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Benard convection in a non-linear magnetic fluid

By

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To my parents and my wife

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Preface

This dissertation is submitted to the University of Glasgow, in accordance with the requirements for the degree of Doctor of Philosophy in Mathematics.

The work presented here has been carried out under the supervision of Dr. K. A. Lindsay. To him I would like to express my deepest gratitude for his guidence, constant interest and encouragement throughout the period of this research. I also wish to thank Professor B. Straughan for his useful discussion and helpful advice on this work.

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Summary

Convective instabilities in the Benard problem have been examined for a magnetohydrodynamic fluid with a non-linear permeability, the model having been proposed by P. H. Roberts [45] in the context of neutron stars. This non-linearity is found to have no effect on the development of instabilities through the mechanism of stationary convection but influences the onset of overstable convection.

The linear stability of the magnetic Benard problem is investigated via a generalized energy theory. The magnetic field is shown to have a stabilising effect using this generalized energy theory.

In chapter (6), the Benard problem is investigated on the assumption that the magnetic field is in the direction of the vertical. The case of a non-vertical magnetic field is considered in chapter (7). The effect of both magnetic field and rotation is examined in chapter (8). In chapter (9) an energy analysis of the Benard problem is considered when the magnetic field is in the direction of the vertical. Chapters (2), (3) and (4) describe and illustrate the numerical methods which were used to solve the related eigenvalue problems. A constitutive analysis for a typical permeability is considered in chapter (5).

Chapter One

Introduction

Recently thermal instability theory has attracted considerable interest and has been recognised as a problem of fundamental importance in many fields of fluid dynamics. The earliest experiments to demonstrate the effect of surface tension on convection are those of Benard [1,2]. In an attempt to examine how far the results obtained by Benard's experiments can be explained theoretically, Rayleigh [3] produced his fundamental paper in which he showed that the numerical value of the non-dimensional parameter

$$R = \frac{g\ \alpha\ \beta}{\kappa\ \nu}\ d^4$$

decides whether a layer of fluid heated from below is stable or not, where g is the acceleration of gravity, β is the uniform adverse temperature gradient, d is the depth of the layer and α, ν, κ are respectively the coefficients of volume expansion, kinematic viscosity and thermal diffusivity.

The theory of thermal instability has been discussed from a more general viewpoint by Jeffreys [4,5,6] and Low [7]. Jeffreys [6] showed that the criterion for the onset of thermal instability derived for incompressible fluids can be used for compressible fluids under certain conditions on the adverse temperature gradient. Rayleigh [3] proved that for a layer of fluid heated from below the condition for marginal stability could be expressed by a principle of exchange of instabilities for the case of two free boundaries and Pellow & Southwell [8] extended this exchange principle to the rigid and mixed boundary value problems.

In the convection of a layer of incompressible viscous fluid only the free boundary eigenvalue problem has an exact solution and this was first obtained by Low [7]. Subsequently other workers eg. Pellow & Southwell [8], Chandrasekhar [9] and Reid & Harris [10] have used other techniques to examine the free boundary, rigid boundary and mixed boundary value problems.

Thermal instability theory has been enlarged by the interest in hydrodynamic flows of electrically conducting fluids in the presence of magnetic fields. The presence of such fields in an electrically conducting fluid usually has the effect of inhibiting the development of instabilities. Benard convection has been examined in the context of magnetohydrodynamic fluid by Thompson [11], Chandrasekhar [12,13,14,15] and others. Chandrasekhar [12] investigated the problem for the cases of stationary convection and overstability. He showed that the principle of the exchange of stability is valid if $P_m < P_r$ where P_r and P_m are respectively the viscous and magnetic Prandtle numbers. This condition is satisfied under most terrestial conditions, but under astrophysical conditions it is not. In fact under astrophysical conditions instability can arise either as cellular convection or as overstability (i.e. by oscillations of increasing amplitude) depending on the magnitude of Chandrasekhar number

$$Q = \frac{\mu^2 \; B^2 \; \sigma^2}{\rho \; \nu} \, d^2$$

where μ denotes the magnetic permeability, σ is the coefficient of electrical conductivity and ρ is the density of the fluid.

The instability of a layer of fluid heated from below and subjected to Coriolis forces has been discussed by Chandrasekhar [16,17]. He showed that the presence of these forces usually has the effect of inhibiting the onset of thermal convection.

The simultaneous effect of both magnetic field and rotation on the thermal instability of a layer of fluid heated from below has been investigated by Chandrasekhar [18] for the case of stationary convection when both boundaries are free. The analysis of this problem has shown that under the influences of rotation and magnetic field, fluid motion can show unexpected patterns of behaviour. The alternative possibility of the instability setting in as overstability has been considered by Chandrasekhar [19] who showed that instability can set in either as convection or as overstability depending on the values of the non-dimensional numbers P_r, P_m, Q and T where

$$T = \frac{4\ \Omega^2\ d^4}{\nu^2}$$

is the Taylor number and where Ω is the angular velocity. Chandrasekhar [19] obtained numerical results for the special case when the physical constants are those applicable to mercury at ordinary room temperature.

The formulation of the energy method began with the consideration of the kinetic energy of a perturbation motion. The method judges stability or instability of a given fluid motion by whether the energy of a disturbance of the given motion grows or decays. The energy method was originated by Reynolds [20] and Orr [21] and use since that time by many other writers e.g. Rayleigh

[22], Thomas [23,24] and others. However the theory of energy has been developed in the variational formulation by Serrin [25] and subsequently refined and extended extensively by Joseph [26,27,28, 29,30].

Serrin [25] applied the theory to calculate accurately the critical value of the Reynolds number for the stability of an arbitrary fluid motion in a bounded region. He also discussed the possibility of assuming periodic disturbances in directions where the flow domain is unbounded. Joseph [26] generalized the method of energy to discuss the stability of thermally-driven convective flows governed by the Boussinesq equations. He also generalized the method to accommodate convective motions governed by the nonlinear equations of Boussinesq [27].

The method has been applied successfully to a number of time-dependent flows, notably by Davis & Kerczek [31] and Homsey [32,33] It has also met with much success in a variety of convection problems, Joseph & Shir [34], Joseph [26], Joseph & Hung [35], Joseph & Carmi [36], Davis & Homsey [37], Galdi [38], Galdi & Straughan [39,40], Mctaggart & Lindsay [41], Straughan [42,43], Payne & Straughan [44] and others.

Moreover, in recent years it has been dramatically demonstrated that the method of energy and linear theory complement each other in delimiting the regions in parameter space for which subcritical instabilities are possible (see Joseph [28] and Galdi & Straughan [39]).

The energy theory provides lower bound for the region of stability whereas linear theory provides the top limit for which any form of stability is possible. Ideally we wish these bounds to be as close as possible and for the classical Benard problem both these limits agree. Thus the result for classical Benard theory is optimal.

Previous work in Benard convection has been done for magnetohydrodynamic fluids which have a linear constitutive relationship between the magnetic field H and the magnetic induction **B**. However a non-linear constitutive relationship between H and **B** may be appropriate for certain classes of materials. The relevence of this criterion to the configuration of a neutron star is discussed by Roberts [45] and Muzikhar & Pethick [46]. Cowley & Rosensweig [47], Gailitis [48] and others use non-linear magnetisation laws to describe the properties of ferrofluids.

This thesis studies convective instabilities in the Benard problem for a conducting magnetohydrodynamic fluid with variable permeability confined between two infinite horizontal surfaces. Temperature conditions on these surfaces generate a transfer of heat and the presence of an external magnetic field induces a non-linear magnetization of the fluid. The problem is first investigated on the assumption that the magnetic field is in the verticle direction. The strength of the non-linearity is measured by a non-dimensional parameter ϵ where the classical case discussed by Chandrasekhar [15] corresponds to $\epsilon = 0$ and in this case our results agree with those of Chandrasekhar [15] for the free, rigid and mixed boundary value problems. Appropriate values of the parameter ϵ have been obtained by using data collected from the work of Kaiser & Miskolczy [49], Chantrell, Popplewell & Charles [50], Charles & Popplewell [51], Popplewell Charles & Hoon [52] and Rosensweig [53] on numerous ferromagnetic fluids.

The case when H and **g** act in different directions is considered for the same model. This problem differs from that of the vertical magnetic field case in the sense that here we have a twelvth order eigenvalue problem and the z-component of the vorticity and the current density do not vanish. The convection which appears at marginal stability is in the form of rolls extended in directions parallel to the plane containing H and g and the critical Rayleigh numbers when $\epsilon = 0$, for the cases of stationary convection and overstability, coincide with those produced in the case when H and **g** are parallel if we interpret H to mean the component of H in the direction of **g** (see Chandrasekhar [13]). In other words the critical Rayleigh numbers does not depend on the inclination of H to the vertical. However when $\epsilon \neq 0$ this is not the case for overstability. In fact when instability sets in as overstability the critical Rayleigh number depends greatly on the parameter ϵ and on the inclination of the magnetic field to the vertical and we obtain critical Rayleigh numbers which are lower than those of the vertical magnetic field case.

The effect of both magnetic field and rotation is examined for the cases of stationary and overstability when both boundaries are rigid. When Q = 0, the inhibiting effect of the rotation is apparent. However when both magnetic field and rotation act together unexpected behaviour happened. In this problem the vanishing of the parameter ϵ produces results which coincide with those of the classical Benard problem under the influence of both magnetic field and rotation. However when $\epsilon \neq 0$ the inhibiting effect of both magnetic field and rotation increases as the parameter ϵ increases.

An energy analysis of the Benard problem is considered when the magnetic field is in the direction of the vertical. The stabilizing effect of the magnetic field is investigated via a generalized energy functional for a non-linear magnetohydrodynamic fluid. It is shown that the magnetic field has a stabilizing effect for this model.

The Compound Matrix Technique and the Chebyshev polynomials method have been used to solve the related eigenvalue problems. These methods shall be described in detail in chapter three and four respectively

Chapter Two

Minimisation / Maximisation Procedures

Introduction

In this chapter we consider the problem of finding the minimum of a continuous function F in a given interval in which the function is known to be unimodal i.e. it only has one turning value in the interval. Clearly there are two aspects to this problem viz.

(i) The determination of the minimum value.

(ii) The determination of the point of attainment.

Analytically these are equivalent problems but numerically this may not be the case. To see this let us consider the situation in which F is also differentiable in the vicinity of $x = c$, the point at which F has its minimum. The Taylor series of $F(x)$ about $x = c$ is

$$F(x) = F(c) + (x - c)\, F'(c) + \frac{1}{2!}\, (x - c)^2\, F''(c) + \ldots\ldots$$

and since $F'(c) = 0$ then

$$F(x) - F(c) = \frac{(x - c)^2}{2!}\, F''(c) + \ldots\ldots\ldots \qquad (2.1)$$

It is clear from (2.1) that it is easier to find $F(c)$ to a prescribed accuracy than to determine c to the same accuracy. In fact if we wish to determine c to a prescribed accuracy then (2.1) indicates that we must determine $F(c)$ to a significantly higher degree of accuracy and this may not be possible within the particular finite precision arithmetic in use.

In the following two sections, we discuss the minimisation procedures used in this thesis. The methods essentially revolve around calculating the value of the function $F(x)$ at two points x,y within the search interval $[a,b]$ and then reducing the size of the interval on the basis that F has a single minimum in $[a,b]$. Suppose that $a < x < y < b$ and that

(i) $F(x) \geqslant F(y)$ then minimum ε $[x,b]$

(ii) $F(x) < F(y)$ then minimum ε $[a,y]$

i.e. in case (i) we replace the lower limit a by the new value x whereas in (ii) the upper limit b is replaced by y and the whole process is repeated until the required accuracy is obtained. At each application of the scheme one point is effectively rejected and so an efficient algorithm is one which makes optimal use of the remaining point and related function value in the iterative scheme. The two schemes used in this thesis have this important property.

Finally, we note that any minimising scheme can be used to maximise $F(x)$ by simply minimising $G(x) = - F(x)$ and using the fact that

$$\underset{x\varepsilon[a,b]}{\text{Max}} \; F(x) = - \underset{x\varepsilon[a,b]}{\text{Min}} \; G(x) \; .$$

The Fibonacci Search

Suppose that $F(x)$ has a minimum in $[a,b]$ and that N applications of the scheme are required to achieve the desired accuracy. After k steps let us assume that the minimum has been isolated to the interval $[a_k,b_k]$ and define points x_k, y_k within this interval by

$$x_k = \frac{\lambda_{N-1-k}}{\lambda_{N+1-k}} (b_k - a_k) + a_k \; ,$$

$$y_k = \frac{\lambda_{N-k}}{\lambda_{N+1-k}} (b_k - a_k) + a_k \; ,$$

where $\{\lambda_n\}$ is a sequence of integers defined inductively by the relation

$$\lambda_{N+2} = \lambda_{N+1} + \lambda_N ,$$

$$\lambda_0 = \lambda_1 = 1 .$$

These integers form the Fibonacci numbers (Cohen [54]) and for this reason, this scheme will be called the Fibonacci Search Algorithm. Moreover, from the definition of x_k and y_k, it is obvious that

$$a_k < x_k < y_k < b_k .$$

<u>Case 1</u> $F(x_k) \geq F(y_k)$

Here the minimum is restricted to $[x_k, b_k]$ and so

$$a_{k+1} = x_k \qquad b_{k+1} = b_k ,$$

$$x_{k+1} = \frac{\lambda_{N-2-k}}{\lambda_{N-k}} (b_{k+1} - a_{k+1}) + a_{k+1} ,$$

$$y_{k+1} = \frac{\lambda_{N-k-1}}{\lambda_{N-k}} (b_{k+1} - a_{k+1}) + a_{k+1} .$$

We direct our attention to the expression for x_{k+1} .

$$x_{k+1} = \frac{\lambda_{N-2-k}}{\lambda_{N-k}} (b_k - x_k) + x_k$$

$$= \frac{\lambda_{N-2-k}}{\lambda_{N-k}} \left[(b_k - a_k) - \frac{\lambda_{N-1-k}}{\lambda_{N+1-k}} (b_k - a_k)\right] + a_k$$

$$+ \frac{\lambda_{N-1-k}}{\lambda_{N+1-k}} (b_k - a_k)$$

$$= \frac{\lambda_{N-2-k}}{\lambda_{N-k}} \frac{(\lambda_{N+1-k} - \lambda_{N-1-k})}{\lambda_{N+1-k}} (b_k - a_k) + a_k$$

$$+ (b_k - a_k) \frac{\lambda_{N-1-k}}{\lambda_{N+1-k}}$$

$$= \frac{\lambda_{N-2-k}}{\lambda_{N-k}} \frac{\lambda_{N-k}}{\lambda_{N+1-k}} (b_k - a_k) + \frac{\lambda_{N-1-k}}{\lambda_{N+1-k}} (b_k - a_k) + a_k$$

i.e. $$x_{k+1} = \frac{(\lambda_{N-1-k} + \lambda_{N-2-k})}{\lambda_{N+1-k}} (b_k - a_k) + a_k$$

$$= \frac{\lambda_{N-k}}{\lambda_{N+1-k}} (b_k - a_k) + a_k$$

$$= y_k \ .$$

Thus $F(x_{k+1}) = F(y_k)$.

Case 2 $F(x_k) < F(y_k)$

Here the minimum is restricted to $[a_k, y_k]$ and so

$$a_{k+1} = a_k \qquad b_{k+1} = y_k \ .$$

The previous expressions for x_{k+1}, y_{k+1} are still valid but in this case, we direct our attention to the investigation of y_{k+1}.

$$y_{k+1} = \frac{\lambda_{N-k-1}}{\lambda_{N-k}} (y_k - a_k) + a_k$$

$$= \frac{\lambda_{N-k-1}}{\lambda_{N-k}} \frac{\lambda_{N-k}}{\lambda_{N+1-k}} (b_k - a_k) + a_k$$

$$= \frac{\lambda_{N-k-1}}{\lambda_{N+1-k}} (b_k - a_k) + a_k$$

$$= x_k \ .$$

Thus $F(y_{k+1}) = F(x_k)$.

In both cases, the value of the function at one of the new points x_{k+1}, y_{k+1} is already known and so the scheme is "efficient" Further, we may easily verify that

$$b_{k+1} - a_{k+1} = \frac{\lambda_{N-k}}{\lambda_{N+1-k}} (b_k - a_k)$$

irrespective of which case is activated. Thus after N repetitions of the scheme the interval [a,b] is reduced to one of length

$$(b - a) / \lambda_{N+1}$$

and so if ϵ is the required accuracy then we need to choose N such that

$$\lambda_{N+1} > (b - a) / 2\epsilon \ .$$

Golden Section Search

This algorithm is very similar to the Fibonacci Search except that the selection of the points x and y does not depend on the number of times the algorithm is to be applied. Suppose that F(x) has a single minimum in [a,b] and define points x and y by

$$x = a + r^2 (b - a)$$

$$y = a + r (b - a)$$

where $r = (\sqrt{5} - 1)/2$ is the inverse of the golden section ratio . In fact $r^2 + r = 1$.

Case 1

If $F(x) \geqslant F(y)$ then the minimum can be further isolated to [x,b] and so the next application of the algorithm requires the evaluation of F at x^*, y^* where

$$x^* = x + r^2 (b - x)$$

$$y^* = x + r (b - x) \ .$$

We direct our attention to the expression for x^* .

$$x^* = a + r^2 (b - a) + r^2 [b - a - r^2 (b - a)]$$

$$= a + (b - a) (2r^2 - r^4) \ .$$

But $2r^2 - r^4 = 1 - (1 - r^2)^2 = 1 - r^2 = r$ and so

$$x^* = a + r\,(b - a)$$

$$= y\ .$$

Thus $F(x^*) = F(y)$.

Case 2

If $F(x) < F(y)$ then the minimum is isolated to $[a,y]$ and so the next application of the algorithm requires the evaluation of F at x^*, y^* where

$$x^* = a + r^2\,(y - a)$$

$$y^* = a + r\,(y - a)\ .$$

Here we direct our attention to the expression for y^* .

$$y^* = a + r\,[a + r\,(b - a) - a]$$

$$= a + r^2\,(b - a)$$

$$= x\ .$$

Thus $F(y^*) = F(x)$.

In both cases the value of the function at one of the new points is already known and so the algorithm is "efficient" . In particular we may verify that

$$(y - a) = (b - x) = r\,(b - a)$$

so that after N repetitions of the scheme, the interval $[a,b]$ is reduced to one of length $r^N\,(b - a)$ and so if ϵ is the required accuracy then we need to choose N such that

$$r^N < 2\epsilon\ /\ (b - a)\ .$$

It is easily verified that the solution of

$$\lambda_N = \lambda_{N-1} + \lambda_{N-2}$$

satisfying $\lambda_0 = \lambda_1 = 1$ is

$$\lambda_N = \frac{1}{1 + r^2} [r^{-N} + (-1)^N \ r^{N+2}]$$

$$= \frac{r^{-N}}{1 + r^2} [1 + (-1)^N \ r^{2N+2}] \ .$$

Thus, after N evaluations of the previous schemes, the ratio of the lengths of the reduced intervals can be computed as follows

$$\frac{\text{Golden Section interval length}}{\text{Fibonacci interval length}} = \frac{(b - a) \ r^N}{(b - a) \ / \ \lambda_{N+1}}$$

$$= r^N \ \lambda_{N+1}$$

$$= \frac{r^N \ r^{-(N+1)}}{1 + r^2} [1 + (-1)^{N+1} \ r^{2(N+2)}]$$

$$= \frac{1}{r} \frac{1}{1 + r^2} [1 + (-1)^{N+1} \ r^{2(N+2)}]$$

$$\longrightarrow \frac{1}{r \ (1 + r^2)} \ \text{as } N \longrightarrow \infty$$

$$= \frac{1}{3r - 1}$$

$$= (5 + 3\sqrt{5}) \ / \ 10$$

$$\simeq 1.17 \ .$$

Thus Golden Section search is technically inferior to the Fibonacci search. Since the ratio of interval lengths tends to a constant which is not greatly larger than unity then both methods have effectively the same efficiency. The greater efficiency of the Fibonacci algorithm is only pronounced for small numbers of iterations. The Golden Section search is used in this thesis instead of the Fibonacci search because the Fibonacci search is more complicated and the number of steps N must be determined initially for the desired accuracy. Further details of these methods can be found in Cheney and Kincaid [55].

Chapter Three

The Compound Matrix Technique

Introduction

The Compound Matrix method has been used by Gilbert and Backus [56] in their discussion of elastic wave problems and it has also been used by Lakin, Ng and Reid [57] to approximate the eigenvalues of the Orr-Sommerfeld problem. This method has been extensively developed by Ng and Reid [58] in their investigation of boundary layer and related stiff problems. For further details see Drazin and Reid [59]. The technique also lends itself naturally to linear and non-linear convective studies. (see Payne and Straughan [44], Straughan [60])

The technique can be applied to differential equations of any order where the boundary conditions are distributed in any fashion between the two boundaries. Suppose that the original equation has order n and that (n-m) boundary conditions are applied at one boundary and m at the other boundary then in this case the Compound Matrix method involves the generation of $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$ differential equations. Since $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$ has its largest value when $m = n/2$ then the worst senario occurs when the boundary conditions are evenly distributed and the original differential equation has even order i.e. $n = 2m$. This is the case which most often appears in applications and also appears in this work.

The Compound Matrix Method

Let us examine the problem of determining an eigenvalue λ of the nth order system

$$\frac{dY}{dx} = A(\lambda,x)\ Y \tag{3.1}$$

subject to the boundary conditions

$$B\ Y = 0 \qquad \text{on } x = 0 \quad , \tag{3.2}$$

$$C\ Y = 0 \qquad \text{on } x = 1 \tag{3.3}$$

where Y is an n vector, $A(\lambda,x)$ is an nxn matrix and **B** and **C** are respectively (n–m)xn and mxn matrices of full rank. i.e. (n – m) conditions are given at x = 0 and m conditions at x = 1. We may assume without loss of generality that $m \leqslant n/2$. Notionally the general solution of the nth order system (3.1) has n degrees of freedom. Clearly (n–m) of these degrees of freedom can be removed by requiring that conditions (3.2) be satisfied and so we may conceptually construct m functionally independent solutions of (3.1) which also satisfy conditions (3.2). Let these solutions be

$$\mathbf{w}_1(\lambda,x),\ \mathbf{w}_2(\lambda,x),\ \ldots\ldots,\ \mathbf{w}_m(\lambda,x)$$

where

$$\mathbf{w}_i(\lambda,x) = (w_i,\ w_i',\ w_i'',\ \ldots\ldots,\ w_i^{(n-1)}\ )^T$$

Let us define the nxm matrix **M** to be the matrix whose rth column is $\mathbf{w}_r(\lambda,x)$ i.e.

$$\mathbf{M} = [\mathbf{w}_1,\ \mathbf{w}_2,\ \ldots\ldots,\ \mathbf{w}_m]$$

then by construction **BM** = 0 on x = 0 since $\mathbf{w}_r(\lambda,0)$ $1 \leqslant r \leqslant m$ all lie in the nullspace of **B**. Let $\mathbf{W}(\lambda,x)$ be any solution of (3.1) satisfying (3.2) then

$$W(\lambda,x) = \sum_{r=1}^{m} \alpha_r \, w_r(\lambda,x) \; . \tag{3.4}$$

We now direct our attention to condition (3.3). If $CW = 0$ on $x = 1$ then

$$\sum_{r=1}^{m} \alpha_r \, C \, w_r(\lambda,x) = 0 \quad \text{on } x = 1. \tag{3.5}$$

However all the α's cannot be zero and so we deduce that the vectors $Cw_1, Cw_2, \ldots\ldots, Cw_m$ must be linearly dependent and hence if we define the mxm matrix D to be the matrix whose rth column is Cw_r. i.e.

$$D = [Cw_1, Cw_2, \ldots\ldots, Cw_m] = C\,M$$

then det D = 0. By definition

$$\det D = e_{k_1 k_2 \ldots\ldots k_m} d_{k_1 1} d_{k_2 2} \ldots\ldots d_{k_m m}$$

$$= e_{k_1 k_2 \ldots\ldots k_m} \Big[\sum_{r_1=1}^{n} c_{k_1 r_1} (w_1)_{r_1} \Big] \ldots\ldots\ldots$$

$$\ldots\ldots\ldots \Big[\sum_{r_m=1}^{n} c_{k_m r_m} (w_m)_{r_m} \Big]$$

$$= \sum_{r_1=1}^{n} \ldots \sum_{r_m=1}^{n} e_{k_1 \ldots k_m} c_{k_1 r_1} \ldots c_{k_m r_m} (w_1)_{r_1} \ldots\ldots (w_m)_{r_m}.$$

The sum

$$e_{k_1 k_2 \ldots\ldots k_m} c_{k_1 r_1} \ldots\ldots c_{k_m r_m}$$

is the determinant of a minor of C where the columns $r_1, r_2, \ldots, r_m$ are selected from the n available columns of C. The number of possible selections of these columns is $\left[\begin{matrix} n \\ m \end{matrix} \right]$ and for each selection, we require to compute the determinant of the corresponding minor of **M**. In terms of these minors

$$|D| = \sum_{k=1}^{\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]} |C_k| \, \Phi_k \qquad (3.6)$$

where Φ_k are the minors of **M** and where the summation is performed over some exhaustive listing of the $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ combinations of m columns out of n. In order for this approach to succeed, we therefore require to compute the value of the $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ minors of the nxm matrix **M** at the endpoint x = 1. The Compound Matrix method achieves this by developing a system of $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ ordinary differential equations for the evaluation of each of these determinants.

Derivation of the Compound Matrix Equations

We start by producing an exhaustive listing of the $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ possible combinations. There is no unique approach to this problem but in the context of this work we shall adopt the following enumeration scheme:- Number the variables in terms of the rows of **M** which appear in the minor and arrange these rows in an increasing order from left to right so that

$$\Phi_1 = (1,\ 2,\ 3,\ \ldots\ldots\ldots,\ m-1,\ m)$$

$$\Phi_2 = (1,\ 2,\ 3,\ \ldots\ldots\ldots,\ m-1,\ m+1)$$

$$\vdots$$

$$\Phi_{n-m+1} = (1,\ 2,\ 3,\ \ldots\ldots\ldots,\ m-1,\ n)$$

$$\Phi_{n-m+2} = (1,\ 2,\ 3,\ \ldots\ldots\ldots,\ m,\ m+1)$$

$$\vdots$$

$$\Phi_{\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]} = (n-m+1,\ \ldots\ldots\ldots,\ n-1,\ n)$$

where the general strategy is to generate further combinations by increasing the righthand parameters gradually drifting the incremental process to the left until no further increments are possible. In this way we obtain an exhaustive list of all possible combinations ordered in increasing order from left to right. In order to determine the differential equations satisfied by the Φ's we observe that

$$\frac{dM}{dx} = \left[\frac{dw_1}{dx}, \; \ldots\ldots\ldots, \; \frac{dw_m}{dx}\right]$$

$$= \left[Aw_1, \; \ldots\ldots\ldots, \; Aw_m\right]$$

$$= AM \, . \qquad (3.7)$$

Let Φ be any minor of **M** then clearly $d\Phi/dx$ is a minor of dM/dx and in view of property (3.7), $d\Phi/dx$ is expressable as a sum of products of the minors of **A** and **M**. Hence we may obtain a set of $\left[{n \atop m} \right]$ differential equations for $\Phi_1, \Phi_2, \ldots\ldots, \Phi_{\left[{n \atop m} \right]}$.

In conclusion we can develop a system of $\left[{n \atop m} \right]$ linear differential equations for the minors of **M**. In order to determine the value of these minors at $x = 1$, it is necessary to derive a set of initial conditions for the minors consistent with conditions (3.2) imposed on the original equation. We recall that the matrix **M** is constructed to have rank m and so we can select a subset of m linearly independent rows of this matrix and assert that the (n−m) remaining rows can be expressed as linear combinations of the m " preferred " rows. As a result, it is clear that there is only one degree of freedom in the initial conditions. We choose the preferred minor to have value +1 and all other non-zero minors of **M** are determined so that any one set of initial conditions is equivalent to any other set modulo a constant multiplying factor. Hence the minors of M are determined as the solution of an initial value problem. We may handle this initial value problem using a differential equation solver and thus directly compute the value of det D at $x = 1$.

Example (1)

The stationary convection of a Navier-Stokes fluid between two heated plates is controlled by the differential equations

$$D^4w - 2\ a^2\ D^2w + a^4\ w - a^2\ \surd(R)\ \theta = 0$$
$$D^2\theta - a^2\theta + \surd(R)\ w = 0 \qquad 0 < z < 1 \qquad (3.8)$$

where D = d/dz, a is a variable parameter and R is the eigenvalue to be determined. Let us consider the situation of perfectly conducting mixed boundaries on z = 0 and z = 1. i.e.

$$w = Dw = \theta = 0 \qquad \text{on } z = 0\ ,$$
$$w = D^2w = \theta = 0 \qquad \text{on } z = 1\ . \qquad (3.9)$$

We begin by converting the given equations into a system. Define six new variables y_1, y_2, ... , y_6 by the rules

$$y_1 = w,\quad y_2 = Dw,\quad y_3 = D^2w,\quad y_4 = D^3w,\quad y_5 = \theta,\quad y_6 = D\theta.$$

Clearly

$$Dy_1 = Dw = y_2$$

$$Dy_2 = D^2w = y_3$$

$$Dy_3 = D^3w = y_4$$

$$Dy_4 = D^4w = 2\ a^2\ D^2w - a^4\ w + a^2\ \surd(R)\ \theta$$
$$= -\ a^4\ y_1 + 2\ a^2\ y_3 + a^2\ \surd(R)\ y_5$$

$$Dy_5 = D\theta = y_6$$

$$Dy_6 = D^2\theta = a^2\ \theta - \surd(R)\ w = -\ \surd(R)\ y_1 + a^2\ y_5$$

and so in terms of the theory previously described

$$DY = A\ Y$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -a^4 & 0 & 2a^2 & 0 & a^2\surd(R) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\surd(R) & 0 & 0 & 0 & a^2 & 0 \end{bmatrix} .$$

The boundary conditions can be written in the form

$$B\,Y = 0 \qquad \text{on } z = 0 \ ,$$

$$C\,Y = 0 \qquad \text{on } z = 1$$

where B and C are the 3x6 matrices

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

In this case we expect $\begin{bmatrix} 6 \\ 3 \end{bmatrix} = 20$ compound matrix equations. The matrix M in this instance is 6x3. We may list the 20 variables in terms of the rows of M. i.e

$\Phi_1 = (1,2,3)$, $\Phi_2 = (1,2,4)$, $\Phi_3 = (1,2,5)$, $\Phi_4 = (1,2,6)$,

$\Phi_5 = (1,3,4)$, $\Phi_6 = (1,3,5)$, $\Phi_7 = (1,3,6)$, $\Phi_8 = (1,4,5)$,

$\Phi_9 = (1,4,6)$, $\Phi_{10} = (1,5,6)$, $\Phi_{11} = (2,3,4)$, $\Phi_{12} = (2,3,5)$,

$\Phi_{13} = (2,3,6)$, $\Phi_{14} = (2,4,5)$, $\Phi_{15} = (2,4,6)$, $\Phi_{16} = (2,5,6)$,

$\Phi_{17} = (3,4,5)$, $\Phi_{18} = (3,4,6)$, $\Phi_{19} = (3,5,6)$, $\Phi_{20} = (4,5,6)$

The compound matrix equations in terms of the variables described above are

$$\dot{\Phi}_1 = \Phi_2$$

$$\dot{\Phi}_2 = 2\ a^2\ \Phi_1 + a^2\ \surd(R)\ \Phi_3 + \Phi_5$$

$$\dot{\Phi}_3 = \Phi_4 + \Phi_6$$

$$\dot{\Phi}_4 = a^2\ \Phi_3 + \Phi_7$$

$$\dot{\Phi}_5 = a^2\ \surd(R)\ \Phi_6 + \Phi_{11}$$

$$\dot{\Phi}_6 = \Phi_7 + \Phi_8 + \Phi_{12}$$

$$\dot{\Phi}_7 = a^2\ \Phi_6 + \Phi_9 + \Phi_{13}$$

$$\dot{\Phi}_8 = 2\ a^2\ \Phi_6 + \Phi_9 + \Phi_{14}$$

$$\dot{\Phi}_9 = 2\ a^2\ \Phi_7 + a^2\ \Phi_8 + a^2\ \surd(R)\ \Phi_{10} + \Phi_{15}$$

$$\dot{\Phi}_{10} = \Phi_{16}$$

$$\dot{\Phi}_{11} = -\ a^4\ \Phi_1 + a^2\ \surd(R)\ \Phi_{12}$$

$$\dot{\Phi}_{12} = \Phi_{13} + \Phi_{14}$$

$$\dot{\Phi}_{13} = -\ \surd(R)\ \Phi_1 + a^2\ \Phi_{12} + \Phi_{15}$$

$$\dot{\Phi}_{14} = a^4\ \Phi_3 + 2\ a^2\ \Phi_{12} + \Phi_{15} + \Phi_{17}$$

$$\dot{\Phi}_{15} = -\ \surd(R)\ \Phi_2 + a^4\ \Phi_4 + 2\ a^2\ \Phi_{13} + a^2\ \Phi_{14} + a^2\ \surd(R)\ \Phi_{16} + \Phi_{18}$$

$$\dot{\Phi}_{16} = -\ \surd(R)\ \Phi_3 + \Phi_{19}$$

$$\dot{\Phi}_{17} = a^4\ \Phi_6 + \Phi_{18}$$

$$\dot{\Phi}_{18} = -\ \surd(R)\ \Phi_5 + a^4\ \Phi_7 + a^2\ \Phi_{17} + a^2\ \surd(R)\ \Phi_{19}$$

$$\dot{\Phi}_{19} = -\ \surd(R)\ \Phi_6 + \Phi_{20}$$

$$\dot{\Phi}_{20} = -\ \surd(R)\ \Phi_8 - a^4\ \Phi_{10} + 2\ a^2\ \Phi_{19}\ .$$

Since BM = 0 then at z = 0, the first, second and fifth rows of M must be zero. The third, fourth and sixth rows of M are unrestricted except in the respect that M must have rank 3 i.e. at z = 0, (3,4,6) is the only non-zero minor of M and so Φ_{18} is assigned the value 1. At the final point z = 1, we need to compute the sum of the products of each minor of C with the corresponding minor of M. However, the trivial nature of C in this instance makes this an easy computation. In this instance the only non-zero minor of C comes from the first, third and fifth columns of C and this we associate with the minor of M derived from the first, third and fifth rows of M i.e. (1,3,5) = Φ_6 in this application.

Example (2)

The linear stability of Poiseuille flow is characterised by the eigenvalue λ of the Orr-Sommerfeld equation

$$(D^2 - \alpha^2)^2 V = i \alpha R [(\bar{u} - \lambda)(D^2 - \alpha^2) - D^2\bar{u}] V$$
$$V(y) = V'(y) = 0 \qquad \text{on } y = \pm 1 . \tag{3.10}$$

where $D = d/dy$, $\bar{u} = 1 - y^2$, α is the wave number and R is the Reynolds number. We illustrate how the compound matrix technique can be used to deal with this problem in the complex situation. Define

$$x_1 = V, \qquad x_2 = DV, \qquad x_3 = D^2V, \qquad x_4 = D^3V .$$

Clearly

$$Dx_1 = DV = x_2$$
$$Dx_2 = D^2V = x_3$$
$$Dx_3 = D^3V = x_4$$
$$Dx_4 = D^4V = [2 a^2 + i \alpha R (1 - y^2 - \lambda)] D^2V$$
$$- [a^4 - 2 i \alpha R + i a^2 \alpha R (1 - y^2 - \lambda)] V$$
$$= \beta_1 x_1 + \beta_2 x_3$$

where

$$\beta_1 = - [a^4 - 2 i \alpha R + i a^2 \alpha R (1 - y^2 - \lambda)]$$
$$\beta_2 = 2 a^2 + i \alpha R (1 - y^2 - \lambda) .$$

Thus

$$DX = A X$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_1 & 0 & \beta_2 & 0 \end{bmatrix}.$$

The boundary conditions $(3.10)_{(ii)}$ can be written in the form

$$B\,X = 0 \qquad \text{on } y = -1 ,$$

$$C\,X = 0 \qquad \text{on } y = 1$$

where B and C are 2×4 matrices (B = C in this particular problem).

i.e.

$$B = C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

In this problem we expect $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 6$ compound matrix equations. The matrix M in this instance is 4×2. We may list the 6 variables in terms of the rows of M. i.e

$$\Phi_1 = (1,2), \quad \Phi_2 = (1,3), \quad \Phi_3 = (1,4),$$

$$\Phi_4 = (2,3), \quad \Phi_5 = (2,4), \quad \Phi_6 = (3,4).$$

The compound matrix equations in terms of the variables described above are

$$\dot{\Phi}_1 = \Phi_2$$

$$\dot{\Phi}_2 = \Phi_3 + \Phi_4$$

$$\dot{\Phi}_3 = \beta_2\,\Phi_2 + \Phi_5$$

$$\dot{\Phi}_4 = \Phi_5$$

$$\dot{\Phi}_5 = -\,\beta_1\,\Phi_1 + \beta_2\,\Phi_4 + \Phi_6$$

$$\dot{\Phi}_6 = -\,\beta_1\,\Phi_2 .$$

For the initial condition at $y = -1$, the first and second rows of M must be zero. The third and fourth rows of M are unrestricted except in the respect that M must have rank 2 i.e. at $y = -1$, (3,4) is the only non-zero minor of M and so Φ_6 is assigned the value $1 + i$. At $y = 1$, the only non-zero minor of C comes from the first and second columns of C and this we associate with the minor of M derived from the first and second rows of M i.e. Φ_1 in this example.

Equations $(3.10)_{(i),(ii)}$ are solved using a Nag routine which uses a Runge-Kutta-Merson method. The secant method is used to vary the eigenvalue parameter until the target condition is satisfied to some prescribed degree of accuracy. To show the effectiveness of the Compound Matrix method we have considered the unstable mode for $\alpha = 1$ and $R = 10000$ as this is a case for which a comparison can be made with existing results.

Accuracy Required	λ
10^{-3}	0.235389033705023398 + 0.00325098801925888969i
10^{-4}	0.237255408173027024 + 0.00367893416944019665i
10^{-5}	0.237495201298997885 + 0.00373546738492939925i
10^{-6}	0.237523090540996074 + 0.00373955537711399025i
10^{-7}	0.237526130495021345 + 0.00373968057880910427i
10^{-8}	0.237526451077050837 + 0.00373967321565089270i
10^{-9}	0.237526484974780253 + 0.00373967101997027041i
10^{-10}	0.237526488430710184 + 0.00373967067271911962i
10^{-11}	0.237526488781399148 + 0.00373967062858209404i
10^{-12}	0.237526488816553583 + 0.00373967062358091217i

So for accuracy of 10^{-11} we obtain the result which Orszag [61] obtained using 50 Chebyshev polynomials. The Fortran77 code which produced the above results is listed in Appendix I.

Chapter Four

Chebyshev Method for Boundary Value Problems

Introduction

The use of Chebyshev polynomials for solving differential equations was first introduced by Lanczos [62] and Clenshaw [63]. The Lanczos method has been developed and extensively applied to ordinary differential equations by Fox [64], Fox & Parker [65] and others. Orszag [66], [61] and Orszag & Kells [67] have shown that expansions in Chebyshev polynomials are better suited to the solutions of hydrodynamic stability problems than expansions in other sets of orthogonal functions. Results of high accuracy are obtained when using Chebyshev approximation to compute solutions of boundary value problems. (see Orszag [61] and Davis, Karageorghis & Phillips [68],[69]).

Some properties of Chebyshev polynomials

Let $T_n(x)$ denotes the nth degree Chebyshev polynomial of the first kind, defined in the interval [-1,1] by

$$T_n(\cos\theta) = \cos(n\theta) \qquad n \geqslant 0 \tag{4.1}$$

then from the trigonometric identity

$$\cos(n+m)\theta + \cos(n-m)\theta = 2\cos(n\theta)\cos(m\theta)$$

we obtain

$$T_{n+m}(x) + T_{|n-m|}(x) = 2\,T_n(x)\,T_m(x) \tag{4.2}$$

and, when m = 1 we obtain the recurrence relation of the Chebyshev polynomials

$$2x\,T_n(x) = T_{n+1}(x) + T_{n-1}(x) \qquad n \geqslant 1. \tag{4.3}$$

Furthermore from the definition of $T_n(x)$ we can see that

$$|T_n(x)| \leqslant 1 \qquad\qquad -1 \leqslant x \leqslant 1 \,. \tag{4.4}$$

In terms of the variable x, we may write (4.1) as

$$T_n(x) = \cos(n \cos^{-1}(x))$$

from which we can show that

$$\begin{aligned} T_n(1) &= 1 & T_n(-1) &= (-1)^n \\ T_n'(1) &= n^2 & T_n'(-1) &= (-1)^{n-1}\, n^2 \,. \end{aligned} \tag{4.5}$$

The Chebyshev polynomials are orthogonal over [-1,1] with respect to the weight function $(1 - x^2)^{-\frac{1}{2}}$. To see this consider the trigonometric integral

$$\int_0^{\pi} \cos(n\theta)\, \cos(m\theta)\, d\theta = C_n\, \delta_{mn}$$

where

$$C_0 = \pi \quad , \quad C_n = \pi/2 \quad (n \neq 0) \,.$$

With the change of variable, $x = \cos\theta$, this becomes

$$\int_{-1}^{1} \frac{T_n(x)\; T_m(x)}{(1 - x^2)^{\frac{1}{2}}}\, dx = C_n\, \delta_{nm} \,. \tag{4.6}$$

Let the Chebyshev expansion of an infinitely differentiable function y(x) in [-1,1] be

$$y(x) = \sum_{n=0}^{\infty} a_{n+1}\, T_n(x)$$

where

$$a_{n+1} = \frac{2}{\pi} \int_{-1}^{1} y(x)\, T_n(x)\, (1 - x^2)^{-\frac{1}{2}}\, dx \qquad n \geqslant 0$$

then the Chebyshev expansion of the derivative dy/dx will be

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_{n+1}\, T_n'(x). \tag{4.7}$$

Clearly $T_n'(x)$ is a polynomial of degree n-1. Let us now write $T_n'(x)$ as an expansion of Chebyshev polynomials. i.e.

$$T_n'(x) = \sum_{s=0}^{\infty} B_{s+1,n+1}\, T_s(x) \qquad n \geq 0. \tag{4.8}$$

then

$$B_{s+1,n+1} = 0 \qquad s \geq n .$$

From (4.8)

$$\int_{-1}^{1} \frac{T_n'(x)\; T_j(x)}{(1-x^2)^{\frac{1}{2}}}\, dx = \sum_{s=0}^{\infty} B_{s+1,n+1} \int_{-1}^{1} \frac{T_s(x)\; T_j(x)}{(1-x^2)^{\frac{1}{2}}}\, dx.$$

Using (4.6) we obtain

$$\int_{-1}^{1} \frac{T_n'(x)\; T_j(x)}{(1-x^2)^{\frac{1}{2}}}\, dx = \frac{\pi}{2} \sum_{s=0}^{\infty} B_{s+1,n+1} \begin{cases} \delta_{sj} & j > 0 \\ 2\delta_{sj} & j = 0 \end{cases}$$

Thus

$$B_{j+1,n+1} = \frac{2}{\pi} \int_{-1}^{1} \frac{T_n'(x)\; T_j(x)}{(1-x^2)^{\frac{1}{2}}}\, dx \qquad j \geq 1$$

$$B_{1,n+1} = \frac{1}{\pi} \int_{-1}^{1} \frac{T_n'(x)}{(1-x^2)^{\frac{1}{2}}}\, dx .$$

Let

$$I_n = \frac{1}{\pi} \int_{0}^{\pi} \frac{\sin(n\theta)}{\sin\theta}\, d\theta$$

and recognize that

$$T_n'(\cos\theta) = \frac{n \sin(n\theta)}{\sin\theta} .$$

Thus

$$B_{j+1,n+1} = \frac{2}{\pi} \int_0^{\pi} \frac{n \sin(n\theta) \cos(j\theta)}{\sin\theta} d\theta$$

$$= \frac{n}{\pi} \int_0^{\pi} \frac{\sin(n+j)\theta}{\sin\theta} d\theta + \frac{n}{\pi} \int_0^{\pi} \frac{\sin(n-j)\theta}{\sin\theta} d\theta$$

$$= n \, (I_{n+j} + I_{n-j})$$

and

$$B_{1,n+1} = \frac{1}{\pi} \int_0^{\pi} \frac{n \sin(n\theta)}{\sin\theta} d\theta = n \, I_n \, .$$

Since

$$I_n = \begin{cases} 0 & \text{if n is even.} \\ 1 & \text{if n is odd.} \end{cases}$$

Then

$$B_{rk} = \begin{cases} 0 & \text{if r+k is even} \\ (k-1)\,(2 - \delta_{1r}) & \text{if r+k is odd .} \end{cases}$$

From (4.7) and (4.8) we obtain

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} B_{j+1,n+1} \, a_{n+1} \, T_j(x).$$

and

$$\frac{d^r y}{dx^r} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} B^r_{j+1,n+1} \, a_{n+1} \, T_j(x) \qquad (4.9)$$

where **B** is the derivative matrix.

To understand why Chebyshev polynomials are so effective in approximating the solution of differential equations we need to introduce the shifted Chebyshev polynomials $T_n^*(x)$ which are defined in terms of the Chebyshev polynomials $T_n(x)$ by the relation

$$T_n^*(x) = T_n(2x - 1) \; . \qquad (4.10)$$

The recurrence relation of $T_n^*(x)$ is given by

$$2\ (2x - 1)\ T_n^*(x) = T_{n+1}^*(x) + T_{n-1}^*(x) \qquad n \geqslant 1\ .$$

When $x = -1$

$$T_{n+1}^*(-1) = -\ 6\ T_n^*(-1) - T_{n-1}^*(-1)$$

from which we can show that

$$T_n^*(-1) = A\ (-\ 3 - 2\ \surd 2)^n + B\ (-\ 3 + 2\ \surd 2)^n\ .$$

Since

$$T_0^*(-1) = T_0(-3) = 1,$$

$$T_1^*(-1) = T_1(-3) = -\ 3$$

then

$$T_n^*(-1) = \frac{1}{2}\ (-1)^n\ [(3 + 2\ \surd 2)^n + (3 - 2\ \surd 2)^n]$$

and

$$\frac{1}{\left| T_n^*(-1) \right|} = \frac{2}{(3 + 2\ \surd 2)^n + (3 - 2\ \surd 2)^n}$$

$$< \frac{2}{(3 + 2\ \surd 2)^n}\ . \qquad (4.11)$$

It is advantageous to approximate solutions in terms of expansions of orthogonal polynomials rather than a power series of the same degree. To understand why, we consider the solution of the differential equation

$$(1 + x)\ \frac{dy}{dx} = 1 \qquad y(0) = 0 \qquad (4.12)$$

which has an exact solution

$$y = \log(1 + x)\ .$$

Suppose we search for a power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n$$

then clearly

$$(1 + x) \sum_{n=1}^{\infty} n\, a_n x^{n-1} - 1 = 0$$

and $a_0 = 0$ in order to satisfy the initial condition. So

$$\sum_{n=1}^{\infty} n\, a_n x^{n-1} + \sum_{n=1}^{\infty} n\, a_n x^n - 1 = 0 \; .$$

Put $m = n - 1$

$$\therefore \quad \sum_{m=0}^{\infty} (m + 1)\, a_{m+1} x^m + \sum_{m=1}^{\infty} m\, a_m x^m - 1 = 0 \quad .$$

$$\text{i.e.} \quad a_1 - 1 + \sum_{m=1}^{\infty} [m\, a_m + (m + 1)\, a_{m+1}]\, x^m = 0 \quad .$$

$$\Rightarrow \quad a_1 = 1 \quad , \quad m\, a_m + (m + 1)\, a_{m+1} = 0 \quad .$$

$$\Rightarrow \quad m\, a_m (-1)^m = (m + 1)\, a_{m+1} (-1)^{m+1} = \text{constant.}$$

$$\Rightarrow \quad m\, a_m (-1)^m = - a_1 = - 1$$

$$\text{i.e.} \quad a_m = (-1)^{m-1} / m \; .$$

Thus solution is

$$y = \sum_{m=1}^{\infty} (-1)^{m-1} x^m / m \quad .$$

Since we know the exact solution then we can see immediately that the error in estimating y(x) by the first n terms of its Maclaurin series is

$$\frac{x^{N+1}}{(N + 1)!}\, y^{N+1}(\xi) \qquad 0 < \xi < x$$

but

$$\left| \, y^{N+1}(\xi) \, \right| = \left| \frac{(-1)^N N!}{(1 + \xi)^{N+1}} \right|$$

$$\therefore \qquad R_N \leqslant \frac{x^{N+1}}{N+1}$$

i.e. the power series solution is very accurate for small x but the accuracy deteriorates as x becomes larger. Consider the partial solution

$$y_N = \sum_{m=1}^{N} (-1)^{m-1} x^m / m \quad . \tag{4.13}$$

We now determine the extent to which the partial solution (4.13) fails to satisfy (4.12).

$$\text{Functional Remainder} = (1 + x)\, y_N'(x) - 1$$

$$= (1 + x) \sum_{m=1}^{N} (-1)^{m-1} x^{m-1} - 1$$

$$= \sum_{m=1}^{N} (-1)^{m-1} x^{m-1} + \sum_{m=1}^{N} (-1)^{m-1} x^m - 1$$

$$= \sum_{m=0}^{N-1} (-1)^m x^m - \sum_{m=1}^{N} (-1)^m x^m - 1$$

$$\text{i.e. Functional Remainder} = \sum_{m=1}^{N-1} [(-1)^m x^m - x^m (-1)^m] + (-1)^{N+1} x^N$$

$$= (-1)^{N+1} x^N \ .$$

Construct

$$Z_N(x) = \frac{\sum_{k=0}^{N} \alpha_k\, y_k(x)}{\sum_{k=0}^{N} \alpha_k}$$

then

$$(1 + x)\, Z_N'(x) - 1 = \frac{\sum_{k=0}^{N} \alpha_k\, (-1)^{k+1} x^k}{\sum_{k=0}^{N} \alpha_k} \ .$$

Suppose that

$$T_N^*(x) = \sum_{k=0}^{N} a_k x^k \quad .$$

and choose the α_k so that

$$\alpha_k (-1)^{k+1} = a_k$$

$$\text{i.e.} \qquad \alpha_k = (-1)^{k+1} a_k \; .$$

Then

$$(1 + x) Z_N'(x) - 1 = \frac{T_N^*(x)}{\sum_{k=0}^{N} (-1)^{k+1} a_k} \quad .$$

Now

$$\sum_{k=0}^{N} a_k (-1)^{k+1} = - \sum_{k=0}^{N} a_k (-1)^k$$

$$= - T_N^*(-1) \; .$$

$$\therefore \qquad (1 + x) Z_N'(x) - 1 = \frac{T_N^*(x)}{- T_N^*(-1)} \quad .$$

By using (4.10) this becomes

$$(1 + x) Z_N'(x) - 1 = \frac{T_N(2x - 1)}{- T_N^*(-1)}$$

and from (4.4) we have

$$\left| (1 + x) Z_N' - 1 \right| = \left| \frac{T_N(2x - 1)}{- T_N^*(-1)} \right| \leqslant \frac{1}{| T_N^*(-1) |} \quad .$$

$$< \frac{2}{(3 + 2 \sqrt{2})^N} \quad .$$

Clearly the effect of the Chebyshev polynomials is to distribute the error evenly over the interval [0,1] and thereby reduce the average error in the estimate of the solution. It is for this reason that orthogonal expansion methods are preferred.

Applications

To illustrate the method of solving differential equations using expansions in Chebyshev polynomials, let us consider the problem of determining an eigenvalue λ of the ordinary differential equation

$$\sum_{r=0}^{2m} \alpha_r(\lambda,x) \frac{d^r y}{dx^r} = 0 \qquad (-1 < x < 1) \qquad (4.14)$$

subject to the boundary conditions

$$\sum_{r=0}^{m-1} D_{ir}(\lambda) \frac{d^r y}{dx^r} = 0 \qquad \text{on } x = -1 \ , \ 0 \leqslant i \leqslant m-1,$$

(4.15)

$$\sum_{r=0}^{m-1} E_{ir}(\lambda) \frac{d^r y}{dx^r} = 0 \qquad \text{on } x = 1 \ , \ 0 \leqslant i \leqslant m-1 \ .$$

where $\alpha_r(\lambda,x)$ are expressible as polynomials in x. Let

$$y(x) = \sum_{n=0}^{\infty} a_{n+1} T_n(x).$$

Apply this to (4.14) and use (4.9) to obtain

$$\sum_{r=0}^{2m} \alpha_r(\lambda,x) \frac{d^r y}{dx^r} = \sum_{r=0}^{2m} \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \alpha_r(\lambda,x) B^r_{j+1,s+1} a_{s+1} T_j(x).$$

$$= \sum_{r=0}^{2m} \sum_{j=0}^{\infty} \alpha_r(\lambda,x) (B^r A)_{j+1} T_j(x).$$

where

$$A = (a_1, a_2, a_3, \ldots\ldots\ldots).$$

Let

$$\alpha_r(\lambda,x) = \sum_{k=0}^{M} C_{rk} T_k(x)$$

then

$$\sum_{r=0}^{2m} \sum_{j=0}^{\infty} \alpha_r(\lambda,x) \; (B^r A)_{j+1} \; T_j(x)$$

$$= \sum_{r=0}^{2m} \sum_{j=0}^{\infty} \sum_{k=0}^{M} C_{rk} \; (B^r A)_{j+1} \; T_j(x) \; T_k(x).$$

Using (4.2) we further simplify the left-hand side of (4.14) to

$$\frac{1}{2} \sum_{r=0}^{2m} \sum_{j=0}^{\infty} \sum_{k=0}^{M} C_{rk} \; (B^r A)_{j+1} \; [T_{j+k}(x) + T_{|j-k|}(x)].$$

This can be written in the form

$$\frac{1}{2} \sum_{r=0}^{2m} \sum_{k=0}^{M} C_{rk} \left[\sum_{j=0}^{\infty} (B^r A)_{j+1} \; T_{j+k}(x) + \sum_{j=0}^{k} (B^r A)_{j+1} \; T_{k-j}(x) \right.$$

$$\left. + \sum_{j=k+1}^{\infty} (B^r A)_{j+1} \; T_{j-k}(x) \right]$$

$$= \frac{1}{2} \sum_{r=0}^{2m} \sum_{k=0}^{M} C_{rk} \left[\sum_{p=k}^{\infty} (B^r A)_{p-k+1} \; T_p(x) + \sum_{p=0}^{k} (B^r A)_{k-p+1} \; T_p(x) \right.$$

$$\left. + \sum_{p=1}^{\infty} (B^r A)_{p+k+1} \; T_p(x) \right]$$

$$= \frac{1}{2} \sum_{r=0}^{2m} \sum_{k=0}^{M} C_{rk} \left\{ \sum_{p=0}^{\infty} \left[(B^r A)_{|p-k|+1} + \delta_{pk} \; (B^r A)_1 \right] T_p(x) \right.$$

$$\left. + \sum_{p=1}^{\infty} (B^r A)_{p+k+1} \; T_p(x) \right\} . \qquad (4.18)$$

The boundary conditions (4.15) become

$$\sum_{r=0}^{m-1} \sum_{j=0}^{\infty} D_{ir}(\lambda) \; (B^r A)_{j+1} \; T_j(x) = 0 \quad \text{on } x = -1 \; , \; 0 \leqslant i \leqslant m-1 \; ,$$

$$(4.19)$$

$$\sum_{r=0}^{m-1} \sum_{j=0}^{\infty} E_{ir}(\lambda) \; (B^r A)_{j+1} \; T_j(x) = 0 \quad \text{on } x = 1 \; , \quad 0 \leqslant i \leqslant m-1.$$

So we need to find A which satisfy (4.18) and also satisfy the boundary conditions (4.19).

In practice we cannot use infinite series of Chebyshev polynomials and have to rely on a finite approximation of suitable accuracy. If we look for an expansion of the solution of the form

$$y(x) = \sum_{n=0}^{N} a_{n+1}\, T_n(x) \tag{4.20}$$

then this will satisfy the differential equation (4.14) with a remainder term of the form $\tau_i\, T_i$ where the number and the nature of the τ terms are determined from the order and form of the coefficients of the differential equation (4.14). If the coefficients $\alpha_r(\lambda,x)$ are expressable exactly as a finite sum of Chebyshev polynomials then k has an upper limit M. Thus $|p-k|$ becomes unbounded as $p \longrightarrow \infty$ and so we only get a remainder which is a finite series of Chebyshev polynomials. If however, any of the coefficients require an infinite expansion in terms of Chebyshev polynomials then there will always be terms such that $|p-k|+1$ has a sufficiently small value that we get a contribution from $B^r A$. Since $A = (a_1, a_2, \ldots, a_{N+1})$ then we have N+1 degrees of freedom. By satisfying the boundary conditions (4.19) we use 2m of these degrees of freedom. Thus the number of degrees of freedom left is N+1-2m. We use these degrees of freedom to make zero the coefficients of

$$T_0,\ T_1,\ T_2,\ \ldots,\ T_{N-2m}\ .$$

Thus the lowest non-zero Chebyshev polynomials is T_{N-2m+1}. For any fixed integer I, it is clear that the highest value of p such that

$$|p - k| + 1 = I_r \quad , \quad p + k + 1 = I_r \quad \text{or} \quad p = k = I_r$$

comes from $|p-k|+1 = I_r$ and hence we associate the highest Chebyshev polynomials in the remainder term with the highest value of p such that $|p-k|+1 = I_r$ and $C_{rk} \neq 0$.

<u>**The Orr-Sommerfeld Problem for Plane Poiseuille Flow**</u>

Here we consider the problem of the stability of plane Poiseuille flow, using expansion in Chebyshev polynomials to approximate the solutions of the Orr-Sommerfeld equation

$$(D^2 - \alpha^2)^2 v = i \alpha R [(\bar{u} - \lambda)(D^2 - \alpha^2) - D^2\bar{u}] v \qquad (4.21)$$

where $D = d/dy$, $\bar{u} = 1 - y^2$, α is the wave number and R is the Reynolds number. This equation is solved subject to the boundary conditions

$$v(y) = v'(y) = 0 \qquad \text{at } y = \pm 1. \qquad (4.22)$$

Equation (4.21) can be written in the form

$$D^4v - [2\alpha^2 + i\alpha R(1 - \lambda - y^2)] D^2v$$
$$+ [\alpha^4 - 2 i \alpha R + i \alpha^3 R (1 - \lambda - y^2)] v = 0. \qquad (4.23)$$

Follwing the theory described earlier (4.22) and (4.23) can be written in the form

$$\sum_{r=0}^{4} \alpha_r(\lambda, y) \frac{d^r v}{dy^r} = 0 \qquad (4.24)$$

$$\sum_{r=0}^{1} D_{ir}(\lambda) \frac{d^r v}{dy^r} = 0 \qquad \text{at } y = 1 \quad 0 \leqslant i \leqslant 1$$

$$(4.25)$$

$$\sum_{r=0}^{1} E_{ir}(\lambda) \frac{d^r v}{dy^r} = 0 \qquad \text{at } y = -1 \quad 0 \leqslant i \leqslant 1.$$

Now

$$\alpha_0 = \alpha^4 - 2 i \alpha R + i \alpha^3 R (1 - \lambda - y^2),$$

$$\alpha_1 = \alpha_3 = 0,$$

$$\alpha_2 = -2\alpha^2 - i \alpha R (1 - \lambda - y^2),$$

$$\alpha_4 = 1.$$

From (4.3)

$$T_2(y) = 2y^2 - 1$$

i.e. $$y^2 = \frac{1}{2}\,[T_0(y) + T_2(y)].$$

In terms of Chebyshev polynomials the coefficients of the Orr-Sommerfeld equation becomes

$$\alpha_r(\lambda, y) = \sum_{k=0}^{2} C_{rk}\, T_k(y) \tag{4.26}$$

where

$$C_{00} = \alpha^4 - 2\,i\,\alpha\,R + i\,\alpha^3\,R\,(\frac{1}{2} - \lambda)\;,$$

$$C_{01} = 0 \;,\quad C_{02} = -\,i\,\alpha^3\,R\,/\,2\;,$$

$$C_{1k} = 0 \;,\quad C_{20} = -\,2\,\alpha^2 - i\,\alpha\,R\,(\frac{1}{2} - \lambda)\;,$$

$$C_{21} = 0 \;,\quad C_{22} = i\,\alpha\,R\,/\,2 \;,\quad C_{3k} = 0 \;,\quad C_{40} = 1.$$

Let

$$v(y) = \sum_{n=0}^{\infty} a_{n+1}\, T_n(y) \qquad , \qquad n \geqslant 0\;.$$

Apply this to equation (4.24) and use (4.9) and (4.26) to obtain

$$\sum_{r=0}^{4}\sum_{j=0}^{\infty}\sum_{k=0}^{2} C_{rk}\,(B^r A)_{j+1}\, T_j(y)\, T_k(y) = 0\;.$$

Using (4.2) we further simplify this to

$$\frac{1}{2}\sum_{r=0}^{4}\sum_{j=0}^{\infty}\sum_{k=0}^{2} C_{rk}\,(B^r A)_{j+1}\,[T_{j+k}(y) + T_{|j-k|}(y)] = 0\;.$$

Following the same procedures as before, this becomes

$$\frac{1}{2} \sum_{r=0}^{4} \sum_{k=0}^{2} C_{rk} \left\{ \sum_{p=0}^{\infty} \left[(B^r A)_{|p-k|+1} + \delta_{pk} (B^r A)_1 \right] T_p(y) + \sum_{p=1}^{\infty} (B^r A)_{p+k+1} T_p(y) \right\} = 0 . \qquad (4.27)$$

By expanding (4.27) we can show that

$$\sum_{p=0}^{\infty} (B^4 A)_{p+1} T_p(y) - 2\alpha^2 \sum_{p=0}^{\infty} (B^2 A)_{p+1} T_p(y) + (\alpha^4 - 2 i \alpha R) \sum_{p=0}^{\infty} A_{p+1} T_p(y) - i \alpha R (\frac{1}{2} - \lambda) \sum_{p=0}^{\infty} [(B^2 A)_{p+1} - \alpha^2 A_{p+1}] T_p(y) + \frac{i \alpha R}{4} \left\{ \sum_{p=0}^{\infty} \left[(B^2 A)_{|p-2|+1} + \delta_{p2} (B^2 A)_1 - \alpha^2 A_{|p-2|+1} - \alpha^2 \delta_{p2} A_1 \right] T_p(y) + \sum_{p=1}^{\infty} \left[(B^2 A)_{p+3} - \alpha^2 A_{p+3} \right] T_p(y) \right\} = 0 \qquad (4.28)$$

i.e.

$$\sum_{p=0}^{\infty} (B^4 A)_{p+1} T_p(y) - 2\alpha^2 \sum_{p=0}^{\infty} (B^2 A)_{p+1} T_p(y) + (\alpha^4 - 2 i \alpha R) \sum_{p=0}^{\infty} A_{p+1} T_p(y) - i \alpha R (\frac{1}{2} - \lambda) \sum_{p=0}^{\infty} [(B^2 A)_{p+1} - \alpha^2 A_{p+1}] T_p(y) + \frac{i \alpha R}{4} Q \sum_{p=0}^{\infty} \left[(B^2 A)_{p+1} - \alpha^2 A_{p+1} \right] T_p(y) = 0$$

where the form of the matrix Q is obvious from (4.28). Clearly

$$B^4 A - 2\alpha^2 B^2 A + (\alpha^4 - 2 i \alpha R) A - i \alpha R (\frac{1}{2} - \lambda)(B^2 - \alpha^2 I) A + \frac{i \alpha R}{4} Q (B^2 - \alpha^2 I) A = 0 .$$

i.e.

$$\left[(B^2 - \alpha^2 I)^2 - 2 i \alpha R I - i \alpha R (\frac{1}{2} - \lambda)(B^2 - \alpha^2 I) + \frac{i \alpha R}{4} Q (B^2 - \alpha^2 I) \right] A = 0 . \qquad (4.29)$$

Let

$$\varphi = X A$$

where

$$X = B^2 - \alpha^2 I$$

then quation (4.29) can be written in the form

$$X A - \varphi = 0$$

$$(X - i \alpha R/2 + i \alpha R Q/4) \varphi - 2 i \alpha R A = - i \alpha R \lambda \varphi \qquad (4.30)$$

i.e.

$$\begin{bmatrix} X & - I \\ - 2i\alpha RI & X + \frac{i\alpha R}{2} (Q/2 - I) \end{bmatrix} \begin{bmatrix} A \\ \varphi \end{bmatrix} = \lambda \begin{bmatrix} 0 & 0 \\ 0 & - i\alpha RI \end{bmatrix} \begin{bmatrix} A \\ \varphi \end{bmatrix} \qquad (4.31)$$

Boundary Conditions

(i) $v(1) = v(-1) = 0 \quad \Leftrightarrow \quad \sum_{n=0}^{\infty} a_{n+1} \; T_n(\pm 1) = 0$.

Using (4.5) we obtain

$$\sum_{n=0}^{\infty} a_{n+1} \; (\pm 1)^n = 0 \; . \qquad (4.32)$$

(ii) $v'(1) = v'(-1) = 0 \quad \Leftrightarrow \quad \sum_{n=0}^{\infty} a_{n+1} \; T_n'(\pm 1) = 0$.

Using (4.5) this becomes

$$\sum_{n=0}^{\infty} n^2 \; (\pm 1)^{n-1} \; a_{n+1} = 0 \; . \qquad (4.33)$$

Now we look for an approximate solution of the form

$$v(y) = \sum_{n=0}^{N} a_{n+1} \; T_n(y) \qquad (4.34)$$

where N is the number of Chebyshev polynomials required. Following the argument mentioned earlier we can see that by satisfying the boundary conditions, the number of degrees of freedom left will be N-3. We use this to make zero the coefficients of

$$T_0, T_1, \ldots, T_{N-4}$$

so that the lowest non-zero Chebyshev polynomials is T_{N-3} . The highest Chebyshev polynomials occurs when p=N+k-r and $C_{rk} \neq 0$. Thus the highest Chebyshev polynomial is T_{N+2}. So it is clear that the remainder terms of this problem are of the form

$$\tau_{N-3} T_{N-3} + \tau_{N-2} T_{N-2} + \tau_{N-1} T_{N-1} + \tau_N T_N + \tau_{N+1} T_{N+1} + \tau_{N+2} T_{N+2}.$$

Equations (4.30)-(4.32) is an eigenvalue problem of the form

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{B}\,\mathbf{x}\,. \tag{4.35}$$

The eigenvalue problem (4.35) is solved using a routine which uses the QZ algorithm. The Fortran77 code which solves the Orr-Sommerfeld problem using expansions in Chebyshev polynomials is listed in Appendix II.

Chapter Five

Constitutive Analysis

Introduction

Consider a conducting magnetohydrodynamic fluid with a non-linear constitutive relationship between the magnetic field H and the magnetic induction B such that (see Roberts [45])

$$H_i = \rho \frac{\partial \psi^*}{\partial B_i} \tag{5.1}$$

where ρ is the density and $\psi^* = \psi^*(\rho, \mathbf{B})$ is the internal energy function. Since ψ^* must be invariant under S.R.B.M. then the dependence of ψ^* on $\mathbf{B}$ is reduced to $\psi^* = \psi(\rho, B)$ where $B = \surd(B_i B_i)$. Thus from (5.1)

$$H_i = \rho \, \varphi \, B_i \tag{5.2}$$

where

$$\varphi = \frac{1}{B} \frac{\partial \psi}{\partial B} .$$

Since

$$|\mathbf{H}| = \rho \, \varphi \, |\mathbf{B}|$$

then

$$\varphi = H \,/\, \rho B .$$

In the following chapters we shall examine the Benard problem for a magnetohydrodynamic fluid with the non-linear constitutive relationship (5.2). The strength of this nonlinearity will be measured by a non-dimensional parameter ϵ which has form

$$\epsilon = B \frac{\partial \varphi}{\partial B} \,/\, \varphi .$$

Now

$$\frac{\partial\varphi}{\partial B} = \frac{1}{\rho}\left(\frac{1}{B}\frac{\partial H}{\partial B} - \frac{H}{B^2}\right)$$

$$\therefore \qquad \epsilon = \frac{B}{\rho\varphi}\left(\frac{1}{B}\frac{\partial H}{\partial B} - \frac{H}{B^2}\right) = \frac{B}{H}\frac{\partial H}{\partial B} - 1 \quad . \qquad (5.3)$$

The magnetic susceptibility χ is defined by the relation

$$M = \chi H$$

where M is the magnetisation. Now since

$$B = \mu_0 (H + M)$$

where μ_0 is the permeability of free space, then

$$B = \mu_0 (1 + \chi) H$$

and

$$|B| = \mu_0 (1 + \chi) |H| \quad .$$

Thus

$$\frac{\partial B}{\partial H} = \mu_0 (1 + \chi) + \mu_0 H \frac{\partial \chi}{\partial H}$$

$$\text{i.e.} \qquad \frac{\partial H}{\partial B} = \frac{1}{\mu_0 (1 + \chi) + \mu_0 H \chi_H} \quad .$$

Substitute for $\partial H/\partial B$ in (5.3) to obtain

$$\epsilon = \frac{\mu_0 (1 + \chi)}{\mu_0 (1 + \chi) + \mu_0 H \chi_H} - 1$$

$$= \frac{\chi - (\chi + H \chi_H)}{1 + \chi + H \chi_H} \quad . \qquad (5.4)$$

The analysis which led to equation (5.4) is in fact general. To obtain numerical values of ϵ, we shall examine a special class of fluids called ferromagnetic fluids. These fluids are formed by a colloidal suspension of ferromagnetic particles such as magnetite in a nonmagnetic carrier fluid. Consider ferromagnetic fluids which have the non-linear constitutive relationship (5.2). To calculate ϵ for this class of fluids, we use Langevin's classical theory which gives the superparamagnetic relationship between the applied field and the resultant magnetisation of the particle collection. Rosensweig [70] suggested that Langevin's equation describes actual magnetisation curves very well.

Langevin's Equation

In the absence of an applied field the particles in a colloidal ferrofluid are randomly oriented. However if a magnetic field is applied then the dipole moments tend to align with the field. The field direction is increasingly favoured with increasing magnetic fields and the magnetisation approaches a saturation value To obtain the relationship between the applied magnetic field and the resultant magnetisation, we consider a unit volume containing N rods. In the absence of an applied field the number of rods lying between θ and θ+dθ is given by (see figure 1)

$$n(\theta)\ d\theta = N\,\frac{2\pi\ \sin\theta\ d\theta}{4\pi}$$

$$= N\ \sin\theta\ d\theta\ /\ 2$$

where $n(\theta)$ is the angular distribution function.

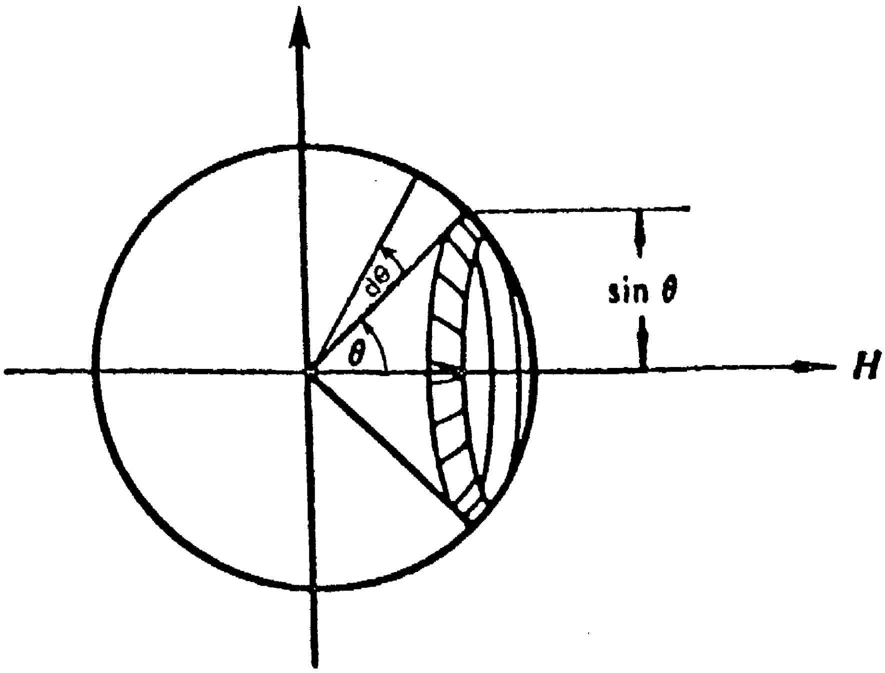

Figure 1

In statistical mechanics the density of states is proportional to $e^{-(W/kT)}$ where $W = - m H \cos\theta$ is the potential energy, m is the magnitude of magnetic moment, k is the Boltzmann constant and T is the absolute temperature. Thus

$$n(\theta)\ d\theta \propto \frac{N}{2}\ e^{(mH\cos\theta/kT)}\ \sin\theta\ d\theta$$

and the constant of proportionality can be calculated by requiring the total number of rods to be equal to N. i.e.

$$\int_0^{\pi} n(\theta)\ d\theta = N\ .$$

The effective dipole moment of a particle is its component along the direction of the field. i.e. $m\cos\theta$. So in terms of $n(\theta)$ the average value of $m\cos\theta$ is given by

$$\bar{m} = \frac{\int_0^\pi m \cos\theta \; n(\theta) \; d\theta}{\int_0^\pi n(\theta) \; d\theta}$$

$$= \frac{\int_0^\pi m \cos\theta \; e^{(mH\cos\theta/kT)} \sin\theta \; d\theta}{\int_0^\pi e^{(mH\cos\theta/kT)} \sin\theta \; d\theta} \; .$$

Let α = mH/kT and put x = α cosθ, then

$$\frac{\bar{m}}{m} = \frac{\int_{-\alpha}^{\alpha} x \; e^x \; dx}{\alpha \int_{-\alpha}^{\alpha} e^x \; dx}$$

$$= \coth(\alpha) - 1/\alpha \; . \qquad (5.5)$$

This is known as the Langevin's equation. The magnitude of the magnetisation **M** of a ferrofluid in the direction of the applied field is the total of the moments of the magnetic particles suspended in a unit volume. i.e

$$\mu_0 \; M = n \; \bar{m} \qquad (5.6)$$

where $\bar{m}$ is the component of the mean magnetic moment per particle along the field direction. Similarly the magnitude of the saturation magnetisation M_s of the fluid is given in terms of the magnitude of the particle moment m by

$$\mu_0 \; M_s = n \; m \qquad (5.7)$$

where n = $6\varphi/\pi d^3$ is the number concentration of particles in a ferrofluid and where d is the particle diameter, $\varphi = M_s/M_d$ is the volume fraction of magnetic solid and M_d is the domain magnetisation of the solid particle. From (5.6) and (5.7) we have

$$\bar{m} / m = M / M_s$$

$$= M / \varphi M_d \quad .$$

Substitute for $\bar{m}/m$ in equation (5.5) to obtain

$$M / \varphi M_d = \coth(\alpha) - 1/\alpha \qquad (5.8)$$

where

$$\alpha = m H / k T$$

$$= \mu_0 M_s H / n k T$$

$$= \mu_0 \pi d^3 M_s H / 6 \varphi k T$$

$$= \mu_0 \pi d^3 M_d H / 6 k T \quad .$$

From equation (5.8) we have

$$M = \varphi M_d [\coth(\alpha) - 1/\alpha]$$

i.e. $\qquad \chi H = \varphi M_d [\coth(\alpha) - 1/\alpha] \quad .$

Differentiate with respect to H to obtain

$$\chi + H \chi_H = \varphi M_d \left[\frac{1}{\alpha^2} - \frac{1}{\sinh^2(\alpha)}\right] \frac{\partial\alpha}{\partial H}$$

$$= \frac{\mu_0 \pi d^3 \varphi M_d^2}{6 K T} \left[\frac{1}{\alpha^2} - \frac{1}{\sinh^2(\alpha)}\right] \quad .$$

Substitute this into (5.4) to obtain

$$\epsilon = \frac{\beta [\alpha \cosh(\alpha) \sinh(\alpha) + \alpha^2 - 2 \sinh^2(\alpha)]}{\alpha^2 \sinh^2(\alpha) + \beta [\sinh^2(\alpha) - \alpha^2]}$$

where

$$\beta = \alpha \varphi M_d / H \ .$$

Numerical results

The numerical values of ϵ depend on several parameters such as particle diameter, domain magnetisation, field temerature, etc.. In the following table we have collected data from the work of Kaiser & Miskolczy [49], Chantrell, Popplewell & Charles [50], Charles & Popplewell [51], Popplewell, Charles & Hoon [52] and Rosensweig [53] on numerous ferromagnetic fluids and have used this data to determine appropriate values for the parameter ϵ. The values given are sensitive to both temperature (T^0 K) and particle diameter d. Most data is presented at 293^0 K but some values at 77^0 K are included.

H (a/m 10^5)	.1	.25	.5	.75	1	2	3	4	d (Å)
Diester (Magnetite)	0.11	0.616	1.47	1.69	1.62	1.09	0.78	0.603	110
Water (Magnetite)	0.189	0.995	1.93	1.99	1.8	1.13	0.798	0.613	120
	0.053	0.309	0.915	1.26	1.33	1.01	0.747	0.586	98
Petroleum (Magnetite)	0.009	0.055	0.202	0.388	0.551	0.719	0.617	0.514	75
Kerosene (Magnetite)	0.006	0.034	0.129	0.259	0.389	0.616	0.566	0.485	70
	0.023	0.136	0.462	0.775	0.946	0.889	0.695	0.557	86
Fluorocarbon (Magnetite)	0.01	0.06	0.22	0.418	0.586	0.738	0.625	0.519	76
Diester (Iron)	1.04	4.29	7.1	7.41	6.87	4.44	3.14	0.044	100
Mercury (Iron)	0.008	0.049	0.192	0.415	0.686	1.55	1.73	1.62	45
	0.015	0.095	0.368	0.769	1.2	2.1	2.06	1.83	50
	0.031	0.191	0.723	1.4	1.98	2.66	2.36	2.0	56
	0.047	0.291	1.07	1.93	2.54	2.97	2.52	2.09	60
T = 77° K	0.124	0.754	2.33	3.44	3.93	3.58	2.8	2.23	45

By using Maclaurin series for $\cosh(\alpha)$ and $\sinh(\alpha)$ we can show that for small α

$$\epsilon = \frac{\alpha^4\ (2 + 2\ \beta\ /\ 3)}{4\ \beta\ \alpha^6\ /\ 45}$$

$$= \frac{2\ \beta}{45 + 15\ \beta}\ \alpha^2$$

from which it is obvious that the graph of ϵ versus H is parabolic for small α. For large α

$$\alpha\ \epsilon = \frac{\beta\ [\alpha^2\ (e^{2\alpha} - e^{-2\alpha}) + 4\ \alpha^3 + 4\ \alpha - 2\ (e^{2\alpha} + e^{-2\alpha})]}{\alpha^2\ (e^{2\alpha} + e^{-2\alpha} - 2) + \beta\ (e^{2\alpha} + e^{-2\alpha} - 2 - 4\ \alpha^2)}$$

$$= \frac{\beta\ [\alpha^2\ (1 - e^{-4\alpha}) + 4\ \alpha\ (1 + \alpha^2)\ e^{-2\alpha} - 2 - 2e^{-4\alpha}]}{\alpha^2\ (1 + e^{-4\alpha} - 2\ e^{-2\alpha}) + \beta\ (1 + e^{-4\alpha} - 2\ e^{-2\alpha} - 4\ \alpha^2\ e^{-2\alpha})}$$

$$= \frac{\beta\ [1 - e^{-4\alpha} + 4\ (\alpha + 1/\alpha)\ e^{-2\alpha} - 2/\alpha^2\ (1 + e^{-4\alpha})]}{1 + e^{-4\alpha} - 2\ e^{-2\alpha} + \beta\ (1/\alpha^2 + e^{-4\alpha}/\alpha^2 - 4\ e^{-2\alpha} - 2e^{-2\alpha}/\alpha^2)}$$

$$\longrightarrow \beta \text{ as } \alpha \longrightarrow \infty$$

$$\therefore \quad \epsilon \simeq \beta\ /\ \alpha = \varphi\ M_d\ /\ H \quad \text{for large } \alpha.$$

Thus the graph of ϵ versus H generally looks like

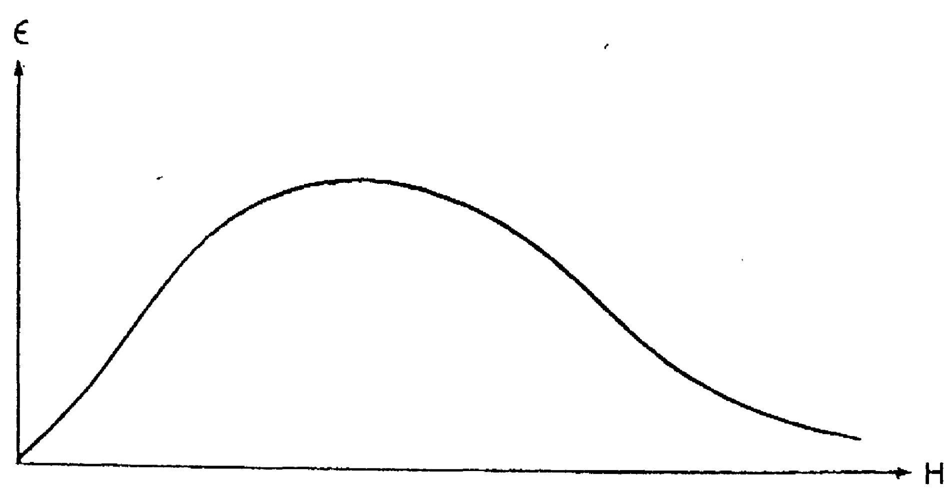

Figure 2

Chapter Six

Benard Convection in a Non-Linear MHD Fluid with Vertical Magnetic Field

Mathematical Formulation

Consider a conducting magnetohydrodynamic fluid of density ρ confined between two parallel planes, distance d apart. In order to fully describe the nature of this fluid we need to discuss the interaction between electromagnetic and mechanical effects and so we introduce the electromagnetic variables **H**, **B**, **E** and **J**, where **H** is the magnetic field, **B** is the magnetic induction, **E** is the electric field and **J** is the current density. If we use the conventional notations, namely, θ is absolute temperature, **V** is velocity, ρ is density, p is hydrostatic pressure and σ_{ik} is the stress tensor then the relevant constitutive relations for a perfectly conducting material take the form (see Roberts [45])

$$H_i = \rho \frac{\partial \psi^*}{\partial B_i} ,$$
$$\sigma_{ik} = -(p + H.B)\,\delta_{ik} + H_i B_k \qquad (6.1)$$

where ψ^* is the internal energy function. As explained in chapter five, we can write

$$H_i = \rho\, \varphi\, B_i$$

where

$$\varphi = \frac{1}{B} \frac{\partial \psi}{\partial B} .$$

The evolution of electromagnetic effects in a stationary material is governed by the Maxwell equations

$$\text{div } B = B_{i,i} = 0 \,,$$

$$(\text{curl } H)_i = e_{ijk} H_{k,j} = J_i \,, \qquad (6.2)$$

$$(\text{curl } E)_i = e_{ijk} E_{k,j} = -\frac{\partial B_i}{\partial t} \,,$$

where the current density J is given by

$$\eta J_k = E_k + e_{krs} V_r B_s \qquad (6.3)$$

and where the resistivity η is assumed to be constant. The relevant equation of motion has form

$$\rho \dot{V}_i = - p_{,i} + 2 \rho \nu \; d_{ij,j} + \rho F_i + (J \times B)_i$$

where ν is the kinematic viscosity and X_i are the external forces. In order to find a form of the electromagnetic stree tensor for a non-perfectly conducting material, we consider the term $(J \times B)_i$. From $(6.2)_{(ii)}$

$$(J \times B)_i = (\text{curl } H \times B)_i = e_{ijk} (\text{curl } H)_j B_k$$

$$= e_{ijk} e_{jrs} H_{s,r} B_k$$

$$= (\delta_{kr} \delta_{is} - \delta_{ks} \delta_{ir}) H_{s,r} B_k$$

$$= H_{i,k} B_k - H_{k,i} B_k$$

$$= H_{i,k} B_k - (H_k B_k)_{,i} + H_k B_{k,i} .$$

Let

$$B^2 = B_i B_i$$

then

$$B B_{,i} = B_j B_{j,i} .$$

Thus

$$(J\times B)_i = (\rho\ \varphi\ B_i)_{,k}\ B_k - (\rho\ \varphi\ B^2)_{,i} + \rho\ \varphi\ B\ B_{,i}.$$

Since

$$\varphi = \frac{1}{B}\ \frac{\partial\psi}{\partial B}$$

then

$$(J\times B)_i = (\rho\ \varphi\ B_i)_{,k}\ B_k - (\rho\ B\ \psi_B - \rho\ \psi)_{,i}.$$

Now

$$\begin{aligned} B\ \psi_B - \psi &= B^2\ \varphi - \int B\ \varphi\ dB \\ &= B^2\ \varphi - \frac{1}{2}\left[\ B^2\ \varphi - \int B^2\ \varphi_B\ dB\ \right] \\ &= \frac{1}{2}\left[\ B^2\ \varphi + \int B^2\ \varphi_B\ dB\ \right] \end{aligned}$$

So

$$(J\times B)_i = -\left[\ \frac{\rho}{2}\ B^2\ \varphi + \frac{\rho}{2}\int B^2\ \varphi_B\ dB\ \right]_{,i} + (\rho\ \varphi\ B_i)_{,k}\ B_k.$$

Thus the electromagnetic stress tensor will have form

$$\sigma_{ki} = -\left[\ \frac{\rho}{2}\ B^2\ \varphi + \frac{\rho}{2}\int B^2\ \varphi_B\ dB\ \right]_{,i} + (\rho\ \varphi\ B_i)_{,k}\ B_k$$

and the equation of motion will have form

$$\begin{aligned} \rho\ \dot{V}_i = -\left[\ p + \frac{\rho}{2}\ B^2\ \varphi + \frac{\rho}{2}\int B^2\ \varphi_B\ dB\ \right]_{,i} + (\rho\ \varphi\ B_i)_{,j}\ B_j \\ + 2\ \rho\ \nu\ d_{ij,j} + \rho\ F_i. \end{aligned}$$

Define

$$P = p + \frac{\rho}{2}\ B^2\ \varphi + \frac{\rho}{2}\int B^2\ \varphi_B\ dB$$

thus

$$\rho\ \dot{V}_i = -P_{,i} + (\rho\ \varphi\ B_i)_{,j}\ B_j + 2\ \rho\ \nu\ d_{ij,j} + \rho\ F_i$$

If we now make the Boussinesq approximation, namely that density is constant everywhere except in the body force term where the density is linearly proportional to temperature, then the governing field equations become

$$\operatorname{div} V = 0$$

$$\frac{DV_i}{Dt} = -\left(\frac{P}{\rho}\right)_{,i} + B_j\ (\varphi\ B_i)_{,j} + \nu\ \Delta V_i - g\ (1 - \alpha\theta)\ \delta_{i3}\ , \qquad (6.4)$$

$$\frac{D\theta}{Dt} = \kappa\ \Delta\theta\ ,$$

together with the Maxwell relations (6.2), where D/Dt is the convected derivative, g is the acceleration of gravity, α is the coefficient of volume expansion and κ is the coefficient of thermal diffusivity. When the constitutive low (6.3) is introduced into $(6.2)_{(iii)}$, the magnetic field is now found to satisfy

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) - \eta\ \nabla \times \mathbf{J} \qquad (6.5)$$

In terms of rectangular cartesian coordinates x, y, z with the z-axis directed vertically upwards, we may observe that equations (6.2), (6.4) and (6.5) have a steady solution in which

$$\mathbf{V} = 0\ ,$$

$$\theta = \theta(z) = T_0 - \beta\ z\ ,$$

$$P = P(z)\ ,$$

$$\mathbf{B} = B\ \mathbf{k} \qquad B = \text{constant}\ ,$$

where the temperature on the planes z = 0, z = d are respectively T_0 and $T_0 - T_d$ so that $\beta = T_d/d$. Although we consider the case in which the external field is normal to the layer of fluid, a similar analysis can be applied to any constant **B**.

Let $\hat{V}$, $\hat{\theta}$, $\hat{P}$, $\hat{b}$ and $\hat{J}$ be the linear perturbation of velocity, temperature, pressure, magnetic induction and current density about the steady state values then calculation reveals that

$$\hat{V}_{i,i} = 0,$$

$$\frac{\partial \hat{V}_i}{\partial t} = -\left(\frac{\hat{P}}{\rho}\right)_{,i} + B^2 \varphi_B \hat{b}_{3,3} \delta_{i3} + B \varphi \hat{b}_{i,3} + \nu \Delta\hat{V}_i + \alpha g \hat{\theta} \delta_{i3}$$

$$\frac{\partial \hat{\theta}}{\partial t} - \beta \hat{V}_3 = \kappa \Delta\hat{\theta}, \qquad (6.6)$$

$$\hat{b}_{i,i} = 0 \quad ,$$

$$\hat{J}_i = e_{ijk} (\rho \varphi \hat{b}_k + \rho B \varphi_B \hat{b}_3 \delta_{k3})_{,j} \quad ,$$

$$\frac{\partial \hat{b}_i}{\partial t} = B \hat{V}_{i,3} - \eta e_{ijk} \hat{J}_{k,j} \quad .$$

At this stage we introduce the dimensionless variables x^*, t^*, V^*, θ^*, P^*, b^* and J^* where

$$x^* = x / d \ ,$$

$$t^* = t \nu / d^2 \ ,$$

$$V^* = \hat{V} d / \kappa \quad ,$$

$$\theta^* = \hat{\theta} \frac{d}{\kappa} \left[\frac{\alpha g}{P_r |\beta|}\right]^{\frac{1}{2}}$$

$$P^* = \hat{P} \frac{d^2}{\rho \kappa \nu} \quad ,$$

$$b^* = \hat{b} \frac{B \varphi d^2}{\kappa \nu} \quad ,$$

$$J^* = \hat{J} \frac{B d^3}{\rho \kappa \nu} \ .$$

After this non-dimensionalization, the field equations simplify to

$$V_{i,i} = 0 \quad ,$$

$$\frac{\partial V_i}{\partial t} = - P_{,i} + \Delta V_i + \surd(R)\ \theta\ \delta_{i3} + b_{i,3} + \epsilon\ b_{3,3}\ \delta_{i3}$$

$$P_r \frac{\partial \theta}{\partial t} + H \surd(R)\ w = \Delta\theta \quad , \qquad (6.7)$$

$$b_{i,i} = 0 \ ,$$

$$J_i = e_{ijk}\ (b_k + \epsilon\ b_3\ \delta_{k3})_{,j} \quad ,$$

$$P_m \frac{\partial b_i}{\partial t} = Q\ V_{i,3} - e_{ijk}\ J_{k,j} \quad ,$$

where the * superscript has been dropped but all variables are now non-dimensional and where the non-dimensional numbers Q, P_m, P_r, R and ϵ are given by

$$Q = \left(\frac{B\ d}{\nu}\right)^2 \frac{\nu}{\rho\ \eta} \quad ,$$

$$P_m = \nu/\rho\eta\varphi \quad ,$$

$$P_r = \nu/\kappa \quad , \qquad (6.8)$$

$$R = \frac{\alpha\ g\ d^4\ |\beta|}{\nu\ \kappa} \quad ,$$

$$\epsilon = B \frac{\partial\varphi}{\partial B} / \varphi$$

where

$$H = - \frac{\beta}{|\beta|} = \begin{cases} +1 & \text{when heating from above} \ , \\ -1 & \text{when heating from below} \ . \end{cases}$$

From equations $(6.7)_{(v),(vi)}$ we have

$$P_m \frac{\partial b_i}{\partial t} = Q\ V_{i,3} - e_{ijk}\ [e_{krs}\ (b_s + \epsilon\ b_3\ \delta_{s3})_{,r}]_{,j}$$

$$= Q\ V_{i,3} - (\delta_{ir}\ \delta_{js} - \delta_{is}\ \delta_{jr})\ (b_s + \epsilon\ b_3\ \delta_{s3})_{,rj}$$

$$= Q\ V_{i,3} + b_{i,jj} - \epsilon\ b_{3,i3} + \epsilon\ b_{3,jj}\ \delta_{i3} .$$

(6.9)

Let ξ be the vorticity of the flow, then

$$\xi = \text{curl } V \ . \tag{6.10}$$

Apply the curl operator to equations (6.7)(ii),(vi) to obtain

$$\frac{\partial \xi_i}{\partial t} = \Delta\xi_i + \surd(R)\ e_{ijk} \frac{\partial \theta}{\partial x_j}\ \delta_{k3} + e_{ijk}\ b_{k,3j} + \epsilon\ e_{ijk}\ b_{3,3j}\ \delta_{k3}$$

$$P_m \frac{\partial J_i}{\partial t} - \epsilon\ P_m\ e_{ijk} \frac{\partial}{\partial t}\ (b_{3,j})\ \delta_{k3} = Q\ \xi_{i,3} - J_{j,ij} + J_{i,jj} \ . \tag{6.11}$$

Apply the curl operator once again to equation $(6.11)_{(i)}$ to obtian

$$\frac{\partial}{\partial t}\ \nabla^2 V_i = \nabla^4 V_i + \surd(R)\ \ (\nabla^2\theta\ \delta_{i3} - \frac{\partial^2\theta}{\partial x_i\ \partial x_j}\ \delta_{j3}) + \nabla^2(b_{i,3})$$
$$- \epsilon\ b_{33,i} + \epsilon\ b_{3,jj}\ \delta_{i3} \ . \tag{6.12}$$

By taking the third components of equations (6.9), (6.11) and (6.12), we obtain

$$P_m \frac{\partial b}{\partial t} = Q\ Dw + b_{,jj} + \epsilon\ (\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2}),$$

$$\frac{\partial \xi}{\partial t} = \Delta\xi + DJ, \tag{6.13}$$

$$P_m \frac{\partial J}{\partial t} = Q\ D\xi + \Delta J,$$

$$\frac{\partial}{\partial t}\ \nabla^2 w = \nabla^4 w + \surd(R)\ \ (\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}) + \nabla^2(Db)$$
$$+ \epsilon\ (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) Db$$

where w, b, J and ξ are the third components of velocity, magnetic induction, current density and vorticity and where D is the operator d/dz.

We now search for a normal mode solution of equations $(6.7)_{(iii)}$ and (6.13) in the form

$$
\begin{aligned}
w &= w(z)\; e^{i(nx+my)+\sigma t} \quad ,\\
\theta &= \theta(z)\; e^{i(nx+my)+\sigma t} \quad ,\\
b &= b(z)\; e^{i(nx+my)+\sigma t} \quad ,\\
P &= P(z)\; e^{i(nx+my)+\sigma t} \quad ,\\
J &= J(z)\; e^{i(nx+my)+\sigma t} \quad ,\\
\xi &= \xi(z)\; e^{i(nx+my)+\sigma t} \quad .
\end{aligned}
\tag{6.14}
$$

Thus the relative equations become

$$
\begin{aligned}
&\sigma\ P_r\ \theta = L\theta - H\ \surd(R)\ w \quad ,\\
&\sigma\ P_m\ b = Lb - \epsilon\ a^2\ b + Q\ Dw\\
&\sigma\ Lw = L^2 w - a^2\ \surd(R)\ \theta + L(Db) - \epsilon\ a^2\ Db\ .\\
&\sigma\ \xi = L\xi + DJ \quad ,\\
&\sigma\ P_m\ J = Q\ D\xi + LJ
\end{aligned}
\tag{6.15}
$$

where L is the operator $(D^2 - a^2)$. Alternatively, we can obtain equation $(6.15)_{(iii)}$ by applying a normal mode solution of the form (6.14) to equations $(6.7)_{(i),(ii)}$.

Thus these equations become

$$
\begin{aligned}
&i\ n\ u + i\ m\ v + Dw = 0\\
&\sigma\ u + i\ n\ P + (2+\epsilon)\ i\ n\ b - Db_1 - Lu = 0\\
&\sigma\ v + i\ m\ P + (2+\epsilon)\ i\ m\ b - Db_2 - Lv = 0\\
&\sigma\ w + DP + Db - \surd(R)\ \theta - Lw = 0
\end{aligned}
\tag{6.16}
$$

If we multiply equations $(6.16)_{(ii),(iii)}$ by (in) and (im) respectively and add, we obtain

$$\sigma \, Dw + a^2 \, P + (2 + \epsilon) \, a^2 \, b + D^2 b + L(Dw) = 0.$$

Differentiate the above equation with respect to z and use $(6.16)_{(iv)}$ to obtain

$$\sigma \, Lw = L^2 w - a^2 \, \surd(R) \, \theta + L(Db) - \epsilon \, a^2 \, Db \; .$$

The advantage of producing equation $(6.15)_{(iii)}$ in this way is that P can be determined in the process whereas this does not occur in the previous analysis.

Equations (6.15) with the boundary conditions form an eighth order eigenvalue problem. By setting $\epsilon = 0$ in equations $(6.15)_{(ii),(iii)}$ we obtain the standard equations of the classical magnetohydrodynamic Benard problem. We may eliminate both θ and b from equations (6.15) and derive an eighth order ordinary differential equation to be satisfied by w. In fact,

$$\begin{aligned}
&\sigma^3 \, P_r \, P_m \, Lw - \sigma^2 \left[L^2 w \, (P_r + P_m + P_r \, P_m) - \epsilon \, a^2 \, P_r \, Lw\right] \\
&\quad + \sigma \left\{(1 + P_r + P_m) \, L^3 w - \left[\epsilon \, a^2 \, (1 + P_r) + P_r \, Q\right] L^2 w \right. \\
&\qquad \left. + P_r \, Q \, a^2 \, (\epsilon - 1) \, Lw + a^2 \, (\epsilon \, a^2 \, P_r \, Q - R \, H \, P_m) \, w \right\} \\
&\qquad\quad - L^4 w + (\epsilon \, a^2 + Q) \, L^3 w - Q \, a^2 \, (\epsilon - 1) \, L^2 w \\
&\qquad\qquad - a^2 \, (\epsilon \, a^2 \, Q - R \, H) \, Lw - \epsilon \, a^4 \, R \, H \, w = 0 \; .
\end{aligned} \tag{6.17}$$

The Boundary Conditions

The fluid is confined between the planes $z = 0$ and $z = 1$ and on these two planes certain boundary conditions must be satisfied. On each boundary we need to specify mechanical, thermal and electromagnetic conditions. From a mechanical viewpoint suitable boundaries are either rigid or free. Regardless of the nature of these boundaries we require

$$w = 0 \qquad \text{on } z = 0,1 \ .$$

i.e. the normal component of velocity must vanish on these surfaces On a rigid boundary no slip occurs and this implies that the horizontal components of the velocity u and v vanish. Thus

$$\mathbf{V} = (u,v,w) = 0 \ . \tag{6.18}$$

From the incompressibility constraint we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + Dw = 0 \ .$$

Thus

$$Dw = 0 \ . \tag{6.19}$$

The normal component of the vorticity is

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \ . \tag{6.20}$$

Thus from (6.18) the condition on ξ for a rigid boundary is

$$\xi = 0 \ . \tag{6.21}$$

On a free boundary the surface is unable to support shear stress so we require the shear stress to be zero. i.e.

$$\sigma_{31} = \sigma_{32} = 0$$

$$\Longrightarrow \qquad d_{31} = d_{32} = 0 \ .$$

Since

$$d_{ij} = \frac{1}{2} (V_{i,j} + V_{j,i})$$

then we require

$$V_{3,1} + V_{1,3} = V_{3,2} + V_{2,3} = 0$$

$$\Longrightarrow \qquad \text{in } w + Du = \text{im } w + Dv = 0 \ .$$

$$\Longrightarrow \qquad Du = Dv = 0. \qquad (6.22)$$

Thus from the incompressibility constraint

$$D^2 w = 0 \qquad (6.23)$$

and from (6.20)

$$D\xi = \frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 u}{\partial y \partial z} \ .$$

Thus from (6.22) the condition on ξ for a free boundary is

$$D\xi = 0 \ . \qquad (6.24)$$

In this work we shall consider only the situation where the temperature of each boundary is specified so that

$$\theta = 0 \qquad \text{on } z = 0,1 \qquad (6.25)$$

but the problem could be posed with Robin type boundary conditions.

Finally, from an electromagnetic point of view, suitable boundaries are either perfectly conducting or insulating. On an insulating boundary no currents may cross the boundary and so the third component of J is zero, i.e. J = 0 and magnetic field is continuous across the boundary with the external magnetic field being derived from a scalar potential since curl H = 0. We shall associate insulating boundaries with a free surface.

On the other hand if the adjoining boundary is a stationary perfect conductor then the normal component of the unsteady magnetic field must be zero i.e. $b = 0$ and there can be no surface components of electric field. Here the surface components of the current vector J are zero. However div $J = 0$ and so $DJ = 0$. Stationary perfect conducting surfaces will be associated with rigid boundaries. In this analysis we shall consider both boundaries to be free but later on we present results for the corresponding rigid and mixed boundary value problems. For the free boundary value problems

$$w = D^2 w = 0 \quad \text{on } z = 0 \text{ and } 1 ,$$

thus, equation (6.17) has eignfunctions

$$w = A \sin(n\pi z) .$$

Consequently $Lw = -\lambda w$ where $\lambda = n^2\pi^2 + a^2$ and σ satisfies the cubic equation

$$\begin{aligned} P_r P_m \sigma^3 + \sigma^2 & \left[\lambda (P_r + P_m + P_r P_m) + \epsilon a^2 P_r\right] \\ & + \sigma \left[\epsilon a^2 \lambda (1 + P_r) + \lambda^2 (1 + P_r + P_m) + \lambda Q P_r - Q P_r a^2 \right. \\ & \left. + \epsilon a^2 Q P_r - \epsilon a^4 P_r Q/\lambda + a^2 P_m R H/\lambda\right] + \lambda^3 + Q \lambda^2 \\ & + \epsilon a^2 \lambda^2 + Q a^2 \lambda (\epsilon - 1) + R H \epsilon a^4/\lambda \\ & - a^2 (\epsilon a^2 Q - R H) = 0 . \end{aligned} \tag{6.26}$$

Since the coefficients of this polynomial are real, then its solutions are either all real or there is one real solution and two complex conjugate pair solutions. In the former case instability happens if any real solution is positive and in the latter case if either the real solution is positive (i.e. stationary instability) or the real part of the complex solution is positive (i.e. oscillatory instability or overstability) .

The solutions of (6.26) are functions of P_r, P_m, ϵ, Q and R and we have to examine how the nature of these solutions depends on P_r, P_m, ϵ, Q and R in the context of heating from above or below.

Heating Fluid From Above

In this situation H = + 1 and so we need to examine the roots of the polynomial equation $f(\sigma) = 0$ where

$$f(\sigma) = P_r\, P_m\, \sigma^3 + \sigma^2 \left[\lambda\, (P_r + P_m + P_r\, P_m) + \epsilon\, a^2\, P_r\right]$$

$$+ \sigma \Big[\epsilon\, a^2\, \lambda\, (1 + P_r) + \lambda^2\, (1 + P_r + P_m) + \lambda\, Q\, P_r$$

$$- Q\, P_r\, a^2 + \epsilon\, a^2\, P_r\, Q + a^2\, P_m\, R/\lambda - \epsilon\, a^4\, P_r\, Q/\lambda\Big]$$

$$+ \lambda^3 + \epsilon\, a^2\, \lambda^2 + \lambda\, Q\, (\epsilon - 1)\, a^2 - \epsilon\, a^4\, Q + Q\, \lambda^2$$

$$+ R\, a^2\, (1 + \epsilon\, a^2/\lambda) \; .$$

Since $\lambda = n^2\pi^2 + a^2$, then the above equation becomes

$$f(\sigma) = P_r\, P_m\, \sigma^3 + \sigma^2 \left[\lambda\, (P_r + P_m + P_r\, P_m) + \epsilon\, a^2\, P_r\right]$$

$$+ \sigma \Big[\epsilon\, a^2\, \lambda\, (1 + P_r) + \lambda^2\, (1 + P_r + P_m) + Q\, P_r\, n^2\, \pi^2$$

$$+ \epsilon\, a^2\, n^2\, \pi^2\, P_r\, Q\, /\, \lambda + a^2\, P_m\, R\, /\, \lambda\Big]$$

$$+ \lambda^3 + \epsilon\, a^2\, \lambda^2 + \epsilon\, a^2\, n^2\, \pi^2\, Q + n^2\, \pi^2\, Q\, \lambda$$

$$+ R\, a^2\, (1 + \epsilon\, a^2/\lambda) \; . \qquad (6.27)$$

Let us make the reasonable assumption that $|B|$ is an increasing function of $|H|$ so that $dB/dH > 0$. Consequently $d(\varphi B)/dB > 0$ and so $\epsilon > -1$. Since $\epsilon > -1$ then

$$\lambda + \epsilon\, a^2 > 0 \qquad (6.28)$$

and (6.27) becomes

$$f(\sigma) = P_r\ P_m\ \sigma^3 + \sigma^2 \left[P_r\ (\lambda + \epsilon\ a^2) + \lambda\ P_m\ (1 + P_r)\right]$$

$$+ \sigma \left[\lambda\ (1 + P_r)\ (\lambda + \epsilon\ a^2) + \lambda^2\ P_m + a^2\ R\ P_m/\lambda\right.$$

$$\left. + Q\ P_r\ n^2\ \pi^2\ (\lambda + \epsilon\ a^2)/\lambda\right] + \lambda^2\ (\lambda + \epsilon\ a^2)$$

$$+ (\lambda + \epsilon\ a^2)\ (Q\ n^2\ \pi^2 + R\ a^2/\lambda) \qquad (6.29)$$

from which it is obvious that the coefficients of $f(\sigma)$ are positive and real. Thus $f(\sigma) = 0$ has either three negative real solutions or one negative real solution and two complex conjugate solutions. Suppose that $f(\sigma)$ has one negative real root and two complex conjugate roots and let Σ be the sum of the roots of $f(\sigma) = 0$ then clearly

$$\Sigma = - \frac{1}{P_r\ P_m} \left[P_r\ (\lambda + \epsilon\ a^2) + \lambda\ P_m\ (1 + P_r)\right]$$

and

$$f(\Sigma) = - \frac{1}{P_r\ P_m} \left[P_r\ (\lambda + \epsilon\ a^2) + \lambda\ P_m\ (1 + P_r)\right] \left[\ \lambda^2\ P_m + \frac{a^2\ R\ P_m}{\lambda}\right.$$

$$\left. + \lambda\ (1 + P_r)\ (\lambda + \epsilon\ a^2) + \frac{Q\ P_r\ n^2\ \pi^2}{\lambda}\ (\lambda + \epsilon\ a^2)\right]$$

$$+ (\lambda + \epsilon\ a^2)\ (\lambda^2 + Q\ n^2\ \pi^2 + R\ a^2\ /\ \lambda)\ .$$

i.e.

$$f(\Sigma) = - \frac{(\lambda + \epsilon\ a^2)^2}{P_m} \left[\frac{Q\ P_r\ n^2\ \pi^2}{\lambda} + \lambda\ (1 + P_r)\right]$$

$$- \frac{\lambda\ (1 + P_r)}{P_r} \left[\lambda\ (1 + P_r)(\lambda + \epsilon\ a^2) + \lambda^2\ P_m + \frac{R\ a^2\ P_m}{\lambda}\right]$$

$$- Q\ n^2\ \pi^2\ P_r\ (\lambda + \epsilon\ a^2)$$

from which it is obvious in view of (6.28) that $f(\Sigma) < 0$. Thus $f(\sigma)$ has a negative real root, σ_{real}, which is greater than Σ. Let σ_1, σ_2 be the two complex conjugate roots of $f(\sigma)$ such that

$$\sigma_1 = \alpha + i\ \beta \quad , \quad \sigma_2 = \alpha - i\ \beta$$

then

$$\Sigma = \sigma_{real} + 2\,\alpha$$

but

$$\Sigma < \sigma_{real} < 0$$

$$\Longrightarrow \quad \Sigma - \sigma_{real} < 0$$

$$\Longrightarrow \quad 2\,\alpha < 0\ .$$

Thus if $f(\sigma)$ has only one real root and two complex conjugate roots then the real part of these conjugate roots is negative. Hence when this fluid is heated from above, no instabilities ensue since all the roots are either negative if they are real or have negative real parts if they are complex.

__Heating Fluid From Below__

In this situation $H = -1$ and so we need to examine the roots of the polynomial equation $f(\sigma) = 0$ where the form of $f(\sigma)$ is determined from (6.26). We start by producing the critical Rayleigh number for the cases of stationary convection and overstability. To find the critical Rayleigh number for the onset of stationary convection we set $\sigma = 0$ in equation (6.26). Thus

$$\lambda^3 + Q\,\lambda\,n^2\,\pi^2 + \epsilon\,a^2\,\lambda^2 + \epsilon\,a^2\,n^2\,\pi^2\,Q - a^2\,R\,(\lambda + \epsilon\,a^2)/\lambda = 0$$

from which we obtain

$$R = \frac{\lambda^3 + \lambda\,Q\,n^2\,\pi^2}{a^2} \tag{6.30}$$

Similarly to obtain the critical Rayleigh number for the onset of overstability, set $\sigma = iy$ in equation (6.26), where $y \in \mathbb{R} - \{0\}$.

Then

$$- i P_r P_m y^3 - y^2 \left[\lambda (P_r + P_m + P_r P_m) + \epsilon a^2 P_r\right]$$

$$+ i y \left[\epsilon a^2 \lambda (1 + P_r) + \lambda^2 (1 + P_r + P_m) + \lambda Q P_r\right.$$

$$\left. - Q P_r a^2 + \epsilon a^2 P_r Q + a^2 P_m R/\lambda - \epsilon a^4 P_r Q/\lambda\right]$$

$$+ \lambda^3 + \epsilon a^2 \lambda^2 + \lambda Q (\epsilon - 1) a^2 - \epsilon a^4 Q + Q \lambda^2$$

$$+ R a^2 (1 + \epsilon a^2/\lambda) .$$

By equating, separately, the real and imaginary parts we obtain

$$R = \frac{\lambda(\lambda^2 + n^2 \pi^2 Q)}{a^2} - \frac{\lambda y^2}{a^2(\lambda + \epsilon a^2)} \left[P_r (\lambda + \epsilon a^2) + \lambda P_m (1 + P_r)\right] \tag{6.31}$$

and

$$y^2 = \frac{(\lambda + \epsilon a^2)}{P_m} \left[\frac{\lambda}{P_r} (1 + P_r) + \frac{n^2 \pi^2 Q}{\lambda}\right] + \frac{1}{P_r} (\lambda^2 - a^2 R / \lambda). \tag{6.32}$$

From (6.31) and (6.32) we can show that

$$R = \frac{\lambda^3}{a^2} + \frac{(\lambda + \epsilon a^2)}{a^2 P_m^2} \left[\lambda P_r (\lambda + \epsilon a^2) + \lambda^2 P_m (1 + P_r)\right.$$

$$\left. + \frac{Q n^2 \pi^2 P_r^2}{\lambda(1 + P_r)} (\lambda + \epsilon a^2 + \lambda P_m)\right] . \tag{6.33}$$

Define the functions g(a,n) and h(a,n) by the expressions on the right of equations (6.30) and (6.33) respectively. Clearly

$$f(\sigma) = \sigma^3 + \alpha \sigma^2 + \beta \sigma + \gamma$$

where the form of α, β and γ are obvious from (6.26). This can be written in the form

$$f(\sigma) = (\sigma + \alpha) (\sigma^2 + \beta) + \delta \tag{6.34}$$

where

$$\delta = \gamma - \alpha \beta .$$

Now

$$\beta = \frac{\lambda^2}{P_r} + \frac{(\lambda + \epsilon\ a^2)}{\lambda\ P_r\ P_m} \left[\lambda^2\ (1 + P_r) + n^2\ \pi^2\ Q\ P_r\right] - \frac{a^2\ R}{\lambda\ P_r}\quad .$$

Define

$$\eta = P_r\ (\lambda + \epsilon\ a^2) + \lambda\ P_m\ (1 + P_r)$$

then

$$\beta = \frac{1}{\lambda\ \eta\ P_r} \left[\lambda^3\ \eta + \frac{\eta\ \lambda^2}{P_m}\ (1 + P_r)\ (\lambda + \epsilon\ a^2) + \frac{n^2\ \pi^2\ Q\ \eta\ P_r}{P_m}\ (\lambda + \epsilon\ a^2)\right] - \frac{a^2\ R}{\lambda\ P_r}$$

$$= \frac{1}{\lambda\ \eta\ P_r} \left\{\lambda\ P_m\ (1 + P_r) \left[\lambda^3 + \frac{\eta\ \lambda\ (\lambda + \epsilon\ a^2)}{P_m^{\ 2}} + \frac{n^2\ \pi^2\ Q\ P_r^{\ 2}}{\lambda\ P_m^{\ 2}\ (1 + P_r)}\ (\lambda + \epsilon\ a^2)\ (\lambda + \epsilon\ a^2 + \lambda\ P_m)\right] + P_r\ (\lambda + \epsilon\ a^2)\ (\lambda^3 + n^2\ \pi^2\ \lambda\ Q)\right\} - \frac{a^2\ R}{\lambda\ P_r}$$

$$= \frac{1}{\lambda\ \eta\ P_r} \left\{\lambda\ P_m\ (1 + P_r) \left[\lambda^3 + \frac{(\lambda + \epsilon\ a^2)}{P_m^{\ 2}}\ [\lambda\ \eta + \frac{n^2\ \pi^2\ Q\ P_r^{\ 2}}{\lambda\ (1 + P_r)}\ (\lambda + \epsilon\ a^2 + \lambda\ P_m)\ ]\ \right] + P_r\ (\lambda + \epsilon\ a^2)\ (\lambda^3 + n^2\ \pi^2\ \lambda\ Q)\right\} - \frac{a^2\ R}{\lambda\ P_r}\quad .$$

In terms of g(a,n) and h(a,n) we can write β in the form

$$\beta = \frac{1}{\lambda\ \eta\ P_r} \left[\lambda\ P_m\ (1 + P_r)\ a^2\ h(a,n) + P_r\ (\lambda + \epsilon\ a^2)\ a^2\ g(a,n)\right] - \frac{a^2\ R}{\lambda\ P_r}$$

$$= \frac{a^2}{\lambda\ \eta\ P_r} \left\{\left[\lambda\ P_m\ (1 + P_r) + P_r\ (\lambda + \epsilon\ a^2)\right]\ h(a,n) + P_r\ (\lambda + \epsilon\ a^2)\ \left[g(a,n) - h(a,n)\right]\right\} - \frac{a^2\ R}{\lambda\ P_r}$$

$$= \frac{a^2}{\lambda\ \eta\ P_r} \left[\eta\ h(a,n) + P_r\ (\lambda + \epsilon\ a^2)\ [g(a,n) - h(a,n)]\right] - \frac{a^2\ R}{\lambda\ P_r}$$

i.e. $\beta = \frac{a^2}{\lambda P_r}[h(a,n) - R] + \frac{a^2}{\lambda \eta}(\lambda + \epsilon a^2)[g(a,n) - h(a,n)]$.

From (6.26)

$$\gamma = \frac{a^2}{\lambda P_r P_m}(\lambda + \epsilon a^2)\left(\frac{\lambda^3 + n^2 \pi^2 \lambda Q}{a^2} - R\right)$$

$$= \frac{a^2}{\lambda P_r P_m}(\lambda + \epsilon a^2)[g(a,n) - R] .$$

So

$$\delta = \gamma - \alpha \beta$$

$$= \frac{a^2 (\lambda + \epsilon a^2)}{\lambda P_r P_m}[g(a,n) - R] - \frac{a^2 \eta}{\lambda P_r P_m}\left[\frac{1}{P_r}[h(a,n) - R + \frac{1}{\eta}(\lambda + \epsilon a^2)[g(a,n) - h(a,n)]\right]$$

$$= \frac{a^2}{{P_r}^2}(1 + P_r)[R - h(a,n)] .$$

Substitute for α, β and δ in equation (6.34) to obtain

$$f(\sigma) = \left[\sigma + \frac{\lambda + \epsilon a^2}{P_m} + \frac{\lambda (1 + P_r)}{P_r}\right]\left[\sigma^2 + \frac{a^2}{\lambda P_r}[h(a,n) - R] + \frac{a^2 (\lambda + \epsilon a^2)[g(a,n) - h(a,n)]}{\lambda [\lambda P_m (1 + P_r) + P_r (\lambda + \epsilon a^2)]}\right] + \frac{a^2 (1 + P_r)}{{P_r}^2}[R - h(a,n)] . \qquad (6.35)$$

There are two cases to consider.

Case (1)

$$\min h(a,n) < \min g(a,n) .$$

Let us select a, n to minimise h(a,n) and consider the nature of the solutions of $f(\sigma) = 0$ as R varies. If $R < \min h(a,n)$ then

$$f(\sigma) = (\sigma + \alpha)(\sigma^2 + \beta) - \delta$$

where $\alpha, \beta, \delta > 0$. Further

$$f(0) = \alpha\,\beta - \delta$$

$$= \frac{a^2\,\eta}{\lambda\,P_r\,P_m}\left[\frac{1}{P_r}\,[h(a,n) - R] + \frac{1}{\eta}\,(\lambda + \epsilon\,a^2)[g(a,n) - h(a,n)]\right]$$

$$- \frac{a^2}{P_r{}^2}\,(1 + P_r)\,[R - h(a,n)]$$

$$= \frac{a^2}{\lambda\,P_r\,P_m}\,(\lambda + \epsilon\,a^2)\,[g(a,n) - h(a,n)]$$

$$+ \frac{a^2}{\lambda\,P_r\,P_m}\,(\lambda + \epsilon\,a^2)\,[h(a,n) - R] > 0$$

and

$$f(-\alpha) = -\delta < 0$$

and so $f(\sigma)$ has a real root in $(-\alpha, 0)$. Also for $\sigma > 0$,

$$f'(\sigma) > f'(0) = \beta > 0$$

and thus in this case $f(\sigma) = 0$ has no positive real roots. Consequently it has three negative real roots all in $(-\alpha, 0)$ or one real root in $(-\alpha, 0)$ and two complex conjugate roots with real parts which are negative since the sum of all the roots is $-\alpha$. In either case we have stability.

On the other hand if $R > \min h(a,n)$ then

$$f(-\alpha) > 0$$

and so $f(\sigma)$ has a real root which is less than $-\alpha$ and since the sum of all roots is $-\alpha$ then $f(\sigma)$ has either real roots at least one of which is positive or one real root and a pair of complex conjugate roots with positive real part. Either way the situation is unstable.

Case (2)

$$\min g(a,n) < \min h(a,n) \ .$$

Let us select a, n to minimise g(a,n) and consider the nature of the solutions of $f(\sigma) = 0$ as R varies. If $R < \min g(a,n)$ then

$$f(-\alpha) < 0 \ .$$

Further,

$$\beta = \frac{a^2}{\lambda P_r} [h(a,n) - R] + \frac{a^2}{\lambda \eta} (\lambda + \epsilon a^2) [g(a,n) - h(a,n)]$$

$$= \frac{a^2}{\lambda \eta} \Big[(\lambda + \epsilon a^2) [g(a,n) - h(a,n)] + \frac{\eta}{P_r} [h(a,n) - R)]\Big]$$

$$= \frac{a^2}{\lambda \eta} \Big[(\lambda + \epsilon a^2)[g(a,n) - R] + \frac{\lambda P_m}{P_r} (1 + P_r)[h(a,n) - R]\Big] > 0$$

and so for positive real σ

$$f'(\sigma) > f'(0) = \beta > 0 \ ,$$

$$f(0) = \frac{a^2 (\lambda + \epsilon a^2)}{\lambda P_m P_r} [g(a,n) - R] > 0 \ . \qquad (6.36)$$

Thus $f(\sigma) = 0$ has no real positive roots and at least one negative real root in $(-\alpha, 0)$. Hence f has either two more negative real roots in $(-\alpha, 0)$ or a pair of complex conjugate roots with negative real parts. In either case the solution is stable. On the other hand, if $R > \min g(a,n)$ then from (6.36) $f(0) < 0$ and so one must have at least one positive real root. Thus this situation is unstable.

In conclusion, the configuration is stable provided

$$R < \min [\min g(a,n), \min h(a,n)] = R_{crit} \ .$$

In particular if $R_{crit} = \min g(a,n)$, then we have stationary instability since the instability is arising through the generation of real eigenvalues whereas if $R_{crit} = \min h(a,n)$, then we have overstability since the instability is ensuing through the generation of complex conjugate pair eigenvalues with positive real parts. The relation between the critical Rayleigh number R_{crit} and the magnetic parameter Q for the cases of stationary convection and overstability is shown in figure (1) for the case of free boundaries when $P_r = 1$, $P_m = 4$ and ϵ varies. The numerical results are illustrated in tables (1)-(6). Figure (2) shows the relation between R_{crit} and Q for the case of mixed boundaries and tables (7)-(12) illustrate the numerical results for this case. The results for the case of two rigid boundaries are given in tables (13)-(16) and the relation between R_{crit} and Q is shown in figure (3). In all cases the graphs show that the critical Rayleigh number for overstability case depends strongly on ϵ. In fact as ϵ increases the critical Rayleigh number for the overstability case increases and this always happens provided a condition on P_m, P_r and ϵ is satisfied and Q is greater than a critical value. This will be explained in a following section.

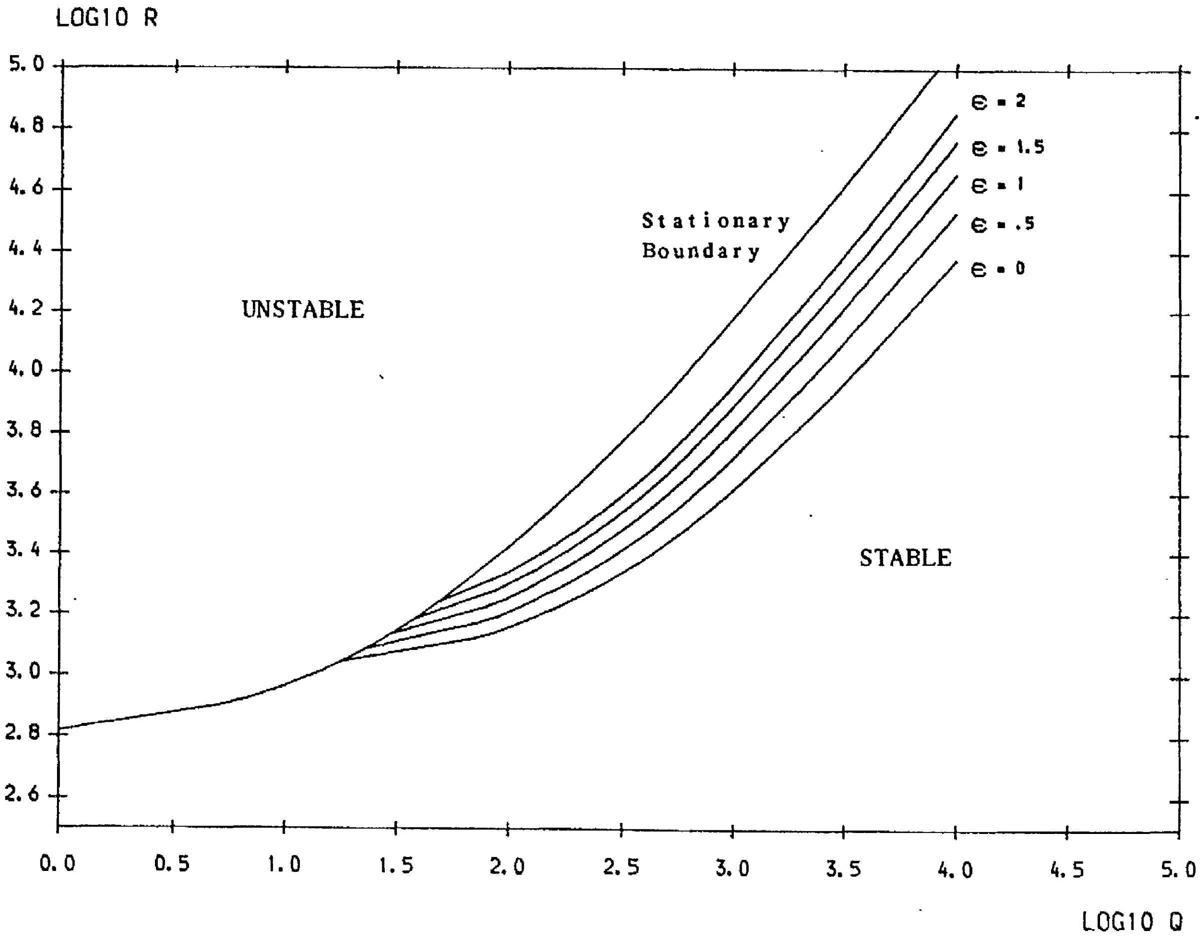

Figure 1.

This figure shows the relation between the critical Rayleigh number R and the magnetic parameter Q for the cases of stationary convection and overstability when both boundaries are free. It shows that the critical Rayleigh number depends strongly on the parameter ϵ for the overstability case.
Here $P_r = 1$ and $P_m = 4$.

Table (1)

The relation between R and Q for the onset of stationary convection when both boundaries are free.

Q	A	R	Q	A	R
0	2.221	657.511	5500	7.741	68896.8
5	2.432	796.573	6000	7.862	74631.7
10	2.59	923.069	6500	7.975	80343.2
15	2.718	1041.50	7000	8.08	86033.7
20	2.826	1154.18	7500	8.18	91705.4
25	2.921	1262.49	8000	8.274	97359.8
50	3.27	1762.04	10000	8.61	119831.0
100	3.7	2653.71	10500	8.68	125418.0
150	3.99	3475.67	11000	8.752	130994.0
200	4.21	4258.49	11500	8.821	136560.0
300	4.543	5752.65	12000	8.887	142116.0
400	4.794	7185.94	13000	9.013	153202.0
500	4.998	8578.89	14000	9.131	164255.0
600	5.171	9942.40	15000	9.242	175279.0
700	5.321	11283.2	16000	9.347	186276.0
800	5.455	12605.6	17000	9.446	197249.0
1000	5.684	15207.0	18000	9.541	208199.0
1500	6.124	21535.2	19000	9.632	219128.0
2000	6.453	27700.0	20000	9.718	230038.0
2500	6.72	33756.5	25000	10.104	284341.0
3000	6.945	39734.2	30000	10.429	338307.0
3500	7.14	45650.7	35000	10.711	392013.0
4000	7.313	51517.9	40000	10.962	445507.0
4500	7.469	57344.1	45000	11.187	498825.0
5000	7.61	63135.5	50000	11.393	551994.0

Table (2)

The relation between R and Q for the overstability case when both boundaries are free. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0$.

Q	A	R	Q	A	R
18	2.305	1108.57	768	3.528	3518.67
68	2.493	1317.58	818	3.568	3656.22
118	2.639	1510.13	868	3.607	3792.45
168	2.759	1691.64	918	3.644	3927.45
218	2.862	1865.04	1000	3.702	4146.42
268	2.952	2032.17	2000	4.21	6653.9
318	3.032	2194.25	3000	3.543	8988.52
368	3.105	2352.14	4000	4.794	11228.0
418	3.172	2506.49	5000	4.998	13404.5
468	3.233	2657.78	6000	5.171	15535.0
518	3.29	2806.41	7000	5.321	17629.9
568	3.343	2952.69	8000	5.455	19696.2
618	3.394	3096.85	9000	5.575	21738.7

Table (3)

The relation between R and Q for the overstability case when both boundaries are free. Here $P_r = 1$, $P_m = 4$ and $\epsilon = .5$.

Q	A	R	Q	A	R
24	2.257	1230.42	774	3.413	4438.49
74	2.432	1493.34	824	3.452	4628.56
124	2.568	1739.82	874	3.488	4817.25
174	2.682	1975.12	924	3.524	5004.66
224	2.779	2202.17	1000	3.575	5287.28
274	2.865	2422.80	2000	4.063	8836.65
324	2.941	2638.24	3000	4.382	12205.0
374	3.01	2849.38	4000	4.623	15474.0
424	3.073	3056.85	5000	4.819	18677.1
474	3.132	3261.17	6000	4.985	21832.2
524	3.186	3462.73	7000	5.129	24950.2
574	3.237	3661.84	8000	5.257	28038.3
624	3.285	3858.78	9000	5.372	31101.6

Table (4)

The relation between R and Q for the overstability case when both boundaries are free. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 1$.

Q	A	R	Q	A	R
30	2.213	1366.27	780	3.294	5429.76
80	2.374	1685.66	830	3.331	5676.87
130	2.501	1989.01	880	3.365	5922.59
180	2.607	2281.33	930	3.399	6167.04
230	2.698	2565.46	1000	3.443	6507.28
280	2.778	2843.20	2000	3.904	11198.3
330	2.85	3115.76	3000	4.205	15707.2
380	2.915	3384.0	4000	4.433	20116.5
430	2.975	3648.59	5000	4.617	24460.2
480	3.03	3910.0	6000	4.773	28756.0
530	3.081	4168.65	7000	4.909	33015.0
580	3.129	4424.85	8000	5.029	37244.4
630	3.173	4678.86	9000	5.137	41449.1

Table (5)

The relation between R and Q for the overstability case when both boundaries are free. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 1.5$.

Q	A	R	Q	A	R
39	2.181	1540.16	789	3.175	6506.98
89	2.327	1917.34	839	3.209	6815.2
139	2.443	2279.54	889	3.241	7122.08
189	2.541	2631.30	939	3.272	7427.72
239	2.624	2975.26	1000	3.307	7799.05
289	2.698	3313.07	2000	3.736	13723.2
339	2.764	3645.9	3000	4.014	19471.4
389	2.824	3974.56	4000	4.224	25124.3
439	2.879	4299.69	5000	4.393	30714.8
489	2.93	4621.76	6000	4.536	36260.1
539	2.977	4941.15	7000	4.661	41770.6
589	3.022	5258.17	8000	4.771	47253.4
639	3.063	5573.07	9000	4.871	52713.2

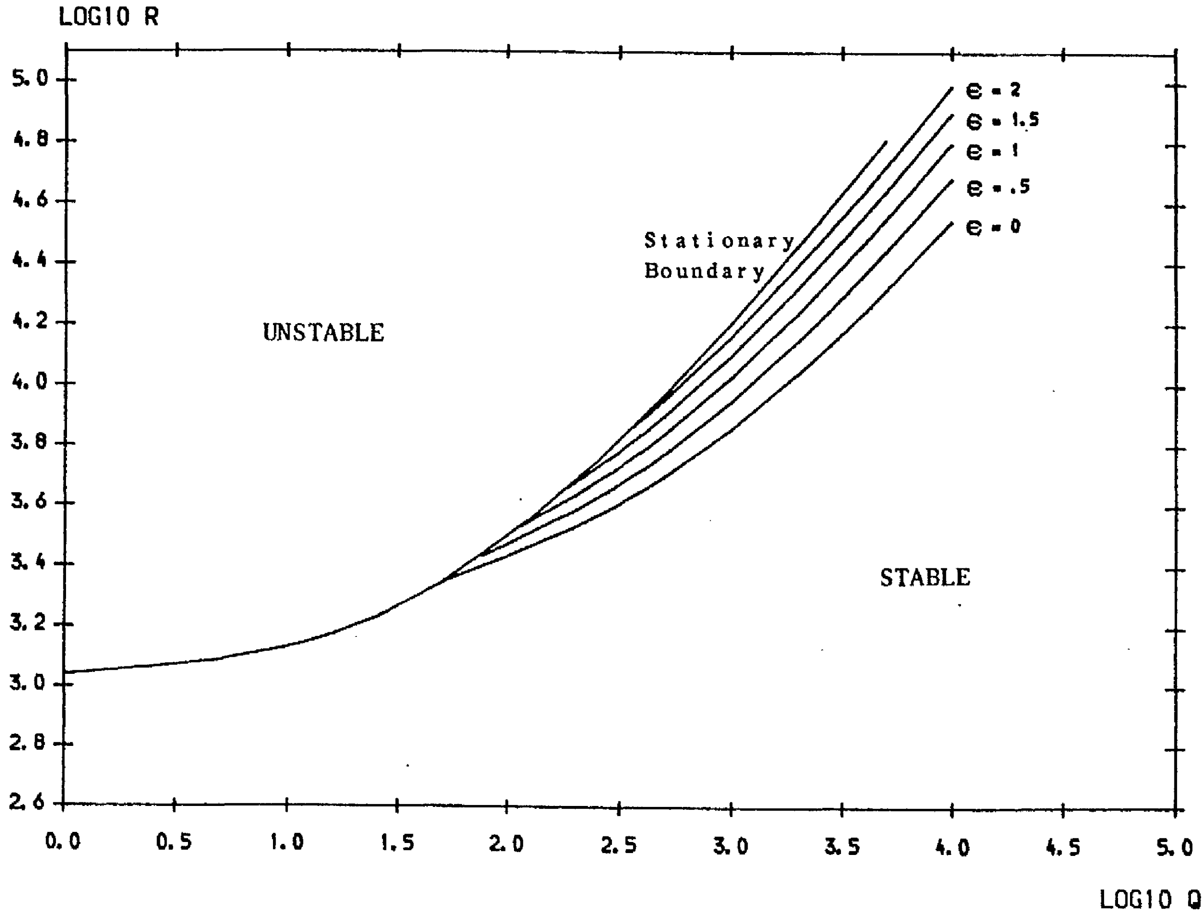

Figure 2.

This figure shows the relation between the critical Rayleigh number R and the magnetic parameter Q for the cases of stationary convection and overstability when the boundaries are mixed. It shows that the critical Rayleigh number depends strongly on the parameter ϵ for the overstability case. Here $P_r = 1$ and $P_m = 4$.

Table (6)

The relation between R and Q for the overstability case when both boundaries are free. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 2$.

Q	A	R	Q	A	R
50	2.156	1754.54	800	3.055	7670.07
100	2.286	2191.34	850	3.086	8043.09
150	2.39	2614.57	900	3.115	8414.85
200	2.478	3028.23	950	3.143	8785.46
250	2.554	3434.65	1000	3.17	9154.99
300	2.621	3835.36	2000	3.56	16394.6
350	2.681	4231.41	3000	3.81	23471.1
400	2.736	4623.57	4000	3.998	30460.8
450	2.786	5012.42	5000	4.149	37394.3
500	2.832	5398.4	6000	4.276	44287.7
550	2.876	5781.87	7000	4.386	51150.4
600	2.916	6163.12	8000	4.484	57988.8
650	2.954	6542.38	9000	4.571	64807.2
700	2.989	6919.85	10000	4.651	71608.8
750	3.023	7295.70			

Table (7)

The relation between R and Q for the onset of stationary convection when the boundaries are mixed.

Q	A	R	Q	A	R
0	2.683	1100.65	500	4.08	9303.66
5	2.81	1231.56	1000	5.748	16118.1
10	2.919	1355.49	1500	6.179	22591.1
15	3.013	1474.11	2000	6.504	28878.3
25	3.171	1699.42	2500	6.766	35043.0
50	3.465	2217.42	5000	7.648	64846.1
250	4.496	5612.9	10000	8.637	122141.0

Table (8)

The relation between R and Q for the overstability case when the boundaries are mixed. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0$.

Q	A	R	Q	A	R
51.241	3.109	2241.92	1000	4.585	7210.32
100	3.287	2721.77	2000	5.181	11005.4
200	3.554	3402.81	3000	5.57	14424.7
300	3.764	3979.18	4000	5.865	17639.2
400	3.934	4506.7	5000	6.105	20718.5
500	4.078	5002.24	10000	6.918	34987.4

Table (9)

The relation between R and Q for the overstability case when the boundaries are mixed. Here $P_r = 1$, $P_m = 4$ and $\epsilon = .5$.

Q	A	R	Q	A	R
76.888	3.13	2730.89	1000	4.479	8889.23
100	3.206	2982.33	2000	5.069	14087.0
200	3.465	3855.52	3000	5.455	18859.6
300	3.669	4600.50	4000	5.748	23399.8
400	3.837	5288.28	5000	5.987	27786.3
500	3.978	5939.41	10000	6.798	48431.1

Table (10)

The relation between R and Q for the overstability case when the boundaries are mixed. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 1$.

Q	A	R	Q	A	R
116.023	3.179	3430.83	2000	4.954	17424.5
200	3.385	4314.68	3000	5.336	23694.7
300	3.582	5241.87	4000	5.626	29706.9
400	3.745	6103.25	5000	5.863	35548.0
500	3.883	6923.29	10000	6.668	63316.3
1000	4.374	10682.9			

Table (11)

The relation between R and Q for the overstability case when the boundaries are mixed. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 1.5$.

Q	A	R	Q	A	R
187.121	3.289	4618.16	2000	4.842	21002.3
200	3.317	4770.91	3000	5.216	28914.2
300	3.506	5892.15	4000	5.502	36544.3
400	3.663	6939.67	5000	5.735	43987.1
500	3.796	7941.36	10000	6.53	79625.3
1000	4.274	12577.2			

Table (12)

The relation between R and Q for the overstability case when the boundaries are mixed. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 2$.

Q	A	R	Q	A	R
364.659	3.547	7345.33	3000	5.105	34489.6
400	3.598	7778.01	4000	5.384	43885.8
500	3.726	8972.04	5000	5.612	53080.2
1000	4.187	14544.8	10000	6.394	97347.1
2000	4.74	24791.0			

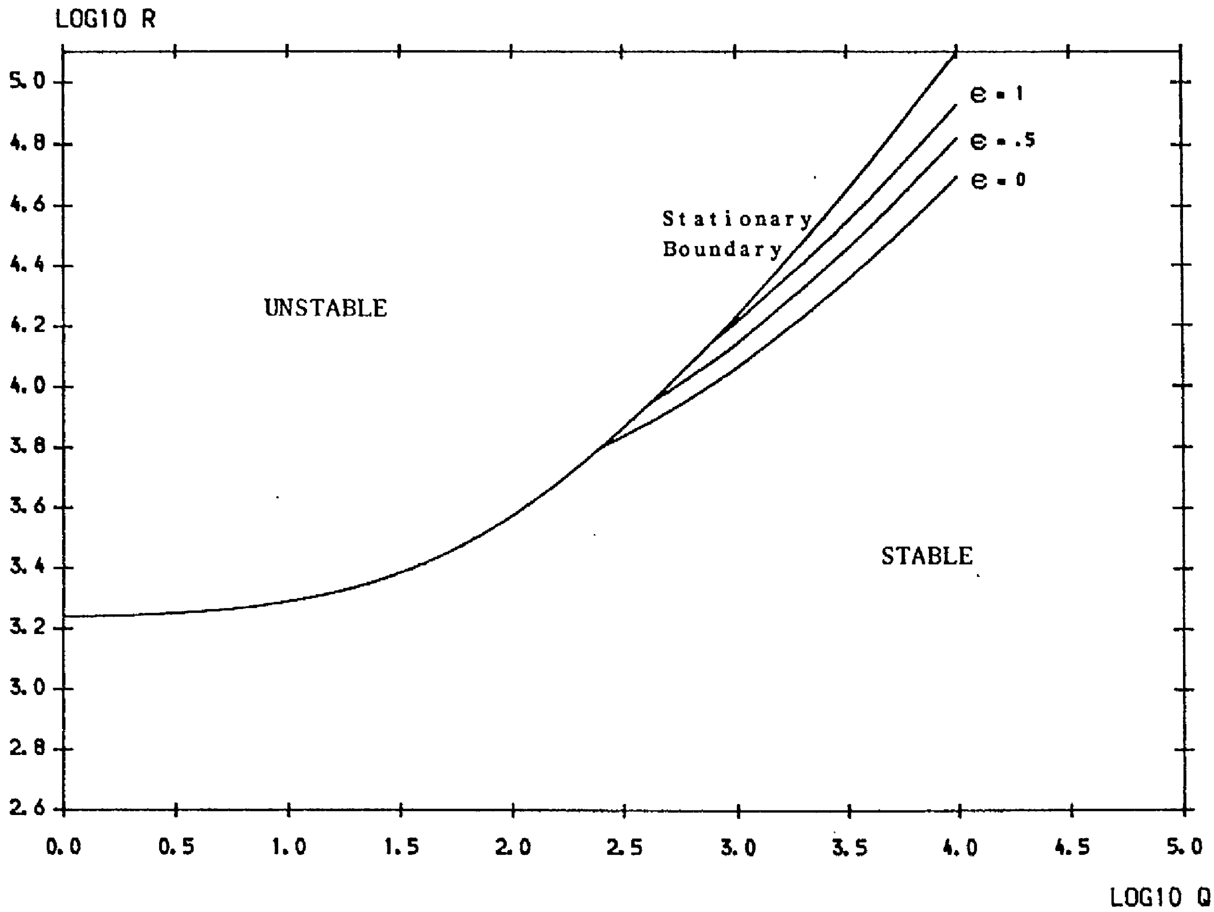

Figure 3.

This figure shows the relation between the critical Rayleigh number R and the magnetic parameter Q for the cases of stationary convection and overstability when both boundaries are rigid. It shows that the critical Rayleigh number depends strongly on the parameter ϵ for the overstability case. Here $P_r = 1$ and $P_m = 4$.

Table (13)

The relation between R and Q for the onset of stationary convection when both boundaries are rigid.

Q	A	R	Q	A	R
0	3.116	1707.76	400	4.975	8626.92
10	3.265	1945.75	900	5.71	15743.9
20	3.391	2171.84	2500	6.814	36396.5
30	3.497	2388.77	4000	7.394	54697.9
40	3.594	2598.42	5000	7.686	66620.1
50	3.679	2802.01	5500	7.814	72522.3
60	3.757	3000.66	6000	7.932	78390.2
70	3.828	3194.99	6500	8.044	84232.3
80	3.894	3385.58	7000	8.147	90047.0
90	3.956	3572.88	7500	8.245	95838.4
100	4.0	3757.23	8000	8.338	101606.0
150	4.253	4644.28	8500	8.426	107360.0
200	4.445	5488.57	9000	8.51	113093.0
250	4.607	6302.79	9500	8.589	118809.0
300	4.745	7094.33	10000	8.662	124508.0
350	4.866	7867.95			

Table (14)

The relation between R and Q for the overstability case when both boundaries are rigid. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0$.

Q	A	R	Q	A	R
251.93	4.29	6333.	2000	6.02	17005.9
300	4.4	6741.03	3000	6.462	21800.1
400	4.597	7538.24	4000	6.795	26233.2
500	4.762	8283.34	5000	7.066	30427.8
600	4.905	8989.12	6000	7.295	34448.7
700	5.031	9663.83	7000	7.495	38335.4
800	5.144	10313.1	8000	7.672	42114.1
900	5.247	10941.1	9000	7.832	45803.3
1000	5.337	11550.9	10000	7.978	49416.6

Table (15)

The relation between R and Q for the overstability case when both boundaries are rigid. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0.5$.

Q	A	R	Q	A	R
414.03	4.504	8837.67	3000	6.334	27599.6
500	4.643	9650.91	4000	6.679	33662.4
600	4.786	10556.32	5000	6.951	39448.7
700	4.912	11426.7	6000	7.181	45032.9
800	5.025	12268.4	7000	7.382	50460.7
900	5.128	13085.9	8000	7.56	55762.2
1000	5.223	13882.8	9000	7.721	60958.9
2000	5.902	21119.9	10000	7.868	66066.6

Table (16)

The relation between R and Q for the overstability case when both boundaries are rigid. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 1$.

Q	A	R	Q	A	R
791.406	4.9	14251.2	5000	6.837	49323.3
900	5.013	15364.4	6000	7.069	56646.6
1000	5.107	16365.5	7000	7.27	63790.5
2000	5.786	25558.7	8000	7.449	70789.5
3000	6.229	33900.2	9000	7.611	77667.9
4000	6.565	41769.2	10000	7.759	84443.7

<u>Stationary Convection</u> (Heating from below)

As mentioned in a previous section that we can derive an eighth order ordinary differential equation from (6.17) and this equation will have form

$$L^4w - \Big[\sigma\ (1 + P_r + P_m) + Q + \epsilon\ a^2\Big]L^3w - \Big\{\Big[\sigma\ Q\ (P_m - P_r) + Q\ a^2 - \sigma^2\ P_r\Big] - (\sigma\ P_m + \epsilon\ a^2)\Big[\sigma\ (1 + P_r) + Q\Big]\Big\}L^2w - \Big[-\ a^2\ \sigma\ P_r\ Q + \epsilon\ a^2\ (\sigma^2\ P_r + \sigma\ P_r\ Q - Q\ a^2) + \sigma^3\ P_r\ P_m - a^2\ R\Big]Lw = \Big[\ \epsilon\ a^4\ (\sigma\ P_r\ Q + R) + \sigma\ P_m\ a^2\ R\Big]w\ . \qquad (6.37)$$

To find the critical Rayleigh number for the onset of stationary convection we set $\sigma = 0$ in equation (6.37) and we can see that it factorizes into

$$(L - \epsilon\ a^2)\ (L^3w - Q\ L^2w - Q\ a^2\ Lw + a^2\ R\ w) = 0$$

i.e. $$L^3w - Q\ L^2w - Q\ a^2\ Lw + a^2\ R\ w = 0\ . \qquad (6.38)$$

Since this equation does not contain P_r, P_m or ϵ then the critical Rayleigh number for stationary convection is independent of P_r, P_m or ϵ for the free, rigid and mixed boundary value problems and we obtain precisely the results of Chandrasekhar [15] .

<u>The case of overstability</u> (Heating from below)

We have seen that instabilities ensue if

$$R > \min\,[R_{stat}, R_{over}]$$

where

$$R_{stat} = \min\, g(a,n)$$

$$R_{over} = \min\, h(a,n)\ .$$

From (6.30) and (6.33) we have

$$h(a,n) - g(a,n) = \frac{\lambda + \epsilon a^2}{a^2 P_m^2}\Big[\lambda P_r (\lambda + \epsilon a^2) + \lambda^2 P_m (1 + P_r) + \frac{n^2 \pi^2 Q P_r^2}{\lambda (1 + P_r)} (\lambda + \epsilon a^2 + \lambda P_m)\Big] - \frac{\lambda Q n^2 \pi^2}{a^2}$$

$$= \frac{P_r (\lambda + \epsilon a^2)}{\lambda P_m}\Big[\frac{\lambda^2}{\lambda P_m} (\lambda + \epsilon a^2) + \frac{\lambda^3 (1 + P_r)}{a^2 P_r}\Big] + \frac{Q n^2 \pi^2}{a^2 P_m^2 \lambda (1 + P_r)}\Big[P_r^2 (\lambda + \epsilon a^2)^2 - \lambda P_r P_m (\lambda + \epsilon a^2) - \lambda^2 P_m^2 (1 + P_r) + \lambda P_r P_m (1 + P_r)(\lambda + \epsilon a^2)\Big]$$

$$= \frac{P_r (\lambda + \epsilon a^2)}{\lambda P_m}\,\frac{\lambda^3 (1 + P_r)}{a^2 P_r}\Big[1 + \frac{P_r (\lambda + \epsilon a^2)}{\lambda P_m (1 + P_r)}\Big] + \frac{Q n^2 \pi^2}{a^2 P_m}\Big\{\frac{P_r (\lambda + \epsilon a^2)}{\lambda P_m (1 + P_r)}\Big[P_r (\lambda + \epsilon a^2) + \lambda P_m (1 + P_r)\Big] - \frac{\lambda P_m}{\lambda P_m (1 + P_r)}\Big[P_r (\lambda + \epsilon a^2) + \lambda P_m (1 + P_r)\Big]\Big\}\ .$$

Let us define

$$\Omega = \frac{P_r (\lambda + \epsilon a^2)}{\lambda P_m (1 + P_r)} > 0$$

then

$$h(a,n) - g(a,n) = \Omega\ (1 + \Omega)\ (1 + P_r)\ \frac{\lambda^3\ (1 + P_r)}{a^2\ P_r}$$

$$+ \frac{Q\ n^2\ \pi^2}{a^2\ P_m}\ (1 + \Omega)\left[P_r\ (\lambda + \epsilon\ a^2) - \lambda\ P_m\right]$$

$$= \frac{\lambda(1 + \Omega)}{a^2\ P_r}\ \Big\{\Omega\ (1 + P_r)\left[\lambda^2\ (1 + P_r) + n^2\pi^2\ Q\ P_r\right]$$

$$- n^2\pi^2\ Q\ P_r\Big\}\ .$$

Define $a^2 = \pi^2 x$ then, bearing in mind that the minima R_{stat} and R_{over} are attained when $n = 1$, we have

$$h(a,1) - g(a,1) = \frac{(1 + x)(1 + \Omega)}{\pi^2\ x\ P_r}\ \Big\{\Omega\ (1 + P_r)\ \Big[n^2\ \pi^2\ Q\ P_r$$

$$+ \pi^4\ (1 + P_r)\ (1 + x)^2\Big] - n^2\ \pi^2\ Q\ P_r\Big\}$$

$$= \Big\{\frac{P_r\ (1 + x + \epsilon\ x)}{P_m\ (1 + x)}\ \left[(1 + P_r)\ (1 + x)^2 + \frac{Q\ P_r}{\pi^2}\right]$$

$$- \frac{Q\ P_r}{\pi^2}\Big\}\ \frac{\pi^4\ (1 + x)(1 + \Omega)}{x\ P_r}$$

$$= \frac{\pi^4}{x\ P_m}\ (1 + \Omega)\ (1 + x)\ f(x) \qquad (6.39)$$

where

$$f(x) = (1 + P_r)\ (1 + x)\ (1 + x + \epsilon\ x) - \frac{Q}{\pi^2}\ (P_m - P_r - \frac{\epsilon\ x\ P_r}{1 + x})$$

We aim to develop some criterion on the values of Q, P_r, P_m and ϵ which will guarantee the existence of a region in which instability by overstability is the preferred mechanism. From (6.39) it is clear that this is only possible provided there is a region of x values for which $f(x) < 0$. Elementary calculus reveals that f(x) is strictly increasing in $x \geq 0$.

Further

$$f(0) = (1 + P_r) - Q\,(P_m - P_r)/\pi^2 \;.$$

In order for overstability to be possible we therefore must ensure that $f(0) < 0$ and this will only occur provided

$$\begin{aligned} &(a) \quad P_m > P_r \\ &(b) \quad Q > \pi^2\,(1 + P_r)/(P_m - P_r) \;. \end{aligned} \qquad (6.40)$$

Assuming that (6.40) is satisfied then we can find $x = x_{crit}$ such that

$$f(x_{crit}) = 0 \qquad x_{crit} > 0 \;.$$

In addition from the definition of $f(x)$

$$\begin{aligned} &(a) \quad P_m - P_r - \epsilon\, P_r \frac{x_{crit}}{1 + x_{crit}} > 0 \\ &(b) \quad h(a_{crit},1) = g(a_{crit},1) \qquad a^2{}_{crit} = \pi^2\, x_{crit} \qquad (6.41) \\ &(c) \quad f(x) \leq 0 \quad \forall\, x \in [0,x_{crit}] \;. \end{aligned}$$

Let x_{stat} be the value of x for which $g(a,1)$ has its minimum then x_{stat} is the critical x value for stationary stability and further

$$2\, x^3{}_{stat} + 3\, x^2{}_{stat} - 1 - Q/\pi^2 = 0 \;.$$

If $x_{stat} \leq x_{crit}$ then overstability will be the preferred instability mechanism and this will be the situation for suitably large Q. At equality we need to find x and Q so that

$$\begin{aligned} &f(x) = 0 \\ &2\, x^3 + 3\, x^2 - 1 - Q/\pi^2 = 0 \end{aligned} \qquad (6.42)$$

i.e. in order to estimate Q at which overstability is guranteed we solve equations (6.42) simultaneously. Equations (6.42) can be written in the forms

$$Q = \frac{\pi^2 \, (1 + P_r) \, (1 + x)^2 \, (1 + x + \epsilon \, x)}{(P_m - P_r) \, (1 + x) - \epsilon \, P_r \, x}$$

$$(2x - 1) \, (x + 1)^2 = Q \, / \, \pi^2 \, . \qquad (6.43)$$

Since

$$x = 1/2 \quad \Longrightarrow \quad h(a,1) - g(a,1) > 0$$

then we must have $x > 1/2$. Eliminate Q from (6.43) to obtain

$$2x - 1 = \frac{(1 + P_r) \, (1 + x + \epsilon \, x)}{(P_m - P_r) \, (1 + x) - \epsilon \, P_r \, x} \quad .$$

After some algebra we obtain

$$2x^2 \, (P_m - P_r - \epsilon \, P_r) + x \, (P_m - 2 \, P_r - 1 - \epsilon) - (1 + P_m) = 0$$

Define

$$\alpha = 1 + \epsilon + P_m$$

$$\beta = P_m - P_r \, (1 + \epsilon) > 0 \, .$$

Let

$$E(x) = 2 \, \beta \, x^2 + x \, (P_m - 2 \, P_r - 1 - \epsilon) - (1 + P_m)$$

and define $\gamma = \alpha \, / \, 2\beta$, then

$$\gamma - \frac{1}{2} = \frac{\alpha - \beta}{2 \, \beta}$$

$$= \frac{(1 + \epsilon) \, (1 + P_r)}{2 \, \beta} > 0$$

and

$$E(\gamma) = \frac{\alpha^2}{2 \, \beta} + \frac{\alpha}{2 \, \beta} \, (P_m - 2 \, P_r - 1 - \epsilon) - (1 + P_m)$$

$$= \frac{\alpha^2}{2 \, \beta} + \frac{\alpha}{2 \, \beta} \, (P_m - 2 \, P_r + P_m - \alpha) - (1 + P_m)$$

$$= \frac{\alpha}{\beta} (P_m - P_r) - (1 + P_m)$$

$$= \frac{\alpha}{\beta} (\beta + \epsilon P_r) - (\alpha - \epsilon)$$

$$= \epsilon (1 + \alpha P_r / \beta) > 0 .$$

Thus

$$1/2 < x < \gamma \tag{6.44}$$

and

$$Q < \frac{\pi^2 (1 + P_r) (1 + \alpha/2\beta)^2 [1 + \epsilon \alpha / (\alpha + 2 \beta)]}{P_m - P_r - \epsilon P_r \alpha / (\alpha + 2 \beta)}$$

$$= \frac{\pi^2 (1 + P_r) (\alpha + 2 \beta)^2 (\alpha + 2 \beta + \epsilon \alpha)}{4 \beta^2 [\alpha \beta + 2 \beta (P_m - P_r)]}$$

$$= \frac{\pi^2 (1 + P_r) (\alpha + 2 \beta)^2 (\alpha + 2 \beta + \epsilon \alpha)}{4 \beta^3 (\alpha + 2 \beta + 2 \epsilon P_r)} . \tag{6.45}$$

The condition $\beta > 0$ automatically ensures that conditions $(6.40)_{(a)}$ and $(6.41)_{(a)}$ are automatically satisfied and the argument used to derive the results (6.44) and (6.45) assumes this to be the case. Finally if $x_{stat} > x_{crit}$ then the preferred mode of instability depends on the relative values of $R_{stat}(Q, P_m, P_r, \epsilon)$ and $R_{over}(Q, P_m, P_r, \epsilon)$ and the stability boundary is

$$R_{stat}(Q, P_m, P_r, \epsilon) = R_{over}(Q, P_m, P_r, \epsilon) . \tag{6.46}$$

The solution of (6.46) can be expressed as

$$Q = Q_{crit}(P_m, P_r, \epsilon) \tag{6.47}$$

so that if $Q < Q_{crit}$ then stationary stability is the preferred mechanism whereas if $Q > Q_{crit}$ then overstability is preferred. The relation between ϵ and the critical values of Q for the cases of free, mixed and rigid boundaries when $P_r = 1$ and P_m varies is shown in figures (4), (5) and (6) respectively. The numerical values of ϵ and Q_{crit} are illustrated in tables 17-21 for the case of free boundaries, tables 22-26 for mixed boundaries and tables 27-29 for rigid boundaries. Moreover if $P_m > P_r\ (1 + \epsilon)$ then clearly

$$\frac{\pi^2\ (1 + P_r)}{P_m - P_r} < Q_{crit}(P_m,\ P_r,\ \epsilon) < \frac{\pi^2\ (1 + P_r)(\alpha + 2\ \beta)^2(\alpha + 2\ \beta + \epsilon\ \alpha)}{4\ \beta^3\ (\alpha + 2\ \beta + 2\ \epsilon\ P_r)}\ . \tag{6.48}$$

Of course overstability is possible in the region

$$P_r < P_m < P_r\ (1 + \epsilon)$$

provided Q is sufficiently large and

$$P_m - P_r - \epsilon\ P_r\ \frac{x_{over}}{1 + x_{over}} > 0$$

where x_{over} is the x value at which R_{over} is attained. The techniques used to obtain the numerical results of this chapter are the Compound matrix method and the Chebyshev polynomials method. The Fortran77 codes used to obtain these results are listed in Appendix III.

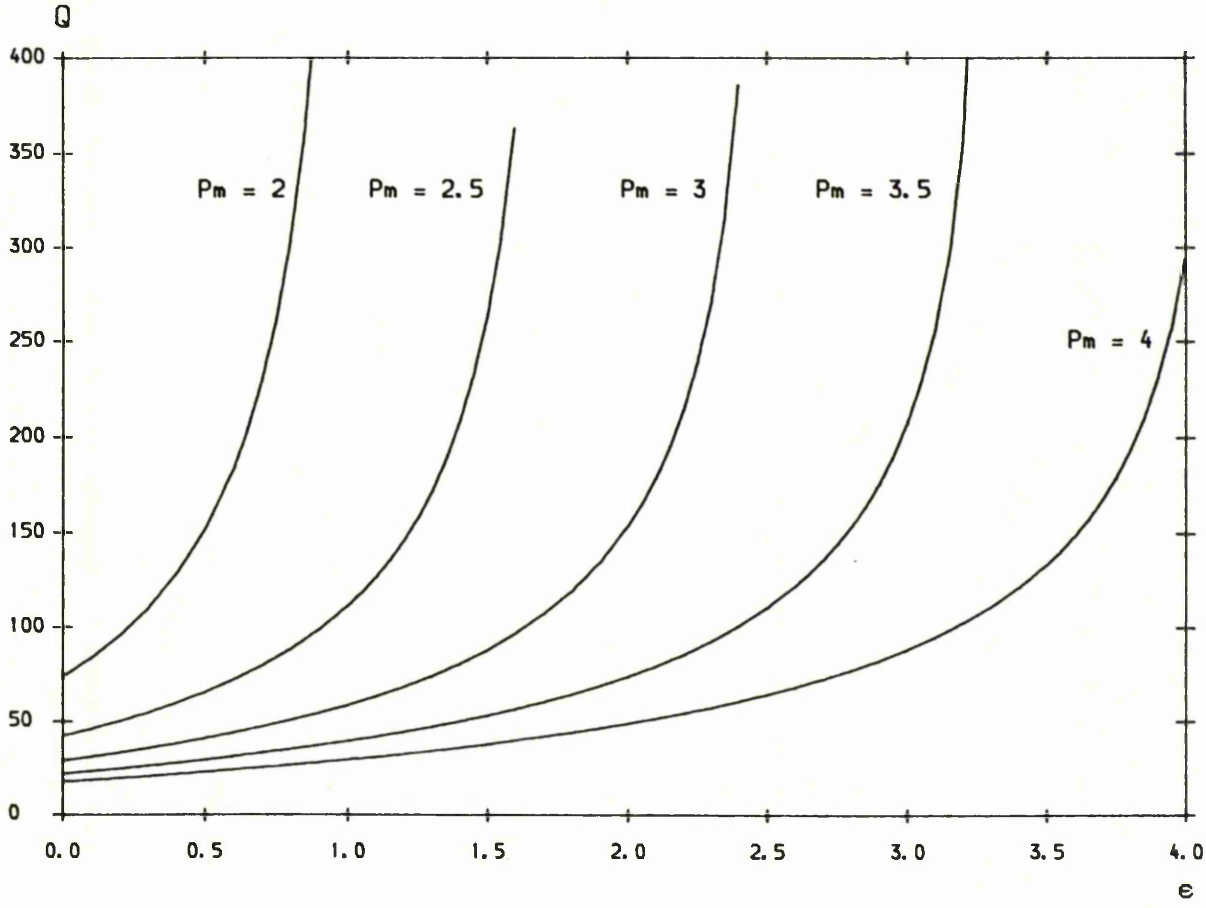

Figure 4.

This figure shows the relation between the parameter ϵ and the critical values of the magnetic parameter Q for various values of P_m. It shows that below critical Q, stationary stability is preferred whereas if $Q > Q_{crit}$ overstability is preferred. Here both bounaries are free.

Table (17)

The relation between ϵ and the critical values of Q when both boundaries are free. Here $P_r = 1$ and $P_m = 2$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	74.1	3.505	2.653	2204.02
0.1	83.8	3.584	2.664	2375.42
0.2	95.5	3.67	2.68	2577.17
0.3	109.8	3.766	2.702	2819.36
0.4	127.8	3.873	2.731	3117.38
0.5	151.3	3.996	2.77	3495.9
0.6	183.1	4.141	2.823	3997.51
0.7	229.2	4.319	2.895	4703.31
0.8	302.7	4.55	2.999	5792.08
0.9	441.3	4.883	3.164	7765.94
1.0	830.2	5.492	3.497	13001.7

Table (18)

The relation between ϵ and the critical values of Q when both boundaries are free. Here $P_r = 1$ and $P_m = 2.5$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	42.4	3.179	2.468	1615.05
0.1	46.1	3.225	2.463	1687.22
0.2	50.2	3.272	2.458	1765.71
0.3	54.8	3.322	2.456	1851.61
0.4	59.9	3.374	2.455	1946.25
0.5	65.6	3.43	2.457	2051.35
0.6	72.1	3.488	2.46	2169.09
0.7	79.7	3.551	2.466	2302.36
0.8	88.4	3.619	2.474	2455.05
0.9	98.8	3.693	2.486	2632.53
1.0	111.2	3.775	2.501	2842.51
1.2	146.1	3.97	2.549	3412.54
1.4	207.9	4.241	2.635	4380.0
1.5	263.3	4.433	2.706	5212.28
1.6	363.3	4.709	2.82	6665.32

Table (19)

The relation between ϵ and the critical values of Q when both boundaries are free. Here $P_r = 1$ and $P_m = 3$.

ϵ	Q	a_{atat}	a_{over}	R
0.0	29.3	2.993	2.382	1352.63
0.2	33.6	3.059	2.363	1440.16
0.4	38.4	3.128	2.347	1538.01
0.6	44.1	3.201	2.335	1648.76
0.8	50.7	3.278	2.326	1776.01
1.0	58.7	3.363	2.322	1924.83
1.2	68.4	3.456	2.321	2102.8
1.4	80.8	3.56	2.326	2321.73
1.6	96.9	3.68	2.338	2601.54
1.8	119.4	3.825	2.361	2979.05
2.0	153.6	4.008	2.4	3533.38
2.2	214.8	4.267	2.47	4484.12
2.4	385.8	4.762	2.64	6984.57

Table (20)

The relation between ϵ and the critical values of Q when both boundaries are free. Here $P_r = 1$ and $P_m = 3.5$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	22.3	2.871	2.334	1204.06
0.4	28.2	2.975	2.294	1329.13
0.8	35.4	3.085	2.262	1477.15
1.2	44.6	3.206	2.238	1658.19
1.6	56.8	3.343	2.222	1889.46
2.0	74.1	3.505	2.218	2203.81
2.4	101.3	3.71	2.23	2675.23
2.8	153.4	4.007	2.274	3529.74
3.0	207.4	4.239	2.326	4372.35
3.1	256.2	4.41	2.371	5107.03
3.2	353.5	4.684	2.452	6524.88

Table (21)

The relation between ϵ and the critical values of Q when both boundaries are free. Here $P_r = 1$ and $P_m = 4$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	17.9	2.784	2.305	1108.29
0.4	22.2	2.869	2.264	1201.64
0.8	27.1	2.957	2.228	1307.23
1.2	33.0	3.051	2.198	1429.02
1.6	40.2	3.151	2.173	1572.87
2.0	49.3	3.262	2.154	1747.91
2.4	61.1	3.387	2.14	1969.49
2.8	77.6	3.535	2.135	2266.26
3.2	102.8	3.72	2.141	2700.69
3.6	148.6	3.983	2.17	3452.95
3.8	192.8	4.181	2.204	4147.28
4.0	294.2	4.526	2.28	5667.56

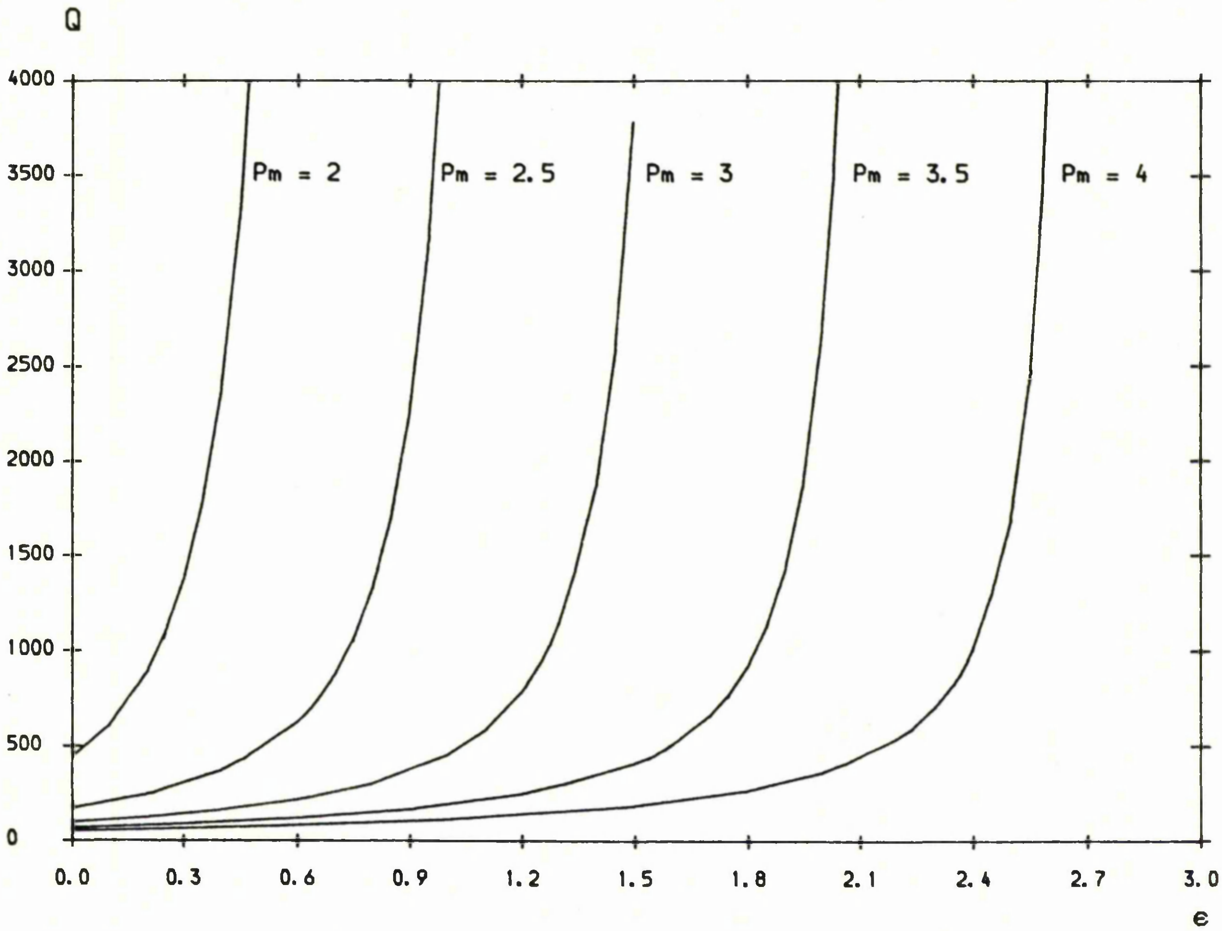

Figure 5.

This figure shows the relation between the parameter ϵ and the critical values of the magnetic parameter Q for various values of P_m. It shows that below critical Q, stationary stability is preferred whereas if $Q > Q_{crit}$ overstability is preferred. Here the bounaries are mixed.

Table (22)

The relation between ϵ and the critical values of Q when the boundaries are mixed. Here $P_r = 1$ and $P_m = 2$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	446.9	4.972	4.362	8543.52
0.1	611.7	5.266	4.577	10871.98
0.2	881.1	5.62	4.853	14538.77
0.3	1366.7	6.078	5.225	20886.58
0.4	2366.1	6.701	5.748	33401.16
0.5	4850.8	7.608	6.534	63098.0

Table (23)

The relation between ϵ and the critical values of Q when the boundaries are mixed. Here $P_r = 1$ and $P_m = 2.5$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	172.2	4.212	3.684	4375.9
0.2	245.8	4.482	3.835	5546.9
0.4	372.3	4.821	4.048	7457.37
0.6	629.9	5.293	4.378	11121.8
0.8	1325.5	6.045	4.952	20357.0
1.0	4623.8	7.554	6.189	60434.8

Table (24)

The relation between ϵ and the critical values of Q when the boundaries are mixed. Here $P_r = 1$ and $P_m = 3$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	99.1	3.845	3.382	3134.13
0.2	126.7	4.002	3.44	3614.64
0.4	164.1	4.18	3.516	4242.67
0.8	303.6	4.651	3.762	6433.17
1.0	455.9	4.998	3.979	8673.71
1.2	789.8	5.512	4.335	13311.1
1.4	1877.2	6.431	5.033	27346.4

Table (25)

The relation between ϵ and the critical values of Q when the boundaries are mixed. Here $P_r = 1$ and $P_m = 3.5$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	68.0	3.571	3.215	2565.31
0.3	90.9	3.792	3.251	2987.13
0.6	122.5	3.98	3.304	3542.57
0.9	169.8	4.205	3.387	4335.93
1.2	249.6	4.494	3.521	5606.33
1.5	413.4	4.912	3.756	8059.45
1.6	513.8	5.105	3.878	9499.09
1.7	666.0	5.346	4.037	11622.5
1.8	920.4	5.664	4.258	15062.9
1.9	1415.7	6.116	4.587	21514.7
2.0	2666.0	6.844	5.141	37068.0

Table (26)

The relation between ϵ and the critical values of Q when the boundaries are mixed. Here $P_r = 1$ and $P_m = 4$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	51.2	3.476	3.109	2241.93
0.5	76.9	3.691	3.13	2730.89
1.0	116.0	3.945	3.179	3430.83
1.5	187.1	4.275	3.289	4618.16
1.8	268.8	4.553	3.415	5903.46
2.0	364.7	4.804	3.547	7345.33
2.2	545.1	5.159	3.757	9940.78
2.3	711.9	5.41	3.918	12252.9
2.4	1009.7	5.758	4.151	16245.1
2.5	1679.6	6.305	4.539	24865.7
2.6	4211.0	7.106	5.134	44661.5

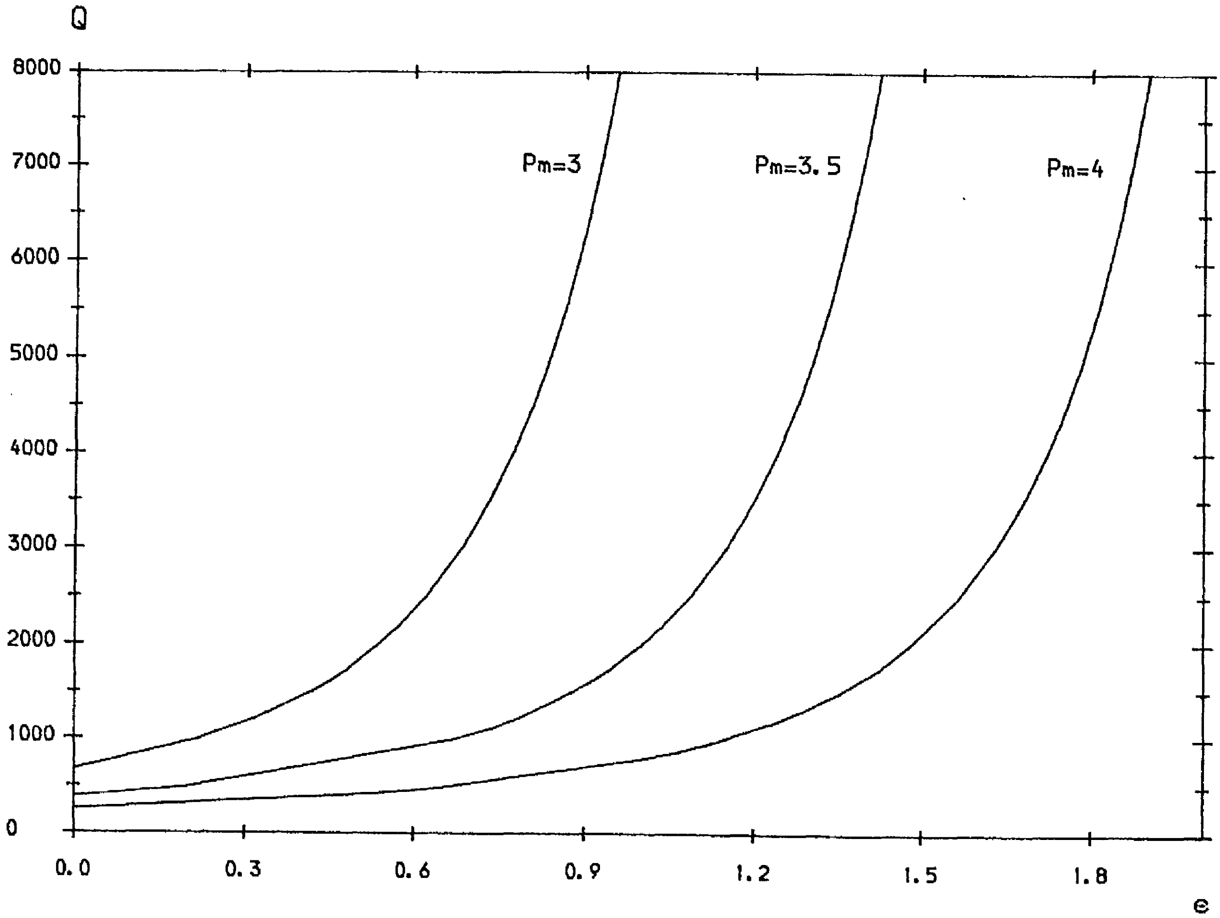

Figure 6.

This figure shows the relation between the parameter ϵ and the critical values of the magnetic parameter Q for various values of P_m. It shows that below critical Q, stationary stability is preferred whereas if $Q > Q_{crit}$ overstability is preferred. Here both bounaries are rigid.

Table (27)

The relation between ϵ and the critical values of Q when both boundaries are rigid. Here $P_r = 1$ and $P_m = 3$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	676.9	5.428	5.199	12649.0
0.22	1000	5.814	5.488	17103.0
0.42	1500	6.236	5.835	23717.1
0.54	2000	6.555	6.05	30125.2
0.62	2500	6.814	6.338	36396.0
0.69	3000	7.034	6.535	42567.9
0.74	3500	7.224	6.708	48663.2
0.78	4000	7.394	6.864	54697.9
0.81	4500	7.515	7.005	60680.3
0.84	5000	7.685	7.135	66618.8
0.87	5500	7.814	7.255	72520.8
0.89	6000	7.932	7.367	78390.2
0.93	7000	8.121	7.571	90045.5
0.96	8000	8.338	7.753	101606.0

Table (28)

The relation between ϵ and the critical values of Q when both boundaries are rigid. Here $P_r = 1$ and $P_m = 3.5$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	382.2	4.937	4.641	8358.42
0.67	1000	5.814	5.261	17103.0
0.88	1500	6.236	5.604	23717.1
1.0	2000	6.555	5.874	30125.2
1.09	2500	6.814	6.098	36396.0
1.15	3000	7.034	6.242	42567.9
1.2	3500	7.224	6.463	48663.2
1.24	4000	7.394	6.616	54697.9
1.28	4500	7.515	6.755	60680.3
1.31	5000	7.685	6.883	66618.8
1.35	6000	7.932	7.111	78390.2
1.39	7000	8.121	7.312	90045.5

Table (29)

The relation between ϵ and the critical values of Q when both boundaries are rigid. Here $P_r = 1$ and $P_m = 4$.

ϵ	Q	a_{stat}	a_{over}	R
0.0	251.9	4.61	4.29	6333.0
0.66	500	5.165	4.606	10109.9
1.14	1000	5.814	5.074	17103.0
1.35	1500	6.236	5.411	23717.1
1.48	2000	6.555	5.677	30125.2
1.57	2500	6.814	5.898	36396.0
1.63	3000	7.034	6.087	42567.9
1.68	3500	7.224	6.257	48663.2
1.72	4000	7.394	6.407	54697.9
1.76	4500	7.515	6.544	60680.3
1.79	5000	7.685	6.67	66618.8
1.81	5500	7.814	6.786	72520.8
1.83	6000	7.932	6.895	78390.2
1.87	7000	8.121	7.092	90045.5
1.90	8000	8.338	7.269	101606.0

Chapter Seven

Benard Convection in a Non-Linear MHD Fluid under the Influence of a Non-vertical Magnetic Field

Mathematical formulation

The effect of an externally impressed magnetic field on the onset of thermal instability in a magnetohydrodynamic fluid was discussed in chapter six when the magnetic field is in the direction of the vertical. In this chapter we shall consider the case in which the external magnetic field is a non-vertical one. i.e. when **H** and **g** act in different directions.

Let **H** acts in a direction inclined at an angle γ to the vertical and let the direction of the x-axis be chosen such that H lies in the xz-plane. Then

$$\mathbf{B} = B\,(\sin\gamma,\ 0,\ \cos\gamma)\ .$$

We may observe that if we bound ourselves to the onset of instability as rolls in the x-direction, then the results derived in chapter six can be applied to this case if we interpret B everywhere to mean the component of **B** in the direction of g. (see Chandrasekhar [13]). To show that this is the case we shall follow the same procedures as in chapter six.

Let $\hat{V}$, $\hat{\theta}$, $\hat{P}$, $\hat{b}$ and $\hat{J}$ be the linear perturbation of velocity, temperature, pressure, magnetic induction and current density about the steady state values then calculation reveals that

$$\hat{V}_{i,i} = 0 \quad ,$$

$$\frac{\partial \hat{V}_i}{\partial t} = - (\hat{P}/\rho)_{,i} + B^2 \varphi_B G_r G_i G_j \hat{b}_{r,j} + B \varphi G_j \hat{b}_{i,j}$$

$$+ \nu \Delta \hat{V}_i + g \alpha \hat{\theta} \quad \delta_{i3} \ ,$$

$$\frac{\partial \hat{\theta}}{\partial t} - \beta \hat{V}_3 = \kappa \Delta \hat{\theta} \quad , \qquad (7.1)$$

$$\hat{b}_{i,i} = 0 \quad ,$$

$$\hat{J}_i = \text{curl } [\rho \varphi \hat{b} + \rho B \varphi_B (\hat{b}. G) G] \ ,$$

$$\frac{\partial \hat{b}_i}{\partial t} = B G_j \hat{V}_{i,j} - \eta e_{ijk} \hat{J}_{k,j} \quad .$$

where

$$G = (\sin\gamma, 0, \cos\gamma).$$

At this stage we introduce the dimensionless variables x^*, t^*, V^*, θ^*, P^*, b^* and J^* where

$$x^* = x \ / \ d,$$

$$t^* = t \ \nu \ / \ d^2 \ ,$$

$$V^* = \hat{V} \ d \ / \ \kappa \ ,$$

$$\theta^* = \hat{\theta} \ \frac{d}{\kappa} \ \left[\frac{\alpha \ g}{P_r \ |\beta|}\right]^{\frac{1}{2}},$$

$$P^* = \hat{P} \ \frac{d^2}{\rho \ \kappa \ \nu} \ ,$$

$$b^* = \hat{b} \ \frac{B \ \varphi \ d^2 \ \cos\gamma}{\nu \ \kappa} \ ,$$

$$J^* = \hat{J} \ \frac{B \ d^3 \ \cos\gamma}{\rho \ \kappa \ \nu} \ .$$

After this non-dimensionalization, the field equations simplify to

$$\text{div } V = 0 \quad ,$$

$$\text{div } b = 0 \quad ,$$

$$\frac{\partial V_i}{\partial t} = - P_{,i} + \frac{G_j}{\cos\gamma} b_{i,j} + \Delta V_i + \frac{\epsilon \, G_i}{\cos\gamma} G_r \, G_j \, b_{r,j} + \surd(R) \, \theta \, \delta_{i3},$$

$$P_r \frac{\partial \theta}{\partial t} = \Delta\theta - H \surd(R) \, w \quad , \qquad (7.2)$$

$$J = \text{curl } [b + \epsilon \, (b \, . \, G) \, G] \quad ,$$

$$P_m \frac{\partial b_i}{\partial t} = \frac{Q}{\cos\gamma} G_j \, V_{i,j} - \text{curl } J_i$$

where the * superscript has been dropped but all variables are now non-dimensional and where the non-dimensional numbers R, P_r, P_m and ϵ are given by (6.8) and where

$$Q = (B \cos\gamma \, d/\nu)^2 \, \nu/\rho\eta \; .$$

Define

$$\text{(i)} \qquad \Psi = G_j \, b_j$$
$$\text{(ii)} \qquad f = b + \epsilon \, \Psi \, G \; . \qquad (7.3)$$

Thus equations $(7.2)_{(iii),(v)}$ become

$$\frac{\partial V_i}{\partial t} = - P_{,i} + \Delta V_i + \surd(R) \, \theta \, \delta_{i3} + \frac{G_j}{\cos\gamma} f_{i,j} \quad , \qquad (7.4)$$

$$J = \text{curl } f \; . \qquad (7.5)$$

Now apply the curl operator to equations $(7.2)_{(vi)}$ and (7.4) to obtain

$$P_m \frac{\partial}{\partial t} (\text{curl } b_i) = \frac{Q}{\cos\gamma} G_j \, \omega_{i,j} + \Delta J_i \qquad (7.6)$$

$$\frac{\partial \omega_i}{\partial t} = \Delta\omega_i + \surd(R) \, \text{curl}(0, \, 0, \, \theta) + \frac{G_k}{\cos\gamma} J_{i,k} \quad , \qquad (7.7)$$

where ω_i = curl V_i is the vorticity. Apply the curl operator again to equation (7.7) to obtain

$$\frac{\partial}{\partial t} \Delta V_i = \Delta^2 V_i + \surd(R)\ [\Delta(0,0,\theta) - \text{grad}(\theta_{,3})] - \frac{G_k}{\cos\gamma}\ (\text{curl } J_i)_{,k} \tag{7.8}$$

If we now take the third components of equations $(7.2)_{(vi)}$, (7.6), (7.7) and (7.8) we obtain

$$\begin{aligned} &P_m \frac{\partial b}{\partial t} = \frac{Q}{\cos\gamma} G_j\ w_{,j} - (\text{curl } J)_3\ , \\ &P_m \frac{\partial}{\partial t} (\text{curl } b)_3 = \frac{Q}{\cos\gamma} G_j\ \xi_{,j} + \Delta J \\ &\frac{\partial \xi}{\partial t} = \Delta\xi + \frac{G_k}{\cos\gamma} J_{,k}\ , \\ &\frac{\partial}{\partial t} \Delta w = \Delta^2 w + \surd(R)\ \left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} \right) - \frac{G_k}{\cos\gamma}\ (\text{curl } J)_{3,k}\ , \end{aligned} \tag{7.9}$$

where w, b, J and ξ are the third components of velocity, magnetic induction, current density and vorticity. If we use equation $(7.9)_{(i)}$ to obtain an expression for $(\text{curl } J)_3$, then equation $(7.9)_{(iv)}$ becomes

$$\frac{\partial}{\partial t} \Delta w = \Delta^2 w + \surd(R)\ \left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} \right) + \frac{G_k}{\cos\gamma}\ \left(P_m \frac{\partial b}{\partial t} - \frac{Q}{\cos\gamma} G_j\ w_{,j}\right)_{,k} \tag{7.10}$$

Apply the curl operator to equation (7.5) to obtain

$$\text{curl } J = \text{curl curl } f$$

i.e

$$\text{curl } J_i = \epsilon\ (G_j\ \Psi_{,j})_{,i} - \Delta b_i - \epsilon\ G_i\ \Delta\Psi\ . \tag{7.11}$$

The third component of this equation yields

$$(\text{curl } J)_3 = \epsilon\ G_j\ (\Psi_{,3})_{,j} - \Delta b - \epsilon \cos\gamma\ \Delta\Psi\ . \tag{7.12}$$

With the aid of equation (7.12) we can eliminate $(\text{curl } J)_3$ from $(7.9)_{(i)}$. Thus

$$P_m \frac{\partial b}{\partial t} = \frac{Q}{\cos\gamma} G_j \, w_{,j} + \Delta b + \epsilon \cos\gamma \, \Delta\Psi - \epsilon \, G_j \, (\Psi_{,3})_{,j} \quad . \tag{7.13}$$

From $(7.3)_{(ii)}$

$$\begin{aligned} \text{curl } \mathbf{b} &= \text{curl } \mathbf{f} - \epsilon \text{ curl } (\Psi \mathbf{G}) \\ &= \mathbf{J} - \epsilon \text{ curl } (\Psi \mathbf{G}) \\ &= \mathbf{J} - \epsilon \, (\Psi \text{ curl } \mathbf{G} - \mathbf{G} \times \nabla\Psi) \\ &= \mathbf{J} + \epsilon \, (\mathbf{G} \times \nabla\Psi) \end{aligned}$$

and the third component of this equation yields

$$(\text{curl } \mathbf{b})_3 = J + \epsilon \sin\gamma \, \Psi_{,2} \quad . \tag{7.14}$$

Using (7.14) we can eliminate $(\text{curl } \mathbf{b})_3$ from equation $(7.9)_{(ii)}$. Thus

$$\frac{\partial}{\partial t} (J + \epsilon \sin\gamma \, \Psi_{,2}) = \frac{Q}{\cos\gamma} G_j \, \xi_{,j} + \Delta J \quad . \tag{7.15}$$

Now we look for a solution of the form

$$\begin{aligned} w &= w(z) \, e^{i(nx+my)+\sigma t} \quad , \\ \theta &= \theta(z) \, e^{i(nx+my)+\sigma t} \quad , \\ \xi &= \xi(z) \, e^{i(nx+my)+\sigma t} \quad , \\ J &= J(z) \, e^{i(nx+my)+\sigma t} \quad , \\ b &= b(z) \, e^{i(nx+my)+\sigma t} \quad . \end{aligned}$$

Thus equations $(7.2)_{(iv)}$, $(7.9)_{(iii)}$, (7.10), (7.13) and (7.15) become

$$\sigma P_r \theta = L\theta - H \surd(R)\, w \quad ,$$

$$\sigma \xi = L\xi + (D + i c)J \quad ,$$

$$\sigma Lw = L^2 w - a^2 \surd(R)\, \theta - Q (D + i c)^2 w + \sigma P_m (D + i c)b$$

$$\sigma P_m b = Q (D + i c)w + Lb - \epsilon \cos\gamma\, (a^2 + i c D)\Psi, \qquad (7.16)$$

$$\sigma P_m [J + i m \epsilon \sin\gamma\, \Psi] = Q (D + i c)\xi + LJ \quad ,$$

where $a^2 = n^2 + m^2$, D is the operator d/dz, L is the operator $(D^2 - a^2)$ and $c = n \tan\gamma$. In fact we have five equations in six unknowns and so we need to write Ψ in terms of any combination of the rest of the variables. Since

$$J = \text{curl } f$$

$$= \text{curl } (b + \Psi G)$$

then the third component yields

$$J = i n b_2 - i m b_1 - \epsilon i m \sin\gamma\, \Psi . \qquad (7.17)$$

From the incompressibility constraint on b

$$b_2 = - (i n b_1 + Db) / im \quad .$$

Thus equation (7.17) becomes

$$J = - i n (i n b_1 + Db) / im - i m b_1 - i m \epsilon \sin\gamma\, \Psi .$$

Multiply both sides by $(i m \sin\gamma)$ to obtain

$$i m \sin\gamma\, J = a^2 \sin\gamma\, b_1 - i n \sin\gamma\, Db + \epsilon m^2 \sin^2\gamma\, \Psi. \qquad (7.18)$$

From definition $(7.3)_{(i)}$

$$\Psi = \sin\gamma \, b_1 + \cos\gamma \, b \ .$$

Thus equation (7.18) becomes

$$\Psi = \frac{i\ m \sin\gamma\ J + a^2 \cos\gamma\ b + i\ n \sin\gamma\ Db}{a^2 + \epsilon\ m^2 \sin^2\gamma} \quad .$$

Substitute for Ψ in equations $(7.16)_{(iv),(v)}$ to obtain

$$\begin{aligned} \sigma P_m b &= Lb + Q (D + i c)w - \epsilon \cos\gamma (a^2 + i c D) (i m \sin\gamma J \\ &\quad + a^2 \cos\gamma\ b + i n \sin\gamma\ Db)/ (a^2 + \epsilon m^2 \sin^2\gamma) \\ \sigma P_m [a^2 J &+ i m \epsilon \sin\gamma \cos\gamma (a^2 + i c D)b] \\ &= [LJ + Q(D + i c)\xi] (a^2 + \epsilon m^2 \sin^2\gamma) . \end{aligned} \qquad (7.19)$$

Equations $(7.16)_{(i),(ii),(iii)}$ together with (7.19) constitute a twelvth order eigenvalue problem whose boundary conditions are similar to those given in chapter six. The reason that we have a twelvth order eigenvalue problem in this case is that the z-component of the vorticity and the current density do not vanish identically as in the vertical magnetic field case. The twelvth order eigenvalue problem is solved numerically using expansions in Chebyshev polynomials and the critical Rayleigh numbers are obtained by minimizing over n and m for various assigned values of Q, P_r, P_m, ϵ and γ. It has been noticed that by minimizing over the wave numbers n and m, the critical Rayleigh numbers obtained always corresponds to $n \longrightarrow 0$. Thus when H and g act in different directions, instability when it first sets in appears as longitudinal rolls. Chandrasekhar [13] explained this conclusion and pointed out that there are several patterns of motion. The longitudinal rolls which corresponds to the case when $n = 0$ and the transverse rolls which is the most difficult pattern to excite and which corresponds to the case when $m = 0$. The difference between the Rayleigh numbers in these cases tends to zero as the inclination of the magnetic field to the direction of the vertical tends to zero.

In this problem we could not obtain an exact solution for the cases of stationary convection and overstability when both boundaries are free as in the vertical magnetic field problem and we had to solve the problem numerically. In order for overstability to be possible, a similar condition on P_r, P_m and ϵ must be satisfied as in the vertical magnetic field case.

When instability sets in as longitudinal rolls, n must be zero and equations (7.16)(i),(ii),(iii) and (7.19) become

$$\sigma P_r \theta = L\theta - H \surd(R) w ,$$

$$\sigma \xi = L\xi + DJ ,$$

$$\sigma Lw = L^2 w - m^2 \surd(R) \theta - Q D^2 w + \sigma P_m Db$$

$$\sigma P_m b = Lb + Q Dw - \epsilon \cos\gamma (i m \sin\gamma J + m^2 \cos\gamma b)/(1 + \epsilon \sin^2\gamma) \qquad (7.20)$$

$$\sigma P_m (J + i m \epsilon \sin\gamma \cos\gamma b) = (LJ + Q D\xi)(1 + \epsilon \sin^2\gamma)$$

where L here is the operator $D^2 - m^2$. When instability sets in as transverse rolls m must be zero and equations (7.16)(i),(ii),(iii) and (7.19) become

$$\sigma P_r \theta = L\theta - H \surd(R) w ,$$

$$\sigma Lw = L^2 w - n^2 \surd(R) \theta - Q (D + i c)^2 w + \sigma P_m (D + i c) b , \qquad (7.21)$$

$$\sigma P_m b = Lb + Q (D + i c) w - \frac{\epsilon \cos^2\gamma}{n^2} (n^2 + i c D)^2 b$$

where L here is the operator $D^2 - n^2$. In this case we have an eighth order eigenvalue problem because the z-component of the vorticity and the current density vanish identically.

Differentiation of the Rayleigh number (Heating from below)

Case (1) (Longitudinal rolls)

Here we shall consider the variation of the Rayleigh number as a function of the magnetic parameter Q and the wave number m for the stationary convection case. i.e. when $\sigma = 0$. When instability sets in as longitudinal rolls n must be zero and (7.20) are the relevant equations. When $\sigma = 0$ in (7.20),

$$\begin{aligned} &L\theta + \surd(R)\ w = 0, \\ &L^2 w - m^2\ \surd(R)\ \theta - Q\ D^2 w = 0. \end{aligned} \qquad (7.22)$$

From $(7.22)_{(i)}$

$$D^2\theta_1 - m_1^2\ \theta_1 + \surd(R_1)\ w_1 = 0.$$

Multiply by θ_2 and integrate to obtain

$$\int D\theta_2\ D\theta_1 + m_1^2 \int \theta_2\ \theta_1 = \surd(R_1) \int \theta_2\ w_1. \qquad (7.23)$$

Similarly,

$$\int D\theta_1\ D\theta_2 + m_2^2 \int \theta_1\ \theta_2 = \surd(R_2) \int \theta_1\ w_2. \qquad (7.24)$$

From $(7.22)_{(ii)}$

$$D^4 w_1 - (2\ m_1^2 + Q_1)\ D^2 w_1 + m_1^4\ w_1 - m_1^2\ \surd(R_1)\ \theta_1 = 0.$$

Multiply by w_2 and integrate to obtain

$$\int D^2 w_2\ D^2 w_1 + (2\ m_1^2 + Q_1)\int Dw_2\ Dw_1 + m_1^4 \int w_2\ w_1$$

$$- m_1^2\ \surd(R_1) \int w_2\ \theta_1 = 0. \qquad (7.25)$$

Similarly,

$$\int D^2w_1\ D^2w_2 + (2\ m_2^2 + Q_2)\int Dw_1\ Dw_2 + m_2^4 \int w_1\ w_2$$

$$- m_2^2\ \surd(R_2) \int w_1\ \theta_2 = 0. \qquad (7.26)$$

By using equation (7.24) we can eliminate $\int \theta_1 w_2$ from (7.25). Thus

$$\int D^2w_2\ D^2w_1 + (2\ m_1^2 + Q_1)\int Dw_2\ Dw_1 + m_1^2 \int w_2\ w_1$$

$$- m_1^2\ \frac{\surd(R_1)}{\surd(R_2)} \left[\int D\theta_1\ D\theta_2 + m_2^2 \int \theta_1\ \theta_2 \right] = 0. \qquad (7.27)$$

Similarly, from (7.23) and (7.26) we can eliminate $\int \theta_2 w_1$. Thus

$$\int D^2w_1\ D^2w_2 + (2\ m_2^2 + Q_2)\int Dw_1\ Dw_2 + m_2^2 \int w_1\ w_2$$

$$- m_2^2\ \frac{\surd(R_2)}{\surd(R_1)} \left[\int D\theta_2\ D\theta_1 + m_1^2 \int \theta_2\ \theta_1 \right] = 0. \qquad (7.28)$$

Subtract (7.28) from (7.27) to obtain

$$(2m_1^2 - 2m_2^2 + Q_1 - Q_2) \int Dw_2\ Dw_1 + (m_1^4 - m_2^4) \int w_2\ w_1$$

$$- \left[m_1^2\ \frac{\surd(R_1)}{\surd(R_2)} - m_2^2\ \frac{\surd(R_2)}{\surd(R_1)} \right] \int D\theta_1\ D\theta_2$$

$$- m_1^2\ m_2^2 \left[\frac{\surd(R_1)}{\surd(R_2)} - \frac{\surd(R_2)}{\surd(R_1)} \right] \int \theta_1\ \theta_2 = 0. \qquad (7.29)$$

Suppose that $w_1 \longrightarrow w_2$, $\theta_1 \longrightarrow \theta_2$, $m_1 \longrightarrow m_2$, $Q_1 \longrightarrow Q_2$ and $R_1 \longrightarrow R_2$. Divide (7.29) by $s_1 - s_2$ and take the limit as $s_1 \longrightarrow s_2$. Thus (7.29) becomes

$$4\ m \frac{dm}{ds} \left[\int (Dw)^2 + m^2 \int w^2 - \frac{1}{2} \int (D\theta)^2 \right] + \frac{dQ}{ds} \int (Dw)^2$$

$$= \frac{m^2}{R}\ \frac{dR}{ds} \left[\int (D\theta)^2 + m^2 \int \theta^2 \right]. \qquad (7.30)$$

Since

$$R = R(m,Q)$$

then

$$\frac{dR}{ds} = \frac{\partial R}{\partial m}\frac{dm}{ds} + \frac{\partial R}{\partial Q}\frac{dQ}{ds} .$$

Thus (7.30) becomes

$$4\, m \frac{dm}{ds} \left[\int (Dw)^2 + m^2 \int w^2 - \frac{1}{2} \int (D\theta)^2 \right] + \frac{dQ}{ds} \int (Dw)^2$$

$$= \frac{m^2}{R} \left[\frac{\partial R}{\partial m}\frac{dm}{ds} + \frac{\partial R}{\partial Q}\frac{dQ}{ds} \right] \left[\int (D\theta)^2 + m^2 \int \theta^2 \right]. \qquad (7.31)$$

From (7.31), we can show that

$$\frac{\partial R}{\partial m} = \frac{4\, R}{m \left[(D\theta)^2 + m^2 \int \theta^2 \right]} \left[\int (Dw)^2 + m^2 \int w^2 - \frac{1}{2} \int (D\theta)^2 \right]$$

so the minimum Rayleigh number occurs when

$$\int (Dw)^2 + m^2 \int w^2 = \frac{1}{2} \int (D\theta)^2 .$$

Also from (7.31), we can show that

$$\frac{\partial R}{\partial Q} = \frac{R \int (Dw)^2}{m^2 \left[(D\theta)^2 + m^2 \int \theta^2 \right]} > 0 .$$

Thus the Rayleigh number is an increasing function of the magnetic parameter Q.

Case (2) (Transverse rolls)

Here we shall consider the variation of the Rayleigh number as a function of the magnetic parameter Q, the wave number n and the variable c for the stationary convection case. i.e. when $\sigma = 0$. For the transverse rolls m must be zero and the relative equations are equations (7.21). When $\sigma = 0$, these equations become

$$L\theta + \surd(R)\, w = 0,$$
$$L^2 w - n^2 \surd(R)\, \theta - Q\,(D + i\,c)^2\, w = 0. \tag{7.32}$$

From $(7.32)_{(i)}$

$$D^2\theta_1 - n_1^2\, \theta_1 + \surd(R_1)\, w_1 = 0.$$

Multiply by the conjugate of θ_2 and integrate to obtain

$$\int D\bar{\theta}_2\, D\theta_1 + n_1^2 \int \bar{\theta}_2\, \theta_1 = \surd(R_1) \int \bar{\theta}_2\, w_1. \tag{7.33}$$

Similarly,

$$\int D\bar{\theta}_1\, D\theta_2 + n_2^2 \int \bar{\theta}_1\, \theta_2 = \surd(R_2) \int \bar{\theta}_1\, w_2. \tag{7.34}$$

From $(7.32)_{(ii)}$

$$D^4 w_1 - (2\, n_1^2 + Q_1)\, D^2 w_1 - 2\, i\, c_1\, Q_1\, D w_1 + (n_1^4 + c_1^2\, Q_1) w_1$$
$$- n_1^2\, \surd(R_1)\, \theta_1 = 0.$$

Multiply by the conjugate of w_2 and integrate to obtain

$$\int D^2\bar{w}_2\, D^2 w_1 + (2\, n_1^2 + Q_1) \int D\bar{w}_2\, D w_1 - 2\, i\, c_1\, Q_1 \int \bar{w}_2\, D w_1$$
$$+ (n_1^2 + c_1^2\, Q_1) \int \bar{w}_2\, w_1 - n_1^2\, \surd(R_1) \int \bar{w}_2\, \theta_1 = 0. \tag{7.35}$$

Similarly,

$$\int D^2\bar{w}_1\, D^2 w_2 + (2\, n_2^2 + Q_2)\int D\bar{w}_1\, Dw_2 - 2\, i\, c_2\, Q_2 \int \bar{w}_1\, Dw_2$$

$$+ (n_2^2 + c_2^2\, Q_2) \int \bar{w}_1\, w_2 - n_2^2\, \surd(R_2) \int \bar{w}_1\, \theta_2 = 0. \tag{7.36}$$

Let us now take the conjugate of equations (7.33) and (7.34). Thus

$$\int D\theta_2\, D\bar{\theta}_1 + n_1^2 \int \theta_2\, \bar{\theta}_1 = \surd(R_1) \int \theta_2\, \bar{w}_1 \quad , \tag{7.37}$$

$$\int D\theta_1\, D\bar{\theta}_2 + n_2^2 \int \theta_1\, \bar{\theta}_2 = \surd(R_2) \int \theta_1\, \bar{w}_2. \tag{7.38}$$

Using (7.38), we can eliminate $\int \theta_1 \bar{w}_2$ from (7.35) to obtain

$$\int D^2\bar{w}_2\, D^2 w_1 + (2\, n_1^2 + Q_1) \int D\bar{w}_2\, Dw_1 - 2\, i\, c_1\, Q_1 \int \bar{w}_2\, Dw_1$$

$$+ (n_1^4 + c_1^2\, Q_1) \int \bar{w}_2\, w_1 - n_1^2\, \frac{\surd(R_1)}{\surd(R_2)} \left[\int D\theta_1\, D\bar{\theta}_2 + n_2^2 \int \theta_1\, \bar{\theta}_2 \right] = 0 \tag{7.39}$$

Similarly, from (7.36) and (7.37) we can eliminate $\int \theta_2 \bar{w}_1$. Thus

$$\int D^2\bar{w}_1\, D^2 w_2 + (2\, n_2^2 + Q_2) \int D\bar{w}_1\, Dw_2 - 2\, i\, c_2\, Q_2 \int \bar{w}_1\, Dw_2$$

$$+ (n_2^4 + c_2^2\, Q_2) \int \bar{w}_1\, w_2 - n_2^2\, \frac{\surd(R_2)}{\surd(R_1)} \left[\int D\theta_2\, D\bar{\theta}_1 + n_1^2 \int \theta_2\, \bar{\theta}_1 \right] = 0 \tag{7.40}$$

Let us take the conjugate of equation (7.40). Thus

$$\int D^2 w_1\, D^2\bar{w}_2 + (2\, n_2^2 + Q_2) \int Dw_1\, D\bar{w}_2 + 2\, i\, c_2\, Q_2 \int w_1\, D\bar{w}_2$$

$$+ (n_2^4 + c_2^2\, Q_2) \int w_1\, \bar{w}_2 - n_2^2\, \frac{\surd(R_2)}{\surd(R_1)} \left[\int D\bar{\theta}_2\, D\theta_1 + n_1^2 \int \bar{\theta}_2\, \theta_1 \right] = 0 \tag{7.41}$$

Subtract (7.41) from (7.39) to obtain

$$(2n_1^2 - 2n_2^2 + Q_1 - Q_2) \int Dw_1 \, D\bar{w}_2 - 2\, i\, (c_1 Q_1 - c_2 Q_2) \int \bar{w}_2 \, Dw_1$$

$$+ (n_1^4 - n_2^4 + c_1^2 Q_1 - c_2^2 Q_2) \int w_1 \, \bar{w}_2$$

$$- \left[n_1^2 \frac{\surd(R_1)}{\surd(R_2)} - n_2^2 \frac{\surd(R_2)}{\surd(R_1)} \right] \int D\theta_1 \, D\bar{\theta}_2$$

$$- n_1^2 n_2^2 \left[\frac{\surd(R_1)}{\surd(R_2)} - \frac{\surd(R_2)}{\surd(R_1)} \right] \int \theta_1 \, \bar{\theta}_2 = 0. \qquad (7.42)$$

Suppose that $w_1 \longrightarrow w_2$, $\theta_1 \longrightarrow \theta_2$, $n_1 \longrightarrow n_2$, $Q_1 \longrightarrow Q_2$ $c_1 \longrightarrow c_2$ and $R_1 \longrightarrow R_2$. Divide (7.42) by $s_1 - s_2$ and take the limit as $s_1 \longrightarrow s_2$. Thus (7.42) becomes

$$4\, n \frac{dn}{ds} \left[\int (Dw)^2 + n^2 \int w^2 - \frac{1}{2} \int (D\theta)^2 \right]$$

$$+ \frac{dQ}{ds} \left[\int (Dw)^2 - 2\, i\, c \int \bar{w}\, Dw + c^2 \int w^2 \right]$$

$$+ 2\, Q \frac{dc}{ds} \left[c \int w^2 - i \int \bar{w}\, Dw \right]$$

$$= \frac{n^2}{R} \frac{dR}{ds} \left[\int (D\theta)^2 + n^2 \int \theta^2 \right]. \qquad (7.43)$$

Since

$$R = R(n, Q, c)$$

then

$$\frac{dR}{ds} = \frac{\partial R}{\partial n}\frac{dn}{ds} + \frac{\partial R}{\partial Q}\frac{dQ}{ds} + \frac{\partial R}{\partial c}\frac{dc}{ds}.$$

Thus we can show that

$$\frac{\partial R}{\partial n} = \frac{4\, R}{n \left[(D\theta)^2 + n^2 \int \theta^2 \right]} \left[\int (Dw)^2 + n^2 \int w^2 - \frac{1}{2} \int (D\theta)^2 \right]$$

so the minimum Rayleigh number occurs when

$$\int (Dw)^2 + n^2 \int w^2 = \frac{1}{2} \int (D\theta)^2 ,$$

$$\frac{\partial R}{\partial Q} = \frac{R\left[\int (Dw)^2 - 2\,i\,c\int \bar{w}\,Dw + c^2\int w^2\right]}{n^2\left[(D\theta)^2 + n^2\int \theta^2\right]}$$

$$= \frac{R\left[\int \left|Dw + i\,c\,w\right|^2\right]}{n^2\left[(D\theta)^2 + n^2\int \theta^2\right]} > 0 \,.$$

Thus the Rayleigh number is an increasing function of the magnetic parameter Q. Also

$$\frac{\partial R}{\partial c} = \frac{2\,Q\,R\left[c\int w^2 - i\int \bar{w}\,Dw\right]}{n^2\left[(D\theta)^2 + n^2\int \theta^2\right]} \,.$$

Numerical results

When instability sets in as longitudinal rolls, we obtained the twelvth order eigenvalue problem (7.20) which was solved using the method of expansions in Chebyshev polynomials. The relation between the critical Rayleigh number R and the magnetic parameter Q for the cases of stationary convection and overstability is shown in figures (1), (2) and (3) when both boundaries are free. Various values of the parameter ϵ and the angle γ have been applied. The parameter ϵ has been given the values 0, 0.5 and 1 in figures (1), (2) and (3) respectively. In each figure the angle γ has been given the values 15^0, 30^0, 45^0, 60^0 and 75^0. When $\epsilon = 0$, changing the angle γ has no effect in the critical Rayleigh number for the cases of stationary convection and overstability and we obtain precisely the same results of the vertical magnetic field case (see figure 1) When $\epsilon = 0.5$ and 1, increasing the angle γ produces a decrease in the critical Rayleigh number for the overstability case only (see figure 2 and 3). i.e. the non-linearity has no effect in the stationary convection case but it has a stronge effect in the overstability case and this becomes clear when the angle γ varied. The numerical results are listed in tables (1)-(11).

A comparison was made between the cases when instability sets in as longitudinal rolls and when it sets in as transverse rolls. The variation of the critical Rayleigh number as a function of the angle γ is shown in figure (4) and (5) for the cases of stationary convection and overstability respectively when $Q = 100$, $P_r = 1$, $P_m = 4$ and $\epsilon = 1$. For the stationary convection case, changing the angle γ has no effect when instability sets in as longitudinal rolls but it has a stronge effect when instability sets in as transverse rolls (see figure 4). In fact, when instability sets in as longitudinal rolls, R = 2653.71 when Q = 100 and this value does not depend on the angle γ. For overstability case, changing the angle γ has a significant effect for both cases of longitudinal rolls and transverse rolls (see figure 5). The numerical results for figures 4 and 5 are listed in tables (12)-(14). The Fortran77 code used to obtian the numerical results is listed in Appendix IV.

Table (1)

The relation between R and Q for the overstability case. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0$.

Q	m	R	Q	m	R
18	2.305	1108.57	500	3.27	2753.19
50	2.432	1244.65	2500	4.39	7836.18
100	2.59	1442.3			

Table (2)

The relation between R and Q for the overstability case when $\gamma = 15^0$. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0.5$.

Q	m	R	Q	m	R
23	2.261	1215.88	500	3.172	3309.1
50	2.361	1358.15	2500	4.251	10273.1
100	2.513	1606.75			

Table (3)

The relation between R and Q for the overstability case when $\gamma = 30^0$. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0.5$.

Q	m	R	Q	m	R
21	2.27	1183.44	500	3.198	3167.83
50	2.38	1329.63	2500	4.288	9633.44
100	2.533	1565.85			

Table (4)

The relation between R and Q for the overstability case when $\gamma = 45^0$. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0.5$.

Q	m	R	Q	m	R
20	2.285	1154.03	500	3.22	3004.58
50	2.4	1296.46	2500	4.329	8912.22
100	2.554	1517.87			

Table (5)

The relation between R and Q for the overstability case when $\gamma = 60^0$. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0.5$.

Q	m	R	Q	m	R
19	2.297	1129.46	500	3.25	2868.37
50	2.417	1268.51	2500	4.36	8324.21
100	2.574	1477.18			

Table (6)

The relation between R and Q for the overstability case when $\gamma = 75^0$. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0.5$.

Q	m	R	Q	m	R
18	2.302	1112.66	500	3.267	2782.27
50	2.428	1250.69	2500	4.384	7958.59
100	2.586	1451.15			

Table (7)

The relation between R and Q for the overstability case when $\gamma = 15^0$. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 1$.

Q	m	R	Q	m	R
29	2.228	1334.83	500	3.077	3862.55
50	2.301	1466.85	2500	4.112	12746.0
100	2.448	1766.5			

Table (8)

The relation between R and Q for the overstability case when $\gamma = 30^0$. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 1$.

Q	m	R	Q	m	R
25	2.252	1256.63	500	3.136	3523.8
50	2.342	1400.35	2500	4.204	11177.8
100	2.49	1670.66			

Table (9)

The relation between R and Q for the overstability case when $\gamma = 45^0$. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 1$.

Q	m	R	Q	m	R
22	2.277	1189.25	500	3.193	3185.76
50	2.381	1333.07	2500	4.288	9675.28
100	2.53	1572.43			

Table (10)

The relation between R and Q for the overstability case when $\gamma = 60^0$. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 1$.

Q	m	R	Q	m	R
20	2.294	1143.79	500	3.236	2939.82
50	2.409	1283.18	2500	4.345	8618.47
100	2.565	1499.1			

Table (11)

The relation between R and Q for the overstability case when $\gamma = 75^0$. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 1$.

Q	m	R	Q	m	R
22	2.301	1114.79	500	3.262	2798.63
50	2.426	1254.09	2500	4.381	8025.09
100	2.584	1456.22			

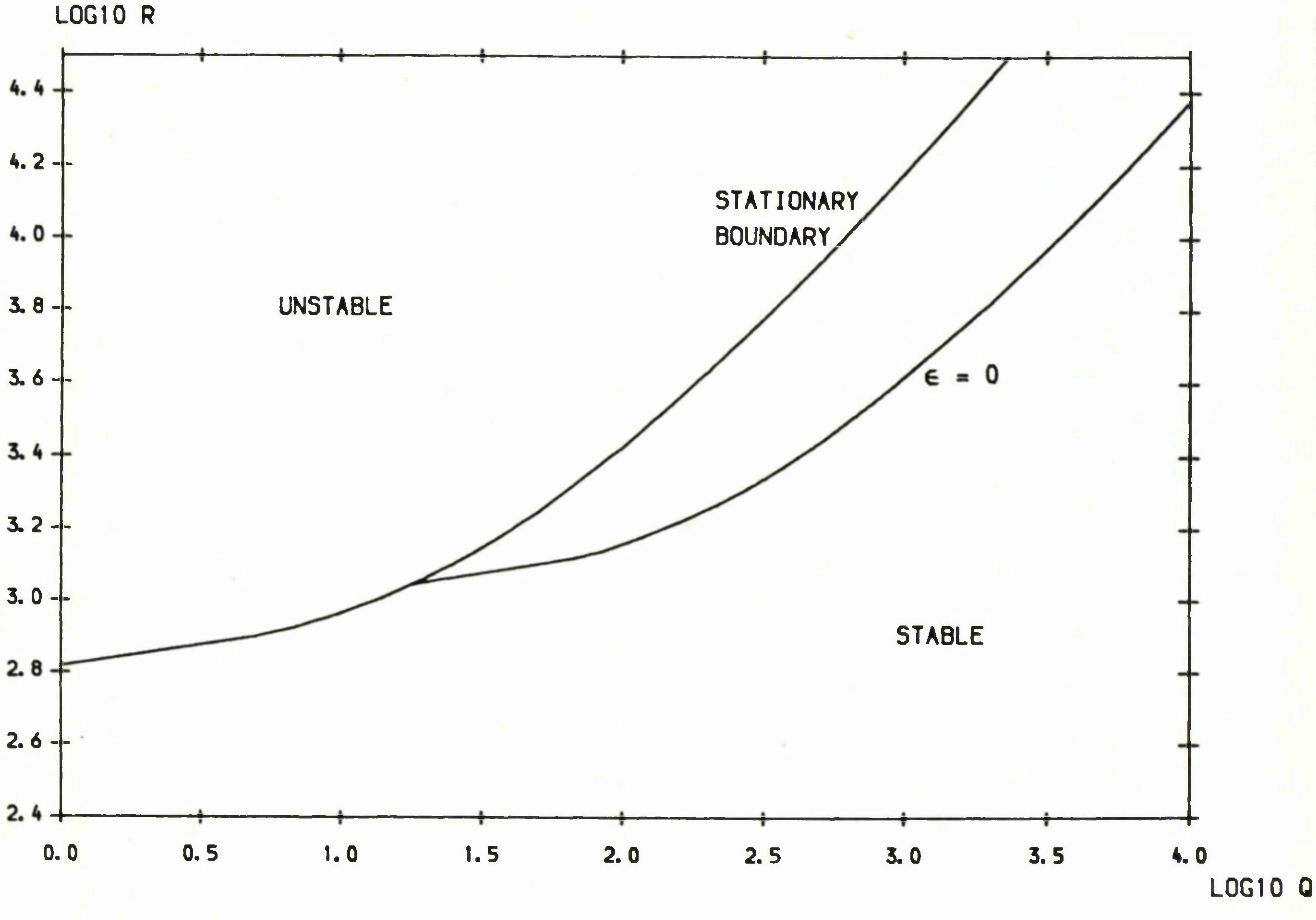

Figure 1.

This figure shows the relation between the critical Rayleigh number R and the magnetic parameter Q for the cases of stationary convection and overstability when both boundaries are free. Here $P_r = 1$, $P_m = 4$ and $\epsilon = 0$.

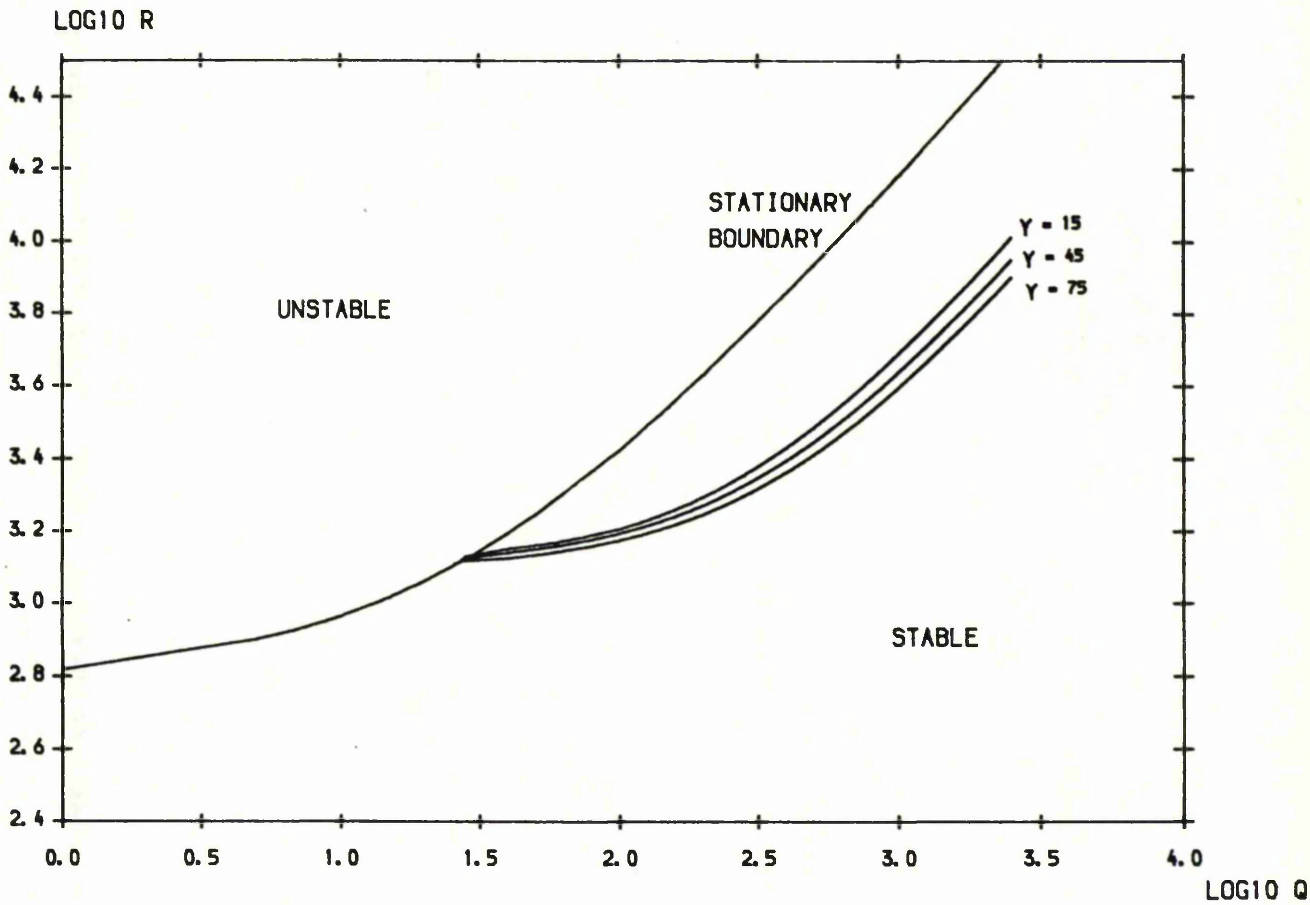

Figure 2.

This figure shows the relation between the critical Rayleigh number R and the magnetic parameter Q for the cases of stationary convection and overstability. Here $P_r = 1$, $P_m = 4$, $\epsilon = 0.5$ and γ varies. It shows that increasing the angle γ produces a decrease in the critical Rayleigh number for the overstability case.

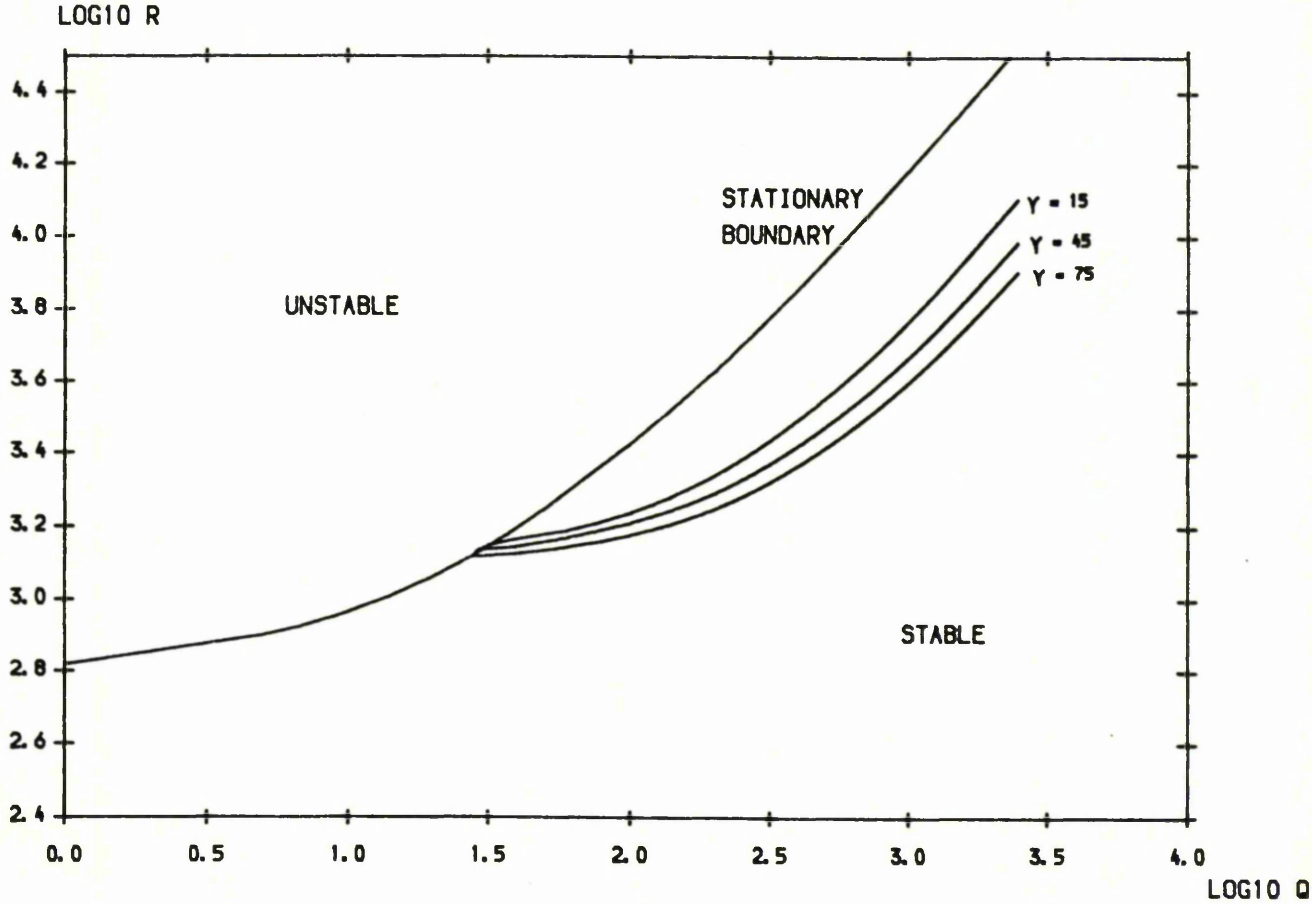

Figure 3.

This figure shows the relation between the critical Rayleigh number R and the magnetic parameter Q for the cases of stationary convection and overstability. Here $P_r = 1$, $P_m = 4$, $\epsilon = 1$ and γ varies. It shows that increasing the angle γ produces a decrease in the critical Rayleigh number for the overstability case.

Table (12)

The relation between R and γ for the stationary convection case when instability sets in as transverse rolls. Here $P_r = 1$, $P_m = 4$, $\epsilon = 1$ and $Q = 100$.

γ	n	R	γ	n	R
15	3.642	2741.37	60	2.526	5811.41
30	3.453	3051.65	75	1.631	15438.2
45	3.1	3795.			

Table (13)

The relation between R and γ for the overstability case when instability sets in as transverse rolls. Here $P_r = 1$, $P_m = 4$, $\epsilon = 1$ and $Q = 100$.

γ	n	R	γ	n	R
15	2.475	1788.47	60	2.489	1980.43
30	2.591	1750.25	75	1.728	3313.93
45	2.669	1758.5			

Table (14)

The relation between R and γ for the overstability case when instability sets in as longitudinal rolls. Here $P_r = 1$, $P_m = 4$ $\epsilon = 1$ and $Q = 100$.

γ	n	R	γ	n	R
15	2.448	1766.5	60	2.565	1499.1
30	2.49	1670.66	75	2.584	1456.22
45	2.53	1572.43			

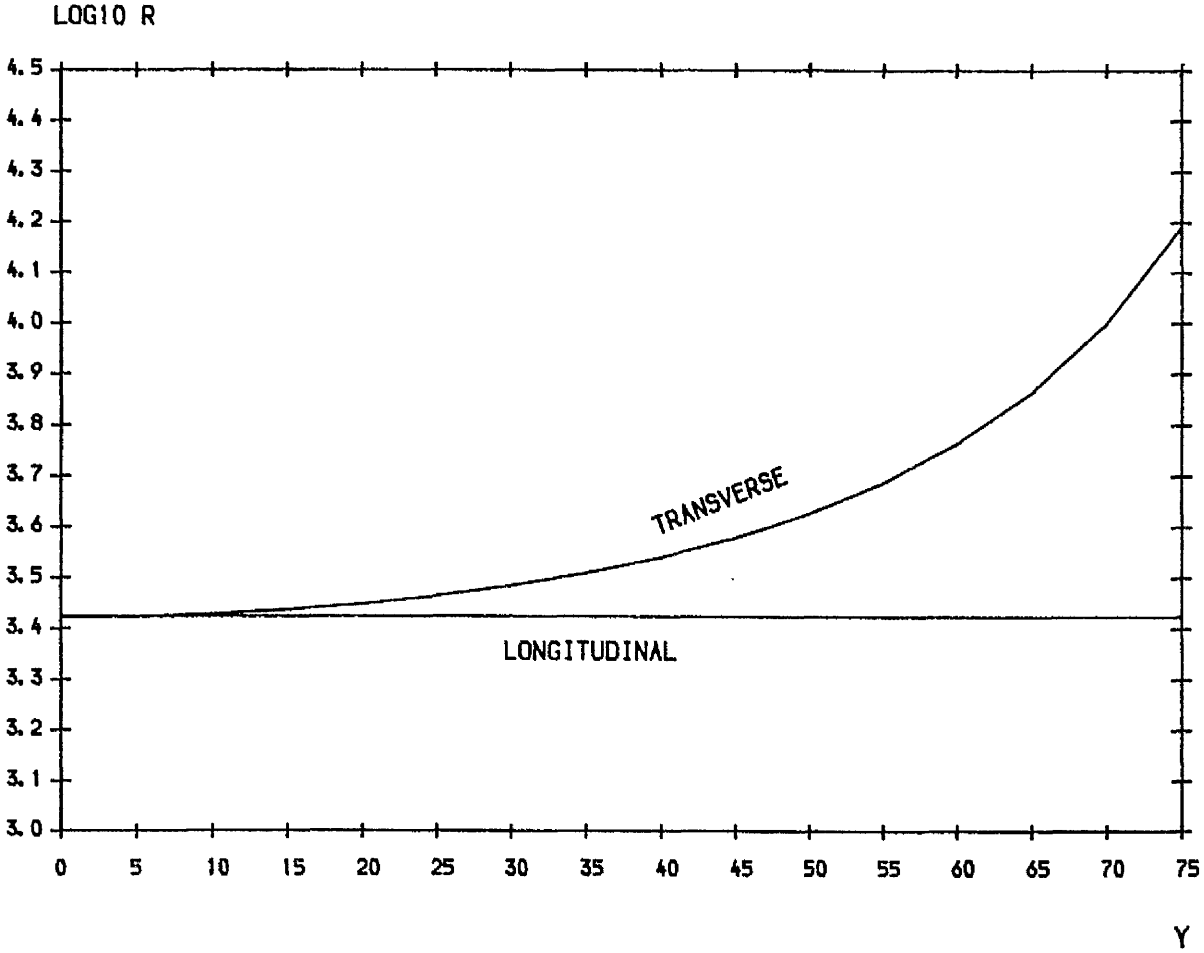

Figure 4.

This figure shows the relation between the critical Rayleigh number R and the angle γ in the stationary convection case for the cases when instability sets in as longitudinal rolls and when it sets in as transverse rolls. It shows that changing the angle γ has an effect only when instability sets as transverse rolls. Here $P_r = 1$, $P_m = 4$, $\epsilon = 1$ and $Q = 100$.

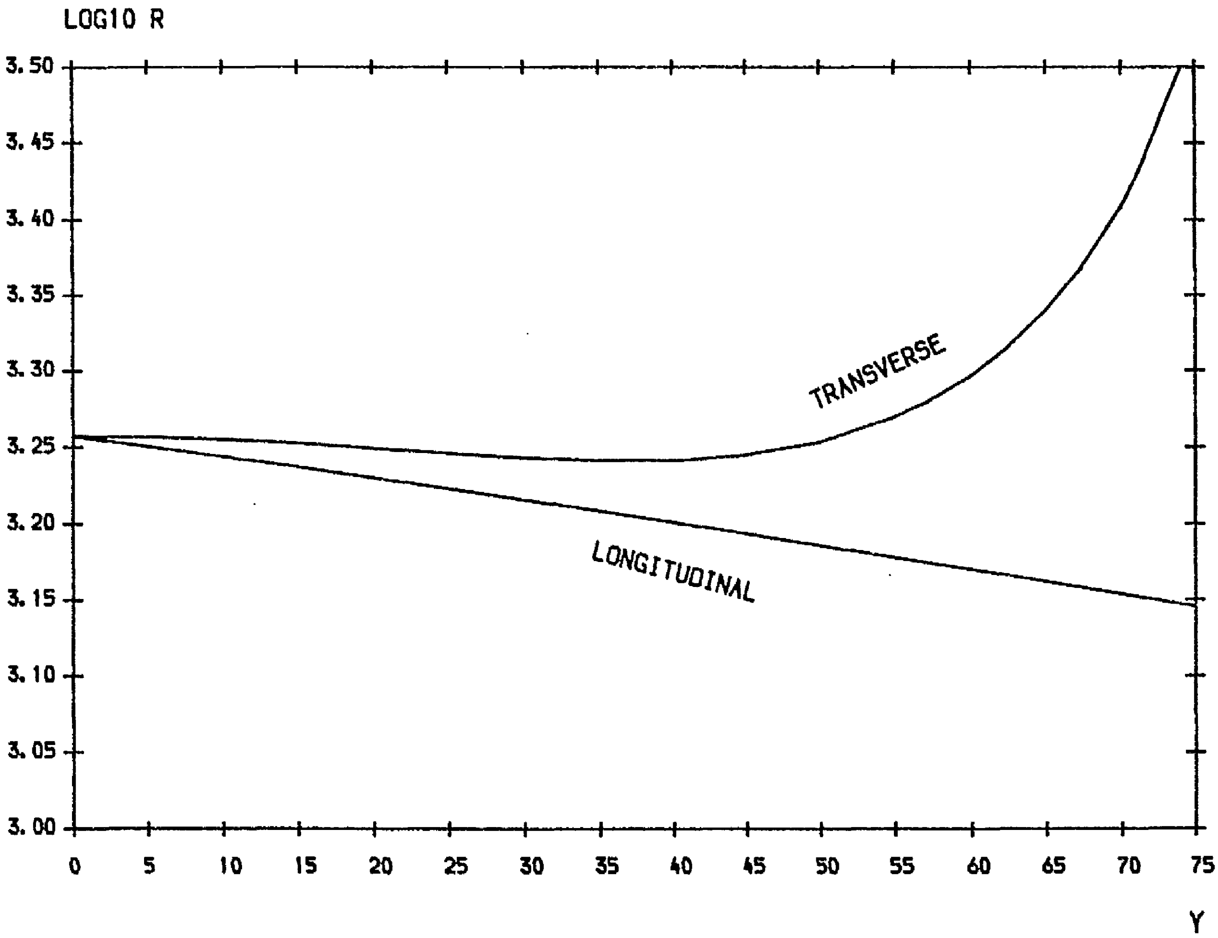

Figure 5.

This figure shows the relation between the critical Rayleigh number R and the angle γ in the case of overstability for the cases when instability sets in as longitudinal rolls and when it sets in as transverse rolls. It shows that changing the angle γ has an effect in both cases. Here $P_r = 1$, $P_m = 4$, $\epsilon = 1$ and $Q = 100$.

Chapter Eight

Benard Convection in a Non-Linear MHD Fluid under the Influence of both Magnetic Field and Rotation

Mathematical Formulation

In chapter six we have discussed the Benard convection under the influence of a vertical magnetic field in a non-linear magnetohydrodynamic fluid. Here we shall discuss the jointly effect of both magnetic field and rotation to the same fluid. The presence of rotation brings two more terms to the equation of motion. These terms are $2\boldsymbol{\Omega} \times \mathbf{V}$ which represents the Coriolis acceleration and $-\frac{1}{2}\frac{\partial}{\partial x_i}(|\boldsymbol{\Omega} \times \mathbf{r}|^2)$ which represents the centrifugal force, where $\boldsymbol{\Omega}$ is the angular velocity and $\mathbf{r} = (x,y,z)$. If we now make the Boussinesq approximation then the governing field equations become

$$\begin{aligned}
&\text{Div } \mathbf{V} = 0 \\
&\frac{DV_i}{Dt} = -\left(\frac{P}{\rho}\right)_{,i} + B_k\,(\varphi\, B_i)_{,k} + \nu\,\Delta V_i - g\,(1 - \alpha\,\theta)\,\delta_{i3} \\
&\qquad + 2\,e_{ijk}\,V_j\,\Omega_k + \frac{1}{2}\frac{\partial}{\partial x_i}(|\Omega \times \mathbf{r}|^2), \\
&\frac{D\theta}{Dt} = \kappa\,\Delta\theta
\end{aligned} \tag{8.1}$$

together with the Maxwell relations

$$\begin{aligned}
&\text{div } \mathbf{B} = B_{i,i} = 0\,, \\
&(\text{curl } \mathbf{H})_i = e_{ijk}\,H_{k,j} = J_i\,, \\
&(\text{curl } \mathbf{E})_i = e_{ijk}\,E_{k,j} = -\frac{\partial B_i}{\partial t}\,.
\end{aligned} \tag{8.2}$$

Following the same procedures as in chapter six, the resulting perturbation equations are

$$\hat{V}_{i,i} = 0 \quad ,$$

$$\frac{\partial \hat{V}_i}{\partial t} = - (\frac{P}{\rho})_{,i} + B^2 \varphi_B \hat{b}_{3,3} \delta_{i3} + B \varphi \hat{b}_{i,3} + \nu \Delta \hat{V}_i + \alpha g \hat{\theta} \delta_{i3}$$
$$+ 2 e_{ijk} V_j \Omega_k$$

$$\frac{\partial \hat{\theta}}{\partial t} - \beta \hat{V}_3 = \kappa \Delta \hat{\theta}, \qquad (8.3)$$

$$\hat{b}_{i,i} = 0 \quad ,$$

$$\hat{J}_i = e_{ijk} (\rho \varphi \hat{b}_k + \rho B \varphi_B \hat{b}_3 \delta_{k3})_{,j} \quad ,$$

$$\frac{\partial \hat{b}_i}{\partial t} = B \hat{V}_{i,3} - \eta e_{ijk} \hat{J}_{k,j} \quad .$$

where $\hat{V}$, $\hat{\theta}$, $\hat{P}$, $\hat{b}$ and $\hat{J}$ are the linear perturbation of velocity, temperature, pressure, magnetic induction and current density. At this stage we introduce the dimensionless variables x^*, t^*, V^*, θ^*, P^*, b^* and J^* where

$$x^* = x / d \ ,$$

$$t^* = t \nu / d^2 \ ,$$

$$V^* = \hat{V} d / \kappa \quad ,$$

$$\theta^* = \hat{\theta} \frac{d}{\kappa} \left[\frac{\alpha g}{P_r |\beta|}\right]^{\frac{1}{2}}$$

$$P^* = \hat{P} \frac{d^2}{\rho \kappa \nu} \quad ,$$

$$b^* = \hat{b} \frac{B \varphi d^2}{\kappa \nu} \quad ,$$

$$J^* = \hat{J} \frac{B d^3}{\rho \kappa \nu} \ .$$

After this non-dimensionalization, the field equations simplify to

$$V_{i,i} = 0 \quad ,$$

$$\frac{\partial V_i}{\partial t} = - P_{,i} + \Delta V_i + \sqrt{R}\, \theta\, \delta_{i3} + b_{i,3} + \epsilon\, b_{3,3}\, \delta_{i3}$$
$$+ \sqrt{T}\, e_{ijk}\, V_j\, \delta_{k3} \quad ,$$

$$P_r \frac{\partial \theta}{\partial t} + H \sqrt{R}\, w = \Delta\theta \quad , \qquad (8.4)$$

$$b_{i,i} = 0 \quad ,$$

$$J_i = e_{ijk}\, (b_k + \epsilon\, b_3\, \delta_{k3})_{,j} \quad ,$$

$$P_m \frac{\partial b_i}{\partial t} = Q\, V_{i,3} - e_{ijk}\, J_{k,j}$$

where the * superscript has been dropped but all variables are now non-dimensional and where the non-dimensional numbers Q, R, P_r, P_m and ϵ are given by (6.8) and where

$$T = (2\, \Omega\, d^2/\nu)^2 \ . \qquad (8.5)$$

From equations $(8.4)_{(v),(vi)}$ we have

$$P_m \frac{\partial b_i}{\partial t} = Q\, V_{i,3} - e_{ijk}\, [e_{krs}\, (b_s + \epsilon\, b_3\, \delta_{s3})_{,r}]_{,j}$$
$$= Q\, V_{i,3} - (\delta_{ir}\, \delta_{js} - \delta_{is}\, \delta_{jr})\, (b_s + \epsilon\, b_3\, \delta_{s3})_{,rj}$$
$$= Q\, V_{i,3} + b_{i,jj} - \epsilon\, b_{3,i3} + \epsilon\, b_{3,jj}\, \delta_{i3} . \qquad (8.6)$$

Let ξ be the vorticity of the flow, then

$$\xi = \text{curl } V \ .$$

Apply the curl operator to equations $(8.4)_{(ii),(vi)}$ to obtain

$$\frac{\partial \xi_i}{\partial t} = \Delta\xi_i + \surd(R)\; e_{ijk} \frac{\partial \theta}{\partial x_j} \delta_{k3} + e_{ijk}\, b_{k,3j} + \surd(T)\; V_{i,3}$$

$$+ \epsilon\; e_{ijk}\, b_{3,3j}\, \delta_{k3} \qquad (8.7)$$

$$P_m \frac{\partial J_i}{\partial t} - \epsilon\, P_m\, e_{ijk} \frac{\partial}{\partial t} (b_{3,j})\, \delta_{k3} = Q\, \xi_{i,3} - J_{j,ij} + J_{i,jj}\; .$$

Apply the curl operator once again to equation $(8.7)_{(i)}$ to obtian

$$\frac{\partial}{\partial t} \nabla^2 V_i = \nabla^4 V_i + \surd(R)\; (\nabla^2\theta\; \delta_{i3} - \frac{\partial^2\theta}{\partial x_i\, \partial x_j} \delta_{j3}) + \nabla^2(b_{i,3})$$

$$- \epsilon\, b_{33,i} + \epsilon\, b_{3,jj}\, \delta_{i3} - \surd(T)\; \xi_{i,3}\; . \qquad (8.8)$$

By taking the third components of equations (8.6), (8.7) and (8.8), we obtain

$$P_m \frac{\partial b}{\partial t} = Q\; Dw + b_{,jj} + \epsilon\; (\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2}),$$

$$\frac{\partial \xi}{\partial t} = \Delta\xi + DJ + \surd(T)\; Dw\;,$$

$$P_m \frac{\partial J}{\partial t} = Q\; D\xi + \Delta J, \qquad (8.9)$$

$$\frac{\partial}{\partial t} \nabla^2 w = \nabla^4 w + \surd(R)\; (\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}) + \nabla^2(Db)$$

$$+ \epsilon\; (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) Db - \surd(T)\; D\xi$$

where w, b, J and ξ are the third components of velocity, magnetic induction, current density and vorticity and where D is the operator d/dz.

We now search for a normal mode solution of equations $(8.4)_{(iii)}$ and (8.9) in the form

$$\begin{aligned} w &= w(z)\ e^{i(nx+my)+\sigma t} \ , \\ \theta &= \theta(z)\ e^{i(nx+my)+\sigma t} \ , \\ b &= b(z)\ e^{i(nx+my)+\sigma t} \ , \\ P &= P(z)\ e^{i(nx+my)+\sigma t} \ , \\ J &= J(z)\ e^{i(nx+my)+\sigma t} \ , \\ \xi &= \xi(z)\ e^{i(nx+my)+\sigma t} \ . \end{aligned} \tag{8.10}$$

Thus the relative equations become

$$\begin{aligned} \sigma P_r\ \theta &= L\theta - H\ \surd(R)\ w \ , \\ \sigma P_m\ b &= Lb - \epsilon\ a^2\ b + Q\ Dw \\ \sigma\ Lw &= L^2 w - a^2\ \surd(R)\ \theta + L(Db) - \epsilon\ a^2\ Db - \surd(T)\ D\xi \\ \sigma\ \xi &= L\xi + DJ + \surd(T)\ Dw \ , \\ \sigma P_m\ J &= Q\ D\xi + LJ \end{aligned} \tag{8.11}$$

where L is the operator $(D^2 - a^2)$. By setting $\epsilon = 0$ in equations (8.11) we obtain the standard equations of the classical magnetohydrodynamic Benard problem under the influence of rotation, which has been discussed by Chandrasekhar [15]. We may eliminate θ, b, ξ and J from equations (8.11) and derive a twelvth order ordinary differential equation to be satisfied by w.

<u>Boundary Conditions</u>

Solutions of equations (8.11) must be sought which satisfy the boundary conditions given in chapter six. Here we shall consider both boundaries to be free but later on we shall present results for the corresponding rigid boundary value problems. For the free boundary problem

$$w = D^2 w = 0 \qquad \text{on } z = 0 \text{ and } 1 \quad , \tag{8.12}$$

thus equations (8.11) has eigenfunctions

$$w = A \sin(n\pi z) \quad . \tag{8.13}$$

Consequently $Lw = -\lambda w$ where $\lambda = n^2\pi^2 + a^2$ and σ satisfies the fifth order equation

$$\left[\sigma^2 P_m + \sigma \lambda (1 + P_m) + \beta\right] \left[H R a^2 (\alpha + \sigma P_m) + (\alpha \beta + \lambda \alpha \sigma + \lambda^2 P_m \sigma + \lambda P_m \sigma^2)(\lambda + \sigma P_r)\right] + n^2 \pi^2 T (\lambda + \sigma P_r)(\alpha + \sigma P_m)(\lambda + \sigma P_m) = 0 \tag{8.14}$$

where

$$\alpha = \lambda + \epsilon a^2$$
$$\beta = n^2 \pi^2 Q + \lambda^2 \quad . \tag{8.15}$$

<u>Differentiation of the Rayleigh number</u> (Heating from below)

Here we shall consider the variation of the Rayleigh number as a function of the magnetic parameter Q, the Taylor number T and the wave number a for the stationary convection case, i.e. when $\sigma = 0$. The relative equations are

$$L\theta + \surd(R)\, w = 0\ ,$$

$$Lb - \epsilon\, a^2\, b + Q\, Dw = 0\ ,$$

$$L^2 w - a^2 \surd(R)\, \theta + L(Db) - \epsilon\, a^2\, Db - \surd(T)\, D\xi = 0\ ,$$

$$L\xi + DJ + \surd(T)\, Dw = 0\ , \qquad (8.16)$$

$$LJ + Q\, D\xi = 0\ .$$

Eliminate b from $(8.16)_{(ii),(iii)}$. Thus

$$L^2 w - a^2 \surd(R)\, \theta - Q\, D^2 w - \surd(T)\, D\xi = 0. \qquad (8.17)$$

From (8.17)

$$D^4 w_1 - (2\, a_1^2 + Q_1)\, D^2 w_1 + a_1^4\, w_1 - a_1^2 \surd(R_1)\, \theta_1 - \surd(T_1)\, D\xi_1 = 0.$$

Multiply by w_2 and integrate to obtain

$$\int D^2 w_2\, D^2 w_1 + (2\, a_1^2 + Q_1)\int Dw_2\, Dw_1 + a_1^4 \int w_2\, w_1$$

$$- a_1^2 \surd(R_1) \int w_2\, \theta_1 - \surd(T_1) \int w_2\, D\xi_1 = 0. \qquad (8.18)$$

Similarly

$$\int D^2 w_1\, D^2 w_2 + (2\, a_2^2 + Q_2)\int Dw_1\, Dw_2 + a_2^4 \int w_1\, w_2$$

$$- a_2^2 \surd(R_2) \int w_1\, \theta_2 - \surd(T_2) \int w_1\, D\xi_2 = 0. \qquad (8.19)$$

From $(8.16)_{(iv)}$

$$D^2 \xi_1 - a_1^2\, \xi_1 + DJ_1 + \surd(T_1)\, Dw_1 = 0.$$

Multiply by ξ_2 and integrate to obtain

$$\int D\xi_1\, D\xi_2 + a_1^2 \int \xi_1\, \xi_2 + \int J_1\, D\xi_2 = \surd(T_1) \int \xi_2\, Dw_1 . \qquad (8.20)$$

Similarly

$$\int D\xi_2\, D\xi_1 + a_2^2 \int \xi_2\, \xi_1 + \int J_2\, D\xi_1 = \surd(T_2) \int \xi_1\, Dw_2 . \qquad (8.21)$$

From $(8.16)_{(v)}$

$$D^2 J_1 - a_1^2\, J_1 + Q\, D\xi_1 = 0 .$$

Multiply by J_2 and integrate to obtain

$$\int DJ_2\, DJ_1 + a_1^2 \int J_2\, J_1 = Q_1 \int J_2\, D\xi_1 . \qquad (8.22)$$

Similarly

$$\int DJ_1\, DJ_2 + a_2^2 \int J_1\, J_2 = Q_2 \int J_1\, D\xi_2 . \qquad (8.23)$$

By using equation (8.23) we can eliminate $\int J_1\, D\xi_2$ from (8.20). Thus

$$\int D\xi_1\, D\xi_2 + a_1^2 \int \xi_1\, \xi_2 + \frac{1}{Q_2} \left[\int DJ_1\, DJ_2 + a_2^2 \int J_1\, J_2 \right]$$

$$= \surd(T_1) \int \xi_2\, Dw_1 . \qquad (8.24)$$

Similarly from (8.21) and (8.22) we can eliminate $\int J_2\, D\xi_1$. Thus

$$\int D\xi_2\, D\xi_1 + a_2^2 \int \xi_2\, \xi_1 + \frac{1}{Q_1} \left[\int DJ_2\, DJ_1 + a_1^2 \int J_2\, J_1 \right]$$

$$= \surd(T_2) \int \xi_1\, Dw_2 . \qquad (8.25)$$

Since

$$\int w_2\, D\xi_1 = \int D(w_2\, \xi_1) - \int \xi_1\, Dw_2 = - \int \xi_1\, Dw_2$$

then using (8.25) we can eliminate $\int w_2 D\xi_1$ from (8.18). Thus

$$\int D^2w_2 D^2w_1 + (2a_1^2 + Q_1)\int Dw_2 Dw_1 + a_1^4 \int w_2 w_1 - a_1^2 \surd(R_1) \int w_2 \theta_1$$
$$+ \frac{\surd(T_1)}{\surd(T_2)} \left\{ \int D\xi_2 D\xi_1 + a_2^2 \int \xi_2 \xi_1 + \frac{1}{Q_1} \left[\int DJ_2 DJ_1 + a_1^2 \int J_2 J_1 \right] \right\} = 0. \qquad (8.26)$$

Similarly from (8.19) and (8.24) we can eliminate $\int w_1 D\xi_2$. Thus

$$\int D^2w_1 D^2w_2 + (2a_2^2 + Q_2)\int Dw_1 Dw_2 + a_2^4 \int w_1 w_2 - a_2^2 \surd(R_2) \int w_1 \theta_2$$
$$+ \frac{\surd(T_2)}{\surd(T_1)} \left\{ \int D\xi_1 D\xi_2 + a_1^2 \int \xi_1 \xi_2 + \frac{1}{Q_2} \left[\int DJ_1 DJ_2 + a_2^2 \int J_1 J_2 \right] \right\} = 0. \qquad (8.27)$$

From $(8.16)_{(i)}$

$$D^2\theta_1 - a_1^2 \theta_1 + \surd(R_1) w_1 = 0.$$

Multiply by θ_2 and integrate to obtain

$$\int D\theta_2 D\theta_1 + a_1^2 \int \theta_2 \theta_1 = \surd(R_1) \int \theta_2 w_1. \qquad (8.28)$$

Similarly,

$$\int D\theta_1 D\theta_2 + a_2^2 \int \theta_1 \theta_2 = \surd(R_2) \int \theta_1 w_2. \qquad (8.29)$$

By using equation (8.29) we can eliminate $\int \theta_1 w_2$ from (8.26). Thus

$$\int D^2w_2 D^2w_1 + (2a_1^2 + Q_1)\int Dw_2 Dw_1 + a_1^4 \int w_2 w_1$$
$$+ \frac{\surd(T_1)}{\surd(T_2)} \left\{ \int D\xi_2 D\xi_1 + a_2^2 \int \xi_2 \xi_1 + \frac{1}{Q_1} \left[\int DJ_2 DJ_1 + a_1^2 \int J_2 J_1 \right] \right\}$$
$$- a_1^2 \frac{\surd(R_1)}{\surd(R_2)} \left[\int D\theta_1 D\theta_2 + a_2^2 \int \theta_1 \theta_2 \right] = 0. \qquad (8.30)$$

Similarly from (8.27) and (8.28) we can eliminate $\int \theta_1 w_2$. Thus

$$\int D^2w_1\, D^2w_2 + (2\, a_2^2 + Q_2)\int Dw_1\, Dw_2 + a_2^4 \int w_1\, w_2$$

$$+ \frac{\sqrt{(T_2)}}{\sqrt{(T_1)}} \left\{ \int D\xi_1\, D\xi_2 + a_1^2 \int \xi_1\, \xi_2 + \frac{1}{Q_2} \left[\int DJ_1\, DJ_2 + a_2^2 \int J_1\, J_2 \right]\right\}$$

$$- a_1^2 \frac{\sqrt{(R_1)}}{\sqrt{(R_2)}} \left[\int D\theta_1\, D\theta_2 + a_2^2 \int \theta_1\, \theta_2 \right] = 0. \qquad (8.31)$$

Subtract (8.31) from (8.30) to obtain

$$(2a_1^2 - 2a_2^2 + Q_1 - Q_2) \int Dw_2\, Dw_1 + (a_1^4 - a_2^4) \int w_2\, w_1$$

$$+ \frac{T_1 - T_2}{\sqrt{(T_1T_2)}} \int D\xi_1\, D\xi_2 + \frac{a_1^2\, a_2^2}{\sqrt{(T_1T_2)}} \int \xi_1\, \xi_2 + \frac{1}{\sqrt{(T_1T_2)}}\left(\frac{T_1}{Q_1} - \frac{T_2}{Q_2}\right) \int DJ_2\, DJ_1$$

$$+ \frac{1}{\sqrt{(T_1T_2)}} \left(\frac{a_1^2\, T_1}{Q_1} - \frac{a_2^2\, T_2}{Q_2}\right) \int J_2\, J_1$$

$$= \frac{1}{\sqrt{(R_1R_2)}} (a_1^2\, R_1 - a_2^2\, R_2) \int D\theta_1\, D\theta_2 + \frac{a_1^2\, a_2^2}{\sqrt{(R_1R_2)}} (R_1 - R_2) \int \theta_1\, \theta_2$$

$$(8.32)$$

Suppose that $w_1 \longrightarrow w_2$, $\theta_1 \longrightarrow \theta_2$, $\xi_1 \longrightarrow \xi_2$, $J_1 \longrightarrow J_2$, $a_1 \longrightarrow a_2$ and $R_1 \longrightarrow R_2$. Divide (8.32) by $s_1 - s_2$ and take the limit as $s_1 \longrightarrow s_2$. Thus (8.32) becomes

$$4\, a \frac{da}{ds} \left[\int (Dw)^2 + a^2 \int w^2 + \frac{1}{2\, Q} \int J^2 - \frac{1}{2} \int (D\theta)^2 - \frac{1}{2} \int \xi^2 \right]$$

$$+ \frac{1}{T} \frac{dT}{ds} \left[(D\xi)^2 + a^2 \int \xi^2 + \frac{1}{Q} \int (DJ)^2 + \frac{a^2}{Q} \int J^2 \right]$$

$$+ \frac{dQ}{ds} \left[\int (Dw)^2 - \frac{1}{Q^2} \int (DJ)^2 - \frac{a^2}{Q^2} \int J^2 \right]$$

$$= \frac{a^2}{R} \frac{dR}{ds} \left[\int (D\theta)^2 + a^2 \int \theta^2 \right]. \qquad (8.33)$$

Since

$$R = R(a,T,Q)$$

then

$$\frac{dR}{ds} = \frac{\partial R}{\partial a}\frac{da}{ds} + \frac{\partial R}{\partial T}\frac{dT}{ds} + \frac{\partial R}{\partial Q}\frac{dQ}{ds} .$$

Thus (8.33) becomes

$$4\, a \frac{da}{ds}\left[\int (Dw)^2 + a^2 \int w^2 + \frac{1}{2\,Q} \int J^2 - \frac{1}{2} \int (D\theta)^2 - \frac{1}{2} \int \xi^2 \right]$$

$$+ \frac{1}{T}\frac{dT}{ds}\left[(D\xi)^2 + a^2 \int \xi^2 + \frac{1}{Q} \int (DJ)^2 + \frac{a^2}{Q} \int J^2 \right]$$

$$+ \frac{dQ}{ds}\left[\int (Dw)^2 - \frac{1}{Q^2} \int (DJ)^2 - \frac{a^2}{Q^2} \int J^2 \right]$$

$$= \frac{a^2}{R}\left[\frac{\partial R}{\partial a}\frac{da}{ds} + \frac{\partial R}{\partial T}\frac{\partial T}{ds} + \frac{\partial R}{\partial Q}\frac{dQ}{ds} \right]\left[\int (D\theta)^2 + a^2 \int \theta^2 \right]. \tag{8.34}$$

From (8.34), we can show that

$$\frac{\partial R}{\partial a} = \frac{4aR}{\alpha}\left[\int (Dw)^2 + a^2 \int w^2 + \frac{1}{2\,Q} \int J^2 - \frac{1}{2} \int (D\theta)^2 - \frac{1}{2} \int \xi^2 \right]$$

where

$$\alpha = a^2 \left[(D\theta)^2 + a^2 \int \theta^2 \right] .$$

Clearly the minimum Rayleigh number occurs when

$$\int (Dw)^2 + a^2 \int w^2 + \frac{1}{2\,Q} \int J^2 \;>\; \frac{1}{2} \int (D\theta)^2 + \frac{1}{2} \int \xi^2 .$$

Also from (8.34), we can show that

$$\frac{\partial R}{\partial T} = \frac{R}{T\,Q\,\alpha}\left[Q\,(D\xi)^2 + a^2\,Q \int \xi^2 + \int (DJ)^2 + a^2 \int J^2 \right] > 0.$$

Thus the Rayleigh number is an increasing function of the Taylor number T. In fact when $Q \longrightarrow \infty$ $\partial R/\partial T \longrightarrow 0$. So for higher values of Q the Rayleigh number does not depend on T.

Moreover from (8.34)

$$\frac{\partial R}{\partial Q} = \frac{R}{\alpha}\left[\int (Dw)^2 - \frac{1}{Q^2}\int (DJ)^2 - \frac{a^2}{Q^2}\int J^2\right]$$

Thus the Rayleigh number is an increasing function of Q provided

$$Q^2 \int (Dw)^2 > \int (DJ)^2 + a^2 \int J^2 .$$

Stationary Convection

As mentioned in a previous section that we can derive a twelvth order ordinary differential equation from (8.11) and this equation will have form

$$L^6 w + A\ L^5 w + B\ L^4 w + C\ L^3 w + D\ L^2 w + E\ Lw + F\ w = 0 \qquad (8.35)$$

where

$$A = -\epsilon\ a^2 - [P_r + 2\ (1 + P_m)]\ \sigma \quad ,$$

$$B = 2\ Q\ n^2\ \pi^2 + \epsilon\ a^2\ (2 + P_r + P_m)\ \sigma + P_m\ (2 + P_r + P_m)\ \sigma^2$$
$$+ [1 + 2\ P_r + P_m(2 + P_r)]\ \sigma^3 \quad ,$$

$$C = -2\ \epsilon\ a^2\ Q\ n^2\ \pi^2 - n^2\ \pi^2\ T + R\ a^2 - 2\ Qn^2\pi^2\ (1 + P_r + P_m)\sigma$$
$$- \epsilon\ a^2\ [1 + 2\ P_r + P_m\ (2 + P_r)]\ \sigma^2$$
$$- [P_r + 2\ P_m\ (1 + P_m) + P_r\ P_m\ (4 + P_m)]\ \sigma^3 \quad ,$$

$$D = Q^2\ n^4\ \pi^4 + \epsilon\ a^2\ n^2\ \pi^2\ T - \epsilon\ a^4\ R + \{\ n^2\ \pi^2\ T\ (P_r + 2\ P_m)$$
$$+ \epsilon\ a^2\ n^2\ \pi^2\ Q\ [P_m + 2\ (1 + P_r)] - R\ a^2\ (1 + 2\ P_m)\ \}\ \sigma$$
$$+ 2\ n^2\ \pi^2\ Q\ (P_r + P_m + P_r\ P_m)\ \sigma^2 + \epsilon\ a^2\ [P_r +$$
$$P_m\ (1 + 2\ P_r)]\ \sigma^3 + [P_m^2 + 2\ P_r\ P_m\ (1 + P_m)]\ \sigma^4 \quad ,$$

$$E = Q\ n^2\ \pi^2\ (R\ a^2 - \epsilon\ a^2\ Q\ n^2\ \pi^2) - [n^4\ \pi^4\ Q^2\ P_r - \epsilon a^4\ R(1+P_m)$$
$$+ \epsilon\ a^2\ n^2\ T\ (P_r + P_m)]\ \sigma - [\epsilon\ a^2\ n^2\pi^2 Q\ (P_m + 2P_r + P_r P_m)$$
$$+ n^2\ \pi^2\ T\ P_m\ (P_m + 2\ P_r) - R\ a^2\ P_m\ (2 + P_m)]\ \sigma^2$$
$$- 2\ n^2\ \pi^2\ Q\ P_r\ P_m\ \sigma^3 - \epsilon\ a^2\ P_r\ P_m\ \sigma^4 - P_r\ P_m^2\ \sigma^5 \quad ,$$

$$F = -\epsilon\ a^4\ n^2\ \pi^2\ Q\ R + n^2\ \pi^2\ Q\ (\epsilon\ a^2\ n^2\ \pi^2\ Q\ P_r - R\ a^2\ P_m)\ \sigma$$
$$+ \epsilon\ a^2\ P_m\ (n^2\ \pi^2\ T\ P_r - R\ a^2)\ \sigma^2 + [\epsilon\ a^2\ n^2\ \pi^2\ Q\ P_r\ P_m$$
$$+ P_m^2\ (n^2\ \pi^2\ T\ P_r - R\ a^2)]\ \sigma^3 \quad .$$

To find the critical Rayleigh number for the onset of stationary convection we set $\sigma = 0$ in equation (8.14) and we can see that it factorizes into

$$(L - \epsilon\, a^2)\ [L^5 w + 2\, Q\, n^2\, \pi^2\, L^3 w + (R\, a^2 - n^2\, \pi^2\, T)\, L^2 w + Q^2\, n^4\, \pi^4\, Lw + Q\, n^2\, \pi^2\, R\, a^2\, w] = 0$$

i.e.
$$L^5 w + 2\, Q\, n^2\, \pi^2\, L^3 w + (R\, a^2 - n^2\, \pi^2\, T)\, L^2 w + Q^2\, n^4\, \pi^4\, Lw + Q\, n^2\, \pi^2\, R\, a^2\, w = 0 \qquad (8.36)$$

from which we have
$$R = \frac{\lambda\ (\beta^2 + n^2\, \pi^2\, \lambda\, T)}{a^2\, \beta} \qquad (8.37)$$

where $\beta = \lambda^2 + n^2\, \pi^2\, Q.$

Since this equation does not contain P_r, P_m or ϵ then the critical Rayleigh number for stationary convection is independent of P_r, P_m or ϵ for the free, rigid and mixed boundary value problems and we obtain precisely the results of Chandrasekhar [15] who obtained results for the free boundary problem only. The critical Rayleigh number is obtained when both boundaries are rigid. The results of this case are summarized in table (1). When $Q = 0$, the inhibiting effect of rotation on the onset of instability is apparent. When $T_1 = (T/\pi^4) = 1,\ 10\ ,100$ the critical Rayleigh number appears to be a monotonic increasing function of $Q_1 = (Q/\pi^2)$. However for higher values of T_1, $T_1 = 1000$ say, the critical Rayleigh number always shows an initial decrease with Q_1 and this feature becomes very clear for higher values of T_1. The effect of rotation appears to be very small for large values of Q_1 and in this case the inhibition due to magnetic field seems to be predominate.

Table (1)

The relation between R and Q for the onset of stationary convection when both boundaries are rigid.

$T_1 = 1$

Q_1	A	R
0	3.16	1755.103
100	5.801	16928.472
1000	8.646	123022.407
10000	12.832	1077630.228

$T_1 = 10$

Q_1	A	R
0	3.447	2140.822
100	5.8	16964.572
1000	8.646	123024.862
10000	12.832	1077630.622

$T_1 = 100$

Q_1	A	R
0	4.763	4653.596
100	5.795	17127.377
1000	8.645	123049.405
10000	12.832	1077634.556

$T_1 = 1000$

Q_1	A	R
0	7.139	16457.621
10	5.745	13814.32
20	5.187	12578.083
30	5.084	12440.009
50	5.22	13625.109
100	5.747	18917.431
1000	8.632	123294.61
10000	12.832	1077673.916

$T_1 = 10000$

Q_1	A	R
0	10.769	69885.353
10	9.61	63677.687
20	8.017	57155.269
30	6.613	50747.508
50	5.724	42048.896
100	5.576	35723.327
1000	8.512	125695.951
10000	12.832	1078067.168

The Case of Overstability

Here we look for a solution of equation (8.14) in which $\sigma = iy$. Because of the complication of the equations obtained in this case we were unable to produce analytic solution for the critical Rayleigh number. By solving the problem numerically we noticed that overstability is possible for certain ranges of the parameters P_r, P_m and ϵ. Results for overstability case are obtained when both boundaries are rigid. These results are summarized in tables (2)-(4) for various values of ϵ. It is apparent from these results that the inhibiting effect of both magnetic field and rotation increases as the non-dimensional parameter ϵ increases. The critical Rayleigh number appears to be a monotonic increasing function of Q_1 for $T_1 = 1$, 10, 100 and 1000. However for higher values of Q_1, $Q_1 = 10000$ say, the critical Rayleigh number becomes a decreasing function of T_1 and this is true for all values of the parameter ϵ. Moreover when $\epsilon = 0$, the critical Rayleigh number shows an initial decrease with Q_1 and this is happened when T_1 is large, $T_1 = 10000$ say, but this is not the case when $\epsilon \neq 0$. In fact when $\epsilon \neq 0$ the critical Rayleigh number is an increasing function of Q_1 for higher values of T_1. Figures (1) and (2) show the relation between the critical Rayleigh number R and the Taylor number T for specific values of ϵ when Q = 100 and 1000 respectively. The numerical values are summarized in tables (2)-(4)

The preferred mode of instability in this problem depends on the relative values of $R_{stat}(T,Q,P_r,P_m,\epsilon)$ and $R_{over}(T,Q,P_r,P_m,\epsilon)$ and the stability boundary is that

$$R_{stat}(T,Q,P_r,P_m,\epsilon) = R_{over}(T,Q,P_r,P_m,\epsilon). \tag{8.38}$$

The solution of (8.38) can be expressed as

$$Q = Q_{crit}(T,P_r,P_m,\epsilon)$$

so that if $Q < Q_{crit}$, then stationary stability is the preferred mechanism whereas if $Q > Q_{crit}$, then overstability is preferred. Critical values of Q for various values of ϵ are summarized in tables (5)-(7). Figures (3) shows the relation between the critical values of Q and T when $\epsilon = 0$, 0.5 and 1 respectively. The Fortran77 code used to obtain the numerical results is listed in Appendix V.

Table (2)

The relation between R and Q for the onset of overstability when both boundaries are rigid. Here $P_m = 4$, $P_r = 1$ and $\epsilon = 0$.

$T_1 = 1$

Q_1	A	R
100	5.33	11479.149
1000	7.959	48949.976
10000	11.949	301167.677

$T_1 = 10$

Q_1	A	R
100	5.331	11540.559
1000	7.959	48955.507
10000	11.95	301153.325

$T_1 = 100$

Q_1	A	R
100	5.334	12135.83
1000	7.952	49010.78
10000	11.95	301130.486

$T_1 = 1000$

Q_1	A	R
100	5.433	17444.656
1000	7.888	49562.699
10000	11.942	301059.14

$T_1 = 10000$

Q_1	A	R	Q_1	A	R
300	6.165	43604.804	400	6.379	43185.296
310	6.185	43469.69	450	6.492	43539.848
320	6.204	43359.554	500	6.603	44134.962
330	6.225	43272.521	1000	7.516	55578.1
350	6.268	43161.141	10000	11.836	300252.932

Table (3)

The relation between R and Q for the onset of overstability when both boundaries are rigid. Here $P_m = 4$, $P_r = 1$ and $\epsilon = 0.5$

$T_1 = 1$

Q_1	A	R
100	5.211	13787.458
1000	7.849	65410.402
10000	11.896	437405.137

$T_1 = 10$

Q_1	A	R
100	5.211	13854.521
1000	7.849	65417.304
10000	11.896	437404.446

$T_1 = 100$

Q_1	A	R
100	5.214	14509.27
1000	7.842	65486.287
10000	11.854	437403.51

$T_1 = 1000$

Q_1	A	R
100	5.799	24794.891
1000	7.779	66175.318
10000	11.885	437329.041

$T_1 = 10000$

Q_1	A	R
500	6.485	55522.552
1000	7.416	73556.147
10000	12.152	437204.886

Table (4)

The relation between R and Q for the onset of overstability when both boundaries are rigid. $P_m = 4$, $P_r = 1$ and $\epsilon = 1$.

$T_1 = 1$

Q_1	A	R
100	5.095	16244.17
1000	7.74	83573.266
10000	11.825	590357.785

$T_1 = 10$

Q_1	A	R
100	5.095	16316.193
1000	7.739	83581.666
10000	11.825	590357.479

$T_1 = 100$

Q_1	A	R
100	5.1	17025.659
1000	7.733	83665.71
10000	11.824	590354.464

$T_1 = 1000$

Q_1	A	R
100	5.698	29649.957
1000	7.677	84510.781
10000	11.815	590340.193

$T_1 = 10000$

Q_1	A	R
500	6.398	67988.626
1000	7.344	93528.176
10000	11.729	590146.708

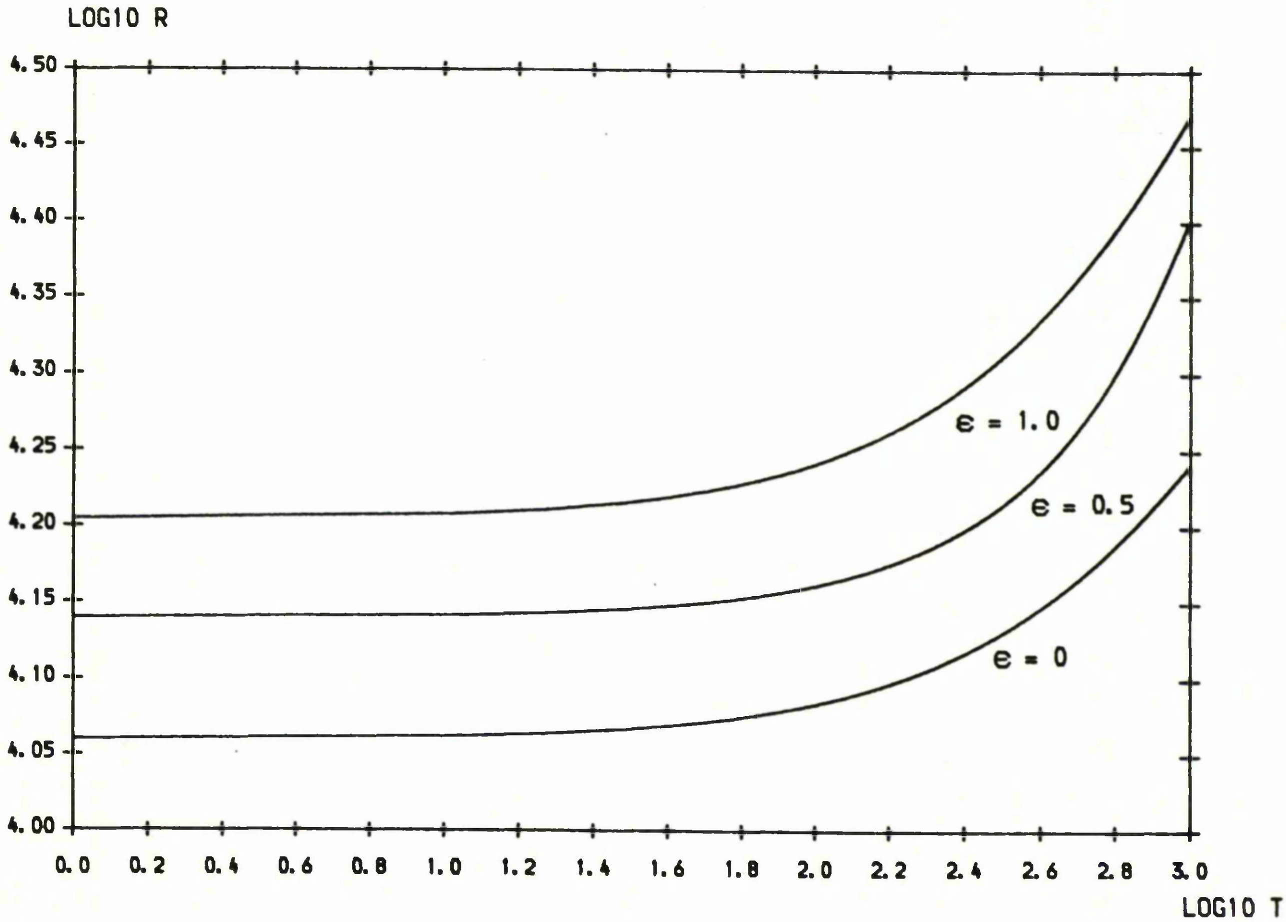

Figure 1.

This figure shows the relation between the critical Rayleigh number R and the Taylor number T when both boundaries are rigid for various values of the parameter ϵ. Here Q = 100. The figure shows that as ϵ increases the critical Rayleigh number increases.

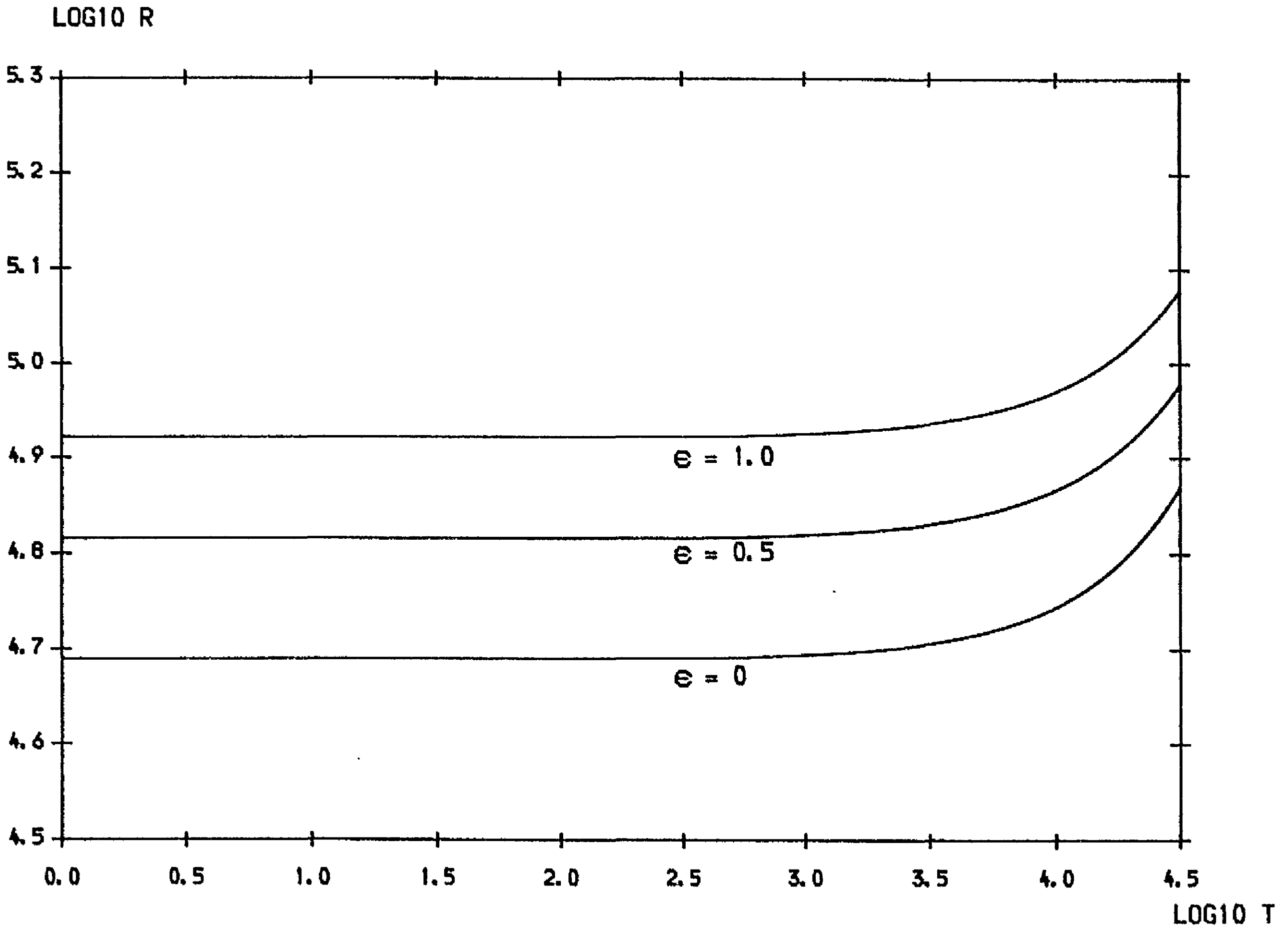

Figure 2.

This figure shows the relation between the critical Rayleigh number R and the Taylor number T when both boundaries are rigid for various values of the parameter ϵ. Here $Q = 1000$. The figure shows that as ϵ increases the critical Rayleigh number increases.

Table (5)

The relation between critical Q and T when both boundaries are rigid. Here $P_m = 4$, $P_r = 1$ and $\epsilon = 0$.

T_1	Q_{crit}	R	A_o	A_s
1	25.69	6366	4.297	4.617
10	27.41	6649	4.353	4.663
100	37.795	8688	4.67	4.939
1000	85.785	17288	5.381	5.606
10000	251.239	44711	6.088	6.429

Table (6)

The relation between critical Q and T when both boundaries are rigid. Here $P_m = 4$, $P_r = 1$ and $\epsilon = 0.5$

T_1	Q_{crit}	R	A_o	A_s
1	42.17	8872	4.494	5.012
10	43.871	9159	4.562	5.043
100	56.932	11334	4.786	5.27
1000	118.705	21090	5.398	5.915
10000	332.763	52745	6.102	6.815

Table (7)

The relation between critical Q and T when both boundaries are rigid. Here $P_m = 4$, $P_r = 1$ and $\epsilon = 1$.

T_1	Q_{crit}	R	A_o	A_s
1	80.802	14324	4.907	5.592
10	82.647	14596	4.928	5.613
100	97.951	16839	5.082	5.774
1000	178.189	28197	5.594	6.353
10000	463.863	66612	6.316	7.304

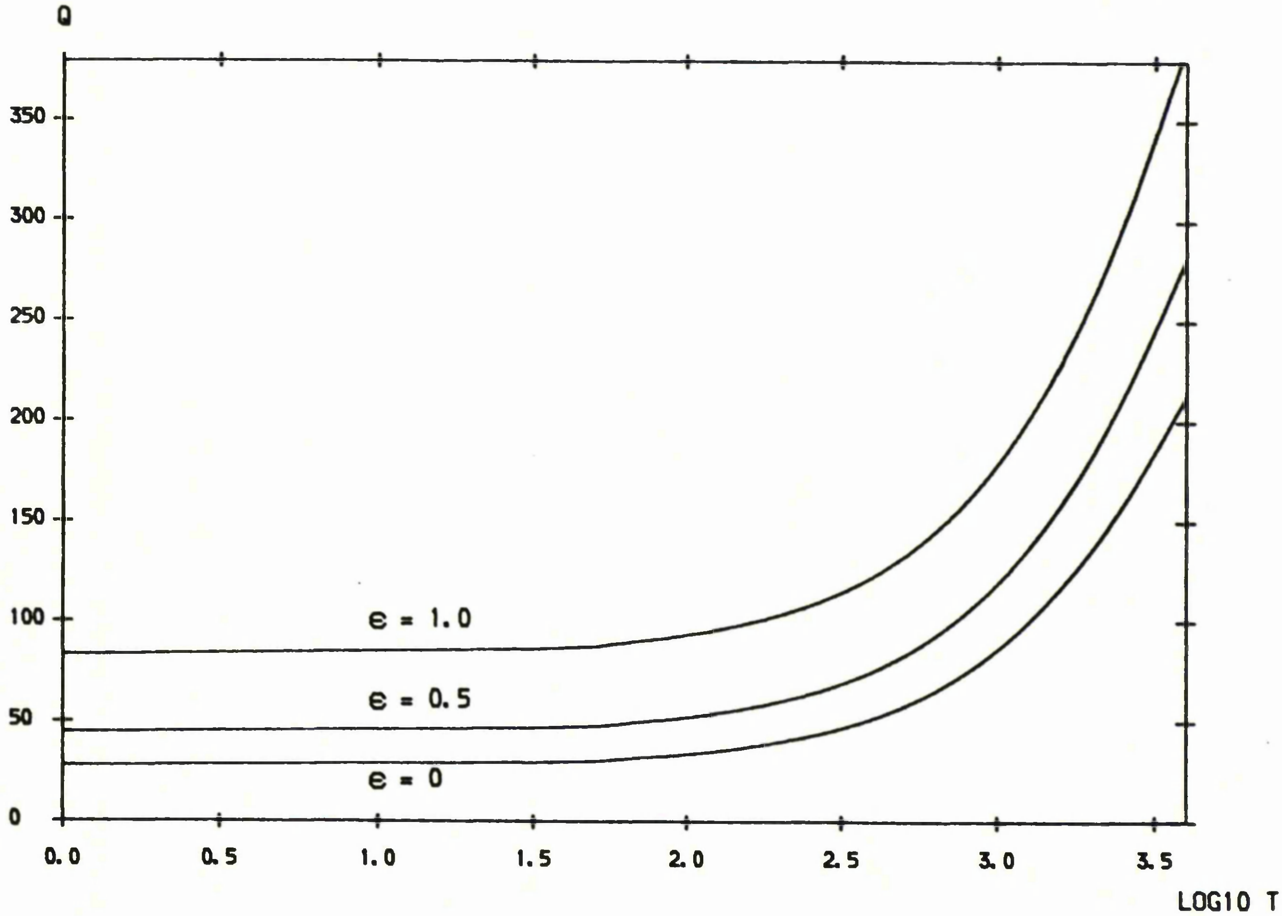

Figure 3.

This figure shows the relation between the critical values of the magnetic parameter Q and the Taylor number T when both boundaries are rigid for various values of the parameter ϵ. It shows that if $Q > Q_{crit}$ then overstability is the preferred mechanism whereas if $Q < Q_{crit}$ stationary stability is preferred.

Chapter Nine

Energy Stability in the Benard Convection for a Non-Linear MHD Fluid with Vertical Magnetic Field

Introduction

The classical magnetic Benard problem has been studied by Rionero [71] and Carmi & Lalas [72] using standard energy method. Their analyses were unable to describe the stabilising effect of the magnetic field. Essentially the problem derives from the fact that the magnetic forces do not directly dissipate total energy. A similar phenomenon appears in the context of rotating systems. Galdi [38] overcomes this difficulty by using a generalized energy method based on a type of energy functional which involves a suitable coupling of the field variables and of their first spatial derivatives. He developed a criterion in which the critical Rayleigh number was bounded above by a factor. In this chapter we shall investigate the linear stability of the magnetic Benard problem in a non-linear magnetohydrodynamic fluid via a generalized energy method when the fluid is heated from below.

Energy stability

We start with the linearized equations which are obtained in chapter six. i.e.

$$V_{i,i} = 0 \quad ,$$

$$\frac{\partial V_i}{\partial t} = - P_{,i} + \Delta V_i + \surd(R)\ \theta\ \delta_{i3} + b_{i,3} + \epsilon\ b_{3,3}\ \delta_{i3}$$

$$P_r \frac{\partial \theta}{\partial t} - \surd(R)\ w = \Delta \theta \quad , \qquad (9.1)$$

$$b_{i,i} = 0 \ ,$$

$$J_i = e_{ijk}\ (b_k + \epsilon\ b_3\ \delta_{k3})_{,j} \quad ,$$

$$P_m \frac{\partial b_i}{\partial t} = Q\ V_{i,3} - e_{ijk}\ J_{k,j} \quad ,$$

where the non-dimensional numbers Q, P_m, P_r, R and ϵ are given by

$$Q = (\frac{B\ d}{\nu})^2\ \frac{\nu}{\rho\ \eta} \quad ,$$

$$P_m = \nu/\rho\eta\varphi \quad ,$$

$$P_r = \nu/\kappa \quad ,$$

$$R = \frac{\alpha\ g\ d^4\ |\beta|}{\nu\ \kappa} \quad ,$$

$$\epsilon = B\ \frac{\partial \varphi}{\partial B}\ /\ \varphi \ .$$

From equations $(9.1)_{(v),(vi)}$ we have

$$P_m \frac{\partial b_i}{\partial t} = Q\ V_{i,3} - e_{ijk}\ [e_{krs}\ (b_s + \epsilon\ b_3\ \delta_{s3})_{,r}]_{,j}$$

$$= Q\ V_{i,3} - (\delta_{ir}\ \delta_{js} - \delta_{is}\ \delta_{jr})\ (b_s + \epsilon\ b_3\ \delta_{s3})_{,rj}$$

$$= Q\ V_{i,3} + b_{i,jj} - \epsilon\ b_{3,i3} + \epsilon\ b_{3,jj}\ \delta_{i3} .$$

(9.2)

Suppose that V, θ, P and **b** are solutions of (9.1) which are periodic in x and y and define the Energy E(t) by

$$E(t) = \frac{1}{2}\,[\langle V^2\rangle + P_r \langle\theta^2\rangle + \frac{P_m}{Q}\langle b_\alpha\, b_\alpha\rangle + \gamma_1 \langle f^2\rangle + \gamma_2 \langle(\theta_{,z})^2\rangle]$$

where < > denotes the integral over a disturbance cell S and where

$$f = P_m\, b - \lambda\, P_r\, \theta_{,z} \quad ,$$

$$\gamma_1 \;,\; \gamma_2 > 0 \;.$$

and λ is a constant to be advantageously chosen in the course of the analysis. Thus

$$\begin{aligned}\frac{dE}{dt} = &- \langle V_i\, P_{,i}\rangle + \langle V_i\, \Delta V_i\rangle + 2\,\surd(R)\,\langle\theta\, w\rangle + \epsilon\,\langle w\, b_{,z}\rangle \\ &+ \langle V_i\, b_{i,z}\rangle + \langle\theta\,\theta_{,ii}\rangle + \langle b_\alpha\, V_{\alpha,z}\rangle + \frac{1}{Q}\langle b_\alpha\, \Delta b_\alpha\rangle \\ &+ \gamma_1 \langle f\,[Q\, w_{,z} + (1+\epsilon)\,\Delta b - \epsilon\, b_{,zz} - \lambda\,(\Delta\theta_{,z} + \surd(R)\, w_{,z})]\rangle \\ &- \frac{\epsilon}{Q}\langle b_\alpha\, b_{,\alpha z}\rangle + \frac{\gamma_2}{P_r}\langle\theta_{,z}\,\Delta\theta_{,z} + \surd(R)\,\theta_{,z}\, w_{,z}\rangle \;.\end{aligned}$$

Following the boundary conditions assumed by Galdi [38], namely

$$V = b = \theta = 0 \qquad \text{on } z = 0,1 \;,$$

we can show that

$$\begin{aligned}\frac{dE}{dt} = &- D(V_i) - D(\theta) - \frac{1}{Q} D(b_\alpha) + 2\,\surd(R)\,\langle\theta\, w\rangle + \epsilon\,\langle w\, b_{,z}\rangle \\ &+ \langle V_i\, b_{i,z}\rangle + \langle b_\alpha\, V_{\alpha,z}\rangle - \frac{\epsilon}{Q}\langle b_\alpha\, b_{,\alpha z}\rangle - \epsilon\,\gamma_1 \langle f\, b_{,zz}\rangle \\ &+ \gamma_1 \langle f\,[Q - \lambda\,\surd(R)]\, w_{,z}\rangle + \gamma_1 \langle f\,[(1+\epsilon)\,\Delta b - \lambda\,\Delta\theta_{,z}]\rangle \\ &+ \frac{\gamma_2}{P_r}\surd(R)\,\langle\theta_{,z}\, w_{,z}\rangle - \frac{\gamma_2}{P_r}\langle\theta_{,iz}\,\theta_{,iz}\rangle\end{aligned}$$

where D( ) denoting the Dirichlet integral.

Since

$$\langle V_i\, b_{i,z} + b_\alpha\, V_{\alpha,z}\rangle = \langle w\, b_{,z} + V_\alpha\, b_{\alpha,z} + b_\alpha\, V_{\alpha,z}\rangle$$
$$= \langle w\, b_{,z} + (V_\alpha\, b_\alpha)_{,z}\rangle$$
$$= \langle w\, b_{,z}\rangle$$

and

$$\langle b_\alpha\, b_{,\alpha z}\rangle = \langle (b_\alpha\, b_{,z})_{,\alpha} - b_{\alpha,\alpha}\, b_{,z}\rangle$$
$$= -\langle b_{,z}\,(b_{1,1} + b_{2,2})\rangle$$
$$= \langle (b_{,z})^2\rangle$$

then

$$\frac{dE}{dt} = -D(V_i) - D(\theta) - \frac{1}{Q} D(b_\alpha) + 2\sqrt{(R)}\,\langle\theta\, w\rangle + (1+\epsilon)\,\langle w\, b_{,z}\rangle$$
$$- \frac{\epsilon}{Q}\langle (b_{,z})^2\rangle - \epsilon\,\gamma_1\,\langle f\, b_{,zz}\rangle + \frac{\gamma_2}{P_r}\sqrt{(R)}\,\langle\theta_{,z}\, w_{,z}\rangle$$
$$+ \gamma_1\,\langle f\,[Q - \lambda\sqrt{(R)}]\, w_{,z}\rangle + \gamma_1\,\langle f\,[(1+\epsilon)\,\Delta b - \lambda\,\Delta\theta_{,z}]\rangle$$
$$- \frac{\gamma_2}{P_r}\langle\theta_{,iz}\,\theta_{,iz}\rangle\,.$$

Now

$$\gamma_1\,\langle f\,[Q - \lambda\sqrt{(R)}]\, w_{,z}\rangle + (1+\epsilon)\,\langle w\, b_{,z}\rangle$$
$$= \gamma_1\,[Q - \lambda\sqrt{(R)}]\,\langle f\, w_{,z}\rangle - (1+\epsilon)\,\langle b\, w_{,z}\rangle$$
$$= \gamma_1\,[Q - \lambda\sqrt{(R)}]\,\langle f\, w_{,z}\rangle - \frac{(1+\epsilon)}{P_m}\langle (f + \lambda\, P_r\,\theta_{,z})\, w_{,z}\rangle$$
$$= \left[\gamma_1\,(Q - \lambda\sqrt{(R)}) - \frac{(1+\epsilon)}{P_m}\right]\langle f\, w_{,z}\rangle - \frac{\lambda\, P_r\,(1+\epsilon)}{P_m}\langle\theta_{,z}\, w_{,z}\rangle$$

Thus

$$\frac{dE}{dt} = -D(V_i) - D(\theta) - \frac{1}{Q} D(b_\alpha) + 2\sqrt{(R)}\,\langle\theta\, w\rangle - \frac{\epsilon}{Q}\langle (b_{,z})^2\rangle$$
$$- \epsilon\,\gamma_1\,\langle f\, b_{,zz}\rangle + \left[\frac{\gamma_2}{P_r}\sqrt{(R)} - \frac{\lambda\, P_r\,(1+\epsilon)}{P_m}\right]\langle\theta_{,z}\, w_{,z}\rangle$$
$$+ \left[\gamma_1\,(Q - \lambda\sqrt{(R)}) - \frac{(1+\epsilon)}{P_m}\right]\langle f\, w_{,z}\rangle$$
$$- \frac{\gamma_2}{P_r}\langle\theta_{,iz}\,\theta_{,iz}\rangle + \gamma_1\,\langle f\,[(1+\epsilon)\,\Delta b - \lambda\,\Delta\theta_{,z}]\rangle.$$

Let us now eliminate the terms $<\theta_{,z}\ w_{,z}>$ and $<f\ w_{,z}>$ by selecting γ_1 and γ_2 to satisfy

$$\frac{\gamma_2}{P_r} \surd(R) = \frac{\lambda\ P_r\ (1+\epsilon)}{P_m} \quad ,$$

$$\gamma_1\ [Q - \lambda\ \surd(R)] = \frac{1+\epsilon}{P_m} \ .$$

Define

$$\lambda = Q\ \xi\ /\ \surd(R)$$

then

$$\gamma_1 = \frac{1+\epsilon}{Q\ \ P_m\ (1-\xi)} \ ,$$

$$\gamma_2 = \frac{\xi\ Q\ P_r^2\ (1+\epsilon)}{R\ P_m} \ .$$

With this choice of γ_1 and γ_2 we have

$$\frac{dE}{dt} = -D(V_i) - D(\theta) - \frac{1}{Q} D(b_\alpha) + 2\ \surd(R)\ <\theta\ w> - \frac{\epsilon}{Q} <(b_{,z})^2>$$
$$- \frac{\gamma_2}{P_r} <\theta_{,iz}\ \theta_{,iz}> + \gamma_1 <f\ [(1+\epsilon)\ \Delta b - \lambda\ \Delta\theta_{,z}]>.$$
$$- \epsilon\ \gamma_1 <f\ b_{,zz}>$$

$$= -D(V_i) - D(\theta) - \frac{1}{Q} D(b_\alpha) + 2\ \surd(R)\ <\theta\ w> - \frac{\gamma_2}{P_r} <\theta_{,iz}\ \theta_{,iz}>$$
$$- \frac{\epsilon}{Q} <(\frac{f_{,z}}{P_m} + \lambda\ \frac{P_r}{P_m}\ \theta_{,zz})^2> - \epsilon\ \gamma_1 <f\ b_{,zz}>$$
$$+ \frac{\gamma_1}{P_m} <f\ [(1+\epsilon)\ \Delta(f + \lambda\ P_r\ \theta_{,z}) - \lambda\ P_m\ \Delta\theta_{,z}]> \ .$$

Since

$$<f\ \Delta f> = -<f_{,\alpha}\ f_{,\alpha}> - <(f_{,z})^2> \ ,$$

$$<f\ \Delta\theta_{,z}> = -<f_{,\alpha}\ \theta_{,\alpha z}> - <f_{,z}\ \theta_{,zz}> \ ,$$

$$<f\ b_{,zz}> = -<f_{,z}\ b_{,z}> = -\frac{1}{P_m} <f_{,z}\ (f_{,z} + \lambda\ P_r\ \theta_{,zz})> \ ,$$

$$<\theta_{,iz}\ \theta_{,iz}> = <\theta_{,\alpha z}\ \theta_{,\alpha z}> + <(\theta_{,zz})^2>$$

then

$$\frac{dE}{dt} = - D(V_i) - D(\theta) - \frac{1}{Q} D(b_\alpha) + 2\ \surd(R) <\theta\ w> - \frac{\gamma_2}{P_r} <\theta_{,\alpha z}\ \theta_{,\alpha z}>$$
$$- \frac{\gamma_1}{P_m} (1 + \epsilon) <f_{,\alpha}\ f_{,\alpha}> + \gamma_1 \lambda \left[1 - \frac{P_r (1+\epsilon)}{P_m}\right] <f_{,\alpha}\ \theta_{,\alpha z}>$$
$$- (\frac{\gamma_2}{P_r} + \frac{\epsilon P_r^2 \lambda^2}{Q P_m^2}) <(\theta_{,zz})^2> - (\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q P_m^2}) <(f_{,z})^2>$$
$$+ \left[\gamma_1 \lambda (1 - \frac{P_r}{P_m}) - \frac{2 \epsilon \lambda P_r}{Q P_m^2}\right] <f_{,z}\ \theta_{,zz}> .$$

We now wish to find conditions under which there exists a positive constant β such that

$$\frac{dE}{dt} \leqslant - \beta E . \qquad (9.3)$$

Let

$$x = \gamma_1 \lambda (1 - \frac{1+\epsilon}{a}) ,$$

$$y = \gamma_1 \lambda (1 - \frac{1}{a}) - \frac{2 \epsilon \lambda}{Q a P_m} ,$$

$$a = \frac{P_m}{P_r}$$

then using Cauchy's inequality, we obtain

$$x <f_{,\alpha}\ \theta_{,\alpha z}> \leqslant \mu <f_{,\alpha}\ f_{,\alpha}> + \frac{x^2}{4\mu} <\theta_{,\alpha z}\ \theta_{,\alpha z}> ,$$
$$y <f_{,z}\ \theta_{,z}> \leqslant \nu <(f_{,z})^2 + \frac{y^2}{4\nu} <(\theta_{,z})^2> .$$

where μ and ν are arbitrary positive constant. Thus

$$\frac{dE}{dt} \leqslant - D(V_i) - D(\theta) - \frac{1}{Q} D(b_\alpha) + 2\ \surd(R) <\theta\ w>$$
$$- \left[\frac{\gamma_1}{P_m} (1+\epsilon) - \mu\right] <f_{,\alpha}\ f_{,\alpha}> - (\frac{\gamma_2}{P_r} - \frac{x^2}{4\mu}) <\theta_{,\alpha z}\ \theta_{,\alpha z}>$$
$$- (\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q P_m^2} - \nu) <(f_{,z})^2>$$
$$- (\frac{\gamma_2}{P_r} + \frac{\epsilon P_r^2 \lambda^2}{Q P_m^2} - \frac{y^2}{4\nu}) <(\theta_{,zz})^2> . \qquad (9.4)$$

In particular, we require to arrange that

$$\frac{\gamma_1}{P_m}(1 + \epsilon) - \mu > 0 \quad , \quad \frac{\gamma_2}{P_r} - \frac{x^2}{4\mu} > 0 \qquad (9.5)$$

and

$$\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q\,P_m^2} - \nu > 0 \quad , \quad \frac{\gamma_2}{P_r} + \frac{\epsilon\,P_r^2\,\lambda^2}{Q\,P_m^2} - \frac{y^2}{4\nu} > 0 \; . \qquad (9.6)$$

Conditions (9.5) and (9.6) impose restrictions on the value of the parameter ξ which as yet is still arbitrary within (0,1). We investigate the implication of these conditions in turn. In view of (9.5),

$$\frac{\gamma_1}{P_m}(1 + \epsilon) > \mu > \frac{P_r\,x^2}{4\,\gamma_2}$$

and so of necessity

$$\frac{\gamma_1}{P_m}(1 + \epsilon) > \frac{P_r\,x^2}{4\,\gamma_2}$$

from which we can show that

$$\xi < \frac{a}{a + \frac{(a - 1 - \epsilon)^2}{4\,(1 + \epsilon)}} \; .$$

Also from (9.6),

$$\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q\,P_m^2} > \nu > \frac{y^2}{4\left(\frac{\gamma_2}{P_r} + \frac{\epsilon\,P_r^2\,\lambda^2}{Q\,P_m^2}\right)}$$

and hence

$$\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q\,P_m^2} > \frac{y^2}{4\left(\frac{\gamma_2}{P_r} + \frac{\epsilon\,P_r^2\,\lambda^2}{Q\,P_m^2}\right)} \; .$$

$$\Leftrightarrow \left[\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q\,P_m^2}\right]\left[\frac{\lambda\,P_r\,(1 + \epsilon)}{\surd(R)\,P_m} + \frac{\epsilon\,P_r^2\,\lambda^2}{Q\,P_m^2}\right]$$

$$> \frac{1}{4}\left[\gamma_1\,\lambda\,(1 - \frac{1 + \epsilon}{a}) + \epsilon\,\gamma_1\,\lambda\,\frac{P_r}{P_m} - 2\,\epsilon\,\lambda\,\frac{P_r}{Q\,P_m^2}\right]^2$$

$$\Leftrightarrow \left[\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q\,P_m^2}\right]\left[\frac{a\,(1 + \epsilon)}{Q\,\xi} + \frac{\epsilon}{Q}\right] - \frac{1}{4}\left[\gamma_1\,(a - 1) - \frac{2\,\epsilon}{Q\,P_m}\right]^2 > 0$$

$$\Leftrightarrow (\epsilon + Q\, P_m\, \gamma_1) \left[\epsilon + \frac{a\,(1+\epsilon)}{\xi}\right] - \frac{1}{4}\left[Q\, P_m\, \gamma_1\, (a-1) - 2\,\epsilon\right]^2 > 0.$$

Substitute for γ_1 to obtain

$$a + \frac{a\,(1-\xi)}{\xi} + \frac{a\,(1+\epsilon)}{\epsilon\,\xi} - \frac{1}{4}\,\frac{(a-1)^2\,(1+\epsilon)}{\epsilon\,(1-\xi)} > 0$$

from which we can show that

$$\xi < \frac{a}{a + \dfrac{(a-1)^2\,(1+\epsilon)}{4\,(1+2\,\epsilon)}} .$$

both these restrictions on ξ must be valid and thus

$$\xi < \operatorname*{Min}_{a \geq 0} \left[\frac{a}{a + \dfrac{(a-1)^2\,(1+\epsilon)}{4\,(1+2\,\epsilon)}} \;,\; \frac{a}{a + \dfrac{(a-1-\epsilon)^2}{4\,(1+\epsilon)}}\right] . \qquad (9.7)$$

Let us define

$$\xi = \operatorname*{Min}_{a \geq 0} \left[\frac{\delta\, a}{a + \eta\, \dfrac{(a-1)^2\,(1+\epsilon)}{(1+2\,\epsilon)}} \;,\; \frac{\delta\, a}{a + \eta\, \dfrac{(a-1-\epsilon)^2}{(1+\epsilon)}}\right] \qquad (9.8)$$

and observe that this expression for ξ satisfies (9.7) provided

$$\delta \in (0,1) \quad , \quad \eta > 1/4 . \qquad (9.9)$$

Now let us write the coefficient of $<f_{,\alpha}\, f_{,\alpha}>$ in (9.4) as

$$-\left[\frac{\gamma_1}{P_m}\,(1+\epsilon) - \mu\right] = -\frac{\gamma_1}{P_m}\,(1+\epsilon)\,(1 - \frac{1}{4\,\eta}) \qquad (9.10)$$

$$\Rightarrow \qquad \mu = \frac{\gamma_1\,(1+\epsilon)}{4\,\eta\, P_m} .$$

Now substitute for μ in the coefficient of $<\theta_{,\alpha z}\, \theta_{,\alpha z}>$ in (9.4). Thus

$$-\left[\frac{\gamma_2}{P_r} - \frac{x^2}{4\,\mu}\right] = -\left[\frac{\lambda\, P_r\,(1+\epsilon)}{P_m\, \surd(R)} - \frac{x^2\,\eta\, P_m}{\gamma_1\,(1+\epsilon)}\right]$$

$$= -\left[\frac{\lambda\,(1+\epsilon)}{a\,\surd(R)} - \frac{\eta\, P_m\, \gamma_1\, \lambda^2\,(a-1-\epsilon)^2}{a^2\,(1+\epsilon)}\right] .$$

$$= -\frac{\lambda^2\,(1+\epsilon)}{a\, Q}\left[\frac{1}{\xi} - \frac{\eta\,(a-1-\epsilon)^2}{a\,(1+\epsilon)\,(1-\xi)}\right]$$

$$= -\frac{\lambda^2 (1+\epsilon)}{a^2 Q \xi (1-\xi)} \left\{a - \xi \left[a + \frac{\eta (a-1-\epsilon)^2}{1+\epsilon}\right]\right\}$$

(9.11)

Similarly, let us write the coefficient of $<(f_{,z})^2>$ in (9.4) as

$$-\left(\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q P_m^2} - \nu\right) = -\left(\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q P_m^2}\right)\left(1 - \frac{1}{4\eta}\right) \quad (9.12)$$

$$\Rightarrow \quad \nu = \frac{1}{4\eta}\left(\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q P_m^2}\right) .$$

Now substitute for ν in the coefficient of $<(\theta_{,zz})^2>$ in (9.4). Thus

$$-\left(\frac{\gamma_2}{P_r} + \frac{\epsilon P_r^2 \lambda^2}{Q P_m^2} - \frac{y^2}{4\nu}\right) = -\frac{Q P_m^2}{\epsilon + Q P_m \gamma_1}\left\{\left(\frac{\gamma_2}{P_r} + \frac{\epsilon P_r^2 \lambda^2}{Q P_m^2}\right)\left(\frac{\gamma_1}{P_m}\right.\right.$$

$$\left.\left. + \frac{\epsilon}{Q P_m^2}\right) - \eta \left[\gamma_1 \lambda \left(1 - \frac{1+\epsilon}{a}\right) + \epsilon \gamma_1 \lambda \frac{P_r}{P_m} - 2 \epsilon \lambda \frac{P_r}{Q P_m^2}\right]^2\right\}$$

$$= -\frac{\lambda^2}{a^2 Q (\epsilon + Q P_m \gamma_1)}\left\{(Q P_m \gamma_1 + \epsilon)\left(\epsilon + a \frac{1+\epsilon}{\xi}\right)\right.$$

$$\left. - \eta \left[Q P_m \gamma_1 (a-1) - 2 \epsilon\right]^2\right\}$$

$$= -\frac{\lambda^2}{a^2 Q (\epsilon + Q P_m \gamma_1)}\left\{\epsilon^2 + \epsilon a \frac{1+\epsilon}{\xi} + \epsilon Q P_m \gamma_1\right.$$

$$+ Q P_m \gamma_1 a \frac{1+\epsilon}{\xi} - \eta \left[Q^2 P_m^2 \gamma_1^2 (a-1)^2 + 4 \epsilon^2\right.$$

$$\left.\left. - 4 \epsilon Q P_m \gamma_1 (a-1)\right]\right\} .$$

After some algebra we can show that the coefficient of $<(\theta_{,zz})^2>$ has form

$$-\frac{\lambda^2}{a^2 Q \xi (1-\xi)(\epsilon + Q P_m \gamma_1)}\left\{\epsilon \xi (4\eta - 1)\left[(1+\epsilon)(a-1)\right.\right.$$

$$\left.\left. - \epsilon (1-\xi)\right] + \frac{(1+\epsilon)(1+2\epsilon)}{1-\xi}\left[a - \xi\left[a + \frac{\eta (1+\epsilon)(a-1)^2}{1+2\epsilon}\right]\right]\right\}$$

(9.13)

From (9.8)

$$\xi \leqslant \frac{\delta a}{a + \eta \frac{(a-1-\epsilon)^2}{(1+\epsilon)}}$$

$$\Rightarrow \quad a - \xi \left[a + \eta \frac{(a-1-\epsilon)^2}{1+\epsilon}\right] \geqslant a (1-\delta) . \quad (9.14)$$

Similarly from (9.8)

$$\xi \leqslant \frac{\delta\ a}{a + \eta\ \frac{(a - 1)^2\ (1 + \epsilon)}{(1 + 2\ \epsilon)}}$$

$$\Rightarrow \quad a - \xi\ \left[a + \eta\ \frac{(a - 1)^2\ (1 + \epsilon)}{1 + 2\ \epsilon}\right] \geqslant a\ (1 - \delta). \quad (9.15)$$

Using (9.14), the coefficient of $<\theta_{,\alpha z}\ \theta_{,\alpha z}>$ given by (9.11) becomes

$$- \frac{\lambda^2\ (1 + \epsilon)}{a^2\ Q\ \xi\ (1 - \xi)}\ \left\{a - \xi\ \left[a + \frac{\eta\ (a - 1 - \epsilon)^2}{1 + \epsilon}\right]\right\}$$

$$\leqslant - \frac{\lambda^2\ (1 + \epsilon)\ (1 - \delta)}{a\ Q\ \xi\ (1 - \xi)}\ . \quad (9.16)$$

Similarly, using (9.15) the coefficient of $<(\theta_{,zz})^2>$ given by (9.13) becomes

$$- \frac{\lambda^2}{a^2\ Q\ \xi\ (1 - \xi)\ (\epsilon + Q\ P_m\ \gamma_1)}\ \left\{\epsilon\ \xi\ (4\ \eta - 1)\ \left[(1 + \epsilon)\ (a - 1)\right.\right.$$

$$\left.\left. - \epsilon\ (1 - \xi)\right] + \frac{(1 + \epsilon)\ (1 + 2\ \epsilon)}{1 - \xi}\ \left[a - \xi\ \left[a + \frac{\eta\ (1 + \epsilon)\ (a - 1)^2}{1 + 2\ \epsilon}\right]\right]\right\}$$

$$\leqslant - \frac{\lambda^2}{a^2\ Q\ \xi\ (1 - \xi)\ (\epsilon + Q\ P_m\ \gamma_1)}\ \left\{\epsilon\ \xi\ (4\ \eta - 1)\ \left[(1 + \epsilon)\ (a - 1)\right.\right.$$

$$\left.\left. - \epsilon\ (1 - \xi)\right] + \frac{a\ (1 + \epsilon)\ (1 + 2\ \epsilon)\ (1 - \delta)}{1 - \xi}\right\}\ . \quad (9.17)$$

Now let us compare the coefficients of $<f_{,\alpha}\ f_{,\alpha}>$ and $<(f_{,z})^2>$ given by (9.10) and (9.12) respectively. The difference between these coefficients is

$$- \frac{\gamma_1}{P_m}\ (1 + \epsilon)\ (1 - \frac{1}{4\ \eta}) + (\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q\ P_m^2})\ (1 - \frac{1}{4\ \eta})$$

$$= - \frac{\gamma_1}{P_m}\ \left[1 + \epsilon - 1 - \frac{\epsilon}{Q\ P_m\ \gamma_1}\right]\ (1 - \frac{1}{4\ \eta})$$

$$= - \epsilon\ (\epsilon + \xi)\ \frac{\gamma_1}{P_m}\ (1 - \frac{1}{4\ \eta}) \quad < \quad 0$$

and thus we can write

$$- \frac{\gamma_1}{P_m} (1 + \epsilon)(1 - \frac{1}{4\eta}) <f_{,\alpha}\, f_{,\alpha}> - (\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q\, P_m{}^2})(1 - \frac{1}{4\eta}) <(f_{,z})^2>$$

$$\leqslant - (\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q\, P_m{}^2})\ (1 - \frac{1}{4\eta})\ <(\nabla f)^2> . \qquad (9.18)$$

Also, let us compare the coefficients of $<\theta_{,\alpha z}\ \theta_{,\alpha z}>$ and $<(\theta_{,zz})^2>$ given by (9.16) and (9.17) respectively. Again, the difference is

$$- \frac{\lambda^2}{a^2\, Q\, \xi\, (1 - \xi)\, (\epsilon + Q\, P_m\, \gamma_1)} \Big\{\epsilon\, \xi\, (4\eta - 1) \Big[(1 + \epsilon)\, (a - 1)$$

$$- \epsilon\, (1 - \xi)\Big] + \frac{a\, (1 + \epsilon)\, (1 + 2\epsilon)\, (1 - \delta)}{1 - \xi}$$

$$- a\, (1 + \epsilon)\, (1 - \delta)\, (\epsilon + Q\, P_m\, \gamma_1)\Big\}$$

$$= - \frac{\lambda^2}{a^2\, Q\, \xi\, (1 - \xi)\, (\epsilon + Q\, P_m\, \gamma_1)} \Big\{\epsilon\, \xi\, (4\eta - 1) \Big[(1 + \epsilon)\, (a - 1)$$

$$- \epsilon\, (1 - \xi)\Big] + \frac{a\, (1 + \epsilon)\, (1 + 2\epsilon)\, (1 - \delta)}{1 - \xi}$$

$$- a\, (1 + \epsilon)\, (1 - \delta)\, (\epsilon + \frac{1 + \epsilon}{1 - \xi})\Big\}$$

$$= - \frac{\lambda^2}{a^2\, Q\, \xi\, (1 - \xi)\, (\epsilon + Q\, P_m\, \gamma_1)} \Big\{\epsilon\, \xi\, (4\eta - 1) \Big[(1 + \epsilon)\, (a - 1)$$

$$- \epsilon\, (1 - \xi)\Big] + \frac{\epsilon\, a\, (1 + \epsilon)\, (1 - \delta)}{1 - \xi} - \epsilon\, a\, (1 + \epsilon)\, (1 - \delta)\Big\}$$

$$= - \frac{\epsilon\, \lambda^2}{a^2\, Q\, (1 - \xi)\, (\epsilon + Q\, P_m\, \gamma_1)} \Big\{(4\eta - 1) \Big[(1 + \epsilon)\, (a - 1)$$

$$- \epsilon\, (1 - \xi)\Big] + \frac{(1 + \epsilon)\, (1 - \delta)}{1 - \xi}\Big\} .$$

We therefore restrict η and δ such that

$$\frac{(1 + \epsilon)\, (1 - \delta)}{1 - \xi} + (4\eta - 1) \Big[(1 + \epsilon)\, (a - 1) - \epsilon\, (1 - \xi)\Big] > 0 \qquad (9.19)$$

and so in this case we guarantee that

$$\text{coefficient of } <(\theta_{,zz})^2>\ <\ \text{coefficient of } <\theta_{,\alpha z}\ \theta_{,\alpha z}>. \qquad (9.20)$$

Notice that when $\eta = 1/4$, inequality (9.19) is satisfied and hence there must be a region in the vicinty of this value at which inequality (9.19) is still satisfied. From (9.20)

$$- \frac{\lambda^2}{a^2 Q \xi (1-\xi)(\epsilon + Q P_m \gamma_1)} \Big\{ \epsilon \xi (4\eta - 1) \Big[(1+\epsilon)(a-1) - \epsilon (1-\xi) \Big] + \frac{a(1+\epsilon)(1+2\epsilon)(1-\delta)}{1-\xi} \Big\} <(\theta_{,zz})^2>$$
$$- \frac{\lambda^2 (1+\epsilon)(1-\delta)}{a Q \xi (1-\xi)} <\theta_{,\alpha z} \theta_{,\alpha z}> \leqslant - \frac{\lambda^2 (1+\epsilon)(1-\delta)}{a Q \xi (1-\xi)} <(\nabla\theta_{,z})^2>. \tag{9.21}$$

By using (9.18) and (9.21), equation (9.4) becomes

$$\frac{dE}{dt} \leqslant - D(V_i) - D(\theta) - \frac{1}{Q} D(b_\alpha) + 2 \surd(R) <\theta w>$$
$$- \Big[\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q P_m^2}\Big](1 - \frac{1}{4\eta}) <(\nabla f)^2> - \frac{Q \xi (1+\epsilon)(1-\delta)}{a R (1-\xi)} <(\nabla\theta_{,z})^2> \tag{9.22}$$

Let

$$\Gamma_1 = \Big[\frac{\gamma_1}{P_m} + \frac{\epsilon}{Q P_m^2}\Big] (1 - \frac{1}{4\eta}) > 0 ,$$

$$\Gamma_2 = - \frac{Q \xi (1+\epsilon)(1-\delta)}{a R (1-\xi)} > 0$$

and define

$$\mathcal{D} = D(V_i) + \frac{1}{Q} D(b_\alpha) + <(\nabla\theta)^2> + \Gamma_1 <(\nabla f)^2> + \Gamma_2 <(\nabla\theta_{,z})^2> ,$$

$$I = 2 <\theta w>$$

then (9.22) becomes

$$\frac{dE}{dt} \leqslant \surd(R) I - \mathcal{D}$$

i.e. $$\frac{dE}{dt} \leqslant (\surd(R) \frac{I}{\mathcal{D}} - 1) \mathcal{D} .$$

Using the Poincare-Sansone inequality

$$\pi^2 \int |\nabla\theta|^2 dS \leqslant \int |\nabla\theta_{,z}|^2 dS$$

and thus we have

$$D \geq D(V_i) + \frac{1}{Q} D(b_\alpha) + \Gamma_1 <(\nabla f)^2> + (1 + \pi^2 \Gamma_2) <(\nabla\theta)^2> .$$

In fact

$$D \geq (1 + \pi^2 \Gamma_2) \left[<(\nabla\theta)^2> + \frac{D(V_i)}{1 + \pi^2 \Gamma_2}\right]$$

where the terms in b_α and f have been discarded. Define

$$V_i = \surd(1 + \pi^2 \Gamma_2) \hat{V}_i .$$

Thus

$$D \geq (1 + \pi^2 \Gamma_2) \left[<(\nabla\theta)^2> + D(\hat{V}_i)\right] = (1 + \pi^2 \Gamma_2) D_1 .$$

Now

$$\surd(R) \frac{I}{D} \leq 2 \surd(R) \frac{\surd(1 + \pi^2 \Gamma_2) <\theta \hat{w}>}{(1 + \pi^2 \Gamma_2) D_1}$$

i.e. $$\surd(R) \frac{I}{D} \leq \surd(R^*) \max_S (\frac{I_1}{D_1})$$

where

$$R^* = \frac{R}{1 + \pi^2 \Gamma_2}$$

and S is the class of kinematically admissible fields **u** and ψ such that

S = {**u**,ψ : ∇ . **u** = 0, w = ψ = 0 on z = 0,1 and **u**,ψ are periodic in x and y}.

Thus

$$\frac{dE}{dt} \leq \left\{\surd \left[\frac{R^*}{R_0}\right] - 1\right\} D$$

where

$$\frac{1}{\surd(R_0)} = \max_S (\frac{I_1}{D_1})$$

and where R_0 is the critical Rayleigh number for the Benard problem in the absence of a magnetic field. Clearly we can write

$$\frac{dE}{dt} \leqslant - \alpha D_1 \quad , \quad \beta > 0 \tag{9.23}$$

provided

$$\surd(R^*) < \surd(R_0)$$

where $\alpha = \surd(R^*/R_0) - 1.$

It is obvious in view of (9.3) and (9.23) that

$$E(t) \leqslant E(0)\ e^{-\beta t}$$

which proves that solutions to the linearized equations (9.1) must decay exponentially in time to zero. A sufficient condition for stability is that

$$\surd(R) < \surd(R_0)\ \left[1 + \frac{\pi^2\ Q\ \xi\ (1+\epsilon)\ (1-\delta)}{a\ R\ (1-\xi)}\right]^{\frac{1}{2}}$$

$$\Rightarrow \qquad R < \frac{R_0}{2} + \left[\frac{R_0{}^2}{4} + \frac{R_0\ \pi^2\ Q\ (1+\epsilon)}{a}\ \frac{\delta\ (1-\delta)}{a\ (1-\delta) + \eta\ \alpha}\right]^{\frac{1}{2}} \tag{9.24}$$

where

$$\alpha = \text{Max}\ \left[\frac{(1+\epsilon)\ (a-1)^2}{1+2\ \epsilon}\ ,\ \frac{(a-1-\epsilon)^2}{1+\epsilon}\right].$$

Now we maximise the right hand side of inequality (9.22) with respect to the parameters δ and η. i.e. we need to maximise

$$\chi = \frac{\delta\ (1-\delta)}{a\ (1-\delta) + \eta\ \alpha} \quad .$$

We first observe that since $\alpha > 0$ then the maximum value of χ occurs when η has its minimum acceptable value. The value of η is essentially 1/4 but since we require Γ_1 to be positive then we choose a value for η close to 1/4 and such that (9.19) is valid. In particular, for fixed η,

$$\frac{\partial\chi}{\partial\delta} = \frac{a\ \delta\ (1-\delta) + (1-2\ \delta)\ [\eta\ \alpha + a\ (1-\delta)]}{[a\ (1-\delta) + \eta\ \alpha]^2} \quad .$$

To maximise χ, we need

$$a\ (1 - \delta)^2 + \eta\ \alpha\ (1 - 2\delta) = 0$$

i.e. $$a\ (1 - \delta)^2 + \eta\ \alpha\ [(1 - \delta)^2 - \delta^2] = 0$$

i.e. $$(1 - \delta)^2\ (a + \eta\ \alpha) = \delta^2\ \eta\ \alpha$$

$$\Rightarrow \quad (1 - \delta)\ \surd(a + \eta\ \alpha) = \pm\ \delta\ \surd(\eta\alpha)\ .$$

Thus

$$\delta = \frac{\surd(a + \eta\ \alpha)}{\surd(a + \eta\ \alpha) + \surd(\eta\ \alpha)}\ ,$$

$$(1 - \delta) = \frac{\surd(\eta\ \alpha)}{\surd(a + \eta\ \alpha) + \surd(\eta\ \alpha)}$$

and so $$\chi = \frac{\surd(\eta\ \alpha)\ \surd(a + \eta\ \alpha)}{[\surd(a+\eta\alpha) + \surd(\eta\alpha)]\ \{a\ \surd(\eta\alpha) + \eta\ \alpha\ [\surd(\eta\alpha) + \surd(a+\eta\alpha)]\}}$$

$$= \frac{\surd(a + \eta\ \alpha)}{[\surd(a + \eta\ \alpha) + \surd(\eta\ \alpha)]\ [a + \eta\ \alpha + \surd(\eta\ \alpha)\ \surd(a + \eta\ \alpha)]}$$

$$= \frac{1}{[\surd(a + \eta\ \alpha) + \surd(\eta\ \alpha)]^2}\ .$$

The value $\eta = 1/4$ maximises χ. i.e.

$$\chi = \frac{4}{[\surd(4\ a + \alpha) + \surd(\alpha)]^2}\ .$$

Now let us compare

$$\frac{(1 + \epsilon)\ (a - 1)^2}{1 + 2\ \epsilon} \quad \text{and} \quad \frac{(a - 1 - \epsilon)^2}{1 + \epsilon}$$

i.e. we need to find the critical value of a such that

$$\frac{(1 + \epsilon)\ (a - 1)^2}{1 + 2\ \epsilon} - \frac{(a - 1 - \epsilon)^2}{1 + \epsilon} = 0$$

$$\Rightarrow \quad (1 + \epsilon)^2\ (a - 1)^2 = (1 + 2\ \epsilon)\ (a - 1 - \epsilon)^2$$

Thus, we can show that

$$a_{crit} = \frac{(1 + \epsilon)}{\epsilon}\ [\surd(1 + 2\ \epsilon) - 1]\ .$$

We further observe that

$$a_{crit} - (1 + \epsilon) = \frac{(1 + \epsilon)}{\epsilon} [\surd(1 + 2 \epsilon) - (1 + \epsilon)] < 0$$

i.e. $\quad a_{crit} < 1 + \epsilon.$

There are two cases to consider.

<u>Case I</u> $\quad a \geqslant a_{crit}$

Here we have

$$\alpha = \frac{(a - 1)^2 (1 + \epsilon)}{1 + 2 \epsilon} ,$$

$$4 a + \alpha = \frac{4 \epsilon a + (1 + \epsilon) (a + 1)^2}{1 + 2 \epsilon}$$

Thus

$$\chi = \frac{4 (1 + 2 \epsilon)}{(1 + \epsilon) \{a - 1 + \surd[(a + 1)^2 + 4 a \epsilon / (1+\epsilon)]\}^2}$$

and (9.24) becomes

$$R < \frac{R_0}{2} + \left[\frac{R_0}{4}^2 + \frac{4 R_0 \pi^2 Q (1 + 2 \epsilon)}{a \{a - 1 + \surd[(a + 1)^2 + 4 a \epsilon / (1+\epsilon)]\}^2}\right]^{\frac{1}{2}}.$$

<u>Case II</u> $\quad a \leqslant a_{crit} < 1 + \epsilon$

Here we have

$$\alpha = \frac{(a - 1 - \epsilon)^2}{1 + \epsilon} ,$$

$$4 a + \alpha = \frac{(a + 1 + \epsilon)^2}{1 + \epsilon} .$$

Thus

$$\chi = \frac{1}{1 + \epsilon}$$

and (9.24) becomes

$$R < \frac{R_0}{2} + \left[\frac{R_0^{\,2}}{4} + \frac{R_0\ \pi^2\ Q}{a}\right]^{\frac{1}{2}}.$$

In both cases, when the magnetic parameter Q vanishes, we reduce to the classical condition of linear stability i.e. $R < R_0$. However when $Q > 0$, both cases show an enlargement of the region of stability and the stabilising effect of the magnetic field comes into play.

Conclusion

Roberts [45] and Muzikar & Pethick [46] have suggetsed that a non-linear constitutive relationship between **H** and **B** may be appropriate for certain types of materials.

A non-linear permeability has no effect on the development of instabilities through the mechanism of stationary convection and from the viewpoint of terrestial applications, this is frequently the preferred process and so we should not expect the non-linear magnetic permeability to manifest itself under terrestial circumstances. However in non-terrestial applications such as the modelling of a type II superconductor, overstability is the preferred mechanism and in this situation, the presence of the non-linear permeability retards the onset of overstable convection.

The linear stability of the magnetic Benard problem is investigated via a generalized energy theory. The magnetic field is shown to have a stabilizing effect using this generalized energy theory.

Appendix I

```
      PROGRAM ORRSOMM

* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
*                                                                       *
*  THIS PROGRAM SOLVES THE ORR-SOMMERFELD PROBLEM USING THE COMPOUND    *
*  MATRIX TECHNIQUE. A NAG ROUTINE CALLED DO2BAF IS USED TO INTEGRATE   *
*  THE SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS USING A    *
*  RUNGE-KUTTA-MERSON METHOD. IN THIS PROBLEM THE COMPOUND MATRIX       *
*  EQUATIONS ARE COMPLEX. TO DEAL WITH IT WE SPLIT IT INTO TWO SETS     *
*  OF EQUATIONS ONE FOR REAL PARTS AND THE OTHER FOR IMAGINARY PARTS.   *
*  SO THE NUMBER OF COMPOUND MATRIX EQUATIONS IS INCREASED TO 12.       *
*                                                                       *
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION E(3),VALUEX(2),VALUEY(2),Y(12),SIGMAX(3),SIGMAY(3)
     *          ,W(12,50)
      EXTERNAL FSUB
      COMMON / ONE / A,RAYNUM,SIGMAR,SIGMAI

C ** HERE THE WAVE NUMBER IS A AND THE RAYNOLDS NUMBER IS RAYNUM. ***

      A = 1.D0
      RAYNUM = 1.D4

C *** WE START BY SPECIFYING TWO VALUES FOR THE EIGENVALUE. HERE      ***
C *** SIGMAX IS THE REAL PART AND SIGMAY IS THE IMAGINARY PART.       ***

      SIGMAX(1) = 0.23D0
      SIGMAY(1) = 0.D0
      SIGMAX(2) = 0.24D0
      SIGMAY(2) = 0.005D0
      JCOUNT = 1
101    IF (JCOUNT.EQ.1) THEN
         SIGMAR = SIGMAX(1)
         SIGMAI = SIGMAY(1)
         WRITE(*,*) ' VALUE OF SIGMA IS ',SIGMAR,SIGMAI
      ELSE
         SIGMAR = SIGMAX(2)
         SIGMAI = SIGMAY(2)
         WRITE(*,*) ' VALUE OF SIGMA IS ',SIGMAR,SIGMAI
      ENDIF
      XBEGIN = -1.D0
      XEND = 1.D0
      TOL = 1.D-12
      IFAIL = 0

C *** WE NOW SET THE INITIAL CONDITIONS.                              ***

      CALL START(Y)

C *** THE NAG ROUTINE DO2BAF IS TO BE CALLED TO INTEGRATE THE         ***
C *** SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS.          ***

      CALL DO2BAF (XBEGIN,XEND,12,Y,TOL,FSUB,W,IFAIL)
```

```
C *** THE TARGET CONDITIONS.                                           ***

      IF (JCOUNT.EQ.1) THEN
         VALUEX(1) = Y(1)
         VALUEY(1) = Y(7)
         JCOUNT = 2
         GOTO 101
      ELSE
         VALUEX(2) = Y(1)
         VALUEY(2) = Y(7)

C *** THE SECANT CONVERGENT SUBROUTINE IS TO BE CALLED TO OBTAIN A ***
C *** NEW EIGENVALUE.                                                  ***

         CALL SECANT(SIGMAX,SIGMAY,VALUEX,VALUEY)
         DIFF = SQRT((SIGMAX(3)-SIGMAX(1))**2+(SIGMAY(3)-SIGMAY(1))**2)
         IF (DIFF.LE.TOL) THEN
            WRITE(*,10)
            WRITE(*,*) SIGMAX(3),SIGMAY(3)
10          FORMAT(10X,'EIGENVALUE IS ')
            STOP
         ELSE
            WRITE(*,*) ' VALUE OF DIFFERENCE IS ' , DIFF
            GOTO 101
         ENDIF
      ENDIF
      END


C *** THIS SUBROUTINE SETS THE INITIAL CONDITIONS.                     ***

      SUBROUTINE START(Y)
      DOUBLE PRECISION Y
      DIMENSION Y(12)
      DO 5 J=1,12
         Y(J) = 0.D0
5     CONTINUE
      Y(6) = 1.D0
      Y(12) = 1.D0
      RETURN
      END


C *** THIS SUBROUTINE USES THE SECANT CONVERGENT TO OBTAIN A NEW     ***
C *** EIGENVALUE.                                                      ***

      SUBROUTINE SECANT(E,F,P,Q)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION E(3),F(3),P(2),Q(2)
      DIFFX = P(2) - P(1)
      DIFFY = Q(2) - Q(1)
      DIFFSX = E(2) - E(1)
      DIFFSY = F(2) - F(1)
      TEMP = DIFFX**2 + DIFFY**2
      TEMPX = (DIFFX*P(2) + DIFFY*Q(2))/TEMP
      TEMPY = (DIFFX*Q(2) - DIFFY*P(2))/TEMP
      E(3) = E(2) - TEMPX*DIFFSX + TEMPY*DIFFSY
      F(3) = F(2) - TEMPX*DIFFSY - TEMPY*DIFFSX
```

```
      P(1) = P(2)
      Q(1) = Q(2)
      E(1) = E(2)
      F(1) = F(2)
      E(2) = E(3)
      F(2) = F(3)
      RETURN
      END

C *** THIS SUBROTINE CONATINS THE COMPOUND MATRIX EQUATIONS.        ***

      SUBROUTINE FSUB(X,Y,V)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION Y(12),V(12)
      COMMON / ONE / A,RAYNUM,SIGMAX,SIGMAY
      ALPHAX = 2.DO*A**2 + A*RAYNUM*SIGMAY
      ALPHAY = A*RAYNUM*(1.DO - SIGMAX - X**2)
      BETAX = - A**4 - RAYNUM*SIGMAY*A**3
      BETAY = 2.DO*A*RAYNUM - RAYNUM*(1.DO - SIGMAX - X**2)*A**3
      V(1) = Y(2)
      V(2) = Y(3)+Y(4)
      V(3) = ALPHAX*Y(2)+Y(5)-ALPHAY*Y(8)
      V(4) = Y(5)
      V(5) = -BETAX*Y(1)+ALPHAX*Y(4)+Y(6)+BETAY*Y(7)-ALPHAY*Y(10)
      V(6) = -BETAX*Y(2)+BETAY*Y(8)
      V(7) = Y(8)
      V(8) = Y(9)+Y(10)
      V(9) = ALPHAY*Y(2)+ALPHAX*Y(8)+Y(11)
      V(10)= Y(11)
      V(11)= -BETAY*Y(1)+ALPHAY*Y(4)-BETAX*Y(7)+ALPHAX*Y(10)+Y(12)
      V(12)= -BETAY*Y(2)-BETAX*Y(8)
      RETURN
      END
```

Appendix II

```
      PROGRAM ORRCHEB

* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
*                                                                         *
*  THIS PROGRAM SOLVES THE ORR-SOMMERFELD PROBLEM USING EXPANSIONS        *
*  IN CHEBYSHEV POLYNOMIALS. BY USING THIS METHOD THE ORR-SOMMERFELD      *
*  PROBLEM CAN BE WRITTEN IN THE FORM                                     *
*                      A X = λ B X                                        *
*  WHERE A AND B ARE COMPLEX SQUARE MATRICES. SINCE A AND B ARE           *
*  COMPLEX THEN WE HAVE TO SPLIT THEM INTO REAL AND IMAGINARY PARTS.      *
*  THE EIGENVALUES λ ARE THEN DETERMINED USING A NAG ROUTINE WHICH        *
*  USES THE QZ ALGORITHM AND THE REQUIRED EIGENVALUE IS THE ONE           *
*  WHICH HAS THE LARGEST IMAGINARY PART.                                  *
*                                                                         *
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

      PARAMETER (L=50)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION B(L,L),V(L,L),Q(L,L),AR(2*L,2*L),AI(2*L,2*L),BR(2*L,2*L)
     *,BI(2*L,2*L),ALFR(2*L),ALFI(2*L),BETA(2*L),VR(2*L,2*L),VI(2*L,2*L)
     *,ITER(2*L),C(L,L)

      LOGICAL MATV
      MATV = .FALSE.

C ***  INPUT THE VALUES OF THE WAVE NUMBER 'A', THE REYNOLDS NUMBER  ***
C ***  'R' AND THE NUMBER OF CHEBYSHEV POLYNOMIALS FOR THE REQUIRED  ***
C ***  ACCURACY 'NORDER'.                                            ***

      WRITE(*,*)  '   ENTER THE VALUE OF THE WAVE NUMBER    '
      READ(*,*) ALPHA
      WRITE(*,*) '   ENTER THE VALUE OF THE REYNOLDS NUMBER    '
      READ(*,*) R
      RALPHA = R*ALPHA
      WRITE(*,*)  'ENTER THE REQUIRED NUMBER OF CHEB. POLYS.'
      READ(*,*) NORDER

C ***  A SUBROUTINE 'DERIV' IS CALLED TO CONSTRUCT THE DERIVATIVE    ***
C ***  MATRIX.

      CALL DERIV (NORDER,L,B)

C ***  A SUBROUTINE 'MULT' IS CALLED TO MULTIPLY TWO MATRICES.       ***

      CALL MULT (NORDER,L,B,B,V)
```

```
C ***  A SUBROUTINE 'QMTRX' IS CALLED TO CONSTRUCT THE MATRIX Q.      ***

      CALL QMTRX (NORDER,L,Q)
      DO 10 I = 1,NORDER
        DO 10 J = 1,NORDER
          Q(I,J) = 0.25D0*RALPHA*Q(I,J)
          C(I,J) = 0.D0
          IF(I.EQ.J) THEN
            V(I,J) = V(I,J) - ALPHA**2
            Q(I,J) = Q(I,J) - RALPHA/2.D0
            C(I,J) = 1.D0
          ENDIF
 10   CONTINUE

C ***  BUILDING THE MATRICES AR, AI, BR AND BI SUCH THAT THE          ***
C ***  PROBLEM IS                                                     ***
C ***                [AR + i AI] X = λ [BR + i BI] X .                **
*

      DO 15 I = 1,2*NORDER
        DO 15 J = 1,2*NORDER
          AR(I,J) = 0.D0
          AI(I,J) = 0.D0
          BR(I,J) = 0.D0
          BI(I,J) = 0.D0
 15   CONTINUE
      DO 20 I = 1,NORDER
        DO 20 J = 1,NORDER
          IF(I.GE.(NORDER - 1)) THEN
            IF(I.EQ.(NORDER - 1)) AR(I,J) = 1.D0
            IF(I.EQ.NORDER) AR(I,J) = (-1.D0)**(J + 1)
          ELSE
            AR(I,J) = V(I,J)
          ENDIF
 20   CONTINUE
      DO 25 I = 1,NORDER - 2
        DO 25 J = NORDER + 1,2*NORDER
            AR(I,J) = - C(I,J - NORDER)
 25   CONTINUE
      DO 30 I = NORDER + 1,2*NORDER
        DO 30 J = 1,NORDER
          NB = (J - 1)**2
          VAL = DBLE(NB)
          IF(I.GE.(2*NORDER - 1)) THEN
            IF(I.EQ.(2*NORDER - 1)) AR(I,J) = VAL
            IF(I.EQ.2*NORDER) AR(I,J) = VAL*(-1.D0)**J
          ELSE
            AI(I,J) = - 2.D0*RALPHA*C(I - NORDER,J)
          ENDIF
 30   CONTINUE
      DO 35 I = NORDER + 1,2*NORDER - 2
       DO 35 J = NORDER + 1,2*NORDER
         AR(I,J) = V(I - NORDER,J - NORDER)
         AI(I,J) = Q(I - NORDER,J - NORDER)
         BI(I,J) = - RALPHA*C(I - NORDER,J - NORDER)
  35  CONTINUE
```

```
      EPS = X02AAF(0.D0)
      IFAIL = 0
      CALL XUFLOW(0)

C ***  THE NAG ROUTINE F02GJF IS CALLED TO SOLVE THE REQUIRED          ***
C ***  EIGENVALUE PROBLEM.

      CALL F02GJF (2*NORDER,AR,2*L,AI,2*L,BR,2*L,BI,2*L,EPS,ALFR,ALFI,
     *BETA,MATV,VR,2*L,VI,2*L,ITER,IFAIL)
      CALL XUFLOW(1)

C ***  WE NOW SPECIFY THE REAL PARTS OF THE EIGENVALUES 'ALFR(M)'     ***
C ***  AND THE IMAGINARY PARTS ALFI(M).                               ***

      M = 0
      DO 70 I = 1,2*NORDER
        IF(BETA(I).NE.0.D0) THEN
          IF(DABS(ALFR(I)/BETA(I)).GT.100.D0) GO TO 70
          IF(DABS(ALFI(I)/BETA(I)).GT.100.D0) GO TO 70
          M = M + 1
          ALFR(M) = ALFR(I)/BETA(I)
          ALFI(M) = ALFI(I)/BETA(I)
        ENDIF
 70   CONTINUE

C ***  WE NOW DETERMINE THE LARGEST IMAGINARY PART OF ALL THE          ***
C ***  EIGENVALUES. THE REQUIRED EIGENVALUE IS THEN THE ONE WHICH     ***
C ***  HAS THE LARGEST IMAGINARY PART.

      ALARGE = ALFI(1)
      AREAL = ALFR(1)
      DO 80 I = 2,M
        IF(ALARGE.GE.ALFI(I)) GO TO 80
        ALARGE = ALFI(I)
        AREAL = ALFR(I)
 80   CONTINUE
      WRITE(*,*) AREAL,ALARGE
      STOP
      END

C ***  THIS SUBROUTINE CONSTRUCTS THE DERIVATIVE MATRIX.                ***

      SUBROUTINE DERIV (NORDER,L,D)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION D(L,L)
      DO 10 I = 1, NORDER
        DO 10 J = 1, NORDER
        IF((J.GT.I).AND.(MOD(J - I,2).EQ.1))  THEN
           D(I,J) = 2.D0*DBLE(J - 1)
        ELSE
           D(I,J) = 0.D0
        ENDIF
  10  CONTINUE
      DO 20  I = 1, NORDER
        D(1,I) = .5D0*D(1,I)
  20  CONTINUE
      RETURN
      END
```

```
C ***  THIS SUBROUTINE MULTIPLIES TWO MATRICES TOGETHER.              ***

      SUBROUTINE  MULT (NORDER,L,A,B,D)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(L,L),B(L,L),D(L,L)
      DO 10 I = 1, NORDER
        DO 10 J = 1, NORDER
          S = 0.D0
          DO 20 K = 1, NORDER
            S = S + A(I,K)*B(K,J)
   20     CONTINUE
          D(I,J) = S
   10 CONTINUE
      RETURN
      END


C ***  THIS SUBROUTINE CONSTRUCTS THE MATRIX 'Q'.                     ***

      SUBROUTINE QMTRX (NORDER,L,C)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION C(L,L)
      DO 10 I = 1, NORDER
        DO 10 J = 1, NORDER
          C(I,J) = 0.D0
 10   CONTINUE
      C(2,2) = C(2,2) + 1.D0
      DO 20 I = 1, NORDER - 2
        C(I,I + 2) = 1.D0
 20   CONTINUE
      DO 30 I = 3, NORDER
        C(I,I - 2) = 1.D0
 30   CONTINUE
      C(3,1) = C(3,1) + 1.D0
      RETURN
      END
```

Appendix III

```
      PROGRAM BENVER

* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
*                                                                     *
*  THIS PROGRAM SOLVES THE BENARD PROBLEM WHEN A VERTICAL MAGNETIC    *
*  FIELD IS IMPRESSED IN THE FLUID. THE TECHNIQUE USED TO SOLVE THE   *
*  RELATED EIGENVALUE PROBLEM IS THE COMPOUND MATRIX METHOD. THE      *
*  CRITICAL RAYLEIGH NUMBER IS OBTAINED BY MINIMIZING OVER THE WAVE   *
*  NUMBER FOR THE STATIONARY CONVECTION CASE. A NAG ROUTINE CALLED    *
*  D02BAF IS USED TO INTEGRATE THE COMPOUND MATRIX EQUATIONS USING    *
*  A RUNGE-KUTTA-MERSON METHOD.                                       *
*                                                                     *
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *


      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON / ONE / Q,PR,PM,EPS
      COMMON / TWO/ Z1,Z2
      EXTERNAL FVALUE
      TOL=1.E-6
      WRITE(*,2)
2     FORMAT(10X,'ENTER TWO VALUES FOR THE WAVE NUMBER')
      READ(*,*) AL,AU
      WRITE(*,3)
3     FORMAT(10X,'ENTER TWO VALUES FOR THE RAYLEIGH NUMBER')
      READ(*,*) Z1,Z2
      WRITE(*,*) 'ENTER THE VALUES OF Q, PR, PM AND EPS  '
      READ(*,*) Q,PR,PM,EPS


C
C  A SUBROUTINE IS CALLED TO MINIMIZE OVER THE WAVE NUMBER.
C

      CALL MIN (AL,AU,TOL,AVAL,VALMIN,FVALUE)
      WRITE(*,10) AVAL
10    FORMAT(//5X,'THE CRITICAL VALUE OF THE WAVE NUMBER IS',4X,F10.5)
      WRITE(*,20)VALMIN
20    FORMAT(/5X,'THE CRITICAL VALUE OF THE RAYLEIGH NUMBER IS',4X
     *,F12.5//)
      STOP
      END
```

```
C
C THIS SUBROUTINE USES THE GOLDEN SECTION SEARCH TO MINIMIZE OVER THE
C WAVE NUMBER.
C
      SUBROUTINE MIN (A,B,TOL,VALUE,VALMIN,RAY)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DATA G/0.61803398874989000D0/, COEFF/-2.078086921235000D0/
      N=ABS(COEFF*LOG(TOL/(B-A)))
      X=A+(B-A)*G**2
      Y=A+(B-A)*G
      CALL RAY(X,VX)
      CALL RAY(Y,VY)
      DO 10 J=1,N
      IF(VX.GE.VY) THEN
      A=X
      X=Y
      VX=VY
      Y=A+(B-A)*G
      CALL RAY(Y,VY)
      ELSE
      B=Y
      Y=X
      VY=VX
      X=A+(B-A)*G**2
      CALL RAY(X,VX)
      ENDIF
      DIFF=ABS(VY-VX)
      IF(DIFF.LE.TOL) GO TO 20
10    CONTINUE
20    VALUE=0.50*(X+Y)
      CALL RAY(VALUE,VALMIN)
      RETURN
      END


C
C  THIS SUBROUTINE FINDS THE CRITICAL RAYLEIGH NUMBER FOR A GIVEN WAVE
C  NUMBER. IT USES A NAG ROUTINE CALLED D02BAF TO INTEGRATE THE
C  COMPOUND MATRIX EQUATIONS.
C
      SUBROUTINE FVALUE(ALFA,RAYVAL)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION VALUE(2),Y(70),W(70,200),RAY(3),S(2)
      EXTERNAL FCN
      COMMON / THREE/ A,R,SIGMA
      COMMON / TWO / Z1,Z2
      NCOUNT = 1
      A = ALFA
      SIGMA = 0.D0
      ERR = 1.D-6
```

```
C
C  WE START BY SPECIFYING TWO VALUES FOR THE RAYLEIGH NUMBER.
C
      RAY(1) = Z1
      RAY(2) = Z2
      JCOUNT = 1
10    IF (JCOUNT.EQ.1) THEN
        R = RAY(1)
      ELSE
        R = RAY(2)
      ENDIF
      XBEGIN = 0.D0
      XEND = 1.D0
      TOL = 1.D-10
      IFAIL = 0

C
C  WE NOW SET THE INITIAL CONDITIONS.
C
      CALL START(Y)

C
C  THE NAG ROUTINE D02BAF IS CALLED TO INTEGRATE THE COMPOUND MATRIX
C  EQUATIONS.
C
      CALL D02BAF (XBEGIN,XEND,70,Y,TOL,FCN,W,IFAIL)

C                    THE TARGET CONDITIONS.
C    (i)  FOR A FREE BOUNDARY, THE TARGET CONDITION IS Y(22).
C    (ii) FOR A RIGID BOUNDARY, THE TARGET CONDITION IS Y(11).
C

      IF(JCOUNT.EQ.1) THEN
        VALUE(1) = Y(22)
C       VALUE(1) = Y(11)
        JCOUNT  = 2
        GO TO 10
      ELSE
        VALUE(2) = Y(22)
C       VALUE(2) = Y(11)

C
C  THE SECANT CONVERGENT ROUTINE IS CALLED TO OBTAIN A NEW RAYLEIGH
C  NUMBER.
C

        CALL SECANT (RAY,VALUE)
        DIFF = ABS(RAY(3) - RAY(1))
        IF(DIFF.LE.ERR) THEN
          WRITE(*,*) ' RAYNUM IS  ',RAY(3)
          WRITE(*,*) ' WAVE NUMBER IS  ',A
          RAYVAL = RAY(3)
          RETURN
        ELSE
          GO TO 10
        ENDIF
      ENDIF
      END
```

```
C
C  THIS SUBROUTINE SETS THE INITIAL CONDITIONS.
C

      SUBROUTINE START(Y)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION Y(70)
      DO 10 I = 1,70
        Y(I) = 0.0
10    CONTINUE

C
C                          INITIAL CONDITIONS.
C    (i)   FOR A FREE BOUNDARY THE INITIAL CONDITION IS Y(49) = 1.
C    (ii)  FOR A RIGID BOUNDARY THE INITIAL CONDITION IS Y(60) = 1.
C
      Y(49) = 1.DO
C     Y(60) = 1.DO
      RETURN
      END


C
C  THIS SUBROUTINE USES THE SECANT CONVERGENT TO OBTAIN A NEW RAYLEIGH
C  NUMBER.
C

      SUBROUTINE SECANT(E,P)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION E(3),P(2)
      E(3) = E(2) + P(2)*(E(1) - E(2))/(P(2) - P(1))
      E(1) = E(2)
      E(2) = E(3)
      P(1) = P(2)
      RETURN
      END


C
C  THIS SUBROUTINE CONTAINS THE COMPOUND MATRIX EQUATIONS.
C

      SUBROUTINE FCN(X,Y,V)
      PARAMETER( LIMIT=70 )
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION Y(LIMIT),V(LIMIT)
      COMMON / ONE / Q,PR,PM,EPS
      COMMON / THREE / A,R,SIGMA
      COEFFA = - A**4 - SIGMA*A**2
      COEFFB = 2.DO*A**2 + Q + SIGMA
      COEFFC = SQRT(R)*A**2
      COEFFD = - PM*SIGMA
      COEFFE = - SQRT(R)
      COEFFF = A**2 + PR*SIGMA
      COEFFG = - Q
      COEFFH = (1.DO + EPS)*A**2 - COEFFD
```

```
  V(1) = COEFFC*Y(2) + COEFFD*Y(5)
  V(2) = Y(3) + Y(6)
  V(3) = COEFFF*Y(2) + Y(7)
  V(4) = Y(5) + Y(8)
  V(5) = COEFFH*Y(4) + Y(9)
  V(6) = COEFFB*Y(2) + Y(7) - COEFFD*Y(12) + Y(16)
  V(7) = COEFFB*Y(3) + COEFFF*Y(6) + COEFFC*Y(10) - COEFFD*Y(14)
*        + Y(17)
  V(8) = COEFFB*Y(4) + Y(9) + COEFFC*Y(11) - COEFFD*Y(15) + Y(18)
  V(9) = COEFFB*Y(5) + COEFFH*Y(8) + COEFFC*Y(12) + Y(19)
 V(10) = Y(20)
 V(11) = Y(12) + Y(13) + Y(21)
 V(12) = COEFFH*Y(11) + Y(14) + Y(22)
 V(13) = COEFFF*Y(11) + Y(14) + Y(23)
 V(14) = COEFFF*Y(12) + COEFFH*Y(13) + Y(24)
 V(15) = Y(25)
 V(16) = Y(17) - COEFFD*Y(22) + Y(36)
 V(17) = COEFFF*Y(16) + COEFFC*Y(20) - COEFFD*Y(24) + Y(37)
 V(18) = Y(19) + COEFFC*Y(21) - COEFFD*Y(25) + Y(38)
 V(19) = COEFFG*Y(1) + COEFFH*Y(18) + COEFFC*Y(22) + Y(39)
 V(20) = Y(26) + Y(40)
 V(21) = Y(22) + Y(23) + Y(27) + Y(41)
 V(22) = COEFFG*Y(2) + COEFFH*Y(21) + Y(24) + Y(28) + Y(42)
 V(23) = COEFFF*Y(21) + Y(24) + Y(29) + Y(43)
 V(24) = COEFFG*Y(3) + COEFFF*Y(22) + COEFFH*Y(23) + Y(30) + Y(44)
 V(25) = COEFFG*Y(4) + Y(31) + Y(45)
 V(26) = COEFFB*Y(20) + COEFFD*Y(33) + Y(46)
 V(27) = COEFFB*Y(21) + Y(28) + Y(29) + COEFFD*Y(34) + Y(47)
 V(28) = COEFFG*Y(6) + COEFFB*Y(22) + COEFFH*Y(27) + Y(30) + Y(48)
 V(29) = COEFFB*Y(23) + COEFFF*Y(27) + Y(30) + COEFFC*Y(32)
*        + COEFFD*Y(35) + Y(49)
 V(30) = COEFFG*Y(7) + COEFFB*Y(24) + COEFFF*Y(28) + COEFFH*Y(29)
*        + COEFFC*Y(33) + Y(50)
 V(31) = COEFFG*Y(8) + COEFFB*Y(25) + COEFFC*Y(34) + Y(51)
 V(32) = Y(33) + Y(52)
 V(33) = COEFFG*Y(10) + COEFFH*Y(32) + Y(53)
 V(34) = COEFFG*Y(11) + Y(35) + Y(54)
 V(35) = COEFFG*Y(13) + COEFFF*Y(34) + Y(55)
 V(36) = COEFFA*Y(2) + Y(37) - COEFFD*Y(42)
 V(37) = - COEFFE*Y(1) + COEFFA*Y(3) + COEFFF*Y(36) + COEFFC*Y(40)
*        - COEFFD*Y(44)
 V(38) = COEFFA*Y(4) + Y(39) + COEFFC*Y(41) - COEFFD*Y(45)
 V(39) = COEFFA*Y(5) + COEFFH*Y(38) + COEFFC*Y(42)
 V(40) = - COEFFE*Y(2) + Y(46)
 V(41) = Y(42) + Y(43) + Y(47)
 V(42) = COEFFH*Y(41) + Y(44) + Y(48)
 V(43) = COEFFE*Y(4) + COEFFF*Y(41) + Y(44) + Y(49)
 V(44) = COEFFE*Y(5) + COEFFF*Y(42) + COEFFH*Y(43) + Y(50)
 V(45) = Y(51)
      V(46)  =  -  COEFFE*Y(6)  -  COEFFA*Y(10)  +  COEFFB*Y(40)  +
COEFFD*Y(53)
*        + Y(56)
 V(47) = - COEFFA*Y(11) + COEFFB*Y(41) + Y(48) + Y(49)
*        + COEFFD*Y(54) + Y(57)
 V(48) = - COEFFA*Y(12) + COEFFB*Y(42) + COEFFH*Y(47) + Y(50)
*        + Y(58)
 V(49) = COEFFE*Y(8) - COEFFA*Y(13) + COEFFB*Y(43) + COEFFF*Y(47)
*        + Y(50) + COEFFC*Y(52) + COEFFD*Y(55) + Y(59)
```

```
 V(50) = COEFFE*Y(9) - COEFFA*Y(14) + COEFFB*Y(44) + COEFFF*Y(48)
*        + COEFFH*Y(49) + COEFFC*Y(53) + Y(60)
 V(51) = - COEFFA*Y(15) + COEFFB*Y(45) + COEFFC*Y(54) + Y(61)
 V(52) = COEFFE*Y(11) + Y(53) + Y(62)
 V(53) = COEFFE*Y(12) + COEFFH*Y(52) + Y(63)
 V(54) = Y(55) + Y(64)
 V(55) = - COEFFE*Y(15) + COEFFF*Y(54) + Y(65)
 V(56) = - COEFFE*Y(16) - COEFFA*Y(20) + COEFFD*Y(63)
 V(57) = - COEFFA*Y(21) + Y(58) + Y(59) + COEFFD*Y(64)
 V(58) = - COEFFA*Y(22) - COEFFG*Y(36) + COEFFH*Y(57) + Y(60)
 V(59) = COEFFE*Y(18) - COEFFA*Y(23) + COEFFF*Y(57) + Y(60)
*        + COEFFC*Y(62) + COEFFD*Y(65)
 V(60) = COEFFE*Y(19) - COEFFA*Y(24) - COEFFG*Y(37) + COEFFF*Y(58)
*        + COEFFH*Y(59) + COEFFC*Y(63)
 V(61) = - COEFFA*Y(25) - COEFFG*Y(38) + COEFFC*Y(64)
 V(62) = COEFFE*Y(21) + Y(63) + Y(66)
 V(63) = COEFFE*Y(22) - COEFFG*Y(40) + COEFFH*Y(62) + Y(67)
 V(64) = - COEFFG*Y(41) + Y(65) + Y(68)
 V(65) = - COEFFE*Y(25) - COEFFG*Y(43) + COEFFF*Y(64) + Y(69)
 V(66) = COEFFE*Y(27) + COEFFA*Y(32) + COEFFB*Y(62) + Y(67)
*        - COEFFD*Y(70)
 V(67) = COEFFE*Y(28) + COEFFA*Y(33) - COEFFG*Y(46) + COEFFB*Y(63)
*        + COEFFH*Y(66)
 V(68) = COEFFA*Y(34) - COEFFG*Y(47) + COEFFB*Y(64) + Y(69)
 V(69) = - COEFFE*Y(31) + COEFFA*Y(35) - COEFFG*Y(49)
*        + COEFFB*Y(65) + COEFFF*Y(68) + COEFFC*Y(70)
 V(70) = - COEFFE*Y(34) - COEFFG*Y(52)
 RETURN
 END
```

```
      PROGRAM  BENVERO

* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
*                                                                     *
*  THIS PROGRAM SOLVES THE BENARD PROBLEM WHEN A MAGNETIC FIELD IS    *
*  IMPRESSED IN THE FLUID. THE RELATED EIGENVALUE PROBLEM IS SOLVED   *
*  USING THE COMPOUND MATRIX METHOD. A NAG ROUTINE CALLED D02BAF IS   *
*  USED TO INTEGRATE THE SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL  *
*  EQUATIONS USING A RUNGE-KUTTA-MERSON METHOD. HERE THE EIGENVALUE   *
*  PROBLEM IS AN EIGHEH ORDER ONE WITH FOUR BOUNDARY CONDITIONS ON    *
*  EACH SIDE. THUS THE NUMBER OF COMPOUND MATRIX EQUATIONS WILL BE    *
*  70 EQUATIONS. USING THESE 70 EQUATIONS WE CAN SOLVE THE PROBLEM    *
*  THE PROBLEM FOR THE CASE OF STATIONARY CONVECTION.                 *
*       FOR OVERSTABILITY CASE THESE EQUATIONS ARE COMPLEX AND        *
*  SINCE THERE IS NO NAG ROUTINE WHICH INTEGRATE THESE COMPLEX        *
*  EQUATIONS, THEN WE HAVE TO SPLIT THEM INTO TWO SETS OF EQUATIONS   *
*  ONE SET FOR THE REAL PARTS AND THE OTHER FOR THE IMAGINARY PARTS.  *
*  THUS THE NUMBER OF THE COMPOUND MATRIX EQUATIONS WILL INCREASE TO  *
*  140 EQUATIONS.                                                     *
*                                                                     *
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON / ONE / Q,PR,PM,EPS
      COMMON / TWO/ Z1,Z2
      EXTERNAL FVALUE
      TOL=1.E-6
      WRITE(*,2)
2     FORMAT(10X,'ENTER TWO VALUES FOR THE WAVE NUMBER')
      READ(*,*) AL,AU
      WRITE(*,3)
3     FORMAT(10X,'ENTER TWO VALUES FOR THE RAYLEIGH NUMBER')
      READ(*,*) Z1,Z2
      WRITE(*,*) 'ENTER THE VALUES OF Q, PR, PM AND EPS  '
      READ(*,*) Q,PR,PM,EPS

C
C  A SUBROUTINE IS CALLED TO MINIMIZE OVER THE WAVE NUMBER.
C

      CALL MIN (AL,AU,TOL,AVAL,VALMIN,FVALUE)
      WRITE(*,10) AVAL
10    FORMAT(//5X,'THE CRITICAL VALUE OF THE WAVE NUMBER IS',4X,F10.5)
      WRITE(*,20) VALMIN
20    FORMAT(/5X,'THE CRITICAL VALUE OF THE RAYLEIGH NUMBER IS',4X
     *,F12.5//)
      STOP
      END
```

```
C
C  THIS SUBROUTINE USES THE GOLDEN SECTION SEARCH TO MINIMIZE OVER THE
C  THE WAVE NUMBER.
C

      SUBROUTINE MIN (A,B,TOL,VALUE,VALMIN,RAY)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DATA G/0.618033988749890000D0/, COEFF/-2.078086921235000D0/
      N=ABS(COEFF*LOG(TOL/(B-A)))
      X=A+(B-A)*G**2
      Y=A+(B-A)*G
      CALL RAY(X,VX)
      CALL RAY(Y,VY)
      DO 10 J=1,N
      IF(VX.GE.VY) THEN
      A=X
      X=Y
      VX=VY
      Y=A+(B-A)*G
      CALL RAY(Y,VY)
      ELSE
      B=Y
      Y=X
      VY=VX
      X=A+(B-A)*G**2
      CALL RAY(X,VX)
      ENDIF
      DIFF=ABS(VY-VX)
      IF(DIFF.LE.TOL) GO TO 20
10    CONTINUE
20    VALUE=0.5D0*(X+Y)
      CALL RAY(VALUE,VALMIN)
      RETURN
      END


C
C  THIS SUBROUTINE FINDS THE CRITICAL RAYLEIGH NUMBER FOR A GIVEN WAVE
C  NUMBER. IT USES A NAG ROUTINE CALLED D02BAF TO INTEGRATE THE
C  COMPOUND MATRIX EQUATIONS.
C

      SUBROUTINE FVALUE (A,RAYVAL)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION Y(140),VALUE(2),E(3),SEGMA(3),S(2),W(140,100)
      COMMON / TWO / Z1,Z2
      COMMON / THREE / G,R,H
      EXTERNAL START,FCN
      PARAMETER(IFAIL=0,B=1.D0,N=140,ERR=1.D-4,TOL=1.D-10)
      G=A

C
C  WE START BY SPECIFYING TWO VALUES FOR THE RAYLEIGH NUMBER.
C

      E(1)=Z1
      E(2)=Z2
```

```
      SEGMA(1)=1.D0
      SEGMA(2)=0.D0
      NCOUNT=1
5     IF (NCOUNT.EQ.1) THEN
        H=SEGMA(1)
      ELSE
        H=SEGMA(2)
      ENDIF
      JCOUNT=1
10    IF (JCOUNT.EQ.1) THEN
        R=E(1)
      ELSE
        R=E(2)
      ENDIF

C
C  WE NOW SET THE INITIAL CONDITIONS.
C

      CALL START(Y)
      A1=0.D0

C
C  THE NAG ROUTINE D02BAF IS CALLED TO INTEGRATE THE COMPOUND MATRIX
C  EQUATIONS
C

      CALL D02BAF(A1,B,N,Y,TOL,FCN,W,IFAIL)

C
C              THE TARGET CONDITIONS FOR THE REAL PARTS.
C  (i)   FOR A FREE BOUNDARY, THE TARGET CONDITION IS Y(21).
C  (ii)  FOR A RIGID BOUNDARY, THE TARGET CONDITION IS
C          Y(11)-(2.D0*G**2+Q)*Y(4)+Y(74)+(Q**2+4.D0*G**2*Q
C             +3.D0*G**4-H)*Y(2)-(2.D0*Q+PM*Q+3.D0*G**2)*Y(72).
C

      IF (JCOUNT.EQ.1) THEN
        VALUE(1)=Y(21)
C      VALUE(1)=Y(11)-(2.D0*G**2+Q)*Y(4)+Y(74)+(Q**2+4.D0*G**2*Q
C     *         +3.D0*G**4-H)*Y(2)-(2.D0*Q+PM*Q+3.D0*G**2)*Y(72)
        JCOUNT=2
        GO TO 10
      ELSE
        VALUE(2)=Y(21)
C      VALUE(1)=Y(11)-(2.D0*G**2+Q)*Y(4)+Y(74)+(Q**2+4.D0*G**2*Q
C     *         +3.D0*G**4-H)*Y(2)-(2.D0*Q+PM*Q+3.D0*G**2)*Y(72)

C
C  THE SECANT CONVERGENT ROUTINE IS CALLED TO OBTAIN A NEW RAYLEIGH
C  NUMBER.
C

        CALL SECANT(E,VALUE)
        DIFF=ABS(E(3)-E(1))
```

```
        IF(DIFF.LE.ERR) THEN

C            THE TARGET CONDITIONS FOR THE IMAGINARY PARTS.
C  (i)    FOR A FREE BOUNDARY, THE TARGET CONDITION IS Y(91).
C  (ii)   FOR A RIGID BOUNDARY, THE TARGET CONDITION IS
C           Y(81)-Y(4)-(2.D0*G**2+Q)*Y(74)+(2.D0*Q+PM*Q+3.D0*G**2)*Y(2)
C               +(-SEGMA(L)+Q**2+4.D0*Q*G**2+3.D0*G**4)*Y(72).
C

        IF (NCOUNT.EQ.1) THEN
          S(1)=Y(91)
C         S(1)=Y(81)-Y(4)-(2.D0*G**2+Q)*Y(74)+(2.D0*Q+PM*Q+3.D0*G**2)
C    *          *Y(2)+(-SEGMA(L)+Q**2+4.D0*Q*G**2+3.D0*G**4)*Y(72)
          NCOUNT=2
          GO TO 5
        ELSE
          S(2)=Y(91)
C         S(2)=Y(81)-Y(4)-(2.D0*G**2+Q)*Y(74)+(2.D0*Q+PM*Q+3.D0*G**2)
C     *         *Y(2)+(-SEGMA(L)+Q**2+4.D0*Q*G**2+3.D0*G**4)*Y(72)

C
C  THE SECANT CONVERGENT ROUTINE IS CALLED TO OBTAIN A NEW SEGMA.
C

          CALL SECANT(SEGMA,S)
          D=ABS(SEGMA(3)-SEGMA(1))
          IF(D.LE.ERR) THEN
            WRITE(*,*) ' RAYLEIGH NO. IS   ',E(3)
            WRITE(*,*) ' EIGENVALUE IS   ',SEGMA(3)
            WRITE(*,*) ' WAVE NO. IS  ',A
            RAYVAL=E(3)
            RETURN
          ELSE
            GO TO 5
          ENDIF
        ENDIF
        ELSE
          GO TO 10
        ENDIF
      ENDIF
      END


C
C  THIS SUBROUTINE SETS THE INITIAL CONDITIONS.
C

      SUBROUTINE START(Y)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION Y(140)
      COMMON / ONE / Q,PR,PM,EPS
      COMMON / THREE / G,R,H
      DO 100 I=1,140
        Y(I)=0.D0
100   CONTINUE
```

```
C
C             THE INITIAL CONDITIONS.
C   (i)    FOR FREE BOUNDARY, THE INITIAL CONDITION IS Y(50)=1.
C   (ii)   FOR RIGID BOUNDARY, THE INITIAL CONDITIONS ARE
C                    Y(60)=1.D0
C                    Y(67)=-(2.D0*G**2+Q)
C                    Y(69)=-H+Q**2+4.D0*G**2*Q+3.D0*G**4
C                    Y(137)=-1.D0
C                    Y(139)=2.D0*Q+PM*Q+3.D0*G**2
C

      Y(50)=1.D0
C      Y(60)=1.D0
C      Y(67)=-(2.D0*G**2+Q)
C      Y(69)=-H+Q**2+4.D0*G**2*Q+3.D0*G**4
C      Y(137)=-1.D0
C      Y(139)=2.D0*Q+PM*Q+3.D0*G**2
      RETURN
      END


C
C  THIS SUBROUTINE USES THE SECANT CONVERGENT TO OBTAIN A NEW VALUE.
C

      SUBROUTINE SECANT (R,S)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION R(3),S(2)
      R(3) = R(2) + S(2)*(R(1) - R(2))/(S(2) -S(1))
      R(1) = R(2)
      R(2) = R(3)
      S(1) = S(2)
      RETURN
      END


C
C  THIS SUBROUTINE CONTAINS THE COMPOUND MATRIX EQUATIONS.
C

      SUBROUTINE FCN(X,Y,F)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION Y(140),F(140)
      COMMON / ONE / Q,PR,PM,EPS
      COMMON / THREE / G,R,H
      ALPH1=Q+G**2*(EPS+4.D0)
      ALPH2=(1.D0+PR+PM)
      BETA1=-G**2*(EPS+2.D0)*(Q+3.D0*G**2)+H*(PR+PM+PR*PM)
      BETA2=-(3.D0*G**2*(1.D0+PR+PM)+EPS*G**2*(1.D0+PR)+PR*Q)
      GAMMA1=G**6*(4.D0+3.D0*EPS)+G**4*Q*(1.D0+EPS)-G**2*R
     *       -G**2*H*(EPS*PR+2.D0*(PR+PM+PR*PM))
      GAMMA2=(2.D0*EPS*G**4*(1+PR)+G**2*PR*Q*(1.D0+EPS)
     *       +3.D0*G**4*(1.D0+PR+PM))-H*PR*PM
      ALAM1=(1.D0+EPS)*(R-G**4)*G**4+G**4*H*(EPS*PR
     *      +PR+PM+PR*PM)
      ALAM2=G**2*PR*PM*H+G**2*PM*R-EPS*G**6*(1.D0+PR)
     *      -G**6*(1.D0+PR+PM)
```

```
 F(1)=Y(2)
 F(2)=Y(3)+Y(6)
 F(3)=Y(4)+Y(7)
 F(4)=Y(5)+Y(8)
 F(5)=Y(9)+ALPH1*Y(4)-ALPH2*Y(74)+BETA1*Y(2)-BETA2*Y(72)
 F(6)=Y(7)+Y(16)
 F(7)=Y(8)+Y(10)+Y(17)
 F(8)=Y(9)+Y(11)+Y(18)
 F(9)=Y(12)+Y(19)+ALPH1*Y(8)-ALPH2*Y(78)+BETA1*Y(6)-BETA2*Y(76)
*     -GAMMA1*Y(1)+GAMMA2*Y(71)
 F(10)=Y(11)+Y(20)
 F(11)=Y(12)+Y(13)+Y(21)
 F(12)=Y(14)+Y(22)+ALPH1*Y(11)-ALPH2*Y(81)-GAMMA1*Y(2)
*      +GAMMA2*Y(72)
 F(13)=Y(14)+Y(23)
 F(14)=Y(15)+Y(24)+ALPH1*Y(13)-ALPH2*Y(83)-BETA1*Y(10)
*      +BETA2*Y(80)-GAMMA1*Y(3)+GAMMA2*Y(73)
 F(15)=Y(25)-BETA1*Y(11)+BETA2*Y(81)-GAMMA1*Y(4)+GAMMA2*Y(74)
 F(16)=Y(17)+Y(36)
 F(17)=Y(18)+Y(20)+Y(37)
 F(18)=Y(19)+Y(21)+Y(38)
 F(19)=Y(22)+Y(39)+ALPH1*Y(18)-ALPH2*Y(88)+BETA1*Y(16)
*      -BETA2*Y(86)
 F(20)=Y(21)+Y(26)+Y(40)
 F(21)=Y(22)+Y(23)+Y(27)+Y(41)
 F(22)=Y(24)+Y(28)+Y(42)+ALPH1*Y(21)-ALPH2*Y(91)
 F(23)=Y(24)+Y(29)+Y(43)
 F(24)=Y(25)+Y(30)+Y(44)+ALPH1*Y(23)-ALPH2*Y(93)
*      -BETA1*Y(20)+BETA2*Y(90)
 F(25)=Y(31)+Y(45)-BETA1*Y(21)+BETA2*Y(91)
 F(26)=Y(27)+Y(46)
 F(27)=Y(28)+Y(29)+Y(47)
 F(28)=Y(30)+Y(48)+ALPH1*Y(27)-ALPH2*Y(97)+GAMMA1*Y(16)
*      -GAMMA2*Y(86)
 F(29)=Y(30)+Y(32)+Y(49)
 F(30)=Y(31)+Y(33)+Y(50)+ALPH1*Y(29)-ALPH2*Y(99)-BETA1*Y(26)
*      +BETA2*Y(96)+GAMMA1*Y(17)-GAMMA2*Y(87)
 F(31)=Y(34)+Y(51)-BETA1*Y(27)+BETA2*Y(97)+GAMMA1*Y(18)
*      -GAMMA2*Y(88)
 F(32)=Y(33)+Y(52)
 F(33)=Y(34)+Y(53)+ALPH1*Y(32)-ALPH2*Y(102)+GAMMA1*Y(20)
*      -GAMMA2*Y(90)
 F(34)=Y(35)+Y(54)+GAMMA1*Y(21)-GAMMA2*Y(91)
 F(35)=Y(55)+BETA1*Y(32)-BETA2*Y(102)+GAMMA1*Y(23)-GAMMA2*Y(93)
 F(36)=Y(37)
 F(37)=Y(38)+Y(40)
 F(38)=Y(39)+Y(41)
 F(39)=Y(42)+ALPH1*Y(38)-ALPH2*Y(108)+BETA1*Y(36)
*      -BETA2*Y(106)-ALAM1*Y(1)+ALAM2*Y(71)
 F(40)=Y(41)+Y(46)
 F(41)=Y(42)+Y(43)+Y(47)
 F(42)=Y(44)+Y(48)+ALPH1*Y(41)-ALPH2*Y(111)-ALAM1*Y(2)
*      +ALAM2*Y(72)
 F(43)=Y(44)+Y(49)
 F(44)=Y(45)+Y(50)+ALPH1*Y(43)-ALPH2*Y(113)-BETA1*Y(40)
*      +BETA2*Y(110)-ALAM1*Y(3)+ALAM2*Y(73)
 F(45)=Y(51)-BETA1*Y(41)+BETA2*Y(111)-ALAM1*Y(4)+ALAM2*Y(74)
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```
 F(46)=Y(47)+Y(56)
 F(47)=Y(48)+Y(49)+Y(57)
 F(48)=Y(50)+Y(58)+ALPH1*Y(47)-ALPH2*Y(117)+GAMMA1*Y(36)
*      -GAMMA2*Y(106)-ALAM1*Y(6)+ALAM2*Y(76)
 F(49)=Y(50)+Y(52)+Y(59)
 F(50)=Y(51)+Y(53)+Y(60)+ALPH1*Y(49)-ALPH2*Y(119)-BETA1*Y(46)
*      +BETA2*Y(116)+GAMMA1*Y(37)-GAMMA2*Y(107)-ALAM1*Y(7)
*      +ALAM2*Y(77)
 F(51)=Y(54)+Y(61)-BETA1*Y(47)+BETA2*Y(117)+GAMMA1*Y(38)
*      -GAMMA2*Y(108)-ALAM1*Y(8)+ALAM2*Y(78)
 F(52)=Y(53)+Y(62)
 F(53)=Y(54)+Y(63)+ALPH1*Y(52)-ALPH2*Y(122)+GAMMA1*Y(40)
*      -GAMMA2*Y(110)-ALAM1*Y(10)+ALAM2*Y(80)
 F(54)=Y(55)+Y(64)+GAMMA1*Y(41)-GAMMA2*Y(111)-ALAM1*Y(11)
*      +ALAM2*Y(81)
 F(55)=Y(65)+BETA1*Y(52)-BETA2*Y(122)+GAMMA1*Y(43)-GAMMA2*Y(113)
*      -ALAM1*Y(13)+ALAM2*Y(83)
 F(56)=Y(57)
 F(57)=Y(58)+Y(59)
 F(58)=Y(60)+ALPH1*Y(57)-ALPH2*Y(127)-ALAM1*Y(16)+ALAM2*Y(86)
 F(59)=Y(60)+Y(62)
 F(60)=Y(61)+Y(63)+ALPH1*Y(59)-ALPH2*Y(129)-BETA1*Y(56)
*      +BETA2*Y(126)-ALAM1*Y(17)+ALAM2*Y(87)
 F(61)=Y(64)+BETA2*Y(127)-BETA1*Y(57)-ALAM1*Y(18)+ALAM2*Y(88)
 F(62)=Y(63)+Y(66)
 F(63)=Y(64)+Y(67)+ALPH1*Y(62)-ALPH2*Y(132)-ALAM1*Y(20)
*      +ALAM2*Y(90)
 F(64)=Y(65)+Y(68)-ALAM1*Y(21)+ALAM2*Y(91)
 F(65)=Y(69)+BETA1*Y(62)-BETA2*Y(132)-ALAM1*Y(23)+ALAM2*Y(93)
 F(66)=Y(67)
 F(67)=Y(68)+ALPH1*Y(66)-ALPH2*Y(136)-GAMMA1*Y(56)+GAMMA2*Y(126)
*      -ALAM1*Y(26)+ALAM2*Y(96)
 F(68)=Y(69)-GAMMA1*Y(57)+GAMMA2*Y(127)-ALAM1*Y(27)
*      +ALAM2*Y(97)
 F(69)=Y(70)+BETA1*Y(66)-BETA2*Y(136)-GAMMA1*Y(59)
*      +GAMMA2*Y(129)-ALAM1*Y(29)+ALAM2*Y(99)
 F(70)=-GAMMA1*Y(62)+GAMMA2*Y(132)-ALAM1*Y(32)+ALAM2*Y(102)
 F(71)=Y(72)
 F(72)=Y(73)+Y(76)
 F(73)=Y(74)+Y(77)
 F(74)=Y(75)+Y(78)
 F(75)=Y(79)+ALPH1*Y(74)+ALPH2*Y(4)+BETA1*Y(72)+BETA2*Y(2)
 F(76)=Y(77)+Y(86)
 F(77)=Y(78)+Y(80)+Y(87)
 F(78)=Y(79)+Y(81)+Y(88)
 F(79)=Y(82)+Y(89)+ALPH1*Y(78)+ALPH2*Y(8)+BETA1*Y(76)
*      +BETA2*Y(6)-GAMMA1*Y(71)-GAMMA2*Y(1)
 F(80)=Y(81)+Y(90)
 F(81)=Y(82)+Y(83)+Y(91)
 F(82)=Y(84)+Y(92)+ALPH1*Y(81)+ALPH2*Y(11)-GAMMA1*Y(72)
*      -GAMMA2*Y(2)
 F(83)=Y(84)+Y(93)
 F(84)=Y(85)+Y(94)+ALPH1*Y(83)+ALPH2*Y(13)-BETA1*Y(80)
*      -BETA2*Y(10)-GAMMA1*Y(73)-GAMMA2*Y(3)
 F(85)=Y(95)-BETA1*Y(81)-BETA2*Y(11)-GAMMA1*Y(74)-GAMMA2*Y(4)
 F(86)=Y(87)+Y(106)
 F(87)=Y(88)+Y(90)+Y(107)
```

```
 F(88)=Y(89)+Y(91)+Y(108)
 F(89)=Y(92)+Y(109)+ALPH1*Y(88)+ALPH2*Y(18)+BETA1*Y(86)
*      +BETA2*Y(16)
 F(90)=Y(91)+Y(96)+Y(110)
 F(91)=Y(92)+Y(93)+Y(97)+Y(111)
 F(92)=Y(94)+Y(98)+Y(112)+ALPH1*Y(91)+ALPH2*Y(21)
 F(93)=Y(94)+Y(99)+Y(113)
 F(94)=Y(95)+Y(100)+Y(114)+ALPH1*Y(93)+ALPH2*Y(23)
*      -BETA1*Y(90)-BETA2*Y(20)
 F(95)=Y(101)+Y(115)-BETA1*Y(91)-BETA2*Y(21)
 F(96)=Y(97)+Y(116)
 F(97)=Y(98)+Y(99)+Y(117)
 F(98)=Y(100)+Y(118)+ALPH1*Y(97)+ALPH2*Y(27)+GAMMA1*Y(86)
*      +GAMMA2*Y(16)
 F(99)=Y(100)+Y(102)+Y(119)
 F(100)=Y(101)+Y(103)+Y(120)+ALPH1*Y(99)+ALPH2*Y(29)
*       -BETA1*Y(96)-BETA2*Y(26)+GAMMA1*Y(87)+GAMMA2*Y(17)
 F(101)=Y(104)+Y(121)-BETA1*Y(97)-BETA2*Y(27)+GAMMA1*Y(88)
*       +GAMMA2*Y(18)
 F(102)=Y(103)+Y(122)
 F(103)=Y(104)+Y(123)+ALPH1*Y(102)+ALPH2*Y(32)+GAMMA1*Y(90)
*       +GAMMA2*Y(20)
 F(104)=Y(105)+Y(124)+GAMMA1*Y(91)+GAMMA2*Y(21)
 F(105)=Y(125)+BETA1*Y(102)+BETA2*Y(32)+GAMMA1*Y(93)
*       +GAMMA2*Y(23)
 F(106)=Y(107)
 F(107)=Y(108)+Y(110)
 F(108)=Y(109)+Y(111)
 F(109)=Y(112)+ALPH1*Y(108)+ALPH2*Y(38)+BETA1*Y(106)+BETA2*Y(36)
*       -ALAM1*Y(71)-ALAM2*Y(1)
 F(110)=Y(111)+Y(116)
 F(111)=Y(112)+Y(113)+Y(117)
 F(112)=Y(114)+Y(118)+ALPH1*Y(111)+ALPH2*Y(41)-ALAM1*Y(72)
*       -ALAM2*Y(2)
 F(113)=Y(114)+Y(119)
 F(114)=Y(115)+Y(120)-BETA1*Y(110)-BETA2*Y(40)+ALPH1*Y(113)
*       +ALPH2*Y(43)-ALAM1*Y(73)-ALAM2*Y(3)
 F(115)=Y(121)-BETA1*Y(111)-BETA2*Y(41)-ALAM1*Y(74)-ALAM2*Y(4)
 F(116)=Y(117)+Y(126)
 F(117)=Y(118)+Y(119)+Y(127)
 F(118)=Y(120)+Y(128)+ALPH1*Y(117)+ALPH2*Y(47)+GAMMA1*Y(106)
*       +GAMMA2*Y(36)-ALAM1*Y(76)-ALAM2*Y(6)
 F(119)=Y(120)+Y(122)+Y(129)
 F(120)=Y(121)+Y(123)+Y(130)+ALPH1*Y(119)+ALPH2*Y(49)
*       -BETA1*Y(116)-BETA2*Y(46)+GAMMA1*Y(107)+GAMMA2*Y(37)
*       -ALAM1*Y(77)-ALAM2*Y(7)
 F(121)=Y(124)+Y(131)-BETA1*Y(117)-BETA2*Y(47)+GAMMA1*Y(108)
*       +GAMMA2*Y(38)-ALAM1*Y(78)-ALAM2*Y(8)
 F(122)=Y(123)+Y(132)
 F(123)=Y(124)+Y(133)+ALPH1*Y(122)+ALPH2*Y(52)+GAMMA1*Y(110)
*       +GAMMA2*Y(40)-ALAM1*Y(80)-ALAM2*Y(10)
 F(124)=Y(125)+Y(134)+GAMMA1*Y(111)+GAMMA2*Y(41)-ALAM1*Y(81)
*       -ALAM2*Y(11)
 F(125)=Y(135)+BETA1*Y(122)+BETA2*Y(52)+GAMMA1*Y(113)+GAMMA2*Y(43)
*       -ALAM1*Y(83)-ALAM2*Y(13)
 F(126)=Y(127)
 F(127)=Y(128)+Y(129)
```

```
      F(128)=Y(130)+ALPH1*Y(127)+ALPH2*Y(57)-ALAM1*Y(86)-ALAM2*Y(16)
      F(129)=Y(130)+Y(132)
      F(130)=Y(131)+Y(133)+ALPH1*Y(129)+ALPH2*Y(59)-BETA1*Y(126)
     *       -BETA2*Y(56)-ALAM1*Y(87)-ALAM2*Y(17)
      F(131)=Y(134)-BETA1*Y(127)-BETA2*Y(57)-ALAM1*Y(88)-ALAM2*Y(18)
      F(132)=Y(133)+Y(136)
      F(133)=Y(134)+Y(137)+ALPH1*Y(132)+ALPH2*Y(62)-ALAM1*Y(90)
     *       -ALAM2*Y(20)
      F(134)=Y(135)+Y(138)-ALAM1*Y(91)-ALAM2*Y(21)
      F(135)=Y(139)+BETA1*Y(132)+BETA2*Y(62)-ALAM1*Y(93)-ALAM2*Y(23)
      F(136)=Y(137)
      F(137)=Y(138)+ALPH1*Y(136)+ALPH2*Y(66)-GAMMA1*Y(126)
     *       -GAMMA2*Y(56)-ALAM1*Y(96)-ALAM2*Y(26)
      F(138)=Y(139)-GAMMA1*Y(127)-GAMMA2*Y(57)-ALAM1*Y(97)-ALAM2*Y(27)
      F(139)=Y(140)+BETA1*Y(136)+BETA2*Y(66)-GAMMA1*Y(129)
     *       -GAMMA2*Y(59)-ALAM1*Y(99)-ALAM2*Y(29)
      F(140)=-GAMMA1*Y(132)-GAMMA2*Y(62)-ALAM1*Y(102)-ALAM2*Y(32)
      RETURN
      END
```

Appendix IV

```
      PROGRAM NVBEN

* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
*                                                                     *
*  THIS PROGRAM SOLVES THE BENARD PROBLEM WHEN A NON-VERTICAL         *
*  MAGNETIC FIELD IS IMPRESSED IN THE FLUID. THE TECHNIQUE USED TO    *
*  SOLVE THE RELATED EIGENVALUE PROBLEM IS THE CHEBYSHEV METHOD.      *
*  HERE WE OBTAIN THE CRITICAL RAYLEIGH NUMBER BY MINIMIZING OVER     *
*  THE WAVE NUMBER. THE RESULTS PRODUCED BY THIS PROGRAM CORRESPOND   *
*  TO THE CASE WHEN INSTABILITY SETS IN AS LONGITUDINAL ROLLS. I.E.   *
*  WHEN AX = 0.                                                       *
*                                                                     *
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *


      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON / ONE / Q,PR,PM,EPS,Z1,Z2,FI,NORDER
      EXTERNAL FVALUE
      OPEN(1,FILE='/NVF2 RESULT')
      TOL=1.D-6
      WRITE(*,1)
1     FORMAT(10X,'ENTER THE NUMBER OF CHEB. POLYS.')
      READ(*,*) NORDER
      WRITE(*,2)
2     FORMAT(10X,'ENTER TWO VALUES FOR THE WAVE NUMBER')
      READ(*,*) AL,AU
      WRITE(*,3)
3     FORMAT(10X,'ENTER TWO VALUES FOR THE RAYLEIGH NUMBER')
      READ(*,*) Z1,Z2
      WRITE(*,4)
4     FORMAT(10X,'ENTER THE VALUES OF PR,PM,EPS AND Q')
      READ(*,*) PR,PM,EPS,Q

*
*  CHOOSE THE REQUIRED ANGLE.
*
      FI=0.261799335D0
C     FI=0.523598671D0
C     FI=0.785398006D0
C     FI=1.04719734D0
C     FI=1.30899620D0

*
*  A SUBROUTINE IS CALLED TO MINIMIZE OVER THE WAVE NUMBER.
*
      CALL MIN (AL,AU,TOL,AVAL,VALMIN,FVALUE)
      WRITE(1,10) AVAL
10    FORMAT(//5X,'THE CRITICAL VALUE OF THE WAVE NUMBER IS',4X,F10.5)
      WRITE(1,20)VALMIN
20    FORMAT(/5X,'THE CRITICAL VALUE OF THE RAYLEIGH NUMBER IS',4X
     *,F12.5//)
      STOP
      END
```

```
C
C  THIS SUBROUTINE USES THE GOLDEN SECTION SEARCH TO MINIMIZE OVER THE
C  WAVE NUMBER.


      SUBROUTINE MIN (A,B,TOL,VALUE,VALMIN,RAY)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DATA G/0.6180339887498900D0/, COEFF/-2.0780869212350OD0/
      N=DABS(COEFF*DLOG(TOL/(B-A)))
      X=A+(B-A)*G**2
      Y=A+(B-A)*G
      CALL RAY(X,VX)
      CALL RAY(Y,VY)
      DO 10 J=1,N
      IF(VX.GE.VY) THEN
      A=X
      X=Y
      VX=VY
      Y=A+(B-A)*G
      CALL RAY(Y,VY)
      ELSE
      B=Y
      Y=X
      VY=VX
      X=A+(B-A)*G**2
      CALL RAY(X,VX)
      ENDIF
      DIFF=DABS(VY-VX)
      IF(DIFF.LE.TOL) GO TO 20
10    CONTINUE
20    VALUE=0.5D0*(X+Y)
      CALL RAY(VALUE,VALMIN)
      RETURN
      END



C
C  THIS SUBROUTINE FINDS THE CRITICAL RAYLEIGH NUMBER FOR A GIVEN
C  WAVE NUMBER. IT USES A NAG ROUTINE CALLED F02GJF TO SOLVE THE
C  EIGENVALUE PROBLEM
C                    A X =   B X
C  WHERE A AND B ARE COMPLEX SQUARE MATRICES.
C

      SUBROUTINE FVALUE(AY,RAYNUM)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      INTEGER SIZE
      PARAMETER (L=30,SIZE=6*L)
      DIMENSION B(L,L),V(L,L),S(L,L),D1(L,L),D2(L,L),D3(L,L),
     *          ALFR(SIZE),AR(SIZE,SIZE),AI(SIZE,SIZE),BR(SIZE,SIZE),
     *          BI(SIZE,SIZE),ALFI(SIZE),BETA(SIZE),ITER(SIZE),R(50),
     *          SEGMA(50),X(2)
     COMMON / ONE / Q,PR,PM,EPS,Z1,Z2,FI,NORDER
      LOGICAL MATV
```

```
      MATV=.FALSE.
      R(1) = Z1
      R(2) = Z2
      AX=0.D0
      C=AX*TAN(FI)
      AV=AX**2+AY**2
      ALFA=AV+EPS*AY**2*SIN(FI)**2
      GAMMA=EPS*AY*SIN(FI)*COS(FI)/ALFA
      ALAMD=EPS*AV**2*COS(FI)**2/ALFA
      DELTA=4.D0*C**2*EPS*COS(FI)**2/ALFA
      ETTA=4.D0*EPS*C*AV*COS(FI)**2/ALFA

C
C  A SUBROUTINE DERIV IS CALLED TO CONSTRUCT THE DERIVATIVE MATRIX.
C

      CALL DERIV (NORDER,B)

C
C  A SUBROUTINE MULT IS CALLED TO MULTIPLY TWO MATRICES.
C

      CALL MULT (NORDER,B,B,S)
      DO 8 I=1,NORDER
        DO 8 J=1,NORDER
          V(I,J)=S(I,J)
          IF(I.EQ.J) THEN
            V(I,J)=S(I,J)-AV/4.D0
          ENDIF
8     CONTINUE
      DO 10 I=1,NORDER
        DO 10 J=1,NORDER
          D1(I,J)=0.D0
          D2(I,J)=-4.D0*Q*S(I,J)
          D3(I,J)=4.D0*V(I,J)+DELTA*S(I,J)
          IF(I.EQ.J) THEN
            D1(I,J)=1.D0
            D2(I,J)=D2(I,J)+Q*C**2
            D3(I,J)=D3(I,J)-ALAMD
          ENDIF
10    CONTINUE
      NS=6*NORDER
      ND=5*NORDER
      NR=4*NORDER
      NO=3*NORDER
      NE=2*NORDER
      DO 130 K=1,100
        IF(K.GE.3) THEN
          R(K)=(R(K-2)*SEGMA(K-1)-R(K-1)*SEGMA(K-2))/
     *              (SEGMA(K-1)-SEGMA(K-2))
        ENDIF
```

```
C
C  BUILDING UP THE MATRICES AR, AI, BR AND BI SUCH THAT THE PROBLEM IS
C          [AR + i AI] =   [BR + i BI]

      DO 20 I=1,NS
        DO 20 J=1,NS
          AR(I,J)=0.D0
          AI(I,J)=0.D0
          BR(I,J)=0.D0
          BI(I,J)=0.D0
 20   CONTINUE
      DO 25 I=1,NORDER
        DO 25 J=1,NORDER
          IF(I.LE.2) THEN
            IF(I.EQ.1) AR(I,J)=1.D0
            IF(I.EQ.2) AR(I,J)=(-1.D0)**(J+1)
          ELSE
            AR(I,J)=4.D0*V(I-2,J)
            BR(I,J)=PR*D1(I-2,J)
          ENDIF
 25   CONTINUE
      DO 30 I=3,NORDER
        DO 30 J=ND+1,NS
          AR(I,J)=SQRT(R(K))*D1(I-2,J-ND)
 30   CONTINUE
      DO 35 I=NORDER+1,NE
        DO 35 J=NORDER+1,NE
          NB=(J-1-NORDER)**2
          VAL=2.D0*DBLE(NB)
          IF(I.LE.(NORDER+2)) THEN
            IF(I.EQ.(NORDER+1)) AR(I,J)=VAL
            IF(I.EQ.(NORDER+2)) AR(I,J)=(-1.D0)**(J-NORDER)*VAL
          ELSE
            AR(I,J)=4.D0*V(I-2-NORDER,J-NORDER)
            BR(I,J)=D1(I-2-NORDER,J-NORDER)
          ENDIF
35    CONTINUE
      DO 40 I=NORDER+3,NE
        DO 40 J=NR+1,ND
            AR(I,J)=2.D0*B(I-2-NORDER,J-NR)
            AI(I,J)=C*D1(I-2-NORDER,J-NR)
40    CONTINUE
      DO 45 I=NE+3,NO
        DO 45 J=1,NORDER
          AR(I,J)=-AV*SQRT(R(K))*D1(I-2-NE,J)
45    CONTINUE
      DO 50 I=NE+1,NO
        DO 50 J=NE+1,NO
          IF(I.LE.NE+2) THEN
            IF(I.EQ.(NE+1)) AR(I,J)=1.D0
            IF(I.EQ.(NE+2)) AR(I,J)=(-1.D0)**(J-NE+1)
          ELSE
            AR(I,J)=4.D0*V(I-2-NE,J-NE)
            BR(I,J)=D1(I-2-NE,J-NE)
          ENDIF
50    CONTINUE
```

```
      DO 55 I=NE+3,NO
        DO 55 J=NO+1,NR
            BR(I,J)=-2.DO*PM*B(I-2-NE,J-NO)
            BI(I,J)=-C*PM*D1(I-2-NE,J-NO)
55    CONTINUE
      DO 60 I=NE+3,NO
        DO 60 J=ND+1,NS
            AR(I,J)=D2(I-2-NE,J-ND)
            AI(I,J)=-4.DO*C*Q*B(I-2-NE,J-ND)
60    CONTINUE
      DO 65 I=NO+1,NR
        DO 65 J=NO+1,NR
          NB=(J-1-NO)**2
          VAL=2.DO*DBLE(NB)
          IF(I.LE.(NO+2)) THEN
            IF(I.EQ.(NO+1)) AR(I,J)=VAL
            IF(I.EQ.(NO+2)) AR(I,J)=(-1.DO)**(J-NO)*VAL
          ELSE
            AR(I,J)=D3(I-2-NO,J-NO)
            AI(I,J)=-ETTA*B(I-2-NO,J-NO)
            BR(I,J)=PM*D1(I-2-NO,J-NO)
          ENDIF
65    CONTINUE
      DO 70 I=NO+3,NR
        DO 70 J=NR+1,ND
            AR(I,J)=2.DO*C*GAMMA*B(I-2-NO,J-NR)
            AI(I,J)=-AV*GAMMA*D1(I-2-NO,J-NR)
70    CONTINUE
      DO 75 I=NO+3,NR
        DO 75 J=ND+1,NS
            AR(I,J)=2.DO*Q*B(I-2-NO,J-ND)
            AI(I,J)=C*Q*D1(I-2-NO,J-ND)
75    CONTINUE
      DO 80 I=NR+3,ND
        DO 80 J=NORDER+1,NE
            AR(I,J)=2.DO*Q*B(I-2-NR,J-NORDER)
            AI(I,J)=C*Q*D1(I-2-NR,J-NORDER)
 80   CONTINUE
      DO 85 I=NR+3,ND
        DO 85 J=NO+1,NR
            BR(I,J)=-2.DO*C*PM*GAMMA*B(I-2-NR,J-NO)
            BI(I,J)=AV*PM*GAMMA*D1(I-2-NR,J-NO)
85    CONTINUE
      DO 90 I=NR+1,ND
        DO 90 J=NR+1,ND
          IF(I.LE.(NR+2)) THEN
            IF(I.EQ.(NR+1)) AR(I,J)=1.DO
            IF(I.EQ.(NR+2)) AR(I,J)=(-1.DO)**(J-NR+1)
          ELSE
            AR(I,J)=4.DO*V(I-2-NR,J-NR)
            BR(I,J)=AV*PM*D1(I-2-NR,J-NR)/ALFA
          ENDIF
90    CONTINUE
      DO 95 I=ND+3,NS
        DO 95 J=NE+1,NO
          AR(I,J)=-D1(I-2-ND,J-NE)
95    CONTINUE
```

```
      DO 100 I=ND+1,NS
        DO 100 J=ND+1,NS
          IF(I.LE.(ND+2)) THEN
            IF(I.EQ.(ND+1)) AR(I,J)=1.D0
            IF(I.EQ.(ND+2)) AR(I,J)=(-1.D0)**(J-ND+1)
          ELSE
            AR(I,J)=4.D0*V(I-2-ND,J-ND)
          ENDIF
100   CONTINUE
      IFAIL=1
      CALL XUFLOW(0)
      EPC=-1.D0

C
C  A NAG ROUTINE F02GJF IS CALLED TO SOLVE THE RELATED EIGENVALUE
C  PROBLEM.

      CALL F02GJF(NS,AR,SIZE,AI,SIZE,BR,SIZE,BI,SIZE,EPC,ALFR,ALFI,
   *             BETA,MATV,VR,SIZE,VI,SIZE,ITER,IFAIL)
      CALL XUFLOW(1)
      M=0
      DO 110 I=1,NS
        IF (BETA(I).NE.0.D0) THEN
          M=M+1
          ALFR(M)=ALFR(I)/BETA(I)
          ALFI(M)=ALFI(I)/BETA(I)
        ENDIF
110   CONTINUE
      ALARG=ALFR(1)
      DO 120 I=2,M
        IF(ALARG.GE.ALFR(I)) GO TO 120
        ALARG=ALFR(I)
120   CONTINUE
      SEGMA(K)=ALARG
      IF(K.EQ.1) GO TO 130
      IF(DABS(R(K)-R(K-1)).LE..01) THEN
        WRITE(*,*) AY,R(K),SEGMA(K)
        RAYNUM=R(K)
        RETURN
      ENDIF
130 CONTINUE
    END


C
C  THIS SUBROUTINE CONSTRUCTS THE DERIVATIVE MATRIX.
C

    SUBROUTINE DERIV (NORDER,D)
    PARAMETER(L=30)
    IMPLICIT DOUBLE PRECISION (A-H,O-Z)
    DIMENSION D(L,L)
    DO 10 I=1,NORDER
      DO 10 J=1,NORDER
```

```
          IF((J.GT.I).AND.(MOD(J-I,2).EQ.1)) THEN
            D(I,J)=2.D0*DBLE(J-1)
          ELSE
            D(I,J)=0.D0
          ENDIF
10    CONTINUE
      DO 20 I=1,NORDER
        D(1,I)=.5D0*D(1,I)
20    CONTINUE
      RETURN
      END


C
C  THIS SUBROUTINE MULTIPLIES TWO MATRICES TOGETHER.
C

      SUBROUTINE MULT (NORDER,A,B,C)
      PARAMETER(L=30)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(L,L),B(L,L),C(L,L),D(L,L)
      DO 10 I=1,NORDER
        DO 10 J=1,NORDER
          S=0.D0
          DO 20 K=1,NORDER
            S=S+A(I,K)*B(K,J)
20        CONTINUE
10    D(I,J)=S
      DO 30 I=1,NORDER
        DO 30 J=1,NORDER
          C(I,J)=D(I,J)
30    CONTINUE
      RETURN
      END
```

Appendix V

```
      PROGRAM ROTATION

* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
*                                                                     *
*  THIS PROGRAM SOLVES THE BENARD PROBLEM FOR A MHD FLUID UNDER       *
*  THE INFLUENCE OF BOTH MAGNETIC FIELD AND ROTATION. THE             *
*  TECHNIQUE USED TO SOLVE THE RELATED EIGENVALUE PROBLEM IS THE      *
*  CHEBYSHEV METHOD. THE EIGENVALUE PROBLEM IS OF ORDER 12. HERE      *
*  WE OBTAIN THE CRITICAL RAYLEIGH NUMBER BY MINIMIZING OVER THE      *
*  WAVE NUMBER.                                                       *
*                                                                     *
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON Q,T,PR,PM,EPS,Z1,Z2,NORDER
      OPEN(1,FILE='/ROTR2 RESULT')
      TOL=1.D-6
      WRITE(*,1)
1     FORMAT(10X,'ENTER THE NUMBER OF CHEB. POLY.')
      READ(*,*) NORDER
      WRITE(*,2)
2     FORMAT(10X,'ENTER TWO VALUES FOR THE WAVE NUMBER')
      READ(*,*) AL,AU
      WRITE(*,3)
3     FORMAT(10X,'ENTER TWO VALUES FOR THE RAYLEIGH NUMBER')
      READ(*,*) Z1,Z2
      WRITE(*,4)
4     FORMAT(10X,'ENTER THE VALUES OF PR,PM,EPS Q AND T')
      READ(*,*) PR,PM,EPS,Q,T
      PI=4.D0*ATAN(1.D0)
      Q=Q*PI**2
      T=T*PI**4

*
*  A SUBROUTINE IS CALLED TO MINIMIZE OVER THE WAVE NUMBER.
*

      CALL MIN (AL,AU,TOL,AVAL,RAYVAL)
      WRITE(*,10) AVAL
      WRITE(1,10) AVAL
10    FORMAT(//5X,'THE CRITICAL VALUE OF THE WAVE NUMBER IS',4X,F10.5)
      WRITE(*,20)RAYVAL
      WRITE(1,20) RAYVAL
20    FORMAT(/5X,'THE CRITICAL VALUE OF THE RAYLEIGH NUMBER IS',4X
     *,F12.5//)
      STOP
      END
```

```
*
*  THIS SUBROUTINE USES THE GOLDEN SECTION SEARCH TO MINIMIZE OVER
*  THE WAVE NUMBER.
*

      SUBROUTINE MIN (A,B,TOL,VALUE,RAY)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DATA G/0.618033988749890D0/, COEFF/-2.078086921235OOD0/
      N=DABS(COEFF*DLOG(TOL/(B-A)))
      X=A+(B-A)*G**2
      Y=A+(B-A)*G
      CALL RAYVAL (X,VX)
      CALL RAYVAL (Y,VY)
      DO 10 J=1,N
      IF(VX.GE.VY) THEN
      A=X
      X=Y
      VX=VY
      Y=A+(B-A)*G
      CALL RAYVAL(Y,VY)
      ELSE
      B=Y
      Y=X
      VY=VX
      X=A+(B-A)*G**2
      CALL RAYVAL(X,VX)
      ENDIF
      DIFF=DABS(VY-VX)
      IF(DIFF.LE.TOL) GO TO 20
10    CONTINUE
20    VALUE=0.5D0*(X+Y)
      CALL RAYVAL(VALUE,RAY)
      RETURN
      END


*
*  THIS SUBROUTINE FINDS THE CRITICAL RAYLEIGH NUMBER FOR A GIVEN
*  WAVE NUMBER. IT USES A NAG ROUTINE CALLED F02BJF TO SOLVE THE
*  EIGENVALUE PROBLEM
*                           A X = L B X
*  WHERE A AND B ARE REAL MATRICES.
*

      SUBROUTINE RAYVAL(AVALUE,RAYNUM)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      INTEGER SIZE
      PARAMETER (L=50,SIZE=6*L)
      DIMENSION
B(L,L),V(L,L),D1(L,L),Z(L,L),BB(SIZE,SIZE),AA(SIZE,SIZE)
     *          ,ALFR(SIZE),ALFI(SIZE),BETA(SIZE),ITER(SIZE),R(100),
     *          SEGMA(100),X(L,L)

      LOGICAL MATV
      COMMON Q,T,PR,PM,EPS,Z1,Z2,NORDER
      MATV=.FALSE.
      R(1)=Z1
      R(2)=Z2
```

```
*
*  A SUBROUTINE 'DERIVE' IS CALLED TO CONSTRUCT THE DERIVATIVE MATRIX.
*

      CALL DERIV (NORDER,B)

*
*  A SUBROUTINE 'MULT' IS CALLED TO MULTIPLY TWO MATRICES.
*

      CALL MULT (NORDER,B,B,V)
      DO 30 I=1,NORDER
        V(I,I)=V(I,I)-(AVALUE*0.5D0)**2
30    CONTINUE
      DO 35 I=1,NORDER
        DO 35 J=1,NORDER
          Z(I,J)=4.D0*V(I,J)
          D1(I,J)=0.D0
          IF(I.EQ.J) THEN
            Z(I,J)=Z(I,J)-EPS*(AVALUE)**2
            D1(I,J)=1.D0
          ENDIF
35    CONTINUE

*
*  THE SUBROUTINE 'MULT' IS CALLED AGAIN TO MULTIPLY TWO MATRICES.
*

      CALL MULT (NORDER,B,B,X)
      N2=2*NORDER
      N3=3*NORDER
      N4=4*NORDER
      N5=5*NORDER
      N6=6*NORDER
      DO 130 K=1,100
        IF(K.GE.3) THEN
          R(K)=(R(K-2)*SEGMA(K-1)-R(K-1)*SEGMA(K-2))/
     *         (SEGMA(K-1)-SEGMA(K-2))
        ENDIF

*
*  BUILDING UP THE MATRICES AA AND BB SUCH
*             AA X =   BB X
*  WHERE AA AND BB ARE REAL MATRICES.
*

        DO 40 I=1,N6
          DO 40 J=1,N6
            AA(I,J)=0.D0
            BB(I,J)=0.D0
40      CONTINUE
        DO 45 I=3,NORDER
          DO 45 J=1,NORDER
              AA(I,J)=4.D0*V(I-2,J)
              BB(I,J)=D1(I-2,J)
 45     CONTINUE
```

```
      DO 50 I=3,NORDER
        DO 50 J=NORDER+1,N2
          AA(I,J)=-(AVALUE)**2*SQRT(R(K))*D1(I-2,J-NORDER)
 50   CONTINUE
      DO 55 I=3,NORDER
        DO 55 J=N2+1,N3
          BB(I,J)=-2.D0*PM*B(I-2,J-N2)
 55   CONTINUE
      DO 60 I=3,NORDER
        DO 60 J=N3+1,N4
          AA(I,J)=-2.D0*SQRT(T)*B(I-2,J-N3)
 60   CONTINUE
      DO 62 I=1,NORDER
        DO 62 J=N5+1,N6
          NB=(J-1-N5)**2
          VAL=2.D0*DBLE(NB)
          IF(I.LE.2) THEN
            IF(I.EQ.1) AA(I,J)=VAL
            IF(I.EQ.2) AA(I,J)=(-1.D0)**(J-N5)*VAL
          ELSE
            AA(I,J)=-4.D0*Q*X(I-2,J-N5)
          ENDIF
 62   CONTINUE
      DO 65 I=NORDER+1,N2
        DO 65 J=NORDER+1,N2
          IF(I.LE.(NORDER+2)) THEN
            IF(I.EQ.(NORDER+1)) AA(I,J)=1.D0
            IF(I.EQ.(NORDER+2)) AA(I,J)=(-1.D0)**(J-NORDER+1)
          ELSE
            AA(I,J)=4.D0*V(I-2-NORDER,J-NORDER)
            BB(I,J)=PR*D1(I-2-NORDER,J-NORDER)
          ENDIF
 65   CONTINUE
      DO 70 I=NORDER+3,N2
        DO 70 J=N5+1,N6
            AA(I,J)=SQRT(R(K))*D1(I-2-NORDER,J-N5)
 70   CONTINUE
      DO 75 I=N2+1,N3
        DO 75 J=N2+1,N3
          IF(I.LE.(N2+2)) THEN
            IF(I.EQ.(N2+1)) AA(I,J)=1.D0
            IF(I.EQ.(N2+2)) AA(I,J)=(-1.D0)**(J-N2+1)
          ELSE
            AA(I,J)=Z(I-2-N2,J-N2)
            BB(I,J)=PM*D1(I-2-N2,J-N2)
          ENDIF
 75   CONTINUE
      DO 80 I=N2+3,N3
        DO 80 J=N5+1,N6
            AA(I,J)=2.D0*Q*B(I-2-N2,J-N5)
 80   CONTINUE
```

```
      DO 85 I=N3+1,N4
        DO 85 J=N3+1,N4
          IF(I.LE.(N3+2)) THEN
            IF(I.EQ.(N3+1)) AA(I,J)=1.D0
            IF(I.EQ.(N3+2)) AA(I,J)=(-1.D0)**(J-N3+1)
          ELSE
            AA(I,J)=4.D0*V(I-2-N3,J-N3)
            BB(I,J)=D1(I-2-N3,J-N3)
          ENDIF
85    CONTINUE
      DO 90 I=N3+3,N4
        DO 90 J=N4+1,N5
            AA(I,J)=2.D0*B(I-2-N3,J-N4)
90    CONTINUE
      DO 95 I=N3+3,N4
        DO 95 J=N5+1,N6
            AA(I,J)=2.D0*SQRT(T)*B(I-2-N3,J-N5)
95    CONTINUE
      DO 100 I=N4+3,N5
        DO 100 J=N3+1,N4
            AA(I,J)=2.D0*Q*B(I-2-N4,J-N3)
100   CONTINUE
      DO 105 I=N4+1,N5
        DO 105 J=N4+1,N5
          NB=(J-1-N4)**2
          VAL=2.D0*DBLE(NB)
          IF(I.LE.(N4+2)) THEN
            IF(I.EQ.(N4+1)) AA(I,J)=VAL
            IF(I.EQ.(N4+2)) AA(I,J)=(-1.D0)**(J-N4)*VAL
          ELSE
            AA(I,J)=4.D0*V(I-2-N4,J-N4)
            BB(I,J)=PM*D1(I-2-N4,J-N4)
          ENDIF
105   CONTINUE
      DO 110 I=N5+3,N6
        DO 110 J=1,NORDER
          AA(I,J)=-D1(I-2-N5,J)
110   CONTINUE
      DO 115 I=N5+1,N6
        DO 115 J=N5+1,N6
          IF(I.LE.(N5+2)) THEN
            IF(I.EQ.(N5+1)) AA(I,J)=1.D0
            IF(I.EQ.(N5+2)) AA(I,J)=(-1.D0)**(J-N5+1)
          ELSE
            AA(I,J)=4.D0*V(I-2-N5,J-N5)
          ENDIF
115   CONTINUE
      EPC=-1.D0
      IFAIL=1
*
*  A NAG ROUTINE F02BJF IS CALLED TO SOLVE THE RELATED EIGENVALUE
*  PROBLEM.
*

      CALL F02BJF(N6,AA,SIZE,BB,SIZE,EPC,ALFR,ALFI,BETA,MATV,
     *            VV,IV,ITER,IFAIL)
```

```
          M=0
          DO 120 I=1,N6
            IF (BETA(I).NE.0.D0) THEN
              M=M+1
              ALFR(M)=ALFR(I)/BETA(I)
              ALFI(M)=ALFI(I)/BETA(I)
            ENDIF
120       CONTINUE
          ALARG=ALFR(1)
          DO 125 I=2,M
            IF(ALARG.GE.ALFR(I)) GO TO 125
            ALARG=ALFR(I)
125       CONTINUE
          SEGMA(K)=ALARG
          WRITE(*,*) ALARG
          IF(K.EQ.1) GO TO 130
          IF(DABS(R(K)-R(K-1)).LE..0001) THEN
            WRITE(*,*) R(K),SEGMA(K)
            WRITE(*,*) AVALUE
            RAYNUM=R(K)
            RETURN
          ENDIF
130     CONTINUE
        END


*
*  THIS SUBROUTINE CONSTRUCTS THE DERIVATIVE MATRIX.
*

        SUBROUTINE DERIV (NORDER,D)
        PARAMETER(L=50)
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        DIMENSION D(L,L)
        DO 10 I=1,NORDER
          DO 10 J=1,NORDER
            IF((J.GT.I).AND.(MOD(J-I,2).EQ.1)) THEN
              D(I,J)=2.D0*DBLE(J-1)
            ELSE
              D(I,J)=0.D0
            ENDIF
10      CONTINUE
        DO 20 I=1,NORDER
          D(1,I)=.5D0*D(1,I)
20      CONTINUE
        RETURN
        END
```

```
*
*  THIS SUBROUTINE MULTIPLIES TWO MATRICES TOGETHER.
*

      SUBROUTINE MULT (NORDER,A,B,C)
      PARAMETER(L=50)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(L,L),B(L,L),C(L,L),D(L,L)
      DO 10 I=1,NORDER
        DO 10 J=1,NORDER
          S=0.D0
          DO 20 K=1,NORDER
            S=S+A(I,K)*B(K,J)
20        CONTINUE
10    D(I,J)=S
      DO 30 I=1,NORDER
        DO 30 J=1,NORDER
          C(I,J)=D(I,J)
30    CONTINUE
      RETURN
      END
```

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