$(f_i - f_j) - t_{ijik}(f_i - f_k)$$

and construct a quantity which is actually the sum of the above mentioned squares over a number of chosen values of the random variable x then the quantity thus constructed can become the basis of a test which decides whether or not one of the f_i's is a proper mixture of the other two.

Let $\hat{f}_1$, $\hat{f}_2$ and $\hat{f}_3$ be kernel estimates of f_1, f_2 and f_3 respectively. Choose a number of points (say m) on the x-axis, $x_1, x_2, \ldots, x_m$. In general, we will assume samples of different sizes underlying the actual densities.

Define the distance function Q(t) as

$$Q(t) = \sum_{i=1}^{m} \left[[(\hat{f}_1(x_i) - \hat{f}_2(x_i)] - t[\hat{f}_1(x_i) - \hat{f}_3(x_i)\ ] \right]^2 . \tag{2}$$

Then

$$Q'(t) = -2 \sum_{i=1}^{m} [\hat{f}_1(x_i) - \hat{f}_3(x_i)] \begin{bmatrix} \hat{f}_1(x_i) - \hat{f}_2(x_i) \\ -t[\hat{f}_1(x_i) - \hat{f}_3(x_i)] \end{bmatrix} .$$

Thus $Q'(t)=0$ iff

$$t = \frac{\sum_{i=1}^{m} [\hat{f}_1(x_i)-\hat{f}_2(x_i)][\hat{f}_1(x_i)-\hat{f}_3(x_i)]}{\sum_{i=1}^{m} [\hat{f}_1(x_i)-\hat{f}_3(x_i)]^2} = \hat{t}. \tag{3}$$

Now $Q''(t) = 2\sum_{i=1}^{m} [\hat{f}_1(x_i)-\hat{f}_3(x_i)]^2$ is positive, therefore $Q(t)$ is minimised by $t=\hat{t}$. Thus

$$Q(\hat{t}) = \sum_{i=1}^{m} [(\hat{f}_1(x_i)-\hat{f}_2(x_i))-\hat{t}(\hat{f}_1(x_i)-\hat{f}_3(x_i))]^2$$

$$= \sum_{i=1}^{m} (\hat{f}_1(x_i)-\hat{f}_2(x_i))^2 - \frac{(\sum_{i=1}^{m} (\hat{f}_1(x_i)-\hat{f}_2(x_i))(\hat{f}_1(x_i)-\hat{f}_3(x_i)))^2}{\sum_{i=1}^{m} (\hat{f}_1(x_i)-\hat{f}_3(x_i))^2}. \tag{4}$$

In (4), the points $x_i (i=1,2,....m)$ may be chosen in a variety of ways. In our simulation study we will choose them to be equally spaced. The points x_i may also be chosen in such a way that the MSE of $\hat{t}$ is minimised. Now, MSE of $\hat{t}$ is given by

$$E(\hat{t}-t)^2 = (\text{bias } \hat{t})^2 + \text{var } \hat{t}.$$

The formulae for $E\hat{t}$ and $E\hat{t}^2$ (and, equivalently, the formula for var $\hat{t}$) as derived in section 4.2 of this chapter, are based on the true f_i's (and the smoothing parameters, h, used to form their estimates, $\hat{f}_i$). Thus the variables for selection on which the formulae for the mean and variance of $\hat{t}$ depend are the points $x_1, x_2, ... x_m$ and the smoothing parameter h.

This is a problem in optimal design where the independent variables are the x_i's and the smoothing parameters h, the function being the MSE of $\hat{t}$.

Denoting MSE of $\hat{t}$ by $M(x_i,........x_m, h's)$, our problem is to find $x_1,......x_m$ and h's to minimise M. Suppose that, given the x_i's, the $\hat{h}$'s minimise M over the h's. Then compute $M(x_1,......x_m, \hat{h}'s)$ and minimise with respect to the x_i's, giving the $\hat{x}_i$'s. Our solution will be $(\hat{x}_1,......\hat{x}_m, \hat{h}'s)$.

In practice, this minimisation procedure will be very difficult because of the large number of variables if m is large.

We will introduce some notation which will enable us to write $\hat{t}$ and $Q(\hat{t})$ in a convenient form. This form will make a number of necessary calculations feasible and will also explain a number of steps, otherwise not possible.

Let $[x_i]i=1,2,...m$ be the chosen values of the variable x, along the x-axis.

Let $\hat{f}_1(x_i)= \mu_{1i}+\delta_{1i}$

where $\mu_{1i}=E\hat{f}_1(x_i)$ and $\delta_{1i}=\hat{f}_1(x_i)-E\hat{f}_1(x_i)$.

Define

$$\hat{f}_1^T=[\hat{f}_1(x_1),\hat{f}_1(x_2),......,\hat{f}_1(x_m)].$$

$\underline{\mu}_1^T = [\mu_{11}, \mu_{12}, \ldots\ldots\ldots\ldots \mu_{1m}]$

and $\underline{\delta}_1^T = [\delta_{11}, \delta_{12}, \ldots\ldots\ldots \delta_{1m}]$.

Thus $\hat{\underline{f}}_1^T = \underline{\mu}_1^T + \underline{\delta}_1^T$. Similar results hold for $\hat{f}_2$ and $\hat{f}_3$.

Thus (3) can be written as

$$\hat{t} = \frac{(\hat{\underline{f}}_1 - \hat{\underline{f}}_2)^T(\hat{\underline{f}}_1 - \hat{\underline{f}}_3)}{(\hat{\underline{f}}_1 - \hat{\underline{f}}_3)^T(\hat{\underline{f}}_1 - \hat{\underline{f}}_3)} \tag{5}$$

and (4) can be written as

$$Q(\hat{t}) = (\hat{\underline{f}}_1 - \hat{\underline{f}}_2)^T(\hat{\underline{f}}_1 - \hat{\underline{f}}_2) - \frac{[(\hat{\underline{f}}_1 - \hat{\underline{f}}_2)^T(\hat{\underline{f}}_1 - \hat{\underline{f}}_3)]^2}{(\hat{\underline{f}}_1 - \hat{\underline{f}}_3)^T(\hat{\underline{f}}_1 - \hat{\underline{f}}_3)} \quad . \tag{6}$$

Further, we introduce the notation

$$\hat{\underline{f}}_{12} = \hat{\underline{f}}_1 - \hat{\underline{f}}_2$$

$$\underline{\mu}_{12} = \underline{\mu}_1 - \underline{\mu}_2$$

$$\underline{\delta}_{12} = \underline{\delta}_1 - \underline{\delta}_2$$

with similar meaning for $\hat{\underline{f}}_{13}$, $\underline{\delta}_{23}$ etc.

Thus (5) and (6) may, respectively, be written as

$$\hat{t} = \frac{\hat{\underline{f}}_{12}^T \hat{\underline{f}}_{13}}{\hat{f}_{13}^T \hat{\underline{f}}_{13}} \tag{7}$$

$$\text{and } Q(\hat{t}) = \hat{\underline{f}}_{12}^T \hat{\underline{f}}_{12} - \frac{(\hat{\underline{f}}_{12}^T \hat{\underline{f}}_{13})^2}{(\hat{\underline{f}}_{13}^T \hat{\underline{f}}_{13})} \quad . \tag{8}$$

Furthermore, breaking up the f_i's in terms of μ and δ we may write down (7) and (8) as

$$\hat{t} = \frac{(\underline{\mu}_{12}+\underline{\delta}_{12})^T(\underline{\mu}_{13}+\underline{\delta}_{13})}{(\underline{\mu}_{13}+\underline{\delta}_{13})^T(\underline{\mu}_{13}+\underline{\delta}_{13})} \quad (9)$$

and

$$Q(\hat{t}) = (\underline{\mu}_{12}+\underline{\delta}_{12})^T(\underline{\mu}_{12}+\underline{\delta}_{12}) - \frac{((\underline{\mu}_{12}+\underline{\delta}_{12})^T(\underline{\mu}_{13}+\underline{\delta}_{13}))^2}{(\underline{\mu}_{13}+\underline{\delta}_{13})^T(\underline{\mu}_{13}+\underline{\delta}_{13})} \quad (10)$$

respectively.

In section 4.2 we write down expressions for the mean and variance of $\hat{t}$ and $Q(\hat{t})$. These expressions were initially obtained uptil order two in δ, but it turned out that for the null case, i.e when equation (1) was true, the leading term of $EQ(\hat{t})$ was of order two in δ and that of var $Q(\hat{t})$ was (as expected) of order four in δ. For the non-null case, the leading term of $EQ(\hat{t})$ was of order zero in δ and that of var $Q(\hat{t})$ of order two in δ. Thus expressions for the mean and variance of $Q(\hat{t})$ were derived uptil order four in δ. The expressions for mean and variance of $\hat{t}$, however, were good enough uptil terms of order two in δ. The reason why, for the null case, leading terms in $EQ(\hat{t})$ and var $Q(\hat{t})$ were of orders 2 and 4, respectively,in δ will be explained in section 4.8.

In section 4.3 and 4.4 the basic quantities comprising the expressions for mean and variance of $\hat{t}$ and $Q(\hat{t})$ are elaborated.

In section 4.5 the computational details for the expressions for mean and variance of $\hat{t}$ and $Q(\hat{t})$ are given.

In section 4.6 simulations were carried out to generate values of $Q(\hat{t})$ for a number of cases (null as well as non-null) . For the $Q(\hat{t})$'s thus obtained, the values of sample mean and sample variance were calculated and compared with the theoretical approximations mentioned above. It was found that the mean and variance of $Q(\hat{t})$ obtained by both methods did not differ much.

Finally, in section 4.7, the distribution of $Q(\hat{t})$ is considered and it is shown, diagrammatically, that $\log Q(\hat{t})$ is approximately normal. A test for the null hypothesis is also suggested.

4.2 Expressions for mean and variance of $\hat{t}$ and $Q(\hat{t})$

In this section we write down the expressions for the mean and variance of $\hat{t}$ and $Q(\hat{t})$. The detailed calculation is done in section 4.5.

We have

$$\text{var } \hat{t} = E(\hat{t})^2 - (E\hat{t})^2$$

$$\text{and} \quad \text{var} Q(\hat{t}) = E[Q(\hat{t})]^2 - [EQ(\hat{t})]^2 .$$

Thus it is enough to write down expressions for $E\hat{t}$, $E(\hat{t})^2$, $EQ(\hat{t})$ and $E[Q(\hat{t})]^2$. Now, since the last two expressions are extremely lengthy, both of them will be written in three parts. Part one is all terms uptil order two in δ, part two is only terms of order three in δ and part three is only terms of order four in δ.

(i) $EQ(\hat{t})$

We have, uptil terms of order two in δ,

$$EQ(\hat{t}) = \underline{\mu}_{12}^T \underline{\mu}_{12} - \frac{(\underline{\mu}_{12}^T \underline{\mu}_{13})^2}{\underline{\mu}_{13}^T \underline{\mu}_{13}} + E\underline{\delta}_1^T \underline{\delta}_1 + E\underline{\delta}_2^T \underline{\delta}_2$$

$$- \frac{1}{\underline{\mu}_{13}^T \underline{\mu}_{13}} \begin{bmatrix} E(\underline{\mu}_{12}^T \underline{\delta}_1)^2 + E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 + 2E(\underline{\mu}_{12}^T \underline{\delta}_1)(\underline{\mu}_{13}^T \underline{\delta}_1) \\ + E(\underline{\mu}_{13}^T \underline{\delta}_2)^2 + E(\underline{\mu}_{12}^T \underline{\delta}_3)^2 + 2\,(\underline{\mu}_{12}^T \underline{\mu}_{13}) E\underline{\delta}_1^T \underline{\delta}_1 \end{bmatrix}$$

$$+ \frac{4\underline{\mu}_{12}^T \underline{\mu}_{13}}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^2} \begin{bmatrix} E(\underline{\mu}_{12}^T \underline{\delta}_1)(\underline{\mu}_{13}^T \underline{\delta}_1) + E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 \\ + E(\underline{\mu}_{13}^T \underline{\delta}_3)\ (\underline{\mu}_{12}^T \underline{\delta}_3) \end{bmatrix}$$

$$\frac{-(\underline{\mu}_{12}^T \underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^3} \begin{bmatrix} 4[E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 + E(\underline{\mu}_{13}^T \underline{\delta}_3)^2 \\ -(\underline{\mu}_{13}^T \underline{\mu}_{13})(E\underline{\delta}_1^T \underline{\delta}_1 + E\underline{\delta}_3^T \underline{\delta}_3) \end{bmatrix}$$

We have, considering terms of order three in δ ,

$$EQ(\hat{t}) = -\frac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\left[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]\right]$$

$$\frac{+2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l} E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 + 2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2 \\ +2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{12}^T\underline{\delta}_3]^2[\underline{\mu}_{13}^T\underline{\delta}_3] \end{array}\right]$$

$$+2\frac{\underline{\mu}_{12}^T\underline{\mu}_{13}}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \\ -E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_3^T\underline{\delta}_3] \end{array}\right]$$

$$\frac{-8[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[\begin{array}{l} E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\mu}_{12}^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 \\ -E[\underline{\mu}_{13}^T\underline{\delta}_3]^2[\underline{\mu}_{12}^T\underline{\delta}_3] \end{array}\right]$$

$$\frac{-4[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] - E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_3^T\underline{\delta}_3]\right]$$

$$+\frac{8[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 - E[\underline{\mu}_{13}^T\underline{\delta}_3]^3\right]$$

we, have, considering terms of order four in δ,

$$EQ(\hat{t}) = -\frac{1}{\underline{\mu}_{13}\underline{\mu}_{13}}\left[E[\underline{\delta}_1^T\underline{\delta}_1]^2 + E[\underline{\delta}_1^T\underline{\delta}_3]^2 + E[\underline{\delta}_1^T\underline{\delta}_2]^2 + E[\underline{\delta}_2^T\underline{\delta}_3]^2\right]$$

$$+\frac{4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1] \\ +E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] \\ +E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] \\ +E[\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_2^T\underline{\delta}_3] + E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_1] \end{array}\right]$$

$$+\frac{1}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1]+2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \\ E[\underline{\mu}_{13}^T\underline{\delta}_2]^2E[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{12}^T\underline{\delta}_3]^2E[\underline{\delta}_1^T\underline{\delta}_1]+2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\delta}_1^T\underline{\delta}_1]^2 \end{array}\right]$$

$$+\frac{4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] \\ +[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\delta}_1^T\underline{\delta}_3]^2 \end{array}\right]$$

$$+\frac{(1/2)}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1]^2\ [\underline{\delta}_3^T\underline{\delta}_3] + E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\delta}_3^T\underline{\delta}_3] \\ +2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]\ E[\underline{\delta}_3^T\underline{\delta}_3] + E[\underline{\mu}_{13}^T\underline{\delta}_2]^2\ E[\underline{\delta}_3^T\underline{\delta}_3] \\ +E[\underline{\mu}_{12}^T\underline{\delta}_3]^2[\underline{\delta}_3^T\underline{\delta}_3] + 2[\underline{\mu}_{12}^T\underline{\mu}_{13}]\ E[\underline{\delta}_1^T\underline{\delta}_1]\ E[\underline{\delta}_3^T\underline{\delta}_3] \end{array}\right]$$

$$
\begin{aligned}
&+\frac{(1/2)}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\Big[E[\underline{\mu}_{12}^T\underline{\delta}_1]^2E[\underline{\delta}_3^T\underline{\delta}_3]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\delta}_3^T\underline{\delta}_3]\\
&\quad+2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\delta}_3^T\underline{\delta}_3]+E[\underline{\mu}_{13}^T\underline{\delta}_2]^2E[\underline{\delta}_3^T\underline{\delta}_3]\\
&\quad+E[\underline{\mu}_{12}^T\underline{\delta}_3]^2[\underline{\delta}_3^T\underline{\delta}_3]+2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\delta}_1^T\underline{\delta}_1]E[\underline{\delta}_3^T\underline{\delta}_3]\Big]\\
&-\frac{4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\Big[E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\mu}_{13}^T\underline{\delta}_1]^2+E[\underline{\mu}_{13}^T\underline{\delta}_1]^4+2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^3\\
&\quad+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{13}^T\underline{\delta}_2]^2+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{12}^T\underline{\delta}_3]^2\\
&\quad+2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_1]\ [\underline{\delta}_1^T\underline{\delta}_1]\Big]\\
&-\frac{16}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\Big[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]\\
&\quad+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]\\
&\quad-2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]\Big]\\
&-\frac{4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\Big[E[\underline{\mu}_{12}^T\underline{\delta}_1]^2E[\underline{\mu}_{13}^T\underline{\delta}_3]^2+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{13}^T\underline{\delta}_3]^2\\
&\quad+2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\mu}_{13}^T\underline{\delta}_3]^2\\
&\quad+E[\underline{\mu}_{13}^T\underline{\delta}_2]^2E[\underline{\mu}_{13}^T\underline{\delta}_3]^2+E[\underline{\mu}_{12}^T\underline{\delta}_3]^2[\underline{\mu}_{13}^T\underline{\delta}_3]^2\\
&\quad+2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_3]^2E[\underline{\delta}_1^T\underline{\delta}_1]\Big]\\
&-\frac{8\underline{\mu}_{12}^T\underline{\mu}_{13}}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\Big[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\delta}_3^T\underline{\delta}_3]\\
&\quad+2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]\Big]\\
&-\frac{8\underline{\mu}_{12}^T\underline{\mu}_{13}}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\Big[E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\delta}_3^T\underline{\delta}_3]\\
&\quad+2E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]\Big]\\
&-\frac{8\underline{\mu}_{12}^T\underline{\mu}_{13}}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\Big[2E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]+E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]\ E[\underline{\delta}_1^T\underline{\delta}_1]\\
&\quad+E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_3^T\underline{\delta}_3]\Big]
\end{aligned}
$$

$$+\frac{16\underline{\mu}_{12}^T\underline{\mu}_{13}}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^3 + 3E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\mu}_{13}^T\underline{\delta}_3]^2\right]$$

$$+\frac{16\underline{\mu}_{12}^T\underline{\mu}_{13}}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1]^4 + 3E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{13}^T\underline{\delta}_3]^2\right]$$

$$+\frac{16\,\underline{\mu}_{12}^T\underline{\mu}_{13}}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[3E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3] + E[\underline{\mu}_{13}^T\underline{\delta}_3]^3[\underline{\mu}_{12}^T\underline{\delta}_3]\right]$$

$$-\frac{[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[\begin{array}{l}E[\underline{\delta}_1^T\underline{\delta}_1]^2 + 4E[\underline{\delta}_1^T\underline{\delta}_3]^2 + E[\underline{\delta}_3^T\underline{\delta}_3]^2\\ +2E[\underline{\delta}_1^T\underline{\delta}_1]E[\underline{\delta}_3^T\underline{\delta}_3]\end{array}\right]$$

$$+\frac{12[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[\begin{array}{l}E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\delta}_3^T\underline{\delta}_3]\\ +E[\underline{\mu}_{13}^T\underline{\delta}_3]^2E[\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_3]^2[\underline{\delta}_3^T\underline{\delta}_3]\\ +4E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]\end{array}\right]$$

$$-\frac{16[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^5}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1]^4 + 6E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{13}^T\underline{\delta}_3]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_3]^4\right].$$

(ii) $E[Q(t)]^2$

We have, uptil terms of order two in δ,

$$E[Q(\hat{t})]^2 = [\underline{\mu}_{12}^T\underline{\mu}_{12}]^2 + \frac{[\underline{\mu}_{12}^T\underline{\mu}_{13}]^4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2} - \frac{2[\underline{\mu}_{12}^T\underline{\mu}_{12}][\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}$$

$$+2[\underline{\mu}_{12}^T\underline{\mu}_{12}]\left[E[\delta_1^T\delta_1] + E[\underline{\delta}_2^T\underline{\delta}_2]\right] + 4\left[E[\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{12}^T\underline{\delta}_2]^2\right]$$

$$+\frac{4[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{12}^T\underline{\delta}_3]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_2]^2 \\ +2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1] \end{array}\right]$$

$$+\frac{2[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^2 + 2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1] \\ +E[\underline{\mu}_{13}^T\underline{\delta}_2]^2 + E[\underline{\mu}_{12}^T\underline{\delta}_3]^2 + 2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\delta}_1^T\underline{\delta}_1] \end{array}\right]$$

$$-\frac{16[\underline{\mu}_{12}^T\underline{\mu}_{13}]^3}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]\right]$$

$$+\frac{12[\underline{\mu}_{12}^T\underline{\mu}_{13}]^4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_3]^2\right]$$

$$\frac{-2[\underline{\mu}_{12}^T\underline{\mu}_{13}]^4}{[\mu_{13}^T\underline{\mu}_{13}]^3}\left[E[\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\delta}_3^T\underline{\delta}_3]\right]$$

$$+\frac{2[\underline{\mu}_{12}^T\underline{\mu}_{12}][\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[E[\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\delta}_3^T\underline{\delta}_3]\right]$$

$$-\frac{8[\underline{\mu}_{12}^T\underline{\mu}_{12}][\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_3]^2\right]$$

$$+\frac{8[\underline{\mu}_{12}^T\underline{\mu}_{13}][\underline{\mu}_{12}^T\underline{\mu}_{12}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1]^2 \\ +E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3] \end{array}\right]$$

$$\frac{-2[\underline{\mu}_{12}^T\underline{\mu}_{12}]}{[\underline{\mu}_{13}\underline{\mu}_{13}]}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^2 + 2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1] \\ E[\underline{\mu}_{13}^T\underline{\delta}_2]^2 + E\ [\underline{\mu}_{12}^T\underline{\delta}_3]^2 + 2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\delta}_1^T\underline{\delta}_1] \end{array}\right]$$

$$+\frac{8[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\mu_{13}]^2}\ E[\underline{\mu}_{12}^T\delta_1][\underline{\mu}_{13}^T\underline{\delta}_1]$$

$$-\frac{8[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]}\left[E[\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1] + E[\underline{\mu}_{12}^T\underline{\delta}_2][\underline{\mu}_{13}^T\underline{\delta}_2]\right]$$

$$-\frac{2[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]}\left[E[\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\delta}_2^T\underline{\delta}_2]\right] .$$

We have, considering terms of order three only, in δ,

$$E[Q(\hat{t})]^2 = 4\left[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] - E[\underline{\mu}_{12}^T\underline{\delta}_2][\underline{\delta}_2^T\underline{\delta}_2]\right]$$

$$+\frac{4[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]\right]$$

$$+\frac{4[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1]^3 + E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2 \\ +2E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\mu}_{13}^T\underline{\delta}_1] + 2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \\ +E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 \\ +2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2 \\ +2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] - E[\underline{\mu}_{12}^T\underline{\delta}_3]^3 \\ -E[\underline{\mu}_{13}^T\underline{\delta}_2]^2 \end{array}\right]$$

$$-\frac{8[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[\begin{array}{l} E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 + 2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2 \\ -E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]^2 \end{array}\right]$$

$$\frac{-8\ [\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[\begin{array}{l} E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 + 2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2 \\ -E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]^2 \end{array}\right]$$

$$\frac{-8[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[\begin{array}{l} E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 \\ + 2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2 + 2[\underline{\mu}_{12}^T\underline{\mu}_{13}]\ E\ [\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \\ - E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]^2 \end{array}\right]$$

$$+\frac{48[\underline{\mu}_{12}^T\underline{\mu}_{13}]^3}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 - E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]^2\right]$$

$$\frac{-8[\underline{\mu}_{12}^T\underline{\mu}_{13}]^3}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] - E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_3^T\underline{\delta}_3]\right]$$

$$+\frac{12[\underline{\mu}_{12}^T\underline{\mu}_{13}]^4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] - E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_3^T\underline{\delta}_3]\right]$$

$$\frac{-32[\underline{\mu}_{12}^T\underline{\mu}_{13}]^4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^5}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 - E[\underline{\mu}_{13}^T\underline{\delta}_3]^3\right]$$

$$-\frac{8[\underline{\mu}_{12}^T\underline{\mu}_{12}][\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] - E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_3^T\underline{\delta}_3]\right]$$

$$+\frac{16[\underline{\mu}_{12}^T\underline{\mu}_{12}][\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 - E[\underline{\mu}_{13}^T\underline{\delta}_3]^3\right]$$

$$+\frac{4[\underline{\mu}_{12}^T\underline{\mu}_{12}][\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l}E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \\ -E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_3^T\underline{\delta}_3]\end{array}\right]$$

$$-\frac{16[\underline{\mu}_{12}^T\underline{\mu}_{12}][\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[\begin{array}{l}E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 \\ -E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]^2\end{array}\right]$$

$$+\frac{4[\underline{\mu}_{12}^T\underline{\mu}_{12}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l}E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1]^2 + E[\underline{\mu}_{13}^T\underline{\delta}_1]^3 + 2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\delta_1]^2 \\ +2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] - E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]^2\end{array}\right]$$

$$\frac{-4[\underline{\mu}_{12}^T\underline{\mu}_{12}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]}\left[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]\right]$$

$$+\frac{4[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2} E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \frac{-16[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3} E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2$$

$$+\frac{16[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\mu}_{13}^T\underline{\delta}_1] + E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2\right]$$

$$\frac{-4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1]^3 + E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^2 \\ + 2E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\mu}_{13}^T\underline{\delta}_1] + 2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \\ - E[\underline{\mu}_{12}^T\underline{\delta}_2]\ [\underline{\mu}_{13}^T\underline{\delta}_2]^2 \end{array}\right]$$

$$+\frac{4[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2} E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]$$

$$-\frac{4[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]}\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \\ -E[\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_2^T\underline{\delta}_2] \end{array}\right] .$$

We have, considering terms of order four in δ,

$$E[\hat{Q}(t)^2] = E[\underline{\delta}_1^T\underline{\delta}_1]^2 + 4E[\underline{\delta}_1^T\underline{\delta}_2]^2 + E[\underline{\delta}_2^T\underline{\delta}_2]^2 + 2E[\underline{\delta}_1^T\underline{\delta}_1]\ E[\underline{\delta}_2^T\underline{\delta}_2]$$

$$
\frac{+\,1}{[\underline{\mu}_{13}^{T}\underline{\mu}_{13}]^{2}}
\Bigg[
E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{4} + E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{4} + 4E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2} + E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{4}
$$

$$
E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{4}
$$

$$
+4\Big[E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}\,E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2} + 2E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2}
$$

$$
+E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}\,E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2}\Big]
$$

$$
+4\Big[E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2} + 2\,E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}][\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}
$$

$$
+E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}\Big]
$$

$$
+4\,E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2}E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}
$$

$$
+4\,[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]^{2}\Big[E[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]^{2} + E[\underline{\delta}_{1}^{T}\underline{\delta}_{3}]^{2} + E\,[\underline{\delta}_{1}^{T}\underline{\delta}_{2}]^{2} + E[\underline{\delta}_{2}^{T}\underline{\delta}_{3}]^{2}\Big]
$$

$$
+2E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2} + 4E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{3}[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]
$$

$$
+2E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2} + 2E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}
$$

$$
+4[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}[\underline{\delta}_{1}^{T}\underline{\delta}_{1}] + 4E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{3}
$$

$$
+2E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2} + 2E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}
$$

$$
+4[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}[\underline{\delta}_{1}\underline{\delta}_{1}]
$$

$$
+4E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2}
$$

$$
+4E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}
$$

$$
+8[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}][\underline{\delta}_{1}^{T}\underline{\delta}_{1}]
$$

$$
+2E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2}E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}
$$

$$
+4[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2}E[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]
$$

$$
+4[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}E[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]
$$

$$\begin{aligned}
&+8[\underline{\mu}_{12}^T\underline{\mu}_{13}]\Big[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2]\Big] \\
&+8[\underline{\mu}_{12}^T\underline{\mu}_{13}]\Big[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]\Big] \\
&+8[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_2^T\underline{\delta}_3]
\end{aligned}$$

$$+\frac{2[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\Big[E[\underline{\delta}_1^T\underline{\delta}_1]^2 + E[\underline{\delta}_1^T\underline{\delta}_3]^2 + E[\underline{\delta}_1^T\underline{\delta}_2]^2 + E[\underline{\delta}_2^T\underline{\delta}_3]^2\Big]$$

$$+\frac{8[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{aligned}
&E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \\
&+E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2] \\
&+E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \\
&+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2] \\
&+E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] \\
&+E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] \\
&+E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] + E[\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_2^T\underline{\delta}_3] \\
&+E[\underline{\mu}_{12}^T\underline{\delta}_3]^2E[\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2] \\
&+E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2] + E[\underline{\mu}_{13}^T\underline{\delta}_2]^2E[\underline{\delta}_1^T\underline{\delta}_1] \\
&+E[\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_2^T\underline{\delta}_3]
\end{aligned}\right]$$

$$-\frac{16[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[\begin{aligned}
&E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] + E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1] \\
&+E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] \\
&+E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] + E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] \\
&+E[\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_2^T\underline{\delta}_3] + E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]E[\underline{\delta}_1^T\underline{\delta}_1]
\end{aligned}\right]$$

$$\frac{+48[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[\begin{array}{l}E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\mu}_{12}^T\underline{\delta}_1]^2+E[\underline{\mu}_{13}^T\underline{\delta}_1]^4+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{12}^T\underline{\delta}_3]^2\\ +E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\mu}_{13}^T\underline{\delta}_2]^2+2E[\underline{\mu}_{13}^T\underline{\delta}_1]^3[\underline{\mu}_{12}^T\underline{\delta}_1]\\ +4E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]\\ +4E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]\\ +E[\underline{\mu}_{13}^T\underline{\delta}_3]^2E[\underline{\mu}_{12}^T\underline{\delta}_1]^2+E[\underline{\mu}_{13}^T\underline{\delta}_3]^2E[\underline{\mu}_{13}^T\underline{\delta}_1]^2\\ +E[\underline{\mu}_{13}^T\underline{\delta}_3]^2[\underline{\mu}_{12}^T\underline{\delta}_3]^2+E[\underline{\mu}_{13}^T\underline{\delta}_3]^2E[\underline{\mu}_{13}^T\underline{\delta}_2]^2\\ +2E[\underline{\mu}_{13}^T\underline{\delta}_3]^2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]\end{array}\right]$$

$$\frac{-16[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\times$$

$$\left[\begin{array}{l}E[\underline{\mu}_{12}^T\underline{\delta}_1]^3[\underline{\mu}_{13}^T\underline{\delta}_1]+E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^3+2E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\mu}_{13}^T\underline{\delta}_1]^2\\ +E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1]E[\underline{\mu}_{13}^T\underline{\delta}_2]^2+E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1]E[\underline{\mu}_{12}^T\underline{\delta}_3]^2\\ +2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\mu}_{12}^T\underline{\delta}_1]^2\\ +E[\underline{\mu}_{13}^T\underline{\delta}_1]^4+2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^3+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{13}^T\underline{\delta}_2]^2\\ +E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{12}^T\underline{\delta}_3]^2+2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1]\\ +2\left[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\mu}_{12}^T\underline{\delta}_3]^2+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{12}^T\underline{\delta}_3]^2\right]\\ +2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\delta_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]\\ +2\left[E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_1]E[\underline{\mu}_{13}^T\underline{\delta}_2]^2+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{13}^T\underline{\delta}_2]^2\right]\\ +2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2]\end{array}\right]$$

$$+2\left[E[\underline{\mu}_{12}^T\underline{\delta}_1]^2E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]+E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]\right]$$

$$+2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]$$

$$+2\left[\begin{array}{l}E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]\\ +E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]\end{array}\right]$$

$$+2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]$$

$$+E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]E[\underline{\mu}_{12}^T\underline{\delta}_1]^2+E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]E[\underline{\mu}_{13}^T\underline{\delta}_1]^2$$

$$+2E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]+E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]E[\underline{\mu}_{13}^T\underline{\delta}_2]^2$$

$$+E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]^3+2[\underline{\mu}_{12}^T\underline{\mu}_3]E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]E[\underline{\delta}_1^T\underline{\delta}_1]$$

$$+2E[\underline{\mu}_{13}^T\underline{\delta}_2]^2E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3]+2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_2^T\underline{\delta}_3]$$

$$\frac{-8[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\times$$

$$\left[\begin{array}{l}E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{12}^T\underline{\delta}_3]^2E[\underline{\delta}_1^T\underline{\delta}_1]\\ +E[\underline{\mu}_{13}^T\underline{\delta}_2]^2E[\underline{\delta}_1^T\underline{\delta}_1]+2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]\\ +4E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]+4E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]\\ +E[\underline{\mu}_{12}^T\underline{\delta}_1]^2E[\underline{\delta}_3^T\underline{\delta}_3]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\delta}_3^T\underline{\delta}_3]\\ +E[\underline{\mu}_{12}^T\underline{\delta}_3]^2[\underline{\delta}_3^T\underline{\delta}_3]+E[\underline{\mu}_{13}^T\underline{\delta}_2]^2E[\underline{\delta}_3^T\underline{\delta}_3]\\ +2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\delta}_3^T\underline{\delta}_3]\end{array}\right]$$

$$\frac{+24[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\times$$

$$\Big[E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}+E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{4}+2E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{3}$$

$$+E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2}+E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}$$

$$+2[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]$$

$$+4\ \times$$

$$\Big[E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]\ [\underline{\mu}_{12}^{T}\underline{\delta}_{3}][\underline{\mu}_{13}^{T}\underline{\delta}_{3}]$$

$$+E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}][\underline{\mu}_{13}^{T}\underline{\delta}_{3}]\Big]$$

$$+4[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{3}][\underline{\delta}_{1}^{T}\underline{\delta}_{3}]$$

$$+E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}]^{2}E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}+E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}]^{2}E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}$$

$$+2E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}]^{2}E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]+E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}]^{2}E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2}$$

$$+E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}]^{2}[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}+2[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}]^{2}E[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]\Big]$$

$$\frac{-4[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]^{2}}{[\underline{\mu}_{13}^{T}\underline{\mu}_{13}]^{3}}\ \times$$

$$\Big[E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]+E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]+2E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}][\underline{\delta}_{1}^{T}\underline{\delta}_{1}]$$

$$+E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2}E[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]+E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}E[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]+2[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]^{2}$$

$$+4\ \big[E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{12}^{T}\underline{\delta}_{3}][\underline{\delta}_{1}^{T}\underline{\delta}_{3}]+E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}][\underline{\mu}_{12}^{T}\underline{\delta}_{3}][\underline{\delta}_{1}^{T}\underline{\delta}_{3}]\big]$$

$$+4[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\delta}_{1}^{T}\underline{\delta}_{3}]^{2}+E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}E[\underline{\delta}_{3}^{T}\underline{\delta}_{3}]+E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}E[\underline{\delta}_{3}^{T}\underline{\delta}_{3}]$$

$$+2E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]E[\underline{\delta}_{3}^{T}\underline{\delta}_{3}]+E[\underline{\mu}_{13}^{T}\underline{\delta}_{2}]^{2}E[\underline{\delta}_{3}^{T}\underline{\delta}_{3}]$$

$$+E[\underline{\mu}_{12}^{T}\underline{\delta}_{3}]^{2}[\underline{\delta}_{3}^{T}\underline{\delta}_{3}]+2[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]E[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]E[\underline{\delta}_{3}^{T}\underline{\delta}_{3}]\Big]$$

$$\frac{+48[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]^{3}}{[\underline{\mu}_{13}^{T}\underline{\mu}_{13}]^{4}}\ \times$$

$$\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\delta}_3^T\underline{\delta}_3] \\ +2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1] \\ +E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\delta}_3^T\underline{\delta}_3]+2E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3] \\ +2E[\underline{\mu}_{13}^T\underline{\delta}_1]\;[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]+E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]E[\underline{\delta}_1^T\underline{\delta}_1] \\ +E[\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\delta}_3^T\underline{\delta}_3] \end{array}\right]$$

$$\frac{-128[\underline{\mu}_{12}^T\underline{\mu}_{13}]^3}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^5}\times$$

$$\left[\begin{array}{l} E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]^3+3E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\mu}_{13}^T\underline{\delta}_3]^2 \\ +E[\underline{\mu}_{13}^T\underline{\delta}_1]^4+3E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{13}^T\underline{\delta}_3]^2 \\ +3E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]+E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]^3 \end{array}\right]$$

$$\frac{+3[\underline{\mu}_{12}^T\underline{\mu}_{13}]^4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[\begin{array}{l} E[\underline{\delta}_1^T\underline{\delta}_1]^2+4E[\underline{\delta}_1^T\underline{\delta}_3]^2+E[\underline{\delta}_3^T\underline{\delta}_3]^2 \\ +2E[\underline{\delta}_1^T\underline{\delta}_1]E[\underline{\delta}_3^T\underline{\delta}_3] \end{array}\right]$$

$$\frac{+80[\underline{\mu}_{12}^T\underline{\mu}_{13}]^4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^6}\left[\begin{array}{l} E[\underline{\mu}_{13}^T\underline{\delta}_1]^4+6E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\mu}_{13}^T\underline{\delta}_3]^2 \\ +E[\underline{\mu}_{13}^T\underline{\delta}_3]^4 \end{array}\right]$$

$$\frac{-48[\underline{\mu}_{12}^T\underline{\mu}_{13}]^4}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^5}\left[\begin{array}{l} E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\delta}_3^T\underline{\delta}_3] \\ +4E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_3][\underline{\delta}_1^T\underline{\delta}_3]+E[\underline{\mu}_{13}^T\underline{\delta}_3]^2E[\underline{\delta}_1^T\underline{\delta}_1] \\ +E[\underline{\mu}_{13}^T\underline{\delta}_3]^2[\underline{\delta}_3^T\underline{\delta}_3] \end{array}\right]$$

$$\frac{-2[\underline{\mu}_{12}^T\underline{\mu}_{12}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]}\left[E[\underline{\delta}_1^T\underline{\delta}_1]^2+E[\underline{\delta}_1^T\underline{\delta}_3]^2+E[\underline{\delta}_1^T\underline{\delta}_2]^2+E[\underline{\delta}_2^T\underline{\delta}_3]^2\right]$$

$$
+\frac{8(\underline{\mu}_{12}^T\underline{\mu}_{12})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}
\left[
\begin{aligned}
& E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1) \\
& +E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \\
& +E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \\
& +E(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3) \\
& +E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3)E(\underline{\delta}_1^T\underline{\delta}_1)
\end{aligned}
\right]
$$

$$
+\frac{2(\underline{\mu}_{12}^T\underline{\mu}_{12})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}
\left[
\begin{aligned}
& E(\underline{\mu}_{12}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1) \\
& +2E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\delta}_1^T\underline{\delta}_1) \\
& +E(\underline{\mu}_{12}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1)+2(\underline{\mu}_{12}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_1)^2 \\
& +4[E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)] \\
& +4(\underline{\mu}_{12}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_3)^2+E(\underline{\mu}_{12}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3) \\
& +E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3)+2E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)E(\underline{\delta}_3^T\underline{\delta}_3) \\
& +E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\delta}_3^T\underline{\delta}_3)+E(\underline{\mu}_{12}^T\underline{\delta}_3)^2(\underline{\delta}_3^T\underline{\delta}_3) \\
& +2(\underline{\mu}_{12}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_1)E(\underline{\delta}_3^T\underline{\delta}_3)
\end{aligned}
\right]
$$

$$
\frac{-8(\underline{\mu}_{12}^T\underline{\mu}_{12})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}\times
$$

$$
\left[
\begin{aligned}
& E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\mu}_{12}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+2E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)^3 \\
& +E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{12}^T\underline{\delta}_3)^2+2(\underline{\mu}_{12}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)
\end{aligned}
\right]
$$

$$\frac{-8(\underline{\mu}_{12}^T\underline{\mu}_{12})}{(\underline{\mu}_{13}\underline{\mu}_{13})^3} \times$$

$$\left[\begin{array}{l}
+4[E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3)] \\
+4(\underline{\mu}_{12}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{12}^T\underline{\delta}_1)^2 \\
+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1) \\
+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\mu}_{12}^T\underline{\delta}_3)^2+2(\underline{\mu}_{12}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1)
\end{array}\right]$$

$$\frac{-16(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3} \times$$

$$\left[\begin{array}{l}
E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1\ (\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)E(\underline{\delta}_3^T\underline{\delta}_3) \\
+2E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1) \\
+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3)+2E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \\
+2E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3)E(\underline{\delta}_1^T\underline{\delta}_1) \\
+E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_3^T\underline{\delta}_3)
\end{array}\right]$$

$$\frac{+32(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^4} \times$$

$$\left[\begin{array}{l}
E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)^3+3E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 \\
+E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+3E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+3E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3) \\
+E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3)^3
\end{array}\right]$$

$$\frac{-2(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}\left[\begin{array}{l}E(\underline{\delta}_1^T\underline{\delta}_1)^2+4E(\underline{\delta}_1^T\underline{\delta}_3)^2+E(\underline{\delta}_3^T\underline{\delta}_3)^2\\ +2E(\underline{\delta}_1^T\underline{\delta}_1)E(\underline{\delta}_3^T\underline{\delta}_3)\end{array}\right]$$

$$+\frac{24(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\mu}_{12}^T\underline{\mu}_{13})^3}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^4}\left[\begin{array}{l}E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3)\\ +4E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)\ (\underline{\delta}_1^T\underline{\delta}_3)\\ +E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_3^T\underline{\delta}_3)\end{array}\right]$$

$$\frac{-32(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^5}\left[\begin{array}{l}E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+6E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\\ +E(\underline{\mu}_{13}^T\underline{\delta}_3)^4\end{array}\right]$$

$$-\frac{8}{(\underline{\mu}_{13}^T\underline{\mu}_{13})}\times$$

$$\left[\begin{array}{l}E(\underline{\mu}_{12}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)\\ +E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)\\ +E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+E(\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_2)E(\underline{\delta}_1^T\underline{\delta}_1)\\ +E(\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)\end{array}\right]$$

$$+\frac{8}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\left[\begin{array}{l}E(\underline{\mu}_{12}^T\underline{\delta}_1)^3(\underline{\mu}_{13}^T\underline{\delta}_1)+E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)^3+2E(\underline{\mu}_{12}^T\underline{\delta}_1)^2(\underline{\mu}_{13}^T\underline{\delta}_1)^2\\ +E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)E(\underline{\mu}_{12}^T\underline{\delta}_3)^2\\ +2(\underline{\mu}_{12}^T\underline{\mu}_{13})E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)\\ +2\ [E(\underline{\mu}_{12}^T\underline{\delta}_1)^2E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3)+E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3)]\\ +2\ [E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_1)E(\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_2)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_2)]\\ +2(\underline{\mu}_{12}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+2E(\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_2)E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3)\\ +2(\underline{\mu}_{12}^T\underline{\mu}_{13})E(\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)\\ +2(\underline{\mu}_{12}^T\underline{\mu}_{13})E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)\end{array}\right]$$

$$\frac{+8[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]}{[\underline{\mu}_{13}^{T}\underline{\mu}_{13}]^{2}}\times$$

$$\left[\begin{array}{l} E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]+E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}][\underline{\delta}_{1}^{T}\underline{\delta}_{1}]+2E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{12}^{T}\underline{\delta}_{3}][\underline{\delta}_{1}^{T}\underline{\delta}_{3}] \\ +2E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}E[\underline{\delta}_{3}^{T}\underline{\delta}_{3}]+E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]E[\underline{\delta}_{3}^{T}\underline{\delta}_{3}] \\ +E[\underline{\mu}_{12}^{T}\underline{\delta}_{2}][\underline{\mu}_{13}^{T}\underline{\delta}_{2}]E[\underline{\delta}_{1}^{T}\underline{\delta}_{1}]+E[\underline{\mu}_{12}^{T}\underline{\delta}_{2}][\underline{\mu}_{13}^{T}\underline{\delta}_{2}]E[\underline{\delta}_{3}^{T}\underline{\delta}_{3}] \end{array}\right]$$

$$\frac{-32[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]}{[\underline{\mu}_{13}^{T}\underline{\mu}_{13}]^{3}}\times$$

$$\left[\begin{array}{l} E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}+E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{3}+2E]\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}][\underline{\mu}_{12}^{T}\underline{\delta}_{3}] \\ +E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}]^{2}E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}]^{2}+E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}]^{2} \\ +E[\underline{\mu}_{12}^{T}\underline{\delta}_{2}][\underline{\mu}_{13}^{T}\underline{\delta}_{2}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{2}+E[\underline{\mu}_{12}^{T}\underline{\delta}_{2}][\underline{\mu}_{13}^{T}\underline{\delta}_{2}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}]^{2} \end{array}\right]$$

$$\frac{-16[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]^{2}}{[\underline{\mu}_{13}^{T}\underline{\mu}_{13}]^{3}}\times$$

$$\left[\begin{array}{l} E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}][\underline{\delta}_{1}^{T}\underline{\delta}_{1}]+E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]E[\underline{\delta}_{3}^{T}\underline{\delta}_{3}] \\ +2[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{3}][\underline{\delta}_{1}^{T}\underline{\delta}_{3}] \end{array}\right]$$

$$\frac{+32[\underline{\mu}_{12}^{T}\underline{\mu}_{13}]^{2}}{[\underline{\mu}_{13}^{T}\underline{\mu}_{13}]^{4}}\left[E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]^{3}+3E[\underline{\mu}_{12}^{T}\underline{\delta}_{1}][\underline{\mu}_{13}^{T}\underline{\delta}_{1}]E[\underline{\mu}_{13}^{T}\underline{\delta}_{3}]^{2}\right]$$

$$\frac{-2}{[\underline{\mu}_{13}^{T}\underline{\mu}_{13}]}\times$$

$$
\begin{aligned}
\Bigg[\; & E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1]+2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1] \\
& +E[\underline{\mu}_{13}^T\underline{\delta}_2]^2E[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{12}^T\underline{\delta}_3]^2E[\underline{\delta}_1^T\underline{\delta}_1]+2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\delta}_1^T\underline{\delta}_1]^2 \\
& +4\,\big[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2]+E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_1]\big] \\
& +4[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\delta}_1^T\underline{\delta}_2]^2+E[\underline{\mu}_{12}^T\underline{\delta}_1]^2E[\underline{\delta}_2^T\underline{\delta}_2]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\delta}_2^T\underline{\delta}_2] \\
& +2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\delta}_2^T\underline{\delta}_2]+E[\underline{\mu}_{13}^T\underline{\delta}_2]^2[\underline{\delta}_2^T\underline{\delta}_2]+E[\underline{\mu}_{12}^T\underline{\delta}_3]^2E[\underline{\delta}_2^T\underline{\delta}_2] \\
& +2[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\delta}_1^T\underline{\delta}_1]E[\underline{\delta}_2^T\underline{\delta}_2] \;\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{8[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2} \\
& \times\Bigg[\; E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}\underline{\delta}_3]E[\underline{\delta}_1^T\underline{\delta}_1] \\
& \quad +2E[\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2]+E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]E[\underline{\delta}_2^T\underline{\delta}_2] \\
& \quad +E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\delta}_2^T\underline{\delta}_2]+E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]E[\underline{\delta}_2^T\underline{\delta}_2] \;\Bigg]
\end{aligned}
$$

$$
+\frac{2[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}
\begin{bmatrix}
E[\underline{\delta}_1^T\underline{\delta}_1]^2+E[\underline{\delta}_1^T\underline{\delta}_1]E[\underline{\delta}_3^T\underline{\delta}_3]+E[\underline{\delta}_1^T\underline{\delta}_1]E[\underline{\delta}_2^T\underline{\delta}_2] \\
+E[\underline{\delta}_2^T\underline{\delta}_2]E[\underline{\delta}_3^T\underline{\delta}_3]
\end{bmatrix}
$$

$$
\frac{-8[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}
\begin{bmatrix}
E[\underline{\mu}_{13}^T\underline{\delta}_1]^2[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\mu}_{13}^T\underline{\delta}_3]^2E[\underline{\delta}_1^T\underline{\delta}_1] \\
+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2E[\underline{\delta}_2^T\underline{\delta}_2]+E[\underline{\mu}_{13}^T\underline{\delta}_3]^2E[\underline{\delta}_2^T\underline{\delta}_2]
\end{bmatrix}.
$$

(iii) $\underline{E\,\hat{t}}$

We have, uptil terms of order two in δ,

$$
E\hat{t}=\frac{[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]}-\frac{[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\Big[E[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\delta}_3^T\underline{\delta}_3]\Big]
$$

$$+\frac{4[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1]^2+E[\underline{\mu}_{13}^T\underline{\delta}_3]^2\right]$$

$$\frac{-2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\;E[\underline{\mu}_{12}^T\underline{\delta}_1]\;\;[\underline{\mu}_{13}^T\underline{\delta}_1]+E(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_3)+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2\right]$$

$$+\frac{1}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]}\;E[\underline{\delta}_1^T\underline{\delta}_1]\;.$$

(iv) $\underline{E[\hat{t}]^2}$

We have, uptil terms of order two in δ,

$$E[\hat{t}]^2 = \frac{[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}$$

$$+12\,\frac{[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^4}\left[E[\underline{\mu}_{13}^T\underline{\delta}_1]^2+E[\underline{\mu}_{13}^T\underline{\delta}_3]^2\right]$$

$$-2\,\frac{[\underline{\mu}_{12}^T\underline{\mu}_{13}]^2}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[\;E[\underline{\delta}_1^T\underline{\delta}_1]+E[\underline{\delta}_3^T\underline{\delta}_3]\;\right]$$

$$\frac{-8[\underline{\mu}_{12}^T\underline{\mu}_{13}]}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^3}\left[E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2+E[\underline{\mu}_{12}^T\underline{\delta}_3][\underline{\mu}_{13}^T\underline{\delta}_3]\right]$$

$$\frac{+1}{[\underline{\mu}_{13}^T\underline{\mu}_{13}]^2}\left[\begin{array}{l}E[\underline{\mu}_{12}^T\underline{\delta}_1]^2+E[\underline{\mu}_{13}^T\underline{\delta}_1]^2+2E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]+E[\underline{\mu}_{13}^T\underline{\delta}_2]^2\\ +E[\underline{\mu}_{12}^T\underline{\delta}_3]^2+2\;[\underline{\mu}_{12}^T\underline{\mu}_{13}]E[\underline{\delta}_1^T\underline{\delta}_1]\end{array}\right]\;.$$

4.3 Elaboration of terms comprising the expressions of means and variance of $\hat{t}$ and $Q(\hat{t})$ in section 4.2

In section 4.2 a few of the typical terms were

(1) $E\underline{\delta}_1^T\underline{\delta}_1$

(2) $E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1]$

(3) $E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]$

(4) $E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\mu}_{13}^T\underline{\delta}_1]$

(5) $E[\underline{\mu}_{12}^T\underline{\delta}_1]^2[\underline{\mu}_{13}^T\underline{\delta}_1]^2$

(6) $E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_1][\underline{\delta}_1^T\underline{\delta}_1]$

(7) $E[\underline{\mu}_{12}^T\underline{\delta}_1][\underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_2]$.

All the terms occuring in the expressions of section 4.2 belong to one of the above-mentioned seven types. In this section we express the types explicitly.

$$E(\delta_1^T\delta_1) = E\sum_{i=1}^{m}[\hat{f}_1(x_i)-E\hat{f}_1(x_i)]^2$$

$$= \sum_{i=1}^{m} E[\hat{f}_1(x_i)-E\hat{f}_1(x_i)]^2 = \sum_{i=1}^{m} \text{var } \hat{f}_1(x_i).$$

We have

$$E(\underline{\mu}_{12}^{T}\underline{\delta}_{1})(\underline{\mu}_{13}^{T}\underline{\delta}_{1})$$

$$= E\left[\sum_{i=1}^{m}[E\hat{f}_1(x_i)-E\hat{f}_2(x_i)][\hat{f}_1(x_i)-E\hat{f}_1(x_i)) \times \sum_{i=1}^{m}[E\hat{f}_1(x_i)-E\hat{f}_3(x_i)][\hat{f}_1(x_i)-E\hat{f}_1(x_i))\right]$$

$$=\sum_i\sum_j\left[[E\hat{f}_1(x_i)-E\hat{f}_2(x_i)][E\hat{f}_1(x_j)-E\hat{f}_3(x_j)] \times E\ [\hat{f}_1(x_i)-E\hat{f}_1(x_i)][\hat{f}_1(x_j)-E\hat{f}_1(x_j)]\right]$$

$$=\sum_i\sum_j\left[[E\hat{f}_1(x_i)-E\hat{f}_2(x_i)][E\hat{f}_1(x_j)-E\hat{f}_3(x_j)] \times [E[\hat{f}_1(x_i)\hat{f}_1(x_j)] - E\hat{f}_1(x_i)E\hat{f}_1(x_j)]\right].$$

Similarly,

$$E(\underline{\mu}_{12}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{1}) = \sum_i\sum_j\left[[E\hat{f}_1(x_i)-E\hat{f}_2(x_i)] \times E\left[[\hat{f}_1(x_i)-E\hat{f}_1(x_i)][\hat{f}_1(x_j)-E\hat{f}_1(x_j)]^2\right]\right].$$

Similarly,

$$E(\underline{\mu}_{12}^{T}\underline{\delta}_{1})^2(\underline{\mu}_{13}^{T}\underline{\delta}_{1})$$

$$=\sum_i\sum_j\sum_k\left[[E\hat{f}_1(x_i)-E\hat{f}_2(x_i)][E\hat{f}_1(x_j)-E\hat{f}_2(x_j)][E\hat{f}_1(x_k)-E\hat{f}_3(x_k)]\ E\left[[\hat{f}_1(x_i)-E\hat{f}_1(x_i)][\hat{f}_1(x_j)-E\hat{f}_1(x_j)][\hat{f}_1(x_k)-E\hat{f}_1(x_k)]\right]\right].$$

Similarly,

$$E(\underline{\mu}_{12}^T\underline{\delta}_1)^2(\underline{\mu}_{13}^T\underline{\delta}_1)^2$$

$$= \sum_{ijk\ell}\left[\begin{array}{l}[E\hat{f}_1(x_i)-E\hat{f}_2(x_i)][E\hat{f}_1(x_j)-E\hat{f}_2(x_j)][E\hat{f}_1(x_k)-E\hat{f}_3(x_k)] \\ \times[E\hat{f}_1(x_\ell)-E\hat{f}_3(x_\ell)] \\ \times\ E\left[\begin{array}{l}[\hat{f}_1(x_i)-E\hat{f}_1(x_i)][\hat{f}_1(x_j)-E\hat{f}_1(x_j)] \\ \times[\hat{f}_1(x_k)-E\hat{f}_1(x_k)][\hat{f}_1(x_\ell)-E\hat{f}_1(x_\ell)]\end{array}\right]\end{array}\right].$$

Similarly,

$$E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)$$

$$= \sum_{ijk}\left[\begin{array}{l}[E\hat{f}_1(x_i)-E\hat{f}_2(x_i)]\ [E\hat{f}_1(x_j-E\hat{f}_3(x_j)] \\ \times E\left[\begin{array}{l}[\hat{f}_1(x_i)-E\hat{f}_1(x_i)]\times\quad[\hat{f}_1(x_k)-E\hat{f}_1(x_k)]^2 \\ \times[\hat{f}_1(x_j)-E\hat{f}_1(x_j)]\end{array}\right]\end{array}\right].$$

Finally,

$$E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)$$

$$= \sum_{ijk}\left[\begin{array}{l}[E\hat{f}_1(x_i)-E\hat{f}_2(x_i)][E\hat{f}_1(x_j)-E\hat{f}_3(x_j)] \\ \times E\left[\begin{array}{l}[\hat{f}_1(x_i)-E\hat{f}_1(x_i)][\hat{f}_2(x_j)-E\hat{f}_2(x_j)] \\ \times[\hat{f}_1(x_k)-E\hat{f}_1(x_k)][\hat{f}_2(x_k)-E\hat{f}_2(x_k)]\end{array}\right]\end{array}\right]$$

$$= \sum\sum\sum_{ijk} \left[[E\hat{f}_1(x_i)-E\hat{f}_2(x_i)][E\hat{f}_1(x_j)-E\hat{f}_3(x_j)] \times E \left[[\hat{f}_1(x_i)-E\hat{f}_1(x_i)][\hat{f}_2(x_j)-E\hat{f}_2(x_j)] \times [\hat{f}_1(x_k)-E\hat{f}_1(x_k)][\hat{f}_2(x_k)-E\hat{f}_2(x_k)] \right] \right]$$

$$= \sum\sum\sum_{ijk} \left[[E\hat{f}_1(x_i)-E\hat{f}_2(x_i)][E\hat{f}_1(x_j)-E\hat{f}_3(x_j)] \times E \left[[\hat{f}_1(x_i)-E\hat{f}_1(x_i)][\hat{f}_1(x_k)-E\hat{f}_1(x_k)] \right] \times E \left[[\hat{f}_2(x_j)-E\hat{f}_2(x_j)][\hat{f}_2(x_k)-E\hat{f}_2(x_k)] \right] \right].$$

Thus, it follows that the expressions for the mean and variance of $\hat{t}$ and $Q(\hat{t})$ are combinations of four basic quantities, which, written for f_1, are

$E\hat{f}_1(x_i), E[\hat{f}_1(x_i)\hat{f}_1(x_j)], E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)]$ and

$E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)\hat{f}_1(x_\ell)]$.

In section 4.4 we will evaluate these quantities with the help of the kernel (a normal kernel will be assumed) of the density estimate.

4.4 Explicit expressions for the basic quantities comprising mean and variance of $\hat{t}$ and $Q(\hat{t})$.

(i) $E\hat{f}_1(x_i)$

We have $E\hat{f}_1(x_i) = E \sum_{p=1}^{n} \frac{1}{nh} k\left[\frac{x_i - v_p}{h}\right]$ where $i \epsilon [1,2,......m]$

and $p \epsilon [1,2,.....n]$. The x_i's are the chosen values of the random variable used to construct $Q(\hat{t})$ and the v_p's are the data points used to construct the density estimate of f_1.

Thus $\hat{Ef_1}(x_i) = \frac{1}{h} \int k \left[\frac{x_i - y}{h}\right] f_1(y)dy$

$$= \frac{1}{h} \int k \left[\frac{x_i - y}{h}\right] f_1(y)dy.$$

where $f_i = p_i N(\mu_1, \sigma_1^2) + (1-p_i) N(\mu_2, \sigma_2^2)$,
$i = 1,2,3$

Using the definition of the function k, $\left[k(x) = \frac{1}{\sqrt{2\Pi}} e^{-(1/2)x^2} \right]$

we have,

$$\hat{Ef_1}(x_i) = \frac{1}{h} \int \frac{1}{\sqrt{2\Pi}} e^{-\frac{1}{2}\left[\frac{x_i - y}{h}\right]^2} \left[\frac{p_1}{\sqrt{2\Pi}\sigma_1} e^{\frac{-(y-\mu_1)^2}{2\sigma_1^2}} + \frac{(1-p_1)}{\sqrt{2\Pi}\sigma_2} e^{\frac{-(y-\mu_2)^2}{2\sigma_2^2}} \right] dy. \quad (11)$$

Consider part of the expression on the right hand side of (11).We have

$$\frac{p_1}{2\Pi h\sigma_1} \int e^{-\frac{1}{2}\left[\frac{x_i - y}{h}\right]^2 - \frac{1}{2}\left[\frac{y - \mu_1}{\sigma_1}\right]^2} dy.$$

$$\frac{p_1}{2\Pi h\sigma_1} \int e^{\frac{-x_i^2}{2h^2} - \frac{\mu_1^2}{2\sigma_1^2}} \times e^{\left[\frac{x_i}{h^2} + \frac{\mu_1}{\sigma_1^2}\right] y} \times e^{\frac{-y^2}{2h^2} - \frac{y^2}{2\sigma_1^2}} dy$$

$$= \frac{p_1}{2\Pi h\sigma_1} e^{\frac{-x_i^2}{2h^2} - \frac{\mu_1^2}{2\sigma_1^2}} \int e^{-\frac{1}{2}\left[\frac{1}{h^2} + \frac{1}{\sigma_1^2}\right] y^2 + \left[\frac{x_i}{h^2} + \frac{\mu_1}{\sigma_1^2}\right] y} dy. \quad (12)$$

The integral in (12) can be written as

$$e^{\frac{(\sigma_1^2 x_i + h^2 \mu_1)^2}{2h^2 \sigma_1^2 (\sigma_1^2 + h^2)}} \int e^{\frac{-(\sigma_1^2 + h^2)}{2h^2 \sigma_1^2} \left[y - \frac{\sigma_1^2 x_i + h^2 \mu_1}{\sigma_1^2 + h^2} \right]^2} dy \,. \qquad (13)$$

In the integral in (13) change the variable y to w by the substitutions

$$v = \quad y - \frac{\sigma_1^2 x_i + h^2 \mu_1}{\sigma_1^2 + h^2}$$

and $$w = \frac{v\sqrt{\sigma_1^2 + h^2}}{h\sigma_1} \,.$$

Thus, the integral in (13) becomes

$$\frac{\sigma_1 h}{\sqrt{\sigma_1^2 + h^2}} \int e^{-\frac{1}{2}w^2} dw = \frac{\sigma_1 h \sqrt{2\Pi}}{\sqrt{\sigma_1^2 + h^2}} \,.$$

Hence (12) becomes

$$\frac{p_1}{2\Pi h \sigma_1} \times \frac{\sigma_1 h \sqrt{2\Pi}}{\sqrt{\sigma_1^2 + h^2}} e^{\frac{-x_i^2}{2h^2} - \frac{\mu_1^2}{2\sigma_1^2}} \times e^{\frac{(\sigma_1^2 x_i + h^2 \mu_1)^2}{2h^2 \sigma_1^2 (\sigma_1^2 + h^2)}}$$

$$= \frac{p_1}{\sqrt{2\Pi}\sqrt{\sigma_1 + h^2}} e^{\frac{-(x_i - \mu_1)^2}{2(\sigma_1^2 + h^2)}} \,.$$

Thus (11) can be written as

$$E\hat{f}_1(x_i) = \frac{1}{\sqrt{2\Pi}} \left[\frac{p_1}{\sqrt{\sigma_1^2+h^2}} e^{\frac{-(x_i-\mu_1)^2}{2(\sigma_1^2+h^2)}} + \frac{(1-p_1)}{\sqrt{\sigma_2^2+h^2}} e^{\frac{-(x_i-\mu_2)^2}{2(\sigma_2^2+h^2)}} \right] . \quad (14)$$

Similarly for $E\hat{f}_2(x_i)$ and $E\hat{f}_3(x_i)$.

(ii) $\underline{E[\hat{f}_1(x_i)\hat{f}_1(x_j)]}$

We have

$$E[\hat{f}_1(x_i)\hat{f}_1(x_j)] = E\left[\sum_r \frac{1}{nh} k\left[\frac{x_i-v_r}{h}\right] \sum_s \frac{1}{nh} k\left[\frac{x_j-v_s}{h}\right]\right]$$

where $i,j\varepsilon[1,2,\ldots..m]$ and $r,s\varepsilon[1,2,\ldots.n]$.

For brevity, we will write $k\left[\frac{x_i-v_r}{h}\right]$ as k_{ir}, $k\left[\frac{x_i-y}{h}\right]$ as k_i and $Ek\left[\frac{x_i-v_r}{h}\right]$ as Ek_{ir} etc. .

Thus $E[\hat{f}_1(x_i)\ \hat{f}_1(x_j)] = E[\sum_r \frac{1}{nh} k_{ir} \sum_s \frac{1}{nh} k_{js}]$

$$= \sum_r \sum_s \frac{1}{n^2h^2} E[k_{ir}k_{js}] = \sum_r \frac{1}{n^2h^2} E[k_{ir}k_{jr}] + \sum_{r\neq s}\sum \frac{1}{n^2h^2} Ek_{ir}Ek_{js}$$

$$= \frac{1}{nh^2}\int k_ik_jf_1(y)dy + \frac{1}{n^2} \sum_{r\neq s}\sum E(\frac{1}{h} k_{ir})E(\frac{1}{h}k_{js})$$

$$= \frac{1}{nh^2}\int k_ik_jf_1(y)dy + \frac{1}{n^2} \sum_{r\neq s}\sum E\hat{f}_1(x_i)E\hat{f}_1(x_j)$$

$$= \frac{1}{nh^2}\int k_ik_jf_1(y)dy + \frac{n(n-1)}{n^2} E\hat{f}_1(x_i)E\hat{f}_1(x_j)$$

$$= \frac{1}{nh^2}\int k_ik_jf_1(y)dy + (1-\frac{1}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_j). \quad (15)$$

In (15), $\int k_ik_jf_1(y)dy$ is given by

$$\int k_i k_j f_1(y)\,dy = \frac{h}{2\Pi}\left[\frac{p_1}{\sqrt{2\sigma_1^2+h^2}}\, e^{\frac{-\sigma_1^2(x_i-x_j)^2}{2h^2(2\sigma^2+h^2)} - \frac{[(x_i-\mu_1)^2+(x_j-\mu_1)]^2}{2(2\sigma^2+h^2)}} \frac{+(1-p_1)e^{\frac{-\sigma_2^2(x_i-x_j)^2}{2h^2(2\sigma_2^2+h^2} - \frac{[(x_i-\mu_2)^2+(x_j-\mu_2)^2}{2(2\sigma_2^2+h^2)}}}{\sqrt{2\sigma_2^2+h^2}}\right].$$

(iii) $E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)]$

We have

$$E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)] = E[\sum_q \frac{1}{nh}k_{iq} \sum_r \frac{1}{nh} k_{jr} \sum_s \frac{1}{nh} k_{ks}]$$

$$= \frac{1}{n^3h^3} \sum_q \sum_r \sum_s E[k_{iq}k_{jr}k_{ks}]$$

$$= \frac{1}{n^3h^3} \sum_q E[k_{iq}k_{jq}k_{kq}] + \frac{1}{n^3h^3} \sum_{q\neq r}\sum E[k_{iq}k_{jq}]Ek_{kr}$$

$$+ \frac{1}{n^3h^3} \sum_{q\neq r}\sum E[k_{iq}k_{kq}]Ek_{jr} + \frac{1}{n^3h^3} \sum_{q\neq r}\sum E[k_{jq}k_{kq}]Ek_{ir}$$

$$+ \frac{1}{n^3h^3} \sum_{q\neq r\neq s}\sum\sum Ek_{iq}Ek_{jr}Ek_{ks}$$

$$= \frac{1}{n^2h^3}\int k_ik_jk_kf_1(y)\,dy + \frac{n(n-1)}{n^2}\left[\frac{1}{nh^2}\int k_ik_jf_1(y)\,dy\right] E\hat{f}_1(x_k)$$

$$+\frac{n(n-1)}{n^2}\left[\frac{1}{nh^2}\int k_ik_kf_1(y)\,dy\right] E\hat{f}_1(x_j) + \frac{n(n-1)}{n^2}\left[\frac{1}{nh^2}\int k_jk_kf_1(y)\,dy\right] E\hat{f}_1(x_i)$$

$$+\frac{n(n-1)(n-2)}{n^3} E\hat{f}_1(x_i)E\hat{f}_1(x_j)E\hat{f}_1(x_k) \cdot \qquad (16)$$

Using (15), (16) can be written as

$$E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)]$$

$$= \frac{1}{n^2h^3}\int k_ik_jk_kf_1(y)dy$$

$$+\left[1-\frac{1}{n}\right][E\hat{f}_1(x_k)][E\ \hat{f}_1(x_i)\hat{f}_1(x_j) - \left[1-\frac{1}{n}\right] E\hat{f}_1(x_i)E\hat{f}_1(x_j)]$$

$$+\left[1-\frac{1}{n}\right][E\hat{f}_1(x_j)][E\ \hat{f}_1(x_i)\hat{f}_1(x_k) - \left[1-\frac{1}{n}\right] E\hat{f}_1(x_i)E\hat{f}_1(x_k)]$$

$$+\left[1-\frac{1}{n}\right][E\hat{f}_1(x_i)][E\ \hat{f}_1(x_j)\hat{f}_1(x_k) - \left[1-\frac{1}{n}\right] E\hat{f}_1(x_j)E\hat{f}_1(x_k)]$$

$$+\left[1-\frac{1}{n}\right]\left[1-\frac{2}{n}\right] E\hat{f}_1(x_i)E\hat{f}_1(x_j)E\hat{f}_1(x_k).$$

Thus

$$E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)]$$

$$= \frac{1}{n^2h^3}\int k_ik_jk_kf_1(y)dy$$

$$+\left[1-\frac{1}{n}\right]\left[\begin{matrix}E\hat{f}_1(x_i)E[\hat{f}_1(x_j)\hat{f}_1(x_k)]+E\hat{f}_1(x_j)E[\hat{f}_1(x_k)\hat{f}_1(x_i)] \\ +E\hat{f}_1(x_k)E[\hat{f}_1(x_i)\hat{f}_1(x_j)]\end{matrix}\right]$$

$$-\left[1-\frac{1}{n}\right]\left[2-\frac{1}{n}\right] E\hat{f}_1(x_i)E\hat{f}_1(x_j)E\hat{f}_1(x_k). \qquad (17)$$

In (17), $\int k_ik_jk_kf_1(y)dy$ is given by

$$\int k_i k_j k_k f_1(y)dy$$

$$= \frac{h}{(2\Pi)^{3/2}} \left[\frac{p_1}{\sqrt{3\sigma_1^2+h^2}} e^{\frac{-\sigma_1^2[(x_i-x_j)^2+(x_i-x_k)^2+(x_j-x_k)^2]}{2h^2(3\sigma_1^2+h^2)}} \times e^{\frac{-[(x_i-\mu_1)^2+(x_j-\mu_1)^2+(x_k-\mu_1)^2]}{2(3\sigma_1^2+h^2)}} + \frac{(1-p_1)}{\sqrt{3\sigma_2^2+h^2}} e^{\frac{-\sigma_2^2[(x_i-x_j)^2+(x_i-x_k)^2+(x_j-x_k)^2]}{2h^2(3\sigma_2^2+h^2)}} \times e^{\frac{-[(x_i-\mu_2)^2+(x_j-\mu_2)^2+(x_k-\mu_2)^2]}{2(3\sigma_2^2+h^2)}} \right] . \quad (18)$$

(iv) $\underline{E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)\hat{f}_1(x_\ell)]}$

We have

$$E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)\hat{f}_1(x_\ell)]$$

$$= E\left[\sum_p \frac{1}{nh} k_{ip} \sum_q \frac{1}{nh} k_{jq} \sum_r \frac{1}{nh} k_{kr} \sum_s \frac{1}{nh} k_{\ell s}\right]$$

$$= \sum_p \sum_q \sum_r \sum_s \frac{1}{n^4h^4} E[k_{ip}k_{jq}k_{kr}k_{\ell s}]$$

$$= \frac{1}{n^4h^4} \left[\sum_p E[k_{ip}k_{jp}k_{kp}k_{\ell p}] + \sum_{p\neq q}\sum E[k_{ip}k_{jp}k_{kp}]Ek_{\ell q} + \sum_{p\neq q}\sum E[k_{ip}k_{jp}k_{\ell p}]Ek_{kq} + \sum_{p\neq q}\sum E[k_{ip}k_{kp}k_{\ell p}]Ek_{jq} + \sum_{p\neq q}\sum E[k_{jp}k_{kp}k_{\ell p}]Ek_{iq} + \sum_{p\neq q\neq r}\sum\sum E[k_{ip}k_{jp}]Ek_{kq}Ek_{\ell r} \right]$$

$$+\frac{1}{n^4h^4}\times$$

$$\Big[+\sum_{p\neq q\neq r}\sum\sum E[k_{ip}k_{kp}]Ek_{jq}Ek_{\ell r}+\sum_{p\neq q\neq r}\sum\sum E[k_{ip}k_{\ell p}]Ek_{jq}Ek_{kr}$$

$$+\sum_{p\neq q\neq r}\sum\sum E[k_{jp}k_{kp}]Ek_{iq}Ek_{\ell r}+\sum_{p\neq q\neq r}\sum\sum E[k_{jp}k_{\ell p}]Ek_{iq}Ek_{kr}$$

$$+\sum_{p\neq q\neq r}\sum\sum E[k_{kp}k_{\ell p}]Ek_{iq}Ek_{jr}+\sum_{p\neq q}\sum E[k_{ip}k_{jp}]E[k_{kq}k_{\ell q}]$$

$$+\sum_{p\neq q}\sum E[k_{ip}k_{kp}]E[k_{jq}k_{\ell q}]+\sum_{p\neq q}\sum E[k_{ip}k_{\ell p}]E[k_{jq}k_{kq}]$$

$$+\sum_{p\neq q\neq r\neq s}\sum\sum\sum Ek_{ip}E_{jq}Ek_{kr}k_{\ell s}\Big].$$

Using (17), we have

$$E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)\hat{f}_1(x_\ell)]$$

$$=\frac{1}{n^3h^4}\int k_ik_jk_kk_\ell f_1(y)dy+\frac{n(n-1)}{n^2}\left[\frac{1}{n^2h^3}\int k_ik_jk_kf_1(y)dy\right]E\hat{f}_1(x_\ell)$$

$$+\frac{n(n-1)}{n^2}\left[\frac{1}{n^2h^3}\int k_ik_jk_\ell f_1(y)dy\right]E\hat{f}_1(x_k)+\frac{n(n-1)}{n^2}\left[\frac{1}{n^2h^3}\int k_ik_kk_\ell f_1(y)dy\right]$$

$$\times E\hat{f}_1(x_j)$$

$$+\frac{n(n-1)}{n^2}\left[\frac{1}{n^2h^3}\int k_jk_kk_\ell f_1(y)dy\right]E\hat{f}_1(x_i)$$

$$+\frac{n(n-1)(n-2)}{n^3}\left[\frac{1}{nh^2}\int k_ik_jf_1(y)dy\right]E\hat{f}_1(x_\ell)E\hat{f}_1(x_k)$$

$$+\frac{n(n-1)(n-2)}{n^3}\left[\frac{1}{nh^2}\int k_ik_kf_1(y)dy\right]E\hat{f}_1(x_j)E\hat{f}_1(x_\ell)$$

$$+\frac{n(n-1)(n-2)}{n^3}\left[\frac{1}{nh^2}\int k_ik_\ell f_1(y)dy\right]E\hat{f}_1(x_j)E\hat{f}_1(x_k)$$

$$+\frac{n(n-1)(n-2)}{n^3}\left[\frac{1}{nh^2}\int k_jk_kf_1(y)dy\right]E\hat{f}_1(x_i)E\hat{f}_1(x_\ell)$$

$$+\frac{n(n-1)(n-2)}{n^3}\left[\frac{1}{nh^2}\int k_jk_\ell f_1(y)dy\right]E\hat{f}_1(x_i)E\hat{f}_1(x_k)$$

$$+\frac{n(n-1)(n-2)}{n^3}\left[\frac{1}{nh^2}\int k_k k_\ell \hat{f}_1(y)dy\right]E\hat{f}_1(x_i)Ef_1(x_j)$$

$$+\frac{n(n-1)}{n^2}\left[\frac{1}{nh^2}\int k_i k_j f_1(y)dy\right]\left[\frac{1}{nh^2}\int k_k k_\ell f_1(y)dy\right]$$

$$+\frac{n(n-1)}{n^2}\left[\frac{1}{nh^2}\int k_i k_k f_1(y)dy\right]\left[\frac{1}{nh^2}\int k_j k_\ell f_1(y)dy\right]$$

$$+\frac{n(n-1)}{n^2}\left[\frac{1}{nh^2}\int k_i k_\ell f_1(y)dy\right]\left[\frac{1}{nh^2}\int k_j k_k f_1(y)dy\right]$$

$$+\frac{n(n-1)(n-2)(n-3)}{n^4}\, E\hat{f}_1(x_i)E\hat{f}_1(x_j)E\hat{f}_1(x_k)E\hat{f}_1(x_\ell),$$

$$=\frac{1}{n^3 h}\int k_i k_j k_k k_\ell f_1(y)dy$$

$$+\frac{n(n-1)}{n^2}\Bigg[E\hat{f}_1(x_\ell)\Bigg[E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)] - (1-\frac{1}{n})\Big[E\hat{f}_1(x_i)E[\hat{f}_1(x_j)\hat{f}_1(x_k)] + E\hat{f}_1(x_j)E[\hat{f}_1(x_k)\hat{f}_1(x_i)] + E\hat{f}_1(x_k)E[\hat{f}_1(x_i)\hat{f}_1(x_j)]\Big] + (1-\frac{1}{n})(2-\frac{1}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_j)E\hat{f}_1(x_k)\Bigg]$$

$$+E\hat{f}_1(x_k)\Bigg[E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_\ell)] - (1-\frac{1}{n})\Big[E\hat{f}_1(x_i)E[\hat{f}_1(x_j)\hat{f}_1(x_\ell)] + E\hat{f}_1(x_j)E[\hat{f}_1(x_\ell)\hat{f}_1(x_i)] + E\hat{f}_1(x_\ell)E[\hat{f}_1(x_i)\hat{f}_1(x_j)]\Big] + (1-\frac{1}{n})(2-\frac{1}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_j)E\hat{f}_1(x_\ell)\Bigg]$$

$$\frac{+n(n-1)}{n^2}\Bigg[+E\hat{f}_1(x_j)\Bigg[E[\hat{f}_1(x_i)\hat{f}_1(x_k)\hat{f}_1(x_\ell)] - (1-\tfrac{1}{n})\Big[E\hat{f}_1(x_i)E[\hat{f}_1(x_k)\hat{f}_1(x_\ell)]+E\hat{f}_1(x_k)E[\hat{f}_1(x_i)\hat{f}_1(x_\ell)] + E\hat{f}_1(x_\ell)E[\hat{f}_1(x_i)\hat{f}_1(x_k)]\Big] + (1-\tfrac{1}{n})(1-\tfrac{2}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_k)E\hat{f}_1(x_\ell)\Bigg]$$

$$+E\hat{f}_1(x_i)\Bigg[E[\hat{f}_1(x_j)\hat{f}_1(x_k)\hat{f}_1(x_\ell)] - (1-\tfrac{1}{n})\Big[E\hat{f}_1(x_j)E[\hat{f}_1(x_k)\hat{f}_1(x_\ell)]+E\hat{f}_1(x_k)E[\hat{f}_1(x_j)\hat{f}_1(x_\ell)] + E\hat{f}_1(x_\ell)E[\hat{f}_1(x_j)\hat{f}_1(x_k)]\Big] + (1-\tfrac{1}{n})(1-\tfrac{2}{n})E\hat{f}_1(x_j)E\hat{f}_1(x_k)E\hat{f}_1(x_\ell)\Bigg]\Bigg]$$

$$\frac{+n(n-1)(n-2)}{n^3}\Bigg[E\hat{f}_1(x_i)E\hat{f}_1(x_j)\Big[E[\hat{f}_1(x_k)\hat{f}_1(x_\ell)]-(1-\tfrac{1}{n})E\hat{f}_1(x_k)E\hat{f}_1(x_\ell)\Big]$$

$$+E\hat{f}_1(x_i)E\hat{f}_1(x_k)\Big[E[\hat{f}_1(x_j)\hat{f}_1(x_\ell)]-(1-\tfrac{1}{n})E\hat{f}_1(x_j)E\hat{f}_1(x_\ell)\Big]$$

$$+E\hat{f}_1(x_i)E\hat{f}_1(x_\ell)\Big[E[\hat{f}_1(x_j)\hat{f}_1(x_k)]-(1-\tfrac{1}{n})E\hat{f}_1(x_j)E\hat{f}_1(x_k)\Big]$$

$$+E\hat{f}_1(x_j)E\hat{f}_1(x_k)\Big[E[\hat{f}_1(x_i)\hat{f}_1(x_\ell)]-(1-\tfrac{1}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_\ell)\Big]$$

$$+E\hat{f}_1(x_j)E\hat{f}_1(x_\ell)\Big[E[\hat{f}_1(x_i)\hat{f}_1(x_k)]-(1-\tfrac{1}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_k)\Big]$$

$$+E\hat{f}_1(x_k)E\hat{f}_1(x_\ell)\Big[E[\hat{f}_1(x_i)\hat{f}_1(x_j)]-(1-\tfrac{1}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_j)\Big]\Bigg]$$

$$\frac{+n(n-1)}{n^2}\Bigg[\Big[E[\hat{f}_1(x_i)\hat{f}_1(x_j)]-(1-\tfrac{1}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_j)\Big]$$

$$\times\Big[E[\hat{f}_1(x_k)\hat{f}_1(x_\ell)]-(1-\tfrac{1}{n})E\hat{f}_1(x_k)E\hat{f}_1(x_\ell)\Big]$$

$$+\Big[E[\hat{f}_1(x_i)\hat{f}_1(x_k)]-(1-\tfrac{1}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_k)\Big]$$

$$+\frac{n(n-1)}{n^2}\left[\begin{array}{l}\times\left[E[\hat{f}_1(x_j)\hat{f}_1(x_\ell)]-(1-\frac{1}{n})E\hat{f}_1(x_j)E\hat{f}_1(x_\ell)\right]\\ +\left[E[\hat{f}_1(x_i)\hat{f}_1(x_\ell)]-(1-\frac{1}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_\ell)\right]\\ \times\left[E[\hat{f}_1(x_j)\hat{f}_1(x_k)]-(1-\frac{1}{n})E\hat{f}_1(x_j)E\hat{f}_1(x_k)\right]\end{array}\right]$$

$$+\frac{n(n-1)(n-2)(n-3)}{n^4}\quad E\hat{f}_1(x_i)E\hat{f}_1(x_j)E\hat{f}_1(x_k)E\hat{f}_1(x_\ell)\ .$$

Thus

$$E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)\hat{f}_1(x_\ell)]$$

$$=\frac{1}{n^3h^4}\int k_ik_jk_kk_\ell f_1(y)dy$$

$$+(1-\frac{1}{n})\left[\begin{array}{l}E\hat{f}_1(x_i)E[\hat{f}_1(x_j)\hat{f}_1(x_k)\hat{f}_1(x_\ell)]+E\hat{f}_1(x_j)E[\hat{f}_1(x_i)\hat{f}_1(x_k)\hat{f}_1(x_\ell)]\\ +E\hat{f}_1(x_k)E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_\ell)]+E\hat{f}_1(x_\ell)E[\hat{f}_1(x_i)\hat{f}_1(x_j)\hat{f}_1(x_k)]\end{array}\right]$$

$$+(1-\frac{1}{n})\left[\begin{array}{l}E[\hat{f}_1(x_i)\hat{f}_1(x_j)]E[\hat{f}_1(x_k)\hat{f}_1(x_\ell)]+E[\hat{f}_1(x_i)\hat{f}_1(x_k)]E[\hat{f}_1(x_j)\hat{f}_1(x_\ell)]\\ +E[\hat{f}_1(x_i)\hat{f}_1(x_\ell)]E[\hat{f}_1(x_j)\hat{f}_1(x_k)]\end{array}\right]$$

$$-(1-\frac{1}{n})(2-\frac{1}{n})\left[\begin{array}{l}E[\hat{f}_1(x_i)\hat{f}_1(x_j)]E\hat{f}_1(x_k)E\hat{f}_1(x_\ell)\\ +E[\hat{f}_1(x_i)\hat{f}_1(x_k)]E\hat{f}_1(x_j)E\hat{f}_1(x_\ell)\\ +E[\hat{f}_1(x_i)\hat{f}_1(x_\ell)]E\hat{f}_1(x_j)E\hat{f}_1(x_k)\\ +E[\hat{f}_1(x_j)\hat{f}_1(x_k)]E\hat{f}_1(x_i)E\hat{f}_1(x_\ell)\\ +E[\hat{f}_1(x_j)\hat{f}_1(x_\ell)]E\hat{f}_1(x_i)E\hat{f}_1(x_k)\\ +E[\hat{f}_1(x_k)\hat{f}_1(x_\ell)]E\hat{f}_1(x_i)E\hat{f}_1(x_j)\end{array}\right]$$

$$+(1-\frac{1}{n})(2-\frac{1}{n})(3-\frac{1}{n})E\hat{f}_1(x_i)E\hat{f}_1(x_j)E\hat{f}_1(x_k)E\hat{f}_1(x_\ell). \qquad (19)$$

In (19), $\int k_ik_jk_kk\ f_1(y)dy$ is given by

$$\int k_ik_jk_kk_\ell f_1(y)dy$$

$$= \frac{h}{(2\Pi)^2}\left[\frac{p_1}{\sqrt{4\sigma_1^2+h^2}}\, e^{\dfrac{-\sigma_1^2\left[(x_i-x_j)^2+(x_i-x_k)^2+(x_i-x_\ell)^2+(x_j-x_k)^2+(x_j-x_\ell)+(x_k-x_\ell)^2\right]}{2h^2(4\sigma_1^2+h^2)}} \times e^{\dfrac{-\left[(x_i-\mu_1)^2+(x_j-\mu_1)^2+(x_k-\mu_1)^2+(x_\ell-\mu_1)^2\right]}{2(4\sigma_1^2+h^2)}} + \frac{1-p_1}{\sqrt{4\sigma_2^2+h^2}}\, e^{\dfrac{-\sigma_2^2\left[(x_i-x_j)^2+(x_i-x_k)^2+(x_i-x_\ell)^2+(x_j-x_k)^2+(x_j-x_\ell)^2+(x_k-x_\ell)^2\right]}{2h^2(4\sigma_2^2+h^2)}} \times e^{\dfrac{-\left[(x_i-\mu_2)^2+(x_j-\mu_2)^2+(x_k-\mu_2)^2+(x_\ell-\mu_2)^2\right]}{2(4\sigma_2^2+h^2)}}\right].$$

4.5 Calculation of mean and variance of $\hat{t}$ and $Q(\hat{t})$

In this section we derive the expressions for the mean and variance of $\hat{t}$ and $Q(\hat{t})$ stated in section 4.2.

Consider (10) of section 4.1, i.e.,

$$Q(\hat{t})=(\underline{\mu}_{12}+\underline{\delta}_{12})^T(\underline{\mu}_{12}+\underline{\delta}_{12}) - \frac{[(\underline{\mu}_{12}+\underline{\delta}_{12})^T(\underline{\mu}_{13}+\delta_{13})]^2}{(\underline{\mu}_{13}+\underline{\delta}_{13})^T(\underline{\mu}_{13}+\underline{\delta}_{13})}$$

$$= A-\frac{B}{C}, \qquad (20)$$

$$A = (\underline{\mu}_{12}+\underline{\delta}_{12})^T(\underline{\mu}_{12}+\underline{\delta}_{12})=\underline{\mu}^T_{12}\underline{\mu}_{12}+2(\underline{\mu}^T_{12}\underline{\delta}_1-\underline{\mu}^T_{12}\underline{\delta}_2)$$
$$+(\underline{\delta}^T_1\underline{\delta}_1-2\underline{\delta}^T_1\underline{\delta}_2+\underline{\delta}_2\underline{\delta}_2)$$

$$B = [(\underline{\mu}_{12}+\underline{\delta}_{12})^T(\underline{\mu}_{13}+\underline{\delta}_{13})]^2$$

$$= (\underline{\mu}_{12}^T\underline{\mu}_{13})^2$$

$$+2(\underline{\mu}_{12}^T\underline{\mu}_{13})[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)]$$

$$+\left[\begin{array}{l}(\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2\\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)]\\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)]+2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3)\\ +2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1-\underline{\delta}_1^T\underline{\delta}_3-\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2\underline{\delta}_3]\end{array}\right]$$

$$+2\left[\begin{array}{l}(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)\\ -(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)+(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)\\ -(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3)\\ -(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)\end{array}\right]$$

$$+\left[\begin{array}{l}(\underline{\delta}_1^T\underline{\delta}_1)^2+(\underline{\delta}_1^T\underline{\delta}_3)^2+(\underline{\delta}_1^T\underline{\delta}_2)^2+(\underline{\delta}_2^T\underline{\delta}_3)^2\\ -2[(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)]\\ +2[(\underline{\delta}_1^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\delta}_1^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3)]-2(\underline{\delta}_1^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)\end{array}\right]$$

and

$$\frac{1}{C} = \frac{1}{(\underline{\mu}_{13}+\underline{\delta}_{13})^T(\underline{\mu}_{13}+\underline{\delta}_{13})}$$

$$= \frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}+2(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)+(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)}$$

$$= \frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\left[1+\frac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)+\frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)\right]^{-1}. \quad (21)$$

Thus, uptil terms of order 4 in δ, we have

$$\frac{1}{C}=\frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\left[1-\frac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)\right.$$

$$-\frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)$$

$$+\frac{4}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2$$

$$+\frac{4}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)$$

$$-\frac{8}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^3$$

$$+\frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)^2$$

$$-\frac{12}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)$$

$$\left.+\frac{16}{(\underline{\mu}_{13}\underline{\mu}_{13})^4}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^4\right].$$

The part of $\frac{B}{C}$ consisting of terms uptil order 2 in δ is

$$\frac{(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}+\frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\left[(\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2\right.$$

$$\left.+(\underline{\mu}_{12}^T\underline{\delta}_3)^2+2(\underline{\mu}_{12}^T\underline{\mu}_{13})(\underline{\delta}_1^T\underline{\delta}_1)\right]$$

$$\frac{-4(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\mu}_{12}^T\underline{\delta}_1+\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_3-\underline{\mu}_{13}^T\underline{\delta}_2)$$

$$\frac{-(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)$$

$$\frac{+4(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2 .$$

The part of $\frac{B}{C}$ consisting only of terms of order 3 in δ is

$$\frac{4(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}\left[\begin{array}{l}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)\\ -\dfrac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^3\end{array}\right]$$

$$\frac{-2(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\left[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)\right]$$

$$\times\left[(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)-\frac{4}{\underline{\mu}_{13}\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$\frac{-2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)$$

$$\times\left[\begin{array}{l}(\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2\\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)]\\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)]+2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3)\\ +2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1-\underline{\delta}_1^T\underline{\delta}_3-\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_3]\end{array}\right]$$

$$+\frac{2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})}\left[\begin{array}{l}(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)\\ -(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)+(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)\\ -(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3)\\ -(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)\end{array}\right].$$

The part of $\frac{B}{C}$ consisting only of terms of order 4 in δ is

$$\frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})}\left[\begin{array}{l}(\underline{\delta}_1^T\underline{\delta}_1)^2+(\underline{\delta}_1^T\underline{\delta}_3)^2+(\underline{\delta}_1^T\underline{\delta}_2)^2+(\underline{\delta}_2^T\underline{\delta}_3)^2\\ +2[(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)]\\ +2[(\underline{\delta}_1^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\delta}_1^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3)]-2(\underline{\delta}_1^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)\end{array}\right]$$

$$
-\frac{4}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)
$$

$$
\times\begin{bmatrix}
(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)\\
-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)+(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)\\
-(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3)\\
-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)
\end{bmatrix}
$$

$$
-\frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\left[[(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)-\frac{4}{(\underline{\mu}_{13}^T\underline{\mu}_{13})}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]
$$

$$
\times\begin{bmatrix}
(\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2\\
-2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)\\
-2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)]\ +2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3)\\
+2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1-\underline{\delta}_1^T\underline{\delta}_3-\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_3]
\end{bmatrix}
$$

$$
+\frac{8\underline{\mu}_{12}^T\underline{\mu}_{13}}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)]
$$

$$
\times(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)\begin{bmatrix}
(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)\\
\frac{-2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2
\end{bmatrix}
$$

$$
+\frac{(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}\qquad(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)^2
$$

$$
\begin{bmatrix}
\frac{-12}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)\\
\frac{+16}{(\underline{\mu}_{13}\underline{\mu}_{13})^2}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^4
\end{bmatrix}.
$$

From (20) of section 4.5 we have

$$
[\hat{Q}(t)]^2=(A-\frac{B}{C})^2=A^2+\frac{B^2}{C^2}-\frac{2AB}{C}.
$$

We have

$$
\begin{aligned}
A^2 &= [(\underline{\mu}_{12}+\underline{\delta}_{12})^T(\underline{\mu}_{12}+\underline{\delta}_{12})]^2 \\
&= (\underline{\mu}_{12}^T\underline{\mu}_{12})^2+4(\underline{\mu}_{12}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_2)^2+(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_2)^2 \\
&+4(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\mu}_{12}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_2)+2(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_2) \\
&+4(\underline{\mu}_{12}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_2).
\end{aligned}
$$

$$
\begin{aligned}
B^2 &= [(\underline{\mu}_{12}+\underline{\delta}_{12})^T(\underline{\mu}_{13}+\underline{\delta}_{13})]^4 \\
&= (\underline{\mu}_{12}^T\underline{\mu}_{13})^4+4(\underline{\mu}_{12}^T\underline{\mu}_{13})^3[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)] \\
&+4(\underline{\mu}_{12}^T\underline{\mu}_{13})^2\left|(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)\right|^2
\end{aligned}
$$

$$
+2(\underline{\mu}_{12}^T\underline{\mu}_{13})^2
\left[
\begin{aligned}
&(\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2 \\
&+(\underline{\mu}_{12}^T\underline{\delta}_3)^2-2\left|(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)\right| \\
&-2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3) \\
&+2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3) \\
&+2(\underline{\mu}_{12}^T\underline{\mu}_{13})\ [\underline{\delta}_1^T\underline{\delta}_1-\underline{\delta}_1^T\underline{\delta}_3-\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_3]
\end{aligned}
\right]
$$

$$
+4(\underline{\mu}_{12}^T\underline{\mu}_{13})^2
\left[
\begin{aligned}
&(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2) \\
&-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3) \\
&+(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_1) \\
&+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3) \\
&-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \\
&-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)
\end{aligned}
\right]
$$

$$
+4(\underline{\mu}_{12}^T\underline{\mu}_{13})[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)]
$$

$$
\begin{aligned}
\times \Big[& (\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2 \\
& -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)] \\
& -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)]+2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3) \\
& +2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1-\underline{\delta}_1^T\underline{\delta}_3-\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_3] \Big]
\end{aligned}
$$

$$
\begin{aligned}
+\Big[& (\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2 \\
& -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)] \\
& +2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3) \\
& +2(\underline{\mu}_{12}^T\underline{\mu}_{13})[(\underline{\delta}_1^T\underline{\delta}_1)-(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\delta}_2^T\underline{\delta}_3)] \Big]^2
\end{aligned}
$$

$$
+2(\underline{\mu}_{12}^T\underline{\mu}_{13})^2
$$

$$
\begin{aligned}
\times \Big[& (\underline{\delta}_1^T\underline{\delta}_1)^2+(\underline{\delta}_1^T\underline{\delta}_3)^2+(\underline{\delta}_1^T\underline{\delta}_2)^2+(\underline{\delta}_2^T\underline{\delta}_3)^2 \\
& -2[(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)] \\
& +2[(\underline{\delta}_1^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\delta}_1^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3)-(\underline{\delta}_1^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)] \Big]
\end{aligned}
$$

$$
+8(\underline{\mu}_{12}^T\underline{\mu}_{13})[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)
$$

$$
\begin{aligned}
\times \Big[& (\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2) \\
& -(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)+(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3) \\
& -(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3) \\
& -(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3) \Big] .
\end{aligned}
$$

Using equation (21) of section (4.5) we have,

$$\frac{1}{C^2} = \frac{1}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^2} \times \left[1 + \frac{2}{\underline{\mu}_{13}^T \underline{\mu}_{13}} (\underline{\mu}_{13}^T \underline{\delta}_1 - \underline{\mu}_{13} \underline{\delta}_3) + \frac{1}{\underline{\mu}_{13}^T \underline{\mu}_{13}} (\underline{\delta}_1^T \underline{\delta}_1 - 2\underline{\delta}_1^T \underline{\delta}_3 + \underline{\delta}_3^T \underline{\delta}_3) \right]^{-2}.$$

Now, uptil terms of order 4 in δ, we have

$$\frac{1}{C^2} = \frac{1}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^2} \times$$

$$\left[1 - \frac{4}{\underline{\mu}_{13}^T \underline{\mu}_{13}} (\underline{\mu}_{13}^T \underline{\delta}_1 - \underline{\mu}_{13}^T \underline{\delta}_3) \right.$$

$$+ \frac{12}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^2} (\underline{\mu}_{13}^T \underline{\delta}_1 - \underline{\mu}_{13}^T \underline{\delta}_3)^2 - \frac{2}{\underline{\mu}_{13}^T \underline{\mu}_{13}} (\underline{\delta}_1^T \underline{\delta}_1 - 2\underline{\delta}_1^T \underline{\delta}_3 + \underline{\delta}_3^T \underline{\delta}_3)$$

$$+ \frac{12}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^2} (\underline{\mu}_{13}^T \underline{\delta}_1 - \underline{\mu}_{13}^T \underline{\delta}_3)(\underline{\delta}_1^T \underline{\delta}_1 - 2\underline{\delta}_1^T \underline{\delta}_3 + \underline{\delta}_3^T \underline{\delta}_3) - \frac{32}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^3} (\underline{\mu}_{13}^T \underline{\delta}_1 - \underline{\mu}_{13}^T \underline{\delta}_3)^3$$

$$+ \frac{3}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^2} (\underline{\delta}_1^T \underline{\delta}_1 - 2\underline{\delta}_1^T \underline{\delta}_3 + \underline{\delta}_3^T \underline{\delta}_3)^2 + \frac{80}{(\underline{\mu}_{13}^T \underline{\mu}_{13})} (\underline{\mu}_{13}^T \underline{\delta}_1 - \underline{\mu}_{13}^T \underline{\delta}_3)^4$$

$$\left. - \frac{48}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^3} (\underline{\mu}_{13}^T \underline{\delta}_1 - \underline{\mu}_{13}^T \underline{\delta}_3)^2 (\underline{\delta}_1^T \underline{\delta}_1 - 2\underline{\delta}_1^T \underline{\delta}_3 + \underline{\delta}_3^T \underline{\delta}_3) \right].$$

The part of $\frac{B^2}{C^2}$ consisting of terms uptil order 2 in δ is

$$\frac{(\underline{\mu}_{12}^T \underline{\mu}_{13})^4}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^2} \left[1 - \frac{4}{\underline{\mu}_{13}^T \underline{\mu}_{13}} (\underline{\mu}_{13}^T \underline{\delta}_1 - \underline{\mu}_{13}^T \underline{\delta}_3) + \frac{12}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^2} (\underline{\mu}_{13}^T \underline{\delta}_1 - \underline{\mu}_{13}^T \underline{\delta}_3)^2 \right.$$

$$\left. \frac{2}{\underline{\mu}_{13}^T \underline{\mu}_{13}} (\underline{\delta}_1^T \underline{\delta}_1 - 2\underline{\delta}_1^T \underline{\delta}_3 + \underline{\delta}_3^T \underline{\delta}_3) \right]$$

$$+\frac{4(\underline{\mu}_{12}^T\underline{\mu}_{13})^3}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\left[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)\right]$$

$$\times\left[1-\frac{4}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)\right]$$

$$+\frac{4(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\left[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)\right]^2 .$$

The part of $\frac{B^2}{C^2}$ consisting of only terms of order 3 in δ is

$$\frac{4(\underline{\mu}_{12}^T\underline{\mu}_{13})^4}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^4}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)\left[3(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)-\frac{8}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$+\frac{8(\underline{\mu}_{12}^T\underline{\mu}_{13})^3}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}(\underline{\mu}_{12}^T\underline{\delta}_1+\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_3-\underline{\mu}_{13}^T\underline{\delta}_2)$$

$$\times\left[\frac{6}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2-(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)\right]$$

$$-\frac{16(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)\left[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)\right]^2$$

$$+\frac{4(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\left[\begin{array}{l}(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)\\ -(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)+(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)\end{array}\right]$$

$$\frac{+4(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}$$

$$\begin{bmatrix} -(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3) \\ -(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_2) \\ +(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3) \end{bmatrix}$$

$$\frac{-8(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}\times(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)$$

$$\times\begin{bmatrix} (\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2 \\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)] \\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)] \\ +2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3) \\ +2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1-\underline{\delta}_1^T\underline{\delta}_3-\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_3] \end{bmatrix}.$$

The part of $\frac{B^2}{C^2}$ consisting of only terms of order 4 in δ is

$$\frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times\begin{bmatrix} (\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1) \\ +(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2 \\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)] \\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)] \\ +2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3)+2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1-\underline{\delta}_1^T\underline{\delta}_3-\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_3] \end{bmatrix}$$

$$
\frac{+2(\underline{\mu}_{12}^{T}\underline{\mu}_{13})^{2}}{(\underline{\mu}_{13}^{T}\underline{\mu}_{13})^{2}}
\left[\begin{array}{l}
(\underline{\delta}_{1}^{T}\underline{\delta}_{1})^{2}+(\underline{\delta}_{1}^{T}\underline{\delta}_{3})^{2}+(\underline{\delta}_{1}^{T}\underline{\delta}_{2})^{2}+(\underline{\delta}_{2}^{T}\underline{\delta}_{3})^{2} \\
-2[(\underline{\delta}_{1}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{2})+(\underline{\delta}_{1}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{3})-(\underline{\delta}_{1}^{T}\underline{\delta}_{1})(\underline{\delta}_{2}^{T}\underline{\delta}_{3})] \\
+2[(\underline{\delta}_{1}^{T}\underline{\delta}_{2})(\underline{\delta}_{1}^{T}\underline{\delta}_{3})-(\underline{\delta}_{1}^{T}\underline{\delta}_{2})(\underline{\delta}_{2}^{T}\underline{\delta}_{3})] \\
-2(\underline{\delta}_{1}^{T}\underline{\delta}_{3})(\underline{\delta}_{2}^{T}\underline{\delta}_{3})
\end{array}\right]
$$

$$
\frac{+8\underline{\mu}_{12}^{T}\underline{\mu}_{13}}{(\underline{\mu}_{13}^{T}\underline{\mu}_{13})^{2}}
[\underline{\mu}_{12}^{T}\underline{\delta}_{1}+\underline{\mu}_{13}^{T}\underline{\delta}_{1}-\underline{\mu}_{12}^{T}\underline{\delta}_{3}-\underline{\mu}_{13}^{T}\underline{\delta}_{2}]
$$

$$
\times\left[\begin{array}{l}
(\underline{\mu}_{12}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{1})+(\underline{\mu}_{13}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{1})-(\underline{\mu}_{12}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{2}) \\
-(\underline{\mu}_{13}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{2})-(\underline{\mu}_{12}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{3})-(\underline{\mu}_{13}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{3})+(\underline{\mu}_{12}^{T}\underline{\delta}_{1})(\underline{\delta}_{2}^{T}\underline{\delta}_{3}) \\
+(\underline{\mu}_{13}^{T}\underline{\delta}_{1})(\underline{\delta}_{2}^{T}\underline{\delta}_{3})-(\underline{\mu}_{13}^{T}\underline{\delta}_{2})(\underline{\delta}_{1}^{T}\underline{\delta}_{1})+(\underline{\mu}_{13}^{T}\underline{\delta}_{2})(\underline{\delta}_{1}^{T}\underline{\delta}_{2})+(\underline{\mu}_{13}^{T}\underline{\delta}_{2})(\underline{\delta}_{1}^{T}\underline{\delta}_{3}) \\
-(\underline{\mu}_{13}^{T}\underline{\delta}_{2})(\underline{\delta}_{2}^{T}\underline{\delta}_{3})-(\underline{\mu}_{12}^{T}\underline{\delta}_{3})(\underline{\delta}_{1}^{T}\underline{\delta}_{1})+(\underline{\mu}_{12}^{T}\underline{\delta}_{3})(\underline{\delta}_{1}^{T}\underline{\delta}_{2}) \\
+(\underline{\mu}_{12}^{T}\underline{\delta}_{3})(\underline{\delta}_{1}^{T}\underline{\delta}_{3})-(\underline{\mu}_{12}^{T}\underline{\delta}_{3})(\underline{\delta}_{2}^{T}\underline{\delta}_{3})
\end{array}\right]
$$

$$
\frac{-16(\underline{\mu}_{12}^{T}\underline{\mu}_{13})^{2}}{(\underline{\mu}_{13}^{T}\underline{\mu}_{13})^{3}}
(\underline{\mu}_{13}^{T}\underline{\delta}_{1}-\underline{\mu}_{13}^{T}\underline{\delta}_{3})
$$

$$
\times\left[\begin{array}{l}
(\underline{\mu}_{12}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{1})+(\underline{\mu}_{13}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{1})-(\underline{\mu}_{12}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{2})-(\underline{\mu}_{13}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{2}) \\
-(\underline{\mu}_{12}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{3})-(\underline{\mu}_{13}^{T}\underline{\delta}_{1})(\underline{\delta}_{1}^{T}\underline{\delta}_{3})+(\underline{\mu}_{12}^{T}\underline{\delta}_{1})(\underline{\delta}_{2}^{T}\underline{\delta}_{3})+(\underline{\mu}_{13}^{T}\underline{\delta}_{1})(\underline{\delta}_{2}^{T}\underline{\delta}_{3}) \\
-(\underline{\mu}_{13}^{T}\underline{\delta}_{2})(\underline{\delta}_{1}^{T}\underline{\delta}_{1})+(\underline{\mu}_{13}^{T}\underline{\delta}_{2})(\underline{\delta}_{1}^{T}\underline{\delta}_{2})+(\underline{\mu}_{13}^{T}\underline{\delta}_{2})(\underline{\delta}_{1}^{T}\underline{\delta}_{3})-(\underline{\mu}_{13}^{T}\underline{\delta}_{2})(\underline{\delta}_{2}^{T}\underline{\delta}_{3}) \\
-(\underline{\mu}_{12}^{T}\underline{\delta}_{3})(\underline{\delta}_{1}^{T}\underline{\delta}_{1})+(\underline{\mu}_{12}^{T}\underline{\delta}_{3})(\underline{\delta}_{1}^{T}\underline{\delta}_{2})+(\underline{\mu}_{12}^{T}\underline{\delta}_{3})(\underline{\delta}_{1}^{T}\underline{\delta}_{3})-(\underline{\mu}_{12}^{T}\underline{\delta}_{3})(\underline{\delta}_{2}^{T}\underline{\delta}_{3})
\end{array}\right]
$$

$$\frac{-16(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}\quad(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)]$$

$$\times\left[\begin{array}{l}(\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2\\-2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)]\\-2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)]+2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3)\\+2\ (\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1-\underline{\delta}_1^T\underline{\delta}_3-\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_3]\end{array}\right]$$

$$\frac{-4(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}\left[(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)-\frac{6}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$\times\left[\begin{array}{l}2[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)]^2\\+\ (\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2\\-2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)]\\-2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)]\ +2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3)\\+2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\ \underline{\delta}_1^T\underline{\delta}_1-\underline{\delta}_1^T\underline{\delta}_3-\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_3]\end{array}\right]$$

$$\frac{+16(\underline{\mu}_{12}^T\underline{\mu}_{13})^3}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^4}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)[\ \underline{\mu}_{12}^T\underline{\delta}_1+\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_3-\underline{\mu}_{13}^T\underline{\delta}_2]$$

$$\times\left[\ 3(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)-\frac{8}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2\ \right]$$

$$+\frac{(\mu_{12}^T\mu_{13})^4}{(\mu_{13}^T\mu_{13})^4}\left[\begin{array}{l}3(\delta_1^T\delta_1 - 2\delta_1^T\delta_3+\delta_3^T\delta_3)^2 \\ - \frac{48}{\mu_{13}^T\mu_{13}}(\mu_{13}^T\delta_1-\mu_{13}^T\delta_3)^2(\delta_1^T\delta_1-2\delta_1^T\delta_3+\delta_3^T\delta_3) \\ + \frac{80}{(\mu_{13}^T\mu_{13})^2}(\mu_{13}^T\delta_1-\mu_{13}^T\delta_3)^4\end{array}\right].$$

The part of $\frac{AB}{C}$ consisting of terms uptil order 2 in δ is

$$\frac{(\mu_{12}^T\mu_{12})(\mu_{12}^T\mu_{13})^2}{(\mu_{13}^T\mu_{13})}$$

$$\times\left[\begin{array}{l}1-\frac{2}{\mu_{13}^T\mu_{13}}(\mu_{13}^T\delta_1-\mu_{13}^T\delta_3) - \frac{1}{\mu_{13}^T\mu_{13}}(\delta_1^T\delta_1-2\delta_1^T\delta_3+\delta_3^T\delta_3) \\ + \frac{4}{(\mu_{13}^T\mu_{13})^2}(\mu_{13}^T\delta_1-\mu_{13}^T\delta_3)^2\end{array}\right]$$

$$+\frac{2(\mu_{12}^T\mu_{12})(\mu_{12}^T\mu_{13})}{\mu_{13}^T\mu_{13}}(\mu_{12}^T\delta_1+\mu_{13}^T\delta_1-\mu_{12}^T\delta_3-\mu_{13}^T\delta_2)$$

$$\times\left[1-\frac{2}{\mu_{13}^T\mu_{13}}(\mu_{13}^T\delta_1-\mu_{13}^T\delta_3)\right]$$

$$+\frac{\mu_{12}^T\mu_{12}}{\mu_{13}^T\mu_{13}}\left[\begin{array}{l}(\mu_{12}^T\delta_1)^2+(\mu_{13}^T\delta_1)^2+2(\mu_{12}^T\delta_1)(\mu_{13}^T\delta_1)+(\mu_{13}^T\delta_2)^2+(\mu_{12}^T\delta_3)^2 \\ -2[(\mu_{12}^T\delta_1)(\mu_{13}^T\delta_2)+(\mu_{13}^T\delta_1)(\mu_{13}^T\delta_2)] \\ -2[(\mu_{12}^T\delta_1)(\mu_{12}^T\delta_3)+(\mu_{13}^T\delta_1)(\mu_{12}^T\delta_3)] +2(\mu_{13}^T\delta_2)(\mu_{12}^T\delta_3) \\ +2(\mu_{12}^T\mu_{13})[\delta_1^T\delta_1-\delta_1^T\delta_3-\delta_1^T\delta_2+\delta_2^T\delta_3]\end{array}\right]$$

$$+\frac{2(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{12}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_2)\times\left[1-\frac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)\right]$$

$$+\frac{4(\underline{\mu}_{12}^T\underline{\mu}_{13})}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{12}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_2)[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)]$$

$$+\frac{(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_2).$$

The part of $\frac{AB}{C}$ consisting only of terms of order 3 in δ is

$$\frac{4(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)\times\left[(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)-\frac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$-\frac{2(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)]\times\left[(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)-\frac{4}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$\frac{-2\underline{\mu}_{12}^T\underline{\mu}_{12}}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)\times\left[\begin{array}{l}(\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2\\-2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)]\end{array}\right]$$

$$\frac{-2\underline{\mu}_{12}^T\underline{\mu}_{12}}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\,(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)$$

$$\times\begin{bmatrix} -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)]+2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3) \\ +2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1 - \underline{\delta}_1^T\underline{\delta}_3 - \underline{\delta}_1^T\underline{\delta}_2 + \underline{\delta}_2^T\underline{\delta}_3] \end{bmatrix}$$

$$+2\,\frac{\underline{\mu}_{12}^T\underline{\mu}_{12}}{\underline{\mu}_{13}^T\underline{\mu}_{13}}$$

$$\times\begin{bmatrix} (\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2) \\ -(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)+(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3) \\ -(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3) \\ -(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3) \end{bmatrix}$$

$$\frac{-2(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\,(\underline{\mu}_{12}^T\underline{\delta}_1 - \underline{\mu}_{12}^T\underline{\delta}_2)$$

$$\times\left[(\underline{\delta}_1^T\underline{\delta}_1 - 2\underline{\delta}_1^T\underline{\delta}_3 + \underline{\delta}_3^T\underline{\delta}_3) - \frac{4}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\,(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$\frac{-8(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\,(\underline{\mu}_{12}^T\underline{\delta}_1 - \underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)$$

$$\times[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)]$$

$$\frac{+2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\,(\underline{\mu}_{12}^T\underline{\delta}_1 - \underline{\mu}_{12}^T\underline{\delta}_2)$$

$$\times\left[(\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2(\underline{\mu}_{12}^T\underline{\delta}_3)^2\right]$$

$$\frac{+2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\,(\underline{\mu}_{12}^T\underline{\delta}_1 - \underline{\mu}_{12}^T\underline{\delta}_2)$$

$$\times\left[\begin{array}{l} -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)] \\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)]+2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3) \\ +2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1 - \underline{\delta}_1^T\underline{\delta}_3 - \underline{\delta}_1^T\underline{\delta}_2 + \underline{\delta}_2^T\underline{\delta}_3] \end{array}\right]$$

$$\frac{-2(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\,(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1 -2\underline{\delta}_1^T\underline{\delta}_2 + \underline{\delta}_2^T\underline{\delta}_2)$$

$$\frac{+2(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})}[\underline{\mu}_{12}^T\underline{\delta}_1 +\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{12}^T\underline{\delta}_3 - \underline{\mu}_{13}^T\underline{\delta}_2][\underline{\delta}_1^T\underline{\delta}_1 -2\underline{\delta}_1^T\underline{\delta}_2 + \underline{\delta}_2^T\underline{\delta}_2] \ .$$

The part of $\frac{AB}{C}$ consisting only of terms of order 4 in δ is

$$\frac{\underline{\mu}_{12}^T\underline{\mu}_{12}}{\underline{\mu}_{13}^T\underline{\delta}_{13}} \times \left[\begin{array}{l} (\underline{\delta}_1^T\underline{\delta}_1)^2+(\underline{\delta}_1^T\underline{\delta}_3)^2+(\underline{\delta}_1^T\underline{\delta}_2)^2+(\underline{\delta}_2^T\underline{\delta}_3)^2 \\ -2[(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\delta}_1^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)] \\ +2[(\underline{\delta}_1^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\delta}_1^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3)]-2(\underline{\delta}_1^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3) \end{array}\right]$$

$$\frac{-4(\underline{\mu}_{12}^T\underline{\mu}_{12})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\,(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)$$

$$\times\left[\begin{array}{l} (\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2) \\ -(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)+(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3) \end{array}\right]$$

$$\frac{-4(\underline{\mu}_{12}^T\underline{\mu}_{12})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)$$

$$\times\begin{bmatrix} -(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3) \\ -(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_2) \\ +(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3) \end{bmatrix}$$

$$\frac{-(\underline{\mu}_{12}^T\underline{\mu}_{12})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\left[[\underline{\delta}_1^T\underline{\delta}_1 - 2\underline{\delta}_1^T\underline{\delta}_3 + \underline{\delta}_3^T\underline{\delta}_3] - \frac{4}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$\times\begin{bmatrix} (\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2 \\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)] \\ -2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)] + 2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3) \\ +2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1 - \underline{\delta}_1^T\underline{\delta}_3 - \underline{\delta}_1^T\underline{\delta}_2 + \underline{\delta}_2^T\underline{\delta}_3] \end{bmatrix}$$

$$\frac{+8(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)]$$

$$\times[(\underline{\delta}_1^T\underline{\delta}_1 - 2\underline{\delta}_1^T\underline{\delta}_3 + \underline{\delta}_3^T\underline{\delta}_3) - \frac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)^2]$$

$$\frac{+(\underline{\mu}_{12}^T\underline{\mu}_{12})(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}\times$$

$$\begin{bmatrix} (\underline{\delta}_1^T\underline{\delta}_1 - 2\underline{\delta}_1^T\underline{\delta}_3 + \underline{\delta}_3^T\underline{\delta}_3)^2 - \frac{12}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_1^T\underline{\delta}_1 - 2\underline{\delta}_1^T\underline{\delta}_3 + \underline{\delta}_3^T\underline{\delta}_3) \\ \frac{+16}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)^4 \end{bmatrix}$$

$$\frac{+4}{(\underline{\mu}_{13}^T\underline{\mu}_{13})}(\underline{\mu}_{12}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_2)\times$$

$$\Big[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_2)$$

$$-(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)+(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_2^T\underline{\delta}_3)$$

$$-(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_3)$$

$$-(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_1)+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_2)$$

$$+(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)-(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)\Big]$$

$$\frac{-4}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\mu}_{12}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)\times$$

$$\Big[(\underline{\mu}_{12}^T\underline{\delta}_1)^2+(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_2)^2+(\underline{\mu}_{12}^T\underline{\delta}_3)^2$$

$$-2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)]$$

$$-2[(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)+(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{12}^T\underline{\delta}_3)]+2(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3)$$

$$+2(\underline{\mu}_{12}^T\underline{\mu}_{13})[\underline{\delta}_1^T\underline{\delta}_1-\underline{\delta}_1^T\underline{\delta}_3-\underline{\delta}_1^T\underline{\delta}_2+\underline{\delta}_2^T\underline{\delta}_3]\Big]$$

$$\frac{-4(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}(\underline{\mu}_{12}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_2)[(\underline{\mu}_{12}^T\underline{\delta}_1)+(\underline{\mu}_{13}^T\underline{\delta}_1)-(\underline{\mu}_{12}^T\underline{\delta}_3)-(\underline{\mu}_{13}^T\underline{\delta}_2)]$$

$$\times\left[(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)-\frac{4}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$\frac{+8(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^3}(\underline{\mu}_{12}^T\underline{\delta}_1-\underline{\mu}_{12}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)$$

$$\times\left[(\underline{\delta}_1^T\underline{\delta}_1-2\underline{\delta}_1^T\underline{\delta}_3+\underline{\delta}_3^T\underline{\delta}_3)-\frac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}(\underline{\mu}_{13}^T\underline{\delta}_1-\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$\frac{+1}{\mu_{13}^T\mu_{13}}(\delta_1^T\delta_1 - 2\delta_1^T\delta_2 + \delta_2^T\delta_2)$$

$$\times\left[\begin{array}{l}(\mu_{12}^T\delta_1)^2+(\mu_{13}^T\delta_1)^2+2(\mu_{12}^T\delta_1)(\mu_{13}^T\delta_1)+(\mu_{13}^T\delta_2)^2+(\mu_{12}^T\delta_3)^2 \\ -2[(\mu_{12}^T\delta_1)(\mu_{13}^T\delta_2)+(\mu_{13}^T\delta_1)(\mu_{13}^T\delta_2)] \\ -2[(\mu_{12}^T\delta_1)(\mu_{12}^T\delta_3)+(\mu_{13}^T\delta_1)(\mu_{12}^T\delta_3)] +2(\mu_{13}^T\delta_2)(\mu_{12}^T\delta_3) \\ +2\ (\mu_{12}^T\mu_{13})[\delta_1^T\delta_1 - \delta_1^T\delta_3 - \delta_1^T\delta_2 + \delta_2^T\delta_3\ ]\end{array}\right]$$

$$\frac{-4(\mu_{12}^T\mu_{13})}{(\mu_{13}^T\mu_{13})^2}\ (\mu_{13}^T\delta_1 - \mu_{13}^T\delta_3)(\delta_1^T\delta_1 - 2\delta_1^T\delta_2 + \delta_2^T\delta_2\ )$$

$$\times\ [(\mu_{12}^T\delta_1)+(\mu_{13}^T\delta_1)-(\mu_{12}^T\delta_3)-(\mu_{13}^T\delta_2)]$$

$$\frac{-(\mu_{12}^T\mu_{13})^2}{(\mu_{13}^T\mu_{13})^2}\ (\delta_1^T\delta_1 - 2\delta_1^T\delta_2 + \delta_2^T\delta_2)$$

$$\times[(\delta_1^T\delta_1 - 2\delta_1^T\delta_3 + \delta_3^T\delta_3) - \frac{4}{\mu_{13}^T\mu_{13}}(\mu_{13}^T\delta_1 - \mu_{13}^T\delta_3)^2] \ .$$

Consider (9) of section (4.1) i.e.

$$\hat{t}= \frac{(\mu_{12}+\delta_{12})^T(\mu_{13}+\delta_{13})}{(\mu_{13}+\delta_{13})^T(\mu_{13}+\delta_{13})} = \frac{D}{C}\ , \qquad (23)$$

where

$$D=(\mu_{12} + \delta_{12})^T(\mu_{13}+\delta_{13})$$

$$=\mu_{12}^T\mu_{13}+[\ \mu_{12}^T\delta_1 - \mu_{12}^T\delta_3 + \mu_{13}^T\delta_1 - \mu_{13}^T\delta_2\ ]$$

$$+(\delta_1^T\delta_1 - \delta_1^T\delta_3 - \delta_1^T\delta_2 + \delta_2^T\delta_3)$$

and $\frac{1}{C}$ is defined by (21) of this section.

The part of $\frac{D}{C}$ consisting of terms uptil order 2 in δ is

$$\frac{\underline{\mu}_{12}^T\underline{\mu}_{13}}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \times \left[1 - \frac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}} (\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3) - \frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}} (\underline{\delta}_1^T\underline{\delta}_1 - 2\underline{\delta}_1^T\underline{\delta}_3 + \underline{\delta}_3^T\underline{\delta}_3) + \frac{4}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2} (\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)^2 \right]$$

$$\frac{+1}{\underline{\mu}_{13}^T\underline{\mu}_{13}} (\underline{\mu}_{12}^T\underline{\delta}_1 - \underline{\mu}_{12}^T\underline{\delta}_3 + \underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_2)$$

$$\times \left[1 - \frac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}} (\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3) \right]$$

$$+ \frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}} [(\underline{\delta}_1^T\underline{\delta}_1) - (\underline{\delta}_1^T\underline{\delta}_3) - (\underline{\delta}_1^T\underline{\delta}_2) + (\underline{\delta}_2^T\underline{\delta}_3)].$$

From (24) of this section we have

$$t = \frac{D^2}{C^2}$$

and

$$D^2 = [(\underline{\mu}_{12} + \underline{\delta}_{12})^T (\underline{\mu}_{13} + \underline{\delta}_{13})]^2 .$$

Note that D^2 is the same as B defined in this section.

Hence, the part of $\frac{D^2}{C^2}$ consisting of terms uptil order 2 in δ is

$$\frac{(\underline{\mu}_{12}^T\underline{\mu}_{13})^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2} \left[1 - \frac{4}{\underline{\mu}_{13}^T\underline{\mu}_{13}} (\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3) + \frac{12}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2} (\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3)^2 \frac{-2}{\underline{\mu}_{13}^T\underline{\mu}_{13}} (\underline{\delta}_1^T\underline{\delta}_1 - 2\underline{\delta}_1^T\underline{\delta}_3 + \underline{\delta}_3^T\underline{\delta}_3) \right]$$

$$\frac{+2(\underline{\mu}_{12}^T\underline{\mu}_{13})}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2} (\underline{\mu}_{12}^T\underline{\delta}_1 + \underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{12}^T\underline{\delta}_3 - \underline{\mu}_{13}^T\underline{\delta}_2) \times \left[1 - \frac{4}{\underline{\mu}_{13}^T\underline{\mu}_{13}} (\underline{\mu}_{13}^T\underline{\delta}_1 - \underline{\mu}_{13}^T\underline{\delta}_3) \right]$$

$$\frac{+1}{(\underline{\mu}^T_{13}\underline{\mu}_{13})^2} \times$$

$$\Big[(\underline{\mu}^T_{12}\underline{\delta}_1)^2 + (\underline{\mu}^T_{13}\underline{\delta}_1)^2 + 2(\underline{\mu}^T_{12}\underline{\delta}_1)(\underline{\mu}^T_{13}\underline{\delta}_1) + (\underline{\mu}^T_{13}\underline{\delta}_2)^2 + (\underline{\mu}^T_{12}\underline{\delta}_3)^2$$

$$-2[(\underline{\mu}^T_{12}\underline{\delta}_1)(\underline{\mu}^T_{13}\underline{\delta}_2) + (\underline{\mu}^T_{13}\underline{\delta}_1)(\underline{\mu}^T_{13}\underline{\delta}_2)]$$

$$-2[(\underline{\mu}^T_{12}\underline{\delta}_1)(\underline{\mu}^T_{12}\underline{\delta}_3) + (\underline{\mu}^T_{13}\underline{\delta}_1)(\underline{\mu}^T_{12}\underline{\delta}_3)] + 2(\underline{\mu}^T_{13}\underline{\delta}_2)(\underline{\mu}^T_{12}\underline{\delta}_3)$$

$$+2(\underline{\mu}^T_{12}\underline{\mu}_{13})[\underline{\delta}^T_1\underline{\delta}_1 - \underline{\delta}^T_1\underline{\delta}_3 - \underline{\delta}^T_1\underline{\delta}_2 + \underline{\delta}_2{}^T\underline{\delta}_3] \quad . \Big] .$$

Expected values of $\hat{t}$, $\hat{t}^2$, $Q(\hat{t})$ and $[Q(\hat{t})]^2$ in terms of μ's and δ's are evaluated using the building blocks established in section 4.3. The basic quantities comprising the building blocks are derived in section 4.4.

Note that since $\hat{f}_1$, $\hat{f}_2$ and $\hat{f}_3$ are constructed from independent data sets, therefore, there is zero covariance between any two of the three corresponding δ's . Thus terms like , say,

$E(\underline{\mu}^T_{12}\underline{\delta}_1)(\underline{\delta}^T_1\underline{\delta}_3)$, $E\underline{\delta}^T_1\underline{\delta}_2$ and $E[(\underline{\mu}^T_{12}\underline{\delta}_1)^2 \; \underline{\delta}^T_2\underline{\delta}_3]$ etc. are all zero.

The expressions for $\hat{t}$, $\hat{t}^2$ uptil order 2 in δ and for $Q(\hat{t})$ and $[Q(\hat{t})]^2$ uptil order 4 in δ have already been stated in section 4.2.

4.6 A simulation study

A simulation study was carried out in order to check the effectiveness of the theoretical approximations, to the mean and variance of $\hat{t}$ and $Q(\hat{t})$, that we have obtained in this chapter. In the study, the theoretical results are compared with corresponding empirical values generated by the simulated data ; see Table 1.

Only a few runs were possible because of the large computation time required for each run. However, it can be seen that theoretical approximations compare very well with the corresponding sample means and variances, even for the null cases.

In the simulation study n(k) is the size of the sample underlying the estimate of the density f_k, where k varies from 1 to 3. M is the number of points on the x-axis used to construct $Q(\hat{t})$ (and its mean and variance as well). Limit points are the points on the x-axis between which all the M points lie. We are including the limit points among the M points. Also, the M points will be chosen to be equidistant.

We will take n(1)=n(2)=n(3)=n for simplicity. In the simulation study M is taken to be 9. n_Q is the number of $Q(\hat{t})$'s used to construct the sample mean and variance of $Q(\hat{t})$.

The following sets of functions were used in the simulation study.

SET 1

f_1=0.1N(1,1)+0.9N(2,1)

f_2=0.3N(1,1)+0.7N(2,1)

f_3=0.7N(1,1)+0.3N(2,1)

SET 2

f_1=0.3N(1,1)+0.7N(3,1)

f_2=0.5N(1,1)+0.5N(3,1)

f_3=0.8N(1,1)+0.2N(3,1)

SET 3

f_1=0.1N(1,1)+0.9N(4,1)

f_2=0.3N(1,1)+0.7N(4,1)

f_3=0.7N(1,1)+0.3N(4,1)

SET 4

f_1=0.3N(1,1)+0.7N(6,1)

f_2=0.5N(1,1)+0.5N6,1)

f_3=0.8N(1,1)+0.2N(6,1)

SET 5

f_1=0.2N(1,1)+0.8N(3,1)

f_2=0.4N(1,1)+0.6N(4,1)

f_3=0.6N(1,1)+0.4N(4,1)

SET 6

f_1=0.2N(1,1)+0.8N(3,1)

f_2=0.4N(1,1)+0.6N(3.5,1)

f_3=0.6N(1,1)+0.4N(4,1)

SET 7

f_1=N(1,1), f_2=N(2,1), f_3=N(3,1).

Note that the first four sets correspond to the null case because the three densities are proper mixtures of the same two components and therefore by Theorems 2 and 3 of Chapter 2 one of the densities is a proper mixture of the other two.

Case	Set No.	n	Lt.pts	n_Q	Theoretical $E\hat{Q}(t)$	Theoretical $Var\ \hat{Q}(t)$	Simulated $E\hat{Q}(t)$	Simulated $Var\ \hat{Q}(t)$	No. of run
Null	1	4000	(-1.5,6.5)	125	3.5162×10^{-4}	1.2667×10^{-7}	3.7401×10^{-4}	7.8567×10^{-8}	1
							3.5352×10^{-4}	8.0475×10^{-8}	2
							4.2172×10^{-4}	1.1360×10^{-7}	3
		1000	(-1.5,6.5)	300	9.5427×10^{-4}	1.0302×10^{-6}	1.3644×10^{-3}	1.3315×10^{-6}	1
							8.0674×10^{-4}	4.1490×10^{-7}	2
							$8.7973\ 10^{-4}$	4.7516×10^{-7}	3
Null	2	4000	(-1.5,6.5)	125	2.9196×10^{-4}	7.3710×10^{-8}	2.6385×10^{-4}	2.9124×10^{-8}	1
							4.4520×10^{-4}	8.0801×10^{-8}	2
Null	3	1000	(-1.5,6.5)	300	1.0900×10^{-3}	8.7954×10^{-7}	1.3371×10^{-3}	7.3907×10^{-7}	
		4000	(-1.5,6.5)	125	3.9058×10^{-4}	1.0112×10^{-7}	4.4224×10^{-4}	8.3545×10^{-8}	
Null	4	1000	(-1.5,8.5)	300	8.5375×10^{-4}	6.0779×10^{-7}	7.6626×10^{-4}	2.1928×10^{-7}	
		3000	(-1.5,8.5)	150	3.7165×10^{-4}	1.0633×10^{-7}	3.9581×10^{-4}	5.5857×10^{-8}	
Non-null	5	1000	(-1.5,6.5)	300	1.6956×10^{-2}	1.8459×10^{-5}	1.6609×10^{-2}	1.1299×10^{-5}	
Non-null	6	1000	(-1.5,6.5)	300	3.0149×10^{-3}	3.0638×10^{-6}	3.0144×10^{-3}	2.0925×10^{-6}	1
							2.9221×10^{-3}	2.2292×10^{-6}	2
Non-null	7	1000	(-1.5,6.5)	300	3.5642×10^{-2}	4.1574×10^{-5}	3.7720×10^{-2}	3.6557×10^{-5}	1
							3.6548×10^{-2}	3.6140×10^{-5}	2
		2000	(-1.5,6.5)	225	3.6473×10^{-2}	2.4936×10^{-5}	3.9835×10^{-2}	2.2780×10^{-5}	
		3000	(-1.5,6.5)	150	3.6910×10^{-2}	1.8584×10^{-5}	3.5539×10^{-2}	1.5337×10^{-5}	

TABLE 1

4.7 The distribution of $Q(\hat{t})$ and the normal distribution

In this section the kernel estimates of the f_i's have been constructed from samples of size 1000.

Consider figures 1 to 5 at the end of this chapter. In figure 1, the normal kernel estimates of f_1, f_2 and f_3, where

$$f_1=0.1N(1,1)+0.9N(2,1)$$

$$f_2=0.3N(1,1)+0.7N(2,1)$$

and $$f_3=0.7N(1,1)+0.3N(2,1)$$

are the underlying densities, are simulated to obtain 250 values of $Q(\hat{t})$. The histogram of $Q(\hat{t})$ is plotted. It is obvious from figure 1 that the distribution of $Q(\hat{t})$ shows deviation from the normal.

In figure 2, the blue curve is the normal kernel density estimate of $\log Q(\hat{t})$, plotted with an appropriate value of the smoothing parameter h. The sample mean and variance of $\log Q(\hat{t})$ is calculated and a normal curve (red curve) drawn with the same mean and variance.

In figures 3 to 5 the blue curves are the kernel density estimates of $\log Q(\hat{t})$ and the red curves are the normal curves, drawn, with mean and variance equal to the sample mean and variance of $\log Q(\hat{t})$.

As is obvious from figures 2 to 5, the plot of $\log Q(\hat{t})$ is approximately normal.

The f_i's used in figures 3 to 5 are

$f_1=N(1,1)$, $f_2=N(2,1)$ and $f_3=N(3,1)$ in figures 3,

$f_1=0.1N(1,1)+0.9N(4,1)$, $f_2=0.3N(1,1)+0.7N(4,1)$

and $f_3=0.7N(1,1)+0.3N(4,1)$ in figure 4

and $f_1=0.1N(1,1)+0.9N(4,1)$, $f_2=0.3N(1,1)+0.7N(4,1)$

and $f_3=N(2.5,1)$ in figure 5.

Note that figures 2 and 4 deal with the null case and figures 3 and 5 with the non-null case. Also note that the sample mean of $\log Q(\hat{t})$ tends to be smaller for the null cases, as expected.

The mean and variance of $\log Q(\hat{t})$, in terms of mean and variance of $Q(\hat{t})$, may be written , approximately, as

$$E \log Q(t) = \log E\, Q(\hat{t}) \qquad (24)$$

$$\text{and } \operatorname{var} \log Q(\hat{t}) = \frac{1}{[EQ(\hat{t})]^2} \operatorname{var} Q(\hat{t}) \; . \qquad (25)$$

To evaluate $EQ(\hat{t})$ and $\operatorname{var} Q(\hat{t})$ in (24) and (25), we use the formulae derived in this chapter.

As mentioned earlier, rename the f_i's, if necessary, such that $\hat{t}$ lies between zero and 1. Since $\underline{\hat{f}}_{12} = \hat{t}\underline{\hat{f}}_{13}$, therefore $\underline{\hat{f}}_2 = (1-\hat{t})\underline{\hat{f}}_1 + \hat{t}\underline{\hat{f}}_3$.

Now evaluate $EQ(\hat{t})$ and $\operatorname{var} Q(\hat{t})$ by replacing, in their formulae, f_1 and f_3 by $\hat{f}_1$ and $\hat{f}_3$, respectively, and f_2 by $(1-\hat{t})\hat{f}_1 + \hat{t}\underline{\hat{f}}_3$.

Refer the observed $\log Q(\hat{t})$ to $N(\tilde{\mu}, \tilde{\sigma}^2)$.

where $\tilde{\mu} = E \log Q(\hat{t})$ and $\tilde{\sigma}^2 = \operatorname{var} \log Q(\hat{t})$ are functions of $\underline{\hat{f}}_1$, $\underline{\hat{f}}_3$ and $(1-\hat{t})\underline{\hat{f}}_1 + \hat{t}\underline{\hat{f}}_3$.

We will reject H_0 if

$\log Q(\hat{t}) > \tilde{\mu} + 1.645\, \tilde{\sigma}$.

This is a 5% test.

4.8 Dominant part of the mean and variance of $\hat{Q}(t)$ under the null hypothesis.

With different meanings for the $\underline{\mu}$'s and $\underline{\delta}$'s, as stated in section 5.1 of chapter 5, it will be shown in section 5.4 of chapter 5 that, under the null hypothesis, the leading terms in the mean and variance of $\hat{Q}(t)$ are of orders 2 and 4, respectively, in δ .

The null hypothesis , in the context of chapter 5, is $\underline{p}_{12} = t\underline{p}_{13}$, i.e, $\underline{p}_1 - \underline{p}_2 = t(\underline{p}_1 - \underline{p}_3)$. Here $\underline{p}_1$, $\underline{p}_2$ and $\underline{p}_3$ are the vectors representing three multinomial densities. The density estimates are $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$. These vectors are the relative frequency vectors.

We have

$$E\underline{r}_i = \underline{p}_i \qquad i \varepsilon [1,2,3].$$

Thus,the means of the estimates are unbiased.

On the other hand, in chapter 4, the means of the density estimates are biased. Consider the density f_1, say. We have, approximately,

$$E\hat{f}_1(x_i)=f_1(x_i)+ \frac{1}{2}h_1^2 I_1 f_1''(x_i). \qquad (26)$$

In equation (26)

I, is given by $\int v^2 k(v)dv$, where, $k(v)= \frac{1}{\sqrt{2\Pi}} e^{-\frac{1}{2}v^2}$ is the kernel function, h_1 is the smoothing parameter in f_1 and $i\varepsilon[1,2,....m]$.

Equation (26), in vector form, may be written as

$$E\hat{\underline{f}}_1 = \underline{f}_1 + \tfrac{1}{2} I_1 h_1^2 \underline{f}_1''.$$

Thus $$E\hat{\underline{f}}_{12} = \underline{f}_{12} + \tfrac{1}{2} I_1 [h_1^2\underline{f}_1'' - h_2^2\underline{f}_2''] . \qquad (27)$$

Also, $$E\hat{\underline{f}}_{13} = \underline{f}_{13} + \tfrac{1}{2} I_1[h_1^2\underline{f}_1'' - h_3^2\underline{f}_3''] . \qquad (28)$$

Now, the null hypothesis, in the context of chapter 4, is $\underline{f}_{12} = t\underline{f}_{13}$, but, unlike chapter 5, where $E\underline{r}_{12} = tE\underline{r}_{13}$, we do not have $E\hat{\underline{f}}_{12} = tE\hat{\underline{f}}_{13}$.

Note that, from the null hypothesis, it follows that $\underline{f}''_{12} = t\underline{f}''_{13}$.

Thus, from (27) and (28), it follows that, if h_1, h_2 and h_3 are not very different then $E\hat{\underline{f}}_{12} = tE\hat{\underline{f}}_{13}$, i.e. $\underline{\mu}_{12} = t\underline{\mu}_{13}$. Thus, for such cases the leading terms in the mean and variance of $Q(\hat{t})$ are of orders 2 and 4, respectively, in δ .

For the non-null cases the leading terms in mean and variance of $Q(\hat{t})$ are of orders zero and 2, respectively, in δ.

The following table shows the smoothing parameters h_i, $i \epsilon [1,2,3]$ for the density estimates $\hat{f}_i$ in sets 1 to 4, mentioned in section 4.6, which correspond to the null case. In all of the four sets mentioned below, the sample sizes underlying the density estimates are 1000.

Set No.	h_1	h_2	h_3
1	0.2760	0.2940	0.2940
2	0.3336	0.3695	0.3076
3	0.2785	0.3027	0.3027
4	0.2944	0.3024	0.2859

Note that the h_i's in each set are not very different. This explains why the leading terms in the mean and variance of $Q(\hat{t})$ are of order 2 and 4, respectively , in δ.

Thus to eliminate any possibility of error the expressions for mean and variance of $Q(\hat{t})$ have been derived till orders 2 and 4 in , respectively.

FIG 1

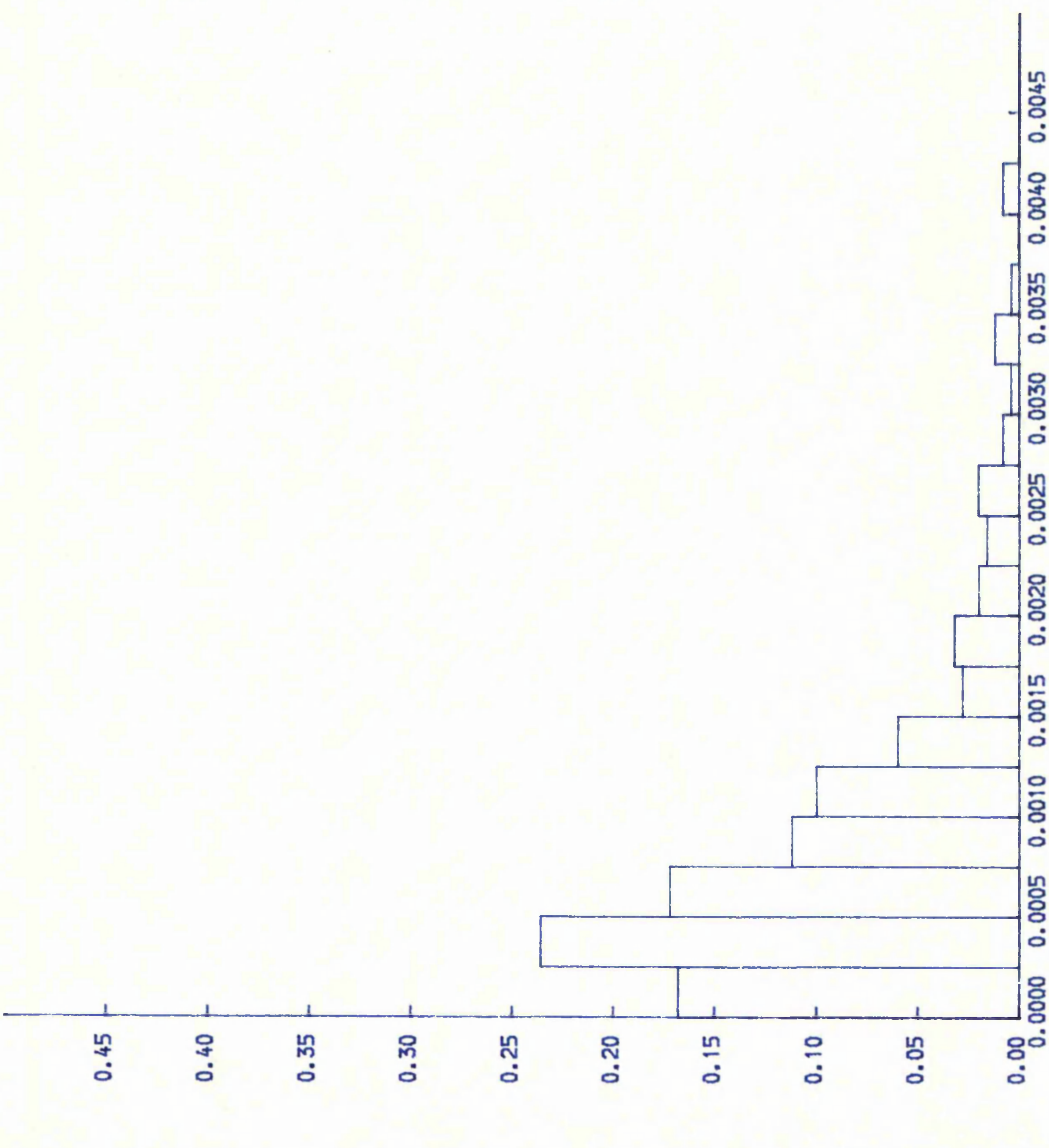
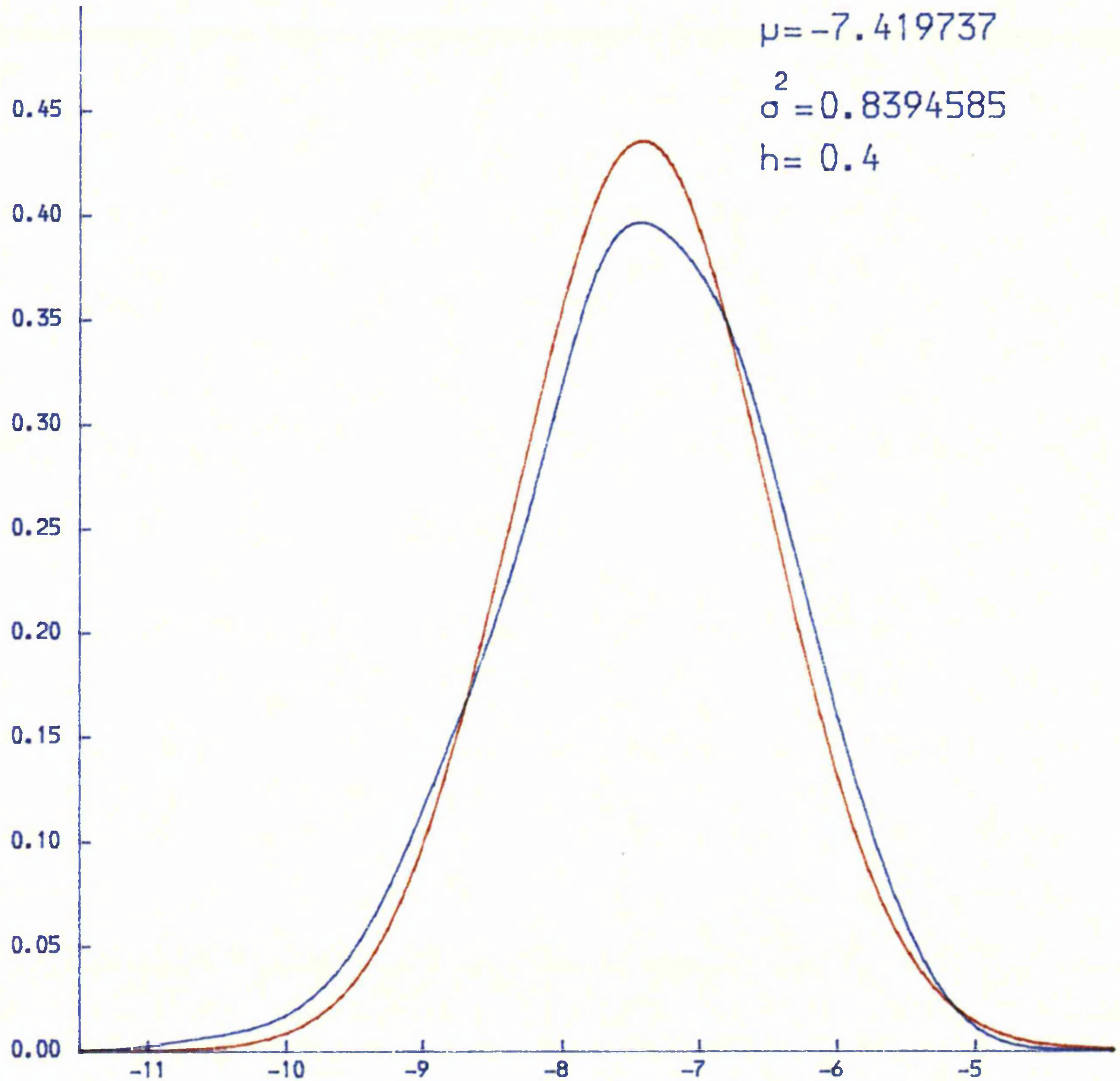

0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0.0000
0.0005
0.0010
0.0015
0.0020
0.0025
0.0030
0.0035
0.0040
0.0045

FIG 2


μ=-7.419737
σ² =0.8394585
h= 0.4
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
-11
-10
-9
-8
-7
-6
-5

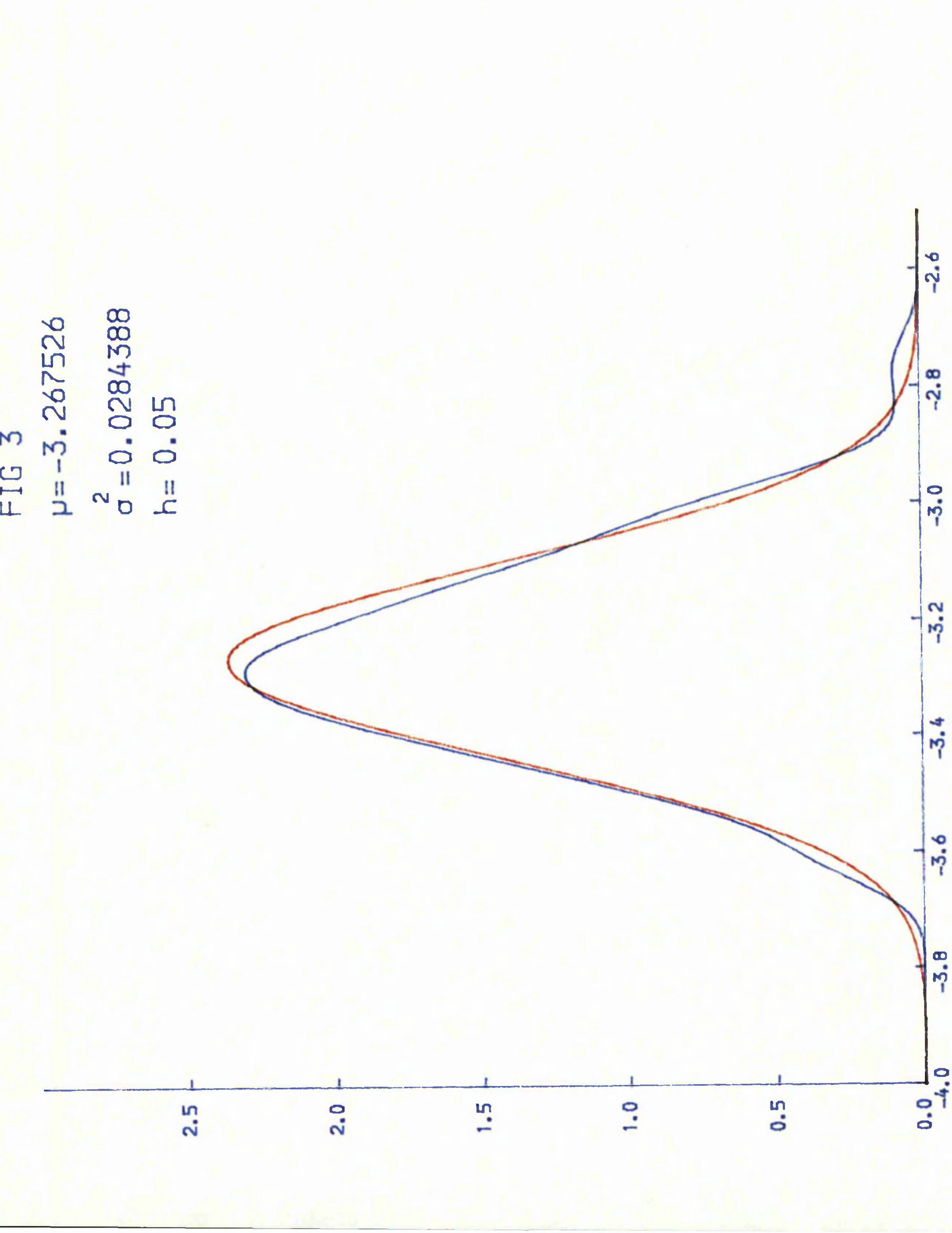


FIG 3
μ=-3.267526
σ² =0.0284388
h= 0.05
2.5
2.0
1.5
1.0
0.5
0.0
-4.0
-3.8
-3.6
-3.4
-3.2
-3.0
-2.8
-2.6

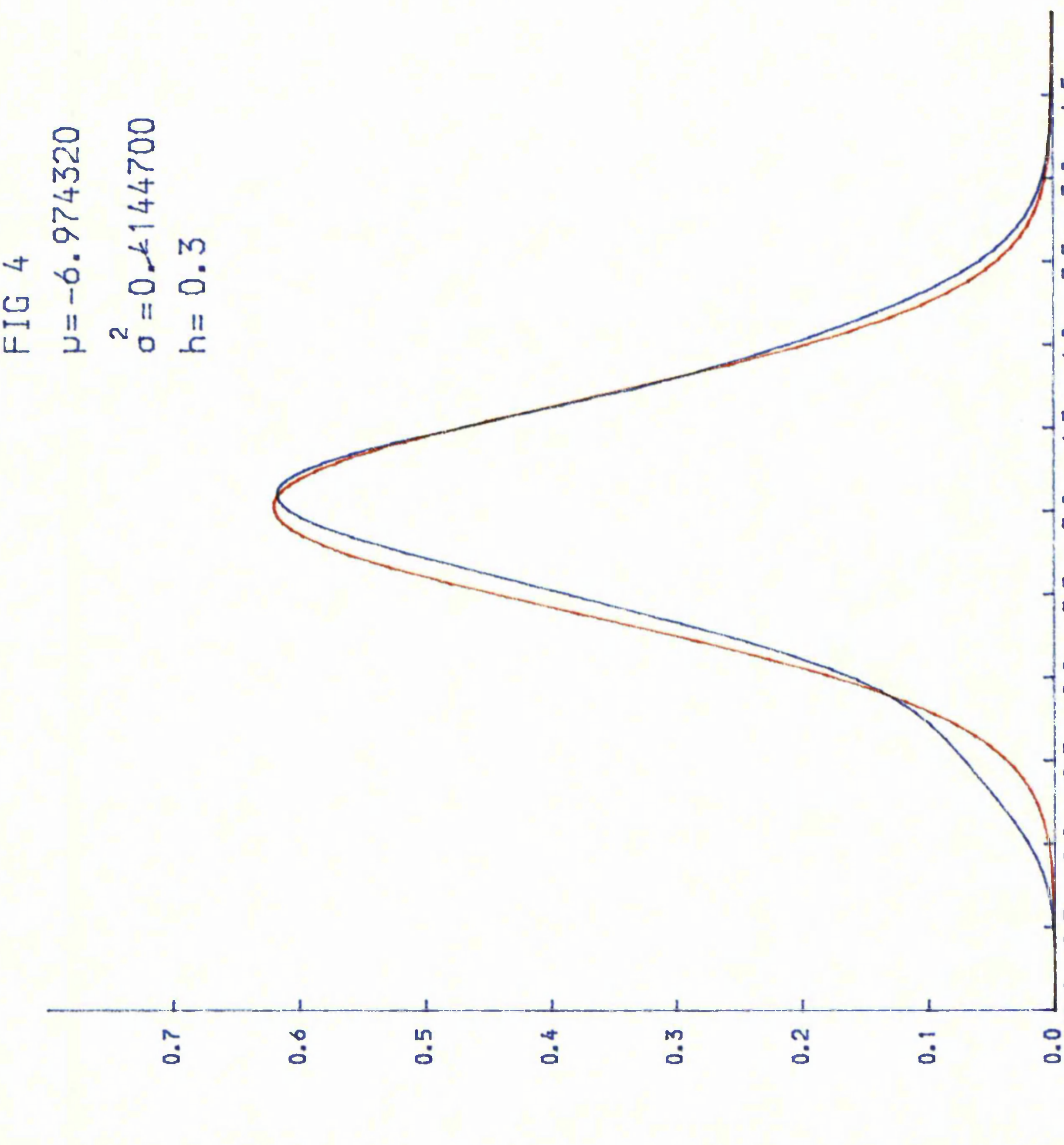


FIG 4
μ=-6.974320
σ² = 0.4144700
h= 0.3
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
-10.0
-9.5
-9.0
-8.5
-8.0
-7.5
-7.0
-6.5
-6.0
-5.5
-5.0
-4.5

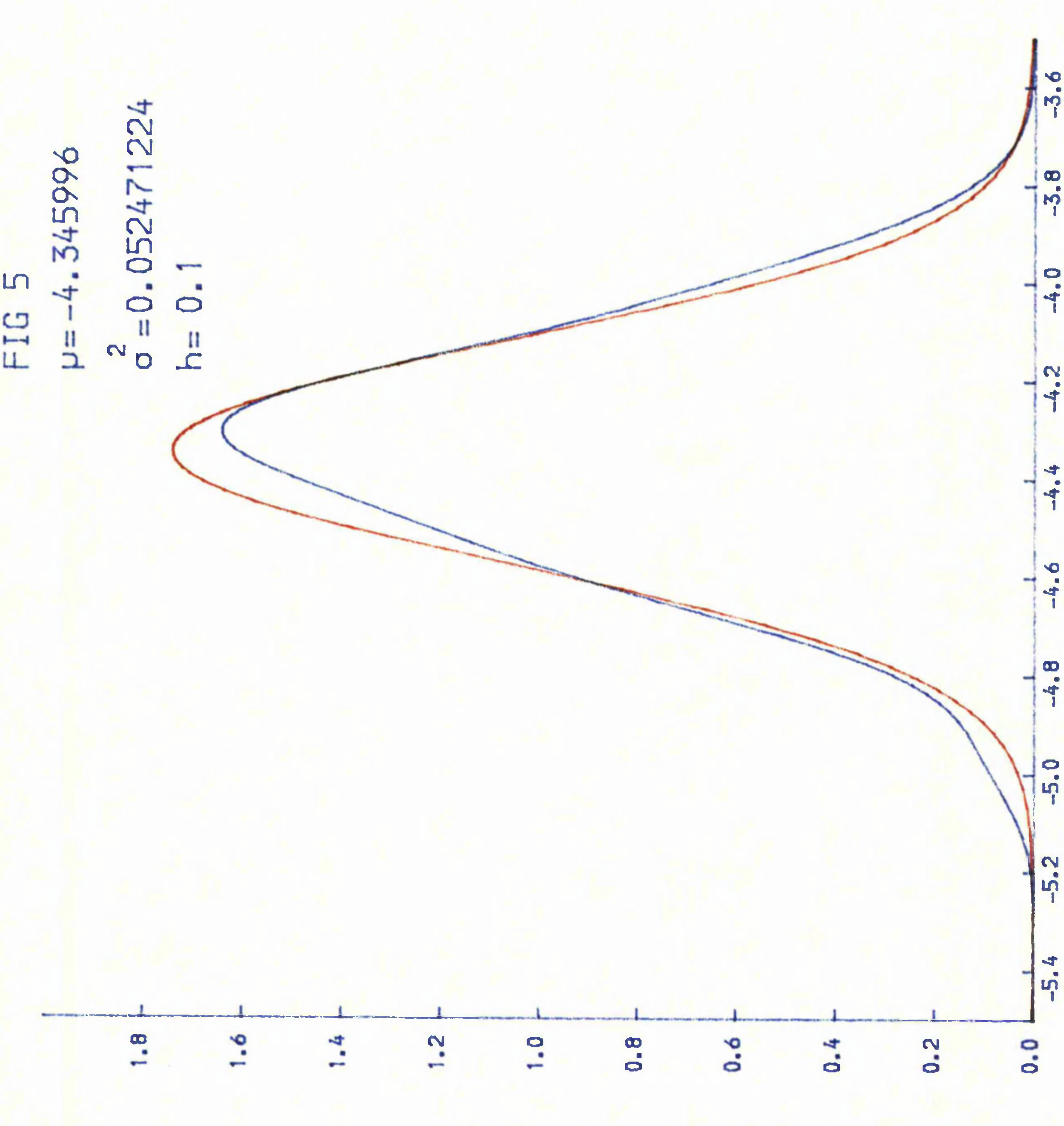


FIG 5
μ=-4.345996
σ^2=0.0524712224
h= 0.1
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-5.4
-5.2
-5.0
-4.8
-4.6
-4.4
-4.2
-4.0
-3.8
-3.6

CHAPTER 5

A TEST STATISTIC: DISCRETE DATA

5.1 Introduction

This chapter is analogous to Chapter 4. Here, instead of three densities we have three multinomial distributions with data in the form of vectors of relative frequencies $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$. In general, we will assume samples of different sizes underlying the "densities" $\underline{p}_1$, $\underline{p}_2$ and $\underline{p}_3$. Let the number of cells in each multinomial be m.

Thus

$$\underline{r}_j = \underline{p}_j + n_j^{-\frac{1}{2}} \underline{z}_j + n_j^{-1} \underline{w}_j \ldots \qquad j=1,2,3 \qquad (1)$$

where $\underline{z}_j \dot{\sim} N(\underline{0}, V_j)$, if n_j is large, and

$V_j = \text{diag}\,(p_{j1}, \ldots\ldots p_{jm}) - \underline{p}_j \underline{p}_j^T$.

We are interested in testing whether or not the p's are related as

$$\underline{p}_i - \underline{p}_j = t_{ijik}(\underline{p}_i - \underline{p}_k) \qquad (2)$$

or, equivalently, whether or not one of the $\underline{p}$'s is a proper mixture of the other two.

Here we define the function $Q(\hat{t})$ as follows.

$$Q(\hat{t}) = \sum_{\ell=1}^{m} [(r_{1\ell} - r_{2\ell}) - t(r_{1\ell} - r_{3\ell})]^2. \qquad (3)$$

r_{ji} (i=1,2,......m and j=1,2,3) are the components of $\underline{r}_j$.

The minimum distance estimator $\hat{t}$ is given by

$$\hat{t} = \frac{\sum_{\ell=1}^{m} (r_{1\ell} - r_{2\ell})(r_{1\ell} - r_{3\ell})}{\sum_{\ell=1}^{m} (r_{1\ell} - r_{3\ell})^2} \, . \qquad (4)$$

Putting $t=\hat{t}$ in (3) we get

$$Q(\hat{t}) = \sum_{\ell=1}^{m} (r_{1\ell} - r_{2\ell})^2 - \frac{\sum_{\ell=1}^{m} \left[(r_{1\ell} - r_{2\ell})(r_{1\ell} - r_{3\ell})\right]^2}{\sum_{\ell=1}^{m} (r_{1\ell} - r_{3\ell})^2} \quad . \tag{5}$$

Using the notation $\underline{r}_{ij} = \underline{r}_i - \underline{r}_j$ we may write (4) and (5) as

$$\hat{t} = \frac{(\underline{r}_1 - \underline{r}_2)^T(\underline{r}_1 - \underline{r}_3)}{(\underline{r}_1 - \underline{r}_3)^T(\underline{r}_1 - \underline{r}_3)} = \frac{\underline{r}_{12}^T\underline{r}_{13}}{\underline{r}_{13}^T\underline{r}_{13}} \tag{6}$$

$$\text{and } Q(\hat{t}) = (\underline{r}_1 - \underline{r}_2)^T(\underline{r}_1 - \underline{r}_2) - \frac{[(\underline{r}_1 - \underline{r}_2)^T(\underline{r}_1 - \underline{r}_3)]^2}{(\underline{r}_1 - \underline{r}_3)^T(\underline{r}_1 - \underline{r}_3)}$$

$$= \underline{r}_{12}^T\underline{r}_{12} - \frac{(\underline{r}_{12}^T\underline{r}_{13})^2}{(\underline{r}_{13}^T\underline{r}_{13})} \quad . \tag{7}$$

From (1) we have

$$\underline{r}_j - \underline{p}_j \cong n_j^{-1/2}\, \underline{z}_j + n_j^{-1}\underline{w}_j .$$

Let $\underline{r}_j - \underline{p}_j$ be denoted by $\underline{\delta}_j$ and $E\underline{r}_j$ by $\underline{\mu}_j$ respectively.

Obviously, $\underline{\mu}_j = \underline{p}_j$. Thus (6) and (7) may be respectively written as

$$\hat{t} = \frac{(\underline{\mu}_{12} + \underline{\delta}_{12})^T(\underline{\mu}_{13} + \underline{\delta}_{13})}{(\underline{\mu}_{13} + \underline{\delta}_{13})^T(\underline{\mu}_{13} + \underline{\delta}_{13})} \tag{8}$$

$$\text{and } Q(\hat{t}) = (\underline{\mu}_{12} + \underline{\delta}_{12})^T(\underline{\mu}_{12} + \underline{\delta}_{12}) - \frac{[(\underline{\mu}_{12} + \underline{\delta}_{12})^T(\underline{\mu}_{13} + \underline{\delta}_{13})]^2}{(\underline{\mu}_{13} + \underline{\delta}_{13})^T(\underline{\mu}_{13} + \underline{\delta}_{13})} \quad . \tag{9}$$

In section 5.3 the terms comprising the expressions of mean and variance of $\hat{t}$ and $Q(\hat{t})$ are expressed explicitly. In section 5.4 the expressions for the mean and variance of $Q(\hat{t})$ under the null hypothesis are stated for different orders in δ. It is shown that for the mean of $Q(\hat{t})$ the term of order zero in δ is equal to zero. It is also shown that for the mean of $[Q(\hat{t})]^2$ the terms of orders two and three in δ are equal to zero. The expressions for the mean and variance of $\hat{t}$ are also mentioned uptil order 2 in δ. The corresponding situation for the non-null case is also discussed at the end of section 5.4.

The expected values for various powers of the normal random variable with zero mean are derived in section 5.5. These results are necessary in evaluating the building blocks defined in section 5.3 and appropriately approximated in section 5.4.

In section 5.6 the expressions for the mean and variance of $\hat{t}$ and $Q(\hat{t})$, under the null hypothesis, are stated in a very convenient and compact form. A simulation study is carried out in section 5.7 and a 'parametric' test of the null hypothesis is suggested and carried out on several data sets. In section 5.8 a non-parametric test of the null hypothesis is suggested and carried out on the same data sets. In section 5.9, the various tests suggested are applied to some fish data.

5.2 Expressions for mean and variance of $\hat{t}$ and $Q(\hat{t})$

With different meanings for the $\underline{\mu}$'s and $\underline{\delta}$'s, the expressions for mean and variance of $\hat{t}$ and $Q(\hat{t})$ in terms of $\underline{\mu}$ and $\underline{\delta}$ are , obviously, exactly the same as those derived in section 4.2 of Chapter 4. Thus this section may be considered to be identical to section 4.2 of Chapter 4.

5.3 Elaboration of the terms comprising the expressions of mean and variance of $\hat{t}$ and $Q(\hat{t})$ in section 5.2

The typical terms in section 5.2 expressed explicitly are those involving up to order -2 in n_1.

(1) $$E(\underline{\delta}_1^T\underline{\delta}_1) = E[n_1^{-\frac{1}{2}}\underline{z}_1^T + n_1^{-1}\underline{w}_1^T][n_1^{-\frac{1}{2}}\underline{z}_1 + n_1^{-1}\underline{w}_1]$$

$$= n_1^{-1}E(\underline{z}_1^T\underline{z}_1) + 2n_1^{-3/2}E(\underline{z}_1^T\underline{w}_1) + n_1^{-2}E(\underline{w}_1^T\underline{w}_1).$$

(2) $$E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_1)$$

$$= E\sum_i\sum_j(p_{1i}-p_{2i})(p_{1j}-p_{3j})[n_1^{-\frac{1}{2}}z_{1i}+n_1^{-1}w_{1i}][n_1^{-\frac{1}{2}}z_{1j}+n_1^{-1}w_{1j}]$$

$$= \sum_i\sum_j(p_{1i}-p_{2i})(p_{1j}-p_{3j})\begin{bmatrix} n_1^{-1}E(z_{1i}z_{1j}) + n_1^{-3/2}E(z_{1i}w_{1j}) \\ +n_1^{-3/2}E(w_{1i}z_{1j}) + n_1^{-2}E(w_{1i}w_{1j}) \end{bmatrix}.$$

(3) $$E(\underline{\mu}_{12}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)$$

$$= E\sum_i\sum_j(p_{1i}-p_{2i})[n_1^{-\frac{1}{2}}z_{1i}+n_1^{-1}w_{1i}][n_1^{-\frac{1}{2}}z_{1j}+n_1^{-1}w_{ij}]$$

$$\times [n_1^{-\frac{1}{2}}z_{ij}+n_1^{-1}w_{1j}]$$

$$= \sum_i\sum_j(p_{1i}-p_{2i})\begin{bmatrix} n_1^{-3/2}E(z_{1i}z_{1j}^2) + n_1^{-2}E(z_{1i}z_{1j}w_{1j}) \\ +n_1^{-2}E(z_{1i}z_{1j}w_{1j}) + n_1^{-2}E(z_{1j}^2w_{1i}) \end{bmatrix}.$$

(4) $$E(\underline{\mu}_{12}^T\underline{\delta}_1)^2(\underline{\mu}_{13}^T\underline{\delta}_1)$$

$$= \sum_i\sum_j\sum_k(p_{1i}-p_{2i})(p_{1j}-p_{2j})(p_{1k}-p_{3k})$$

$$\times \left[n_1^{-3/2} E(z_{1i}z_{1j}z_{1k}) + n_1^{-2} E(z_{1i}z_{1j}w_{1k}) + n_1^{-2} E(z_{1i}w_{1j}z_{1k}) + n_1^{-2} E(z_{1j}z_{1k}w_{1i}) \right].$$

(5) $$E(\underline{\mu}_{12}^T \underline{\delta}_1)^2 (\underline{\mu}_{13}^T \underline{\delta}_1)^2$$

$$= \sum_i \sum_j \sum_k \sum_\ell (p_{1i} - p_{2i})(p_{1j} - p_{2j})(p_{1k} - p_{3k})(p_{1\ell} - p_{3\ell})$$

$$\times n_1^{-2} E(z_{1i}z_{1j}z_{1k}z_{1\ell}).$$

(6) $$E(\underline{\mu}_{12}^T \underline{\delta}_1)(\underline{\mu}_{13}^T \underline{\delta}_1)(\underline{\delta}_1^T \underline{\delta}_1)$$

$$= \sum_i \sum_j \sum_k (p_{1i}-p_{2i})(p_{1j}-p_{3j}) n_1^{-2} E(z_{1i}z_{1j}z_{1k}^2).$$

(7) $$E(\underline{\mu}_{12}^T \underline{\delta}_1)(\underline{\mu}_{13}^T \underline{\delta}_2)(\underline{\delta}_1^T \underline{\delta}_2)$$

$$= \sum_i \sum_j \sum_k (p_{1i}-p_{2i})(p_{1j}-p_{3j}) E(\delta_{1i}\delta_{2j}\delta_{1k}\delta_{2k})$$

$$= \sum_i \sum_j \sum_k (p_{1i}-p_{2i})(p_{1j}-p_{3j}) E(\delta_{1i}\delta_{1k}) E(\delta_{2j}\delta_{2k})$$

$$= \sum_i \sum_j \sum_k (p_{1i}-p_{2i})(p_{1j}-p_{3j}) n_1^{-1} n_2^{-1} E(z_{1i}z_{1k}) E(z_{2j}z_{2k}).$$

5.4 Expressions for mean and variance of $\hat{t}$ and $Q(\hat{t})$ (for different orders of δ) under the null hypothesis

The null hypothesis means that $\underline{p}_{12}=t\underline{p}_{13}$, where

$\underline{p}_{ij}=\underline{p}_i-\underline{p}_j$. $i,j\epsilon[1,2,3]$.

In terms of $\underline{\mu}$ the relationship is $\underline{\mu}_{12} = t\underline{\mu}_{13}$.

Thus under the null hypothesis, we have the following.

(1) $EQ(\hat{t})$, uptil terms of order 2 in δ , is given by

$$EQ(\hat{t})=(1-t)^2 E(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\delta}_2^T\underline{\delta}_2)+t^2 E(\underline{\delta}_3^T\underline{\delta}_3)$$

$$-\frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\left[(1-t)^2 E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+t^2 E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\right] . \qquad (10)$$

(2) $E[Q(\hat{t})]^2$, uptil terms of order 2 in δ , is given by

$$E[Q(\hat{t})]^2 = t^4(\underline{\mu}_{13}^T\underline{\mu}_{13})^2+4t^2\left[E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\right]$$

$$+2t^2(\underline{\mu}_{13}^T\underline{\mu}_{13})\left[E(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\delta}_2^T\underline{\delta}_2)\right]+t^4(\underline{\mu}_{13}^T\underline{\mu}_{13})^2$$

$$+4t^2\begin{bmatrix} t^2 E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+t^2 E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 \\ +E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2 \end{bmatrix}$$

$$+2t^2\begin{bmatrix} t^2 E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 \\ t^2 E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_1) \end{bmatrix}$$

$$-16t^3\left[tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$+12t^4\left[E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$-2t^4(\underline{\mu}_{13}^T\underline{\mu}_{13})\left[E(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\delta}_3^T\underline{\delta}_3)\right]$$

$$+2t^4(\underline{\mu}_{13}^T\underline{\mu}_{13})\left[E(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\delta}_3^T\underline{\delta}_3)\right]$$

$$-8t^4\left[E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]-2t^4(\underline{\mu}_{13}^T\underline{\mu}_{13})^2$$

$$+8t^3\left[tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2\right]$$

$$-2t^2 \begin{bmatrix} t^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 + E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 + 2tE(\underline{\mu}_{13}^T \underline{\delta}_1)^2 \\ + E(\underline{\mu}_{13}^T \underline{\delta}_2)^2 + t^2 E(\underline{\mu}_{13}^T \underline{\delta}_3)^2 \, 2t(\underline{\mu}_{13}^T \underline{\mu}_{13}) E(\underline{\delta}_1^T \underline{\delta}_1) \end{bmatrix}$$

$$+8t^3 E(\underline{\mu}_{13}^T \underline{\delta}_1)^2$$

$$-8t^2 [tE(\underline{\mu}_{13}^T \underline{\delta}_1)^2 + E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 + E(\underline{\mu}_{13}^T \underline{\delta}_2)^2]$$

$$-2t^2 (\underline{\mu}_{13}^T \underline{\mu}_{13}) [E(\underline{\delta}_1^T \underline{\delta}_1) + E(\underline{\delta}_2^T \underline{\delta}_2)] \; .$$

It turns out that everything cancels out, giving, upto terms of order two in δ ,

$$E[Q(\hat{t})]^2 = 0.$$

(3) $E[Q(\hat{t})]^2$, for terms of order three only in δ , is given by

$$E[Q(\hat{t})]^2 = 4t[E(\underline{\mu}_{13}^T \underline{\delta}_1)(\underline{\delta}_1^T \underline{\delta}_1) - E(\underline{\mu}_{13}^T \underline{\delta}_2)(\underline{\delta}_2^T \underline{\delta}_2)]$$

$$+4t^2 [tE(\underline{\mu}_{13}^T \underline{\delta}_1)(\underline{\delta}_1^T \underline{\delta}_1) + E(\underline{\mu}_{13}^T \underline{\delta}_1)(\underline{\delta}_1^T \underline{\delta}_1)]$$

$$\frac{+4t}{\underline{\mu}_{13}^T \underline{\mu}_{13}} \begin{bmatrix} t^3 E(\underline{\mu}_{13}^T \underline{\delta}_1)^3 + tE(\underline{\mu}_{13}^T \underline{\delta}_1)^3 + 2t^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)^3 \\ +2t^2 (\underline{\mu}_{13}^T \underline{\mu}_{13}) E(\underline{\mu}_{13}^T \underline{\delta}_1)(\underline{\delta}_1^T \underline{\delta}_1) + t^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)^3 \\ +E(\underline{\mu}_{13}^T \underline{\delta}_1)^3 + 2tE(\underline{\mu}_{13}^T \underline{\delta}_1)^3 + 2t(\underline{\mu}_{13}^T \underline{\mu}_{13}) E(\underline{\mu}_{13}^T \underline{\delta}_1)(\underline{\delta}_1^T \underline{\delta}_1) \\ -t^3 E(\underline{\mu}_{13}^T \underline{\delta}_3)^3 - E(\underline{\mu}_{13}^T \underline{\delta}_2)^3 \end{bmatrix}$$

$$\frac{-16t^2}{\underline{\mu}_{13}^T \underline{\mu}_{13}} \begin{bmatrix} t^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)^3 + E(\underline{\mu}_{13}^T \underline{\delta}_1)^3 + 2tE(\underline{\mu}_{13}^T \underline{\delta}_1)^3 - t^2 E(\underline{\mu}_{13}^T \underline{\delta}_3)^3 \end{bmatrix}$$

$$\frac{-8t^2}{\underline{\mu}_{13}^T \underline{\mu}_{13}} \begin{bmatrix} t^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)^3 + E(\underline{\mu}_{13}^T \underline{\delta}_1)^3 + 2tE(\underline{\mu}_{13}^T \underline{\delta}_1)^3 \\ +2t(\underline{\mu}_{13}^T \underline{\mu}_{13}) E(\underline{\mu}_{13}^T \underline{\delta}_1)(\underline{\delta}_1^T \underline{\delta}_1) - t^2 E(\underline{\mu}_{13}^T \underline{\delta}_3)^3 \end{bmatrix}$$

$$\frac{+48t^3}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\ [tE(\underline{\mu}_{13}^T\underline{\delta}_1)^3+E(\underline{\mu}_{13}^T\underline{\delta}_1)^3-tE(\underline{\mu}_{13}^T\underline{\delta}_3)^3]$$

$$-8t^3\ \times\ [tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-tE(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_3^T\underline{\delta}_3)]$$

$$+12t^4\ \times\ [E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-E(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_3^T\underline{\delta}_3)]$$

$$\frac{-32t^4}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\ [E(\underline{\mu}_{13}^T\underline{\delta}_1)^3-E(\underline{\mu}_{13}^T\underline{\delta}_3)^3]$$

$$-8t^4\ \times\ [E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-E(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_3^T\underline{\delta}_3)]$$

$$\frac{+16t^4}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\ [E(\underline{\mu}_{13}^T\underline{\delta}_1)^3-E(\underline{\mu}_{13}^T\underline{\delta}_3)^3]$$

$$+4t^3\ \times\ [tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-tE(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_3^T\underline{\delta}_3)]$$

$$\frac{-16t^3}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\ [tE(\underline{\mu}_{13}^T\underline{\delta}_1)^3+E(\underline{\mu}_{13}^T\underline{\delta}_1)^3-tE(\underline{\mu}_{13}^T\underline{\delta}_3)^3]$$

$$\frac{+4t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\left[\begin{array}{l} t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^3+E(\underline{\mu}_{13}^T\underline{\delta}_1)^3+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^3 \\ +2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^3 \end{array}\right]$$

$$-4t^2\,[tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)]$$

$$+4t^3E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-\frac{16t^3}{\underline{\mu}_{13}^T\underline{\mu}_{13}}E(\underline{\mu}_{13}^T\underline{\delta}_1)^3$$

$$\frac{+16t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}[tE(\underline{\mu}_{13}^T\underline{\delta}_1)^3+E(\underline{\mu}_{13}^T\underline{\delta}_1)^3]$$

$$\frac{-4t}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\left[\begin{array}{l} t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^3+E(\underline{\mu}_{13}^T\underline{\delta}_1)^3+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^3 \\ +2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-E(\underline{\mu}_{13}^T\underline{\delta}_2)^3 \end{array}\right]$$

$$+4t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)$$

$$-4t[tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_1)-E(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_2)].$$

Again, all terms cancel out so that, for terms of order 3 in δ, $E[Q(\hat{t})]^2=0$.

(4) $E[Q(\hat{t})]^2$, for terms of order four only in δ, is given by

$$E[Q(\hat{t})]^2=E(\underline{\delta}_1^T\underline{\delta}_1)^2+4E(\underline{\delta}_1^T\underline{\delta}_2)^2+E(\underline{\delta}_2^T\underline{\delta}_2)^2+2E(\underline{\delta}_1^T\underline{\delta}_1)E(\underline{\delta}_2^T\underline{\delta}_2)$$

$$+\frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\left[\begin{array}{l}
t^4E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+4t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_2)^4+t^4E(\underline{\mu}_{13}^T\underline{\delta}_3)^4\\
+4[t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2]\\
+4t^2[t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{12}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2]\\
+4t^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2
\end{array}\right]$$

$$+\frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\left[\begin{array}{l}
4t^2(\underline{\mu}_{13}^T\underline{\mu}_{13})^2[E(\underline{\delta}_1^T\underline{\delta}_1)^2+E(\underline{\delta}_1^T\underline{\delta}_3)^2+E(\underline{\delta}_1^T\underline{\delta}_2)^2+E(\underline{\delta}_2^T\underline{\delta}_3)^2]\\
+2t^2\left[\begin{array}{l}E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\\
+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)\end{array}\right]\\
+4tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4+2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+2t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\\
+2t\left[\begin{array}{l}2(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\\
+2t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{12}^T\underline{\delta}_3)^2+4t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)\end{array}\right]
\end{array}\right]$$

$$+\frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\left[\begin{aligned}
&2t\left[\begin{aligned}&+tE(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+2(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\delta}_1^T\underline{\delta}_1)\\&+2t^2(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1)\end{aligned}\right]\\
&\quad+8t(\underline{\mu}_{13}^T\underline{\mu}_{13})[tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)]\\
&\quad+8t^2(\underline{\mu}_{13}^T\underline{\mu}_{13})\left[\begin{aligned}&tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)\\&+E(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)\end{aligned}\right]
\end{aligned}\right]$$

$$+2t^2[E(\underline{\delta}_1^T\underline{\delta}_1)^2+E(\underline{\delta}_1^T\underline{\delta}_3)^2+E(\underline{\delta}_1^T\underline{\delta}_2)^2+E(\underline{\delta}_2^T\underline{\delta}_3)^2]$$

$$+\frac{8t}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\times$$

$$\left[\begin{aligned}
&t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)\\
&+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)\\
&+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)\\
&+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)\\
&+tE(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1)\\
&+tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)\\
&+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\delta}_1^T\underline{\delta}_1)+tE(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{12}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)
\end{aligned}\right]$$

$$\frac{-16t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\times$$

$$\left[\begin{aligned}
&tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)\\
&+tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)
\end{aligned}\right]$$

$$\frac{-16t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \times$$

$$\left[\begin{array}{l} +E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3) \\ +tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1) \end{array}\right]$$

$$\frac{-16t}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2} \times$$

$$\left[\begin{array}{l}
t^3E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4+2t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 \\
+t^2[[tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+2(\underline{\mu}_{13}^T\underline{\delta}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)]+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^4 \\
+E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 \\
+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1) \\
+2t^2\left[\begin{array}{l} tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 \\ +(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \end{array}\right] \\
+2\left[\begin{array}{l} tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 \\ +t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2) \end{array}\right] \\
+2t^2\left[\begin{array}{l} tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 \\ +(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \end{array}\right] \\
+t^3E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 \\
+2t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 \\
+t^2[tE(\underline{\mu}_{13}^T\underline{\delta}_3)^4+2(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1)] \\
+2t[E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)]
\end{array}\right]$$

$$\frac{32t^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\left[\begin{array}{l} tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 \\ +(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \end{array}\right]$$

$$\frac{+48t^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\left[\begin{array}{l} t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 \\ +2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4+4t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\ 4tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 \\ +t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^4 \\ +E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+2tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 \end{array}\right]$$

$$\frac{-\ 8t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\times$$

$$\left[\begin{array}{l} t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1) \\ +E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\delta}_1^T\underline{\delta}_1)+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+4t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \\ +4tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3) \\ +E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3)+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_3^T\underline{\delta}_3) \\ +E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\delta}_3^T\underline{\delta}_3)+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3) \end{array}\right]$$

$$\frac{+\ 24t^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\left[t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\right]$$

$$\frac{+24t^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\left[\begin{array}{l}
+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)\\
+4t\quad tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\\
\qquad+(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)\\
+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2\\
+2tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\\
+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^4+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1)
\end{array}\right]$$

$$\frac{-4t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\times$$

$$\left[\begin{array}{l}
t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2\underline{\delta}_1^T\underline{\delta}_1\\
+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\delta}_1^T\underline{\delta}_1)+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E\underline{\delta}_1^T\underline{\delta}_1+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_1)^2\\
+4t\ [tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)]\\
+4t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_3)^2+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_3^T\underline{\delta}_3\\
+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\delta}_3^T\underline{\delta}_3)\\
+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_3^T\underline{\delta}_3)+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_1)E(\underline{\delta}_3^T\underline{\delta}_3)
\end{array}\right]$$

$$\frac{+48t^3}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\times$$

$$\left[\begin{array}{l}
tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3)\\
\\
+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)
\end{array}\right]$$

$$\frac{+48t^3}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \times \left[E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E(\underline{\delta}_3^T\underline{\delta}_3) + 2E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) + 2tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) + tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2 E(\underline{\delta}_1^T\underline{\delta}_1) + tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_3^T\underline{\delta}_3) \right]$$

$$\frac{-128t^3}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2} \times \left[tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4 + 3tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 + E(\underline{\mu}_{13}^T\underline{\delta}_1)^4 + 3E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 + 3tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 + tE(\underline{\mu}_{13}^T\underline{\delta}_3)^4 \right]$$

$$+3t^4\left[E(\underline{\delta}_1^T\underline{\delta}_1)^2 + 4E(\underline{\delta}_1^T\underline{\delta}_3)^2 + E(\underline{\delta}_3^T\underline{\delta}_3)^2 + 2E(\underline{\delta}_1^T\underline{\delta}_1)E(\underline{\delta}_3^T\underline{\delta}_3)\right]$$

$$\frac{+80t^4}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\left[E(\underline{\mu}_{13}^T\underline{\delta}_1)^4 + 6E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 + E(\underline{\mu}_{13}^T\underline{\delta}_3)^4\right]$$

$$\frac{-48t^4}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \times \left[E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1) + E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E(\underline{\delta}_3^T\underline{\delta}_3) + 4E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) + E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 E(\underline{\delta}_1^T\underline{\delta}_1) + E(\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_3^T\underline{\delta}_3) \right]$$

$$-2t^2\left[E(\underline{\delta}_1^T\underline{\delta}_1)^2 + E(\underline{\delta}_1^T\underline{\delta}_3)^2 + E(\underline{\delta}_1^T\underline{\delta}_2)^2 + E(\underline{\delta}_2^T\underline{\delta}_3)^2\right]$$

$$\frac{+8t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \times \left[tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1) + E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1) + E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2) + tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) + tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \right]$$

$$\frac{+8t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \times$$

$$\left[+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_2^T\underline{\delta}_3) \right.$$
$$\left. +tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1) \right]$$

$$\frac{+2t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \times$$

$$\left[t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1) \right.$$
$$+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 E\underline{\delta}_1^T\underline{\delta}_1+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1)+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_1)^2$$
$$+4t[tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)]$$
$$+4t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_3)^2+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_3^T\underline{\delta}_3+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3)$$
$$+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\delta}_3^T\underline{\delta}_3)$$
$$\left. +t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_3^T\underline{\delta}_3)+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_1)E(\underline{\delta}_3^T\underline{\delta}_3) \right]$$

$$\frac{-8t^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2} \times$$

$$\left[t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 \right.$$
$$t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)$$
$$+4t\left[tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 + (\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \right]$$
$$+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2$$
$$+2tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2$$
$$\left. +t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^4+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E\underline{\delta}_1^T\underline{\delta}_1 \right]$$

$$\frac{-\,16t^3}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \times \left[\begin{array}{l} tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_3^T\underline{\delta}_3)+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \\ +E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_3^T\underline{\delta}_3+2E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \\ +2tE(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\delta}_1^T\underline{\delta}_3)+tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1) \\ +tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_3^T\underline{\delta}_3) \end{array} \right]$$

$$\frac{+32t^3}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2} \times \left[\begin{array}{l} tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4+3tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^4 \\ +3E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+3tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 \\ +tE(\underline{\mu}_{13}^T\underline{\delta}_3)^4 \end{array} \right]$$

$$-2t^4[E(\underline{\delta}_1^T\underline{\delta}_1)^2+4E(\underline{\delta}_1^T\underline{\delta}_3)^2+E(\underline{\delta}_3^T\underline{\delta}_3)^2+2E(\underline{\delta}_1^T\underline{\delta}_1)E(\underline{\delta}_3^T\underline{\delta}_3)]$$

$$\frac{+24t^4}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \left[\begin{array}{l} E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_3^T\underline{\delta}_3 \\ +4E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_1^T\underline{\delta}_1) \\ +E(\underline{\mu}_{13}^T\underline{\delta}_3)^2(\underline{\delta}_3^T\underline{\delta}_3) \end{array} \right]$$

$$\frac{-32t^4}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}[E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+6E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_3)^4]$$

$$\frac{-8t}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \times \left[\begin{array}{l} tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2\underline{\delta}_1^T\underline{\delta}_1+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2) \\ +tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)+tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2) \\ +E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E\underline{\delta}_1^T\underline{\delta}_1 \\ +tE(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3) \end{array} \right]$$

$$\frac{+\ 8t}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\Big[\, t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2$$

$$+t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2$$

$$+2t^2\Big[(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)^2\underline{\delta}_1^T\underline{\delta}_1+tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2$$

$$+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)\Big]$$

$$+2t\Big[tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\Big]$$

$$+2t^2\Big[(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2$$

$$+(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3)\Big]\Big]$$

$$\frac{+8t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\times$$

$$\Big[tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)$$

$$+tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_3^T\underline{\delta}_3+tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_3^T\underline{\delta}_3+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_3^T\underline{\delta}_3$$

$$+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E\underline{\delta}_1^T\underline{\delta}_1+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E\underline{\delta}_3^T\underline{\delta}_3\Big]$$

$$\frac{-32t^2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\times$$

$$\Big[tE(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2$$

$$+tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2$$

$$+E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\Big]$$

$$\frac{-16t^3}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\times$$

$$[E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_3^T\underline{\delta}_3+2E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3)]$$

$$+\frac{32t^3}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}[E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+3E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2]$$

$$\frac{-2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\times$$

$$\left[\begin{array}{l} t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1) \\ +E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E\underline{\delta}_1^T\underline{\delta}_1+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E\underline{\delta}_1^T\underline{\delta}_1+2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_1)^2 \\ +4\left[\begin{array}{l} tE(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)+E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2) \\ +t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_2)^2 \end{array}\right] \\ +t^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_2^T\underline{\delta}_2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\delta}_2^T\underline{\delta}_2)+2tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_2^T\underline{\delta}_2 \\ +E(\underline{\mu}_{13}^T\underline{\delta}_2)^2(\underline{\delta}_2^T\underline{\delta}_2)+t^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E(\underline{\delta}_2^T\underline{\delta}_2) \\ +2t(\underline{\mu}_{13}^T\underline{\mu}_{13})E(\underline{\delta}_1^T\underline{\delta}_1)E(\underline{\delta}_2^T\underline{\delta}_2) \end{array}\right]$$

$$+\frac{8t}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\times\left[\begin{array}{l} tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1) \\ +tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2E\underline{\delta}_1^T\underline{\delta}_1+2E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2) \\ +tE(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_2^T\underline{\delta}_2+E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_2^T\underline{\delta}_2 \\ +tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2E\underline{\delta}_2^T\underline{\delta}_2 \end{array}\right]$$

$$+2t^2[E(\underline{\delta}_1^T\underline{\delta}_1)^2+E(\underline{\delta}_1^T\underline{\delta}_1)E(\underline{\delta}_3^T\underline{\delta}_3)+E(\underline{\delta}_1^T\underline{\delta}_1)E(\underline{\delta}_2^T\underline{\delta}_2)+E(\underline{\delta}_2^T\underline{\delta}_2)E\underline{\delta}_3^T\underline{\delta}_3]$$

$$\frac{-8t^2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\left[\begin{array}{l} E(\underline{\mu}_{13}^T\underline{\delta}_1)^2(\underline{\delta}_1^T\underline{\delta}_1)+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E\underline{\delta}_1^T\underline{\delta}_1 \\ +E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E\underline{\delta}_2^T\underline{\delta}_2+E(\underline{\mu}_{13}^T\underline{\delta}_3)^2E\underline{\delta}_2^T\underline{\delta}_2 \end{array}\right].$$

After considerable algebra, we have, $E[Q(t)]^2$, uptil terms of order 4 in δ only , given by

$E[Q(\hat{t})]^2$

$$= \frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\left[\begin{array}{l} (1-t)^4E(\underline{\mu}_{13}^T\underline{\delta}_1)^4+E(\underline{\mu}_{13}^T\underline{\delta}_2)^4+t^4E(\underline{\mu}_{13}^T\underline{\delta}_3)^4 \\ +6(1-t)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 \end{array}\right.$$

$$= \frac{1}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^2} \left[\begin{array}{l} 6t^2 E(\underline{\mu}_{13}^T \underline{\delta}_2)^2 E(\underline{\mu}_{13}^T \underline{\delta}_3)^2 \\ +6t^2 (1-t)^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 E(\underline{\mu}_{13}^T \underline{\delta}_3)^2 \end{array} \right]$$

$$\frac{-2}{\underline{\mu}_{13}^T \underline{\mu}_{13}} \left[\begin{array}{l} (1-t)^4 E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 (\underline{\delta}_1^T \underline{\delta}_1) + E(\underline{\mu}_{13}^T \underline{\delta}_2)^2 \underline{\delta}_2^T \underline{\delta}_2 \\ +t^4 E(\underline{\mu}_{13}^T \underline{\delta}_3)^2 \underline{\delta}_3^T \underline{\delta}_3 \\ +(1-t)^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 E\underline{\delta}_2^T \underline{\delta}_2 \\ +t^2 (1-t)^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 E\underline{\delta}_3^T \underline{\delta}_3 \\ +(1-t)^2 E(\underline{\mu}_{13}^T \underline{\delta}_2)^2 E\underline{\delta}_1^T \underline{\delta}_1 \\ +t^2 (1-t)^2 E(\underline{\mu}_{13}^T \underline{\delta}_3)^2 E\underline{\delta}_1^T \underline{\delta}_1 \\ +t^2 E(\underline{\mu}_{13}^T \underline{\delta}_3)^2 E\underline{\delta}_2^T \underline{\delta}_2 \\ +t^2 E(\underline{\mu}_{13}^T \underline{\delta}_2)^2 E\underline{\delta}_3^T \underline{\delta}_3 \\ +4(1-t)^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)(\underline{\mu}_{13}^T \underline{\delta}_2)(\underline{\delta}_1^T \underline{\delta}_2) \\ -4t^2 (1-t)^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)(\underline{\mu}_{13}^T \underline{\delta}_3)(\underline{\delta}_1^T \underline{\delta}_3) \\ +4t^2 E(\underline{\mu}_{13}^T \underline{\delta}_2)(\underline{\mu}_{13}^T \underline{\delta}_3)(\underline{\delta}_2^T \underline{\delta}_3) \end{array} \right]$$

$$+(1-t)^4 E(\underline{\delta}_1^T \underline{\delta}_1)^2 + E(\underline{\delta}_2^T \underline{\delta}_2)^2 + t^4 E(\underline{\delta}_3^T \underline{\delta}_3)^2 + 4(1-t)^2 E(\underline{\delta}_1^T \underline{\delta}_2)^2$$
$$+4t^2 E(\underline{\delta}_2^T \underline{\delta}_3)^2 + 4t^2 (1-t)^2 E(\underline{\delta}_1^T \underline{\delta}_3)^2 + 2(1-t)^2 E(\underline{\delta}_1^T \underline{\delta}_1) E(\underline{\delta}_2^T \underline{\delta}_2)$$
$$+2t^2 E(\underline{\delta}_2^T \underline{\delta}_2) E(\underline{\delta}_3^T \underline{\delta}_3) + 2t^2 (1-t)^2 E(\underline{\delta}_1^T \underline{\delta}_1) E(\underline{\delta}_3^T \underline{\delta}_3). \quad (11)$$

From equation (10) of this section on page 98 we have

$$[E\ Q(\hat{t})]^2$$
$$=(1-t)^4 (E(\underline{\delta}_1^T \underline{\delta}_1))^2 + (E(\underline{\delta}_2^T \underline{\delta}_2))^2 + t^4 (E(\underline{\delta}_3^T \underline{\delta}_3))^2$$
$$+2(1-t)^2 E\underline{\delta}_1^T \underline{\delta}_1\, E\underline{\delta}_2^T \underline{\delta}_2 + 2t^2 (1-t)^2 E\underline{\delta}_1^T \underline{\delta}_1\, E\underline{\delta}_3^T \underline{\delta}_3$$
$$+2t^2 E\underline{\delta}_2^T \underline{\delta}_2\, E\underline{\delta}_3^T \underline{\delta}_3$$

$$+\frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\left[\begin{array}{l}(1-t)^4[E(\underline{\mu}_{13}^T\underline{\delta}_1)^2]^2+[E(\underline{\mu}_{13}^T\underline{\delta}_2)^2]^2\\ +t^4[E(\underline{\mu}_{13}^T\underline{\delta}_3)^2]^2\\ +2(1-t)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\\ +2t^2(1-t)^2E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\\ +2t^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\end{array}\right]$$

$$\frac{-2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\left[\begin{array}{l}(1-t)^4E\underline{\delta}_1^T\underline{\delta}_1E(\underline{\mu}_{13}^T\underline{\delta}_1)^2\\ +(1-t)^2E\underline{\delta}_1^T\underline{\delta}_1E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\\ +t^2(1-t)^2E\underline{\delta}_1^T\underline{\delta}_1E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\\ +(1-t)^2E(\underline{\delta}_2^T\underline{\delta}_2)E(\underline{\mu}_{13}^T\underline{\delta}_1)^2\\ +E(\underline{\delta}_2^T\underline{\delta}_2)E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\\ +t^2E(\underline{\delta}_2^T\underline{\delta}_2)E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\\ +t^2(1-t)^2E\underline{\delta}_3^T\underline{\delta}_3E(\underline{\mu}_{13}^T\underline{\delta}_1)^2\\ +t^2E\underline{\delta}_3^T\underline{\delta}_3E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\\ +t^4E\underline{\delta}_3^T\underline{\delta}_3E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\end{array}\right]. \qquad (12)$$

Thus, using equations (11) and (12) we have

$$\text{var } Q(\hat{t})=E[Q(\hat{t})]^2-[EQ(\hat{t})]^2$$

$$=(1-t)^4[E(\underline{\delta}_1^T\underline{\delta}_1)^2-(E\underline{\delta}_1^T\underline{\delta}_1)^2]$$
$$+[E(\underline{\delta}_2^T\underline{\delta}_2)^2-(E\underline{\delta}_2^T\underline{\delta}_2)^2]$$
$$+t^4[E(\underline{\delta}_3^T\underline{\delta}_3)^2-(E\underline{\delta}_3^T\underline{\delta}_3)^2]$$
$$+4(1-t)^2E(\underline{\delta}_1^T\underline{\delta}_2)^2+4t^2E(\underline{\delta}_2^T\underline{\delta}_3)^2$$
$$+4t^2(1-t)^2E(\underline{\delta}_1^T\underline{\delta}_3)^2$$

$$\frac{-2}{\underline{\mu}_{13}^T\underline{\mu}_{13}} \left[\begin{array}{l} (1-t)^4[E(\underline{\mu}_{13}^T\underline{\delta}_1)^2\underline{\delta}_1^T\underline{\delta}_1 - E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E\underline{\delta}_1^T\underline{\delta}_1] \\ +[E(\underline{\mu}_{13}^T\underline{\delta}_2)^2\underline{\delta}_2^T\underline{\delta}_2 - E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 E\underline{\delta}_2^T\underline{\delta}_2] \\ +t^4[E(\underline{\mu}_{13}^T\underline{\delta}_3)^2\underline{\delta}_3^T\underline{\delta}_3 - E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 E\underline{\delta}_3^T\underline{\delta}_3] \\ +4(1-t)^2 E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2) \\ +4t^2(1-t)^2 E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_1^T\underline{\delta}_3) \\ +4t^2 E(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\mu}_{13}^T\underline{\delta}_3)(\underline{\delta}_2^T\underline{\delta}_3) \end{array} \right]$$

$$\frac{+1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2} \left[\begin{array}{l} (1-t)^4[E(\underline{\mu}_{13}^T\underline{\delta}_1)^4 - [E(\underline{\mu}_{13}^T\underline{\delta}_1)^2]^2]] \\ +[E(\underline{\mu}_{13}^T\underline{\delta}_2)^4 - [E(\underline{\mu}_{13}^T\underline{\delta}_2)^2]^2] \\ +t^4[E(\underline{\mu}_{13}^T\underline{\delta}_3)^4 - [E(\underline{\mu}_{13}^T\underline{\delta}_3)^2]^2] \\ +4(1-t)^2 E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 \\ +4t^2 E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 \\ +4t^2(1-t)^2 E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 \end{array} \right] \quad (13)$$

(5) $E\hat{t}$ and var $\hat{t}$, uptil terms of order 2 in δ are given by

$$E\hat{t} = t + \frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}}[(1-t)E\underline{\delta}_1^T\underline{\delta}_1 - tE\underline{\delta}_3^T\underline{\delta}_3]$$

$$- \frac{2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}[(1-t)E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 - tE(\underline{\mu}_{13}^T\underline{\delta}_3)^2]$$

and

$$E(\hat{t})^2 = t^2 + \frac{2t}{\underline{\mu}_{13}^T\underline{\mu}_{13}}[(1-t)E\underline{\delta}_1^T\underline{\delta}_1 - tE\underline{\delta}_3^T\underline{\delta}_3]$$

$$+ \frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2} \left[\begin{array}{l} (5t^2-6t+1)E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 \\ +E(\underline{\mu}_{13}^T\underline{\delta}_2)^2 + 5t^2 E(\underline{\mu}_{13}^T\underline{\delta}_3)^2 \end{array} \right] .$$

Thus

var $\hat{t}$, uptil terms of order 2 in δ is given by

$$\text{var } \hat{t} = E(\hat{t})^2 - (E\hat{t})^2$$

$$= \frac{1}{(\underline{\mu}_{13}^T \underline{\mu}_{13})^2} \left[(1-t)^2 E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 + E(\underline{\mu}_{13}^T \underline{\delta}_2)^2 + t^2 E(\underline{\mu}_{13}^T \underline{\delta}_3)^2 \right] .$$

For the non-null case there is no reason why the zero order term in δ, and the second and third order terms in δ, in the mean and variance of $Q(\hat{t})$, respectively,are zero. Thus the leading terms are of order zero in δ for the mean of $Q(\hat{t})$, and of order two in δ for the variance of $Q(\hat{t})$. Hence the expressions for the mean of $Q(\hat{t})$ uptil order zero in δ and for the variance of $Q(\hat{t})$ uptil order two in δ, as mentioned in section 5.2, are sufficient for the non-null case. The basic components of the mean and variance of $Q(\hat{t})$ are elaborated in sections 5.3 and 5.5.

(Continued on next page)

Since it is enough to consider terms uptil order n^{-2} in var $\hat{Q}(t)$, therefore, the building blocks, defined in section 5.3, are given by

$$E(\underline{\delta}_1^T\underline{\delta}_1)^2 = En_1^{-2}(\underline{z}_1^T\underline{z}_1)^2 = n_1^{-2}E(\underline{z}_1^T\underline{z}_1)^2$$

$$E\underline{\delta}_1^T\underline{\delta}_1 = n_1^{-1}E\underline{z}_1^T\underline{z}_1$$

$$E(\underline{\delta}_1^T\underline{\delta}_2)^2 = n_1^{-1}n_2^{-1}E(\underline{z}_1^T\underline{z}_2)^2$$

$$E(\underline{\mu}_{13}^T\underline{\delta}_1)^2\underline{\delta}_1^T\underline{\delta}_1 = n_1^{-2}\sum_i\sum_j\sum_k (p_{1i}-p_{3i})(p_{1j}-p_{3j})\times E(z_{1i}z_{1j}z_{1k}^2)$$

$$E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E\underline{\delta}_1^T\underline{\delta}_1 = n_1^{-2}E(\underline{z}_1^T\underline{z}_1)\times\sum_i\sum_j (p_{1i}-p_{3i})(p_{1j}-p_{3j})E(z_{1i}z_{1j})$$

$$E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)$$

$$= n_1^{-1}n_2^{-1}\sum_i\sum_j\sum_k (p_{1i}-p_{3i})(p_{1j}-p_{3j})E(z_{1i}z_{1k})E(z_{2j}z_{2k})$$

$$E(\underline{\mu}_{13}^T\underline{\delta}_1)^4$$

$$= n_1^{-2}\sum_i\sum_j\sum_k\sum_l (p_{1i}-p_{3i})(p_{1j}-p_{3j})(p_{1k}-p_{3k})(p_{1l}-p_{3l})$$

$$\times E(z_{1i}z_{1j}z_{1k}z_{1l})$$

and

$$E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 = n_1^{-1}\sum_i\sum_j (p_{1i}-p_{3i})(p_{1j}-p_{3j})E(z_{1i}z_{1j}).$$

In the next section we determine the expected values of various powers of the normal random variable z with zero mean.

5.5 Relevant moments of $\underline{z}\sim N(\underline{0},V)$

Let $\underline{z}=(z_1,z_2,\ldots\ldots z_m)$.

The moment generating function of $\underline{z}$ is

$M(\underline{t})=\exp\frac{1}{2}(\underline{t}^TV\underline{t})$

Thus $\frac{\partial M}{\partial t_i} = (\sum_r V_{ir} t_r) M$

$$\frac{\partial^2 M}{\partial t_i \partial t_j} = (\sum_r V_{ir} t_r)(\sum_r V_{jr} t_r) M + V_{ij} M$$

$$\frac{\partial^3 M}{\partial t_i \partial t_j \partial t_k} = (\sum_r V_{ir} t_r)(\sum_r V_{jr} t_r)(\sum_r V_{kr} t_r) M$$

$$+ V_{ij}(\sum_r V_{kr} t_r) M + V_{ik}(\sum_r V_{jr} t_r) M$$

$$+ V_{jk}(\sum_r V_{ir} t_r) M$$

and

$$\frac{\partial^4 M}{\partial t_i \partial t_j \partial t_k \partial t_\ell}$$

$$= (\sum_r V_{ir} t_r)(\sum_r V_{jr} t_r)(\sum_r V_{kr} t_r)(\sum_r V_{\ell r} t_r) M$$

$$+ V_{i\ell}(\sum_r V_{jr} t_r)(\sum_r V_{kr} t_r) M$$

$$+ (\sum_r V_{ir} t_r)(\sum_r V_{kr} t_r) V_{j\ell} M$$

$$+ (\sum_r V_{ir} t_r)(\sum_r V_{jr} t_r) V_{k\ell} M$$

$$+ (\sum_r V_{kr} t_r)(\sum_r V_{\ell r} t_r) V_{ij} M$$

$$+ V_{ij} V_{k\ell} M + V_{ik}(\sum_r V_{jr} t_r)(\sum_r V_{\ell r} t_r) M$$

$$+ V_{ik} V_{j\ell} M + V_{jk}(\sum_r V_{ir} t_r)(\sum_r V_{\ell r} t_r) M$$

$$+ V_{jk} V_{i\ell} M.$$

Thus

$$E(z_i z_j) = \left. \frac{\partial^2 M}{\partial t_i \partial t_j} \right|_{\underline{t}=0} = V_{ij}$$

$$E(z_i z_j z_k) = \left. \frac{\partial^3 M}{\partial t_i \partial t_j \partial t_k} \right|_{\underline{t}=0} = 0$$

and

$$E(z_i z_j z_k z_\ell) = \left. \frac{\partial^4 M}{\partial t_i \partial t_j \partial t_k \partial t_\ell} \right|_{\underline{t}=\underline{0}}$$

$$= V_{ij} V_{k\ell} + V_{ik} V_{j\ell} + V_{jk} V_{i\ell} .$$

Thus

$$E(\underline{\delta}_1^T \underline{\delta}_1)^2 = n_1^{-2} E(\underline{z}_1^T \underline{z}_1)^2 = n_1^{-2} \sum_i \sum_j E z_{1i}^2 z_{1j}^2$$

$$= 2n_1^{-2} \sum_i \sum_j [V_{1ij}^2 + V_{1ii} V_{1jj}]$$

and $E\underline{\delta}_1^T \underline{\delta}_1 = n_1^{-1} E(\underline{z}_1^T \underline{z}_1) = n_1^{-1} \sum_i E z_{1i}^2 = n_1^{-1} \sum_i V_{1ii} = n_1^{-1} \mathrm{tr} V_1 .$

Hence $E(\underline{\delta}_1^T \underline{\delta}_1)^2 - [E\underline{\delta}_1^T \underline{\delta}_1]^2 = n_1^{-2} \sum_i \sum_j [2V_{1ij}^2 + V_{1ii} V_{1jj} - V_{1ii} V_{1jj}]$

$$= 2n_1^{-2} \sum_i \sum_j V_{1ij}^2 = 2n_1^{-2} \mathrm{tr} V_1^2 .$$

Also, $E(\underline{\delta}_1^T \underline{\delta}_2)^2 = n_1^{-1} n_2^{-1} E(\underline{z}_1^T \underline{z}_2)^2$

$$= n_1^{-1} n_2^{-1} \sum_i \sum_j E z_{1i} z_{2i} z_{1j} z_{2j}$$

$$= n_1^{-1} n_2^{-1} \sum_i \sum_j E(z_{1i} z_{1j}) E(z_{2i} z_{2j})$$

$$= n_1^{-1} n_2^{-1} \sum_i \sum_j V_{1ij} V_{2ij}$$

$$= n_1^{-1} n_2^{-1} \mathrm{tr} V_1 V_2 .$$

$$E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 \underline{\delta}_1^T \underline{\delta}_1 = n_1^{-2} \sum_i \sum_j \sum_k (p_{1i} - p_{3i})(p_{1j} - p_{3j}) E(z_{1i} z_{1j} z_{1k})^2$$

$$= 2n_1^{-2} \sum_i \sum_j \sum_k (p_{1i} - p_{3i})(p_{1j} - p_{3j}) V_{1ki} V_{1kj} .$$

$$E(\underline{\mu}_{13}^T \underline{\delta}_1)^2 = n_1^{-1} \sum_i \sum_j (p_{1i} - p_{3i})(p_{1j} - p_{3j}) E(z_{1i} z_{1j})$$

$$= n_1^{-1} \sum_i \sum_j (p_{1i} - p_{3i})(p_{1j} - p_{3j}) V_{1ij} = n_1^{-1} \underline{p}_{13}^T V_1 \underline{p}_{13} .$$

Thus, we have,

$$E(\underline{\mu}_{13}^T\underline{\delta}_1)^2\underline{\delta}_1^T\underline{\delta}_1 - E(\underline{\mu}_{13}^T\underline{\delta}_1)^2 E\underline{\delta}_1^T\underline{\delta}_1$$

$$= n_1^{-2}\sum_{ijk}\sum\sum (p_{1i}-p_{3i})(p_{1j}-p_{3j})\begin{bmatrix} 2V_{1ki}V_{1kj}+V_{1kk}V_{1ij} \\ -V_{1kk}V_{1ij} \end{bmatrix}$$

$$= 2n_1^{-2}\sum_{ijk}\sum\sum (p_{1i}-p_{3i})(p_{1j}-p_{3j})V_{1ki}V_{1kj}$$

$$= 2n_1^{-2}\,\underline{p}_{13}^T V_1^2 \underline{p}_{13}^3 .$$

Also,

$$E(\underline{\mu}_{13}^T\underline{\delta}_1)(\underline{\mu}_{13}^T\underline{\delta}_2)(\underline{\delta}_1^T\underline{\delta}_2)$$

$$= n_1^{-1}n_2^{-1}\sum_{ijk}\sum\sum (p_{1i}-p_{3i})(p_{1j}-p_{3j})E(z_{1i}z_{1k})E(z_{2j}z_{2k})$$

$$= n_1^{-1}n_2^{-1}\sum_{ijk}\sum\sum (p_{1i}-p_{3i})(p_{1j}-p_{3j})V_{1ik}V_{2jk}$$

$$= n_1^{-1}n_2^{-1}\underline{p}_{13}^T V_1 V_2 \underline{p}_{13}.$$

$$E(\underline{\mu}_{13}^T\underline{\delta}_1)^4 = n_1^{-2}\sum_{ijk\ell}\sum\sum\sum (p_{1i}-p_{3i})(p_{1j}-p_{3j})(p_{1k}-p_{3k})(p_{1\ell}-p_{3\ell})$$

$$\times E(z_{1i}z_{1j}z_{1k}z_{1\ell})$$

$$= n_1^{-2}\sum_i\sum_j\sum_k\sum_\ell (p_{1i}-p_{3i})(p_{1j}-p_{3j})(p_{1k}-p_{3k})(p_{1\ell}-p_{3\ell})$$

$$\times[V_{1ij}V_{1k\ell}+V_{1ik}V_{1j\ell}+V_{1jk}V_{1i\ell}] .$$

Thus

$$E(\underline{\mu}_{13}^T\underline{\delta}_1)^4 - [E(\underline{\mu}_{13}^T\underline{\delta}_1)^2]^2$$

$$= n_1^{-2}\sum_{ijk\ell}\sum\sum\sum (p_{1i}-p_{3i})(p_{1j}-p_{3j})(p_{1k}-p_{3k})(p_{1\ell}-p_{3\ell})$$

$$\times[V_{1ij}V_{1k\ell}+V_{1ik}V_{1j\ell}+V_{1jk}V_{1i\ell}-V_{1ij}V_{1k\ell}]$$

$$=n_1^{-2}\sum_{ijkl}\Sigma\Sigma\Sigma (p_{1i}-p_{3i})(p_{1j}-p_{3j})(p_{1k}-p_{3k})(p_{1l}-p_{3l})$$

$$\times[V_{1ik}V_{1jl}+V_{1jk}V_{1il}]$$

$$=2n_1^{-2}(\underline{p}_{13}^T V_1\underline{p}_{13})^2=2n_1^{-2}D_1^2$$

where $\underline{p}_{13}^T V_i\underline{p}_{13}=D_i$ for i=1,2,3.

Finally

$$E(\underline{\mu}_{13}^T\underline{\delta}_1)^2E(\underline{\mu}_{13}^T\underline{\delta}_2)^2$$

$$=n_1^{-1}n_2^{-1}(\underline{p}_{13}^T V_1\underline{p}_{13})(\underline{p}_{13}^T V_2\underline{p}_{13})$$

$$=n_1^{-1}n_2^{-1}D_1D_2.$$

5.6 Expressions for mean and variance of $\hat{t}$ and $Q(\hat{t})$ (under the null hypothesis) in terms of the covariance matrices V_i

Using the results derived in section 5.5 equations (10) and (13) of section 5.4 for the mean and variance of $Q(\hat{t})$, respectively, can be written as

$$EQ(\hat{t})=(1-t)^2n_1^{-1}trV_1+n_2^{-1}trV_2+t^2\,trV_3$$

$$-\frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}}[(1-t)^2n_1^{-1}\underline{p}_{13}^TV_1\underline{p}_{13}+n_2^{-1}\underline{p}_{13}^TV_2\underline{p}_{13}+t^2n_3^2\underline{p}_{13}^TV_3\underline{p}_{13}]$$

i.e.

$$EQ(\hat{t})=\sum_{i=1}^{3}n_i^{-1}\alpha_i[trV_i-\frac{D_i}{\underline{\mu}_{13}^T\underline{\mu}_{13}}]$$

where $D_i=\underline{p}_{13}^T V_i\underline{p}_{13}$ for i=1,2,3

$\alpha_1=(1-t)^2$, $\alpha_2=1$, $\alpha_3=t^2$, and

$\operatorname{var} Q(\hat{t})$

$$=2\left[\begin{array}{l}(1-t)^4 n_1^{-2}\operatorname{tr}V_1^2+n_2^{-2}\operatorname{tr}V_2^2+t^4 n_3^{-2}\operatorname{tr}V_3^2\\ +2(1-t)^2 n_1^{-1}n_2^{-1}\operatorname{tr}V_1V_2+2t^2n_2^{-1}n_3^{-1}\operatorname{tr}V_2V_3\\ +2t^2(1-t)^2 n_1^{-1}n_3^{-1}\operatorname{tr}V_1V_3\end{array}\right]$$

$$\frac{-4}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\left[\begin{array}{l}(1-t)^4 n_1^{-2}\underline{p}_{13}^T V_1^2\underline{p}_{13}+n_2^{-2}\underline{p}_{13}^T V_2^2\underline{p}_{13}\\ +t^4 n_3^{-2}\underline{p}_{13}^T V_3^2\,\underline{p}_{13}\\ +2(1-t)^2 n_1^{-1}n_2^{-1}\underline{p}_{13}^T\,V_1\,V_2\,\underline{p}_{13}\\ +2t^2(1-t)^2 n_1^{-1}n_3^{-1}\underline{p}_{13}^T V_1V_3\underline{p}_{13}\\ +2t^2 n_2^{-1}n_3^{-1}\underline{p}_{13}^T V_2V_3\underline{p}_{13}\end{array}\right]$$

$$\frac{+2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}\left[\begin{array}{l}(1-t)^4 n_1^{-2}D_1^2+n_2^{-2}D_2^2\;+t^4n_3^{-2}D_3^2\\ +2(1-t)^2 n_1^{-1}n_2^{-1}D_1D_2\\ +2t^2n_2^{-1}n_3^{-1}D_2D_3\\ +2t^2(1-t)^2D_1D_2\end{array}\right],$$

i.e.,

$\operatorname{var} Q(\hat{t})$

$$=2\sum_i\sum_j n_i^{-1}n_j^{-1}\alpha_i\alpha_j\left[\operatorname{tr}V_iV_j-\frac{2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\underline{p}_{13}^T V_iV_j\underline{p}_{13}+\frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}D_iD_j\right].$$

Similarly, equations (14) and (15) of section 5.4 for the mean and variance of $\hat{t}$, respectively, can be written as

$$\hat{Et}=t+\frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}}[(1-t)n_1^{-1}trV_1-tn_3^{-1}trV_3]$$

$$\frac{-2}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}[(1-t)n_1^{-1}\underline{p}_{13}^TV_1\underline{p}_{13}-tn_3^{-1}\underline{p}_{13}^TV_3\underline{p}_{13}]$$

$$=t+\frac{1}{\underline{\mu}_{13}^T\underline{\mu}_{13}}[\sum_i n_i^{-1}\beta_i[trV_i\frac{-2}{\underline{\mu}_{13}^T\underline{\mu}_{13}}\underline{p}_{13}^TV_i\underline{p}_{13}]] \qquad (14)$$

where $\beta_1=1-t$, $\beta_2=0$ and $\beta_3=-t$.

Also

$$\text{var }\hat{t}=\frac{1}{(\underline{\mu}_{13}^T\underline{\mu}_{13})^2}[\sum_i n_i^{-1}\beta_i'\underline{p}_{13}^TV_i\underline{p}_{13}]$$

where $\beta_1'=(1-t)^2$, $\beta_2'=1$ and $\beta_3'=t^2$. (15)

5.7 A simulation study

Given three multinomial data sets our problem is to determine whether, or not, one of the three underlying multinomial densities is a proper mixture of the other two. In the simulation study, the cell probabilities of the three multinomial densities are approximated in the density estimates by the relative frequencies of the cells. $E\hat{t}$ was evaluated using (14) of section 5.6. Since we are working with the density estimates, therefore, t and the p_{ij}'s (i=1,2,3 and j=1,2,3,...m) which comprise the right hand side of (14) were replaced by $\hat{t}$ and the relative frequencies of the cells.

For reasons mentioned later, it is important that $E\hat{t}$ lies between zero and one. This can always be managed by appropriately renaming the densities. As obtained in section 5.4, under the null hypothesis, the minimum order terms in $EQ(\hat{t})$ are of order two in δ and the minimum order terms in $\text{var}Q(\hat{t})$ are of order four in δ. When the null hypothesis is not true then the minimum order terms in $EQ(\hat{t})$ are of order zero in δ and the minimum order terms in var $Q(\hat{t})$ are of order two in δ.

When the null hypothesis is not true then, since the leading term in $EQ(\hat{t})$ is of order zero in δ , therefore, the term of order two in δ is negligible, comparatively. Also, the leading term in $\mathrm{var}Q(\hat{t})$ being of order two in δ, the term of order four in δ is negligible, comparatively. In the computer programme for the simulation, the expression for $EQ(\hat{t})$ is taken as the sum of the zero order term in δ (not under the null hypothesis) and the second order term in δ (under the null hypothesis). Similarly, the expression for $\mathrm{var}Q(\hat{t})$ is the sum of the second order term in δ (not under the null hypothesis) and the fourth order term in δ (under the null hypothesis). After suitably renaming the densities so that $E\hat{t}$ lies between zero and one, let the null relation be

$$\underline{p}_1 - \underline{p}_2 = t(\underline{p}_1 - \underline{p}_3)$$

i.e. $\underline{p}_2 = (1-t)\underline{p}_1 + t\underline{p}_3$.

where the mixing weight t is replaced by $E\hat{t}$. After the first simulation, for the purposes of all the other simulations the densities considered were $\underline{p}_1, \underline{p}_3$ and $(1-E\hat{t})\underline{p}_1 + (E\hat{t})\underline{p}_3$. Each simulation yielded a value of $EQ(\hat{t})$ and $\mathrm{var}Q(\hat{t})$.

For each case of three densities one thousand simulations were performed. Three null and four non-null cases were considered. Each run of the programme yielded one value each of $Q(\hat{t})$, $EQ(\hat{t})$ and $\mathrm{var}\,Q(\hat{t})$. Two densities, i.e., the gamma and the lognormal fitted the null situation very well. An algorithm to compute the incomplete gamma integral was used in the computer programme, see Moore (1982). For each run the mean and variance of $Q(\hat{t})$ obtained were equated to the mean and variance, respectively, of the gamma distribution . The null-hypothesis was rejected at 90% if $Q(\hat{t})$ did not lie below the 90% quantile of that gamma distribution.

Similarly for rejections at the 95% and 99% levels. Similar procedure was followed for the lognormal distribution where the mean μ and variance σ^2 (of the corresponding normal distribution) are related to $E\hat{Q}(t)$ and var $\hat{Q}(t)$ as

$$\mu = [\log(E\hat{Q}(t))] - \tfrac{1}{2}\log\left[\frac{\mathrm{var}\hat{Q}(t)}{[E\hat{Q}(t)]^2} + 1\right]$$

and $$\sigma^2 = 2[\log[E\hat{Q}(t)] - \mu]$$

and the new random variable is $\dfrac{\log[\hat{Q}(t)] - \mu}{\sigma}$

whose values for the 90%, 95% and 99% quantiles are 1.2815, 1.6448 and 2.3263 respectively.

The following sets of densities were used.

SET 1 (Null case)

$\underline{p}_1^T = (0.1, 0.1, 0.1, 0.1, 0.1, 0.5)$

$\underline{p}_3^T = (0.2, 0.2, 0.2, 0.2, 0.1, 0.1)$

$\underline{p}_2^T = (1-t)\underline{p}_1^T + t\underline{p}_3^T$ where $0<t<1$.

SET 2 (Null case)

$\underline{p}_1^T = (0.05, 0.15, 0.30, 0.30, 0.15, 0.05)$

$\underline{p}_3^T = (0.10, 0.35, 0.25, 0.15, 0.10, 0.05)$

$\underline{p}_2^T = (1-t)\underline{p}_1^T + t\underline{p}_3^T$ where $0<t<1$.

SET 3 (Null case)

$\underline{p}_1^T = (0.5, 0.35, 0.10, 0.05, 0.00, 0.00)$

$\underline{p}_3^T = (0.25, 0.20, 0.20, 0.15, 0.10, 0.10)$

$\underline{p}_2^T = (1-t)\underline{p}_1^T + t\underline{p}_3^T$ where $0<t<1$.

<u>SET 4</u> (Non-null case)

$\underline{p}_1^T$= (0.50,0.35,0.10,0.05,0.00,0.00)

$\underline{p}_3^T$= (0.25,0.20,0.20,0.15,0.10,0.10)

$\underline{p}_2^T$= (0.375,0.275,0.15,0.10,0.10,0.00).

Here $\underline{p}_2^T = 0.5\underline{p}_1^T + 0.5\underline{p}_3^T$ for the first four cells only.

<u>SET 5</u> (Non-null case)

$\underline{p}_1$ and $\underline{p}_3$ same as in set 4,

$\underline{p}_2^T$= (0.375,0.275,0.15,0.20,0.00,0.00).

Here $\underline{p}_2^T = 0.5\underline{p}_1^T + 0.5\underline{p}_3^T$ for the first three cells only.

<u>SET 6</u> $\underline{p}_1$ and $\underline{p}_3$ same as in set 4,

$\underline{p}_2^T$= (0.375,0.200,0.225,0.20,0.00,0.00).

Here $\underline{p}_2^T = 0.5\underline{p}_1^T + 0.5\underline{p}_3^T$ for the first cell only.

<u>SET 7</u> $\underline{p}_1$ and $\underline{p}_3$ same as in set 4,

$\underline{p}_2^T$= (0.00,0.65,0.15,0.10,0.10,0.00).

Here $\underline{p}_2^T = 0.5\underline{p}_1^T + 0.5\underline{p}_3^T$ for third and fourth cells only.

In tables 1 to 4 , n denotes the common sample size of the density estimates. For sets 1 to 3, i.e. the null case , the cases tried are for t=0.25,0.50 and 0.75.

TABLE 1

n=800

Total number of simulations=1000

Set No.	t	Gamma Distribution No. of rejections of null hypothesis Level= 10%	5%	1%	Lognormal Distribution No. of rejections of null hypothesis Level= 10%	5%	1%
1	0.25	93	44	10	105	45	5
	0.50	104	52	13	124	57	8
	0.75	89	43	10	108	43	9
2	0.25	99	45	9	122	49	8
	0.50	95	57	17	112	60	11
	0.75	78	37	12	96	41	11
3	0.25	92	43	13	123	53	9
	0.50	78	51	17	99	58	12
	0.75	90	49	16	108	55	9
4	–	998	990	888	1000	992	829
5	–	1000	1000	1000	1000	1000	1000
6	–	1000	1000	1000	1000	1000	1000
7	–	1000	1000	1000	1000	1000	1000

TABLE 2

n=400

Total number of simulations=1000

Set No.	t	Gamma Distribution			Lognormal Distribution		
		No. of rejections of null hypothesis			No. of rejections of null hypothesis		
		Level=10%	5%	1%	Level=10%	5%	1%
1	0.25	97	45	14	116	50	8
	0.50	99	54	10	115	58	6
	0.75	108	54	10	124	60	9
2	0.25	103	52	17	111	58	11
	0.50	74	40	8	95	41	6
	0.75	90	51	16	106	56	12
3	0.25	83	41	11	110	54	9
	0.50	111	59	24	136	67	20
	0.75	100	52	14	114	56	11
4	-	845	670	268	887	717	203
5	-	1000	999	980	1000	999	970
6	-	1000	1000	1000	1000	1000	999
7	-	1000	1000	1000	1000	1000	1000

<u>TABLE 3</u>

n=200

Total number of simulations=1000

Set No.	t	Gamma Distribution			Lognormal Distribution		
		No. of rejections of null hypothesis			No. of rejections of null hypothesis		
		Level= 10%	5%	1%	Level=10%	5%	1%
1	0.25	95	40	8	107	47	7
	0.50	111	54	18	120	60	12
	0.75	81	41	6	95	44	6
2	0.25	89	45	14	105	47	9
	0.50	95	48	16	113	55	13
	0.75	75	29	4	92	35	2
3	0.25	105	60	16	120	71	12
	0.50	77	40	11	95	47	7
	0.75	88	47	16	112	53	12
4	-	420	245	50	493	275	39
5	-	935	824	514	951	848	448
6	-	998	988	871	999	989	829
7	-	1000	1000	1000	1000	1000	1000

TABLE 4

n=100

Total number of simulations=1000

Set No.	t	Gamma Distribution			Lognormal Distribution		
		No. of rejections of null hypothesis			No. of rejections of null hypothesis		
		Level=10%	5%	1%	Level=10%	5%	1%
1	0.25	90	48	5	108	56	1
	0.50	87	44	14	105	48	8
	0.75	108	63	12	127	66	4
2	0.25	77	39	9	93	46	7
	0.50	84	45	14	102	47	8
	0.75	92	44	9	110	50	5
3	0.25	85	39	10	106	52	7
	0.50	78	44	10	104	47	8
	0.75	89	51	5	109	56	3
4	-	224	132	27	267	148	17
5	-	559	410	166	614	439	125
6	-	839	682	387	871	718	322
7	-	1000	1000	999	1000	1000	999

5.8 The likelihood ratio test

For the multinomial densities $\underline{p}_1$, $\underline{p}_2$ and $\underline{p}_3$, the general loglikelihood function $\mathcal{L}(\underline{p}_1,\underline{p}_2,\underline{p}_3)$ is given by

$$\mathcal{L}(\underline{p}_1,(1-t)\underline{p}_1+t\underline{p}_3,\underline{p}_3)=\sum_{j=1}^{m} f_{1j}\log p_{1j}+\sum_{j=1}^{m} f_{3j}\log p_{3j}$$

$$+\sum_{j=1}^{m} f_{2j}\log\left[(1-t)p_{1j}+tp_{3j}\right]$$

where p_{1j} is the probability of the jth cell in $\underline{p}_1$

f_{1j} is the frequency of the jth cell in $\underline{p}_1$.

and $p_{2j}=(1-t)p_{1j}+tp_{3j}$.

Define $$w_j=\frac{(1-t)p_{1j}}{(1-t)p_{1j}+tp_{3j}}.$$

Maximum likelihood estimates of $\underline{p}_1$, $\underline{p}_3$ and t can be found using the EM algorithm which generates a sequence of estimates of p_{1j}, p_{2j} and t starting with the initial values,

$$\underline{p}_1^{(0)}=\underline{r}_1,\ \underline{p}_3^{(0)}=\underline{r}_3 \text{ and } \hat{t}^{(0)}= \text{value used already.}$$

The iterative formulae are

$$p_{1j}^{(r+1)}=\frac{f_{1j}+f_{2j}w_j^{(r)}}{n_1+\sum_j f_{2j}w_j^{(r)}}$$

$$p_{3j}^{(r+1)}=\frac{f_{3j}+f_{2j}(1-w_j^{(r)})}{n_3+(n_2-\sum_j f_{2j}w_j^{(r)})}$$

$$\text{and } \hat{t}^{(r+1)}=1-\frac{\sum_j f_{2j}w_j^{(r)}}{n_2} \qquad (16)$$

$$\text{where } w_j^{(r)} = \frac{(1-\hat{t}^{(r)})\, p_{1j}^{(r)}}{(1-\hat{t}^{(r)})p_{1j}^{(r)} + \hat{t}^{(r)} p_{3j}^{(r)}} \; . \qquad (17)$$

In the above, n_1, n_2 and n_3 are the sizes of samples forming the estimates of $\underline{p}_1, \underline{p}_2$ and $\underline{p}_3$ respectively.

We thus evaluate $L^{(r)} = \mathcal{L}(\underline{p}_1^{(r)}, (1-t^{(r)})\underline{p}_1^{(r)} + t^{(r)}\underline{p}_3^{(r)}, \underline{p}_3^{(r)})$.

Our stopping rule will be to stop when

$$(L^{(r+1)} - L^{(r)}) < 0.0001 \text{ (say)}.$$

The likelihood ratio test is based on

$$2 \log\Lambda = 2[\mathcal{L}(\underline{r}_1, \underline{r}_2, \underline{r}_3) - \mathcal{L}(\underline{p}_1^*, (1-t^*)\underline{p}_1^* + t^*\underline{p}_3^*, \underline{p}_3^*)]$$

where $\underline{p}_1^*$, $\underline{p}_3^*$ and t^* are the final estimates from the EM algorithm.

The approximate null distribution of 2 $\log\Lambda$ is obviously $\chi^2(4)$ in our examples, in which the number of multinomial cells is 6. In the general k-cell case, the number of degrees of freedom would be

$$3(k-1)-[2(k-1)+1] = k-2.$$

Similarly, as in section 5.7, for each of the seven sets of densities mentioned there , one thousand simulations were performed. Using the EM algorithm, as explained, the maximum likelihood estimates of $\underline{p}_1, \underline{p}_3$ and t, i.e. $\underline{p}_1^*, \underline{p}_3^*$ and t^*, respectively were obtained .Thus using the value of 2 $\log\Lambda$, given by equation (17), and the fact that the approximate null distribution of 2 $\log\Lambda$ is, obviously , $\chi^2(4)$, as mentioned earlier, we will reject the null hypothesis at 90% if the value of 2 $\log\Lambda$ did not lie below the 90% quantile of the chi-square distribution with four degrees of freedom. Similarly, we will reject the null hypothesis at 95% and 99% levels. In tables 5 and 6, n is the common size of the samples underlying the density estimates.

TABLE 5

Total number of simulations=1000

Log likelihood method

Set No.	t	Case 1 n=800 No. of rejections of null hypothesis			Case 2 n=400 No. of rejections of null hypothesis		
		Level=10%	5%	1%	Level=10%	5%	1%
1	0.25	95	47	9	98	51	18
	0.50	106	52	14	94	51	10
	0.75	84	39	10	113	54	12
2	0.25	89	47	9	100	47	14
	0.50	103	47	10	83	36	7
	0.75	83	40	7	94	52	15
3	0.25	112	56	14	92	48	11
	0.50	77	36	7	113	67	15
	0.75	99	54	9	99	56	13
4	-	1000	1000	1000	1000	1000	1000
5	-	1000	1000	1000	1000	1000	1000
6	-	1000	1000	1000	1000	1000	1000
7	-	1000	1000	1000	1000	1000	1000

TABLE 6

Total number of simulations=1000

Loglikelihood Method

Set No.	t	Case 3 n=200 No. of rejections of null hypothesis Level=10%	5%	1%	Case 4 n=100 No. of rejections of null hypothesis Level=10%	5%	1%
1	0.25	91	45	12	95	52	4
	0.50	105	57	17	89	48	13
	0.75	80	34	8	110	63	9
2	0.25	82	46	18	89	36	9
	0.50	84	40	9	73	37	10
	0.75	79	39	8	106	50	10
3	0.25	.111	65	5	116	59	10
	0.50	85	42	9	102	50	3
	0.75	99	57	12	101	51	13
4	-	1000	1000	981	904	788	468
5	-	997	995	960	891	814	604
6	-	1000	1000	1000	985	965	876
7	-	1000	1000	1000	1000	1000	1000

It was mentioned in the beginning of section 5.7, that it is important that $E\hat{t}$ lies between 0 and 1. It is obvious that this can always be managed by appropriately renaming the densities. When fitting the gamma and lognormal distributions, in all cases, more correct rejection rates of the null hypothesis were obtained for $E\hat{t}$ lying between 0 and 1. On the other hand, when applying the likelihood ratio test, if we did not start the EM algorithm with $\hat{t}$ lying between 0 and 1 then the final estimate, t^*, always turned out to be either 0 or 1, thus introducing a serious error.

Consider equations (16) and (17). If $\hat{t}^{(0)}$ lies between 0 and 1, then, obviously, $w_j^{(0)}$'s, for all j, lie between 0 and 1 and so $\hat{t}^{(1)}$ lies between 0 and 1. The same is true for all the subsequent estimates of t generated by the EM algorithm.

From (17) it is clear that if an estimate of t gets close to 1 then the corresponding w_j's get close to 0. Consider (16). It is clear that the estimate of t starts converging to 1. Similarly if an estimate of t gets close to 0 then the w_j's get close to 1 and so the estimate of t starts converging to 0. It was noted in a number of examples that, when the actual value of t was greater than one, the value of $\hat{t}$ started converging to one. Also, when the actual t was negative, $\hat{t}$ started converging to 0.

5.9 Application to some fish data.

In this chapter, for the case of three multinomial densities, three methods of testing the null hypothesis were suggested. The methods were also applied to several sets of simulated data. In this section we apply those methods to some sets of actual data. The data were obtained courtesy of Dr. C.W. Anderson of Sheffield University. They consist of the measurement of the number of plates for fish (stickleback) from various sites not too far from Sheffield (though there are a couple of Scottish outliers). The number of plates on a fish is a skeletal characteristic.

The two subpopulations in each case are fish with and without a carina (rudder). We are interested in determining whether, or not, the data obtained from various sites are proper mixtures of the same two components. The two components are the distributions of fish with carina and fish without carina.

Consider Theorem 2 in section 2.2 of Chapter 2. If the necessary condition of Theorem 2 does not hold, then, clearly, the three densities are not proper mixtures of the same two components. In three different sets of data tested in this chapter it turned out that the necessary condition of Theorem 2 did not hold.

In two of the three methods mentioned, the necessary condition is tested through the distribution of the function $Q(\hat{t})$. In the third method, the relation in $\underline{p}_1$, $\underline{p}_2$ and $\underline{p}_3$, i.e. $\underline{p}_2=(1-t)\underline{p}_1+\underline{p}_3$, is used in the maximum likelihood method.

In all of the three sets of fish data it was not necessary to mention the two subpopulations in each frequency vector. The reason is, as mentioned earlier in this section, that, if the necessary condition of Theorem 2, of Chapter 2, which does not involve the subpopulations, is demonstrated to be not true then we conclude right there that the three multinomials are not mixtures of the same two components.

Also, in the data, the number of plates on a fish varied from 1 to 30. But since the data were very sparse for number of plates nearer 1 and nearer 30, therefore, a few plates near 1 and, also, a few plates near 30 were pooled together to make the results meaningful. Also as mentioned in section 5.8, the densities were renamed, if necessary, to keep the estimate of t between 0 and 1.

In Table 7, the three densities of set 1 are defined together with their sources and sample sizes. In the raw data, the data points were classified according to the number of lateral plates. Thus the number of the cell and number of lateral plates were synonymous. In Table 7 we have pooled together the first four cells and the last seven cells. This reduces the number of cells to 21, and the number of the cell and the number of lateral plates do not remain synonymous. Dividing the frequency of each cell by the sample size of the data set gives us the density estimates of set 1. Similarly, Tables 8 and 9 describe the density estimates of sets 2 and 3 respectively.

In Table 8, the first four cells and the last nine cells have been pooled. In Table 9, the first three cells and the last thirteen cells have been pooled.

For each of the three sets of data, all of the three methods of this chapter rejected the null hypothesis.

The null hypothesis was rejected at 90%, 95% or 99% level if, for the method of the lognormal distribution, the log of the value of the standardised normal random variable was greater than 1.28,1.64 or 2.32 respectively. The variable was represented by the symbol LN in Table 10.

In the method of the gamma distribution the algorithm (mentioned in section 5.7) used in the programme yielded the quantile position of the gamma random variable. The variable was represented by the symbol GD in Table 10.In the likelihood ratio test method the distribution was chi-square. The degrees of freedom of the distribution being given by k-2 where k is the number of cells in the multinomial density estimates.

Density 1 , Source (Conisbrough ,year 1982) , Sample size 219

Cell No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Frequency	27	59	35	10	6	2	4	10	6	12	8	6	5	3	2	1	5	6	6	3	3

Density 2 , Source (Rawcliffe bridge , year 1981), Sample size 215

Cell No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Frequency	18	40	13	9	3	10	13	17	12	13	12	7	6	7	4	5	3	1	4	7	11

Density 3 , Source (Swinton , year 1982) , Sample size 250

Cell No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Frequency	3	17	47	40	7	3	4	13	12	14	13	8	11	5	5	4	7	5	9	14	9

Table 7

Density 1 , Source (York (Foss), year 1980), Sample size 236

Cell No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Frequency	34	88	66	13	4	1	4	4	6	3	2	3	1	3	1	0	0	1	2

Density 2, Source (Appleby-Lincs, year 1981) , Sample size 188

Cell No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Frequency	0	12	21	16	12	8	7	9	7	5	3	7	11	8	4	8	4	5	41

Density 3 , Source (Thorne , year 1980) , Sample size 165

Cell No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Frequency	7	31	16	2	2	4	2	5	8	8	8	6	6	10	3	6	1	5	35

Table 8

Density 1, Source (York (Foss), year 1980), sample size 236

Cell No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Frequency	4	30	88	66	13	4	1	4	4	6	3	2	3	1	3	4

Density 2, Source (Stainforth, year 1980), sample size 300

Cell No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Frequency	2	15	55	39	9	8	8	10	15	12	12	10	14	13	6	72

Density 3, Source (Bramwith bridge, year 1980), sample size 345

Cell No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Frequency	0	19	46	48	16	9	11	16	25	20	20	19	15	7	14	60

TABLE 9

TABLE 10

	Set 1	Set 2	Set 3
$\hat{Et}$	0.3013	0.8535	0.5819
LN	3.3953	4.3099	7.5171
GD	0.9999	0.9999	1.0000
Chi square	49.3512	63.8296	139.1566

CHAPTER 6

A BOOTSTRAP VERSION OF THE TEST

6.1 Introduction

In this chapter we illustrate the use of a Monte Carlo test based on the test statistic $Q(\hat{t})$ introduced in chapter four.

The distance function $Q(\hat{t})$ is constructed from the normal kernel estimates $\hat{f}_1$, $\hat{f}_2$ and $\hat{f}_3$ of the underlying densities f_1, f_2 and f_3 respectively.

Using the 'smoothed' bootstrap technique explained in the next section we construct another data set from $\hat{f}_1$, $\hat{f}_3$ and $(1-\hat{t})\hat{f}_1 + \hat{t}\hat{f}_3$, respectivly, where, for reasons to be mentioned later, the functions have been named such that $\hat{t}$ lies between zero and one.

The new three data sets give another set of estimates for the densities and hence lead to another value of $Q(\hat{t})$.

This procedure is repeated nine times in the first instance.Thus,we obtain, in all,ten different values for $Q(\hat{t})$.If the first value is greater than the other nine then we reject the hypothesis that one of the densities is a proper mixture of the other two, the level of rejection being ten percent. By repeating this procedure another ten times, we obtain, in all, twenty different values for $Q(\hat{t})$. If the first value of $Q(\hat{t})$ is greater than the remaining nineteen then we reject the hypothesis, the level of rejection being five percent.

6.2 The smoothed bootstrap technique

Let there be some interesting property of a distribution function that depends on it in some complicated way. Now, even if the distribution function is unknown,the property can most easily be estimated by repeatedly simulating samples from it.

In many statistical problems the distribution function itself is unknown but a sample of observations from the distribution is available. The bootstrap technique, introduced by Efron(1982), consists of simulating samples from the empirical distribution function of the observed data. A sample from the empirical distribution function is generated by successively selecting uniformly with replacement from the sample of observations.

In bootstrap simulation, all members of the samples will be drawn from the original sample and nearly every sample will contain repeated values.

The smoothed bootstrap technique is a variation which will not lead to samples with repeated values. Here the simulations are constructed not from the empirical distribution function but by using an algorithm (as explained later in this section) to simulate from a smoothed version of the empirical distribution function.

This technique obtains a new data set from the original one used to form the kernel estimates of the density. We simulate a value z of the random variable from the standard normal distribution. The interval between zero and one is divided into n equal parts where n is the size of the sample underlying the density. Generate a number from the uniform distribution. If the number lies in the rth of the n sub-intervals, then, the first new data point will be defined as

$$y=x_r+hz \qquad r\epsilon(1,2,3.....n). \qquad (1)$$

In equation (1), x_r is the rth point (in the order of generation) in the original data set. Continuing this procedure n times we obtain a new set of data points.

The new data set is constructed from $(1-\hat{t})\hat{f}_1+\hat{t}\hat{\hat{f}}_3$ as follows. A number is generated from the uniform distribution. If the generated number u is greater than $1-\hat{t}$ then the previously mentioned procedure is repeated with $\hat{f}_3$ or otherwise with $\hat{f}_1$. Note that it is necessary that

$\hat{t}$ lies between zero and one because, otherwise, all of the new data set will be generated from $\hat{f}_1$ alone or $\hat{f}_3$ depending on whether $\hat{t}$ is negative or greater than one.

6.3 A simulation study

The procedure mentioned in 6.1 was repeated for three null as well as two non-null cases. It is clear from Table 1 that the rejection rates(%ages) were approximately correct for the null cases. For distinctly non-null cases the number of rejections was extremely high.

In Table 1 the f's are all mixtures of normals or are pure normals. Data sets of sizes 100 and 400 (n in the table) were used. In the construction of $Q(\hat{t})$ the limit points (i.e., the extreme left and the extreme right of the chosen points on the x-axis) were taken to be -1.5 and 6.5. Finally m, the number of points used to construct $Q(\hat{t})$,was taken to be 9. The reason for choosing this number was that there was not enough workspace on the computer for a bigger number.

6.4 A drawback of the test

In its use of density estimates, the Monte Carlo test introduced in this chapter is "non-parametric".A parametric version was used by Aitkin, Anderson and Hinde (1981) to try to assess the structure of latent class models, using the standard likelihood ratio statistic. In neither their case nor ours can the rejection rates under the null model be regarded as exactly known, because the null hypothesis is not simple. In contrast to that case (see Hope 1968), data-based estimation is required of some features of the null model and this destroys the exactness of the test.

f_1	f_2	f_3	n	Rejection Levels 10%	5%
0.1N(1,1)+0.9N(4,1)	0.3N(1,1)+0.7N(4,1)	0.7N(1,1)+0.3N(4,1)	100	8.70	4.5
,,	,,	,,	400	9.58	4.47
0.1N(1,1)+0.9N(2,1)	0.3N(1,1)+0.7N(2,1)	0.7N(1,1)+0.3N(2,1)	100	7.80	3.70
,,	,,	,,	400	9.78	5.83
0.2N(1,1)+0.8N(3,1)	+0.4N(1,1)+0.6N(3,1)	+0.6N(1,1)+0.4N(3,1)	100	7.50	3.20
,,	,,	,,	400	8.98	4.38
0.2N(1,1)+0.8N(3,1)	0.4N(1,1)+0.6N(3,1)	0.6N(1,1)+0.4N(2,1)	100	12.30	6.50
,,	,,	,,	400	19.74	12.59
0.1N(1,1)+0.9N(4,1)	0.3N(1,1)+0.7N(4,1)	N(2.5,1)	100	61.19	48.95
,,	,,	,,	400	98.21	96.43

Table 1

CHAPTER 7

A NECESSARY CONDITION FOR THREE BINARY MIXTURES TO HAVE THE SAME TWO COMPONENTS

7.1 Introduction

In section 2.2 of Chapter 2, Theorem 2 states a necessary condition for three densities to be proper mixtures of the same two components. In this chapter we determine another necessary condition for three densities to be proper mixtures of the same two components.

7.2 Infimum and supremum of the ratio of two densities

Let $f_1 = p_1 g + (1-p_1)h$ (1)

and $f_2 = p_2 g + (1-p_2)h,$ (2)

where f_1 and f_2 are the densities, g and h the component densities and $0 < p_1, p_2 < 1, p_1 > p_2$.

Referring to (12) of Chapter 2 we have

$$\frac{1-p_1}{1-p_2} \leq \frac{f_1}{f_2} \leq \frac{p_1}{p_2} . \tag{3}$$

We now show with the help of a counter example that it is not necessary that $\frac{1-p_1}{1-p_2}$ and $\frac{p_1}{p_2}$ are the infimum and supremum of $\frac{f_1}{f_2}$ respectively.

Example

$$f_1(x) = p_1 \frac{1}{\sqrt{2\Pi}\sigma_1} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} + (1-p_1)\frac{1}{\sqrt{2\Pi}\sigma_2} e^{-\frac{1}{2\sigma_2^2}(x-\mu_2)^2} \tag{4}$$

$$f_2(x) = p_2 \frac{1}{\sqrt{2\Pi}\sigma_1} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} + (1-p_2)\frac{1}{\sqrt{2\Pi}\sigma_2} e^{-\frac{1}{2\sigma_2^2}(x-\mu_2)^2} . \tag{5}$$

Then $\frac{d}{dx}(f_1/f_2) = \frac{f_2 f_1' - f_1 f_2'}{f_2^2} = 0$, where

$f_1' f_2 - f_1 f_2' = 0$. Thus the extreme points of the ratio of f_1 and f_2 are given by the equation

$$\frac{1}{2\Pi\sigma_1\sigma_2}\left[\frac{x-\mu_1}{\sigma_1^2} - \frac{x-\mu_2}{\sigma_2^2}\right] e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(x-\mu_2)^2}{2\sigma_2^2}} = 0 . \tag{6}$$

Equation (6) is satisfied when

$$\frac{x-\mu_1}{\sigma_1^2} = \frac{x-\mu_2}{\sigma_2^2} . \tag{7}$$

(7) has no solution for the case when $\sigma_1 = \sigma_2$. For the case $\sigma_1 \neq \sigma_2$ we have

$$x = \frac{\mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_2^2 - \sigma_1^2} .$$

Consider $\frac{d^2}{dx^2}(f_1/f_2) = \frac{f_2^2(f_2 f_1'' - f_1 f_2'') - 2f_2 f_2'(f_2 f_1' - f_1 f_2')}{f_2^4}$

Thus, at the extreme point,

$$\frac{d^2}{dx^2}(f_1/f_2) = \frac{f_2 f_1'' - f_1 f_2''}{f_2^2}$$

$$= \frac{1}{f_2^2}(p_1 - p_2) \times \frac{1}{2\Pi\sigma_1\sigma_2}\left[\frac{(x-\mu_1)^2 - \sigma_1^2}{\sigma_1^4} - \frac{(x-\mu_2)^2 - \sigma_2^2}{\sigma_2^4}\right]$$

$$\times e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2 - \frac{1}{2\sigma_2^2}(x-\mu_2)^2}$$

$$= \frac{(p_2-p_1)\ (\sigma_2^2-\sigma_1^2)}{2\Pi\sigma_1^3\sigma_2^3 f_2^2}\, e^{\frac{-(x-\mu_1)^2}{\sigma_1^2}} \quad . \tag{8}$$

Now, at the point where (7) is true,

$$\frac{f_1}{f_2} = \frac{\sigma_2 p_1+\sigma_1(1-p_1)}{\sigma_2 p_2+\sigma_1(1-p_2)} \quad . \tag{9}$$

We have

$$\frac{\sigma_2 p_1+\sigma_1(1-p_1)}{\sigma_2 p_2+\sigma_1(1-p_2)} = \frac{p_1}{p_2} + \frac{\sigma_1(1-(p_1/p_2))}{\sigma_2 p_2+\sigma_1(1-p_2)} \tag{10}$$

and also

$$\frac{\sigma_2 p_1+\sigma_1(1-p_1)}{\sigma_2 p_2+\sigma_1(1-p_2)} = \frac{1-p_1}{1-p_2} + \frac{\sigma_2 \dfrac{p_1-p_2}{1-p_2}}{\sigma_2 p_2+\sigma_1(1-p_2)} \quad . \tag{11}$$

If $p_1>p_2$ then from (10)

$$\frac{\sigma_2 p_1+\sigma_1(1-p_1)}{\sigma_2 p_2+\sigma_1(1-p_2)} < \frac{p_1}{p_2}$$

and from (11)

$$\frac{\sigma_2 p_1+\sigma_1(1-p_1)}{\sigma_2 p_2+\sigma_1(1-p_2)} \quad \frac{1-p_2}{1-p_2} \quad .$$

Thus

$$\frac{1-p_1}{1-p_2} < \frac{\sigma_2 p_1+\sigma_1(1-p_1)}{\sigma_2 p_2+\sigma_1(1-p_2)} < \frac{p_1}{p_2} \quad . \tag{12}$$

Since $\frac{d^2}{dx^2}(f_1/f_2)$ is non-zero as is obvious from (8) therefore, at

$x = \dfrac{\mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_2^2-\sigma_1^2}$, $\dfrac{f_1}{f_2}$ has either a maximum or a minimum.

Hence, it is clear from (9) and (12), that, in the example, either the infimum or the supremum will not be attained if $x=\frac{\mu_1\sigma_2^2-\mu_2\sigma_1^2}{\sigma_2^2-\sigma_1^2}$ is an extreme point.

Note that, in the absence of an extreme point, i.e. when $\sigma_1=\sigma_2$ but $\mu_1\neq\mu_2$, $\frac{f_1}{f_2}$ is a monotonic function.

This case is illustrated in Figure 1 on page 148 where

$$f_1= 0.45N(1,1)+0.55N(4,1)$$

and $f_2= 0.55N(1,1)+0.45N(4,1)$.

Note that $\frac{f_1}{f_2}$ is monotonic.

In figure 2 on page 149

$$f_1=0.2N(4,1)+0.8N(3,2)$$

and $f_2=0.4N(4,1)+0.6N(3,2)$.

Note that, according to theorem 4 of Chapter 2, $\frac{f_1}{f_2}$ lies between finite (and non-zero) limits. Again, since $\sigma_1\neq\sigma_2$, therefore, $x=\frac{\mu_1\sigma_2^2-\mu_2\sigma_1^2}{\sigma_2^2-\sigma_1^2}=5$ is an extreme point (a minimum, since $\frac{d^2}{dx^2}(f_1/f_2)>0$) of $\frac{f_1}{f_2}$. The value of $\frac{f_1}{f_2}$ at x=5 is evaluated to be 0.8262308. From the definition of $\frac{f_1}{f_2}$, we have,

$$\frac{1}{2}<\frac{f_1}{f_2}\leq\frac{4}{3}.$$

Note that the supremum is attained but the infimum is not.

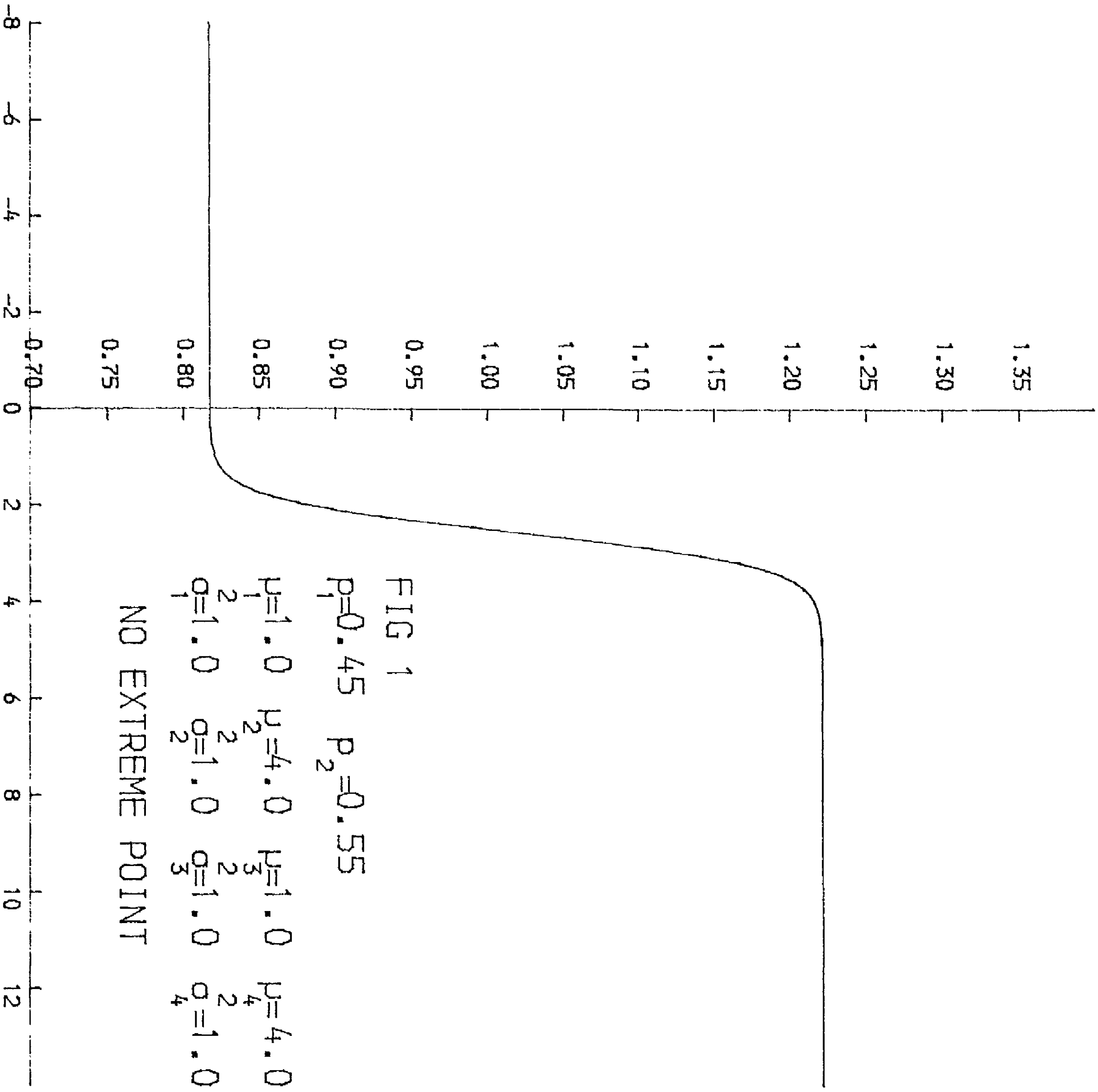


FIG 1
P1=0.45 P2=0.55
μ1=1.0 μ2=4.0 μ3=1.0 μ4=4.0
σ1²=1.0 σ2²=1.0 σ3²=1.0 σ4²=1.0
NO EXTREME POINT
0.70
0.75
0.80
0.85
0.90
0.95
1.00
1.05
1.10
1.15
1.20
1.25
1.30
1.35
-8
-6
-4
-2
0
2
4
6
8
10
12

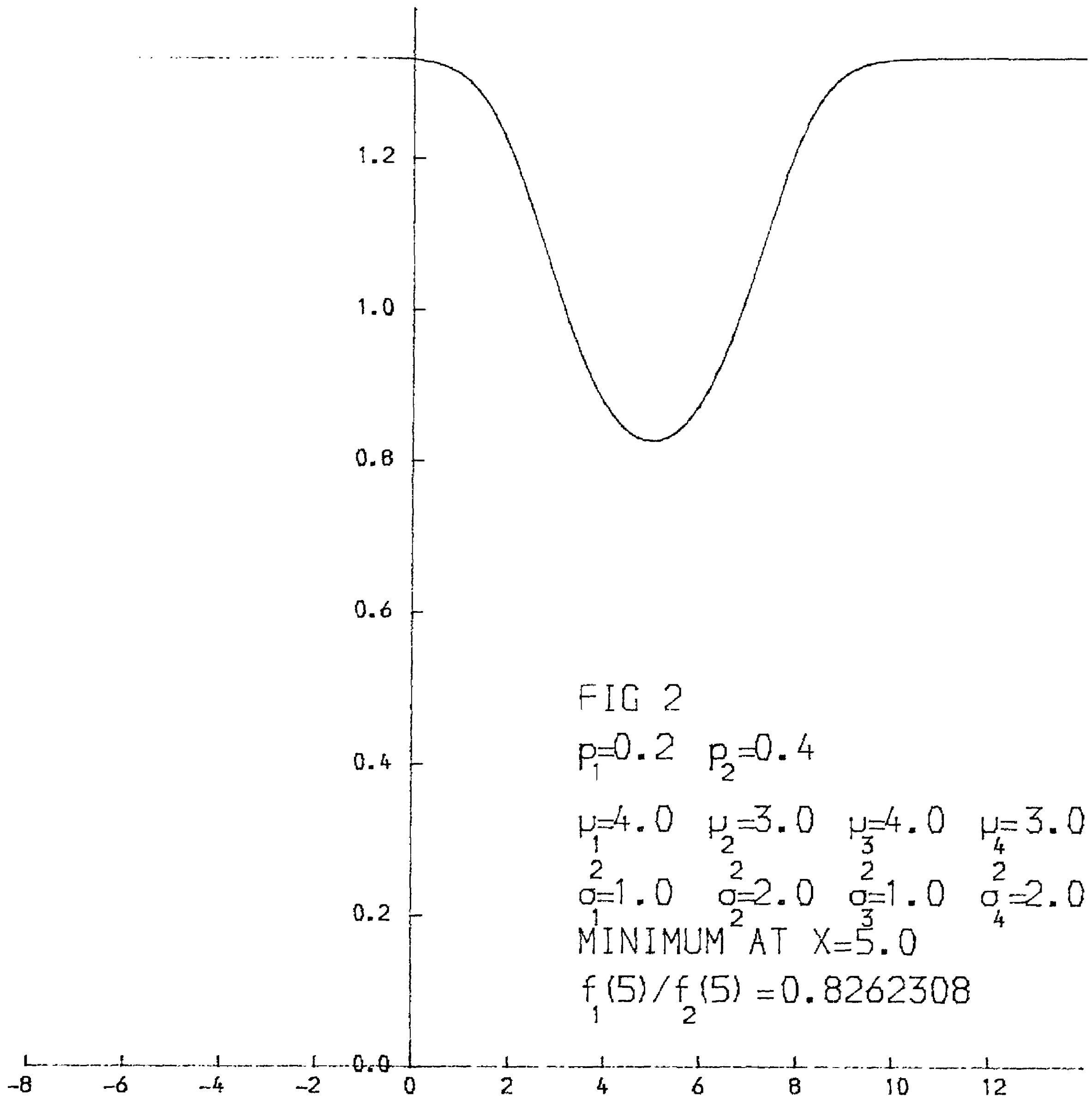


FIG 2
p_1=0.2 p_2=0.4
μ_1=4.0 μ_2=3.0 μ_3=4.0 μ_4=3.0
σ_1^2=1.0 σ_2^2=2.0 σ_3^2=1.0 σ_4^2=2.0
MINIMUM AT X=5.0
$f_1(5)/f_2(5)$ =0.8262308
0.0
0.2
0.4
0.6
0.8
1.0
1.2
-8
-6
-4
-2
0
2
4
6
8
10
12

7.3 Two preliminary theorems

In this section we establish two theorems which are of relevance in section 7.4 The first theorem is a slightly different version of Theorem 1 of Chapter 2.

Theorem 1

If f_1, f_2 and f_3 are mixtures of the same two components with mixing weights p_1, p_2 and p_3 respectively such that $p_1 > p_2 > p_3$, then f_2 is a proper mixture of the other two.

Proof

Let $f_1 = p_1 g + (1-p_1)h$

$f_2 = p_2 g + (1-p_2)h$

and $f_3 = p_3 g + (1-p_3)h$,

where, $0 < p_1, p_2, p_3 < 1$, $p_1 > p_2 > p_3$ and g and h are the components.

Thus $f_1 - f_2 = (p_1-p_2)(g-h)$

and $f_1 - f_3 = (p_1-p_3)(g-h)$.

Thus $f_1 - f_2 = t(f_1-f_3)$, where $t = \dfrac{p_1-p_2}{p_1-p_3}$.

Thus $f_2 = (1-t)f_1 + tf_3$.

Note that with our choice of ordering of p_i's, t lies between 0 and 1.

Thus f_2 is a proper mixture of f_1 and f_3.

Theorem 2

If f_1 and f_2 are proper mixtures of the same two components then the extreme points of $\dfrac{f_1}{f_2}$ are dependent on the components and not on the mixing weights.

Proof

Let $f_1 = p_1 g + (1-p_1)h$

and $f_2 = p_2 g + (1-p_2)h$,

where g and h are the components and p_1, p_2 the mixing weights.

Then

$$\frac{d}{dx}(f_1 / f_2) = 0 \qquad \text{when } f_2 f_1' - f_1 f_2' = 0,$$

$$\text{i.e where} \quad g h' - g' h = 0 \; . \tag{13}$$

Note that (13) also gives the extreme points of g/h.

From Theorem 2 it follows that if f_1, f_2, ... are densities which are mixtures of the same two components then the ratio of any two of the densities has the same extreme points (if any).

7.4 A necessary condition for three densities to be proper mixtures of the same two components.

Let f_1,f_2 and f_3 be proper mixtures of the same two components, i.e,

$$f_1 = p_1 g + (1-p_1)h \tag{14}$$

$$f_2 = p_2 g + (1-p_2)h \tag{15}$$

$$f_3 = p_3 g + (1-p_3)h \tag{16}$$

where g and h are the components and $0 < p_1, p_2, p_3 < 1$. We assume, without loss of generality, that $p_1 > p_2 > p_3$.

Now, according to Theorem 1, f_2 is a proper mixture of f_1 and f_3.

Consider f_1 and f_3. From the example in 7.2 it is clear that it is not necessary that $\frac{1-p_1}{1-p_3}$ and $\frac{p_1}{p_3}$ are the infimum and supremum, respectively, of $\frac{f_1}{f_3}$. Let a and b be the infimum and supremum, respectively, of $\frac{f_1}{f_3}$.

Let $a = \frac{1-\bar{p}_1}{1-\bar{p}_3}$ and $b = \frac{\bar{p}_1}{\bar{p}_3}$.

Since $0 < a < 1$ and $b > 1$, therefore,

$\bar{p}_1 = \frac{b(1-a)}{b-a}$ and $\bar{p}_3 = \frac{1-a}{b-a}$ satisfy $0 < \bar{p}_1, \bar{p}_3 < 1$

and our choice of a and b ensures that $\bar{p}_1 > \bar{p}_3$. Thus (14) and (16) may be written as

$$f_1 = p_1 g + (1-p_1) h = \bar{p}_1 \bar{g} + (1-\bar{p}_1)\bar{h} \qquad (17)$$

$$f_3 = p_3 g + (1-p_3) h = \bar{p}_3 \bar{g} + (1-\bar{p}_3)\bar{h} \qquad (18)$$

where $\bar{g} = \dfrac{(1-\bar{p}_3)f_1 - (1-\bar{p}_1)f_3}{\bar{p}_1 - \bar{p}_3}$

and $\bar{h} = \dfrac{\bar{p}_1 f_3 - \bar{p}_3 f_1}{\bar{p}_1 - \bar{p}_3}$.

Note that $\bar{g}$ and $\bar{h}$ satisfy the requirements of a density because

$$\frac{1-\bar{p}_1}{1-\bar{p}_3} \leq \frac{f_1}{f_3} \leq \frac{\bar{p}_1}{\bar{p}_3} .$$

In the region where $\frac{f_1}{f_3} \cong \inf\left(\frac{f_1}{f_3}\right)$, i.e. a, we have $\bar{g} \cong 0$ as follows from

$$\frac{f_1}{f_3}\left[= \frac{\bar{p}_1\bar{g} + (1-\bar{p}_1)\bar{h}}{\bar{p}_3\bar{g} + (1-\bar{p}_3)\bar{h}}\right] \cong \frac{1-\bar{p}_1}{1-\bar{p}_3} .$$

Similarly, in the region where $\frac{f_1}{f_3} \cong \sup\left(\frac{f_1}{f_3}\right)$, i.e. b,

we have $\bar{h} \cong 0$.

Now, f_2, being a proper mixture of f_1 and f_3, can be written as

$$f_2 = tf_1 + (1-t)f_3 \qquad \text{(where } 0<t<1\text{)}$$

$$= [t\bar{p}_1 + (1-t)\bar{p}_3]\bar{g} + [1-(t\bar{p}_1 + (1-t)\bar{p}_3)]\ \bar{h}$$

$$= \bar{p}_2\bar{g} + (1-\bar{p}_2)\bar{h},$$

where $\bar{p}_2 = t\bar{p}_1 + (1-t)\bar{p}_3$.

Since $0<t,\bar{p}_1,\bar{p}_3<1$ therefore it is obvious that $0<\bar{p}_2<1$. Thus f_2 is also a proper mixture of $\bar{g}$ and $\bar{h}$. It follows from Theorem 1 that $\bar{p}_1>\bar{p}_2>\bar{p}_3$.

In the region where $\bar{g} \simeq 0$ we have $\frac{f_2}{f_3} \simeq \frac{1-\bar{p}_2}{1-\bar{p}_3}$

and in the region where $\bar{h} \simeq 0$ we have

$\frac{f_2}{f_3} \simeq \frac{\bar{p}_2}{\bar{p}_3}$. Hence $\frac{1-\bar{p}_2}{1-\bar{p}_3}$ and $\frac{\bar{p}_2}{\bar{p}_3}$ are the infimum and supremum, respectively, of $\frac{f_2}{f_3}$.

Hence solving the infimum and supremum of $\frac{f_1}{f_3}$ for $\bar{p}_1$ and $\bar{p}_3$, and the infimum and supremum of $\frac{f_2}{f_3}$ for $\bar{p}_2$ and $\bar{p}_3$ gives the same value for the weight $\bar{p}_3$.

Now we state the procedure for determining the necessary condition for three densities to be proper mixtures of the same two components.

Let $a \leq \frac{f_1}{f_2} \leq b$, where a is the infimum and b the supremum of $\frac{f_1}{f_2}$.

Let $t \leq \frac{f_2}{f_3} \leq d$, where t is the infimum and d the supremum of $\frac{f_2}{f_3}$.

Put $a = \frac{1-\bar{p}_1}{1-\bar{p}_2}$ and $b = \frac{\bar{p}_1}{\bar{p}_2}$.

Solve for $\bar{p}_1$ and $\bar{p}_2$ in terms of a and b.

Now if a and t (or equivalently b and d) are attained at the same point then put

$$t = \frac{1-\bar{\bar{p}}_2}{1-\bar{p}_3} \quad \text{and} \quad d = \frac{\bar{\bar{p}}_2}{\bar{p}_3} .$$

If, however, a and d are attained at the same point (or, equivalently, b and t at the same point) then put

$$t = \frac{\bar{\bar{p}}_2}{\bar{p}_3} \quad \text{and} \quad d = \frac{1-\bar{\bar{p}}_2}{1-\bar{p}_3} .$$

Solve for $\bar{\bar{p}}_2$ and $\bar{p}_3$. Thus if $\bar{\bar{p}}_2 \neq \bar{p}_2$ then f_1, f_2 and f_3 are not proper mixtures of the same two components.

7.5 Geometrical view of the extreme points of the ratio of two densities f_1 and f_2 which are mixtures of the same two components for the case of unimodal components.

$$\text{Let } f_1 = p_1 g + (1-p_1)h$$
$$f_2 = p_2 g + (1-p_2)h.$$

From before we have that the ratio of f_1 and f_2 has extreme points where

$$g(x)h'(x)-g'(x)h(x) = 0 \tag{19}$$

Case 1

$g'(x)$ and $h'(x)$ are both zero at some point. It follows that $\frac{f_1}{f_2}$ has an extreme value at the same point.

Case II

$g'(x)$ and $h'(x)$ are not both zero at the same point. Let $g'(x) = 0$ at $x = x_a$. Then (19) can only be satisfied at $x = x_a$ if

$$g(x)h'(x) = 0 \text{ at } x = x_a. \text{ Now } h'(x) \neq 0$$

at $x = x_a$ i.e $g(x_a) = 0$. This is not possible because $g'(x)$ at $x = x_a$ is zero and $g(x)$ is given to be unimodal.

Hence the extreme points of (f_1/f_2) will not lie at the point where $g'(x)=0$ and the point where $h'(x)=0$. Thus in Case II, the extreme points of (f_1/f_2) lie where (19) holds, i.e, where

$$\frac{g(x)}{g'(x)} = \frac{h(x)}{h'(x)}.$$

This geometrically means that the points where the tangents to the curve g and h intersect the x-axis at the same point.

7.6 A graphical study

Let $f_1 = p_1 N(\mu_1, \sigma_1^2) + (1-p_1)N(\mu_2, \sigma_2^2)$

and $f_2 = p_2 N(\mu_3, \sigma_3^2) + (1-p_2)N(\mu_4, \sigma_4^2)$.

In figures 1 to 12 the red curve is the plot of (f_1/f_2). The blue curve is the plot of $(\hat{f}_1/\hat{f}_2)$, where $\hat{f}_1$ and $\hat{f}_2$ are normal kernel estimates of f_1 and f_2 respectively, constructed using appropriate window widths.

The following table illustrates the three cases considered (for four different sample sizes) in figures 1 to 12. In the table, n is the sample size underlying the estimates $\hat{f}_1$ and $\hat{f}_2$.

Table 1

Fig.No.	n	p_1	p_2	μ_1	μ_2	μ_3	μ_4	σ_1^2	σ_2^2	σ_3^2	σ_4^2
1	10000	0.45	0.60	1	4	1	4	1	1	1	1
2	4000										
3	1000		,,			,,			,,		
4	500										
5	10000	0.75	0.25	1	2	1	2	1	1	1	1
6	4000										
7	1000		,,			,,			,,		
8	500										
9	10000	0.50	0.30	1	3	1	5	1	1	1	1
10	4000										
11	1000		,,			,,			,,		
12	500										

Note that, in cases one and two, f_i's are mixtures of the same two components. Also note that the agreement between (f_1/f_2) and $(\hat{f}_1/\hat{f}_2)$ is not very good in the tails. As expected, for a bigger value of n, (f_1/f_2) and $(\hat{f}_1/\hat{f}_2)$ are closer.

Figures 1 to 8 illustrate Theorem 4 of Chapter 1 which states that $\frac{f_1}{f_2}$ will have finite (and non-zero) limits iff f_1 and f_2 are mixtures of the same two components. Note that the condition of theorem 4 is not fulfilled for case 3, hence the pattern is different in figures 9 to 12.

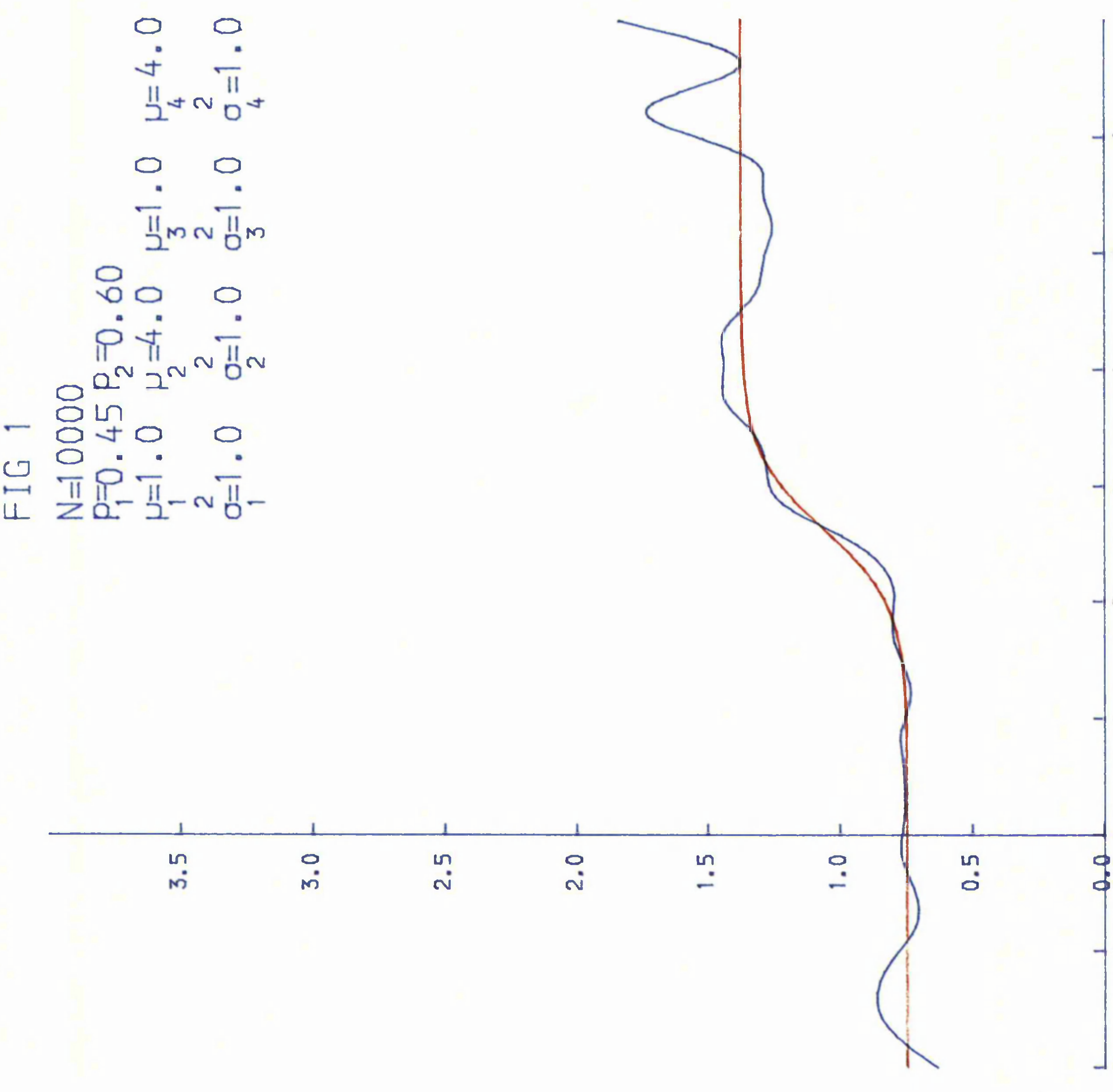


FIG 1
N=10000
P1=0.45 P2=0.60
μ1=1.0 μ2=4.0 μ3=1.0 μ4=4.0
σ²1=1.0 σ²2=1.0 σ²3=1.0 σ²4=1.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-2
-1
0
1
2
3
4
5
6

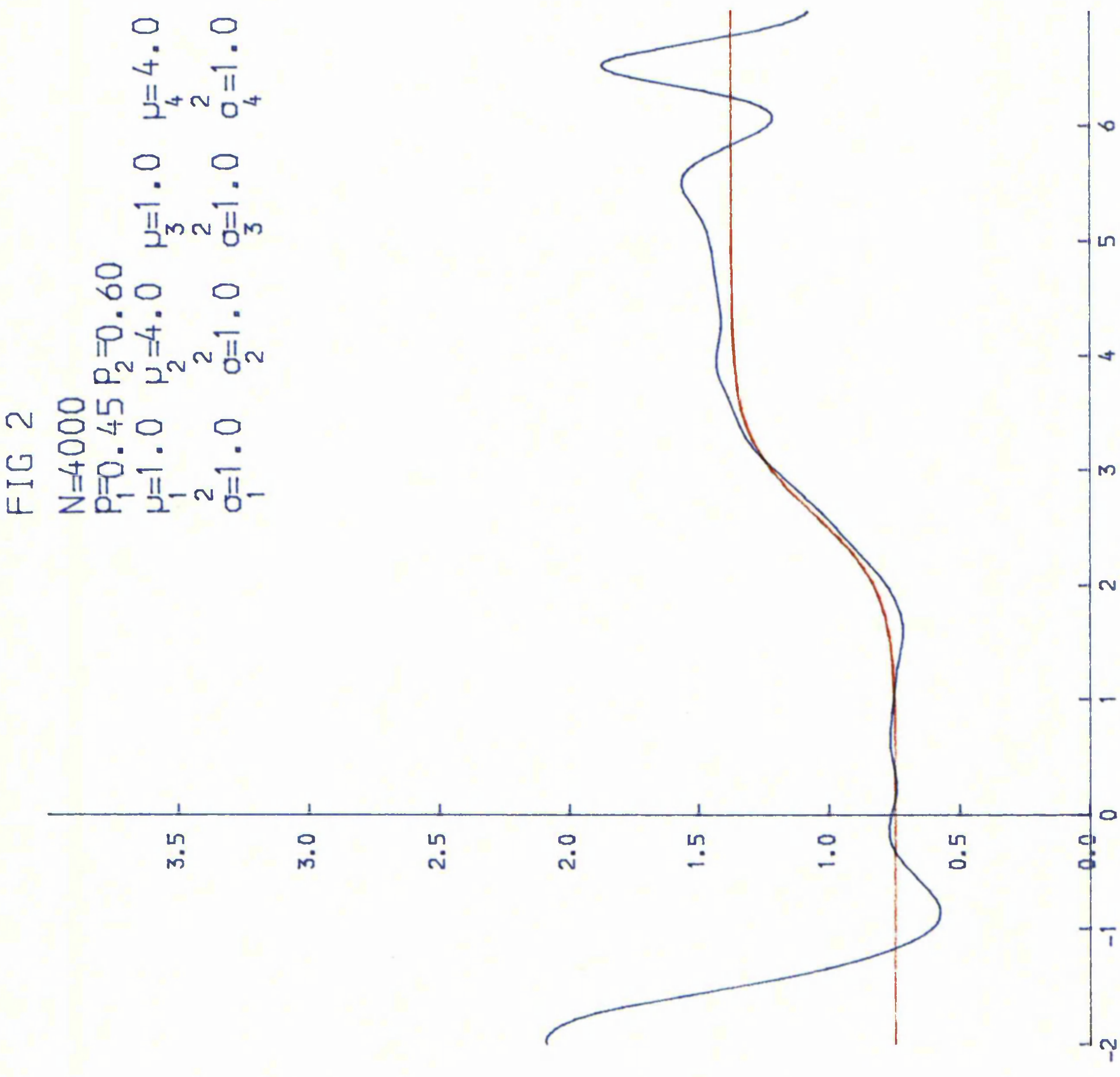
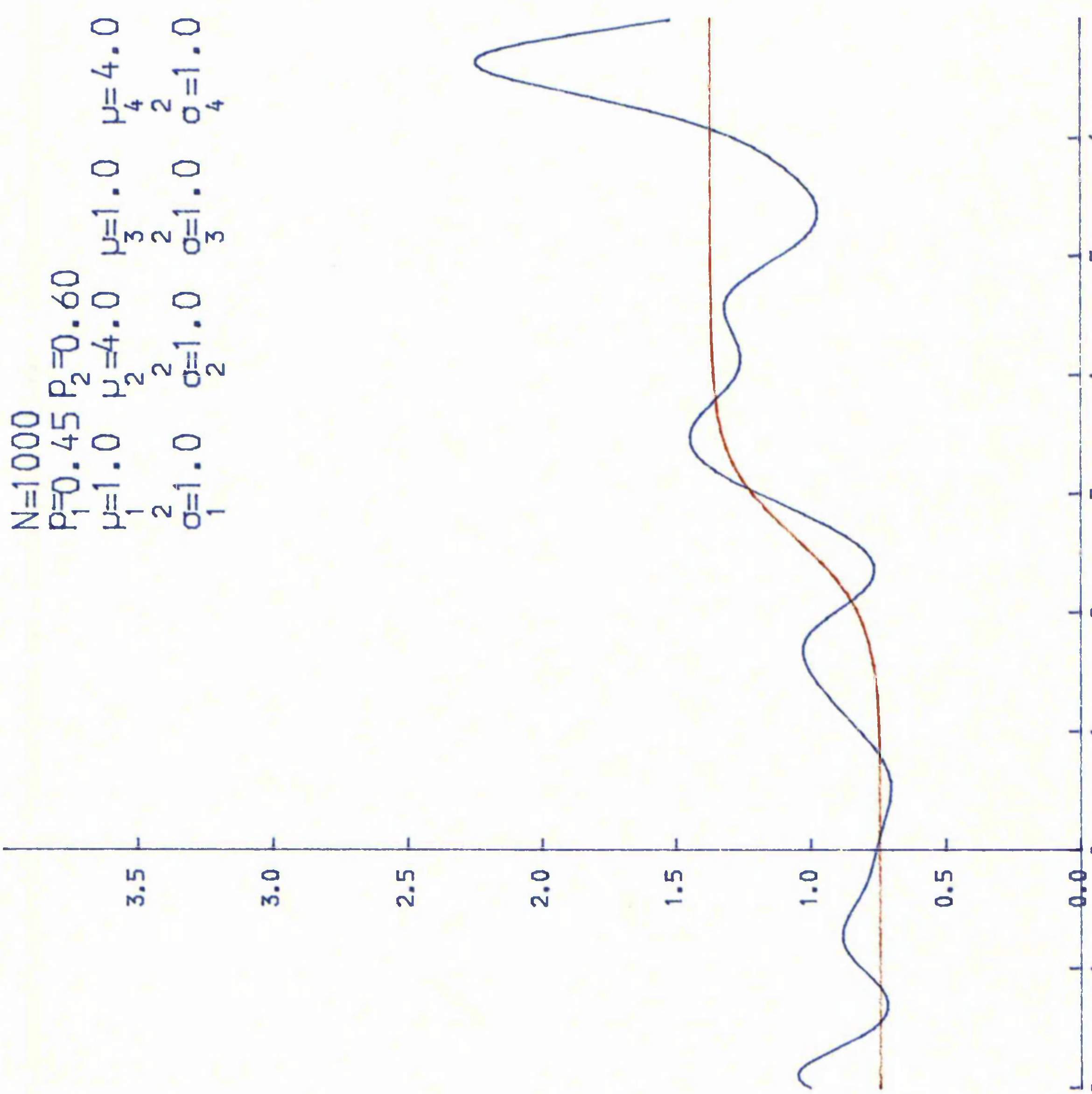


FIG 2
N=4000
P1=0.45 P2=0.60
μ1=1.0 μ2=4.0 μ3=1.0 μ4=4.0
σ²1=1.0 σ²2=1.0 σ²3=1.0 σ²4=1.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-2
-1
0
1
2
3
4
5
6



N=1000
P1=0.45 P2=0.60
μ1=1.0 μ2=4.0 μ3=1.0 μ4=4.0
σ²1=1.0 σ²2=1.0 σ²3=1.0 σ²4=1.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-2
-1
0
1
2
3
4
5
6

FIG 3

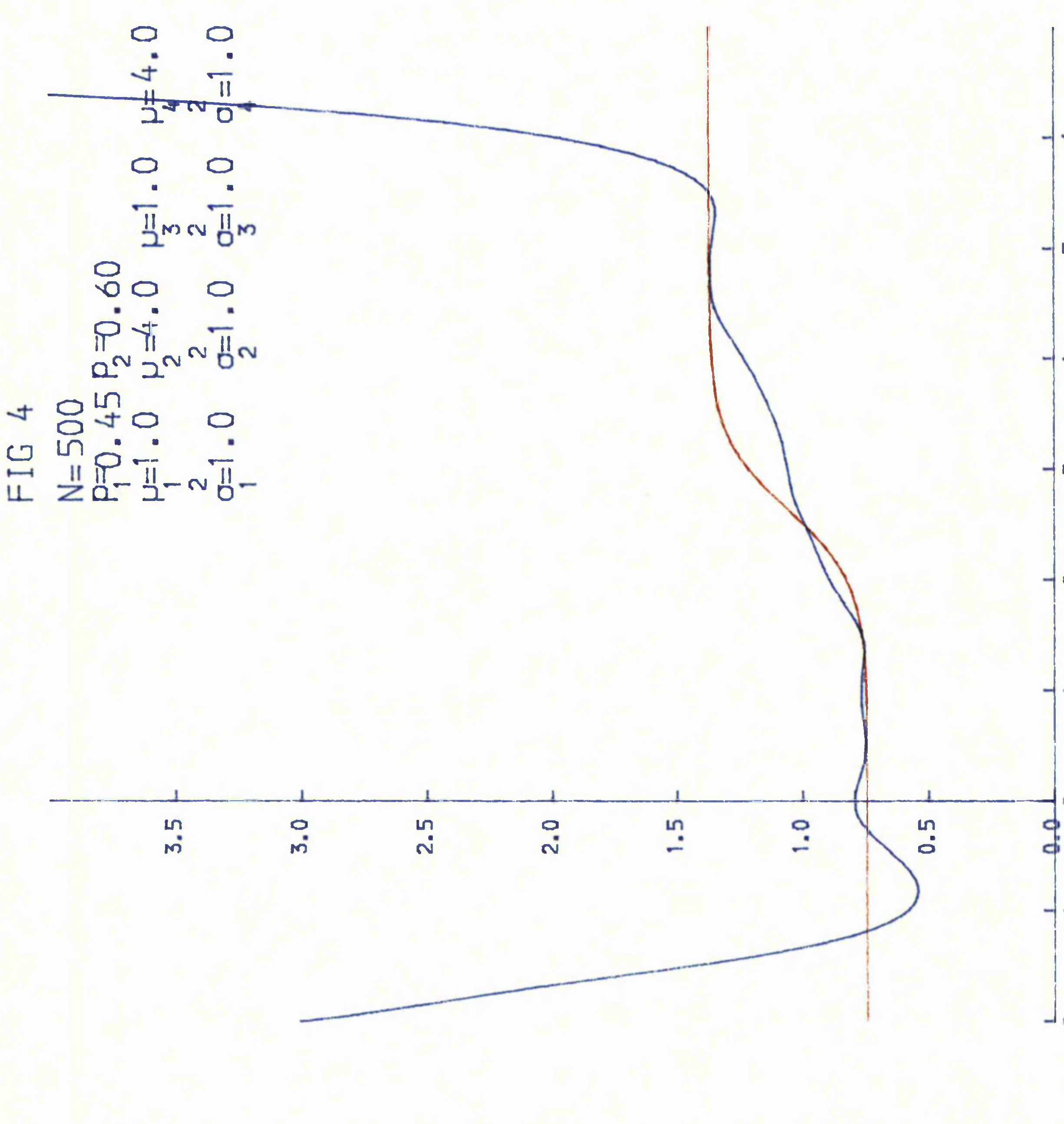


FIG 4
N=500
P_1=0.45 P_2=0.60
μ_1=1.0 μ_2=4.0 μ_3=1.0 μ_4=4.0
σ_1^2=1.0 σ_2^2=1.0 σ_3^2=1.0 σ_4^2=1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
-2
-1
0
1
2
3
4
5
6

FIG 5

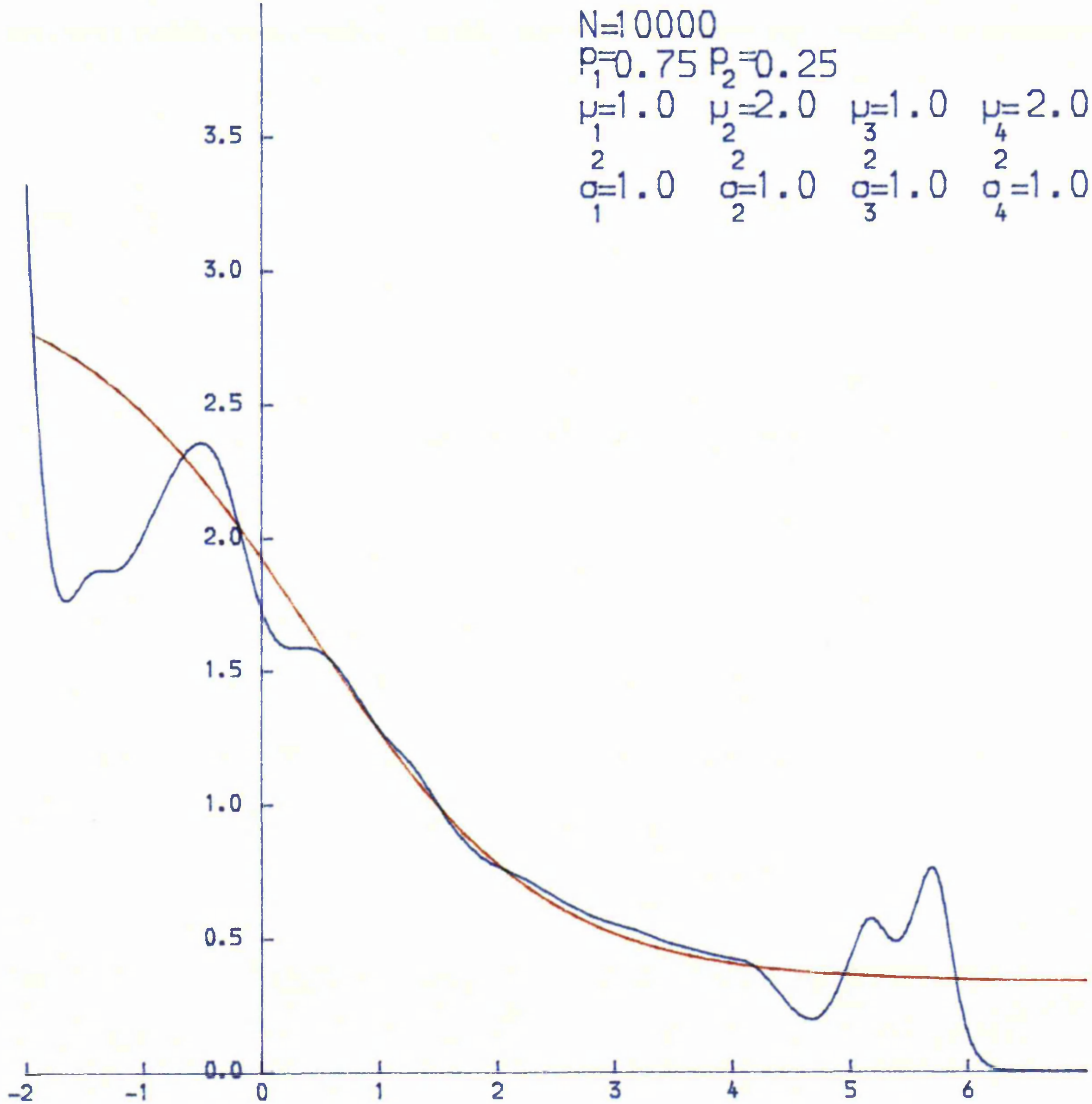
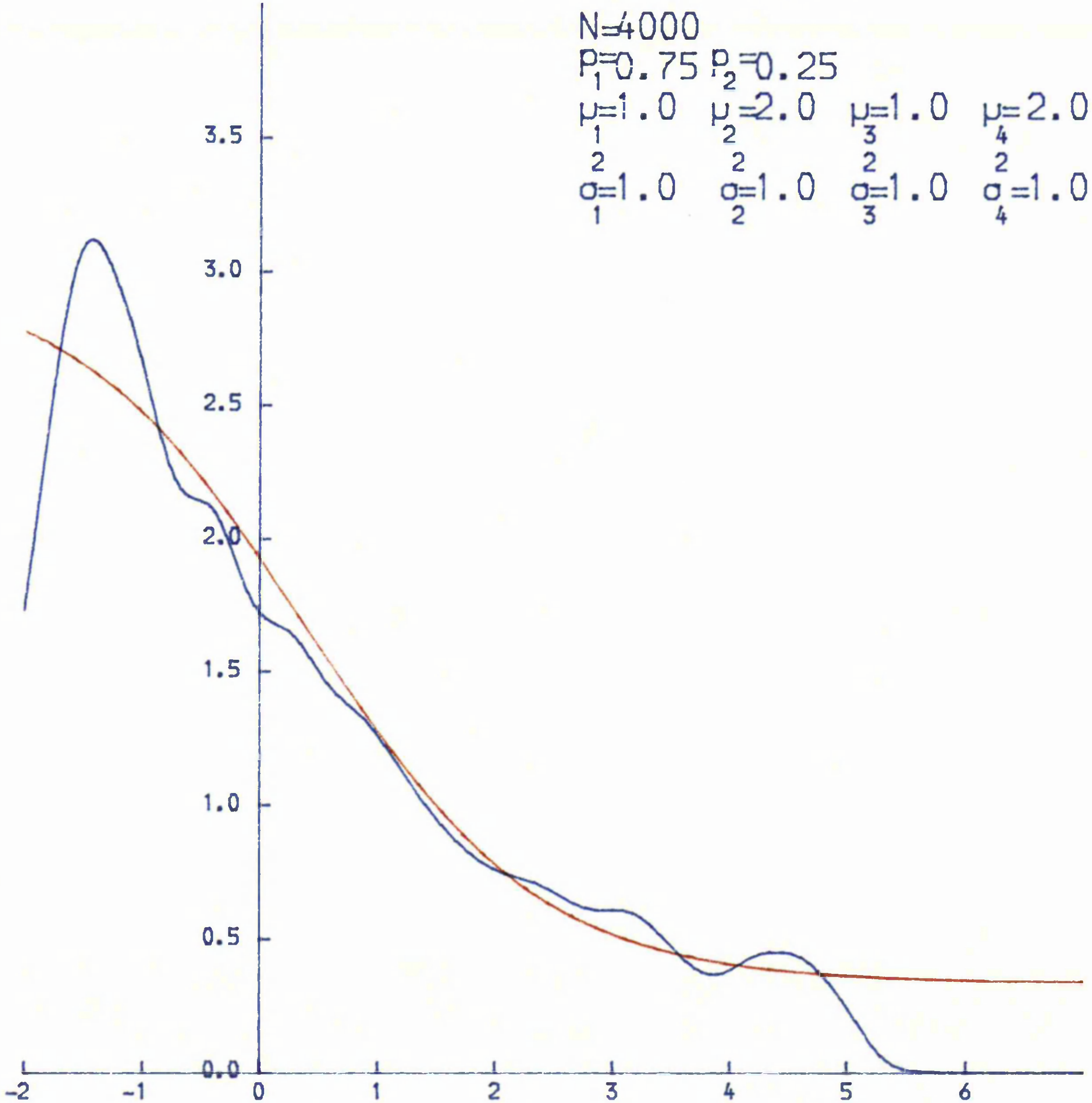

N=10000
P_1=0.75 P_2=0.25
μ_1=1.0 μ_2=2.0 μ_3^2=1.0 μ_4^2=2.0
σ_1^2=1.0 σ_2^2=1.0 σ_3^2=1.0 σ_4^2=1.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-2
-1
0
1
2
3
4
5
6

FIG 6


N=4000
P_1=0.75 P_2=0.25
μ_1=1.0 μ_2=2.0 μ_3=1.0 μ_4=2.0
σ_1^2=1.0 σ_2^2=1.0 σ_3^2=1.0 σ_4^2=1.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-2
-1
0
1
2
3
4
5
6

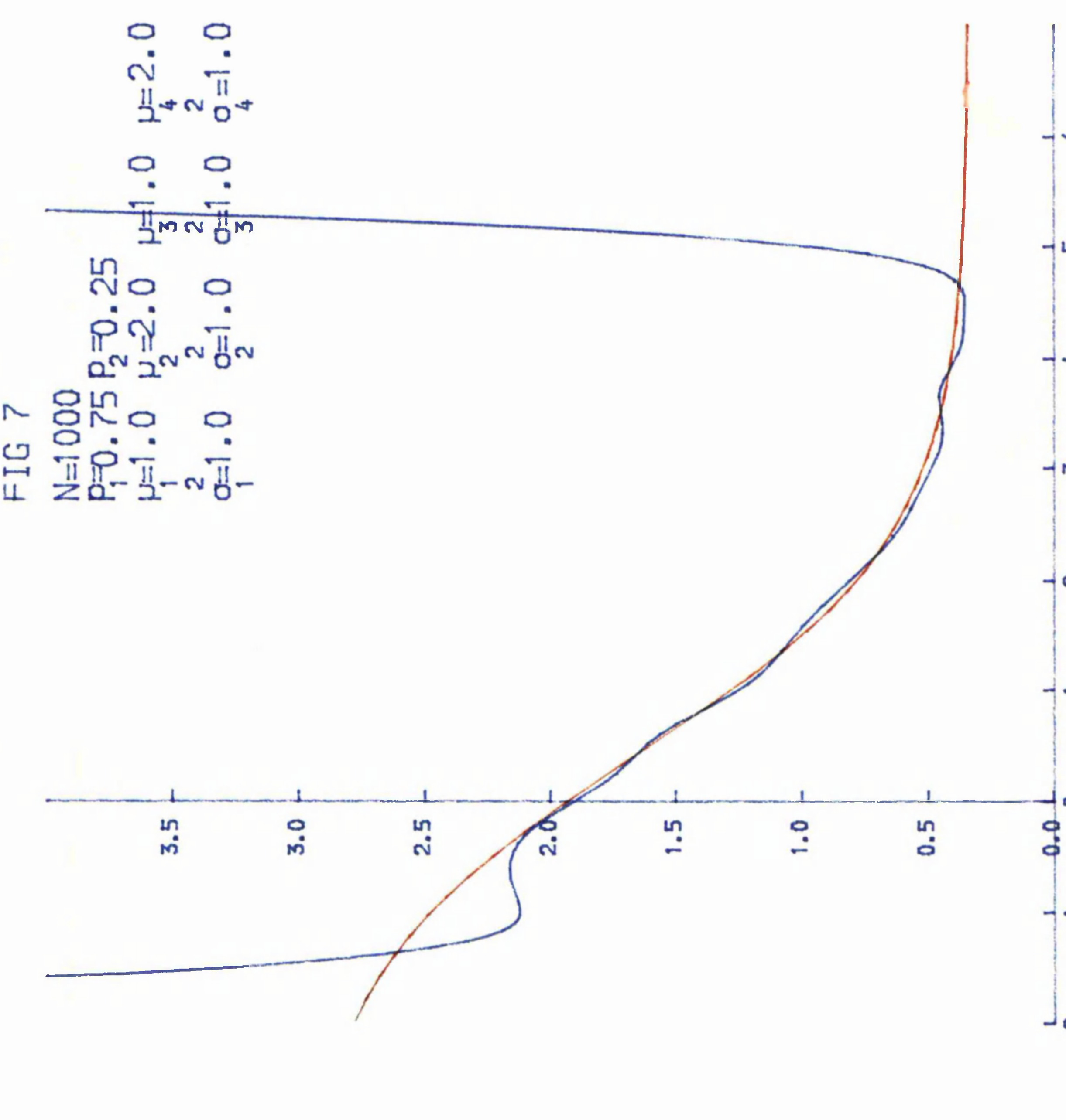

N=1000
P_1=0.75 P_2=0.25
μ_1=1.0 μ_2=2.0 μ_3=1.0 μ_4=2.0
σ_1^2=1.0 σ_2^2=1.0 σ_3^2=1.0 σ_4^2=1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
-2
-1
0
1
2
3
4
5
6

FIG 7

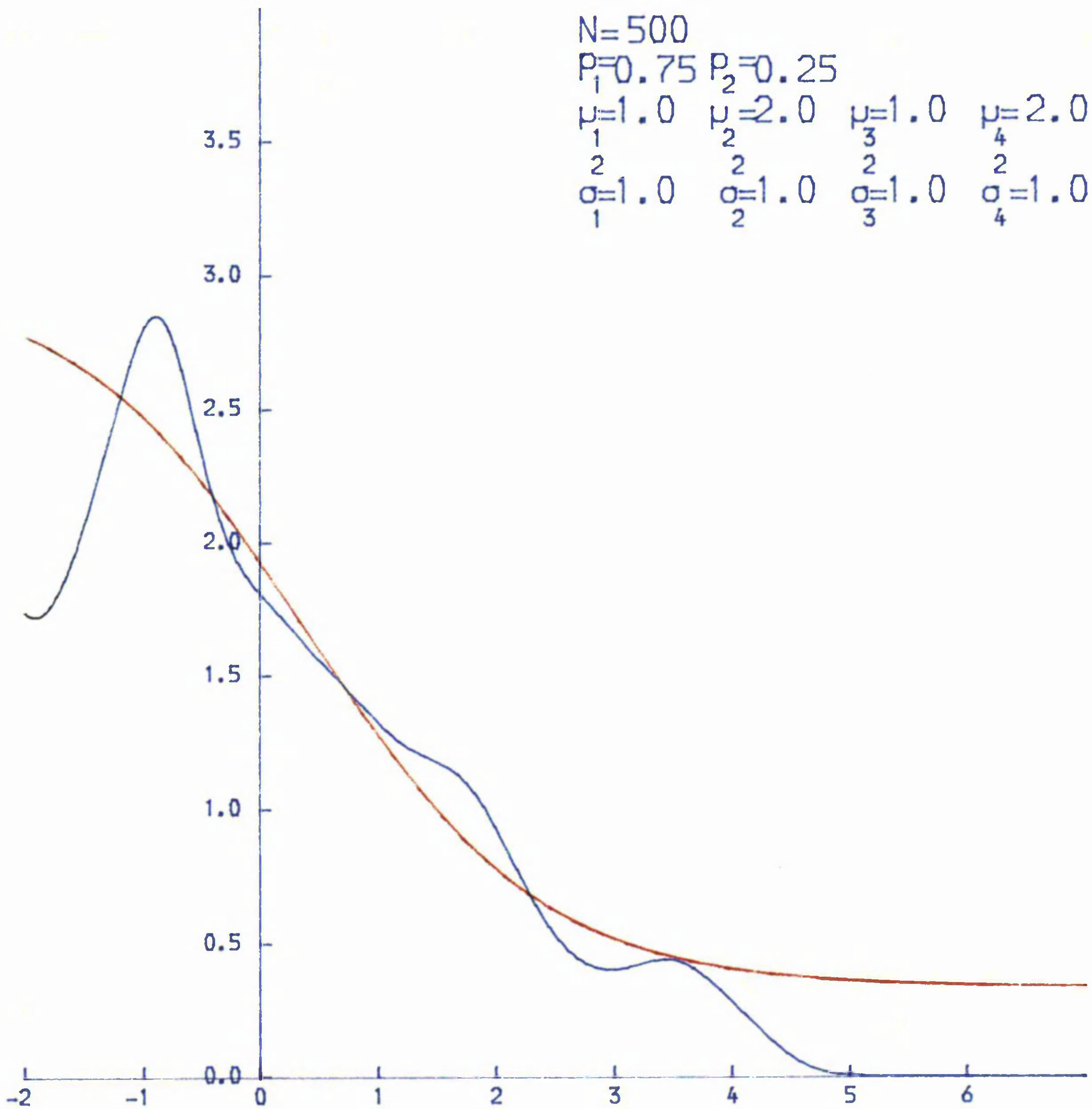
FIG 8

N=500

$P_1=0.75$ $P_2=0.25$

$\mu_1^2=1.0$ $\mu_2^2=2.0$ $\mu_3^2=1.0$ $\mu_4^2=2.0$

$\sigma_1^2=1.0$ $\sigma_2^2=1.0$ $\sigma_3^2=1.0$ $\sigma_4^2=1.0$



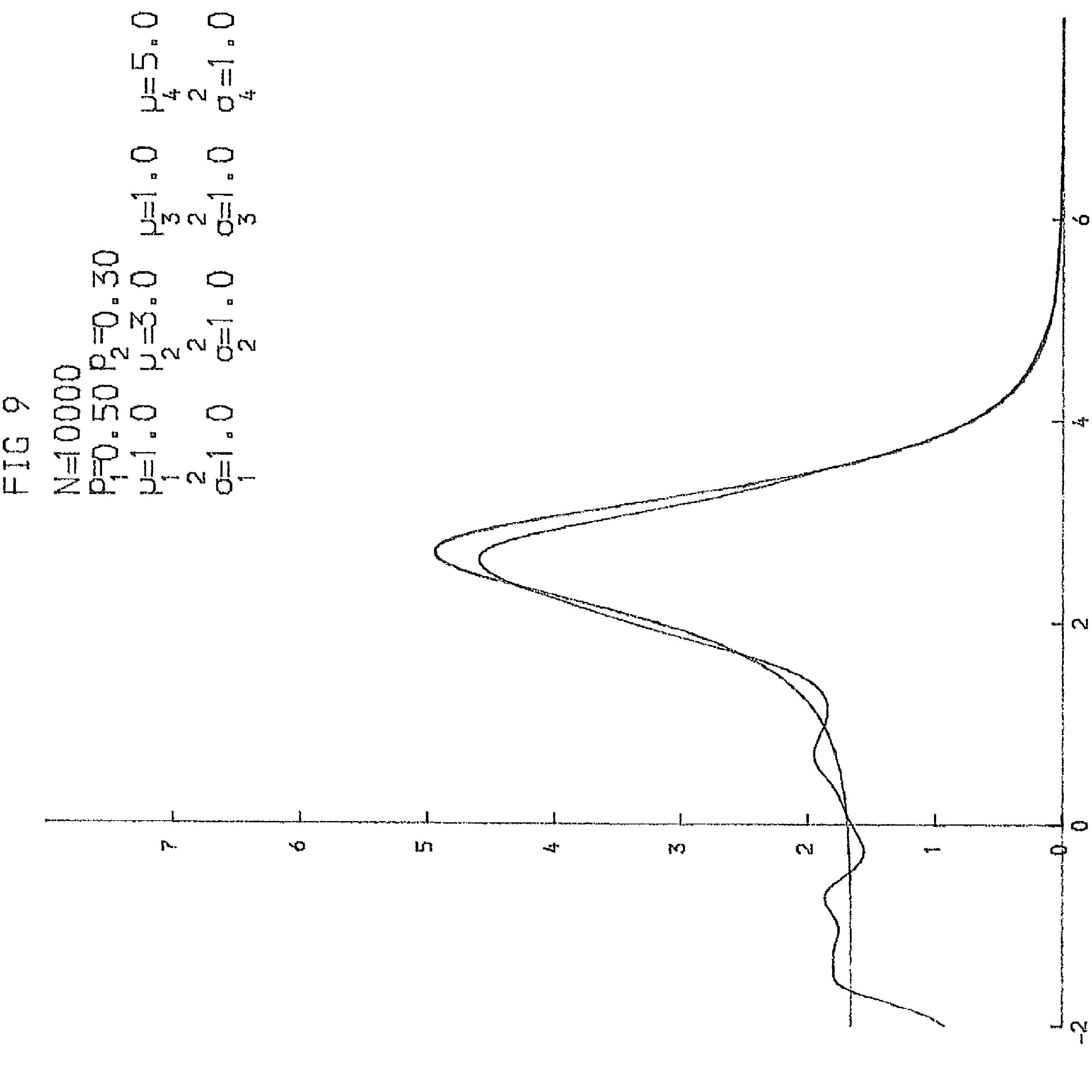


FIG 9
N=10000
P_1=0.50 P_2=0.30
μ_1=1.0 μ_2=3.0 μ_3=1.0 μ_4=5.0
σ_1^2=1.0 σ_2^2=1.0 σ_3^2=1.0 σ_4^2=1.0
0 1 2 3 4 5 6 7
-2 0 2 4 6

FIG 10

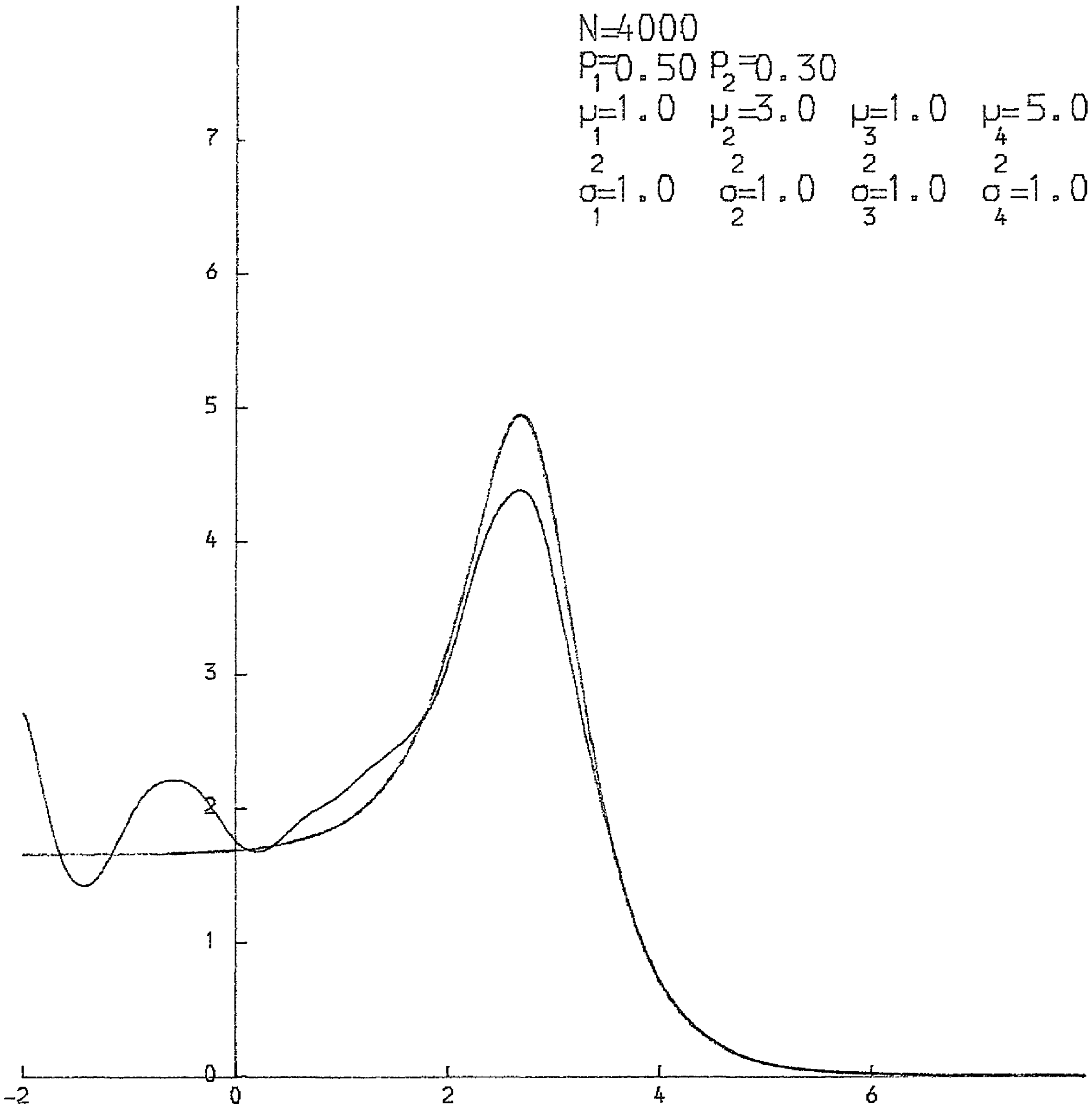
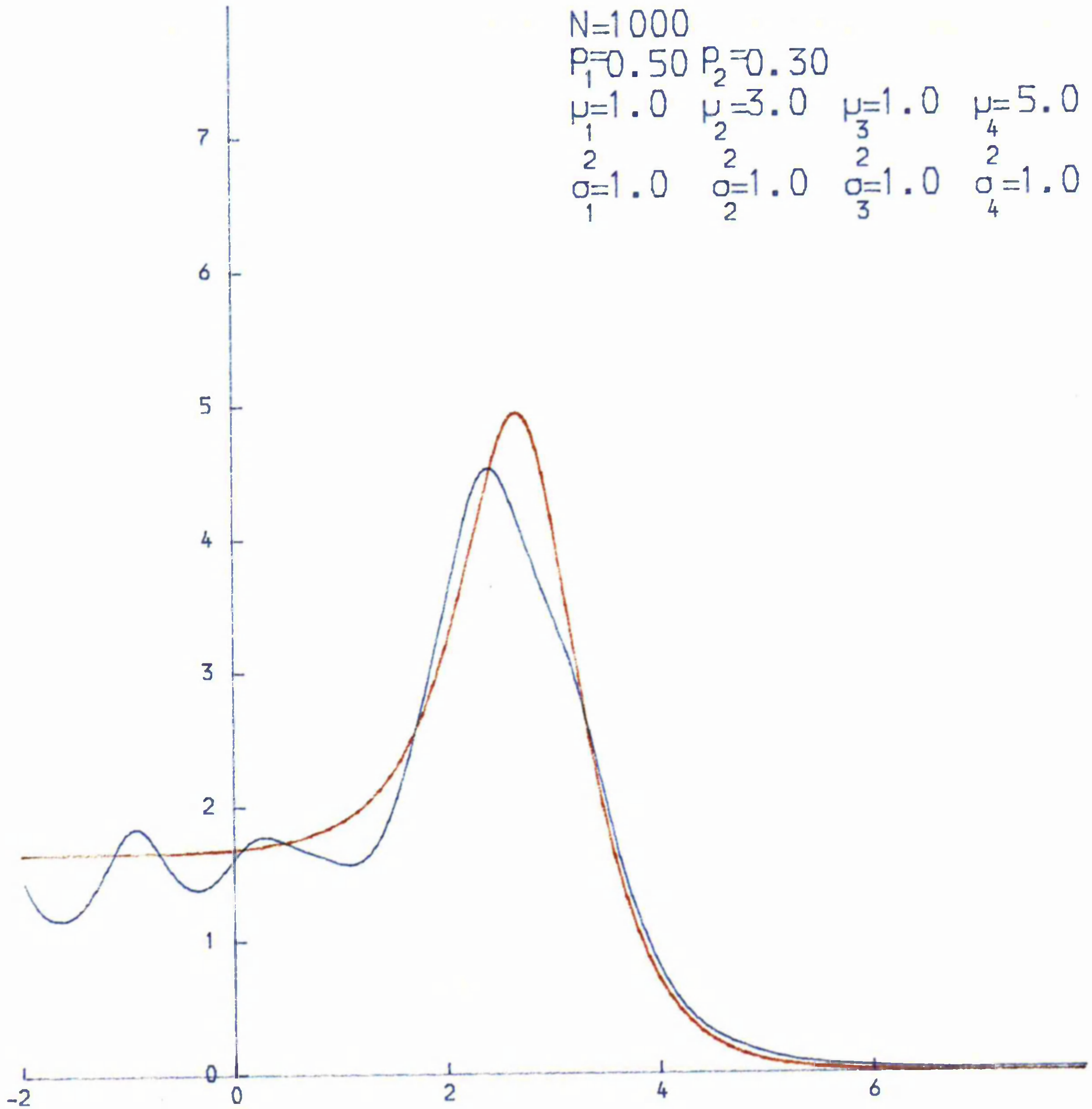

N=4000
P1=0.50 P2=0.30
μ1=1.0 μ2=3.0 μ3=1.0 μ4=5.0
σ1 2=1.0 σ2 2=1.0 σ3 2=1.0 σ4 2=1.0
7
6
5
4
3
2
1
0
-2
0
2
4
6

FIG 11


N=1000
P_1=0.50 P_2=0.30
μ_1=1.0 μ_2=3.0 μ_3=1.0 μ_4=5.0
σ_1^2=1.0 σ_2^2=1.0 σ_3^2=1.0 σ_4^2=1.0
7
6
5
4
3
2
1
0
-2
0
2
4
6

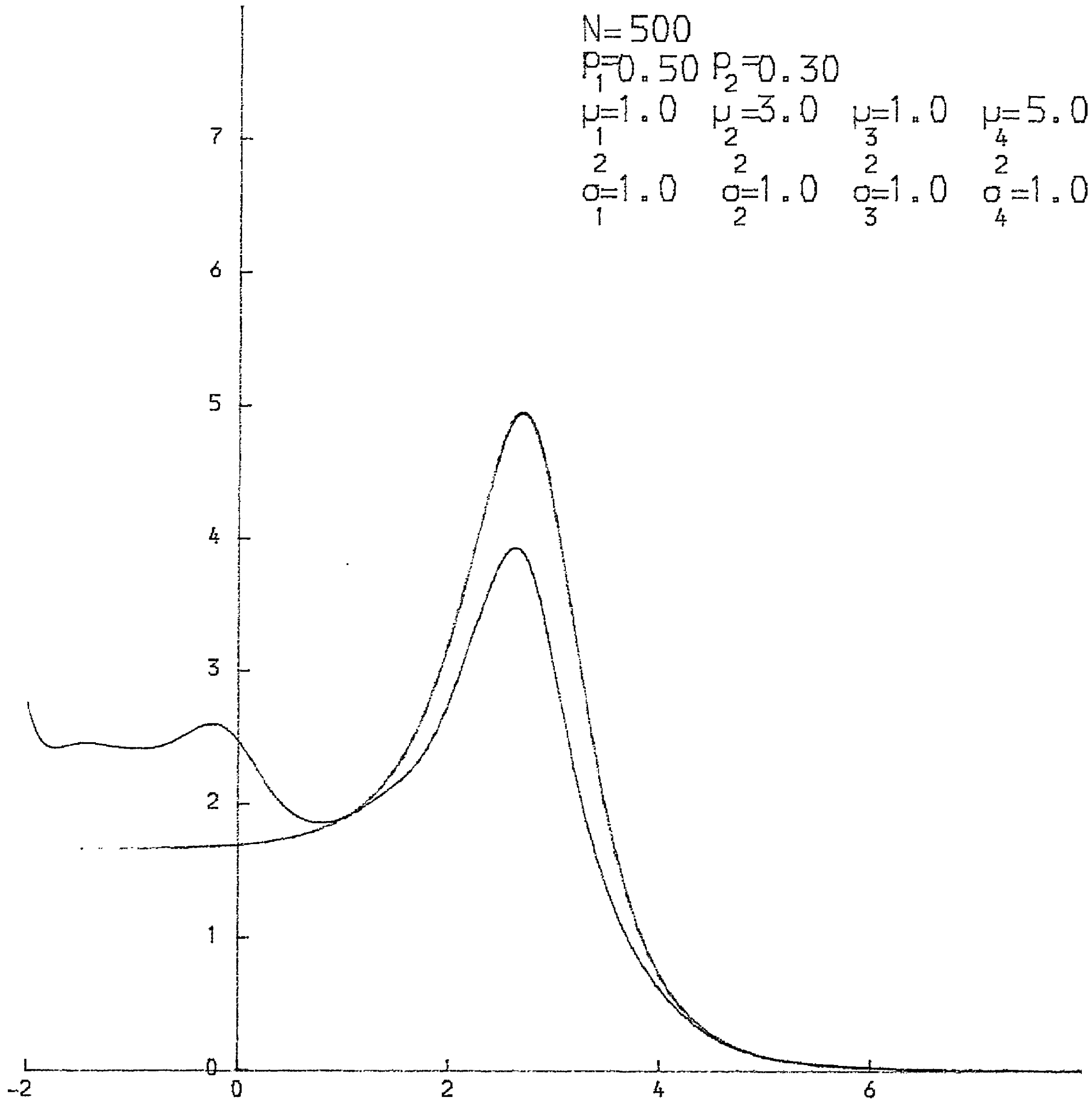

N=500
P_1=0.50 P_2=0.30
μ_1=1.0 μ_2=3.0 μ_3=1.0 μ_4=5.0
σ_1^2=1.0 σ_2^2=1.0 σ_3^2=1.0 σ_4^2=1.0
7
6
5
4
3
2
1
0
-2
0
2
4
6

FIG 12

CHAPTER 8

FURTHER REMARKS AND SUGGESTIONS

8.1 Introduction

In section 8.2 we give two graphical representations of the null hypothesis for the case of three density functions. In section 8.3, the theory and methods developed in this thesis (which are applicable only to the case of three density functions) are modified to cover the general case, i.e., when the number of density functions is greater than three. We start with Theorem 1 of Chapter 7 and state and prove its generalisation. It is pointed out that the converse of this theorem is not true always. Given n (where n is greater than 3) density functions it is explained how to check whether n-2 of them are proper mixtures of the other two.

In section 8.4, the case of three densities is considered where each one of the three is a mixture of the same three components. It is shown that a particular property of binary mixtures is not unconditionally true for the case of mixtures having more than two components. It is also shown that a mixture of more than three components can be 'mathematically' reduced to a binary mixture. Finally, in this section it is demonstrated that if the densities are mixtures of the same components then to obtain a linear relationship between the densities, the number of densities should be one more than the number of components.

8.2 Two graphical representations of the null hypothesis for the case of three densities.

Consider three density functions f_1, f_2 and f_3 which are such that one of them (say f_2) is a proper mixture of the other two. In other words, the null hypothesis is true.

Let $x_i, i\varepsilon[1,2,\ldots m]$ be the m chosen points which were used to construct the 'distance' function $Q(\hat{t})$ earlier in the thesis. Also, let

$$f_{12r} \overset{\text{defn.}}{=} f_1(x_r)-f_2(x_r)$$

$$\text{and } f_{13r} \overset{\text{defn.}}{=} f_1(x_r)-f_3(x_r)$$

where $r\varepsilon[1,2,\ldots,m]$.

In Fig.1 below, the m values of $|f_{12r}|$ are plotted on the vertical axis and, similarly, the m values of $|f_{13r}|$ are plotted on the horizontal axis. The corresponding points are joined together by straight lines.

Now, since f_2 is a proper mixture of f_1 and f_3, therefore, it follows that f_{12r}= constant $\times f_{13r}$

$\forall r\varepsilon[1,2,\ldots,m]$. Thus, the transverse lines in Fig 1 are all parallel.

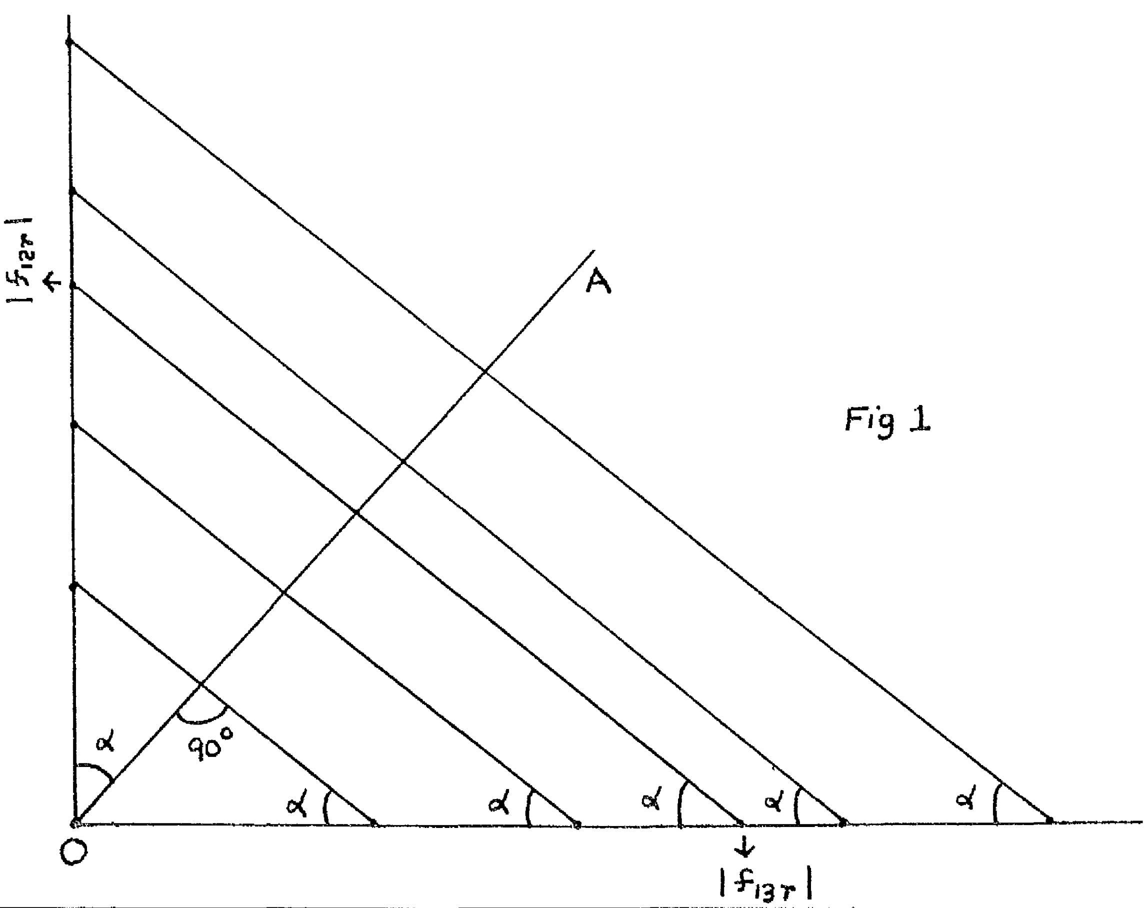


Fig 1

A line OA is drawn from the origin of the axes, perpendicular to the transverse lines.

Note that we have

$$\alpha = \tan^{-1} \frac{|f_{12r}|}{|f_{13r}|} = \tan^{-1}|t|$$

where $\frac{f_{12}}{f_{13}} = t$, i.e., $f_2 = (1-t)f_1 + tf_3$.

Ideally, all the transverse lines are parallel but, in practice, when we are dealing with density estimates, the perpendiculars from O to each line will be different lines. The distribution of $\hat{\alpha}$, where

$$\hat{\alpha} = \tan^{-1} \frac{|\hat{f}_{12r}|}{|\hat{f}_{13r}|}$$ may prove to be of some use.

The second graphical representation is given by Figure 2.

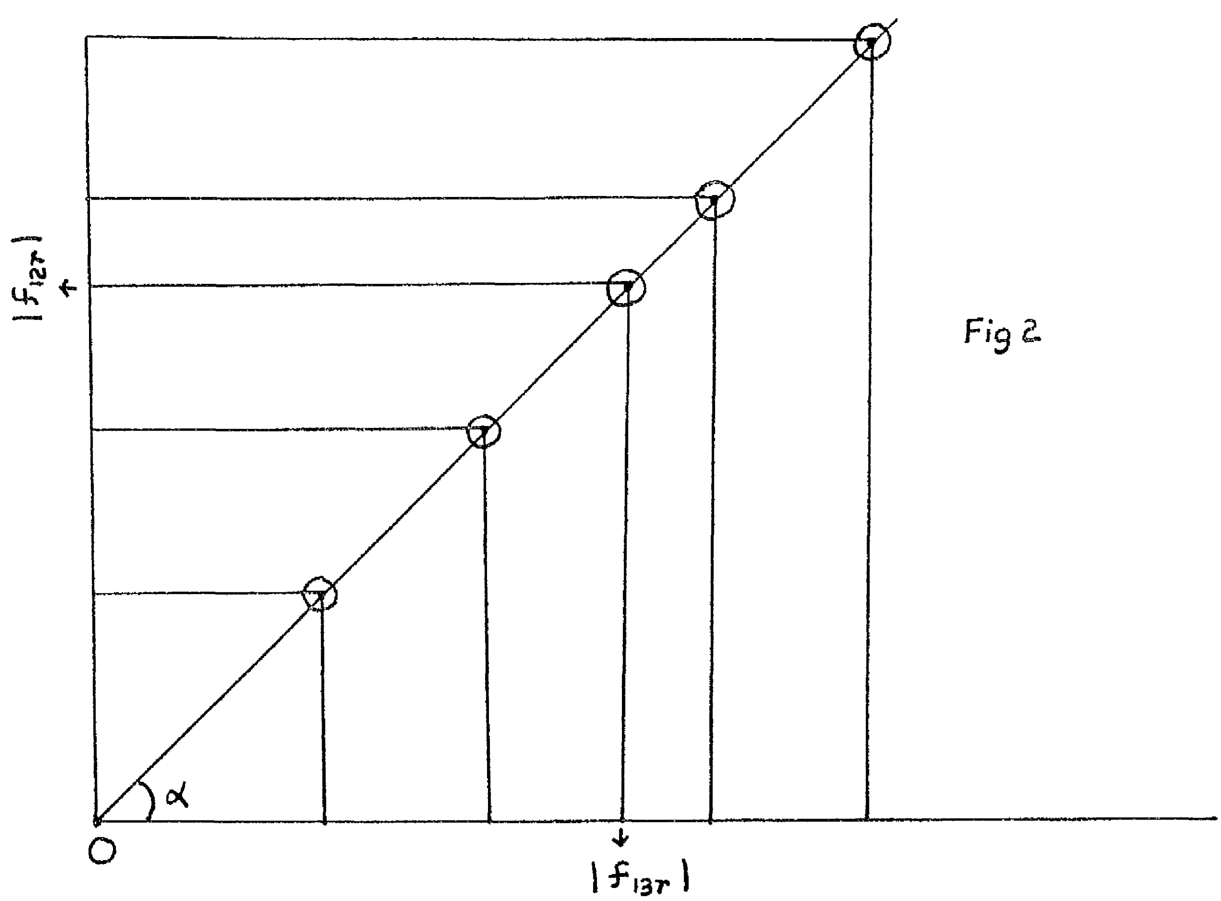


Fig 2

Figure 2 is different from Figure 1 in that the line through O is drawn through the m points (f_{13r},f_{12r}). Thus, lines OA and OB are different. When we are dealing with density estimates then the circled points will not lie on a single line and a least squares estimate will need to be obtained.

8.3 A general theorem and an associated method

For the case of n $(n>3)f_i$'s, $i\epsilon[1,2,....n]$,

Theorem 1 (on page 150), in Chapter 7, generalises as follows.

If $f_1,f_2,.....,f_n$ are proper mixtures of the same two components with mixing weights $p_1,p_2,...p_n$ respectively, such that $p_1>p_2>...>p_n$, then $f_2,f_3,....f_{n-1}$ are proper mixtures of f_1 and f_n.

Proof

Consider f_1, f_n and one of the other f_i's. We have $p_1>p_i>p_n$ where $i\epsilon[2,3,...n-1]$, From Theorem 1 on page 150 it is clear that f_i is a proper mixture of f_1 and f_n. Hence it follows that f_2, $f_3,....f_{n-1}$ are proper mixtures of f_1 and f_n.

Note that the same result follows when the p_i's are ordered as $p_1<p_2<......<p_n$.

As demonstrated by the example in section 2.2, the converse of Theorem 1 of Chapter 7 is not necessarily true. Hence, the converse of its generalised version is also not necessarily true. The converse is,

"If n-2 of the n f_i's are proper mixtures of the remaining two then all of the f_i's are proper mixtures of the same two components."

Note that if the 'component' f_i's are removed from the set of the f_i's then the remaining n-2 f_i's can be considered to be mixtures of the same two 'components'. This way of looking at it is not very meaningful unless the number of f_i's is more than three.

Suppose there are $n(n>3)$ density functions f_i which are proper mixtures of the same two components. Using any two of the n relations express the components in terms of two of the f_i's. Substituting the expressions for the components (in terms of the f_i's) in the other relations, we obtain n-2 independent relations in the f_i's. Thus, as we know very well, three f_i's will give one independent relation in the f_i's. Similarly, four f_i's give two independent relations.

Finally, in this section, suppose we are given four densities f_i and we want to determine the 'component' f_i's (if any). Consider any three of the f_i's. Using the tests developed in this thesis if we conclude that one of the f_i's is a proper mixture of the other two then dropping that f_i we apply the same test again to the remaining three f_i's. If, again, we conclude that one of the f_i's is a proper mixture of the other two then it follows that two of the f_i's are proper mixtures of the other two. Explaining in more detail, suppose, of f_1,f_2 and f_3, f_3 is a proper mixture of the other two. Thus, dropping f_3 consider f_1, f_2 and f_4. If f_4 is a proper mixture of f_1 and f_2 then it is clear that f_3 and f_4 are proper mixtures of f_1 and f_2. If f_1 is a proper mixture of f_2 and f_4 then, the fact that f_3 is a proper mixture of f_1 and f_2 implies that f_3 is a proper mixture of f_2 and f_4. Thus f_1 and f_3 are proper mixtures of f_2 and f_4. Similarly, if f_2 is a proper mixture of f_1 and f_4 then f_2 and f_3 are proper mixtures of f_1 and f_4.

8.4 Finite mixtures having more than two components

Consider three densities f_i which are proper mixtures of the same three components g,h and j.

Let $f_1=p_1g+q_1h+(1-p_1-q_1)j$

$f_2=p_2g+q_2h+(1-p_2-q_2)j$

and $f_3=p_3g+q_3h+(1-p_3-q_3)j$

where p_1, p_2 and p_3 and q_1, q_2 and q_3 are the mixing weights.
Unlike the binary case (as explained in Chapter 2), here, it is not necessary that one of the f_i's is a proper mixture of the other two unless the mixing weights are related in a particular way. Suppose f_2 is a proper mixture of f_1 and f_3 with mixing weight α, given by

$$f_2=\alpha f_1+(1-\alpha)f_3,$$

then, it follows that

$$p_2=\alpha p_1+(1-\alpha)p_3 \qquad \text{i.e. } \frac{p_2-p_3}{p_1-p_3}=\alpha$$

and

$$q_2=\alpha q_1+(1-\alpha)q_3 \qquad \text{i.e. } \frac{q_2-q_3}{q_1-q_3}=\alpha \ .$$

Thus $\dfrac{p_2-p_3}{p_1-p_3}=\dfrac{q_2-q_3}{q_1-q_3}$.

This is the necessary condition for f_2 to be a proper mixture of f_1 and f_3.
For the cases where this necessary condition holds, given three densities, (f_i, i=1,2,3) we will conclude that one of the f_i's is a proper mixture of the other two.
Note that a mixture with three (or more) components may be 'mathematically' reduced to a binary mixture. For example the f_1 mentioned above may be written as

$$f_1=p_1g+q_1h+(1-p_1-q_1)j$$

$$=p_1g+(1-p_1)\left[\frac{q_1}{1-p_1}h+\frac{1-p_1-q_1}{1-p_1}j\right]$$

$$=p_1g+(1-p_1)w,$$

where

$$w=\frac{q_1}{1-p_1}h+\frac{1-p_1-q_1}{1-p_1}j$$

obviously satisfies the conditions of a density.

Lastly, if we have four f_i's which are proper mixtures of the same three components then solving for the components in terms of the f_i's,using any three relations, we may substitute the expressions,thus obtained, in the fourth f_i to obtain a linear relation between the four f_i's where it is easy to check that one of the f_i's is a mixture of the other three. Hence to obtain a linear relation between the f_i's the number of f_i's required is one more than the number of components comprising each f_i.

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