Decomposable approximations and coloured isomorphisms for C^*-algebras

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A thesis submitted to the
College of Science and Engineering
at the University of Glasgow
for the degree of
Doctor of Philosophy

October 2016

Abstract

In this thesis we introduce nuclear dimension and compare it with a stronger form of the completely positive approximation property. We show that the approximations forming this stronger characterisation of the completely positive approximation property witness finite nuclear dimension if and only if the underlying C^*-algebra is approximately finite dimensional. We also extend this result to nuclear dimension at most omega.

We review interactions between separably acting injective von Neumann algebras and separable nuclear C^*-algebras. In particular, we discuss aspects of Connes' work and how some of his strategies have been used by C^*-algebraist to estimate the nuclear dimension of certain classes of C^*-algebras.

We introduce a notion of coloured isomorphisms between separable unital C^*-algebras. Under these coloured isomorphisms ideal lattices, trace spaces, commutativity, nuclearity, finite nuclear dimension and weakly pure infiniteness are preserved. We show that these coloured isomorphisms induce isomorphisms on the classes of finite dimensional and commutative C^*-algebras. We prove that any pair of Kirchberg algebras are 2-coloured isomorphic and any pair of separable, simple, unital, finite, nuclear and $\mathcal{Z}$-stable C^*-algebras with unique trace which satisfy the UCT are also 2-coloured isomorphic.

I would like to thank:

Stuart for all his guidance, support and friendship during my PhD. It has been a great privilege and honor to share all these years with him.

My parents for all their love and support throughout all these years.

Paty for sharing this adventure with me.

All the members of the Analysis group in Glasgow.

Aaron Tikuisis, Christian Voigt, Misha Feigin and Stuart for their help with my job applications.

All my friends in Glasgow and Muenster, especially: Gabriele, Gerardo, Haris, Joan, Liam, Sam and Søren.

Andy, Gabriele, Joan, Liam, Luke, Sam and Stuart for their help with the proofreading of this thesis.

Xin Li and Joachim Zacharias for taking time out of their busy schedules to read this thesis.

Wilhelm Winter for his kind support during my time in Muenster.

CONACYT and SEP for providing the funding necessary to complete this PhD.

The College of Science and Engineering for providing support for my research visit to Germany.

Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Introduction

Operator algebras arise as *-subalgebras of the algebra of bounded operators on a Hilbert space which are closed with respect to certain topologies. If they are closed with respect to the norm, they are called C^*-algebras, and if they are closed with respect to the weak operator topology, they are called von Neumann algebras.

A C^*-algebra A is *nuclear* if there is a unique way to complete its algebraic tensor product with any other C^*-algebra. Kirchberg and Choi-Effros obtained a useful characterisation of nuclearity in terms of *completely positive finite rank approximations* [21, 53]. In this context, an approximation is a triple (F, ψ, φ) consisting of a finite dimensional algebra F and completely positive contractions $\psi : A \longrightarrow F$, $\varphi : F \longrightarrow A$ such that $\varphi \circ \psi$ is equal to id_A on a finite set up to some positive ε. This characterisation is known as the *completely positive approximation property*. It turns out that nuclear C^*-algebras have strong connections with *injective* von Neumann algebras: A is nuclear if and only if A^{**} is injective (Theorem 4.6.5). In particular, the weak closure of every GNS representation of A is injective and it has a separable predual if A is separable. Since injective von Neumann algebras with separable predual were classified by Connes and Haagerup [22, 46], it seems reasonable to ask if we can also classify nuclear and separable C^* -algebras. Many endeavours towards this classification have been made in the last 30 years.

This classification programme initiated by Elliott seeks to classify nuclear separable and infinite dimensional C^*-algebras by means of K-theoretical invariants. Glimm provided a starting point. He gave a complete classification of UHF-algebras using supernatural numbers [42, Theorem 1.12]. During the 70's, Elliott classified all AF-algebras by their ordered K_0-group [31, Theorem 4.3] and in the 90's he extended this classification to simple $\mathrm{A}\mathbb{T}$-algebras of real rank zero using their graded K-theory [33, Theorem 7.3]. Based on this, Elliott conjectured that an invariant constructed from K-theory and traces might be useful to classify a larger class of nuclear C^*-algebras [32]. This invariant has been called the *Elliott invariant*, denoted by $\mathrm{Ell}(A)$, and it has undergone some modifications over time. This programme has been particularly focused on the classification of simple separable nuclear C^* algebras (see [84] for a more comprehensive overview).

The Elliott conjecture asserts that isomorphisms between the Elliott invariants can be lifted to actual isomorphisms between the corresponding C^*-algebras. This programme has had a lot of success: Elliot, Gong and Li showed the conjecture is true for simple AH-algebras with very slow dimension growth [35, Theorem 4.9], Lin proved that C^*-algebras of tracial rank zero in the UCT class satisfy the Elliott conjecture [61, Theorem 5.2], and Kirchberg and Phillips proved that the conjecture is valid for Kirchberg algebras in the UCT class [75, Theorem 4.2.1].

Unfortunately the conjecture, as it stands, is not true. At the end of the last century, Jiang and Su introduced a C^*-algebra, denoted by $\mathcal{Z}$, which is simple, separable, nuclear, infinite dimensional, and projectionless with the same Elliott invariant as $\mathbb{C}$ [51, Theorem 1]. This algebra has become highly important in the classification programme. Gong, Jiang and Su proved that for simple unital C^*-algebras with weakly unperforated K_0-groups, the Elliott invariant remains the same after tensoring with $\mathcal{Z}$; precisely, $\mathrm{Ell}(A) \cong \mathrm{Ell}(A \otimes \mathcal{Z})$ [43, Theorem 1]. This immediately shows that a success in the Elliott conjecture for this particular class of C^*-algebras would imply $\mathcal{Z}$-stability. However, remarkable examples of C^*-algebras with weakly unperforated K_0-groups which are not $\mathcal{Z}$-stable were constructed by Rørdam and Toms ([82, Theorem 6.10], [97, Theorem 1.1] and [98, Theorem 1.1]).

In order to fix the Elliott conjecture, we must reduce the scope of it. Some regularity conditions have materialised as the solution for this problem. These conditions, now known as the *regularity properties*, are of very different flavours: topological, analytical and algebraic (c.f. [38]). Naturally, one of them, the analytical one originates from our previous discussion: $\mathcal{Z}$-stability. The algebraic one corresponds to strict comparison [74, Definition 2.3]. Roughly speaking, strict comparison allows us to determine the order of positive elements using traces. Finally, the topological property is nuclear dimension which is a non commutative analogue of covering dimension for topological spaces. Toms and Winter conjectured that all these three conditions are equivalent for simple, separable, unital, infinite dimensional nuclear C^*-algebras [111, Conjecture 9.3]. There has been many progress in the resolution of the Toms-Winter conjecture in the recent years. In particular, it has been proved that finite nuclear dimension implies $\mathcal{Z}$-stability and that $\mathcal{Z}$-stability implies strict comparison for separable and simple C^*-algebras [83, 107, 109].

The reverse implications have been verified only under certain hypotheses [8, 57, 63, 89, 90, 99]. In particular, the conjecture holds if the trace simplex $T(A)$ is Bauer and its extreme boundary $\partial_e T(A)$ has finite covering dimension. Very recently, after the work of many people during many years, the classification of simple, separable, unital, infinite dimensional nuclear C^*-algebras with finite nuclear dimension which satisfy the UCT has been completed [36, 44, 96].

Nuclear dimension theory has been a very active area of research and it is an important topic for this thesis. Originally, precursor concepts were introduced by Winter in [104, 106] and years later, Winter and Kirchberg introduced a refinement called decomposition rank [58]. It turns out that algebras with finite decomposition rank are stably finite and quasidiagonal. Since finite decomposition rank imposes such strong restrictions on the algebras enjoying this property, a further refinement known as nuclear dimension was introduced by Winter and Zacharias in [111] which circumvents these obstructions. Roughly speaking, a C^*-algebra A has nuclear dimension at most n if there exist completely positive finite rank approximations (F, ψ, σ), where φ is the sum of $n+1$ maps which preserve orthogonality (these maps are known as *order zero maps*). Observe that we only know that the norm of φ is bounded by $n+1$. If the norm of φ is at most one, then the algebra has decomposition rank at most n.

In [49], a refined form of the completely positive approximation property was proved. The algebra A is nuclear if and only if there exist completely positive finite rank approximations (F, ψ, φ) such that φ is a finite convex combination of order zero maps. Notice that the number of summands is not assumed to be uniformly bounded (in contrast with the definition of nuclear dimension). The similarities between this strengthened version of the completely positive approximation property and the definition of nuclear dimension immediately trigger questions about their relations (if there is any). In this thesis, we will show that in general the approximations coming from this form of the completely positive approximation property are not useful to determine nuclear dimension of the algebra unless it is approximately finite dimensional (Theorem 3.1.5).

A breakthrough in the classification programme was achieved by Matui and Sato in [64]. They showed that separable, simple, unital, quasidiagonal, $\mathcal{Z}$-stable and nuclear C^*-algebras with unique trace have decomposition rank at most three. The importance of their work lies in the way they used Connes' and Haagerup's proof of injectivity implies hyperfiniteness for II_1-factors with separable predual [22, 45]. Connes showed that an injective II_1-factor M with separable predual absorbs tensorially the hyperfinite II_1-factor $\mathcal{R}$ (factors satisfying this condition are known as McDuff), M embeds into $\mathcal{R}^\omega$ (the von Neumann ultrapower of $\mathcal{R}$), and the flip of M is strongly approximately inner. These three deep facts are the main ingredients in Connes' proof. Matui and Sato observed that Connes' proof provides a natural framework to prove, under certain conditions, that $\mathcal{Z}$-stable C^*-algebras have finite decomposition rank. Indeed, the Jiang-Su algebra $\mathcal{Z}$ arises as the C^*-analogue of the hyperfinite II_1-factor $\mathcal{R}$ and so following this analogy we might think of simple $\mathcal{Z}$-stable algebras as the C^*-analogue of McDuff factors. In the same way, we can view finite nuclear dimension and finite decomposition rank as C^*-forms of hyperfiniteness. Therefore, Connes's proof supplies a strategy to show that $\mathcal{Z}$-stability implies finite nuclear dimension or decomposition rank.

In the work of Matui and Sato, the embedding of M into $\mathcal{R}^\omega$ is replaced with quasidiagonality of A since this provides an embedding of A into the C^*-ultrapower of the universal UHF algebra $\mathcal{Q}$. Sato, White and Winter were able to remove quasidiagonality of A by instead constructing a c.p.c. order zero map $A \longrightarrow \mathcal{Q}_\omega$ [90].

The other ingredient is of high relevance for this work. The flip of A is an automorphism on $A \otimes A$ given by $a \otimes b \mapsto b \otimes a$ (since we are working with nuclear C^*-algebras there is no need to specify which tensor product we are using). As mentioned before, Connes proved that the flip of injective II_1-factors with separable predual are strongly approximately inner. Effros and Rosenberg, motivated by the work of Connes, investigated approximately inner flips for C^*-algebras [30]. It turns out that having an approximately inner flip is a strong condition for a C^*-algebra, since it forces the algebra to be simple, nuclear and with at most one trace. Even more, there are some topological obstructions. They observed this by proving, using K-theory, that approximately finite dimensional algebras with an approximately inner flip are forced to be UHF. Matui and Sato circumvented the use of approximately inner flips by using techniques developed by Haagerup in [45]. Instead they used what now is called an "approximately 2-coloured flip".

These coloured flips avoid topological obstructions since we are breaking the flip in a sum of two order zero maps and this type of maps do not carry topological data. However, these type flips still force the underlying C^*-algebra to be simple, nuclear and with at most one trace. Thus, in order to extend this result outside from the realm of the unique trace, a replacement for the flip needs to be found. This was carried out in [8] by Bosa, Brown, Sato, Tikuisis, White and Winter. The authors of [8] were able to show that separable, simple, unital, $\mathcal{Z}$-stable and nuclear C^*-algebras whose trace simplices are Bauer (*i.e.* the extreme boundary of $T(A)$ is closed) have nuclear dimension at most 1. They also introduced a notion that they called "coloured equivalent maps".

Based on these notions of coloured flips and coloured maps, we develop a notion of coloured isomorphism between separable unital C^*-algebras in this thesis. This is a joint work with A. Tikuisis and S. White that will be published in [17]. Loosely speaking, two algebras A and B are n-coloured isomorphic if there are c.p.c. order zero maps between these two algebras in such a way that using both compositions we can express id_A and id_B as sum of n order zero maps. This notion allows us to transfer some information between equivalent algebras such as ideal lattices and tracial information, while at the same time, circumvents topological obstructions. We show that for the case $n = 1$, coloured isomorphic algebras are in fact isomorphic. The main theorems are the following: any two Kirchberg algebras are 2-coloured isomorphic and any two separable, simple, unital, finite, $\mathcal{Z}$-stable and nuclear C^*-algebras with unique trace that satisfy the UCT are 2-coloured isomorphic. The proofs of these theorems are in spirit the same but the technicalities are rather different; in the finite case we rely in the tracial behaviour of the maps inducing the coloured isomorphisms yet in the Kirchberg case there are no traces. The key idea is to split the identity map of A as the sum of two order zero maps using $\mathcal{Z}$-stability and a positive element $h \in \mathcal{Z}$ with spectrum $[0, 1]$. Precisely, $\mathrm{id}_A \otimes 1_{\mathcal{Z}} = \mathrm{id}_A \otimes h + \mathrm{id}_A \otimes (1_{\mathcal{Z}} - h)$.

Let us finish this introduction with an outline of this thesis. In Chapter 1 we will review some preliminaries that will be needed throughout this thesis. We review important properties of completely positive order zero maps and nuclearity, and introduce the Jiang-Su algebra $\mathcal{Z}$. The last section presents basic facts about Cuntz comparison and Cuntz semigroups. We also compute important examples needed in the last chapter of this thesis.

In Chapter 2, we review the covering dimension of topological spaces and present the definitions of decomposition rank and nuclear dimension. We explain the main differences between these two notions and provide some examples. We also present a detailed analysis of the commutative case and the zero dimensional objects for these dimension theories.

Chapter 3 is based on [16] which was published by the author. This chapter is devoted to the study of decomposable approximations and nuclear dimension. It is shown that approximations coming from the refined form of the completely positive approximation property proved in [49] witness nuclear dimension if and only if A is approximately finite dimensional. In the last two sections of this chapter, we discuss the notion of nuclear dimension at most omega and we extend the previous result to this case.

In Chapter 4, we present some important interactions between von Neumann algebras and C^*-algebras. In the first section we state some basic facts about von Neumann algebras and we review the relations between nuclearity, semidiscreteness and injectivity. We discuss aspects of Connes' proof of his celebrated theorem stating that injectivity implies hyperfiniteness and we also explain how these ideas were implemented by Matui and Sato, and Sato, White and Winter. We study the analogies between the Jiang-Su algebra $\mathcal{Z}$ and the hyperfinite II_1-factor $\mathcal{R}$ and, after introducing the idea of "colourings", we explained why we view nuclear dimension as a "coloured" form of hyperfiniteness. We finish this chapter by reviewing part of the work carried out in [8]. We study the notion of coloured equivalent maps and we state some important theorems that will be used in the final chapter.

Finally in Chapter 5, we discuss the ideas that lead to the definition of a coloured isomorphism of separable unital C^*-algebras. After this, we present the definition and its basic properties. It is showed that any two Kirchberg algebras are 2-coloured isomorphic and the last section is devoted to proving that any two separable, simple, unital, $\mathcal{Z}$-stable and nuclear C^*-algebras with unique trace that satisfies the UCT are 2-coloured isomorphic. An appendix dedicated to the basic properties of ultraproducts is also included.

Contents

Chapter 1

Preliminaries

In this chapter we will introduce some of the basic concepts we will use throughout this thesis. We also present some technical statements that will be used later. This introduction is not self-contained and some knowledge of operator algebras is assumed. This chapter is mostly based on [13, 84].

1.1 Notation

We will denote the *unitisation* of a C^*-algebra A by $\tilde{A}$. The unit ball of A will be denoted by A^1 and the set of positive elements of A will be denoted by A_+. Similarly, the set of unitaries will be denoted by $\mathcal{U}(A)$.

In this thesis, a functional $\tau : A \longrightarrow \mathbb{C}$ is *tracial* if $\tau(ab) = \tau(ba)$ for all $a, b \in A$ and a *trace* on A is a tracial state $\tau : A \longrightarrow \mathbb{C}$. The set of traces will be denoted by $T(A)$. Actually, the space of traces $T(A)$ is a Choquet simplex [88, Theorem 3.1.18].

For $a \in A$ and $X \subset A$, we will denote $\inf\limits_{x \in X} \|a - x\|$ by $\mathrm{dist}(a, X)$. In this thesis, zero is not considered to be a natural number. We will write $\mathbb{N}_0$ to denote the set $\mathbb{N} \cup \{0\}$.

Normally, H will denote a Hilbert space and $B(H)$ denotes the algebra of bounded operators on H. We will write $\mathbb{K}$ to denote the compact operators on a separable Hilbert space. The symbol $A^{\otimes n}$ will denote the tensor product of A with itself n times.

1.2 Multipliers

The multiplier algebra is the C^*-analogue of the Stone-Čech compactification. This analogy is justified by the fact that the multiplier algebra of $C_0(X)$ is $C(\beta X)$, where βX is the Stone-Čech compactification of X. For our purposes, we use the original construction using multipliers, due to Busby.

Definition 1.2.1 ([14, Definition 2.1])**.** Let A be a C^*-algebra. A *multiplier* is a pair (L, R) of maps $L, R : A \longrightarrow A$ such that $aL(b) = R(a)b$ for all $a, b \in A$. $\mathcal{M}(A)$ will denote the set of multipliers of A.

Multipliers are also called *double centralisers*. Observe that continuity of the maps is not assumed since the definition already implies that the maps L and R are bounded. In fact, $\|L\| = \|R\|$ [14, Proposition 2.5, Lemma 2.6]. We can define operations and a norm on $\mathcal{M}(A)$ in order to equip it with the structure of a unital C^*-algebra [14, Definition 2.10, Theorem 2.11]. Precisely, for a bounded map $L : A \longrightarrow A$, let L^* be the map given by

$$L^*(a) = \left(L\left(a^*\right)\right)^*.$$

Since

$$aR^*(b) = a\left(R(b^*)\right)^* = \left(R(b^*)a^*\right)^* = \left(b^*L(a^*)\right) = \left(L(a^*)\right)^* b = L^*(a)b \tag{1.1}$$

for all $a, b \in A$, we have that (R^*, L^*) is a multiplier if (L, R) is a multiplier. Then, the operations and norm on $\mathcal{M}(A)$ are given by

$$\begin{aligned}
&\bullet \left(L_1, R_1\right) + \left(L_2, R_2\right) = \left(L_1 + L_2, R_1 + R_2\right), \\
&\bullet \lambda \left(L, R\right) = \left(\lambda L, \lambda R\right), \qquad \lambda \in \mathbb{C}, \\
&\bullet \left(L_1, R_1\right)\left(L_2, R_2\right) = \left(L_1 L_2, R_2 R_1\right), \\
&\bullet \left(L, R\right)^* = \left(R^*, L^*\right), \\
&\bullet \|(L, R)\| = \|L\| = \|R\|.
\end{aligned} \tag{1.2}$$

With these operations and norm, $\mathcal{M}(A)$ is a unital C^*-algebra where the unit is the multiplier $(\mathrm{id}_A, \mathrm{id}_A)$ [14, Theorem 2.11]. The algebra $\mathcal{M}(A)$ is called the *multiplier algebra* of A. Observe there is a canonical embedding $\mathfrak{M} : A \longrightarrow \mathcal{M}(A)$ given by

$$\mathfrak{M}_a = (L_a, R_a), \tag{1.3}$$

where L_a and R_a are defined as left and right multiplication by a respectively,

$$L_a(b) = ab, \qquad R_a(b) = ba, \qquad a, b \in A.$$

Via this canonical embedding, A is an ideal of $\mathcal{M}(A)$. Indeed, let $a, b, c \in A$. Thus

$$cL(ab) = R(c)ab = \left(R(c)a\right)b = c\left(L(a)b\right). \tag{1.4}$$

After doing the same computation for R, we obtain

$$L(ab) = L(a)b, \qquad R(ab) = aR(b), \qquad a, b \in A. \tag{1.5}$$

With these identities in hand, it is easy to verify that A is an ideal of $\mathcal{M}(A)$. Consider $(L', R') \in \mathcal{M}(A)$ and $a \in A$, then by definition $(L', R')(L_a, R_a) = (L'L_a, R_aR')$. After observing the following identities,

$$L'L_a(b) = L'(ab) = L'(a)b, \qquad R_aR'(b) = R'(b)a = bL'(a), \qquad b \in A, \tag{1.6}$$

we obtain

$$\left(L', R'\right)(L_a, R_a) = \left(L_{L'(a)}, R_{L'(a)}\right). \tag{1.7}$$

Similarly, we have

$$(L_a, R_a)\left(L', R'\right) = \left(L_{R'(a)}, R_{R'(a)}\right). \tag{1.8}$$

This shows A is an ideal of $\mathcal{M}(A)$.

We have mentioned before that $\mathcal{M}(C_0(X)) \cong C(\beta X)$, where βX is the Stone-Čech compactification of the locally compact space X. Another important example is the following: $\mathcal{M}(\mathbb{K}) \cong B(H)$ where H is a Hilbert space with countable basis.

Before stating a technical lemma, let us recall a basic fact about extreme points of the unit ball of a C^*-algebra.

Theorem 1.2.2 ([4, Theorem II.3.2.17])**.** *Let A be a C^*-algebra. If A is non unital, then there are no extreme points in the closed unit ball of A. If A is unital, then the extreme points of the closed unit ball of A are precisely the elements x such that*

$$(1_A - xx^*)\, A\, (1_A - x^*x) = 0.$$

In particular, every unitary is an extreme point of the unit ball of A.

The reason why we introduced multipliers is the following lemma which will be used in Chapter 3.

Lemma 1.2.3 ([16, Lemma 4])**.** *Let A be a C^*-algebra and $a_1, a_2 \in A_+^1$. Let B be a C^*-subalgebra of A and let λ_1 and λ_2 be strictly positive real numbers satisfying $\lambda_1 + \lambda_2 = 1$. If $a_1b \in B$ and $(\lambda_1 a_1 + \lambda_2 a_2)\, b = b$ for all $b \in B$, then $a_1b = a_2b = b$ for all $b \in B$.*

Proof. By hypothesis, we have that $L_{a_1}(b) = a_1 b \in B$ and $R_{a_1}(b) = ba_1 \in B$ for all $b \in B$. Then the maps $L_{a_1}\big|_B$ and $R_{a_1}\big|_B$ are maps from B to B and they satisfy

$$bL_{a_1}\big|_B(b') = R_{a_1}\big|_B(b)b', \qquad b, b' \in B. \tag{1.9}$$

Thus $\mathfrak{M}_{a_1} = \left(L_{a_1}\big|_B, R_{a_1}\big|_B\right) \in \mathcal{M}(B)$.

Similarly if $a = \lambda_1 a_1 + \lambda_2 a_2$ then $\mathfrak{M}_a \in \mathcal{M}(B)$. In fact, $\mathfrak{M}_a = 1_{\mathcal{M}(B)}$ since a is positive and $ab = b$ for all $b \in B$. We have

$$\lambda_2 a_2 b = b - \lambda_1 a_1 b, \qquad b \in B. \tag{1.10}$$

By the hypothesis, the right side of the previous equation is in B, therefore $a_2 b \in B$ for all $b \in B$ and this yields $\mathfrak{M}_{a_2} \in \mathcal{M}(B)$. It is also straightforward to see that

$$1_{\mathcal{M}(B)} = \mathfrak{M}_a = \lambda_1 \mathfrak{M}_{a_1} + \lambda_2 \mathfrak{M}_{a_2}. \tag{1.11}$$

By Theorem 1.2.2, $1_{\mathcal{M}(B)}$ is an extreme point of the unit ball of $\mathcal{M}(B)$. Since $\mathfrak{M}_{a_1}$ and $\mathfrak{M}_{a_2}$ also lie in the unit ball, we have

$$1_{\mathcal{M}(B)} = \mathfrak{M}_{a_1} = \mathfrak{M}_{a_2}. \tag{1.12}$$

This finishes the proof. □

1.3 Approximately finite dimensional algebras

It is well known that finite dimensional C^*-algebras are nothing more than finite direct sums of matrix algebras. One can build more complicated C^*-algebras from these finite dimensional blocks. This was done by Bratteli, who introduced the approximately finite dimensional C^*-algebras ([9, Definition 1.1]).

A separable C^*-algebra A is *approximately finite dimensional* (AF) if it contains an increasing sequence of finite dimensional C^*-algebras $\{A_n\}_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} A_n$ is dense in A. It is also well known that, alternatively, we can define AF-algebras as the inductive limit of finite dimensional C^*-algebras. One important class of AF-algebras are the UHF algebras, which are inductive limits of matrix algebras with unital connecting maps [42, Theorem 1.13]. Bratteli proved the following theorem, known as the local characterisation of AF-algebras.

Theorem 1.3.1 ([9, Theorem 2.2]). *A separable C^*-algebra A is AF if and only if for every finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exists a finite dimensional C^*-algebra $B \subset A$ such that*

$$dist\,(a, B) < \varepsilon$$

for all $a \in \mathfrak{F}$.

There are two possible definitions of non separable AF-algebras, either as algebras containing a directed family of finite dimensional C^*-subalgebras with dense union (equivalently as the direct limit of finite dimensional C^*-algebras over general directed sets) or via the local characterisation. These are not the same ([40, Theorem 1.5]) and in this thesis we choose to work with the local characterisation as the definition of AF since it is better suited to our purposes.

Definition 1.3.2. A C^*-algebra is AF if for every finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exists a finite dimensional C^*-algebra $B \subset A$ such that

$$\mathrm{dist}\,(a, B) < \varepsilon$$

for all $a \in \mathfrak{F}$.

1.4 Completely positive maps

One important class of maps for C^*-algebras are the so-called completely positive maps. These maps are fundamental for the approximation theory of C^*-algebras. Remember that a *positive* map between C^*-algebras is a bounded linear map which sends positive elements to positive elements.

Let A and B be C^*-algebras. A map $\varphi : A \longrightarrow B$ is *completely positive* if for every $n \in \mathbb{N}$ the map $\varphi_n : M_n(A) \longrightarrow M_n(B)$, given by

$$\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})],$$

is positive. For brevity, we will refer to completely positive maps as c.p. maps and, in the same way, u.c.p. and c.p.c. map will stand for unital and completely positive map and completely positive and contractive map respectively.

As examples of c.p. maps we have *-homomorphisms and compression of *-homomorphisms, *i.e.* maps of the form $\varphi(a) = V^*\pi(a)V$ where π is a *-homomorphism and V is an operator. It turns out that any c.p. map looks like this. We present this result only for unital C^*-algebras.

Theorem 1.4.1 (Stinespring)**.** *Let A be a unital C^*-algebra and let $\varphi : A \longrightarrow B(H)$ be a c.p.c. map where H is a Hilbert space. Then there exists a Hilbert space $\widetilde{H}$, a *-representation $\pi : A \longrightarrow B(\widetilde{H})$ and an operator $V : H \longrightarrow \widetilde{H}$ such that*

$$\varphi(a) = V^*\pi(a)V$$

for all $a \in A$. In particular $\|\varphi\| = \|V^*V\| = \|\varphi(1_A)\|$.

The following proposition summarises some basic facts about c.p. maps which are straightforward consequences of Stinespring's theorem.

Proposition 1.4.2 ([13, Proposition 1.5.6])**.** *Let A and B be C^*-algebras and $\varphi : A \longrightarrow B$ a c.p. map.*

(i) The inequality $\varphi(a^*)\varphi(a) \leq \varphi(a^*a)$ holds for every $a \in A$.

(ii) If $a \in A$ satisfies $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ and $\varphi(aa^*) = \varphi(a)\varphi(a)^*$, then $\varphi(ba) = \varphi(b)\varphi(a)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for every $b \in A$.

(iii) The subspace

$$A_\varphi = \{a \in A \mid \varphi(a^*a) = \varphi(a)^*\varphi(a) \text{ and } \varphi(aa^*) = \varphi(a)\varphi(a)^*\}$$

is a C^*-subalgebra of A.

Let us introduce an important class of c.p.c. maps.

Definition 1.4.3. Consider two C^*-algebras A and B with $B \subset A$. A *conditional expectation* is a c.p.c. map $E : A \longrightarrow B$ such that $E\big|_B = \mathrm{id}_B$ and

$$E\left(bab'\right) = bE(a)b'$$

for all $a \in A$ and $b, b' \in B$.

We finish our collection of results about c.p.c. maps by stating the following useful inequality. Its proof is a consequence of Stinespring's theorem and the C^*-identity.

Lemma 1.4.4 ([104, Lemma 3.1])**.** *Let $\varphi : A \longrightarrow B$ be a c.p.c. map between C^*-algebras and let $a, b \in A$. Then*

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| \leq \|\varphi(aa^*) - \varphi(a)\varphi(a^*)\|^{1/2}\|b\|. \tag{1.13}$$

1.4.1 Order zero maps

An important class of c.p.c. maps are the ones which preserve orthogonality. Originally these maps were introduced by Winter when the domain is a finite dimensional algebra [104, Definition 2.1] and they were examined in full generality by Winter and Zacharias in [110].

Definition 1.4.5 ([110, Definition 2.3])**.** Let A and B be C^*-algebras. A c.p. map $\varphi : A \longrightarrow B$ is of *order zero* if it preserves orthogonality, *i.e.* if $a, b \in A_+$ satisfy $ab = 0$ then $\varphi(a)\varphi(b) = 0$.

Easy examples of c.p. order zero maps are *-homomorphisms. Let us provide another example. Let $\pi : A \longrightarrow B$ be a *-homomorphisms between C^*-algebras and consider $h \in B$ which commutes with $\pi(a)$ for all $a \in A$. Then the map given by $\varphi(a) := h\pi(a)$ is c.p. of order zero. Like in the c.p. case, order zero maps essentially look like this. The following theorem is an order zero version of Stinespring's theorem and it was proved by Winter and Zacharias ([110, Theorem 3.3]) based on previous work by Wolff [112, Theorem 2.3].

Theorem 1.4.6. *Let $\varphi : A \longrightarrow B$ be a c.p. map of order zero between C^*-algebras and set $C := C^*(\varphi(A))$. Then there exist a positive $h \in \mathcal{M}(C) \cap C'$ with $\|h\| = \|\varphi\|$ and a *-homomorphism*

$$\pi_\varphi : A \longrightarrow \mathcal{M}(C) \cap \{h\}'$$

such that

$$\varphi(a) = h\pi_\varphi(a) \tag{1.14}$$

for all $a \in A$. If A is unital, then one may take $h = \varphi(1_A)$.

The *-homomorphism π_φ is called the *support *-homomorphism* of the order zero map φ. Let us give an explicit description of π_φ. Consider H as the universal Hilbert space of $C^*(\varphi(A))$ and we can assume that $C^*(\varphi(A))$ acts nondegenerately on H. The support *-homomorphism π_φ of φ is given by

$$\pi_\varphi(a) = \text{s.o.}\lim_{n\to\infty} \left(\varphi(1_A) + \frac{1}{n}\mathbf{1}_H\right)^{-1} \varphi(a), \qquad a \in A, \tag{1.15}$$

where s.o. lim stands for the limit in the strong operator topology. Throughout this thesis, we will denote the support *-homomorphism of φ by π_φ, unless otherwise stated. As a straightforward consequence of the previous theorem we have that if A is unital and $\varphi(1_A)$ is a projection, the order zero map φ is in fact a *-homomorphism.

Let $\varphi : A \longrightarrow B$ be a c.p.c. order zero map. Then it induces a *-homomorphism $\rho_\varphi : C_0(0,1] \otimes A \longrightarrow B$ by

$$\rho_\varphi\left(\mathrm{id}_{(0,1]} \otimes a\right) = \varphi(a).$$

This indeed defines a *-homomorphism because $C(0,1]$ is isomorphic to the universal C^*-algebra generated by a positive contraction, identifying this generator with $\mathrm{id}_{C(0,1]}$ [62]. Similarly, every *-homomorphism $\rho : C_0(0,1] \otimes A \longrightarrow B$ induces a c.p.c. order zero map $\varphi_\rho : A \longrightarrow B$ by

$$\varphi_\rho(a) = \rho\left(\mathrm{id}_{(0,1]} \otimes a\right).$$

These two assignations are in fact the inverse of each other. These facts lead to the following corollary.

Corollary 1.4.7 ([110, Corollary 4.1])**.** *Let A and B be C^*-algebras. There is a canonical bijection between the space of c.p.c. order zero maps $A \longrightarrow B$ and *-homomorphisms $C_0(0,1] \otimes A \longrightarrow B$.*

Another useful application of the structure of order zero maps is the positive functional calculus for c.p.c. order zero maps.

Corollary 1.4.8 ([110, Corollary 4.2])**.** *Let $\varphi : A \longrightarrow B$ be a c.p.c. order zero map between C^*-algebras and let $f \in C_0(0,1]$ be a positive function. Let h and π_φ be as in Theorem 1.4.6, then the map $f(\varphi) : A \longrightarrow B$ given by*

$$f(\varphi)(a) = f(h)\pi_\varphi(a), \qquad a \in A,$$

is a well defined order zero map. If the norm of f is at most one, then $f(\varphi)$ is also contractive.

Another form of writing functional calculus is using the corresponding map $C_0(0,1] \otimes A \longrightarrow B$. Precisely, if $\varphi : A \longrightarrow B$ is a c.p.c. order zero map and $\rho_\varphi : C_0(0,1] \otimes A \longrightarrow B$ is the *-homomorphism induced by φ, we have

$$f(\varphi)(a) = \rho_\varphi(f \otimes a)$$

for every positive $f \in C_0(0,1]$.

Order zero maps are well behaved with respect to the minimal and maximal tensor product (see Section 1.5).

Corollary 1.4.9 ([110, Corollary 4.3])**.** *Let A, B, C and D be C^*-algebras and let $\varphi : A \longrightarrow B$ and $\psi : C \longrightarrow D$ be c.p.c. order zero maps. Then the induced map*

$$\varphi \otimes \psi : A \otimes_\alpha C \longrightarrow B \otimes_\alpha D$$

has order zero, where $\otimes_\alpha$ denotes the minimal or maximal tensor product.

An important feature of order zero maps is that the composition of an order zero map with a positive tracial functional is also tracial.

Corollary 1.4.10 ([110, Corollary 4.4])**.** *Let A and B be C^*-algebras, $\varphi : A \longrightarrow B$ a c.p.c. order zero map and τ a positive tracial functional. Then the composition $\tau \circ \varphi$ is a positive tracial functional.*

Support *-homomorphisms of order zero maps are useful tools while working with order zero maps. The downside of this map is that its image is contained in some multiplier algebra and sometimes this might be inconvenient. Let us introduce now another useful tool for order zero maps. This is another map which has the feature that its image is contained in the ultraproduct of the codomain C^*-algebra but the price to pay is that this map is no longer a *-homomorphism. The proper definition of ultrapowers can be found in the Appendix A.

Proposition 1.4.11 ([90, Lemma 2.2], [8, Lemma 1.14])**.** *Let A and B be unital C^*-algebras such that A is separable. Suppose $S \subset B_\omega$ is separable and self-adjoint and let $\varphi : A \longrightarrow B_\omega \cap S'$ be a c.p.c. order zero map. Then there exists a c.p.c. order zero map $\hat{\varphi} : A \longrightarrow B_\omega \cap S'$ such that*

$$\varphi(ab) = \hat{\varphi}(a)\varphi(b) = \varphi(a)\hat{\varphi}(b), \qquad a, b \in A.$$

The c.p.c. order zero map $\hat{\varphi}$ is called a support order zero map of φ.

A proof of this proposition can be found in A.1.8. The map $\hat{\varphi}$ is not unique, however in many applications this downside will not be relevant. We can recover the positive functional calculus for order zero maps using these support order zero maps. Precisely

$$f\left(\varphi\right)(a) = f\left(\varphi(1_A)\right)\hat{\varphi}(a)$$

for every positive $f \in C(0,1]$ [8, Lemma 1.14].

Now let us present some basic results which are part of the folklore. We include their proofs since we were unable to find a reference.

Lemma 1.4.12. *Let A and B be unital C^*-algebras and let $\varphi : A \longrightarrow B$ be a c.p.c. order zero map. If I is a closed two sided ideal of B, then $\varphi^{-1}(I)$ is a closed two sided ideal of A.*

Proof. It is immediate that $\varphi^{-1}(I)$ is closed in A. Let $a, x \in A$ with $a \in \varphi^{-1}(I)$. Then $\varphi(ax) = h\pi_\varphi(ax) = h\pi_\varphi(a)\pi_\varphi(x) = \varphi(a)\pi_\varphi(x)$. Since $I \cap C^*(\varphi(A))$ is an ideal of $C^*(\varphi(A))$ which is an ideal of $\mathcal{M}\left(C^*(\varphi(A))\right)$, we have that $I \cap C^*(\varphi(A))$ is an ideal of $\mathcal{M}\left(C^*(\varphi(A))\right)$. We consider $a \in \varphi^{-1}(I)$, thus $\varphi(a) \in I \cap C^*(\varphi(A))$ and therefore we have $\varphi(ax) = \varphi(a)\pi_\varphi(x) \in I$. Similarly $xa \in \varphi^{-1}(I)$. This shows $\varphi^{-1}(I)$ is an ideal of A. □

Lemma 1.4.13. *There are no non zero order zero maps from $M_k(\mathbb{C})$ to any commutative C^*-algebra for $k \geq 2$.*

Proof. Let A be a commutative C^*-algebra and let $\varphi : M_k(\mathbb{C}) \longrightarrow A$ be a non zero c.p. order zero map for some $k \in \mathbb{N}$. By Lemma 1.4.12, the kernel of φ is an ideal of $M_k(\mathbb{C})$. Since $M_k(\mathbb{C})$ is simple, φ is injective. For any $x, y \in M_k(\mathbb{C})$, we have

$$\varphi(xy) = \varphi^{\frac{1}{2}}(x)\varphi^{\frac{1}{2}}(y) \overset{(*)}{=} \varphi^{\frac{1}{2}}(y)\varphi^{\frac{1}{2}}(x) = \varphi(yx) \tag{1.16}$$

where the equality $(*)$ is given by the commutativity of A. Since φ is injective, we obtain $xy = yx$ for any $x, y \in M_k(\mathbb{C})$. As a consequence, $M_k(\mathbb{C})$ is commutative and this forces k to be equal to 1. □

An alternative proof for the previous lemma can be obtained in the following way. The existence of a non zero c.p.c. order zero map $\varphi : M_k(\mathbb{C}) \longrightarrow A$ yields the existence of an injective *-homomorphism $\pi_\varphi : M_k(\mathbb{C}) \longrightarrow \mathcal{M}\left(C^*(\varphi(A))\right)$. Since $C^*(\varphi(A))$ is commutative, its multiplier algebra is also commutative. This immediately implies $M_k(\mathbb{C})$ is commutative (so $k = 1$) since it is isomorphic to a commutative subalgebra of $\mathcal{M}\left(C^*(\varphi(A))\right)$.

Corollary 1.4.14. *Let A, B and C be unital C^*-algebras and let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ be c.p. order zero maps. Then the following identity holds*

$$\pi_{\psi\varphi}(a)x = \pi_\psi\pi_\varphi(a)x.$$

for all $x \in C^*(\psi\varphi(A))$.

Proof. By Theorem 1.4.6, we have

$$\varphi(a) = \varphi(1_A)\pi_\varphi(a), \qquad \psi(b) = \psi(1_B)\pi_\psi(b)$$

for all $a \in A$ and $b \in B$. In particular, we have

$$\psi\varphi(a) = \psi(1_B)\pi_\psi(\varphi(a)) = \psi(1_B)\pi_\psi(\varphi(1_A))\pi_\psi\pi_\varphi(a)$$

for all $a \in A$. Let $D = C^*(\psi\varphi(A))$ and consider H as the universal Hilbert space of D and, in fact, D acts nondegenerately on H. The support order zero map $\pi_{\psi\varphi}$ of $\psi\varphi$ is given by

$$\pi_{\psi\varphi}(a) = \text{s.o.} \lim_{n\to\infty} \left(\psi\varphi(1_A) + \frac{1}{n}\mathbf{1}_H\right)^{-1} \psi\varphi(a).$$

This leads to the following

$$\begin{aligned}
\pi_{\psi\varphi}(a)x &= \text{s.o.} \lim_{n\to\infty} \left(\psi\varphi(1_A) + \frac{1}{n}\mathbf{1}_H\right)^{-1} \psi\varphi(a)x \\
&= \text{s.o.} \lim_{n\to\infty} \left(\psi(1_B)\pi_\psi(\varphi(1_A)) + \frac{1}{n}\mathbf{1}_H\right)^{-1} \psi(1_B)\pi_\psi(\varphi(1_A))\pi_\psi\pi_\varphi(a)x \\
&= \pi_\psi\pi_\varphi(a)x. \qquad (1.17)
\end{aligned}$$

The last identity holds since

$$\text{s.o.} \lim_{n\to\infty} \left(\psi(1_B)\pi_\psi(\varphi(1_A)) + \frac{1}{n}\mathbf{1}_H\right)^{-1} \psi(1_B)\pi_\psi(\varphi(1_A)) = \mathbf{1}_H.$$

This finishes the proof. □

1.5 Nuclearity

We will denote the algebraic tensor product of the C^*-algebras A and B as $A \odot B$. A *C^*-norm* $\|\cdot\|_\alpha$ on $A \odot B$ is a norm which satisfies

$$\|xy\|_\alpha \leq \|x\|_\alpha\|y\|_\alpha, \qquad \|x^*\|_\alpha = \|x\|_\alpha, \qquad \|x^*x\|_\alpha = \|x\|_\alpha^2, \qquad x, y \in A \odot B.$$

The completion of $A \odot B$ with respect to this norm will be denoted as $A \otimes_\alpha B$. There are two important C^*-norms: the maximal C^*-norm $\|\cdot\|_{\max}$ and the spatial C^*-norm $\|\cdot\|_{\min}$ (which is also called the minimal C^*-norm). As the names are indicating, these two norms are the largest and the smallest C^*-norms on $A \odot B$, *i.e.*

$$\|x\|_{\min} \leq \|x\|_\alpha \leq \|x\|_{\max}, \qquad x \in A \otimes B,$$

for every C^*-norm $\|\cdot\|_\alpha$ on $A \odot B$ [13, Corollary 3.8, Theorem 4.8].

Another important fact is that every C^*-norm $\|\cdot\|_\alpha$ on $A \odot B$ is a *cross norm*, *i.e.* $\|a \otimes b\|_\alpha = \|a\|_A\|b\|_B$ for all elementary tensors $a \otimes b \in A \odot B$ [13, Lemma 3.4.10].

Definition 1.5.1. A C^*-algebra A is *nuclear* if for every other C^*-algebra B, there is a unique C^*-norm on $A \odot B$.

The tensor product of a nuclear C^*-algebra A with any other C^*-algebra B will be simply denoted as $A \otimes B$. Examples of nuclear C^*-algebras are matrix algebras, AF-algebras, commutative algebras and group C^*-algebras of amenable groups [13, Proposition 4.1, Proposition 4.2, Theorem 6.8]. The majority of the C^*-algebras in this thesis are nuclear.

We will now provide a characterisation of nuclearity through finite rank completely positive approximations.

Definition 1.5.2. A C^*-algebra A has the *completely positive approximation property* (CPAP) if for all finite subsets $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exist a finite dimensional algebra F and c.p.c. maps $\psi : A \longrightarrow F, \varphi : F \longrightarrow A$ such that

$$\|a - \varphi\psi(a)\| < \varepsilon \tag{1.18}$$

for all $a \in A$,

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\psi"'] & & A. \\ & F \arrow[ur, "\varphi"'] & \end{tikzcd}$$

The triple (F, ψ, φ) is called a *c.p.c. approximation* for $\mathfrak{F}$ within ε. A *system of c.p.c. approximations* for A will be a net of c.p.c. approximations $\left\{\left(F^{(r)}, \psi^{(r)}, \varphi^{(r)}\right)\right\}_{r\in I}$ converging to id_A in the point-norm topology, *i.e.* $\varphi^{(r)}\psi^{(r)}(a) \longrightarrow a$ for all $a \in A$. If A is separable, it is enough to consider a sequence of c.p.c. approximations. It follows from the definition that A has the completely positive approximation property if and only if there exists a system of c.p.c. approximations for A.

An striking theorem due to Choi and Effros [21], and Kirchberg [53] states that the completely positive approximation property and nuclearity are (surprisingly!) equivalent conditions.

Theorem 1.5.3 ([21, Theorem 3.1], [53, Corollary 1])**.** *A C^*-algebra A is nuclear if and only if A has the completely positive approximation property.*

The proof of the following folklore proposition is very similar to the proof of [13, Proposition 2.2.6]. We include it for completeness.

Proposition 1.5.4. *Let A be a unital nuclear C^*-algebra and let $M \in \mathbb{R}$ be a positive constant. Suppose that for every finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exist a finite dimensional algebra F and c.p. maps $\psi : A \longrightarrow F, \varphi : F \longrightarrow A$ such that*

$$\|a - \varphi\psi(a)\| < \varepsilon$$

for all $a \in A$ and $\|\psi\|, \|\varphi\| < M$. Then A is nuclear.

Proof. We will show A has the completely positive approximation property. In particular, if $M \leq 1$, the algebra already has it. So let us assume $M > 1$. Let $\mathfrak{F}$ be a finite subset of A and $\varepsilon > 0$. Without loss of generality we can suppose $\varepsilon < 1$, the elements of $\mathfrak{F}$ are postitive contractions and $1_A \in \mathfrak{F}$. Observe that the approximations given by the hypothesis do not entail the nuclearity because such maps are not necessarily contractions.

The idea of the proof is the following. We will replace the c.p. maps given by the hypothesis with different u.c.p. maps which also approximate id_A. The replacement of the map ψ is immediately given by [13, Lemma 2.2.5] and the replacement of the other map needs more technical details but essentially it boils down to finding approximations (F, ψ, φ) such that $\varphi\psi(1_A)$ is almost 1_A.

Using the continuity of the real valued function $t \mapsto t^{-\frac{1}{2}}$ at 1, we can find $\delta > 0$ such that if $|1 - t| < \delta$ then

$$\left|1 - t^{-\frac{1}{2}}\right| < \frac{\varepsilon}{4M}. \tag{1.19}$$

By hypothesis, there exist a matrix algebra F and c.p. maps $\psi : A \longrightarrow F, \varphi : F \longrightarrow A$ such that

$$\|a - \psi\varphi(a)\| < \min\left\{\delta, \frac{\varepsilon}{4}\right\} \tag{1.20}$$

for all $a \in \mathfrak{F}$. In particular, $\|1_A - \psi\varphi(1_A)\| < \delta$. We can assume δ is sufficiently small in such a way that $\psi\varphi(1_A)$ is a positive invertible element. By continuous functional calculus we have

$$\left\|1_A - \psi\varphi(1_A)^{-\frac{1}{2}}\right\| < \frac{\varepsilon}{4M}. \tag{1.21}$$

By [13, Lemma 2.2.5], there exists a u.c.p. map $\vartheta : A \longrightarrow F$ such that

$$\psi(a) = \psi(1_A)^{\frac{1}{2}}\vartheta(a)\psi(1_A)^{\frac{1}{2}}, \qquad a \in A. \tag{1.22}$$

Let us define a u.c.p. $\zeta : F \longrightarrow A$ that will replace the c.p. map $\varphi : F \longrightarrow A$. Set $b = \psi\varphi(1_A)^{-\frac{1}{2}}$ and define

$$\zeta(x) = b\varphi\left(\psi(1_A)^{\frac{1}{2}} x \psi(1_A)^{\frac{1}{2}}\right) b, \qquad x \in F. \tag{1.23}$$

Observe that this map is unital. Indeed,

$$\begin{aligned}\zeta(1_F) &= b\varphi\left(\psi(1_A)^{\frac{1}{2}} 1_F \psi(1_A)^{\frac{1}{2}}\right) b \\ &= \psi\varphi(1_A)^{-\frac{1}{2}} \varphi\psi(1_A) \psi\varphi(1_A)^{-\frac{1}{2}} \\ &= 1_A. \end{aligned} \tag{1.24}$$

In order to finish, let us prove these u.c.p. maps approximate the finite subset $\mathfrak{F}$. By construction we have

$$\begin{aligned}\zeta\vartheta(a) &= b\varphi\left(\psi(1_A)^{\frac{1}{2}} \vartheta(a) \psi(1_A)^{\frac{1}{2}}\right) b \\ &\overset{(1.22)}{=} b\varphi\psi(a)b. \end{aligned} \tag{1.25}$$

Observe

$$\begin{aligned}\|\varphi\psi(a) - \zeta\vartheta(a)\| &\overset{(1.25)}{=} \|\varphi\psi(a) - b\varphi\psi(a)b\| \\ &\leq \|\varphi\psi(a) - b\varphi\psi(a)\| + \|\varphi\psi(a)b - b\varphi\psi(a)b\| \\ &\leq \|1_A - b\| \, \|\varphi\psi(a)\| + \|1_A - b\| \, \|\varphi\psi(a)\| \, \|b\| \\ &\overset{(1.21)}{<} \frac{\varepsilon}{4M}M + \frac{\varepsilon}{4M}M\left(1 + \frac{\varepsilon}{4M}\right) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4}(1+\varepsilon) \\ &< \frac{3\varepsilon}{4}. \end{aligned} \tag{1.26}$$

Finally we have

$$\begin{aligned}\|a - \zeta\vartheta(a)\| &\leq \|a - \varphi\psi(a)\| + \|\varphi\psi(a) - \zeta\vartheta(a)\| \\ &\overset{(1.20)(1.26)}{<} \frac{\varepsilon}{4} + \frac{3\varepsilon}{4} \\ &= \varepsilon \end{aligned} \tag{1.27}$$

for all $a \in \mathfrak{F}$. Since ϑ and ζ are unital, these maps are contractive. Then, by Theorem 1.5.3, A is nuclear. □

A refinement of the completely positive approximation was proved in [49]. The approximations can be taken to be convex combinations of order zero maps. Very recently, it was proved in [12] that the first map ψ can be taken to be approximately order zero. This theorem is one of the key motivations for this thesis. We will discuss its connections with nuclear dimension in Chapter 3 and we will sketch its proof in Section 4.3.

Theorem 1.5.5 ([49, Theorem 1.4], [12, Theorem 3.1])**.** *Let A be a nuclear C^*-algebra. Then for any finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exist a finite dimensional C^*-algebra F and c.p.c. maps $\psi : A \longrightarrow F, \varphi : F \longrightarrow A$ such that*

(i) $\|a - \varphi\psi(a)\| < \varepsilon$ *for all* $a \in \mathfrak{F}$*,*

(ii) the c.p.c. map φ is a convex combination of finitely many order zero maps,

(iii) $\|\psi(a)\psi(b)\| < \varepsilon$ *if* $a, b \in \mathfrak{F}$ *are orthogonal positive elements.*

1.6 Finite algebras

In this section we will recall some basic facts about finite algebras. A more detailed study can be found in [84, Section 1.1].

Two projections p and q in a C^*-algebra A are *Murray-von Neumann equivalent* if there exists $v \in A$ such that

$$p = v^*v, \qquad q = vv^*.$$

Definition 1.6.1. Let A be a C^*-algebra and let $p \in A$ be a projection.

(i) If p is Murray-von Neumann equivalent to a proper sub-projection of itself, $p \in A$ is called *infinite*. A C^*-algebra is *infinite* if it contains an infinite projection.

(ii) The projection p is *finite* if it is not infinite. A C^*-algebra is *finite* if it admits an approximate unit of projections and all projections are finite.

A unital C^*-algebra A is *stably finite* if its stabilisation $A \otimes \mathbb{K}$ is finite and a non unital C^*-algebra is stably finite if its unitisation is. Similarly, A is *stably projectionless* if $A \otimes \mathbb{K}$ is projectionless.

A *quasitrace* is a continuous function $\tau : A_+ \longrightarrow \mathbb{R}_+$ that satisfies $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$, $\tau(\lambda a) = \lambda\tau(a)$ for all $\lambda \geq 0$ and $a \in A_+$, and $\tau(a + b) = \tau(a) + \tau(b)$ if $a, b \in A_+$ commute, and such that τ extends to a map $M_2(A)_+ \longrightarrow \mathbb{R}_+$ with the same properties. It was proved by Blackadar and Handelman that every unital stably finite C^*-algebra admits a quasitrace [5]. Morever, Haagerup proved that in *exact* C^*-algebras quasitraces are actually traces [47, Theorem 5.11]. These two results add up to the following theorem.

Theorem 1.6.2 (Blackadar-Handelman-Haagerup, [84, Theorem 1.1.10])**.** *Every unital, stably finite, exact C^*-algebra admits a trace, and every unital stably finite C^*-algebra admits a quasitrace.*

Quasitraces and traces might be unbounded if the C^*-algebra is not unital. Instead, they are defined only on a dense subset of A_+ or A. However, densely defined traces are not important for this thesis since we will be working with unital C^*-algebras most of the time.

1.7 Purely infiniteness and Kirchberg algebras

In this section we will introduce an important class of C^*-algebras. Historically, pure infiniteness was introduced as a property for simple C^*-algebras by Cuntz [25]. This definition has been extended by Kirchberg and Rørdam to non simple algebras [55].

Definition 1.7.1 ([55, Definition 4.1])**.** A C^*-algebra A is *purely infinite* if it has no non zero abelian quotients and for every pair of non zero positive elements $a, b \in A$, where b belongs to the closed two sided ideal generated by a, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A such that $x_n^* a x_n \longrightarrow b$.

Purely infinite and simple C^*-algebras have been studied extensively in the past. We present some properties of this particular class.

Proposition 1.7.2 ([84, Proposition 4.1])**.** *Let A be a simple C^*-algebra. Then the following conditions are equivalent:*

(i) A is purely infinite,

(ii) for every pair of non zero positive elements $a, b \in A$ there exists $x \in A$ such that $a = x^* b x$,

(iii) for every pair of non zero positive elements $a, b \in A$ there exist $x, y \in A$ such that $a = xby$,

(iv) the real rank of A is zero and every non zero projection in A is properly infinite,

(v) every non zero hereditary subalgebra of A contains an infinite projection.

The Cuntz algebras are examples of simple and purely infinite algebras. We have the following dichotomy result for purely infinite and simple C^*-algebras.

Proposition 1.7.3 ([113, Theorem 1.2]). *Every separable, purely infinite, simple C^*-algebra is either unital or stable.*

Kirchberg obtained spectacular results concerning simple and purely infinite C^*-algebras. Let us recall some of them. A C^*-algebra is *elementary* if it is isomorphic either to $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ or $\mathbb{K}$. A simple C^*-algebra A is called *tensorially non-prime* if it is not isomorphic to the minimal tensor product of two non-elementary algebras, otherwise A is called *tensorially prime*.

Theorem 1.7.4 (Kirchberg, [84, Theorem 4.1.10]). *Let A and B be simple C^*-algebras.*

(i) Suppose that A is not stably finite and that B is not elementary. Then the minimal tensor product $A \otimes B$ is simple and purely infinite.

(ii) Suppose that D is a simple, exact C^*-algebra that is tensorially non-prime. Then D is either stably finite or purely infinite.

Theorem 1.7.5 (Kirchberg's Absorption Theorems). *The following statements are true.*

(i) [84, Theorem 7.1.2] *A is a simple, separable, unital, and nuclear C^*-algebra if and only if $\mathcal{O}_2 \cong A \otimes \mathcal{O}_2$.*

(ii) [84, Theorem 7.2.2] *Let A be a simple, separable, and nuclear C^*-algebra. Then $A \cong A \otimes \mathcal{O}_\infty$ if and only if A is purely infinite.*

The class of C^*-algebras described in part (ii) of the previous theorem have been named after Kirchberg.

Definition 1.7.6. A *Kirchberg algebra* is a purely infinite, simple, nuclear and separable C^*-algebra.

Now let us focus on a weaker form of pure infiniteness that was introduced in [56].

Definition 1.7.7 ([7, Definition 1.2]). Let n be a natural number. A C^*-algebra A is *n-purely infinite* if the following conditions hold.

(i) For every pair of non zero positive elements $a, b \in A$ such that b lies in the closed two-sided ideal of A generated by a, and for every $\varepsilon > 0$, there exist $d_1, \dots, d_n \in A$ such that
$$\left\| b - \sum_{k=1}^{n} d_k^* a d_k \right\| < \varepsilon.$$

(ii) There is no non zero quotient algebra of $\ell^\infty(A)$ of dimension less or equal to n^2.

The algebra A is called *weakly purely infinite* if A is n-purely infinite for some $n \in \mathbb{N}$.

Remark 1.7.8. Another definition of n-pure infiniteness was introduced by Kirchberg and Rørdam in [56, Definition 4.1]. It was proved in [7, Proposition 4.12] by Blanchard and Kirchberg that the definition introduced here is weaker than the former definition from [56]. However, both definitions induce the same notion of weakly purely infinite C^*-algebras [7, Proposition 4.12].

It immediately follows from the definition that 1-pure infiniteness is just pure infiniteness. It turns out that if a C^*-algebra is weakly purely infinite then its ultrapower is *traceless*.

Definition 1.7.9. A C^*-algebra A is *traceless* if no algebraic ideal of A admits a non-zero quasitrace.

Theorem 1.7.10 ([56, Theorem 4.8])**.** *Let A be a C^*-algebra.*

(i) For each free filter ω on $\mathbb{N}$ the following three conditions are equivalent.

(a) A_ω is traceless.

(b) A_ω is weakly purely infinite.

(c) A is weakly purely infinite.

(ii) If A is weakly purely infinite, then A is traceless.

In the simple case, the notion of weakly pure infiniteness is equivalent to pure infiniteness.

Theorem 1.7.11 ([56, Corollary 4.6])**.** *Any weakly purely infinite C^*-algebra which is simple is purely infinite.*

1.8 Quasidiagonality

In this section we will review the notion of quasidiagonality. For a more complete discussion see [13, Chapter 7].

Definition 1.8.1. Let H be a Hilbert space and let $\Omega \subset B(H)$ be an arbitrary collection of operators. Then Ω is called a *quasidiagonal set* if for each finite set $\mathfrak{F} \subset \Omega$, each finite set $\chi \subset H$ and each $\varepsilon > 0$ there exists a finite-rank projection $P \in B(H)$ such that

$$\|PT - TP\| < \varepsilon, \qquad T \in \mathfrak{F},$$

and

$$\|Pv - v\| < \varepsilon, \qquad v \in \chi.$$

A separable C^*-algebra A is *quasidiagonal* if it has a faithful representation as a set of quasidiagonal operators on some Hilbert space. The C^*-algebra A is called *strongly quasidiagonal* if $\pi(A)$ is a quasidiagonal set of operators for every representation π of A.

It is immediate from the definition that matrix algebras are quasidiagonal. Other more interesting examples are AF-algebras. Like with nuclearity, we can characterise quasidiagonality using c.p.c. maps.

Theorem 1.8.2 (Voiculescu [103])**.** *A C^*-algebra A is quasidiagonal if and only if there exists a net of c.p.c maps $\varphi_i : A \longrightarrow M_{k_i}(\mathbb{C})$ such that*

(i) $\|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\| \longrightarrow 0$

(ii) $\|\varphi_i(a)\| \longrightarrow \|a\|$

for all $a, b \in A$.

If the C^*-algebra is unital, we can assume the c.p.c. maps φ_n are unital. As a straightforward application of this theorem, we obtain that commutative C^*-algebras are quasidiagonal. In this case it is rather simple to construct *-homomorphisms from a commutative C^*-algebra to finitely many copies of $\mathbb{C}$ just by evaluating at some points of its spectrum. If we select the appropriate points, we obtain a net of *-homomorphisms satisfying conditions (i) and (ii) of Theorem 1.8.2.

Another consequence of this theorem is that a separable C^*-algebra A is quasidiagonal if and only if it embeds in the ultrapower of the universal UHF algebra, $\mathcal{Q}_\omega$, and this embedding has a c.p. lift $A \longrightarrow \ell^\infty(\mathcal{Q})$,

$$\begin{tikzcd} & \ell^\infty(\mathcal{Q}) \arrow[d, two heads] \\ A \arrow[ur, dashed] \arrow[r, hook] & \mathcal{Q}_\omega. \end{tikzcd}$$

If the algebra is nuclear we do not need to ask for a c.p. lift since the Choi-Effros lifting theorem automatically provides it [90, Proposition 1.4.(i)].

Definition 1.8.3. A C^*-algebra A *homotopically dominates* another C^*-algebra B if there are *-homomorphisms $\pi : A \longrightarrow B, \sigma : B \longrightarrow A$ such that $\pi \circ \sigma$ is homotopic to id_B. The algebras A and B are *homotopy equivalent* if there are *-homomorphisms $\pi : A \longrightarrow B, \sigma : B \longrightarrow A$ such that $\sigma \circ \pi$ is homotopic to id_B and $\pi \circ \sigma$ is homotopic to id_A.

Voiculescu proved that quasidiagonality is a homotopy invariant. This fact will produce many examples of quasidiagonal C^*-algebras.

Theorem 1.8.4 (Voiculescu [103])**.** *Let A and B be C^*-algebras. If A homotopically dominates B and A is quasidiagonal, then B is also quasidiagonal.*

It is immediate that the 0 algebra is quasidiagonal. Let $\sigma_t : C_0(0,1] \longrightarrow C_0(0,1]$ be given by $\sigma_t(f)(s) = f(ts)$. This defines a homotopy between the zero map and $\mathrm{id}_{C_0(0,1]}$. After tensoring this homotopy with id_A, we obtain that $C_0(0,1]$ is homotopic to zero. By Voiculescu's theorem, the cone over A, $C_0(0,1] \otimes A$ is quasidiagonal. Finally, since quasidiagonality passes to subalgebras, the suspension $C(0,1) \otimes A$ is also quasidiagonal.

Corollary 1.8.5. *For any C^*-algebra, both the cone over A, $C_0(0,1] \otimes A$, and the suspension of A, $C(0,1) \otimes A$, are quasidiagonal.*

There exist two known obstructions to quasidiagonality. One is stable finiteness, which was introduced in Section 1.6. The other obstruction relates to the existence of an *amenable trace*.

Definition 1.8.6. Let A be a C^*-algebra represented in $B(H)$. A state τ is called an *amenable trace* if there exists a state φ on $B(H)$ such that

1. $\varphi|_A = \tau$,
2. $\varphi(uTu^*) = \varphi(T)$ for every unitary $u \in A$ and $T \in B(H)$.

The definition of amenable trace does not depend on the representation [13, Proposition 6.2.2].

Proposition 1.8.7 ([13, Proposition 7.1.15, Proposition 7.1.16])**.** *Let A be a quasidiagonal C^*-algebra. Then A is stably finite. If additionally A is unital, then A has an amenable trace.*

This immediately implies that C^*-algebras which are not stably finite cannot be quasidiagonal, for example the Toeplitz algebra, $B(H)$ and Cuntz algebras. An important example is the reduced group C^*-algebra $C^*_\lambda(\mathbb{F}_2)$, it is stably finite but it does not have an amenable trace and hence it is not quasidiagonal.

In the appendix of [48], Rosenberg showed that if the reduced group C^*-algebra of a discrete group Γ is quasidiagonal then Γ is amenable and he conjectured the converse was true. This question remained unsolved for many year. A partial solution was found by Ozawa, Rørdam and Sato who verified this conjecture for elementary amenable groups [72, Theorem 3.8]. Very recently, Tikuisis, White and Winter proved that this conjecture is true for any amenable group [96, Corollary C].

Corollary 1.8.8 (Rosenberg [48, Appendix], [90, Corollary C])**.** *Let Γ be a discrete group. Then the reduced group C^*-algebra is quasidiagonal if and only if Γ is amenable.*

We finish this section by giving one last family of examples of quasidiagonal algebras.

Theorem 1.8.9 ([19, Theorem 7])**.** *For $n = \infty, 1, 2, \dots$ the full group C^*-algebra $C^*(\mathbb{F}_n)$ is quasidiagonal.*

In fact, Choi proved this theorem for $n = 2$. But the result is true for $n = \infty, 1, 3, \dots$ since $C^*(\mathbb{F}_n)$ sits as a subalgebra of $C^*(\mathbb{F}_2)$ and quasidiagonality passes to subalgebras.

1.9 The Jiang-Su algebra $\mathcal{Z}$

In this section we will recall an important C^*-algebra that was introduced by Jiang and Su in [51]. As explained in the introduction, this algebras has become extremely important for the classification program. The Jiang-Su algebra $\mathcal{Z}$ is highly relevant for this thesis and more properties of this algebra will be discussed in the forthcoming chapters. This section is mostly based on Sections 2 and 3 of [85].

Let $p, q \in \mathbb{N}$. The *dimension drop algebra* $Z_{p,q}$ is given by

$$Z_{p,q} = \left\{ f \in C\left([0,1], M_p(\mathbb{C}) \otimes M_q(\mathbb{C})\right) \mid f(0) \in M_q(\mathbb{C}) \otimes 1_{M_q(\mathbb{C})},\ f(1) \in 1_{M_p(\mathbb{C})} \otimes M_q(\mathbb{C}) \right\}.$$

If q and q are relatively prime numbers, the dimension drop algebra $Z_{p,q}$ is called prime. It turns out that in this case, $Z_{p,q}$ has no non trivial projections [51, Lemma 2.2], and $K_0\left(Z_{p,q}\right) \cong \mathbb{Z}$ and $K_1\left(Z_{p,q}\right) \cong 0$ [51, Lemma 2.3].

Theorem 1.9.1 ([51, Theorem 1])**.** *There exists a separable, simple, nuclear, infinite dimensional C^*-algebra $\mathcal{Z}$ with unique trace and with no non trivial projections such that $K_0(\mathcal{Z}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(\mathcal{Z}) \cong K_1(\mathbb{C}) \cong 0$.*

The algebra $\mathcal{Z}$ is called the *Jiang-Su algebra*. It was constructed by Jiang and Su as an inductive limit of prime dimension drop algebras. However this construction is not canonical and infinitely many choices are made when choosing the connecting maps. Let us recall some important properties of the Jiang-Su algebra.

Theorem 1.9.2 ([51, Theorem 3, Theorem 4])**.** *Let $\mathcal{Z}$ be the Jiang-Su algebra.*

(i) Any unital endomorphism on $\mathcal{Z}$ is approximately inner.

(ii) For any $n \in \mathbb{N}$, $\mathcal{Z} \cong \mathcal{Z}^{\otimes n}$.

We will finish this section by providing an intrinsic description of the Jiang-Su algebra $\mathcal{Z}$. Let p be a supernatural number and let us denote by M_p the UHF algebra associated to p. If $p^\infty = p$, M_p and p are called of infinite type.

Consider supernatural numbers p and q. The *generalised dimension drop algebra* $\mathcal{Z}_{p,q}$ is given by

$$\mathcal{Z}_{p,q} = \left\{ f \in C\left([0,1], M_p \otimes M_q\right) \mid f(0) \in M_p \otimes 1_{M_q},\ f(1) \in 1_{M_p} \otimes M_q \right\}. \tag{1.28}$$

Like in the case of the dimension drop algebras, if p and q are relatively prime supernatural numbers, the generalised dimension drop algebra $\mathcal{Z}_{p,q}$ is called prime. It was proved by Rørdam and Winter that we can also characterise the Jiang-Su algebra $\mathcal{Z}$ using generalised dimension drop algebras which are prime.

Theorem 1.9.3 ([85, Corollary 3.2, Proposition 3.3])**.** *Let p and q be infinite supernatural numbers of infinite type.*

(i) The generalised dimension drop algebra $\mathcal{Z}_{p,q}$ tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$, i.e. $\mathcal{Z}_{p,q} \otimes \mathcal{Z} \cong \mathcal{Z}_{p,q}$.

(ii) If p and q are relatively prime, then $\mathcal{Z}_{p,q}$ embeds unitally into $\mathcal{Z}$.

Remark 1.9.4. Let p, q be natural numbers and consider the dimension drop algebra $Z_{p,q}$. A unital *-homomorphism $\psi : Z_{p,q} \longrightarrow \mathcal{Z}$ is called *standard* if

$$\tau_{\mathcal{Z}}\left(\psi(f)\right) = \int_0^1 \operatorname{tr}\left(f(t)\right) dt, \qquad f \in Z_{p,q},$$

where $\tau_{\mathcal{Z}}$ is the unique trace on the Jiang-Su algebra $\mathcal{Z}$ and tr is the normalized trace on M_{pq}. Observe that standard homomorphisms induce the *Lebesgue trace* on its image; *i.e.*

$$\tau_{\mathcal{Z}}\left(\psi\left(f\otimes 1_{M_p}\otimes 1_{M_q}\right)\right)=\int_0^1 f(t)dt, \qquad f\in C[0,1].$$

By the previous theorem, the generalised dimension drop algebra $\mathcal{Z}_{p^\infty,q^\infty}$ embeds unitally in the Jiang-Su algebra $\mathcal{Z}$ if p and q are relatively prime. Furthermore, this embedding can be taken to be an inductive limit of standard unital embeddings. This shows that this embedding can be taken in such a way that the trace on $\mathcal{Z}$ induces the *Lebesgue trace* on $\mathcal{Z}_{p^\infty,q^\infty}$ [83, Proposition 2.2].

A unital endomorphism φ on a unital C^*-algebra is *trace collapsing* if $\tau\circ\varphi=\tau'\circ\varphi$ for every pair of traces τ and τ' on A.

Theorem 1.9.5 ([85, Theorem 3.4])**.** *Let p and q be infinite supernatural numbers that are relatively prime.*

(i) There exists a trace collapsing unital endomorphism on $\mathcal{Z}_{p,q}$.

(ii) Let φ be any trace collapsing unital endomorphism on $\mathcal{Z}_{p,q}$. Then $\mathcal{Z}$ is isomorphic to the indictive limit of the sequence

$$\mathcal{Z}_{p,q}\xrightarrow{\varphi}\mathcal{Z}_{p,q}\xrightarrow{\varphi}\mathcal{Z}_{p,q}\xrightarrow{\varphi}\cdots.$$

The (canonical) trace collapsing endomorphism on $\mathcal{Z}_{p,q}$ is given by Theorem 1.9.3; precisely, since p and q are relatively prime there exist an embedding $\mathcal{Z}_{p,q}\hookrightarrow\mathcal{Z}$ and we also have an embedding $\mathcal{Z}\hookrightarrow\mathcal{Z}_{p,q}$ given by the second factor embedding $1_{\mathcal{Z}_{p,q}}\otimes\mathrm{id}_{\mathcal{Z}}$ followed by an isomorphism between $\mathcal{Z}_{p,q}\otimes\mathcal{Z}$ and $\mathcal{Z}_{p,q}$. A trace collapsing endomorphism on $\mathcal{Z}_{p,q}$ is obtained by the composition

$$\mathcal{Z}_{p,q}\hookrightarrow\mathcal{Z}\hookrightarrow\mathcal{Z}_{p,q}.$$

It turns out that the Jiang-Su algebra $\mathcal{Z}$ can be characterised as the unique C^*-algebra satisfying Theorem 1.9.2 such that there exist relatively prime infinite supernatural numbers p and q and embeddings $\mathcal{Z}_{p,q}\hookrightarrow\mathcal{Z}\hookrightarrow\mathcal{Z}_{p,q}$ [85, Proposition 3.5].

1.10 Cuntz comparison

We will present a quick overview of Cuntz comparison and the Cuntz semigroup. We will also compute the Cuntz semigroup of some examples we will use later. A more comprehensive review of this topic can be found in [2].

Consider a C^*-algebra A. Let $M_\infty(A)$ denote the algebraic limit of the sequence

$$A \xrightarrow{\varphi_1} M_2(A) \xrightarrow{\varphi_2} M_3(A) \xrightarrow{\varphi_3} \dots \xrightarrow{\varphi_{n-1}} M_n(A) \xrightarrow{\varphi_n} \dots$$

where $\varphi_n : M_n(A) \longrightarrow M_{n+1}(A)$ is given by

$$\varphi_n(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

For positive elements $a, b \in M_\infty(A)$, let us denote

$$a \oplus b := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

It is immediate that $a \oplus b$ is also positive.

Definition 1.10.1 ([24])**.** Let A be a C^*-algebra and let $x, y \in M_\infty(A)$ be positive elements. The element x is *Cuntz subequivalent* to y, denoted by $x \preceq y$, if there exists a sequence $(z_n)_n$ in $M_\infty(A)$ such that $x = \lim\limits_{n\to\infty} z_n^* y z_n$. The element x is called *Cuntz equivalent* to y, denoted by $x \sim y$, if $x \preceq y$ and $y \preceq x$.

For $a \in A_+$ and $\varepsilon > 0$, the element $(a - \varepsilon)_+$ will denote $g_\varepsilon(a) \in A$ given by functional calculus where $g_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $g_\varepsilon(t) = \max\{0, t - \varepsilon\}$. The next proposition is a useful statement whilst working with $\preceq$.

Proposition 1.10.2 ([81, Proposition 2.4])**.** *Let A be a C^*-algebra and $x, y \in M_\infty(A)_+$. The following conditions are equivalent.*

(i) $x \preceq y$.

(ii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $(x - \varepsilon)_+ \preceq (y - \delta)_+$.

(iii) For every $\varepsilon > 0$ there exists $\delta > 0$ and $r \in A$ such that $(x - \varepsilon)_+ = r(y - \delta)_+ r^*$.

The Cuntz equivalence is in fact an equivalence relation ([4, Definition II.3.4.3]) and we will denote the class of a by $\langle a \rangle$.

Definition 1.10.3 ([24, Section 4],[23, Appendix])**.** Let A be a C^*-algebra. The *Cuntz semigroup* of A, denoted as $W(A)$, is the quotient

$$W(A) := M_\infty(A)/\sim.$$

The completed Cuntz semigroup of A is defined as

$$\mathrm{Cu}(A) = W(A \otimes \mathbb{K}).$$

We can equip the Cuntz semigroup $W(A)$ with the structure of a positively ordered abelian monoid; *i.e.* $\langle 0\rangle \leq x$ for any $x \in W(A)$. The sum is given by

$$\langle a\rangle \oplus \langle b\rangle = \langle a \oplus b\rangle$$

and the partial order is defined by

$$\langle a\rangle \leq \langle b\rangle \quad \text{if} \quad a \precsim b.$$

The Murray-von Neumann semigroup $V(A)$ is defined as the set of Murray-von Neumann equivalence classes of projections of $M_\infty(A)$. There is a natural map $V(A) \longrightarrow W(A)$, given by sending the Murray-von Neumann equivalence class of each projection to its Cuntz-equivalence class. This map is in fact injective if the C^*-algebra is stably finite [2, Lemma 2.20].

Example 1.10.4. (i) [2, Proposition 2.38] $W(M_n(\mathbb{C})) = \mathbb{N}_0$ and $\mathrm{Cu}(M_n(\mathbb{C})) = \mathbb{N}_0 \cup \{\infty\}$. The class of each positive element is determined by its rank.

(ii) [2, Proposition 2.41] If A is a Kirchberg algebra, then $W(A) = \mathrm{Cu}(A) = \{0, \infty\}$. This follows from the fact that any two non zero elements in a Kirchberg algebra are Cuntz-equivalent (Proposition 1.7.2) and because $A \otimes \mathbb{K}$ is a Kirchberg algebra by Proposition 1.7.3.

(iii) [2, Theorem 2.45] $W(\mathbb{K}) = \mathrm{Cu}(\mathbb{K}) = \mathbb{N}_0 \cup \{\infty\}$. The class of each positive element is determined by its rank.

From these examples, we can observe that W is not continuous with respect to inductive limits. In [23], Cu was introduced in order to amend this inconvenience.

Let $\pi : A \longrightarrow B$ be a *-homomorphism between two C^*-algebras, π induces a map $\mathrm{Cu}(\pi) : \mathrm{Cu}(A) \longrightarrow \mathrm{Cu}(B)$ in the following way:

$$\mathrm{Cu}(\pi)(\langle a\rangle) = \langle \pi(a)\rangle.$$

In this way, Cu is a functor from the category of C^*-algebras to a category which is called `Cu`. Very roughly speaking, the objects of `Cu` are ordered abelian semigroups which satisfy that the order is compatible with the sum, every countable upward directed set has a supremum and some other technical axioms (see [2, Definition 4.2] for a proper description of the category `Cu`). The maps in `Cu` are semigroup homomorphisms which preserve the zero element, order, suprema of countable upward directed sets and the "way below" order relation $\ll$. This relation is defined in the following way: Let $(S, \leq)$ be an ordered abelian semigroup and $x, y \in S$. The element x is *way below* y, denoted as $x \ll y$ if whenever $(y_n)_{n\in\mathbb{N}}$ is an increasing sequence with $\sup y_n \geq y$, then there exists n such that $x \leq y_n$.

Let A be a unital C^*-algebra. An state on $\mathrm{Cu}(A)$ is an order preserving morphism $s : \mathrm{Cu}(A) \longrightarrow \mathbb{R}$ such that $s(\langle 1_A \rangle) = 1$.

Definition 1.10.5. Every quasi trace τ on A determines a *dimension function* $d_\tau : A_+ \longrightarrow [0, \infty]$ in the following way,

$$d_\tau(a) = \lim_{n\to\infty} \tau\left(a^{\frac{1}{n}}\right), \qquad a \in A_+.$$

Similarly, a quasitrace τ on $A \otimes \mathbb{K}$ determines a dimension function $d_\tau : \mathrm{Cu}(A) \longrightarrow [0, \infty]$

$$d_\tau(\langle a \rangle) = \lim_{n\to\infty} \tau\left(a^{\frac{1}{n}}\right), \qquad \langle a \rangle \in \mathrm{Cu}(A).$$

It turns out that every state on $\mathrm{Cu}(A)$ arises in this way [37, Theorem 4.4]. Let $(W, \leq)$ be an ordered semigroup. An element $x \in W$ is *compact* if $x \ll x$ and an element $y \in W$ is soft if $y' \ll y$ then there exists $k \in \mathbb{N}$ such that $(k+1)y' \leq ky$. With these definitions in hand, let us present the following theorem which will allow us to compute Cu of some C^*-algebras. The symbol $\sqcup$ denotes disjoint union.

Theorem 1.10.6 ([37, Corollary 6.8, Remark 6.9])**.** *Let A be separable, unital, simple, stably finite, $\mathcal{Z}$-stable and exact C^*-algebra with unique trace τ_A. Then*

$$\mathrm{Cu}(A) \cong V(A) \sqcup (0, \infty]. \tag{1.29}$$

The order $\leq$ of $V(A) \sqcup (0, \infty]$ restricted to $V(A)$ or $(0, \infty]$ agrees with their usual orders. Let $[p] \in V(A)$ and $t \in (0, \infty]$, then $[p] \leq t$ if $\tau_A(p) < t$. In the same way, $t \leq [p]$ if $t \leq \tau_A(p)$.

Remark 1.10.7. The elements in $V(A)$ correspond to the compact elements of $\mathrm{Cu}(A)$ and the elements in $(0, \infty]$ correspond to the soft elements. Let us describe this identification in more detail. We will describe the compact part of $\mathrm{Cu}(A)$ first. Since A is stably finite, there is a canonical map from $V(A)$ to $\mathrm{Cu}(A)$ which is injective, this map just takes the Murray-von Neumann class of a projection p and sends it to its Cuntz-equivalence class $[p]$. If $\langle a \rangle \in \mathrm{Cu}(A)$ is compact, then there exists a projection $p \in M_\infty(A)$ such that $a \sim p$ and $\langle a \rangle$ is identified with the Murray-von Neumann class of p. On the other hand, if $\langle a \rangle \in \mathrm{Cu}(A)$ is soft, it is identified with $d_{\tau_A}(\langle a \rangle)$.

Let us present an important example.

Example 1.10.8. By Theorem 1.9.1, the Jiang-Su algebra $\mathcal{Z}$ has the same K-theoretical data of $\mathbb{C}$. This yields $V(A) \cong \mathbb{N}_0$. Then

$$\mathrm{Cu}(\mathcal{Z}) \cong \mathbb{N}_0 \sqcup (0, \infty]. \tag{1.30}$$

Let X be a topological space and S an object of $\mathtt{Cu}$. A function $f : X \longrightarrow S$ is *lower semicontinuous* if for all $s \in S$ the set $f^{-1}(s^{\ll}) = \{x \in X \mid s \ll f(x)\}$ is open in X. The set of lower semicontinuous functions from X to S will be denoted as $\mathrm{Lsc}(X, S)$. The following theorem gives us a precise description of Cu for cones of AF-algebras.

Theorem 1.10.9 ([1, Theorem 3.4])**.** *Let X be a locally compact Hausdorff space which is second countable and one dimensional. Let A be a simple AF-algebra. Then*

$$\mathrm{Cu}\left(C_0(X) \otimes A\right) \cong \mathrm{Lsc}\left(X, \mathrm{Cu}(A)\right). \tag{1.31}$$

It is important to note that Theorem 1.10.6 and 1.10.9 apply to a broader class of C^*-algebras. They are presented here only in the form we will need them in Chapter 5.

Let us briefly recall the constructions of pullbacks of C^*-algebras and pullbacks of semigroups in $\mathtt{Cu}$. A more complete description can be found in [1, Section 3]. Let A, B and C be C^*-algebras. Let $\pi : A \longrightarrow C$ and $\varphi : B \longrightarrow C$ be *-homomorphisms. The pullback $A \oplus_C B$ is the C^*-algebra given by

$$A \oplus_C B = \{(a, b) \in A \oplus B \mid \pi(a) = \varphi(b)\}. \tag{1.32}$$

Normally pullbacks are represented with the following diagram,

$$\begin{tikzcd} A \oplus_C B \arrow[r, dashed] \arrow[d, dashed] & B \arrow[d, "\varphi"] \\ A \arrow[r, "\pi"'] & C. \end{tikzcd}$$

We can also construct pullbacks in the category of $\mathtt{Cu}$. Let R, S and T be elements of $\mathtt{Cu}$ and consider Cu-morphisms $\rho : R \longrightarrow T$ and $\sigma : S \longrightarrow T$. The pullback $R \oplus_T S$ is the semigroup given by

$$R \oplus_T S = \{(r, s) \in R \oplus S \mid \rho(r) = \sigma(s)\}, \tag{1.33}$$

$$\begin{tikzcd} R \oplus_T S \arrow[r, dashed] \arrow[d, dashed] & S \arrow[d, "\sigma"] \\ R \arrow[r, "\rho"'] & T. \end{tikzcd}$$

Theorem 1.10.10 ([1, Corollary 3.5])**.** *Let X be a compact Hausdorff space that is second countable and one dimensional. Let A be a separable C^*-algebra with stable rank one such that $K_1(I) = 0$ for every closed two sided ideal I of A. Let B be any C^*-algebra and suppose $\varphi : B \longrightarrow C(Y, A)$ is a *-homomorphism, where $Y \subset X$ is a closed subset of X. Then*

$$\mathrm{Cu}\left(C\left(X, A\right) \oplus_{C(Y,A)} B\right) \cong \mathrm{Lsc}\left([0,1], \mathrm{Cu}(A)\right) \oplus_{\mathrm{Lsc}(Y,\mathrm{Cu}(A))} \mathrm{Cu}(B) \tag{1.34}$$

in the category of `Cu`*, where*

$$\begin{tikzcd} \mathrm{Lsc}\left([0,1], \mathrm{Cu}(A)\right) \oplus_{\mathrm{Lsc}(Y,\mathrm{Cu}(A))} \mathrm{Cu}(B) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Cu}(B) \arrow[d, "\mathrm{Cu}(\varphi)"] \\ \mathrm{Lsc}\left([0,1], \mathrm{Cu}(A)\right) \arrow[r, "f \mapsto f|_Y"'] & \mathrm{Lsc}\left(Y, \mathrm{Cu}(A)\right). \end{tikzcd}$$

With this theorem in hand, let us compute an important example.

Example 1.10.11. Let A be a separable, simple, unital AF-algebra with unique trace. The unitisation of $C_0(0,1] \otimes A$ is given by

$$(C_0(0,1] \otimes A)^{\sim} \cong \{f \in C([0,1], A) \mid f(0) \in \mathbb{C}1_A\}. \tag{1.35}$$

We can write $(C_0(0,1] \otimes A)^{\sim}$ as a pullback given by the diagram

$$\begin{tikzcd} C([0,1], A) \oplus_{C(\{0\},A)} \mathbb{C} \arrow[r, dashed] \arrow[d, dashed] & \mathbb{C} \arrow[d, "\varphi"] \\ C([0,1], A) \arrow[r, "\pi"'] & C(\{0\}, A), \end{tikzcd}$$

where $\pi : C([0,1], A) \longrightarrow C(\{0\}, A)$ is given by $\pi(f) = f|_{\{0\}}$ and $\varphi : \mathbb{C} \longrightarrow C(\{0\}, A)$ is given by $\varphi(\lambda)(0) = \lambda 1_A$. It is immediate that

$$(C_0(0,1] \otimes A)^{\sim} \cong C([0,1], A) \oplus_{C(\{0\},A)} \mathbb{C}. \tag{1.36}$$

Then, by Theorem 1.10.10, we have

$$\mathrm{Cu}\left((C_0(0,1] \otimes A)^{\sim}\right) \cong \mathrm{Cu}\left(C[0,1] \otimes A\right) \oplus_{\mathrm{Cu}(A)} \mathrm{Cu}(\mathbb{C}). \tag{1.37}$$

By Theorem 1.10.9, we obtain

$$\mathrm{Cu}\left(C([0,1],A)\right) \cong \mathrm{Lsc}\left([0,1],\mathrm{Cu}(A)\right), \qquad \mathrm{Cu}\left(C\left(\{0\},A\right)\right) \cong \mathrm{Cu}(A). \tag{1.38}$$

We have the following diagram

$$\begin{tikzcd}
\mathrm{Lsc}\left([0,1],\mathrm{Cu}(A)\right) \oplus_{\mathrm{Cu}(A)} \mathrm{Cu}(\mathbb{C}) \arrow[r, dashed] \arrow[d, dashed] & \mathbb{N}_0 \cup \{\infty\} \arrow[d, "\mathrm{Cu}(\varphi)"] \\
\mathrm{Lsc}\left([0,1],\mathrm{Cu}(A)\right) \arrow[r, "\mathrm{Cu}(\pi)"'] & \mathrm{Cu}(A),
\end{tikzcd}$$

which shows that

$$\mathrm{Lsc}\left([0,1],\mathrm{Cu}(A)\right) \oplus_{\mathrm{Cu}(A)} \mathbb{N}_0 \cup \{\infty\} \cong \left\{f \in \mathrm{Lsc}\left([0,1],\mathrm{Cu}(A)\right) \mid f(0) \in n\langle 1_A\rangle, n \in \mathbb{N}_0 \cup \{\infty\}\right\}.$$

Before moving forward, let us prove the following lemma that will be needed in Chapter 5. Remember that the support of a Borel measure υ on $\mathbb{R}$ is given by

$$\operatorname{supp} \upsilon = \overline{\{x \in \mathbb{R} \mid \upsilon((x-\varepsilon, x+\varepsilon)) > 0 \text{ for all } \varepsilon > 0\}}.$$

In particular, if U is an open set such that $U \cap \operatorname{supp} \upsilon \neq \emptyset$, then $\upsilon(U) > 0$.

Proposition 1.10.12. *Let A be a separable, simple AF-algebra with unique trace τ and let υ be a Borel measure on $[0,1]$ with support $[0,1]$. Then the map $\sigma : \mathrm{Lsc}\left([0,1],\mathrm{Cu}(A)\right) \longrightarrow \mathbb{N}_0 \sqcup (0,\infty] \cong \mathrm{Cu}(\mathcal{Z})$ given by*

$$\sigma(f) = \int_0^1 d_\tau\left(f(t)\right) d\upsilon(t) \in (0,\infty], \qquad f \in \mathrm{Lsc}\left([0,1],\mathrm{Cu}(A)\right), \tag{1.39}$$

is a Cu*-map (i.e. a morphism in the category* $\mathtt{Cu}$*).*

We remark that the element $\sigma(f)$ is regarded as an element of the soft part of $\mathrm{Cu}(\mathcal{Z})$, *i.e.* $(0,\infty]$.

Proof. We have that d_τ is additive, order preserving and preserve suprema of increasing sequences [37, Section 4.1]. Then it is immediate that σ preserves the order and the semigroup structure. Indeed, let $f, g \in \mathrm{Lsc}\left([0,1],\mathrm{Cu}(A)\right)$. Since d_τ is additive we have

$$\begin{aligned}
\sigma(f+g) &= \int_0^1 d_\tau\left((f+g)(t)\right) d\upsilon(t) \\
&= \int_0^1 d_\tau\left(f(t)\right) d\upsilon(t) + \int_0^1 d_\tau\left(g(t)\right) d\upsilon(t) \\
&= \sigma(f) + \sigma(g).
\end{aligned} \tag{1.40}$$

Similarly, if $f \leq g$ then $f(t) \leq g(t)$ for all $t \in (0,1]$. Thus $d_\tau(f(t)) \leq d_\tau(g(t))$ and we obtain

$$\sigma(f) = \int_0^1 d_\tau\left(f(t)\right) dv(t) \leq \int_0^1 d_\tau\left(f(t)\right) dv(t) = \sigma(g). \tag{1.41}$$

In order to finish we have to show σ preserves suprema of increasing sequences and the way below order relation.

Let $f_n \in \mathrm{Lsc}\left([0,1], \mathrm{Cu}(A)\right)$ be an increasing sequence with supremum f. This means

$$f(t) = \sup_{n\in\mathbb{N}} f_n(t), \qquad t \in [0,1]. \tag{1.42}$$

Then, since d_τ preserves suprema of increasing sequences, we have

$$d_\tau\left(f(t)\right) = \sup_{n\in\mathbb{N}} d_\tau\left(f_n(t)\right) = \lim_{n\in\mathbb{N}} d_\tau\left(f_n(t)\right), \qquad t \in [0,1]. \tag{1.43}$$

By the monotone convergence theorem we have

$$\int_0^1 d_\tau\left(f(t)\right) dv(t) = \lim_{n\to\infty} \int_0^1 d_\tau\left(f_n(t)\right) dv(t) = \sup_{n\in\mathbb{N}} \int_0^1 d_\tau\left(f_n(t)\right) dv(t). \tag{1.44}$$

This proves σ preserves suprema of increasing sequences.

Now let us show σ preserves the way below order relation $\ll$. Let us note that the relation $\ll$ is just the strict order $<$ in $(0,\infty]$. It is enough to check that if $f \in \mathrm{Lsc}\left([0,1], \mathrm{Cu}(A)\right)$ is non zero then $\sigma(f) > 0$. Indeed, if $g \ll h$ then $0 \ll h - g$ and if $\sigma(h-g) > 0$, we will obtain that $\sigma(g) < \sigma(h)$.

Let us suppose $f \in \mathrm{Lsc}\left([0,1], \mathrm{Cu}(A)\right)$ is non zero. Then there exists $t_0 \in [0,1]$ such that $f(t_0) \neq 0$ and hence $d_\tau\left(f(t_0)\right) > 0$. In particular, $f(t_0) = \langle a \rangle$ for some $a \in A \otimes \mathbb{K}$ and for every $\varepsilon > 0$ we have $\langle (a-\varepsilon)_+ \rangle \ll \langle a \rangle$. We can choose a sufficiently small $\varepsilon > 0$ such that $d_\tau\left((a-\varepsilon)_+\right) > 0$.

Since the function f is lower semicontinuous, we have that

$$U = f^{-1}\left(\langle (a-\varepsilon)_+ \rangle^{\ll}\right) = \{t \in [0,1] \mid \langle (a-\varepsilon)_+ \rangle \ll f(t)\}$$

is a non empty open set in $[0,1]$. Notice that for every $t \in U$ we have

$$d_\tau\left((a-\varepsilon)_+\right) < d_\tau(f(t)).$$

Using that the measure has support $[0,1]$ we have $\upsilon(U) > 0$. Hence we obtain the following,

$$\begin{aligned}
0 &< d_\tau\left((a-\varepsilon)_+\right)\upsilon(U) \\
&= \int_U d_\tau\left((a-\varepsilon)_+\right) d\upsilon(t) \\
&\leq \int_U d_\tau(f(t)) d\upsilon(t) \\
&\leq \int_0^1 d_\tau(f(t)) d\upsilon(t) \\
&= \sigma(f). \qquad (1.45)
\end{aligned}$$

This finishes the proof. □

We will finish this section by presenting an important classification result due to Robert. This theorem uses another construction which is related to Cu and its construction is very similar to the definiton of K_0.

Let A be a unital C^*-algebra. We will denote by $\mathrm{Cu}^\sim$ the ordered semigroup of formal differences $\langle a\rangle - n\langle 1_A\rangle$, with $\langle a\rangle \in \mathrm{Cu}(A)$ and $n \in \mathbb{N}_0$. Precisely, $(\langle a\rangle, n) \sim (\langle b\rangle, m)$ if $\langle a\rangle + m\langle 1_A\rangle + k\langle 1_A\rangle = \langle b\rangle + n\langle 1_A\rangle + k\langle 1_A\rangle$ for some $k \in \mathbb{N}_0$ and

$$\mathrm{Cu}^\sim(A) = \left(\mathrm{Cu}(A) \times \mathbb{N}_0\right) / \sim . \qquad (1.46)$$

The class of $(\langle a\rangle, n)$ will be denoted as $\langle a\rangle - n\langle 1_A\rangle$. The order is given in the following way: $\langle a\rangle - n\langle 1_A\rangle \leq \langle b\rangle - m\langle 1_A\rangle$ if for some $k \in \mathbb{N}_0$ the inequality $\langle a\rangle + m\langle 1_A\rangle + k\langle 1_A\rangle \leq \langle a\rangle + m\langle 1_A\rangle + k\langle 1_A\rangle$ holds in $\mathrm{Cu}(A)$.

Let us define $\mathrm{Cu}(A)^\sim$ for a non unital C^*-algebra A. Let $\pi : \tilde{A} \longrightarrow \mathbb{C}$ be the quotient map from the unitization of A to $\mathbb{C}$. This map produces morphisms

$$\begin{aligned}
\mathrm{Cu}(\pi) &: \mathrm{Cu}\left(\tilde{A}\right) \longrightarrow \mathrm{Cu}(\mathbb{C}) \cong \mathbb{N}_0 \cup \{\infty\}, \\
\mathrm{Cu}^\sim(\pi) &: \mathrm{Cu}^\sim\left(\tilde{A}\right) \longrightarrow \mathrm{Cu}^\sim(\mathbb{C}) \cong \mathbb{Z} \cup \{\infty\}. \qquad (1.47)
\end{aligned}$$

Then $\mathrm{Cu}^\sim(A)$ is defined as the subsemigroup of $\mathrm{Cu}^\sim\left(\tilde{A}\right)$ consisting of the elements $\langle a\rangle - n\langle 1_A\rangle$, with $\langle a\rangle \in \mathrm{Cu}\left(\tilde{A}\right)$ such that $\mathrm{Cu}(\pi)(\langle a\rangle) = n < \infty$.

We will explicitly state the $\mathrm{Cu}^\sim$ semigroups of some C^*-algebras we will need later.

Example 1.10.13. (i) We will compute $\mathrm{Cu}^\sim$ of the Jiang-Su algebra $\mathcal{Z}$. Since it is unital, it is enough to know $\mathrm{Cu}(\mathcal{Z}) \cong \mathbb{N}_0 \sqcup (0,\infty]$. Hence

$$\begin{aligned}
\mathrm{Cu}^\sim(\mathcal{Z}) &= \{x - n \cdot 1_{\mathbb{N}} \mid x \in \mathbb{N}_0 \sqcup (0,\infty],\, n \in \mathbb{N}\} \\
&\cong \mathbb{Z} \sqcup (\infty, \infty]. \qquad (1.48)
\end{aligned}$$

(ii) Let us compute $\mathrm{Cu}^{\sim}$ of the cone of a simple, separable, unital AF-algebra A with unique trace. Since $C_0(0,1]\otimes A$ is not unital, we need to compute Cu of the unitisation of the cone, $(C(0,1]\otimes A)^{\sim}$. By Example 1.10.11, we have

$$\mathrm{Cu}\left((C(0,1]\otimes A)^{\sim}\right) \cong \left\{f \in \mathrm{Lsc}\left([0,1], \mathrm{Cu}(A)\right) \mid f(0) \in n\langle 1_A\rangle, n \in \mathbb{N}_0 \cup \{\infty\}\right\}.$$

Moreover, by Theorem 1.10.6, we have

$$\mathrm{Cu}(A) = V(A) \sqcup (0,\infty]. \tag{1.49}$$

Consider $\mathbf{1} \in \mathrm{Lsc}\left([0,1], \mathrm{V}(A) \sqcup (0,\infty]\right)$ as the constant function $\langle 1_A\rangle$. In this case, the quotient map $\pi : (C(0,1]\otimes A)^{\sim} \longrightarrow \mathbb{C}$ is given by the evaluation at 0. Hence, $\mathrm{Cu}(\pi)(f) = f(0)$ for $f \in \mathrm{Lsc}\left([0,1], \mathrm{V}(A) \sqcup (0,\infty]\right)$. Putting all together yields

$$\begin{aligned}\mathrm{Cu}^{\sim}(C_0(0,1]\otimes A) &= \left\{f - n\cdot\mathbf{1} \mid f \in \mathrm{Lsc}\left([0,1], \mathrm{V}(A) \sqcup (0,\infty]\right), f(0) = n\langle 1_A\rangle, n \in \mathbb{N}_0\right\} \\ &\cong \left\{f \in \mathrm{Lsc}\left([0,1], K_0(A) \sqcup (-\infty,\infty]\right) \mid f(0) = n\langle 1_A\rangle, n \in \mathbb{Z}\right\}.\end{aligned}$$

We have introduced all the previous machinery in order to state the following remarkable theorem. Using the same approach as before, we will state this theorem in some particular case. In general, this theorem classifies maps from C^*-algebras which are inductive limits of 1-dimensional non commutative CW complexes with trivial K_1 to C^*-algebras with stable rank one. As you can imagine, we will enunciate this theorem in the particular case when the domain is a cone of a simple AF-algebra and the codomain is the Jiang-Su algebra $\mathcal{Z}$.

Theorem 1.10.14 ([80, Theorem 1.0.1])**.** *Let A be a unital AF-algebra. Then for every morphism $\alpha : \mathrm{Cu}^{\sim}(C_0(0,1]\otimes A) \longrightarrow \mathrm{Cu}^{\sim}(\mathcal{Z})$ in the category* `Cu` *such that $\alpha(\langle s_A\rangle) \leq \langle s_B\rangle$, where $s_A \in (C_0(0,1]\otimes A)_+$ and $s_B \in B_+$ are strictly positive elements, there exists a *-homomorphism $\pi : A \longrightarrow \mathcal{Z}$ such that $\mathrm{Cu}^{\sim}(\pi) = \alpha$. Moreover, this map is unique up to approximate unitary equivalence.*

Chapter 2

Nuclear dimension

The purpose of this chapter is to review nuclear dimension and decomposition rank. The first section is devoted to the covering dimension of topological spaces. In the second one, we will present the definition of nuclear dimension and decomposition rank. We will also explain the differences between these two notions. After this, we will focus on the commutative case and we will finish this chapter with the study of the zero dimensional objects.

2.1 Covering dimension

In this section we will present a brief introduction to the covering dimension of topological spaces. A more comprehensive explanation can be found in [73].

It is natural to ask how we can define dimension for general topological spaces, and during time, there have been several notions of dimension. Historically, probably the first notions were the small inductive dimension, the large inductive dimension and the covering dimension. It is well known that all these dimension theories agree on separable metrizable spaces [73, Proposition 4.5.9].[1] However, in general, we cannot say much about the relation between these dimensions. For our purposes, we will focus on the covering dimension.

Lebesgue observed that the n-dimensional cube can be covered with finitely many arbitrarily small closed sets such that the intersection of any $n+2$ sets of this cover is always empty. Lebesgue's discovery suggested one way to define the dimension for euclidean spaces. Following this idea, Čech introduced the notion of covering dimension

[1] This can be strengthened to strongly pseudo-metrizable spaces.

(also called Lebesgue covering dimension) for normal spaces in [18].

Definition 2.1.1 ([18, Definition 3])**.** Let X be a set. The *order* of a family $\mathcal{U} = \{U_i\}_{i\in I}$ of subsets of X is defined to be the largest integer n for which there exist elements $U_{i_0}, U_{i_1}, \ldots, U_{i_n} \in \mathcal{U}$ such that $\bigcap_{k=0}^{n} U_{i_k} \neq \emptyset$. If this integer does not exist, the order of $\mathcal{U}$ is infinite.

Observe that if the order of $\mathcal{U}$ is n, then the intersection of any $n+2$ elements of $\mathcal{U}$ is empty. In other words, if the order of $\mathcal{U}$ is n, then every point is in at most $n+1$ elements of $\mathcal{U}$.

Definition 2.1.2 ([73, Definition 3.1.1])**.** The *covering dimension* of a topological space X, denoted as $\dim X$, is the least integer n such that every finite open cover of X has an open refinement of order at most n.

Example 2.1.3. The covering dimension of $[0,1]$ is 1.

Proof. Let $\mathcal{U}$ be a finite open cover of $[0,1]$. Let $t \in [0,1]$, so there is $U \in \mathcal{U}$ containing t. In particular, there exists an open interval I_t such that $t \in I_t \subset U$. Then, the family $\{I_t \mid t \in [0,1]\}$ is an open cover of $[0,1]$ refining $\mathcal{U}$. By compactness, there exists a finite subcover $\{I_{t_1}, \ldots, I_{t_n}\}$. Consider the set $\{s_1, \ldots, s_{2n}\}$ of the end points of the intervals $I_{t_1}, \ldots, I_{t_n}$. We can assume $0 = s_1 \leq s_2 \leq \ldots \leq s_{2n} = 1$.

Observe that the family of intervals

$$\mathcal{W}_1 = \{(s_i, s_{i+1}) \mid i = 1, \ldots, 2n-1\}$$

are pairwise disjoint. Consider now the family of intervals

$$\mathcal{W}_2 = \left\{ \left[0, \frac{s_1}{2}\right), \left(\frac{s_{2n-1}}{2}, 1\right] \right\} \cup \left\{ \left(\frac{s_i + s_{i-1}}{2}, \frac{s_i + s_{i+1}}{2} \right) \mid i = 2, \ldots, 2n-1 \right\}.$$

Like before, the elements in $\mathcal{W}_2$ are pairwise disjoint. Hence, the set $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ has order 1. By construction, $\mathcal{W}$ is a finite open cover of $[0,1]$ with order 1 which refines $\mathcal{U}$. This shows that $\dim[0,1] \leq 1$. Since $[0,1]$ is not totally disconnected then $\dim[0,1] \geq 1$ by Proposition 2.1.5. Therefore $\dim[0,1] = 1$. $\square$

A family of sets is *discrete* if each point in X has a neighbourhood which meets at most one member of the family. The following proposition gives a useful characterisation of the covering dimension in terms of discrete families of open sets.

Proposition 2.1.4 ([70, Theorem 2])**.** *A metrizable topological space X has covering dimension at most n if and only if for each open cover $\mathcal{U}$ of X and each integer $k \geq n+1$, there exist k discrete families of open sets $\mathcal{V}_1, \ldots, \mathcal{V}_k$ such that the union of any $n+1$ of these families is a cover of X which refines $\mathcal{U}$.*

Observe that the previous proposition is very strong. It caracterises covering dimension using any open cover rather than only finite ones. However, if we start with a finite open cover $\mathcal{U}$, the families $\mathcal{V}_i$ given by Proposition 2.1.4 are also finite (this is not immediate from Proposition 2.1.4, however it follows from its proof).

Hence Proposition 2.1.4 allows us to think in covering dimension as colouring open covers if X is metrizable. Precisely, the covering dimension of X is at most n if for any finite open cover there exists a finite refinement such that we can colour this refinement using $n+1$ colours, assigning one colour to each element of this refinement, in such a way that any two open sets with the same colour do not intersect each other. Now, we briefly summarize some of the most important properties about covering dimension.

Proposition 2.1.5. *Let X and Y be normal topological spaces.*

(a) [73, Proposition 3.1.3] *If* $\dim X = 0$ *then X is totally disconnected.*[2] *If X is compact and Hausdorff, then the converse is also true.*

(b) If A is a closed subset of X then

  i) [73, Proposition 3.1.5] $\dim A \leq \dim X$*;*

  ii) [73, Corollary 3.5.8] $\dim X \leq \max\{\dim A, \dim(X \setminus A)\}$*. In particular,* $\dim \alpha X = \dim X$ *where αX denotes the one point compactification of X.*

(c) [73, Theorem 3.2.5] *If X is normal and $X = \bigcup_{i \in \mathbb{N}} A_i$, where each A_i is closed and* $\dim A_i \leq n$ *for all $i \in \mathbb{N}$, then* $\dim X \leq n$*.*

(d) [73, Proposition 3.2.6] $\dim X \times Y \leq \dim X + \dim Y$*.*

(e) [73, Proposition 3.5.11] *If $X = A \cup B$ then* $\dim X \leq \dim A + \dim B + 1$*.*

(f) [73, Proposition 6.4.3] *If additionally X is Hausdorff,* $\dim \beta X = \dim X$ *where βX denotes the Stone-Čech compactification of X.*

[2]This implication does not need normality of the space.

In practice, computing the covering dimension can be a very challenging task mostly because of the combinatorial flavour of the definition. For example, we can guess (and expect) that the covering dimension of $\mathbb{R}^n$ is n. Even though this seems completely natural, the proof of this fact is far from trivial when $n > 1$. Showing that the covering dimension of $\mathbb{R}^n$ is bounded above by n is straightforward using Proposition 2.1.5 (d). The difficulties arise in trying to show that the covering dimension of $\mathbb{R}^n$ is exactly n.

Theorem 2.1.6 ([73, Theorem 3.2.7]). *The covering dimension of $\mathbb{R}^n$ is equal to n.*

Now we recall a remarkable theorem proved independently by Lefschetz, Nöbeling, and Pontrjagyn and Tolstowa.

Theorem 2.1.7 ([60, Theorem 14], [68, Theorem 1] and [76, Theorem 11]). *Let X be a second countable normal Hausdorf space with $\dim X = n$. Then X can be embedded in the cube $[0,1]^{2n+1}$.*

We have seen that covering dimension passes to closed subsets and, since covering dimension is preserved under countable unions, it also passes to F_σ subsets. However if the subset is not of this kind, this is not true in general.

Example 2.1.8. [73, Example 3.6.1] Consider $X = [0,1] \cup \{2\}$ equipped with the following topology: A subset U of X is open if $U = X$ or if U is a usual open set of $[0,1]$. This topology is not the relative topology induced by the usual topology of $\mathbb{R}$. Observe that, by construction, any open cover of X must contain X itself. Thus $\{X\}$ is always a refinement of any open cover and we can conclude that $\dim X = 0$. Therefore X contains a subset, namely $[0,1]$, of covering dimension equal to 1.

The previous example is not very interesting because it is not even Hausdorff nor normal. However, there are examples of normal spaces containing subsets of higher dimension. In fact, totally disconnected compact Hausdorff spaces may contain subspaces of arbitrarily large covering dimension (c.f. [73, Remark 5.4.6]).

2.2 Nuclear dimension and decomposition rank

Wilhelm Winter initiated the study of a notion of covering dimension for C^*-algebras in [104]. This notion has been refined in the last fifteen years in several papers ([58, 106, 111]), leading to the concepts of nuclear dimension and decomposition rank. This theory was a breakthrough in the classification programme for C^*-algebras and has been a driving force for the research in this area during the last years.

Definition 2.2.1. ([58, Definition 3.1], [111, Definition 2.1]) Let A be C^*-algebra and let n be a natural number. The *nuclear dimension* of A is at most n, denoted as $\dim_{\text{nuc}} A \leq n$, if for any finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$, there exist finite dimensional C^*-algebras $F^{(0)}, \dots, F^{(n)}$, and maps $\psi : A \longrightarrow \bigoplus\limits_{k=0}^{n} F^{(k)}$ and $\varphi : \bigoplus\limits_{k=0}^{n} F^{(k)} \longrightarrow A$ such that:

(a) The map ψ is a c.p.c. map.

(b) The restriction $\varphi_k := \varphi|_{F^{(k)}}$ is a c.p.c. order zero map for $k = 0, \dots, n$.

(c) $\|a - \psi \circ \varphi(a)\| < \varepsilon$ for all $a \in \mathfrak{F}$.

The *decomposition rank* of A is at most n, denoted as $\operatorname{dr} A \leq n$, if additionally the map φ is contractive.

Remark 2.2.2. We can rephrase nuclear dimension and decomposition rank in terms of *decomposable approximations*. An approximation (F, ψ, φ) is *n-decomposable* if φ is the sum of $n+1$ order zero maps.

(i) A C^*-algebra A has nuclear dimension at most n if for every finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exists an n-decomposable approximation (F, ψ, φ) for $\mathfrak{F}$ within ε.

(ii) A C^*-algebra A has decomposition rank at most n if for every finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exists an n-decomposable approximation (F, ψ, φ) for $\mathfrak{F}$ within ε such that φ is contractive.

Hence, the difference between nuclear dimension and decomposition rank appears in the size of the norm of the second map φ. Decomposition rank always asks for $\|\varphi\| \leq 1$ meanwhile nuclear dimension at most n requires $\|\varphi\| \leq n+1$.

Remark 2.2.3 ([111, Remark 2.2 (iv)]). In general we can not choose the second map φ to be contractive in the definition of nuclear dimension; however, we can arrange the composition $\varphi\psi$ to be contractive.

Let A be a C^*-algebra. Consider a finite subset $\mathfrak{F}$ and $\varepsilon > 0$. By the continuity of $t \mapsto t^{-1}$ around 1, there exists $\delta > 0$ such that if $|t-1| < \delta$ then $\left|t^{-1} - 1\right| < \frac{\varepsilon}{3}$. Using an approximate unit, find a positive contraction h such that

$$|\|h\| - 1| < \delta/2 \tag{2.1}$$

$$\begin{tikzcd}
 & F^{(0)} \arrow[ddddddr, "\varphi_0"] & \\
 & \oplus & \\
 & F^{(1)} \arrow[ddddr, "\varphi_1"] & \\
 & \oplus & \\
 & \vdots & \\
 & \oplus & \\
\mathcal{A} \arrow[uuuuuur, "\psi_0"] \arrow[uuuur, "\psi_1"'] \arrow[r, "\psi_k"] \arrow[ddddr, "\psi_{n-1}"] \arrow[ddddddr, "\psi_n"'] & F^{(k)} \arrow[r, "\varphi_k"] & \mathcal{A} \\
 & \oplus & \\
 & \vdots & \\
 & \oplus & \\
 & F^{(n-1)} \arrow[uuuur, "\varphi_{n-1}"] & \\
 & \oplus & \\
 & F^{(n)} \arrow[uuuuuur, "\varphi_n"'] &
\end{tikzcd}$$

Figure 2.1: The nuclear dimension of a C^*-algebra.

and

$$\left\| h^{\frac{1}{2}} a h^{\frac{1}{2}} - a \right\| < \frac{\varepsilon}{3} \tag{2.2}$$

for all $a \in \mathfrak{F}$. Then find an n-decomposable approximation $\left(F, \hat{\psi}, \hat{\varphi}\right)$ for $\mathfrak{F} \cap \{h\}$ within $\min\{\varepsilon/3, \delta/2\}$. In particular, this entails $|\|\varphi\psi(h)\| - 1| < \delta$, and hence

$$\left| \left\| \hat{\varphi}\hat{\psi}(h) \right\|^{-1} - 1 \right| < \frac{\varepsilon}{3}.$$

Set

$$\varphi(x) = \left\| \hat{\varphi}\hat{\psi}(h) \right\|^{-1} \hat{\varphi}(x) \tag{2.3}$$

and

$$\psi(a) = \hat{\psi}\left(h^{\frac{1}{2}} a h^{\frac{1}{2}} \right) \tag{2.4}$$

for $x \in F$ and $a \in A$. Then, for any positive contraction $a \in A$, we have

$$\begin{aligned}
\|\varphi\psi(a)\| &= \left\| \varphi\hat{\psi}\left(h^{\frac{1}{2}} a h^{\frac{1}{2}} \right) \right\| \\
&\leq \left\| \varphi\hat{\psi}(h) \right\| \\
&\overset{(2.3)}{=} \left\| \hat{\varphi}\hat{\psi}(h) \right\|^{-1} \left\| \hat{\varphi}\hat{\psi}(h) \right\| = 1.
\end{aligned} \tag{2.5}$$

This shows the composition $\varphi\psi$ is a c.p.c. map. Finally, notice that (F, ψ, φ) is a good approximation since

$$\begin{aligned}\|a-\varphi\psi(a)\| &= \left\|a-\left\|\hat{\varphi}\hat{\psi}(h)\right\|^{-1}\hat{\varphi}\hat{\psi}\left(h^{\frac{1}{2}}ah^{\frac{1}{2}}\right)\right\| \\ &\leq \left\|a-\hat{\varphi}\hat{\psi}(a)\right\|+\left\|\hat{\varphi}\hat{\psi}(a)-\hat{\varphi}\hat{\psi}\left(h^{\frac{1}{2}}ah^{\frac{1}{2}}\right)\right\| \\ &\quad+\left\|\hat{\varphi}\hat{\psi}\left(h^{\frac{1}{2}}ah^{\frac{1}{2}}\right)-\left\|\hat{\varphi}\hat{\psi}(h)\right\|^{-1}\hat{\varphi}\hat{\psi}\left(h^{\frac{1}{2}}ah^{\frac{1}{2}}\right)\right\| \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon \end{aligned} \tag{2.6}$$

for all $a \in \mathfrak{F}$.

It is clear from Theorem 1.5.4 that a C^*-algebra is nuclear if its nuclear dimension is finite. However the converse is not true; there are nuclear C^*-algebras with infinite nuclear dimension, for example $C_0\left(\mathbb{R}^{\mathbb{N}}\right)$. It also follows from the definition that $\dim_{\mathrm{nuc}} A \leq \mathrm{dr}\, A$. We would like to point out, even though decomposition rank and nuclear dimension might look quite similar, they in general do not coincide. This will be explained in more detail in Subsection 2.2.1.

The following theorem is helpful in the non separable case as it allows us to reduce many proofs to the separable case.

Theorem 2.2.4 ([111, Proposition 2.6])**.** *Let A be a C^*-algebra. For any countable subset $\mathcal{S}$ there exists a separable C^*-subalgebra B of A containing $\mathcal{S}$ such that*

$$\dim_{\mathrm{nuc}} B \leq \dim_{\mathrm{nuc}} A.$$

Now, we briefly summarize some of the main properties of nuclear dimension and decomposition rank. In order to simplify the notation, we will write $\dim^{+1}_{\mathrm{nuc}} A$ instead of $\dim_{\mathrm{nuc}} A + 1$.

Proposition 2.2.5. *Let A, B and C be C^*-algebras. Then*

(i) [111, Remark 2.2] *A is an AF-algebra if and only if* $\dim_{\mathrm{nuc}} A = 0$.

(ii) [111, Proposition 2.3 (i)] $\dim_{\mathrm{nuc}} A \oplus B \leq \max\{\dim_{\mathrm{nuc}} A, \dim_{\mathrm{nuc}} B\}$.

(iii) [111, Proposition 2.3 (ii)] $\dim^{+1}_{\mathrm{nuc}} A \otimes B \leq \dim^{+1}_{\mathrm{nuc}} A \cdot \dim^{+1}_{\mathrm{nuc}} B$.[3]

(iv) [111, Proposition 2.3 (iii)] *If $A = \varinjlim A_n$ then* $\dim_{\mathrm{nuc}} A \leq \liminf\left(\dim_{\mathrm{nuc}} A_n\right)$.

[3]Since this expression can be finite only for nuclear C^*-algebras, there is no need to specify the tensor product.

(v) [111, Proposition 2.3 (iiii)]*If B is a quotient of A then* $\dim_{\mathrm{nuc}} B \leq \dim_{\mathrm{nuc}} A$.

(vi) [111, Proposition 2.4] $\dim_{\mathrm{nuc}} C_0(X) = \dim X$ *for every locally compact Hausdorff space X.*

(vii) [111, Proposition 2.5] *If B is a hereditary subalgebra of A then* $\dim_{\mathrm{nuc}} B \leq \dim_{\mathrm{nuc}} A$. *Moreover, if B is a full hereditary subalgebra then* $\dim_{\mathrm{nuc}} B = \dim_{\mathrm{nuc}} A$.

(viii) [111, Remark 2.11] $\dim_{\mathrm{nuc}} A = \dim_{\mathrm{nuc}} \widetilde{A}$, *where $\widetilde{A}$ is the unitization of A.*

(ix) [111, Proposition 2.9] *Let $0 \to A \to B \to C \to 0$ be a exact sequence. Then*

$$\max\{\dim_{\mathrm{nuc}} A, \dim_{\mathrm{nuc}} C\} \leq \dim_{\mathrm{nuc}} B \leq \dim_{\mathrm{nuc}} A + \dim_{\mathrm{nuc}} C + 1.$$

Properties (i)-(xiii) are also true for decomposition rank [58, Before Proposition 3.3, Corollary 3.10, Proposition 3.11, Example 4.1].

Observe that these properties of nuclear dimension correspond to analogous properties of covering dimension. In particular, the countable union theorem corresponds to inductive limits, product of spaces corresponds to tensor products and Proposition 2.1.5 (b) corresponds to short exact sequences. In subsequent sections, we will explore the commutative and zero dimensional case in more detail. In the same way, in the next subsection we will explain why Property 2.2.5 (ix) is not true for decomposition rank in general.

2.2.1 Nuclear dimension vs Decomposition rank

As we mentioned before, nuclear dimension and decomposition rank look quite similar but there are deep differences between them. Probably, the most dramatic difference between these two notions is quasidiagonality. The approximations (F, ψ, φ) witnessing decomposition rank and finite nuclear dimension can be chosen with different behaviours in the limit: the map ψ is approximately multiplicative for finite decomposition rank and approximately order zero for finite nuclear dimension. This shows that algebras with finite decomposition rank have to be quasidiagonal (plus the well known obstructions to quasidiagonality) meanwhile there are algebras with finite nuclear dimension which are not quasidiagonal. Finally, we will see that decomposition rank does not behave well with respect to extensions in contrast with nuclear dimension.

Proposition 2.2.6. *Let A be a C^*-algebra.*

(i) [58, Proposition 5.1]*Suppose* $\operatorname{dr} A \leq n$*. Then for all finite subsets* $\mathfrak{F}$ *and* $\varepsilon > 0$ *there exists an* n*-decomposable approximation* (F, ψ, φ) *for* $\mathfrak{F}$ *within* ε*, with* φ *contractive, such that*

$$\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon \tag{2.7}$$

for all $a, b \in \mathfrak{F}$*.*

(ii) [111, Proposition 3.2]*Suppose* $\dim_{\mathrm{nuc}} A \leq n$*. Then for all finite subsets* $\mathfrak{F}$ *and* $\varepsilon > 0$ *there exists an* n*-decomposable approximation* (F, ψ, φ) *for* $\mathfrak{F}$ *within* ε *such that*

$$\|\psi(a)\psi(b)\| < \varepsilon \tag{2.8}$$

whenever $a, b \in \mathfrak{F}$ *satisfy* $\|ab\| < \varepsilon$*.*

Remark 2.2.7. We can restate the previous proposition in the language of ultraproducts:

(i) If $\operatorname{dr} A \leq n$ then there exists a system of n-decomposable approximations $\{(F_i, \psi_i, \varphi_i)\}_{i \in I}$, with each φ_i contractive for all $i \in I$, such that the induced map $\Psi : A \longrightarrow \prod_{\mathcal{U}} F_i$ is multiplicative, where $\mathcal{U}$ is a free ultrafilter on I.

(ii) If $\dim_{\mathrm{nuc}} A \leq n$ then there exists a system of n-decomposable approximations $\{(F_i, \psi_i, \varphi_i)\}_{i \in I}$ such that the induced map $\Psi : A \longrightarrow \prod_{\mathcal{U}} F_i$ is order zero, where $\mathcal{U}$ is a free ultrafilter on I.

If A is separable, using the previous proposition and [6, Theorem 5.2.2], we can see that if $\operatorname{dr} A < \infty$ then A is quasidiagonal (see Definition 1.8.1). Thus in particular, A is stably finite and has an amenable trace. Moreover, since decomposition rank passes to quotients, we can obtain a stronger obstruction for finite decomposition rank.

Theorem 2.2.8 ([58, Theorem 5.3])**.** *Let* A *be a separable* C^**-algebra with* $\operatorname{dr} A < \infty$*. Then* A *is strongly quasidiagonal.*

As we will see in Section 2.4, there are C^*-algebras with finite nuclear dimension which are not quasidiagonal. Furthermore, there are examples of group C^*-algebras with finite nuclear dimension which are quasidiagonal but not strongly quasidiagonal [15, Corollary 3.5]. This immediately shows that decomposition rank and nuclear dimension do not coincide in general.

In Proposition 2.2.5, properties (i)-(xiii) are true for nuclear dimension and decomposition rank. However this is not true for extensions. Precisely, Proposition 2.2.5 (ix) states nuclear dimension is well behaved under extensions but this is not the case for decomposition rank. Of course, the first inequality is true for decomposition rank, since it is preserved for ideals and quotients. The problem arises in the second inequality. We will illustrate this with one example.

The Toeplitz algebra $\mathcal{T}$ is defined as the C^*-algebra generated by the (simple) unilateral shift. It is well known that $\mathcal{T}$ is an extension of the continuous functions on the circle by the compact operators [4, Example II.8.3.2 (v)].

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{T}) \longrightarrow 0.$$

Of course, $\operatorname{dr} C(\mathbb{T}) = 1$ and, by Theorem 2.5.4, $\operatorname{dr} \mathbb{K} = 0$. However, since $\mathcal{T}$ is not quasidiagonal, $\operatorname{dr} \mathcal{T} = \infty$. Thus the decomposition rank of the Toeplitz algebra $\mathcal{T}$ is not bounded in terms of decomposition rank of $C(\mathbb{T})$ and $\mathbb{K}$.

Once we have spent some time talking about the Toeplitz algebra, we can also compute its nuclear dimension. Using Proposition 2.2.5 (ix):

$$\dim_{\text{nuc}} \mathcal{T} \leq \dim_{\text{nuc}} C(\mathbb{T}) + \dim_{\text{nuc}} \mathbb{K} + 1 = 2.$$

The question about the exact value of the nuclear dimension of the Toeplitz algebra has been open for several years. Recently, Wilhelm Winter has announced that $\dim_{\text{nuc}} \mathcal{T} = 1$.

Nevertheless, there are some particular kind of extensions which are well behaved under decomposition rank. These extensions are called quasidiagonal extensions.

Proposition 2.2.9 ([58, Proposition 6.1])**.** *Let A be a C^*-algebra and let J be an ideal of A containing a quasicentral approximate unit consisting of projections. Then*

$$\operatorname{dr} A = \max\{\operatorname{dr} A, \operatorname{dr} A/J\}.$$

2.3 The commutative case

The following proposition establishes that in the commutative case nuclear dimension and decomposition rank agree with the covering dimension of the spectrum. Because of this, we regard nuclear dimension as a non commutative version of the covering dimension. This is not a surprise since, from the beginning, the definition of nuclear dimension was constructed having the covering dimension as a model in the commutative case.

Theorem 2.3.1 ([111, Proposition 2.4])**.** *Let X be a locally compact Hausdorff space. Then*

$$\dim_{\text{nuc}} C_0(X) = \operatorname{dr} C_0(X) = \dim X.$$

Originally, Winter proved that the covering dimension of a locally compact Hausdorff space X agrees with the completely positive rank of $C_0(X)$, a precursor concept of decomposition rank and nuclear dimension. Later on, when decomposition rank and nuclear dimension were introduced, a direct proof of Proposition 2.3.1 was not given; instead, its proof was based on the original proof concerning completely positive rank and some connections between the new concepts and the previous ones. It is important to notice that the inequality $\dim_{\mathrm{nuc}} C_0(X) \leq \mathrm{dr}\, C_0(X) \leq \dim X$ is relatively straightforward and its proof is essentially Winter's original proof. The reverse inequality is the one which while known to experts has not precisely appeared in the literature, at least to the author's knowledge. Because of this, we include a proof here. We will split the proof of Theorem 2.3.1 in two parts, one for each inequality.

Proposition 2.3.2. *Let X be a locally compact Hausdorff space. Then*

$$\mathrm{dr}\, C_0(X) \leq \dim X.$$

Proof. We will prove it for the metrizable compact case only. Suppose $\dim X = n$ and let X be compact and metrizable. Let $\mathfrak{F} \subset C(X)$ be a finite subset and $\varepsilon > 0$. Then we can find

i) A finite open cover $\mathcal{U} = \{U_1, \ldots, U_m\}$ of order at most n and disjoint subsets $\mathcal{U}_0, \ldots, \mathcal{U}_n \subset \mathcal{U}$ such that $\mathcal{U} = \bigcup_{k=0}^{n} \mathcal{U}_k$ and $U_i \cap U_j = \emptyset$ if $U_i, U_j \in \mathcal{U}_k$ for some k and $i \neq j$.

ii) Elements $x_i \in U_i$ such that $|f(x_i) - f(x)| < \varepsilon$ for all $x \in U_i$ and $f \in \mathfrak{F}$.

Let $(h_i)_{i=1}^m$ be a partition of unity subordinated to the open cover $\mathcal{U}$, *i.e.* $h_i \in C(X)$ such that $\sum_{i=1}^{m} h_i = 1_{C(X)}$ and the support of h_i is contained in U_i.

Set $\psi : C(X) \longrightarrow \mathbb{C}^m$ by

$$\psi(f) = (f(x_1), \ldots, f(x_m)).$$

Similarly, set $\varphi : \mathbb{C}^m \longrightarrow C(X)$ by

$$\varphi(t_1, \ldots, t_m) = \sum_{i=1}^{m} t_i h_i.$$

Let $x \in X$. By construction, we have

$$|\varphi\psi(f)(x) - f(x)| = \left| \sum_{i=1}^{m} f(x_i) h_i(x) - f(x) \right| = \left| \sum_{i=1}^{m} \left(f(x_i) - f(x) \right) h_i(x) \right| < \varepsilon$$

for all $f \in \mathfrak{F}$. This yields

$$\|\varphi\psi(f) - f\| < \varepsilon$$

for all $f \in \mathfrak{F}$. It immediately follows from the definitions that ψ is c.p.c. and φ is a c.p. map. Let us check that φ is in fact contractive. Consider $(t_1, \ldots, t_m) \in \mathbb{C}^m$ and observe

$$|\varphi((t_1, \ldots, t_m))(x)| = \left|\sum_{i=1}^{m} t_i h_i(x)\right| \leq \|(t_1, \ldots, t_m)\|_\infty .$$

Now, using the colouring of the open cover $\mathcal{U}$, we can express φ as the sum of $n+1$ order zero maps. We will define an order zero map using the subset $\mathcal{U}_k$ of $\mathcal{U}$. Precisely, let $\varphi_k : \mathbb{C}^m \longrightarrow C(X)$ be given by

$$\varphi_k(t_1, \ldots, t_m) = \sum_{i \,|\, U_i \in \mathcal{U}_k} t_i h_i.$$

It is immediate that $\varphi = \sum\limits_{k=0}^{n} \varphi_k$, so it only remains to check that φ_k is order zero for $k = 0, 1, \ldots, n$. Consider $(t_1, \ldots, t_m), (s_1, \ldots, s_m) \in \mathbb{C}^m$ such that

$$(t_1, \ldots, t_m)(s_1, \ldots, s_m) = (t_1 s_1, \ldots, t_m s_m) = (0, \ldots, 0). \tag{2.9}$$

By construction, the support of h_i is contained in U_i. Thus, if U_i and U_j are in $\mathcal{U}_k$, the intersection $U_i \cap U_j$ is empty. Hence the supports of h_i and h_j are disjoint, so

$$h_i(x) h_j(x) = 0 \tag{2.10}$$

for all $x \in X$. After this observation, we obtain

$$\begin{aligned} \varphi_k((t_1, \ldots, t_m))\, \varphi_k((s_1, \ldots, s_m)) &= \left(\sum_{i \,|\, U_i \in \mathcal{U}_k} t_i h_i(x)\right) \left(\sum_{i \,|\, U_j \in \mathcal{U}_k} s_j h_j(x)\right) \\ &= \sum_{i \,|\, U_i \in \mathcal{U}_k} t_i s_i h_i(x)^2 + \sum_{i \neq j} t_i s_j h_i(x) h_j(x) \\ &\overset{(2.9, 2.10)}{=} 0. \end{aligned} \tag{2.11}$$

This shows φ_k is order zero and ultimately, $\operatorname{dr} C(X) \leq \dim X$. $\square$

Proposition 2.3.3. *Let X be a locally compact Hausdorff space. Then*

$$\dim X \leq \dim_{\mathrm{nuc}} C_0(X)$$

Proof. Let us sketch the proof first. We consider any finite open cover $\mathcal{U}$ of X and a partition of unity $(h_r)_{r=1}^m$ subordinated to $\mathcal{U}$. For a sufficiently small ε, we find an n-decomposable approximation (F, ψ, φ) for $\{h_1, \ldots, h_m\}$ within ε. Let us write $\varphi = \sum\limits_{k=0}^{n} \varphi_k$

where each φ_k is order zero. We will show that in fact $F \cong \mathbb{C}^r$ for some $r \in \mathbb{N}$. Consider $f_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^r$ where 1 is in the ith position. Then the family $\mathcal{W}_0 = \left\{\varphi_k\left(f_i\right)^{-1}\left(\left(\frac{1}{m(n+1)} - \varepsilon, \infty\right)\right) \mid i = 1, \ldots, r;\ k = 0, \ldots, n\right\}$ is a cover of X of order at most n. We will finish by showing that a subcover of $\mathcal{W}_0$ will refine the cover $\mathcal{U}$.

Suppose $\dim_{\text{nuc}} C(X) = n$. Let $\mathcal{U} = \{U_1, \ldots, U_m\}$ be a finite open cover of X and let $(h_r)_{r=1}^m$ be a partition of unity subordinated to $\mathcal{U}$. Consider $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{3m(n+1)} \tag{2.12}$$

and let $\mathfrak{F} = \left\{1_{C(X)}, h_1, \ldots, h_m\right\}$. Find an n-decomposable approximation $\left(\bigoplus_{k=0}^{n} F^{(k)}, \psi, \varphi\right)$ for $\mathfrak{F}$ within $\frac{\varepsilon}{m(n+1)}$. The choice of ε yields the following inequality:

$$2\varepsilon < \frac{1}{m(n+1)} - \varepsilon. \tag{2.13}$$

So, we have

$$\frac{5\varepsilon}{3} < 2\varepsilon < \frac{1}{m(n+1)} - \varepsilon \leq \frac{1}{m(n+1)} - \frac{\varepsilon}{m(n+1)^2}. \tag{2.14}$$

The first observation is that $F^{(k)} \cong \mathbb{C}^{m_k}$ for some $m_k \in \mathbb{N}$. This follows from the commutativity of $C(X)$ and Lemma 1.4.13. Since $F^{(k)}$ is finite dimensional, it is a finite direct sum of matrix algebras. Thus we may restrict φ_k to each summand, say $M_d(\mathbb{C})$, and, by Lemma 1.4.13, we obtain $d = 1$. This implies $F^{(k)}$ is in fact a finite direct sum of copies of $\mathbb{C}$, *i.e.* $F^{(k)} \cong \mathbb{C}^{m_k}$ for some $m_k \in \mathbb{N}$.

Let us denote $1^{(k)}$ as the unit of $\mathbb{C}^{m_k}$ and $e_i^{(k)} = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^{m_k}$ where the 1 is in the ith position. Hence $1^{(k)} = \sum_{i=1}^{m_k} e_i^{(k)}$. In particular, for each $r \leq m$,

$$\psi(h_r) = \left(\lambda_r^{(0)}, \ldots, \lambda_r^{(n)}\right) \in \bigoplus_{k=0}^{n} \mathbb{C}^{m_k}$$

where

$$\lambda_r^{(k)} = \left(\lambda_{1,r}^{(0)}, \ldots, \lambda_{m_k,r}^{(n)}\right) \in \mathbb{C}^{m_k}.$$

This allows us to express the image of h_r under $\varphi\psi$ in the following form:

$$\varphi\psi(h_r) = \sum_{k=0}^{n}\sum_{i=1}^{m_k} \lambda_{i,r}^{(k)}\varphi_k\left(e_i^{(k)}\right).$$

Similarly, since $1_{C(X)} = \sum_{r=1}^{m} h_r$,

$$\varphi\psi\left(1_{C(X)}\right) = \sum_{r=1}^{m}\sum_{k=0}^{n}\sum_{i=1}^{m_k} \lambda_{i,r}^{(k)}\varphi_k\left(e_i^{(k)}\right). \tag{2.15}$$

It is important to notice, that due to φ_k being order zero, the functions $\varphi_k\left(e_i^{(k)}\right)$ have disjoint supports. In other words, if $\varphi_k\left(e_i^{(k)}\right)(x) > 0$ then $\varphi\left(e_j^{(k)}\right)(x) = 0$ for $j \neq i$. Hence, pointwise, the sum in (2.15) is, in fact, at most the sum of $m(n+1)$ strictly positive summands. For each $e_i^{(k)}$, define

$$\begin{aligned} W_i^{(k)} &= \varphi_k\left(e_i^{(k)}\right)^{-1}\left(\left(\frac{1}{m(n+1)} - \varepsilon, \infty\right)\right) \\ &= \left\{x \in X \mid \varphi_k\left(e_i^{(k)}\right)(x) > \frac{1}{m(n+1)} - \varepsilon\right\}. \end{aligned} \tag{2.16}$$

Set $\mathcal{W}_0 = \left\{W_i^{(k)} \mid 0 \leq k \leq n;\ 1 \leq i \leq m_k\right\}$. Observe that the order of $\mathcal{W}_0$ is at most n because φ is the sum of $n+1$ order zero maps. Precisely, for each $k = 0, 1, \ldots, n$ let $\mathcal{W}_0^{(k)} = \left\{W_i^{(k)} \mid i = 1, \ldots, m_k\right\}$ and since the functions $\varphi_k\left(e_i^{(k)}\right)$ have disjoint supports, we have that $W_i^{(k)} \cap W_j^{(k)} = \emptyset$ for $i \neq j$. It is immediate that $\mathcal{W}_0 = \bigcup_{k=0}^{n} \mathcal{W}_0^{(k)}$.

Consider some open set $W_j^{(s)}$ for some j and s. We will show that if it is not contained in some element U_r of the original cover $\mathcal{U}$, then the coefficient $\lambda_{j,r}^{(s)}$ is "small". More precisely, suppose there exists $U_r \in \mathcal{U}$ such that $W_j^{(s)} \cap (X \setminus U_r) \neq \emptyset$, we will show that $\lambda_{j,r}^{(s)} < 5\varepsilon/3$. Consider $x \in W_j^{(s)} \cap (X \setminus U_r)$. Since the support of h_r is contained in U_r, we have $h_r(x) = 0$. Since

$$\|h_r - \varphi\psi(h_r)\| < \frac{\varepsilon}{m(n+1)}, \tag{2.17}$$

we obtain

$$\sum_{k=0}^{n}\sum_{i=1}^{m_k} \lambda_{i,r}^{(k)}\varphi_k\left(e_i^{(k)}\right)(x) = \varphi\psi(h_r)(x) < \frac{\varepsilon}{m(n+1)}. \tag{2.18}$$

Since all summands are positive, we obtain

$$\lambda_{j,r}^{(s)}\varphi_s\left(e_j^{(s)}\right)(x) < \frac{\varepsilon}{m(n+1)}. \tag{2.19}$$

Now using $x \in W_j^{(s)}$, (2.19) and

$$\varepsilon \overset{(2.12)}{<} \frac{1}{3m(n+1)} < \frac{2}{5m(n+1)}, \tag{2.20}$$

we find

$$\begin{aligned} \frac{\varepsilon}{m(n+1)} &> \lambda_{j,r}^{(s)}\varphi_s\left(e_j^{(s)}\right)(x) \geq \lambda_{j,r}^{(s)}\left(\frac{1}{m(n+1)} - \varepsilon\right) \\ &> \lambda_{j,r}^{(s)}\left(\frac{1}{m(n+1)} - \frac{2}{5m(n+1)}\right) = \frac{3\lambda_{j,r}^{(s)}}{5m(n+1)}. \end{aligned} \tag{2.21}$$

Thus

$$\lambda_{j,r}^{(s)} < \frac{5\varepsilon}{3}. \tag{2.22}$$

The last key point is the following: Since for any $x \in X$ there exists some function of the partition of unity h_r such that $h_r(x) \geq \frac{1}{m}$, then there are coefficients $\lambda_{j,r}^{(s)}$ which are "large enough", *i.e.* $\lambda_{j,r}^{(s)} \geq 5\varepsilon/3$. Let $x \in X$ and h_r such that $h_r(x) \geq \frac{1}{m}$. Then, by (2.17), we have

$$\sum_{k=0}^{n} \sum_{i=1}^{m_k} \lambda_{i,r}^{(k)} \varphi_k \left(e_i^{(k)}\right)(x) = \varphi\psi(h_r)(x) \geq \frac{1}{m} - \frac{\varepsilon}{m(n+1)}. \tag{2.23}$$

Then, there exists s such that

$$\sum_{i=1}^{m_s} \lambda_{i,r}^{(s)} \varphi_s \left(e_i^{(s)}\right)(x) = \varphi_s\psi(h_r)(x) \geq \frac{1}{m(n+1)} - \frac{\varepsilon}{m(n+1)^2}. \tag{2.24}$$

Since the functions $\varphi_s\left(e_i^{(s)}\right)$ have disjoint supports, there exists j such that

$$\lambda_{j,r}^{(s)} \varphi_s \left(e_j^{(s)}\right)(x) = \sum_{i=1}^{m_s} \lambda_{i,r}^{(s)} \varphi_s \left(e_i^{(s)}\right)(x) \geq \frac{1}{m(n+1)} - \frac{\varepsilon}{m(n+1)^2}. \tag{2.25}$$

Since $\lambda_{j,r}^{(s)}$ and $\varphi_s\left(e_j^{(s)}\right)(x)$ are positive numbers less than 1, we obtain

$$\lambda_{j,r}^{(s)} \geq \frac{1}{m(n+1)} - \frac{\varepsilon}{m(n+1)^2} \tag{2.26}$$

and

$$\varphi_s \left(e_j^{(s)}\right)(x) \geq \frac{1}{m(n+1)} - \frac{\varepsilon}{m(n+1)^2} \geq \frac{1}{m(n+1)} - \varepsilon. \tag{2.27}$$

This immediately shows $x \in W_j^{(s)}$. By (2.14), (2.22) and (2.26), we have $W_j^{(s)} \cap (X \setminus U_r) = \emptyset$. Thus $W_j^{(s)} \subset U_r$.

Summarising, we have shown that for any $x \in X$, there exists some U_r and $W_j^{(s)}$ such that $x \in W_j^{(s)} \subset U_r$. Set

$$\mathcal{W} = \left\{ W_i^{(k)} \in \mathcal{W}_0 \;\middle|\; W_i^{(k)} \subset U_r \text{ for some } r = 1, \ldots, m \right\}. \tag{2.28}$$

Our previous arguments show that $\mathcal{W}$ is a cover of X. Since it is contained in $\mathcal{W}_0$, its order is at most n and, by construction, $\mathcal{W}$ refines $\mathcal{U}$. Therefore

$$\dim X \leq \dim_{\mathrm{nuc}} A. \qquad \square$$

Proof of Theorem 2.3.1. By Propositions 2.3.2 and 2.3.3, we have

$$\operatorname{dr} C(X) \leq \dim X \leq \dim_{\text{nuc}} C(X).$$

Since $\dim_{\text{nuc}} C(X) \leq \operatorname{dr} C(X)$, we obtain

$$\operatorname{dr} C(X) = \dim X = \dim_{\text{nuc}} C(X).$$

This finishes the proof. □

2.4 Examples

So far, we know that the nuclear dimension and decomposition rank of commutative C^*-algebras agree with the covering dimension of the spectrum. In this section we collect other examples.

Example 2.4.1 ([58, Example 4.2])**.** A C^*-algebra A is homogeneous if there is $N \in \mathbb{N}$ such that every irreducible representation of A is of dimension N. It is a well known fact that homogeneous algebras are continuous trace [4, Proposition IV.1.4.14]. Hence, by [111, Corollary 2.10]

$$\dim_{\text{nuc}} A = \operatorname{dr} A = \dim \hat{A}$$

for every homogeneous algebra A.

A C^*-algebra A is approximately homogeneous (AH) if it is isomorphic to a inductive limit of homogeneous algebras A_i with $\sup \dim \hat{A}_i < \infty$. Therefore, by Proposition 2.2.5 (iv),

$$\dim_{\text{nuc}} A \leq \operatorname{dr} A \leq \liminf \left(\dim_{\text{nuc}} A_i\right) = \liminf \left(\dim \hat{A}_i\right).$$

Example 2.4.2. Consider θ in $[0, 1]$. The rotation algebra $\mathcal{A}_\theta$ is the universal C^*-algebra generated by two unitaries u and v satisfying

$$vu = e^{2\pi\theta} uv.$$

The rotation algebra $\mathcal{A}_\theta$ is called rational or irrational depending if θ is rational or irrational respectively.

The rational rotational algebra $\mathcal{A}_\theta$ is a homogeneous C^*-algebra over the two-torus $\mathbb{T}^2$ with constant fibre a matrix algebra (c.f. [10]). Hence

$$\dim_{\text{nuc}} \mathcal{A}_\theta = \operatorname{dr} \mathcal{A}_\theta = 2$$

if θ is rational.

On the other hand, it was shown in [34, Theorem 4] that irrational rotation algebras are AH-algebras where each base space $\hat{A}_i$ is equal to $\mathbb{T}$ (these algebras are usually called A$\mathbb{T}$-algebras). Therefore

$$\dim_{\text{nuc}} A_\theta = \operatorname{dr} \mathcal{A}_\theta = 1$$

if θ is irrational.

Example 2.4.3. Let X be the Cantor set and let φ be a homeomorphism of X. The homeomorphism φ naturally induces an action of $\mathbb{Z}$ on $C(X)$. The homeomorphism φ is called minimal if $\emptyset$ and X are the only closed invariants sets under φ. It was shown in [78, Corollary 2.2] that if φ is minimal, then the crossed product $C(X) \rtimes_\varphi \mathbb{Z}$ is a simple A$\mathbb{T}$-algebra. As before, we obtain

$$\dim_{\text{nuc}} C(X) \rtimes_\varphi \mathbb{Z} = \operatorname{dr} C(X) \rtimes_\varphi \mathbb{Z} = 1.$$

Alternatively, Example 2.4.7 provides a different approach to compute the nuclear dimension of this crossed product.

Example 2.4.4 ([105, Theorem 1.6])**.** A C^*-algebra A is subhomogeneous if there is $N \in \mathbb{N}$ such that every irreducible representation of A is of dimension at most N. Then

$$\dim_{\text{nuc}} A = \operatorname{dr} A = \max_{k \in \mathbb{N}} \dim \left(\text{Prim}_k A\right)$$

where $\text{Prim}_k A$ is the space of kernels of k-dimensional irreducible representations.

Example 2.4.5 ([58, Example 4.5])**.** Consider natural numbers p and q. The dimension drop algebra $Z_{p,q}$ is defined as

$$Z_{p,q} = \left\{ f \in C\left([0,1], M_p(\mathbb{C}) \otimes M_q(\mathbb{C})\right) \mid f(0) \in M_p(\mathbb{C}) \otimes \mathbb{C},\ f(1) \in \mathbb{C} \otimes M_q(\mathbb{C}) \right\}.$$

If p and q are relative prime, $Z_{p,q}$ is called prime. It is immediate that dimension drop algebras are subhomogeneous and

$$\dim_{\text{nuc}} Z_{p,q} = \operatorname{dr} Z_{p,q} = 1.$$

The Jiang-Su algebra $\mathcal{Z}$ can be defined as an inductive limit of prime dimension drop algebras (Theorem 1.9.1). Therefore

$$\dim_{\text{nuc}} \mathcal{Z} = \operatorname{dr} \mathcal{Z} = 1.$$

Of course, Proposition 2.2.5 (iv) only shows that nuclear dimension is bounded by 1. However, the nuclear dimension of $\mathcal{Z}$ is exactly equal to 1 since the Jiang-Su algebra $\mathcal{Z}$ is projectionless and, hence, it is not AF (c.f. Theorem 2.5.4).

Example 2.4.6. Remember that a C^*-algebra A is Kirchberg if it is nuclear, purely infinite, simple and separable. Research concerning the nuclear dimension of Kirchberg algebras has been very active in the last years.

Winter and Zacharias showed that the nuclear dimension of Kircherg algebras satisfying the Universal Coefficient theorem (UCT) is at most 5 ([111, Therem 7.5]). After this, Enders proved that the nuclear dimension is exactly 1 for Kirchberg algebras in the UCT class with torsion free K_1-groups ([39, Theorem 4.1]). Ruiz, Sims and Sørensen settled the question in the UCT case: Any Kirchberg algebra in the UCT class has nuclear dimension equal to 1 ([86, Theorem 6.7]).

Matui and Sato proved that nuclear dimension of Kirchberg algebras is at most 3 without the UCT hypothesis ([64, Theorem 4.1]). Another proof of this was given in [3, Corollary 3.4] using $\mathcal{O}_\infty$-absorption. Finally, Bosa, Brown, Sato, Tikuisis, White and Winter obtained the exact value in the general case using a 2-coloured technique: Any Kirchberg algebra has nuclear dimension 1 [8, Corollary 9.9].

On the other hand, since Kirchberg algebras are not quasidiagonal, their decomposition rank is infinite.

Example 2.4.7 ([50, Theorem 5.1])**.** Let X be a locally compact metrizable space with finite covering dimension and let α be an automorphism of $C_0(X)$. Then

$$\dim_{\mathrm{nuc}} C_0(X) \rtimes_\alpha \mathbb{Z} \leq 2\,(\dim X)^2 + 6 \dim X + 4. \tag{2.29}$$

If the homeomorphism is minimal or free, we can improve the upper bound ([102, Theorem C] and [92, Corollary 5.2]). Precisely,

$$\dim_{\mathrm{nuc}} C_0(X) \rtimes_\alpha \mathbb{Z} \leq 2\,(\dim X) + 1$$

if α is minimal or free.

In particular, the inequality (2.29) shows that the nuclear dimension of the group C^*-algebra generated by the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ is finite. This follows from the fact that this group C^*-algebra can be viewed as a crossed product of the form $C(X) \rtimes \mathbb{Z}$ with X as the Cantor set. Therefore

$$\dim_{\mathrm{nuc}} C^*\left((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}\right) \leq 4.$$

However, it is known that the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ is not strongly quasidiagonal ([15, Corollary 3.5]) and hence

$$\operatorname{dr} C^* \left((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}\right) = \infty.$$

This last example also shows that a general estimate of the form (2.29) is false for decomposition rank.

Example 2.4.8 ([28, Theorem 4.4])**.** Let Γ be a discrete finitely generated nilpotent group. Then

$$\dim_{\mathrm{nuc}} C^*(\Gamma) \leq 10^{h(\Gamma)-1} h(\Gamma)!$$

where $h(\Gamma)$ is the Hirsch number of Γ (see [91, Section 1.C]).

2.5 The zero dimensional case

It follows immediately from the definition that finite dimensional C^*-algebras have nuclear dimension zero. Thus, by Proposition 2.2.5 (iv), separable AF-algebras have nuclear dimension zero as these algebras can be expressed as inductive limit of finite dimensional algebras. The natural question to ask is if there are other type of C^*-algebras with nuclear dimension zero. This was done by Winter in [104] using the concept of completely positive rank. In this section, we will rewrite Winter's proof using the structure of order zero maps which was proved by himself and Zacharias several years later.

The idea of the proof is simple. We would like to use the local characterisation of AF-algebras and, to this end, we will produce finite dimensional subalgebras using the approximations given by nuclear dimension. This basically boils down to replacing the corresponding order zero maps in the approximations with *-homomorphisms. We would like to point out that the proof in the unital case is rather simple.

Let A be a unital C^*-algebra with $\dim_{\mathrm{nuc}} A = 0$. Let $\mathfrak{F} \subset A$ be a finite subset and $\varepsilon > 0$. Then there exists a 0-decomposable approximation (F, ψ, φ) for $\mathfrak{F} \cup \{1_A\}$ within $\varepsilon/4$, *i.e.* the map $\varphi : F \longrightarrow A$ is c.p.c. order zero and $\|a - \varphi\psi(a)\| < \varepsilon/4$ for all $a \in \mathfrak{F}$. We can assume $\varepsilon < 1/2$ and since $\varphi\psi(1_A) \leq \varphi(1_F) \leq 1_A$, we obtain $\|\varphi(1_F) - 1_A\| < \varepsilon/2 < 1$ so $\varphi(1_F)$ is invertible. We can consider now the map $\pi : F \longrightarrow A$ given by

$$\pi(x) = \varphi(1_F)^{-1}\varphi(x). \tag{2.30}$$

From the structure of order zero maps, we obtain that π is in fact the support *-homomorphism of φ and its image is contained in A. In this way, we obtain a finite dimensional subalgebra of A, namely $\pi(F)$. In order to finish we have to show it is "close" to $\mathfrak{F}$. But this follows from the following inequality

$$\begin{aligned}\|a-\pi\psi(a)\| &\le \|a-\varphi\psi(a)\|+\|\varphi\psi(a)-\pi\psi(a)\| \\ &< \frac{\varepsilon}{4}+\|\varphi(1_F)-1_A\|\|\pi\psi(a)\| \\ &< \frac{\varepsilon}{4}+\frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned} \tag{2.31}$$

Hence A is an AF-algebra.

In the non-unital case, we need to do some extra work in order to find a good replacement for the order zero map φ.

Proposition 2.5.1 ([104, Proposition 2.17])**.** *Let A be a C^*-algebra and let $a \in A_+$ be a contraction satisfying*

$$\left\|a-a^2\right\| < \varepsilon < \frac{1}{4}.$$

Then there exists a projection $p \in C^*(a) \subset A$ such that

$$\|p-a\| < 2\varepsilon.$$

The proof of the following lemma is essentially the proof of [104, Proposition 3.2 (c)].

Lemma 2.5.2. *For every $\delta > 0$ there exists $\gamma > 0$ such that for any c.p.c. order zero map $\varphi : A \longrightarrow B$ between C^*-algebras, with A unital, satisfying*

$$\left\|\varphi(1_A)-\varphi(1_A)^2\right\| < \gamma$$

there exists a *-homomorphism $\pi : A \longrightarrow B$ such that

$$\|\varphi-\pi\| < \delta.$$

Proof. Consider $\gamma < \min\{\delta/2, 1/4\}$. Then by Proposition 2.5.1 there exists a projection $p \in C^*(\varphi(1_A))$ such that $\|p-\varphi(1_A)\| < \delta$. By Theorem 1.4.6, there exists a *-homomorphism $\rho : A \longrightarrow \mathcal{M}(C^*(\varphi(A))) \cap \{\varphi(1_A)\}'$ such that $\varphi(a) = \varphi(1_A)\rho(a)$ for all $a \in A$. Set $\pi : A \longrightarrow B$ as $\pi(a) = \rho(a)p$. As $p \in C^*(\varphi(1_A)) \subset \rho(A)'$, this defines an order zero map with $\pi(1_A) = p$ and

$$\|\varphi-\pi\| \le \|\varphi(1_A)-p\| < \delta. \tag{2.32}$$

Finally, by Corollary 1.4.14, π is a *-homomorphism. □

A more general version of the following lemma was proved in the case of finite decomposition rank in [58, Lemma 3.7]. That lemma, for the particular case of $\operatorname{dr} A = \dim_{\mathrm{nuc}} A = 0$, will be a key step in the proof of the main theorem of this section, and for completeness, we include its proof in this particular case.

Lemma 2.5.3. *Let A be a separable nuclear C^*-algebra with $\dim_{\mathrm{nuc}} A = 0$ and let $\varepsilon > 0$. Let $\mathfrak{F} \subset A_+$ be a finite subset of contractions and suppose there is a positive contraction $h \in A$ such that $ha = a$ for all $a \in \mathfrak{F}$. Then there exist a 0-decomposable approximation (F, ψ, φ) for $\mathfrak{F} \cap \{h\}$ within ε and a projection $p \in F$ such that*

$$\|\varphi\left(p\psi(a)p\right) - a\| < \varepsilon \tag{2.33}$$

for all $a \in \mathfrak{F} \cap \{h\}$, and

$$\|\varphi(p)^2 - \varphi(p)\| < \varepsilon. \tag{2.34}$$

Proof. By hypothesis, there exists a sequence of approximations $\left(F^{(n)}, \psi^{(n)}, \varphi^{(n)}\right)$, with $\varphi^{(n)}$ order zero, such that $\lim \varphi^{(n)}\psi^{(n)}(a) = a$ for all $a \in A$. Consider a free ultrafilter ω on $\mathbb{N}$. Let $\psi^{(\omega)} : A_\omega \longrightarrow \prod_\omega F^{(n)}$ and $\varphi^{(\omega)} : \prod_\omega F^{(n)} \longrightarrow A_\omega$ be the c.p.c. maps induced by $\left(\psi^{(n)}\right)$ and $\left(\varphi^{(n)}\right)$, respectively. In particular, we obtain

$$\varphi^{(\omega)}\psi^{(\omega)}(a) = a \tag{2.35}$$

for all $a \in A$. Hence, Proposition 1.4.2 yields the following

$$\begin{aligned} a^*a &= \varphi^{(\omega)}\psi^{(\omega)}(a^*)\varphi^{(\omega)}\psi^{(\omega)}(a) \\ &\leq \varphi^{(\omega)}\left(\psi^{(\omega)}(a^*)\psi^{(\omega)}(a)\right) \\ &\leq \varphi^{(\omega)}\psi^{(\omega)}(a^*a) \\ &= a^*a \end{aligned} \tag{2.36}$$

for $a \in A$. This implies that $\varphi^{(\omega)}$ is multiplicative on $C^*\left(\psi^{(\omega)}(A)\right)$ by Proposition 1.4.2. Similarly, by Lemma 1.4.4

$$\varphi^{(\omega)}\left(x\psi^{(\omega)}(a)\right) = \varphi^{(\omega)}(x)\varphi^{(\omega)}\psi^{(\omega)}(a) = \varphi^{(\omega)}(x)a \tag{2.37}$$

for all $a \in A$ and $x \in \prod_\omega F^{(n)}$. In particular,

$$\varphi^{(\omega)}\left(1_{\prod_\omega F^{(n)}}\right)a = \varphi^{(\omega)}\psi^{(\omega)}(a) = a. \tag{2.38}$$

Let χ be the characteristic function of $[1-\frac{\varepsilon}{4},\infty)$. Set

$$q^{(n)} = \chi\left(\psi^{(n)}(h)\right) \in F^{(n)} \tag{2.39}$$

and, by construction,

$$q^{(n)} f_{1-\frac{\varepsilon}{4},1}\left(\psi^{(n)}(h)\right) = f_{1-\frac{\varepsilon}{4},1}\left(\psi^{(n)}(h)\right) \tag{2.40}$$

where $f_{1-\frac{\varepsilon}{4},1}$ is identical 0 in $(-\infty, 1-\frac{\varepsilon}{4}]$, identically 1 in $[1,\infty)$ and linear elsewhere.

Let $q \in \prod_\omega F^{(n)}$ be the projection induced by the sequence $\left(q^{(n)}\right)$. Since $f_{1-\frac{\varepsilon}{4},1}(1) = 1$ and $ha = a$ for all $a \in \mathfrak{F}$, by functional calculus we have

$$f_{1-\frac{\varepsilon}{4},1}(h)a = a \tag{2.41}$$

for each $a \in \mathfrak{F}$. Moreover, by (2.40), we have

$$q f_{1-\frac{\varepsilon}{4},1}\left(\psi^{(\omega)}(h)\right) = f_{1-\frac{\varepsilon}{4},1}\left(\psi^{(\omega)}(h)\right). \tag{2.42}$$

Notice that, since $\varphi^{(\omega)}$ is multiplicative on $C^*\left(\psi^{(\omega)}(A)\right)$, we have

$$g\left(\varphi^{(\omega)}\psi^{(\omega)}(a)\right) = \varphi^{(\omega)}\left(g\left(\psi^{(\omega)}(a)\right)\right) \tag{2.43}$$

for all $a \in A$ and $g \in C(\sigma(a))$. Now, we are ready to show that $\varphi^{(\omega)}(q)$ behaves like a unit on $\mathfrak{F}$,

$$\begin{aligned}
\varphi^{(\omega)}(q)a &\overset{(2.41)}{=} \varphi^{(\omega)}(q) f_{1-\frac{\varepsilon}{4},1}(h)a \\
&\overset{(2.35)}{=} \varphi^{(\omega)}(q) f_{1-\frac{\varepsilon}{4},1}(\varphi^{(\omega)}\psi^{(\omega)}(h))a \\
&\overset{(2.43)}{=} \varphi^{(\omega)}(q)\,\varphi^{(\omega)}\left(f_{1-\frac{\varepsilon}{4},1}\left(\psi^{(\omega)}(h)\right)\right)a \\
&\overset{(2.37)}{=} \varphi^{(\omega)}\left(q f_{1-\frac{\varepsilon}{4},1}\left(\psi^{(\omega)}(h)\right)\right)a \\
&\overset{(2.42)}{=} \varphi^{(\omega)}\left(f_{1-\frac{\varepsilon}{4},1}\left(\psi^{(\omega)}(h)\right)\right)a \\
&\overset{(2.43)}{=} f_{1-\frac{\varepsilon}{4},1}\left(\varphi^{(\omega)}\psi^{(\omega)}(h)\right)a \\
&\overset{(2.35)}{=} f_{1-\frac{\varepsilon}{4},1}(h)a \\
&\overset{(2.41)}{=} a
\end{aligned} \tag{2.44}$$

for all $a \in \mathfrak{F}$.

Combining the last identity with (2.37) we obtain

$$\begin{aligned}
a &\overset{(2.44)}{=} \varphi^{(\omega)}(q)a\varphi^{(\omega)}(q) \\
&= \left(a^{\frac{1}{2}}\varphi^{(\omega)}(q)\right)^{*}\left(a^{\frac{1}{2}}\varphi^{(\omega)}(q)\right) \\
&\overset{(2.37)}{=} \varphi^{(\omega)}\left(\psi^{(\omega)}\left(a^{\frac{1}{2}}\right)q\right)^{*}\varphi^{(\omega)}\left(\psi^{(\omega)}\left(a^{\frac{1}{2}}\right)q\right) \\
&\overset{(1.4.2)}{\leq} \varphi^{(\omega)}\left(\left(\psi^{(\omega)}\left(a^{\frac{1}{2}}\right)q\right)^{*}\psi^{(\omega)}\left(a^{\frac{1}{2}}\right)q\right) \\
&= \varphi^{(\omega)}\left(q\psi^{(\omega)}\left(a^{\frac{1}{2}}\right)\psi^{(\omega)}\left(a^{\frac{1}{2}}\right)q\right) \\
&\overset{(1.4.2)}{\leq} \varphi^{(\omega)}\left(q\psi^{(\omega)}(a)q\right) \\
&\leq \varphi^{(\omega)}\psi^{(\omega)}(a) = a.
\end{aligned} \tag{2.45}$$

This yields

$$a = \varphi^{(\omega)}\left(q\psi^{(\omega)}(a)q\right) \tag{2.46}$$

for all $a \in \mathfrak{F}$.

Now we are going to show that h is almost a unit for $\varphi^{(\omega)}(q)$. By construction,

$$\left\|q\psi^{(\omega)}(h) - q\right\| \leq \frac{\varepsilon}{4}.$$

Thus

$$\left\|\varphi^{(\omega)}(q)h - \varphi^{(\omega)}(q)\right\| \overset{(2.37)}{=} \|\varphi^{(\omega)}\left(q\psi^{(\omega)}(h)\right) - \varphi^{(\omega)}(q)\| \leq \frac{\varepsilon}{4}. \tag{2.47}$$

By Proposition A.1.7, $\varphi^{(\omega)}$ is order zero, and hence,

$$\varphi^{(\omega)}(x)\varphi^{(\omega)}(y) = \varphi^{(\omega)}(1_F)\varphi^{(\omega)}(xy) \qquad x, y \in \prod_{\omega} F^{(n)}. \tag{2.48}$$

In particular, $\varphi^{(\omega)}(q)^2 = \varphi^{(\omega)}\left(1_{\prod_{\omega} F^{(n)}}\right)\varphi^{(\omega)}(q)$. Then

$$\begin{aligned}
\|\varphi^{(\omega)}(q)^2 - \varphi^{(\omega)}(q)\| &\leq \left\|\varphi^{(\omega)}(1_F)\varphi^{(\omega)}(q) - \varphi^{(\omega)}(q)h\right\| + \|\varphi^{(\omega)}(q)h - \varphi^{(\omega)}(q)\| \\
&\overset{(2.38)}{=} \|\varphi^{(\omega)}(1_F)\varphi^{(\omega)}(q) - \varphi^{(\omega)}(q)\varphi^{(\omega)}(1_F)h\| \\
&\quad + \|\varphi^{(\omega)}(q)h - \varphi^{(\omega)}(q)\| \\
&\overset{(2.47)}{\leq} \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned} \tag{2.49}$$

By equations (2.35), (2.46) and (2.49), there exists $n \in \mathbb{N}$ such that

(i) $\|a - \varphi^{(n)}\psi^{(n)}(a)\| < \varepsilon$,

(ii) $\|a - \varphi^{(n)}\left(q^{(n)}\psi^{(n)}(a)q^{(n)}\right)\| < \varepsilon$,

(iii) $\|\varphi^{(n)}(q^{(n)})^2 - \varphi^{(n)}(q^{(n)})\| < \varepsilon$,

for all $a \in \mathfrak{F} \cap \{h\}$. Therefore the approximation $\left(F^{(n)}, \psi^{(n)}, \varphi^{(n)}\right)$ and the projection $q^{(n)} \in F^{(n)}$ satisfy the required properties. □

Now we have all the tools we need to prove the main theorem of this section. Observe this theorem does not assume separability.

Theorem 2.5.4 ([58, Example 4.1],[111, Remark 2.2 (iii)]). *Let A be a C^*-algebra. Then $\dim_{\text{nuc}} A = 0$ if and only if A is an AF-algebra.*

Proof. Let us suppose A is an AF-algebra. We will show the nuclear dimension of A is bounded by zero and hence it has to be equal to zero. Consider a finite set $\mathfrak{F} \subset A$ and $\varepsilon > 0$. By hypothesis there is a finite dimensional C^*-subalgebra F of A such that

$$\text{dist}(a, F) < \varepsilon/2$$

for all $a \in \mathfrak{F}$. By Arveson's Extension theorem, we can extend the identity of F, id_F, to a c.p.c. map $\psi : A \longrightarrow F$ such that $\varphi|_F = \text{id}_F$. Set $\varphi : F \longrightarrow A$ as the inclusion map. Hence φ is a *-homomorphism and, in particular, it is an order zero map.

For each $a \in \mathfrak{F}$ let $b_a \in F$ be such that $\|a - b_a\| < \frac{\varepsilon}{2}$. Then

$$\|a - \varphi\psi(a)\| \leq \|a - b_a\| + \|\varphi\psi\left(b_a\right) - \varphi\psi(a)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $a \in \mathfrak{F}$. Therefore $\dim_{\text{nuc}} A = 0$.

Conversely, suppose $\dim_{\text{nuc}} A = 0$. Let $\mathfrak{F} = \{a_1, \ldots, a_r\}$ be a finite subset of A_+ and $\varepsilon > 0$. We can assume there exists a separable subalgebra of A containing $\mathfrak{F}$ with nuclear dimension zero (Proposition 2.2.4). Hence we can assume A is separable. Then, using a strictly positive element of B, we can construct an approximate unit $(h_n)_{n\in\mathbb{N}}$ such that $h_n h_m = h_m$ if $m \leq n$. Find k sufficiently large such that

$$\|h_k a_i - a_i\| \leq \frac{\varepsilon}{4} \tag{2.50}$$

for all $i = 1, \ldots, r$. Set $b_i = h_k a_i$ for $i = 1, \ldots, r$ and $h = h_{k+1}$. By construction $hb_i = b_i$ and consider $\mathfrak{F}' = \{h, b_1, \ldots, b_r\}$.

Let γ be given by Lemma 2.5.2 using $\delta = \frac{\varepsilon}{4}$. Set $\eta = \min\{\frac{\varepsilon}{4}, \gamma\}$. By Lemma 2.5.3, there exist a 0-decomposable approximation (F, ψ, φ) for $\mathfrak{F}'$ within η and a projection $p \in F$

such that

$$\|\varphi\left(p\psi(b_i)p\right) - b_i\| < \eta \leq \frac{\varepsilon}{4} \tag{2.51}$$

for all $b_i \in \mathfrak{F}'$ and

$$\|\varphi(p)^2 - \varphi(p)\| < \eta \leq \gamma. \tag{2.52}$$

By Lemma 2.5.2 there exists a *-homomorphism $\pi : pFp \longrightarrow A$ such that

$$\|\pi - \varphi|_{pFp}\| < \frac{\varepsilon}{4}. \tag{2.53}$$

We finish the proof by showing that the distance between the finite dimensional C^*-subalgebra $\pi(pFp)$ and $\mathfrak{F}$ is at most ε. Let $a_i \in \mathfrak{F}$, then

$$\begin{aligned} \|a_i - \pi(p\psi(a_i)p)\| \quad &\leq \quad \|a_i - b_i\| + \|b_i - \varphi(p\psi(b_i)p)\| + \|\varphi(p\psi(b_i)p) - \varphi(p\psi(a_i)p)\| \\ &\quad\quad + \|\varphi(p\psi(a_i)p) - \pi(p\psi(a_i)p)\| \\ &\overset{(2.51, 2.53)}{<} \quad \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned} \tag{2.54}$$

Therefore A is an AF-algebra. $\square$

Chapter 3

Nuclear dimension and decomposable approximations

As shown by Kirchberg [53] and Choi-Effros [21], nuclearity can be defined using the completely positive approximation property (CPAP). A C^*-algebra has the CPAP if there is a system of completely positive approximations (F, ψ, φ) (Theorem 1.5.3). In particular, for commutative C^*-algebras the CPAP is established from partitions of unity subordinate to suitable open covers of the spectrum of the algebra. A stronger version of the CPAP was established in 2012 in [49, Theorem 1.4]. This shows that the maps φ can always be taken to be decomposable, though the size of the decomposition may vary with the tolerances in the approximation. Moreover, this theorem shows that the map φ can be taken as a convex combination of contractive order zero maps, which is a crucial ingredient in obtaining a near inclusion type perturbation result for separable nuclear C^*-algebras [49, Section 2].

Very recently, Carrion, Brown and White proved that the map ψ can be taken to be asymptotically order zero [12, Theorem 3.1]. Precisely, every nuclear C^*-algebra has a system of completely positive approximations (F, ψ, φ) where ψ is approximately order zero and φ is a convex combination of order zero maps, *i.e.* $\varphi = \sum \lambda_k \varphi_k$ for some finite number of positive scalars λ_k adding up to 1 and each φ_k being order zero.

At first glance, these approximations for nuclear C^*-algebras look very similar to the approximations witnessing finite nuclear dimension introduced in Chapter 2. Remember that the approximations witnessing nuclear dimension at most n are of the form (F, ψ, φ) where $F = \bigoplus\limits_{k=0}^{n} F_k$ and $\varphi = \sum\limits_{k=0}^{n} \varphi_k$ with the restrictions $\varphi_k = \varphi|_{F_k}$ being order zero.

It is important to notice that decomposition rank was introduced before Theorem 1.5.5 was published. As explained in Chapter 2, the approximations witnessing finite nuclear dimension were constructed as a non-commutative analogue of covering dimension rather than from this stronger form of CPAP.

Despite these approximations having different origins, one might expect that approximations coming from the CPAP are useful to witness finite nuclear dimension if we impose an upper bound in the number of summands appearing in the convex combinations. We could think that nuclear dimension, ultimately, is only asking for the existence of an upper bound in the number of summands appearing in the convex combinations.

Thus, as suggested by Winter in the NSF/CBMS conference in Louisiana 2012, it is natural to investigate the situation when the completely positive and contractive approximations are decomposable as a convex combination with a uniformly bounded number of summands. In this chapter, we show that such approximations force the underlying C^*-algebra to be approximately finite dimensional (Theorem 3.1.5), and hence, the approximations coming from the stronger CPAP are not useful to witness nuclear dimension in general. The results from Section 3.1 were published by the author in [16].

In the last part of this chapter, we review the notion of nuclear dimension at most omega introduced by Robert [79, Definition 3.1]. Roughly speaking, nuclear dimension at most omega asks for approximations (F, ψ, φ) where ψ is approximately order zero and φ is the sum of countable many approximately order zero maps. Like in the finite nuclear dimension case, we can ask if the approximations coming from the CPAP can be useful to witness nuclear dimension at most omega (with no upper bound in the number of summands of the convex combinations). As an application of the techniques developed in this chapter, we obtain a similar result concerning nuclear dimension at most omega: the approximations coming from the CPAP witness nuclear dimension at most omega if and only if the C^*-algebra is approximately finite dimensional. The main result in Section 3.3 is original.

3.1 Decomposable approximations vs Nuclear dimension

This section is an extended version of the results in published in [16]. As explained before, we want to study the approximations for nuclear C^*-algebras given by Theorem 1.5.5.

These approximations are of the form displayed in the following diagram:

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi"'] & & A \\ & F \arrow[ur, "\varphi=\sum \lambda_k \varphi_k"'] & \end{tikzcd}$$

where φ is a convex combination of contractive order zero maps φ_k, ψ is c.p.c. and the diagram commutes on a finite subset $\mathfrak{F} \subset A$ up to $\varepsilon > 0$. It is important to note that the number of summands in the convex combinations depends on $\mathfrak{F}$ and ε, and in general this number is unbounded. Precisely, we want to study these approximations when the number of summands in the convex combination is uniformly bounded, *i.e.* there exists $n \in \mathbb{N}$ such that $\varphi = \sum\limits_{k=1}^{n} \lambda_k \varphi_k$ for all $\mathfrak{F}$ and $\varepsilon > 0$.

If a C^*-algebra A has the approximations described above, it is almost immediate that its decomposition rank is at most $n-1$. However, we would like to determine the dimension of this algebra exactly. The reason why we want to do this is because we want to compare the nuclear dimension of the algebra with the upper bound of the number of summands in the convex combinations.

First of all, let us show that the approximations described above can be slightly modified in order to witness decomposition rank at most $n-1$. The reason why the original approximations (F, ψ, φ) are not necessarily witnessing decomposition rank is because we do not have a decomposition $\bigoplus\limits_{k=1}^{n} F_k$ of F such that each restriction $\varphi|_{F_k}$ is order zero. Remember that a c.p.c. approximation (F, ψ, φ) for $\mathfrak{F}$ within ε means that $\|a - \varphi\psi(a)\| < \varepsilon$ for all $a \in \mathfrak{F}$.

Proposition 3.1.1. *Let A be a C^*-algebra and let n be a natural number. Suppose that for every finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exists a c.p.c. approximation (F, ψ, φ) for $\mathfrak{F}$ within ε such that φ is a convex combination of at most n order zero maps, i.e.*

$$\varphi = \sum_{k=1}^{n} \lambda_k \varphi_k$$

where each $\varphi_k : F \longrightarrow A$ is an order zero map and the coefficients $(\lambda_k)_{k=1}^{n}$ are positive scalars adding up to 1. Then

$$\mathrm{dr}\, A \leq n-1.$$

Proof. Using the approximations provided by the hypothesis, we will construct new approximations $\left(\bigoplus\limits_{k=1}^{n} F_k, \widehat{\psi}, \widehat{\varphi}\right)$ such that the restrictions $\widehat{\varphi}\big|_{F_k}$ are order zero maps.

Let $\mathfrak{F} \subset A$ be a finite subset and $\varepsilon > 0$. Suppose there exists a c.p.c. approximation (F, ψ, φ) for $\mathfrak{F}$ within ε as described in the proposition. Set $F_k = F$ for $k = 1, \cdots, n$ and define c.p.c. maps $\widehat{\psi} : A \longrightarrow \bigoplus_{k=1}^{n} F_k$, $\widehat{\varphi} : \bigoplus_{k=1}^{n} F_k \longrightarrow A$ as

$$\widehat{\psi}(a) = \psi(a) \oplus \cdots \oplus \psi(a)$$

and

$$\widehat{\varphi}(x_1 \oplus \cdots \oplus x_n) = \sum_{k=1}^{n} \lambda_k \varphi_k (x_k) .$$

Since $\varphi\psi(a) = \widehat{\varphi}\widehat{\psi}(a)$ for all $a \in A$, $\left(\bigoplus_{k=1}^{n} F_k, \widehat{\psi}, \widehat{\varphi}\right)$ is a c.p.c. approximation for $\mathfrak{F}$ within ε, and for each k, the restriction $\widehat{\varphi}|_{F_k}$ is φ_k so is order zero. Therefore

$$\operatorname{dr} A \leq n - 1. \qquad \square$$

Our aim is to reduce this estimate and show that in fact the decomposition rank of algebras with this type of approximation is zero, so they have to be AF.

The next technical lemma will be used in the proof of the main theorem and it will allow us to work with one map instead of a convex combination.

Lemma 3.1.2. *Let A be a C^*-algebra, $\varepsilon > 0$, $n \in \mathbb{N}$ and consider $\lambda_1, \ldots, \lambda_n \in (0, 1)$ such that $\sum_{k=1}^{n} \lambda_k = 1$. If $p \in A$ is a projection and contractions $a_k \in A_+1$, $k = 1, \ldots, n$, satisfy*

$$\left\| p - \sum_{k=1}^{n} \lambda_k a_k \right\| \leq \varepsilon. \tag{3.1}$$

Then

$$\|p - a_k\| \leq \sqrt{\lambda_k^{-1}\varepsilon}\left(\sqrt{\lambda_k^{-1}\varepsilon} + 1\right) \tag{3.2}$$

for $k = 1, \ldots, n$.

Proof. We may suppose $A \subset B(H)$ for some Hilbert space H. For fixed k consider

$$b = \frac{1}{1 - \lambda_k} \sum_{i \neq k} \lambda_i a_i \in A_+^1. \tag{3.3}$$

With this construction we can treat the sum as the convex combination of only two summands, precisely

$$\sum_{i=1}^{n} \lambda_i a_i = \lambda_k a_k + (1 - \lambda_k)\, b. \tag{3.4}$$

By (3.1), we get

$$p - (\lambda_k a_k + (1 - \lambda_k)\, b) \leq \varepsilon 1_{B(H)}. \tag{3.5}$$

Thus

$$\lambda_k\left(p-pa_kp\right)+\left(1-\lambda_k\right)\left(p-pbp\right)\leq\varepsilon p. \tag{3.6}$$

Since $p-pa_kp$ and $p-pbp$ are positive, the previous inequality leads to

$$0\leq p-pa_kp\leq\lambda_k^{-1}\varepsilon p-\left(\lambda_k^{-1}-1\right)\left(p-pbp\right)\leq\lambda_k^{-1}\varepsilon p. \tag{3.7}$$

Thus

$$\|p-pa_kp\|\leq\lambda_k^{-1}\varepsilon \tag{3.8}$$

and similarly we obtain

$$\|(1_{B(H)}-p)a_k(1_{B(H)}-p)\|\leq\lambda_k^{-1}\varepsilon. \tag{3.9}$$

We can write any $h\in H$ as h_1+h_2 where $h_1=p(h)$ and $h_2=(1_{B(H)}-p)(h)$. Since a_k is positive, we have

$$\begin{aligned}0&\leq\langle a_kh,h\rangle\\&=\langle pa_kp(h_1),h_1\rangle+2\mathrm{Re}\langle pa_k(1_{B(H)}-p)\left(h_2\right),h_1\rangle\\&\quad+\langle(1_{B(H)}-p)a_k(1_{B(H)}-p)(h_2),h_2\rangle.\end{aligned} \tag{3.10}$$

Let us suppose that $\|pa_k(1_{B(H)}-p)\|>\sqrt{\lambda_k^{-1}\varepsilon}$. Then there exists $h_2\in\left(1_{B(H)}-p\right)(H)$ with $\|h_2\|=1$ such that $\|pa_k(1_{B(H)}-p)(h_2)\|>\sqrt{\lambda_k^{-1}\varepsilon}$. Set $h_1=pa_k(1_{B(H)}-p)(h_2)$ and considering $h=-h_1+h_2$ in (3.10) we obtain

$$\begin{aligned}0&\leq\langle pa_kp(-h_1),-h_1\rangle+2Re\langle pa_k(1_{B(H)}-p)(h_2),-h_1\rangle\\&\quad+\langle(1_{B(H)}-p)a_k(1_{B(H)}-p)(h_2),h_2\rangle\\&\overset{(3.9)}{\leq}\langle p(h_1),h_1\rangle-2\langle h_1,h_1\rangle+\lambda_k^{-1}\varepsilon\\&=-\|h_1\|^2+\lambda_k^{-1}\varepsilon\\&<-\lambda_k^{-1}\varepsilon+\lambda_k^{-1}\varepsilon=0\end{aligned} \tag{3.11}$$

which is clearly a contradiction. Therefore

$$\|(1_{B(H)}-p)a_kp\|=\|pa_k(1_{B(H)}-p)\|\leq\sqrt{\lambda_k^{-1}\varepsilon}. \tag{3.12}$$

Finally we obtain

$$\begin{aligned}\|p-a_k\|&\leq\max\left\{\|p-pa_kp\|,\|(1_{B(H)}-p)a_k(1_{B(H)}-p)\|\right\}\\&\quad+\max\left\{\|pa_k(1_{B(H)}-p)\|,\|(1_{B(H)}-p)a_kp\|\right\}\\&\leq\lambda_k^{-1}\varepsilon+\sqrt{\lambda_k^{-1}\varepsilon}=\sqrt{\lambda_k^{-1}\varepsilon}\left(\sqrt{\lambda_k^{-1}\varepsilon}+1\right).\end{aligned} \tag{3.13}$$

□

We will need a lemma from [58]. We will rewrite it here in the form in which we need it. This lemma was enunciated in Lemma 2.5.3 in the particular case of $n = 0$. With it, we will be able slightly perturb the approximations by cutting down with projections. This will allow us to show that the image of some of these projections under the order zero maps are almost projections.

Lemma 3.1.3 ([58, Lemma 3.7])**.** *Let A be a separable nuclear C^*-algebra and suppose $\operatorname{dr} A \leq n$ and let $0 < \varepsilon < 1$. Let $\mathfrak{F} \subset A_+$ be a finite subset of contractions and suppose there is a positive contraction $h \in A$ such that $ha = a$ for all $a \in \mathfrak{F}$. Then there exists an n-decomposable approximation (F, ψ, φ), with φ a contraction, for $\mathfrak{F} \cup \{h\}$ within ε and a projection $p \in F$ such that*

(i) $\|\varphi(p\psi(a)p) - a\| < \varepsilon$ *for all* $a \in \mathfrak{F} \cup \{h\}$,

(ii) if $F = \bigoplus_{k=1}^{n} F_k$ *is a decomposition of* F *such that* $\varphi|_{F_k}$ *is order zero, then*

$$\|\varphi(p_k) - \varphi(p_k)\varphi(1_F)\| < \varepsilon \tag{3.14}$$

for $k = 1, \ldots, n$ *where* $p_k = p1_{F_k}$.

A slightly more general version of this lemma will be proved in Section 3.3. Because of this, we delay the proof of this lemma until the proof of Lemma 3.3.2.

We will now proceed to prove the main theorem. We will split the proof in two steps. Firstly, we show that the order zero maps appearing in the convex combinations can be replaced by *-homomorphisms. Secondly, by approximating twice in a suitable way, we will be able to produce finite dimensional subalgebras contained in the original C^*-algebra.

Lemma 3.1.4. *Let A be a separable C^*-algebra and $n \in \mathbb{N}$. Consider $\lambda_1, \ldots, \lambda_n \in (0, 1)$ such that $\sum_{k=1}^{n} \lambda_k = 1$ and let $\{a_i\}_{i \in \mathbb{N}}$ be a dense countable subset of A. Suppose A has a system of c.p.c. approximations $\left\{\left(F^{(r)}, \psi^{(r)}, \varphi^{(r)}\right)\right\}_{r \in \mathbb{N}}$ satisfying the following conditions:*

(a) For every $r \in \mathbb{N}$ there exist a decomposition $F^{(r)} = \bigoplus_{k=1}^{n} F_k^{(r)}$, as an internal direct sum, and order zero maps $\varphi_k^{(r)} : F^{(r)} \longrightarrow A$, with $k = 1, \ldots, n$, satisfying

$$\varphi^{(r)} = \sum_{k=1}^{n} \lambda_k \varphi_k^{(r)}. \tag{3.15}$$

Moreover, $\bigoplus_{i \neq k} F_i^{(r)} \subset \ker \varphi_k$.

(b) $\left\|\varphi^{(r)}\psi^{(r)}(a_i) - a_i\right\| < r^{-1}$ *for every* $r \in \mathbb{N}$ *and* $i \leq r$.

(c) For every $r \in \mathbb{N}$ *there exist projections* $p_k^{(r)} \in F_k^{(r)}$ *satisfying*

> *(I)* $\left\|\varphi^{(r)}\left(p^{(r)}\psi^{(r)}\left(a_i\right)p^{(r)}\right) - a_i\right\| < r^{-1}$ *for* $i \leq r$ *with* $p^{(r)} = \sum\limits_{k=1}^{n} p_k^{(r)}$.
>
> *(II)* $\left\|\varphi^{(r)}\left(p_k^{(r)}\right) - \varphi^{(r)}\left(p_k^{(r)}\right)\varphi^{(r)}\left(1_{F^{(r)}}\right)\right\| < r^{-1}$ *where* $1_{F^{(r)}}$ *denotes the unit of* $F^{(r)}$.

Then for every finite subset $\mathfrak{F} \subset A$ *and every* $\varepsilon > 0$ *there exists a c.p.c. approximation* $\left(\bigoplus\limits_{k=1}^{n} \widetilde{F}_k, \psi, \pi\right)$ *for* $\mathfrak{F}$ *within* ε *such that* $\pi = \sum\limits_{k=1}^{n} \lambda_k \pi_k$ *with each* $\pi_k : \bigoplus\limits_{k=1}^{n} \widetilde{F}_k \longrightarrow A$ *a* **-homomorphism satisfying* $\bigoplus\limits_{i \neq k} \widetilde{F}_i \subset \ker \pi_k$.

Proof. Let $\mathfrak{F} \subset A$ and $\varepsilon > 0$. Without loss of generality we can assume the elements of $\mathfrak{F}$ are in the dense subset $\{a_n\}$ and are positive contractions. Consider γ given by Lemma 2.5.2 using $\delta = \varepsilon/2$.

Let ω be a free ultrafilter on $\mathbb{N}$. We will show below that $\varphi_k^{(\omega)}(p_k)$ is a projection for $k = 1, \ldots, n$, where $p_k \in \prod_\omega F^{(r)}$ is represented by $\left(p_k^{(r)}\right)_{r \in \mathbb{N}}$ and $\varphi_k^{(\omega)}$ is the induced map at the level of ultrapowers (see Appendix A for the proper definitions). Once this is done, for $k = 1, \ldots, n$, there exists $U_k \in \omega$ such that

$$\left\|\varphi_k^{(r)}\left(p_k^{(r)}\right) - \varphi_k^{(r)}\left(p_k^{(r)}\right)^2\right\| < \gamma \tag{3.16}$$

for all $r \in U_k$. Similarly, since $\lim\limits_{r \to \omega} \varphi^{(r)}\psi^{(r)}\left(p^{(r)}a_i p^{(r)}\right) = a_i$ for all $i \in \mathbb{N}$, there exists $V \in \omega$ such that

$$\left\|a - \varphi^{(r)}\left(p^{(r)}\psi^{(r)}\left(a\right)p^{(r)}\right)\right\| < \frac{\varepsilon}{2} \tag{3.17}$$

for all $r \in V$ and for all $a \in \mathfrak{F}$.

Fix $r \in U_1 \cap \cdots \cap U_n \cap V$ and set $\widetilde{F}_k = p_k^{(r)} F^{(r)} p_k^{(r)}$. Hence, by the choice of the constant γ and (3.16), there exist *-homomorphisms $\pi_k : \widetilde{F}_k \longrightarrow A$ such that

$$\left\|\varphi_k^{(r)}|_{\widetilde{F}_k} - \pi_k\right\| < \frac{\varepsilon}{2} \tag{3.18}$$

for $k = 1, \ldots, n$. Extend π_k to $\widetilde{F} := \bigoplus\limits_{i=1}^{n} \widetilde{F_k} = p^{(r)}F^{(r)}p^{(r)}$ linearly by defining $\pi_k(x_1 \oplus \cdots \oplus x_{k-1} \oplus 0 \oplus x_{k+1} \oplus \cdots \oplus x_n) = 0$ for $x_i \in \widetilde{F}_i$ with $i \neq k$.

Define $\psi : A \longrightarrow \widetilde{F}$ as $\psi(a) = p^{(r)}\psi^{(r)}(a)p^{(r)}$ and set $\pi : \widetilde{F} \longrightarrow A$ as $\pi = \sum\limits_{k=1}^{n} \lambda_k \pi_k$. Then $\left(\widetilde{F}, \psi, \pi\right)$ is a completely positive and contractive approximation with the required properties since, using (3.17) and (3.18), we obtain

$$\begin{aligned}\|a-\pi\psi(x)\| &\leq \left\|a-\varphi^{(r)}\left(p^{(r)}\psi^{(r)}(a)p^{(r)}\right)\right\| + \left\|\sum_{k=1}^{n}\lambda_k\left(\varphi_k^{(r)}-\pi_k\right)\left(p^{(r)}\psi^{(r)}(a)p^{(r)}\right)\right\| \\ &< \frac{\varepsilon}{2}+\sum_{k=1}^{n}\lambda_k\left(\frac{\varepsilon}{2}\right)=\varepsilon \end{aligned} \tag{3.19}$$

for all $a \in \mathfrak{F}$.

To finish the proof, we will show $\varphi_k^{(\omega)}(p_k)$ is a projection for $k=1,\dots,n$. Due to the hypotheses, we have $\varphi^{(\omega)} = \sum\limits_{k=1}^{n}\lambda_k\varphi_k^{(\omega)}$ and $\varphi^{(\omega)}\psi^{(\omega)}(a) = a$ for all $a \in A$. Recall $p_k \in \prod\limits_{\omega} F^{(r)}$ is represented by $\left(p_k^{(r)}\right)_r$ and consider $p \in \prod\limits_{\omega} F^{(r)}$ represented by $\left(p^{(r)}\right)_r$ with $p^{(r)} = \sum\limits_{k=1}^{n} p_k^{(r)}$. Then by hypothesis (cI) we have

$$\varphi^{(\omega)}(p\psi^{(\omega)}(a)p) = a \tag{3.20}$$

for all $a \in A$ and by hypothesis (cII),

$$\varphi^{(\omega)}(p_k) = \varphi^{(\omega)}(p_k)\varphi^{(\omega)}(1_{\prod_\omega F^{(r)}}) \tag{3.21}$$

where $1_{\prod_\omega F^{(r)}}$ denotes the unit of $\prod\limits_{\omega} F^{(r)}$. Taking adjoints in (3.21) we get

$$\varphi_k^{(\omega)}(p_k) = \varphi_k^{(\omega)}(p_k)\varphi^{(\omega)}(1_{\prod_\omega F^{(r)}}) = \varphi^{(\omega)}(1_{\prod_\omega F^{(r)}})\varphi_k^{(\omega)}(p_k). \tag{3.22}$$

Fix k and consider $B := \overline{\varphi_k^{(\omega)}(p_k)A_\omega\varphi_k^{(\omega)}(p_k)}$. Then we have

$$\varphi^{(\omega)}(1_{\prod_\omega F^{(r)}})b = b \tag{3.23}$$

for all $b \in B$. By Proposition A.1.7, the map $\varphi_k^{(\omega)} : \prod\limits_{\omega} F^{(r)} \longrightarrow A_\omega$ is order zero and, by the structure of order zero maps given in Theorem 1.4.6, we can write

$$\varphi_k^{(\omega)}(x) = \varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}})\rho(x) = \rho(x)\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}}), \qquad x \in \prod_{\omega} F^{(r)}, \tag{3.24}$$

for a *-homomorphism

$$\rho : \prod_{\omega} F^{(r)} \longrightarrow \mathcal{M}\left(C^*\left(\varphi_k^{(\omega)}\left(\prod_{\omega} F^{(r)}\right)\right)\right) \bigcap \left\{\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}})\right\}'.$$

Thus

$$\begin{aligned}\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}})\varphi_k^{(\omega)}(p_k) &= \varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}})^2\rho(p_k) = \rho(p_k)\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}})^2 \\ &= \varphi_k^{(\omega)}(p_k)\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}}). \end{aligned} \tag{3.25}$$

Using this, we obtain

$$\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}})\varphi_k^{(\omega)}(p_k)x\varphi_k^{(\omega)}(p_k) = \varphi_k^{(\omega)}(p_k)\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}})x\varphi_k^{(\omega)}(p_k) \in B \tag{3.26}$$

for any $x \in A_\omega$. Thus

$$\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}})b \in B \tag{3.27}$$

for all $b \in B$. Set

$$h = \frac{1}{1-\lambda_k}\sum_{j\neq k}\lambda_j\varphi_j^{(\omega)}(1_{\prod_\omega F^{(r)}}). \tag{3.28}$$

By construction h is a positive contraction and

$$\varphi^{(\omega)}(1_{\prod_\omega F^{(r)}}) = \lambda_k\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}}) + (1-\lambda_k)h. \tag{3.29}$$

By Lemma 1.2.3, (3.27) and (3.23) we have

$$\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}})b = b \tag{3.30}$$

for all $b \in B$. By [4, Proposition II.3.4.2 (ii)] $\varphi_k^{(\omega)}(p_k)$ is in B, so in particular we obtain

$$\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}})\varphi_k^{(\omega)}(p_k) = \varphi_k^{(\omega)}(p_k). \tag{3.31}$$

Using the last identity and the fact that $\varphi_k^{(\omega)}$ is order zero, we obtain

$$\begin{aligned} 0 &= \varphi_k^{(\omega)}(p_k)\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}} - p_k) \\ &= \varphi_k^{(\omega)}(p_k)\varphi_k^{(\omega)}(1_{\prod_\omega F^{(r)}}) - \varphi_k^{(\omega)}(p_k)^2 \\ &= \varphi_k^{(\omega)}(p_k) - \varphi_k^{(\omega)}(p_k)^2, \end{aligned} \tag{3.32}$$

which means that $\varphi_k^{(\omega)}(p_k)$ is a projection as required. □

The following theorem is the main result of this section.

Theorem 3.1.5 ([16, Theorem 14]). *Let A be a C^*-algebra. Suppose there exists $n \in \mathbb{N}$ such that for every finite subset $\mathfrak{F} \subset A$ and every $\varepsilon > 0$ there exist c.p.c. maps $\psi : A \longrightarrow F$, $\varphi : F \longrightarrow A$ where F is a finite dimensional C^*-algebra and φ is a convex combination of n contractive order zero maps such that*

$$\|a - \varphi\psi(a)\| < \varepsilon \tag{3.33}$$

for all $a \in \mathfrak{F}$. Then A is AF.

Proof. If $n = 1$, the result follows from Theorem 2.5.4. Thus, we can suppose $n \geq 2$. By the proof of Proposition 2.2.4, any countable subset of A is contained in a separable subalgebra satisfying the hypotheses of the theorem. Therefore, without loss of generality we may assume A is separable.

From the hypotheses, for any finite subset $\mathfrak{F}$ and any $\varepsilon > 0$ there exist a c.p.c. approximation $\left(F, \psi^{(\mathfrak{F},\varepsilon)}, \varphi^{(\mathfrak{F},\varepsilon)}\right)$ for $\mathfrak{F}$ within ε, order zero maps $\varphi_k^{(\mathfrak{F},\varepsilon)} : F \longrightarrow A$ and coefficients $\lambda_k^{(\mathfrak{F},\varepsilon)} \geq 0$, for $k = 1, \cdots, n$, such that $\sum\limits_{k=1}^{n} \lambda_k^{(\mathfrak{F},\varepsilon)} = 1$ and $\varphi^{(\mathfrak{F},\varepsilon)} = \sum\limits_{k=1}^{n} \lambda_k^{(\mathfrak{F},\varepsilon)} \varphi_k^{(\mathfrak{F},\varepsilon)}$. By compactness of $[0,1]^n$, we may assume there are constants $\lambda_1, \cdots, \lambda_n \in [0,1]$ satisfying $\sum\limits_{k=1}^{n} \lambda_k = 1$ such that $\lambda_k^{(\mathfrak{F},\varepsilon)} = \lambda_k$ for any finite subset $\mathfrak{F}$ and $\varepsilon > 0$. Indeed, consider the net $\left\{\left(\lambda_1^{(\mathfrak{F},\varepsilon)}, \ldots, \lambda_n^{(\mathfrak{F},\varepsilon)}\right)\right\}_{\mathfrak{F},\varepsilon} \subset [0,1]^n$. By compactness of $[0,1]^n$ and after passing to a subnet if necessary, there exists an element $(\lambda_1, \ldots, \lambda_n) \in [0,1]^n$ such that

$$\left(\lambda_1^{(\mathfrak{F},\varepsilon)}, \ldots, \lambda_n^{(\mathfrak{F},\varepsilon)}\right) \longrightarrow (\lambda_1, \ldots, \lambda_n). \tag{3.34}$$

In particular, $\lambda_k^{(\mathfrak{F},\varepsilon)} \longrightarrow \lambda_k$ for $k = 1, \ldots, n$. Moreover, by continuity of the sum, we have $\sum\limits_{k=1}^{n} \lambda_k = 1$. Observing that

$$\begin{aligned}\left\|\varphi^{(\mathfrak{F},\varepsilon)}\psi^{(\mathfrak{F},\varepsilon)}(a) - \sum_{k=1}^{n} \lambda_k \varphi_k^{(\mathfrak{F},\varepsilon)}\psi^{(\mathfrak{F},\varepsilon)}(a)\right\| &= \left\|\sum_{k=1}^{n} \lambda_k^{(\mathfrak{F},\varepsilon)}\varphi^{(\mathfrak{F},\varepsilon)}\psi^{(\mathfrak{F},\varepsilon)}(a) - \sum_{k=1}^{n} \lambda_k \varphi_k^{(\mathfrak{F},\varepsilon)}\psi^{(\mathfrak{F},\varepsilon)}(a)\right\| \\ &\leq \sum_{k=1}^{n} \left|\lambda_k^{(\mathfrak{F},\varepsilon)} - \lambda_k\right| \left\|\varphi_k^{(\mathfrak{F},\varepsilon)}\psi^{(\mathfrak{F},\varepsilon)}(a)\right\| \longrightarrow 0 \end{aligned} \tag{3.35}$$

for all $a \in A$, we obtain that we can replace the coefficients $\left(\lambda_1^{(\mathfrak{F},\varepsilon)}, \ldots, \lambda_n^{(\mathfrak{F},\varepsilon)}\right)$ with $(\lambda_1, \ldots, \lambda_n)$ and these new approximations still converge to id_A in the point-norm topology. This entails that we can assume $\lambda_k^{(\mathfrak{F},\varepsilon)} = \lambda_k$ for any finite subset $\mathfrak{F}$ and $\varepsilon > 0$.

Additionally we can suppose (renaming n if necessary) that each λ_k is strictly positive. Thus, by Proposition 3.1.1, for any finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exists a c.p.c. approximation $\left(\bigoplus\limits_{k=1}^{n} F_k, \psi^{(\mathfrak{F},\varepsilon)}, \varphi^{(\mathfrak{F},\varepsilon)}\right)$ for $\mathfrak{F}$ within ε with $\varphi^{(\mathfrak{F},\varepsilon)} = \sum\limits_{k=1}^{n} \lambda_k \varphi_k^{(\mathfrak{F},\varepsilon)}$ where each $\varphi_k^{(\mathfrak{F},\varepsilon)} : F \longrightarrow A$ is an order zero map and $\bigoplus\limits_{i \neq k} F_i \subset \ker \varphi_k^{(\mathfrak{F},\varepsilon)}$.

By Lemma 3.1.3, there exist projections $p_k^{(\mathfrak{F},\varepsilon)} \in F_k$ for $1 \leq k \leq n$ such that:

(I) $\left\|\varphi^{(\mathfrak{F},\varepsilon)}\left(p^{(\mathfrak{F},\varepsilon)}\psi^{(\mathfrak{F},\varepsilon)}(a)p^{(\mathfrak{F},\varepsilon)}\right) - a\right\| < \varepsilon$ for all $a \in \mathfrak{F}$ with $p^{(\mathfrak{F},\varepsilon)} = \sum\limits_{k=1}^{n} p_k^{(\mathfrak{F},\varepsilon)}$,

(II) $\left\|\varphi^{(\mathfrak{F},\varepsilon)}\left(p_k^{(\mathfrak{F},\varepsilon)}\right) - \varphi^{(\mathfrak{F},\varepsilon)}\left(p_k^{(\mathfrak{F},\varepsilon)}\right)\varphi^{(\mathfrak{F},\varepsilon)}(1_F)\right\| < \varepsilon$ where 1_F denotes the unit of F.

Then we can produce, using a countable dense subset of A, a sequence of completely positive and contractive approximations

$$A \xrightarrow{\psi^{(r)}} F^{(r)} \xrightarrow{\varphi^{(r)}} A$$

satisfying the hypothesis of Lemma 3.1.4.

We will apply Lemma 3.1.4 to replace the convex combination of order zero maps with convex combination of *-homomorphisms. After this, we will proceed to replace the convex combination of *-homomorphisms with exactly one of them. The choice of such *-homomorphism is not important by Lemma 3.1.2 and, in order to simplify the notation, we will choose the first one.

Fix $\mathfrak{F}$ and $\varepsilon > 0$ such that $\sqrt{\lambda_1^{-1}\varepsilon} < 1$. We can assume that any element in $\mathfrak{F}$ is positive with norm at most 1. By Lemma 3.1.4, there exists a completely positive and contractive approximation $\left(\bigoplus_{k=1}^{n} F_k, \psi, \pi\right)$ such that

$$\|a - \pi\psi(a)\| < \frac{\varepsilon}{3} \tag{3.36}$$

for all $a \in \mathfrak{F}$ and $\pi = \sum_{k=1}^{n} \lambda_k \pi_k$, where each $\pi_k : \bigoplus_{k=1}^{n} F_k \longrightarrow A$ is a *-homomorphism satisfying $\bigoplus_{i\neq k} F_i \subset \ker \pi_k$.

Since the set of all minimal projections of F_k, $\mathcal{P}(F_k)$, is compact, we can find minimal projections $p_1, ..., p_r \in \mathcal{P}(F)$ such that for all $p \in \mathcal{P}(F_k)$ and all k there exists some $j \in \{1, \cdots, r\}$ such that

$$\|p - p_j\| < \frac{\lambda_1 \varepsilon^2}{3\,(6M)^2} \tag{3.37}$$

for some $j \in \{1, ..., r\}$, where $M = \dim F$. Assume $p_j \in \mathcal{P}\left(F_{k_j}\right)$ and set

$$\mathfrak{F}' = \mathfrak{F} \cup \{\pi_{k_j}(p_j) : 1 \leq j \leq r\}. \tag{3.38}$$

By Lemma 3.1.4 again, we find c.p.c. maps $\psi' : A \longrightarrow \bigoplus_{k=1}^{n} F'_k$ and $\theta : \bigoplus_{k=1}^{n} F'_k \longrightarrow A$ with $\theta = \sum_{k=1}^{n} \lambda_k \theta_k$, F'_k finite dimensional C^*-algebras and each θ_k a *-homomorphism satisfying $\bigoplus_{i\neq k} F'_i \subset \ker \theta_k$, such that

$$\left\|a - \theta\psi'(a)\right\| < \frac{\lambda_1 \varepsilon^2}{3\,(6M)^2} \tag{3.39}$$

for all $a \in \mathfrak{F}'$. In particular for $p \in \mathcal{P}(F_k)$, let p_j satisfy (3.37) so that

$$\begin{aligned} \|\pi_k(p) - \theta\psi'(\pi_k(p))\| &\leq \|\pi_k(p) - \pi_k(p_j)\| + \|\pi_k(p_j) - \theta\psi'(\pi_k(p_j))\| \\ &\quad + \|\theta\psi'(\pi_k(p_j)) - \theta\psi'(\pi_k(p))\| \\ &< \frac{\lambda_1 \varepsilon^2}{(6M)^2}. \end{aligned} \tag{3.40}$$

Using that $\sqrt{\lambda_1^{-1}\varepsilon} < 1$, we obtain

$$\frac{\varepsilon}{6M} < 1. \tag{3.41}$$

Then, by Lemma 3.1.2, we have

$$\begin{aligned}\|\pi_k(p) - \theta_1\psi'(\pi_k(p))\| &\leq \sqrt{\frac{\lambda_1\varepsilon^2}{\lambda_1(6M)^2}}\left(\sqrt{\frac{\lambda_1\varepsilon^2}{\lambda_1(6M)^2}} + 1\right) \\ &= \frac{\varepsilon}{6M}\left(\frac{\varepsilon}{6M} + 1\right) \\ &< \frac{\varepsilon}{3M} \end{aligned}\tag{3.42}$$

for all k. For any $a \in \mathfrak{F}$, by the spectral theorem for Hermitian matrices, we can write

$$\psi(a) = \sum_{i=1}^{d} t_i q_i \tag{3.43}$$

with $0 \leq t_i \leq 1$, where $\{q_i \in F : 1 \leq i \leq d\}$ is some set of minimal projections, and $d \leq M$. Using the last identity and (3.42) we have

$$\begin{aligned}\left\|\pi\psi(a) - \theta_1\psi'\pi\psi(a)\right\| &= \left\|\sum_{i,k} t_i\lambda_k\pi_k(q_i) - \sum_{i,k} t_i\lambda_k\theta_1\psi'\pi_k(q_i)\right\| \\ &\leq \sum_{k=1}^{n}\lambda_k\left(\sum_{i=1}^{d}\|\pi_k(q_i) - \theta_1\psi'\pi_k(q_i)\|\right) \\ &\leq \sum_{k=1}^{n}\lambda_k\left(\frac{\varepsilon d}{3M}\right) \leq \frac{\varepsilon}{3}\end{aligned}\tag{3.44}$$

for all $a \in \mathfrak{F}$.

Finally, using the last inequality and (3.36) we obtain

$$\begin{aligned}\|a - \theta_1\psi'(a)\| &\leq \|a - \pi\psi(a)\| + \|\pi\psi(a) - \theta_1\psi'\left(\pi\psi(a)\right)\| \\ &\quad + \|\theta_1\psi'(\pi\psi(a) - a)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}\tag{3.45}$$

Thus $\mathrm{dist}(a, \theta_1(F_1')) < \varepsilon$ for all $a \in \mathfrak{F}$. Since $\theta_1 : F_1' \longrightarrow A$ is a *-homomorphism and F_1' is a finite dimensional C^*-algebra, $\theta_1(F_1')$ is also a finite dimensional algebra. Therefore A is an AF-algebra. □

By the previous theorem, the decomposable approximations of a nuclear C^*-algebra A given by Theorem 1.5.5 can witness finite nuclear dimension (in fact, finite decomposition rank since φ is forced to be contractive) if and only if A is an approximately finite dimensional C^*-algebra. Thus, in general, the approximations given by Theorem 1.5.5 are not useful to compute nuclear dimension.

3.2 Nuclear dimension at most omega

In [79], Robert generalised the notion of nuclear dimension by allowing arbitrarily almost decomposable approximations which persist at the level of the sequence algebras. This is a relatively mild regularity property, but does impose structural requirements: a σ-unital stable C^*-algebra A with nuclear dimension at most omega has the *corona factorisation property*, *i.e.* every full projection in $\mathcal{M}(A)$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(A)}$ [79, Corollary 3.5]. In this section we extend Theorem 3.1.5 to show that the maps constructed in [49, Theorem 1.4] can only witness the weaker property of nuclear dimension at most omega for approximately finite dimensional C^*-algebras. For reasons of simplicity, in this section we restrict to separable C^*-algebras.

Consider a C^*-algebra with nuclear dimension equal to n. Then there is a system of n-decomposable approximations $\left\{\left(F^{(r)}, \psi^{(r)}, \varphi^{(r)}\right)\right\}_{r\in\mathbb{N}}$. By hypothesis, $F^{(r)} = \bigoplus_{k=0}^{n} F_k^{(r)}$ and the restrictions $\varphi_k^{(r)} := \varphi^{(r)}\big|_{F_k^{(r)}}$ are order zero for all $r \in \mathbb{N}$ and $k = 0, \ldots, n$. Let us denote the kth-component of $\psi^{(r)}$ by $\psi_k^{(r)} : A \longrightarrow F_k^{(r)}$; in other words $\psi_k^{(r)}(a) = 1_{F_k^{(r)}}\psi^{(r)}(a)\,1_{F_k^{(r)}}$ where $1_{F_k^{(r)}}$ is the unit of $F_k^{(r)}$. We have seen in Proposition 2.2.6 that we can choose these approximations in such a way that the maps $\psi^{(r)}$ are almost order zero; hence, the induced map at the level of sequence algebras is order zero. By [111, Proposition 3.1], the components of this last map are also order zero. Robert observed that we have the same conclusion for the order zero maps $\varphi_k^{(r)}$.

Proposition 3.2.1 ([79, Proposition 2.2])**.** *Let A be a C^*-algebra such that* $\dim_{\mathrm{nuc}} A = n$. *Then, for $k = 0, \ldots, n$ there exist order zero maps*

$$\psi_k^{(\infty)} : A \longrightarrow \prod_{r=1}^{\infty} F_k^{(r)} \Big/ \bigoplus_{r=1}^{\infty} F_k^{(r)} \quad \textit{and} \quad \varphi_k^{(\infty)} : \prod_{r=1}^{\infty} F_k^{(r)} \Big/ \bigoplus_{r=1}^{\infty} F_k^{(r)} \longrightarrow A_\infty,$$

with $F_k^{(r)}$ finite dimensional C^*-algebras for all $r \in \mathbb{N}$, such that

$$a = \sum_{k=0}^{n} \varphi_k \psi_k(a) \tag{3.46}$$

for all $a \in A$.

It was proved that nuclear dimension is connected to some properties of the Cuntz semigroup (see Section 1.10). Before going further, let us say something about these properties. Let $(W, \leq)$ be an ordered semigroup in the category Cu and consider $x, y \in W$. The element x is *stably dominated* by y, denoted by $x \leq_s y$, if there exists $k \in \mathbb{N}$ such that $(k+1)x \leq ky$. The ordered semigroup W is *almost unperforated* if $x \leq_s y$ implies $x \leq y$. In particular, if A is a $\mathcal{Z}$-stable C^*-algebra then $\mathrm{Cu}(A)$ is almost unperforated [83, Theorem 4.5].

Weaker forms of almost unperforation were introduced in [69] by Ortega, Perera and Rørdam. Let $(W, \leq)$ be an ordered semigroup in $\mathtt{Cu}$. The ordered semigroup W has *n-comparison* if whenever $x, y_0, \ldots, y_n \in W$ such that $x \leq_s y_k$, for $k = 0, \ldots, n$, then $x \leq \sum_{k=0}^{n} y_k$. Robert showed that if a C^*-algebra A has nuclear dimension equal to n, then its Cuntz semigroup has n-comparison [79, Theorem 1.3]. Ortega, Perera and Rørdam also introduced a weaker form of n-comparison called omega comparison [69, Definition 2.11]. The ordered semigroup W in the category $\mathtt{Cu}$ has *omega comparison* if whenever $x, y_k \in W$ with $k \in \mathbb{N}$ such that $x \leq_s y_k$ for every $k \in \mathbb{N}$, then $x \leq \sum_{k \in \mathbb{N}} y_k$.

As mentioned before, these comparison properties are weaker forms of the almost unperforation of semigroups and the salient feature of these comparison properties for $\mathrm{Cu}(A)$ is that they imply the corona factorisation property of A [69, Proposition 2.17, Theorem 5.11]. Robert used Proposition 3.2.1 and omega comparison as models for his definition of nuclear dimension at most omega.

Definition 3.2.2 ([79, Definition 3.3])**.** A C^*-algebra A has *nuclear dimension at most omega* if for $k = 0, 1, 2, \ldots$ there are sequences of c.p.c. maps $\psi_k^{(r)} : A \longrightarrow F_k^{(r)}$ and $\varphi_k^{(r)} : F_k^{(r)} \longrightarrow A$, with $F_k^{(r)}$ finite dimensional C^*-algebras and $r \in \mathbb{N}$, such that:

(i) for each $k \in \mathbb{N}$ the induced maps, at the level of sequence algebras,

$$\psi_k^{(\infty)} : A \longrightarrow \prod_{r=1}^{\infty} F_k^{(r)} \Big/ \bigoplus_{r=1}^{\infty} F_k^{(r)} \text{ and } \varphi_k^{(\infty)} : \prod_{r=1}^{\infty} F_k^{(r)} \Big/ \bigoplus_{r=1}^{\infty} F_k^{(r)} \longrightarrow A_\infty$$

are c.p.c. of order zero;

(ii) $a = \sum_{k=0}^{\infty} \varphi_k^{(\infty)} \psi_k^{(\infty)}(a)$ for all $a \in A$, where the series on the right hand side is understood to be convergent in the norm topology.

We will denote a system of approximations witnessing the nuclear dimension at most omega as $\left\{ \left(\left(F_k^{(r)} \right)_{k \in \mathbb{N}}, \left(\psi_k^{(r)} \right)_{k \in \mathbb{N}}, \left(\varphi_k^{(r)} \right)_{k \in \mathbb{N}} \right) \right\}_{r \in \mathbb{N}}$. Let us present some easy examples.

Example 3.2.3. Any C^*-algebra with finite nuclear dimension has nuclear dimension at most omega by Proposition 3.2.1.

Example 3.2.4. Any countable direct sum of C^*-algebras with finite nuclear dimension has nuclear dimension at most omega.

As one could expect, we have the following theorem.

Theorem 3.2.5 ([79, Theorem 3.4])**.** *Let A be a C^*-algebra with nuclear dimension at most omega. Then* $\mathrm{Cu}(A)$ *has the omega comparison property.*

With this theorem in hand, we can obtain examples of C^*-algebras without nuclear dimension at most omega.

Example 3.2.6. Let X be a countable infinite cartesian product of spheres. It was proved in [59, Proposition 5.3] that $C(X) \otimes \mathbb{K}$ does not have the omega comparison property. Therefore, $C(X) \otimes \mathbb{K}$ does not have nuclear dimension at most omega.

3.3 Decomposable approximations vs Nuclear dimension at most omega

Previously we have investigated the relation between the c.p.c. approximations (F, ψ, φ) given by the stronger form of CPAP (Theorem 1.5.5) and nuclear dimension if we impose an upper bound on the number of summands in the convex combinations $\varphi = \sum \lambda_k \varphi_k$. We have proved that approximations of this type force the nuclear dimension of the algebra to be equal to zero (or equivalently force the algebra to be approximately finite dimensional).

In this section we will investigate the relationship between the c.p.c. approximations provided by Theorem 1.5.5 and nuclear dimension at most omega. Of course, if there is an upper bound in the number of summands in the convex combinations we already know that the nuclear dimension is zero; hence, the C^*-algebra has nuclear dimension at most omega. So instead of asking for an upper bound in the number of summands, we will ask for these approximations to witness nuclear dimension at most omega.

Similarly to the previous situation, we will obtain that if the approximations coming from the stronger form of the CPAP witness nuclear dimension at most omega, then the underlying algebra has to be approximately finite dimensional. The proof is very similar to the finite case. However, some technical lemmas need certain modifications to handle this more general situation. We will rewrite and prove these lemmas in the form we will need them here.

Before proceeding, let us explain this problem in more detail. Let A be a nuclear C^*-algebra. By Theorem 1.5.5, there exists a system of approximations $\{(F^{(r)}, \psi^{(r)}, \varphi^{(r)})\}$ such that $\varphi^{(r)}$ is a convex combination of order zero maps, *i.e.* for each $r \in \mathbb{N}$ there exists

$n^{(r)} \in \mathbb{N}$ such that for $k = 1, \ldots, n^{(r)}$ there are c.p.c. order zero maps $\varphi_k^{(r)} : F^{(r)} \longrightarrow A$, scalars $\lambda_k^{(r)} \in [0,1]$ satisfying $\sum\limits_{k=1}^{n^{(r)}} \lambda_k^{(r)} = 1$ and

$$\varphi^{(r)} = \sum_{k=1}^{n^{(r)}} \lambda_k \varphi_k^{(r)}. \tag{3.47}$$

Moreover,

$$\lim_{r\to\infty} \varphi^{(r)}\psi^{(r)}(a) = \lim_{r\to\infty} \sum_{k=1}^{n^{(r)}} \lambda_k^{(r)} \varphi_k^{(r)} \psi_k^{(r)}(a) = a \tag{3.48}$$

for all $a \in A$.

For each $r \in \mathbb{N}$ set $\lambda_k^{(r)} := 0$ if $k > n^{(r)}$. Now define $F_k^{(r)} = F^{(r)}$, $\psi_k^{(r)} := \psi^{(r)}$ and $\widehat{\varphi}_k^{(r)} = \lambda_k^{(r)} \varphi_k^{(r)}$ for $k \in \mathbb{N}$. Note that $\widehat{\varphi}_k^{(r)} = 0$ if $k > n^{(r)}$. One is tempted to think that the system we have constructed, $\left\{ \left(\left(F_k^{(r)}\right)_{k\in\mathbb{N}}, \left(\psi_k^{(r)}\right)_{k\in\mathbb{N}}, \left(\widehat{\varphi}_k^{(r)}\right)_{k\in\mathbb{N}} \right) \right\}_{r\in\mathbb{N}}$, witnesses nuclear dimension at most omega because we have equation (3.48) and the maps $\psi_k^{(r)}$ and $\varphi_k^{(r)}$ are approximately order zero and order zero respectively. However this is false in general. For example suppose the sequence $\left(n^{(r)}\right)_r$ diverges and suppose $\lambda_k^{(r)} = \frac{1}{n^{(r)}}$ for $k \leq n^{(r)}$. Then the induced maps at the level of sequence algebras, $\widehat{\varphi}_k^{(\infty)} : \prod\limits_{r\to\infty} F_k^{(r)} \longrightarrow A_\infty$, are the zero map since $\lambda_k^{(r)} \to 0$. Hence condition 3.2.2 (ii) is not fulfilled. Observe that individually, each map $\widehat{\varphi}_k^{(\infty)}$ becomes zero while the sum of those maps does persists at the level of the sequence algebras.

We are going to investigate when the system $\left\{ \left(\left(F_k^{(r)}\right)_{k\in\mathbb{N}}, \left(\psi_k^{(r)}\right)_{k\in\mathbb{N}}, \left(\widehat{\varphi}_k^{(r)}\right)_{k\in\mathbb{N}} \right) \right\}_{r\in\mathbb{N}}$ actually witnesses nuclear dimension at most omega. In particular, this implies that at least one sequence of coefficients, say $\left(\lambda_k^{(r)}\right)_{k\in\mathbb{N}}$, does not converge to zero. Notice that this new system is essentially the original one, we are only presenting it in a different form.

As in Section 3.1, the next technical lemma will allow us to work with one order zero map instead of a convex combination. The proof is essentially the proof of Lemma 3.1.2. We briefly explain why this is the case.

Lemma 3.3.1. *Let A be a C^*-algebra, $\varepsilon > 0$ and let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence contained in $(0,1)$ such that $\sum\limits_{k=1}^{\infty} \lambda_k = 1$. If $p \in A$ is a projection and $a_k \in A_+^1$, $k \in \mathbb{N}$, satisfy*

$$\left\| p - \sum_k \lambda_k a_k \right\| \leq \varepsilon. \tag{3.49}$$

Then

$$\|p - a_k\| \leq \sqrt{\lambda_k^{-1}\varepsilon}\left(\sqrt{\lambda_k^{-1}\varepsilon} + 1\right) \tag{3.50}$$

for k such that $\lambda_k \neq 0$.

Proof. We may suppose $A \subset B(H)$ for some Hilbert space H. For fixed k consider

$$b = \frac{1}{1-\lambda_k} \sum_{i \neq k} \lambda_i a_i \in A_+^1. \tag{3.51}$$

With this construction we can treat the sum as the convex combination of only two summands, precisely

$$\sum_i \lambda_i a_i = \lambda_k a_k + (1 - \lambda_k)\, b. \tag{3.52}$$

From here, the exact same arguments from the proof of Lemma 3.1.2 finish the proof. □

The following lemma is one of the key steps. This lemma gives us sufficiently close approximations that remain close after being cut down with a projection. Its proof is contained in the proof of Lemma 3.1.3. We now present a proof.

Lemma 3.3.2. *Let A be a separable nuclear C^*-algebra. Let $\left\{\left(F^{(r)}, \psi^{(r)}, \varphi^{(r)}\right)\right\}_{r\in\mathbb{N}}$ be a system of decomposable c.p.c. approximations for A with $F^{(r)}$ finite dimensional. Let $\mathfrak{F} \subset A_+$ be a finite subset and $0 < \varepsilon \leq 1$. Suppose there is a positive contraction $h \in A$ such that $ha = a$ for all $a \in \mathfrak{F}$. Then there exists $r \in \mathbb{N}$ and a projection $p \in F^{(r)}$ such that*

(i) $\left\|\varphi^{(r)}\psi^{(r)}(a) - a\right\| < \varepsilon$ *for all* $a \in \mathfrak{F} \cup \{h\}$,

(ii) $\left\|\varphi^{(r)}\left(p\psi^{(r)}(a)p\right) - a\right\| < \varepsilon$ *for all* $a \in \mathfrak{F} \cup \{h\}$,

(iii) if $F^{(r)} = \bigoplus_{k=1}^{n^{(r)}} F_k^{(r)}$ *is a decomposition of* $F^{(r)}$ *such that* $\varphi|_{F_k^{(r)}}$ *is order zero, then*

$$\left\|\varphi^{(r)}\left(p_k\right) - \varphi^{(r)}\left(p_k\right)\varphi^{(r)}\left(1_{F^{(r)}}\right)\right\| < \varepsilon \tag{3.53}$$

for $k = 1, \ldots, n$ *where* $p_k = p1_{F_k}$.

Proof. Consider the induced maps at the level of the sequence algebra,

$$\psi^{(\infty)} : A \longrightarrow \prod_{r=1}^{\infty} F^{(r)} \Big/ \bigoplus_{k=1}^{\infty} F^{(r)} \quad \text{and} \quad \varphi^{(\infty)} : \prod_{r=1}^{\infty} F^{(r)} \Big/ \bigoplus_{k=1}^{\infty} F^{(r)} \longrightarrow A_\infty. \tag{3.54}$$

From the hypothesis $\varphi^{(\infty)}\psi^{(\infty)}(a) = a$ for all $a \in A$. Moreover, we have that $\varphi^{(\infty)}$ is multiplicative on $C^*(\psi(A))$ as in the proof of Lemma 2.5.3. By Lemma 1.4.4 we obtain

$$\varphi^{(\infty)}\left(x\psi^{(\infty)}(a)\right) = \varphi^{(\infty)}(x)\,\varphi^{(\infty)}\psi^{(\infty)}(a) = \varphi^{(\infty)}(x)\,a \tag{3.55}$$

for all $x \in \prod_{\infty} F^{(r)}$ and $a \in A$. Set $p^{(r)} = \chi_{[1-\frac{\varepsilon^2}{4},\infty)}\left(\psi^{(r)}(h)\right)$. Let $p^{(\infty)} \in \prod_{\infty} F^{(r)}$ be represented by $\left(p^{(r)}\right)$. As in the proof of Lemma 2.5.3, we have

$$a = \varphi^{(\infty)}\left(p^{(\infty)}\right) a \tag{3.56}$$

and

$$a = \varphi^{(\infty)}\left(p^{(\infty)}\psi^{(\infty)}(a)p^{(\infty)}\right) \tag{3.57}$$

for all $a \in \mathfrak{F}$. Observe that by construction

$$\left\| p^{(\infty)}\psi^{(\infty)}(h) - p^{(\infty)} \right\| < \frac{\varepsilon^2}{4}, \tag{3.58}$$

and by (3.55),

$$\varphi^{(\infty)}\left(p^{(\infty)}\psi^{(\infty)}(h)\right) = \varphi^{(\infty)}\left(p^{(\infty)}\right) h. \tag{3.59}$$

Hence

$$\left\| \varphi^{(\infty)}\left(p^{(\infty)}\right) - \varphi^{(\infty)}\left(p^{(\infty)}\right) h \right\| < \frac{\varepsilon^2}{4}. \tag{3.60}$$

Therefore we can choose $r \in \mathbb{N}$ such that

(i) $\left\|\varphi^{(r)}\left(p^{(r)}\psi^{(r)}(a)p^{(r)}\right) - a\right\| < \frac{\varepsilon^2}{2}$ for all $a \in \mathfrak{F}$,

(ii) $\left\|\varphi^{(r)}\left(p^{(r)}\right) - \varphi^{(r)}\left(p^{(r)}\right) h\right\| < \frac{\varepsilon^2}{2}$,

(iii) $\left\|\varphi^{(r)}\psi^{(r)}(h) - h\right\| < \frac{\varepsilon^2}{2}$.

This proves the first two parts of the lemma if we choose $p = p^{(r)}$.

Suppose now that $F^{(r)} = \bigoplus_{k=1}^{n^{(r)}} F_k$ for some $n^{(r)} \in \mathbb{N}$ such that $\varphi|_{F_k^{(r)}}$ is order zero. Set $p_k = p1_{F_k}$, then

$$\begin{aligned}
\left\|\varphi^{(r)}(p_k) - \varphi^{(r)}(p_k)\varphi^{(r)}(1_{F^{(r)}})\right\|^2 &= \left\|\left(1_{\tilde{A}} - \varphi^{(r)}(1_{F^{(r)}})\right)\varphi^{(r)}(p_k)^2\left(1_{\tilde{A}} - \varphi^{(r)}(1_{F^{(r)}})\right)\right\| \\
&\leq \left\|\left(1_{\tilde{A}} - \varphi^{(r)}(1_{F^{(r)}})\right)\varphi^{(r)}(p)^2\left(1_{\tilde{A}} - \varphi^{(r)}(1_{F^{(r)}})\right)\right\| \\
&= \left\|\varphi^{(r)}(p)\left(1_{\tilde{A}} - \varphi^{(r)}(1_{F^{(r)}})\right)^2\varphi^{(r)}(p)\right\| \\
&\leq \left\|\varphi^{(r)}(p)\left(1_{\tilde{A}} - \varphi^{(r)}(1_{F^{(r)}})\right)\varphi^{(r)}(p)\right\| \\
&\leq \left\|\varphi^{(r)}(p)\left(1_{\tilde{A}} - \varphi^{(r)}\psi^{(r)}(h)\right)\varphi^{(r)}(p)\right\| \\
&\leq \left\|\varphi^{(r)}(p)\left(1_{\tilde{A}} - h\right)\right\| + \left\|\varphi^{(r)}(p)\left(h - \varphi^{(r)}\psi^{(r)}(h)\right)\right\| \\
&< \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2.
\end{aligned} \tag{3.61}$$

Therefore $\left\|\varphi^{(r)}(p_k) - \varphi^{(r)}(p_k)\varphi^{(r)}(1_{F^{(r)}})\right\| < \varepsilon$. □

The next lemma is an infinite version of Lemma 3.1.4.

Lemma 3.3.3. *Let A be a separable C^*-algebra and let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence contained in $(0,1)$ such that $\sum\limits_{k=1}^{\infty}\lambda_k = 1$ and let $\{a_n\}_{n\in\mathbb{N}}$ be a dense countable subset of A. Suppose A has a system of c.p.c. approximations $\left\{\left(F^{(r)},\psi^{(r)},\varphi^{(r)}\right)\right\}_{r\in\mathbb{R}}$ satisfying the following conditions:*

(a) For every $r\in\mathbb{N}$ there exist $n^{(r)}\in\mathbb{N}$, a decomposition $F^{(r)} = \bigoplus\limits_{k=1}^{n^{(r)}} F_k^{(r)}$, as internal direct sum, and a family $\left\{\varphi_k^{(r)} : F^{(r)} \longrightarrow A : k\in\mathbb{N}\right\}$ of contractive order zero maps such that $\varphi_k^{(r)} = 0$ if $k > n^{(r)}$ satisfying

$$\varphi^{(r)} = \sum_{k=1}^{n^{(r)}} \lambda_k \varphi_k^{(r)}. \tag{3.62}$$

Moreover, $\bigoplus\limits_{i\neq k} F_i^{(r)} \subset \ker\varphi_k$.

(b) $\left\|\varphi^{(r)}\psi^{(r)}(a_n) - a_n\right\| < r^{-1}$ for every $r\in\mathbb{N}$ and $n\le r$.

(c) For every $r\in\mathbb{N}$ there exist projections $p_k^{(r)}\in F_k^{(r)}$ satisfying

(I) $\left\|\varphi^{(r)}\left(p^{(r)}\psi^{(r)}\left(a_n\right)p^{(r)}\right) - a_n\right\| < r^{-1}$ for $n\le r$ with $p^{(r)} = \sum\limits_{k=1}^{n^{(r)}} p_k^{(r)}$.

(II) $\left\|\varphi^{(r)}\left(p_k^{(r)}\right) - \varphi^{(r)}\left(p_k^{(r)}\right)\varphi^{(r)}\left(1_{F^{(r)}}\right)\right\| < r^{-1}$ where $1_{F^{(r)}}$ denotes the unit of $F^{(r)}$.

Then for every finite subset $\mathfrak{F}\subset A$ and every $\varepsilon>0$ there exist $N\in\mathbb{N}$ and a c.p.c. approximation $\left(\bigoplus\limits_{k=1}^{N}\widetilde{F}_k,\psi,\pi\right)$ for $\mathfrak{F}$ within ε such that $\pi = \sum\limits_{k=1}^{N}\lambda_k\pi_k$ with each $\pi_k : \bigoplus\limits_{k=1}^{N}\widetilde{F}_k \longrightarrow A$ a *-homomorphism satisfying $\bigoplus\limits_{i\neq k}\widetilde{F}_i \subset \ker\pi_k$.

Proof. Let $\mathfrak{F}\subset A$ and $\varepsilon>0$. Without loss of generality we can assume the elements of $\mathfrak{F}$ are in the dense subset $\{a_n\}$ and are positive contractions. Consider γ given by Lemma 2.5.2 using $\delta=\varepsilon/3$. Since $\sum\limits_{k=1}^{\infty}\lambda_k = 1$ there exists $N\in\mathbb{N}$ such that

$$\sum_{k>N}^{\infty}\lambda_k < \frac{\varepsilon}{3}. \tag{3.63}$$

In the proof of Lemma 3.1.4 we worked with ultrafilters, but since we did not use any special feature of ultraproducts the same proof is still valid for the cofinite filter, *i.e.* for sequence algebras. By the proof of Lemma 3.1.4, $\varphi_k^{(\infty)}(p_k)$ is a projection for every k, where $p_k \in \prod_{r=1}^{\infty} F^{(r)} \Big/ \bigoplus_{k=1}^{\infty} F^{(r)}$ is represented by $\left(p_k^{(r)}\right)_{r\in\mathbb{N}}$ if $\lambda_k \neq 0$. Hence, for all $k \in \mathbb{N}$ there exists $m_k \in \mathbb{N}$ such that

$$\left\|\varphi_k^{(r)}\left(p_k^{(r)}\right) - \varphi_k^{(r)}\left(p_k^{(r)}\right)^2\right\| < \gamma \tag{3.64}$$

for all $r \geq m_k$. Similarly, since $\lim_{r\to\infty} \varphi^{(r)}\psi^{(r)}\left(p^{(r)}a_n p^{(r)}\right) = a_n$ for all $n \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that

$$\left\|a - \varphi^{(r)}\left(p^{(r)}\psi^{(r)}\left(a\right)p^{(r)}\right)\right\| < \varepsilon/3 \tag{3.65}$$

for all $r \geq M$ and for all $a \in \mathfrak{F}$.

Fix $r \geq \max\{m_1, m_2, \ldots, m_N, M\}$ and set $\widetilde{F}_k = p_k^{(r)}F^{(r)}p_k^{(r)}$. Hence, by the choice of the constant γ and (3.64), there exists a *-homomorphism $\pi_k : \widetilde{F}_k \longrightarrow A$ such that

$$\left\|\varphi_k^{(r)}|_{\widetilde{F}_k} - \pi_k\right\| < \frac{\varepsilon}{3} \tag{3.66}$$

for $k \leq N$. Extend π_k to $\widetilde{F} := \bigoplus_{i=1}^{n^{(r)}} \widetilde{F_k} = p^{(r)}F^{(r)}p^{(r)}$ linearly by defining $\pi_k(x_1 \oplus \cdots \oplus x_{k-1} \oplus 0 \oplus x_{k+1} \oplus \cdots \oplus x_{n^{(r)}}) = 0$ for $x_i \in \widetilde{F}_i$ with $i \neq k$.

Define $\psi : A \longrightarrow \widetilde{F}$ as $\psi(a) = p^{(r)}\psi^{(r)}(a)p^{(r)}$ and set $\pi : \widetilde{F} \longrightarrow A$ as $\pi = \sum_{k=1}^{N} \lambda_k \pi_k$, then $\left(\widetilde{F}, \psi, \pi\right)$ is a completely positive and contractive approximation with the required properties since, using (3.65), (3.66) and (3.63), we obtain

$$\begin{aligned}\|a - \pi\psi(x)\| &\leq \left\|a - \varphi^{(r)}\left(p^{(r)}\psi^{(r)}(a)p^{(r)}\right)\right\| + \left\|\sum_{k=1}^{N}\lambda_k\left(\varphi_k^{(r)} - \pi_k\right)\left(p^{(r)}\psi^{(r)}(a)p^{(r)}\right)\right\| \\ &\quad + \left\|\sum_{k>N}\lambda_k\varphi_k^{(r)}\left(p^{(r)}\psi^{(r)}(a)p^{(r)}\right)\right\| \\ &< \frac{\varepsilon}{3} + \sum_{k=1}^{N}\lambda_k\left(\frac{\varepsilon}{3}\right) + \frac{\varepsilon}{3} < \varepsilon \end{aligned} \tag{3.67}$$

for all $a \in \mathfrak{F}$. □

We can reformulate Theorem 3.1.5 in terms of finite nuclear dimension as follows: If a C^*-algebra A has nuclear dimension equal to n such that the system of contractible approximations witnessing the nuclear dimension are decomposable as convex combinations of n summands, then the nuclear dimension of the algebra A is in fact 0. We now extend this to the case of nuclear dimension at most omega.

Theorem 3.3.4. *Let A be a separable nuclear C^*-algebra. If A has a system of approximations $\left\{\left(\left(F_k^{(r)}\right)_{k\in\mathbb{N}}, \left(\psi_k^{(r)}\right)_{k\in\mathbb{N}}, \left(\widetilde{\varphi}_k^{(r)}\right)_{k\in\mathbb{N}}\right)\right\}_{r\in\mathbb{N}}$ witnessing nuclear dimension at most omega satisfying the following property:*

(i) *For all* $k, r \in \mathbb{N}$ *there exist* $\lambda_k^{(r)} \geq 0$ *such that* $\sum_{k \in \mathbb{N}} \lambda_k^{(r)} = 1$ *for all* $r \in \mathbb{N}$*, and a c.p.c. order zero map* $\varphi_k^{(r)} : F_k^{(r)} \longrightarrow A$ *such that*

$$\widetilde{\varphi}_k^{(r)} = \lambda_k^{(r)} \varphi_k^{(r)}. \tag{3.68}$$

Then A *is AF.*

Proof. By the weak*-compactness of the unit ball of ℓ^1, we can assume (probably after passing to a subsequence) that $\left(\lambda_k^{(r)}\right)_k \to (\lambda_k)_k$ in the weak*-topology when $r \to \infty$ for some $(\lambda_k)_k \in \ell^1$ in the unit ball. In particular we have that $\lambda_k^{(r)} \to \lambda_k$ when $r \to \infty$ for all $k \in \mathbb{N}$ (which also implies $\lambda_k \geq 0$). Consider the induced maps at the level of the sequence algebra,

$$\psi_k^{(\infty)} : A \longrightarrow \prod_{r=1}^{\infty} F^{(r)} \Big/ \bigoplus_{k=1}^{\infty} F^{(r)} \quad \text{and} \quad \varphi_k^{(\infty)} : \prod_{r=1}^{\infty} F^{(r)} \Big/ \bigoplus_{k=1}^{\infty} F^{(r)} \longrightarrow A_\infty. \tag{3.69}$$

Since $\lambda_k^{(r)} \to \lambda_k$ when $r \to \infty$ for all $k \in \mathbb{N}$, we obtain $\widetilde{\varphi}_k^{(\infty)} = \lambda_k \varphi_k^{(\infty)}$ and, by the hypotheses, we have

$$a = \sum_{k=1}^{\infty} \widetilde{\varphi}_k^{(\infty)} \psi_k^{(\infty)}(a) = \sum_{k=1}^{\infty} \lambda_k \varphi_k^{(\infty)} \psi_k^{(\infty)}(a) \tag{3.70}$$

for all $a \in A$. Then we can slightly modify the system of approximations by replacing each $\lambda_k^{(r)}$ with λ_k for all $r \in \mathbb{N}$. Precisely, replace $\widetilde{\varphi}_k^{(\infty)}$ with $\hat{\varphi}_k^{(\infty)}$ where $\hat{\varphi}_k^{(r)} := \lambda_k \varphi_k^{(r)}$ and this new system also witnesses the nuclear dimension at most omega because it induces the same maps at the level of the sequence algebra, *i.e.* $\hat{\varphi}_k^{(\infty)} = \lambda_k \varphi_k^{(\infty)} = \widetilde{\varphi}_k^{(\infty)}$.

We know that $(\lambda_k)_{k \in \mathbb{N}}$ is an element of the unit ball of ℓ^1, hence $\sum_{k=1}^{\infty} \lambda_k \leq 1$. Let us show the value of the series is exactly 1. Consider a nonzero $a \in A$, then

$$\|a\| = \left\| \sum_{k=1}^{\infty} \lambda_k \varphi_k \psi_k(a) \right\| \leq \sum_{k=0}^{\infty} \lambda_k \|a\|. \tag{3.71}$$

This yields $1 \leq \sum_{k=1}^{\infty} \lambda_k$. Hence

$$\sum_{k=1}^{\infty} \lambda_k = 1. \tag{3.72}$$

If there is only a finite number of coefficients λ_k different from zero, by Theorem 3.1.5, A is an AF-algebra. So let us assume there exists an infinite number of coefficients λ_k different from 0. Deleting terms if necessary, we can also assume $\lambda_k > 0$ for all $k \in \mathbb{N}$.

For each $r \in \mathbb{N}$, there exists $n^{(r)} \in \mathbb{N}$ such that $\sum_{k=n^{(r)}+1}^{\infty} \lambda_k < \frac{1}{r}$. Let us perturb once again the system of approximations. Set $\overline{\varphi}_k^{(r)} := \lambda_k \varphi_k^{(r)}$ if $k \leq n^{(r)}$ and $\overline{\varphi}_k^{(r)} := 0$ if

$k > n^{(r)}$ Since for every $k \in \mathbb{N}$ there is a sufficiently large R such that $\overline{\varphi}_k^{(r)} = \lambda_k \varphi_k^{(r)}$ if $r \geq R$, this new system induces the same maps at the level of the sequence algebra, *i.e.* $\overline{\varphi}_k^{(\infty)} = \lambda_k \varphi_k^{(\infty)} = \hat{\varphi}_k^{(\infty)}$.

Set $\psi^{(r)} = \sum\limits_{k=1}^{n^{(r)}} \psi_k^{(r)}$ and $\varphi^{(r)} = \sum\limits_{k=1}^{n^{(r)}} \lambda_k \varphi_k^{(r)}$. Notice that $\left(\bigoplus\limits_{k=1}^{n^{(r)}} F_k^{(r)}, \psi^{(r)}, \varphi^{(r)} \right)$ is $n^{(r)}$-decomposable since $\psi^{(r)}\big|_{F_k^{(r)}} = \lambda_k \varphi^{(r)_k}$ and hence order zero for $k = 1, \ldots, n^{(r)}$. By Lemma 3.3.2, possible after passing to a subsequence, there exist projections $p_k^{(r)} \in F_k^{(r)}$ for all $k, r \in \mathbb{N}$ such that the system of approximations $\left\{ \left(\bigoplus\limits_{k=1}^{n^{(r)}} F_k^{(r)}, \psi^{(r)}, \varphi^{(r)} \right) \right\}_{r \in \mathbb{N}}$ satisfies the hypotheses of Lemma 3.3.3. From this point the proof follows the same steps as the proof of Theorem 3.1.5. The reason for this is because the number of coefficients different from zero is not relevant, the key fact is the existence of at least one coefficient different from zero. We include the details for completeness.

Let $\mathfrak{F} \subset A$ be a finite subset of A and $\varepsilon > 0$ such that $\sqrt{\lambda_1^{-1}\varepsilon} < 1$. We can assume that any element in $\mathfrak{F}$ is positive of norm at most 1. By Lemma 3.3.3, there exists a completely positive and contractive approximation $\left(\bigoplus\limits_{k=1}^{n} F_k, \psi, \pi \right)$ such that

$$\|a - \pi\psi(a)\| < \frac{\varepsilon}{3} \tag{3.73}$$

for all $a \in \mathfrak{F}$ and $\pi = \sum_{k=1}^{n} \lambda_k \pi_k$ where each $\pi_k : \bigoplus\limits_{k=1}^{n} F_k \longrightarrow A$ is a *-homomorphism satisfying $\bigoplus_{i \neq k} F_i \subset \ker \pi_k$.

Since the set of all minimal projections of F_k, $\mathcal{P}(F_k)$, is compact, we can find minimal projections $p_1, ..., p_r \in \mathcal{P}(F)$ such that for all $p \in \mathcal{P}(F_k)$ and all k there exists some $j \in \{1, \cdots, r\}$ such that

$$\|p - p_j\| < \frac{\lambda_1 \varepsilon^2}{3\,(6M)^2} \tag{3.74}$$

for some $j \in \{1, ..., r\}$ where $M = \dim F$. Assume $p_j \in \mathcal{P}\left(F_{k_j}\right)$ and set

$$\mathfrak{F}' = \mathfrak{F} \cup \{\pi_{k_j}(p_j) : 1 \leq j \leq r\}. \tag{3.75}$$

By Lemma 3.3.3 again, we find c.p.c. maps $\psi' : A \longrightarrow \bigoplus_{k=1}^{n} F_k'$ and $\theta : \bigoplus_{k=1}^{n} F_k' \longrightarrow A$ with $\theta = \sum_{k=1}^{n} \lambda_k \theta_k$, F_k' finite dimensional C^*-algebras and each θ_k is a *-homomorphism satisfying $\bigoplus\limits_{i \neq k} F_i' \subset \ker \theta_k$, such that

$$\left\|a - \theta\psi'(a)\right\| < \frac{\lambda_1 \varepsilon^2}{3\,(6M)^2} \tag{3.76}$$

for all $a \in \mathfrak{F}'$. In particular for $p \in \mathcal{P}(F_k)$, let p_j satisfy (3.74) so that

$$\begin{aligned} \|\pi_k(p) - \theta\psi'(\pi_k(p))\| &\leq \|\pi_k(p) - \pi_k(p_j)\| + \|\pi_k(p_j) - \theta\psi'(\pi_k(p_j))\| \\ &\quad + \|\theta\psi'(\pi_k(p_j)) - \theta\psi'(\pi_k(p))\| \\ &< \frac{\lambda_1\varepsilon^2}{(6M)^2}. \end{aligned} \tag{3.77}$$

Using that $\sqrt{\lambda_1^{-1}\varepsilon} < 1$, we obtain

$$\frac{\varepsilon}{6M} < 1. \tag{3.78}$$

Then, by Lemma 3.3.1, we have

$$\begin{aligned} \|\pi_k(p) - \theta_1\psi'(\pi_k(p))\| &\leq \sqrt{\frac{\lambda_1\varepsilon^2}{\lambda_1(6M)^2}}\left(\sqrt{\frac{\lambda_1\varepsilon^2}{\lambda_1(6M)^2}} + 1\right) \\ &= \frac{\varepsilon}{6M}\left(\frac{\varepsilon}{6M} + 1\right) \\ &< \frac{\varepsilon}{3M} \end{aligned} \tag{3.79}$$

for all k. This last step is a key point because it is allowing us to work with exactly one *-homomorphisms instead of a convex combination of *-homomorphisms. For any $a \in \mathfrak{F}$, by the spectral theorem for Hermitian matrices, we can write

$$\psi(a) = \sum_{i=1}^{d} t_i q_i \tag{3.80}$$

with $0 \leq t_i \leq 1$ where $\{q_i \in F : 1 \leq i \leq d\}$ is some set of minimal projections and $d \leq M$. Using the last identity and (3.79) we have

$$\begin{aligned} \left\|\pi\psi(a) - \theta_1\psi'\pi\psi(a)\right\| &= \left\|\sum_{i,k} t_i\lambda_k\pi_k(q_i) - \sum_{i,k} t_i\lambda_k\theta_1\psi'\pi_k(q_i)\right\| \\ &\leq \sum_{k=1}^{n}\lambda_k\left(\sum_{i=1}^{d}\|\pi_k(q_i) - \theta_1\psi'\pi_k(q_i)\|\right) \\ &\leq \sum_{k=1}^{n}\lambda_k\left(\frac{\varepsilon d}{3M}\right) \leq \frac{\varepsilon}{3} \end{aligned} \tag{3.81}$$

for all $a \in \mathfrak{F}$. Finally, using the last inequality and (3.73) we obtain

$$\begin{aligned} \|a - \theta_1\psi'(a)\| &\leq \|a - \pi\psi(a)\| + \|\pi\psi(a) - \theta_1\psi'\left(\pi\psi(a)\right)\| \\ &\quad + \|\theta_1\psi'(\pi\psi(a) - a)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{3.82}$$

Thus $\mathrm{dist}(a, \theta_1(F_1')) < \varepsilon$ for all $a \in \mathfrak{F}$. Since $\theta_1 : F_1' \longrightarrow A$ is a *-homomorphism and F_1' is a finite dimensional C^*-algebra, $\theta_1(F_1')$ is also a finite dimensional algebra. Therefore A is an AF-algebra. $\square$

As a straightforward corollary, we get that the approximations given by [49, Theorem 1.4] can only witness nuclear dimension at most omega for AF-algebras.

Corollary 3.3.5. *Let A be a nuclear C^*-algebra with nuclear dimension at most omega witnessed with a system of approximations $\left\{\left(F^{(r)}, \psi^{(r)}, \varphi^{(r)}\right)\right\}_{r\in\mathbb{N}}$ such that $\varphi^{(r)}$ is a convex combination of order zero maps. Then A is an AF-algebra.*

Chapter 4

Interactions between von Neumann algebras and nuclear C^*-algebras

In this chapter we will review some well known relations between von Neumann algebras and C^*-algebras. These examples are motivation for the coloured theory we will introduce later in this chapter and Chapter 5. Our goal is to explain some aspects of Connes' proof of injectivity implies hyperfiniteness and how these provide a strategy to estimate nuclear dimension for certain classes of C^*-algebras.

4.1 Von Neumann algebras

We will recall some facts about von Neumann algebras. So far we have been working with separable C^*-algebras and, in the same spirit, we will be working with von Neumann algebras acting in a separable Hilbert space, *i.e.* M can be represented faithfully on a separable Hilbert space. First of all, we should observe that separability in norm is not a useful property for von Neumann algebras.

Proposition 4.1.1. *Let M be an infinite dimensional von Neumann algebra. Then M is not separable in norm.*

This can be proved by showing that infinite dimensional von Neumann algebras contain an infinite family of non zero pairwise orthogonal projections. From this family we can construct an uncountable set of elements such that the distance between any two is at least 1.

Since we cannot ask for separability in norm, one reasonable condition is to ask for separability of M in one of the following locally convex topologies: weak, strong, strong*, ultraweak, ultrastrong or ultrastrong* (we refer to [93, Section II.2] for the definitions). It turns out that separability in any of these locally convex topologies implies separability in all of them.

To begin with let us observe these locally convex topologies behave similarly on convex subsets. Let $B_r(0)$ denote the ball of radius r centered at 0.

Theorem 4.1.2 ([93, Theorem 2.6.(iv)])**.** *Let M be a von Neumann algebra. For a convex subset K of M, the following statements are equivalent.*

(i) K is ultraweakly closed.

(ii) K is ultrastrongly closed

(iii) K is closed in the ultrastrong topology.*

(iv) $K \cap B_r(0)$ is weakly closed for every $r > 0$.

(v) $K \cap B_r(0)$ is strongly closed for every $r > 0$.

(vi) $K \cap B_r(0)$ is strongly closed for every $r > 0$.*

It is well known that we can give an axiomatic characterisation of C^*-algebras. This was done by Gelfand and Naimark in [41]. In a similar fashion, Sakai characterised von Neumann algebras axiomatically using their predual.

Definition 4.1.3 ([93, Definition III.2.13])**.** The *predual* of a von Neumann algebra M is the space of all ultraweakly continuous linear functionals on M (also called normal functionals). We will denote it as M_*.

By definition, M_* is contained in the dual M^* and, in fact, it is a norm closed subset. Thus M_* is a Banach space. In particular, the canonical bilinear map on $M \times M^*$, $(a, \varphi) \mapsto \varphi(a)$, induces a bilinear map on $M \times M_*$. This bilinear map produces an isomorphism between M and the dual of M_*.

Theorem 4.1.4 ([26, Theorem 1])**.** *Let M be a von Neumann algebra. Then $M \cong (M_*)^*$, as Banach spaces.*

We can characterise von Neumann algebras as C^*-algebras which are isometrically isomorphic to the dual of some Banach space. This space is also called a predual and it turns out that von Neumann algebras have a unique predual [26, 87]. Observe that general Banach spaces can have more than one predual; for example $c^* \cong (c_0)^* \cong \ell^1$.

At this point it is important to notice that the ultraweak topology is nothing more than the weak* topology $\sigma(M, M_*)$. This immediately shows, with the help of Banach-Alaoglu theorem, that the closed unit ball of M is ultraweakly compact. This is one of the key features of von Neumann algebras.

Now let us introduce another relevant concept for this discussion.

Definition 4.1.5 ([52, Definition 5.5.14])**.** A projection p in a von Neumann algebra is *countably decomposable* if every orthogonal family of non zero subprojections of p is countable. A von Neumann algebra is *countably decomposable* if the identity is countably decomposable.

It follows from the definition that any separably acting von Neumann algebra is countably decomposable but the converse is false [4, III.3.1.5]. Now we are ready to state the following theorem.

Theorem 4.1.6. *Let M be a von Neumann algebra. The following statements are equivalent.*

(i) M can be represented faithfully as a von Neumann algebra on a separable Hilbert space.

(ii) M_* is separable in norm.

(iii) M is countably generated and countably decomposable.

(iv) M is countably decomposable and separable in one of the following locally convex topologies: weak, strong, strong, ultraweak, ultrastrong and ultrastrong*.*

(v) M is countably decomposable and separable in all of the following locally convex topologies: weak, strong, strong, ultraweak, ultrastrong and ultrastrong*.*

The proof of this theorem is a combination of [88, Proposition 2.1.9, Proposition 2.1.10], [27, Proposition 1.6, Proposition 3.1], the double commutant theorem and Theorem 4.1.2.

We finish this part by pointing out a very useful property of finite von Neumann algebras. Remember that a von Neumann algebra is *finite* if 1_M is finite. Let M be a finite von Neumann algebra with faithful trace τ. We can introduce a new norm in M,

$$\|a\|_2 = \sqrt{\tau(a^*a)}, \qquad a \in M.$$

The following lemma states important equivalences between $\|\cdot\|_2$-norm and the ultrastrong topology.

Lemma 4.1.7 ([87, Lemma 7.1])**.** *Let M be a von Neumann algebra with faithful finite trace τ and let N be a *-subalgebra of M containing 1_M. For $a \in M$, the following conditions are equivalent:*

(i) a is limit, in $\|\cdot\|_2$-norm, of elements of N;

(ii) a is limit, in $\|\cdot\|_2$-norm, of a bounded sequence (in $\|\cdot\|$-norm) of elements of N;

(iii) a is limit, in the ultrastrong topology, of elements of N.

4.1.1 Type decomposition and factors

Needless to say, projections are very important in the study of von Neumann algebras, in contrast with C^*-algebras where projections might not exist. In particular, for a von Neumann algebra M projections form a lattice, $\mathcal{P}(M)$, and the Murray-von Neumann equivalence, $\precsim$, defines a partial order on $\mathcal{P}(M)$. We can give a rough classification into types based on structural behaviour of $(\mathcal{P}(M), \precsim)$ [88, Proposition 1.10.2]. This was originally done by Murray and von Neumann in their seminal paper [66]. They classified von Neumann algebras into different types: $\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty$ and III.

Let M be a von Neumann algebra. Remember that two projections p and q in M are Murray-von Neumann equivalent if there exists $v \in M$ such that $p = v^*v$ and $q = vv^*$. A projection $p \in M$ is *finite* if it is not Murray-von Neumann equivalent to any of its subprojections. Otherwise p is called infinite. A projection $p \in M$ is called *purely infinite* if there is no non zero finite projection $q \in M$ such that $q \leq p$. If qp is infinite for every central projection $q \in M$ with $qp \neq 0$, then p is *properly infinite.* If pMp is a commutative von Neumann algebra, then p is called *abelian.* A von Neumann algebra M is called *finite, infinite, purely infinite* or *properly infinite* accordingly to the property of the identity 1_M.

Definition 4.1.8. Let M be a von Neumann algebra.

(i) M is of *type* I if every non zero central projection majorizes a non zero abelian projection in M.

(ii) If M has no non zero abelian projections and if every non central projection in M majorizes a non zero finite projection of M, then it is said to be of *type* II.

(iii) If M is finite and of type II, then M is of *type* II_1.

(iv) If M is type II and has no non zero central finite projections, then M is of *type* II_∞.

(v) A von Neumann algebra M is of *type* III if it is purely infinite.

Theorem 4.1.9 ([93, Theorem V.1.19])**.** *Every von Neumann algebra M has a unique decomposition*

$$M \cong M_{\mathrm{I}} \oplus M_{\mathrm{II}_1} \oplus M_{\mathrm{II}_\infty} \oplus M_{\mathrm{III}}$$

as a direct sum of von Neumann algebras of type $\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty$ *and* III*.*

Before introducing factors, let us describe ideals in von Neumann algebras. In contrast with C^*-algebras, strongly closed ideals can be easily be described.

Theorem 4.1.10 ([87, Proposition 1.10.5])**.** *Let M be a von Neumann algebra and let I be a strongly closed ideal in M. Then there is a unique central projection p in I such that $I = Mp$.*

In light of the previous theorem, we can describe "simple" von Neumann algebras using central projections.

Definition 4.1.11. A von Neumann algebra M is a *factor* if its center is trivial, *i.e.* $\mathcal{Z}(M) = M' \cap M = \mathbb{C}1_M$.

It can be proved from Theorem 4.1.9 that each factor is of exactly one type. Von Neumann showed that, to some extent, the study of separably acting von Neumann algebras can be reduced to the study of factors (any von Neumann algebra on a separable Hilbert space is a direct integral of factors [93, Theorem IV.8.21]). For the purpose of this thesis, we will focus on II_1-factors. We view simple C^*-algebras as the C^*-analogue of factors.

Following this analogy, we also view stably finite and unital C^*-algebras as C^*-analogues of von Neumann algebras of type II_1. The following theorem provides a very useful way to detect II_1-factors.

Theorem 4.1.12 ([93, Theorem V.2.15])**.** *A factor M is of type II_1 if and only if it is infinite dimensional and admits a faithful normal (i.e. ultraweakly continuous) trace.*

Since any two non zero projections in a separably acting type III factor are Murray-von Neumann equivalent ([88, Proposition 2.2.14]) similarly to the situation of purely infinite C^*-algebras where any two non zero positive elements are Cuntz equivalent, we also regard simple and purely infinite C^*-algebras as analogues of type III factors.

4.1.2 The double dual of a C^*-algebra

The *universal representation* $\pi_{\mathcal{U}}$ of a C^*-algebra A is the direct sum of all GNS-representations and the *enveloping von Neumann algebra* of A is the double commutant $\pi_{\mathcal{U}}(A)''$. It is important to notice that even when A is separable, the double dual A^{**} is generally not a separably acting von Neumann algebra.

Theorem 4.1.13 ([93, Proposition III.2.4])**.** *Let A be a C^*-algebra. The enveloping von Neumann algebra of A is isometrically isomorphic to the double dual A^{**}. In particular, the ultraweak topology on $\pi_{\mathcal{U}}(A)''$ restricts to the weak topology $\sigma(A, A^*)$ on A.*

This theorem is very useful when we are working in the double dual A^{**} and we want to go back to A. By the Hahn-Banach theorem we know that the weak closure and the norm closure of any convex set are the same. Thus, by Theorem 4.1.13 the ultraweak closure and norm closure of any convex subset of A agree. An example of this application is given in the proof of Theorem 4.3.2. Another useful tool which allows us to return to A from A^{**} is Kaplansky's density theorem.

4.2 Hyperfiniteness and injectivity

The notion of hyperfiniteness was introduced by Murray and von Neumann in one of their seminal papers about von Neumann algebras [67, Definition 4.1.1].

Definition 4.2.1. A von Neumann algebra M is *hyperfinite* if for all finite subsets $\mathfrak{F} \subset M$ and all ultrastrong* open neighborhoods U of 0 there exists a finite dimensional subalgebra $F \subset M$ such that $\mathfrak{F} \subset F + U$.

Remark 4.2.2. If the von Neumann algebra M is finite with faithful trace τ, then M is hyperfinite if and only if for every finite subset $\mathfrak{F} \subset M$ and $\varepsilon > 0$ there exists a finite dimensional algebra $F \subset M$ such that $\mathfrak{F} \subset F + B_\varepsilon$, where B_ε denotes the open ball of radius ε centered at 0 with respect to the $\|\cdot\|_2$-norm. This follows from Theorem 4.1.6 and Lemma 4.1.7.

Example 4.2.3 (The hyperfinite II_1-factor)**.** Let $A = M_{2^\infty}(\mathbb{C})$ be the CAR-algebra. This algebra has a unique faithful trace τ. Let π_τ be the GNS-representation associated to τ and define

$$\mathcal{R} := \pi_\tau(A)''.$$

This algebra is a factor with a faithful trace, thus $\mathcal{R}$ is a II_1-factor. Furthermore, by construction, $\mathcal{R}$ is hyperfinite.

Murray and von Neumann carried out a detailed study of hyperfinite von Neumann algebras in [67] and they showed that there is exactly one hyperfinite II_1-factor.

Theorem 4.2.4 ([67, Theorem XIV])**.** *Let M be a separably acting hyperfinite* II_1*-factor. Then M is isomorphic to $\mathcal{R}$.*

Approximately finite dimensional C^*-algebras are the C^*-analogue of hyperfinite algebras; however, in contrast with the von Neumann case, there are uncountably many simple separable unital approximately finite dimensional algebras (this follows from Glimm's classification of UHF algebras [42, Theorem 1.12] or Elliott's classification of AF-algebras [31, Theorem 4.3]).

We now state a theorem about type III hyperfinite von Neumann algebras.

Theorem 4.2.5 ([94, Theorem XVI.1.4])**.** *Let M be separably acting von Neumann algebra of type* III*. The following conditions are equivalent.*

(i) M is hyperfinite.

(ii) There exists an increasing sequence $\{N_n\}$ of finite dimensional *-subalgebras of M such that $M = \left(\bigcup_{n=1}^{\infty} N_n\right)''$.

(iii) There exists an increasing sequence $\{N_{2^{k_n}}\}$ of subfactors such that $M = \left(\bigcup_{n=1}^{\infty} N_{2^{k_n}}\right)''$.

As a consequence of Theorem 4.2.5 and Theorem 4.2.4, we have that any hyperfinite separably acting factor contains a dense separable approximately finite dimensional C^*-algebra.

We proceed now to introduce the notion of injectivity. This concept was introduced by Effros and Lance in [29].

Definition 4.2.6. A von Neumann algebra M is *injective* if for some faithful representation $\pi : M \longrightarrow B(H)$ there is a conditional expectation from $B(H)$ onto M.

It turns out that this definition does not depend on the representation. The hyperfinite II_1-factor $\mathcal{R}$ is an example of an injective von Neumann algebra. In fact, Effros and Lance showed that any hyperfinite factor is injective [29, Corollary 5.8].

We finish this section by enunciating a celebrated theorem due to Connes. This theorem states that the hyperfinite II_1-factor $\mathcal{R}$ is the unique separably acting injective II_1-factor.

Theorem 4.2.7 ([22, Theorem 6])**.** *Let M be an injective II_1-factor acting on a separable Hilbert space. Then M is isomorphic to the hyperfinite II_1-factor $\mathcal{R}$.*

As a consequence of this theorem, we can think of injectivity as an abstract characterisation of hyperfiniteness. Aspects of the proof of this theorem will be discussed throughout this chapter and in Section 4.6 we will sketch the last part of Connes' proof.

4.3 Semidiscreteness and nuclearity

We can define nuclearity of a C^*-algebra A using the completely positive approximation property (Definition 1.5.2). This means there exist a system of c.p.c. maps $\psi_i : A \longrightarrow F_i, \varphi_i : F_i \longrightarrow A$, where F_i is a finite dimensional algebra, such that $\varphi_i\psi_i$ converges to id_A in the point-norm topology, *i.e.* $\varphi_i\psi_i(a) \longrightarrow a$ in norm for all $a \in A$. The completely positive approximation property is normally represented with the following diagram

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\psi_i"'] & & A \\ & F_i \arrow[ur, "\varphi_i"'] & \end{tikzcd}$$

which commutes on a finite subset of A up to some positive ε in norm.

The natural analogue of this definition for von Neumann algebras is called semidiscreteness and is obtained by asking these maps to converge to the identity in the point-ultraweak topology. This definition is due to Effros and Lance [29].

Definition 4.3.1 ([29, Section 3])**.** A von Neumann algebra M is *semidiscrete* if for every finite subsets $\mathfrak{F} \subset M$ and $\mathfrak{S} \subset M_*$, and $\varepsilon > 0$ there exist a finite dimensional von Neumann algebra F and c.p.c. maps $\psi : M \longrightarrow F, \varphi : F \longrightarrow M$ such that

$$|\eta\left(\varphi\psi(a)\right) - \eta(a)| < \varepsilon \tag{4.1}$$

for all $a \in \mathfrak{F}$ and $\eta \in \mathfrak{S}$. We also represent semidiscreteness with diagrams

$$\begin{tikzcd} M \arrow[rr, "\mathrm{id}_M"] \arrow[dr, "\psi"'] & & M \\ & F \arrow[ur, "\varphi"'] & \end{tikzcd}$$

which commutes on a finite subset of M up to some positive ε in the ultraweak topology.

The following theorem is an important connection between nuclearity and semidiscreteness. The proof of this theorem is very beautiful and it is an excellent example of how we can pass from the von Neumann world to the C^*-level.

Theorem 4.3.2 ([11, Proposition 1.3.3])**.** *Let A be a C^*-algebra. Then A is nuclear if and only if A^{**} is semidiscrete.*

We sketch the proof of the "easy" direction in the unital case. Suppose A^{**} is semidiscrete. Then there exists a net of c.p.c. maps $\psi_n : M \longrightarrow F_n, \varphi_n : F_n \longrightarrow M$ such that $\varphi_n \circ \psi_n$ converges to $\mathrm{id}_{A^{**}}$ in the point-ultraweak topology, *i.e.* $\varphi_n\psi_n(x) \to x$ in the ultraweak topology for all $x \in A^{**}$. We can assume that each F_n is in fact a matrix algebra $M_{k_n}(\mathbb{C})$. Since there is a bijection between c.p. maps $M_{k_n}(\mathbb{C}) \longrightarrow A^{**}$ and elements of $M_{k_n}(A^{**})$, we have $[\varphi_n(e_{i,j})]_{i,j} \in M_{k_n}(A^{**})_+$ where $\{e_{i,j}\}$ is a system of matrix units of $M_{k_n}(\mathbb{C})$ [13, Proposition 1.5.12]. Moreover, using that $M_{k_n}(A)_+$ is ultraweakly dense in $M_{k_n}(A^{**})_+$, we can replace each map φ_n with a new c.p. map $\varphi'_n : M_{k_n}(\mathbb{C}) \longrightarrow A^{**}$ such that $\varphi'_n(M_{k_n}(\mathbb{C})) \subset A$ in such a way that $\varphi'_n \circ \psi_n$ still converges to $\mathrm{id}_{A^{**}}$ in the point-ultraweak topology.

Remember that by Theorem 4.1.13, the ultraweak topology restricts to the weak topology $\sigma(A, A^*)$ in A. Consider now a finite subset $\mathfrak{F} = \{a_1, \ldots, a_k\}$ of A and $\varepsilon > 0$. In particular we have that $(\varphi_n\psi_n(a_1) \oplus \ldots \oplus \varphi_n\psi_n(a_k)) \in A^{\oplus n}$ converges ultraweakly to $(a_1 \oplus \ldots \oplus a_k) \in A^{\oplus n}$. Hence $(a_1 \oplus \ldots \oplus a_k)$ belongs to the ultraweak closure of the convex hull of $\{\varphi_n\psi_n(a_1) \oplus \ldots \oplus \varphi_n\psi_n(a_k)\}_n$, and by the Hahn-Banach theorem, it also belongs to the norm closure of this convex hull.

Therefore for $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exist c.p. maps $\sigma : A \longrightarrow F, \rho : F \longrightarrow A$ such that $\|\rho\sigma(a) - a\| < \varepsilon$ for all $a \in \mathfrak{F}$ where $\sigma = \bigoplus_{k=1}^r \psi_{n_k}$ and $\rho = \sum_{k=1}^r \lambda_k\varphi'_{n_k}$ for some finite set of indices $n_1, \dots, n_r$ and positives constants $\lambda_1, \dots, \lambda_r$ adding up to 1,

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\sigma = \bigoplus_{k=1}^{r} \psi_{n_k}"'] & & A. \\ & \bigoplus_{k=1}^{r} M_{n_k}(\mathbb{C}) \arrow[ur, "\rho = \sum_{k=1}^{r} \lambda_k \varphi'_{n_k}"'] & \end{tikzcd}$$

This shows A is nuclear. The other direction uses deep results about von Neumann algebras. It relies on the fact that injectivity implies hyperfiniteness (Theorem 4.2.7).

As an application of the last theorem, we obtain the next corollary which follows from the fact that $A^{**} = J^{**} \oplus (A/J)^{**}$ for any ideal J of A.

Corollary 4.3.3 ([11, Corollary 3.2.3])**.** *Let A be a C^*-algebra and let I be an ideal of A. Then A is nuclear if and only if I and A/I are nuclear.*

We finish this section with the following characterisation of nuclearity. In the proof of Theorem 4.3.2, we saw that if we assume semidiscreteness of the double dual we can construct c.p.c. approximations at the C^*-level which are convex combinations of another c.p.c. maps. The following theorem characterises nuclearity via approximations arising as convex combinations of order zero maps.

Theorem 4.3.4 ([49, Theorem 1.4])**.** *Let A be a nuclear C^*-algebra. Then for any finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exist a finite dimensional C^*-algebra F and c.p.c. maps $\psi : A \longrightarrow F, \varphi : F \longrightarrow A$ such that*

$$\|a - \varphi\psi(a)\| < \varepsilon$$

for all $a \in \mathfrak{F}$ and φ is a convex combination of finitely many order zero maps.

We also provide an sketch of its proof which shows how it fits in the same framework of Theorem 4.3.2. Let A be a nuclear C^*-algebra and we can assume A is separable. By Theorem 4.3.2, A^{**} is semidiscrete and, as a consequence of Theorem 4.2.7, we obtain that A^{**} is hyperfinite as well [94, Chapter XVI]. Using local reflexivity[1] of nuclear C^*-algebras [13, Corollary 9.4.1] and Arveson's extension theorem one can show that there exists a net of finite dimensional C^*-subalgebras $\{F_\lambda\}$ of A^{**} and a net of c.p.c. maps $\psi_\lambda : A^{**} \longrightarrow F_\lambda$

[1]Definition. A C^*-algebra A is locally reflexive if for any finite dimensional operator system $E \subset A^{**}$ there exists a net of c.p.c. maps $\psi_\lambda : E \longrightarrow A$ such that $\psi_\lambda(a) \to a$ ultraweakly for all $a \in A$.

such that $\psi_\lambda(x) \to x$ ultraweakly for all $x \in A^{**}$ [49, Lemma 1.2]. Let $\iota_\lambda : F_\lambda \longrightarrow A^{**}$ be the inclusion map. Then the net $(\iota_\lambda \psi_\lambda)$ converges to $\mathrm{id}_{A^{**}}$ in the point-ultraweak topology [49, Lemma 1.3],

$$\begin{tikzcd}
A^{**} \arrow[rr, "\mathrm{id}_{A^{**}}"] \arrow[dr, "\psi_\lambda"'] & & A^{**}. \\
& F_\lambda \arrow[ur, "\iota_\lambda"'] &
\end{tikzcd}$$

Using Kaplansky's density theorem and the fact that cones on finite dimensional C^*-algebras are projective[2] [62, Proposition 10.2.1], it follows that c.p.c. order zero maps $F \longrightarrow A^{**}$, with F finite dimensional, can be approximated with c.p.c. order zero maps $F \longrightarrow A$ in the strong*-topology [49, Lemma 1.1]. Hence, we can replace each ι_λ with a c.p.c. order zero map $\varphi_\lambda : F_\lambda \longrightarrow A$ in such a way that $\varphi_\lambda \psi_\lambda(a) \to a$ in the ultraweak topology for all $a \in A$,

$$\begin{tikzcd}
A \arrow[rrrr, "\mathrm{id}_A"] \arrow[dr, "\iota_A"] \arrow[ddrr, bend right, "\psi_\lambda|_A"'] & & & & A. \arrow[dl, "\iota_A"'] \\
& A^{**} \arrow[rr, "\mathrm{id}_{A^{**}}"] \arrow[dr, "\psi_\lambda"'] & & A^{**} & \\
& & F_\lambda \arrow[ur, "\iota_\lambda"'] \arrow[uurr, dashed, bend right, "\varphi_\lambda"'] & &
\end{tikzcd}$$

Again, by Theorem 4.1.13, the composition $\varphi_\lambda \psi_\lambda|_A$ converges to id_A in the weak topology $\sigma(A, A^*)$. Thus, using Hahn-Banach theorem as in the proof of Theorem 4.3.2, for each finite subset $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exists a finite dimensional C^*-algebra F and c.p.c. maps $\psi : A \longrightarrow F, \varphi : F \longrightarrow A$ such that

$$\|a - \varphi\psi(a)\| < \varepsilon$$

for all $a \in \mathfrak{F}$ and φ is a convex combination of finitely many order zero maps.

4.4 The hyperfinite factor, flips and strongly self-absorbing algebras

In this section we will review one particular automorphism of C^*-algebras called the flip. This map plays an important role in Connes' classification of injective factors. Motivated

[2] Definition. A C^*-algebra P is projective if for every pair of C^*-algebras B, C such that $\pi : B \longrightarrow C$ is a surjective *-homomorphism, and for each *-homomorphism $\varphi : P \longrightarrow C$, there is a *-homomorphism $\psi : P \longrightarrow B$ such that $\pi \circ \psi = \varphi$.

by Connes' work, Effros and Rosenberg studied flips in C^*-algebras and, many years later, flips have also become extremely useful in the computation of the nuclear dimension for an important class of C^*-algebras.

Definition 4.4.1. Let A be a C^*-algebra. The *flip* on A is the automorphism $\sigma_A : A \otimes_{\min} A \longrightarrow A \otimes_{\min} A$ given by

$$\sigma_A(a \otimes b) = b \otimes a, \qquad a, b \in A.$$

Let M be a von Neumann algebra. The *flip* on M is the automorphism $\sigma_M : M \mathbin{\overline{\otimes}} M \longrightarrow M \mathbin{\overline{\otimes}} M$ given by

$$\sigma_M(a \otimes b) = b \otimes a, \qquad a, b \in M.$$

4.4.1 Flips and II_1-factors

One of the main steps in Connes' proof of Theorem 4.2.7 is that the flip of an injective II_1-factor M is (strongly) approximately inner. This means that there exists a sequence of unitaries $(u_n) \subset M \mathbin{\overline{\otimes}} M$ such that

$$u_n \,(a \otimes b)\, u_n^* \longrightarrow b \otimes a, \qquad a, b \in M,$$

in the strong operator topology when $n \to \infty$. In fact, as a consequence of his work, Connes obtained that (strongly) approximately inner flips are equivalent to hyperfiniteness for factors.

Theorem 4.4.2 ([22, Theorem 5.1])**.** *Let M be a separable acting II_1-factor. Then M is isomorphic to the hyperfinite II_1-factor $\mathcal{R}$ if and only if the flip σ_M is (strongly) approximately inner.*

4.4.2 Flips and C^*-algebras

Subsequently, Effros and Rosenberg studied C^*-algebras A with an approximately inner flip. This means there exists a sequence of unitaries $(u_n) \subset \mathcal{M}(A \otimes_{\min} A)$ such that

$$u_n \,(a \otimes b)\, u_n^* \longrightarrow b \otimes a, \qquad a, b \in A,$$

in the norm topology when $n \to \infty$. In the separable setting, one can rephrase approximately inner flips in the language of ultraproducts by asking for a unitary $u \in \mathcal{M}(A \otimes_{\min} A)_\omega$ such that

$$b \otimes a = u(a \otimes b)u^*.$$

Effros and Rosenberg showed that having an approximately inner flip is a strong requirement for C^*-algebras.

Proposition 4.4.3 ([30, Propositions 2.7, 2.8, 2.10])**.** *Let A be a C^*-algebra with an approximately inner flip. Then A is simple, nuclear and admits at most one normalized trace.*

Moreover, Effros and Rosenberg proved that separable AF-algebras with an inner flip have to be inductive limits of matrix algebras by means of K-theory (if the algebra is additionally unital then it is UHF). In particular, this shows that approximately inner flips impose restrictions at the level of K-theory. Recently, Tikuisis determined exactly which classifiable C^*-algebras have an approximately inner flip in [95] by K-theoretical means.

Theorem 4.4.4 ([30, Theorem 3.9])**.** *Let A be a separable AF-algebra. Then the flip σ_A is approximately inner if and only if A is an inductive limit of matrix algebras.*

Following Connes' strategy in the proof of Theorem 4.2.7, Effros and Rosenberg obtained an analogous result for C^*-algebras. The main ingredients in Connes' proof are:

(i) M embeds in $\mathcal{R}^\omega$.

(ii) M tensorially absorbs the hyperfinite II$_1$-factor $\mathcal{R}$, *i.e* $M \cong M \mathbin{\overline{\otimes}} \mathcal{R}$.

(iii) M has (strongly) approximately inner flip.

Connes' theorem will be discussed in detail in Section 4.6. After stating the main ingredients we now state Effros and Rosenberg's theorem in which they replace the hyperfinite II$_1$-factor with the most natural candidate: the universal UHF algebra $\mathcal{Q}$.

Theorem 4.4.5 ([30, Theorem 5.1])**.** *Let A be a separable unital C^*-algebra. Then A is isomorphic to the universal UHF algebra $\mathcal{Q}$ if and only if*

(i) A embeds in $\mathcal{Q}_\omega$,

(ii) A tensorially absorbs the universal UHF algebra $\mathcal{Q}$, i.e. $A \cong A \otimes \mathcal{Q}$,

(iii) the flip of A is approximately inner.

As a consequence of the previous theorem, Effros and Rosenberg proved that if A is a separable unital C^*-algebra with approximately inner flip which can be embedded in $\mathcal{Q}_\omega$ then $A \otimes \mathcal{Q} \cong \mathcal{Q}$ [30, Corollary 5.2].

4.4.3 Strongly self-absorbing algebras

We have seen that the hyperfinite II_1-factor $\mathcal{R}$ has some striking properties. These properties were essential in Connes work. An important task for C^*-algebraist was to find the right replacement for $\mathcal{R}$ in the C^*-context. Probably, the most natural candidate among the strongly self-absorbing algebras to replace $\mathcal{R}$ is the universal UHF algebra $\mathcal{Q}$. This was explored by Effros and Rosenberg in [30] but this has disadvantages: for example $\mathcal{Q}$-stability is very difficult to achieve, not even the CAR-algebra is $\mathcal{Q}$-stable.

Before proceeding, let us recall and state some of the most important properties of the hyperfinite II_1-factor.

Theorem 4.4.6 ([22, Corollary 3.2, Theorem 5.1]). *Let $\mathcal{R}$ be the hyperfinite II_1-factor. Then*

(i) $\mathcal{R}$ is isomorphic to $\mathcal{R}\overline{\otimes}\mathcal{R}$.

(ii) The flip $\sigma_{\mathcal{R}}$ is (strongly) approximately inner.

(iii) Any automorphism of $\mathcal{R}$ is approximately inner.

It follows, after the work of Murray, von Neumann and Connes, that the hyperfinite II_1-factor is the unique infinite dimensional factor that can be embedded in all infinite dimensional factors.

Toms and Winter initiated the study of strongly self-absorbing C^*-algebras motivated by properties of the following examples: UHF algebras of infinite type, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ and the Jiang-Su algebra $\mathcal{Z}$. These algebras have properties that resemble those of the hyperfinite II_1-factor $\mathcal{R}$.

Definition 4.4.7 ([100, Definition 1.3.(iv)]). Let $\mathcal{D}$ be a separable unital C^*-algebra. $\mathcal{D}$ is *strongly self-absorbing* if $\mathcal{D} \ncong \mathbb{C}$ and there is an isomorphism $\varphi : \mathcal{D} \longrightarrow \mathcal{D} \otimes_{\min} \mathcal{D}$ which is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$.

By Theorem 1.7.4, it follows that strongly self-absorbing C^*-algebras are stably finite or purely infinite. In the stably finite case, by Theorem 1.6.2, there exists one trace and the proof of Theorem 4.4.3 shows this trace is unique.

Theorem 4.4.8 ([100, Theorem 1.7]). *A separable unital strongly self-absorbing C^*-algebra $\mathcal{D}$ is either purely infinite or stably finite with a unique trace.*

Let A be a separable unital C^*-algebra and let $A^{\otimes\infty}$ denote the inductive limit of the sequence

$$A \xrightarrow{\mathrm{id}_A \otimes 1_A} A^{\otimes 2} \xrightarrow{\mathrm{id}_{A^{\otimes 2}} \otimes 1_A} A^{\otimes 3} \xrightarrow{\mathrm{id}_{A^{\otimes 3}} \otimes 1_A} \cdots .$$

Corollary 4.4.9 ([100, Corollary 1.11])**.** *If $\mathcal{D}$ is separable, unital and strongly self-absorbing, then*

$$\mathcal{D} \cong \mathcal{D}^{\otimes k} \cong \mathcal{D}^{\otimes\infty}$$

for any $k \in \mathbb{N}$ and $\mathcal{D}$ has approximately inner flip.

This corollary, in light of Theorem 4.4.3, implies that strongly self-absorbing C^*-algebras are simple and nuclear.

Corollary 4.4.10 ([100, Corollary 1.12])**.** *Let A and $\mathcal{D}$ be separable unital C^*-algebras, with $\mathcal{D}$ strongly self-absorbing. Then, any two unital *-homomorphisms $\alpha, \beta : \mathcal{D} \longrightarrow A \otimes \mathcal{D}$ are approximately unitarily equivalent. In particular, any two unital endomorphisms of $\mathcal{D}$ are approximately unitarily equivalent.*

With the insight provided by all the information given abode, we can see that strongly self-absorbing C^*-algebras are suitable candidates to replace the hyperfinite II_1-factor at the C^*-level. But, before choosing the right option, we should know which C^*-algebras are strongly self-absorbing. Examples of strongly self-absorbing C^*-algebras are the UHF algebras of infinite type, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, tensor products products of $\mathcal{O}_\infty$ with UHF algebras of infinite type and the Jiang-Su algebra $\mathcal{Z}$ [100, Examples 1.14]. In fact, these are the only known examples and among the algebras satisfying the UCT these are the only ones [96, Corollary 6.7].

Theorem 4.4.11 ([100, Corollary 5.2],[96, Corollary 6.7])**.** *The class of strongly self-absorbing C^*-algebras in the UCT class consists of $\mathcal{O}_2$, $\mathcal{O}_\infty$, tensor products of $\mathcal{O}_\infty$ with UHF-algebras of infinite type, UHF algebras of infinite type and the Jiang-Su algebra $\mathcal{Z}$.*

From the previous theorem and Theorem 1.7.5, it follows that in the category of strongly self-absorbing C^*-algebras (with unital *-homomorphisms) the final object is the Cuntz algebra $\mathcal{O}_2$. The following theorem together with [100, Proposition 5.12] establishes that the initial object in this category is the Jiang-Su algebra $\mathcal{Z}$.

Theorem 4.4.12 ([108, Theorem 3.1])**.** *Any strongly self-absorbing C^*-algebra absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially.*

The list of known strongly self-absorbing C^*-algebras is represented in the following diagram (the arrows represent unital embeddings). It is still unknown if this list contains all the strongly self-absorbing C^*-algebras,

$$\begin{tikzcd}
 & \mathcal{O}_2 \\
 & \mathcal{O}_\infty \otimes \mathcal{Q} \arrow[u] \\
\mathcal{Q} \arrow[ur] & \\
 & \mathcal{O}_\infty \otimes UHF^\infty \arrow[uu] \\
UHF^\infty \arrow[uu] \arrow[ur] & \\
 & \mathcal{O}_\infty . \arrow[uu] \\
\mathcal{Z} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

4.5 McDuff factors and $\mathcal{Z}$-stability

At the beginning of the 70's, McDuff studied central sequence algebras of II_1-factors. She gave a characterisation of II_1-factors which tensorially absorb the hyperfinite II_1-factor $\mathcal{R}$. In Appendix A, we explain the construction of ultraproducts of II_1-factors (Definition A.1.2). We present a modern reformulation of McDuff's results.

Theorem 4.5.1 ([65])**.** *Let M be a separably acting II_1-factor. Then the relative commutant $M^\omega \cap M'$ is either of type II_1 or an abelian algebra. Moreover, the following statements are equivalent:*

(i) The relative commutant $M^\omega \cap M'$ is not abelian.

(ii) There exists an embedding $\mathcal{R} \hookrightarrow M^\omega \cap M'$.

(iii) The factor M absorbs $\mathcal{R}$ tensorially, i.e. $M \cong M \,\overline{\otimes}\, \mathcal{R}$.

(iv) For all $k \in \mathbb{N}$ there exists an embedding $M_k(\mathbb{C}) \hookrightarrow M^\omega \cap M'$.

Factors satisfying one of these properties (and hence all of them) have been named after McDuff.

Definition 4.5.2. A separably acting II_1-factor M is *McDuff* if it tensorially absorbs the hyperfinite II_1-factor $\mathcal{R}$, *i.e.* $M \cong M \,\overline{\otimes}\, \mathcal{R}$.

McDuff factors played an important role in Connes' proof of Theorem 4.2.7. Following the strategy of [84, Section 7.2], one can prove the following property of McDuff factors.

Proposition 4.5.3. *Let M be a separably acting McDuff factor. Then for every free ultrafilter on $\mathbb{N}$ there exist a sequence of *-homomorphisms $\varphi_n : M \overline{\otimes} \mathcal{R} \longrightarrow M$ such that*

$$\lim_{n\to\omega} \|\varphi_n(x \otimes 1_{\mathcal{R}}) - x\|_2 = 0$$

for all $x \in M$.

4.5.1 $\mathcal{Z}$-stability

We have seen that strongly self-absorbing C^*-algebras share many similarities with the hyperfinite II_1-factor $\mathcal{R}$, for example any unital endomorphisms and the flip are approximately inner. McDuff factors give more information about which properties the substitute of $\mathcal{R}$ should have: if it embeds in the central sequence algebra of C^*-algebra A then A must tensorially absorb it.

Before going further, let us state en equivalent characterisation for strongly self-absorbing algebras.

Theorem 4.5.4 ([100, Theorem 2.3])**.** *Let A and $\mathcal{D}$ be separable C^*-algebras. Suppose that $\mathcal{D}$ is unital and strongly self-absorbing and let ω be a free ultrafilter on $\mathbb{N}$. Then there is an isomorphism $\varphi : A \longrightarrow A \otimes \mathcal{D}$ if and only if there is a *-homomorphism*

$$\sigma : A \otimes \mathcal{D} \longrightarrow A_\omega \cap A'$$

satisfying

$$\sigma(a \otimes 1_{\mathcal{D}}) = a$$

for all $a \in A$, and in this case the maps φ and $\mathrm{id}_A \otimes 1_{\mathcal{D}}$ are approximately unitarily equivalent.

We remark that the original result has the extra hypothesis that the canonical map $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \longrightarrow K_1(\mathcal{D})$ is injective. However, by [108, Remark 3.3] this condition turns to be unnecessary. The alternative use of ultraproducts instead of sequence algebras is found in [54].

It should now be evident that the right candidate to replace $\mathcal{R}$ is in the category of strongly self-absorbing C^*-algebras and the most reasonable candidate is the minimal object in this category: the Jiang-Su algebra $\mathcal{Z}$.

The following theorem adds even more evidence to this choice. Before enunciating this theorem, we have to introduce a particular class of order zero maps. Let A be a unital C^*-algebra. A contractive order zero map $\varphi : M_k(\mathbb{C}) \longrightarrow A$ is *large*, if for some $v \in A$, the following conditions are satisfied:

$$v^*v = 1_A - \varphi\left(1_{M_k(\mathbb{C})}\right), \qquad \varphi\left(e_{1,1}\right) v = v.$$

The important feature of this type of maps is that if $\varphi : M_k(\mathbb{C}) \longrightarrow A$ is large, then $Z_{k,k+1} \cong C^*\left(\varphi\left(M_k(\mathbb{C})\right)\right)$ where $Z_{k,k+1}$ is a dimension drop algebra ([85, Proposition 5.1]).

Theorem 4.5.5 ([85, Proposition 5.1],[100, Theorem 2.2])**.** *Let A be a unital separable C^*-algebra. The following statements are equivalent:*

(i) The algebra A tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$, i.e. $A \cong A \otimes \mathcal{Z}$.

(ii) The Jiang-Su algebra $\mathcal{Z}$ embeds unitally in the relative commutant $A_\omega \cap A'$.

(iii) For all $k \in \mathbb{N}$, there exists a large order zero map $M_k(\mathbb{C}) \to A_\omega \cap A'$.

Toms and Winter showed (i) is equivalent to (ii) [100, Theorem 2.2]. As explained before, Rørdam and Winter showed (iii) is true if and only if for all $k \in \mathbb{N}$ there is a unital *-homomorphisms from the dimension drop algebra $Z_{k,k+1}$ to $A_\omega \cap A'$ [85, Proposition 5.1]. Hence, (iii) implies (i) (via a result of Toms and Winter [101, Proposition 2.2] and the uniqueness of $\mathcal{Z}$) and (ii) implies (iii).

Regarding the Jiang-Su algebra $\mathcal{Z}$ as the substitute of the hyperfinite II_1-factor $\mathcal{R}$ in the C^*-context, we view simple $\mathcal{Z}$-stable C^*-algebras as the C^*-analogue of McDuff factors.

4.6 Connes' proof and nuclear dimension

Throughout this chapter, we have been discussing aspects of the proof of Theorem 4.2.7. In this section we will sketch the last part of Connes' proof. As mentioned earlier, this proof relies on three fundamental facts which are deep and difficult to proof. We will omit their proofs but instead we will explain how using these facts it can be showed that an injective factor is hyperfinite.

4.6.1 Injectivity implies hyperfiniteness

In [22], Connes obtained a classification of injective factors except for the type III_1. Haagerup completed this classification in [46]. In particular we are interested in the following implication, which is probably the main and most challenging part of Connes' work.

Theorem 4.6.1 ([22, Theorem 6])**.** *Let M be an injective factor acting on a separable Hilbert space. Then M is hyperfinite.*

Alternative proofs were obtained by Haagerup and Popa years later [45, 77], but we will focus on Connes' proof in this thesis. Connes reduced the proof of this theorem to the case when M is a II_1-factor. In this situation, he proved the following fundamental and deep facts:

(i) There exists a unital embedding $\theta : M \hookrightarrow R^\omega$.

(ii) M is a McDuff factor.

(iii) M has (strongly) approximately inner flip.

We will show how we can use these facts to produce *-homomorphisms $\varphi_n : M \mathbin{\overline{\otimes}} \mathcal{R} \longrightarrow M$ satisfying that $\varphi_n(a \otimes 1_{\mathcal{R}})$ is close to a and with these homomorphisms we can "move" finite dimensional algebras from $\mathcal{R}$ to M in order to show the hyperfiniteness of M. We briefly sketch this last part of Connes' proof.

Sketch. Let $\mathfrak{F} \subset M$ be a finite subset and $\varepsilon > 0$. First of all, since M is finite, by Remark 4.2.2, in order to demonstrate hyperfiniteness we need to exhibit a finite dimensional subalgebra which is ε-close to $\mathfrak{F}$ in $\|\cdot\|_2$-norm. By Theorem 4.1.6 and Lemma 4.1.7, we can work in $\|\cdot\|_2$-norm instead of any other locally convex topology appearing in Theorem 4.1.6. Since the flip of M is (strongly) approximately inner, there exist a sequence $(u_n) \subset \mathcal{U}\,(M \mathbin{\overline{\otimes}} M)$ such that

$$\lim_{n\to w} \left\|u_n^*\,(a\otimes b)\,u_n - b\otimes a\right\|_2 = 0 \tag{4.2}$$

for all $a, b \in M$. For any $a \in M$, using the diagram below, we can represent $1_M \otimes \theta(a) \in$

$M \overline{\otimes} \mathcal{R}^\omega$ with a sequence $\left(1_M \otimes y_n^{(a)}\right)_{n\in\mathbb{N}} \subset \ell^\infty (M \overline{\otimes} \mathcal{R})$,

$$\begin{tikzcd} & & & \ell^\infty (M \overline{\otimes} \mathcal{R}) \arrow[d, two heads] \\ M \overline{\otimes} M \arrow[urrr, dashed] \arrow[r, "\mathrm{id}\otimes\theta"'] & M \overline{\otimes} \mathcal{R}^\omega \arrow[rr, hook] & & (M \overline{\otimes} \mathcal{R})^\omega . \end{tikzcd}$$

Let $w_n = (\mathrm{id}_M \otimes \theta)(u_n) \in (M \overline{\otimes} \mathcal{R})^\omega$ and define $\Psi_n : (M \overline{\otimes} \mathcal{R})^\omega \longrightarrow (M \overline{\otimes} \mathcal{R})^\omega$ by

$$\Psi_n(x) = w_n^* x w_n, \qquad x \in (M \overline{\otimes} \mathcal{R})^\omega . \tag{4.3}$$

Notice

$$\begin{aligned} \Psi_n(1_M \otimes a) &= w_n^*(1_M \otimes \theta(a))w_n \\ &= (\mathrm{id}_M \otimes \theta)(u_n)^*(1_M \otimes \theta(a))(\mathrm{id}_M \otimes \theta)(u_n) \\ &= (\mathrm{id}_M \otimes \theta)(u_n^* (1_M \otimes a) u_n) \end{aligned} \tag{4.4}$$

for all $a \in M$. Since $\lim_{n\to w} \|u_n^* (1_M \otimes a) u_n - a \otimes 1_M\|_2 = 0$ and θ is unital (and hence trace preserving), we have

$$\lim_{n\to w} \|\Psi_n(1_M \otimes a) - (\mathrm{id}_M \otimes \theta)(a \otimes 1_M)\|_2 = \lim_{n\to w} \|\Psi_n(1_M \otimes a) - a \otimes 1_\mathcal{R}\|_2 = 0 \tag{4.5}$$

for all $a \in M$. In particular there exists $U \in \omega$ such that

$$\|\Psi_n (1_M \otimes x) - x \otimes 1_M\|_2 < \frac{\varepsilon}{2} \tag{4.6}$$

for all $n \in U$ and $x \in \mathfrak{F}$.

Since M is McDuff, by Proposition 4.5.3, there exist *-homomorphisms $\varphi_n : M \overline{\otimes} \mathcal{R} \longrightarrow M$ such that

$$\lim_{n\to\omega} \|\varphi_n(a \otimes 1_\mathcal{R}) - a\|_2 = 0 \tag{4.7}$$

for all $a \in M$. Then, there exists $V \in \omega$ such that

$$\|\varphi_n (x \otimes 1_\mathcal{R}) - x\|_2 < \frac{\varepsilon}{2} \tag{4.8}$$

for $n \in V$ and all $x \in \mathfrak{F}$.

For $x \in \mathfrak{F}$, let $\left(1_M \otimes y_n^{(x)}\right)_{n\in\mathbb{N}} \subset \ell^\infty (M \overline{\otimes} \mathcal{R})$ represent $1_M \otimes \theta(x)$. Similarly, suppose each w_n is represented by the sequence $\left(1_M \otimes v_k^{(n)}\right)_{k\in\mathbb{N}} \subset \ell^\infty (M \overline{\otimes} \mathcal{R})$ where each $v_k^{(n)} \in \mathcal{R}$ is a unitary element. Define $\psi_k^{(n)} : M \overline{\otimes} \mathcal{R} \longrightarrow M \overline{\otimes} \mathcal{R}$ by

$$\psi_k^{(n)}(a \otimes b) = \left(1_M \otimes v_k^{(n)}\right)^* (a \otimes b) \left(1_M \otimes v_k^{(n)}\right), \qquad a \otimes b \in M \overline{\otimes} \mathcal{R}.$$

It follows from the definitions that the sequence of maps $\left(\psi_k^{(n)}\right)_{k\in\mathbb{N}}$ induces the map Ψ_n. Hence, for each $n \in V$, there exists $W_n \in \omega$ such that if $k \in W_n$ then

$$\left\|\psi_k^{(n)}\left(1_M \otimes y_k^{(x)}\right) - x \otimes 1_{\mathcal{R}}\right\|_2 < \frac{\varepsilon}{4} \tag{4.9}$$

for all $x \in \mathfrak{F}$.

Let us fix $N \in U \cap V$ and $K \in W_N$. Since $\mathcal{R}$ is hyperfinite, there exists a finite dimensional von Neumann algebra $F \subset \mathcal{R}$ such that dist $\left(1_M \otimes y_K^{(x)}, 1_M \otimes F\right) < \varepsilon/4$ for all $x \in \mathfrak{F}$ in $\|\cdot\|_2$-norm. In other words, for every $x \in \mathfrak{F}$, there exists $f_x \in F$ such that

$$\left\|1_M \otimes y_K^{(x)} - 1_M \otimes f_x\right\|_2 < \frac{\varepsilon}{4}. \tag{4.10}$$

Hence, by (4.9) and (4.10), we have

$$\begin{aligned}\left\|x \otimes 1_{\mathcal{R}} - \psi_K^{(N)}\left(1_M \otimes f_x\right)\right\|_2 &\leq \left\|x \otimes 1_{\mathcal{R}} - \psi_K^{(N)}\left(1_M \otimes y_K^{(x)}\right)\right\|_2 \\ &\quad + \left\|\psi_K^{(N)}\left(1_M \otimes y_K^{(x)}\right) - \psi_K^{(N)}\left(1_M \otimes f_x\right)\right\|_2 \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \qquad (4.11)\end{aligned}$$

for all $x \in \mathfrak{F}$.

Then, by (4.8) and (4.11), we obtain

$$\begin{aligned}\left\|x - \varphi_n\psi_N^{(n)}\left(1_M \otimes f_x\right)\right\|_2 &\leq \left\|x - \varphi_n\left(x \otimes 1_{\mathcal{R}}\right)\right\|_2 + \left\|\varphi_n\left(x \otimes 1_{\mathcal{R}}\right) - \varphi_n\psi_k^{(n)}\left(1_M \otimes f_x\right)\right\|_2 \\ &< \frac{\varepsilon}{2} + \left\|\varphi_n\left(x \otimes 1_{\mathcal{R}} - \psi_k^{(n)}\left(1_M \otimes f_x\right)\right)\right\|_2 \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad (4.12)\end{aligned}$$

for all $x \in \mathfrak{F}$. Since the image of $1_M \overline{\otimes} F$ under the *-homomorphisms $\varphi_n\psi_k^{(n)}$ is a finite dimensional von Neumann algebra, we conclude that M is hyperfinite. □

As explained before, Murray and von Neumann showed that there is a unique hyperfinite II_1-factor (up to isomorphism). Therefore we have the following theorem.

Theorem 4.6.2 ([22, Theorem 7.1])**.** *The hyperfinite II_1-factor $\mathcal{R}$ is the unique separably acting injective II_1-factor up to isomorphism.*

Let M be a factor. A *subfactor* of M is a factor which is contained in M. Due to the uniqueness of $\mathcal{R}$ as hyperfinite II_1-factor, we obtain that any hyperfinite subfactor of $\mathcal{R}$ is isomorphic to $\mathcal{R}$. The question about the existence of non finite dimensional subfactors of $\mathcal{R}$ which are isomorphic to $\mathcal{R}$ remained open until Connes' work in [22].

Corollary 4.6.3 ([22, Corollary 2]). *All subfactors of the hyperfinite* II_1*-factor* $\mathcal{R}$ *are isomorphic to* $\mathcal{R}$ *or finite dimensional.*

Effros and Lance showed that any semidiscrete von Neumann algebra is injective [29, Corollary 5.10]. Another consequence of Connes' work is the following remarkable theorem.

Theorem 4.6.4 ([13, Theorem 9.3.3]). *Let* M *be a separably acting factor. The following properties are equivalent.*

(i) M *is hyperfinite.*

(ii) M *is injective.*

(iii) M *is semidiscrete.*

Now we are ready to state the following important theorem which was originally conjectured and partially proved by Effros and Lance in [29, Theorem 6.4]. This theorem contains the work of many people, particularly Connes, Choi, Effros, Kirchberg and Lance ([20, 21, 22, 29, 53]).

Theorem 4.6.5 ([11, Theorem 3.2.2]). *Let* A *be a nuclear* C^**-algebra. The following statements are equivalent.*

(i) A *is nuclear.*

(ii) A *has the completely positive approximation property.*

(iii) A^{**} *is semidiscrete.*

(iv) A^{**} *is injective.*

4.6.2 $\mathcal{Z}$-stability implies finite nuclear dimension

In the introduction, we mentioned the Toms-Winter conjecture and its relationship with the classification programme. This conjecture predicts that all regularity properties (finite nuclear dimension, $\mathcal{Z}$-stability and strict comparison) are equivalent for some class of C^*-algebras. In this section we will explore in more detail the equivalence between finite nuclear dimension and $\mathcal{Z}$-stability. Winter proved that finite decomposition rank implies $\mathcal{Z}$-stability in [107], and some years later, he extended his result to finite nuclear dimension [109].

The converse has striking connections with Connes' proof of Theorem 4.2.7 for II_1-factors. Remember nuclearity can be viewed as injectivity of von Neumann algebras and $\mathcal{Z}$-stable C^*-algebras as the analogue of McDuff factors. Similarly, it follows from the definiton of nuclear dimension or decomposition rank (see Definition 2.2.1) that we can view them as C^*-analogues of hyperfiniteness (this will explore in more depth in Section 4.7). Following these analogies, Connes' proof provides a strategy to show that $\mathcal{Z}$-stable C^*-algebras have finite nuclear dimension. Roughly speaking, Connes' proof is explained in the following diagram,

$$\begin{tikzcd}
M \arrow[rrrr, hook] \arrow[dr, hook] & & & & M. \\
& 1_M \overline{\otimes} M \arrow[dr, "\text{“approx. embedding”}"'] & & M \overline{\otimes} \mathcal{R} \arrow[ur, "\cong"'] & \\
& & M \overline{\otimes} \mathcal{R} \arrow[ur, "\text{“flip”}"'] & &
\end{tikzcd}$$

A breakthrough was achieved by Matui and Sato in [64]. Using Connes' approach, Matui and Sato showed that the decomposition rank of separable unital simple nuclear quasidiagonal $\mathcal{Z}$-stable C^*-algebras with unique trace is at most three. But of course, not everything works exactly the same. The approximately inner flip is an essential tool in Connes' proof but, as we saw in Section 4.4.2, asking a C^*-algebra to have approximately inner flip is a very strong requirement. Using Haagerup's techniques from his proof of Connes' theorem [45, Theorem 4.2], Matui and Sato circumvented the use of approximately inner flips. Instead, they implicitly used what we now call “2-coloured approximately inner flip” which is essentially the sum of two order zero maps (we will explain these coloured ideas in the Section 4.7.2).

Let us briefly explain Matui and Sato's proof. Initially they proved that separable unital simple nuclear quasidiagonal C^*-algebras with unique trace which are $\mathcal{Q}$-stable have decomposition rank at most one. Matui and Sato's proof is based on Connes' after replacing $\mathcal{R}$ by $\mathcal{Q}$ with the very new ingredient of the “2-coloured approximately inner flip”. Let A be a C^*-algebra satisfying the previous hypothesis. Quasidiagonality of A yields an embedding $A \hookrightarrow \mathcal{Q}_\omega$ and the 2-coloured flip allows to “move” a finite dimensional C^*-subalgebra of $\mathcal{Q}$ to A via a c.p.c. map which is the sum of two order zero maps. The following diagram explains how this proof is similar to Connes' proof,

$$
\begin{tikzcd}
A \arrow[rrrr] \arrow[dr, hook] & & & & A. \\
& 1_A \otimes A \arrow[dr, "\text{“approx. embedding”}"'] & & A \otimes \mathcal{Q} \arrow[ur, "\cong"'] & \\
& & A \otimes \mathcal{Q} \arrow[ur, "\text{“2-flip”}"'] & &
\end{tikzcd}
$$

As explained before, from the "2-coloured approximately flip", they obtained two order zero maps which entails that decomposition rank of A is at most one.

In order to pass from $\mathcal{Q}$-stability to $\mathcal{Z}$-stability, Matui and Sato also produced two order zero maps $\Lambda_1, \Lambda_2 : \mathcal{Q} \longrightarrow \mathcal{Z}$ such that $\Lambda_1 (1_{\mathcal{Q}}) + \Lambda_2 (1_{\mathcal{Q}})$ is almost $1_{\mathcal{Z}}$. They basically finished their proof by composing the coloured flip, $\mathrm{id}_A \otimes (\Lambda_1 + \Lambda_1)$ and φ. In the end, with this composition they obtained four order zero maps which entails that decomposition rank of A is at most three. The following diagram summarizes their proof in this case,

$$
\begin{tikzcd}
A \arrow[rr] \arrow[d, hook] & & A \\
1_A \otimes A \arrow[d, hook] \arrow[rr, phantom, "\approx"] & & A \otimes \mathcal{Z} \arrow[u, "\cong"'] \\
A \otimes \mathcal{Q} \arrow[rr, "\text{“2-flip”}"'] & & A \otimes \mathcal{Q}. \arrow[u, "\mathrm{id}_A \otimes \Lambda_1 + \mathrm{id}_A \otimes \Lambda_2"']
\end{tikzcd}
$$

Sato, White and Winter extended Matui and Sato's result in [90] following again Connes' proof as a model. They showed that the nuclear dimension of separable unital simple nuclear $\mathcal{Z}$-stable C^*-algebras with unique trace is at most 3.

They consider two UHF-algebras of infinite type U and V and they proved $A \otimes V$ has 2-coloured approximately inner flip. They were able to remove quasidiagonality by constructing an order zero embedding $A \longrightarrow U_\omega$ and, as in Matui and Sato's proof, they produced two order zero maps $\Lambda_1, \Lambda_2 : U \otimes V \longrightarrow \mathcal{Z}$ such that $\Lambda_1 (1_U \otimes 1_V) + \Lambda_2 (1_U \otimes 1_V)$ is almost $1_{\mathcal{Z}}$. The following diagram summarizes their proof,

$$
\begin{tikzcd}
A \arrow[rr] \arrow[d, hook] & & A \\
1_A \otimes A \otimes 1_V \arrow[d, hook] \arrow[rr, phantom, "\approx"] & & A \otimes \mathcal{Z} \arrow[u, "\varphi"'] \\
A \otimes A \otimes V \arrow[r, "\text{“2-flip”}"'] & A \otimes A \otimes V \arrow[r, "\mathrm{id}_A \otimes \psi \otimes \mathrm{id}_V"'] & A \otimes U \otimes V. \arrow[u, "\mathrm{id}_A \otimes \Lambda_1 + \mathrm{id}_A \otimes \Lambda_2"']
\end{tikzcd}
$$

4.7 Colouring C^*-algebras

Motivated by the deep connections between von Neumann algebras and C^*-algebras, we will try to obtain C^*-analogues of von Neumann algebras results by adding "colours". Following the analogy introduced in nuclear dimension, we will refer to order zero maps as "colours".

We aim to replace von Neumann algebras statements involving projections and *-homomorphisms with statements using finite sums of positive elements and finite sum of order zero maps whose sum is unital.

The reasons behind this idea are the following: In contrast with von Neumann algebras, there exist projectionless C^*-algebras. Hence, the natural framework are positive elements rather than projections. By using *-homomorphisms, we have topological obstructions since these maps carry K-theoretical data. One example of this is Theorem 4.4.4. By Corollary 1.4.7, order zero maps $A \longrightarrow B$ correspond to *-homomorphisms $C_0(0,1]\otimes A \longrightarrow B$. Since the cone $C_0(0,1] \otimes A$ is contractive, the K-theory of $C_0(0,1] \otimes A$ is 0. Hence, order zero maps $A \longrightarrow B$ do not contain K-theoretical data and this justifies our choice.

This strategy is motivated by the work carried out in [8, 90]. Before proceeding, we provide some examples of this colouring.

4.7.1 Hyperfiniteness and nuclear dimension

Let M be a separable acting hyperfinite von Neumann algebra. Thus, there exists an increasing family of finite dimensional algebras $\{F_r\}_{r\in\mathbb{N}}$ such that $M = \overline{\bigcup_{r\in\mathbb{N}} F_r}^{SOT}$. Using Arveson's extension theorem, we can obtain c.p.c. maps $\psi_r : M \longrightarrow F_r$ such that $\psi_r|_{F_r} = \mathrm{id}_{F_r}$. Set $\iota_r : F_r \longrightarrow M$ as the inclusion map. By construction, we have the following approximately commuting diagram, which is an equivalent formulation of hyperfiniteness,

$$\begin{tikzcd} M \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_r"'] & & M \\ & F_r \arrow[ur, "\iota_r"'] & \end{tikzcd} .$$

We are going to "colour" this diagram. We will replace the von Neumann algebra M with a C^*-algebra A, the *-homomorphism ι_r with the sum $\varphi^{(r)} = \sum_{k=1}^{n} \varphi_k^{(r)}$, where each $\varphi_k^{(r)} : F_r \longrightarrow A$ is an order zero map. The n-coloured form of the previous diagram is the following

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_r"'] & & A. \\ & F \arrow[ur, "\varphi^{(r)}=\sum_{k=1}^{n}\varphi_k^{(r)}"'] & \end{tikzcd}$$

This diagram corresponds to nuclear dimension at most $n-1$ (Definition 2.2.1). Hence, nuclear dimension is a coloured form of hyperfiniteness.

4.7.2 Coloured flips

As mentioned before, the existence of approximately inner flips in C^*-algebras is restrictive since it implies nuclearity, simplicity and at most one trace. Moreover, they have K-theoretical obstructions (c.f. Theorem 4.4.4). In order to avoid such obstructions, we might consider coloured flips. This was done explicitly by Sato, White and Winter in [90]. They extracted this idea from [64, Theorem 4.2].

Definition 4.7.1 ([90])**.** Let A be a separable unital C^*-algebra. The flip of A is n*-coloured approximately inner* if there exist contractions $u_1, \ldots, u_n \in (A \otimes A)_\omega$ such that

(i) $b \otimes a = \sum_{k=1}^{n} u_k \, (a \otimes b) \, u_k^*$ for $a, b \in A$.

(ii) $\sum_{k=1}^{n} u_k^* u_k = 1_{A \otimes A}$.

(iii) the elements $u_k^* u_k \in (A \otimes A)_\omega$ commute with $A \otimes A$ for $k = 1, \ldots, n$.

Observe that the flip is expressed as sum of n order zero maps and such maps are given in a very precise form: $a \otimes b \mapsto u_k(a \otimes b)u_k^*$. These maps are order zero since $u_k^* u_k$ is in the relative commutant of $A \otimes A$ (see [8, Remark 6.3]). Although coloured flips avoid topological obstructions, they also impose strong conditions in algebras with this type of flips.

Proposition 4.7.2 ([90, Proposition 4.3])**.** *Let A be a separable unital C^*-algebra with n-coloured approximately inner flip. Then A is simple, nuclear and has at most one trace.*

Using the approach used in [64], Sato, White and Winter demonstrated the existence of 2-coloured approximately inner flips in some particular classes of C^*-algebras.

Lemma 4.7.3 ([90, Lemma 4.2])**.** *Let A be simple, separable, unital and nuclear C^*-algebra with a unique trace which absorbs a UHF-algebra of infinite type. Then the flip of A is* 2*-coloured approximately inner.*

4.8 Coloured maps

In [90], with the help of coloured flips, the authors showed $\mathcal{Z}$-stability implies finite nuclear dimension in the simple, separable, unital, nuclear setting. This implication forms part of the Toms-Winter conjecture.

Theorem 4.8.1 ([90, Theorem B])**.** *Let A be a simple, separable, unital, nuclear and $\mathcal{Z}$-stable C^*-algebra with unique trace. Then*

$$\dim_{\mathrm{nuc}} A \leq 3.$$

However, if we want to extend this result outside the monotracial case a replacement for the coloured flip is needed (since it implies the algebra has at most one trace). The task of finding a replacement was carried out in [8]. The authors of [8] generalised the previous theorem to the case where the trace simplex $T(A)$ is Bauer, *i.e.* the extreme boundary $\partial_e T(A)$ is closed.

Theorem 4.8.2 ([8, Theorem 7.5])**.** *Let A be a simple, separable, unital, nuclear and $\mathcal{Z}$-stable C^*-algebra such that $T(A)$ is a Bauer simplex. Then*

$$\dim_{\mathrm{nuc}} A \leq 1.$$

Moreover, if all traces are quasidiagonal then

$$\mathrm{dr}\, A \leq 1.$$

While proving this theorem, the authors of [8] generalised coloured flips to coloured equivalence of unital maps in the following way.

Definition 4.8.3 ([8, Definition 6.1])**.** Let A and B be unital C^*-algebras, and let $\varphi_1, \varphi_2 : A \longrightarrow B$ be unital *-homomorphisms, and $n \in \mathbb{N}$. The maps φ_1 and φ_2 are *approximately n-coloured equivalent* if there exist $u_1, \ldots, u_n \in B_\omega$ such that

$$\varphi_1(a) = \sum_{k=1}^{n} u_k \varphi_2(a) u_k^*, \tag{4.13}$$

$$\varphi_2(a) = \sum_{k=1}^{n} u_k^* \varphi_1(a) u_k, \tag{4.14}$$

for all $a \in A$ and the element $u_k^* u_k \in B_\omega$ commutes with $\varphi_2(A)$ and $u_k u_k^* \in B_\omega$ commutes with $\varphi_1(A)$ for all $k = 1, \ldots, n$.

Remark 4.8.4. As a consequence of the definition, we have

$$\sum_{k=1}^{n} u_k u_k^* = \sum_{k=1}^{n} u_k \varphi_2(1_A) u_k^* = \varphi_1(1_A) = 1_B = \varphi_2(1_A) = \sum_{k=1}^{n} u_k^* u_k. \tag{4.15}$$

It follows from the definition and the previous identity that approximately n-coloured equivalent unital *-homomorphisms agree on traces. Indeed, suppose φ_1 and φ_2 are approximately n-coloured equivalent and let τ be a trace on B. Then

$$\begin{aligned} \tau \circ \varphi_1(a) &= \tau \left(\sum_{k=1}^{n} u_k \varphi_2(a) u_k^* \right) \\ &= \tau \left(\varphi_2(a) \left(\sum_{k=1}^{n} u_k^* u_k \right) \right) \\ &= \tau \circ \varphi_2(a). \end{aligned} \tag{4.16}$$

for all $a \in A_+$.

Notice that 1-coloured equivalence of maps is nothing more than unitary equivalence of maps. Thus coloured equivalence is nothing more than a coloured form of unitary equivalence in the sense of Section 4.7.

In [8], apart from their nuclear dimension estimates, the authors obtained a coloured classification of certain classes of C^*-algebras. One of the key lemmas is a generalization of Connes' 2×2 trick, which we will need in Section 5.6.

Let A and B be C^*-algebras, an *-homomorphisms $\pi : A \longrightarrow B$ is *totally full* if if for every non zero element $a \in A$, $\pi(a)$ is *full* (*i.e.* $\pi(a)$ generates B as a closed two-sided ideal). Likewise, a positive element $b \in B_+$ is *totally full* if $b \neq 0$ and the *-homomorphism $C_0((0, \|b\|]) \longrightarrow B$ given by $\mathrm{id}_{(0,\|b\|]} \mapsto b$ is totally full.

Lemma 4.8.5 ([8, Lemma 2.3])**.** *Let A and B be separable C^*-algebras with A unital and let $\varphi_1, \varphi_2 : A \longrightarrow B_\omega$ be order zero maps and $\hat{\varphi}_1, \hat{\varphi}_2 : A \longrightarrow B_\omega$ be a supporting order zero maps (see Proposition 1.4.11). Suppose that either:*

(i) B has stable rank one; or

(ii) B is a Kirchberg algebra, and $\varphi_i(1_A)$ is totally full in B_ω for $i = 1, 2$.

Let $\pi : A \longrightarrow M_2(B_\omega)$ be given by

$$\pi(a) = \begin{pmatrix} \hat{\varphi}_1(a) & 0 \\ 0 & \hat{\varphi}_2(a) \end{pmatrix}. \tag{4.17}$$

If

$$\begin{pmatrix} \varphi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & \varphi_2(1_A) \end{pmatrix} \tag{4.18}$$

are unitarily equivalent in the unitisation of $C := M_2(B) \cap \pi(A)' \cap \left\{1_{M_2(B_\omega)} - \pi\left(1_A\right)\right\}^{\perp}$, *then* φ_1 *and* φ_2 *are unitarily equivalent.*

Now we will state some theorems about coloured equivalent maps. We will work separately two different settings: the finite case and the purely infinite case. The results are similar but the process of achieving them are different (even though similar in spirit). In the finite case, traces will play a prominent role while in the purely infinite setting there are no traces.

4.8.1 Finite case

The following theorem is a key step in the nuclear dimension estimate of Theorem 4.8.2 and the coloured classification of finite algebras by traces (Theorem 4.8.7).

Theorem 4.8.6 ([8, Theorem 5.5])**.** *Let* A *be a separable, unital and nuclear* C^**-algebra, and let* B *be a simple, separable, unital, finite and exact* C^**-algebra such that* B *is* $\mathcal{Z}$*-stable and* $T(B)$ *is a Bauer simplex. Let* $\varphi_1 : A \longrightarrow B_\omega$ *be a totally full* **-homomorphism and let* $\varphi_2 : A \longrightarrow B_\omega$ *be a c.p.c. order zero map such that*

$$\tau \circ \varphi_1 = \tau \circ \varphi_2^m \tag{4.19}$$

for all $\tau \in T(B_\omega)$ *and all* $m \in \mathbb{N}$*, where order zero functional calculus is used to interpret* φ_2^m*. Let* $k \in \mathcal{Z}_+$ *be a positive contraction with spectrum* $[0,1]$ *and set* $\psi_i := \varphi_i(\cdot) \otimes k : A \longrightarrow (B \otimes \mathcal{Z})_\omega$ *for* $i = 1, 2$*. Then* ψ_1 *is unitarily equivalent to* ψ_2 *in* $(B \otimes \mathcal{Z})_\omega$*.*

The following theorem states that two unital *-homomorphisms are n-coloured equivalent if and only if these maps carry the same tracial data. Moreover, we can always take $n = 2$.

Theorem 4.8.7 ([8, Corollary 6.5])**.** *Let* A *be a separable, unital and nuclear* C^**-algebra, and let* B *be a simple, separable, unital, finite and exact* C^**-algebra such that* B *is* $\mathcal{Z}$*-stable and* $T(B)$ *is a Bauer simplex and nonempty. Let* $\varphi_1, \varphi_2 : A \longrightarrow B_\omega$ *be unital* **-homomorphisms such that* φ_1 *is injective. The following statements are equivalent.*

(i) $\tau \circ \varphi_1 = \tau \circ \varphi_2$ *for all* $\tau \in T(B)$*.*

(ii) φ_1 *and* φ_2 *are approximately* n*-coloured equivalent for some* $n \in \mathbb{N}$.

(iii) φ_1 *and* φ_2 *are approximately* 2*-coloured equivalent.*

We briefly explain the idea of the proof. It is clear that (iii) and (ii) imply (i). Similarly, (iii) implies (ii). To complete the proof, it is enough to prove (i) implies (iii).

After some reductions, we can identity φ_i with $\varphi_i \otimes 1_{\mathcal{Z}}$ for $i = 1, 2$. Using a positive element $k \in \mathcal{Z}$ with spectrum $[0, 1]$, we can split $\varphi_1(a) \otimes 1_{\mathcal{Z}}$ as the sum of two order zero maps. Precisely

$$\varphi_1(a) \otimes 1_{\mathcal{Z}} = \varphi_1(a) \otimes k + \varphi_1(a) \otimes (1 - k), \qquad a \in A.$$

Now, we can use Theorem 4.8.6 to obtain a unitary w_1 implementing the unitary equivalence between $\varphi_1(\cdot) \otimes k$ and $\varphi_2(\cdot) \otimes k$. We can repeat the process to obtain another unitary w_2 if we replace k with $1 - k$. Then we have the following

$$\begin{aligned} \varphi_1(a) \otimes 1_{\mathcal{Z}} &= \varphi_1(a) \otimes k + \varphi_1(a) \otimes (1 - k) \\ &= w_1 \left(\varphi_2(a) \otimes k\right) w_1^* + w_2 \left(\varphi_2(a) \otimes (1 - k)\right) w_2^* \\ &= w_1 \left(1_A \otimes k^{1/2}\right) \left(\varphi_2(a) \otimes 1_{\mathcal{Z}}\right) \left(1_A \otimes k^{1/2}\right) w_1^* \\ &\quad + w_2 \left(1_A \otimes (1 - k)^{1/2}\right) \left(\varphi_2(a) \otimes 1_{\mathcal{Z}}\right) \left(1_A \otimes (1 - k)^{1/2}\right) w_2^* \end{aligned} \tag{4.20}$$

for all $a \in A$. Setting $u_1 := w_1 \left(1_A \otimes k^{1/2}\right)$ and $w_2 := w_2 \left(1_A \otimes (1 - k)^{1/2}\right)$, we obtain

$$\begin{aligned} \varphi_1(a) \otimes 1_{\mathcal{Z}} &= u_1 \left(\varphi_2(a) \otimes 1_{\mathcal{Z}}\right) u_1^* + u_2 \left(\varphi_2(a) \otimes 1_{\mathcal{Z}}\right) u_2^*, \\ \varphi_2(a) \otimes 1_{\mathcal{Z}} &= u_1^* \left(\varphi_1(a) \otimes 1_{\mathcal{Z}}\right) u_1 + u_2^* \left(\varphi_1(a) \otimes 1_{\mathcal{Z}}\right) u_2, \end{aligned} \tag{4.21}$$

for all $a \in A$.

Finally, notice $u_1^* u_1 = u_1 u_1^* = 1_A \otimes k$ and $u_2^* u_2 = u_2 u_2^* = 1_A \otimes (1 - k)$. This establishes the approximately 2-coloured equivalence between φ_1 and φ_2.

We can also work with one order zero map in the previous theorem. However the decomposition is not symmetric and more technical conditions are required on the maps. This will be highly relevant for the coloured isomorphisms that will be introduced in Chapter 5.

Theorem 4.8.8 ([8, Theorem 6.6])**.** *Let* A *be a separable, unital and nuclear* C^**-algebra, and let* B *be a simple, separable, unital, finite, exact* $\mathcal{Z}$*-stable* C^**-algebra such that the extreme boundary* $\partial_e T(B)$ *is closed and non empty. Let* $\varphi_1 : A \longrightarrow B_\omega$ *be a totally full*

*-homomorphism and $\varphi_2 : A \longrightarrow B_\omega$ a c.p.c. order zero map with $\tau \circ \varphi_1 = \tau \circ \varphi_2^m$ for all $\tau \in T(B_\omega)$ and all $m \in \mathbb{N}$. Then, there exist contractions $u_1, u_2, v_1, v_2 \in B_\omega$ satisfying

$$\varphi_1(a) = u_1\varphi_2(a)u_1^* + u_2\varphi_2(a)u_2^*, \tag{4.22}$$

$$\varphi_2(a) = v_1\varphi_1(a)v_1^* + v_2\varphi_1(a)v_2^*, \tag{4.23}$$

for all $a \in A$, with $u_1^*u_1, u_2^*u_2 \in B_\omega$ commuting with $\varphi_2(A)$ and $v_1^*v_1, v_2^*v_2 \in B_\omega$ commuting with $\varphi_1(A)$, and

$$u_1^*u_1 + u_2^*u_2 = v_1^*v_1 + v_2^*v_2 = 1_{B_\omega}. \tag{4.24}$$

4.8.2 Kirchberg case

The previous results about finite C^*-algebras remain true for Kirchberg algebras. In this setting, we do not have traces but the following uniqueness theorem for order zero maps into the ultraproducts of Kirchberg algebras will play the role of Theorem 4.8.6.

Theorem 4.8.9 ([8, Theorem 9.1])**.** *Let A be a separable, unital, nuclear C^*-algebra, and let B be a unital Kirchberg algebra. Let $\varphi_1, \varphi_2 : A \longrightarrow B_\omega$ be c.p.c. order zero maps such that $f(\varphi_i)$ is injective for every non zero $f \in C_0(0,1]_+$ for $i = 1, 2$. Then φ_1 and φ_2 are unitarily equivalent.*

Similarly, the following theorem is a purely infinite version of Theorem 4.8.7. It is important to notice that this theorem does not require the use of the UCT and it applies to any pair of injective *-homomorphisms.

Theorem 4.8.10 ([8, Corollary 9.11])**.** *Let A be a separable, unital, nuclear C^*-algebra, and let B be a Kirchberg algebra. Let $\varphi_1, \varphi_2 : A \longrightarrow B_\omega$ be a pair of c.p.c. order zero maps such that $(\varphi_i - t)_+(a)$ is non zero for all $0 \leq t < 1$, non zero $a \in A$ and $i = 1, 2$. Then there exist contractions $u_1, u_2 \in B_\omega$ such that*

$$\varphi_1(a) = u_1\varphi_2(a)u_1^* + u_2\varphi_2(a)u_2^* \tag{4.25}$$

for all $a \in A$ and

$$u_1^*u_1 + u_2^*u_2 = 1_{B_\omega}, \tag{4.26}$$

and $u_1^*u_1, u_2^*u_2 \in B_\omega$ commute with $\varphi_2(A)$.

If moreover both φ_1 and φ_2 are *-homomorphisms, then there exist $w_1, w_2 \in B_\omega$ such that

$$\varphi_1(a) = w_1\varphi_2(a)w_1^* + w_2\varphi_2(a)w_2^*, \tag{4.27}$$

$$\varphi_2(a) = w_1^*\varphi_1(a)w_1 + w_2^*\varphi_1(a)w_2, \tag{4.28}$$

and such that $w_i^*w_i$ commutes with $\varphi_2(A)$ and $w_iw_i^*$ commutes with $\varphi_1(A)$. In the case that φ_1 and φ_2 are unital *-homomorphisms, this says that φ_1 and φ_2 are 2-coloured equivalent in the sense of Definition 4.8.3, and in this case, w_i can be chosen to be normal.

Chapter 5

Coloured isomorphism between C^*-algebras

The aim of this chapter is to introduce a notion of coloured isomorphism for separable unital C^*-algebras. This concept is motivated by the coloured equivalence of maps introduced in [8]. Our goal is to design a sequence of relations $(R_n)_{n\in\mathbb{N}}$ for separable C^*-algebras, which we will call n-coloured isomorphisms, in such a way that coloured isomorphic algebras must share structural properties but at the same time these relations must be sufficiently mild so that we can avoid topological obstructions. In particular, 1-coloured isomorphic algebras must be isomorphic.

We also expect the family of relations $(R_n)_{n\in\mathbb{N}}$ to satisfy the following *coloured transitive identity*, for at least some class of separable unital C^*-algebras,

$$xR_my \text{ and } yR_nz \implies xR_{f(n,m)}z, \quad x,y,z\in X,\ m,n\in\mathbb{N} \tag{5.1}$$

for some transition function $f:\mathbb{N}\times\mathbb{N}\longrightarrow\mathbb{N}$. The most natural (and expected) candidate for such transition function is $f(n,m)=nm$. All the results from this section are part of a joint work with A. Tikuisis and S. White that will be published in [17].

5.1 Idea

In this section we will explain the ideas that will lead to the definition of coloured isomorphisms. Consider two separable unital C^*-algebras. One way to show that A is isomorphic to B is by constructing *-homomorphisms $\varphi:A\longrightarrow B$ and $\psi:B\longrightarrow A$ such that $\varphi\circ\psi$ is approximately unitarily equivalent to id_A and $\psi\circ\varphi$ is approximately unitarily equivalent to id_B. This is essentially one particular case of the so-called Elliott's approximate intertwinings [33]. In the following diagram, the triangles approximately commute up to unitary equivalence

$$\begin{tikzcd}
& A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi"] & & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi"] & & A \arrow[r] & \cdots \arrow[r] & A \arrow[d, dashed, shift left] \arrow[d, phantom, shift left=4, "\cong"] \\
B \arrow[ur, "\psi"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur, "\psi"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur, "\psi"] \arrow[rr] & & \cdots \arrow[r] & B. \arrow[u, dashed, shift left]
\end{tikzcd}$$

Theorem 5.1.1 ([84, Corollary 2.3.4])**.** *Let A and B be separable C^*-algebras, and suppose that there are *-homomorphisms $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ such that $\psi\varphi$ is approximately unitarily equivalent to id_A and $\varphi\psi$ is approximately unitarily equivalent to id_B. Then A is isomorphic to B and there are *-isomorphisms $\rho : A \longrightarrow B$ and $\sigma : B \longrightarrow A$ with $\sigma = \rho^{-1}$ satisfying that ρ and σ are approximately unitarily equivalent to φ and ψ respectively.*

We aim to develop a coloured form of this fact in the sense of Section 4.7 by making use of coloured equivalent maps. This in particular explains why this concept mostly applies to separable C^*-algebras since Elliott's intertwinings only apply to this class.

Roughly speaking, the idea is to construct two maps, $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$, such that $\psi \circ \varphi$ is approximately n-coloured equivalent to id_A and $\varphi \circ \psi$ is approximately n-coloured equivalent to id_B,

$$A \xrightarrow{\ \varphi\ } B \xrightarrow{\ \psi\ } A.$$

However, this clean approach has some flaws. Let us illustrate them with one example.

Consider the CAR-algebra M_{2^∞} and the Jiang-Su algebra $\mathcal{Z}$. First of all, notice M_{2^∞} has many projections whereas $\mathcal{Z}$ is projectionless. Hence, there are no non trivial *-homomorphisms between M_{2^∞} and $\mathcal{Z}$. However, there do exist order zero maps between M_{2^∞} and $\mathcal{Z}$ (see Example 5.6.3). This suggests we might consider order zero maps instead of *-homomorphisms in our definition of coloured isomorphisms as we do in other definitions of coloured properties. In the same way, we could try to use order zero maps instead of *-homomorphisms in Definition 4.8.3 to define coloured equivalent order zero maps. It is important to observe that, like with *-homomorphisms, coloured equivalent order zero maps (in the sense of Definition 4.8.3) must agree on traces. Another important reason why we would like to use order zero maps is that, since they are implemented by *-homomorphisms from the cone of the domain to the codomain algebra and cones are contractible, they do not transfer any K-theoretical data. This helps us to avoid topological obstructions.

Let $\varphi : M_{2^\infty} \longrightarrow \mathcal{Z}$ be a c.p.c. order zero map. Since there is a unique trace $\tau_{M_{2^\infty}}$ in M_{2^∞} and $\tau_{\mathcal{Z}} \circ \varphi$ is a positive tracial functional on M_{2^∞} (by Corollary 1.4.10), we have

$$\tau_{\mathcal{Z}} \circ \varphi = \lambda\, \tau_{M_{2^\infty}} \tag{5.2}$$

for some scalar $0 < \lambda \leq 1$. Observe $\varphi(1_A) \neq 1_{\mathcal{Z}}$ (otherwise φ would be a *-homomorphism by Theorem 1.4.6), hence

$$\lambda = \tau_{\mathcal{Z}} \circ \varphi\, (1_A) < 1. \tag{5.3}$$

This shows any c.p.c. order zero map $\varphi : M_{2^\infty} \longrightarrow \mathcal{Z}$ reduces the trace, *i.e.*

$$\tau_{\mathcal{Z}} \circ \varphi(a) = \lambda \tau_{M_{2^\infty}}(a) < \tau_{M_{2^\infty}}(a) \tag{5.4}$$

for all $a \in A$. In particular, it is not possible to find c.p.c. order zero maps $\psi : \mathcal{Z} \longrightarrow M_{2^\infty}$ and $\varphi : M_{2^\infty} \longrightarrow \mathcal{Z}$,

$$\begin{tikzcd} \mathcal{Z} \arrow[rr, "\mathrm{id}_{\mathcal{Z}}"] \arrow[dr, "\psi"'] & & \mathcal{Z} \\ & M_{2^\infty} \arrow[ur, "\varphi"'] & \end{tikzcd}$$

such that the identity map $\mathrm{id}_{\mathcal{Z}}$ and $\varphi \circ \psi$ agree on the trace $\tau_{\mathcal{Z}}$; therefore, $\mathrm{id}_{\mathcal{Z}}$ and $\varphi \circ \psi$ cannot be approximately coloured equivalent. Hence, the CAR algebra M_{2^∞} and the Jiang-Su algebra $\mathcal{Z}$ would not be coloured isomorphic if we require coloured equivalent maps. This suggests we could try to slightly modify the notion of coloured equivalence of maps in order to avoid trace preserving maps in such a way that we still have some control on the traces.

Summarising, in order to avoid these flaws, we might define n-coloured isomorphism between A and B by asking for c.p.c. order zero maps $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ which are close to being n-coloured equivalent but do not necessarily preserve traces. We will provide precise definitions in Section 5.3 which remove these difficulties.

In particular, since this notion is a coloured form of [84, Corollary 2.3.4], this definition should satisfy that 1-coloured isomorphic C^*-algebras are isomorphic. In general, we are trying to design mild relations which allow to transfer some structural properties while avoiding K-theoretic obstructions.

5.2 Coloured maps revisited

Suppose we have two coloured approximately equivalent unital *-homomorphisms $\rho_1, \rho_2 : A \longrightarrow B$. Then there exist $w_1, \dots w_n \in B_\omega$ such that

$$\rho_1(a) = \sum_{k=1}^{n} w_k \rho_2(a) w_k^*, \qquad \rho_2(a) = \sum_{k=1}^{n} w_k^* \rho_2(a) w_k \tag{5.5}$$

for all $a \in A$, $w_k^* w_k \in B_\omega \cap \rho_2(A)'$ and $w_k w_k^* \in B_\omega \cap \rho_1(A)'$ for $k = 1, \dots, n$.

As discussed before, coloured equivalent unital *-homomorphisms agree on traces (see Remark 4.8.4). This is true because we have the following identity

$$\sum_{k=1}^{n} w_k^* w_k = \sum_{k=1}^{n} w_k w_k^* = 1_B. \tag{5.6}$$

Observe this identity is obtained from (5.5) and the fact that ρ_1 and ρ_2 are unital.

This suggests one simple way to extend the definition of coloured equivalence of maps to non unital *-homomorphisms: by asking for w_k's which satisfy equation (5.6). Precisely, the *-homomorphisms $\rho_1, \rho_2 : A \longrightarrow B$ are *n-coloured approximately equivalent* if there exist $w_1, \dots, w_n \in B_\omega$ such that

$$\rho_1(a) = \sum_{k=1}^{n} w_k \rho_2(a) w_k^*, \qquad \rho_2(a) = \sum_{k=1}^{n} w_k^* \rho_2(a) w_k \tag{5.7}$$

for all $a \in A$,

$$\sum_{k=1}^{n} w_k^* w_k = \sum_{k=1}^{n} w_k w_k^* = 1_B, \tag{5.8}$$

and the elements $u_k^* u_k$ commute with $\varphi_2(A)$ and $v_k^* v_k$ commutes with $\varphi_1(A)$ for $k = 1, \dots n$.

Like in the unital case, coloured equivalent maps agree on traces. We aim to slightly weaken this condition and we can do this by adding a constant λ in (5.8) such that

$$\sum_{k=1}^{n} w_k^* w_k = \lambda 1_B, \qquad \sum_{k=1}^{n} w_k w_k^* = \lambda^{-1} 1_B. \tag{5.9}$$

This condition allows us to have coloured equivalent maps which do not necessarily agree on traces but we can still control their tracial data. Notice this last condition implies

$$\tau \circ \rho_1(a) = \lambda \tau \circ \rho_2(a) \tag{5.10}$$

for all $a \in A$ and $\tau \in T(A)$.

Our last goal in this section is to extend the definition of coloured equivalence to order zero maps between unital C^*-algebras. In this situation, Theorems 4.8.8 and 4.8.10 are indicating a viable option with the downside that the decomposition is not symmetric any more. Additionally, if we want to avoid agreement of coloured equivalent maps on traces we might add some constants λ_1 and λ_2 in the same way we did in (5.9). Precisely, one possible definition is the following:

Definition 5.2.1 (Alternative definition of coloured equivalence of maps)**.** Let A and B be unital separable C^*-algebras. The c.p.c. order zero maps $\varphi_1, \varphi_2 : A \longrightarrow B$ are *n-coloured equivalent* if there exist $u_1, \ldots, u_n, v_1, \ldots, v_n \in B_\omega$ and a constant $\lambda \geq 1$ such that

$$\varphi_1(a) = \sum_{k=1}^{n} u_k \varphi_2(a) u_k^*, \qquad \varphi_2(a) = \sum_{k=1}^{n} v_k \varphi_1(a) v_k^*, \tag{5.11}$$

$$\sum_{k=1}^{n} u_k^* u_k = \lambda 1_B, \qquad \sum_{k=1}^{n} v_k^* v_k = \lambda^{-1} 1_B, \tag{5.12}$$

the elements $u_k^* u_k$ commute with $\varphi_2(A)$ and $v_k^* v_k$ commutes with $\varphi_1(A)$ for $k = 1, \ldots n$.

With this definition, Theorems 4.8.8 and 4.8.10 provide conditions to obtain 2-coloured equivalent maps in the sense of Definition 5.2.1 when the codomain is a Kirchberg algebra or a finite algebra satisfying the hypotheses of Theorem 4.8.8.

5.3 Coloured isomorphisms

We have discussed some disadvantages of the original form of coloured equivalence of unital *-homomorphisms when used to define coloured isomorphisms of C^*-algebras and we have also discussed some ways to amend them. Now, the next natural step is to define coloured isomorphisms between unital separable C^*-algebras. After all of our previous discussion, we might expect to use the alternative definition of coloured equivalence of maps (Definition 5.2.1).

As explained before, our original goal was to define coloured isomorphisms between A and B by asking c.p.c. order zero maps $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are close to be approximately n-coloured equivalent to id_A and id_B. This basically means we can express id_A as the sum of n order zero maps of the form $u_i \psi \circ \varphi(\cdot) u_i^*$ and $\psi \circ \varphi$ as the sum of n order zero maps of the form $v_i \mathrm{id}_A(\cdot) v_i^*$. After studying this potential definition, we realised it is enough to express id_A as the sum of n order zero maps rather than id_A and $\psi \circ \varphi$.

Definition 5.3.1. Let A and B be unital C^*-algebras. We say A is *n-coloured isomorphic* to B, denoted as $A \cong_{(n)} B$, if there exist c.p.c. order zero maps $\varphi : A \longrightarrow B$, $\psi : B \longrightarrow A$ and unitaries $u_1, \dots, u_n \in A_\omega$, $v_1, \dots, v_n \in B_\omega$ such that

(i) $a = \sum\limits_{k=1}^{n} u_k \psi\varphi(a) u_k^*$ for all $a \in A$,

(ii) $b = \sum\limits_{k=1}^{n} v_k \varphi\psi(b) v_k^*$ for all $b \in B$.

Remark 5.3.2. It immediately follows from the definition that the c.p.c. order zero maps φ and ψ are injective. Indeed, for example suppose $\varphi(a) = 0$. Then

$$\sum_{k=1}^{n} u_k \psi\varphi(a) u_k^* = 0 \tag{5.13}$$

and we obtain $a = 0$.

We have also omitted the condition of the sum of $u_k^* u_k$ and $v_k^* v_k$ being scalar multiples of 1_A and 1_B, respectively. Instead, we ask the u_k's and v_k's to be unitaries. As a consequence of this, the commutation relations are automatically satisfied and this is why they are not included in the definition any more. We point out that the choice of unitaries in the definition is motivated by our examples and, in particular, it implies

$$\sum_{k=1}^{n} u_k^* u_k = n1_A, \qquad \sum_{k=1}^{n} v_k^* v_k = n1_B.$$

Hence, in the presence of traces, we have

$$\tau \circ \psi \circ \varphi(a) = n\tau(a), \qquad \tau' \circ \varphi \circ \psi(b) = n\tau'(b)$$

for all $a \in A, b \in B, \tau \in T(A)$ and $\tau' \in T(B)$.

We point out that Definition 5.3.1 was formulated for any unital C^*-algebra but, by design, we will use it mostly for separable C^*-algebras. However, we will be able to obtain some simple facts for ultrapowers of separable C^*-algebras.

Alternative definition

Of course, another viable alternative to Definition 5.3.1 is asking for the sums of $u_k^* u_k$ and $v_k^* v_k$ to be scalar multiples of 1_A and 1_B, respectively.

Definition 5.3.3 (Alternative definition)**.** Let A and B be unital separable C^*-algebras. We say A is *n-coloured equivalent* to B if there exist c.p.c. order zero maps $\varphi : A \longrightarrow B$, $\psi : B \longrightarrow A$, constants $\lambda_A, \lambda_B \geq 1$ and unitaries $u_1, \dots, u_n \in A_\omega$, $v_1, \dots, v_n \in B_\omega$ such that

(i) $a = \sum_{k=1}^{n} u_k \psi\varphi(a) u_k^*$ for all $a \in A$,

(ii) $b = \sum_{k=1}^{n} v_k \varphi\psi(b) v_k^*$ for all $b \in B$,

(iii) $\sum_{k=1}^{n} u_k^* u_k = \lambda_A 1_A$ and $\sum_{k=1}^{n} v_k^* v_k = \lambda_B 1_B$,

(iv) $u_k^* u_k$ commutes with $\psi(B)$, and $v_k^* v_k$ commutes with $\varphi(A)$ for $k = 1, \dots, n$.

The theory of coloured isomorphisms works essentially the same with this alternative definition (some proofs have to be slightly modified) but probably the main difference is the coloured transitivity identity (5.1). We will see that with the Definition 5.3.1 the coloured transitive identity holds for the class of separable stable rank one unital C^*-algebras with transition function $(n, m) \mapsto nm$ meanwhile for the alternative Definition 5.3.3 the coloured transitive identity holds for the larger class of separable unital C^*-algebras with transitive function $(n, m) \mapsto nm + 1$.

5.4 Properties

In this section we will prove some basic properties of coloured isomorphisms. We will show it satisfies the coloured transitive relation for the class of unital separable C^*-algebras of stable rank one. Other properties as commutativity, nuclearity and pure infiniteness are preserved. Probably two of the key features of these type of isomorphisms is that the trace simplices are homeomorphic and ideal lattices of coloured isomorphic C^*-algebras are ordered isomorphic.

Proposition 5.4.1. *(i) The relation $\cong_{(n)}$ is reflexive and symmetric for all $n \in \mathbb{N}$.*

(ii) If A, B and C are unital separable and stable rank one, $A \cong_{(n)} B$ and $B \cong_{(m)} C$ then $A \cong_{(nm)} C$.

Proof. It is immediate from the definition that the relations $\cong_{(n)}$ are reflexive and symmetric for all $n \in \mathbb{N}$. Let us prove the coloured transitive identity. By hypothesis there exist c.p.c. order zero maps $\varphi_1 : A \longrightarrow B, \psi_1 : B \longrightarrow A$ and unitaries $u_1, \dots u_n \in A_\omega, v_1, \dots, v_n \in B_\omega$ such that

$$a = \sum_{i=1}^{n} u_i \psi_1 \varphi_1(a) u_i^*, \qquad a \in A, \tag{5.14}$$

$$b = \sum_{i=1}^{n} v_i \varphi_1 \psi_1(b) v_i^*, \qquad b \in B. \tag{5.15}$$

Similarly, there exist c.p.c. order zero maps $\varphi_2 : B \longrightarrow C, \psi_2 : C \longrightarrow B$ and unitaries $x_1, \ldots, x_m \in B_\omega, y_1, \ldots, y_n \in C_\omega$ such that

$$b = \sum_{j=1}^{m} x_j \psi_2 \varphi_2(b) x_j^*, \qquad b \in B, \tag{5.16}$$

$$c = \sum_{j=1}^{m} y_j \varphi_2 \psi_2(c) y_j^*, \qquad c \in C. \tag{5.17}$$

The maps ψ_i and φ_i induce each a map at the level of the ultrapowers, which we continue to denote by ψ_i and φ_i. By Lemma A.1.7, these induced maps

$$\varphi_1 : A_\omega \longrightarrow B_\omega,\ \varphi_2 : B_\omega \longrightarrow C_\omega,\ \psi_1 : B_\omega \longrightarrow A_\omega,\ \psi_2 : C_\omega \longrightarrow B_\omega$$

are order zero. Let $D \subset B_\omega$ be the separable unital C^*-subalgebra generated by B and $x_1, \ldots, x_m \in B_\omega$. By Corollary A.1.9, applied to the c.p.c. order zero map $\psi_1\big|_D : D \longrightarrow A_\omega$, there exist unitaries $r_1, \ldots, r_m \in A_\omega$ such that

$$\psi_1\left(x_j b x_j^*\right) = r_j \psi_1(b) r_j^* \tag{5.18}$$

for all $b \in B$ and $j = 1, \ldots, m$. Then

$$\begin{aligned}
a &\overset{(5.14)}{=} \sum_{i=1}^{n} u_i \psi_1 \varphi_1(a) u_i^* \\
&\overset{(5.16)}{=} \sum_{i=1}^{n} u_i \psi_1 \left(\sum_{j=1}^{m} x_j \psi_2 \varphi_2 \left(\varphi_1(a)\right) x_j^* \right) u_i^* \\
&= \sum_{i,j} u_i \psi_1 \left(x_j \psi_2 \varphi_2 \varphi_1(a) x_j^*\right) u_i^* \\
&\overset{(5.18)}{=} \sum_{i,j} u_i r_j \psi_1 \psi_2 \varphi_2 \varphi_1(a) r_j^* u_i^*
\end{aligned} \tag{5.19}$$

for all $a \in A$. In the same way, using Corollary A.1.9, there exist unitaries $s_1, \ldots, s_n \in C_\omega$ such that

$$\varphi_2(v_i b v_i^*) = s_i \varphi_2(b) s_i^* \tag{5.20}$$

for all $b \in B$ and $i = 1, \ldots, n$. Then

$$\begin{aligned}
c &\overset{(5.17)}{=} \sum_{j=1}^{m} y_j \varphi_2 \psi_2(c) y_j^* \\
&\overset{(5.15)}{=} \sum_{j=1}^{m} y_j \varphi_2 \left(\sum_{i=1}^{n} v_i \varphi_1 \psi_1 \left(\psi_2(c)\right) v_i^* \right) y_j^* \\
&= \sum_{i,j} y_j \varphi_2 \left(v_i \varphi_1 \psi_1 \psi_2(c) v_i^*\right) y_j^* \\
&\overset{(5.20)}{=} \sum_{i,j} y_j s_i \varphi_2 \varphi_1 \psi_1 \psi_2(c) s_i^* y_j^*
\end{aligned} \tag{5.21}$$

for all $c \in C$. This shows the c.p.c. order zero maps $\varphi_1\varphi_2 : A \longrightarrow C, \psi_1\psi_2 : C \longrightarrow A$ and the unitaries $u_i r_j \in A_\omega, y_j s_i \in B_\omega$, with $i = 1, \ldots, n$ and $j = 1, \ldots, m$, implement an nm-coloured isomorphism between A and B. □

Remark 5.4.2. The coloured transitivity identity also holds for the alternative definition of coloured equivalence of C^*-algebras (Definition 5.3.3). Precisely, if A, B and C are unital separable C^*-algebras such that A is n-coloured equivalent to B and B is m-coloured equivalent to C, then A is $(nm+1)$-coloured equivalent to C.

Proof. The proof is essentially the same as the proof of Proposition 5.4.1. By hypothesis there exist c.p.c. order zero maps $\varphi_1 : A \longrightarrow B, \psi_1 : B \longrightarrow A$, positive constants λ_A, λ_B and elements $u_1, \ldots u_n \in A_\omega, v_1, \ldots, v_n \in B_\omega$ satisfying $\sum_{i=1}^{n} u_i^* u_i = \lambda_A 1_A$ and $\sum_{i=1}^{n} v_i^* v_i = \lambda_B 1_B$ such that

$$a = \sum_{i=1}^{n} u_i \psi_1 \varphi_1(a) u_i^*, \qquad a \in A, \tag{5.22}$$

$$b = \sum_{i=1}^{n} v_i \varphi_1 \psi_1(b) v_i^* \qquad b \in B, \tag{5.23}$$

and $u_i^* u_i$ commutes with $\psi_1(B)$, and $v_k^* v_k$ commutes with $\varphi_1(A)$ for $i = 1, \ldots, n$.

Similarly, there exist c.p.c. order zero maps $\varphi_2 : B \longrightarrow C, \psi_2 : C \longrightarrow B$, positive constants μ_B, μ_C and elements $x_1, \ldots, x_m \in B_\omega, y_1, \ldots, y_n \in C_\omega$ satisfying $\sum_{j=1}^{m} x_j^* x_j = \mu_B 1_B$ and $\sum_{j=1}^{m} y_j^* y_j = \mu_C 1_C$ such that

$$b = \sum_{j=1}^{m} x_j \psi_2 \varphi_2(b) x_j^*, \qquad b \in B, \tag{5.24}$$

$$c = \sum_{j=1}^{m} y_j \varphi_2 \psi_2(c) y_j^*, \qquad c \in C, \tag{5.25}$$

and $x_j^* x_k$ commutes with $\psi_2(C)$, and $y_j^* y_j$ commutes with $\varphi_2(B)$ for $j = 1, \ldots, n$.

Let $D \subset B_\omega$ be the separable unital C^*-subalgebra generated by B and $x_1, \ldots, x_m \in B_\omega$. By Lemma A.1.8 there exists a c.p.c. order zero map $\hat{\psi}_1 : D \longrightarrow A_\omega$ such that

$$\psi_1(d_1 d_2) = \hat{\psi}_1(d_1)\psi_1(d_2) = \psi_1(d_1)\hat{\psi}_1(d_2). \tag{5.26}$$

for all $d_1, d_2 \in D$. Then

$$\begin{aligned} a &\overset{(5.22)}{=} \sum_{i=1}^{n} u_i \psi_1 \varphi_1(a) u_i^* \\ &\overset{(5.24)}{=} \sum_{i=1}^{n} u_i \psi_1 \left(\sum_{j=1}^{m} x_j \psi_2 \varphi_2 \left(\varphi_1(a)\right) x_j^* \right) u_i^* \\ &= \sum_{i,j} u_i \psi_1 \left(x_j \psi_2 \varphi_2 \varphi_1(a) x_j^* \right) u_i^* \\ &\overset{(5.26)}{=} \sum_{i,j} u_i \hat{\psi}_1(x_j) \psi_1 \psi_2 \varphi_2 \varphi_1(a) \hat{\psi}_1(x_j)^* u_i^* \end{aligned} \tag{5.27}$$

for all $a \in A$.

Let us compute the following sum

$$\begin{aligned} \sum_{i,j} \left(u_i \hat{\psi}_1(x_j) \right)^* u_i \hat{\psi}_1(x_j) &= \sum_{i,j} \hat{\psi}_1(x_j)^* u_i^* u_i \hat{\psi}_1(x_j) \\ &= \sum_{j=1}^{m} \hat{\psi}_1(x_j)^* \left(\sum_{i=1}^{n} u_i^* u_i \right) \hat{\psi}_1(x_j) \\ &= \lambda_A \sum_{j=1}^{m} \hat{\psi}_1\left(x_j\right)^* \hat{\psi}_1\left(x_j\right) \\ &\overset{(*)}{=} \lambda_A \sum_{j=1}^{m} \hat{\psi}_1\left(1_{B_\omega}\right) \hat{\psi}_1\left(x_j^* x_j\right) \\ &= \lambda_A \hat{\psi}_1(1_{B_\omega}) \hat{\psi}_1 \left(\sum_{j=1}^{m} x_j^* x_j \right) \\ &= \lambda_A \mu_B \hat{\psi}_1(1_{B_\omega})^2, \end{aligned} \tag{5.28}$$

where the equality in $(*)$ follows from the structure of order zero maps (Theorem 1.4.6).

Since $\lambda_A \mu_B \hat{\psi}_1(1_B)^2$ is in general different from $\lambda_A \mu_B 1_A$, we have to add an extra colour. Set

$$h := \left(1_A - \hat{\psi}_1(1_B)^2\right)^{\frac{1}{2}} \tag{5.29}$$

and define

$$r_0 = \left(\lambda_A \mu_B\right)^{\frac{1}{2}} h, \qquad r_{i,j} := u_i \hat{\psi}_1(x_j) \tag{5.30}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. First of all, since $\hat{\psi}_1(1_B)$ acts as a unit for $\psi(B)$, we have $h\psi_1(b) = 0$ for all $b \in B$. In particular

$$r_0 \psi_1 \psi_2 \varphi_2 \varphi_1(a) r_0^* = 0. \tag{5.31}$$

This leads to the following

$$\begin{aligned} a &\overset{(5.27)}{=} \sum_{i,j} u_i\hat{\psi}_1(x_j)\psi_1\psi_2\varphi_2\varphi_1(a)\hat{\psi}_1(x_j)^*u_i^* \\ &\overset{(5.31)}{=} r_0\psi_1\psi_2\varphi_2\varphi_1(a)r_0^* + \sum_{i,j} u_i\hat{\psi}_1(x_j)\psi_1\psi_2\varphi_2\varphi_1(a)\hat{\psi}_1(x_j)^*u_i^* \\ &\overset{(5.30)}{=} r_0\psi_1\psi_2\varphi_2\varphi_1(a)r_0^* + \sum_{i,j} r_{i,j}\psi_1\psi_2\varphi_2\varphi_1(a)r_{i,j}^* \end{aligned} \tag{5.32}$$

for all $a \in A$. Similarly, we obtain

$$\begin{aligned} r_0^*r_0 + \sum_{i,j} r_{i,j}^*r_{i,j} &\overset{(5.30)}{=} \lambda_A\mu_B\left(1_A - \hat{\psi}_1(1_B)^2\right) + \sum_{i,j}\left(u_i\hat{\psi}_1(x_j)\right)^* u_i\hat{\psi}_1(x_j) \\ &\overset{(5.28)}{=} \lambda_A\mu_B\left(1_A - \hat{\psi}_1(1_B)^2 + \hat{\psi}_1(1_B)^2\right) \\ &= \lambda_A\mu_B 1_A. \end{aligned} \tag{5.33}$$

Let us prove the commutation relations. It is immediate that $r_0^*r_0$ commutes with $\psi_1\psi_2$ since h annihilates ψ_1. Remember that $u_i^*u_i$ commutes with the image of ψ_1 and $x_j^*x_j$ commutes with the image ψ_2 for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Hence, for all $c \in C$, we have

$$\begin{aligned} r_{i,j}^*r_{i,j}\psi_1\psi_2(c) &\overset{(5.30)}{=} \left(u_i\hat{\psi}_1(x_j)\right)^* u_i\hat{\psi}_1(x_j)\psi_1\psi_2(c) \\ &= \hat{\psi}_1\left(x_j^*\right)u_i^*u_i\hat{\psi}_1(x_j)\psi_1\psi_2(c) \\ &\overset{(5.26)}{=} \hat{\psi}_1\left(x_j^*\right)u_i^*u_i\psi_1(x_j\psi_2(c)) \\ &= \hat{\psi}_1\left(x_j^*\right)\psi_1\left(x_j\psi_2(c)\right)u_i^*u_i \\ &\overset{(5.26)}{=} \psi_1\left(x_j^*x_j\psi_2(c)\right)u_i^*u_i \\ &= \psi_1\left(\psi_2(c)x_j^*x_j\right)u_i^*u_i \\ &= u_i^*u_i\psi_1\left(\psi_2(c)x_j^*x_j\right) \\ &\overset{(5.26)}{=} u_i^*u_i\psi_1\left(\psi_2(c)x_j^*\right)\hat{\psi}_1(x_j) \\ &= \psi_1\left(\psi_2(c)x_j^*\right)u_i^*u_i\hat{\psi}_1(x_j) \\ &\overset{(5.26)}{=} \psi_1(\psi_2(c))\hat{\psi}_1(x_j)^*u_i^*u_i\hat{\psi}_1(x_j) \\ &\overset{(5.30)}{=} \psi_1\psi_2(c)r_{i,j}^*r_{i,j}. \end{aligned} \tag{5.34}$$

Similarly, let E be the separable C^*-subalgebra of B_ω generated by B and $v_1, \ldots, v_n \in B_\omega$. By Lemma A.1.8 there exists a c.p.c. order zero map $\hat{\varphi}_2 : E \longrightarrow C_\omega$ such that

$$\varphi_2(e_1e_2) = \hat{\varphi}_2(e_1)\varphi_2(e_2) \tag{5.35}$$

for all $e_1, e_2 \in E$. Set

$$s_0 := (\lambda_B \mu_C)^{\frac{1}{2}} \left(1_A - \hat{\varphi}_2(1_B)^2\right)^{\frac{1}{2}}, \qquad s_{i,j} = y_j\left(\hat{\varphi}_2(v_i)\right), \tag{5.36}$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$. Then, as above, we can verify

$$c = s_0 \varphi_2 \varphi_1 \psi_1 \psi_2(c) s_0 + \sum_{i,j} s_{i,j}^* \varphi_2 \varphi_1 \psi_1 \psi_2(c) s_{i,j} \tag{5.37}$$

for all $c \in C$, each $s_{i,j}^* s_{i,j}$ commutes with $\varphi_2\varphi_1$ and

$$s_0^* s_0 + \sum_{i,j} s^* i, j s_{i,j} = \lambda_B \mu_C 1_{C_\omega}. \tag{5.38}$$

Therefore the c.p.c. order zero maps $\varphi_2\varphi_1 : A \longrightarrow C, \psi_1\psi_2 : C \longrightarrow A$, the positive constants $\lambda_A\mu_B, \lambda_B\mu_C$ and the elements $r_0, r_{i,j} \in A_\omega, s_0, s_{i,j} \in C_\omega$ induce a $(nm+1)$-coloured equivalence between A and C (in the alternative sense of Definition 5.3.3). □

Observe that in the previous proof, we added a colour because (5.28) is in general different from a multiple of 1_A but, by (5.31), this extra colour is essentially zero and this is not satisfactory. This is the reason why we prefer Definition 5.3.1.

As explained before, the definition of coloured isomorphism was constructed as a coloured version of some particular form of the intertwining argument (see Section 5.1), so that 1-coloured isomorphism induces an isomorphism. We now confirm this here.

Proposition 5.4.3. *Let A and B be unital separable C^*-algebras. Suppose $A \cong_{(1)} B$, then $A \cong B$.*

Proof. By hypothesis there exist c.p.c. order zero maps $\varphi : A \longrightarrow B, \psi : B \longrightarrow A$ and unitaries $u \in A_\omega, v \in B_\omega$ such that

$$\psi\varphi(a) = uau^*, \qquad \varphi\psi(b) = vbv^* \tag{5.39}$$

for all $a \in A$ and $b \in B$. In particular

$$\psi\varphi(1_A) = 1_A, \qquad \varphi\psi(1_B) = 1_B. \tag{5.40}$$

Since $\varphi(1_A) \leq 1_B$, we obtain

$$1_A = \psi\varphi(1_A) \leq \psi(1_B) \leq 1_A. \tag{5.41}$$

This shows ψ is unital and, by the structure of order zero maps (Definition 1.4.6), ψ is a *-homomorphism. In the same way we can show φ is a unital *-homomorphism. By

(5.39), we have the following diagram where each triangle is approximately commuting up to unitary equivalence

$$\begin{tikzcd}
& A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi"] && A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi"] && A \arrow[r] & \cdots \arrow[r] & A \arrow[d, dashed, shift left, "\cong"] \\
B \arrow[ur, "\psi"] \arrow[rr, "\mathrm{id}_B"'] && B \arrow[ur, "\psi"] \arrow[rr, "\mathrm{id}_B"'] && B \arrow[ur, "\psi"] \arrow[rr] && \cdots \arrow[r] & B. \arrow[u, dashed, shift left]
\end{tikzcd}$$

Therefore, by Theorem 5.1.1, A is isomorphic to B. $\square$

Observe that the key point of the previous proof is the unitality of φ and ψ. In fact, this proposition remains valid for Definition 5.3.3 because we also obtain that the maps have to be unital in the one colour case. This is the reason why, in Definition 5.3.3, we have to ask for elements $u_1, \ldots, u_n$ such that $\sum_{i=1}^{n} u_k^* u_k$ is a multiple of 1_A.

We now proceed to prove the following permanence properties.

Proposition 5.4.4. *Let A, B, C and D be separable unital C^*-algebras such that $A \cong_{(n)} B$ and $C \cong_{(m)} D$. Then*

(i) $A \oplus C \cong_{(k)} B \oplus D$ with $k = \operatorname{lcm}\{n, m\}$,

(ii) $\ell^\infty(A) \cong_{(n)} \ell^\infty(B)$,

(iii) $A \otimes_\alpha C \cong_{(nm)} B \otimes_\alpha D$, where $\otimes_\alpha$ denotes the minimal or maximal tensor product.

Proof. By hypothesis there exist c.p.c. order zero maps $\varphi_1 : A \longrightarrow B, \psi_1 : B \longrightarrow A$ and unitaries $u_1, \ldots, u_n \in A_\omega, v_1, \ldots, v_n \in B_\omega$ such that

$$a = \sum_{i=1}^{n} u_i \psi_1 \varphi_1(a) u_i^*, \qquad b = \sum_{i=1}^{n} v_i \varphi_1 \psi_1(b) v_i^* \tag{5.42}$$

for all $a \in A, b \in B$. Similarly, there exist c.p.c. order zero maps $\varphi_2 : C \longrightarrow D, \psi_2 : D \longrightarrow C$ and unitaries $x_1, \ldots, x_m \in C_\omega, y_1, \ldots, y_m \in D_\omega$ such that

$$c = \sum_{j=1}^{m} x_j \psi_2 \varphi_2(c) x_j^*, \qquad b = \sum_{j=1}^{m} y_j \varphi_2 \psi_2(b) y_j^* \tag{5.43}$$

for all $c \in C, d \in D$.

(i) Consider $k = \operatorname{lcm}\{n, m\}$ and define c.p.c. order zero maps $\rho : A \oplus C \longrightarrow B \oplus D$, $\sigma : B \oplus D \longrightarrow A \oplus C$ in the following way.

$$\rho(a \oplus c) = \frac{n}{k}\varphi_1(a) \oplus \frac{m}{k}\varphi_2(c), \qquad a \oplus c \in A \oplus C, \tag{5.44}$$

$$\sigma(b \oplus d) = \frac{m}{k}\psi_1(b) \oplus \frac{n}{k}\psi_2(d), \qquad b \oplus d \in B \oplus D. \tag{5.45}$$

For $i = n+1, \ldots, k$ set

$$u_i := u_j, \qquad v_i := v_j \tag{5.46}$$

if $i \equiv j \bmod n$ for some $j \in \{1, \ldots, n\}$. Similarly, for $i = m+1, \ldots, k$ set

$$x_i := x_j, \qquad y_i := y_j \tag{5.47}$$

if $i \equiv j \bmod m$ for some $j \in \{1, \ldots, m\}$. Now, we can define the following unitaries

$$r_i := u_i \oplus x_i \in (A \oplus C)_\omega\,, \qquad s_i := v_i \oplus y_i \in (B \oplus D)_\omega\,, \tag{5.48}$$

for $i = 1, \ldots, k$. Observe that by construction we have

$$\begin{aligned}
\sigma\rho(a \oplus c) &\overset{(5.44)}{=} \sigma\left(\frac{n}{k}\varphi_1(a) \oplus \frac{m}{k}\varphi_2(c)\right) \\
&\overset{(5.45)}{=} \left(\frac{m}{k} \cdot \frac{n}{k}\psi_1\varphi_1(a)\right) \oplus \left(\frac{n}{k} \cdot \frac{m}{k}\psi_2\varphi_2(c)\right) \\
&= \frac{nm}{k^2}\left(\psi_1\varphi_1(a) \oplus \psi_2\varphi_2(c)\right)
\end{aligned} \tag{5.49}$$

for all $a \oplus c \in A \oplus C$. By (5.42) and (5.43) we have

$$\sum_{i=1}^{k} u_i\psi_1\varphi_1(a)u_i^* = \frac{k}{n}a, \qquad \sum_{i=1}^{k} x_i\psi_2\varphi_2(c)x_i^* = \frac{k}{m}c \tag{5.50}$$

for all $a \in A, c \in C$. Then we obtain

$$\begin{aligned}
\sum_{i=1}^{k} r_i\sigma\rho(a \oplus c)r_i^* &\overset{(5.49)}{=} \frac{nm}{k^2}\sum_{i=1}^{k} r_i^*\left(\psi_1\varphi_1(a) \oplus \psi_2\varphi_2(c)\right) r_i^* \\
&\overset{(5.48)}{=} \frac{nm}{k^2}\sum_{i=1}^{k} (u_i \oplus x_i)\left(\psi_1\varphi_1(a) \oplus \psi_2\varphi_2(c)\right)(u_i \oplus x_i)^* \\
&= \frac{nm}{k^2}\sum_{i=1}^{k} u_i\psi_1\varphi_1(a)u_i^* \oplus x_i\psi_2\varphi_2(c)x_i^* \\
&= \frac{nm}{k^2}\left(\sum_{i=1}^{k} u_i\psi_1\varphi_1(a)u_i^*\right) \oplus \left(\sum_{i=1}^{k} x_i\psi_2\varphi_2(c)x_i^*\right) \\
&\overset{(5.50)}{=} \frac{nm}{k^2}\left(\frac{k}{n}a\right) \oplus \left(\frac{k}{m}c\right) \\
&= a \oplus c.
\end{aligned} \tag{5.51}$$

Similarly we can prove

$$b \oplus d = \sum_{i=1}^{k} s_i\rho\sigma(b \oplus d)s_i^* \tag{5.52}$$

for all $b \oplus d \in B \oplus D$.

Therefore the c.p.c. order zero maps $\rho : A \oplus C \longrightarrow B \oplus D, \sigma : B \oplus D \longrightarrow A \oplus C$ and the unitaries $r_1, \ldots, r_k \in (A \oplus C)_\omega, s_1, \ldots, s_k \in (B \oplus D)_\omega$ induce a k-coloured isomorphism between $A \oplus C$ and $B \oplus D$.

(ii) By Proposition A.1.4, we have a canonical inclusion $\ell^\infty(A_\omega) \hookrightarrow \ell^\infty(A)_\omega$. Then it is immediate that the induced maps $\varphi : \ell^\infty(A) \longrightarrow \ell^\infty(B)$, $\psi : \ell^\infty(B) \longrightarrow \ell^\infty(A)$ and the constant sequences $(u_1), \ldots, (u_n) \in \ell^\infty(A)_\omega, (v_1), \ldots, (v_n) \in \ell^\infty(B)_\omega$ implement an n-coloured isomorphism between $\ell^\infty(A)$ and $\ell^\infty(B)$.

(iii) By Corollary 1.4.9, the tensor product of c.p.c. order zero maps is order zero. Thus the c.p.c. maps $\varphi_1 \otimes \varphi_2 : A \otimes_\alpha C \longrightarrow B \otimes_\alpha D$ and $\psi_1 \otimes \psi_2 : B \otimes_\alpha D \longrightarrow A \otimes_\alpha C$ are order zero. Then

$$\begin{aligned} a \otimes c &= \left(\sum_{i=1}^n u_i \psi_1 \varphi_1(a) u_i^*\right) \otimes \left(\sum_{j=1}^m x_j \psi_2 \varphi_2(c) x_j^*\right) \\ &= \sum_{i=1}^n \sum_{j=1}^m u_i \psi_1 \varphi_1(a) u_i^* \otimes x_j \psi_2 \varphi_2(c) x_j^* \\ &= \sum_{i=1}^n \sum_{j=1}^m (u_i \otimes x_j)\left((\psi_1 \otimes \psi_2) \circ (\varphi_1 \otimes \varphi_2)(a \otimes c)\right)(u_i \otimes x_j)^* \end{aligned} \tag{5.53}$$

for any elementary tensor $a \otimes c \in A \otimes_\alpha C$. Similarly

$$b \otimes d = \sum_{i=1}^n \sum_{j=1}^m (v_i \otimes y_j)\left((\varphi_1 \otimes \varphi_2) \circ (\psi_1 \otimes \psi_2)(b \otimes d)\right)(v_i \otimes y_j)^* \tag{5.54}$$

for any elementary tensor $b \otimes d \in B \otimes_\alpha D$. Observe (5.53) and (5.54) extend to $A \otimes C$ and $B \otimes C$ by linearity and density of the span of elementary tensors.

Therefore the c.p.c. order zero maps $\varphi_1 \otimes \varphi_2 : A \otimes_\alpha C \longrightarrow B \otimes_\alpha D$, $\psi_1 \otimes \psi_2 : B \otimes_\alpha D \longrightarrow A \otimes_\alpha C$ and the unitaries $u_i \otimes x_j \in (A \otimes_\alpha C)_\omega, v_i \otimes y_j \in (B \otimes_\alpha D)_\omega$, with $i = 1, \ldots, n$ and $j = 1, \ldots, m$, induce a nm-coloured isomorphism between $A \otimes_\alpha C$ and $B \otimes_\alpha D$. $\square$

As a straightforward corollary of the coloured transitive identity (Proposition 5.4.1) and the previous proposition, we obtain the following.

Corollary 5.4.5. *Let A, B and D be unital separable C^*-algebras. Suppose A and B are stable rank one. If A satisifes $A \cong A \otimes_\alpha D$ and $A \cong_{(n)} B$, then $B \cong_{(n^2)} B \otimes_\alpha D$ where $\otimes_\alpha$ denotes the minimal or maximal tensor product.*

Proof. By hypothesis $A \otimes_\alpha D \cong A$. Moreover, by Proposition 5.4.4, $A \otimes_\alpha D \cong_{(n)} B \otimes_\alpha D$. Hence $A \otimes_\alpha D \cong_{(n)} B$ and $A \otimes_\alpha D \cong_{(n)} B \otimes D$. By the coloured transitive identity (Proposition 5.4.1), $B \cong_{(n^2)} B \otimes_\alpha D$. □

We will prove now that nuclearity is also preserved under coloured isomorphisms. The same proof can be extended without too much effort to show that finite nuclear dimension is also preserved with multiplicative estimates. Remember $\dim_{\text{nuc}}^{+1} A$ stands for $\dim_{\text{nuc}} A + 1$.

Proposition 5.4.6. *Let A and B be separable unital C^*-algebras. Suppose A is nuclear and $A \cong_{(n)} B$, then B is nuclear. Moreover, if $\dim_{\text{nuc}} A$ is finite then*

$$\dim_{\text{nuc}}^{+1} B \leq n \cdot \dim_{\text{nuc}}^{+1} A.$$

Proof. By hypothesis there exist c.p.c. order zero maps $\varphi : A \longrightarrow B, \psi : B \longrightarrow A$ and unitaries $u_1, \ldots, u_n \in A_\omega, v_1, \ldots, v_n \in B_\omega$ such that

$$a = \sum_{i=1}^{n} u_i \psi\varphi(a) u_i^*, \qquad b = \sum_{i=1}^{n} v_i \varphi\psi(b) v_i^* \tag{5.55}$$

for all $a \in A, b \in B$. The idea proof of this proposition is basically contained in the following diagram,

$$\begin{tikzcd}
B \arrow[rr, "\mathrm{id}_B"] \arrow[d, "\psi"'] & & B \\
A \arrow[rr, "\mathrm{id}_A"'] \arrow[dr, "\rho"'] & & A \arrow[u, "\zeta"'] \\
& F \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

where ρ and σ are c.p.c. maps coming from the nuclearity of A and $\zeta : A \longrightarrow B$ is given by $\zeta(a) = \sum_{i=1}^{n} b_i \varphi(a) b_i^*$ for some unitaries $b_1, \ldots, b_n \in B$.

Let $\mathfrak{F} \subset B$ be a finite subset and $\varepsilon > 0$. By Proposition A.1.3, we can lift each v_i to a sequence $\left(v_i^{(k)}\right)_{k \in \mathbb{N}}$ where each $v_i^{(k)}$ is a unitary in B. By (5.55), there exists $k \in \mathbb{N}$ such that

$$\left\| b - \sum_{i=1}^{n} v_i^{(k)} \varphi\psi(b) v_i^{(k)*} \right\| < \frac{\varepsilon}{2} \tag{5.56}$$

for all $b \in \mathfrak{F}$. Using the completely positive approximation property for A, applied to the finite subset $\psi(\mathfrak{F}) = \{\psi(b) \mid b \in \mathfrak{F}\} \subset A$ and $\varepsilon/2$, there exist a finite dimensional

C^*-algebra F and c.p.c. maps $\rho : A \longrightarrow F, \sigma : F \longrightarrow A$ such that

$$\|\psi(b) - \sigma\rho(\psi(b))\| < \frac{\varepsilon}{2n} \tag{5.57}$$

for all $b \in \mathfrak{F}$. Then

$$\begin{aligned}
\left\| b - \sum_{i=1}^{n} v_i^{(k)} \varphi\sigma\rho\psi(b) v_i^{(k)*} \right\| &\leq \left\| b - \sum_{i=1}^{n} v_i^{(k)} \varphi\psi(b) v_i^{(k)*} \right\| \\
&\quad + \left\| \sum_{i=1}^{n} v_i^{(k)} \varphi\psi(b) v_i^{(k)*} - \sum_{i=1}^{n} v_i^{(k)} \varphi\sigma\rho\psi(b) v_i^{(k)*} \right\| \\
&\overset{(5.56)}{<} \frac{\varepsilon}{2} + \left\| \sum_{i=1}^{n} v_i^{(k)} \varphi\left(\psi(b) - \sigma\rho\psi(b)\right) v_i^{(k)*} \right\| \\
&\leq \frac{\varepsilon}{2} + \sum_{i=1}^{n} \left\| v_i^{(k)} \right\| \left\| \varphi\left(\psi(b) - \sigma\rho\psi(b)\right) \right\| \left\| v_i^{(k)*} \right\| \\
&= \frac{\varepsilon}{2} + n \left\| \psi(b) - \sigma\rho(\psi(b)) \right\| \\
&\overset{(5.57)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned} \tag{5.58}$$

for all $b \in \mathfrak{F}$. Notice that the map $a \mapsto v_i^{(k)} \varphi(a) v_i^{(k)*}$ is c.p. and, since φ is contractive and $v_i^{(k)}$ is a unitary, this map is contractive as well. Set $\zeta : A \longrightarrow B$ as

$$\zeta(a) = \sum_{i=1}^{n} v_i^{(k)} \varphi(a) v_i^{(k)*}. \tag{5.59}$$

Since ζ is the sum of n c.p.c. maps, ζ is c.p. and $\|\zeta\| < n$. Summarising, for $\mathfrak{F}$ and $\varepsilon > 0$, the c.p.c. maps $\rho\sigma : B \longrightarrow F, \zeta\sigma : F \longrightarrow B$ satisfy

$$\|b - \zeta\sigma\rho\psi(b)\| < \varepsilon \tag{5.60}$$

for all $b \in \mathfrak{F}$ and $\|\zeta\sigma\| \leq n$,

$$\begin{tikzcd}
B \arrow[rr, "\mathrm{id}_B"] \arrow[dr, "\rho\psi"'] & & B \, . \\
& F \arrow[ur, "\zeta\sigma"'] &
\end{tikzcd}$$

Therefore, by Proposition 1.5.4, B is nuclear.

Let us prove the second part of the proposition, so suppose $\dim_{\mathrm{nuc}} A = d$. The outline of the proof is the following: By the first part, we can produce an approximation for $\mathfrak{F}$ within ε. We will finish the proof if, using the extra properties of the approximations for A, we show that the map $\eta\sigma$ going back to B is the sum of $n(d+1)$ order zero maps.

Let $\mathfrak{F} \subset B$ and $\varepsilon > 0$ and let us keep the same notation as before. From the first part of the proof, we know that there is an approximation $(F, \rho\sigma, \eta\sigma)$ for $\mathfrak{F}$ within ε. Using that

$\dim_{\text{nuc}} A = d$, we can further assume that, for the chosen $\mathfrak{F}$ and ε, there is decomposition $F = \bigoplus_{j=0}^{d} F_j$ and the c.p.c. map $\sigma : F \longrightarrow A$ satisfies that the restrictions $\sigma_j := \sigma|_{F_j}$ are order zero for $j = 1, \ldots, d$. Thus σ is the sum of $d+1$ order zero maps; precisely $\sigma = \sum_{j=0}^{d} \sigma_j$. Expressing σ in this form, we obtain

$$\begin{aligned}\eta\sigma(x) &= \sum_{i=1}^{n} v_i^{(k)} \varphi\sigma(x) v_i^{(k)*} \\ &= \sum_{i=1}^{n} v_i^{(k)} \varphi \left(\sum_{j=0}^{d} \sigma_j(x) \right) v_i^{(k)*} \\ &= \sum_{i=1}^{n} \sum_{j=0}^{d} v_i^{(k)} \varphi\sigma_j(x) v_i^{(k)*} \end{aligned} \tag{5.61}$$

for all $x \in F$. This shows $\eta\sigma$ is the sum of $n(d+1)$ maps of the form $v_i^{(k)}\varphi\sigma_j(x)v_i^{(k)*}$. Since φ and σ_j are order zero and $\mathrm{Ad}\left(v_i^{(k)*}\right)$ is a *-homomorphism, the map $v_i^{(k)}\varphi\sigma_j(x)v_i^{(k)*}$ is order zero. Therefore $\eta\sigma$ is the sum of $n(d+1)$ order zero maps. This shows

$$\dim_{\text{nuc}}^{+1} B \leq n \cdot \dim_{\text{nuc}}^{+1} A.$$ □

The ideal lattice of a C^*-algebra A is just the set of ideals of A. We will denote it by $\mathcal{I}(A)$. As the name suggests, this set is in fact an ordered lattice where the order is given by inclusion. Our next goal is to show that ideal lattices of coloured isomorphic C^*-algebras are ordered isomorphic. We need some preparation lemmas first. Observe that in these lemmas we do not need to assume separability.

Lemma 5.4.7. *Let A and B be unital C^*-algebras such that the c.p.c. order zero maps $\varphi : A \longrightarrow B$, $\psi : B \longrightarrow A$ and the unitaries $u_1, \ldots, u_n \in A_\omega, v_1, \ldots, v_n \in B_\omega$ induce an n-coloured isomorphism between A and B. Then $\varphi^{-1}(\psi^{-1}(I)) = I$ and $\psi^{-1}(\varphi^{-1}(J)) = J$ for any ideal $I \trianglelefteq A$ and $J \trianglelefteq B$.*

Proof. Since this property is symmetric, it is enough to prove it for ideals of A only. By hypothesis $a = \sum_{k=1}^{n} u_k\psi\varphi(a)u_k^*$ for all $a \in A$. Let I be an ideal of A, if $a \in \varphi^{-1}(\psi^{-1}(I))$ then $\psi\varphi(a) \in I$. Hence $a = \sum_{k=1}^{n} u_k\psi\varphi(a)u_k^* \in I$. This shows

$$\varphi^{-1}(\psi^{-1}(I)) \subset I.$$

Conversely, let $a \in I$ and without loss of generality we can assume a is a positive contraction. Since $a = \sum_{k=1}^{n} u_k\psi\varphi(a)u_k^*$ and each summand $u_k\psi\varphi(a)u_k^*$ is positive, we have

$$u_i\psi\varphi(a)u_i^* \leq \sum_{k=1}^{n} u_k\psi\varphi(a)u_k^* = a \tag{5.62}$$

for $i = 1, \ldots, n$. Hence

$$\psi\varphi(a) \leq u_i^* a u_i \in I. \tag{5.63}$$

Since $u_i^* a u_i \in I$ and I is hereditary, $\psi\varphi(a) \in I$ and therefore $a \in \varphi^{-1}(\psi^{-1}(I))$. This shows

$$I \subset \varphi^{-1}(\psi^{-1}(I)).$$

This finishes the proof. $\square$

Lemma 5.4.8. *Let A, B, C and D be unital C^*-algebras. Suppose the c.p.c. order zero maps $\varphi : A \longrightarrow B$, $\psi : B \longrightarrow A$ induce an n-coloured isomorphism between A and B. If there exist surjective *-homomorphisms $q : A \longrightarrow C$ and $r : B \longrightarrow D$ such that $\varphi(\ker q) = \ker r$ and $\psi(\ker r) = \ker q$. Then $C \cong_{(n)} D$.*

Proof. By hypothesis, there exist unitaries $u_1, \ldots, u_n \in A_\omega, v_1, \ldots, v_n \in B_\omega$ such that

$$a = \sum_{k=1}^{n} u_k\psi\varphi(a)u_k^*, \qquad b = \sum_{k=1}^{n} v_k\psi\varphi(a)v_k^* \tag{5.64}$$

for all $a \in A$ and $b \in B$. Let us define c.p.c. order zero maps $\rho : C \longrightarrow D$ and $\sigma : D \longrightarrow C$ in the following way:

$$\rho(c) = r \circ \varphi(a_c), \qquad \sigma(d) = q \circ \psi(b_d) \tag{5.65}$$

where $a_c \in q^{-1}(\{c\})$ and $b_d \in r^{-1}(\{d\})$,

$$\begin{tikzcd} A \arrow[r, shift left, "\varphi"] \arrow[d, "q"'] & B \arrow[l, shift left, "\psi"] \arrow[d, "r"] \\ C \arrow[r, shift left, dashed, "\rho"] & D. \arrow[l, shift left, dashed, "\sigma"] \end{tikzcd}$$

First of all, let us show these maps are well defined. Consider $c \in C$ and let $a_1, a_2 \in q^{-1}(\{c\})$. Hence $q(a_1 - a_2) = 0$. Since $\varphi(\ker q) = \ker r$, we have $\varphi(a_1 - a_2) \in \ker r$. This yields $r \circ \varphi(a_1) = r \circ \varphi(a_2)$ and the map ρ is well defined.

Now let us show ρ is order zero. Suppose $c_1, c_2 \in C$ satisfy $c_1c_2 = 0$. Let $a_1, a_2 \in A$ such that $q(a_1) = c_1$ and $q(a_2) = c_2$. Then

$$q(a_1a_2) = q(a_1)q(a_2) = c_1c_2 = 0. \tag{5.66}$$

Thus $a_1a_2 \in \ker q$. Since $\varphi(\ker q) = \ker r$ we have that

$$r \circ \varphi(a_1a_2) = 0. \tag{5.67}$$

Since φ is order zero, we have $\varphi(a_1)\varphi(a_2) = \varphi(1_A)\varphi(a_1a_2)$. This yields

$$\begin{aligned} \rho(c_1)\rho(c_2) &= r \circ \varphi(a_1) \cdot r \circ \varphi(a_2) \\ &= r\left(\varphi(a_1)\varphi(a_2)\right) \\ &= r\left(\varphi(1_A)\varphi(a_1a_2)\right) \\ &= r(\varphi(1_A)) \cdot r(\varphi(a_1a_2)) \\ &\overset{(5.67)}{=} 0. \end{aligned} \tag{5.68}$$

This shows ρ is order zero. In the same way, we can show σ is a well defined injective order zero map.

Suppose $u_k \in A_\omega$ and $v_k \in B_\omega$ are represented by the sequences of unitaries $\left(u_k^{(i)}\right)_{i\in\mathbb{N}} \subset A$ and $\left(v_k^{(i)}\right)_{i\in\mathbb{N}} \subset B$ respectively. Let $x_k \in C_\omega$ and $y_k \in D_\omega$ be the elements represented by the sequences $\left(q\left(u_k^{(i)}\right)\right)_{i\in\mathbb{N}}$ and $\left(r\left(v_k^{(i)}\right)\right)_{i\in\mathbb{N}}$. Observe x_k and y_k are unitaries as well. Since

$$\lim_{i\to\omega}\left(\sum_{k=1}^{n} u_k^{(i)}\psi\varphi(a)u_k^{(i)*}\right) = a$$

for every $a \in A$, we have

$$\lim_{i\to\omega}\left(\sum_{k=1}^{n} q\left(u_k^{(i)}\right) q\circ\psi\varphi(a) q\left(u_k^{(i)*}\right)\right) = q(a). \tag{5.69}$$

This entails

$$\sum_{k=1}^{n} x_k \cdot q\circ\psi\varphi(a)\cdot x_k^* = q(a) \tag{5.70}$$

for every $a \in A$. Observe that, since $\varphi(a) \in r^{-1}\left(r\circ\varphi(a)\right)$, we have

$$\sigma\left(r\circ\varphi(a)\right) \overset{(5.65)}{=} q\circ\psi\varphi(a) \tag{5.71}$$

for any $a \in A$. Then, the following identity holds

$$\begin{aligned}\sum_{k=1}^{n} x_k \sigma\rho(c) x_k^* &\overset{(5.65)}{=} \sum_{k=1}^{n} x_k \sigma\left(r \circ \varphi(a_c)\right) x_k^* \\ &\overset{(5.71)}{=} \sum_{k=1}^{n} x_i q \circ \psi\varphi(a_c) x_k^* \\ &\overset{(5.70)}{=} q\left(a_c\right) \\ &= c \end{aligned} \tag{5.72}$$

for every $c \in C$. Similarly we have

$$d = \sum_{k=1}^{n} y_i \rho\sigma(d) y_i^* \tag{5.73}$$

for all $d \in D$. Therefore $C \cong_{(n)} D$. □

Now we are ready to prove that the ideal lattices of coloured isomorphic C^*-algebras are order isomorphic. We will also show that this induces coloured isomorphisms between quotients of the coloured isomorphic C^*-algebras.

Theorem 5.4.9. *Let A and B be unital C^*-algebras and suppose $A \cong_{(n)} B$. Then there exists an order preserving isomorphism $\Psi : \mathcal{I}(A) \longrightarrow \mathcal{I}(B)$ satisfying*

$$A/I \cong_{(n)} B/\Psi(I) \tag{5.74}$$

for every $I \trianglelefteq A$.

Proof. Suppose the c.p.c. order zero maps $\varphi : A \longrightarrow B$, $\psi : B \longrightarrow A$ and the unitaries $u_1, \ldots, u_n \in A_\omega, v_1, \ldots, v_n \in B_\omega$ induce an n-coloured isomorphism between A and B. Let $\mathcal{I}(A)$ and $\mathcal{I}(B)$ be the ideal lattices of A and B and remember that their orders are given by inclusion. By Lemma 1.4.12, the maps $\Psi : \mathcal{I}(A) \longrightarrow \mathcal{I}(B)$ and $\Phi : \mathcal{I}(B) \longrightarrow \mathcal{I}(A)$ given by

$$\Psi(I) = \psi^{-1}(I), \qquad \Phi(J) = \varphi^{-1}(J)$$

are well defined, with $I \in \mathcal{I}(A), J \in \mathcal{I}(B)$. By Lemma 5.4.7, the maps Ψ and Φ are mutual inverses. This shows Ψ and Φ are bijections. It is immediate that if $I_1 \subset I_2$, then $\Psi(I_1) \subset \Psi(I_2)$. This shows in particular that Ψ is an order isomorphism.

Finally, we would like to finish the proof using Lemma 5.4.8 with $C = A/I, D =$

$B/\psi^{-1}(I)$, and q and r as the corresponding quotient maps,

$$\begin{tikzcd} A \arrow[r, shift left, "\varphi"] \arrow[d, "q"'] & B \arrow[l, shift left, "\psi"] \arrow[d, "r"] \\ A/I \arrow[r, dashed, shift left] & B/\psi^{-1}(I). \arrow[l, dashed, shift left] \end{tikzcd}$$

In order to do this, we need to show $\varphi(I) = \psi^{-1}(I)$. Observe that the other condition, $\ker q = \psi(\ker r)$, is automatically satisfied since $I = \psi\left(\psi^{-1}(I)\right)$. By Lemma 5.4.7, $\varphi^{-1}(\psi^{-1}(I)) = I$. Thus

$$\varphi(I) = \varphi\left(\varphi^{-1}(\psi^{-1}(I))\right) = \psi^{-1}(I). \tag{5.75}$$

Therefore, by Lemma 5.4.8, A/I is n-coloured isomorphic to $B/\psi^{-1}(I)$. □

In Proposition 5.4.4, we established that $\ell^\infty(A)$ and $\ell^\infty(B)$ are coloured isomorphic if A and B are. As a corollary of the previous theorem, we will show that the ultrapowers are coloured isomorphic as well.

Corollary 5.4.10. *Let A and B be separable unital C^*-algebras and let ω be a free ultrafilter on $\mathbb{N}$. If $A \cong_{(n)} B$, then $A_\omega \cong_{(n)} B_\omega$.*

Proof. Suppose the c.p.c. order zero maps $\varphi : A \longrightarrow B$, $\psi : B \longrightarrow A$ and the unitaries $u_1, \ldots, u_n \in A_\omega, v_1, \ldots, v_n \in B_\omega$ induce an n-coloured isomorphism between A and B. By Proposition 5.4.4, $\ell^\infty(A) \cong_{(n)} \ell^\infty(B)$. Let us denote by ψ and φ the induced maps at level of $\ell^\infty(A)$ and $\ell^\infty(B)$. Set

$$I = \left\{(a_n) \in \ell^\infty(A) \mid \lim_{n\to\omega} a_n = 0\right\}, \qquad J = \left\{(b_n) \in \ell^\infty(B) \mid \lim_{n\to\omega} b_n = 0\right\}.$$

Since $A_\omega = \ell^\infty(A)/I$ and $B_\omega = \ell^\infty(A)/J$, in order to prove this corollary it is enough to show that $\psi^{-1}(I) = J$,

$$\begin{tikzcd} \ell^\infty(A) \arrow[r, shift left, "\varphi"] \arrow[d, "q"'] & \ell^\infty(B) \arrow[l, shift left, "\psi"] \arrow[d, "r"] \\ A_\omega = \ell^\infty(A)/I \arrow[r, dashed, shift left] & B_\omega = \ell^\infty(B)/J. \arrow[l, dashed, shift left] \end{tikzcd}$$

From (5.75) and the continuity of φ, we have

$$\psi^{-1}(I) = \varphi(I) \subset J. \tag{5.76}$$

For the same reasons we have

$$\varphi^{-1}(J) = \psi(J) \subset I. \tag{5.77}$$

Remember that by Lemma 5.4.7 we have

$$J = \psi^{-1}\left(\varphi^{-1}(J)\right). \tag{5.78}$$

Hence

$$J \overset{(5.78)}{=} \psi^{-1}\left(\varphi^{-1}(J)\right) \overset{(5.77)}{\subset} \psi^{-1}(I) = \varphi(I) \overset{(5.76)}{\subset} J. \tag{5.79}$$

Therefore $\psi^{-1}(I) = J$ and the result follows from Theorem 5.4.9. $\square$

Remark 5.4.11. Essentially the same proofs remain valid for general ultraproducts instead of ultrapowers (see Appendix A). Precisely, let $\{A_m\}_{m\in\mathbb{N}}$ and $\{B_m\}_{m\in\mathbb{N}}$ be families of separable unital C^*-algebras such that $A_m \cong_{(n)} B_m$ for all $m \in \mathbb{N}$. Then

$$\ell^\infty\left(\{B_m\}_{m\in\mathbb{N}}\right) \cong_{(n)} \ell^\infty\left(\{A_m\}_{m\in\mathbb{N}}\right)$$

and

$$\prod_{m\to\omega} B_m \cong_{(n)} \prod_{m\to\omega} A_m.$$

It is an straightforward consequence of the definition that if two matrix algebras are coloured isomorphic then they have to be isomorphic (this follows from the fact that order zero maps inducing the coloured isomorphism are injective). Using that the ideal lattices of coloured isomorphic algebras are isomorphic, this can be extended to general finite dimensional algebras.

Proposition 5.4.12. *Let A and B be unital separable C^*-algebras. Suppose $A \cong_{(n)} B$. If A is finite dimensional then $A \cong B$.*

Proof. By hypothesis there exist c.p.c. order zero maps $\varphi : A \longrightarrow B, \psi : B \longrightarrow A$ and unitaries $u_1, \ldots, u_n \in A_\omega, v_1, \ldots, v_n \in B_\omega$ such that

$$a = \sum_{i=1}^{n} u_i \psi\varphi(a) u_i^*, \qquad b = \sum_{i=1}^{n} v_i \varphi\psi(b) v_i^* \tag{5.80}$$

for all $a \in A, b \in B$. This immediately shows that if A is finite dimensional then B is also finite dimensional. If fact, we have $\dim B \leq n \dim A$. Since φ and ψ are injective maps, we conclude $\dim A = \dim B$ and that φ and ψ are surjective maps.

This implies A and B are direct sums of matrix algebras, say $A \cong \bigoplus_{j=1}^{r} M_{k_j}(\mathbb{C})$ and $B \cong \bigoplus_{j=1}^{s} M_{m_j}(\mathbb{C})$. By Theorem 5.4.9, the ideal lattices are isomorphic and this proves $r = s$. We will prove this proposition by induction over r.

Let $r = 1$. Then A and B are matrix algebras and we have shown that $\dim A = \dim B$. Hence $A \cong B$. Suppose we have proved that this is true if $r = p$. Let $r = p+1$ and set $I_1 = \bigoplus_{j=1}^{p} M_{k_j}(\mathbb{C})$ and $I_2 = M_{k_{p+1}}(\mathbb{C})$. By Lemma 5.4.7 we have $\varphi(I_2) = \psi^{-1}(I_2)$. Hence, using that φ is surjective and $A = I_1 \oplus I_2$, we have $\varphi(I_1) \oplus \varphi(I_2) = B$. Then, by Theorem 5.4.9, we obtain

$$I_1 \cong A/I_2 \cong_{(n)} B/\psi^{-1}(I_2) \cong \varphi(I_1). \tag{5.81}$$

By inductive hypothesis we obtain $I_1 \cong \varphi(I_1)$. Similarly, $I_2 \cong \varphi(I_2)$. Therefore $A \cong B$. $\square$

We will show now that coloured isomorphisms are rigid for the class of commutative C^*-algebras. This will follow from the fact that commutativity is preserved under coloured isomorphisms and this will give a colour reduction: n-coloured isomorphisms between separable unital C^*-algebras induce isomorphisms.

Proposition 5.4.13. *Let A and B be unital separable C^*-algebras. Suppose $A \cong_{(n)} B$. If A is commutative, then $A \cong B$.*

Proof. Let us show B is commutative. After this, the proof will be similar to the proof of Proposition 5.4.3. By hypothesis there exist c.p.c. order zero maps $\varphi : A \longrightarrow B, \psi : B \longrightarrow A$ and unitaries $u_1, \ldots, u_n \in A_\omega, v_1, \ldots, v_n \in B_\omega$ such that

$$a = \sum_{i=1}^{n} u_i \psi\varphi(a) u_i^*, \qquad b = \sum_{i=1}^{n} v_i \varphi\psi(b) v_i^* \tag{5.82}$$

for all $a \in A, b \in B$. From the commutativity of A and positive functional calculus for order zero maps, (Corollary 1.4.8), we have

$$\begin{aligned} \psi(b_1 b_2) &= \psi^{\frac{1}{2}}(b_1)\psi^{\frac{1}{2}}(b_2) \\ &= \psi^{\frac{1}{2}}(b_2)\psi^{\frac{1}{2}}(b_1) \\ &= \psi(b_2 b_1) \end{aligned} \tag{5.83}$$

for all $b_1, b_2 \in B$. Since ψ is injective (Remark 5.3.2), we obtain $b_1 b_2 = b_2 b_1$. This shows B is commutative.

Notice that from commutativity, we can simplify the previous identities. Precisely

$$a \overset{(5.82)}{=} \sum_{i=1}^{n} u_i \psi\varphi(a) u_i^* = \sum_{i=1}^{n} u_i u_i^* \psi\varphi(a) = n\psi\varphi(a) \tag{5.84}$$

for all $a \in A$ and, similarly, we obtain

$$b = n\varphi\psi(b) \tag{5.85}$$

for all $b \in B$. We will denote the support *-homomorphisms of $\varphi, \psi, \psi\varphi$ and $\varphi\psi$ as $\pi_\varphi, \pi_\psi, \pi_{\psi\varphi}$ and $\pi_{\varphi\psi}$ respectively. Remember that the support homomorphism π_φ maps A to $\mathcal{M}\left(C^*\left(\varphi(A)\right)\right)$. Since $n\varphi\psi(1_B) = 1_B$, we have that 1_B is an element of the C^*-algebra generated by $\varphi(A)$. Hence $\mathcal{M}\left(C^*\left(\varphi(A)\right)\right) = C^*\left(\varphi(A)\right) \subset B$. This shows π_φ is a map from A to B. Similarly π_ψ is a map from B to A.

By equations (5.84) and (5.85), we have

$$\psi\varphi = \frac{1}{n}\mathrm{id}_A, \qquad \varphi\psi = \frac{1}{n}\mathrm{id}_B. \tag{5.86}$$

Hence the support *-homomorphism of $\psi\varphi$ and $\varphi\psi$ are the identity maps. Precisely

$$\pi_{\psi\varphi} = \mathrm{id}_A, \qquad \pi_{\varphi\psi} = \mathrm{id}_B. \tag{5.87}$$

By Corollary 1.4.14, we have

$$\pi_\psi\pi_\varphi = \pi_{\psi\varphi} = \mathrm{id}_A, \qquad \pi_\varphi\pi_\psi = \pi_{\varphi\psi} = \mathrm{id}_B. \tag{5.88}$$

Therefore $A \cong B$. $\square$

Let us prove now that the trace simplices are homeomorphic. Recall that we endow the trace simplex $T(A)$ of a C^*-algebra A with the relative weak*-topology $\sigma(A^*, A)|_{T(A)}$.

Proposition 5.4.14. *Let A and B be separable unital C^*-algebras. Suppose $A \cong_{(n)} B$. Then the trace spaces of A and B are homeomorphic.*

Proof. By hypothesis there exist c.p.c. order zero maps $\varphi : A \longrightarrow B, \psi : B \longrightarrow A$ and unitaries $u_1, \ldots, u_n \in A_\omega, v_1, \ldots, v_n \in B_\omega$ such that

$$a = \sum_{i=1}^{n} u_i \psi\varphi(a) u_i^*, \qquad b = \sum_{i=1}^{n} v_i \varphi\psi(b) v_i^* \tag{5.89}$$

for all $a \in A, b \in B$. Let us denote the trace simplices of A and B as $T(A)$ and $T(B)$ respectively.

Let us define maps $\Phi : T(B) \longrightarrow T(A)$ and $\Psi : T(A) \longrightarrow T(B)$ by

$$\Phi(\tau) = \frac{1}{\tau \circ \varphi(1_A)} \tau \circ \varphi, \qquad \Psi(\rho) = \frac{1}{\rho \circ \psi(1_B)} \rho \circ \psi, \tag{5.90}$$

for all $\tau \in T(B)$ and $\rho \in T(A)$. First of all, let us show these maps are well defined; this basically will follow from Corollary 1.4.10 but we include these details for completeness. Let $\tau \in T(B)$ and $a_1, a_2 \in A$ be positive contractions, then

$$\begin{aligned}
\Phi(\tau)(a_1 a_2) &\overset{(5.90)}{=} \frac{1}{\tau \circ \varphi(1_A)} \tau \circ \varphi(a_1 a_2) \\
&= \frac{1}{\tau \circ \varphi(1_A)} \tau \left(\varphi^{\frac{1}{2}}(a_1) \varphi^{\frac{1}{2}}(a_2) \right) \\
&= \frac{1}{\tau \circ \varphi(1_A)} \tau \left(\varphi^{\frac{1}{2}}(a_2) \varphi^{\frac{1}{2}}(a_1) \right) \\
&= \frac{1}{\tau \circ \varphi(1_A)} \tau \circ \varphi(a_2 a_1) \\
&\overset{(5.90)}{=} \Phi(\tau)(a_2 a_1).
\end{aligned} \tag{5.91}$$

It is immediate that $\Phi(\tau)$ is a positive linear functional and $\Phi(\tau)(1_A) = 1$ for all $\tau \in T(B)$. Hence, $\Phi(\tau)$ is a trace on A. Similarly Ψ is well defined.

We will show that Ψ is the right inverse of Φ. Let $\rho \in T(A)$ and observe

$$\begin{aligned}
\Phi\Psi(\rho)(a) &\overset{(5.90)}{=} \frac{1}{\Psi(\rho)\left(\varphi(1_A)\right)} \Psi(\rho)\left(\varphi(a)\right) \\
&\overset{(5.90)}{=} \left(\frac{1}{\frac{\rho\psi\varphi(1_A)}{\rho\psi(1_B)}} \right) \left(\frac{1}{\rho\psi(1_B)} \rho\psi\varphi(a) \right) \\
&= \frac{1}{\rho\psi\varphi(1_A)} \rho\psi\varphi(a)
\end{aligned} \tag{5.92}$$

for all $a \in A$. By hypothesis we have

$$\rho(a) = \rho\left(\sum_{i=1}^{n} u_i \psi\varphi(a) u_i^* \right) = n\rho\left(\psi\varphi(a)\right) \tag{5.93}$$

for all $a \in A$. In particular

$$\rho\left(\psi\varphi(1_A)\right) = \frac{1}{n}\rho(1_A) = \frac{1}{n}. \tag{5.94}$$

Then

$$\begin{aligned}
\Phi\Psi(\rho)(a) &\overset{(5.92)}{=} \frac{1}{\rho\left(\psi\varphi(1_A)\right)} \rho\left(\psi\varphi(a)\right) \\
&\overset{(5.94)}{=} n\rho\left(\psi\varphi(a)\right) \\
&\overset{(5.93)}{=} \rho(a)
\end{aligned} \tag{5.95}$$

for all $a \in A$. This shows Ψ is the right inverse of Φ. In the same way we can prove

$$\Psi\Phi(\tau) = \tau \tag{5.96}$$

for all $\tau \in T(B)$. Therefore Ψ is the inverse of Φ (not only the right inverse).

To finish the proof, we have to show the maps Φ and Ψ are continuous with respect to the weak*-topologies. Let us prove Φ is continuous. In order to do this we to show that for every $a \in A$ there exists a positive constant M and $b \in B$ such that

$$|\Phi(\tau)(a)| \leq M\,|\tau(b)|\,. \tag{5.97}$$

Since $\tau(\varphi\psi(1_B)) \leq \tau(\varphi(1_A))$ and $\tau(\varphi\psi(1_B)) = 1/n$, we obtain

$$|\Phi(\tau)(a)| = \left|\frac{1}{\tau(\varphi(1_A))}\tau\varphi(a)\right| \leq n\,|\tau\varphi(a)| \tag{5.98}$$

for all $a \in A$ and $\tau \in T(B)$. This entails the continuity of Φ in the weak*-topology and, similarly, Ψ is weak*-continuous. Therefore $T(A)$ and $T(B)$ are homeomorphic. □

Remark 5.4.15. The homeomorphism Φ between $T(A)$ and $T(B)$ is not affine. Of course, if one of the trace simplices is finite then the homeomorphism Φ induces an affine isomorphism between $T(A)$ and $T(B)$. However, it is unknown to the author if the trace simplices are in general affinely isomorphic.

5.5 Kirchberg algebras and coloured isomorphisms

In this section we will prove that any two Kirchberg algebras are 2-coloured isomorphic. We will rely on the machinery developed in [8]. In particular, Theorem 4.8.9 will be fundamental for this Chapter.

Firstly, we will show coloured isomorphisms preserve weak pure infiniteness (see Section 1.7). This, in conjunction with Theorem 5.4.9, will show that pure infiniteness is preserved if the algebra is simple.

Proposition 5.5.1. *Let A and B be separable unital C^*-algebras such that $A \cong_{(n)} B$. If A is weakly purely infinite then B is weakly purely infinite.*

Proof. Since A is weakly purely infinite then A_ω is traceless by Theorem 1.7.10. By Corollary 5.4.10, $A_\omega \cong_{(n)} B_\omega$. Since the trace spaces of A_ω and B_ω are homeomorphic by Proposition 5.4.12, B_ω is also traceless. Finally, again by Theorem 1.7.10, B is weakly purely infinite. □

Corollary 5.5.2. *Let A and B be separable unital C^*-algebras such that $A \cong_{(n)} B$. If A is simple and purely infinite then B is simple and purely infinite. Moreover, if A is a Kirchberg algebra then B is a Kirchberg algebra as well.*

Proof. If A is simple then B is simple as well by Proposition 5.4.9. Since A is purely infinite, in particular, A is weakly purely infinite. By Proposition 5.5.1, B is weakly purely infinite. By Theorem 1.7.11, simplicity and weak pure infiniteness imply pure infiniteness. Hence B is purely infinite. If additionally A is nuclear (so A is a Kirchberg algebra), then B is nuclear as well by Proposition 5.4.6. Therefore B is a Kirchberg algebra. □

In light of Corollary 5.5.2, we already know that if a Kirchberg algebra A is coloured isomorphic to some other C^*-algebra B then B has to be a Kirchberg algebra as well. We will proceed to prove the converse.

As mentioned earlier, Theorem 4.8.9 will be highly important for this section. In order to have access to this theorem, we must be able to verify when a c.p.c. order zero map ϕ satisfies that, for any non zero $f \in (C_0(0,1])_+$, $f(\phi)$ is injective. The following lemma, which is an straightforward application of [8, Lemma 9.8], will indicate us when certain type of c.p.c. order zero maps satisfy this technical condition.

Lemma 5.5.3 ([8, Lemma 9.8])**.** *Let A be a unital Kirchberg algebra and let $h \in A_+$ with spectrum $[0,1]$. Suppose $\vartheta : A \longrightarrow A$ is a c.p.c. order zero map such that $\|\vartheta(a)\| = \|a\|$ for all $a \in A$ and define $\phi : A \longrightarrow A \otimes \mathcal{Z}$ by*

$$\phi(a) = \vartheta(a) \otimes h, \qquad a \in A.$$

Then ϕ is a c.p.c. order zero map with the property that, for any non zero $f \in (C_0(0,1])_+$, $f(\phi)$ is injective.

Proof. Let $t \in [0,1)$. Since ϑ is isometric, we have

$$\big\|(\vartheta(1_A) - t)_+\big\| = 1 - t. \tag{5.99}$$

This shows that the order zero map $(\vartheta - t)_+$ has norm equal to $1 - t$ by Corollary 1.4.8. Since B is simple, by Lemma 1.4.12, $(\vartheta - t)_+$ is injective. Then, by [8, Lemma 9.8], the map $\tilde{\phi} : A \longrightarrow C_0(0,1] \otimes A$ given by

$$\tilde{\phi}(a) = \mathrm{id}_{(0,1]} \otimes \vartheta(a), \qquad a \in A, \tag{5.100}$$

is a c.p.c. order zero map with the property that, for any non zero $f \in (C_0(0,1])_+$, $f(\tilde{\phi})$ is injective. By [62, Lemma 3.3.2], $C_0(0,1]$ is canonically isomorphic to the universal C^*-algebra generated by a positive contraction and we identify $\mathrm{id}_{(0,1]}$ with the universal generator. Hence, by functional calculus, we have that the C^*-algebra generated by h is isomorphic to $C_0(0,1]$. After identifying $\mathrm{id}_{(0,1]}$ with h, we obtain that ϕ is a c.p.c. order zero map with the property that, for any non zero $f \in (C_0(0,1])_+$, $f(\phi)$ is injective. $\square$

The proof of the the following theorem is based on the work carried out in [8]. It is well known that by Kirchberg's embedding theorems, we can always embed any Kirchberg algebra into another Kirchberg algebra. Then, using a positive element h of $\mathcal{Z}$ with spectrum $[0,1]$, we consider the maps given by the tensor product of the composition of the embeddings with this positive element. By Theorem 4.8.9, these maps are approximately unitarily equivalent to the corresponding identity map tensorised with h. We will finish by repeating this process with $1_{\mathcal{Z}} - h$ instead of h. It is important to notice that this theorem does not require the UCT.

Theorem 5.5.4. *Let A be a Kirchberg algebra and let B be a separable unital C^*-algebra. Then $A \cong_{(2)} B$ if and only if B is a Kirchberg algebra.*

Proof. Firstly, if A is a Kirchberg algebra and $A \cong_{(2)} B$, by Proposition 5.5.2, B is a Kirchberg algebra as well.

Conversely, let A and B be Kirchberg algebras. By Kirchberg's embedding theorems (Theorem 1.7.5), there are embeddings $A \hookrightarrow \mathcal{O}_2$ and $B \hookrightarrow \mathcal{O}_2$. Let $p \in \mathcal{P}(A)$ and $q \in \mathcal{P}(B)$ be properly infinite projections of K_0-class 0. By [84, Proposition 4.2.3.(ii)], there exist embeddings $\mathcal{O}_2 \hookrightarrow pAp \subset A$ and $\mathcal{O}_2 \hookrightarrow qBq \subset B$. Let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ given by composing the previous embeddings,

$$\begin{tikzcd} A \arrow[r, hook] \arrow[rr, bend right, "\varphi"'] & \mathcal{O}_2 \arrow[r, hook] & B, \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, hook] \arrow[rr, bend right, "\psi"'] & \mathcal{O}_2 \arrow[r, hook] & A. \end{tikzcd}$$

Notice that A and B are $\mathcal{Z}$-stable because, by Kirchberg embedding theorems (Theorem 1.7.5), Kirchberg algebras are $\mathcal{O}_\infty$-stable and, by Theorem 4.4.8, $\mathcal{O}_\infty$ tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$. Then by Theorem 4.5.4 there exist isomorphisms $\sigma : A \otimes \mathcal{Z} \longrightarrow A$ and $\rho : B \otimes \mathcal{Z} \longrightarrow B$, and unitaries $x \in A_\omega, y \in B_\omega$ such that

$$a = x^*\sigma(a \otimes 1_{\mathcal{Z}})x, \tag{5.101}$$

$$b = y^*\rho(b \otimes 1_{\mathcal{Z}})y \tag{5.102}$$

for all $a \in A$ and $b \in B$. Let $h \in \mathcal{Z}$ be a positive element with spectrum $[0,1]$ and let us define c.p.c. order zero maps $\theta : A \longrightarrow B$ and $\zeta : B \longrightarrow A$ by

$$\theta(a) = \rho\left(\varphi(a) \otimes h\right), \qquad a \in A, \tag{5.103}$$

$$\zeta(b) = \sigma(\psi(b) \otimes h), \qquad b \in B. \tag{5.104}$$

In particular we have

$$\zeta\theta(a) = \sigma(\psi(\theta(a)) \otimes h), \qquad a \in A. \tag{5.105}$$

We would like to show, using Theorem 4.8.9, that $\psi(\theta(\cdot)) \otimes h$ and $\mathrm{id}_A \otimes h$ are approximately unitarily equivalent. In order to do this, we have to verify that $f\left(\psi(\theta(\cdot)) \otimes h\right)$ and $f\left(\mathrm{id}_A \otimes h\right)$ are injective for all $f \in \left(C_0(0,1]\right)_+$.

By Lemma 5.5.3, this boils down to checking that $\psi\theta$ and id_A are isometric. Since it is obvious that id_A is isometric, we only have to show $\psi\theta$ is isometric but this immediately follows from the fact that the minimal norm is a cross norm, $\|h\| = 1$ and ψ, ρ and φ are isometric. We include these details for completeness. Since ψ, φ and ρ are *-homomorphisms, they are isometric. Hence

$$\begin{aligned}
\|\psi\left(\theta(a)\right)\| &= \|\theta(a)\| \\
&\overset{(5.103)}{=} \|\rho\left(\varphi(a) \otimes h\right)\| \\
&= \|\varphi(a) \otimes h\| \\
&= \|\varphi(a)\|\|h\| \\
&= \|\varphi(a)\| \\
&= \|a\|
\end{aligned} \tag{5.106}$$

for all $a \in A$. This shows we can apply Theorem 4.8.9 to the pair $\psi(\theta(\cdot)) \otimes h$ and $\mathrm{id}_A \otimes h$. Therefore there exists a unitary $w_1 \in \left(A \otimes \mathcal{Z}\right)_\omega$ such that

$$a \otimes h = w_1\left(\psi(\theta(a)) \otimes h\right) w_1^*, \qquad a \in A. \tag{5.107}$$

Observe that $1_{\mathcal{Z}} - h$ also has spectrum $[0,1]$ and, because of this, we can repeat the previous arguments after replacing $\mathrm{id}_A \otimes h$ with $\mathrm{id}_A \otimes (1_{\mathcal{Z}} - h)$. Hence, by Theorem 4.8.9 applied now to the maps $\psi(\theta(\cdot)) \otimes h$ and $\mathrm{id}_A \otimes (1_{\mathcal{Z}} - h)$, there exists a unitary $w_2 \in \left(A \otimes \mathcal{Z}\right)_\omega$ such that

$$a \otimes (1_{\mathcal{Z}} - h) = w_2\left(\psi(\theta(a)) \otimes h\right) w_2^*, \qquad a \in A. \tag{5.108}$$

Then, by identities (5.107) and (5.108), we have

$$\begin{aligned} a \otimes 1_{\mathcal{Z}} &= a \otimes h + a \otimes (1_{\mathcal{Z}} - h) \\ &= w_1 \left(\psi(\theta(a)) \otimes h\right) w_1^* + w_2 \left(\psi(\theta(a)) \otimes h\right) w_2^* \end{aligned} \tag{5.109}$$

for all $a \in A$. Thus

$$\begin{aligned} a &\overset{(5.101)}{=} x^* \sigma(a \otimes 1_{\mathcal{Z}}) x \\ &= x^* \sigma\left(a \otimes h + a \otimes (1_{\mathcal{Z}} - h)\right) x \\ &\overset{(5.109)}{=} x^* \sigma\left(w_1 \left(\psi(\theta(a)) \otimes h\right) w_1^* + w_2 \left(\psi(\theta(a)) \otimes h\right) w_2^*\right) x \\ &= x^* \sigma(w_1) \sigma\left(\left(\psi(\theta(a)) \otimes h\right)\right) \sigma(w_1)^* x \\ &\quad + x^* \sigma(w_2) \sigma\left(\psi(\theta(a)) \otimes h\right) \sigma(w_2)^* x \end{aligned} \tag{5.110}$$

for all $a \in A$. Notice that implicitly we are extending σ to $(A \otimes \mathcal{Z})_\omega$. After setting $u_i = x^* \sigma(w_i)$, we obtain

$$a = u_1 \zeta\theta(a) u_1^* + u_2 \zeta\theta(a) u_2^* \tag{5.111}$$

for all $a \in A$. Similarly there exist unitaries $v_1, v_2 \in B_\omega$ such that

$$b = v_1 \theta\zeta(b) v_1^* + v_2 \theta\zeta(b) v_2^* \tag{5.112}$$

for all $b \in B$. Therefore the c.p.c order zero maps $\theta : A \longrightarrow B, \zeta : B \longrightarrow A$ and the unitaries $u_1, u_2 \in A_\omega, v_1, v_2 \in B_\omega$ induce a 2-coloured isomorphism between A and B. $\square$

5.6 Finite algebras and coloured isomorphisms

In this section we will show an anologue result for separable, simple, unital, finite, nuclear and $\mathcal{Z}$-stable C^*-algebras with unique trace which satisfy the UCT. In general, the idea of the proofs of this section is the same idea we used for Kirchberg algebras. Unfortunately, there are more technical details we have to deal in this case. In this setting, the tracial behavior of maps will play a prominent role. Firstly, we will establish an analogue to Theorem 4.8.9 in this setting, which is built from Theorem 4.8.8 but with additional hypothesis (this will simplify its proof). With this theorem in hand, we will show UHF algebras are 2-coloured equivalent to the Jiang-Su algebra $\mathcal{Z}$. The proof we provide is constructive and illustrate in a nutshell the key ideas behind the more general results to follow.

In order to prove our first technical lemma, we will make use of the heavy machinery developed in [8]. Let us recall the definition of totally full *-homomorphisms and totally full elements first.

Definition 5.6.1 ([8, Definition 1.1])**.** Let A and B be C^*-algebras, a *-homomorphism $\pi : A \longrightarrow B$ is *totally full* if if for every non zero element $a \in A$, $\pi(a)$ is *full* (*i.e.* $\pi(a)$ generates B as a closed two-sided ideal). Likewise, a positive element $b \in B_+$ is *totally full* if $b \neq 0$ and the *-homomorphism $C_0((0, \|b\|]) \longrightarrow B$ given by $\mathrm{id}_{(0,\|b\|]} \mapsto b$ is totally full.

The following lemma is a slight variation of [8, Theorem 5.2]. Notice that the algebra B has a unique trace instead of $T(B)$ being a Bauer simplex. This will simplify its proof but the idea is the same. This lemma will play the role of Theorem 4.8.9 in the previous section.

Lemma 5.6.2. *Let A be a separable, unital, nuclear C^*-algebra and let B be a simple, separable, unital, finite, $\mathcal{Z}$-stable, exact C^*-algebra with unique trace τ_B. Let $\varphi_1, \varphi_2 : A \longrightarrow B_\omega$ be c.p.c. order zero maps such that $\varphi_1(a)$ is full for every non zero $a \in A$, and*

$$\tau \circ \varphi_1^n(a) = \tau \circ \varphi_2^n(a) \neq 0 \tag{5.113}$$

for all $a \in A$ and all $n \in \mathbb{N}$, where φ_i^n is understood as order zero functional calculus. Then there exists a unitary $w \in B_\omega$ such that

$$\varphi_1(a) = w\varphi_2(a)w^* \tag{5.114}$$

for all $a \in A$.

Proof. Let us sketch the proof first. Using support order zero maps of φ_1 and φ_2 we will define a new c.p.c. order zero map $\pi : A \longrightarrow M_2(B_\omega)$. We will use the 2×2 trick (Lemma 4.8.5) with this map to show φ_1 and φ_2 are unitarily equivalent. In order to do this, we have to see

$$\begin{pmatrix} \varphi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & \varphi_2(1_A) \end{pmatrix}$$

are unitarily equivalent in some relative commutant C and, by the machinery of [8], this boils down to checking that h_1 and h_2 are totally full in C and that all powers of these elements agree on all traces on C.

First of all, by Ozawa's density theorem [71, Theorem 8], B_ω has a unique trace τ_{B_ω} given by

$$\tau_{B_\omega}\left(\overline{(x_n)_{n=1}^\infty}\right) = \lim_{n \to \omega} \tau_B(x_n), \tag{5.115}$$

where $\overline{(x_n)}_{n=1}^{\infty}$ denotes the class of $(x_n)_{n=1}^{\infty}$ in B_ω. Let $J_{B_\omega} \trianglelefteq B_\omega$ be the trace kernel ideal of B, *i.e.*

$$J_{B_\omega} = \left\{ \overline{(x_n)}_{n=1}^{\infty} \in B_\omega \mid \lim_{n \to \omega} \tau_B(x_n^* x_n) = 0 \right\}.$$

By [8, Lemma 1.14], there are support order zero maps $\hat{\varphi}_1, \hat{\varphi}_2 : A \longrightarrow B_\omega$ of φ_1 and φ_2 such that the induced maps $\overline{\hat{\varphi}}_1, \overline{\hat{\varphi}}_2 : A \longrightarrow B_\omega / J_{B_\omega}$ are *-homomorphisms. Observe that [8, Lemma 1.14] requires the maps $\tau \mapsto d_\tau(\varphi_1(1_A))$ and $\tau \mapsto d_\tau(\varphi_2(1_A))$, from $T(B_\omega)$ to $[0,1]$, to be continuous but since B_ω has a unique trace this condition is automatically satisfied.

Let us define a c.p.c. map $\pi : A \longrightarrow M_2(B_\omega)$ by

$$\pi(a) = \begin{pmatrix} \hat{\varphi}_1(a) & 0 \\ 0 & \hat{\varphi}_2(a) \end{pmatrix}, \qquad a \in A, \tag{5.116}$$

and set $C := M_2(B_\omega) \cap \pi(A)' \cap \left\{1_{M_2(B_\omega)} - \pi(1_A)\right\}^{\perp}$. Let us denote the unique trace of $M_2(B_\omega)$ as $\overline{\tau}$ (observe that $\overline{\tau} = \tau_{M_2(\mathbb{C})} \otimes \tau_{B_\omega}$). By hypothesis, $\varphi_1(a)$ is full for all non zero $a \in A_+$, then $\pi(a)$ is also full since

$$0 \leq \begin{pmatrix} \varphi_1(a) & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} \hat{\varphi}_1(a) & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} \hat{\varphi}_1(a) & 0 \\ 0 & \hat{\varphi}_2(a) \end{pmatrix} = \pi(a). \tag{5.117}$$

Hence, by [8, Theorem 4.1], C has *strict comparison (of positive elements with respect to traces)* and the set of traces

$$T_0 = \{\overline{\tau}\left(\pi(a)(\cdot)\right) \mid a \in A_+,\ \overline{\tau}(\pi(a)) = 1\} \tag{5.118}$$

has weak*-closed convex hull equal to $T(C)$. Recall that strict comparison means that if $d_\rho(c_1) < d_\rho(c_2)$ for all $\rho \in T(C)$, then $c_1 \preceq c_2$ for $k \in \mathbb{N}$ and $c_1, c_2 \in M_k(C)_+$ (see Definition 1.10.5).

Set

$$h_1 = \begin{pmatrix} \varphi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & \varphi_2(1_A) \end{pmatrix}. \tag{5.119}$$

Let us verify h_1 and h_2 are elements of C. Observe

$$\begin{aligned} h_1\pi(a) &= \begin{pmatrix} \varphi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \hat{\varphi}_1(a) & 0 \\ 0 & \hat{\varphi}_2(a) \end{pmatrix} \\ &= \begin{pmatrix} \varphi_1(1_A)\hat{\varphi}_1(a) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \varphi_1(a) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.120}$$

Similarly

$$\begin{aligned} \pi(a)h_1 &= \begin{pmatrix} \hat{\varphi}_1(a) & 0 \\ 0 & \hat{\varphi}_2(a) \end{pmatrix} \cdot \begin{pmatrix} \varphi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \hat{\varphi}_1(a)\varphi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \varphi_1(a) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.121}$$

This shows $h_1\pi(a) = \pi(a)h_1$ for all $a \in A$. Now, let us check $\left(1_{M_2(B_\omega)} - \pi(1_A)\right) h_1 = 0$,

$$\begin{aligned} \left(1_{M_2(B_\omega)} - \pi(1_A)\right) h_1 &= \begin{pmatrix} 1_A - \hat{\varphi}_1(1_A) & 0 \\ 0 & 1_A - \hat{\varphi}_2(1_A) \end{pmatrix} \begin{pmatrix} \varphi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \left(1_A - \hat{\varphi}_1(1_A)\right)\varphi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \varphi_1(1_A) - \varphi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.122}$$

This proves $h_1 \in C$. The same computations with h_2 instead of h_1 show $h_2 \in C$. By hypothesis, $\varphi_1(1_A)$ is full in B, so h_1 is full in $M_2(B_\omega)$ and hence C is full in $M_2(B_\omega)$. Let us verify now that $\rho\left(h_1^n\right) = \rho\left(h_2^n\right)$ for all $\rho \in T(C)$ and $n \in \mathbb{N}$. Consider $\rho = \overline{\tau}(\pi(a)\cdot) \in T_0$

where $a \in A_+$ is such that $\overline{\tau}(\pi(a)) = 1$. Hence

$$\begin{aligned}
\rho\left(h_1^n\right) &= \overline{\tau}\left(\pi(a)h_1^n\right) \\
&= \overline{\tau}\left(\begin{pmatrix} \hat{\varphi}_1(a) & 0 \\ 0 & \hat{\varphi}_2(a) \end{pmatrix}\begin{pmatrix} \varphi_1(1_A)^n & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&= \overline{\tau}\left(\begin{pmatrix} \hat{\varphi}_1(a)\varphi_1(1_A)^n & 0 \\ 0 & \end{pmatrix}\right) \\
&= \overline{\tau}\left(\begin{pmatrix} \varphi_1^n(a) & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&= \frac{\tau_{B_\omega}(\varphi_1^n(a))}{2}.
\end{aligned} \tag{5.123}$$

Similarly we obtain

$$\rho\left(h_2^n\right) = \frac{\tau_{B_\omega}\left(\varphi_2^n(a)\right)}{2}. \tag{5.124}$$

By hypothesis $\tau_{B_\omega}\left(\varphi_1^n(a)\right) = \tau_{B_\omega}\left(\varphi_2^n(a)\right)$ for all $a \in A$. Hence

$$\rho\left(h_1^n\right) = \rho\left(h_2^n\right) \tag{5.125}$$

for all $n \in \mathbb{N}$ and $\rho \in T_0$. Since the convex hull of T_0 is weak*-dense in $T(C)$ we have

$$\rho\left(h_1^n\right) = \rho\left(h_2^n\right) \tag{5.126}$$

for all $n \in \mathbb{N}$ and $\rho \in T(C)$. Since $\tau\left(\varphi_1^n(a)\right) \neq 0$ for all non zero $a \in A$ and $n \in \mathbb{N}$, then

$$\tau_{B_\omega}\left(f(\varphi_1(a))\right) = \tau_{B_\omega}\left(f(\varphi_2(a))\right) \neq 0 \tag{5.127}$$

for all $a \in A$ and $f \in C((0,1])_+$. Hence

$$\rho(f(h_1)) = \rho(f(h_2)) \neq 0 \tag{5.128}$$

for all $\rho \in T(B_\omega)$ and $f \in C_0((0,1])_+$. Let us prove now that h_1 and h_2 are totally full elements in C. In order to do this, by Definition 5.6.1, we need to show that $f(h_1)$ and $f(h_2)$ are full in C for $f \in C_0((0,1])_+$. Consider a non zero positive contraction $c \in C$. Since $\rho(f(h_1)) \neq 0$ and $T(C)$ is compact, we have

$$\inf_{\rho \in T(C)} \rho(f(h_1)) > 0. \tag{5.129}$$

Thus, there exists $n \in \mathbb{N}$ such that

$$nd_\rho\left(f(h_1)\right) \geq n\rho(f(h_1)) > 1 \geq d_\rho(c) \tag{5.130}$$

for all $\rho \in T(C)$. By strict comparison c is Cuntz below $f(h_1)^{\oplus n}$ in C. This immediately shows that for every $\varepsilon > 0$ there exist elements $y_1, \ldots, y_n \in C$ such that

$$\left\| c - \sum_{k=1}^{n} y_k^* f(h_1) y_k \right\| < \varepsilon. \tag{5.131}$$

Therefore c is in the closed two sided ideal generated by $f(h_1)$ in C. Since c was arbitrary, this shows $f(h_1)$ is full. This shows h_1 is totally full. Analogously, we can show h_2 is totally full.

By [8, Theorem 5.1], h_1 and h_2 are unitarily equivalent in the unitisation of C. Rørdam showed that simple, separable, unital, finite $\mathcal{Z}$-stable C^*-algebras have stable rank one [83, Theorem 6.7]. Then, by Lemma A.1.6, B_ω has stable rank one. Finally, by the 2×2 trick (Lemma 4.8.5), φ_1 and φ_2 are unitarily equivalent. □

Now we are ready to establish the following example.

Example 5.6.3. The CAR algebra M_{2^∞} is 2-coloured equivalent to the Jiang-Su algebra $\mathcal{Z}$.

Proof. Let $\mathcal{Z}_{2^\infty,3^\infty}$ be the generalized dimension drop algebra. Recall

$$\mathcal{Z}_{2^\infty,3^\infty} = \left\{ f \in C\left([0,1], M_{2^\infty} \otimes M_{3^\infty}\right) \mid f(0) \in M_{2^\infty} \otimes \mathbb{C}1_{M_{3^\infty}},\, f(1) \in \mathbb{C}1_{M_{2^\infty}} \otimes M_{3^\infty} \right\}.$$

By [85, Proposition 3.3], $\mathcal{Z}_{2^\infty,3^\infty}$ embeds unitally into $\mathcal{Z}$. We can regard $\mathcal{Z}_{2^\infty,3^\infty}$ as a C^*-subalgebra of $C[0,1] \otimes M_{2^\infty} \otimes M_{3^\infty}$ and consider $h : [0,1] \longrightarrow \mathbb{R}$ as $h(t) = 1 - t$. Define a c.p.c. order zero map $\tilde{\varphi} : M_{2^\infty} \longrightarrow \mathcal{Z}_{2^\infty,3^\infty}$ by

$$\tilde{\varphi}(a) = h \otimes a \otimes 1_{M_{3^\infty}}, \qquad a \in M_{2^\infty}. \tag{5.132}$$

Observe that since $h(1) = 0$, $\tilde{\varphi}(a)$ is an element of $\mathcal{Z}_{2^\infty,3^\infty}$. By Remark 1.9.4, there is a unital standard embedding $\iota : \mathcal{Z}_{2^\infty,3^\infty} \longrightarrow \mathcal{Z}$, *i.e.* the trace of $\mathcal{Z}$ induces the Lebesgue trace on $\mathcal{Z}_{2^\infty,3^\infty}$.

Let $\varphi : M_{2^\infty} \longrightarrow \mathcal{Z}$ be the order zero map given by the composition of $\tilde{\varphi}$ and the standard embedding ι of $\mathcal{Z}_{2^\infty,3^\infty}$ into $\mathcal{Z}$,

$$\begin{tikzcd} M_{2^\infty} \arrow[r, "\tilde{\varphi}"] \arrow[rr, bend right, "\varphi"'] & \mathcal{Z}_{2^\infty,3^\infty} \arrow[r, hook, "\iota"] & \mathcal{Z}. \end{tikzcd}$$

In particular

$$\begin{aligned}\tau_{\mathcal{Z}} \circ \varphi^n(a) &= \tau_{\mathcal{Z}}(\iota\,(h^n \otimes a \otimes 1_{M_{3^\infty}})) \\ &= \tau_{M_{2^\infty}}(a) \cdot \tau_{M_{3^\infty}}(1_{M_{3^\infty}}) \cdot \int_0^1 h^n(t)dt \\ &= \frac{\tau_{M_{2^\infty}}(a)}{n+1}, \end{aligned} \tag{5.133}$$

where φ^n is understood as order zero functional calculus. Recall that the CAR algebra M_{2^∞} is $\mathcal{Z}$-stable. Then, by Proposition 4.5.4, there exists an isomorphism $\sigma : M_{2^\infty} \otimes \mathcal{Z} \longrightarrow M_{2^\infty}$ and a unitary $x \in (M_{2^\infty})_\omega$ such that

$$a = x^* \sigma(a \otimes 1_{\mathcal{Z}})x \tag{5.134}$$

for all $a \in M_{2^\infty}$. Let $\psi : \mathcal{Z} \longrightarrow M_{2^\infty}$ be the composition of the second factor embedding of $\mathcal{Z}$ into $M_{2^\infty} \otimes \mathcal{Z}$ and the isomorphism σ. Notice that ψ is a *-homomorphism,

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, "1_{M_{2^\infty}} \otimes \mathrm{id}_{\mathcal{Z}}"] \arrow[rr, bend right, "\psi"'] & M_{2^\infty} \otimes \mathcal{Z} \arrow[r, "\sigma"] & M_{2^\infty}. \end{tikzcd}$$

Observe that since ψ is a *-homomorphism, its support *-homomorphisms is ψ itself. Recall that the support *-homomorphism of the composition is the composition of the support *-homomomorphisms (Corollary 1.4.14). Then we have

$$\begin{aligned}(\psi\varphi)^n(a) &= (\psi\varphi(1_A))^n\, \psi\pi_\varphi(a) \\ &= \psi\left(\varphi(1_A)^n\right) \psi\pi_\varphi(a) \\ &= \psi\left(\varphi(1_A)^n \pi_\varphi(a)\right) = \psi\varphi^n(a) \end{aligned} \tag{5.135}$$

for all $a \in A$.

Now, since ψ is unital, it is trace preserving. Hence

$$\begin{aligned}\tau_{M_{2^\infty}} \circ (\psi\varphi)^n(a) &\overset{(5.135)}{=} \tau_{M_{2^\infty}} \circ \psi\varphi^n(a) \\ &= \tau_{\mathcal{Z}} \circ \varphi^n(a) \\ &\overset{(5.133)}{=} \frac{\tau_{M_{2^\infty}}(a)}{n+1}. \end{aligned} \tag{5.136}$$

Set c.p.c. order zero maps $\rho_1, \rho_2 : M_{2^\infty} \longrightarrow M_{2^\infty}$ by

$$\rho_1(a) = \sigma(a \otimes \iota\,(h \otimes 1_{M_{2^\infty}} \otimes 1_{M_{3^\infty}})), \qquad a \in M_{2^\infty}, \tag{5.137}$$

$$\rho_2(a) = \sigma(a \otimes \iota((1-h) \otimes 1_{M_{2^\infty}} \otimes 1_{M_{3^\infty}})), \qquad a \in M_{2^\infty}. \tag{5.138}$$

Observe that

$$\rho_1(a) + \rho_2(a) = \sigma\left(a \otimes \iota\left(1_{C[0,1]} \otimes 1_{M_{2^\infty}} \otimes 1_{M_{3^\infty}}\right)\right) = \sigma(a \otimes 1_{\mathcal{Z}}) \tag{5.139}$$

for all $a \in M_{2^\infty}$. We also have

$$\begin{aligned}\tau_{M_{2^\infty}} \circ \rho_1^n(a) &= \tau_{M_{2^\infty}} \circ \sigma\left(a \otimes \iota\left(h^n \otimes 1_{M_{2^\infty}} \otimes 1_{M_{3^\infty}}\right)\right) \\ &= \tau_{M_{2^\infty}} \otimes \tau_{\mathcal{Z}}(a \otimes \iota\left(h^n \otimes 1_{M_{2^\infty}} \otimes 1_{M_{3^\infty}}\right)) \\ &= \tau_{M_{2^\infty}}(a) \cdot \tau_{\mathcal{Z}} \circ \iota\left(h^n \otimes 1_{M_{2^\infty}} \otimes 1_{M_{3^\infty}}\right) \\ &\overset{(*)}{=} \tau_{M_{2^\infty}}(a) \cdot \int_0^1 h^n(t)dt \\ &= \frac{\tau_{M_{2^\infty}}(a)}{n+1}\end{aligned} \tag{5.140}$$

for all $a \in M_{2^\infty}$. Notice that $(*)$ is given because the trace of the Jiang-Su algebra induces the Lebesgue trace. Similarly

$$\begin{aligned}\tau_{M_{2^\infty}} \circ \rho_2^n(a) &= \tau_{M_{2^\infty}}(a) \cdot \int_0^1 (1-h)^n(t)dt \\ &= \frac{\tau_{M_{2^\infty}}(a)}{n+1}\end{aligned} \tag{5.141}$$

for all $a \in M_{2^\infty}$. This shows ρ_1 and ρ_2 tracially agree with $\psi\varphi$ on each power. Precisely,

$$\tau_{M_{2^\infty}} \circ \rho_1^n(a) = \tau_{M_{2^\infty}} \circ (\psi\varphi)^n(a), \qquad \tau_{M_{2^\infty}} \circ \rho_2^n(a) = \tau_{M_{2^\infty}} \circ (\psi\varphi)^n(a) \tag{5.142}$$

for all $a \in M_{2^\infty}$ and $n \in \mathbb{N}$. By Lemma 5.6.2, there exist unitaries $w_1, w_2 \in (M_{2^\infty})_\omega$ such that

$$\rho_1(a) = w_1\psi\varphi(a)w_1^*, \qquad \rho_2(a) = w_2\psi\varphi(a)w_2^* \tag{5.143}$$

for all $a \in M_{2^\infty}$. Set unitaries $u_1 = x^*w_1$ and $u_2 = x^*w_2$. Then

$$\begin{aligned}u_1\psi\varphi(a)u_1^* + u_1\psi\varphi(a)u_1^* &= x^*w_1\psi\varphi(a)w_1^*x + x^*w_2\psi\varphi(a)w_2^*x \\ &= x^*\left(w_1\psi\varphi(a)w_1^* + w_2\psi\varphi(a)w_2^*\right)x \\ &\overset{(5.143)}{=} x^*\left(\rho_1(a) + \rho_2(a)\right)x \\ &\overset{(5.139)}{=} x^*\sigma(a \otimes 1_{\mathcal{Z}})x \\ &\overset{(5.134)}{=} a\end{aligned} \tag{5.144}$$

for all $a \in M_{2^\infty}$.

Now notice that

$$\begin{aligned}\tau_{\mathcal{Z}} \circ (\varphi\psi)^n(b) &\overset{(5.133)}{=} \tau_{M_{2^\infty}}(\psi(b)) \cdot \int_0^1 h^n(t)dt \\ &= \frac{\tau_{M_{2^\infty}}(\sigma(1_{M_{2^\infty}} \otimes b))}{n+1} \\ &= \frac{\tau_{\mathcal{Z}}(b)}{n+1} \end{aligned} \tag{5.145}$$

for all $b \in \mathcal{Z}$. In order to finish we have to repeat the same constructions for $\varphi\psi$. We include those details for completeness. Since $\mathcal{Z}$ is strongly self-absorbing, by Theorem 4.5.4, there exists an *-isomorphism $\theta : \mathcal{Z} \otimes \mathcal{Z} \longrightarrow \mathcal{Z}$ and a unitary $y \in \mathcal{Z}_\omega$ such that $b = y^*\theta(b \otimes 1_{\mathcal{Z}})y$ for all $b \in \mathcal{Z}$. Define c.p.c. order zero maps $\zeta_1, \zeta_2 : \mathcal{Z} \longrightarrow \mathcal{Z}$ by

$$\zeta_1(b) = \theta(b \otimes \iota(h \otimes 1_{M_{2^\infty}} \otimes 1_{M_{3^\infty}})), \qquad b \in \mathcal{Z}, \tag{5.146}$$

$$\zeta_2(b) = \theta(b \otimes \iota((1_{\mathcal{Z}} - h) \otimes 1_{M_{2^\infty}} \otimes 1_{M_{3^\infty}})), \qquad b \in \mathcal{Z}. \tag{5.147}$$

By construction, we have

$$\tau_{\mathcal{Z}} \circ \zeta_1^n(b) = \tau_{\mathcal{Z}} \circ (\varphi\psi)^n(b), \qquad \tau_{\mathcal{Z}} \circ \zeta_2^n(b) = \tau_{\mathcal{Z}} \circ (\varphi\psi)^n(b) \tag{5.148}$$

for all $b \in \mathcal{Z}$ and $n \in \mathbb{N}$. Then, by Lemma 5.6.2, there exist unitaries $\tilde{w}_1, \tilde{w}_2 \in \mathcal{Z}_\omega$ such that

$$\zeta_1(b) = \tilde{w}_1\varphi\psi(b)\tilde{w}_1^* \qquad \zeta_2(b) = \tilde{w}_2\varphi\psi(b)\tilde{w}_2^* \tag{5.149}$$

for all $b \in \mathcal{Z}$. After setting unitaries $v_1 = y^*\tilde{w}_1$ and $v_2 = y^*\tilde{w}_2$, we obtain

$$b = v_1\varphi\psi(b)v_1^* + v_2\varphi\psi(b)v_2^* \tag{5.150}$$

for all $b \in \mathcal{Z}$. Therefore, the c.p.c. order zero maps $\varphi : M_{2^\infty} \longrightarrow \mathcal{Z}$, $\psi : \mathcal{Z} \longrightarrow M_{2^\infty}$ and unitaries $u_1, u_2 \in (M_{2^\infty})_\omega$, $v_1, v_2 \in \mathcal{Z}_\omega$ implement a 2-coloured isomorphism between M_{2^∞} and $\mathcal{Z}$. □

From the previous proof, we can see that one way of obtaining 2-coloured isomorphisms between separable, simple, unital, finite, nuclear and $\mathcal{Z}$-stable C^*-algebras with unique trace boils down to finding c.p.c. order zero maps $\varphi : A \longrightarrow B, \psi : B \longrightarrow A$ such that

$$\tau_A \circ (\psi\varphi)^n(a) = \frac{\tau_A(a)}{n+1}, \qquad \tau_B \circ (\varphi\psi)^n(b) = \frac{\tau_B(b)}{n+1},$$

for all $a \in A, b \in B$ and $n \in \mathbb{N}$. Then Lemma 5.6.2 will give us that $\psi\varphi$ is approximately unitarily equivalent to $\mathrm{id}_A \otimes h$ and $\varphi\psi$ is approximately unitarily equivalent to $\mathrm{id}_B \otimes h$

for any positive element of $\mathcal{Z}$ with spectrum $[0,1]$ and we can finish as in the previous example. The following lemmas will help us to construct these c.p.c. order zero maps in a much more generality.

Lemma 5.6.4. *There exists a Borel measure μ on $[0,1]$ with support $[0,1]$ such that*

$$\int_0^1 t^n d\mu(t) = \frac{1}{\sqrt{n+1}} \tag{5.151}$$

for all $n \in \mathbb{N}$.

Proof. Let us prove the following well known identity

$$\int_0^\infty e^{-nt^2} dt = \frac{\sqrt{\pi}}{2\sqrt{n}}. \tag{5.152}$$

We have

$$\begin{aligned}\left(\int_0^\infty e^{-nx^2} dx\right)^2 &= \left(\int_0^\infty e^{-nx^2} dx\right)\left(\int_0^\infty e^{-ny^2} dy\right)\\ &= \int_0^\infty \int_0^\infty e^{-n\left(x^2+y^2\right)} dxdy\\ &\overset{\text{(I)}}{=} \int_0^\infty \int_0^{\frac{\pi}{2}} re^{-nr^2} d\theta dr\\ &\overset{\text{(II)}}{=} \frac{\pi}{4n}\int_0^\infty e^{-u} du\\ &= \frac{\pi}{4n}. \end{aligned} \tag{5.153}$$

Observe that in (I) we changed to polar coordinates and in (II) we made the substitution $u = nr^2$. Then we have showed

$$\int_0^\infty e^{-nt^2} dt = \frac{\sqrt{\pi}}{2\sqrt{n}} \tag{5.154}$$

for all $n \in \mathbb{N}$.

With this identity in hand, let us define a measure on $[0,\infty)$ in the following way,

$$\tilde{\mu}(U) = \frac{2}{\sqrt{\pi}} \int_U e^{-t^2} dt \tag{5.155}$$

for every Borel set $U \subset [0,\infty)$. Hence

$$\int_{[0,\infty)} f(t) d\tilde{\mu}(t) = \frac{2}{\sqrt{\pi}} \int_{[0,\infty)} f(t) e^{-t^2} dt. \tag{5.156}$$

Using the homeomorphism $h : [0,\infty) \longrightarrow (0,1]$, given by $h(t) = e^{-t}$, we can define a measure μ on $(0,1]$ by

$$\mu(U) = \tilde{\mu}\left(h^{-1}(U)\right) \tag{5.157}$$

with U a Borel set of $[0,1]$. Hence we obtain

$$\int_{(0,1]} f(t)d\mu(t) = \int_{[0,\infty)} (f\circ h(t))\, d\tilde{\mu}(t). \tag{5.158}$$

We can extend this measure to $[0,1]$ by considering $\hat{\mu}(U) = \mu(U\cap(0,1])$. Let us rename this extension $\hat{\mu}$ as μ to simplify the notation. By construction, the support of μ is $[0,1]$ and we have

$$\begin{aligned}\int_0^1 t^n d\mu(t) &\overset{(5.158)}{=} \int_0^\infty e^{-nt^2}\, d\tilde{\mu}(t)\\ &\overset{(5.156)}{=} \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-nt^2}\cdot e^{-t^2}\, d(t)\\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-(n+1)t^2} dt\\ &\overset{(5.152)}{=} \frac{2}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2\sqrt{n+1}}\right)\\ &= \frac{1}{\sqrt{n+1}} \end{aligned} \tag{5.159}$$

for all $n\in\mathbb{N}$ as wanted. □

Lemma 5.6.5. *Let A be a simple AF-algebra with unique trace τ_A. Then there exists a c.p.c. order zero map $\varphi : A \longrightarrow \mathcal{Z}$ such that*

$$\tau_{\mathcal{Z}}\circ\varphi^n(a) = \frac{\tau_A(a)}{\sqrt{n+1}}, \qquad a\in A, \tag{5.160}$$

for all $n\in\mathbb{N}$.

Let us sketch the proof first. By Lemma 5.6.4, there exists a Borel measure μ on $[0,1]$ with support $[0,1]$ such that

$$\int_0^1 t^n d\mu(t) = \frac{1}{\sqrt{n+1}}.$$

With this measure in hand we will define a $\mathrm{Cu}^\sim$-map σ between $\mathrm{Cu}^\sim(C_0(0,1]\otimes A)$ and $\mathrm{Cu}^\sim(\mathcal{Z})$. With the help of Theorem 1.10.14, we will be able to lift this map to a *-homomorphism $\pi : C_0(0,1]\otimes A \longrightarrow \mathcal{Z}$ such that $\mathrm{Cu}^\sim(\pi) = \sigma$. This *-homomorphism will induce an order zero map $\varphi : A \longrightarrow B$. Using that A has real rank zero, we will be able to show

$$\tau_{\mathcal{Z}}\circ\varphi^n(a) = \frac{\tau_A(a)}{\sqrt{n+1}}.$$

Proof. Recall that, by Theorems 1.10.6 and 1.10.9, we have

$$\mathrm{Cu}(A) \cong V(A)\sqcup(0,\infty], \tag{5.161}$$

$$\mathrm{Cu}(\mathcal{Z}) \cong \mathbb{N}_0\sqcup(0,\infty], \tag{5.162}$$

$$\mathrm{Cu}(C[0,1]\otimes A) \cong \mathrm{Lsc}\left((0,1], \mathrm{V}(A)\sqcup(0,\infty]\right). \tag{5.163}$$

By Lemma 5.6.4 there exists a Borel measure μ on $[0,1]$ with support $[0,1]$. Then by Lemma 1.10.12 the map $\tilde{\sigma} : \mathrm{Cu}(C[0,1]\otimes A) \longrightarrow \mathrm{Cu}(\mathcal{Z})$ given by

$$\tilde{\sigma}(f) = \int_0^1 d_{\tau_A}(f(t))\, d\mu(t) \tag{5.164}$$

is a Cu-map where we regard the image of $\tilde{\sigma}$ contained in the soft part of $\mathcal{Z}$, *i.e.* $(0,\infty]$. Consider the unit $\mathbf{1}$ of $\mathrm{Cu}(C[0,1]\otimes A)$, precisely $\mathbf{1} \in \mathrm{Lsc}([0,1], \mathrm{V}(A)\sqcup(0,\infty])$ is the constant function $\langle 1_A\rangle \in V(A)$. Notice

$$\tilde{\sigma}(1) = \int_0^1 d\mu(t) = \frac{2}{\sqrt{\pi}}\int_0^1 e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}\right) = 1_{\mathbb{R}_+}. \tag{5.165}$$

We will use this Cu-map to define a map at the level of $\mathrm{Cu}^\sim$. By Example 1.10.13, we have

$$\begin{aligned}\mathrm{Cu}^\sim(C_0(0,1]\otimes A) &= \{f - n\cdot\mathbf{1} \mid f\in \mathrm{Lsc}([0,1],\mathrm{V}(A)\sqcup(0,\infty]),\ f(0)<\infty\},\\ &\cong \{f\in \mathrm{Lsc}([0,1],K_0(A)\sqcup(-\infty,\infty]) \mid f(0) = n\langle 1_A\rangle, n\in\mathbb{Z}\} \end{aligned}\tag{5.166}$$

and

$$\begin{aligned}\mathrm{Cu}^\sim(\mathcal{Z}) &= \{x - n\cdot 1_{\mathbb{N}} \mid x\in\mathbb{N}_0\sqcup(0,\infty],\ n\in\mathbb{N}\}\\ &\cong \mathbb{Z}\sqcup(\infty,\infty]. \end{aligned}\tag{5.167}$$

Then the Cu-map $\tilde{\sigma}$ extends to a Cu-map $\sigma : \mathrm{Cu}^\sim(C_0(0,1]\otimes A)\longrightarrow \mathrm{Cu}^\sim(\mathcal{Z})$. Precisely,

$$\sigma(f - n\cdot\mathbf{1}) = \tilde{\sigma}(f) - n\cdot 1_{\mathbb{R}_+}. \tag{5.168}$$

By Theorem 1.10.14, there exists a *-homomorphism $\pi : C_0(0,1]\otimes A\longrightarrow\mathcal{Z}$ such that $\mathrm{Cu}^\sim(\pi) = \sigma$. Observe that equation (5.165) guarantees we can apply Theorem 1.10.14. Via the duality between c.p.c order zero maps and cones over *-homomorphisms, we obtain a c.p.c. order zero map $\varphi : A\longrightarrow\mathcal{Z}$ by

$$\varphi(a) = \pi\left(\mathrm{id}_{(0,1]}\otimes a\right), \qquad a\in A. \tag{5.169}$$

In order to finish the proof, we need to show this map satisfies the trace condition of equation (5.160). Let us analyse the map π at the level of Cu. To this end, consider $f\otimes a\in (C_0(0,1]\otimes A)_+$, by [1, Corollary 2.7] its Cu-class, which will be denoted as $[f\otimes a]$, is

$$[f\otimes a] = [a]\cdot\chi_{\mathrm{supp} f}\in \mathrm{Lsc}((0,1], V(A)\sqcup(0,\infty]), \tag{5.170}$$

where $[a] \cdot \chi_{\mathrm{supp} f} : (0,1] \longrightarrow V(A) \sqcup (0,\infty]$ is given by

$$[a] \cdot \chi_{\mathrm{supp} f}(t) = \begin{cases} [a] & t \in \mathrm{supp} f \\ 0 & \text{otherwise} \end{cases}. \tag{5.171}$$

Recall $\mathrm{Cu}(\mathcal{Z}) \cong \mathbb{N}_0 \sqcup (0,1]$. In Remark 1.10.7, we explain that if $\langle a \rangle \in \mathrm{Cu}(\mathcal{Z})$ is a soft element, we identify it with $d_{\tau_{\mathcal{Z}}}(\langle a \rangle)$. Remember the image of $\tilde{\sigma}$ is contained in the soft part of $\mathrm{Cu}(\mathcal{Z})$ and since $\mathrm{Cu}(\pi) = \sigma$, we obtain

$$\begin{aligned} d_{\tau_{\mathcal{Z}}}([\pi(f \otimes a)]) &= \mathrm{Cu}(\pi)([f \otimes a]) \\ &= \sigma([f \otimes a]) \\ &\overset{(5.170)}{=} \sigma([a] \cdot \chi_{\mathrm{supp} f}) \\ &\overset{(5.164)}{=} \int_0^1 d_{\tau_A}([a]) \cdot \chi_{\mathrm{supp} f}(t) d\mu(t) \\ &= d_{\tau_A}(a) \int_0^1 \chi_{\mathrm{supp} f}(t) d\mu(t) \\ &= d_{\tau_A}(a) \cdot \mu(\mathrm{supp} f). \end{aligned} \tag{5.172}$$

For each $a \in A$, set $\eta_a : C_0(0,1] \longrightarrow \mathcal{Z}$ by

$$\eta_a(f) = \pi(f \otimes a). \tag{5.173}$$

Since the composition $\tau_{\mathcal{Z}} \circ \eta_a$ defines a linear functional on $C_0(0,1]$, there exists a measure υ_a on $(0,1]$ such that

$$\tau_{\mathcal{Z}} \circ \eta_a(f) = \int_{(0,1]} f(t) d\upsilon_a(t) \tag{5.174}$$

for all $f \in C_0(0,1]$. Moreover, for a contraction $f \in C_0(0,1]_+$ we have

$$\begin{aligned} d_{\tau_{\mathcal{Z}} \circ \eta_a}(f) &= \lim_{n\to\infty} \tau_{\mathcal{Z}} \circ \eta_a\left(f^{\frac{1}{n}}\right) \\ &= \lim_{n\to\infty} \int_{(0,1]} f^{\frac{1}{n}}(t) d\upsilon_a(t) \\ &= \upsilon_a(\mathrm{supp} f). \end{aligned} \tag{5.175}$$

For a projection $p \in A$, observe $d_{\tau_A}(p) = \tau(p)$. Then we have

$$\begin{aligned} d_{\tau_{\mathcal{Z}} \circ \eta_p}(f) &= \lim_{n\to\infty} \tau_{\mathcal{Z}} \circ \eta_p\left(f^{\frac{1}{n}}\right) \\ &= \lim_{n\to\infty} \tau_{\mathcal{Z}} \circ \pi\left(f^{\frac{1}{n}} \otimes p\right) \\ &= \lim_{n\to\infty} \tau_{\mathcal{Z}}\left(\pi(f \otimes p)^{\frac{1}{n}}\right) \\ &= d_{\tau_{\mathcal{Z}}}(\pi(f \otimes p)) \\ &\overset{(5.172)}{=} \tau_A(p) \cdot \mu(\mathrm{supp} f). \end{aligned} \tag{5.176}$$

This entails

$$\upsilon_p(\mathrm{supp} f) \overset{(5.175)}{=} d_{\tau_{\mathcal{Z}}\circ\eta_p}(f) \overset{(5.176)}{=} \tau_A(p)\cdot\mu(\mathrm{supp} f) \tag{5.177}$$

for each positive $f \in C_0(0,1]\otimes A$. This shows

$$\upsilon_p = \tau_A(p)\mu. \tag{5.178}$$

Let us now consider the case when $a \in A_+$ is spanned by projections, say $a = \sum_{i=1}^{m}\lambda_i p_i$ with $\{p_1,\dots,p_m\}$ a family of pairwise orthogonal projections and each λ_i positive. Then

$$\begin{aligned}
d_{\tau_{\mathcal{Z}}\circ\eta_a}(f) &= \lim_{n\to\infty}\tau_{\mathcal{Z}}\circ\eta_a\left(f^{\frac{1}{n}}\right)\\
&= \lim_{n\to\infty}\tau_{\mathcal{Z}}\circ\pi\left(f^{\frac{1}{n}}\otimes a\right)\\
&= \lim_{n\to\infty}\tau_{\mathcal{Z}}\circ\pi\left(f^{\frac{1}{n}}\otimes\left(\sum_{i=1}^{m}\lambda_i p_i\right)\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{m}\tau_{\mathcal{Z}}\circ\pi\left(f^{\frac{1}{n}}\otimes\lambda_i p_i\right)\\
&= \sum_{i=1}^{m}\lambda_i\lim_{n\to\infty}\tau_{\mathcal{Z}}\left(\left(\pi\left(f\otimes p_i\right)\right)^{\frac{1}{n}}\right)\\
&= \sum_{i=1}^{m}\lambda_i d_{\tau_{\mathcal{Z}}}\left(\pi\left(f\otimes p_i\right)\right)\\
&\overset{(5.176)}{=} \sum_{i=1}^{m}\lambda_i\tau_A(p_i)\cdot\mu(\mathrm{supp} f)\\
&= \tau_A\left(\sum_{i=1}^{m}\lambda_i p_i\right)\cdot\mu(\mathrm{supp} f)\\
&= \tau_A(a)\cdot\mu(\mathrm{supp} f).
\end{aligned}\tag{5.179}$$

This shows

$$\upsilon_a(\mathrm{supp} f) \overset{(5.175)}{=} d_{\tau_{\mathcal{Z}}\circ\eta_a}(f) \overset{(5.176)}{=} \tau_A(a)\cdot\mu(\mathrm{supp} f) \tag{5.180}$$

for any $f \in C_0(0,1]_+$. This entails $\upsilon_a = \tau(a)\mu$ if a is spanned by projections; *i.e.* $a = \sum_{i=1}^{m}\lambda_i p_i$ where $\{p_1,\dots,p_m\}$ is a family of pairwise orthogonal projections in A. By continuity, we can extend this to any positive element $a \in A$ which can be approximated by linear combinations of projections. We include those details for completeness.

Let $a \in A$, $\varepsilon > 0$ and suppose it can be approximated by linear combinations of projections. Then there exists $b = \sum_{i=1}^{r}\lambda_i p_i$, where $\{p_1,\dots,p_m\}$ is a family of pairwise orthogonal projections in A, such that

$$\|a-b\| < \frac{\varepsilon}{2}. \tag{5.181}$$

Let $f \in C_0(0,1]$ be a function of norm at most 1, then

$$\begin{aligned}\left|\int_0^1 f d\upsilon_a - \int_0^1 f d\upsilon_b\right| &\overset{(5.174)}{=} |\tau_{\mathcal{Z}} \circ \pi\left(f \otimes a\right) - \tau_{\mathcal{Z}} \circ \pi\left(f \otimes b\right)| \\ &= |\tau_{\mathcal{Z}} \circ \pi\left(f \otimes (a-b)\right)| \\ &< \frac{\varepsilon}{2}. \end{aligned} \tag{5.182}$$

This yields that for any open interval I we have

$$|\upsilon_a(I) - \upsilon_b(I)| \leq \frac{\varepsilon}{2}. \tag{5.183}$$

Since b is a linear combination of projections, we know $\upsilon_b = \tau(b)\mu$. Then

$$\begin{aligned}|\upsilon_a(I) - \tau(a)\mu(I)| &\leq |\upsilon_a(I) - \upsilon_b(I)| + |\tau(b)\mu(I) - \tau(a)\mu(I)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{5.184}$$

Since ε was arbitrary, we conclude $\upsilon_a = \tau(a)\mu$.

Finally, A is real rank zero by [84, Proposition 1.2.4]. Thus

$$\upsilon_a = \tau_A(a)\mu \tag{5.185}$$

for every $a \in A_{sa}$. Therefore

$$\begin{aligned}\tau_{\mathcal{Z}} \circ \varphi^n(a) &= \tau_{\mathcal{Z}}\left(\pi(t^n \otimes a)\right) \\ &\overset{(5.173)}{=} \tau_{\mathcal{Z}} \circ \eta_a(t^n) \\ &\overset{(5.174)}{=} \int_0^1 t^n d\upsilon_a(t) \\ &\overset{(5.185)}{=} \tau_A(a) \int_0^1 t^n d\mu(t) \\ &\overset{(5.159)}{=} \frac{\tau_A(a)}{\sqrt{n+1}} \end{aligned} \tag{5.186}$$

for all $a \in A$. This finishes the proof. □

As an easy application of the previous lemma let us prove the next example.

Example 5.6.6. Any two UHF-algebras are 2-coloured equivalent.

Proof. Let A and B two UHF algebras. We will denote the unique traces of A and B by τ_A and τ_B. By Theorem 4.5.4, there exist *-isomorphisms $\sigma : A \otimes \mathcal{Z} \longrightarrow A$ and $\theta : B \otimes \mathcal{Z} \longrightarrow B$, and unitaries $x \in A_\omega, y \in B_\omega$ such that

$$a = x^*\sigma(a \otimes 1_{\mathcal{Z}})x, \qquad b = y^*\theta(b \otimes 1_{\mathcal{Z}})y \tag{5.187}$$

for all $a \in A$ and $b \in B$. By Lemma 5.6.5, there exist order zero maps $\tilde{\varphi} : A \longrightarrow \mathcal{Z}$ and $\tilde{\psi} : B \longrightarrow \mathcal{Z}$ such that

$$\tau_{\mathcal{Z}} \circ (\tilde{\varphi})^n(a) = \frac{\tau_A(a)}{\sqrt{n+1}}, \qquad \tau_{\mathcal{Z}} \circ (\tilde{\psi})^n(b) = \frac{\tau_B(b)}{\sqrt{n+1}} \tag{5.188}$$

for all $a \in A$ and $b \in B$. Define order zero maps $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ where φ is given by the following compositions,

$$\begin{tikzcd} A \arrow[r, "\tilde{\varphi}"] \arrow[rrr, bend right, "\varphi"'] & \mathcal{Z} \arrow[r, "1_B \otimes \mathrm{id}_{\mathcal{Z}}"] & B \otimes \mathcal{Z} \arrow[r, "\theta"] & B \end{tikzcd}$$

and ψ is given similarly by the following compositions.

$$\begin{tikzcd} B \arrow[r, "\tilde{\psi}"] \arrow[rrr, bend right, "\psi"'] & \mathcal{Z} \arrow[r, "1_A \otimes \mathrm{id}_{\mathcal{Z}}"] & A \otimes \mathcal{Z} \arrow[r, "\sigma"] & A \end{tikzcd}$$

Observe that since $1_B \otimes \mathrm{id}_{\mathcal{Z}}, 1_A \otimes \mathrm{id}_{\mathcal{Z}}, \sigma$ and θ are unital *-homomorphisms, the maps φ and ψ satisfy the following trace conditions.

$$\tau_B \circ \varphi^n(a) = \frac{\tau_A(a)}{\sqrt{n+1}}, \qquad \tau_A \circ \psi^n(b) = \frac{\tau_B(b)}{\sqrt{n+1}} \tag{5.189}$$

for all $a \in A$ and $b \in B$. Hence

$$\begin{aligned} \tau_A \circ (\psi\varphi)^n\,(a) &= \tau_A \circ \psi^n\,(\varphi^n(a)) \\ &= \frac{\tau_B \circ \varphi^n(a)}{\sqrt{n+1}} \\ &= \frac{\tau_A(a)}{\sqrt{n+1}\sqrt{n+1}} \\ &= \frac{\tau_A(a)}{n+1}. \end{aligned} \tag{5.190}$$

Similarly

$$\tau_B \circ (\varphi\psi)^n\,(b) = \frac{\tau_B(n)}{n+1}. \tag{5.191}$$

We will finish exactly as in Example 5.6.3. Let $h \in \mathcal{Z}$ be a positive element of spectrum $[0,1]$ such that

$$\tau_{\mathcal{Z}}(h^n) = \frac{1}{n+1}. \tag{5.192}$$

Define c.p.c. order zero maps $\rho_1, \rho_2 : A \longrightarrow A$ by

$$\rho_1(a) = \sigma(a \otimes h), \qquad a \in A \tag{5.193}$$

$$\rho_2(a) = \sigma(a \otimes (1_{\mathcal{Z}} - h)), \qquad a \in A. \tag{5.194}$$

Similarly, define c.p.c. order zero maps $\zeta_1, \zeta_2 : B \longrightarrow B$ by

$$\zeta_1(b) = \theta(b \otimes h), \qquad b \in B, \tag{5.195}$$

$$\zeta_2(b) = \theta(b \otimes (1_{\mathcal{Z}} - h)), \qquad b \in B. \tag{5.196}$$

By construction, we have

$$\tau_A \circ \rho_1^n(a) = \tau_A \circ (\psi\varphi)^n (a) = \tau_A \circ \rho_2^n(a) \tag{5.197}$$

for all $a \in A$ and $n \in \mathbb{N}$. Similarly

$$\tau_B \circ \zeta_1^n(b) = \tau_B \circ (\varphi\psi)^n (b) = \tau_B \circ \zeta_2^n(b) \tag{5.198}$$

for all $b \in B$ and $n \in \mathbb{N}$. By Lemma 5.6.2, there exist unitaries $w_1, w_2 \in A_\omega$ and $\tilde{w_1}, \tilde{w_2} \in B_\omega$ such that

$$\rho_1(a) = w_1\psi\varphi(a)w_1^*, \qquad \rho_2(a) = w_2\psi\varphi(a)w_2^*, \tag{5.199}$$

$$\zeta_1(b) = \tilde{w}_1^*\varphi\psi(b)\tilde{w}_1^*, \qquad \zeta_2(b) = \tilde{w}_2\varphi\psi(b)\tilde{w}_2^*. \tag{5.200}$$

After setting $u_1 = x^*w_1, u_2 = x^*w_2 \in A_\omega$ and $v_1 = y^*\tilde{w}_1, v_2 = y^*\tilde{w}_2 \in B_\omega$ we obtain

$$a = u_1\psi\varphi(a)u_1^* + u_2\psi\varphi(a)u_2^*, \tag{5.201}$$

$$b = v_1\varphi\psi(b)v_1^* + v_2\varphi\psi(b)v_2^* \tag{5.202}$$

for all $a \in A$ and $b \in B$. Therefore the c.p.c. order zero maps $\varphi : A \longrightarrow B, \psi : B \longrightarrow A$ and the unitaries $u_1, u_2 \in A_\omega, v_1, v_2 \in B_\omega$ implement a 2-coloured isomorphism between A and B. □

The following result from [96] will allow us to extend the previous proof much more generally.

Theorem 5.6.7 ([96, Corollary 6.5])**.** *Let A be a separable, simple, unital and nuclear C^*-algebra with unique trace which satisfy the UCT. Then A embeds unitally into a simple AF algebra with unique trace.*

The following theorem is the main result of this section. It is an equivalent version of Theorem 5.5.4 for separable, simple, unital, finite, nuclear and $\mathcal{Z}$-stable C^*-algebras with unique trace which satisfy the UCT algebras.

Theorem 5.6.8. *Let A and B be separable, simple, unital, finite, nuclear and $\mathcal{Z}$-stable C^*-algebras with unique trace which satisfy the UCT. Then $A \cong_{(2)} B$.*

Proof. By Corollary 5.6.7, there exist simple monotracial AF algebras C and D, and unital embeddings $A \hookrightarrow C$ and $B \hookrightarrow D$. Let us denote the unique traces of C and D by τ_C and τ_D. By Lemma 5.6.5, there exist c.p.c. order zero maps $\tilde{\varphi} : C \longrightarrow \mathcal{Z}$ and $\tilde{\psi} : D \longrightarrow \mathcal{Z}$ such that

$$\tau_{\mathcal{Z}} \circ (\tilde{\varphi})^n (c) = \frac{\tau_C(c)}{\sqrt{n+1}}, \qquad \tau_{\mathcal{Z}} \circ \left(\tilde{\psi}\right)^n (d) = \frac{\tau_D(d)}{\sqrt{n+1}} \tag{5.203}$$

for all $c \in C$ and $d \in D$.

By Theorem 4.5.4, there exist *-isomorphisms $\sigma : A \otimes \mathcal{Z} \longrightarrow A$ and $\theta : B \otimes \mathcal{Z} \longrightarrow B$, and unitaries $x \in A_\omega, y \in B_\omega$ such that

$$a = x^* \sigma(a \otimes 1_{\mathcal{Z}}) x, \qquad b = y^* \theta(b \otimes 1_{\mathcal{Z}}) y \tag{5.204}$$

for all $a \in A$ and $b \in B$.

Let us define order zero maps $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ where φ is given by the following compositions,

$$\begin{tikzcd} A \arrow[r, hook] \arrow[rrrr, bend right, "\varphi"'] & C \arrow[r, "\tilde{\varphi}"] & \mathcal{Z} \arrow[r, "1_B \otimes \mathrm{id}_{\mathcal{Z}}"] & B \otimes \mathcal{Z} \arrow[r, "\theta"] & B \end{tikzcd}$$

and ψ is given similarly by the following compositions.

$$\begin{tikzcd} B \arrow[r, hook] \arrow[rrrr, bend right, "\psi"'] & D \arrow[r, "\tilde{\psi}"] & \mathcal{Z} \arrow[r, "1_A \otimes \mathrm{id}_{\mathcal{Z}}"] & A \otimes \mathcal{Z} \arrow[r, "\sigma"] & A \end{tikzcd}$$

From this point the proof is identical to the proof of Example 5.6.6. We include the proof for completeness. Observe that since A and B embeds unitally in C and D, and $1_B \otimes \mathrm{id}_{\mathcal{Z}}, 1_A \otimes \mathrm{id}_{\mathcal{Z}}, \sigma$ and θ are unital *-homomorphisms, the maps φ and ψ satisfy the following conditions,

$$\tau_B \circ \varphi^n(a) = \frac{\tau_A(a)}{\sqrt{n+1}}, \qquad \tau_A \circ \psi^n(b) = \frac{\tau_B(b)}{\sqrt{n+1}} \tag{5.205}$$

for all $a \in A$ and $b \in B$. Hence

$$\begin{aligned} \tau_A \circ (\psi\varphi)^n (a) &= \tau_A \circ \psi^n (\varphi^n(a)) \\ &= \frac{\tau_B \circ \varphi^n(a)}{\sqrt{n+1}} \\ &= \frac{\tau_A(a)}{n+1}. \end{aligned} \tag{5.206}$$

In the same way we can show

$$\tau_B \circ (\varphi\psi)^n (b) = \frac{\tau_B(n)}{n+1}. \tag{5.207}$$

for all $b \in B$. Consider a positive contraction $h \in \mathcal{Z}$ with spectrum $[0,1]$ such that

$$\tau_{\mathcal{Z}}(h^n) = \frac{1}{n+1}. \tag{5.208}$$

Let us define c.p.c. order zero maps $\rho_1, \rho_2 : A \longrightarrow A$ by

$$\rho_1(a) = \sigma(a \otimes h), \qquad a \in A, \tag{5.209}$$

$$\rho_2(a) = \sigma(a \otimes (1_{\mathcal{Z}} - h)), \qquad a \in A. \tag{5.210}$$

Similarly, define c.p.c. order zero maps $\zeta_1, \zeta_2 : B \longrightarrow B$ by

$$\zeta_1(b) = \theta(b \otimes h), \qquad b \in B, \tag{5.211}$$

$$\zeta_2(b) = \theta(b \otimes (1_{\mathcal{Z}} - h)), \qquad b \in B. \tag{5.212}$$

By construction, we have

$$\tau_A \circ \rho_1^n(a) = \tau_A \circ (\psi\varphi)^n(a) = \tau_A \circ \rho_2^n(a), \tag{5.213}$$

$$\tau_B \circ \zeta_1^n(b) = \tau_B \circ (\varphi\psi)^n(b) = \tau_B \circ \zeta_2^n(b) \tag{5.214}$$

for all $a \in A, b \in B$ and $n \in \mathbb{N}$. By Lemma 5.6.2, there exist unitaries $w_1, w_2 \in A_\omega$ and $\tilde{w_1}, \tilde{w_2} \in B_\omega$ such that

$$\rho_1(a) = w_1\psi\varphi(a)w_1^*, \qquad \rho_2(a) = w_2\psi\varphi(a)w_2^* \tag{5.215}$$

$$\zeta_1(b) = \tilde{w}_1^*\varphi\psi(b)\tilde{w}_1^*, \qquad \zeta_2(b) = \tilde{w}_2\varphi\psi(b)\tilde{w}_2^*. \tag{5.216}$$

Set $u_1 = x^*w_1, u_2 = x^*w_2 \in A_\omega$ and $v_1 = y^*\tilde{w}_1, v_2 = y^*\tilde{w}_2 \in B_\omega$. This yields

$$a = u_1\psi\varphi(a)u_1^* + u_2\psi\varphi(a)u_2^* \tag{5.217}$$

$$b = v_1\varphi\psi(b)v_1^* + v_2\varphi\psi(b)v_2^* \tag{5.218}$$

for all $a \in A$ and $b \in B$. Therefore the c.p.c. order zero maps $\varphi : A \longrightarrow B, \psi : B \longrightarrow A$ and the unitaries $u_1, u_2 \in A_\omega, v_1, v_2 \in B_\omega$ implement a 2-coloured isomorphism between A and B. □

5.7 Questions

We will finish this chapter by stating some questions about coloured isomorphisms.

- What is the right notion of coloured isomorphisms for non unital C^*-algebras?

- Coloured isomorphisms are very rigid for finite dimensional and commutative C^*-algebras. In a work in progress with D. McConell we have observed that this is also true for C^*-algebras of the form $C_0(X) \otimes M_n(\mathbb{C})$. Is this true for a more general class of C^*-algebras, for example type I?

- We have seen that if A and B are stable rank one separable unital C^*-algebras such that $A \cong_{(n)} B$ and A is D-stable then $B \cong_{n^2} B \otimes D$. If A and B are simple separable and nuclear such that $A \cong_{(2)} B$ and A is $\mathcal{Z}$-stable. Does B tensorially absorb $\mathcal{Z}$ as well?

- Is the UCT preserved under coloured isomorphisms? If this is true, then we will obtain that any nuclear C^*-algebra satisfy the UCT so we expect this question to be extremely difficult.

- Can we remove the UCT assumption from the hypothesis of Theorem 5.6.8?

- We explained that the trace simplices of coloured isomorphic C^*-algebras are homeomorphic (as topological spaces). Are the trace simplices of coloured isomorphic C^*-algebras affinely isomorphic? If this is true, can we extend Theorem 5.6.8 to the case where $T(A)$ and $T(B)$ are affinely isomorphic?

- Which properties of Cu are preserved under coloured isomorphisms? For example, if $A \cong_{(n)} B$ and $\mathrm{Cu}(A)$ has m-comparison, does $\mathrm{Cu}(B)$ have nm-comparison?

Appendix A

Ultraproducts

In this appendix we will recall the constructions of ultraproducts and ultrapowers. We will also state some properties of them.

A *filter* ω on $\mathbb{N}$ is a subset of $2^{\mathbb{N}}$ such that

(i) $\emptyset \notin \omega$,

(ii) if $U, V \in \omega$, then $U \cap V \in \omega$,

(iii) if $U \in \omega$ and $U \subset V$, then $V \in \omega$.

A filter is *free* if the intersection of all of its elements is empty. A maximal filter is called *ultrafilter*. Let (x_n) be a sequence of real numbers. The sequence $(x_n)_n$ converges to x *along* ω, denoted as $\lim_{n\to\omega} x_n = x$, if for every $\varepsilon > 0$ there exists $U \in \omega$ such that $|x_n - x| < \varepsilon$ if $n \in U$.

A.1 Ultraproducts of C^*-algebras

Given a sequence of C^*-algebras $(A_n)_{n\in\mathbb{N}}$, set

$$\ell^\infty\left((A_n)_{n\in\mathbb{N}}\right) = \left\{(a_n)_{n\in\mathbb{N}} \;\middle|\; a_n \in A_n\,,\ \sup_{n\in\mathbb{N}} \|a_n\| < \infty\right\}. \tag{A.1}$$

Definition A.1.1. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of C^*-algebras and ω a filter on $\mathbb{N}$. We define the sequence algebra of $(A_n)_{n\in\mathbb{N}}$ as

$$\prod A_n \Big/ \bigoplus A_n = \ell^\infty\left((A_n)_{n\in\mathbb{N}}\right) \Big/ \left\{(a_n)_{n\in\mathbb{N}} \in \ell^\infty\left((A_n)_{n\in\mathbb{N}}\right) \;\middle|\; \lim_{n\to\infty} \|a_n\| = 0\right\}. \tag{A.2}$$

We also define $\prod\limits_{n\to\omega} A_n$ as

$$\prod_{n\to\omega} A_n = \ell^\infty\left((A_n)_{n\in\mathbb{N}}\right) \Big/ \left\{(a_n)_{n\in\mathbb{N}} \in \ell^\infty\left((A_n)_{n\in\mathbb{N}}\right) \;\middle|\; \lim_{n\to\mathcal{U}} \|a_n\| = 0\right\}. \tag{A.3}$$

We will omit the n when there is no risk of confusion. If A is a C^*-algebra and $A_n = A$ for all $n \in \mathbb{N}$, we denote them as A_∞ and A_ω. When ω is an ultrafilter, $\prod\limits_{\omega} A_n$ is called an ultraproduct and A_ω an ultrapower.

We will denote the class of the sequence $(a_n)_n$ in $\prod\limits_{\omega} A_n$ as $\overline{(a_n)_n}$. Precisely,

$$a \longmapsto \overline{(a, a, a, \ldots)} \in A_\omega.$$

We will focus on ultraproducts of C^*-algebras but let us introduce the ultraproduct of II_1-factors first. Let M be a II_1-factor with trace τ. The trace τ induces a norm called the 2-norm on M in the following way,

$$\|x\|_2 = \sqrt{\tau(x^*x)}, \qquad x \in M.$$

Definition A.1.2. Let M be a II_1-factor with trace τ and let ω be an ultrafilter on $\mathbb{N}$. The ultrapower M^ω is given by

$$M^\omega = \ell^\infty(M) \Big/ \left\{ (x_n)_n \mid \lim_{n\to\omega} \|x_n\|_2 = 0 \right\}. \tag{A.4}$$

An important fact of ultrapowers is that projections and unitaries can be lifted to a sequence formed by projections and unitaries respectively.

Proposition A.1.3 ([84, Lemma 6.2.4])**.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of C^*-algebras and let $p \in \prod\limits_{\omega} A_n$ be a projection. Then p can be represented by a sequence formed by projections. Moreover, if each A_n is unital and $u \in \prod\limits_{\omega} A_n$ is a unitary, then u can be represented by a sequence formed by unitaries.*

Proposition A.1.4. *Let A and B be separable C^*-algebras and ω a free ultrafilter on . Then*

(i) $(A \oplus B)_\omega \cong A_\omega \oplus B_\omega$.

(ii) There is a canonical embedding $\ell^\infty(A_\omega) \hookrightarrow \ell^\infty(A)_\omega$.

While working with sequences, one of the most standard tools is the diagonal argument. The following lemma is the analogue technique for ultraproducts. This lemma is commonly referred as Kirchberg's ε-test or the reindexing argument.

Lemma A.1.5 (Kirchberg's ε-test, [54, Lemma A.1])**.** *Let $X_1, X_2, \ldots$ be a sequence of nonempty sets, and for each $k, n \in \mathbb{N}$, let $f_n^{(k)} : X_n \longrightarrow [0,\infty)$ be a function. Define $f_\omega^{(k)} : \prod\limits_{n=1}^{\infty} X_n \longrightarrow [0,\infty]$ by*

$$f_\omega^{(k)}\left((s_n)_{n=1}^\infty\right) = \lim_{n\to\omega} f_n^{(k)}(s_n)$$

for $(s_n) \in \prod_{n=1}^{\infty} X_n$*. Suppose that for all* $m \in \mathbb{N}$ *and* $\varepsilon > 0$*, there exists* $(s_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} X_n$ *with* $f_\omega^{(k)}((s_k)) < \varepsilon$ *for* $k = 1, \ldots, m$*. Then there exists* $(t_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} X_n$ *such that* $f_\omega^{(k)}((t_n)) = 0$ *for all* $k \in \mathbb{N}$*.*

One important fact about ultrapowers is that they preserved stable rank one.

Lemma A.1.6 ([62, Lemma 1.21])**.** *Let* A *be unital* C^**-algebra with stable rank one. Then* A_ω *has stable rank one and for any element* $x \in A_\omega$ *there exists a unitary* $u \in A_\omega$ *such that*

$$x = u|a|. \tag{A.5}$$

Let $\varphi : A \longrightarrow B$ be map between C^*-algebras. This map induces a map $\varphi^{(\omega)} : A_\omega \longrightarrow B_\omega$ at the level of ultrapowers in the following way. Let $a \in A_\omega$ be represented by the sequence $(a_n)_n$, then

$$\varphi^{(\omega)}(a) = \overline{(\varphi(a_n))}. \tag{A.6}$$

The proof of the following lemma is contained in the proof of [80, Proposition 2.2].

Proposition A.1.7 (Robert)**.** *Let* $(A_n)_{n\in\mathbb{N}}$ *and* $(B_n)_{n\in\mathbb{N}}$ *be a sequence of* C^**-algebras and let* $\varphi_n : A_n \longrightarrow B_n$ *be c.p.c. order zero maps. Then the induced c.p.c. map*

$$\varphi : \prod_{n\to\mathcal{U}} A_n \longrightarrow \prod_{n\to\mathcal{U}} B_n$$

is also order zero.

A.1.1 Support order zero maps

In this part we will present some technical lemmas we need in Chapter 5. In order to simplify the notation, for any map $\varphi : A \longrightarrow B$ we will also call φ to the induced map at the level of ultraproducts; *i.e.* $\varphi : A_\omega \longrightarrow B_\omega$. We include its proof for completeness.

Lemma A.1.8 ([8, Lemma 1.14])**.** *Let* A *and* B *be separable unital* C^**-algebras and let* ω *be a free ultrafilter on* $\mathbb{N}$*. Suppose* C *is a separable* C^**-subalgebra of* A_ω*,* S *is a separable and selfadjoint subset of* B_ω *and* $\varphi : C \longrightarrow B_\omega \cap S'$ *is a c.p.c. order zero map. Then there exists a c.p.c. order zero map* $\hat{\varphi} : C \longrightarrow B_\omega \cap S'$ *such that*

$$\varphi(xy) = \hat{\varphi}(x)\varphi(y) = \varphi(x)\hat{\varphi}(y) \tag{A.7}$$

for all $x, y \in C$*.*

Proof. Fix a countable dense $\mathbb{Q}[i]$-*-subalgebra C_0 of C and let $(s_n)_{n\in\mathbb{N}}$ be a dense subset of S. Lift each s_n to a bounded sequence $\left(s_i^{(n)}\right)_{i\in\mathbb{N}} \subset B$ and set X_n as the set of *-linear maps from C_0 to B.

By [8, Lemma 1.12], there exist functions $g_n^{(k)} : X_n \longrightarrow [0,\infty)$ indexed by $n \in \mathbb{N}$ and $k \in I$, with I a countable index set, such that the sequence $(\sigma_n) \in \prod_{n=1}^{\infty} X_n$ induces a c.p.c. order zero map $C \longrightarrow B_\omega$ if and only if

$$\lim_{n\to\omega} g_n^{(k)}(\sigma_n) = 0 \tag{A.8}$$

for all $k \in I$. Hence, we can represent φ with a sequence of linear *-maps $\varphi_n : C_0 \longrightarrow B$ such that

$$\lim_{n\to\omega} g_n^{(k)}(\varphi_n) = 0. \tag{A.9}$$

Let $(a_i)_{i\in\mathbb{N}}$ be a dense subset of the unit ball of C. For $i,j \in \mathbb{N}$ set

$$f_n^{(0,i,j)}(\sigma) = \|\varphi_n(a_i a_j) - \sigma(a_i)\varphi_n(a_j)\| + \|\varphi_n(a_i a_j) - \varphi_n(a_i)\sigma(a_j)\|, \tag{A.10}$$

$$f_n^{(1,i,j)}(\sigma) = \left\|\sigma(a_i)s_n^{(j)} - s_n^{(j)}\sigma(a_i)\right\| \tag{A.11}$$

for $\sigma \in X_n$.

Fix $\varepsilon > 0$ and, using functional calculus for order zero maps, we define $\psi = f_\varepsilon(\varphi) : C \longrightarrow B_\omega \cap S'$ where $f_\varepsilon : (0,\infty) \longrightarrow \mathbb{R}$ is identically 0 in $(0,\varepsilon/2]$, 1 in $[\varepsilon,\infty)$ and linear elsewhere. By Corollary 1.4.7, there exists an *-homomorphism $\pi : C_0(0,1]\otimes C \longrightarrow B_\omega \cap S'$ such that $\pi(\mathrm{id}_{(0,1]} \otimes a) = \varphi(a)$ for all $a \in C$. Observe that by construction

$$\psi(c) = \pi(f_\varepsilon \otimes a), \qquad a \in C. \tag{A.12}$$

Thus

$$\begin{aligned}
\psi(a)\varphi(b) &= \pi(f_\varepsilon \otimes a)\,\pi\left(\mathrm{id}_{(0,1]} \otimes b\right) \\
&= \pi\left(f_\varepsilon \cdot \mathrm{id}_{(0,1]} \otimes ab\right) \\
&= \pi\left(\mathrm{id}_{(0,1]} \cdot f_\varepsilon \otimes ab\right) \\
&= \pi\left(\mathrm{id}_{(0,1]} \otimes a\right)\pi(f_\varepsilon \otimes a) \\
&= \varphi(a)\psi(b)
\end{aligned} \tag{A.13}$$

for all $a,b \in C$. Observing that $\|f_\varepsilon \cdot \mathrm{id}_{(0,1]} - \mathrm{id}_{(0,1]}\| < \varepsilon$, we obtain

$$\left\|\pi(f_\varepsilon \otimes a) - \pi\left(\mathrm{id}_{(0,1]} \otimes a\right)\right\| \leq \left\|f_\varepsilon \cdot \mathrm{id}_{(0,1]} - \mathrm{id}_{(0,1]}\right\| < \varepsilon \tag{A.14}$$

for any contraction $a \in A$ and $x \in A_\omega$. Putting all these together shows

$$\begin{aligned}\|\varphi(ab) - \varphi(a)\psi(b)\| &\overset{(A.13)}{=} \|\varphi(ab) - \varphi(a)\psi(b)\| \\ &= \left\|\pi\left(\mathrm{id}_{(0,1]} \otimes ab\right) - \pi\left(\mathrm{id}_{(0,1]} \otimes a\right)\pi\left(f_\varepsilon \otimes b\right)\right\| \\ &= \left\|\pi\left(\mathrm{id}_{(0,1]} \otimes ab\right) - \pi(f_\varepsilon \cdot \mathrm{id}_{(0,1]} \otimes ab)\right\| \\ &\overset{(A.14)}{<} \varepsilon. \end{aligned} \tag{A.15}$$

Hence any sequence $(\psi_n) \in \prod_{n=1}^{\infty} X_n$ representing ψ satisfies

$$\lim_{n\to\omega} f_n^{(0,i,j)}(\psi_n) \leq \varepsilon \tag{A.16}$$

for all $i, j \in \mathbb{N}$. Moreover, since the image of ψ is contained in S', we have

$$\lim_{n\to\omega} f_n^{(1,i,j)}(\psi_n) = 0 \tag{A.17}$$

for all $i, j \in \mathbb{N}$.

Then, for each $\varepsilon > 0$ there exists $(\psi_n) \in \prod_{n=1}^{\infty} X_n$ such that

$$\lim_{n\to\omega} f_n^{(0,i,j)}(\psi_n) \leq \varepsilon, \qquad \lim_{n\to\omega} f_n^{(1,i,j)}(\psi_n) = 0, \qquad \lim_{n\to\omega} g_n^{(k)}(\psi_n) = 0, \tag{A.18}$$

for all $i, j \in \mathbb{N}$ and $k \in I$. By Kirchberg's ε-test (Lemma A.1.5), there exists $(\hat{\varphi}_n) \in \prod_{n=1}^{\infty} X_n$ such that

$$\lim_{n\to\omega} f_n^{(0,i,j)}(\hat{\varphi}_n) = 0, \qquad \lim_{n\to\omega} f_n^{(1,i,j)}(\hat{\varphi}_n) = 0, \qquad \lim_{n\to\omega} g_n^{(k)}(\hat{\varphi}_n) = 0 \tag{A.19}$$

for all $i, j \in \mathbb{N}$ and $k \in I$. By Lemma [8, Lemma 1.12], the sequence $(\hat{\varphi}_n)$ induces a c.p.c. order zero map $\hat{\varphi} : C \longrightarrow B_\omega$ and, by construction, this map satisfies

$$\varphi(xy) = \hat{\varphi}(x)\varphi(y) = \varphi(x)\hat{\varphi}(y), \tag{A.20}$$

and $\varphi(x) \in S'$ for all $x, y \in C$. □

Corollary A.1.9. *Let A and B be separable unital C^*-algebras and let ω be a free ultrafilter on $\mathbb{N}$. Suppose C is a separable unital C^*-subalgebra of A_ω, B is stable rank one and $\varphi : C \longrightarrow B_\omega$ is a c.p.c. order zero map. Then for any unitary $u \in C$ there exists a unitary $w \in B_\omega$ such that*

$$\varphi(ux) = w\varphi(x) \tag{A.21}$$

for any $x \in C$.

Proof. By Lemma A.1.8, there exists a c.p.c. order zero map $\hat{\varphi} : C \longrightarrow B_\omega$ such that

$$\varphi(ab) = \hat{\varphi}(a)\varphi(b) = \varphi(a)\hat{\varphi}(b) \tag{A.22}$$

for all $a, b \in C$. In particular, $\hat{\varphi}(1_C)$ acts like a unit for $\varphi(C)$. By Lemma A.1.6, B_ω is stable rank one and we have polar decomposition in B_ω. In particular, for any unitary $u \in A_\omega$ there exists a unitary $w \in B_\omega$ such that

$$\hat{\varphi}(u) = w|\hat{\varphi}(u)|. \tag{A.23}$$

Since

$$\begin{aligned} |\hat{\varphi}(u)| &= (\hat{\varphi}(u)^*\hat{\varphi}(u))^{\frac{1}{2}} \\ &= (\hat{\varphi}(1_C)\hat{\varphi}(u^*u))^{\frac{1}{2}} \\ &= \left(\hat{\varphi}(1_C)^2\right)^{\frac{1}{2}} \\ &= \hat{\varphi}(1_C). \end{aligned} \tag{A.24}$$

This shows

$$\hat{\varphi}(u) = w\hat{\varphi}(1_C). \tag{A.25}$$

Finally, by putting everything together we obtain

$$\begin{aligned} \varphi(ux) &\overset{(A.22)}{=} \hat{\varphi}(u)\varphi(x) \\ &\overset{(A.25)}{=} w\hat{\varphi}(1_C)\varphi(x) \\ &\overset{(A.22)}{=} w\varphi(x). \end{aligned} \tag{A.26}$$

This finishes the proof. □

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