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BOOLEAN ALGEBRAS OF PROJECTIONS ON BANACH SPACES

by

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PREFACE

This dissertation is submitted in accordance with the regulations of the degree of Master of Science in the University of Glasgow. No part of it has been previously submitted by the author for a degree at any other University.

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Summary

The object of this thesis is the study of the Boolean algebras of projections on Banach spaces and their properties.

Chapter one is devoted to the study of the integration theory needed in the other chapters. The main characteristic features of the theory of normed Köthe spaces are introduced first. Next, we investigate the conditions on ρ which ensure that every function $\phi \in L^1(\mu)$ can be factorized as $\phi = f.g$, with $f \in L\rho$ and $g \in L\rho'$.

In chapter two, which is the main part of the thesis, we introduce the concept of Boolean algebras of projections. We prove that a Boolean algebra of projections is σ-complete if and only if it is the range of a countably additive regular spectral measure. We show that a bounded Boolean algebra of projections on a weakly complete Banach space X can be embedded in a σ-complete Boolean algebra of projections on X. Next, the main structure theorem for a cyclic Banach space X is proved. It is shown that the weakly closed algebra generated by a σ-complete Boolean algebra of projections on a Banach space X is reflexive. A representation theorem for a complete Boolean algebra of projections is included.

In chapter three, we consider the necessary and sufficient conditions for the restrictions of normal operators on Hilbert space to be normal operators, and the restrictions of scalar-type spectral operators on Banach spaces to be of scalar-type.

The results of chapter one are due to T.A. Gillespie [13] and

[14]. The majority of the results in chapter two are due to W.G. Bade [1] and [9], but we show how Gillespie's results can be used to obtain stronger results and give more satisfactory proofs of these results. The results of Chapter three are due to Wermer [18] and Dowson [5].

Contents

LIST OF SYMBOLS

$\mathbb{C}$:	The complex plane
Σ :	The σ-algebra of Borel subsets of $\mathbb{C}$
$B(X)$:	Banach algebra of bounded linear operators on X
$C(K)$:	If K is a compact set, $C(K)$ denotes the continuous complex valued functions on K under the supremum norm
$\langle x,y \rangle$:	Inner product of x and y
spanA:	The linear span of A
$\overline{\text{span}}$A:	The closure of spanA
T/Y :	Denotes the restriction of T to Y
χ_τ :	Denotes the Characteristic function of the set τ
$\ll$:	Means: is absolutely continuous with respect to
$\equiv$:	Means : identically equal to
$\sigma(T)$:	Denotes the spectrum of the linear operator T
$\rho(T)$:	Denotes the resolvent set of the linear operator T
$\perp$:	This symbol means: is orthogonal to
$\{a\}$:	Denotes the set consisting of a alone
$\subset, \supset$:	Denote the strict inclusion
$\cap, \cup$:	Denote the intersection and union respectively
Iff :	Means if and only if
LatA :	Denotes all the closed A-invariant subspaces of X
Alg LatA :	Denotes the subalgebra of $B(X)$ of all operators that leave invariant every subspace in LatA.
spA, $\overline{\text{sp}}$A :	spanA, $\overline{\text{span}}$A respectively
$f^{\frac{1}{2}}$:	$f^{1/2}$
$\mathbb{N}$:	$\{0, 1, 2, 3, ...\}$
$\mathbb{N}^*$:	$\mathbb{N} - \{0\}$

X^* : The dual of X

Φ: The empty set

a.e.: Means : almost everywhere

Ref, Imf : Real and imaginary parts of f respectively

Chapter 1 NORMED KOTHE SPACES AND FACTORIZATION THEOREM

1.1 NORMED KOTHE SPACES

1.1.1 Introduction: In this chapter, we investigate the properties of certain normed linear spaces, the elements of which are complex valued measurable functions. Several characteristic features of the theory are discussed: Riesz-Fischer property and Fatou property of the norm, the associate spaces and, the subspaces of all functions of absolutely continuous norm. Several of the notions and results which follow can be traced back to the work of G.Köthe and O.Toeplitz in the thirties, and for that reason the normed function spaces which will be introduced in the present chapter are sometimes called Normed Köthe Spaces. Next, we investigate conditions on ρ which ensure that every function ϕ in $L^1(\mu)$ can be factorized as $\phi = fg$, with $f \in L_\rho$ and $g \in L_{\rho'}$. We also consider the extent to which the size of the product $\rho(f).\rho'(g)$ for such factorizations can be controlled. The condition that ρ be a saturated norm is not in itself sufficient for factorization always to be possible, but it is a necessary condition for a factorization of this type to exist for every function in $L^1(\mu)$.

Let (Ω,Σ,μ) be a measure space.

1.1.2 Definition: A function seminorm on (Ω,Σ,μ) is a function ρ

$$\rho : M^+ \longrightarrow [0,\infty]$$

such that

$$\text{i) } \rho(f+g) \leqslant \rho(f) + \rho(g) \quad \forall\, f,\ g \in M^+$$

$$\text{ii) } \rho(\alpha f) = \alpha\, \rho(f) \qquad \forall\, f \in M^+,\ \alpha \in [0,\infty]$$

iii) $\rho(f) \leqslant \rho(g)$ if $f \leqslant g$ a.e., $\forall$ $f, g \in M^+$

iv) $\rho(f) = o$ if $f = o$ a.e. $\forall$ $f \in M^+$.

The function seminorm ρ is called a function norm if it has the additional property:

$$\rho(f) = o \text{ iff } f = o \text{ a.e. } (f \in M^+)$$

1.1.3 Theorem: if ρ is a function norm, and $u \in M^+$ satisfies $\rho(u) \angle \infty$, then $u(x) = \infty$ holds only on a set of a null measure.

Proof: Writing $E = \{x: u(x) = \infty\}$, we have $\chi_E \leqslant u/n$ for $n = 1,2,3,\ldots$, and so $\rho(\chi_E) \leqslant \rho(u)/n$, $n = 1,2,3,\ldots$, it follows that $\rho(\chi_E) = o$, so $\chi_E = o$, a.e. Q.E.D.

One can extend the domain of any function seminorm to the whole of M. Then, given the function norm ρ we extend its domain to the whole of M by setting $\rho(f) = \rho(|f|)$ for any $f \in M$.

given such a norm, let

$$\ell_\rho = \{f \in M : \rho(|f|) \angle \infty\},$$

and define

$$N = \{ f \in M : f = o \vee f = o, \text{a.e.} \},$$

let

$$L_\rho = \ell_\rho / N.$$

Then ρ induces a norm, also denoted by ρ, on L_ρ defined by

$$\rho(f+N) = \rho(|f|) \quad \forall f \in \ell_\rho.$$

Theorem 1.1.3 guarantees that any function $f \in L_\rho$ is finite μ-a.e. on Ω, and hence that L_ρ is a linear subspace of the linear space of all μ-a.e. finite functions in M. The resulting normed space is called a normed Köthe space. This space L_ρ is not necessarily norm complete.

1.1.4 Example: Let μ be the discrete measure in $X = \{1,2,3\ldots\}$ and for every $u = \{u(1), u(2),\ldots\} \in M^+$ let the function norm $\rho(u)$ be defined by

$$\rho(u) = \sum_{n=1}^{\infty} 2^{-n} u(n) + \lim_{n} \sup u(n).$$

To prove that L_ρ is not norm complete, let $v_1 = (1,0,0,\dots)$, $v_2 = (1,1,0,0,\dots)$, and so on . For $k > p$ we have $\rho(v_k - v_p) = \sum_{n=p+1}^{k} 2^{-n} \longrightarrow 0$ as $k, p \longrightarrow \infty$ so v_k is a Cauchy sequence. Assume the existence of a function $f \in L_\rho$ such that $\rho(f - v_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Let χ_{E_n} be the characteristic function of $E_n = \{n\}$; then $\rho\{(f - v_k)\chi_{E_n}\} \longrightarrow 0$ as $k \longrightarrow \infty$. Hence, since $\rho(\chi_{E_n}) = 2^{-n}$, we have $v_k \longrightarrow f(n)$ as $k \longrightarrow \infty$ for every n. This implies that $f(n) = 1$ for every n. But $\rho(f - v_k) = \sum_{n=k+1}^{\infty} 2^{-n} + 1 \longrightarrow 1$ as $k \longrightarrow \infty$. This contradicts $\rho(f - v_k) \longrightarrow 0$ as $k \longrightarrow \infty$ and hence L_ρ is not norm complete.

<u>**1.1.5 Definition:**</u> The function norm ρ is said to have the <u>Riesz-Fischer property</u> if, for any sequence $(u_n) \in L_\rho^+$ $(= M^+ \cap L_\rho)$ such that $\sum_{n=1}^{\infty} \rho(u_n) < \infty$, we have $\sum_{n=1}^{\infty} u_n \in L_\rho$, i.e., $\rho(\sum_{n=1}^{\infty} u_n) < \infty$.

<u>**1.1.6 Theorem:**</u> The normed linear space L_ρ, derived from the function norm ρ is a Banach space iff ρ has the Riesz-Fischer property.

For a proof of this result the reader is referred to theorem 2 of (19, 445).

<u>**1.1.7 Definition:**</u> The function seminorm ρ is said to have the <u>Fatou property</u>, whenever it follows from $0 \leq u_1 \leq u_2 \leq \dots\dots \longrightarrow u$, with all $u_n \in M^+$, that $\rho(u_n) \longrightarrow \rho(u)$

<u>**1.1.8 Theorem:**</u> If ρ is a function norm with the Fatou property, then ρ has the Riesz-Fischer property.

<u>**Proof:**</u> Assume that ρ is a function norm with the Fatou property, and let $u_n \in M^+$ for $n = 1,2,3,\dots\dots$, and $\sum_{n=1}^{\infty} \rho(u_n) < \infty$. We want to prove that $\sum_{n=1}^{\infty} u_n \in L\rho$, i.e. $\rho\,(\sum_{n=1}^{\infty} u_n) < \infty$. The functions $S_n = u_1 + u_2 + u_3 + \dots + u_n$, $(n = 1,2,\dots\dots)$ satisfy $0 \leq S_1 \leq S_2 \leq \dots\dots$, and $\rho(S_n) \leq \sum_{k=1}^{n} \rho(u_k)$, so $\lim_n \rho(S_n) < \infty$. Hence, since ρ has the Fatou property, $\lim_n \rho(S_n) = \rho(S) = \rho(\sum_{n=1}^{\infty} u_n) < \infty$. Q.E.D.

<u>**1.1.9 Definition:**</u> Given the function seminorm ρ, let $\rho^{(0)} = \rho$, and for any $u \in M^+$, let $\rho^{(n)}(u) = \sup\{\int uv\, d\mu : \rho^{(n-1)}(v) \leq 1,\ v \in M^+\}$ for $n = 1,2,\dots\dots$, where the integral denotes integration over the whole set

Ω. Instead of $\rho^{(1)}$, $\rho^{(2)}$, we swall write ρ', ρ''.

1.1.10 Theorem: For $n = 1,2$, $\rho^{(n)}$ is a function seminorm having the Fatou property.

Proof: It will be sufficient to present the proof for ρ', the proof for ρ'' follows by induction. For a fixed $v \in M^+$ satisfying $\rho(v) \leq 1$, it is evident that $\rho_v = \int_\Omega uv\, d\mu$ is a function seminorm. We show that ρ_v has the Fatou property. Let $u_n \in M^+$ $(n = 1, 2, \dots)$ be such that $o \leq u_1 \leq u_2 \leq \dots\dots \longrightarrow u$, and let $v \in M^+$ be such that $\rho(v) \leq 1$. Then v is finite a.e., and $o \leq u_1 v \leq u_2 v \leq \dots\dots \longrightarrow uv$, a.e. Using the dominated convergence theorem,

$$\lim_n \rho_v(u_n) = \lim_n \int u_n v\, d\mu = \int uv\, dv = \rho_v(u).$$

Then ρ_v has the Fatou property. We now show that $\rho'(u) = \sup\{\rho_v(u) : \rho(v) \leq 1,\ v \in M^+\}$ has the Fatou property. It is evident that ρ is a function seminorm. Let $o \leq u_1 \leq u_2 \leq \dots \longrightarrow u$ with all $u_n \in M^+$, and let $\rho'(u_n) \longrightarrow \alpha$. We have to show that $\rho'(u) = \alpha$. Since $\alpha = \lim_n \rho'(u_n) \leq \rho'(u)$, it is sufficient to show that $\alpha \geq \rho'(u)$, i.e. $\alpha > B$ for all $B < \rho'(u)$. Let $B < \rho'(u)$. Then there exists $v \in M^+$ with $\rho(v) \leq 1$ and such that $\rho_v(u) > B$. Since ρ_v has the Fatou property, then $\rho_v(u_1) \leq \rho_v(u_2) \leq \dots \longrightarrow \rho_v(u)$ for this particular v. Then there exist $n_o \in N$ such that $\rho_v(u_n) > B$ for $n \geq n_o$. It follows that $\rho'(u_n) > B$ for all $n \geq n_o$ and so

$$\alpha = \lim \rho'(u_n) > B.$$

Hence $\alpha = \rho'(u)$. Q.E.D.

1.1.11 Definition: The function seminorm ρ is said to be saturated if for any set E of positive measure there exists a subset $F \subset E$ such that $\mu(F) > o$ and $\rho(\chi_F) < \infty$.

1.1.12 Definition: Let ρ be a function seminorm. A set $E \subset \Omega$ such that $\mu(E) > o$ is called a ρ-purely infinite set if $\rho(\chi_E) = \infty$ and $\rho(\chi_F) = \infty$ for all $F \subset E$ with positive measure.

1.1.13 Theorem: Let ρ be a function seminorm.

a) (Hölder's inequality) If $\rho(u)$ and $\rho'(v)$ are finite, then

$$\int_\Omega uv\, d\mu \leq \rho(u).\rho'(v)$$

b) We have $\rho'' \leq \rho$.

Proof: *a*) For $0 < \rho(u) < \infty$. By defintion $\rho'(v) = \sup\{ \int_\Omega uv\, d\mu : \rho(u) \leq 1, u \in M^+\} < \infty$. Put $w = u/\rho(u)$. Then $\rho'(v) \geq \int wv\, d\mu = \int v.u / \rho(u)\, d\mu = 1/\rho(u) .\int uv\, d\mu$. Then $\int uv\, d\mu \leq \rho'(v).\rho(u)$. Now let $\rho(u) = 0$. if $u = 0$ a.e., it is clear that the inequality holds. Assume, therefore, that $E = \{ x : u(x) > 0 \}$ is of positive measure, and let $E_n = \{x : u(x) > n^{-1} \}$ for $n = 1,2,\ldots,$ Since $E_n \longrightarrow E$, there exists an index n_0 such that $\mu(E_n) > 0$ for all $n \geq n_0$. We will show that E_n is ρ'-purely infinite for all $n > n_0$. Let $n \geq n_0$, and let $F \subset E_{n_0}$ with $\mu(F) > 0$. Since $u(x) > n^{-1}$ on F, we have $n.u(x) \geq \chi_F$, so $\rho(\chi_F) \leq n\, \rho(u(x))\}$, then $\rho(\chi_F) = 0$. It follows that $\rho(k\, \chi_F) = 0$ for all $k = 1,2,\ldots,$ and

$$\rho'(\chi_F) = \sup \{ \int_F w\, d\mu : \rho(w) \leq 1,\ w \in M^+ \}$$

$$\geq \sup_{k=1,2,\cdots} k \int_F d\mu = \sup k\, \mu(F) = \infty$$

This shows that E_n is ρ'-purely infinite for $n \geq n_0$. But then $E = \{ x : u(x) > 0 \} = \bigcup_{n=1}^{\infty} E_n$ is also ρ'-purely infinite. Hence since $\rho'(v) < \infty$, we have $\rho' < \infty$, and $v = 0$ a.e. on E. For, if E is ρ'-purely infinite and $v \cdot \chi_E > \varepsilon$ on a set of positive measure, there exists a number $\varepsilon > 0$ such that $v.\chi_E > \varepsilon$ on a set of positive measure, i.e. $v > \varepsilon \cdot \chi_F$ for some $F \subset E$ satisfying $\mu(F) > 0$. But then $\rho'(\chi_F) < \rho'(v)/\varepsilon$, and that contradicts the fact that E is ρ'-purely infinite. It follows that $\int uv\, d\mu = \rho(u).\rho'(v)$.

b) It will be shown that $\rho''(u) \leq \rho(u)$ for every $u \in M^+$. Assume that $\rho(u) < \infty$. Then

$$\rho'' = \sup \{\int uv\, d\mu : \rho'(v) \leq 1,\ v \in M^+ \}.$$

$$\leq \sup \{ \rho(u).\rho'(v) : \rho'(v) \leq 1,\ v \in M^+ \}$$

$$= \rho(u) \quad (\text{ by Holder's inequality })$$

It follows that

$$\rho''(u) \leq \rho(u) < \infty.$$

If $\rho(u) = \infty$, then $u = \infty$ a.e. Hence,

$$\rho''(u) = \sup\{ \int uv \, d\mu \, : \, \rho'(v) \leq 1, \, v \in M^+\}$$
$$= \infty$$

Hence

$$\rho'' \leq \rho. \qquad \text{Q.E.D.}$$

1.1.14.Theorem: The following statements are equivalent

a) ρ is saturated, i.e., for any set E of positive measure there is a subset $F \subset E$ of positive measure such that $\rho(\chi_F)$ is finite.

b) For any sequence of measurable sets $Y_n \nearrow \Omega$ there exists a sequence of measurable sets $X_n \nearrow \Omega$ such that $X_n \subset Y_n$ and $\rho(\chi_{X_n}) < \infty$ for all n.

Proof: *a*) $\Rightarrow$ *b*) By hypothesis, ρ is saturated. Let Y_n $(1,2,\ldots)$ be a sequence of measurable subsets such that $Y_n \nearrow \Omega$ and $\mu(Y_n) < \infty$ for all n. Since Y_n is of positive measure, then there exists a subset $E \subset Y_n$ of positive measure such that $\rho(\chi_E) < \infty$ and $\mu(E) \leq \mu(Y_n) < \infty$. Let Γ be the collection of all subsets of E such that their characteristic functions are in $L\rho$. We set $\alpha = \sup \{ \mu(F) : F \in \Gamma \}$. There exists a sequence of sets $F_n \in \Gamma$ such that $\mu(F_n) \longrightarrow \alpha$, and by i) ~~in the definition of a function seminorm,~~ Without loss of generality we may assume that F_n is increasing. The set $F_\infty = \bigcup_{n=1}^{\infty} F_n$ satisfies $\mu(F_\infty) = \alpha$. If $\alpha < \mu(E)$, the set $E - F_\infty$ is of positive measure, and so, since ρ is saturated, there exists a subset $F_o \subset E - F_\infty$, $F_o \in \Gamma$, $\mu(F_o) > o$. It follows that $F_n + F_o \in \Gamma$ for all n, and $\mu(F_n + F_o) > \mu(F_n) + \mu(F_o) > \alpha$ for all n. This contradicts the definition of α. Hence

$$\sup \{\mu(F) : F \in \Gamma \} = \mu(E) = \alpha,$$

it follows that there exists a sequence of sets X_n $(n = 1,2,\ldots)$ such that $X_n \subset Y_n$ and $\rho(\chi_{X_n}) < \infty$, and $\mu(Y_n - X_n) < 1/n$ for all n. It may be assumed that X_n is increasing by i) in the definition of a function seminorm. The set $\Omega' = \sum_{n=1}^{\infty} X_n$ satisfies $Y_n - \Omega' \subset Y_n - X_n$, so $\mu(Y_n - \Omega')$ $< 1/n$ for all n. But $Y_n - \Omega' \nearrow \Omega - \Omega'$, so $\mu(\Omega - \Omega') = \lim \mu(Y_n - \Omega') =$ 0. Thus the sequence X_n has the required properties. Conversely, it is evident that the existence of at least one sequence $X_n \nearrow \Omega$ such that $\rho(\chi_{X_n}) < \infty$ implies that ρ is saturated. Q.E.D.

1.1.15- Theorem: The following statements are equivalent

i) ρ is saturated

ii) ρ' is a norm

iii) ρ'' is saturated.

Proof: Since $\rho'' \le \rho$, it follows that, if ρ is saturated, ρ'' is saturated as well. Conversely, assume that ρ'' is saturated; then, if E is any set of positif measure, there exists a set $F \subset E$ of positif measure such that $\rho''(\chi_F) < \infty$. Assume that ρ is not saturated, then there exists a set E of positive measure which is ρ-purely - infinite, so $u = 0$ on E for any function $u \in M^+$ such that $\rho(u) < \infty$. then

$$\rho'(k\,\chi_E) = \sup \{ \int_E ku \, d\mu : \rho(u) \le 1, u \in M^+\} = 0. \quad k=$$

$1,2,\ldots,$ then

$$\rho'(\infty.\chi_E) = 0,$$

it follows that, if $F \subset E$ with $\mu(F) > 0$, then

$$\rho''(\chi_F) = \sup \{ \int_F v \, d\mu : \rho'(v) \le 1, v \in M^+\} \ge \int_F k \, d\mu$$

$(k=1,2,..)$

Hence

$$\rho''(\chi_F) = \sup \{ \int_F v \, d\mu : \rho'(v) \le 1, v \in M^+\} \ge \int_F \infty \, d\mu = \infty$$

Therefore

$$\rho''(\chi_F) = \infty.$$

Hence ρ'' is not saturated. Now, if ρ is saturated, suppose that $\rho'(u) = 0$. We want to prove that $u = 0$ a.e. Then

$$\rho'(u) = \sup \{ \int uv \, d\mu : \rho(v) \le 1, v \in M^+\} = 0.$$

It follows that $\int uv \, d\mu = 0$ for every $v \in L\rho$ such that $\rho(v) \le 1$. Choose v such that $\rho(v) = 1$. Since ρ is saturated, there exists a set

E of positive measure such that $\rho(\chi_E) = 1$. Then

$$0 = \int u \, \chi_E \, d\mu = \int_E u \, d\mu \geq 0,$$

it follows that $u = 0$ on E. Hence $u = 0$ on any set F of positive measure such that $\rho(\chi_F) < \infty$. Since ρ is saturated, there exists a sequence $X_n \nearrow \Omega$ with $\mu(X_n) > 0$ and such that $\rho(\chi_{X_n}) < \infty$ for all n. Then $u = 0$ on X_n for all n. Since $X_n \nearrow \Omega$, $u = 0$ a.e. Hence ρ' is a norm. Conversely, assume that ρ' is a norm, but ρ is not saturated. Then there exists a ρ-purely infinite set. Hence $u = 0$ a.e. on E for any $u \in M^+$ such that $\rho(u) < \infty$, and so

$$\rho'(\chi_E) = \sup \{ \int_E u \, d\mu : \rho(u) \leq 1, \ u \in M^+ \} = 0$$

since $\mu(E) > 0$, this contradicts the fact that ρ' is a norm. Q.E.D.

1-2-The L^a_ρ space

<u>1.2.1- Definition:</u> Let ρ be a saturated function norm. The function $f \in L\rho$ is said to be of <u>absolutely continuous norm</u> whenever $\rho(f_n) \searrow 0$ for every sequence $f_n (n = 1, 2, ..)$ in $L\rho$ such that $|f| \geq f_1 \geq f_2 \geq .... \searrow 0$ pointwise a.e.on Ω.

<u>1.2.2- Definition:</u> Let ρ be a saturated function norm. <u>An order ideal</u> Y in $L\rho$ is a linear subspace of $L\rho$ with the property that if $f \in Y$, $g(x)$ is measurable and $|g(x)| \leq |f(x)|$ on Ω, then $g \in Y$.

<u>1.2.3-</u> Let ℓ^∞ denote the set of bounded functions in M and, for each $\phi \in \ell^\infty$, let M_ϕ be a bounded linear operator on $L\rho$ defined by $M_\phi \cdot f = \phi f$ $(f \in L\rho)$. For the boundedness of M_ϕ; since ϕ is bounded, there exists a constant $k > 0$ such that $|\phi| \leq k$, $\phi f \in M$ and $|\phi f| \leq k|f|$, then $\rho(\phi f) \leq k\rho(f)$. Hence, M_ϕ is bounded.

<u>1.2.4-Theorem:</u> A linear subspace Y of $L\rho$ is an ideal iff it is M_ϕ-invariant for every $\phi \in \ell^\infty$.

<u>Proof:</u> assume that Y is an ideal of $L\rho$. Let $\phi \in \ell^\infty$ and let $f \in Y$.

$|M_\phi f| = |\phi f| \leq k|f| \in Y$ since Y is a linear subspace. Then $M_\phi f = \phi f \in Y$. Conversely, assume that Y is a linear subspace M_ϕ-invariant for every $\phi \in \ell^\infty$. Then for every $f \in Y$, for every $\phi \in \ell^\infty$, $\phi f \in Y$. Let $g \in L\rho$ be such that $|g| \leq |f|$ a.e. There exists a measurable function θ with $|\theta| \leq 1$ and such that $g = \theta f$. But $\theta f \in Y$ for $M_\Theta f = \theta f$. Hence $g \in Y$. It follows that Y is an ideal. Q.E.D.

1.2.5-Theorem: The set $L^a{}_\rho$ of all absolutely continuous norm functions is a norm closed order ideal in $L\rho$.

Proof: We first prove that $L^a{}_\rho$ is an ideal. Let $f \in L^a{}_\rho$ and, let $g \in L\rho$ be such that $|g| \leq |f|$ a.e. We want to prove that g is of absolutely continuous norm. Then, let f_n ($n = 1,2,\ldots$) be a sequence in $L\rho$ such that $|g| \geq f_1 \geq f_2 \geq \ldots \searrow 0$ pointwise a.e. But $|f| \geq |g| \geq f_1 \geq f_2 \geq \ldots \searrow 0$ pointwise a.e., and f is of absolutely continuous norm. Hence, $\rho(f_n) \longrightarrow 0$ as $n \longrightarrow \infty$, and then $g \in L^a{}_\rho$. It follows that $L^a{}_\rho$ is an ideal in $L\rho$. It remains to prove that $L^a{}_\rho$ is norm closed. Let $f_n \in L^a{}_\rho$ for $n = 1,2,\ldots$, and $\rho(f-f_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Then given $\varepsilon > 0$, there exists an index N such that $\rho(f-f_N) < (1/2)\,\varepsilon$. Now, let E_n ($n = 1,2,\ldots$) be a decreasing sequence of sets with empty intersection. Since

$$\rho(f\cdot\chi_{E_n}) \leq \rho(f-f_N)\cdot\chi_{E_n} + \rho(f_N\cdot\chi_{E_n}) \leq (1/2)\cdot\varepsilon + \rho(f\cdot\chi_{E_n})$$

and since $\rho(f_N\,\chi_{E_n}) \searrow 0$ as $n \longrightarrow \infty$, we have $\rho(f_N\cdot\chi_{E_n}) < \varepsilon$ for all n sufficiently large. Next we prove that f is of absolutely continuous norm. Let $|f| \geq f_1 \geq f_2 \geq \ldots \searrow 0$ on Ω, and let $\varepsilon > 0$. Since ρ is saturated, there exists a sequence of measurable sets $X_n \nearrow \Omega$, $\mu(X_n) < \infty$ and $0 < \rho(\chi_{X_n}) < \infty$ for all n. Since $\Omega-X_n$ decreases to the empty set, there exists an index N such that $\rho(f\chi_{\Omega-X_N}) < (1/2)\cdot\varepsilon$ and so $\rho(f_n\cdot\chi_{\Omega-X_N}) < (1/2)\cdot\varepsilon$ for all n. Furthermore, since $\rho(f\cdot\chi_{E_n}) \searrow 0$ for every sequence E_n ($n = 1,2,\ldots$) of measurable subsets of Ω decreasing to a set of measure zero, then for every $\varepsilon > 0$, there exists a positive number δ and a positive integer N such that $\mu(E_n) < \delta$ implies that $\rho(f\cdot\chi_{E_n}) < 1/4\,.\varepsilon$. Let $\alpha =$

$\varepsilon(4\ \rho(\chi_{X_N}))^{-1}$ and set $E_n = \{x: f_n(x) \geq \alpha\} \cap X_N$ for $n = 1,2,\ldots.$ The sets E_n decrease to a set of measure zero, since $f_n \searrow o$. Hence $\mu(E_n) < \delta$ for $n \geq n_o$. We have $f_n(x) \leq \alpha$ on $X_N - E_{n_o}$ for $n \geq n_o$, and so

$$\rho(f_n \cdot \chi_{X_N}) \leq \rho(f_n \cdot \chi_{E_{n_o}}) + \rho(f_n \cdot \chi_{X_n - E_n})$$

$$\leq \rho(f \cdot \chi_{E_{n_o}}) + \alpha\ \rho(\chi_{X_n - E_n}) < (1/4)\ \varepsilon + (1/4)\ \varepsilon.$$

This shows that $\rho(f_n) \leq \rho(f_n \cdot \chi_{X_N}) + \rho(f_n \cdot \chi_{\Omega - X_N}) < \varepsilon$ for $n \geq n_o$, and so $\rho(f_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence f is of absolutely continiuous norm and so $L^a{}_\rho$ is norm closed. Q.E.D.

We conclude, from this theorem, that if ρ has the Fatou property, $L\rho$ is complete, and hence $L^a{}_\rho$ is norm complete (ρ saturated).

<u>1.2.6-</u> Since $L^a{}_\rho$ is an order ideal, it follows that, if $f \in L^a{}_\rho$, then the positive and the negative parts of Re f and Im f also belong to $L^a{}_\rho$.

Let 1 denote the function with constant value 1 a.e., and let Ω' be the set where it takes on the value 1 everywhere. Then, if $\rho(1) < \infty$ it follows that $\rho(\chi_{\Omega'}) < \infty$. Hence ρ is saturated.

Let L^∞ denote ℓ^∞ / N and let 1 be the coset in L^∞ containing the constant function with value 1.

<u>1.2.7-Lemma:</u> Let $1 \in L^a{}_\rho$. Then L^∞ is a dense subspace of $L^a{}_\rho$.

<u>Proof:</u> Since $L^a{}_\rho$ is an ideal containing the constant function, $L^a{}_\rho$ also contains L^∞. Fix $f \in L^a{}_\rho$ with $f \geq o$ a.e. There exists a sequence f_n of simple measurable functions such that $o \leq f_n \nearrow f$ pointwise a.e. We have

$$f \geq f - f_1 \geq f - f_2 \geq \ldots \searrow o.$$

pointwise a.e.on Ω, and so $\rho(f - f_n) \longrightarrow o$, since f is of absolutely continuous norm. Since $f_n \leq f$ for all n, this shows that every non-negative element of $L^a{}_\rho$ belongs to the closure of L^∞. By considering the positive and negative parts of the real and imaginary parts of an arbitrary element of $L^a{}_\rho$, which all belong to the closure of L^∞, and since L^∞ is a vector space, then the finite linear combinations of these parts also belong to the closure of L^{∞}. It follows immediately that L^{∞} is a dense subspace of $L^{a}{}_{\rho}$. Q.E.D.

1.3- Factorization theorem

1.3.1: The elementary observation that every L^1-function can be expressed as the product of two L^2-functions has important consequences in the study of self-adjoint spectral measures on Hilbert space. In this section, a generalization of this factorization theorem to certain normed Köthe spaces will be proved. See [14] and [13]

Preliminary lemmas are required.

1.3.2-Lemma: Let (Ω,Σ,μ) be a σ-finite measure space and let $\{f_n\}$ be a sequence in M^+, with each f_n finite and strictly positive a.e., such that either

$$\text{i) } \int \mathrm{Log}\{(1/2)(f_n+f_m)f_n^{-1/2}.f_m^{-1/2}\}\, d\mu \longrightarrow 0$$

or

$$\text{ii) } \int \{f_n f_m^{-1} + f_m f_n^{-1} - 2\}\, d\mu \longrightarrow 0$$

as $n,m \longrightarrow \infty$. Then $\{f_n\}$ has a subsequence which is convergent a.e.

Proof: We have $t + t^{-1} \geq 2$ for all $t > 0$. The convexity of the function t^p $(1 < p < \infty)$ for $0 < t < \infty$ shows that

$$1/2\ (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \geq \{(t + t^{-1}) / 2\}^{\frac{1}{2}}$$

and the monotonicity of the function Log shows that

$$\mathrm{Log}\ (1/2)\ (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \geq \mathrm{Log}\ \{(t + t^{-1})/ 2\}^{\frac{1}{2}}$$

Since $t + t^{-1} \geq 2$, then $(t + t^{-1})/2 \geq 1$ and therefore

$$\mathrm{Log}\ \{(t + t^{-1})/2\}^{\frac{1}{2}} = (1/2)\ \mathrm{Log}\ \left(\frac{t + t^{-1}}{2}\right) \geq 0.$$

It follows that the above integrands are non-negative a.e., and so, the integrals are well defined in $[0,\infty]$. We show that each of the hypothesis i) and ii) implies that the subsequence $\{\mathrm{Log}\ f_n\}$ is Cauchy in measure and hence has a subsequence convergent a.e. Then suppose that i) holds and let $\varepsilon > 0$. Since the functions

$$t \longrightarrow |\mathrm{Log}\, t| \quad \text{and} \quad t \longrightarrow \mathrm{Log}\,(1/2)(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$$

are strictly decreasing when $0 < t < 1$, take the value 0 when $t = 1$, and are strictly increasing for $t > 1$, there exists $\eta > 0$ such that, for $t > 0$,

$$|\mathrm{Log}\, t| \geq \varepsilon \quad \text{then} \quad \mathrm{Log}\,(1/2)\,(t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \geq \eta$$

Hence

$$\mu\{|\mathrm{Log}\,(f_n/f_m)| \geq \varepsilon\} \leq \mu\{\mathrm{Log}[(1/2)(f_n + f_m)(f_n^{-\frac{1}{2}} \,.\, f_m^{-\frac{1}{2}}] \geq \eta\}$$

$$\leq 1/\eta \int\{\mathrm{Log}\,[(1/2)(f_n + f_m)\, f_n^{-\frac{1}{2}}\, f_m^{-\frac{1}{2}}]\}d\mu \longrightarrow 0$$

$$\text{as } n,m \longrightarrow \infty$$

and so, $\{\mathrm{Log}\, f_m\}$ is Cauchy in measure. The required result follows by taking expenentials. Next, suppose that ii) holds, and let $\varepsilon > 0$. Since

$$t \longrightarrow |\mathrm{Log}\, t| \geq \varepsilon \quad \text{then} \quad t + t^{-1} - 2 \geq \eta$$

we have

$$\mu\{|\mathrm{Log}\,(f_n f_m^{-1})| \geq \varepsilon\} \leq \mu\{f_n f_m^{-1} + f_m f_n^{-1} - 2 \geq \eta\}$$

$$\leq 1/\eta \int \{f_n f_m^{-1} + f_m f_n^{-1} - 2\}d\mu \longrightarrow 0 \text{ as } n,m \longrightarrow \infty$$

and so, both i) and ii) imply that $\{\mathrm{Log} f_m\}$ is Cauchy in measure. Q.E.D.

Suppose throughout the rest of the section that ρ is a saturated function norm based on the probability measure space (Ω,Σ,μ)

__1.3.3-Lemma:__ Let $\sigma \in \Sigma$ and let $\phi \in M^+$ satisfy

$$\int_\sigma \mathrm{Log}(1 + h\phi)d\mu \leq \mu(\sigma).\rho(h)$$

for all h in L^+_ρ. Then $\rho'(\phi.\chi_\sigma) \leq \mu(\sigma)$

__Proof:__ Fix h in L^+_ρ. Since

$$\int_\sigma \mathrm{Log}\,(1 + h\phi)d\mu \leq \mu(\sigma)\cdot\rho(h) < \infty,$$

the function $h\phi$ is finite a.e. on σ. Hence, there exists an increasing sequence $\{\sigma_n\}$ in Σ such that

$$\sigma_n = \{t : (h\phi)(t) \leq n\} \quad (n \in \mathbb{N})$$

Then $\sigma_n \in \Sigma$ and $\cup\, \sigma_n = \sigma$. Let χ_{σ_n} denote the characteristic function of σ_n. Since

$$\mathrm{Log}(1 + t) \geq t - (1/2)\, t^2 \quad (t \geq 0)$$

then $$t \leq \mathrm{Log}(1 + t) + (1/2)\, t^2 \quad (t \geq 0)$$

it follows that, for $0 < s < \infty$ and $n \in N$,

$$\begin{aligned}\int s\, h\, \phi.\chi_{\sigma_n}\, d\mu = s \int h\phi.\chi_{\sigma_n}\, d\mu &\leq \int \mathrm{Log}(1 + s\, h\, \phi.\chi_{\sigma_n}) d\mu + \\ &\quad + (s^2/2) \int h^2\, \phi^2\, d\mu \\ &\leq \int_\sigma \mathrm{Log}(1 + s\, h\, \phi)\, d\mu + (s^2/2)\, n^2 \mu(\sigma_n) \\ &\leq s\, \mu(\sigma).\rho(h) + (1/2)\, s^2\, n^2.\end{aligned}$$

Then

$$\int h\phi.\chi_{\sigma_n}\, d\mu \leq \mu(\sigma).\rho(h) + (1/2) n^2 s.$$

Letting $s \to 0^+$, with n fixed, it follows that

$$\int h\phi.\chi_{\sigma_n}\, d\mu \leq \mu(\sigma)\rho(h) \quad (n \in N)$$

Since $h\phi.\chi_{\sigma_n} \nearrow h\phi\, \chi_\sigma$, Lebesgue's monotone convergence theorem gives

$$\lim_{n \to \infty} \int h\phi.\chi_{\sigma_n}\, d\mu = \int h\phi.\chi_\sigma\, d\mu \leq \mu(\sigma)\rho(h)$$

since, by Hölder's inequality

$$\int h\phi.\chi_\sigma\, d\mu \leq \rho(h).\rho'(\chi_\sigma\, .\phi),$$

it follows that

$$\rho'(\phi.\chi_\sigma) \leq \mu(\sigma)$$

as required. Q.E.D.

<u>**1.3.4 Lemma:**</u> Suppose that ρ has the Fatou property and let $\sigma \in \Sigma$ with $\chi_\sigma \in L\rho \cap L\rho'$. Then there exist $f \in L\rho$ and $g \in L\rho'$ such that

$$fg = \chi_\sigma, \quad f = g = 0 \text{ off } \sigma$$

and $\rho(f)\rho'(g) = \mu(\sigma)$.

Proof: Assume that $\mu(\sigma) > 0$, the result being trivial when $\mu(\sigma) = 0$. Given $f \in L\rho^+$ with $\rho(f) \leq 1$, we have

$$\int_\sigma \mathrm{Log}\, f\, d\mu \leq \int_\sigma f\, d\mu = \int f\, .\chi_\sigma\, d\mu \leq \rho(f).\rho'(\chi_\sigma) < \infty$$

Hence $\int_\sigma \mathrm{Log}\, f\, d\mu$ is well-defined in the range

$$-\infty \leq \int_\sigma \mathrm{Log}\, f\, d\mu \leq \rho'(\chi_\sigma)$$

for each f in $L\rho^+$ with $\rho(f) \leq 1$. Also by considering an appropriate multiple of χ_σ, it is seen that there exists $f \in L\rho^+$ with $\rho(f) \leq 1$ such that

$$\int_\sigma \operatorname{Log} f \, d\mu > -\infty$$

Defining

$$\alpha = \sup \{ \int_\sigma \operatorname{Log} f \, d\mu : f \in L\rho^+, \ \rho(f) \leq 1\},$$

we thus have

$$-\infty < \alpha \leq \rho'(\chi_\sigma) < \infty.$$

For each integer n, let $f_n \in L\rho^+$ with $\rho(f_n) \leq 1$ and

$$\int_\sigma \operatorname{Log} f_n \, d\mu \geq \alpha - 2^{-n}$$

Since $\rho\{(1/2)(f_n+f_m)\} \leq 1$, we have

$$\alpha \geq \int_\sigma \operatorname{Log}\{(1/2)(f_n+f_m)\} \, d\mu \geq \int_\sigma \operatorname{Log} (f_n^{1/2} f_m^{1/2}) \, d\mu$$

The last inequality follows from the convexity of the function $t \to t^p$ $(0 < t < \infty)$ which implies that

$$f_n^{1/2} f_m^{1/2} \leq (1/2)(f_n + f_m)$$

Therefore

$$\alpha \geq \int_\sigma \operatorname{Log}(f_n^{1/2} . f_m^{1/2}) d\mu = (1/2) \int_\sigma \{\operatorname{Log} f_m + \operatorname{Log} f_n\} d\mu$$
$$\geq \alpha - 2^{-n-1} - 2^{-m-1}$$

for all n, m $\in$ N.

Since Log f_m is integrable over σ, each f_n is strictly positive and finite a.e. on σ. Thus for all n, m $\in$ N,

$$0 \leq \int_\sigma \{ \operatorname{Log}(1/2)(f_n + f_m)(f_n^{-1/2} . f_m^{-1/2})\} d\mu =$$
$$= \int_\sigma \{\operatorname{Log}(1/2)(f_m + f_n)\} d\mu - \int_\sigma \{\operatorname{Log} f_m^{1/2} . f_n^{1/2} \} d\mu$$
$$\leq 2^{-n-1} + 2^{-m-1}$$

therefore, by lemma 1.3.2 (with the restiction of (Σ, μ) to σ as the underlying measure space), there is a subsequence $\{f_{nk}\}$ which converges a.e. on σ. Let $f \in M^+$ be defined by putting

$$f = \lim f_{nk} \quad \text{on } \sigma$$
$$= 0 \quad \text{off } \sigma$$

Then $0 \leq f \leq \liminf_k f_{nk}$ and so $\rho(f) \leq \liminf \rho(f_{nk}) \leq 1$, by theorem 3 of (19, 447) since ρ has the Fatou property. Fix $h \in L\rho^+$. Since

$\mu(\sigma) > 0$ and f_n is non-zero a.e. on σ, $\rho(f_n) > 0$ for all n. Hence $\rho(f_n+h) > 0$ for all n. The definition of α implies that

$$\begin{aligned}\alpha &\geq \int_\sigma \mathrm{Log}[(\rho(f_n + h))^{-1}(f_n + h)d\mu \\ &= \int_\sigma \mathrm{Log}\, f_n\, d\mu + \int_\sigma \mathrm{Log}(1+ hf_n^{-1})d\mu - \mu(\sigma)\mathrm{Log}\rho(f_n+h) \\ &\geq \alpha - 2^{-n} + \int_\sigma \mathrm{Log}(1 + h f_n^{-1})\, d\mu - \mu(\sigma)\, \mathrm{Log}\, \rho(f_n + h)\end{aligned}$$

for all n. Hence

$$\int_\sigma \mathrm{Log}\,(1+hf_n^{-1})d\mu \leq \mu(\sigma)\mathrm{Log}\,\rho(f_n+h) + 2^{-n} \leq \mu(\sigma)\rho(h) + 2^{-n}$$

the second inequality following from the fact that

$$\begin{aligned}\mathrm{Log}(\rho(f_n+h)d\mu &\leq \mathrm{Log}[\rho(f_n) + \rho(h)] \leq \mathrm{Log}[1 + \rho(h)] \\ &\leq \rho(h)\end{aligned}$$

Since each function $\mathrm{Log}\,(1 + hf_n^{-1})$ is non-negative a.e. on σ, Fatou's lemma implies that

$$\begin{aligned}\int_\sigma \mathrm{Log}\,(1 + hf^{-1})d\mu &= \int_\sigma \lim \mathrm{Log}\,(1 + hf_{nk}^{-1})d\mu \\ &\leq \liminf \int_\sigma \mathrm{Log}(1+ hf_{nk}^{-1})d\mu \leq \mu(\sigma).\rho(h).\end{aligned}$$

Since h is an arbitrary element of $L\overset{+}{\rho}$, lemma 1.3.3 implies that $g = \chi_\sigma f^{-1}$ satisfies $\rho'(g) \leq \mu(\sigma)$. Hence

$$\rho(f).\rho'(g) \leq \mu(\sigma).$$

Since $\rho(f) < \infty$, f is finite a.e. Also, taking $h = \chi_\sigma$ in the above inequality it is seen that $\mathrm{Log}(1+g)$ is integrable over σ and so g is finite a.e. on σ. Hence f and g are both finite, and therefore both are strictly positive a.e. on σ, and so $fg = \chi_\sigma$

$$\mu(\sigma) = \int \chi_\sigma\, d\mu = \int fg\, d\mu \leq \rho(f).\rho'(g)$$

Thus $\rho(f).\rho'(g) = \mu(\sigma)$ as required. Finelly, it is clear that both f and g vanish off σ. Q.E.D.

1.3.5 - Lemma: Let $\sigma \in \Sigma$, then, given $\varepsilon > 0$, there exist f in $L\rho^+$, g in $L^+\rho'$ and δ in Σ such that

i) $\delta \subset \sigma$ and $\mu(\sigma\backslash\delta) \leq \varepsilon$,

ii) $f.g = 1$ on δ,

iii) $\rho(f)\rho'(g) \leq (1+\varepsilon)\mu(\sigma)$,

iv) f and g vanish off δ.

Proof: Since ρ is a saturated function norm, by theorem 1.1.15, both ρ and ρ'' are saturated. By theorem 1.1.14, there exists an increasing sequence $\{\sigma_n\}$ in Σ such that $\cup\, \sigma_n = \Omega$ and $\chi_{\sigma n} \in L\rho \cap L\rho'$ for all n. By taking n large enough and considering $\sigma \cap \sigma_n$ in place of σ, it clearly suffices to prove the present result in the case when $\chi_\sigma \in L\rho \cap L\rho'$.

Accordingly, suppose that $\chi_\sigma \in L\rho \cap L\rho'$ and let $\varepsilon > 0$. Assume that $\mu(\sigma) > 0$, the required result being trivial when σ is μ-null. Since ρ has the Fatou property, $\rho = \rho''$ and $\rho''(\chi_\sigma) < \infty$ (see theorem 2 b) of (19, 457)). The preceding theorem gives the existence of f_0 in $L^+\rho''$ and g_0 in $L^+\rho'$ such that

$$f_0 . g_0 = \chi_\sigma, \quad \rho''(f_0).\rho'(g_0) = \mu(\sigma)$$

with each function vanishing off σ. By the arguments in (19, 451-471), there exists a sequence $\{h_n\}$ in M^+ such that

$$h_n \nearrow f_0, \quad \rho(h_n) \nearrow \rho''(f_0).$$

Noting that f_0 is non-zero a.e. on σ, it is seen that $h_n\, f_0^{-1} \nearrow 1$ on σ and therefore, by Egoroff's theorem there exists a measurable subset δ of σ and a positive integer k such that

$$\mu(\sigma \backslash \delta) \leq \varepsilon, \; h_k\, f_0^{-1} \geq (1+\varepsilon)^{-1} \text{ on } \delta.$$

let $f = h_k\, \chi_\delta$ and put $g = h_k^{-1}$ on δ, $g = 0$ on $\Omega \backslash \delta$. Then f and g vanish off δ and $f.g = \chi_\delta$. Also, since $\rho(h_n) \leq \rho''(f_0)$ for all n, then

$$\rho(f) \leq \rho''(f_0)$$

and since $g = f_0\, g_0\, h_k^{-1}$ on δ and g vanishes off δ, we have

$$\rho'(g) = \rho'(f_0 g_0\, h_k^{-1}) \leq \rho'\{g_0(1+\varepsilon)\} = (1+\varepsilon)\rho'(g_0)$$

Hence

$$\rho(f).\rho'(g) \leq (1+\varepsilon)\, \mu(\sigma).$$

Q.E.D.

1.3.6 Theorem: Let ρ be a saturated function norm based on the probability measure space (Ω, Σ, μ).

i) If ρ has the Fatou property, then there exist f in $L^+\rho$ and

g in $L^{+}\rho'$ such that

$$fg = 1, \ \rho(f)\,\rho'(g) \leq 1$$

ii) If ρ has the Riesz-Fischer property then, given $\varepsilon > 0$, there exist f in $L^{+}\rho$ and g in $L^{+}\rho'$ such that

$$f.g = \chi_{\delta}, \qquad \rho(f)\rho'(g) \leq 1 + \varepsilon .$$

iii) In general, given $\varepsilon > 0$, there exist f in $L^{+}\rho$, g in $L^{+}\rho'$ and δ in Σ with $\mu(\Omega/\delta) \leq \varepsilon$ such that $f.g = \chi_{\delta}$, $\rho(f)\,\rho'(g) \leq 1 + \varepsilon$

Proof: iii) Since ρ and ρ'' are saturated, then $1 \in L\rho'' \cap L\rho$. Take $\sigma = \Omega$ in lemma 1.3.5 and the proof follows immediately.

ii) Assume now that ρ has the Riesz-Fischer property and let $0 < \varepsilon < 1$. By repeated application of lemma 1.3.5, we can define sequences $\{\delta_n\}$ in Σ, $\{f_n\}$ in $L^{+}\rho$ and $\{g_n\}$ in $L^{+}\rho'$ inductively such that, for $n = 1,2,\ldots$

$$\text{i) } \delta_n \subset \Omega \setminus \bigcup_{j=0}^{n-1} \delta_j \qquad \text{ii) } \mu(\Omega \setminus \bigcup_{j=0}^{n-1} \delta_j) \leq \varepsilon^{2n}$$

$$\text{iii) } f_n g_n = \chi_{\delta_n} \qquad \text{iv) } \rho(f_n)\rho'(g_n) \leq \mu(\Omega \setminus \bigcup_{j=0}^{n-1} \delta_j)(1 + \varepsilon^{2n}).$$

v) f_n and g_n vanish off δ_n,

where δ_0 is the empty set. By ii) and iv),

$$\rho(f_n)\,\rho'(g_n) \leq \varepsilon^{2(n-1)}(1 + \varepsilon^{2n}), \qquad (n \in \mathbb{N}).$$

Hence, by multiplying f_n by suitable positive constant and g_n by its inverse, we can arrange that

$$\rho(f_n) \leq \varepsilon^{n-1}, \ \rho'(g_n) \leq \varepsilon^{n-1}(1 + \varepsilon^{2n}) \ (n \in \mathbb{N}).$$

Properties i) and ii) imply that the sets $\delta_1, \delta_2, \ldots,$ are mutually disjoint and that

$$\mu(\Omega \setminus \bigcup_{n=0}^{\infty} \delta_n) = 0.$$

We can therefore define f and g in M^{+} by putting $f = f_n$ and $g = g_n$ on δ_n for $n = 1,2,\ldots$. Then $fg = 1$ and

$$f = \sum_{n=1}^{\infty} f_n, \qquad g = \sum_{n=1}^{\infty} g_n .$$

The Riesz-Fischer property for ρ gives

$$\rho(f) \leq \sum_{n=1}^{\infty} \rho(f_n) \leq (1-\varepsilon)^{-1}$$

and, since ρ' has the Riesz-Fischer property we also have

$$\rho'(g) \leq \sum_{n=1}^{\infty} \rho'(g_n) \leq \sum_{n=1}^{\infty} \varepsilon^{n-1}(1 + \varepsilon^{2n})$$
$$\leq (1 + \varepsilon) \sum \varepsilon^{n-1}$$
$$= (1 + \varepsilon)(1 - \varepsilon)^{-1}$$

Thus

$$\rho(f)\rho'(g) \leq (1 + \varepsilon)(1-\varepsilon)^{-2}.$$

Since ε is arbitrary in the range $0<\varepsilon<1$, it follows that

$$\rho(f)\rho'(g) \leq (1+\varepsilon).$$

i) suppose that ρ has the Fatou property. Then it has the Riesz-Fischer property and so, by what has been proved ,there are sequences $\{f_n\}$ in $L^+\rho$ and $\{g_n\}$ in $L^+\rho'$ such that

$$f_n g_n = 1, \ \rho(f_n)\ \rho'(g_n) \leq 1+2^{-n}.$$

By appropriate scaling, we may assume that

$$\rho(f_n) = 1, \ \rho'(g_n) \leq 1+2^{-n}.$$

Note that f_n is strictly positive a.e.(since $fg = 1$) and is finite a.e. since $\rho(f_n) < \infty$ and ρ is a norm. Furthermore,

$$0 \leq \int \{\ f_n\ f_m^{-1} + f_m f_n^{-1} - 2\ \} d\mu$$
$$= \int \{f_n g_m + f_m g_n - 2\}\ d\mu$$
$$\leq \rho(f_n \rho'(g_m) + \rho(f_m)\rho'(g_n) - 2\mu(\Omega)$$
$$\leq 1 + 2^{-m} + 1 + 2^{-n} - 2 = 2^{-m} + 2^{-n},$$

for all $n,m \geq 1$. Therefore by lemma 1.3.2 there is a subsequence $\{f_{nk}\}$ converging a.e. on Ω, to f say, and the Fatou property implies that

$$\rho(f) \leq \liminf \rho(f_{nk}) = 1.$$

See theorem 3 of (19 , 447,448). Interpreting 0^{-1} as ∞ and $0.\infty$ as 0 in the usual way, we have

$$g_{nk} \longrightarrow g \text{ a.e.}$$

where $g = f^{-1}$, and the product fg equals the characteristic function of the set

$$\{\ \omega \in \Omega : f(\omega) > 0\} = \{\ \omega \in \Omega : g(\omega) < \infty\}.$$

Now, since ρ' has the Fatou property,

$$\rho'(g) \leq \liminf_{K} \rho'(g_{nK}) \leq 1.$$

This implies that g is finite a.e. since ρ' is a norm. Hence, $fg = 1$. Finally

$$1 = \int gf \, d\mu \leq \rho(f)\rho'(g) \leq 1.$$

Hence

$$\rho(f)\rho'(g) = 1. \qquad \text{Q.E.D.}$$

1.3.7- Theorem: Let ρ be a saturated function norm based on the σ-finite measure space (Ω,Σ,μ) and let $\phi \in L^1(\mu)$

i) If ρ has the Fatou property, there exist f in $L^+\rho$ and g in $L^+\rho'$ such that

$$\phi = fg, \quad \rho(f).\rho'(g) = \|\phi\|_1.$$

ii) If ρ has the Riesz-Fischer property then, given $\varepsilon > 0$, there exist f in $L\rho$ and g in $L\rho'$ such that

$$\phi = fg \;, \quad \rho(f)\rho'(g) \leq (1 + \varepsilon)\|\phi\|_1.$$

iii) In general, given $\varepsilon > 0$, there exist f in $L\rho$ and g in $L\rho'$ and δ in Σ such that

$$\phi \chi_\delta = fg, \quad \rho(f).\rho'(g) \leq (1+\varepsilon)\|\phi\|_1.$$

and

$$\int_{\Omega\setminus\delta} |\phi| \, d\mu \leq \varepsilon.$$

Proof: Assume that ϕ is non-zero, the required result being trivial when $\phi = 0$ a.e. By homogeneity, we may take $\|\phi\|_1 = 1$.
First, suppose that $\phi \geq 0$ a.e. and define the probability measure ν by $d\nu = \phi d\mu$ on (Ω,Σ) and let

$$\sigma_o = \{\omega \in \Omega : \phi(\omega) > 0 \}$$

Note that $\sigma \in \Sigma$ is ν-null iff $\sigma \cap \sigma_o$ is μ-null. Let

$$\tau(f) = \rho(\chi_{\sigma_o} f) \quad (f \in M^+)$$

It is easily verified that τ has the Riesz-Fischer property or the Fatou property if ρ has the corresponding property, and the associate norm τ' is calculated as follows. Let $g \in M^+$, if $f \in M^+$ with $\tau(f) = \rho(\chi_{\sigma_o} f) \leq 1$, then

$$\int gf\,d\nu = \int \phi\, fg\,d\mu \leq \rho'(\phi g).$$

Thus $\tau'(g) \leq \rho'(\phi g)$. Also if $f \in M^+$ with $\rho(f) \leq 1$, then $\bar{\rho}(f) = \rho(\chi_{\sigma_0} f)$, and so

$$\int \phi\, fg\,d\mu = \int gf\,d\nu \leq \tau(f).\tau'(g) \leq \tau'(g).$$

Thus $\rho'(\phi g) \leq \tau'(g)$, and hence $\rho'(\phi g) = \tau'(g)$ $(g \in M^+)$

i) Suppose that ρ has the Fatou property. Then so has τ. Also, τ is a saturated function norm based on (Ω,Σ,ν). By theorem 1.3.6 i) there exist f_0 in $L^+\tau$ and g_0 in $L^+\tau'$ such that

$$f_0 g_0 = 1, \quad \tau(f_0).\tau'(g_0) \approx 1.$$

If we take f to be a representative of f_0 such that $f = 0$ a.e. on $\Omega\backslash\sigma_0$, and $\phi g_0 = g$ we have

$$fg = \phi, \quad \rho(f)\rho'(g) = 1 = \|\phi\|_1.$$

ii) Assume that ρ has the Riesz-Fischer property, then so has τ. Given $\varepsilon > 0$, by theorem 1.3.6 ii) there exist f_0 in $L^+\rho$ and g_0 in $L^+\tau'$, such that

$$f_0 g_0 = 1, \; \tau(f_0).\tau'(g_0) \leq 1 + \varepsilon$$

By taking f to be a representative of f_0 which vanishes on the ν-null set $\Omega \setminus \sigma_0$ and $\phi g_0 = g$, we get

$$fg = \phi, \quad \rho(f).\rho'(g) \leq 1 + \varepsilon.$$

The third part of this theorem is obtained in a similar way.

Now, for $\phi \in L^1(\mu)$, $\phi \neq 0$ a.e. Then the theorem is proved for $|\phi| = \psi \geq 0$ a.e. Let θ be a unimodular function with $|\theta| = 1$. Then $\phi = \theta\psi$. Since ρ and ρ' are invariant under multiplication by unimodular functions the required results are easily deduced. Q.E.D.

<u>1.3.8 Definition:</u> The ideal $L^a{}_\rho$ is said to have <u>carrier Ω</u> if, given $\sigma \in \Sigma$, with $\mu(\sigma) > 0$, there exists $\tau \in \Sigma$ with $\tau \subset \sigma$ and $\mu(\tau) > 0$ such that $\chi_\tau \in L^a{}_\rho$.

<u>1.3.9 Theorem:</u> Let ρ be a function norm based on the σ-finite measure space (Ω,Σ,μ). Suppose that ρ has the Riesz-Fischer property and that the carrier of $L^a{}_\rho$ is Ω. Then, given $\phi \in L^1(\mu)$ and $\varepsilon > 0$, there exist f

in $L^a{}_\rho$ and g in $L\rho'$ such that

$$\Phi = fg, \ \rho(f)\rho'(g) \leq (1+\varepsilon)\ \|\Phi\|_1.$$

Proof: Given $f \in M^+$, let

$$\begin{aligned} \tau(f) &= \rho(f) \quad \text{if } f \in L^a{}_\rho \\ &= \infty \quad \text{otherwise} \end{aligned}$$

Then τ is a function norm. Since $L^a{}_\rho$ has carrier Ω, ρ is saturated and hence τ is saturated. Let $\{u_n\}$ be a sequence in $L^+{}_\tau$ such that $\sum_{n=1}^{\infty} \tau(u_n) < \infty$. Then $\sum_{n=1}^{\infty} \rho(u_n) < \infty$ and $\{u_n\}$ is a sequence in $L^a{}_\rho$. Put $S_n = \sum_{K=1}^{n} u_K$. Then $S_n \in L^a{}_\rho$ and hence $\tau(S_n) = \rho(S_n)$. It follows that

$$\lim_n \tau(S_n) = \lim_n \rho(S_n).$$

Since $L^a{}_\rho$ is closed in $L\rho$, there exists $u \in L^a{}_\rho$ such that $\lim \rho(S_n) = \rho(u)$

Hence

$$\tau(u) = \tau(\sum_{n=1}^{\infty} u_n) = \rho(u) < \infty.$$

Then τ has the Riesz-Fischer property.

Furthermore, let $f \in M^+$

$$\begin{aligned} \tau'(f) &= \sup\{\textstyle\int fg\, d\mu : g \in M^+, \tau(g) \leq 1\} \\ &= \sup\{\textstyle\int fg\, d\mu : g \in M^+, \rho(g) \leq 1\} = \rho'(f) \end{aligned}$$

The result now follows from theorem 1.3.7 ii) Q.E.D.

1.3.10 -Example: In general, it is not possible to take $\varepsilon = 0$ in theorem 1.3.7 ii). For instance, let μ be the counting measure on $\mathbb{N}$ and let

$$\begin{aligned} \rho(f) &= \|f\|_\infty \quad \text{if } f(n) \to 0 \text{ as } n \to \infty \\ &= \infty \quad \text{otherwise.} \end{aligned}$$

for f in M^+. Let E be a non-empty subset of $\mathbb{N}$, and let k be an element of E. Put $F = \{k\}$. Then

$$\rho(\chi_F) = \|\chi_F\| = 1$$

since $\lim \chi_F(k) = 0$. Then ρ is saturated. Let $\{f_n\}$ be a sequence in $L^+\rho$ such that $\sum_{n=1}^{\infty} \rho(f_n) < \infty$. Then $\sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$. It follows that

$$\|\Sigma_1^n f_k\|_\infty \leq \sum_1^n\|f_k\|_\infty \leq \sum_1^\infty\|f_k\|_\infty = \sum_1^\infty \rho(f_k)$$

Then $\lim_n \rho(\sum_1^n f_k) = \lim_{n\to\infty} \| \sum_1^n f_k\|_\infty \leq \sum_1^\infty \|f_k\| \angle \infty$.

Then $\rho(\Sigma_1^\infty f_n) \angle \infty$, and hence $\sum_1^\infty f_n \in L\rho$.

Therefore ρ has the Riesz-Fischer property.

$$\rho'(f) = \sum_1^\infty |f(n)| = \|f\|_1.$$

Let ϕ be a function in $L^1(\mu)$ with infinite support. By the definition of the elements of $L\rho$, clearly ϕ cannot be factorized as a product of an element of $L\rho$ and one of $L\rho'$. Also, in general, it is not possible to take $\varepsilon = 0$ in theorem 1.3.7 ii). If ϕ is the same as above and if δ is the set discribed in theorem 1.3.7 iii), then

$$\phi\, \chi_\delta = fg.$$

with $f \in L\rho$, and $g \in L\rho'$. Then δ is finite. But since $\int |\phi|\, d\mu = \sum_{N/\delta} |\phi(n)| = 0$. Then $\mathbb{N} \setminus \delta$ is finite as well, and this gives a contradiction. Q.E.D.

1.4 Uniqueness of factorization

Let ρ be a function seminorm based on the σ-finite measure space (Ω,Σ,μ), and let $\phi \in L^1(\mu)$. Define the support of ϕ, supp ϕ, by

$$\text{supp}\ \phi = \{\ \omega \in \Omega\ :\ \phi(\omega) \neq 0\}$$

This set being well defined up to a μ-null set.

1.4.1- Definition: A factorization $\phi = fg$ is called an exact ρ-factorization of ϕ if $f \in L\rho$, $g \in L\rho'$ and

$$\|\phi\|_1 = \rho(f).\rho'(g).$$

1.4.2 Remarks If $\phi = fg$ is an exact ρ-factorization and if f_o and g_o are defined to be

$$f_o = f \quad \text{on supp } \phi \quad ; \quad g_o = g \quad \text{on supp } \phi$$

$$\quad = 0 \quad \text{off supp } \phi \qquad\quad = 0 \quad \text{off supp } \phi$$

Then an exact ρ-factorization is still obtained.

$$\|\phi\|_1 = \int |\phi|\, d\mu = \int_{\mathrm{supp}\phi} |\phi|\, d\mu = \int_{\mathrm{supp}\phi} |fg|\, d\mu = \int |fg|\, d\mu$$
$$= \int |\phi|\, d\mu = \rho(f)\rho'(g) = \int |f_o g_o|\, d\mu = \int |f\chi_\phi| . |g\chi_\phi|\, d\mu$$
$$\leq \rho(f_o)\rho'(go)$$

where χ_ϕ is the charactiristic function of supp ϕ. But $\rho(f) \geq \rho(f_o)$ and $\rho'(g) \geq \rho'(g_o)$. Then $\rho(f).\rho'(g) = \rho(f_o).\rho'(g_o) = \|\phi\|_1$.

Also, given an exact factorization $\phi = fg$, others can be obtained by writing

$$\phi = (\alpha\ \theta\ f)(\alpha^{-1}\theta^{-1}g)$$

where α is any positive constant and θ any unimodular measurable function. Subject to the proviso that the factors vanish off supp ϕ, any two ρ-factorizations are related in the following way.

<u>**1.4.3- Theorem:**</u> Let $\phi = fg = hk$ be two ρ-factorizations of ϕ with f, g, h and k vanishing off supp ϕ. Then there exists a unimodular measurable function θ and a positive contant α such that

$$h = \alpha\ \theta f, \quad k = \alpha^{-1}\ \theta^{-1}\ g.$$

<u>**Proof:**</u> Assume $\|\phi\|_1 > 0$, the result being trivial when $\phi = 0$. Let $\alpha = \rho(h) / \rho(f)$ and define θ by

$$\begin{aligned} \theta &= \alpha^{-1}\ h / f \quad \text{on supp } \phi \\ &= 1 \quad \text{off supp } \phi. \end{aligned}$$

Then

$$h = \alpha\ f\ \theta\ , \quad k = \alpha^{-1}\ \theta^{-1}\ g\ .$$

Also

$$\int |\phi\ \theta|\, d\mu = \int |\phi\alpha^{-1}(h/f)|\, d\mu = \alpha^{-1}\int |\phi(h/f)|\, d\mu = \alpha^{-1}\int |gh|\, d\mu$$
$$\leq \alpha^{-1}\ \rho(h).\rho'(g) = \rho < f).\rho'(g) = \int |\phi|\, d\mu = \|\phi\|_1$$

and, similarly

$$\int |\phi\theta^{-1}|\, d\mu = \int |\phi\alpha\ (f/h)|\, d\mu = \int |hk(f/h).\alpha|\, d\mu$$
$$= \alpha\int |\ kf|\, d\mu \leq \alpha\ \rho(f).\rho'(k) = \rho(h).\rho'(k)$$
$$= \|\phi\|_1.$$

Hence

$$\int |\phi|\{|\theta| + |\theta|^{-1}\}d\mu \leq 2\|\phi\|_1.$$

Since $t + t^{-1} \geq 2$ for $t \geq 0$, with equality only when $t = 1$, it follows that $|\theta| = 1$ on supp ϕ as required. Q.E.D.

Chapter 2 Boolean algebras of projections and reflexive algebras of operators

2.1.0- Introduction: In the first section of this chapter, an attempt will be made to characterize the strong closure of the operator algebra generated by a Boolean algebra of projections *B*. It will next be proved that a bounded Boolean algebra of projections on a weakly complete Banach space X can be embedded in a σ-complete Boolean algebra of projections on X. Then, it will be shown that, given a cyclic Banach space X, there exists a normed Köthe space L_ρ, the norm of which has the Fatou property such that X is linearly homeomorphic to the subspace $L^a{}_\rho$ and such that, under this homeomorphism the projections in *B* correspond to operators consisting of multiplication by characteristic functions. In section 4 , it will be proved that, given a σ-complete Boolean algebra of projections on a complete Banach space X, the weakly closed subalgebra *A* of B(X) is reflexive. As a generalization of this result, it will be shown that, in fact, every weakly closed subalgebra *A* containing the identity operator is reflexive. As an immediate corollary of the reflexivity result, it will follow that every scalar-type spectral operator is reflexive. Finaly, the chapter is concluded by a representation theorem for a complete Boolean algebra of projections.

2.1.1- Definition: A Boolean algebra of projections in X, where X is a complex Banach space, is a set of projections in X which is a Boolean algebra under the operations A ∪ B and A ∩ B which has for its zero and unit elements the operators 0 and I in X. A Boolean algebra *B* of projections in a Banach space X is said to be complete (σ-complete) if each subset (sequence) of $\mathcal{B}$ has a greatest lower bound and a least upper bound in $\mathcal{B}$ and if for every set (sequence) B_0 in $\mathcal{B}$

$$\Big(\bigcup_{E\in B_0} E\Big)X = \overline{sp}\,\{\, EX : E \in B_0\} , \quad \Big(\bigcap_{E\in B_0} E\Big)X = \bigcap_{E\in B_0} EX .$$

where the supremum $A \cup B$ and the infimum $A \cap B$ of two commuting projections are defined as follows

$$A \cup B = A + B - AB$$

$$A \cap B = AB$$

The Boolean algebra $\mathcal{B}$ is said to be bounded if there is a constant M with

$$|E| \leq M \qquad (E \in \mathcal{B})$$

Let $B(X)$ be the Banach algebra of bounded linear operators in X.

2.1.2- Theorem: A linear functional on $B(X)$ is continuous with respect to the weak operator topology iff it is continuous with respect to the strong operator topology.

Proof: Since the strong operator topology is stronger than the weak operator topology, a functional continuous in the weak operator topology is continuous in the strong operator topology. Conversely, let F be a functional on $B(X)$ which is continuous in the strong topology, then, for $\varepsilon > 0$, there exist a finite subset $\{x_1, x_2, x_3, \ldots, x_n\}$ of X and a $\delta > 0$ such that

$$|Tx_i| < \delta,\ T \in B(X),\ i = 1,2,\ldots,n \text{ implies that } |F(T)| < \varepsilon$$

Consider the Banach space $X_n = X + X + X + \ldots + X$ of all n-tuples $\eta = (x_1, x_2, \ldots, x_n)$ with $x_i \in X$, $i = 1,2,\ldots n$. The norm in X_n is $\|\eta\| = \max\limits_{1\leq i\leq n} |x_i|$. Define $H : B(X) \longrightarrow X_n$ by $H(T) = (Tx_1, Tx_2, \ldots, Tx_n)$ and put $f(\eta) = F(T)$ if $\eta \in H(B(X))$ and $\eta = H(T)$. Since $\|H(T)\| = \|(Tx_1, Tx_2, \ldots, Tx_n)\| = \max\limits_{1\leq i\leq n} |Tx_i| < \delta$ implies that $|F(T)| < \varepsilon$, f is well defined and continuous on $H(B(X))$. By definition H is continuous in the strong operator topology. Then $H(B(X))$ is closed in the strong operator topology . Hence, by the Hahn-Banach theorem, f has a continuous linear extention f_1 defined on all of X_n. Clearly f_1 has the form

$$f_1(x_1,x_2,\dots,x_n) = \sum_{i=1}^{n} x_i^*(x_i),$$

where $x_i^* \in X, i = 1,2,\dots,n$. Conversely, $F(T) = f_1(H(T))$ has the form

$$F(T) = f_1(Tx_1,Tx_2,\dots,Tx_n) = \sum_{i=1}^{n} x_i^* Tx_i,$$

and obviously F is continuous in the weak operator topology. Q.E.D.

2.1.3- Corollary: A convex set has the same closure in the strong operator topology as it does in the weak operator topology.

Proof: This follows from the preceding theorem, since the closure of a convex set of B(X) in the strong operator topology is the same as its closure in the weak operator topology induced by the set of all strongly closed continuous linear functionals on B(X). Q.E.D.

2.1.4- Proposition: If a Boolean algebra of projections is σ-complete, it is bounded.

Proof: Suppose the Boolean algebra of projection is not bounded. A projection E is said to have property (α) if $\sup_{F\leq E}|F| = \infty$. Clearly, for any E in $\mathcal{B}$, either E or I-E has the property (α), for if both E and I-E do not have the property (α), then, since

$$EF + (I-E)F = F$$

and

$$\|F\| = \|EF + (I-E)F\| \leq \|EF\| + \|(I-E)F\| .$$

This implies that for any $F \in \mathcal{B}$

$$\|F\| \leq \sup_{G\leq E}\|G\| + \sup_{H\leq I-E}\|H\| < \infty,$$

which gives a contradiction. If E has the property (α) and $F \leq E$, then either F or E-F has the property (α). Let E_1 have the property (α). THen there is an $F_1 \leq E_1$ such that $|F_1| \geq 2 + 2|E_1|$. Let E_2 be a member of the pair F_1, E_1-F_1 with the property (α). The equality

$$|E_1-F_1| \geq |F_1| - |E_1| \geq 2 + |E_1|.$$

shows that $|E_2| \geq 2 + |E_1|$. An F_2 is now selected in E_2 such that $|F_2| \geq 3 + 2|E_2|$, and so on. The construction proceeds by induction. Now, for each n, let $G_n = E_n - E_{n+1}$. The projections G_n are disjoint and $\lim |G_n| = \infty$. By selecting subsequences from the sequence $\{G_n\}$, a collection of mutually disjoint sequences of projections $\{H_{jk}\}$, $j,k = 1,2,\ldots.,$ is obtained such that

$$\lim_{K\to\infty} |H_{jk}| = \infty, \quad j = 1,2,\ldots..$$

Define $P_j = \bigcup_{K=1}^{\infty} H_{jk}$, then $P_m . P_n = \bigcup_{K=1}^{\infty} H_{mk} \bigcup_{K=1}^{\infty} H_{nk} = \bigcup_{K=1}^{\infty} H_{mk} . H_{nk} = 0$ for $m \neq n$

The relation

$$\frac{|H_{mn}x|}{|x|} = \frac{|H_{mn}\ P_m x|}{|x|} = \frac{|H_{mn} \bigcup_{K=1}^{\infty} H_{mk}\ x|}{|x|}$$

$$= \frac{|P_m x|\ |H_{mn}\ P_m\ x|}{|P_m x|\ |x|} \leq \frac{|P_m|\ |H_{mn} P_m\ x|}{|P_m x|}, \quad P_m x \neq 0$$

shows that

$$|P_m|\ |H_{mn}|_{P_m X} \geq |H_{mn}|$$

Then

$$|H_{mn}|_{P_m X} \geq \frac{|H_{mn}|}{|P_m|},$$

where the left side is the norm of H_{mn} as an operator in $P_m X$. Consequently

$$\lim_{n\to\infty} |H_{mn}|_{P_m X} = \infty, \quad m = 1,2,\ldots$$

Select a subsequence $\{n_i\}$ and unit vectors x_i in $P_i X$ such that $|H_{in_i}\ x_i| \geq i$, $i = 1,2,\ldots.$ The projection $Q = \bigcup_{i=1}^{\infty} H_{in_i}$ cannot be bounded since

$$|Qx_i| = |Q\ P_i\ x_i| = |Q \bigcup_{j=1}^{\infty} H_{in_j}\ x_i| = |H_{in_{\overset{i}{\cancel{j}}}}\ x_i| \geq 1.$$

Then *B* contains a sequence which has not a least upper bound. This contradicts the σ-completeness of the Boolean algebra *B*. Q.E.D.

<u>2.1.5- Lemma:</u> Let *B* be a complete (σ-complete) Boolean algebra of projections in the Banach space X and let $\{E_\alpha\}$ be a monotone generalized sequence (a monotone sequence) in *B*. Then if $\{E_\alpha\}$ is increasing,

$$\lim_{\alpha\to\infty} E_\alpha x = (\bigcup_{\alpha} E_\alpha)x, \quad x \in X.$$

while if $\{E_\alpha\}$ is decreasing, then

$$\lim_{\alpha\to\infty} E_\alpha x = (\bigcap_{\alpha} E_\alpha)x, \quad x \in X.$$

Conversely, if every monotone increasing generalized sequence (monotone increasing sequence) of elements of a Boolean algebra of projections converges strongly to an element of B, then B is complete (σ-complete).

Proof: Let $E_o = \cup E_\alpha$ be the supremum of the increasing generalized sequence $\{E_\alpha\}$ and let $\varepsilon \searrow 0$ and $x \in X$ be arbitrary. Since $E_oX = \overline{\text{span}} E_\alpha X$ there is a vector $y = \sum_1^n z_i$ and indices α_i such that $E_{\alpha i} z_i = z_i$ and $|E_ox-y| \leq \varepsilon$. If $\alpha \geq \alpha_i$, $i = 1,2,\ldots,n$, then $E_{\alpha i}y = y = E_\alpha \sum_1^n z_i = \sum_1^n E_\alpha z_i = \sum_{i=1}^n z_i = y$. Thus , since $E_\alpha E_o = E_\alpha$ it follows that, for $\alpha \geq \alpha_i$, we have

$$|E_\alpha x - E_ox| \leq |E_\alpha x - y| + |E_ox - y| = |E_\alpha(E_ox - y)| +$$

$$|E_ox - y| \angle (M + 1)\varepsilon,$$

where $M = \sup\{ \|E\| : E \in B \}$. This proves that $\lim E_\alpha x = E_ox$.

Now assume that $\{E_\alpha\}_\alpha$ is a decreasing generalized sequence . Since

$$\bigcap_\alpha E_\alpha = (I - \bigcup_\alpha(I - E_\alpha))$$

Then $\{(I-E_\alpha)\}$ is increasing. If we put $E_o = \cup (I - E_\alpha)$ we obtain

$$|(I - E_\alpha)x - E_ox| \leq (M + 1)x$$

Clearly $I - E_o = \cap E_\alpha$. It follows from the above inequality that

$$|(I - E_\alpha)x - E_ox| = |(I - E_o)x - E_\alpha x| \leq (M + 1)\varepsilon.$$

The proof in the σ-complete case is quite s~~e~~imilar. To prove the converse, let $\{E\}$ be a subset of B and let $\{F_\alpha\}$ be the generalized sequence of finite unions of elements of $\{E\}$ ordered in the natural order of projections. A projection F is an upper bound for $\{E\}$ if and only if it is an upper bound for $\{F_\alpha\}$, and since

$$\{E_1 \cup \ldots \cup E_n\}X = \overline{\text{span}}\{ \bigcup_1^n E_iX\}$$

for any finite set of projections, to construct a least upper bound for $\{E\}$ in B, it suffices to make the corresponding construction for $\{F_\alpha\}$. Let $F_\infty = \lim_\alpha F_\alpha$ in the strong operator topology. Since

$$F_\alpha F_\beta = F_\alpha \quad \text{if } \beta \geq \alpha,$$

we have

$$F_\alpha F_\infty = \lim_{\beta\to\infty} F_\alpha F_\beta = F_\alpha.$$

Then $F_\infty \geq F_\alpha$ and thus F_∞ is an upper bound for $\{F_\alpha\}$. If F is another upper bound, then

$$FF_\infty = \lim_{\alpha\to\infty} FF_\alpha = \lim F_\alpha = F_\infty.$$

Then $F \geq F_\infty$. It follows that F_∞ is the least upper bound $\cup F_\alpha$ of $\{F_\alpha\}$, since

$$F_\infty X = \lim_{\alpha\to\infty} F_\alpha X \in \overline{\text{span}}\{ \cup F_\alpha X \}.$$

It follows that $F_\infty X \subseteq \overline{\text{span}}\{ \cup F_\alpha X \}$. On the other hand, since F_∞ is an upper bound for $\{F_\alpha\}$, then $F_\alpha X \subseteq F_\infty X$, for all α, which shows that

$$F_\infty X = \overline{\text{span}}\{ \bigcup_\alpha F_\alpha X \}$$

If every increasing generalized sequence $\{F_\alpha\}$ of projections in $\mathcal{B}$ converges strongly, then $\{(I-F_\alpha)\}$ is a monotone decreasing generalized sequence of projections in $\mathcal{B}$ which converges in $\mathcal{B}$ strongly. A greatest lower bound for $\{E\}$ in $\mathcal{B}$ may be constructed in a fashion analogous to that used for constructing the least upper bound. The proof for the σ-complete case follows semilar lines. Q.E.D.

2.1.6- Definition: A spectral measure in X is a hom~~e~~omorphic map of a Boolean algebra of sets into a Boolean algebra of projection operators in X which has the additional property that it maps the unit in its domain into the unit in its range. A spectral measure is said to be bounded if the norms of the projections in its range are bounded.

The structure space of a commutative Banach algebra is the set Λ of all maximal ideals in X with the topology determined by all the neighbourhoods of the form

$$N(\lambda_0,\varepsilon,A) = \{\lambda, : \lambda \in \Lambda,\ \overset{|x(\lambda)-x(\lambda_0)|}{\cancel{|X(\lambda_0)}} \angle\ \varepsilon,\ x \in A\}$$

where A is an arbitrary finite set of elements in X and $\varepsilon \backslash 0$. Furthermore, Λ is a compact Hausdorff space in this topology.

2.1.7- Lemma: The uniformly closed algebra of operators generated by a bounded Boolean algebra of projection operators is equivalent to the algebra of continuous functions on its own space of maximal ideals.

<u>**Proof:**</u> Let B be a Boolean algebra of projections, and $U_o(B)$ the set of all operators U of the form

$$\text{i)} \quad U = \sum_{1}^{n} \alpha_i E_i,$$

where

$$\text{ii)} \quad 0 \neq E_i \in B \ , \ \sum_{1}^{n} E_i = I \ , \ E_iE_j = 0, \ i \neq j, \ i,j = 1,2,\ldots,n.$$

Then $U_o(B)$ is clearly an algebra of operators containing B. Hence, if $U(B) = \overline{U_o(B)}$, then $U(B)$, is the uniformly closed algebra of operators generated by B.

Now let ψ be the canonical mapping of $U(B)$ into the ring of continuous functions on its own space Λ of maximal ideals. To show that ψ establishes an equivalence of $U(B)$ with all of $C(\Lambda)$, it is sufficient to show that $\psi^{-1} = s$ is bounded, and that $\psi(U(B))$ is dense in $C(\Lambda)$. To show that $\psi^{-1} = s$ is bounded, the existence of a finite constant M such that

$$|U| \leq 4 \ M \sup_{\lambda \in \Lambda} |U(\lambda)|$$

will be established. Since both sides of this inequality are continuous functions of U, it is sufficient to establish the inequality in $U_o(B)$. Thus, let U have the form given in equality i) with the auxiliary condition ii) satisfied. For $1 \leq k \leq n$ it is possible, since E_k is not quasi-nilpotent, to find a maximal ideal λ in Λ such that $E_k(\lambda) \neq 0$. Since $E_kE_j = 0$ for $j \neq k$, we have $E_j(\lambda) = 0$ for $j \neq k$ and since $E_k^2 = E_k(\lambda)$ we have $E_k(\lambda) = 1$. It follows that $U(\lambda) = \alpha_k$. On the other hand, it is clear that every maximal ideal λ in Λ has the property that $E_i(\lambda) = 1$ for one and only one integer $i \leq n$ and that for other integers $j \leq n$ we have $E_j(\lambda) = 0$. Thus

$$\sup_{\lambda \in \Lambda} |U(\lambda)| = \sup_{1 \leq i \leq n} |\alpha_i|$$

Let M be an upper bound for the norms $|E|$ of the projection E in B. It will be shown that

$$\text{i)} \ \sum_{1}^{n} |x^* E_i x| \leq 4M|x||x^*|, \ x \in X, \ x^* \in X^*.$$

To see this, note that

$$\begin{aligned}\Sigma\ |\mathrm{Re}(x^*E_ix)| &= \Sigma^+\mathrm{Re}(x^*E_ix)| - \Sigma^-\mathrm{Re}(x^*E_ix) \\ &= \mathrm{Re}(x^*\ \Sigma^+\ E_ix) - \mathrm{Re}(x^*\ \Sigma^-\ E_ix) \\ &\leq 2\ M\ |x|\,|x^*|.\end{aligned}$$

Where Σ^+ (Σ^-) represents the sum over those i for which $\mathrm{Re}(x^*E_ix) \geq 0$ (≤ 0). By treating the imaginary part in the same way and adding, it is seen that

$$\Sigma\ |x^*E_ix| \leq 4\ M|x|\,|x^*|,$$

so that i) is proved. It follows immediately that

$$\begin{aligned}|U| = |\sum_1^n \alpha_iE_i| &= \sup_{|x|,|x^*|\leq 1} |\ \Sigma\ \alpha_ix^*E_ix| \\ &\leq 4\ M\ \sup\ |\alpha_i| \leq \sup\ |U(\lambda)|\,4M\end{aligned}$$

Proving that ψ^{-1} is bounded.

To show that $\psi(U(B))$is dense in $C(\Lambda)$, it suffices to show, in view of the Weierstrass theorem, that $\psi(U_0(B))$ distinguishes between points of Λ (i.e. for every λ, $\gamma \in \Lambda$, there exists f in $\psi(U_0(B))$ with $f(\lambda) \neq f(\gamma)$) and contains the complex conjugate of each of its elements. Since $\psi(U_0(B))$ is dense in $\psi(U(B))$ and Λ is the space of maximal ideals of $U(B)$, it follows immediately that $\psi(U_0(B))$ distinguishes between elements of Λ. If λ is in Λ and E is in B, then $E^2(\lambda) = E(\lambda)$, so that $E(\lambda)$ is 0 or 1, and hence real. Thus.

$$\overline{\sum_1^n \alpha_i\ E_i\ (\lambda)} = \sum_1^n \bar{\alpha}_iE_i(\lambda)$$

Proving that ψ $(U_0(B))$, and hence $\psi(U(B))$, contains the conjugate of each of its elements. Q.E.D.

The following result can be found in [10, 2204].

2.1.8-Lemma: A Boolean algebra of projections in a Banach space X is σ-complete iff it is the range of a countably additive regular spectral measure defined on a σ-field of subsets of a compact space.

Proof: Let B be a σ-complete Boolean algebra of projections. Then B is bounded by proposition 2.1.4 and, by lemma 2.1.7 $U(B)$ is equivalent to the algebra of continuous functions on its own structure space Λ. For each x in X and x* in X^*, the number $x^*(s(f)x)$ depends linearly and continuously on x and upon the function f in $C(\Lambda)$ and hence, by the Riesz representation theorem $x^*(s(f)x)$ determines a unique regular countably additive measure defined on the Borel sets in Λ for which

$$\text{i) } x^*(s(f)x) = \int_\Lambda f(\lambda)\, \mu(d\lambda, x, x^*), \quad f \in C(\Lambda)$$

the left hand side of this equality is bilinear in x and x* and since the measure $\mu(\cdot, x, x^*)$ is uniquely determined by this equation, the number $\mu(\delta, x, x^*)$ is consequently bilinear in x and x*. Furthermore, since

$$|\mu(\delta,x,x^*)| \le |v(\Lambda,\mu(\cdot,x,x^*)| = \sup_{\|f\|\le 1} x^*(s(f)x)| \le M|x|\,|x^*|.$$

Clearly $\mu(\delta,x,x^*)$ is continuous in x and x*. Thus for each Borel set δ in Λ and each x* in X^*, by the Riesz representation theorem for bounded bilinear functionals on $C(\Lambda)$, there is a vector $A(\delta)x^*$ in X^* with

$$\mu(\delta,x,x^*) = \langle x, A(\delta)x^* \rangle, \quad (x \in X).$$

It follows from the bilinearity and the boundedness of μ that $A(\delta)x^*$ is linear and continuous in x*; that is, $A(\delta)$ exists as a bounded linear operator in X^*. Since every function f in $C(\Lambda)$ is bounded and Borel measurable, the integral $\int_\Lambda f(\lambda)\, A(d\lambda)$ exists and it is seen from i) that

$$\text{ii) } s^*(f) = \int_\Lambda f(\lambda)\, A(d\lambda), \quad f \in C(\Lambda).$$

It will now be shown that A is a spectral measure in X^*. By placing $f = 1$ in ii), we see that $I^* = A(\Lambda)$ and since $A(\delta)$ is additive in δ, that

$$A(\delta)' = I^* - A(\delta) = A(\Lambda) - A(\delta) = A(\delta').$$

To prove that A is a spectral measure, it will therefore suffice to show that $A(\delta \cap \sigma) = A(\delta)A(\sigma)$ for every pair δ, σ of Borel subsets of Λ. Now, for every pair f, g of functions in $C(\Lambda)$

$$\int_\Lambda f(\lambda) \int_\Lambda g(\mu)\, A(d\mu \cap d\lambda) = \int_\Lambda f(\lambda)\, g(\lambda)\, A(d\lambda) = s^*(fg)$$

$$= s^*(f)s^*(g)$$

$$=\int_\Lambda f(\lambda)\; s^*(g)\; A(d\lambda) = \int_\Lambda f(\lambda) \int_\Lambda g(\mu)\; A(d\mu)\; A(d\lambda)$$

Thus for x in X and x^* in X^*

$$\text{iii)}\quad \int_\Lambda f(\lambda) \int_\Lambda g(\mu) x\, A(d\mu \cap d\lambda) x^*$$

$$= \int_\Lambda f(\lambda) \int_\Lambda g(\mu)\; x\; A(d\mu)\; A(d\lambda) x^*.$$

Since the measures $xA(\,.\,)x^*$ are all regular, the integral(as a functional on $C(\Lambda)$) uniquely determines the regular measure. It follows from iii) that

$$\int_\Lambda g(\mu)\; x\; a(d\mu \cap \delta) x^* = \int_\Lambda g(\mu) x\; A(d\mu) A(\delta) x^*,$$

where $g \in C(\Lambda)$

This uniqueness argument may be repeated to conclude that

$$A(\delta \cap \sigma) = A(\delta) A(\sigma)$$

for every δ, σ of Borel subsets of Λ. Then A is spectral measure. It will now be shown that, because of the σ-completeness of B, the operator $A(e)$ is, for every Borel set e, the adjoint of a projection in B. Let Σ be the family of those Borel sets e in Λ for which $A(e) = E(e)^*$ for some projection $E(e)$ in B. Since B is a Boolean algebra, the family Σ is a field. To see that Σ is a σ-field, let $\{e_n\}$ be an increasing sequence of sets in Σ. Since

$$E(e_{n+1})^* E(e_n)^* = A(e_{n+1}\; e_n) = A(e_n) = A(e_n) = E(e_n)^*.$$

The sequence~~s~~ $\{E(e_n)\}$ is increasing, and since B is σ-complete it follows from proposition 2.1.4 that the strong limit $E = \underset{\frown}{\lim}\, E(e_n)$ exists and is a projection in B. Thus

$$\langle x, A(\bigcup_{n=1}^{\infty} e_n) x^* \rangle = \lim_{n \to \infty} \langle x, A(e_n) x^* \rangle$$

$$= \lim_{n \to \infty} \langle x, E(e_n)^* x^* \rangle = \langle x, E^* x^* \rangle, \quad (x \in X,\; x^* \in X^*),$$

which shows that the union $\bigcup_{n=1}^{\infty} e_n$ is in Σ and proves that Σ is a σ-field. Hence, to see that Σ contains all Borel sets it suffices to show that Σ contains every open set. Let e be an open subset of Λ, let x and x^* be elements of X, X^* respectively, and let $\varepsilon > 0$. Then, because of the regularity of $(x, A(\,.\,)x^*)$ there is a closed subset δ of e such that

$$\text{i)}\quad \left|(x, A(\delta_1) x^*) - (x, A(e) x^*)\right| \angle\; \varepsilon,$$

for every Borel set δ_1 with $\delta \subseteq \delta_1 \subseteq e$. To each E in B corresponds a continuous function f_E which is determined by the equation $E = s(f_E)$. Since $E^2 = E$ we have $f^2{}_E = f_E$, that is, f_E is the characteristic function of a set $\sigma(E)$. Since f_E is continuous, the set $\sigma(E)$ is both open and closed. The structure space Λ of $U(B)$ is homeomorphic to a closed subset of the Cartesian product $P\sigma(E)$ where E varies over B. The mapping $E \longleftrightarrow \sigma(E)$ is an isomorphism between B and the Boolean algebra of all open and closed subsets of Λ. Sets of the form $\{\lambda \,/\, |(s^{-1})(- A)(\lambda)| \angle \alpha\}$ with A in $U(B)$ form, by definition a subbasis for the topology of Λ. Since $\sigma(EF) = \sigma(E)\sigma(F)$, the sets $\sigma(E)$ actually form a basis for the topology of Λ. Consider now a closed set δ_1 with $\delta \subseteq \delta_1 \subseteq e$. Each point λ in δ_1 is interior to some set $\sigma(E) \subseteq e$. Since δ_1 is compact, a finite number of sets $\sigma(E_1)$, $\sigma(E_1), \ldots, \sigma(E_n)$ cover δ_1, and thus if E_0 is the union of the projections $E_1, E_2, \ldots, E_n$, then

$$\delta \subseteq \delta_1 \subseteq \bigcup_{i=1}^{n} \sigma(E_i) = \sigma(E_0) \subseteq e.$$

Thus if E is a projection in B with $\sigma(E_0) \subseteq \sigma(E) \subseteq e$, it follows from i) that

$$\text{ii)}\ A\,\sigma(E) = \int_{\sigma(E)} A(d\lambda) = \int_\Lambda \chi_{\sigma(E)}\, A(d\lambda)$$

since

$$A(\sigma(E)) = \int_{\sigma(E)} A(d\lambda) = \int_\Lambda \chi_{\sigma(E)}\, A(d\lambda) = s(\chi_{\sigma(E)}) = E.$$

It follows from ii) that

$$\text{iii)}\ |\overset{(x,\, A(\sigma(E))x^*)}{\cancel{(x^*, Ex)}} - (x,\ A(e)x^*)| \angle \varepsilon,$$

for every projection E in B with $\sigma(E_0) \subseteq \sigma(E) \subseteq e$. Thus if $\{E_h\}$ is the generalized sequence of projections E in B with $\sigma(E) \subseteq e$, directed in the increasing order of projections, it is seen from iii) that

$$(x,\ A(e)x^*) = \lim_h x^*(E_h x).$$

On the other hand, it follows from lemma 2.1.5 that

$$E_\infty = \bigcup_h E_h = \lim_h E_h,$$

in the strong operator topology. Thus $x^*(E_\infty x) = (x,\ A(e)x^*)$, which proves that e is in Σ and shows that Σ consists of all Borel sets in Λ. This means that for every Borel set e in Λ there is a projection $E(e)$ in B with $A(e) = E(e)^*$. Conversely, assume that B is the range of a countably additive spectral measure E which is defined on a σ-field Σ of subsets of a set Λ. Let E_n be an increasing sequence of elements of B. If $E_n = E(e_n)$ then

$$E(e_n - e_{n+1}) = E(e_n) - E(e_{n+1}e_n) = E_n - E_{n+1}E_n = E_n - E_n = 0,$$

and so, except for a set of E-measure zero, we have $e_n \subseteq e_{n+1}$. Since E_n is countably additive, we have the limit

$$\lim_n E_n = \lim_n E(e_n) = E\left(\bigcup_{n=1}^{\infty} e_n\right)$$

existing in the strong operator topology and therefore, it is in B. Then B is σ-complete. Q.E.D.

2.1.9- Corollary: The restriction to an invariant subspace of a σ-complete Boolean algebra of projections is σ-complete.

Proof: This follows immediately from the criterion for σ-completeness as given in lemma 2.1.8. Q.E.D.

2.1.10- Lemma: A strongly closed bounded Boolean algebra of projections in a weakly complete Banach space X is complete.

Proof: Assume that the strongly closed bounded Boolean algebra of projections is not complete. Then by lemma 2.1.5, there is a vector x and an increasing generalized sequence $\{E_\alpha\}$ of projections in B such that the limit $\lim_\alpha E_\alpha x$ does not exist. It follows that $\{E_\alpha x\}$ is not a generalized Cauchy sequence. Consequently there is an $\varepsilon > 0$, and, for each α, a $\beta(\alpha) \leq \alpha$ such that

$$|E_{\beta(\alpha)}x - E_\alpha x| > \varepsilon.$$

Let α_1 be arbitrary and, for $n \geq 1$, let $\alpha_{n+1} = \beta(\alpha_n)$ and $E_n = E_{\alpha n}$. Then $\{E_n\}$ is an increasing sequence of elements in B for which the limit $\lim E_n x$ does not exist. A contradiction of this statement will now be obtained by considering the uniformly closed operator algebra $U(B)$. From the preceding lemma, $U(B)$ is equivalent, under the isomorphism $s(f) \longleftrightarrow f$, to $C(\Lambda)$, the algebra of continuous functions on the structure space Λ of $U(B)$. The projection E_n is the image under s of a continuous function f_n, and since $E_n^2 = E_n$ we have $f_n^2 = f_n$ which shows that f_n assumes only the values zero and one and is therefore the characteristic function of a set e_n. Since f_n is continuous, then e_n is both open and closed. Furthermore, since $E_{n+1}E_n = E_n$ we have $f_{n+1}f_n = f_n$ and $e_{n+1}e_n = e_n$. Then the sequence $\{e_n\}$ is increasing. Thus, since

$$E_n = s(f_n) = \int_\Lambda f_n(\lambda)\, E(d\lambda) = \int_{e_n} E(d\lambda) = E(e_n),$$

then

$$\lim_{n\to\infty} E_n x = \lim_{n\to\infty} E(e_n)x$$

because of the countably additivity of the spectral measure E, this limit exists. This contradicts the fact that $\{E_n x\}$ does not converge. Q.E.D.

2.1.11- Lemma: A complete Boolean algebra of projections contains every projection in the weakly closed operator algebra it generates.

Proof: Since the strongly closed and the weakly closed operator algebras generated by the complete Boolean algebra B of projections in X are the same, to prove the theorem it will suffice to show that every projection F in the strong closure of B is in B. The proof that F is in B will be made by showing that to each pair (y,z) where $y \in M = FX$ and $z \in N = (I-F)X$, there can be associated a projection E_{yz} in B such that $E_{yz}y = y = Fy$, and $E_{yz}z = 0 = Fz$. For if this is granted, the projection

$$E = \cap_{y\in N} \cup_{y\in M} E_{yz}$$

is in B. If $x_o = y_o + z_o$, $y_o \in M$, $z_o \in N$, then $(\cup_{y\in M} E_{yz})y_o = y_o$ for each $z \in N$, and $(\cup_{y\in M} E_{yz_o})z_o = 0$. Thus $Ey_o = y_o$, $Ez_o = 0$ and $E = F$. The projection E_{yz} will now be constructed. Let y and z be fixed elements of M and N, respectively, and let ε be a given positive number. Since F is in the strongly closed operator algebra generated by B, there is an operator A having the form $A = \Sigma\, \alpha_i E_i$, where

$$0 \neq E_i \in B,\ \sum_{1}^{n} E_i = I,\ E_i E_j = 0,\ 1 \leq i \neq j \leq n$$

$$\|y - Ay\| \angle\ \varepsilon,\ |Az| \angle\ \varepsilon.$$

Since B is σ-complete, it is bounded. Let k be a bound for the norms of the projections in B; thus if

$$E = \sum_{|\alpha_i| > 1/2} E_i,$$

it follows that

$$|Ez| = |(\sum_{|\alpha_i|>1/2} E_i)z| = |(\sum_{|\alpha_i|>1/2} \alpha_i^{-1} E_i)Az| \leq 4k\varepsilon.$$

In the same way it is seen that

$$|y - Ey| = |\sum_{|\alpha_i| \leq 1/2} E_i y| = |\ (\sum_{|\alpha_i| \leq 1/2} (1 - \alpha_i)^{-1}\, E_i)(y - \overset{A}{\cancel{E}}y)| \leq 8k\varepsilon.$$

By choosing ε so that $8k\varepsilon \angle\ 2^{-n}$, we see that there is a sequence, $\{E_n\}$ in B with

$$\text{i) } |y - E_n y| \angle\ 2^{-n}\ ,\qquad |E_n z| \angle\ 2^{-n}$$

Let the projection $E_{n,m}$ be defined by the equation

$$E_{n,m} = \bigcup_{n}^{n+m} E_k.$$

Then $E_{n,m} \geq E_n$, $(I - E_{n,m}) \leq I - E_n$,and thus

$$(I - E_{n,m})y = (I - E_{n,m})(I - E_n)y$$

from which it follows that

$$\text{ii) } |y - E_{n,m}y| \leq k\ 2^{-n}.$$

Since $\qquad E_{n,m+1} = E_{n,m} + (I - E_{n,m})E_{n+m+1},$

it follows inductively from i) that

$$\text{iii) } \|E_{n,m}z\| \leq 2^{-n} + k.2^{-(n+1)} + \ldots + k.2^{-(n+m)} \angle\ k.2^{-(n+1)}.$$

Now the sequence $\{E_{n,m}\}$ is increasing in m and the sequence

$$\{\bigcup_{m=0}^{\infty} E_{n,m}\} = \{\bigcup_{k=n}^{\infty} E_k\}$$

is decreasing. Thus, the limit

$$E_{yz} = \bigcap_{n=0}^{\infty} \bigcup_{m=0}^{\infty} E_{n,m} = \lim_{n}\ \lim_{m}\ E_{n,m}$$

exists in the strong operator topology. It follows from ii) and iii) that this limit has the desired properties as expressed by the equation

$$E_{yz}y = y\ ;\ E_{yz} = 0.\ \text{Q.E.D.}$$

2.1.12- Corollary: A complete Boolean algebra of projections is strongly closed.

Proof: Since every operator in the strong closure of a bounded Boolean algebra of projections is itself a projection, and , since B is bounded, by lemma 2.1.10, this projection is in B. Hence B is stongly closed. Q.E.D.

2.2.complete Boolean algebras of projections

2.2.1- Definition: A topological space Ω is said to be extremally disconnected if the closure of every open set is open.

2.2.2- The Stone representation theorem: Every complete Boolean algebra B of projections in a Banach space X is isomorphic to the Boolean algebra of all open and closed sets of an extremally disconnected compact Hausdorff space.

Proof: If the Boolean algebra of projections in B contains only one element so that $I = 0$, the theorem is trivial. Consider the Boolean algebra $\Phi_2 = \{0,1\}$ and let Ω be the set of all non-zero algebra hom~~e~~omorphisms of B into the Boolean algebra Φ_2. For each $E \in B$, let $\Omega(E) = \{h \setminus h \in \Omega,\ h(E) = 1\}$. If $E \in B$, $E' = I - E$, and then $\Omega(E') = \{ h \setminus h \in \Omega,\ h(I-E) = h(I) - \overset{h}{\cancel{H}}(E) = 1\} = \{h \in \Omega,\ h(E) = 0\} = \Omega(E)'$, where the prime denotes the complement of $\Omega(E)$ in Ω. The relations

$\Omega(E \cap F) = \Omega(EF) = \{h \setminus h \in \Omega : h(EF) = h(E).h(F) = 1\} = \Omega(E)\cap\Omega(F)$.

$\Omega(E \cup F) = \Omega(E+F-EF)$ ~~[illegible]~~ $= \Omega(E) \cap \Omega(F)' \cup \Omega(E)' \cap \Omega(F)$

and

$$\Omega(E') = \Omega(I-E) = \Omega(I) - \Omega(E) = \Omega - \Omega(E) = \Omega(E)'$$

show that the mapping $E \longrightarrow \Omega(E)$ is a hom~~e~~omorphism of B into a collection of subsets of Ω.

Let $B_1 \subseteq \mathcal{B}$ have the following properties

a) if $E, F \in B_1$, then $EF \in B_1$

b) if $E \in B_1$, then $E \neq 0$.

We will show that there is a hom~~e~~omorpfism $h_1 : \mathcal{B} \to \Phi_2$ such that $h_1(E) = 1$ for $E \in B_1$. Let I_1 be the set of all elements of the form AE' with $A \in \mathcal{B}$, $E \in B_1$. To see that I_1 is an ideal, observe that:

$$\begin{aligned}(AE' + BF')(EF)' &= (AE' + BF')(I - EF)\\ &= (AE' + BF') - (AE' + BF')(EF).\\ &= AE' + BF' - (A - AE + B - BF)EF\\ &= (AE' + BF') - (AEF - AEF + BEF - BEF)\\ &= (AE' + BF')\end{aligned}$$

so that the sum of two elements in I_1 is also in I_1. Since it is clear that I_1 is invariant under multiplications by elements in $\mathcal{B}$, I_1 is an ideal. Further, I_1 is a proper ideal, for if $AE' = I$, then

$$I = AE' = AE'E' = IE' = E'$$

from which it follows that $E = 0$ contradicting *b*). Since $\mathcal{B}$ has a unit element, I_1 is contained in some maximal ideal M_1. Let h_1 be the hom~~e~~omorphism $h_1 : \mathcal{B} \to \mathcal{B} \setminus M_1$. Since M_1 is a maximal ideal, it follows that $\mathcal{B} \setminus M_1 = \Phi_2$. Now if $E \in B_1$, $E' = I - E \in I_1 \subseteq M_1$, and hence

$$h_1(E') = h_1(I - E) = h_1(I) - h_1(E) = 0,$$

showing that $h_1(I) = h_1(E) = 1$. This proves the statement made above.

The mapping $E \to \Omega(E)$ is a hom~~e~~omorphism. To show that it is an isomorphism, let $E_o \neq F_o$. We will prove that there exists an $h_o \in \Omega$ with $h_o(E_o) \neq h_o(F_o)$. If $E_o \neq F_o$ then either $E_o \neq E_oF_o$ or $F_o \neq E_oF_o$; we suppose that the former is valid. Let $B_o = E_o - E_oF_o$ so that $B_o \neq 0$. If $B_1 = \{B_o\}$, then by the preceding paragraph there is an $h_o \in \Omega$ with $h_o(B_o) = 1$. Now $B_oF_o = E_oF_o - E_oF_o = 0$, and so ~~$h_o(F_o)$~~ 0 $= h_o(B_oF_o) = h_o(B_o)h_o(F_o) = h_o(\overset{F_o}{\cancel{0}})$. Also, $h_o(B_o) = h_o(E_o - E_oF_o) = 1 = h_o(E_o) - h_o(E_o)h_o(F_o) = h_o(E_o) = 1 \neq h_o(F_o) = 0$. This proves that $\mathcal{B}$ is isomorphic to a Boolean algebra of subsets of Ω. It remains to prove that Ω may be topologized in such a way that it becomes a totally disconnected compact Hausdorff space (totally disconnected if its topology has a base consisting of sets which are both open and closed) in which the sets $\Omega(E)$, $E \in \mathcal{B}$, are precisely the collection of subsets of Ω which are both open and closed. Then, the completeness of the Boolean algebra $\mathcal{B}$ guarantees that the representation space is extremally disconnected. We have seen that $\Omega(EF) = \Omega(E) \cap \Omega(F)$ and $\Omega(I) = \Omega = \{ h \setminus h \in \Omega : h(I) = 1\}$. Then the collection $\{\Omega(E) : E \in \mathcal{B}\}$ is a base for a topology. Since $(\Omega(E))' = \Omega(I - E)$, each set in the base is both open and closed. So Ω is totally disconnected. To prove that Ω is compact in this topology, observe that since each closed set in Ω is the intersection of sets in $\{\Omega(E) : E \in \mathcal{B}\}$, it is sufficient to prove that if A_1 is a subset of $\mathcal{B}$ such that $\bigcap_{i=1}^{n} \Omega(E_i) \neq \Phi$, then $\bigcap\{E : E \in A_1\} \neq \emptyset$. Otherwise stated, the compactness of a topological space is equivalent to the fact that every family of closed sets with the finite intersection property has a non-void intersection. If B_1 is the set of all finite products of elements in A_1, then B_1 evidently satisfies condition *a*). For condition *b*), if $E \in B_1$, it is clear that $E \neq 0$, for if $E = 0$, $\Omega(E) = \Omega(0) = \Phi$ which contradicts the hypothesis of the finite intesection property. Hence there exists $h_1 \in \Omega$ with $h_1(E) = 1$ with $h_1 \in A_1$; so that h_1 is in each $\Omega(E)$, $E \in A_1$. Therefore Ω is compact.

Next, if G is any set in Ω which is both open and closed in Ω then, since G is open we have $G = \bigcup_\alpha \Omega(E_\alpha)$, since G is closed and hence is compact in Ω, then a finite covering $G = \bigcup_{i=1}^{n} \Omega(E_i) = (\bigcap_{i=1}^{n} \Omega(E_i'))' = \Omega(E_1'E_2'\ldots E_n')' = \Omega((E_1'E_2'\ldots E_n')')$ can be extracted. Thus every open and closed set in Ω has the form $\Omega(E)$ for some $E \in \mathcal{B}$.

Finally, if G is an open set in Ω, then $G = \bigcup_{E_0 \in B_0} \Omega(E_0)$, where B_0 is in $\mathcal{B}$. From the completeness of $\mathcal{B}$ follows the existence of a least upper bound for B_0 in $\mathcal{B}$, E say. therefore $\Omega(E)$ is the smallest closed set containing $\cup\ \Omega(E_o)$. Hence $\Omega(E) = \overline{\bigcup_{E_o \in \mathcal{B}_o} \Omega(E_o)} = \bar{G}$. From the preceding, $\Omega(E)$ is open. It follows that the closure of any open set is open. Hence Ω is extremally disconnected. Q.E.D.

2.2.3- Proposition: Let $\mathcal{B}$ be a bounded Boolean algebra of projections on X and let A be the closed subalgebra of B(X) generated by $\mathcal{B}$. Then, if Ω is the Stone representation space of $\mathcal{B}$, there is a bicontinuous algebra isomorphism ψ from $C(\Omega)$ onto A.

Proof: Let $E(\tau)$ be the projection corresponding to the open and closed subset τ of Ω under the isomorphism of the Stone representation theorem. Suppose that M satisfies:

$$\|E(\tau)\| \leq M < \infty \qquad (\tau \text{ open and closed in } \Omega\ )$$

Let $\tau_1, \tau_2, \ldots, \tau_n$ be a finite pairwise disjoint non-empty open and closed subsets of Ω. Define

$$\psi\left(\sum_{r=1}^{n} \alpha_r\, \chi_{\tau r}\right) = \sum_{r=1}^{n} \alpha_r\, E(\tau_r).$$

From the proof of lemma 2.1.7 it follows that

$$\sup\ \{\ |\alpha_i|,\ 1 \leq i \leq n\ \} \leq \left\|\sum_{i=1}^{n} \alpha_i E(\tau_i)\right\|$$
$$\leq 4M\sup\ \{\ |\alpha_i|,\ 1 \leq i \leq n\}$$

Since Ω is totally disconnected, by the Stone Weierstrass theorem, the set of all finite sums of all characteristic functions of open and closed sets is dense in $C(\Omega)$. Hence ψ can be extended to all of $C(\Omega)$ by continuity and the proof is complete. Q.E.D.

2.2.4- Theorem: A bounded Boolean algebra $\mathcal{B}$ of projections on a weakly complete Banach space X can be embedded in a σ-complete Boolean algebra of projections on X.

Proof: Let Ω be the Stone representation space of $\mathcal{B}$. By the preceding proposition, there exists an algebra isomorphism ψ from $C(\Omega)$ onto A, the closed subalgebra of B(X) generated by $\mathcal{B}$. From the proof of lemma 2.1.8, this isomorphism has the form

$$\psi(f)^* = \int_\Omega f(\lambda)\ E(d\lambda), \quad f \in C(\Omega).$$

where E is a spectral measure in X^*, the dual space of X, defined on the Borel sets in Ω. Let Σ be the algebra of all Borel sets in Ω and let Σ_0 be the class of sets in Σ such that there is a spectral measure F defined by $F(e)^* = E(e)$ for every e in Σ_0. Now we show that Σ_0 is a field. Let $e_1, e_2 \in \Sigma_0$

$$\begin{aligned} E(e_1 \cup e_2) &= E(e_1) + E(e_2) - E(e_1)E(e_2) \\ &= F(e_1)^* + F(e_2)^* - F(e_1)^*F(e_2)^* \\ &= F(e_1 \cup e_2)^* \end{aligned}$$

and $\quad E(e_1 \cap e_2) = E(e_1)\, E(e_2) = F(e_1)^*F(e_2)^* = F(e_1 \cap e_2)^*$

then $e_1 \cap e_2$, $e_1 \cup e_2$, Φ(the empty set) $\in \Sigma_0$.

Let $e_1 \in \Sigma_0$ $\qquad E(e_1') = I - E(e_1) = I - F(e_1)^* = F(e_1')^*$

then $e_1' \in \Sigma_0$ and hence $\Omega \in \Sigma_0$. It follows that Σ_0 is a field. To see that Σ_0 is a σ-field, let $\{e_n\}$ be a sequence of sets in Σ_0 then, for $x \in X$, $y \in X^*$

$$\langle x, E(\bigcup_1^n e_i)y\rangle = \langle x, F(\bigcup_1^n e_i)^*y\rangle = \langle F(\bigcup_1^n e_i)x, y\rangle$$

Since E is a countably additive spectral measure in X^*, it follows that $\langle x, E(\bigcup_1^n e_i)y\rangle$ converges when $n \longrightarrow \infty$. Hence $\{\langle x, E(\bigcup_1^n e_i)y\rangle\}$ is a Cauchy sequence. It follows that $\langle F(\bigcup_{n=1}^{m} e_n)x, y\rangle$ is a Cauchy sequence. Since X is weakly complete, $F(\bigcup_1^m e_n)x$ is weakly convergent. We now show that it converges strongly. Assume that the sequence $\{F(\bigcup_1^m e_n)\}$ converges weakly to a projection F in B, but for some x_0, the sequence $F(\bigcup_1^m e_i)x_0$ does not converge. Hence $F(\bigcup_1^m e_n)x_0$ is not a Cauchy sequence. Consequently, there is an $\varepsilon > 0$ and $m,n \in N^*$, $m \geq n$ such that

$$|F(\bigcup_1^m e_i)x_0 - F(\bigcup_1^m e_i)x_0| > \varepsilon,$$

then $F(\bigcup_1^n e_i)$ is an increasing sequence of elements of B for which the limit $\lim\limits_n F(\bigcup_1^n e_i)x_0$ does not exist. A contradiction of this statement can now be obtained by following the similar lines in the proof of lemma 2.1.10. Then the sequence $\{F(\bigcup_1^n e_i)\}$ is convergent in the strong operator topology, which shows that $\bigcup_1^\infty e_i$ is in Σ_0 and proves that Σ_0 is a σ-field. Hence $\{F(e), e \in \Sigma_0\}$ is a σ-complete Boolean algebra of projections in $B(X)$ and $\mathcal{B}$ is embedded in it. Q.E.D.

See [15].

2.3 Cyclic Banach spaces

2.3.1- Definition: Let λ, μ be countably additive set functions defined on a σ-field Σ, and let λ be finite. Then λ is said to be μ-continuous if and only if $v(\mu,E) = 0$ implies $\lambda(E) = 0$.

The function λ is said to be μ-singular if there exists a set E_0 of Σ such that

$$v(\mu,E_0) = 0,\ \lambda(E) = \lambda(E \cap E_0)\ ,\ E \in \Sigma.$$

By the Lebesgue decomposition theorem (see theorem 14 of [8, 132]) every finite countably additive measure λ defined on Σ is uniquely representable as a sum $\lambda = \alpha + \beta$, where α is μ-continuous and β is μ-singular.

Let g be an integrable function with respect to μ; then for $E \in \Sigma$ define

$$G(E) = \int_E g(s)\ \mu(ds).$$

then G is additive on Σ and has a total variation

$$v(G,\ E) = \int_E |g(s)|\ v(\mu,ds),\quad E \in \Sigma.$$

For a proof of this result, the reader is referred to theorem 20 of [8, 114].

2.3.2- Proposition: Let $\mathcal{B}$ be a σ-complete Boolean algebra of projections in a Banach space X. Then for each x_0 in X, there exists a linear functional x_0^* in X^* with the properties

i) $x_0^*\ E\ x_0 \geq 0\ ,\ E \in \mathcal{B}.$

ii) if $x_0^*\ E\ x_0 = 0$ for some E in $\mathcal{B}$, then also $Ex_0 = 0$.

Proof: By lemma 2.1.8 , $\mathcal{B}$ is the range of a countably additive spectral measure E defined on a σ-field Σ. First, assume that $X = \text{span}\{Ex_0 \ / \ E \in \mathcal{B}\}$. For every functional y^* ~~in~~ X^*, let the measure μ_{y^*} be defined on Σ by the equation

$$\mu_{y^*}(e) = y^* E(e)x_o, \quad e \in \Sigma,$$

and call a set δ in Σ a y^*-carrier if $\mu_{y^*}(\delta) = y^*E(\delta)x_o \neq 0$ and if every measurable subset e of δ upon which the total variation $v(\mu_{y^*},e) = 0$ has $E(e) = 0$. With this definition, a y^*-carrier may have proper subsets which are y^*-carriers. An application of Zorn's lemma yields a maximal family $\{\delta_\alpha\}$ of disjoint sets each of which is a y_α^*-carrier for some y_α^* in X^*. It will first be observed that $\{\delta_\alpha\}$ is at most denumerable. For, since the spectral measure E is strongly countably additive on Σ, every series $\Sigma_\alpha E(\delta_\alpha)x_0$ of a countable number of terms converges and hence contains at most a finite number of terms whose norms exceed a given positive number. Since

$$0 \neq \mu_{y^*}(\delta_\alpha) = y^* E(\delta_\alpha)x_0$$

for all α, it follows that the sequence $\{\delta_\alpha\}$ is at most countable and it will therefore be written as $\{\delta_n\}$ in what follows. Let $\Delta = \cup \delta_n$, so that the complementary set Δ' contains no carrier. It will next be shown that $E(\Delta') = 0$. If $E(\Delta') \neq 0$, it follows readily from the fact that $\{E(e)x_0 \ / \ e \in \mathcal{B}\}$ spans X, that $E(\Delta')x_0 \neq 0$ and hence, for some functional y_0^*, that $y_0^*E(\Delta')x_0 \neq 0$. Let $E(\Delta')^*y_0^* = y^*$ so that μ_{y^*} vanishes on subsets of Δ, and thus the total variation $v(\mu_{y^*},\Delta) = 0$. From the measure theory follows the existence of a finite positive measure ν such that

$$\nu(e) \leq \sup\left| \sum_i^n \alpha_i E(e_i) \right|, \quad e \in \Sigma,$$

where the supremum is taken over all finite collections of scalars with $|\alpha_i| \leq 1$ and all partitions of e into a finite family of disjoint sets in Σ. Then the vector measure $E(\cdot)x_0$ is ν-continuous. The measure ν cannot be μ_{y^*}-singular, for if it were there would be a set e in Σ with $v(\mu_{y^*},e) = 0$ and $\nu(e') = 0$, in which case $E(e')x_0 = 0$,

$v(\mu_{y^*},e') = 0$. It would follow that $v(\mu_{y^*},\Omega) = v(\mu_{y^*},e) + v(\mu_{y^*},e') = 0$ which contradicts the fact that $\mu_{y^*} \neq 0$. It follows from the Lebesgue decomposition theorem that there is a set e_1 with $\mu_{y^*}(e_1) \neq 0$ and such that ν, and hence E vanishes on any subset δ of e_1 upon which the variation $v(\mu_{y^*},\delta) = 0$. If $\delta = e_1 \cap \Delta'$ we have $\mu_{y^*}(\delta) = \mu_{y^*}(e_1) \neq 0$ and at the same time $\delta_1 \subseteq \delta$ and $v(\mu_{y^*},\delta_1) = 0$ imply $E(\delta_1) = 0$. Thus δ is a y^*-carrier contained in Δ' which contradicts the fact that Δ' contains no carrier. Hence $E(\Delta') = 0$.

Let y_n^* be a functional such that δ_n is a y_n^*-carrier. Consider now any measure of the form

$$\mu(e) = \sum_{n=1}^{\infty} c_n \, v(\mu_{y_n^*},e), \quad e \in \Sigma, \qquad (\alpha)$$

where $c_n \searrow 0$, $n = 1,2,\ldots.$ If $\mu(e) = 0$, then $v(\mu_{y_n^*},e) = 0$, for all n and thus $E(e \cap \delta_n) = 0 = E(e \cap \Delta\ )$. Since $E(\Delta') = 0$ we have

$$E(e) = E(e \cap \Delta) + E(e \cap \Delta') = 0.$$

Since the total variation $v(\mu_{y_n^*},\ .)$ is $\mu_{y_n^*}$-continuous, the Radon-Nikodým theorem yields a function f_n for which

$$v(\mu_{y_n^*},e) = \int_e f_n(\lambda)\, \mu_{y_n^*}(d\lambda)\ , \ (e \in \Sigma,\ n \geq 1),$$

and since, by what has been observed after definition 2.3.1, we have

$$v(\mu_{y_n^*},e) = \int_e |f_n(\lambda)| v(\mu_{y_n^*},d\lambda),\ e \in \Sigma$$

it follows that $|f_n(\lambda)| = 1$ for $\mu_{y_n^*}$-almost all λ. It may thus be assumed that $|f_n(\lambda)| = 1$ for all n in Ω. Let $Z_n^* = T_n^* y_n^*$ where the operators T_n are defined by the formula

$$T_n = \int_\Omega f_n(\lambda)\, E(d\lambda),\ n \geq 1,$$

since the functions f_n are bounded, so are the operators T_n.

Then

$$\begin{aligned} \mu_{Z_n^*}(e) &= (T_n^* y_n^*\ E(e)x_o = y_n^*\ E(e)T_n x_o \\ &= y_n^* \int_e f_n(\lambda)\ E(d\lambda)x_o = \int_e f_n(\lambda)\ y_n^*\ E(d\lambda)x_o \\ &= \int_e f_n(\lambda)\mu_{y_n^*}(d\lambda) = v(\mu_{y_n^*},e). \end{aligned}$$

Thus the functional

$$x_o^* = \sum_{n=1}^{\infty} \frac{1}{2^n(1+|Z_n^*|)}\ Z_n^*$$

satisfies

$$\mu_{xo^*} = \sum_{n=o}^{\infty} \frac{1}{2^n(1+|Z_n^*|} \mu_{Zn^*} .$$

Hence

$$\mu_{xo^*}(e) = \sum_{n=o}^{\infty} \frac{1}{2^n(1+|Z_n^*|} \mu_{Zn^*}(e) = \sum_{n=o}^{\infty} \frac{1}{2^n(1+|Z_n^*|} v(\mu_{y_n^*}, e),$$

which is the form expressed in equation (α) and x_o^* has the required properties.

Now, assume that X is not spanned by $\{Ex_o \ / \ E \in \mathcal{B}\}$. Since $\overline{\text{span}}\ \{Ex_o \ / \ E \in \mathcal{B}\}$ is $\mathcal{B}$-invariant, by corollary 2.1.9, the restriction of $\mathcal{B}$ to $\overline{\text{span}}\ \{Ex_o / \ E \in \mathcal{B}\}$ is σ-complete. It follows that this restriction is the range of the spectral measure $E \ / \ \overline{\text{span}}\ \{Ex_o/E \in \mathcal{B}\}$. From what has been proved, there is a functional also denoted by x_o^* with the required properties. By the Hahn-Banach theorem, this functional can be extended to X. Q.E.D.

(See lemma 12 of [10, 2205]).

<u>2.3.3- Remarks:</u>

Let E be a bounded spectral measure defined on the σ-field Σ of subsets of a set Ω. The integral $\int_\Omega \phi(\lambda)E(d\lambda)$ denoted by T_ϕ may be defined for every bounded Σ-measurable scalar function ϕ on Ω. This integral is a bounded hom~~e~~omorphism of the Banach alg$\overset{e}{\wedge}$bra $B(\Omega,\Sigma)$ of Σ-measurable functions on Ω into the Banach algbra B(X) of bounded linear operators in X.

Let $\mathcal{B}$ be the σ-complete Boolean algbra of projections and let e be an element of X. Define X by

$$X = \overline{\text{span}}\ \{Ee \ / \ E \in \mathcal{B}\}$$

Thus X is a cyclic Banach space. By the preceding proposition, there exists a continuous linear functional e* in X^*. This functional e* is normalized so that (e,e*) = 1.

Set

$$\mu(\sigma) = (\ E(\sigma)e,e^*), \quad \sigma \in \Sigma,$$

μ is a probability measure defined on the measure space (Ω, Σ).

Given $f \in M^+$, define

$$\rho(f) = \sup \{\|T_\phi e\| : \phi \in \ell^\infty, |\phi| \leq f\}.$$

2.3.4- Lemma: ρ is a function norm on (Ω, Σ, μ).

Proof: For $\alpha \geq 0$, ii) $\rho(\alpha f) = \sup \{\|T_\phi e\| : \phi \in \ell^\infty, |\phi| \leq \alpha f\}$

$$= \sup \{ \|T_{\alpha\phi} e\| : \alpha\phi \in \ell^\infty, \alpha|\phi| \leq f\}$$

$$= \alpha \sup \{\|T_\phi e\| : \phi \in \ell^\infty, |\phi| \leq f\} = \alpha \rho(f).$$

For i) in the definition of a function norm, let f, $g \in M^+$ and let $\phi \in \ell^\infty$ with $|\phi| \leq f + g$. Let $\theta \in M$ be a unimodilar function with $\theta|\phi| = \phi$, and put

$$\phi_1 = \theta\{ |\phi| \chi_\sigma + f(1 - \chi_\sigma)\}$$

where $\sigma = \{ t \in \Omega : |\phi(t)| \leq f(t)$. If $t \in \sigma$, then $\phi_1(t) = \phi(t)$; ~~then $\phi_1 \in \ell^\infty$, and~~ and if $t \in \Omega / \sigma$, then $f(t) \leq |\phi(t)|$, ~~and hence~~ $\phi_1(t) = \theta f(t)$ ~~[illegible]~~ $|\phi_1(t)| <$ ~~[illegible]~~ $\phi(t)$; thus $\phi_1 \in \ell^\infty$. Also the difference of two elements of ℓ^∞ is in ℓ^∞, that is $\phi_1 - \phi \in \ell^\infty$. Clearly

$$|\phi_1| \leq f, \; |\phi - \phi_1| \leq g.$$

Hence

$$\|T_\phi e\| \leq \|T_{\phi_1} e\| + \|T_{\phi - \phi_1} e\| \leq \rho(f) + \rho(g).$$

from which it follows that

$$\rho(f + g) \leq \rho(f) + \rho(g).$$

Now, suppose that $\rho(f) = 0 = \sup \{\|T_\phi e\| : \phi \in \ell^\infty, |\phi| \leq f\} = 0$. Let $\{\phi_n\}$ be an increasing sequence of simple functions in M^+ converging pointwise to f. Then $\|T_{\phi_n} e\| \leq \rho(f) = 0$, for all n, and hence

$$\int \phi_n d\mu = \langle T_{\phi_n} e, e^* \rangle = 0.$$

Therefore each $\phi_n = 0$ μ-a.e., and so $f = 0$ μ-a.e. Conversely, suppose that $f \in M^+$ with $f = 0$ μ-a.e., and let $\phi \in \ell^\infty$ with $|\phi| \leq f$. Then $\phi = 0$ μ-a.e., and hence ϕ is of the form $\sum_{i=1}^{n} \alpha_i \chi_{\sigma_i}$ with $\mu(\sigma_i) = 0$ for $i = 1, 2, \ldots, n$, and each $\alpha_i \in C$. Then, for $i = 1, 2, \ldots, n$,

$$\langle E(\sigma_i) e, e^* \rangle = \mu(\sigma_i) = 0,$$

and so $E(\sigma_i) e = 0$. Thus

$$\|T_{\phi}e\| = \| \sum_{i=1}^{n} \alpha_i \, E(\sigma_i)e \| = 0,$$

which shows that $\rho(f) = 0$. Consequently

$$\rho(f) = 0 \text{ iff } f = 0.$$

Now, for $\alpha = 0$ in condition i) in the definition of a function norm, let $f \in M^+$, then $\alpha f = 0$, and so $\rho(\alpha f) = 0 = \alpha\rho(f)$, then

$$\rho(\alpha f) = \alpha\rho(f), \ \alpha \geq 0.$$

Finally, let $f, g \in M^+$, if $f \geq g$ a.e., then it follows from i) that

$$\rho(f) \geq \rho(f-g) + \rho(g);$$

that is

$$\sup\{\|T_{\phi}e\| : \phi \in \ell^{\infty}, \ |\phi| \leq f\} \leq \sup\{\|T_{\psi}e\| : \psi \in \ell^{\infty}, \ |\psi| \leq f-g\} + \\ + \sup\{\|T_{\theta}e\| : \theta \in \ell^{\infty}, \ |\theta| \leq g\}.$$

For $|\phi| \leq f$, $|\psi| \leq f - g$ and $|\theta| \leq g$ we have

$$\|T_{\phi}e\| \leq \|T_{\psi}e\| + \|T_{\theta}e\|;$$

then, in particular for $\psi = 0$ a.e., we have $\|T_{\phi}e\| \leq \|T_{\theta}e\|$. Hence $\rho(f) \leq \rho(g)$.

Now, if $f = g$, a.e., then $f - g = 0$ and hence $\rho(f - g) = 0$ which is a consequence of the condition iv). Then

$$\rho(f) = \rho(f-g+g) \leq \rho(f-g) + \rho(g)$$

hence

$$\rho(f) \leq \rho(g)$$

and then $\rho(f) = \sup \{\|T_{\phi}e\| : \phi \in \ell^{\infty}, \ |\phi| \leq f\}$ is a function norm.

Q.E.D.

2.3.5- Lemma: The function norm ρ has the Fatou property.

Proof: Let $\{f_n\}$ be a sequence in M^+ such that $f_n \nearrow f \in M^+$ pointwise a.e. on Ω. The monotonocity of ρ implies that $\alpha = \lim \rho(f_n)$ exists and that $\alpha \leq \rho(f)$. Hence, if $\alpha = \infty$, then $\rho(f_n) \longrightarrow \rho(f)$, as required. Suppose now that $\alpha < \infty$ and put $\tau_{\infty} = \{ t \in \Omega : f(t) = \infty\}$. We first show that $\mu(\tau_{\infty}) = 0$. i.e., $f(t) = \infty$ holds only on a μ-null set. Since $f_n \nearrow f$ pointwise a.e. on Ω, then $f_n \longrightarrow \infty$ pointwise a.e. on τ_{∞} and hence, by Egoroff's theorem, $f_n \longrightarrow \infty$ almost uniformly on τ_{∞}. Put

$$\tau_k = \bigcup_{k=n}^{\infty} \{\, t \in \tau_\infty : |f_n(t)| > m\}, \ \forall\, m.$$

Clearly $\tau_k \subset \tau_{k+1}$, and $f_n \longrightarrow \infty$ uniformly on τ_k for each k and

$$\mu(\tau_\infty \setminus \bigcup_{k=1}^{\infty} \tau_k) = 0,$$

then

$$E(\tau_\infty \setminus \bigcup_{k=1}^{\infty} \tau_k)e = 0,$$

and so $E(\tau_k)e \longrightarrow E(\tau_\infty)e$ as $k \longrightarrow \infty$. Thus, if $E(\tau_\infty)e \neq 0$, then there exists an integer k such that $E(\tau_k)e \neq 0$. This implies that $\rho(\chi_{\tau k}) = \|E(\tau_k)e\| \neq 0$ and hence, by the uniform divergence of f_n on τ_k, that $\rho(f_n) \longrightarrow \infty$, contradicting the fact that $\alpha < \infty$. Therefore $E(\tau_\infty)e = 0$, and so $\mu(\tau_\infty) = 0$.

Since τ_∞ is μ-null, Egoroff's theorem now ~~e~~implies that there exists an increasing sequence $\{\sigma_k : k = 1,2,\ldots\}$ of measurable subsets of Ω with

$$\mu(\Omega \setminus \bigcup_{k=1}^{\infty} \sigma_k) = 0,$$

such that, for each k, f is finite on σ_k and $f_n \longrightarrow f$ uniformly on σ_k. As above, it follows that $E(\sigma_k)e \longrightarrow E(\Omega)e = e$ as $k \longrightarrow \infty$.

Now let $\phi \in \ell^\infty$ with $|\phi| \leq f$, and let $\varepsilon > 0$. There exists a positive integer $k(\varepsilon)$ such that

$$\|e - E(\sigma_{k(\varepsilon)})e\| < \varepsilon,$$

and, since $f_n \longrightarrow f$ uniformly on $\sigma_{k(\varepsilon)}$, there exists a positive integer $n(\varepsilon)$ such that

$$|\chi_{\sigma k(\varepsilon)}\, \phi| \leq f_{n(\varepsilon)} + \varepsilon$$

pointwise on Ω. Hence

$$\|E(\sigma_{k(\varepsilon)})T_\phi e\| \leq \rho(f_{n(\varepsilon)} + \varepsilon) \leq \alpha + \varepsilon\rho(1).$$

Also,

$$\|(I - E(\sigma_{k(\varepsilon)})T_\phi e\| = \|T_\phi\{I - E(\sigma_{k(\varepsilon)})\}e\| =$$

$$\leq \|T_\phi\|.\|e - E(\sigma_{k(\varepsilon)})e\| < \|T_\phi\|.\varepsilon,$$

and so

$$\|T_\phi e\| \leq \|T_\phi\{I - E(\sigma_{k(\varepsilon)})e\| + \|T_\phi E(\sigma_{k(\varepsilon)})e\|$$

$$(*) \qquad \|T_\phi e\| \leq \alpha + \varepsilon\ (\ \rho(1) + \|T_\phi\|),$$

since

$$|e^* Ee| \leq 4k|e^*|.\|e\|,$$

where k is the bound of the projections in $\mathcal{B}$. It follows that $\rho(1) \leq 4k\|e\| \angle \infty$. Thus $\rho(f) \leq \alpha$. Q.E.D.

2.3.6- Lemma: The constant function 1 belongs to $L^a{}_\rho$.

Proof: Clearly $1 \in L\rho$. Let $\{f_n\}$ be a sequence in ℓ_ρ with

$$1 \geq f_1 \geq f_2 \geq \ldots\ldots \searrow 0.$$

pointwise on Ω. Then, there exists $\sigma \in \Sigma$ such that $f_n \longrightarrow 0$ uniformly on σ and $\|e - E(\sigma)e\| \angle \varepsilon$. Then $\chi_\sigma f_n \longrightarrow 0$ uniformly on Ω and hence

$$T_{\chi_\sigma f_n} = E(\sigma)T_{f_n} \longrightarrow 0 \text{ in norm.}$$

Choose m such that $\|T_{\chi_\sigma f_m}\| \angle \varepsilon$, and let $\phi \in \ell^\infty$ with $|\phi| \leq f_m$. Then $\phi = \theta\, f_m$ for some measurable function θ with $|\theta| \leq 1$, and so

$$\|T_\phi e\| \leq \|T_\theta\|.\|T_{f_m} e\| \leq \|T_\theta\|[\|T_{\chi_\sigma f_m} e\| + \|T_{f_m}(e - E(\sigma)e)\|]$$

$$\leq 4k[\varepsilon\|e\| + \varepsilon]$$

since $T_{f_m} = \int_\Omega f_m E(d\lambda) \leq \int_\Omega E(d\lambda) = E(\Omega) = 1$. It follows that

$$\rho(f_m) \leq 4k(\|e\| + 1)\varepsilon.$$

Hence $\rho(f_m) \longrightarrow 0$ since $\{\rho(f_m)\}$ is decreasing. Q.E.D.

2.3.7- Lemma: The consant function 1 satisfies $\rho'(1) \angle \infty$.

Proof: Let $f \in M^+$ with $\rho(f) \leq 1$ and let $\{\phi_n\}$ be a sequence of simple measurable functions in M such that $0 \leq \phi_n \nearrow f$ pointwise on Ω. Then

$$\int \phi_n \, d\mu \longrightarrow \int f \, d\mu.$$

From the definition of the mapping $\phi \longrightarrow T_\phi$,

$$\int \phi_n \, d\mu = \int \phi_n(E(d\lambda)e, e^*) = (T_{\phi_n}e, e^*)$$

since $\|T_{\phi_n}e\| \leq \rho(f) \leq 1$, it follows that $|\int f \, d\mu| \leq \|e^*\|$ and hence that

$$\rho'(1) = \sup\{\int f d\mu \,:\, f \in M^+,\ \rho(f) \leq 1\}$$

$$\leq \|e^*\| \angle \infty$$

as required. Q.E.D.

The main structure theorem for the cyclic Banach space X can now be proved. See [12, 231].

Let M_ϕ be the bounded linear operator on $L\rho$ defined by $M_\phi f = \phi f$, $\phi \in \ell^\infty$, $f \in L\rho$

2.3.8- Theorem: There is a bicontinuous linear isomorphism U of L^a_ρ onto X such that

$$(*) \qquad U(M_\phi f) = T_\phi Uf$$

for all $\phi \in \mathcal{L}^\infty$ and all $f \in L^a\rho$.

Proof: Define $U_0 : \mathcal{L}^\infty \longrightarrow X$ by setting

$$U_0 f = T_f e \qquad (f \in \mathcal{L}^\infty).$$

Clearly U_0 is linear and satisfies $\|U_0 f\| \leqslant \rho(f)$ $(f \in \mathcal{L}^\infty)$. In particular if $f \in \mathcal{L}^\infty$ is μ-null, $U_0 f = T_f e = \int f\, E(d\lambda)e = 0$. Hence U_0 induces a linear mapping $U_1 : L^\infty \longrightarrow X$ defined by

$$U_1(f + N) = U_0 f \qquad (f \in \mathcal{L}^\infty)$$

and satisfying

$$\|U_1 f\| \leqslant \rho(f), \quad f \in L^\infty.$$

By lemmas 2.3.6 and 1.2.7, L^∞ is a dense subspace of L^a_ρ, and so U_1 extends by continuity to give a continuous linear mapping U of L^a_ρ into X. Since

$$U_0(\phi f) = T_{\phi f} e = T_\phi T_f e = T_\phi(U_0 f)$$

for $\phi, f \in \mathcal{L}^\infty$,

$$U_1(M_\phi f) = T_\phi(U_1 f)$$

for $\phi \in \mathcal{L}^\infty$, $f \in L^\infty$. Hence (*) holds by continuity and by extension for $\phi \in \mathcal{L}^\infty$ and $f \in L^a_\rho$.

Finally, let $f \in \mathcal{L}^\infty$ and let $\phi \in \mathcal{L}^\infty$ with $|\phi| \leqslant |f|$. Then there exists a measurable function θ with $|\theta| \leqslant 1$ such that $\phi = \theta f$ and hence

$$\|T_\phi e\| = \|T_{\theta f} e\| = \|T_\theta\| . \|T_f e\| \leqslant 4k\|T_f e\|.$$

Thus

$$\rho(f) \leqslant 4k\|T_f e\| = 4k\|U_0 f\|.$$

It will follow that

$$\rho(g) \leqslant 4k\|Ug\| \qquad (g \in L^a_\rho).$$

On the other hand

$$\|Ug\| \leqslant \rho(g) \qquad (g \in L^a_\rho);$$

that is

$$\rho(g) \leq \|Ug\| \leq 4k\rho(g).$$

By lemma 2.3.6 and theorem 1.2.7, L^a_ρ is a Banach space. Therefore, U is a bicontinuous isomorphism with closed range. Since

$$U_0\chi_\sigma = E(\sigma)e \qquad (\sigma \in \Sigma),$$

the range of U_1 contains the dense subspace

$$\text{span}\{\, E(\sigma)e,\ \sigma \in \Sigma\}$$

of X, the range of U equals X. Q.E.D.

2.3.9- Example: L^a_ρ is a cyclic Banach space.

Proof: For $\Phi \in \ell^\infty$, let M_Φ be the bounded linear operator on $L\rho$ defined by

$$M_\Phi f = \Phi f \qquad (f \in L\rho).$$

For each $\sigma \in \Sigma$, let $E(\sigma)$ denote the restriction of $M\chi_\sigma$ to L^a_ρ. Let $\Phi \in L^\infty$, $\Phi \geq 0$; then there exists a sequence of simple functions f_n such that $f_n \longrightarrow \Phi$ pointwise a.e. on Ω. Every f_n is of the form $\sum_{i=1}^{m_n}\alpha^n_i\chi_{\sigma_i} = \sum_{i=1}^{m_n}\alpha^n_i M\chi_{\sigma_i}1 = \sum_{i=1}^{m_n}\alpha^n_i E(\sigma_i)1$, $\sigma_i \in \Sigma$, $(i = 1,2,\dots,)$ $\alpha^n_i \in \mathbb{C}$, $(i = 1,2,\dots.)$ By considering the negative and positive parts of the real and the imaginary parts of an arbitrary element of L^∞, we conclude that

$$L^a_\rho = \overline{\text{span}}\{\, E(\sigma)1,\ \sigma \in \Sigma\}.$$

Define

$$\mathcal{B} = \{E(\sigma) : \sigma \in \Sigma\}.$$

Hence, to obtain the required result, it remains to show that on L^a_ρ, $\mathcal{B}$ acts as a σ-complete Boolean algebra of projections. ($\mathcal{B}$ has a cyclic vector 1). Then if we show that $E(.)$ is a countably additive spectral measure on L^a_ρ, the required result follows immediately. Let $f \in L^a_\rho$, and let δ, σ be two disjoint sets of Σ.

$$E(\sigma \cup \delta)f = M\chi_{\sigma\cup\delta}\, f = (M\chi_\sigma + M\chi_\delta)f = [E(\sigma) + E(\delta)]f.$$

Then

$$E(\sigma \cup \delta) = E(\sigma) + E(\delta).$$

$$E(\Omega)f = M\chi_\Omega\, f = f.$$

Hence $E(\Omega) = I$

Let $\sigma \in \Sigma$, $f \in L^a{}_\rho$

$$E(\sigma')f = E(\Omega\backslash\sigma)f = M\chi_{\Omega\backslash\sigma}\ f = M\chi_\Omega - M\chi_\sigma\ f = (I - E(\sigma))f.$$

The boundedness of $E(\sigma)$ follows from that of $M\chi_\sigma$, $\sigma \in \Sigma$. Now, let $\{\sigma_n\}$ be a sequence of mutually disjoint elements of Σ, let $\sigma = \bigcup_{i=1}^{\infty} \sigma_n$ and let $f \in L^a{}_\rho$ with $f \geq 0$ a.e. Then

$$f \geq E(\sigma)f - E(\bigcup_{i=1}^{n} \sigma_i)\ f \searrow 0$$

pointwise a.e. on Ω, and since ρ has the Fatou property

$$\sum_{i=1}^{n} E(\sigma_i)f = E(\bigcup_{i=1}^{n} \sigma_i)f \longrightarrow E(\sigma)f$$

in norm as $n \longrightarrow \infty$. By considering the positive and negative parts of the real and imaginary parts of an arbitrary element of $L^a{}_\rho$. It follows that

$$E(\bigcup_{i=1}^{n} \sigma_i)f \longrightarrow E(\sigma)f.$$

Hence $E(.)$ is a countably additive spectral measure on $L^a{}_\rho$.Q.E.D.

Let A be a non-empty subset of B(X), let A' denotes the commutant of A, i.e.,

$$A' = \{\ S \in B(X)\ :\ ST = TS \text{ for all } T \in A\ \}.$$

<u>**2.3.10- Theorem:**</u> Let $\mathcal{B}$ be the σ-complete Boolean algebra of projections on a Banach space X and let $e \in X$ be such that $X = \overline{\text{span}}\{\ Be\ :\ B \in \mathcal{B}\}$. Let A denote the uniformly closed subalgebra of B(X) generated by $\mathcal{B}$. Then

i) $A = A' = \{\ T_\phi\ :\ \phi \in \ell^\infty\}$.

ii) A is a strongly closed subalgebra of B(X) containing I.

<u>**Proof:**</u> i) It is clear that every element in the uniformly closed operator algebra generated by $\mathcal{B}$ commutes with every element of $\mathcal{B}$ and so to prove that $A = A'$, it suffices to show that every operator F in A' is in the uniformly closed operator algebra A. Since $\mathcal{B}$ is σ-complete, then $\mathcal{B}$ is the range of a countably additive spectral measure E on a σ-field Σ of subsets of a set Ω. Let e^* be the Bade functional associated with $e \in X$. Then $\langle FE(\sigma)e, e^*\rangle$, where F is an

operator which commutes with every element in $\mathcal{B}$, is a finite measure. By the Radon-Nikodým theorem, there is a measurable function ϕ with

$$\langle FE(\sigma)e,\ e^*\rangle = \int_\sigma \phi(\lambda)e^*E(d\lambda)e, \quad \sigma \in \Sigma.$$

Let

$$\sigma_n = \{\ \lambda\ /\ |\phi(\lambda)| \leq n\},\ A_n = \int_{\sigma n} \phi(\lambda)E(d\lambda),$$

so that

$$\langle\{E(\delta)E(\sigma_n)F\ E(\sigma) - E(\delta)A_nE(\sigma)\}e,\ e^*\rangle =$$

$$\int_{\sigma\cap\delta\cap\sigma n} [\phi(\lambda) - \phi(\lambda)]e^*E(d\lambda)e = 0,\ \sigma,\ \delta \in \Sigma.$$

Thus since the set $\{E(\sigma)e\ /\ \sigma \in \Sigma\}$ is fundamental in X, we have

$$\langle E(\sigma_n)Fx,\ E(\sigma)^*e^*\rangle = \langle A_nx,\ E(\sigma)^*e^*\rangle,\ x\in X.$$

One can easily prove that the manifold in X^* spanned by vectors of the form $E(\sigma)^*e^*$ with σ in Σ is X-dense in X^*. It follows that $E(\sigma_n)\ F = A_n$ and thus, since F is bounded, the sequence $\{|A_n|\}$ is bounded. By the countable additivity of E and the boundedness of the sequence $\{|A_n|\}$, there follows the existence of àn integer n_o such that :

$$\inf_{E(\sigma_n)=1,\lambda\in\sigma_n} \sup |\phi(\lambda)| = \sup_{\lambda\in\sigma_{no}} |\phi(\lambda)| \angle \infty.$$

Consequently it may be assumed that ϕ is bounded and therefore, that $\|E(\sigma_n)\| = \overset{1}{\cancel{\Omega}}$ for large n. This means that

$$F = A_n = \int_\Omega \phi(\lambda)E(d\lambda).$$

Now, since

$$\mathcal{B} = \{\ E(\sigma)\ :\ \sigma \in \Sigma\} = \{T\chi_\sigma = \int_\Omega \chi_\sigma E(d\lambda),\ \sigma \in \Sigma\};$$

and the simple measurable functions are uniformly dense in ℓ^∞, the continuity of the mapping $\phi \longrightarrow T_\phi$ implies that

$$\{\ T_\phi:\ \phi \in \ell^\infty\} \subseteq A \subseteq A'$$

from which it follows that A is the uniformly closed algebra generated by $\mathcal{B}$ and that $A = A' = \{\ T_\phi\ :\ \phi \in \ell^\infty\}$ since every operator $F \in A'$ has the form $T_\phi = \int_\Omega \phi(\lambda)E(d\lambda)$.

ii) It is clear that every operator in the weakly closed operator algebra generated by $\mathcal{B}$ commutes with every element of $\mathcal{B}$. Hence the uniformly closed algebra generated by $\mathcal{B}$ contains the weakly closed operator algebra. On the other hand, since the uniform operator topology is stronger than the weak operator topology, it follows that the uniformly closed algebra generated by B is contained in the weakly closed algebra generated by B. Q.E.D.

2.4 Reflexive operator algebras

Let X be a complex Banach space and let $\mathcal{B}$ be a σ-complete Boolean algebra of projections on X. Given $e \in X$, let

$$M(e) = \overline{\text{span}}\{ Ee,\ E \in \mathcal{B} \}.$$

Then M(e) is $\mathcal{B}$-invariant and by the proof of proposition 2.3.2 the restiction of $\mathcal{B}$ to M(e) is a σ-complete Boolean algebra of projections on M(e).

The main results of this section can be found in [13].

Let E(.) be the countably additive spectral measure defined on (Ω,Σ)

2.4.1- Theorem: Let $e \in X$ and let ν be a complex measure on (Ω,Σ) such that $\nu \ll E(.)e$. Then there exist $x \in M(e)$ and $x^* \in X^*$ such that

$$\nu(\cdot) = \langle E(\cdot)\overset{x}{\cancel{e}},\ \overset{x^*}{\cancel{e^*}}\rangle.$$

Proof: Assume without any real loss of generality that $e \neq 0$. Let μ be the probability measure on (Ω,Σ) defined by

$$\mu(\cdot) = \langle E(\cdot)e,\ e^*\rangle$$

where e^* is a normalized Bade functional for e. Let ρ be the function norm

$$\rho(f) = \sup\{\|T_\phi e\|,\ \phi \in \ell^\infty,\ |\phi| \le f\},\quad f \in M^+$$

and let $U: L^a_\rho \longrightarrow M(e)$ be the linear hom~~e~~omorphism satisfying

$$U(\phi f) = T_\phi U f \qquad (\phi \in \ell^\infty,\ f \in L^a_\rho)$$

$$U(1) = T_1 e = \int_\Omega E(d\lambda)e = E(\Omega)e = e.$$

It is clear that $\nu \ll \mu$ and hence, by the Radon-Nikodým theorem, there exists $\phi \in L^1(\mu)$ such that $d\nu = \phi d\mu$. By theorem 1.3.9, there exist $f \in L^a_\rho$ and $g \in L\rho'$ such that $\phi = fg$ a.e. Let $x = Uf$ and define $x_0^* \in M(e)^*$ by

$$(z, x_0^*) = \int (U^{-1}z) g\,d\mu \qquad (z \in M(e))$$

Let $x^* \in X^*$ be an extension of x_0^*. Then for $\sigma \in \Sigma$

$$\nu(\sigma) = \int_\sigma \phi\, d\mu = \int (\chi_\sigma f) g\, d\mu$$

$$= \int (U^{-1}E(\sigma)\, Uf) g\, d\mu$$

$$= \langle E(\sigma)Uf,\ x^*\rangle = \langle E(\sigma)x,\ x^*\rangle$$

using the relation

$$U(\chi_\sigma f) = T\chi_\sigma\, Uf = E(\sigma)\, Uf \qquad (\ f \in L^a{}_\rho,\ \sigma \in \Sigma).$$

Hence

$$\nu(\sigma) = \langle E(\sigma)x,\ x^*\rangle,\ \sigma \in \Sigma.$$

Q.E.D.

2.4.2- Corollary: Let $x_1, x_2, \ldots, x_n \in X$ and $x_1^*,\ x_2^*, \ldots, x_n^* \in X^*$ be given. Then there exist $x \in X$ and $x^* \in X^*$ such that

$$\langle E(\sigma)x, x^*\rangle = \sum_{i=1}^{n} \langle\ E(\sigma)x_i,\ x_i^*\rangle\ ,\ \sigma \in \Sigma.$$

Proof: Let $Y = \overline{\text{span}}\ \{\ Ex_i\ :\ E \in B,\ 1 \leq i \leq n\}$. Then Y is B-invariant and the restriction B_0 of B to Y is a σ-complete Boolean algebra of projections. We first show that B_0 is complete. Let $\{E_\alpha\}$ be a family of disjoint elements of B_0. It follows from lemma 2.1.5 that every series $\sum_{i=1}^{\infty} E_{\alpha i}$, with $\alpha_i \neq \alpha_j$ if $i \neq j$, converges strongly. Hence , for every integer $1 \leq i \leq n$ and $\varepsilon > 0$, only a finite number of the vectors $E_\alpha x_i$ have norms greater than ε. This shows that, for all but a countable number of α, $E_{\alpha} x_i = 0$, for every integer $1 \leq i \leq n$. Thus with the exception of a countable number of α, $E_\alpha E x_i = 0$ for every $E \in B$ and $1 \leq i \leq n$. Since the set $\{Ex_i\ :\ E \in B,\ 1 \leq i \leq n\}$ is fundamental in Y, it follows that $E_\alpha = 0$ for all but a countable number of α. Thus $\{E_\alpha\}$ is a countable set. Since B_0 is σ-complete, the family $\{E_\alpha\}$ has a least upper bound. To establish that B_0 is complete, it suffices to show that every set in B_0 has a least upper bound which is the least upper bound of a countable subset. Let A be a subset of B_0 which has an upper bound, and let A_1 be the collection of least upper bounds of all countable subsets of A. Consider the family W of subsets of A_1 which are well ordered under the ordering of B_0. We shall order W by defining the relation $F \leq G$ between elements F , G in W to mean that $F \subseteq G$ and that each element x such that $x \in G$

but $x \notin F$ is an upper bound for F. We first show that W satisfies the hypothesis of Zorn's lemma. Let W_0 be the totally ordered subset of W and let $C \subseteq \cup W_0 = \{ x \in F : F \in W\}$. Then for some $F \in W_0$, $C \cap F \neq \Phi$. Let x be the smallest element of $C \cap F$ and let y be any other element of C. If $y \in G \in W_0$, we have either $F \leqslant G$ or $G \leqslant F$. If $G \leqslant F$, then, since $y \in G$, we have also $y \in F$ and thus $y \in C \cap F$ and thus $y \leqslant x$. If $F \leqslant G$ and $y \in F$ then $y \in C \cap F$ and $y \geqslant x$. Finally, if $F \leqslant G$ and $y \notin F$, then y is an upper bound for F, hence, $y \geqslant x$. Thus x is the smallest element of C. This shows that $\cup W_0$ is well-ordered and thus is an upper bound for W_0 in W. By Zorn's lemma, there exists a maximal element G_0 in W. Then G_0 is a countable subset of A and hence $y_0 = \sup G_0$ exists and is in A_1. If, for some $y_1 \in A_1$, we have $y_0 \leqslant y_1$, then $\sup\{y_0,y_1\} \geq y_0$; but $y_2 = \sup\{ y_0,y_1\} \in A_1$, $G_0 \leqslant G_0 \cup \{y_2\}$ and so G_0 is not maximal, which is a contradiction. Therfore, $y \leqslant y_0$, $y \in A_1$, and since $y_0 \in F$, we conclude that $y_0 = \sup A_1 = \sup F$. In a similar way, one can prove that the infimum of a set with a lower bound exists. Then $\mathcal{B}_0$ is complete. Let $E \in \mathcal{B}_0$, a complete Boolean algebra of projections, be a projection on Y; E is called a carrier projection of a vector $e \in Y$ if $E = \cap\{ E / E \in \mathcal{B}_0, Ee = e\}$; we claim that the identity operator I on Y is a carrier projection relative to $\mathcal{B}_0$ of an element $e \in Y$. For, if $G \in \mathcal{B}_0$ with $Ge = e$, then $GIe =$ $e = \overset{I}{\cancel{G}}e$ and hence $G \geqslant I$. Let $F \in \mathcal{B}$ with $Fe = 0$, then, $(I-F)e = e$, it follows that $(I-F)E \geqslant E$ and then $F \leqslant I$. From the equalities

$$(I - F)I = I$$

$$FI = F$$

we conclude that $F / Y = 0$. Therefore, if $E(\sigma)e = 0$ for some $\sigma \in \Sigma$, it follows that

$$E(\sigma)x_1 = E(\sigma)x_2 = \dots = E(\sigma)x_n = 0.$$

Thus the complex measure

$$\nu(\cdot) = \sum_{i=1}^{n} \langle E(\cdot)x_i, x_i^*\rangle$$

satisfies $\nu(\cdot) \ll E(\cdot)e$ and therefore, theorem 2.4.1 gives the required result. Q.E.D.

2.4.3- Corollary: If X is separable and ν is a complex measure on (Ω,Σ) satisfying $\nu(.) \ll E(.)$. there exist $x \in X$ and $x^* \in X^*$ such that

$$\nu(.) = \langle E(.)x,\ x^*\rangle .$$

Proof: Since X is separable , then it contains a dense denumerable set F. Let $Y = \widetilde{\text{span}}\ \{Ex\ :\ E \in \mathcal{B},\ x \in F\}$. The required result is established by following the s~~e~~imilar lines in the proof of corollary 2.4.2. Q.E.D.

Given a subalgebra A of B(X), let Lat A denote the lattice of all closed A-invariant subspaces of X and let

$$\text{Alg Lat } A = \{\ T \in B(X)\ :\ TM \subset M \text{ for every } M \in \text{Lat } A\ \}.$$

The algebra A is said to be reflexive if Alg Lat A = A.

Note that Alg Lat A is closed in the weak operator topology on B(X).

Also , a single operator $T \in B(X)$ is reflexive if the weakly closed subalgebra of B(X) generated by T and I is reflexive.

2.4.4- Lemma: Let C be a subalgebra of B(X) containing the identity operator and let $T \in B(X)$. Then

i) $T \in$ Alg Lat C if and only if the following condition is satisfied:

$$\langle Tx,\ x^*\rangle = 0$$

whenever $x \in X$, $x^* \in X^*$, and $\langle Ax,x^*\rangle = 0$ for all $A \in C$;

ii) T belongs to the weakly closed subalgebra C if and only if the following condition is satisfied:

$$\Sigma\ \langle Tx_i,\ x^*\rangle = 0$$

whenever $x_1,\ x_2,\ \ldots,\ x_n \in X$, $x_1^*,\ x_2^*,\ldots,x_n^* \in X^*$, and $\sum_{i=1}^{n} \langle Ax_i,\ x_i^*\rangle = 0$ for all $A \in C$.

Proof: i) Let $T \in$ Alg Lat C and let, for $x \in X$ and $x^* \in X^*$, $\langle Ax,\ x^*\rangle = 0$ for all $A \in C$. Then if $Y = \overline{\text{span}}\ \{\ Ax:\ A \in C\}$ we have $x^* \in Y^{\perp}$ and $TY \subseteq Y$. Hence $\langle Tx,x^*\rangle = 0$. Conversely, Let Y be a closed subspace of X

C-invariant. Let $x \in X$ and $x^* \in X^*$. Then $\langle Ax, x^*\rangle = 0$ for all $A \in C$. and so $\langle Tx, x^*\rangle = 0$. Therefore $TY \subseteq Y$.

ii) Let T belong to the weakly closed subalgebra C, and let $\{T_n\}$ be a sequence in C converging to T in the weak operator topology. Let x_1, $x_2, \ldots, x_n \in X$, $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ such that

$$\sum_{i=1}^{n} \langle Ax_i, x_i^*\rangle = 0 \text{ for all } A \in C$$

and so $\sum_{i=1}^{k} \langle T_k x_i, x_i^*\rangle = 0$ for $k \geq 1$. Since $\sum_{i=1}^{n} \langle T_k x_i, x_i^*\rangle \longrightarrow \sum_{i=1}^{n} \langle Tx_i, x_i^*\rangle$ then $\sum_{i=1}^{n} \langle Tx_i, x_i^*\rangle = 0$. Conversely, let $x_1, x_2, \ldots, x_n \in X$, $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and $\sum_{i=1}^{n} \langle Ax_i, x_i^*\rangle = 0$ and so $\sum_{i=1}^{n} \langle Tx_i, x_i^*\rangle = 0$, and suppose that T does not belong to the weakly closed subalgebra C. By the Hahn-Banach separation theorem, there exists a functional F such that $F(T) \neq F(A)$ for all $A \in C$. This functional has the form, for $x_1, x_2, \ldots, x_n \in X$, $x_1^*, x_2^*, \ldots, x_n^* \in X^*$

$$F(T) = \sum_{i=1}^{n} \langle Tx_i, x_i^*\rangle \neq \sum_{i=1}^{n} \langle Ax_i, x_i^*\rangle \text{ for all } A \in C$$

which is a contradiction, since whenever $x_1, x_2, \ldots x_n \in X$, x_1^*, $x_2^*, \ldots, x_n^* \in X^*$ and $\sum_{i=1}^{n} \langle Ax_i, x_i^*\rangle = 0$ for all $A \in C$, we have $\sum_{i=1}^{n} \langle Tx_i, x_i^*\rangle = 0$. Q.E.D.

2.4.5- Theorem: Let $\mathcal{B}$ be a σ-complete Boolean algebra of projections on X and let A be the weakly closed subalgebra of B(X) generated by $\mathcal{B}$ Then A is reflexive.

Proof: Let $T \in \text{Alg Lat } A$. Since each $G \in \mathcal{B}$ is A-reducing and hence T-reducing, $TG = GT$ $(G \in \mathcal{B})$. Given $x \in X$, the cyclic subspace $M(x)$ is T-invariant and $T / M(x)$ commutes with every restriction of each $G \in \mathcal{B}$ to $M(x)$. Hence, by theorem 2.3.10 i), there exists for each $x \in X$ a function $\phi_x \in \ell^\infty$ such that

$$T / M(x) = T_{\phi x} / M(x).$$

Fix $x_1, x_2, \ldots, x_n \in X$, $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ such that

$$\sum_{i=1}^{n} \langle Fx_i, x_i^*\rangle = 0 \ (F \in A).$$

Then, we must show that $\sum_{i=1}^{n} \langle Tx_i, x_i^*\rangle = 0$. The proof of corollary 2.4.2 shows that there exists $e \in X$ such that

$$E(\sigma)x_1 = E(\sigma)x_2 = \dots = E(\sigma)x_n = 0$$

whenever $\sigma \in \Sigma$ and $E(\sigma)e = 0$. Let e^* be the normalized Bade functional for e and let μ be the measure $\langle E(.)e, e^*\rangle$. Let ϕ be the function in ℓ^∞ such that

$$T / M(e) = T_\phi / M(e).$$

It will be shown that

$$Tx_i = T_\phi x_i$$

for $i = 1,2,3,\dots, n$. Since $T_\phi \in A$, this will imply that

$$\sum_{i=1}^{n} \langle Tx_i, x_i^*\rangle = \sum_{i=1}^{n} \langle T_\phi x_i, x_i^*\rangle = 0.$$

Fix $i \in \{1,2,3,\dots, n\}$ and let ψ, $\theta \in \ell^\infty$ satisfy

$$T / M(x_i) = T_\psi / M(x_i), \quad T / M(x_i+e) = T_\theta / M(x_i+e)$$

Then

$$T_\theta x_i + T_\theta e = T(x_i + e) = T_\psi x_i + T_\phi e$$

and so

$$T_{\theta-\psi}x_i = T_{\phi-\theta}e = z$$

say, and z belongs to $M(x_i) \cap M(e)$. Thus

$$T_\phi z = T_z = T_\psi(z)$$

and hence

$$0 = T_{\phi-\psi}z = T_{\phi-\psi}(T_{\phi-\theta})e = T_{(\phi-\psi)(\phi-\theta)}e = 0.$$

Let k be a unimodular function in ℓ^∞ such that

$$k(\phi - \psi)(\phi - \theta) = |(\phi - \psi)(\phi - \theta)|.$$

Then

$$\int k(\phi - \psi)(\phi - \theta)d\mu = \int |(\phi - \psi)(\phi - \theta)|d\mu =$$

$$= \langle T_k T_{(\phi-\psi)(\phi-\theta)}e, e^*\rangle = 0$$

which shows that $(\phi - \psi)(\phi - \theta) = 0$ μ-a.e. Thus there exist disjoint sets σ, τ and δ with $\Omega = \tau \cup \sigma \cup \delta$ such that $\psi = \phi$ on τ, $\phi = \theta$ on δ, and $\mu(\sigma) = 0$. Since e^* is the normalized Bade functional for e, $E(\sigma)e = 0$. Therefore, $E(\sigma)x_i = 0$. Now

$$TE(\tau)x_i = T_\psi T_{\chi_\tau} x_i = T_{\psi\chi_\tau} x_i = T_{\phi\chi_\tau} x_i = T_\phi E(\tau) x_i.$$

Similarly, $T_\theta E(\delta)x_i = T_\theta T_{\chi_\delta} x_i = T_{\theta\chi_\delta} x_i = T_{\phi\chi_\delta} x_i = T_\phi E(\delta)x_i$ and

$TE(\delta)e = T_\phi E(\delta)e = T_\phi T_{\chi_\delta} e = T_{\theta\chi_\delta} e = T_\theta E(\delta)e$. Hence

$$\begin{aligned} T\,E(\delta)x_i &= T\,E(\delta)(x_i + e) - T\,E(\delta)e \\ &= T_\theta E(\delta)(x_i + e) - T_\theta E(\delta)e \\ &= T_\phi E(\delta)x_i. \end{aligned}$$

Since $E(\delta)x_i = 0$, it follows that

$$\begin{aligned} Tx_i = T\,E(\Omega)x_i &= T[E(\tau) + E(\delta)]x_i \\ &= T\,E(\tau)x_i + T\,E(\delta)x_i \\ &= T_\phi E(\tau)x_i + T_\phi E(\delta)x_i \\ &= T_\phi E(\Omega)x_i = T_\phi x_i, \end{aligned}$$

as required. Since $T_\phi \in A$, it follows that

$$\sum_{i=1}^{n} \langle Tx_i, x_i^* \rangle = \sum_{i=1}^{n} \langle T_\phi x_i, x_i^* \rangle = 0.$$

Then $T \in A$. Hence A is reflexive, for $A = \text{Alg Lat } A$.

<u>2.4.6- Theorem:</u> Let A be the weakly closed subalgebra of $B(X)$ generated by the σ-complete Boolean algebra $\mathcal{B}$ of projections on X. Then every weakly closed subalgebra of A containing the identity operator is reflexive.

<u>Proof:</u> Let C be the weakly closed subalgbra of A containing the identity operator and let $T \in \text{Alg Lat } C$. We shall use lemma 2.4.4, ii) to show that $T \in \text{w-cl } C = C$ and hence that C is reflexive. Accordingly, fix $x_1, x_2, \ldots, x_n \in X$ and $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ with

$$\sum_{i=1}^{n} \langle Fx_i, x_i^* \rangle = 0, \text{ for all } F \in C.$$

We must show that

$$\sum_{i=1}^{n} \langle Tx_i, x_i^* \rangle = 0.$$

Since $\mathcal{B}$ is σ-complete, then it is the range of the spectral measure $E(.)$. By corollary 2.4.2, there exist $x \in X$ and $x^* \in X^*$ such that

$$\langle Ex, x^* \rangle = \sum_{i=1}^{n} \langle Ex_i, x_i^* \rangle , \quad (E \in \mathcal{B})$$

Since A is the weakly closed linear span of $\mathcal{B}$, it follows that

$$\langle Fx, x^* \rangle = \sum_{i=1}^{n} \langle Fx_i, x_i^* \rangle, \quad (F \in A).$$

Now

$$T \in \text{Alg Lat } C \subset \text{Alg Lat } A.$$

By theorem 2.4.5, $T \in A$. Hence

$$\langle Tx, x^*\rangle = \sum_{i=1}^{n} \langle Tx_i, x_i^*\rangle.$$

But since $\langle Ax, x^*\rangle = \sum_{i=1}^{n} \langle Ax_i, x_i^*\rangle = 0$ for all $A \in C$. Hence $\langle Tx, x^*\rangle = 0$ by lemma 2.4.4, i), and therefore

$$\langle Tx, x^*\rangle = \sum_{i=1}^{n} \langle Tx_i, x_i^*\rangle = 0.$$

as required. Q.E.D.

2.4.7- Corollary: Let S be a scalar type spectral operator on X. Then S is reflexive.

Proof: Let E(.) be the resolution of the identity of S. Let $\mathcal{B} = \{E(\sigma) : \sigma \in \Sigma\}$ and let A be the weakly closed subalgebra of B(X) generated by $\mathcal{B}$ which is a σ-complete Boolean algebra of projections. We take C to be the weakly closed subalgebra generated by S and the identity operator. Since $S \in A$ and hence $C \subset A$, by theorem 2.4.6, it follows that S is Reflexive. Q.E.D.

2.5 A representation theorem for a complete Boolean algebra of projections

Given a Boolean algebra of projections on X, a Banach space, Let $U(\mathcal{B})$ denote the norm closed subalgebra of $B(X)$ generated by $\mathcal{B}$.

The results of this section can be found in [6].

2.5.1- Theorem: Let $\mathcal{B}$ be a complete Boolean algebra of projections on a complex Banach space X. Then there exist a complex Hilbert space H, a complete Boolean algebra B_0 of self-adjoint projections on H, and an algebraic isomorphism θ of $U(\mathcal{B})$ onto $U(B_0)$ such that

i) $\theta(\mathcal{B}) = B_0$, so that the restriction of θ to $\mathcal{B}$ is a Boolean algebra isomorphism of $\mathcal{B}$ onto B_0;

ii) θ is bicontinuous when $U(\mathcal{B})$ and $U(B_0)$ are endowed with

a) the norm topologies

b) the weak operator topologies

c) the ultraweak operator topologies;

iii) θ is bicontinuous on bounded sets when $U(\mathcal{B})$ and $U(B_0)$ are endowed with the strong operator topologies.

It will be convenient to make some preliminary definitions before proceeding with the proof of the theorem.

2.5.2- Definition: Let H be a Hilbert space.

The ultraweak operator topology is the topology generated by the seminorms

$$T \longrightarrow \left|\sum_{i=1}^{\infty} \langle Tx_i, y_i\rangle\right| , \quad (T \in B(H))$$

where $\{x_i\}$ and $\{y_i\}$ range over the pairs of sequences in H for which $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$ and $\sum_{i=1}^{\infty} \|y_i\|^2 < \infty$.

The ultrastrong operator topology is the topology generated by the seminorms

$$T \longrightarrow \{\sum_{i=1}^{\infty} \|Tx_i\|^2\}^{\frac{1}{2}}, \quad T \in B(H)$$

where $\{x_i\}$ ranges over the sequences in H for which $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$.

This topology is natural in the sense that it is associated with the strong operator topology corresponding to the Hilbert space direct sum of countably many copies of H.

Proof of the theorem:

***) construction of H and B_0**

Let $\{x_\alpha : \alpha \in A\}$ be a subset of X which is maximal with respect to the properties: $C(x_\alpha) \neq 0$ $(\alpha \in A)$, $C(x_\alpha)C(x_\beta) = 0$ $(\alpha \neq \beta)$

where the carrier projection C(x) of a vector x in X, is the element of $\mathcal{B}$ given by

$$C(x) = \cap\{E \in \mathcal{B} : Ex = x\}.$$

We may and shall assume without loss of generality that $\|x_\alpha\| = 1$ $(\alpha \in A)$. Let $E_0 = \cup \{C(x_\alpha) : \alpha \in A\}$. We show that $I = E_0$. Suppose not. Then there exists a unit vector y such that $(I - E_0)y = y$. Observe that

$$(I - E_0)C(x_\alpha) = 0 \quad (\alpha \in A)$$

and

$$(I - E_0)C(y) = C(y).$$

Thus

$$(I - E_0)C(y)C(x_\alpha) = C(y)C(x_\alpha) = (I - E_0)C(x_\alpha)C(y) = 0,$$

for all $\alpha \in A$; and this contradicts the maximality of $\{x_\alpha : \alpha \in A\}$.

Thus

$$E_0 = \cup \{ C(x_\alpha): \alpha \in A\} = I.$$

Since, for $E \in \mathcal{B}$, $Ex = 0 \Longleftrightarrow (I - E)x = x$

$$\Longleftrightarrow (I - E)C(x) = C(x)$$

$$\Longleftrightarrow EC(x) = 0,$$

it follows that, $E = 0$ if and only if $Ex_\alpha = 0$ for all $\alpha \in A$.

For each $\alpha \in A$, Let x_α^* be a Bade functional for x_α such that $\|x_\alpha^*\| =$

1 and let $\mu_\alpha = \langle E(\,.\,)x_\alpha, x_\alpha^*\rangle$ be the measure on (Ω, Σ) $(\alpha \in A)$. Let H be the direct sum of the complex Hilbert spaces $L^2(\mu_\alpha)$ $(\alpha \in A)$. Then, an element $\zeta = \{\zeta_\alpha\}$, $\zeta_\alpha \in L^2(\mu_\alpha)$, is in H if and only if $\sum_{\alpha\in A} \|\zeta_\alpha\|^2 < \infty$. It is elementary that $\|\zeta_\alpha\| = 0$ for all but countably many $\alpha \in A$. Clearly H is a vector space relative to the coordinatewise linear operations $(\zeta + \eta) = (\zeta_\alpha + \eta_\alpha)$, $\lambda\zeta = (\lambda\zeta_\alpha)$.

From the relation

$$4(\zeta_\alpha,\eta_\alpha) = \|\zeta_\alpha + \eta_\alpha\|^2 - \|\zeta_\alpha - \eta_\alpha\|^2 + i\|\zeta_\alpha + i\eta_\alpha\|^2 - i\|\zeta_\alpha - i\eta_\alpha\|^2,$$

it follows that $\sum_{\alpha\in A} |(\zeta_\alpha, \eta_\alpha)| < \infty$; the formula

$$\langle\zeta, \eta\rangle = \sum_{\alpha\in A} (\zeta_\alpha,\eta_\alpha)$$

defines a sesquilinear form on H such that

$$\langle\zeta, \zeta\rangle = \sum_{\alpha\in A} \|\zeta_\alpha\|^2.$$

In particular, $\langle\zeta, \zeta\rangle > 0$ when $\zeta \neq 0$. Thus H is an inner product space; the norm derived from the inner product is given by

$$\|\zeta\| = \langle\zeta, \zeta\rangle^{1/2} = \{\sum_{\alpha\in A} \langle\zeta_\alpha, \zeta_\alpha\rangle\}^{1/2}.$$

To show that H is norm complete, let ζ_n be a Cauchy sequence in H, say $\zeta_n = (\zeta_{n\alpha})$ for $n = 1, 2, \ldots$ For each $\alpha \in A$:

$$\|\zeta_{m\alpha} - \zeta_{n\alpha}\|^2 \leq \|\zeta_m - \zeta_n\|^2 \qquad \text{for all m and n.}$$

Thus the sequence $\zeta_{1\alpha}, \zeta_{2\alpha}, \ldots, \zeta_{n\alpha}, \ldots$, is Cauchy in $L^2(\mu_\alpha)$. We define

$$\zeta_\alpha = \lim \zeta_{n\alpha} \quad (\alpha \in A),$$

we show that $\zeta_n \longrightarrow (\zeta_\alpha) \in H$. Let $\varepsilon > 0$ and choose an index N such that $\|\zeta_m - \zeta_n\| \leq \varepsilon$ whenever $m, n \geq N$. If $m, n \geq N$ and B is a finite subset of A, then $\sum_{\alpha\in B} \|\zeta_{m\alpha} - \zeta_{n\alpha}\|^2 < \varepsilon^2$.

Keeping n and B fixed and letting $m \longrightarrow \infty$, we have

$$\sum_{\alpha\in B} \|\zeta_\alpha - \zeta_{n\alpha}\|^2 \leq \varepsilon^2.$$

Keeping n fixed and varying B, we see that the family $(\zeta_\alpha, \zeta_{n\alpha})$ is in H and has norm $\leq \varepsilon$. Therefore, the family $(\zeta_\alpha - \zeta_{n\alpha}) + (\zeta_{n\alpha}) = (\zeta_\alpha)$ is also in H and setting $\zeta = (\zeta_\alpha)$, we have

$$\|\zeta - \zeta_n\|^2 \leq \varepsilon^2;\ \forall n \geq N.$$

Hence $\zeta_n \longrightarrow \zeta$ as $n \longrightarrow \infty$. Therefore H is complete.

For each $\tau \in \Sigma$, define

$$F(\tau)(\zeta_\alpha) = (\chi_\tau \zeta_\alpha) \qquad \{ (\zeta_\alpha) \in H\},$$

where χ_τ denotes the characteristic function of τ. We show that $F(.)$ is a countably additive spectral measure on Σ whose values are self-adjoint projections on H.

Let $(\zeta_\alpha) \in H$

$$F(\Omega)(\zeta_\alpha) = (\chi_\Omega \zeta_\alpha) = I(\zeta_\alpha).$$

Hence

$$F(\Omega) = I.$$

Let $\sigma, \delta \in \Sigma$,

$$F(\sigma \cup \delta)(\zeta_\alpha) = (\chi_{\sigma\cup\delta}\, \zeta_\alpha) = (\chi_\sigma\, \zeta_\alpha) + (\chi_\delta \zeta_\alpha) - (\chi_{\sigma\cap\delta}\, \zeta_\alpha)$$
$$= \{F(\sigma) + F(\delta) - F(\sigma\cap\delta)\}(\zeta_\alpha),$$

and

$$F(\sigma)(F(\delta)\zeta_\alpha) = (F(\sigma)F(\delta))\zeta_\alpha$$

$F(\sigma \cap \delta)(\zeta_\alpha) = (\chi_\sigma \cdot \chi_\delta\ \zeta_\alpha) =$ ~~$(\chi_\sigma\ \zeta_\alpha \cdot \chi_\delta\ \zeta_\alpha) = (\chi_\sigma\ \zeta_\alpha) + (\chi_\delta\ \zeta_\alpha)$~~

~~$= F(\sigma)(\zeta_\alpha) + F(\delta)(\zeta_\alpha).$~~

$$F(\delta')(\zeta_\alpha) = (\chi_{\delta'} \zeta_\alpha) = ([1 - \chi_\delta]\ \zeta_\alpha) = I\ (\zeta_\alpha) - F(\delta)(\zeta_\alpha).$$

Let $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots,$ be a countable family of disjoint sets in Σ.

$$(F(\bigcup_{i=1}^{n} \sigma_i)\ \zeta,\ \zeta) = \langle\{\chi_{\bigcup_{i=1}^{n}\sigma_i}\ \zeta_\alpha\},\ \zeta\rangle = \sum_{i=1}^{n} \langle\{\chi_{\sigma_i}\ \zeta_\alpha\},\ \zeta\rangle$$
$$= \sum_{i=1}^{n} \sum_{\alpha\in A} \langle \chi_{\sigma_i}\zeta_\alpha, \zeta_\alpha\rangle = \sum_{\alpha\in A} \sum_{i=1}^{n} \langle \chi_{\sigma_i}\zeta_\alpha, \zeta_\alpha\rangle$$

Since Σ is a σ-field, there exists $\sigma \in \Sigma$ such that $\sigma = \bigcup_{i=1}^{\infty} \sigma_i$.

Therefore

$$\sum_{i=1}^{n} \langle\ \chi_{\sigma_i}\ \zeta_\alpha,\ \zeta_\alpha\ \rangle \longrightarrow \sum_{i=1}^{\infty} \langle\ \chi_{\sigma_i}\ \zeta_\alpha,\ \zeta_\alpha\ \rangle = \langle\ \chi_\sigma\ \zeta_\alpha,\ \zeta_\alpha\rangle,$$

then

$$\sum_{\alpha\in A} \sum_{i=1}^{n} \langle\ \chi_{\sigma_i}\ \zeta_\alpha,\ \zeta_\alpha\ \rangle \longrightarrow \sum_{\alpha\in A} \langle\ \chi_\sigma\ \zeta_\alpha,\ \zeta_\alpha\rangle = \langle\ F(\sigma)\zeta,\ \zeta\rangle.$$

Hence

$$F(\bigcup_{i=1}^{n} \sigma_i) \longrightarrow F(\bigcup_{i=1}^{\infty} \sigma_i) = \sum_{i=1}^{\infty} F(\sigma_i).$$

Clearly $\|F(\sigma)\| \leq 1$, $\sigma \in \Sigma$, and since $F(.)$ is indempotent and $F(.) \neq 0$ it follows that $\|F(\sigma)\| = 1$. Hence $F(.)$ is a countably additive spectral measure whose values are self-adjoint projections on H. let

$$B_o = \{\ F(\sigma)\ :\ \sigma \in \Sigma\}.$$

We show that B_0 is a complete Boolean algebra of projections on H which is naturally isomorphic to $\mathcal{B}$. Define

$$\theta_0 : E(\tau) \longrightarrow F(\tau) \ (\tau \in \Sigma);$$

then

$$\theta_0(E(\tau) + E(\delta)) = \theta_0(E(\tau \cup \delta \setminus \tau \cap \delta)) = F(\tau \cup \delta \setminus \tau \cap \delta)$$

$$= F(\tau) + F(\delta) = \theta_0(E(\tau)) + \theta_0(E(\tau)).$$

Let $\tau' \in \Sigma$ be the complement of τ in Ω.

$$\theta_0(E(\tau')) = F(\tau') = I - F(\tau) = I - \theta_0(E(\tau)) = \theta_0(E(\tau))'.$$

Noting that, for each $\tau \in \Sigma$

$$E(\tau) = 0 \iff E(\tau)x_\alpha = 0 \ (\alpha \in A)$$

$$\iff \mu_\alpha(\tau) = 0 \ (\alpha \in A)$$

$$\iff F(\tau) = 0,$$

it follows that θ_0 is a one-to-one Boolean algebra homeomorphism. Hence θ_0 is a Boolean algebra isomorphism of $\mathcal{B}$ onto B_0. In particular, since $\mathcal{B}$ is complete as an abstract Boolean algebra, then so is B_0. To show that B_0 is a complete Boolean algebra of projections on H, let $\{F(\tau_\delta), \ \delta \in \Delta\}$ be a subset of B_0. Choose τ in Σ such that

$$E(\tau) = \cup \{E(\tau_\delta) : \delta \in \Delta\}.$$

Then

$$E(\tau)X = \overline{\text{span}}\{E(\tau_\delta)X : \delta \in \Delta\},$$

and

$$(I - E(\tau))X = \bigcap_{\delta \in \Delta} (I - E(\tau_\delta))X.$$

First we show that $F(\tau)H = \overline{\text{span}}\{F(\tau_\delta)H : \delta \in \Delta\}$. Since $E(\tau_\delta)X \subseteq E(\tau)X$, $\delta \in \Delta$, and θ_0 is an isomorphism, then

$$\theta_0(E(\tau_\delta) \subseteq \theta_0(E(\tau)) \ (\delta \in \Delta) \iff F(\tau_\delta)H \subseteq F(\tau)H \quad (\delta \in \Delta).$$

Hence

$$F(\tau)H \supseteq \overline{\text{span}} \ \{F(\tau_\delta)H : \delta \in \Delta\}.$$

To obtain the reverse inclusion, let ζ be in the orthogonal complement of $\overline{\text{span}}\{ F(\tau_\delta)H : \delta \in \Delta\}$ in H. We must show that $F(\tau)\zeta = 0$. Let $\zeta = (\zeta_\alpha)$. We show that $\chi_\tau \zeta_\alpha = 0$ (μ_α a.e., $\alpha \in A$), from which it follows that $F(\tau)\zeta = 0$. To do this, fix α in A and consider ζ_α as a function. Define

$$\sigma = \{ t \in \Omega : \zeta_\alpha(t) \neq 0\}.$$

Since $F(\tau_\delta)\zeta = 0$, $\chi_{\tau\delta}\zeta_\alpha = 0$, μ_α a.e., and hence

$$\mu_\alpha(\tau_\delta \cap \sigma) = 0 \ (\alpha \in A).$$

Thus

$$E(\tau_\delta)E(\sigma)x_\alpha = 0 \quad (\alpha \in A)$$

and hence

$$E(\tau)E(\sigma)x_\alpha = 0$$

by the completeness of B on X. Therefore

$$\mu_\alpha(\tau_\delta \cap \sigma) = 0, \quad (\alpha \in A)$$

giving $F(\tau)\zeta = 0$ as required.

Since the value of $F(.)$ are self-adjoint projections, it follows that $F(\tau)H$ and $(I - F(\tau))H$ are orthogonal subspaces. Therefore, $(I-F(\tau))H = \bigcap_{\delta\in\Delta}(I - F(\tau_\delta))H$, since $F(\tau)H = \overline{\text{span}}\{ F(\tau_\delta)H : \delta \in \Delta\}$. Hence B_o is complete.

The isomorphism Θ

Define the mapping, from $C(\Omega)$ into $U(B_o)$, by

$$K : \phi \longrightarrow \int \phi(\lambda)\, F(d\lambda).$$

It is clear that K is an algebraic isomorphism of $C(\Omega)$ into $U(B_o)$. Since $F(.)$ is a self-adjoint spectral measure, it follows that K is an algebraic *-isomorphism. Moreover,

$$\|K(\phi)\| \leq \|\phi\|_\infty. \quad (\phi \in C(\Omega)).$$

Then K is a continuous algebraic *_isomorphism of $C(\Omega)$ into $U(B_o)$. Notice that $E(\tau) \neq 0$ for every open and closed subset τ of Ω, since Ω is the Stone representation space of B. It follows that $F(.)$ has the same property. From the proof of lemma 2.1.7 and the fact that $\|K(\chi_\tau)\| \leq \|\chi_\tau\|_\infty$, where τ is open and closed and hence χ_τ is continuous, it follows that $\|K(\chi_\tau)\| = \|\chi_\tau\|_\infty$. Therefore K is isometric on the linear span of

$$\{\chi_\tau : \tau \text{ open and closed in } \Omega\}.$$

By the Stone–Weierstrass theorem and the continuity of K, K is isometric on $C(\Omega)$. Now

$$B_o = \{ F(\tau) : \tau \text{ open and closed in } \Omega\}$$

since the corresponding result holds for $\mathcal{B}$ and $E(.)$. Since $F(\tau) = K(\chi_\tau)$ for τ open and closed, it follows that the range of K contains B_o. By the continuity of K, it follows that K maps $C(\Omega)$ onto $U(B_o)$. Composing K with the inverse of the map

$$\phi \longrightarrow \int \phi(\lambda)E(d\lambda) \ , \quad \phi \in C(\Omega),$$

we thus obtain an algebraic isomorphism

$$\theta : \int \phi(\lambda)E(d\lambda) \longrightarrow \int \phi(\lambda)F(d\lambda), \quad \phi \in C(\Omega)$$

of $U(\mathcal{B})$ onto $U(B_o)$ which is bicontinuous when each operator algebra is given the norm topology. Let $E(\tau)$ be an element of $\mathcal{B}$, τ open and closed; then

$$\theta : \int \chi_\tau \, E(d\lambda) = E(\tau) \longrightarrow \int \chi_\tau \, F(d\lambda) = F(\tau) = \theta_o(E(\tau)).$$

Hence θ extends the map θ_o of $\mathcal{B}$ onto B_o. We shall write S_ϕ for $\int \phi(\lambda)F(d\lambda)$ and T_ϕ for $\int \phi(\lambda)E(d\lambda)$ when convenient. Note that with these notations we have

$$\theta(T_\phi) = S_\phi \quad (\phi \in C(\Omega)).$$

The following lemma establishes both the weak and the ultraweak bicontinuity of θ.

<u>2.5.3- Lemma:</u> i) Given x in X and x^* in X^*, there exist ζ and η in H with $\|\zeta\| . \|\eta\| \leq 4k\|x\| . \|x^*\|$ such that

$$\langle Tx, x^* \rangle = \langle \theta T\zeta, \eta \rangle \quad (T \in U(\mathcal{B}))$$

ii) Given ζ and η in H and $\varepsilon \geq 0$, there exist x in X and x^* in X^* with $\|x\| . \|x^*\| \leq 4k(1 + \varepsilon)\|\zeta\| . \|\eta\|$ such that

$$\langle Tx, x^* \rangle = \langle \theta T\eta, \eta \rangle \quad (T \in U(\mathcal{B})).$$

<u>Proof:</u> Let $x \in X$ and $x^* \in X^*$. The identity operator I can be expressed as $I = \cup \{ C(x_\alpha) : \alpha \in A\}$ where $C(x_\alpha)\ C(x_\beta) = 0$ if $\alpha \neq \beta$. By the completeness of $\mathcal{B}$, it follows that $\lim_{\alpha} \sum_{\alpha\in\sigma} C(x_\alpha)x = x$, where σ runs over the finite subsets of A, ordered by inclusion. For each $\varepsilon \searrow 0$ there exists σ such that $\|x - \sum_{\sigma} C(x_\alpha)x\| \angle \varepsilon k^{-1}$, so if $\beta \notin \sigma$, then $|C(x_\beta)x| = |C(x_\beta)(x - \sum_{\sigma} C(x_\alpha)x)| \angle \varepsilon$. It follows that at most countably many of the vectors $C(x_\alpha)x$, $\alpha \in A$ are not zero. Since $C(x_\alpha)x \neq 0 \Longleftrightarrow C(x_\alpha)C(x) \neq 0$, it follows that $C(x)C(x_\alpha) \neq 0$ for at most countably many $\alpha \in A$. Denote this countable family by $\{ \alpha(n) : n = 1,2,....\}$. Define an element $\beta = \{\beta_\alpha\}$ as follows

$$\beta_\alpha = [2^{-n}] \quad \text{if } \alpha = \alpha(n) \text{ for some n.}$$
$$= 0 \quad \text{otherwise}$$

where $[2^{-n}]$ is the equivalence class in $L^2(\mu_{\alpha(n)})$ of the function on Ω with constant value 2^{-n}.

The measure $\nu(.) = \langle E(.)x, x^*\rangle$ is absolutely continuous with respect to $\langle F(.)\beta, \beta\rangle$ and the total variation $|\nu|$ of ν satisfies

$$|\nu|(\Omega) = \sup_{\tau\in\Sigma} \langle E(\tau)x, x^*\rangle \leq 4k\|x\|.\|x^*\|.$$

We next show that the cyclic subspace $Z(\beta) = \overline{\text{span}} \{ F(\tau)\beta : \tau \in \Sigma\}$ is isomorphic to $L^2(\mu)$, where $\mu(.) = \langle F(.)\beta, \beta\rangle$. We write $U\chi_\tau = F(\tau)\beta$ for every $\tau \in \Sigma$. If the definition of U is extended from characteristic functions by the requirement of linearity, then in view of the definition of $Z(\beta)$, U becomes a linear transformation from a dense subset of $L^2(\mu)$ onto a dense subset of $Z(\beta)$. The additivity of $F(.)$ guarantees that U is well defined. Since the equations

$$\|\chi_\tau\|^2 = \mu(\tau) = \langle F(\tau)\beta, \beta\rangle = \|F(\tau)\beta\|^2 = \|U\chi_\tau\|^2.$$

show that U is norm-preserving, U may be extended to an isomorphism. If τ, $\sigma \in \Sigma$, then

$$U(\chi_\tau.\chi_\sigma) = U(\chi_{\tau\cap\sigma}) = F(\tau \cap \sigma)\beta = F(\tau)F(\sigma)\beta = F(\tau)\, U\chi_\sigma.$$

This means that $U(\chi_\tau f) = F(\tau)\, Uf$ whenever $f = \chi_\sigma$. Since the simple functions are norm dense in $L^2(\mu)$ the desired conclusion follows. Then, by the Radon-Nikodým theorem, there exists a non-negative function g in $L^1(\mu)$ such that

$$\nu(\tau) = \int_\tau g(\lambda)\, \mu(d\lambda) \qquad (\tau \in \Sigma).$$

If f is the non-negative square root of g, then $f \in L^2(\mu)$. If we put $\zeta = Uf$, then

$$\nu(\tau) = \int_\tau |f(\lambda)|^2 d\mu = \int |\chi_\tau f|^2 \mu(d\lambda)$$

$$= \|\chi_\tau f\|^2 = \|F(\tau)Uf\|^2$$

$$= \|F(\tau)\zeta\|^2 = \langle F(\tau)\zeta,\ \eta\rangle,$$

where $\eta = F(\tau)\zeta$, and

$$|\nu|(\Omega) = \sup_{\tau \in \Sigma} \|F(\tau)\zeta\|.\|\eta\| = \|\zeta\|.\|\eta\|.$$

Then

$$\langle Ex,\ x^*\rangle = \langle(\theta E)\zeta,\ \eta\rangle$$

and so, by linearity and norm continuity,

$$\langle Tx,\ x^*\rangle = \langle(\theta T)\zeta,\ \eta\rangle \qquad (T \in U(B)).$$

ii) Let $\varepsilon \searrow 0$ and let $\zeta = (\zeta_\alpha)$ and η belong to H. Then $\zeta_\alpha \neq 0$ for at most countably many α in A, say for $\alpha(n)$ $(n = 1,2,\ldots\ldots)$. Write x_n for $x_{\alpha(n)}$ and μ_n for $\mu_{\alpha(n)}$. Define

$$e = \sum_{n=1}^{\infty} 2^{-n} x_n\ ,$$

and let $\gamma(.) = \langle F(.)\zeta,\ \eta\rangle$. We have

$$\gamma(.) \ll \langle F(.)\zeta,\ \zeta\rangle \ll \sum_{n=1}^{\infty} 2^{-n} \langle E(.)x_n,\ x_n^*\rangle \ll E(.)e$$

the final relationship following from the fact that

$$C(x_n)E(\sigma)e = 2^{-n}E(\sigma)x_n \qquad (\sigma \in \Sigma,\ n = 1,2,\ldots).$$

Also, the total variation of $\gamma(.)$ satisfies $|\gamma|(\Omega) \leq \|\zeta\|.\|\eta\|$. Since $\gamma(.) \ll E(.)e$, put $\mu(.) = \langle E(.)e,\ e^*\rangle$, where e^* is the Bade functional such that $\langle e.e^*\rangle = 1$. Let ρ be the function norm given by

$$\rho(f) = \sup\{\|T_\phi e\| : \phi \in \mathcal{L}^\infty,\ \|\phi\| \leq f\}$$

and let U be the homeomorphism in lemma 2.3.8, where it is assumed that $X = M(e)$. Clearly $\gamma \ll \mu$ and hence, by the Radon-Nikodým theorem, there exists $\phi \in L^1(\mu)$ such that $\gamma(\tau) = \int_\tau \phi(\lambda)\mu(d\lambda)$, then by theorem 1.3.7 ii) there exist $f \in L^a_\rho$ and $g \in L\rho'$ such that $\phi = fg$ μ-a.e. Let $x = Uf$ and define $x_0^* \in M(e^*)$ by

$$\langle z,\ x_0^*\rangle = \int U^{-1}zg\, d\mu \qquad (z \in M(e))$$

$$|\langle z,\ x_o^*\rangle| = |\int U^{-1}z g\, d\mu| \leq \rho(U^{-1}z)\ \rho'(g)$$

By the proof of lemma 2.3.8 $\|x\| = \|Uf\| \leq \rho(f)$

and $\rho(U^{-1}z)\ \rho'(g) \leq 4k\|z\|\ \rho'(g)$

it follows that

$$\|z\|.\|x_o^*\| \leq 4k\|z\|.\rho'(g).$$

Hence, by theorem 1.3.7 ii)

$$\|x\|.\|x_o^*\| \leq 4k\rho'(g).\rho(f) \leq 4k(1 + \varepsilon).\rho(f).$$

But $\rho(f) \leq \int \phi\, d\mu \leq |\nu|(\Omega)$, and if x^* is the extension of x_o^*, by the Hahn-Banach theorem $\|x^*\| = \|x_o^*\|$. Hence

$$\|x\|.\|x^*\| \leq 4k(1 + \varepsilon)|\nu|(\Omega) \leq 4k(1 + \varepsilon)\|\zeta\|.\|\eta\|.$$

Therefore $\gamma(\tau) = \langle E(\tau)x,\ x^*\rangle \quad (\tau \in \Sigma).$

As in the proof of i), it follows that

$$\langle Tx,\ x^*\rangle = \langle(\Theta T)\zeta,\ \eta\rangle \quad (T \in U(B)).$$

This completes the proof of ii)

It only remains to prove the strong bicontinuity of Θ on bounded sets. To see this, first fix ζ in H and let x in X and x^* in X^* satisfy

$$\langle Tx,\ x^*\rangle = \langle(\Theta T)\zeta,\ \zeta\rangle \quad (T \in U(B)).$$

Let $\phi \in C(\Omega)$ and let ψ be a unimodular Borel function on Ω such that $|\phi| = \psi\phi$. Then

$$\begin{aligned}
\|(\Theta T_\phi)\zeta\|^2 &= \|S_\phi\zeta\|^2 = \langle S_\phi\zeta,\ S_\phi\zeta\rangle = \langle \int \phi(\lambda)F(d\lambda)\zeta,\ \int \phi(\lambda)F(d\lambda)\zeta\rangle \\
&= \int |\phi(\lambda)|^2\langle F(d\lambda)\zeta,\ \zeta\rangle = \langle S_{|\phi|^2}\ \zeta,\ \zeta\rangle \\
&= \langle T_{|\phi|^2}\ x,\ x^*\rangle = \int |\phi|^2\langle E(d\lambda)x,\ x^*\rangle \\
&\leq \|\phi\|_\infty\ |\langle T_{|\phi|}\ x,\ x^*\rangle| = \|\phi\|_\infty\ \langle T_{|\phi|}\ x,\ x^*\rangle \\
&= \|\phi\|_\infty\langle T_{\psi\phi}x,\ x^*\rangle \leq \|\phi\|_\infty\|T_\psi\|.\|T_\phi x\|.\|x^*\| \\
&\leq 4k\|\phi\|_\infty\|x^*\|.\|T_\phi x\|,
\end{aligned}$$

the last inequality following from the unimodularity of ψ. Since

$$\|\phi\|_\infty \leq \|T_\phi\| \quad (\phi \in C(\Omega))$$

we thus have

$$\|(\Theta T)\zeta\|^2 \leq 4k\|T\|.\|x^*\|.\|Tx\|\ ,\quad (T \in U(B)).$$

The strong continuity of Θ on bounded subsets of $U(B)$ is immediate.

To obtain the corresponding result for θ^{-1}, fix $e \neq 0$ in X and let $\varepsilon \searrow 0$. Let e^* be the Bade functional for $e \in X$, $\mu = \langle E(.)e, e^*\rangle$, $\rho = \sup\{\|T_\phi e\| : \phi \in \ell^\infty, |\phi| \leq f\}$ and U the linear isomorphism of $L^a{}_\rho$ onto $M(e)$. Since the constant functon $1 \in L^a{}_\rho$, there exists δ with $0 \angle \delta \angle \varepsilon$ such that $\rho(\chi_\tau) \angle \varepsilon$ whenever $\tau \in \Sigma$ and $\mu(\tau) \angle \delta$. Also, since μ is positive, the proof of part i) of the obove lemma gives the existence of ζ in H such that

$$\langle Te, e^*\rangle = \langle(\theta T)\zeta, \zeta\rangle \qquad (T \in U(\mathcal{B})).$$

Let $S \in U(B_o)$ with $\|S\zeta\| \leq \delta^2 / \{\mu(\Omega)\}^{\frac{1}{2}}$. It is readily seen that $\langle E(\Omega)e, e^*\rangle \neq 0$ if $e \neq 0$. Then

$$S = \theta T_\phi = S_\phi \text{ for some } \phi \in C(\Omega)$$

and setting

$$\tau = \{t \in \Omega : |\phi(t)| \overset{\geq}{\cancel{\leq}} \delta\},$$

we have

$$\begin{aligned}
\delta\mu(\tau) &= \delta \int_\tau \langle E(d\lambda)e, e^*\rangle = \int_\tau \delta\mu(d\lambda) \\
&\leq \int 1.|\phi(\lambda)|d\mu \leq \{\int |\phi(\lambda)|^2 d\lambda\}^{\frac{1}{2}}\{\mu(\Omega)\}^{\frac{1}{2}} \\
&= \langle T_{|\phi|^2} e, e^*\rangle^{\frac{1}{2}}. \{\mu(\Omega)\}^{\frac{1}{2}} \\
&= \langle S_{|\phi|^2} \zeta, \zeta\rangle^{\frac{1}{2}}\{\mu(\Omega)\}^{\frac{1}{2}} \\
&= \|S_\phi\zeta\| \{\mu(\Omega)\}^{\frac{1}{2}} \leq \delta^2.
\end{aligned}$$

Thus $\mu(\tau) \leq \delta$ and hence $\rho(\chi\tau) \leq \varepsilon$. Therefore

$$\begin{aligned}
\|\theta^{-1}Se\| &= \|T_\phi e\| = \|T_\phi U1\| = \|U\phi\| \\
&\leq \rho(\phi) = \rho(|\phi|) = \rho(|\phi|\chi_\tau + |\phi|\chi_{\tau'}) \\
&\leq \rho(\chi_\tau)\|\phi\|_\infty + \rho(|\phi|\chi_{\tau'}) \\
&\leq \|\phi\|_\infty \varepsilon + \delta\rho(1) \leq \{\|\phi\|_\infty + \rho(1)\} \varepsilon \\
&= \{\|S\| + \rho(1)\} \varepsilon.
\end{aligned}$$

The last equality follows from the fact that $\|\phi\|_\infty \leq \|S_\phi\| = \|S\|$, and

$$\|S_\phi\| = \|S\| = |\int \phi F(d\lambda)| \leq |\|\phi\|_\infty.$$

We have thus proved that

$$\|(\theta^{-1}S)e\| = \{\|S\| + \rho(1)\} \varepsilon$$

whenever $S \in U(B_o)$ and $\|S\zeta\| \leq \delta^2 / \{\mu(\Omega)\}^{\frac{1}{2}}$. This establishes the strong continuity of Θ^{-1} on bounded sets and completes the proof of the theorem. Q.E.D.

2.5.4- Remark: A representation theorem for σ-complete Boolean algebra of projections can be obtained from the main result above. Let C be a σ-complete Boolean algebra of projections on X and let $\mathcal{B}$ be the strong closure of C in B(X). We first show that $\mathcal{B}$ is complete. Since C is bounded, then $\mathcal{B}$ is also a bounded Boolean algebra of projections in X. Suppose that $\mathcal{B}$ is not complete. By lemma 2.1.5, there is a monotone increasing generalized sequence $\{E_\alpha\}$ in $\mathcal{B}$ and x in X such that the limit $\lim_\alpha\{E_\alpha x\}$ does not exist. Let X(x) and $X_1(x)$ be the closed linear manifolds generated by the sets $\{Ex/E \in C\}$, $\{Ex/E \in \mathcal{B}\}$, respectively. Since $\mathcal{B}$ is the strong closure of C, each element Ex with E in $\mathcal{B}$ is contained in X(x) and thus $X_1(x) \subseteq X(x)$. Evidently $X(x) \subseteq X_1(x)$ and so $X(x) = X_1(x)$, from which it follows that $EX(x) \subseteq X(x)$ for each E in $\mathcal{B}$. For each E in $\mathcal{B}$, let $\tilde{E}$ denote the restriction of E to X(x). It is clear that the set $\{\tilde{E} / E \in \mathcal{B}\}$ is contained in the strong closure of $C(x) = \{\tilde{E} / E \in C\}$. Since the limit $\lim_\alpha E_\alpha x$ does not exist, the limit $\lim_\alpha \tilde{E}_\alpha x$ fails to exist in the strong topology. It follows from lemma 2.1.5 that the strong closure of C(x) is not complete. Thus, by corollary 2.1.11, C(x) is not complete. On the other hand, since the restriction to an invariant subspace of a σ-complete Boolean algebra of projections is σ-complete, it follows that C(x) is σ-complete. Thus, by the proof of corollary 2.4.2, C(x) is complete, which contradicts the first statement and proves the completeness of $\mathcal{B}$. Now, let B_0 and Θ be as in the main theorem above and let $C_0 = \Theta(C)$. Since every monotone increasing sequence of elements of C of projections converges strongly to an element in C that is bounded and, since Θ is strongly continuous on bounded sets. it follows that C_0 is σ-complete. By restricting Θ to U(C), we thus obtain a representation of U(C) as the C^*-algebra generated by a σ-complete Boolean algebra of self-adjoint projections. This representation has the appropriate continuity properties. Q.E.D.

2.5.5- Definition: A von-Ne~~y~~Uman algebra is a *-algebra of operators on H which is closed in the weak operator topology of B(H).

Since the weak and the strong closures of a complete Boolean algebra of projections are the same, it follows that $U(B_o)$ is a von-Ne~~y~~Uman algebra.

Chapter 3 Restrictions of normal operators and scalar type-spectral operators

3.1- Restrictions of Normal operators

3.1.1- Introduction: The purpose of this chapter is to investigate Fixman's problem in more detail. Fixman has shown that the restriction of a spectral operator to an arbitrary closed invariant subspace is spectral if and only if the resolution of the identity of the spectral operator leaves the subspace invariant.

Wermer and Halmos have obtained some conditions which are necessary and sufficient for a restriction of a normal operator to be normal. It is shown that this problem is a particular case of Dowson's problem on the restrictions of scalar-type spectral operators. Finally, we give a generalization of the necessary and sufficiency of the condition for a property (P) to fail to the case of a scalar-type spectral operator on a separable Banach space.

3.1.2- Definition: Let X be a complex Banach space. An operator T in B(X) is called a spectral operator if and only if the following two conditions α) and β) are satisfied:

α) There exists a spectral measure E with values in B(X) such that

$$TE(\delta) = E(\delta)T, \qquad \delta \in \Sigma,$$

β) $\sigma(T / E(\delta)X) \subseteq \bar{\delta}, \qquad \delta \in \Sigma.$

A spectral operator S is said to be of scalar-type if and only if it is of the form:

$$S = \int_{\sigma(S)} \lambda E(d\lambda)$$

3.1.3- Definition: Let H be a complex Hilbert space and let T be in B(H). Then T is called a normal operator if $TT^* = T^*T$.

Consider a polynomial $p(z,\bar{z})$ in the two complex variables z and $\bar{z}$. In view of the Weierstrass polynomial theorem, the map

$$\psi : p(z,\bar{z}) \longrightarrow p(T,T^*)$$

can be extended to an isometric algebra isomorphism of $C(\sigma(T))$ into a subalgebra of B(H) consisting of normal operators such that

$$\psi(\bar{f}) = \psi(f)^*, \qquad (f \in C(\sigma(T))).$$

3.1.4- Theorem: Let T be a normal operator on H. Then T is a scalar-type spectral operator. The values of the resolution of the identity of T are self-adoint projections.

Proof: Let ψ be the isometric algebra isomorphism discribed above. Let x, y $\in$ H. The map $f \longrightarrow \langle\psi(f)x, y\rangle$ is a bounded linear functional on $C(\sigma(T))$ and so, by the Reisz representation theorem, there is a uniquely determined regular Borel measure $\mu(.,x,y)$ such that, for all f in $C(\sigma(T))$

$$\langle\psi(f)x,y\rangle = \int_{\sigma(T)} f(\lambda)\mu\langle d\lambda,x,y\rangle. \qquad (*)$$

Let Σ_0 denote the σ-algebra of Borel subsets of $\sigma(T)$. Since $\mu(\tau,x,y\rangle$ is uniquely determined by x, y and τ it is, for each τ in Σ_0, bilinear in x and y. Also, since

$$\langle\psi(f)x,y) = \langle x, \psi(f)^*y\rangle = \overline{\langle\psi(\bar{f})y,x\rangle} \qquad (f \in C(\sigma(T)))$$

$\mu(\tau,x,y)$ is symmetric. Since $|\mu(\tau,x,y)| \leq \|x\|.\|y\|$, these bilinear functionals are bounded. It follows that, for each τ in Σ_0 there is a unique self-adjoint operator $E(\sigma)$ on H such that

$$\mu(\tau,x,y) = \langle E(\tau)x,y\rangle. \quad (x,y \in H)$$

It will be shown that E(.) is a spectral measure. By putting $f = 1$ in (*) we see that $I = E(\sigma(T))$ and, since E is additive, that

$$I - E(\delta) = E(\sigma(T)) - E(\delta) = E(\sigma(T) / \delta), \qquad (\delta \in \Sigma_0).$$

To prove that E(.) is a spectral measure, it will therefore suffice to show that $E(\sigma \cap \tau) = E(\tau).E(\sigma)$ for every pair σ,τ of Borel subsets of

$\sigma(T)$. For fixed x, y in H and g in $C(\sigma(T))$ define

$$\nu(\tau) = \int_{\tau} g(\lambda)\,\mu(d\lambda,x,y), \qquad (\tau \in \Sigma_0).$$

If $f \in C(\sigma(T))$, then

$$\int_{\sigma(T)} f(\lambda)\nu(d\lambda) = \int_{\sigma(T)} f(\lambda)g(\lambda)\mu(d\lambda,x,y) = \langle\Psi(f)\Psi(g)x,y\rangle$$

$$= \int_{\sigma(T)} f(\lambda)\mu(d\lambda,\Psi(g)x,y).$$

Since $\nu(.)$ and $\mu(.,\overset{\psi(g)x}{\cancel{x}},y)$ are regular measures

$$\nu(\tau) = \int_{\sigma(T)} g(\lambda)\chi(\tau,\lambda)\,\mu(d\lambda,x,y)$$

$$= \langle E(\tau)\Psi(g)x,y\rangle$$

$$= \int_{\sigma(T)} g(\lambda)\mu(d\lambda,E(\tau)x,y) \qquad (\tau \in \Sigma_0)$$

since g is an arbitrary element of $C(\sigma(T))$,

$$\langle E(\tau \cap \delta)x,y\rangle = \int_{\sigma(T)\cap\delta\cap\tau} \mu(d\lambda,x,y)$$

$$= \int_{\delta} \chi(\tau,\lambda)\,\mu(d\lambda,x,y)$$

$$= \langle E(\delta).E(\tau)x,y\rangle,$$

it follows that, for all pairs of Borel sets δ, τ of $\sigma(T)$

$$E(\tau).E(\delta) = E(\delta).E(\tau) = E(\tau \cap \delta).$$

Hence, $E(.)$ is a spectral measure.

Let $\{\tau_n\}$ be a sequence of pairwise disjoint sets in Σ_0 with union τ. Then

$$\langle E(\tau_r)x,E(\tau_k)x\rangle = \langle E(\tau_k)E(\tau_r)x,x\rangle = 0 \quad (k \neq r)$$

and so, for every x in H, $\{E(\tau_r)x\}$ is an orthogonal sequence of vectors. Since

$$\sum_{n=1}^{\infty} \|E(\tau_n)x\|^2 = \sum_{n=1}^{\infty} \langle E(\tau_n)x,x\rangle\| = \langle E(\tau)x,x\rangle = \|E(\tau)x\|^2,$$

it follows that $E(.)$ is countably additive on Σ_0 in the strong operator topology. Define

$$E(\tau) = E(\tau \cap \sigma(T)) \quad (\tau \in \Sigma, \text{ the } \sigma\text{-algebra of Borel subsets of the complex plane})$$

It follows from (*) and the definition of the integral with respect to a spectral measure that

$$T = \int_{\sigma(T)} \lambda E(d\lambda).$$

Clearly T commutes with all values of $E(.)$. Let $\tau \in \Sigma$ and suppose that $\lambda_0 \in \mathbb{C} / \bar{\tau}$. Define

$$U = \int_{\sigma(T)} (\lambda_0 - \lambda)^{-1} \chi(\tau,\lambda)E(d\lambda).$$

Observe that this operator satisfies the equations

$$(\lambda_0 I - T)U = U(\lambda_0 I - T) = E(\tau).$$

This shows that

$$\sigma(T / E(\tau)H) \subseteq \bar{\tau}. \qquad (\tau \in \Sigma)$$

Q.E.D.

We state the following result, valid for a general bounded linear operator on a Banach space. See lemma 1.28 of [3, 20].

3.1.5- Lemma: Let X be a complex Banach space and let T be a bounded operator on X. Suppose that T^{-1} exists as a bounded linear operator on X. Let Y be a closed subspace of X with $TY \subseteq Y$. Then $(T / Y)^{-1}$ exists as a bounded linear operator on Y if and only if $T^{-1}Y \subseteq Y$.

Proof: If $T^{-1}Y \subseteq Y$, then clearly T^{-1}/Y is a bounded linear operator and inverse to T / Y. Conversely, suppose that $(T/Y)^{-1} \in B(Y)$ and let $y_0 \in Y$. Then there is a unique element y in Y such that $Ty = y_0$. However, $T(T^{-1}y_0) = y_0$. Hence $T^{-1}y_0 = y \in Y$ and so $T^{-1}Y \subseteq Y$. Q.E.D.

The next result was proved by Fixman for a spectral operator [11]; a much shorter proof was given by Dowson (Theorem 2 of [4, 441]). We state here the case where T is of scalar-type.

3.1.6- Theorem: Let X be a Banach space, and let T, in B(X) be a scalar-type spectral operator with resolution of the identity E(.). Let Y be a closed subspace of X invariant under T.

a) If T / Y is a scalar-type spectral operator with resolution of the identity F(.) then $E(\tau)Y \subseteq Y$ and $F(\tau) = E(\tau)/Y$, for each τ in Σ.

b) If $E(\tau)Y \subseteq Y$ and $F(\tau) = E(\tau)/Y$, then T/Y is a scalar-type spectral operator.

Proof: Before proving this theorem we need the following definition and remarks.[10].

3.1.7- Definition: Let $T \in B(X)$. An X-valued function *f* defined and analytic on an open set $D(f) \subseteq \mathbb{C}$ is called an analytic extension of $T(\zeta) = (\zeta - T)^{-1}x, \zeta \in \rho(T)$ if

$$(\zeta - T)f(\zeta)x = x, \quad \zeta \in D(f),$$

$f(\zeta) = T(\zeta)$ on $D(f) \cap \rho(T)$. ~~for otherwise $\{(\zeta - T)f(\zeta) - T(\zeta)\}x = x - x = 0$ would $\overset{i}{\text{emply}}$ $\zeta \in \sigma(T)$~~.

The resolvent set of x, which is a maximal open set and may serve as a domain of definition of an analytic extension of $T(\zeta)x$, is denoted by $\rho(x)$ or $\rho_T(x)$. Its complement $\sigma(x)$ is called the spectrum of x.

Let T be a bounded spectral operator on X with resolution of the identity $E(.)$ and let $\sigma \subseteq \mathbb{C}$ be closed, then

$$E(\sigma)X = \{ x : \sigma(x) \subseteq \sigma\}.$$

For a proof of this result the reader is referred to [10, 1934].

Proof of theorem 3.1.6:

a) Let $y \in Y$. The function $y_{T/Y}(\zeta)$ is an analytic extension of $T(\zeta)y$ with domain $\rho_{T/Y}(y)$. Thus $\rho_{T/Y}(y) \subseteq \rho_T(Y)$, or

$$1) \qquad \sigma_T(y) \subseteq \sigma_{T/Y}(y).$$

Let F denote the resolution of the identity for T/Y. If σ is a closed subset of the complex plane, we have by the remark following definition 3.1.7

$$\sigma_{T/Y}(F(\sigma)y) \subseteq \sigma$$

and by 1)

$$\sigma_T(F(\sigma)y) \subseteq \sigma$$

and again by the same remark

$$2) \qquad E(\sigma)F(\sigma)y = F(\sigma)y.$$

If τ is a closed set disjoint from σ, we get, operating with $E(\sigma)$ on $E(\sigma)F(\sigma)y = F(\sigma)y$

$$3) \qquad E(\sigma)E(\tau)F(\tau)y = 0 = E(\sigma)F(\tau)y.$$

3) and the σ-additivity of F in the strong operator topology show that $E(\sigma)F(\sigma') = 0$. (σ' denotes the complement of σ with respect to $\mathbb{C}$) This together with 2) gives

$$E(\sigma)y = F(\sigma)y, \quad \sigma \text{ closed.}$$

The properties of E and F now yield the same properties for every Borel set. [11].

b) It is easily verified that T/Y satisfies the axiom α) of definition 3.1.2. In the case $Y = E(\sigma)X$ we have

$$\sigma(T/E(\sigma)E(\tau)X) = \sigma(T/E(\delta \cap \tau) \subseteq \overline{\delta \cap \tau} \subseteq \bar{\tau}.$$

Hence $T/E(\delta)X$ is spectral. Now observe that, since T is of scalar-type, $T = \int_{\sigma(T)} \lambda E(d\lambda)$, $(\lambda_0 I - T)^{-1} = \int (\lambda_0 - \lambda)^{-1} E(d\lambda)$, and since $E(\tau)Y \subseteq Y$, it follows that $(\zeta I - T)^{-1}$, $\zeta \in \rho(T)$ leaves Y invariant. Hence $\sigma(T/Y) \subseteq \sigma(T)$, by lemma 3.1.5. This argument may be applied to the spectral operator $T/E(\tau)X$ and its restriction to the closed invariant subspace $E(\tau)Y$. Therefore

$$\sigma(T/E(\tau)Y) \subseteq \sigma(T/E(\tau)X) \subseteq \bar{\tau}, \quad \tau \in \Sigma.$$

Since

$$T = \int_{\sigma(T)} \lambda E(\lambda),$$

then

$$T/Y = \{\int_{\sigma(T)} \lambda E(d\lambda)\}/Y = \int_{\sigma(T)} \lambda\, E(d\lambda)/Y = \int_{\sigma(T)} \lambda F(d\lambda)$$

where $F(.) = E(.)/Y$.

Hence T/Y is of scalar-type. Q.E.D.

<u>3.1.8- Theorem:</u> Let H be a Hilbert space, and let T, in B(H) be a normal operator. Let Y be a closed subspace of H invariant under T. Then the following are equivalent

a) T/Y is normal

b) T/Y is spectral

c) $T^*/Y \subseteq Y$, i.e. Y is reducing.

<u>Proof:</u> *c*) ⇒ *a*) Suppose Y is reducing. Let $R = T/Y$ and $T^*/Y = S$. Since Y is invariant under both T and T^*, the operators R and S are both defined on Y. Also, we have for all x, y ∈ Y

$$\langle R^*x, y\rangle = \langle x, Ry\rangle = \langle x, Ty\rangle = \langle T^*x, y\rangle = \langle Sx, y\rangle.$$

Since R and S are operators on Y we obtain R = S.

~~Moreover, If $y \in Y$, $RSy = TT^*y = T^*Ty = SRy$~~

~~and so~~ T/Y is a normal operator, since $T^*Y \subseteq Y$

a) ⇒ *b*) Since T/Y is normal then, it is a spectral operator of scalar-type.

b) ⇒ c) Since $T^* = \int_{\sigma(T)} \bar{\lambda} E(d\lambda)$ if $T = \int_{\sigma(T)} \lambda E(d\lambda)$, it follows that, by theorem 3.1.6 a), Y is reducing. Q.E.D.

3.1.9- Theorem: Let X be a Banach space, and let T in $B(X)$ be a spectral operator. Let Y be a closed subspace of X invariant under T and such that T/Y is spectral. Then $\sigma(T/Y) \subseteq \sigma(T)$.

Proof: Y is invariant under the resolution of the identity of T, by theorem 3.1.6 *a*). The argument in the proof of theorem 3.1.6 *b*) shows that $\sigma(T/Y) \subseteq \sigma(T)$.

3.1.10- Corollary: Let H be a Hilbert space, and let T in $B(H)$ be normal. Let Y be a closed subspace of H invariant under T. If T/Y is normal then $\sigma(T/Y) \subseteq \sigma(T)$.

Proof: Follows at once from theorems 3.1.8 and 3.1.9.

3.1.11- Lemma: Let T be a normal operator on H with resolution of the identity $E(\,.\,)$. Let $x \in H$ and let

$$M(x) = \overline{\text{span}}\{E(\tau)x : \tau \in \Sigma\}.$$

Then $M(x)$ is separable.

Proof: By the proof of Lemma 2.4.3, $M(x)$ is isomorphic to $L^2(\mu)$, where $\mu(\,.\,) = \langle E(\,.\,)x, x\rangle$. Therefore $M(x)$ is separable.

3.1.12- Lemma: Let S be a scalar-type spectral operator on X, a Banach space, with resolution of the identity $E(\,.\,)$. Let $x \in X$ and let

$$M(x) = \overline{\text{span}}\{E(\tau)x, \tau \in \Sigma\}.$$

Then $M(x)$ is separable.(Lemma 1 of [5,305])

Proof: Define $\quad \Psi(f) = \int_{\sigma(S)} f(\lambda)E(d\lambda) \qquad (f \in C(\sigma(S)))$.

Then $f \longrightarrow \Psi(f)$ is a bicontinuous algebra isomorphic of $C(\sigma(S))$ into a subalgebra of $B(X)$. Let $\{P_n : n = 1,2,3,\ldots.\}$ be an enumeration of all polynomials in z and $\bar{z}$ with rational coefficients and defined on

$\sigma(S)$. The set $\{P_n\}$ is dense in $C(\sigma(S))$ by the Stone-Weierstrass theorem. Define

$$Y = \overline{\text{span}}\{ \Psi(P_n)x : n = 1,2,\ldots.\}.$$

Then Y is separable and $Y \subseteq M(x)$. Let $z \in Y^{\perp}$. Observe that

$$\langle\Psi(f)x, z\rangle = 0 \qquad (f \in C(\sigma(S)).$$

Define $\mu(.) = \langle E(.)x,z\rangle$. Then

$$\int_{\sigma(S)} f(\lambda)\mu(d\lambda) = 0 \qquad (f \in C(\sigma(S))).$$

By the Riesz representation theorem, μ is identically zero and so

$$\langle E(\tau)x, z\rangle = 0 \quad (z \in Y^{\perp}, \tau \in \Sigma)$$

since the support of $E(.)$ is $\sigma(S)$. Thus $Y = M(x)$ is separable.

3.1.13- Lemma: Let S be a scalar-type spectral operator on X, a Banach space, with resolution of the identity $E(.)$. Then there is a closed separable subspace Y of X invariant under S such that S/Y is spectral and $\sigma(S/Y) = \sigma(S)$.

Proof: Let $\{\lambda_r \,, r = 1,2,3,\ldots\}$ be a countable dense subset of $\mathbb{C}$ containing a dense subset of $\sigma(S)$. Let λ_k be a particular element of the sequence. With λ_k as centre construct a sequence of open discs, say $\{S_j\}_{j=1}^{\infty}$, so that, if r_j is the radius of S_j then $\lim_{j\to\infty} r_j = 0$. For each disc, choose a vector x_{kj} in $E(S_j)X$. If $S_j \cap \sigma(S) = 0$, then $x_{kj} = 0$; otherwise we may and shall choose $x_{kj} \neq 0$. In this way we obtain an infinite sequence of vectors $\{x_{kj}\}_{j=1}^{\infty}$ associated with λ_k. Repeating this process for each λ_k $(k = 1,2,\ldots)$ we obtain a doubly indexed sequence of vectors $\{x_{ij}\}_{i,j=1}^{\infty}$. By lemma 3.1.12, $M(x_j)$ is a separable subspace of X. Hence

$$Y = \overline{\text{span}}\{ \cup M(x_{ij}), i, j = 1,2,\ldots.\}$$

is separable. Since each of the subspaces $M(x_{ij})$ is invariant under $E(\tau)$ $(\tau \in \Sigma)$, it follows, from the linearity and continuity of S that Y is invariant under $E(\tau)$ $(\tau \in \Sigma)$. Hence $S/Y \subseteq Y$. Since

$$S^* = \int_{\sigma(S)} \bar{\lambda}E(d\lambda)$$

it follows that $S^*Y \subseteq Y$. By theorem 3.1.8 b), S/Y is spectral.

Finally, we show that $\sigma(S/Y) = \sigma(S)$. By theorem 3.1.9 $\sigma(S/Y) \subseteq \sigma(S)$, then it is enough to show that $\sigma(S) \subseteq \sigma(S/Y)$. Suppose that $\lambda_k \in \sigma(S)$. Then, given any neighbourhood $N(\lambda_k)$ of λ_k, some $S_j(\lambda_k)$ has the following properties

$$1)\ S_j(\lambda_k) \subseteq N(\lambda_k); \qquad 2)\ S_j(\lambda_k) \cap \sigma(S) \neq \Phi.$$

By construction there is an $x_{ki} \neq 0$ such that $E(S_i)x_{ki} = x_{ki}$. This means that $S_j(\lambda_k)$ and hence $N(\lambda_k)$ contains a point of $\sigma(S/Y)$. If this point is not λ_k, then this shows that λ_k is a limit point of $\sigma(S/Y)$, and hence $\lambda_k \in \sigma(S/Y)$, since this set is closed. Thus $\sigma(S) \subseteq \sigma(S/Y)$ since $\sigma(S)$ has a dense subset consisting of points λ_k. Q.E.D.

3.1.14-Corollary: Let T be a normal operator on a Hilbert space H with a resolution of the identity E(.). Then there is a closed separable subspace Y of H invariant under T such that T/Y is normal and $\sigma(T/Y) = \sigma(T)$.

Proof: Follows from theorem 3.1.8. and corollary 3.1.10.

3.1.15- Lemma: Let Y be a separable Banach space and let S be a scalar-type spectral operator on Y with resolution of the identity E(.). There is a vector y in Y such that whenever $\tau \in \Sigma$ and $E(\tau)y = 0$ then also $E(\tau) = 0$.

Proof: $\{E(\tau) : \tau \in \Sigma\}$ forms a σ-complete Boolean algebra of projections. By the proof of corollary 2.4.2, $\mathcal{B}$, the closure in the strong operator topology of this Boolean algebra of projections is complete. Thus corresponding to each x in Y, there is a carrier projection C(x) of x given by

$$C(x) = \cap \{ E \in \mathcal{B} : Ex = x \}.$$

Let $\{ C(x_\alpha) : \alpha \in A\}$ be a maximal family of carrier projections with the properties $C(x_\alpha) \neq 0$ $(\alpha \in A)$ and $C(x_\alpha)C(x_\beta) = 0$ if $\alpha \neq \beta$. We claim that $\{C(x_\alpha)\}$ is countable. Observe that $\mathcal{B}$ is a bounded Boolean algebra of projections because $\{E(\tau) : \tau \in \Sigma\}$ is bounded. We can assume without loss of generality that $\|x_\alpha\| = 1$ $(\alpha \in A)$.If $C(x_\alpha)C(x_\beta)$

= 0, then

$$\|x_\alpha - x_\beta\| \geq 1/M \; \|C(x_\alpha)(x_\alpha - x_\beta)\| = \|x_\alpha\| \; / \; M = 1 \; / \; M,$$

where M is a positive bound for the norms of the projections in B. It follows from the separability of Y that $\{C(x_\alpha)\}$ is countable and so we may take the set of positive integers as an index set A. Define

$$y = \sum_{r=1}^{\infty} 2^{-r} x_r.$$

Let $\tau \in \Sigma$ and suppose that $E(\tau)y = 0$. Then $C(x_r)E(\tau)y = 2^{-r}E(\tau)x_r = 0$ and so $E(\tau)x_r = 0$. Because $C(x_r)$ is the carrier projection of x_r

$$(I - E(\tau))C(x_r) = C(x_r)$$

$$E(\tau)C(x_r) = 0 \quad (r = 1,2,\ldots.).$$

If $E(\tau) \neq 0$ there is a vector x in $E(\tau)Y$ with $\|x\| = 1$. From the last equation

$$C(x)C(x_r) = 0 \quad (r = 1,2,\ldots)$$

contradicting the maximality of the family $\{C(x_r) : r = 1,2,\ldots.\}$. We deduce that $E(\tau) = 0$. Q.E.D.

3.1.16- Corollary: Let H be a separable Hilbert space and let T be a normal operator on H with resolution of the identity $E(.)$. Then there is a vector y in H such that whenever $\tau \in \Sigma$ and $E(\tau)y = 0$ then also $E(\tau) = 0$.

Proof: By theorem 3.1.4, T is a scalar-type spectral operator. The conclusion follows at once from theorem 3.1.15, since $M(x)$, $x \in H$ is separable by lemma 3.1.11. Q.E.D.

3.1.17- Definition: Given a sequence of simple closed rectifiable curves $\{C_n \; , \; n = 1, 2, 3, \ldots, \}$ in $\mathbb{C}$, we say that the sequence increases monotonically, if each C_n lies inside C_{n+1}.

We recall that a measure μ is finite if $|\mu(\mathbb{C})|$ is finite.

Throughout the work that follows we write $\Sigma(\mu)$ for the support of μ. [16, 54-64] and [17].

3.1.18- Lemma: Let Γ_1, Γ_2 and C be non-intersecting simple closed rectifiable Jordan curves such that C is the interior of Γ_2 and Γ_1 is in the interior of C. Let $f(\lambda)$ be a function which vanishes at infinity and is holomorphic for λ outside or on Γ_1. Suppose that μ is a compact measure such that C is contained in $\Sigma(\mu)$. Then given $\varepsilon > 0$, arbitrary, there exists a complex measure ν with $\nu \ll \mu$ such that for λ on or exterior to Γ_2,

$$|f(\lambda) - \int (t - \lambda)^{-1} d\nu(t)| < \varepsilon.$$

Proof: Since $f(\lambda)$ is holomorphic for λ outside Γ_1, and vanishes at infinity, then by the Cauchy integral formula, for λ exterior to C

$$f(\lambda) = 1/(2\pi i) \int (t - \lambda)^{-1} f(t)\, dt,$$

where the integral is taken around C in a positive direction with respect to the exterior of C. Let L be the length of C and let $M >$ $\max|f(\lambda)|$ for λ on or exterior to C. For t on C and λ on or exterior to Γ_2, $(t - \lambda)^{-1} f(t)$ is a uniformly continuous function of t independent of λ, i.e., given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|(t_1 - \lambda)^{-1} f(t_1) - (t_2 - \lambda)^{-1} f(t_2)| < \varepsilon / 2L$$

for λ on or exterior to Γ_2 and for $|t_1 - t_2| < \delta$. Since C is a rectifiable Jordan curve, there is a partition of C such that the norm of the partition is less than δ. Let $C_1, C_2, C_3, \ldots, C_n$ be the partition points on C, where the subcripts of C_i's increase as we proceed around C in a positive direction with respect to the exterior of C. Let

$$a_i = - \frac{f(C_i)(C_{i+1} - C_i)}{2\pi i}, \quad i = 1, 2, \ldots, n,$$

where we take $C_{n+1} = C_1$. Note that, from the definition of the a_i,

$$\Sigma\, |a_i| < ML.$$

Define
$$g_1(\lambda) = f(\lambda) - \sum_{j=1}^{n} (\lambda - C_j)^{-1} a_j$$

Then $g_1(\lambda)$ is holomorphic on or exterior to Γ_2.[In fact, $g_1(\lambda)$ is holomorphic exterior to C]. Furthermore for λ on or exterior to Γ_2,

$$|g_1(\lambda)| = \left| \frac{1}{2\pi i} \int \frac{f(t)}{t - \lambda}\, dt - \sum_{i=1}^{n} \frac{a_i}{\lambda - C_i} \right|$$

$$= \left| \frac{1}{2\pi i} \sum_{i=1}^{n} \int_{C_i}^{C_{i+1}} \left[\frac{f(t)}{t - \lambda} - \frac{f(C_i)}{C_i - \lambda} \right] dt \right|$$

$$\le \frac{1}{2\pi}\sum_{i=1}^{n}\int_{C_i}^{C_{i+1}}\left|\frac{f(t)}{t-\lambda}-\frac{f(C_i)}{C_i-\lambda}\right|dt < \varepsilon/2$$

About each C_j we can construct a closed disc of radius δ_j such that for $|t - C_j| \le \delta_j$, we have

$$|(C_j-\lambda)^{-1}-(t-\lambda)^{-1}| \le \varepsilon(2ML)^{-1}$$

for any λ on or exterior to Γ_2. Call this disc F_j. We assume that δ_j are cho~~o~~sen so that $F_j \cap F_i = \Phi$ (the empty set) for $i \ne j$.

Define ν on F_j by

$$\nu(E) = \int_E \frac{-a_j}{\mu(F_j)}\,d\mu \quad \text{for } E \subseteq F_j \ .$$

This defines ν for any subset of the set $F = \bigcup_{j=1}^{n} F_j$. Define $\nu(E) = 0$ if $E \cap F = \Phi$. Then $\nu \ll \mu$. Consider

$$f_1(\lambda) = f(\lambda) - \int\frac{d\nu(t)}{t-\lambda}\ .$$

For λ on or exterior to Γ_2,

$$|f_1(\lambda)| \le |g_1(\lambda)| + \left|\sum_{i=1}^{n}\frac{a_i}{(\lambda-C_i)} - \int\frac{d\nu(t)}{t-\lambda}\right|$$

The first term on the right has already been shown to be less than $\varepsilon/2$. Consider

$$\left|\sum_{j=1}^{n}\frac{a_j}{\lambda-C_j}-\int\frac{d\nu(t)}{t-\lambda}\right| = \left|\sum_{j=1}^{n}\left[\frac{a_j}{\lambda-C_j}+\int_{F_j}\frac{1}{t-\lambda}\,d\nu(t)\right]\right|$$

$$\le \sum_{j=1}^{n}\int_{F_j}\left|\frac{1}{\lambda-C_j}-\frac{1}{\lambda-t}\right|\cdot\frac{|a_j|}{\mu(F_j)}\,d\mu(t)$$

$$< \frac{\varepsilon}{2ML}\sum_{j=1}^{n}\int_{F_j}\frac{|a_j|}{\mu(F_j)}\,d\mu \le \frac{\varepsilon}{2}\ .$$

Therefore, $|f_1(\lambda)| < \varepsilon$ on or exterior to Γ_2. According to the definition of ν, it follows that C is measurable and $|\nu(C)| \le \sum_{j=1}^{n}|a_j| < ML$. Thus ν is finite. Q.E.D.

The following well-known lemma is stated without proof.

3.1.19- Lemma: The function $f_1(\lambda) = f(\lambda) - \int (t - \lambda)^{-1} d\nu(t)$, defined in the preceding lemma, is holomorphic for λ on or exterior to Γ_2, and $f_1(\lambda)$ vanishes at infinity.

3.1.20- Lemma: Let μ be a compact measure and suppose that $\Sigma(\mu)$ includes a monotonically increasing sequence of closed rectifiable Jordan curves $\{C_n\}_{n=0}^{\infty}$. Then there is a finite complex measure ν with $\nu \ll \mu$ such that $\int \lambda^n d\nu = 0$ for $n \geq 0$, but $\nu \not\equiv 0$.

Proof: Let $f(\lambda)$ be a function which is holomorphic for λ on or exterior to C_0 and such that $f(\lambda)$ vanishes at infinity. Assume also that $f(\lambda) \not\equiv 0$. Since μ is compact, there is a positive number r such that $|\lambda| \leq r$ for λ in $\Sigma(\mu)$. The Runge method of approximation is used to give a representation of $f(\lambda)$ as a Borel series for $|\lambda| \searrow r$. We use essentially the same method of approximation to show that there is a complex measure ν and a set S, with $\nu \ll \mu$, such that

$$f(\lambda) = \int_S \frac{1}{t - \lambda} \, d\nu(t), \quad \text{for} \quad |\lambda| \searrow r.$$

Let L_n be the length of C_n. Choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\varepsilon_n \searrow 0$ for n = 1, 2, ... and $\sum_{n=1}^{\infty} \varepsilon_n L_n \angle 0$.

We proceed by applying Lemma 3.1.18 to each of the curves with an odd subscript. Let $C = C_1$, $\Gamma_1 = C_0$, $\Gamma_2 = C_2$ and $\varepsilon = \varepsilon_3$.

Then, by lemma 3.1.18 there exists a complex measure $\nu_1 \ll \mu$ such that $|f(\lambda) - \int (t - \lambda)^{-1} d\nu_1(t)| \angle \varepsilon_3$. for λ on or exterior to C_2. To indicate the curves of the sequence to which they belong, we add a superscript to the coefficients in the rational approximation and the symbol for the closed sets. Thus for some real M

$$\sum_{j=1}^{n_1} |a^1{}_j| \angle ML_1, \text{ and } F^1 = \bigcup_{j=1}^{n_1} F^1{}_j \,.$$

Consider the function

$$f_1(\lambda) = f(\lambda) - \int (t - \lambda)^{-1} d\nu_1(t)$$

by lemma 3.1.19, $f_1(\lambda)$ is holomorphic for λ on or exterior to C_2. Let $C = C_3$, $\Gamma_1 = C_2$, $\Gamma_2 = C_4$, and $\varepsilon = \varepsilon_5$. Then by applying lemma 3.1.18, again, we obtain a function

$$f_2(\lambda) = f_1(\lambda) - \int (t - \lambda)^{-1} d\nu_2(t)$$

where ν_2 is a complex measure with $\nu_2 \ll \mu$ and with $|f_2(\lambda)| \angle \varepsilon_5$ for λ on or exterior to C_4. By lemma 3.1.19, $f_2(\lambda)$ is a holomorphic function for λ on or outside C_4 and it vanishes at infinity. We note that

$$\sum_{j=1}^{n_3} |a^3{}_j| \angle \varepsilon_3 L_3.$$

Now, $\Sigma(\nu_1)$ is in the annulus bounded by C_0 and C_2 while $\Sigma(\nu_2)$ is in the annulus bounded by C_2 and C_4. Furthermore, from the definition of $\Sigma(\nu_1)$ it follows that C_2 does not intersect $\Sigma(\nu_1)$. Hence $\Sigma(\nu_1) \cap \Sigma(\nu_2) = \Phi$. Thus without ambiguity, we define a measure ν such that

$$\nu(E) = \nu_1(E) \text{ for } E \text{ in } \Sigma(\nu_1)$$

and

$$\nu(E) = \nu_2(E) \text{ for } E \text{ in } \Sigma(\nu_2) .$$

Then we have

$$f_2(\lambda)\cancel{m} = f(\lambda) - \int_{F^1 \cap F^3} (t\cancel{1} - \lambda)^{-1} d\nu(t).$$

Proceeding by induction, suppose $f_{m-1}(\lambda)$ has been defined and ν has been defined on $\bigcup_{j=1}^{m-1} F^{2j-1}$. We apply lemma 3.1.18 to the function $f_{m-1}(\lambda)$ with $C = C_{2m-1}$, $\Gamma_1 = C_{2m-2}$, $\Gamma_2 = C_{2m}$, and $\varepsilon = \varepsilon_{2m+1}$. We obtain a function

$$f_m(\lambda) = f_{m-1}(\lambda) - \int (t - \lambda)^{-1} d\nu_m(t)$$

with $|f_m(\lambda)| \angle \varepsilon_{2m+1}$ for λ on or exterior to C_{2m},

$$\sum_{j=1}^{n_{2m-1}} |a^{2m-1}{}_j| \angle \varepsilon_{2m-1} \cdot L_{2m-1}$$

and $\Sigma(\nu_m)$ is contained in F^{2m-1}. By lemma 3.1.19, $f_m(\lambda)$ is holomorphic for λ on or exterior to C_{2m} and vanishes at infinity. Defining $\nu(E) = \nu_m(E)$ for E in $\Sigma(\nu_m)$, we have

$$f_m(\lambda) = f(\lambda) - \int_K (t - \lambda)^{-1} d\nu(t), \text{ where } K = \bigcup_{j=1}^{m} F^{2j-1}.$$

Since $\lim_{n\to\infty} \varepsilon_n = 0$, it follows that $\lim_{n\to\infty} f_m(\lambda) = 0$, if $|\lambda| \leftthreetimes r$. Hence

$$f(\lambda) = \int_S (t - \lambda)^{-1} d\nu(t)$$

for $|\lambda| \leftthreetimes r$, where $S = \bigcup_{j=1}^{\infty} F^{2j-1}$.

We now repeat the approximation process. This time we use the curves with even subscripts and approximate $- f(\lambda)$. The function $- f(\lambda)$ is holomorphic for λ on or exterior to C_1 and $- f(\lambda)$ vanishes at infinity. We apply lemmas 3.1.18 and 3.1.19. Let $C = C_2$, $\Gamma_1 = C_1$, $\Gamma_2 = C_3$ and $\varepsilon = \varepsilon_4$. Then there is a function

$$g_1(\lambda) = - f(\lambda) - \int (t - \lambda)^{-1} \, d\gamma_1(t)$$

with $g_1(\lambda)$ holomorphic for λ outside or on C_3, $g_1(\lambda)$ vanishes at infinity, and $|g_1(\lambda)| \angle \varepsilon_4$ for λ outside or on C_3. Also $\sum_{j=1}^{n_2} |a^2{}_j| \angle ML_2$. We choose the $F^2{}_j$ so that $F^2 \cap S = \Phi$. Then we define ν on F^2 by $\nu(E) = \gamma_1(E)$ for $E \subseteq F^2$. After applying the preceding lemma m times, we have

$$g_m(\lambda) = - f(\lambda) - \int_M (t - \lambda)^{-1} \, d\nu(t)$$

where $M = \bigcup_{j=1}^{m} F^{2j}$ and $M \cap S = \Phi$. For λ on or outside C_{2m+1}, $g_m(\lambda)$ is holomorphic and $|g_m(\lambda)| \angle \varepsilon_{2m+2}$. Since $\lim_{n \to \infty} \varepsilon_n = 0$, it follows that $\lim_{n \to \infty} g_n(\lambda) = 0$ for $|\lambda| > r$. Thus

$$- f(\lambda) = \int_A (t - \lambda)^{-1} d\nu(t)$$

where $A = \bigcup_{j=1}^{\infty} F^{2j}$. Define $\nu(E) = 0$ if $E \cap (A \cup S) = \Phi$. Then we have

$$\text{*)} \quad \int (t - \lambda)^{-1} d\nu(t) = \int_S (t - \lambda)^{-1} d\nu(t) + \int_A (t - \lambda)^{-1} d\nu(t)$$

$$= f(\lambda) - f(\lambda) = 0$$

for $|\lambda| > r$.

From the definition of ν, it follows that $\nu \ll \mu$ and $\nu \not\equiv 0$. Also

$$|\nu(\mathbb{C})| \leq \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} |a^i{}_j| .$$

It was pointed out above that

$$\sum_{j=1}^{n_1} |a^1{}_j| \leq ML_1, \ \sum_{j=1}^{n_2} |a^2{}_j| \angle ML_2, \text{ and } \sum_{j=1}^{n_i} |a^i{}_j| \leq \varepsilon_i L_i$$

for i = 3, 4, ... Hence, from the choice of the ε_n,

$$|\nu(\mathbb{C})| \leq ML_1 + ML_2 + \sum_{n=3}^{\infty} \varepsilon_n L_n \angle \infty.$$

Therefore ν is finite. Then, ν is of compact support , and if $t \in \Sigma(\nu)$, then $|t| \leq r$. Thus for $|\lambda| > r$ and $t \in \Sigma(\nu)$, we have

$$\text{**)} \quad (t - \lambda)^{-1} = - \sum_{n=0}^{\infty} t^n \lambda^{-n-1}$$

Since this series is uniformly convergent in t for $|\lambda| > r$ and $t \in$

$\Sigma(\nu)$, where ν is finite, integration term by term is just~~e~~ified. Therefore

$$***) \quad \int (t - \lambda)^{-1} \, d\nu(t) = -\sum_{n=0}^{\infty} \int \lambda^{-(n+1)} t^n \, d\nu(t)$$

for $t \in \Sigma(\nu)$ and $|\lambda| > r$. By equation *), the left side of the equation ***) is identically zero for $|\lambda| > r$. However, the right hand side of the equation ***) cannot be identically zero for $|\lambda| > r$ unless the coefficient of each power of λ is zero. Hence, we have

$$\int t^n d\nu(t) = 0 \text{ (for } n = 0, 1, 2, \ldots\text{).}$$

Q.E.D.

3.1.21- Lemma: Let T be a normal operator on H, a Hilbert space, with resolution of the identity E(.). Suppose that $\sigma(T)$ contains an infinite sequence of simple closed rectifiable curves which increases monotonically. Then there is a closed subspace H_0 of H such that $TH_0 \subseteq H_0$ but T / H_0 is not a normal operator.

Before proving this lemma we need the following definition

3.1.22- Definition: According to the spectral theorem for normal operators

$$\langle Tx, y \rangle = \int \lambda \, d\mu_{xy}$$

where T is a normal operator on H , x, y $\in$ H and μ_{xy} is a complex measure defined for Borel sets in the plane. Also, for any Borel set M of the complex plane, there is defined a projection E(M) on H such that

$$\langle E(M)x, y \rangle = \mu_{xy}(M).$$

The set function μ_{xy} is a finite measure with compact support. We say that μ_{xy} has property (P) if and only if the statement "ν is a complex measure with $\nu \ll \mu_{xy}$, and $\int \lambda^n \, d\nu = 0$ $(n = 0, 1, 2, \ldots)$ implies $\nu = 0$.

The next result is a generalization of lemma 3.1.20 .

3.1.23- Lemma: Let K be the subspace of H spanned by vectors of the form E(M)x for each Borel set M. If μ_{xx} does not have property (P), then there are y, z $\in$ K such that

$$\int \lambda^n \, d\mu_{yz} = 0 \quad (n = 0, 1, 2, \ldots)$$

but the complex measure $\mu_{yz} \neq 0$.

3.1.24- Theorem: Let T be a normal operator on the Hilbert space H. Then a necessary and sufficient condition that property (P) fails for T is that there is an $x \in H$ such that property (P) fails for μ_{xx}. [17, 96].

Proof: Suppose that property (P) fails for T. There is a subspace $K \subseteq H$ such that for each $y \in K$, $Ty \in K$ but for some $x \in K$, $T^*x \notin K$. Thus there is $z \in K^\perp$ such that $\langle T^n x, z\rangle = 0$, $(n = 1,2,..)$, but $\langle T^*x, z\rangle \neq 0$. Hence $\int \lambda^n \mu_{xz} = 0$ $(n = 0,1,2,...)$ but $\int \bar{\lambda} \, d\mu_{xz} \neq 0$. Since $\mu_{xz} \ll \mu_{xx}$ it follows, from the definition 3.1.22, that μ_{xx} does not have property (P). Conversely, if there is x in H such that μ_{xx} does not have property (P), then by lemma 3.1.23 there are y and z in H such that

$$\int \lambda^n d\mu_{yz} = 0 \ (n = 0,1,2,\ldots)$$

but $\langle E(M)y, z\rangle = \mu_{yz}(M) \neq 0$ for some Borel set M. Now since $\langle T^n y, z\rangle = \int \lambda^n \, d\mu_{yz} = 0$ $(n = 0,1,2,..)$, we see that z is orthogonal to the smallest invariant subspace containing y. Now the set of all vectors of the form E(M)y belong to the smallest reducing subspace containing y. Then, since $\langle E(M)y, z\rangle \neq 0$ for some Borel set M, it follows that z is not orthogonal to the smallest subspace of H which reduces T and contains y. Hence the smallest invariant subspace containing y is not reducing. Therefore property (P) fails for T. Q.E.D.

Proof of lemma 3.1.21: By corollary 3.1.14 there is a subspace Y of H such that $\sigma(T/Y) = \sigma(T)$. Also by theorem 3.1.16 we may choose a vector, x say, in Y, such that $E(.) / Y \ll \mu_{xx}$, where E(.) is the spectral measure for T. Now, $\sigma(T/M(x))$ is the support of the spectral measure E(.)/ M(x) and this is the same as the support of the spectral measure E(.)/ Y, which is equal to $\sigma(T/Y)$. Hence $\sigma(T/M(x)) = \sigma(T)$. Also the support of μ_{xx} is precisely $\sigma(T)$. By lemma 3.1.20, it follows that μ_{xx} does not have property (P). The desired result then follows from theorem 3.1.24. Q.E.D.

3.1.25- corollary: Let T be a normal operator on a Hilbert space H. If the interior of $\sigma(T)$ is non-empty, then property (P) fails for T.

Proof: If the interior of $\sigma(T)$ is non-empty, then $\sigma(T)$ contains an increasing sequence $(C_n)_{n=1}^{\infty}$ of simple closed rectifiable Jordan curves. Hence by theorem 3.1.21 property (P) fails for T. Q.E.D.

Next we prove the following result of Wermer (theorem 9 of [18, 276]) under the additional assumption that the underlying Hilbert space is separable.[7].

3.1.26- Theorem: Let H be a separable complex Hilbert space, and let U, in B(H), be a unitary operator. Let E(.) be the resolution of the identity for U. Let m(.) denote Lebesgue lineal measure on $T = \{ z : |z| = 1\}$. Then property (P) fails for U if and only if $m(.) \ll E(.)$.

Proof: Suppose the property (P) fails for U. Then by theorem 3.1.24 we can find x in H such that $x \neq 0$ and $\langle U^k x, x\rangle = 0$ $(k \neq 0)$. Let $\mu(.) = \langle E(.)x, x\rangle$. Then

$$\int_T \zeta^k d\mu(\zeta) = 0 \quad (k \neq 0).$$

Observe that

$$\int_T \zeta^k \, dm(\zeta) = \int_0^{2\pi} e^{ik\theta} \, d\theta = 0 \ (k \neq 0).$$

The set of finite linear combinations of functions of the form $\{\zeta^k : k = 0, \pm 1, \pm 2, \dots\}$ is, by the Stone-Weierstrass theorem, dense in C(T). Hence, there is $\lambda \geq 0$ such that

$$\int_T \lambda \, f(\zeta) \, dm(\zeta) = \int_T f(\zeta) \, d\mu(\zeta), \ (f \in C(T))$$

and so by the Riesz representation theorem, $\mu(.) = \zeta m(.)$. Hence, since $\mu(.) \ll E(.)$, we have also $m(.) \ll E(.)$. Conversely, suppose that $m(.) \ll E(.)$. Then, by the proof of Lemma 2.5.3, there is an x in H with $m(.) = \langle E(.)x, x\rangle$. Hence

$$\langle U^k x, x\rangle = \int_T \zeta^k \, dm(\zeta) = \int_0^{2\pi} e^{ik\theta} \, d\theta = 0 \ (k \neq 0)$$

and $\langle x\ x\rangle = 2\pi$. Therefore $x \neq 0$, and property (P) fails for U by theorem 3.1.24. Q.E.D.

Note that the proof of the preceding theorem shows that, in a general complex Hilbert space, If the property (P) fails for U then $m(.) \ll E(.)$.

A conterexample will be given to show that, in general the converse is false.

3.1.27- Theorem: Let Γ be a simple closed rectifiable curve in $\mathbb{C}$ and let $m(.)$ denote lineal measure on Γ. Let ϕ be a function with the following properties:

i) ϕ is an injective mapping of $\{z : |z| \leq 1\}$ onto $\Gamma \cup I(\Gamma)$, where $I(\Gamma)$ is the inside of Γ, such that ϕ and ϕ^{-1} are continuous.

ii) ϕ is holomorphic on $\{ z : |z| \angle 1\}$.

Let $\ell(.)$ denote lineal measure on $\Gamma_o = \{ z : |z| = 1\}$. If δ is a Borel subset of Γ_o then $\phi(\delta)$ is a Borel subset of ~~ϕ~~ Γ and $m(\phi(\delta)) = 0$ if and only if $\ell(\delta) = 0$.

Outline of proof: Let Γ be parametrized by $\Gamma = \ell\{\phi(e^{it}); 0 \leq t \leq 2\pi\}$. Introduce the family of curves $\Gamma_r = \ell\{\phi(re^{it}) : 0 \leq t \leq 2\pi \}$, where $0 \angle r \angle 1$. Let μ_r be Lebesgue lineal measure on Γ_r. We show that for every Borel subset δ of Γ_o we have

i

$$\lim_{r\to 1} \mu_r(\phi(r\delta) = m(\phi(\delta)) \qquad (1)$$

D~~e~~vide $[o, 2\pi]$ into 2^n equal subintervals $[t_{m-1}, t_m]$ $(m= 1,2, \ldots,2^n)$

$$I_n(r) = \sum_{m=0}^{2^n} |\phi(re^{it_m}) - \phi(re^{it_{m-1}})|$$

Observe that

$$\mu_r(\Gamma_r) = \lim_{n\to\infty} I_n(r)$$

and

$$m(\Gamma) = \lim_{r\to 1} \mu_r(\Gamma_r).$$

It follows from the E.H.Moore theorem on interchange of double limits that

$$m(\Gamma) = \lim_{n\to\infty}\lim_{r\to 1} I_n(r) = \lim_{r\to 1}\lim_{n\to\infty} I_n(r) = \lim_{r\to 1} \mu_r(\Gamma_r)$$

A s~~e~~imilar argument proves the corresponding results for open arcs of the curves Γ and Γ_r (corresponding to the same subintervals of $[o,2\pi]$) and the extension to open sets follows, since an open set is a countable union of disjoint open arcs. Note also that if δ is a Borel set such that

$$\lim_{r\to 1} \mu_r(\Phi(r\delta)) = m(\Phi(\delta)) \qquad (*)$$

Then

$$\lim_{r\to 1}\mu_r(\Phi(\Gamma_r\backslash r\delta) = m(\Phi(\Gamma\backslash\delta)).$$

The class of sets δ for which (*) holds forms, by the preceding argument a σ-algebra containing all Borel sets and hence (1) is proved. It follows that $m(\Phi(.))$ is absolutely continuous with respect to $\ell(.)$. To see this, obsrve that

$$\begin{aligned} \ell(\delta) = 0 \quad &\Rightarrow \int_\delta r|\Phi'(re^{it})|\, dt = 0 \\ &\Rightarrow \mu_r(\Phi r(\delta)) = 0 \qquad (0 \angle r \angle 1) \\ &\Rightarrow m(\Phi(\delta)) = 0 \end{aligned}$$

using (*)

Hence there is an absolutely continuous function ζ on $[o, 2\pi]$ such that

$$m(\Phi(\delta)) = \int_\delta \zeta'(t)\, dt,$$

where ζ' exists a.e. on $[o, 2\pi]$ and $\zeta'(t) \geq 0$, since both $m(\Phi(.))$ and $\ell(.)$ are positive measures. Since the inverse function Φ^{-1} is continuous, it follows that

$$\zeta'(t) > 0 \text{ a.e. on } [\, o, 2\pi]$$

Let σ be the subset of $[o, 2\pi]$ on which $\zeta'(t)$ exists and is positive. If $\ell(\delta) > 0$, then $m(\Phi(\delta)) = \int_\delta \zeta'(t)dt = \int_{\delta\cap\sigma} \zeta'(t)dt > 0$. Q.E.D.

The follwing result is due to J.Wermer. (Theorem 9 of [18, 276]).

3.1.28- Theorem: Let T be a normal operator on a separable Hilbert space H. Let the resolution of the identity of T be E(.). Suppose that the spectrum of T is a simple closed rectifiable curve Γ. Then there is a closed subspace Y of H such that $TY \subseteq Y$ but $T \backslash Y$ is not normal if and only if Lebesgue lineal measure $\mu(.)$ on Γ is absolutely continuous with respect to the spectral measure $E(.)$.

Before proving the theorem we need the following lemma (see lemma 8 of [10, 2188]).

3.1.29- Lemma: Let E be a spectral measure in the complex Banach space X which is defined and countably additive on a σ-field Σ of a set Ω, and let g be a bounded Borel function defined on the complex plane. Then

$$\int_\Omega g(f(\lambda))E(d\lambda) = \int_{f(\Omega)} g(\lambda)E(f^{-1}(d\mu)),$$

for every E-essentially bounded Σ_measurable function on Ω. (Noted $EB(\Omega,\Sigma)$).

Proof: Let f be in $EB(\Omega,\Sigma)$, and for every Borel set δ in the complex plane let $F(\delta) = E(f^{-1}(\delta))$. If g is the characteristic function of such a set δ, then, since F vanishes outside $f(\Omega)$,

$$\int_{f(\Omega)} g(\lambda)F(d\lambda) = F(\delta) = E(f^{-1}(\delta)) = \int_\Omega g(f(\lambda))\ E(d\lambda).$$

Now the set of bounded Borel functions g for which

$$\int_{f(\Omega)} g(\lambda)F(d\lambda) = \int_\Omega g(f(\lambda))E(d\lambda)$$

is clearly linear and closed in the set of all bounded Borel functions. Since this set contains every characteristic function of Borel set, it contains every Borel function, Q.E.D.

Proof of theorem 3.1.28: Let D be the inside of Γ and let Φ map D conformally onto $\{z : |z| \angle 1\}$. Then Φ maps $\Gamma \cup D$ homeomorphically onto $\{ z : |z| \leq 1\}$, and a Borel subset δ of Γ has $\mu(\delta) = 0$ if and only if $m(\Phi(\delta)) = 0$ by theorem 3.1.27, where $m(.)$ denotes Lebesgue lineal measure on the unit circle. Let ψ be the function inverse to Φ so that ψ maps $\{z : |z| \leq 1\}$ homeomorphically onto $\Gamma \cup D$. Define

$$U = \int_{\sigma(T)} \Phi(\lambda)\ E(d\lambda)$$

since $|\Phi(\lambda)| = 1$ if $\lambda \in \Gamma$, U is unitary. By lemma 3.1.29, the resolution of the identity $F(.)$ for U is given by

$$F(\Phi(\tau)) = E(\tau)$$

for any Borel subset τ of Γ. Let δ be a Borel subset of the unit circle. ~~If~~ Then $F(\delta) = 0$ if and only if $E(\psi(\delta)) = 0$.

1) Property (P) fails for T if and only if (P) fails for U. An application of lemma 3.1.29 gives us

$$U = \int_{\lambda=1} \lambda F(d\lambda).$$

Now let property (P) fails for T. Then there exist x, y with

$$\langle T^n x, y\rangle = \int_\Gamma \lambda^n \langle E(d\lambda)x, y\rangle = 0, \ n \geq 0$$

and $\langle E(.)x, y\rangle$ is not identically zero. Therefore $\langle F(.)x, y\rangle$ is not identically zero and

$$\langle U^n x, y\rangle = \int_{\lambda=1} \langle F(d\lambda)x, y\rangle = 0, \quad n \geq 0$$

by lemma 3.1.23 and theorem 3.1.24 property (P) fails for U. A dual argument shows the converse.

2) (P) fails for U if and only if Lebesgue lineal measure m(.) on the unit circle is absolutely continuous with respect to the spectral measure of U follows from theorem 3.1.26

3) Lebesgue lineal measure m(.) is absolutely continuous with respect to F(.) if and only if $m(.) \ll E(.)$. Let δ be a Borel subset of the unit circle.

If now $F(\delta) = 0$ then $E(\psi(\delta)) = 0$ and so $\mu(\phi(\delta)) = 0$, by assumption. By theorem 3.1.29, this implies that $m(\delta) = 0$. Thus $m \ll F(.)$. The converse follows s~~e~~imilarly.

1),2) and 3) together now give the assertion.

3.2 A counterexample

The purpose of this section is to show that theorems 3.1.~~19~~ 26 and 3.1.2~~3~~9 may both fail if the hypothesis of separability on the Hilbert space is omitted. See [7, 163].

3.2.1- Example: Let $T = \{ z : |z| = 1 \}$. Consider the set H consisting of all complex functions f, defined on T, vanishing off some countable set $G(f) \subseteq T$, and such that : $\sum_{t \in T} |f(t)|^2 \angle \infty$ with the usual vector space operations, defined by setting

$$(\alpha_1 f_1 + \alpha_2 f_2)(t) = \alpha_1 f_1(t) + \alpha_2 f_2(t) \quad (t \in T,\ f_1, f_2 \in H).$$

Inner product and norm defined respectively by

$$\langle f, g\rangle = \sum_{t \in T} f(t)\, g(t), \qquad \|f\| = (\sum_{t \in T} |f(t)|^2)^{\frac{1}{2}} \qquad (f, g \in H)$$

One can prove that H is a Hilbert space. Define a unitary operator U on H by

$$(Uf)(t) = tf(t) \qquad (t \in T,\ f \in H).$$

The resolution of the identity E(.) for U is given by

$$(E(\tau)f)(t) = \chi_\tau(t)f(t) \qquad (t \in T,\ f \in H,\ \ \tau \in \Sigma_T)$$

Next we show that if $\tau \in \Sigma_T$ and $\tau \neq \Phi$ then $E(\tau) \neq 0$. Suppose that $\lambda \in \tau$. Then $E(\tau)\chi(\lambda) = \chi(\lambda)$. It follows immediately that if m(.) denotes Lebesgue lineal measure on T, then $m(.) \ll E(.)$. Also if $x \in H$, then $\langle E(.)x, x\rangle$ vanishes off some countable subset of T. Therefore, for all x in H, $m(.) \neq \langle E(.)x, x\rangle$. This shows that lemma 3.1.1$\overset{5}{\cancel{6}}$ and hence theorems 3.1.$\overset{26}{\cancel{19}}$ and 3.1.28 may fail if the hypothesis of separability is omitted. Suppose now that property (P) fails for U. The arguments of theorems 3.1.$\overset{26}{\cancel{19}}$ and 3.1.28 show that there is a vector y in H such that $m(.) = \langle E(.)y, y\rangle$. As explained above, this is impossible. Hence although we have $m(.) \ll E(.)$, property (P) holds in this case, and so theorem 3.1.$\overset{26}{\cancel{19}}$ and 3.1.28 may also fail if the hypothesis of separability is dropped.

3.3 Restrictions of scalar-type spectral operators.

3.3.1-Theorem: Let S be a scalar-type spectral operator on X, a Banach space, with resolution of the identity E(.). Suppose that $\sigma(S)$ contains an infinite sequence of simple closed rectifiable curves which increases monotonically. Then there is a closed subspace X_0 of X such that $SX_0 \subseteq X_0$ but $S \setminus X_0$ is not spectral operator.

Proof: By lemma 3.1.13, there is a closed separable subspace Y of X invariant under S, such that $S \setminus Y$ is a scalar-type spectral operator and $\sigma(S) = \sigma(S \setminus Y)$. By theorem 3.1.6 a), the resolution of the identity of $S \setminus Y$ is given by $F(.) = E(.) \setminus Y$. By lemma 3.1.15, there is a vector y in Y such that if $\tau \in \Sigma$ and $F(\tau)y = 0$, then $F(\tau) = 0$. By proposition 2.3.2, there is a functional z in Y^* such that

i) $\langle F(\tau)y, z\rangle \geq 0 \quad (\tau \in \Sigma)$;

ii) If $\langle F(\tau)y, z\rangle = 0$ for some τ in Σ, then $F(\tau)y = 0$.

Thus the support of the measure $\langle F(\tau)y, z\rangle$ is $\sigma(S)$. By lemma 3.1.20 there is a complex measure $\nu(.) \ll \langle F(.)y, z\rangle$ such that ν is not identically zero but

$$\int_{\sigma(S)} \lambda^n \, d\nu(\lambda) = 0 \qquad (n = 0, 1, 2, \ldots).$$

By corollary 2.4.3, there are x in Y and u in Y^* such that

$$\nu(\tau) = \langle F(\tau)x, u\rangle \qquad (\tau \in \Sigma)$$

Thus $\langle S^n x, u\rangle = 0$, $(n = 0, 1, 2, \ldots)$ but for some τ in Σ we have $\langle F(\tau)x, u\rangle \neq 0$. Hence, if $X_0 = \overline{\text{span}}\{ S^n x : n = 0,1,2,\ldots\}$, then $SX_0 \subseteq X_0$ but $F(\tau)x \notin X_0$ so $F(\tau)$ does not leave X_0 invariant. By theorem 3.1.6 a), $S \setminus X_0$ is not of scalar-type. Q.E.D.

3.3.2- Corollary: Let S be a scalar-type spectral operator on X. Suppose that $\sigma(S)$ has non-empty interior. Then there is a closed subspace X_0 of X invariant under S and such that $S \setminus X_0$ is not a spectral operator.

3.3.3- Theorem: Let S be a scalar-type spectral operator on X with resolution of the identity $E(.)$. Suppose that the spectrum of S is a simple closed rectifiable curve Γ. If there is a closed subspace Y of X such that $SY \subseteq Y$ but $S \setminus Y$ is not spectral, then Lebesgue lineal measure on Γ is absolutely continuous with respect to the spectral measure $E(.)$.

Proof: By corollary 3.3.2 and the proof of theorem 3.3.1, there exists a closed subspace X_0 of X and an $x \in X$ such that $X_0 = \overline{\text{span}} \{ S^n x : n = 0$ $1, 2, \ldots\}$ is invariant under S and such that $S \setminus X_0$ is not spectral. Let $M(x) = \overline{\text{span}} \{ E(\tau)x, \tau \in \Sigma\}$, where $E(.)$ is the resolution of the identity of S. By theorem 3.1.6 b), $S \setminus M(x)$ is a scalar-type spectral operator. since $\sigma(S\setminus M(x)) \subseteq \sigma(S)$ by theorem 3.1.9, it follows that, by corollary 3.3.2, property (P) fails for $S \setminus M(x)$. Let $S_0 = S \setminus M(x)$ and by theorem 3.1.6, $E_0(.) = E(.) \setminus M(x)$, where $E_0(.)$ is the resolution of the identity of S_0. Let $Y = M(x_0)$. By proposition 2.3.2 there exists x^* in Y^* such that

i) $\langle E_0(\tau)x, x^* \rangle \geq 0, \quad \tau \in \Sigma$

ii) if, for some τ in Σ, $\langle E_0(\tau)x, x^* \rangle = 0$, then also $E_0(\tau) = 0$.

Let

$$\mu_0(\tau) = \langle E_0(\tau)x, x^* \rangle , \quad \tau \in \Sigma.$$

It is easily verified that

$$E_0(.) \ll E_0(.)x \ll \mu_0 \ll E_0(.).$$

Therefore the support of μ_0 is equal to $\sigma(S_0) = \Gamma$. Also, by corollary 3.3.2 and lemma 3.1.20, there is a measure μ_1 with $\mu_1 \ll \mu_0$ such that

$$\int_\Gamma \lambda^n \, \mu_1(d\lambda) = 0 \quad , (n = 0, 1, 2, \ldots.)$$

but μ_1 is not identically zero.

by corollary 2.4.3, there are x_0 in Y and y in Y^* such that

$$\mu_1(\tau) = \langle E_0(\tau)x_0, y \rangle.$$

Thus $\langle S^n x_0, y \rangle = 0$ $(n = 0, 1, 2, \ldots)$ but for some τ in Σ we have $\langle E_0(\tau)x_0, y \rangle \neq 0$. Now consider the Hilbert space $H = L^2(\mu_0)$, and let A, in B(H), be the normal operator defined by

$$(Af)(t) = tf(t), \quad t \in \Gamma, \ f \in H.$$

Now, $\sigma(A) = \Gamma$, by theorem 3.1.21, property (P) fails for A. For let $\{F(\tau) : \tau \in \Sigma\}$ be the resolution of the identity of A, where

$$(F(\tau)f)(t) = \chi_\tau(t)f(t), \ t \in \Gamma, \ f \in H.$$

$$\chi_\tau(t) = 1, \quad t \in \tau$$

$$\chi_\tau(t) = 0, \; t \notin \tau.$$

Then $\mu_0(\tau) = \langle F(\tau)\chi_\Gamma, \chi_\Gamma\rangle$, $\tau \in \Sigma$, and so

$$F(.) \ll \mu_0 \ll F(.).$$

$\sigma(A) = \Gamma$. Now, by theorem 3.1.21, there is a closed subspace H_0 of H such that $AH_0 \subseteq H_0$ but $S \setminus H_0$ is not normal. By theorem 3.1.28 Lebesgue lineal measure on Γ, μ say, is absolutely continuous with respect to $F(.)$. Therefore

$$m \ll F(.) \ll \mu_0 \ll E_0(.) \ll E(.).$$

Q.E.D.

3.3.4- Theorem: Let S be a scalar-type spectral operator on a separable Banach space X. Let the resolution of the identity of S be E(.). Suppose that the spectrum of S is a simple closed rectifiable curve Γ. Then there is a closed subspace Y of X such that $SY \subseteq Y$ but $S \setminus Y$ is not spectral if and only if Lebesgue lineal measure $\mu(.)$ on Γ is absolutely continuous with respect to the spectral measure E(.). (Theorem 3 of [5, 308]).

Proof: The necessity of the condition $\mu(.) \ll E(.)$ follows from theorem 3.3.3. Now, suppose that $\mu(.) \ll E(.)$. Let D be the inside of Γ and let ϕ map D conformally onto $\{ z : |z| < 1\}$. Then ϕ maps $\Gamma \cup D$ homeomorphically onto $\{z : |z| \leq 1\}$, and a Borel subset δ of Γ has $\mu(\delta) = 0$ if and only if $m(\phi(\delta)) = 0$ by theorem 3.1.27, where $m(.)$ denotes Lebesgue lineal measure on the unit circle. Let ψ be the function inverse to ϕ so that ψ maps $\{ z : |z| \leq 1\}$ homeomorphically onto $\Gamma \cup D$. Define

$$U = \int_{\sigma(S)} \phi(\lambda)\, E(d\lambda)$$

U is a scalar-type spectral operator on X. Also $\sigma(U) \subseteq \{z : |z| = 1\}$. The resolution of the identity F(.) of U is given by

$$F(\phi(\tau)) = E(\tau)$$

for any Borel subset τ of Γ. Let δ be a Borel subset of the unit circle. If $F(\delta) = 0$, then $E(\psi(\delta)) = 0$. Hence $m(\delta) = 0$, and so

$m(.) \ll F(.)$.

Observe that

$$\int_0^{2\pi} e^{in\theta}\, d\theta = 0 \quad (n = 1, 2, \ldots).$$

By corollary 2.4.3, there are u in X and v in X^* such that $m(.) = \langle F(.)u, v\rangle$. From above

$$\langle U^n u, v\rangle = \int_\Gamma \lambda^n \langle F(d\lambda)u, v\rangle = \int_0^{2\pi} e^{in\theta}\, d\theta = 0,\ n = 1, 2, \ldots$$

Let $Uu = x$. Then

$$\langle U^n x, v\rangle = 0 \ (n = 0, 1, 2, \ldots)$$

But for some Borel set τ of the unit circle $\langle F(\tau)x, v\rangle \neq 0$. Observe that

$$\langle S^n x, v\rangle = \int_{|\zeta| 1} (\psi(\zeta))^n \langle F(d\zeta)x, v\rangle = 0,\ n = 0, 1, 2, \ldots$$

This follows since ψ is continuous on $\{z : |z| \leq 1\}$ and analytic in $\{z : |z| \angle 1\}$ and so it can be approximated uniformaly on the unit circle by a sequence of polynomials. Thus

$$\langle S^n x, v\rangle = \int_\Gamma \lambda^n \langle E(d\lambda)x, v\rangle = 0 \quad (n = 0, 1, 2, \ldots)$$

But $\langle E(\tau)x, v\rangle \neq 0$ for some Borel subsets τ of Γ, it follows that the subspace $Y = \overline{\text{span}}\ \{ S^n x : n = 0, 1, 2, \ldots\}$ is closed and invariant under S but not under $\{ E(\tau) : \tau \in \Sigma\}$. By theorem 3.1.~~1~~6, $S \setminus Y$ is not a scalar-type spectral operator. Q.E.D.

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