ESTIMATES OF THE NUMBER OF PRIMES REPRESENTABLE BY QUADRATIC POLYNOMIALS IN TWO VARIABLES

by

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A thesis presented to the

University of Glasgow

Faculty of Science

for the degree of

Doctor of Philosophy

May 1990

CONTENTS

PREFACE

The thesis is submitted in accordance with the degree of Doctor of Philosophy in the University of Glasgow. It presents the results of research undertaken by the author between October 1985 and October 1888.

All the results of the thesis are the original work of the author except for the instances indicated within the text.

I would like to express my gratitude to my supervisor Dr. M.K.N Nair for suggesting the subject of the thesis and for his assistance and encouragement throughout the research period.

Also to Professor Ogden for all assistance and patience.

I should like to thank the Science and Engineering Research Council for financing the research through an SERC studentship.

Last but by no means least many thanks to my family for listening.

SUMMARY

The objective of this thesis is to estimate the functions

$$F(x,y,z) = \left|\{(n,m);\ 0<n\leq x,\ n\equiv \ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv \ell_2 \bmod k_2,\ ((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i), \prod_{p<z} p)=1\}\right|$$

for n and m integers, and

$$P(x,y,z) = \left|\{(q,r);\ 0<q\leq x,\ q\equiv \ell_1 \bmod k_1,\ 0<r\leq y,\ r\equiv \ell_2 \bmod k_2,\ ((aq^2+bq+c)r^2+(dq^2+eq+f)r+(gq^2+hq+i), \prod_{p<z} p)=1\}\right|$$

for q and r primes.

In Chapter One we give a series of lemmas relating to the ensuing chapters. In Chaper Two we deal with the function F(x,y,z) for a=b=c=0, and in Chapter Three with P(x,y,z) for a=b=c=0.

In Chapters Four and Five the major theorems of the thesis are presented.

INTRODUCTION

Nair and Perelli in their paper "Sieve Methods and class-number problems I" derived an asymptotic formula for the function

$$S(x,y,z) = \left|\{(n,m);\ 0<n\leq x,\ 0<m\leq y,\ (n^2+m, \prod_{p<z} p)=1\}\right|$$

where the product Π ranges over all primes less than z, and where $z\leq\max(x,y)$. Their approach was based on the observation that S(x,y,z) can be written in two different ways ie.

$$\sum_{0<n\leq x} \left|\{m;\ 0<m\leq y,\ (n^2+m, \prod_{p<z} p)=1\}\right| = S(x,y,z)$$

$$= \sum_{0<m\leq y} \left|\{n;\ 0<n\leq x,\ (n^2+m, \prod_{p<z} p)=1\}\right| .$$

A simple and explicit estimate of the function within the first summation sign may be given whenever $z\leq y$. This immediately gives an initial estimate of the second version of S(x,y,z). But to complete the theorem it is required that we extend the estimate to z within the range $y\leq z\leq x$. The best available estimate of $\left|\{n;\ 0<n\leq x,\ (n^2+m, \prod_{p<z} p)=1\}\right|$ for $z\leq x$, given by Halberstam and Richert [2] involves the product $\prod_{p<z} \left(1-\frac{\rho_m(p)}{p}\right)$ where $\rho_m(p)=|\{n: n^2\equiv -m \mod p\}|$.

The aim of this thesis is to try and extend these arguments to the most general quadratic case

$$F(x,y,z) = \left|\{(n,m);\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \mod k_1,\ \beta<m\leq\beta+y,\ m\equiv\ell_2 \mod k_2,\right.$$
$$\left.((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i), \prod_{p<z} p)=1\}\right|$$

and then the same involving primes.

Rather than launch into the complexities of the most general case which is quadratic in both n and m, it was decided that a simpler approach would be taken whereby we begin with the most general case with the qualification, as in the case dealt with by Nair and Perelli, that m is linear only.

We examine the function

$$S(x,y,z) = \left|\{(n,m);\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leqslant y,\ m\equiv\ell_2 \bmod k_2,\ ((an^2+bn+c)m+(dn^2+en+f),\prod_{p<z}p)=1\}\right|.$$

This way many of the arguments that will subsequently be used in an evaluation of F(x,y,z) can be developed with a minimum of complication. Other benefits to this approach include the fact that although subsequently we are only able to find an upper bound on F(x,y,z), an asymptotic formula for S(x,y,z) may be found. Furthermore the associated error terms are effectively computable. The resulting theorem is Theorem One of the thesis.

The approach to finding an asymptotic formula for S(x,y,z) is in essence that of Nair and Perelli's. In the following I aim both to clarify the general direction and at the same time to highlight points of departure from the original paper.

As explained above we write S(x,y,z) in two different ways, namely

$$\sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv\ell_1 \bmod k_1}} \left|\{m;\ 0<m\leqslant y,\ m\equiv\ell_2 \bmod k_2,\ ((an^2+bn+c)m+(dn^2+en+f),\prod_{p<z}p)=1\}\right|$$

$$= S(x,y,z) =$$

$$\sum_{\substack{0<m\leqslant y \\ m\equiv\ell_2 \bmod k_2}} \left|\{n;\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ ((an^2+bn+c)m+(dn^2+en+f),\prod_{p<z}p)=1\}\right|.$$

In Step One of the proof of Theorem One we find an asymptotic formula for S(x,y,z) whenever $z\leqslant y/k_2$ using the first of these formulations. We firstly remove from the sum any cases trivially equal to zero. An asymptotic formula for

$$\left|\{m;\ 0<m\leqslant y,\ m\equiv\ell_2 \bmod k_2,\ ((an^2+bn+c)m+(dn^2+en+f),\prod_{p<z}p)=1\}\right|$$

in all other cases may then be given explicitely. Summation over n gives a formula for S(x,y,z) whenever $z\leqslant y/k_2$.

Were $y/k_2 > x/k_1$ then the theorem would be complete. If however $x/k_1 > y/k_2$ then in Step Two we turn to the second formulation of $S(x,y,z)$ and attempt to find an asymptotic formula for

$$\left|\{n;\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ ((an^2+bn+c)m+(dn^2+en+f), \prod_{p<z} p)=1\}\right|$$

whenever $z \leq x/k_1$.

This attempt leaves us with the sum

$$\sum_{\substack{0<m\leq y \\ m\equiv\ell_2 \bmod k_2 \\ (m,z)\ \text{app}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \qquad (1)$$

to evaluate if we are to complete the theorem where

$\rho_m(p)=|\{s \bmod p:\ s^2\equiv g_m \bmod p\}|$

for a quadratic function g_m, and where "(m,z) app" is some set of conditions given explicitely in the text. We do however have some information on (1).

If we assume that $z \leq y/k_2 \leq x/k_1$ then a comparison with the formulation of $S(x,y,z)$ given in Step One gives an asymptotic formula for (1). This is the springboard from which we develop the rest of the theorem.

Now $\rho_m(p)$ is closely related to the Legendre symbol, a relationship made explicit in Step Three. Excluding the cases where g_m is a square (Step Four), the observation is made that $\prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$ may be written as

$$\prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) c(g_m,z)$$

for some function $c(g_m,z)$, and where $\chi(p)$ is the Kronecker Symbol. (Step Five)

In this way we reduce the problem to one whereby we must find an asymptotic formula for

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\text{ app}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m,z)$$

whenever $z\geq y/k_2$.

We now see that if we were able to write this sum in terms of the sum

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z_0)\text{ app}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m,z_0)$$

for some $z_0 \leq y/k_2$ (in the proof taken to be $\exp(27(\ln y/k_2)^{\frac{1}{2}})$) then we would have our asymptotic formula as required.

Straightforward arguments alone are required to show that the dependence of $c(g_m,z)$ on z may be removed (Step Six), and it is easily demonstrated that the dependence of the conditions "(m,z) app" on z may be removed. This leaves only the dependence on z of the product

$$\prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)$$

as a problem.

Fortunately, for z relatively large, this product may be written in terms of the "smaller" product

$$\prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right)$$

in the majority of cases. (Step Seven). These cases we denote "good". The minority that resist such rewriting we denote "bad". The remainder of the theorem is essentially concerned with trying to find an upper bound on

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\text{ app}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m,z)$$

for these "bad" cases.

We can find an upper bound sufficient for our purposes if we place an upper bound on z, namely $z \leq \exp(y^{1/17})$. (Step Eight) However to make the theorem as broad as possible we really require a bound covering a wider range.

In Step Nine we make use of the fact that

$$\prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) \ll \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right)$$

with at most one exceptional modulus to reduce the problem yet further. It leaves us with the relatively narrow problem of finding an upper bound on

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \mathrm{app}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)^{c(g_m,z)}$$

for $z>\exp(y^{1/17})$ for this one possible exceptional modulus. Unfortunately this is the most stubborn case of all. To tackle it we firstly find an upper bound on

$$\prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)^{c(g_m,z)}$$

involving the product

$$\prod_{N\beta<z} \left(1-\frac{1}{N\beta}\right)$$

where the $N\beta$ represent the norms of prime ideals in $\mathbb{Q}(\sqrt{g_m})$. (Step Ten). But

$$\prod_{N\beta<z} \left(1-\frac{1}{N\beta}\right) \ll \frac{1}{L(1,\chi_D)\ln z}$$

whenever $z \gg D^6$. So to find an upper bound on

$$\prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)^{c(g_m,z)}$$

for this final case we must find an upper bound on $L(1,\chi_D)^{-1}$. Such a bound is given by the class number formula together with the Gross-Zagier theorem [11] which gives an upper bound on h(d), the class number, for d<0.

This effectively completes the theorem. The final piecing together of all the various strands is completed in Step Twelve.

It is convenient in Theorem One to assume that the polynomials in n of S(x,y,z) ie an^2+bn+c and dn^2+en+f, have no common factors. Chapter Two concludes with an examination of the alternative cases. The results are summarised in Theorem Two.

Having concluded the integer case involving a linear variable it is natural that we should consider whether the same arguments may be applied to the function involving primes,

$$P(x,y,z) = \left|\{(q,r);\ \alpha<q\leq\alpha+x,\ q\equiv\ell_1 \bmod k_1,\ 0<r\leq y,\ r\equiv\ell_2 \bmod k_2,\ ((aq^2+bq+c)r+(dq^2+eq+f),\prod_{p<z}p)=1\}\right|$$

where both the qs and rs are prime.

Following the route of Theorem One and writing P(x,y,z) in two different forms namely

$$\sum_{\substack{\alpha<q\leq\alpha+x \\ q\equiv\ell_1 \bmod k_1}} \left|\{r;\ 0<r\leq y,\ r\equiv\ell_2 \bmod k_2,\ ((aq^2+bq+c)r+(dq^2+eq+f),\prod_{p<z}p)=1\}\right|$$

$$= P(x,y,z) =$$

$$\sum_{\substack{0<r\leq y \\ r\equiv\ell_2 \bmod k_2}} \left|\{q;\ \alpha<q\leq\alpha+x,\ q\equiv\ell_1 \bmod k_1,\ ((aq^2+bq+c)r+(dq^2+eq+f),\prod_{p<z}p)=1\}\right|$$

quickly leads to difficulties, as a study of the right hand side of this equation requires that we take α to be 0 and furthermore the subsequent error terms turn out to be non-computable. As an alternative approach we study the function

$$T(x,y,z) = \Big|\{(n,q);\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ 0<q\leq y,\ q\equiv\ell_2 \bmod k_2,$$
$$(((an^2+bn+c)q+(dn^2+en+f))n, \prod_{p<z} p)=1\}\Big|.$$

We may derive an upper bound on $T(x,y,z)$ following the method of proof of Theorem One. Then an application of the observation that

$$P(x,y,z) \leq T(x,y,z) + O\left[\frac{y}{\varphi(k_2)\ln y}\cdot\frac{z}{\varphi(k_1)\ln^{z}/k_1}\right]$$

completes our estimate of $P(x,y,z)$. This is stated in Theorem Four.

In Chapter Four we turn to the most general integer case. Here the function we desire an upper bound on is

$$F(x,y,z) = \Big|\{(n,m);\ 0<n\leq x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2,$$
$$((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i), \prod_{p<z} p)=1\}\Big|$$

for $z\leq\max({}^{y}/k_2, {}^{x}/k_1)$.

Writing, as previously, $F(x,y,z)$ in two different ways, ie

$$\sum_{\substack{0<n\leq x\\ n\equiv\ell_1 \bmod k_1}} \Big|\{m;\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2,\ ((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i), \prod_{p<z} p)=1\}\Big|$$

$$= F(x,y,z) =$$

$$\sum_{\substack{0<m\leq y\\ m\equiv\ell_2 \bmod k_2}} \Big|\{n;\ 0<n\leq x,\ n\equiv\ell_1 \bmod k_1,\ ((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i), \prod_{p<z} p)=1\}\Big|$$

we would, if we were to follow the argument of Theorem One, require an asymptotic formula for one of the functions within the summation sign for $z\leq\min({}^{y}/k_2, {}^{x}/k_1)$. However either function gives an asymptotic formula involving the product

$$\prod_{p<z} \left(1-\frac{\rho(p)}{p}\right)$$

where $\rho(p)$ is a function of the form

$$\rho(p)=|\{s(\bmod p):\ s^2\equiv A \bmod p\}|$$

for A some quartic function in either m or n. Previously we had, for $z<y/k_2$, an asymptotic formula involving the uncomplicated product

$$\prod_{p<z}\left(1-\frac{1}{p}\right)$$

from which to begin the proof and we should have liked the same in this instance.

However if we assume that $z<y/k_2<x/k_1$ for instance, then we may find an upper bound on the function

$$\sum_{\substack{0<n<x \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid k_2}}\left(1-\frac{\rho_n(p)}{p}\right) \qquad (2)$$

appearing in the estimation of the first formulation of F(x,y,z). We may use this upper bound as a starting point for a general theorem. The construction of the upper bound uses many of the arguments developed in Theorem One.

Firstly we write (2) in terms of

$$\sum_{\substack{0<n<y \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid k_2}}\left(1-\frac{\rho_n(p)}{p}\right)$$

which is permissable so long as we assume that $z<y$. We then write this latter sum in terms of

$$\sum_{\substack{0<n<\exp(\ln^{\frac{1}{2}} y) \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<(5\ln y)^{50} \\ p\nmid k_2}}\left(1-\frac{\rho_n(p)}{p}\right);$$

and this in terms of

$$\sum_{\substack{0<n\leq\exp(\ln^{\frac{1}{4}}y)\\ n\equiv\ell_1\bmod k_1\\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<(5\ln y)^{25}\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$$

etc. gradually reducing the range over which we extend both the sum and the product. In this way the sum is eventually brought to a manageable form, so that we may find a reasonable upper bound on the sum (2) as we require. From this starting point we are able to construct an upper bound on $F(x,y,z)$ using the methods developed in Theorem One. (Theorem Five) Unfortunately the proof introduces non-computable error terms into the upper bound.

The final few pages of the chapter are concerned with demonstrating how the ideas outlined above may be adapted to cover the case where n and m within $F(x,y,z)$ are not restricted to $0<n\leq x$ and $0<m\leq y$. Here we examine the function

$$F(x,y,z) = \Big|\{(n,m);\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1\bmod k_1,\ \beta<m\leq\beta+y,\ m\equiv\ell_2\bmod k_2,\ ((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i),\prod_{p<z}p)=1\}\Big|.$$

Chapter Five covers the same ground as Chapter Four but for primes rather than integers. The function we are concerned with here is

$$P(x,y,z) = \Big|\{(q,r);\ 0<q\leq x,\ q\equiv\ell_1\bmod k_1,\ 0<r\leq y,\ r\equiv\ell_2\bmod k_2,\ ((aq^2+bq+c)r^2+(dq^2+eq+f)r+(gq^2+hq+i),\prod_{p<z}p)=1\}\Big|$$

for q and r both primes.

Here, for reasons that are given within the text, to find our starting point we examine instead the function

$$T(x,y,z) = \left|\{(n,q);\ 0<n\leq x,\ n\equiv \ell \bmod k,\ 0<q\leq y,\ q\equiv \alpha \bmod \beta,\ ((an^2+bn+c)q^2+(dn^2+en+f)q+(gn^2+hn+i), \prod_{p<z} p)=1\}\right|$$

which clearly has much in common with P(x,y,z). Then by adapting the methods of Chapters Three and Four an upper bound on P(x,y,z) may be constructed.

In Chapter Six we make the observation that the methods employed throughout the previous five chapters may be applied to functions of the type

$$\Phi_k(x,y) = \left|\{(n,m);\ n\leq x,\ m\leq y,\ ((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i),\ k)=1\}\right|.$$

A general theorem is not given but a short outline of the direction a proof might take is included.

Finally a note on the layout of the thesis. Chapter Two and onwards covers topics as considered in this introduction. Chapter One however is of a different format. It consists of a somewhat disparate collection of lemmas, each of which (apart from Lemma 5.2) is referred to at some point in the rest of the thesis. Although to an extent these lemmas are ordered as they appear in the ensuing chapters, whenever lemmas are considered to follow similar themes they are grouped together. Since Chapter One follows no apparent rational progression the reader may prefer to begin with Chapter Two and refer back to the lemmas as they arise in the proof. (The penalty paid for this is that the continuity of the proofs of the theorems will be broken.) Should this approach be taken attention is drawn to Lemma 5.2 of page 61. Although Lemma 5.2 makes no further appearance in the thesis it is included as a natural successor to Lemma 5.1.. It is also considered to be of interest in its own right. We show that, whenever $2 \leq D \leq x$,

$$L(1,\chi_D) = \prod_{p \leq \ln^{2\alpha} x} \left(1 - \frac{\chi_D(p)}{p}\right)^{-1} \left(1 + O(\exp(-c(\ln\ln x)^{\frac{1}{4}}))\right)$$

holds with at most $O\left(\left[\frac{x}{\ln\ln x}\right]^{\frac{1}{2}}\right)$ exceptions. The proof is an optimisation of the methods of proof of Elliott in Lemma 22.8. [8].

NOTATION

For symbols that occur frequently within the proof of theorems it may be helpful to have a page reference denoting where that symbol is introduced. A word of caution; the same symbols are often used within different theorems but their definitions may not be completely consistent across theorems. Consequently we subdivide into theorems.

THEOREM ONE

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$S(x,y,z)$	71	$M(y,z,n)$	75	$\chi_D(n), D$	89
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$\gamma(w)$	72	u	75	z_0	91
$\Gamma_z(w)$	72	$N(y,z,m)$	79	"m app"	93
F	72	s_m	79	$c(g_m)$	94
$G(x,\alpha)$	73	v	80	g_m good, bad	100
λ	73	$\rho_m(p)$	80	$\overline{s}, \overline{D}$	114
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ξ, η, θ	73	g_m	84		

THEOREM TWO

	Page		Page		Page
$S(x,y,z)$	130	M	130	$M(y,z)$	131
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δ	130	v	131		

THEOREM THREE

	Page		Page		Page
$S(x,y,z)$	135	$M(y,z,n)$	136	"(m,z) app"	140
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$\Gamma_z(w)$	135	$\gamma(w)$	138	$c(m,z)$	142
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THEOREM FOUR

THEOREM FIVE

THEOREM SIX

CHAPTER ONE

INTRODUCTION

As explained in the Introduction, Chapter One consists almost entirely of lemmas each of which (apart from Lemma 5.2) is referred to in the theorems of the following chapters. The lemmas are grouped where common themes exist but otherwise are roughly ordered as they appear in the ensuing theorems. The major exception to the above is Lemma 5.2. Lemma 5.2 is a consequence and generalisation of Lemma 5.1. It is independent of the rest of the thesis and it's arguments may be understood without a knowledge of lemmas and theorems other than Lemma 5.1.

It is again suggested that the reader may go straight to Chapter Two and refer back to the lemmas of Chapter One as they occur in the theorems. However "grouped lemmas" may refer to each other so it is also suggested that should the first member of a group be read the simplest approach would be to read the other members of the same group at the same time.

LEMMA 1.1

Let $F(n)$ *be a polynomial of degree* g *with integer coefficients. Let* $\rho(p)$ *denote the number of solutions of the congruence*

$$F(n) \equiv 0 \bmod p$$

and assume that

$$\rho(p) < p \quad \textit{for all primes } p. \tag{1}$$

Let $x/k \geq z$

and set $u = \dfrac{\ln x/k}{\ln z}$.

Write $k' = \prod_{\substack{p<z \\ p \mid k}} p$.

Then,

$$\left| \left\{ n:\ \alpha<n\leq\alpha+x,\ n\equiv \ell \ (\bmod k\ ),\ (\ F(n),\prod_{p<z} p\ )=1 \right\} \right|$$

$$= \begin{cases} x/k \prod\limits_{\substack{p<z \\ p\nmid k}} \left(1-\dfrac{\rho(p)}{p} \right) \{ 1+O(\exp(-u(\ln u-\ln\ln 3u-\ln g-2))) \\ \qquad\qquad +O(\exp(-(\ln x/k)^{\frac{1}{2}}\ )) \} & ;\quad (\ F(\ell),\prod\limits_{\substack{p<z \\ p \mid k}} p)=1 \\ 0 & ;\quad (\ F(\ell),\prod\limits_{\substack{p<z \\ p \mid k}} p)>1 \end{cases}$$

The O*-constants are effectively computable and depend on, at most,* g.

PROOF

The proof consists of an application of Theorem 2.5 of Halberstam -Richert's "Sieve Methods"[2]. We begin with an explanation of some of the notation used in their book, which we will consequently adopt here. Our proof will be an estimate of the *sifting function*,

$$S(A,B,z) = \left| \{\ a:\ a\epsilon A;\ (a,\prod_{\substack{p<z \\ p\epsilon B}} p) = 1\ \} \right|$$

where A = { a:.... } denotes a sequence of integers; where B is a set of primes; $\bar{B}$ the complement of B.

We define

$$A_d := \{ a: a\epsilon A,\ a\equiv 0 \bmod d \}$$

for d a squarefree integer,

and the number of elements in A_d to be $|A_d|$.

We choose a convenient function X which approximates to $|A|$, the number of elements in A, and for each prime p we choose a function $w_0(p)$ such that $\left(\frac{w_0(p)}{p}\right)X$ approximates to $|A_p|$.

The remainder we write as

$$r_p := |A_p| - \frac{w_0(p)}{p}X$$

Consequently we define, for each squarefree d,

$$w_0(1) := 1, \qquad w_0(d) := \prod_{p|d} w_0(p)$$

and $$r_d := |A_d| - \frac{w_0(d)}{d}X \quad .$$

Finally we define

$$w(p) = \begin{cases} w_0(p) & ;\ p\epsilon B \\ 0 & ;\ p\epsilon \bar{B} \end{cases}$$

and extend this to

$$w(1) := 1, \qquad w(d) := \prod_{p|d} w(p) \qquad\qquad (\ \mu(d)\neq 0\ )$$

With the function w(p) we form the product

$$W(z) := \prod_{p<z} \left(1 - \frac{w(p)}{p}\right) \quad .$$

Similarly

$$R_d := |A_d| - \frac{w(d)}{d}X \qquad\qquad (\ \mu(d)\neq 0\ )$$

Theorem 2.5 of [2] states that under conditions (Ω_1), $(\Omega_2(k))$, and (R) (which will be explained during the proof below), assuming that $X \geqslant z$ and setting $u = \frac{\ln X}{\ln z}$,

$$S(A,B,z) = X\,W(z)\{\ 1+O(\exp(-u(\ln u-\ln\ln 3u-\ln k-2)) + O(\exp(-(\ln X)^{\frac{1}{2}}))\}\ .$$

With regards to the sifting function

$$\left|\{\ n:\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell \bmod k,\ (F(n),\prod_{p<z}p) = 1\ \}\right|$$

we firstly observe, writing $F(n)=a_g n^g+a_{g-1}n^{g-1}+\ldots+a_0$, that for $n\equiv\ell \bmod k$, recalling that $k':=\prod_{\substack{p<z\\ p\mid k}}p$,

$$\begin{aligned}(F(n),k')>1 &\iff \exists\ p\mid k' \text{ such that } F(n)\equiv 0 \bmod p\\ &\iff \exists\ p\mid k' \text{ such that}\\ &\qquad a_g n^g+a_{g-1}n^{g-1}+\ldots+a_0\equiv 0 \bmod p\\ &\iff \exists\ p\mid k' \text{ such that}\\ &\qquad a_g \ell^g+a_{g-1}\ell^{g-1}+\ldots+a_0\equiv 0 \bmod p\\ &\iff (\ F(\ell),\ k'\ )>1\end{aligned}$$

So for $(F(\ell),k')>1$

$$\left|\{\ n:\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell \bmod k,\ (F(n),\prod_{p<z}p)=1\ \}\right| = 0$$

Assume henceforth that $(F(\ell),k')=1$. The above now implies that $(F(n),k')=1$ so that

$$\begin{aligned}&\left|\{\ n:\alpha<n\leqslant\alpha+x,\ n\equiv\ell \bmod k,\ (F(n),\prod_{p<z}p)=1\ \}\right|\\ &= \left|\{\ n:\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell \bmod k,\ (F(n),\prod_{\substack{p<z\\ p\nmid k}}p)=1\ \}\right|\end{aligned}$$

and it is this final sifting function which we will apply Theorem 2.5 to. Using the notation described above we take

$$A = \{\ F(n):\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell \bmod k\ \}$$

and $\qquad B = \{\ p:\ p\nmid k\ \}$.

Then if $(d,k)=1$,

$$\begin{aligned}|A_d| &= \left|\{\ n:\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell \bmod k,\ F(n)\equiv 0 \bmod d\ \}\right|\\ &= \sum_{\substack{m=1\\ F(m)\equiv 0 \bmod d}}^{d} \left|\{\ n:\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell \bmod k,\ n\equiv m \bmod d\ \}\right|\\ &= \rho(d)\left\{\ \frac{x}{kd} + \theta\ \right\}\ . \qquad (|\theta|\leqslant 1)\end{aligned}$$

Accordingly we choose

$$X = {}^{x}/_{k}, \qquad w_0(d) = \rho(d) \qquad \text{for } (d,k)=1$$

and it follows that

$$|r_d| \leqslant w_0(d). \tag{2}$$

We have now, for these choices of X and $w_0(d)$, to show that the conditions (Ω_1), $(\Omega_2(k))$, and (R) are satisfied.

We take them in order:

(Ω_1) states $0 \leqslant \frac{w(p)}{p} \leqslant 1 - \frac{1}{A_1}$ for some suitable constant $A_1 \geqslant 1$.

But here

$$w(p) = \begin{cases} \rho(p) & ; \quad (p,k)=1 \\ 0 & ; \quad (p,k)>1 \end{cases}$$

and if $(p,k)=1$ then $w(p)=\rho(p)\leqslant g$ by Lagranges Theorem together with (1). Certainly $\frac{w(p)}{p} \geqslant 0$, and it is easily seen that

$\frac{w(p)}{p} \leqslant 1 - \frac{1}{g+1}$ using $w(p) \leqslant g$ whenever $p \geqslant g+1$ and $w(p) \leqslant p-1$ otherwise.

So taking $A_1 = g+1$ ensures that (Ω_1) is satisfied for all p.

$(\Omega_2(\kappa))$ states $\sum\limits_{w \leqslant p < z} \frac{w(p) \ln p}{p} \leqslant \kappa \ln \frac{z}{w} + A_2$ if $2 \leqslant w \leqslant z$

for suitable constants $\kappa (>0)$ and $A_2 (\geqslant 1)$.

However Lemma 2.2 of [2] implies that, if condition (Ω_0) holds then $(\Omega_2(\kappa))$ holds also with $\kappa = A_2 = A_0$ where (Ω_0) is the condition $w(p) \leqslant A_0$.

But $w(p) \leqslant \rho(p) \leqslant g$ so $(\Omega_2(\kappa))$ holds with $\kappa = A_2 = g$.

(R) is the condition $|R_d| \leqslant w(d)$ if $\mu(d) \neq 0$ and $(d, \bar{B})=1$

But, by the definition of $|R_d|$ this is simply (2).

We are now in a position to apply Theorem 2.6 stated above to give

$$\left| \left\{ n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell\ (\bmod\ k),\ (F(n),\prod_{\substack{p<z\\p\nmid k}} p)=1 \right\} \right|$$

$$= \frac{x}{k}\prod_{\substack{p<z\\p\nmid k}}\left(1-\frac{\rho(p)}{p}\right)\left\{1+O(\exp(-u(\ln u-\ln\ln 3u-\ln g-2))) + O(\exp(-(\ln x/k)^{\frac{1}{2}}))\right\}$$

as required.

□

<u>LEMMA 1.2</u>

Let $F(n)$ *be a polynomial of degree* g *with integer coefficients. Let* $\rho(p)$ *denote the number of solutions of the congruence*

$$F(n)\equiv 0 \bmod p$$

and assume that

(i) $\rho(p)<p$ *for all primes* p

(ii) $\rho(p)<p-1$ *if* $p\nmid F(0)$.

Let

$$\rho'(p) = \begin{cases} \rho(p)+1 & ;\ p\nmid F(0) \\ \rho(p) & ;\ p\mid F(0) \end{cases}$$

and set $u = \dfrac{\ln x/k}{\ln z}$ *with* $x/k\geq z$.

Then, for $(\ell,k)=1$,

$$\left| \left\{ n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell\ (\bmod\ k),\ (nF(n),\prod_{p<z} p)=1 \right\} \right|$$

$$= \begin{cases} \dfrac{x}{k}\displaystyle\prod_{\substack{p<z\\p\nmid k}}\left(1-\frac{\rho'(p)}{p}\right)\left\{1+O(\exp(-u(\ln u-\ln\ln 3u-\ln(g+1)-2))) + O(\exp(-(\ln x/k)^{\frac{1}{2}}))\right\} & ;\ (F(\ell),\displaystyle\prod_{\substack{p<z\\p\mid k}} p)=1 \\ 0 & ;\ (F(\ell),\displaystyle\prod_{\substack{p<z\\p\mid k}} p)>1 \end{cases}$$

The O*-constants are effectively computable and depend on, at most,* g.

PROOF

From Lemma 1.1 we have

$$\left| \left\{ n:\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell(\bmod k),\ (nF(n),\prod_{p<z}p)=1 \right\} \right|$$

$$= \begin{cases} x/_k \prod\limits_{\substack{p<z\\ p\nmid k}} \left(1-\frac{\rho'(p)}{p}\right)\left\{1+0(\exp(-u(\ln u-\ln\ln 3u-\ln(g+1)-2))) +0(\exp(-(\ln x/k)^{\frac{1}{2}}))\right\} & ;\ (F(\ell)\ell,\prod\limits_{\substack{p<z\\ p\mid k}}p)=1 \\ 0 & ;\ (F(\ell)\ell,\prod\limits_{\substack{p<z\\ p\mid k}}p)>1 \end{cases}$$

with $\rho'(p) = \left|\{ n \bmod p : F(n)n\equiv 0 \bmod p \}\right|$.

Certainly $\rho'(p) = \left|\{ n \bmod p : F(n)\equiv 0 \bmod p \}\right|$

$$+ \left|\{ n \bmod p : n\equiv 0 \bmod p \}\right| \quad ;\ \text{if } p\nmid F(0)$$

and $\rho'(p) = \left|\{ n \bmod p : F(n)\equiv 0 \bmod p \}\right|$

$$;\ \text{if } p\mid F(0)$$

and so

$$\rho'(p) = \begin{cases} \rho(p)+1 & ;\ p\nmid F(0) \\ \rho(p) & ;\ p\mid F(0). \end{cases}$$

Further, for $(\ell,k)=1$, $(F(\ell)\ell,\prod\limits_{\substack{p<z\\ p\mid k}}p)=1 \Leftrightarrow (F(\ell),\prod\limits_{\substack{p<z\\ p\mid k}}p)=1$

which completes the lemma. □

LEMMA 1.3

Let F(n) *be a polynomial of degree* g *with integer coefficients. Let* $\rho(p)$ *denote the number of solutions of the congruence*

$$F(n)\equiv 0 \bmod p$$

and assume that

(i) $\rho(p)<p$ *for all primes* p

(ii) $\rho(p)<p-1$ *if* $p\nmid F(0)$.

Let
$$\rho'(p) = \begin{cases} \rho(p)+1 & ; \ p\nmid F(0) \\ \rho(p) & ; \ p\mid F(0) \end{cases}$$

and set $u = \dfrac{\ln {}^{x}/_{k}}{\ln z}$ *with* ${}^{x}/_{k} \geq z$.

Write $k' = \prod_{\substack{p<z \\ p\mid k}} p$.

Then, for $(\ell,k)=1$, *and* q *prime,*

$$\left| \left\{ q:\ \alpha<q\leq\alpha+x,\ q\equiv\ell(\bmod k),\ (F(q),\prod_{p<z} p)=1 \right\} \right|$$

$$< \begin{cases} {}^{x}/_{k} \prod\limits_{\substack{p<z \\ p\nmid k}} \left(1-\dfrac{\rho'(p)}{p}\right)\left\{1+O(\exp(-u(\ln u-\ln\ln 3u-\ln(g+1)-2))) + O(\exp(-(\ln x/k)^{\frac{1}{2}}))\right\} + O(A) & ;\ (F(\ell),\prod\limits_{\substack{p<z \\ p\mid k}} p)=1 \\ 0 & ;\ (F(\ell),\prod\limits_{\substack{p<z \\ p\mid k}} p)>1 \end{cases}$$

where
$$A = \begin{cases} \dfrac{z}{\varphi(k)\ln^{z}/_{k}} & ;\ z>k \\ 1 & ;\ z\leq k \end{cases}$$

The 0*-constants are effectively computable, and depend on, at most*, g.

PROOF

Certainly if $(F(\ell),k')>1$ then $(F(q),k')>1$ and

$$\left|\{ q:\ \alpha<q\leq\alpha+x,\ q\equiv\ell \bmod k,\ (F(q),\prod_{p<z}p)=1 \}\right| = 0.$$

Assume instead that $(F(\ell),k')=1$.

Clearly the function

$$\left|\{ q:\ \alpha<q\leq\alpha+x,\ q\equiv\ell \bmod k,\ (F(q),\prod_{p<z}p)=1 \}\right|$$

counts the integers, n, satisfying $\alpha<n\leq\alpha+x$, $n\equiv\ell \bmod k$ for which n is a prime and $(F(n),\prod_{p<z}p)=1$. If, however, in addition $n\geq z$, then n is counted in

$$\left|\{ n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell \bmod k,\ (F(n)n,\prod_{p<z}p)=1 \}\right|.$$

Otherwise $n \leqslant z$ and as there are $O\left[\frac{z}{\varphi(k)\ln^{z}/_{k}}\right]$ primes $\leqslant z$ which are congruent to ℓ mod k (by the Brun–Titchmarsh inequality) if $z>k$, and $O(1)$ primes if $z \leqslant k$, it follows that

$$\left|\{ q:\ \alpha<q\leqslant\alpha+x,\ q\equiv\ell \bmod k,\ (F(q),\prod_{p<z}p)=1 \}\right|$$

$$\leqslant \left|\{ n:\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell \bmod k,\ (F(n)n,\prod_{p<z}p)=1 \}\right| + O(A)$$

if $(F(\ell),k')=1$.

The lemma follows immediately by an application of Lemma 1.2.

□

<u>LEMMA 1.4</u>

Let F(n) *be a polynomial of degree* g *with integer coefficients. Let* $\rho(p)$ *denote the number of solutions of the congruence*

$$F(n) \equiv 0 \bmod p.$$

Let

$$\rho_1(p) = \begin{cases} \rho(p) & \text{if } p\nmid F(0) \\ \rho(p)-1 & \text{if } p\mid F(0) \end{cases}$$

and assume that $\rho_1(p)<p-1$ *for all primes* p. (1)

Let $\frac{\mathrm{li}\ x}{\varphi(k)}>z$ *and set* $u=\frac{\ln(\mathrm{li}\ x/\varphi(k))}{\ln z}$. *Write* $k'=\prod_{\substack{p<z\\ p\mid k}}p$.

Then, for q *prime, and* $k\leqslant \ln x$,

$$\left|\{q:\ 0<q\leqslant x,\ q\equiv\ell \bmod k,\ (F(q),\prod_{p<z}p)=1\}\right|$$

$$= \begin{cases} \frac{x}{\varphi(k).\ln x} \prod_{\substack{p<z\\ p\nmid k}} \left(1-\frac{\rho_1(p)}{p-1}\right) \Big(1 + O(\ln^{-1}x) \\ \qquad + O(\exp(-^{u}/_{3}(\ln u-\ln\ln 3u-\ln 6g-2)))\Big) \\ \qquad\qquad ;\ (F(\ell),\prod_{\substack{p<z\\ p\mid k}}p)=1 \\ 0 \qquad\qquad ;\ (F(\ell),\prod_{\substack{p<z\\ p\mid k}}p)>1 \end{cases}$$

The 0-constants depend on, at most, g.

REMARK: Lemma 1.4 stands in contrast to Lemma 1.3. Though with fundamentally the same function, namely

$$\left|\{q:\ 0<q\leq x,\ q\equiv \ell \bmod k,\ (F(q),\prod_{p<z} p)=1\}\right|,$$

in Lemma 1.4 we are able to give an asymptotic formula rather than an upper bound on this function. The price we pay for this apparently stronger lemma is, fistly that we no longer have effectively computable 0-constants, and secondly that the range of values over which q varies is restricted to $0<q\leq x$, whereas in Lemma 1.3 we were able to take the more flexible range, $\alpha<q\leq\alpha+x$.

PROOF

As in Lemma 1.3, if $(F(\ell),k')>1$ then $(F(q),k')>1$ and

$$\left|\{q:\ 0<q\leq x,\ q\equiv \ell \bmod k,\ (F(q),\prod_{p<z} p)=1\}\right| = 0.$$

Assume instead that $(F(q),k')=1$ so that the function becomes

$$\left|\{q:\ 0<q\leq x,\ q\equiv \ell \bmod k,\ (F(q),\prod_{\substack{p<z\\ p\nmid k}} p)=1\}\right| .$$

The proof is an application of Theorem 2.5' of Halberstam-Richert [2] which reads

"(Ω_1), $(\Omega_2(\kappa))$, (R_0), $(R_1(\kappa,\alpha))$: Let $X\geq z$ and write

$$u = \frac{\ln X}{\ln z}.$$

Then

$$S(A;B,z) = X\,W(z)\{1 + O(\exp(-\alpha u(\ln u-\ln\ln 3u-\ln^{\kappa}/_{\alpha}-2))) + O_U(L.\ln^{-U}X)\}$$

where the 0-constants may depend on U as well as on the usual constants A_0', A_1, A_2, κ and α."

However the details of the proof follow to a large extent the proof of Theorem 4.2 of the same.

Take $A = \{F(q): q \leq x \;\; q \equiv \ell \bmod k\}$ and $B = \{p: p \nmid k\}$.

Following the analysis of Example 6 of Chapter 1 of [2] we take

$$X = \frac{\text{li } x}{\varphi(k)} \quad \text{and} \quad w_0(d) = \rho_1{*}(p).\varphi([k,d]).\frac{d}{\varphi(d)}$$

where

$$\rho_1{*}(p) = \rho_1({}^{d}/(d,k))$$

and where $\rho_1(d)$ is the number of solutions of

$$F(m) \equiv 0 \bmod d \quad \text{for } (m,d)=1.$$

For $E(x,q)$ defined as

$$E(x,q) = \max_{2 \leq y \leq x} \max_{\substack{1 \leq \ell \leq q \\ (\ell,q)=1}} \left| \Pi(y;q,\ell) - \frac{\text{li } y}{\varphi(q)} \right|$$

it is demonstrated that

$$|r_d| \leq \rho(d)\{ E(x,kd) + 1 \} \quad \text{if } \mu(d)\neq 0,\; (d,k)=1 \tag{2}$$

and $$w_0(p) = \frac{\rho_1(p).p}{p-1} \quad \text{if } p \nmid k. \tag{3}$$

Further
$$\rho_1(p) = \begin{cases} \rho(p) & \text{if } p \nmid F(0) \\ \rho(p)-1 & \text{if } p \mid F(0) \end{cases} \tag{4}$$

and $\rho(p) \leq g$ if $\rho(p) < p$. (5)

Finally $\rho_1(d) \leq \rho(d) \leq g^{\gamma(d)}$ for $\mu(d)\neq 0$ (6)

where $\gamma(d)$ denotes the number of prime factors of d.

Given all this information we must show that the conditions (Ω_1), $(\Omega_2(\kappa))$, (R_0), $(R_1(\kappa,\alpha))$ are satisfied. We take them in turn:

(Ω_1) states $0 \leq \frac{w(p)}{p} \leq 1-\frac{1}{A_1}$ for some suitable constant $A_1 \geq 1$.

But here

$$w(p) = \begin{cases} \frac{\rho_1(p)p}{p-1} & \text{if } (p,k)=1 \\ 0 & \text{if } (p,k)>1 \;. \end{cases}$$

It is easily seen, from (5), that $\frac{w(p)}{p} < 1-\frac{1}{g+1}$ if $p \geq g+2$,

and, from (1) that $\frac{w(p)}{p} \leq 1-\frac{1}{g}$ if $p \leq g+1$.

So taking $A_1=g+1$ ensures that (Ω_1) is satisfied for all primes p.

$(\Omega_2(\kappa))$ states $\sum_{w \leq p < z} \frac{w(p).\ln p}{p} \leq \kappa \ln {}^z/_w + A_2$ if $2 \leq w \leq z$.

However it is enough to show that $w(p) \leq A_0$ in which case $(\Omega_2(\kappa))$ holds with $A_2=\kappa=2g$.

(R_0) is the condition that

$$|R_d| \leq L\left[\frac{X\ln X}{d} + 1\right]A_0'^{\gamma(d)} \quad \text{for } \mu(d) \neq 0,$$

for L a real number ≥ 1 and A_0' a constant ≥ 1.

From (2) and (6),

$$|R_d| \leq \{ E(x,kd) + 1 \} g^{\gamma(d)} \quad \text{if } \mu(d) \neq 0. \tag{7}$$

But

$$E(x,kd) = \max_{2 \leq y \leq x} \max_{\substack{1 \leq \ell \leq kd \\ (\ell,kd)=1}} \left| \Pi(y;kd,\ell) - \frac{\text{li } y}{\varphi(kd)} \right|$$

$$\leq \frac{x}{kd} + 1$$

trivially.

So

$$|R_d| \leq \left\{ \frac{x}{dk} + 2 \right\} g^{\gamma(d)} \quad \text{if } \mu(d) \neq 0.$$

However, as $X=\frac{\text{li } x}{\varphi(k)}$, and assuming that x is large, we have

$$\varphi(k)X = \int_2^x \frac{du}{\ln u} \geq \int_2^x \frac{\ln u-1}{\ln^2 u}\, du = \left[\frac{u}{\ln u}\right]_2^x \geq \frac{2}{3}\cdot \frac{x}{\ln x}.$$

So

$$2X\ln X \geq \frac{4}{3} \frac{x}{\varphi(k)\ln x} \ln\left[\frac{2}{3} \frac{x}{\varphi(k)\ln x}\right] \geq \frac{x}{\varphi(k)} \geq \frac{x}{k}$$

and

$$|R_d| < \left\{\frac{2X\ln X}{d} + 2\right\} g^{\gamma(d)} \quad \text{if } \mu(d)\neq 0$$

implying that (R_0) holds with $L=2$ and $A_0'=g$.

Finally we look at $(R_1(\kappa,\alpha))$ which reads

"For some constant α $(0<\alpha\leq 1)$ there exists corresponding to any given constant $U\geq 1$ a positive constant c_0 such that

$$\sum_{\substack{d<X^{\alpha}\ln^{-c_0}X \\ (d,\overline{B})=1}} \mu^2(d)|R_d| = 0_U\left[\frac{X}{\ln^{\kappa+U}X}\right] \text{"} .$$

In our case, as $\overline{B} = \{p: p|k\}$,

$$\sum_{\substack{d<X^{\alpha}\ln^{-c_0}X \\ (d,\overline{B})=1}} \mu^2(d)|R_d| = \sum_{\substack{d<X^{\alpha}\ln^{-c_0}X \\ (d,k)=1}} \mu^2(d)|R_d| .$$

Taking $\alpha={}^1/_3$ and $U=1$, $\kappa=2g$ we need only show

$$\sum_{\substack{d<X^{1/3}\ln^{-c_0}X \\ (d,k)=1}} \mu^2(d)|R_d| = 0\left[\frac{X}{\ln^{2g+1}X}\right] .$$

By (7) above

$$\sum_{\substack{d<X^{1/3}\ln^{-c_0}X \\ (d,k)=1}} \mu^2(d)|R_d| \leq \sum_{\substack{d<X^{1/3}\ln^{-c_0}X \\ (d,k)=1}} \mu^2(d)E(x,kd)g^{\gamma(d)}$$

$$+ \sum_{\substack{d<X^{1/3}\ln^{-c_0}X \\ (d,k)=1}} \mu^2(d)g^{\gamma(d)} \tag{8}$$

To find upper bounds on the sums on the right of (8) we use respectively Lemmas 3.5 and 3.4 of [2] which read:

"LEMMA 3.4 For any natural number h and for $x\geq 1$ we have

$$\sum_{d<x} \mu^2(d)h^{\gamma(d)} \leq x(\ln x+1)^h.\text{"}$$

"LEMMA 3.5 Let h and k be positive integers and suppose that $k\leq \ln^A x$. Then, given any positive constant U, there exists a positive constant $c=c(U,h,A)$ such that

$$\sum_{d<\frac{x^{\frac{1}{2}}}{k\ln^{c}x}} \mu^2(d)h^{\gamma(d)}E(x,kd) = 0_{U,h,A}\left[\frac{x}{\varphi(k)\ln^{U}x}\right]. \text{"}$$

Unfortunately the 0-constant of Lemma 3.5 is not computable with current knowledge.

For $X=\frac{\text{li } x}{\varphi(k)}$, and for $k\leqslant\ln x$ say,

$$\sum_{\substack{d<X^{1/3}\ln^{-c_0}X\\(d,k)=1}} \mu^2(d)g^{\gamma(d)}E(x,kd) \leqslant \sum_{d<\frac{X^{\frac{1}{2}}\ln^{-c_0}X}{k}} \mu^2(d)g^{\gamma(d)}E(x,kd).$$

Taking $h=g$, $A=1$, and $U=2g+1$ in Lemma 3.5 we thus have

$$\sum_{\substack{d<X^{1/3}\ln^{-c_0}X\\(d,k)=1}} \mu^2(d)g^{\gamma(d)}E(x,kd) = 0_g\left[\frac{X}{\varphi(k)\ln^{2g+1}X}\right]$$

$$= 0_g\left[\frac{X}{\ln^{2g+1}X}\right].$$

Further Lemma 3.4 gives

$$\sum_{\substack{d<X^{1/3}\ln^{-c_0}X\\(d,k)=1}} \mu^2(d)g^{\gamma(d)} \leqslant \frac{X^{1/3}}{\ln^{c_0}X}(\ln X+1)^g$$

$$\ll X^{1/3}(\ln X)^g$$

$$\ll \frac{X}{(\ln X)^{2g+1}} \qquad \text{for } g\ll\ln^{\frac{1}{2}}X \text{ say.}$$

Substitution into (8) gives

$$\sum_{\substack{d<X^{\alpha}\ln^{-c_0}X\\(d,k)=1}} \mu^2(d)|R_d| = 0_g\left[\frac{X}{\ln^{2g+1}X}\right].$$

so that $(R_1(\kappa,\alpha))$ is satisfied with $\alpha=1/3$ and $\kappa=2g$.

We are now in a position to apply Theorem 2.5' stated above to give

$$\left|\{q:\ 0<q\leq x,\ q\equiv \ell \bmod k,\ (F(q),\prod_{p<z}p)=1\}\right|$$

$$= \frac{\text{li } x}{\varphi(k)} \prod_{\substack{p<z\\ p\nmid k}} \left(1-\frac{\rho_1(p)}{p-1}\right)\Big\{1 + O_g(\exp(-{}^{u}/_{3}(\ln u-\ln\ln 3u-\ln(6g)-2)))$$

$$+ O_g(\ln^{-1}x)\Big\}.$$

Since $\text{li } x = \frac{x}{\ln x}\left\{1 + O\left[\frac{1}{\ln x}\right]\right\}$ this becomes

$$\left|\{q:\ 0<q\leq x,\ q\equiv \ell \bmod k,\ (F(q),\prod_{p<z}p)=1\}\right|$$

$$= \frac{x}{\varphi(k)\ln x} \prod_{\substack{p<z\\ p\nmid k}} \left(1-\frac{\rho_1(p)}{p-1}\right)\Big\{1 +$$

$$O_g(\exp(-{}^{u}/_{3}(\ln u-\ln\ln 3u-\ln(6g)-2))) + O_g(\ln^{-1}x)\Big\}$$

which completes the lemma.

□

LEMMA 2.1

Let $an^2 + bn + c$ *and* $dn^2 + en + f$ *be polynomials with integer coefficients, and having no common factors. Then there exists an integer* $F(\neq 0)$ *defined by* $F=|ce-fb|$ *if* $a=d=0$, *and* $F=|(cd-af)^2-(bd-ea)(ce-fb)|$ *otherwise, for which, for all* n,

$$(an^2 + bn + c,\ dn^2 + en + f)=w$$

$$\Leftrightarrow\ (a(n+F)^2 + b(n+F) + c,\ d(n+F)^2 + e(n+F) + f)=w.$$

Furthermore, if there exists an integer n *for which*

$$(an^2 + bn + c,\ dn^2 + en + f)=w,$$

then $w|F$.

PROOF

By definition $(an^2 + bn + c,\ dn^2 + en + f)>1$ if and only if there exists an integer m such that

$$an^2 + bn + c \equiv 0 \bmod m \quad \text{and} \quad dn^2 + en + f \equiv 0 \bmod m.$$

We will show that for any such m it follows that $m \mid F$ where

$$F = \begin{cases} |ce-fb| & ; \ a=0,\ d=0 \\ |(cd-af)^2 - (bd-ea)(ce-fb)| & ; \ \text{otherwise} \end{cases}$$

with $F \neq 0$.

(i) If $a = d = 0$ and $be = 0$ then it is clear that

$$an^2 + bn + c \equiv 0 \bmod m \quad \text{and} \quad dn^2 + en + f \equiv 0 \bmod m$$

implies $m \mid F$ with $F \neq 0$.

(ii) If $a = d = 0$ and $be \neq 0$ then

$$an^2 + bn + c \equiv 0 \bmod m \quad \text{and} \quad dn^2 + en + f \equiv 0 \bmod m$$

if and only if

$$bn + c \equiv 0 \bmod m \quad \text{and} \quad en + f \equiv 0 \bmod m.$$

This implies

$$(bn + c)e - (en + f)b \equiv 0 \bmod m$$

i.e. $ce - fb \equiv 0 \bmod m$.

Certainly $ce - fb \neq 0$ for otherwise $^b/_e = {}^c/_f$ contradicting our assumption that $an^2 + bn + c$, and $dn^2 + en + f$ have no common factors.

(iii) If at least one of a and d is not zero then

$$an^2 + bn + c \equiv 0 \bmod m \quad \text{and} \quad dn^2 + en + f \equiv 0 \bmod m$$

implies

$$(an^2 + bn + c)d - (dn^2 + en + f)a \equiv 0 \bmod m$$

i.e. $(bd - ea)n + (cd - fa) \equiv 0 \bmod m.$ (1)

(iv) If $bd - ea = 0$ then $cd - fa \equiv 0 \bmod m$ and in this instance $cd - fa \neq 0$, for otherwise we would have $^a/_d = {}^b/_e = {}^c/_f$ or $^d/_a = {}^e/_b = {}^f/_c$.

Clearly, $m \mid F$ and $F \neq 0$ as required.

(v) Assuming that $bd - ea \neq 0$, from which it follows that e and b are not both zero,

$$an^2 + bn + c \equiv 0 \bmod m \quad \text{and} \quad dn^2 + en + f \equiv 0 \bmod m$$

implies

$$(an^2 + bn + c)e - (dn^2 + en + f)b \equiv 0 \bmod m$$

i.e. $(ae - db)n^2 + (ce - fb) \equiv 0 \bmod m.$ (2)

But (1) gives

$$(bd - ea)n^2 + (cd - fa)n \equiv 0 \bmod m.$$

This, in conjunction with (2), gives

$$(cd - fa)n + (ce - fb) \equiv 0 \bmod m. \qquad (3)$$

If $cd - fa = 0$ then $ce - fb \equiv 0 \bmod m$ and certainly $ce - fb \neq 0$. Again $m|F$ and $F \neq 0$.

(vi) Assuming finally that $bd - ea \neq 0$ and $cd - af \neq 0$, (1) gives

$$(cd - fa)(bd - ea)n + (cd - fa)^2 \equiv 0 \bmod m \qquad (4)$$

and (3) gives

$$(cd - fa)(bd - ea)n + (ce - fb)(bd - ea) \equiv 0 \bmod m. \qquad (5)$$

Together these imply

$$(cd - fa)^2 - (ce - fb)(bd - ea) \equiv 0 \bmod m$$

or $\qquad F \equiv 0 \bmod m$

as required.

This, however, gives no information if $F=0$, that is, if

$$(cd - fa)^2 = (ce - fb)(bd - ea).$$

If it were the case that

$$(cd - fa)^2 = (ce - fb)(bd - ea)$$

then writing $an^2 + bn + c = g(n)$ and $dn^2 + en + f = h(n)$, and arguing as above, we see that the equations

$$g(n)e - h(n)b = (ae - bd)n^2 + (ce - fb) \qquad (6)$$

and

$$(g(n)d - h(n)a)n = (bd - ae)n^2 + (cd - fa)n \qquad (7)$$

hold, for all n.

These imply

$$g(n)e - h(n)b + g(n)nd - h(n)na = (cd - fa)n + (ce - fb)$$

and consequently that

$$[g(n)e - h(n)b + g(n)nd - h(n)na](bd - ae)$$

$$= (cd - fa)(bd - ae)n + (ce - fb)(bd - ae). \qquad (8)$$

But, from (7),

$$[g(n)d - h(n)a](cd - fa)$$

$$= (bd - ae)(cd - fa)n + (cd - fa)^2$$

and, as $(cd - fa)^2 = (bd - ae)(ce - fb)$

we have

$$[g(n)e - h(n)b + g(n)nd - h(n)na](bd - ea)$$

$$= [g(n)d - h(n)a](cd - fa)$$

i.e. $h(n)\{(bd - ea)(an + b) - a(cd - fa)\}$

$$= g(n)\{(bd - ea)(dn + e) - d(cd - fa)\}. \qquad (9)$$

Hence, there exist integers $\alpha,\beta,\gamma,\delta$ such that

$$h(n)(\alpha n + \beta) = g(n)(\gamma n + \delta) \quad \text{for all } n. \qquad (10)$$

There does not exist a constant, k, such that

$$(\alpha n + \beta) = k(\gamma n + \delta)$$

for this would imply that g(n) and h(n) have a common factor.

The alternative is that

$$g(n) = (\alpha n + \beta)(sn + t)$$

say, with $(sn + t) | h(n)$. But again this would imply that g(n) and h(n) have a common factor.

Hence, as required, $F \neq 0$.

It is clear, then, that

$$(an^2 + bn + c,\ dn^2 + en + f) = w$$

$$\Leftrightarrow (a(n+F)^2 + b(n+F) + c,\ d(n+F)^2 + e(n+F) + f) = w$$

and, furthermore, that $w \mid F$.

This completes the lemma. □

The arguments used in Lemmas 2.2-2.7 below are specific examples of lemmas from W.Schwarz's paper.[3]. As he frequently gives only partial proofs we give them here in their full form for completeness.

Lemmas 2.9-2.12 are extensions of his argument for finding an asymptotic formula for

$$\sum_{n \leq x} \frac{\varphi(f(n))}{f(n)} .$$

LEMMA 2.2

$$\sum_{1 \leq m \leq M} \tau(m)^{\ell} = O(M (\ln M)^{\lambda})$$

where $\lambda = 2^{\ell} - 1$ *and where* $\tau(n)$ *denotes the number of divisors of* n.

PROOF

Hua[4] pg.111. □

LEMMA 2.3

If $\alpha > 1$ *and* $c = \left[\frac{\ln \alpha}{\ln 2} \right] + 1$

then

$$\sum_{1 \leq m \leq M} \alpha^{\omega(m)} = O(M \ln^{\lambda} M)$$

where $\lambda = 2^{c} - 1$ *and where* $\omega(m)$ *denotes the number of prime divisors of* m.

PROOF

Clearly $2^{\omega(m)} \leq \tau(m)$, for if $m = p_1^{\nu_1} \ldots p_{\omega(m)}^{\nu_{\omega(m)}}$, then

$$\tau(m) = (\nu_1 + 1) \ldots (\nu_{w(m)} + 1).$$

So

$$\alpha^{\omega(m)} = [\, 2^{\omega(m)} \,]^{\ln\alpha/\ln 2} \leq (\tau(m))^c .$$

An application of Lemma 2.2 completes the lemma. □

LEMMA 2.4

Let $f(n)$ *be a polynomial of degree* k, *with discriminant* $D \neq 0$. *Let* g *denote the highest common factor of the coefficients of* $f(n)$. *Then, whenever* $r \geq 1$, *and* $(p^r, g) = 1$, *the congruence*

$$f(n) \equiv 0 \bmod p^r$$

has at most $k.D^2$ *solutions.*

Furthermore if $\rho(d)$ *denotes the number of solutions of*

$$f(n) \equiv 0 \bmod d$$

then, for $(d,g)=1$,

$$\rho(d) \leq (k.D^2)^{\omega(d)}.$$

PROOF

Nagell, T [5].

□

LEMMA 2.5

$$\sum_{1 \leq m \leq M} \frac{\alpha^{\omega(m)}}{m} = O((\ln M)^{2^c})$$

where $c = \left[\frac{\ln \alpha}{\ln 2} \right] + 1$.

PROOF

By Abel's identity,

$$\sum_{1<m\leqslant M} \frac{\alpha^{\omega(m)}}{m} = \frac{1}{M} \sum_{1<m\leqslant M} \alpha^{\omega(m)} + \int_1^M \frac{1}{t^2} \sum_{1\leqslant m<t} \alpha^{\omega(m)}\, dt$$

$$= O((\ln M)^{2^c-1}) + O\left(\int_1^M \frac{(\ln t)^{2^c-1}}{t}\, dt \right)$$

$$= O((\ln M)^{2^c-1}) + O\left((\ln M)^{2^c-1} \int_1^M \frac{dt}{t} \right)$$

$$= O((\ln M)^{2^c})$$

□

LEMMA 2.6

Using the notation of Lemma 2.4, if $2^c \ll M^2$,

$$\sum_{\substack{d>M\\(d,g)=1}} \frac{\rho(d)}{\varphi(d)d} = O\left\{ \frac{\ln\ln M\,(\ln M)^{2^c}}{M} \right\}$$

where $c = \left[\frac{\ln (k.D^2)}{\ln 2} \right] + 1$.

PROOF

$$\sum_{\substack{d>M\\(d,g)=1}} \frac{\rho(p)}{\varphi(d)d} \ll \sum_{d>M} \frac{(k.D^2)^{\omega(d)} \ln\ln d}{d^2}$$

$$= -\sum_{d\leqslant M} \frac{(k.D^2)^{\omega(d)} \ln\ln M}{d.M}$$

$$+ \int_M^\infty \sum_{d\leqslant t} \frac{(k.D^2)^{\omega(d)}}{d} . \frac{\ln\ln t - {}^1/_{\ln t}}{t^2} dt.$$

$$= O\left\{ \frac{\ln\ln M}{M} . (\ln M)^{2^c} \right\}$$

$$+ O\left\{ \int_M^\infty \frac{(\ln t)^{2^c}}{t^2} (\ln\ln t - {}^1/_{\ln t})\, dt \right\}.$$

But $\frac{d}{dt} \left\{ \frac{-\ln\ln t}{t} (\ln t)^A \right\} > \frac{(\ln t)^A}{2t^2} . (\ln\ln t - {}^1/_{\ln t})$

whenever $A<t^2$.

So $O\left\{ \int_M^\infty \frac{(\ln t)^{2^c}}{t^2} (\ln\ln t - {}^1/_{\ln t}) \right\}$

$$= O\left\{ \frac{\ln\ln M}{M} (\ln M)^{2^c} \right\} \quad \text{if } 2^c \ll M^2$$

and $\sum_{\substack{d>M\\(d,g)=1}} \frac{\rho(d)}{\varphi(d)d} = O\left\{ \frac{\ln\ln M}{M} (\ln M)^{2^c} \right\}$ as required.

□

LEMMA 2.7

Using the notation of Lemma 2.4,

$$\sum_{\substack{d\leq M\\(d,g)=1}} \frac{\rho(d)}{\varphi(d)} = O(\ln\ln M.\ (\ln M)^{2^c}) \quad \textit{where} \quad c = \left[\frac{\ln(k.D^2)}{\ln 2}\right] + 1$$

PROOF

$$\sum_{\substack{d\leq M\\(d,g)=1}} \frac{\rho(d)}{\varphi(d)} \ll \sum_{\substack{d\leq M\\(d,g)=1}} \frac{\rho(d)\ \ln\ln d}{d} \ll \sum_{d\leq M} \frac{(k.D^2)^{w(d)}\ln\ln d}{d}$$

$$= \sum_{d\leq M} \frac{(k.D^2)^{w(d)}\ln\ln M}{d} - \int_0^M \sum_{d\leq t} \frac{(k.D^2)^{w(d)}}{d} \cdot \frac{1}{t\ln t}\, dt$$

$$\ll \ln\ln M \sum_{d\leq M} \frac{(k.D^2)^{w(d)}}{d}$$

$$\ll \ln\ln M.\ (\ln M)^{2^c}.$$

□

LEMMA 2.8

For any constants a,b, a>b>0, *we have*

$$\frac{1}{\varphi([a,b])} < \frac{a^{\frac{1}{2}}}{\varphi(a)} \cdot \frac{b^{\frac{1}{2}}}{\varphi(b)}$$

PROOF

Firstly we show that $\dfrac{1}{\varphi([a,b])} = \dfrac{\varphi((a,b))}{\varphi(a)\varphi(b)}$. (1)

This follows from the observation that

$$\varphi(ab) = \varphi([a,b](a,b)) = \varphi([a,b])\varphi((a,b))\frac{d}{\varphi(d)}$$

where $d = ([a,b],(a,b))$. Since $d=(a,b)$ it follows that

$$\frac{1}{\varphi([a,b])} = \frac{(a,b)}{\varphi(ab)} .$$

But $\varphi(ab) = \dfrac{\varphi(a)\varphi(b)(a,b)}{\varphi((a,b))}$

so that $\dfrac{(a,b)}{\varphi(ab)} = \dfrac{\varphi((a,b))}{\varphi(a)\varphi(b)}$ which completes (1).

Since $\varphi((a,b)) < (a,b) < a^{\frac{1}{2}}b^{\frac{1}{2}}$ the lemma follows.

□

LEMMA 2.9

Let an^2+bn+c *and* dn^2+en+f *be two polynomials with integer coefficients and having no common factors. Let* D *denote the discriminant of the polynomial* $an^2 + bn + c$.

Then

$$\sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (an^2+bn+c,\ dn^2+en+f)=w \\ \left[\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right]=1}} \prod_{\substack{p<z \\ p\mid(an^2+bn+c)k_2}} \left(1-\frac{1}{p}\right)^{-1}$$

$$= \frac{x}{[k_1,Fk_2]}\ \Gamma_z(w) \left\{ 1 + O\left[\frac{a_1[k_1,Fk_2]^2 \ln\ln G(x,\alpha)\ \ln^{\lambda}G(x,\alpha)}{G(x,\alpha)}\right] + O\left[\frac{a_1[k_1,Fk_2]^3\ A}{x}\right]\right\}$$

where

(i)
$$F = \begin{cases} |ce-fb| & ;\ \textit{if } a=0,\ d=0 \\ |(cd-fa)^2-(bd-ea)(ce-fb)| & ;\ \textit{otherwise} \end{cases}$$

(ii)
$$\Gamma_z(w) = \sum_{\substack{\alpha_i \bmod Fk_2 \\ \alpha_i\equiv\ell_1 \bmod (k_1,Fk_2)}} \prod_{p<z} \left(1+\frac{\rho(p)}{p(p-1)}\right)$$

where $\alpha_1,\ldots,\alpha_\mu$ *denote the integers* n, *in the interval* $1\leq n\leq Fk_2$ *for which both*

$$(an^2+bn+c,\ dn^2+en+f)=w$$

and
$$\left[\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right]=1$$

hold.

(iii) *the unique solution, mod* $[k_1,Fk_2]$, *of the congruences*

$$n\equiv\ell_1 \bmod k_1 \quad \textit{and} \quad n\equiv\alpha_i \bmod F\kappa_2$$

is denoted, if it exists, by $\beta_i=\beta_i(\ell_1,\alpha_i)$. *Letting*

$$h=(a,\ b,\ c);\ a=a_1h,\ b=b_1h,\ c=c_1h,$$

then,

$$\rho(p) = \begin{cases} \left|\{ n:n \bmod p;\ a_1([k_1,Fk_2]t+\beta_i)^2+b_1([k_1,Fk_2]t+\beta_i) +c_1 \equiv 0 \bmod p\}\right| & ; p\nmid k_2h \\ p & ; p\mid k_2h \end{cases}$$

(iv)
$$G(x,\alpha)=\begin{cases} \max\limits_{\alpha<n\leq\alpha+x} |a_1n^2+b_1n+c_1| & ; D\neq 0 \\ \max\limits_{\alpha<n\leq\alpha+x} |a_1n^2+b_1n+c_1|^{\frac{1}{2}} & ; D=0 \end{cases}$$

(v)
$$\ln\lambda = \begin{cases} \left(\left[\dfrac{\ln 2D^2}{\ln 2}\right]+1\right)\ln 2 & ; D\neq 0 \\ 0 & ; D=0. \end{cases}$$

and finally,

(vi) $A = \max(\ln\ln G(x,\alpha)\ln^{\lambda}G(x,\alpha),\ \ln^2 z)$

The term $\prod\limits_{p<z} \dfrac{(1+\rho(p))}{p(p-1)}$ *is convergent.*

PROOF

Denote the sum under consideration S.

i.e.

$$S = \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (an^2+bn+c,\ dn^2+en+f)=w \\ \left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{p<z,\ p\mid k_2} p\right)=1}} \prod_{\substack{p<z \\ p\mid(an^2+bn+c)k_2}} \left(1-\frac{1}{p}\right)^{-1}$$

and assume, for now, that $D\neq 0$.

By Lemma 2.1, the integers, n, in the interval $\alpha<n\leq\alpha+x$ for which $(an^2+bn+c,\ dn^2+en+f)=w$ lie in an arithmetic progression

$\{ n: n\equiv\gamma_i \bmod F : i=1,\dots r\}$

where $\gamma_i\leq F$ and where $w\mid F$, for F a constant dependent only on the constants a, b, c, d, e and f. (If there are no n for which $(an^2+bn+c,\ dn^2+en+f)=w$ then we write F=0.)

Similarly, every integer n for which

$$\left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{p<z,\ p\mid k_2} p\right) = 1$$

lies in an arithmetic progression

$\{ n: n\equiv\delta_j \bmod k_2F\ ;\ j=1,\dots,s \}$

where $\delta_j \leq k_2F$.

This follows from the observation that, if $m|k_2$, then

$$\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right] \equiv 0 \bmod m$$

$$\Leftrightarrow \left[\frac{a(n+k_2F)^2+b(n+k_2F)+c}{w}\right]\ell_2+\left[\frac{d(n+k_2F)^2+e(n+k_2F)+f}{w}\right] \equiv 0 \bmod m$$

Let $\alpha_1,\ldots,\alpha_\mu$ denote the integers n in the interval $1\leq n\leq k_2F$ for which both

$$(an^2+bn+c,\ dn^2+en+f)=w$$

and $$\left\{ \left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{\substack{p<z\\ p|k_2}} p \right\}=1$$

hold.

Then S becomes

$$S=\sum_{\alpha_i \bmod Fk_2}\ \sum_{\substack{\alpha<n\leq\alpha+x\\ n\equiv\ell_1 \bmod k_1\\ n\equiv\alpha_i \bmod Fk_2}}\ \prod_{\substack{p<z\\ p|(an^2+bn+c)k_2}} \left(1-\frac{1}{p}\right)^{-1} \tag{1}$$

A necessary and sufficient condition that the two congruences $n\equiv\ell_1 \bmod k_1$ and $n\equiv\alpha_i \bmod Fk_2$ have a common solution is that

$$\ell_1\equiv\alpha_i \bmod (k_1,Fk_2).$$

The solution, if it exists, is unique mod $[k_1,Fk_2]$ and we denote it $\beta_i=\beta_i(\ell_1,\alpha_i)$.

Hence

$$S=\sum_{\substack{\alpha_i \bmod Fk_2\\ \alpha_i\equiv\ell_1 \bmod (k_1,Fk_2)}}\ \sum_{\substack{\alpha<n\leq\alpha+x\\ n\equiv\beta_i \bmod [k_1,Fk_2]}}\ \prod_{\substack{p<z\\ p|(an^2+bn+c)k_2}} \left(1-\frac{1}{p}\right)^{-1}$$

It is clear that the internal product

$$\prod_{\substack{p<z\\ p|(an^2+bn+c)k_2}} \left(1-\frac{1}{p}\right)^{-1} = \prod_{\substack{p<z\\ p|k_2h}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z\\ p|(a_1n^2+b_1n+c_1)\\ p\nmid k_2h}} \left(1-\frac{1}{p}\right)^{-1}$$

where $h=(a,\ b,\ c)$; $a=ha_1$; $b=hb_1$; $c=hc_1$,

so

$$S = \prod_{\substack{p<z \\ p | k_2 h}} \left(1-\frac{1}{p}\right)^{-1} \times$$

$$\sum_{\substack{\alpha_i \bmod Fk_2 \\ \alpha_i \equiv \ell_1 \bmod (k_1, Fk_2)}} \quad \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\beta_i \bmod [k_1, Fk_2]}} \quad \prod_{\substack{p<z \\ p | (a_1 n^2+b_1 n+c_1) \\ p \nmid k_2 h}} \left(1-\frac{1}{p}\right)^{-1} \tag{2}$$

To estimate S, therefore, it is sufficient that we estimate

the inner sum

$$\sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\beta_i \bmod [k_1, Fk_2]}} \quad \prod_{\substack{p<z \\ p | (a_1 n^2+b_1 n+c_1) \\ p \nmid k_2 h}} \left(1-\frac{1}{p}\right)^{-1} = S_1 \quad \text{say.} \tag{3}$$

The product

$$\prod_{\substack{p<z \\ p | (a_1 n^2+b_1 n+c_1) \\ p \nmid k_2 h}} \left(1-\frac{1}{p}\right)^{-1} = 1 + \sum_{\substack{p_1 \ldots p_\gamma | a_1 n^2+b_1 n+c_1 \\ p_1, \ldots, p_\gamma \nmid k_2 h \\ p_1 < \ldots < p_\gamma < z}} \frac{1}{(p_1-1)\ldots(p_\gamma-1)}$$

$$= \sum_{\substack{a_1 n^2+b_1 n+c_1 \equiv 0 \bmod m \\ (k_2 h, m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)}$$

where P(m) denotes the largest prime factor of m.

Consequently,

$$S_1 = \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\beta_1 \bmod [k_1, Fk_2]}} \quad \sum_{\substack{a_1 n^2+b_1 n+c_1 \equiv 0 \bmod m \\ (k_2 h, m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)}$$

which on changing the order of summation gives

$$S_1 = \sum_{\substack{1\leq m\leq G(x,\alpha) \\ (k_2 h, m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\beta_i \bmod [k_1, Fk_2] \\ a_1 n^2+b_1 n+c_1 \equiv 0 \bmod m}} 1$$

where $G(x,\alpha)$ denotes $\max\limits_{\alpha<n\leq\alpha+x} |a_1 n^2+b_1 n+c_1|$.

Further

$$S_1 = \sum_{\substack{1\leq m\leq G(x,\alpha) \\ (k_2 h, m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\frac{\alpha-\beta_i}{[k_1, Fk_2]} < t \leq \frac{\alpha+x-\beta_i}{[k_1, Fk_2]} \\ g(t)\equiv 0 \bmod m}} 1$$

where

$$g(t)=a_1([k_1,Fk_2]t+\beta_i)^2 + b_1([k_1,Fk_2]t+\beta_i) + c_1$$

$$=a_1[k_1,Fk_2]^2t^2 + [k_1,Fk_2](2a_1\beta_i+b_1)t + (a_1\beta_i^2+b_1\beta_i+c_1).$$

Denoting $\gamma_1(m)$, $\gamma_2(m),\ldots,\gamma_{\rho(m)}$ as the $\rho(m)$ solutions of $g(t)\equiv 0 \bmod m$, we have

$$S_1 = \sum_{\substack{1\leq m\leq G(x,\alpha)\\(k_2h,m)=1\\P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{j=1}^{\rho(m)} \sum_{\substack{\frac{\alpha-\beta_i}{[k_1,Fk_2]}<t\leq\frac{\alpha+x-\beta_i}{[k_1,Fk_2]}\\t\equiv\gamma_j(m) \bmod m}} 1$$

$$= \sum_{\substack{1\leq m\leq G(x,\alpha)\\(k_2h,m)=1\\P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \rho(m)\left\{\frac{x}{[k_1,Fk_2]m} + 0(1)\right\}$$

$$= \frac{x}{[k_1,Fk_2]} \sum_{\substack{1\leq m\leq G(x,\alpha)\\(k_2h,m)=1\\P(m)<z}} \frac{\mu^2(m)\rho(m)}{\varphi(m)m} + 0\left\{\sum_{\substack{1\leq m\leq G(x,\alpha)\\(k_2h,m)=1\\P(m)<z}} \frac{\mu^2(m)\rho(m)}{\varphi(m)}\right\}.$$

Further,

$$S_1 = \frac{x}{[k_1,Fk_2]} \sum_{\substack{(k_2h,m)=1\\P(m)<z}} \frac{\mu^2(m)\rho(m)}{\varphi(m)m} + 0\left\{\frac{x}{[k_1,Fk_2]}\sum_{m>G(x,\alpha)} \frac{\rho(m)}{\varphi(m)m}\right\}$$

$$+ 0\left\{\sum_{\substack{1\leq m\leq G(x,\alpha)\\(k_2h,m)=1\\P(m)<z}} \frac{\mu^2(m)\rho(m)}{\varphi(m)}\right\}. \qquad (4)$$

Our first step from here is to simplify the 0-terms. Recall

$$\rho(m)= \left|\{t:t \bmod m;\ a_1[k_1,Fk_2]^2t^2 + [k_1,Fk_2](2a_1\beta_i+b_1)t + (a_1\beta_i^2+b_1\beta_i+c_1)\equiv 0 \bmod m\}\right|.$$

Writing

$$v(\beta_i)=(\ a_1[k_1,Fk_2]^2,\ [k_1,Fk_2](2a_1\beta_i+b_1),\ (a_1\beta_i^2+b_1\beta_i+c_1)\ )$$

and denoting the divisors of $v(\beta_i)$ to be

$1=e_0,\ e_1,\ldots,e_r\ ;\ e_0<e_1<\ldots<e_r$ gives

$$\sum_{m>G(x,\alpha)} \frac{\rho(m)}{\varphi(m)m} = \sum_{j=0}^{r} \sum_{\substack{m>G(x,\alpha)\\(m,v(\beta_i))=e_j}} \frac{\rho(m)}{\varphi(m)m}.$$

Now, by Lemma 2.6, for j=0,

$$\sum_{\substack{m>G(x,\alpha)\\(m,v(\beta_i))=1}} \frac{\rho(m)}{\varphi(m)m} = O\left\{\frac{\ln\ln G(x,\alpha)\ \ln^\lambda G(x,\alpha)}{G(x,\alpha)}\right\}$$

where $\ln \lambda=\left\{\left[\frac{\ln(2.D^2)}{\ln 2}\right]+1\right\}\ln 2.$

On the other hand, if $j\neq 0$, then $m=m_je_j$ say, and $(m,v(\beta_i))=e_j$ implies $(m_j,\frac{v(\beta_i)}{e_j})=1$.

In this case

$$\rho(m) = \left|\{t:t\bmod m;\ a_1[k_1,Fk_2]^2t^2+[k_1,Fk_2](2a_1\beta_i+b_1)t+(a_1\beta_i^2+b_1\beta_i+c_1)\equiv 0 \bmod m\}\right|$$

$$= \left|\{t:t\bmod m_je_j\ ;\ e_j(At^2+Bt+C)\equiv 0 \bmod m_je_j\}\right|$$

say, where $Ae_j=a_1[k_1,Fk_2]^2,\ldots$ and $(A,B,C)=\frac{v(\beta_i)}{e_j}$.

So $\rho(m)=\left|\{t:t\bmod m_j;\ (At^2+Bt+C)\equiv 0 \bmod m_j\}\right|e_j$

$=\rho(m_j)e_j$.

Consequently

$$\sum_{\substack{m>G(x,\alpha)\\(m,v(\beta_i))=e_j}} \frac{\rho(m)}{\varphi(m)m} = \sum_{\substack{m_je_j>G(x,\alpha)\\(m_j,\frac{v(\beta_i)}{e_j})=1}} \frac{e_j\rho(m_j)}{\varphi(m_je_j)m_je_j}$$

$$< \frac{1}{\varphi(e_j)} \sum_{\substack{m_j>\frac{G(x,\alpha)}{e_j}\\(m_j,\frac{v(\beta_i)}{e_j})=1}} \frac{\rho(m_j)}{m_j\varphi(m_j)}$$

$$\ll \frac{\ln\ln G(x,\alpha)\ \ln^\lambda G(x,\alpha)\ e_j}{\varphi(e_j)\ G(x,\alpha)}.$$

Summing over j gives

$$\sum_{m>G(x,\alpha)} \frac{\rho(m)}{\varphi(m)m} = O\left\{\frac{\ln\ln G(x,\alpha)\ \ln^\lambda G(x,\alpha)}{G(x,\alpha)} \sum_{j=0}^{r}\frac{e_j}{\varphi(e_j)}\right\}$$

$$= O\left\{\frac{\ln\ln G(x,\alpha)\ \ln^\lambda G(x,\alpha)}{G(x,\alpha)} \sum_{n=1}^{e_r}\frac{n}{\varphi(n)}\right\}$$

$$= O\left\{\frac{e_r\ln\ln G(x,\alpha)\ \ln^\lambda G(x,\alpha)}{G(x,\alpha)}\right\}$$

$$= 0\left\{ \frac{a_1[k_1,Fk_2]^2 \ln\ln G(x,\alpha)\ \ln^{\lambda}G(x,\alpha)}{G(x,\alpha)} \right\}. \quad (5)$$

Similarly, the second error term,

$$\sum_{\substack{1\le m\le G(x,\alpha)\\(k_2h,m)=1\\P(m)<z}} \frac{\mu^2(m)\rho(m)}{\varphi(m)} = 0\left\{ \sum_{1\le m\le G(x,\alpha)} \frac{\mu^2(m)\rho(m)}{\varphi(m)} \right\}$$

$$= 0\left\{ \sum_{j=0}^{r} \sum_{\substack{1\le m\le G(x,\alpha)\\(m,v(\beta_i))=e_j}} \frac{\mu^2(m)\rho(m)}{\varphi(m)} \right\}$$

and

$$\sum_{\substack{1\le m\le G(x,\alpha)\\(m,v(\beta_i))=e_j}} \frac{\mu^2(m)\rho(m)}{\varphi(m)} = \sum_{\substack{1\le m_j e_j\le G(x,\alpha)\\(m_j,\frac{v(\beta_i)}{e_j})=1}} \frac{\mu^2(m_j e_j)\ e_j\ \rho(m_j)}{\varphi(m_j e_j)}$$

$$< \frac{e_j}{\varphi(e_j)} \sum_{\substack{1\le m_j\le \frac{G(x,\alpha)}{e_j}\\(m_j,\frac{v(\beta_i)}{e_j})=1}} \frac{\rho(m_j)}{\varphi(m_j)}$$

$$\ll \frac{e_j}{\varphi(e_j)} \ln\ln G(x,a)\ \ln^{\lambda}G(x,\alpha) \quad \text{by Lemma 2.7.}$$

So

$$\sum_{\substack{1\le m\le G(x,\alpha)\\(k_2h,m)=1\\P(m)<z}} \frac{\mu^2(m)\rho(m)}{\varphi(m)} = 0(\ a_1[k_1,Fk_2]^2 \ln\ln G(x,\alpha)\ \ln^{\lambda}G(x,\alpha)\ ) \quad (6)$$

However we may also write the second error term as

$$\sum_{\substack{1\le m\le G(x,\alpha)\\(k_2h,m)=1\\P(m)<z}} \frac{\mu^2(m)\rho(m)}{\varphi(m)} = 0\left\{ \sum_{P(m)<z} \frac{\mu^2(m)\rho(m)}{\varphi(m)} \right\}$$

$$= 0\left\{ \sum_{j=0}^{r} \sum_{\substack{P(m)<z\\(m,v(\beta_i))=e_j}} \frac{\mu^2(m)\rho(m)}{\varphi(m)} \right\}$$

and

$$\sum_{\substack{P(m)<z \\ (m,v(\beta_i))=e_j}} \frac{\mu^2(m)\rho(m)}{\varphi(m)} = \sum_{\substack{P(m_j e_j)<z \\ (m_j, \frac{v(\beta_i)}{e_j})=1}} \frac{\mu^2(m_j e_j)\ e_j\ \rho(m_j)}{\varphi(m_j e_j)}$$

$$< \frac{e_j}{\varphi(e_j)} \sum_{\substack{P(m_j)<z \\ (m_j, \frac{v(\beta_i)}{e_j})=1}} \frac{\mu^2(m_j)\rho(m_j)}{\varphi(m_j)}$$

$$\ll \frac{e_j}{\varphi(e_j)} \prod_{\substack{p<z \\ p \nmid \frac{v(\beta_i)}{e_j}}} \left(1+\frac{\rho(p)}{p-1}\right)$$

$$\ll \frac{e_j}{\varphi(e_j)} \prod_{p<z} \left(1+\frac{2}{p-1}\right) \ll \frac{e_j}{\varphi(e_j)} \ln^2 z.$$

So in comparison with (6) we also have

$$\sum_{\substack{1\le m\le G(x,\alpha) \\ (k_2 h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)\rho(m)}{\varphi(m)} = O(\ a_1 [k_1, Fk_2]^2 \ln^2 z\ ) \qquad (7)$$

This concludes the simplification of the O-terms.

Now, turning to the leading term of (4), we have,

$$\sum_{\substack{(k_2 h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)\rho(m)}{\varphi(m)m} = \prod_{\substack{p<z \\ p\nmid k_2 h}} \left(\ 1+\frac{\rho(p)}{p(p-1)}\ \right).$$

We note that by an argument similar to that used in deriving (5), (6) and (7) we have

$$\sum_{\substack{(k_2 h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)\rho(m)}{\varphi(m)m} = O\left\{ \sum_{j=0}^{r} \sum_{\substack{P(m)<z \\ (m,v(\beta_i))=e_j}} \frac{\rho(m)}{\varphi(m)m} \right\}$$

$$= O\left\{ \sum_{j=0}^{r} \frac{1}{\varphi(e_j)} \right\} = O\left\{ \sum_{n=1}^{e_r} \frac{1}{\varphi(n)} \right\}$$

$$= O(\ \ln e_r\ ) = O(\ln\ (a_1[k_1, Fk_2])) \qquad (8)$$

So the leading term of S_1 is certainly convergent.

Hence, via (5), (6), (7) and (8),

$$S_1 = \frac{x}{[k_1, Fk_2]} \prod_{\substack{p<z \\ p\nmid k_2 h}} \left(\ 1+\frac{\rho(p)}{p(p-1)}\ \right) \left\{ 1 + \right.$$

$$+ \; O\left[\frac{a_1[k_1,Fk_2]^2 \ln\ln G(x,\alpha) \ln^{\lambda} G(x,\alpha)}{G(x,\alpha)}\right\}$$

$$+ \; O\left[\frac{a_1[k_1,Fk_2]^3 \; A}{x}\right] \Bigg\} \qquad (9)$$

where $A = \max(\ln\ln G(x,\alpha)\ln^{\lambda}G(x,\alpha),\ \ln^2 z)$. A arises from equations (6) and (7).

This, on substitution back into (2) gives

$$S = \frac{x}{[k_1,Fk_2]} \prod_{\substack{p<z \\ p \mid k_2 h}} \left(1-\frac{1}{p}\right)^{-1} \sum_{\substack{\alpha_i \bmod Fk_2 \\ \alpha_i \equiv \ell_1 \bmod (k_1,Fk_2)}} \prod_{\substack{p<z \\ p \nmid k_2 h}} \left(1+\frac{\rho(p)}{p(p-1)}\right) \Bigg\{1+$$

$$+ \; O\left[\frac{a_1[k_1,Fk_2]^2 \ln\ln G(x,\alpha) \ln^{\lambda} G(x,\alpha)}{G(x,\alpha)}\right]$$

$$+ \; O\left[\frac{a_1[k_1,Fk_2]^3 \; A}{x}\right] \Bigg\} . \qquad (10)$$

This completes the lemma for $D \neq 0$.

If $D=0$, which may occur only if an^2+bn+c has a repeated factor, so that we may write $an^2+bn+c=\theta(\gamma n+\delta)^2$ say, then S becomes

$$\sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (\theta(\gamma n+\delta)^2,\, dn^2+en+f)=w \\ \left\{ \left[\frac{\theta(\gamma n+\delta)^2}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{p<z,\ p\nmid k_2} p \right\}=1}} \prod_{\substack{p<z \\ p \mid \theta(\gamma n+\delta)k_2}} \left(1-\frac{1}{p}\right)^{-1}$$

The proof of the lemma in this instance is very similar to that for $D\neq 0$. □

LEMMA 2.10

Let an^2+bn+c *and* dn^2+en+f *be two polynomials with integer coefficients having no common factors. Then, for* q *a prime,*

$$\sum_{\substack{\alpha<q\leq\alpha+x \\ q\equiv \ell_1 \bmod k_1 \\ (aq^2+bq+c,\,dq^2+eq+f)=w \\ \left(\left[\frac{aq^2+bq+c}{w}\right]\ell_2+\left[\frac{dq^2+eq+f}{w}\right],\ \prod_{p<z,\ p\mid k_2} p\right)=1}} \quad \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid(aq^2+bq+c)(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)^{-1}$$

$$< \frac{2x}{\ln x} \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid h}} \left(1-\frac{1}{p}\right)^{-1} \frac{[k_1,Fk_2]^{\frac{1}{2}} \ln[k_1,Fk_2]}{\varphi([k_1,Fk_2])} \prod_{\substack{p<z \\ p\nmid k_2 h}} \left(1+\frac{4p^{2/3}}{(p-1)^2}\right)$$

$$\times\ \Upsilon_z(w) \left\{ 1 + 0\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \cdot \frac{M}{x} \cdot \ln x \right] \right\}$$

where

(i) $$F = \begin{cases} |ce-fb| & ;\ a=d=0 \\ |(cd-fa)^2-(bd-ea)(ce-fb)| & ;\ \textit{otherwise} \end{cases}$$

(ii) $h = (ad, ae+bd, af+be+cd, bf+ce, cf)$

and $A={}^{ad}/_h$, $B={}^{(ae+bd)}/_h$, $C={}^{(af+be+cd)}/_h$, $D={}^{(bf+ce)}/_h$, $E={}^{cf}/_h$.

(iii) $M = \max(\ln\ln G(x,\alpha)\ln^{\lambda}G(x,\alpha),\ \ln^2 z)$

where $G(x,\alpha) = \max_{\alpha<q\leq\alpha+x} |Aq^4+Bq^3+Cq^2+Dq+E|$

and $\ln\lambda = \left\{ \left[\frac{\ln(2.\Delta^2)}{\ln 2}\right]+1 \right\} \ln 2$

where Δ *denotes the discriminant of* $(aq^2+bq+c)(dq^2+eq+f)$ *if neither* aq^2+bq+c *nor* dq^2+eq+f *have repeated factors. If* aq^2+bq+c *has a repeated factor, say* $aq^2+bq+c=\theta(\gamma q+\delta)^2$ *and* dq^2+eq+f *does not have a repeated factor then* Δ *is the discriminant of* $\theta(\gamma q+\delta)(dq^2+eq+f)$. *Similarly if* dq^2+eq+f *has a repeated factor. Clearly with this definition* $\Delta\neq 0$.

and where

(iv) $\Upsilon_z(w)$ *denotes the number of integers* n *in the interval* $1\leq n\leq Fk_2$ *for which both*

$$(an^2+bn+c,\ dn^2+en+f)=w$$

and $\left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right] , \prod_{\substack{p<z\\ p|k_2}} p \right)=1$

PROOF

Assume firstly that neither an^2+bn+c nor dn^2+en+f have repeated factors. Denote the sum under consideration S.

i.e.

$$S = \sum_{\substack{\alpha<q\leq\alpha+x\\ q\equiv\ell_1 \bmod k_1\\ (aq^2+bq+c,\,dq^2+eq+f)=w\\ \left(\left[\frac{aq^2+bq+c}{w}\right]\ell_2+\left[\frac{dq^2+eq+f}{w}\right],\ \prod_{p<z,\ p|k_2} p\right)=1}} \prod_{\substack{p<z\\ p\nmid k_2\\ p|(aq^2+bq+c)(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)^{-1} .$$

The proof is essentially the same as that of Lemma 2.9. Certainly the argument follows almost identically until statement (2) of Lemma 2.9 so that we may write

$$S = \prod_{\substack{p<z\\ p\nmid k_2\\ p|h}} \left(1-\frac{1}{p}\right)^{-1} \sum_{\substack{\alpha_i \bmod Fk_2\\ \alpha_i\equiv\ell_1 \bmod (k_1,Fk_2)}} \ \sum_{\substack{\alpha<q\leq\alpha+x\\ q\equiv\beta_i \bmod [k_1,Fk_2]}} \ \prod_{\substack{p<z\\ p\nmid k_2 h\\ p|(Aq^4+Bq^3+Cq^2+Dq+E)}} \left(1-\frac{1}{p}\right)^{-1} \tag{1}$$

where given that

$(aq^2+bq+c)(dq^2+eq+f)=adq^4+(ae+bd)q^3+(af+be+cd)q^2+(bf+ce)q+cf$

we write $h=(ad,ae+bd,af+be+cd,bf+ce,cf)$ and

$A={}^{ad}/_h$, $B={}^{(ae+bd)}/_h$, etc. so that $(A,B,C,D,E)=1$:

where $\alpha_1,\ldots\alpha_\mu$ denote the integers n in the interval $1\leq n\leq Fk_2$ for which both $(an^2+bn+c,dn^2+en+f)=w$ and

$\left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{\substack{p<z\\ p|k_2}} p\right)=1$;

and where $\beta_i=\beta_i(\ell_1,\alpha_i)$ is the unique solution, if it exists, of the pair of congruences $q\equiv\ell_1 \bmod k_1$ and $q\equiv\alpha_i \bmod Fk_2$.

Writing the inner sum of (1) as S_1,

i.e. $$S_1 = \sum_{\substack{\alpha<q\leq\alpha+x\\ q\equiv\beta_i \bmod [k_1,Fk_2]}} \ \prod_{\substack{p<z\\ p\nmid k_2 h\\ p|Aq^4+Bq^3+Cq^2+Dq+E}} \left(1-\frac{1}{p}\right)^{-1}$$

we have

$$S_1 = \sum_{\substack{\alpha<q\leqslant\alpha+x \\ q\equiv\beta_i \bmod [k_1, Fk_2]}} \sum_{\substack{Aq^4+Bq^3+Cq^2+Dq+E\equiv 0 \bmod m \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)}$$

where P(m) denotes the largest prime factor of m.

Changing the order of summation gives

$$S_1 = \sum_{\substack{1\leqslant m\leqslant G(x,\alpha) \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\alpha<q\leqslant\alpha+x \\ q\equiv\beta_i \bmod [k_1, Fk_2] \\ Aq^4+Bq^3+Cq^2+Dq+E\equiv 0 \bmod m}} 1$$

where $G(x,\alpha) = \max_{\alpha<q\leqslant\alpha+x} |Aq^4+Bq^3+Cq^2+Dq+E|$.

Writing $\gamma_1(m), \gamma_2(m), \ldots, \gamma_r(m)$ as the $\rho(m)$ solutions of $An^4+Bn^3+Cn^2+Dn+E\equiv 0 \bmod m$, gives

$$S_1 = \sum_{\substack{1\leqslant m\leqslant G(x,\alpha) \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\gamma_j(m) \bmod m} \sum_{\substack{\alpha<q\leqslant\alpha+x \\ q\equiv\beta_i \bmod [k_1, Fk_2] \\ q\equiv\gamma_j(m) \bmod m}} 1 .$$

Denoting $\delta_{ij} = \delta_{ij}(\beta_i, \gamma_j(m))$ as the unique solution $\bmod [k_1, Fk_2, m]$, if it exists, of the pair of congruences $n\equiv\beta_i \bmod [k_1, Fk_2]$ and $n\equiv\gamma_j(m) \bmod m$ we have

$$S_1 = \sum_{\substack{1\leqslant m\leqslant G(x,\alpha) \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\gamma_j(m) \bmod m \\ \gamma_j(m)\equiv\beta_i \bmod ([k_1, Fk_2], m)}} \sum_{\substack{\alpha<q\leqslant\alpha+x \\ q\equiv\delta_{ij} \bmod [k_1, Fk_2, m]}} 1 .$$

Splitting S_1 into two sums we have

$$S_1 = \sum_{\substack{1\leqslant m\leqslant G(x,\alpha) \\ [k_1, Fk_2, m]<x \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\gamma_j(m) \bmod m \\ \gamma_j(m)\equiv\beta_i \bmod ([k_1, Fk_2], m)}} \sum_{\substack{\alpha<q\leqslant\alpha+x \\ q\equiv\delta_{ij} \bmod [k_1, Fk_2, m]}} 1$$

$$+ \sum_{\substack{1\leq m\leq G(x,\alpha)\\ [k_1,Fk_2,m]\geq x\\ (k_2h,m)=1\\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\gamma_j(m)\bmod m\\ \gamma_j(m)\equiv\beta_i \bmod([k_1,Fk_2],m)}} \sum_{\substack{\alpha<q\leq\alpha+x\\ q\equiv\delta_{ij}\bmod[k_1,Fk_2,m]}} 1$$

Using the estimate of Montgomery-Vaughan [6], namely

$$\Pi(x;k,\ell)-\Pi(x-y;k,\ell) < \frac{2y}{\varphi(k)\ \ln {}^{y}/_{k}} \quad ; \ 1\leq k<y<x$$

in the first of these sums and noting that

$\sum_{\substack{\alpha<q\leq\alpha+x\\ q\equiv\delta_{ij}\bmod[k_1,Fk_2,m]}} 1 < 1$ in the second gives

$$S_1 < \frac{2x}{\ln x} \sum_{\substack{1\leq m\leq G(x,\alpha)\\ [k_1,Fk_2,m]<x\\ (k_2h,m)=1\\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)}{\varphi(m)} \frac{\ln\ [k_1,Fk_2,m]}{\varphi([k_1,Fk_2,m])}$$

$$+ \sum_{\substack{1\leq m\leq G(x,\alpha)\\ [k_1,Fk_2,m]\geq x\\ (k_2h,m)=1\\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)}{\varphi(m)} \quad .$$

By Lemma 2.8 we have $\frac{1}{\varphi([k_1,Fk_2,m])} < \frac{[k_1,Fk_2]^{\frac{1}{2}}}{\varphi([k_1,Fk_2])}\cdot\frac{m^{\frac{1}{2}}}{\varphi(m)}$ so that

$$S_1 < \frac{2x}{\ln x} \frac{[k_1,Fk_2]^{\frac{1}{2}}\ \ln[k_1,Fk_2]}{\varphi([k_1,Fk_2])} \sum_{\substack{1\leq m\leq G(x,\alpha)\\ [k_1,Fk_2,m]<x\\ (k_2h,m)=1\\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)\ \ln m\ m^{\frac{1}{2}}}{\varphi(m)^2}$$

$$+ \sum_{\substack{1\leq m\leq G(x,\alpha)\\ [k_1,Fk_2,m]\geq x\\ (k_2h,m)=1\\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)}{\varphi(m)} \quad .$$

Arguing as in Lemma 2.9, we have

$$\sum_{\substack{1\le m\le G(x,\alpha)\\ [k_1,Fk_2,m]<x\\ (k_2h,m)=1\\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)\ \ln m\ m^{\frac12}}{\varphi(m)^2} < \sum_{\substack{(k_2h,m)=1\\ P(m)<z}} \frac{\rho(m)\ m^{2/3}}{\varphi(m)^2}$$

$$= \prod_{\substack{p<z\\ p\nmid k_2h}} \left(1+\frac{\rho(p)p^{2/3}}{(p-1)^2} \right)$$

and
$$\sum_{\substack{1\le m\le G(x,\alpha)\\ [k_1,Fk_2,m]\ge x\\ (k_2h,m)=1\\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)}{\varphi(m)} < \sum_{\substack{1\le m\le G(x,\alpha)\\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)}{\varphi(m)} = 0(M)$$

where $M=\max(\ln\ln G(x,\alpha)\ln^{\lambda}G(x,\alpha),\ \ln^4 z)$ and where

$$\ln\lambda = \left\{ \left[\frac{\ln(2.\Delta^2)}{\ln 2}\right]+1 \right\}\ln 2 .$$

So as $\rho(p)\le 4$

$$S_1 < \frac{2x}{\ln x}\frac{[k_1,Fk_2]^{\frac12}\ \ln[k_1,Fk_2]}{\varphi([k_1,Fk_2])} \prod_{\substack{p<z\\ p\nmid k_2h}} \left(1+\frac{4p^{2/3}}{(p-1)^2} \right) + 0(M)$$

$$= \frac{2x}{\ln x}\frac{[k_1,Fk_2]^{\frac12}\ \ln[k_1,Fk_2]}{\varphi([k_1,Fk_2])} \prod_{\substack{p<z\\ p\nmid k_2h}} \left(1+\frac{4p^{2/3}}{(p-1)^2} \right)\left\{1+0\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac12}}.\frac{M\ \ln x}{x}\right]\right\}$$

Substitution back into (1) gives

$$S < \prod_{\substack{p<z\\ p\nmid k_2\\ p\mid h}} (1-\frac{1}{p})^{-1} \quad \frac{2x}{\ln x}\frac{[k_1,Fk_2]^{\frac12}\ \ln[k_1,Fk_2]}{\varphi([k_1,Fk_2])} \prod_{\substack{p<z\\ p\nmid k_2h}} \left(1+\frac{4p^{2/3}}{(p-1)^2} \right)$$

$$\times \sum_{\substack{\alpha_i \bmod Fk_2\\ \alpha_i\equiv \ell_1 \bmod Fk_2}} 1 \quad \left\{1 + 0\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^2}.\ \frac{M\ \ln x}{x}\right] \right\} .$$

This completes the lemma.

If aq^2+bq+c has a repeated factor, say $aq^2+bq+c=\theta(\gamma q+\delta)^2$

then S becomes

$$S = \sum_{\substack{\alpha<q\leq\alpha+x \\ q\equiv \ell_1 \bmod k_1 \\ (aq^2+bq+c,\, dq^2+eq+f)=w \\ \left(\left[\frac{aq^2+bq+c}{w}\right]\ell_2+\left[\frac{dq^2+eq+f}{w}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid \theta(\gamma q+\delta)^2(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)^{-1} .$$

The proof of the lemma in this instance is very similar. The same reasoning applies if both aq^2+bq+c and dq^2+eq+f have repeated factors.

□

LEMMA 2.11

Let an^2+bn+c *and* dn^2+en+f *be two polynomials with integer coefficients, having no common factors. Then, for* $z\leq\exp((\ln {}^{x}/k_1)^{1-\epsilon})$, for ϵ some constant $\frac{1}{2}>\epsilon>0$,

$$\sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (n,\prod_{p<z} p)=1 \\ (an^2+bn+c,\, dn^2+en+f)=w \\ \left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid (an^2+bn+c)(dn^2+en+f)}} \left(1-\frac{1}{p}\right)^{-1}$$

$$\leq \quad x \prod_{p<z}\left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid h}} \left(1-\frac{1}{p}\right)^{-1} \frac{[k_1,Fk_2]^{\frac{1}{2}}}{\varphi([k_1,Fk_2])} \prod_{\substack{p<z \\ p\nmid k_2 h}} \left(1+\frac{4p^{2/3}}{(p-1)^2}\right)$$

$$\times\ \Upsilon_z(w)\ \left\{1+0(\exp(-(\ln x)^{\epsilon}))\ +0\left[\frac{\varphi([k_1,Fk_2])\ln\ln x \ln^{\lambda+1}x}{x^{\frac{1}{2}}}\right]\right\}$$

where

(i) $F = \begin{cases} |ce-fb| & ;\ a=d=0 \\ |(cd-fa)^2-(bd-ea)(ce-fb)| & ;\ \textit{otherwise} \end{cases}$

(ii) $h = (ad, ae+bd, af+be+cd, bf+ce, cf)$

and $A={}^{ad}/_{h}$, $B={}^{(ae+bd)}/_{h}$, $C={}^{(af+be+cd)}/_{h}$, $D={}^{(bf+ce)}/_{h}$ $E={}^{cf}/_{h}$.

(iii) $G(x,\alpha) = \max_{\alpha<n\leq\alpha+x} |An^4+Bn^3+Cn^2+Dn+E|$

and $\ln\lambda = \left\{ \left[\frac{\ln(2.\Delta^2)}{\ln 2}\right]+1 \right\} \ln 2$

where Δ *denotes the discriminant of* $(an^2+bn+c)(dn^2+en+f)$ *if neither* an^2+bn+c *nor* dn^2+en+f *have repeated factors. If* an^2+bn+c *has a repeated factor, say* $an^2+bn+c=\theta(\gamma n+\delta)^2$ *and* dn^2+en+f *does not have a repeated factor then* Δ *is the discriminant of* $\theta(\gamma n+\delta)(dn^2+en+f)$. *Similarly if* dn^2+en+f *has a repeated factor.*

and where

(iv) $\Upsilon_z(w)$ *denotes the number of integers* n *in the interval* $1\leq n\leq Fk_2$ *for which both*

$$(an^2+bn+c,dn^2+en+f)=w$$

and $\left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right], \prod_{\substack{p<z\\ p|k_2}} p \right)=1$

PROOF

Assume firstly that neither an^2+bn+c nor dn^2+en+f have repeated factors. Denote the sum under consideration S.

i.e.

$$S = \sum_{\substack{\alpha<n\leq\alpha+x\\ n\equiv\ell_1 \bmod k_1\\ (n,\prod_{p<z}p)=1\\ (an^2+bn+c,dn^2+en+f)=w\\ \left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{\substack{p<z\\ p|k_2}} p\right)=1}} \prod_{\substack{p<z\\ p\nmid k_2\\ p|(an^2+bn+c)(dn^2+en+f)}} \left(1-\frac{1}{p}\right)^{-1} .$$

We argue exactly as in Lemma 2.10. A very rough sketch of the proof is given here. Certainly

$$S = \prod_{\substack{p<z\\ p\nmid k_2\\ p|h}} \left(1-\frac{1}{p}\right)^{-1} \sum_{\substack{\alpha_i \bmod Fk_2\\ \alpha_i\equiv\ell_1 \bmod (k_1,Fk_2)}} S_1 \qquad (1)$$

where

$$S_1 = \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\beta_i \bmod [k_1,Fk_2] \\ (n,\prod_{p<z}p)=1}} \prod_{\substack{p<z \\ p\nmid k_2h \\ p\mid An^4+Bn^3+Cn^2+Dn+E}} (1-\tfrac{1}{p})^{-1}$$

and where $\alpha_1,\ldots,\alpha_\mu$ denote the integers n in the interval $1\leq n\leq Fk_2$ for which both $(an^2+bn+c,dn^2+en+f)=w$ and

$$\left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1\ ;$$

where $h=(ad,ae+bd,af+be+cd,bf+ce,cf)$ and $A={}^{ad}/_h$, $B={}^{(ae+bd)}/_h$, etc. such that $(A,B,C,D,E)=1$;

and where $\beta_i=\beta_i(\ell_1,\alpha_i)$ is the unique solution, if it exists, of the pair of congruences $n\equiv\ell_1 \bmod k_1$ and $n\equiv\alpha_i \bmod Fk_2$. We further have

$$S_1 = \sum_{\substack{1\leq m\leq G(x,\alpha) \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\gamma_j(m) \bmod m \\ \gamma_j(m)\equiv\beta_i \bmod ([k_1,Fk_2],m)}} \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\delta_{ij} \bmod [k_1,Fk_2,m] \\ (n,\prod_{p<z}p)=1}} 1$$

where $\gamma_1(m),\ldots,\gamma_r(m)$ are the $\rho(m)$ solutions of

$$An^4+Bn^3+Cn^2+Dn+E\equiv 0 \bmod m$$

and $\delta_{ij} = \delta_{ij}(\beta_i,\gamma_j(m))$ is the unique solution $\bmod [k_1,Fk_2,m]$, if it exists, of the pair of congruences $n\equiv\beta_i \bmod [k_1,Fk_2]$ and $n\equiv\gamma_j(m) \bmod m$; and where

$$G(x,\alpha) = \max_{\alpha<n\leq\alpha+x} |An^4+Bn^3+Cn^2+Dn+E|.$$

We divide the sum S_1 into two to read

$$S_1 = \sum_{\substack{1\leq m\leq \frac{x^{\frac12}}{[k_1,Fk_2]} \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\gamma_j(m) \bmod m \\ \gamma_j(m)\equiv\beta_i \bmod ([k_1,Fk_2],m)}} \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\delta_{ij} \bmod [k_1,Fk_2,m] \\ (n,\prod_{p<z}p)=1}} 1$$

$$+ \sum_{\substack{\frac{x^{\frac12}}{[k_1,Fk_2]}\leq m\leq G(x,\alpha) \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\gamma_j(m) \bmod m \\ \gamma_j(m)\equiv\beta_i \bmod ([k_1,Fk_2],m)}} \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\delta_{ij} \bmod [k_1,Fk_2,m] \\ (n,\prod_{p<z}p)=1}} 1 \quad . \qquad (2)$$

Now if $m \leq \frac{x^{\frac{1}{2}}}{[k_1, Fk_2]}$ and $z \leq \exp((\ln {}^{x}/k_1)^{1-\epsilon})$ then $z \leq \frac{x}{[k_1, Fk_2, m]}$ and we may apply Lemma 1.1 to the sum $\sum\limits_{\substack{\alpha < n \leq \alpha + x \\ n \equiv \delta_{ij} \bmod [k_1, Fk_2, m] \\ (n, \prod_{p<z} p) = 1}} 1$ to give

$$\sum_{\substack{\alpha < n \leq \alpha + x \\ n \equiv \delta_{ij} \bmod [k_1, Fk_2, m] \\ (n, \prod_{p<z} p) = 1}} 1 \leq \frac{x}{[k_1, Fk_2, m]} \prod_{\substack{p<z \\ p \nmid [k_1, Fk_2, m]}} (1 - \frac{1}{p}) \{ 1 +$$

$$O(\exp(-(\ln\left[\frac{x}{[k_1, Fk_2, m]}\right])^{\frac{1}{2}})) + O(\exp(-u(\ln u - \ln\ln u - 2))) \}$$

where $u = \frac{\ln \left[\frac{x}{[k_1, Fk_2, m]}\right]}{\ln z}$.

We have by our assumptions above that $u \geq \frac{1}{2}(\ln x)^{\epsilon}$, and that $\ln\left[\frac{x}{[k_1, Fk_2, m]}\right] \geq \frac{\ln x}{2}$ and so

$$\sum_{\substack{\alpha < n \leq \alpha + x \\ n \equiv \delta_{ij} \bmod [k_1, Fk_2, m] \\ (n, \prod_{p<z} p) = 1}} 1 \leq \frac{x}{\varphi([k_1, Fk_2, m])} \prod_{p<z} (1 - \frac{1}{p}) \{ 1 + O(\exp(-(\ln x)^{\epsilon})) \}$$

$$\leq \frac{x \; [k_1, Fk_2]^{\frac{1}{2}}}{\varphi([k_1, Fk_2])} \cdot \frac{m^{\frac{1}{2}}}{\varphi(m)} \cdot \prod_{p<z} (1 - \frac{1}{p}) \{ 1 + O(\exp(-(\ln x)^{\epsilon})) \}. \tag{3}$$

If on the other hand $m > \frac{x^{\frac{1}{2}}}{[k_1, Fk_2]}$ then we use the comparatively weak upper bound

$$\sum_{\substack{\alpha < n \leq \alpha + x \\ n \equiv \delta_{ij} \bmod [k_1, Fk_2, m] \\ (n, \prod_{p<z} p) = 1}} 1 \leq \sum_{\substack{\alpha < n \leq \alpha + x \\ n \equiv \delta_{ij} \bmod [k_1, Fk_2, m]}} 1 \leq \frac{x}{[k_1, Fk_2, m]} + 1 \tag{4}$$

Substitution of (3) and (4) into (2) gives

$$S_1 < x \prod_{p<z}\left(1-\frac{1}{p}\right) \frac{[k_1,Fk_2]^{\frac12}}{\varphi([k_1,Fk_2])} \sum_{\substack{1\leq m\leq \frac{x^{\frac12}}{[k_1,Fk_2]} \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)\ m^{\frac12}}{\varphi^2(m)} \left(1 + O(\exp(-(\ln x)^{\epsilon})) \right)$$

$$+ O\left[x \sum_{\substack{\frac{x^{\frac12}}{[k_1,Fk_2]}<m\leq G(x,\alpha) \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)}{\varphi(m)[k_1,Fk_2,m]} \right]$$

$$+ O\left[\sum_{\substack{\frac{x^{\frac12}}{[k_1,Fk_2]}<m\leq G(x,\alpha) \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)}{\varphi(m)} \right].$$

As previously we have

$$\sum_{\substack{1\leq m\leq \frac{x^{\frac12}}{[k_1,Fk_2]} \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)\ m^{\frac12}}{\varphi^2(m)} < \prod_{\substack{p<z \\ p\nmid k_2h}} \left(1+\frac{4p^{\frac12}}{(p-1)^2}\right) \quad ;$$

$$\sum_{\substack{\frac{x^{\frac12}}{[k_1,Fk_2]}\leq m\leq G(x,\alpha) \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)}{\varphi(m)\ [k_1,Fk_2,m]} < \frac{1}{[k_1,Fk_2]^{\frac12}} \sum_{m>\frac{x^{\frac12}}{[k_1,Fk_2]}} \frac{\rho(m)}{\varphi(m)m^{\frac12}}$$

$$= O\left[\frac{\ln\ln x\ \ln^{\lambda}x\ [k_1,Fk_2]^{\frac12}}{x^{\frac12}} \right]$$

where $\ln \lambda = \left\{ \left[\frac{\ln(2\Delta^2)}{\ln 2}\right]+1 \right\}\ln 2$; and

$$\sum_{\substack{\frac{x^{\frac12}}{[k_1,Fk_2]}\leq m\leq G(x,\alpha) \\ (k_2h,m)=1 \\ P(m)<z}} \frac{\mu^2(m)\ \rho(m)}{\varphi(m)} = O(\ln^4 z) = O((\ln {}^{x}/_{k})^{4(1-\epsilon)}) = O(\ln^4 x)$$

So

$$S_1 < x \prod_{p<z}\left(1-\frac{1}{p}\right) \frac{[k_1,Fk_2]^{\frac{1}{2}}}{\varphi([k_1,Fk_2])} \prod_{\substack{p<z \\ p\nmid k_2h}} \left(1+\frac{4p^{\frac{1}{2}}}{(p-1)^2}\right) \Big\{1+O(\exp(-(\ln x)^{\epsilon}))$$

$$+ O\left[\varphi([k_1,Fk_2]) \frac{\ln\ln x \ \ln^{\lambda+1} x}{x^{\frac{1}{2}}}\right] + O\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \cdot \frac{\ln^5 x}{x}\right]\Big\}. \quad (5)$$

The third error term is absorbed into the second. Substituting (5) into (1) completes the lemma. As in the previous lemmas if an^2+bn+c or dn^2+en+f have any common factors then the proof is similar. □

Finally we have Lemma 2.12. The proof is not included as it is almost identical to that of Lemma 2.11.

Although Lemma 2.12 is applied at an earlier stage in the following chapters than either Lemma 2.11 or Lemma 2.10 it is included here as the proof is slightly less complicated than that of Lemma 2.10.

<u>LEMMA 2.12</u>

Let $an+b$, $cn+d$ *and* $en+f$ *be polynomials with integer coefficients. Assume that* $an+b$ *and* $cn+d$ *have no common factors. Then for* $z \leqslant \exp(10(\ln x)^{\frac{1}{2}})$

$$\sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (an+b,cn+d)=w \\ \left(\left[\frac{an+b}{w}\right]\ell_2+\left[\frac{cn+d}{w}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1 \\ (en+f,\ \prod_{p<z} p)=1}} \prod_{\substack{p<z \\ p\mid an+b \\ p\nmid k_2}} \left(1-\frac{1}{p}\right)^{-1}$$

$$< \quad x.\frac{e}{\varphi(e)} \prod_{p<z}\left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid h}} \left(1-\frac{1}{p}\right)^{-1} \frac{[k_1,Fk_2]^{\frac{1}{2}}}{\varphi([k_1,Fk_2])}$$

$$\times \prod_{\substack{p<z \\ p\nmid k_2 h}} \left(1+\frac{p^{\frac12}}{(p-1)^2}\right) \gamma_Z(w)\left\{1 + 0(\exp(-(\ln x)^{\frac12})) + 0\left[\frac{\varphi([k_1,Fk_2]).\ln\ln x.\ln^{3/2}x}{x^{\frac12}}\right]\right\}$$

where

(i) $F=bc-ad$

(ii) $h=(a,b)$ *and* $a_1={}^a/_h$, $b_1={}^b/_h$

(iii) $\gamma_Z(w)$ *denotes the number of integers* n *in the interval* $1\leqslant n\leqslant Fk_2$ *for which both*

$$(an+b,cn+d)=w$$

and

$$\left(\left[\frac{an+d}{w}\right]\ell_2+\left[\frac{cn+d}{w}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1 \quad .$$

<u>LEMMA 3</u>

Let S, T, U *be positive real numbers, and suppose*

$S = T\left\{1 + 0\left[\frac{1}{x}\right]\right\}$, and $S = U\left\{1 + 0\left[\frac{1}{y}\right]\right\}$.

Then

(i) $T = S\left\{1 + 0\left[\frac{1}{x}\right]\right\}$

and

(ii) $T = U\left\{1 + 0\left[\frac{1}{x}\right] + 0\left[\frac{1}{y}\right]\right\}$.

<u>PROOF</u>

(i) Given $S = T(1+0({}^1/_x))$ we have $|S-T| \leqslant {}^{kT}/_x$ for some positive constant k. If $T \leqslant S$ then $|T-S| \leqslant {}^{kS}/_x$ giving $T-S = 0({}^S/_x)$ or $T = S + 0({}^S/_x) = S\{1+0({}^1/_x)\}$.

On the other hand, if $T > S$ then $|S-T| = T-S \leqslant {}^{kT}/_x$ and $T(1-{}^k/_x) \leqslant S$ i.e. $T \leqslant S\{1+{}^k/_{x-k}\}$. Hence

$|T-S| \leq {}^{Sk}/_{x-k} = O({}^{S}/_{x})$ and $T = S\{1+O({}^{1}/_{x})\}$ as required.

(ii) As $S = T\{1+O({}^{1}/_{x})\}$ we have by (i)

$$T = S\{1+O({}^{1}/_{x})\} = U\{1+O({}^{1}/_{y})\}\{1+O({}^{1}/_{x})\} = U\{1+O({}^{1}/_{y})+O({}^{1}/_{x})\}$$

as required.

□

<u>LEMMA 4</u>

Suppose a, b, c, d ϵ $\mathbb{Z}$ *with* $b^2-ac\neq0$ *and* $b^2-ac\equiv0 \bmod 4$.

Then

$$\left|\{(x,y):\ 0\leq x\leq A,\ ax^2+2bx+c=dy^2\ \}\right|$$

$$\ll \begin{cases} \tau\left[\frac{b^2-ac}{4}\right]\ln\left[\frac{A}{|ad|}\right]; & |ad|<A,\ ad>0\ \&\ ad \text{ not a perfect square} \\ \tau\left[\frac{b^2-ac}{4}\right] & ;\ \text{otherwise} \end{cases}$$

where $\tau(n)$ *denotes the number of divisors of* n.

<u>PROOF</u>

Solving the quadratic

$$ax^2+2bx+c=dy^2 \tag{1}$$

for x gives

$$x= \frac{-2b\pm\sqrt{(4b^2-4a(c-dy^2))}}{2a}.$$

So for (1) to have integer solutions we require that $b^2-a(c-dy^2)$ be a square, say z^2, and that either $-b+z$ or $b+z$ be divisible by a. (We may assume that z is positive.)

Now

$$b^2-a(c-dy^2)=z^2$$

if and only if

$$z^2-ady^2=b^2-ac. \tag{2}$$

The proof of the lemma is divided into four steps, Step 1 dealing with the case where ad is negative. Obviously if both ad and b^2-ac are negative, then (2), and consequently (1), has no solutions.

<u>STEP 1</u>: *Number of positive integer solutions of* $Ax^2+By^2=g$ *with* $A,B>0$.

(For convenience we denote the number of positive integer solutions of $Ax^2+By^2=g$ as $N(g, A, B)$).

Clearly we may assume that $(A,B,g)=1$. We may further assume that $(AB,g)=1$, for if there exists a prime p such that $A\equiv 0 \bmod p$ and $g\equiv 0 \bmod p$ say, then $y\equiv 0 \bmod p$ and the number of positive integer solutions of $Ax^2+By^2=g$ equals the number of positive integer solutions of $({}^{A}/_{p})x^2+Bpy^2=({}^{g}/_{p})$. Similarly if there exists a prime p such that $B\equiv 0 \bmod p$ and $g\equiv 0 \bmod p$. Continuing in this way an equation $A'x^2+B'y^2=g'$ is reached for which $(A'B',g')=1$, having the same number of solutions as our original equation.

The solutions of $Ax^2+By^2=g$ may be derived from the solutions of the equations

$$\left\{\begin{array}{ll} Ax^2+By^2=g & ;\ (x,y)=1 \\ Ax^2+By^2=\dfrac{g}{g_1} & ;\ (x,y)=1 \\ \vdots & \\ Ax^2+By^2=\dfrac{g}{g_r} & ;\ (x,y)=1 \end{array}\right. \qquad (3)$$

where $g_1,\ldots,g_r$ denote the square integers dividing g. For completeness we write $g_0=1$ and the equation $Ax^2+By^2=g$ as $Ax^2+By^2=\dfrac{g}{g_0}$.

From section 11.3 of Hua[4], Theorems (4.1), (4.2) and (4.3) it follows that the number of solutions of

$$Ax^2+By^2=\frac{g}{g_i} \quad ; \ (x,y)=1$$

is $$O\left\{ \left| \{ 0\leqslant \ell < 2\frac{g}{g_i} \ : \ \ell^2 \equiv -4AB \bmod 4\frac{g}{g_i} \ \} \right| \right\}$$

and we have

$$N(g,A,B) \leqslant \sum_{i=0}^{r} \left| \{ 0\leqslant \ell < 2\frac{g}{g_i} \ : \ \ell^2 \equiv -4AB \bmod 4\frac{g}{g_i} \ \} \right|. \tag{4}$$

Writing $g = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$; $p_1<p_2<\ldots<p_s$ we will show, by induction on s, that

$$N(g,A,B) \leqslant \tau(g). \tag{5}$$

Assuming initially that g has just one prime factor and writing $g=p_1^{\alpha_1}$, we have

$$N(g,A,B) \leqslant \left| \{ 0\leqslant \ell < 2p_1^{\alpha_1} \ ; \ \ell^2 \equiv -4AB \bmod 4p_1^{\alpha_1} \ \} \right|$$

$$+ \left| \{ 0\leqslant \ell < 2p_1^{\alpha_1 -2} \ ; \ \ell^2 \equiv -4AB \bmod 4p_1^{\alpha_1 -2} \ \} \right|$$

$$+ \ldots + \left| \{ 0\leqslant \ell < 2 \ ; \ \ell^2 \equiv -4AB \bmod 4 \ \} \right|$$

if α_1 is even;

$$N(g,A,B) \leqslant \left| \{ 0\leqslant \ell < 2p_1^{\alpha_1} \ ; \ \ell^2 \equiv -4AB \bmod 4p_1^{\alpha_1} \ \} \right|$$

$$+ \ldots + \left| \{ 0\leqslant \ell < 2p_1 \ ; \ \ell^2 \equiv -4AB \bmod 4p_1 \ \} \right|$$

if α_1 odd.

Taking into account the possibility of p_1 being 2,

$\left| \{ 0\leqslant \ell < 2p_1^{\beta} \ ; \ \ell^2 \equiv -4AB \bmod 4p_1^{\beta} \ \} \right|$ is at most 4.

So $N(g,A,B) \leqslant 4(\alpha_1/2 +1) \leqslant 3\tau(p_1^{\alpha_1})$ giving us our starting case.

Assuming now that whenever g has k primes or fewer in its factorization

$$N(g,A,B) \leqslant 3(\alpha_1+1)(\alpha_2+1)\ldots(\alpha_k+1) = 3\tau(g)$$

we turn our attention to the case

$$g = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} p_{k+1}^{\alpha_{k+1}}.$$

Writing $h_0,\ldots,h_t$ as the squares dividing $p_1^{\alpha_1}\ldots p_k^{\alpha_k}=g'$ say

$$N(g,A,B) \ll \sum_{i=1}^{r} \left|\{\ 0\leq \ell < 2\frac{g}{g_i}\ ;\ \ell^2 \equiv -4AB \bmod 4\frac{g}{g_i}\ \}\right|$$

$$= \sum_{j=1}^{t} \left|\{\ 0\leq \ell < 2\frac{g' p_{k+1}^{\alpha_{k+1}}}{h_j}\ ;\ \ell^2 \equiv -4AB \bmod 4\frac{g' p_{k+1}^{\alpha_{k+1}}}{h_j}\}\right|$$

$$+ \sum_{j=1}^{t} \left|\{\ 0\leq \ell < 2\frac{g' p_{k+1}^{\alpha_{k+1}-2}}{h_j}\ ;\ \ell^2 \equiv -4AB \bmod 4\frac{g' p_{k+1}^{\alpha_{k+1}-2}}{h_j}\ \}\right|$$

$$+\ldots+ \sum_{j=1}^{t} \left|\{\ 0\leq \ell < 2\frac{g'}{h_j}\ ;\ \ell^2 \equiv -4AB \bmod 4\frac{g'}{h_j}\ \}\right|$$

for α_{k+1} even;

$$N(g,A,B) \ll$$

$$\sum_{j=1}^{t} \left|\{\ 0\leq \ell < 2\frac{g' p_{k+1}^{\alpha_{k+1}}}{h_j}\ ;\ \ell^2 \equiv -4AB \bmod 4\frac{g' p_{k+1}^{\alpha_{k+1}}}{h_j}\ \}\right|$$

$$+\ldots+ \sum_{j=1}^{t} \left|\{\ 0\leq \ell < 2\frac{g' p_{k+1}}{h_j}\ ;\ \ell^2 \equiv -4AB \bmod 4\frac{g' p_{k+1}}{h_j}\ \}\right|$$

for α_{k+1} odd.

Now

$$\sum_{j=1}^{t} \left|\{\ 0\leq \ell < 2\frac{g' p_{k+1}^{\beta}}{h_j}\ ;\ \ell^2 \equiv -4AB \bmod 4\frac{g' p_{k+1}^{\beta}}{h_j}\ \}\right|$$

$$= \sum_{j=1}^{t} \left|\{\ 0\leq \ell < 2\frac{g'}{h_j}\ ;\ \ell^2 \equiv -4AB \bmod 4\frac{g'}{h_j}\ \}\right| \times$$

$$\left|\{\ 0\leq \ell < p_{k+1}^{\beta}\ ;\ \ell^2 \equiv -4AB \bmod p_{k+1}^{\beta}\ \}\right|$$

$$\leq \sum_{j=1}^{t} \left|\{\ 0\leq \ell < 2\frac{g'}{h_j}\ ;\ \ell^2 \equiv -4AB \bmod 4\frac{g'}{h_j}\ \}\right|$$

$$\leq 3(\alpha_1+1)\ldots(\alpha_k+1) \quad \text{by the inductive hypothesis.}$$

Applying this $\frac{\alpha_{k+1}}{2}$ times whenever α_{k+1} is even; and $\frac{\alpha_{k+1}+1}{2}$ times when α_{k+1} is odd gives

$$N(g,A,B) \leq 3(\alpha_1+1)\ldots(\alpha_k+1)(\alpha_{k+1}+1) = 3\tau(g)$$

as required.

STEP 2: *Number of positive integer solutions of*

$$x^2-Dy^2=4N \qquad (6)$$

for $\alpha<x\leq\beta$, *with* $D>0$, $N>0$.

Denote the number of positive integer solutions of $x^2-Dy^2=4N$ with $\alpha<x\leq\beta$, as $M(N,D,\alpha,\beta)$.

If D is a perfect square then, since the number of ways in which 4N can be decomposed into two factors is at most $\frac{1}{2}\tau(4N)$, in this instance

$$M(N,D,\alpha,\beta) \leq \frac{\tau(4N)}{2}. \qquad (7)$$

Assuming that D is not a perfect square, suppose that $x^2-Dy^2=4N$ is solvable and let (u,v) be a solution. If (x,y) is a solution of the Pellian equation

$$x^2-Dy^2=4 \qquad (8)$$

then (u_1,v_1) defined by

$$(u_1+v_1\sqrt{D}) = (u+v\sqrt{D})\left(\frac{x+y\sqrt{D}}{2}\right)$$

so that $u_1 = \frac{ux+vyD}{2}$, $v_1 = \frac{xv+uy}{2}$

is also a solution of (6). Certainly u_1 and v_1 are integers as, for u^2-Dv^2 even and x^2-Dy^2 even, both $ux+vyD$ and $xv+uy$ are even. Following the notation used by B. Stolt [7] we say that the solution (u_1,v_1) is *associated* with the solution (u,v).

Now, if $(u+v\sqrt{D})\left(\frac{x+y\sqrt{D}}{2}\right) = u_1+v_1\sqrt{D}$,

then $(u+v\sqrt{D})\left(\frac{x+y\sqrt{D}}{2}\right)\left(\frac{x-y\sqrt{D}}{2}\right)=(u_1+v_1\sqrt{D})\left(\frac{x-y\sqrt{D}}{2}\right)$

giving $(u+v\sqrt{D}) = (u_1+v_1\sqrt{D})\left(\frac{x-y\sqrt{D}}{2}\right)$.

So we see that if (u_1,v_1) is associated with the solution (u,v)

then conversely (u,v) is associated with the solution (u_1,v_1), and we say that (u_1,v_1) and (u,v) are *associated with each other*. The set of all solutions associated with each other we term a *class of solutions*.

Let $\mathbb{C}$ denote a class of solutions of (6), consisting of the solutions

$$(u_i,v_i) \ ; \ i=0,1,2,\ldots$$

If (x_0,y_0) denotes the fundamental solution of (8) such that $x_0>0$, $y_0>0$ it is well known that all the positive solutions of (8) are given by

$$\left[\frac{x_0 + y_0\sqrt{D}}{2}\right]^n 2 \quad ; \ n=1,2,\ldots$$

Let (u_0,v_0) denote the fundamental solution of the class $\mathbb{C}$ defined as the smallest non-negative u belonging to the class $\mathbb{C}$. Then the members of $\mathbb{C}$ (if we regard positive and negative solutions of equal modulus as being the same), are given by

$$u_n+v_n\sqrt{D} = (u_0+v_0\sqrt{D}\ )\left[\frac{x_0+y_0\sqrt{D}}{2}\right]^n \quad ;n=1,2,\ldots \qquad (9)$$

It is generally the case that (u_0,v_0) and $(u_0,-v_0)$ generate different classes so we cannot at this point assume anything about the sign of v_0.

Our first step towards an upper bound for $M(N,D,\alpha,\beta)$ is to show that

$$u_1>0\ . \qquad (10)$$

From (9), $u_1 = \dfrac{u_0x_0+v_0y_0D}{2}$.

If $v_0\geq 0$ then it is obvious that $u_1>0$. If however $v_0<0$ then

$$\begin{aligned} u_1 &= \frac{u_0x_0-|v_0|y_0D}{2} \\ &= \frac{u_0}{2}\left\{x_0-\frac{|v_0|y_0D}{u_0}\right\} \\ &= \frac{u_0}{2}\left\{x_0-y_0\sqrt{D} + y_0\sqrt{D}\left[1-\frac{|v_0|\sqrt{D}}{u_0}\right]\right\} \\ &> 0 \end{aligned}$$

as $u_0>0$, $x_0-y_0\sqrt{D}>0$ and $\left\{1-\frac{|v_0|\sqrt{D}}{u_0}\right\}>0$ for N positive.

We are now in a position to prove, by induction, that

$$u_{n+1} > u_n > 0 \quad \text{for all } n. \tag{11}$$

From the definition of u_0, and as $u_1>0$, it follows that

$$u_1 > u_0 > 0$$

and we have our starting case.

Suppose $u_k > u_{k-1} > 0$.

As
$$u_k+v_k\sqrt{D} = (u_0+v_0\sqrt{D})\left\{\frac{x_0+y_0\sqrt{D}}{2}\right\}^k$$
$$= (u_{k-1}+v_{k-1}\sqrt{D})\left\{\frac{x_0+y_0\sqrt{D}}{2}\right\}$$

we see that

$$u_k = \frac{u_{k-1}x_0 + v_{k-1}y_0D}{2}$$

and as $u_k > u_{k-1} > 0$,

$$\frac{u_{k-1}x_0 + v_{k-1}y_0D}{2} > u_{k-1}$$

so that

$$u_{k-1} > \frac{-v_{k-1}y_0D}{x_0-2} \tag{12}$$

Now

$$u_{k+1} + v_{k+1}\sqrt{D} = (u_{k-1} + v_{k-1}\sqrt{D})\left\{\frac{x_0 + y_0\sqrt{D}}{2}\right\}^2$$

gives

$$u_{k+1} = u_{k-1}\left\{\frac{x_0^2 + y_0^2D}{4}\right\} + \frac{x_0\, y_0\, v_{k-1}\, D}{2}$$

and in order to show $u_{k+1} > u_k > 0$ we require the inequality

$$u_{k-1}\left\{\frac{x_0^2 + y_0^2D}{4}\right\} + \frac{x_0\, y_0\, v_{k-1}\, D}{2} > \frac{u_{k-1}\, x_0 + v_{k-1}\, y_0\, D}{2}$$

to hold. This occurs whenever

$$u_{k-1}\left\{\frac{x_0^2 + y_0^2D - 2x_0}{4}\right\} > \frac{v_{k-1}\, y_0\, D\, (1-x_0)}{2}. \tag{13}$$

(13) holds trivially if $v_{k-1}\geqslant 0$. If $v_{k-1}<0$, (13) becomes

$$u_{k-1} > \frac{2|v_{k-1}|y_0D(x_0-1)}{x_0^2 + y_0^2D - 2x_0} \tag{14}$$

But as, by (12),

$$u_{k-1} > \frac{|v_{k-1}|y_0D}{x_0-2}$$

(14) is satisfied if

$$\frac{|v_{k-1}|y_0D}{x_0-2} > \frac{2|v_{k-1}|y_0D(x_0-1)}{x_0^2 + y_0^2D - 2x_0}$$

an inequality easily seen to be satisfied whenever $x_0>2$, which, by our definition of x_0, we may assume to be the case.

So (11) follows as required.

Further relations, similar to (10) and (11) hold for v_n. Namely

(i) If $v_k>0$ for some k, then $v_n>0$ for all $n\geq k$. (16)

(ii) If $v_{k-1}<0$ and $v_k<0$ for some k then $v_n<0$ for all $n\geq k$. (17)

The proof of (16) follows immediately from the relation

$$u_n+v_n\sqrt{D} = (u_k+v_k\sqrt{D})\left\{\frac{x_0+y_0\sqrt{D}}{2}\right\}^{n-k}.$$

The proof of (17) is similar to that of (11). Suppose

$v_k<0$ and $v_{k-1}<0$.

It is clear that in this case $v_k<v_{k-1}<0$ for since u_n increases as n increases it follows that $|v_n|$ must increase accordingly. For (17) it is enough to show that

$$v_{k+1} < v_k$$

for then the result will follow by induction.

Now

$$v_k = \frac{v_{k-1}x_0 + u_{k-1}y_0}{2} < v_{k-1}$$

implies that

$$\frac{-v_{k-1}x_0 - u_{k-1}y_0}{2} > -v_{k-1}$$

implying in turn that

$$-v_{k-1} > \frac{u_{k-1}y_0}{x_0 - 2} \tag{18}$$

On the other hand

$$v_{k+1} = \frac{x_0\, y_0\, u_{k-1}}{2} + v_{k-1}\left(\frac{x_0^2 + y_0^2 D}{4}\right)$$

so that for

$$v_{k+1} < v_k$$

to hold it is sufficient to show that

$$\frac{x_0 y_0 u_{k-1}}{2} + v_{k-1}\left(\frac{x_0^2 + y_0^2 D}{4}\right) < \frac{v_{k-1}x_0 + u_{k-1}y_0}{2}$$

or

$$\frac{-v_{k-1}x_0 - u_{k-1}y_0}{2} < \frac{-x_0 y_0 u_{k-1}}{2} - v_{k-1}\left(\frac{x_0^2 + y_0^2 D}{4}\right).$$

This is the case whenever

$$\frac{2u_{k-1}y_0(x_0-1)}{x_0^2 + y_0^2 D - 2x_0} < -v_{k-1}. \tag{19}$$

From (18) it follows that we have only to show

$$\frac{2u_{k-1}y_0(x_0-1)}{x_0^2 + y_0^2 D - 2x_0} \leq \frac{u_{k-1}y_0}{x_0 - 2}$$

an inequality which is satisfied whenever $x_0>2$. This completes the proof of (17).

Suppose the solutions u_n belonging to the class $\mathbb{C}$, lying within the range $\alpha<u_n\leq\beta$ are given by the equations

$$u_n + v_n\sqrt{D} = (u_0 + v_0\sqrt{D})\left[\frac{x_0 + y_0\sqrt{D}}{2}\right]^n \quad ; \; n=r,r+1,\ldots,r+s.$$

(Our proof that $u_n>u_{n-1}>0$ for all n ensures that some consecutive sequence of integers will give exactly the solutions in the range $\alpha<u_n\leq\beta$.)

Now

$$u_r + v_r\sqrt{D} = (u_0 + v_0\sqrt{D})\left[\frac{x_0 + y_0\sqrt{D}}{2}\right]^r$$

implies

$$\frac{(u_r + v_r\sqrt{D})(u_0 - v_0\sqrt{D})}{4N} = \left[\frac{x_0 + y_0\sqrt{D}}{2}\right]^r$$

and consequently that

$$r = \frac{\ln\left\{\frac{(u_r + v_r\sqrt{D})(u_0 - v_0\sqrt{D})}{4N}\right\}}{\ln\left\{\frac{x_0 + y_0\sqrt{D}}{2}\right\}}$$

Similarly

$$r + s = \frac{\ln\left\{\frac{(u_{r+s} + v_{r+s}\sqrt{D})(u_0 - v_0\sqrt{D})}{4N}\right\}}{\ln\left\{\frac{x_0 + y_0\sqrt{D}}{2}\right\}}$$

giving

$$s = \frac{\ln\left\{\frac{u_{r+s} + v_{r+s}\sqrt{D}}{u_r + v_r\sqrt{D}}\right\}}{\ln\left\{\frac{x_0 + y_0\sqrt{D}}{2}\right\}}.$$

Certainly $u_{r+s} > u_r > 0$.

If $v_r > 0$ then by (16), $v_{r+s} > 0$ and

$$s \leqslant \frac{\ln\left\{\frac{2u_{r+s}}{u_r}\right\}}{\ln\left\{\frac{x_0 + y_0\sqrt{D}}{2}\right\}} \leqslant \frac{2\ln\left\{\frac{2\beta}{\alpha}\right\}}{\ln\left\{\frac{D}{4}\right\}} \qquad (20)$$

as $\ln\left\{\frac{x_0 + y_0\sqrt{D}}{2}\right\} \geqslant \ln\left\{\frac{3 + \sqrt{D}}{2}\right\} \geqslant \frac{1}{2}\ln\left\{\frac{D}{4}\right\}$.

If $v_r < 0$ and $v_{r+s} < 0$ then

$$s = \frac{\ln\left\{\frac{4N}{u_{r+s} - v_{r+s}\sqrt{D}} \cdot \frac{u_r - v_r\sqrt{D}}{4N}\right\}}{\ln\left\{\frac{x_0 + y_0\sqrt{D}}{2}\right\}}$$

$$= \frac{\ln\left\{\frac{u_r + |v_r|\sqrt{D}}{u_{r+s} + |v_{r+s}|\sqrt{D}}\right\}}{\ln\left\{\frac{x_0 + y_0\sqrt{D}}{2}\right\}}$$

$$\leqslant \frac{2\ln\left\{\frac{2\beta}{\alpha}\right\}}{\ln\left\{\frac{D}{4}\right\}}. \qquad (21)$$

If $v_r < 0$ and $v_{r+s} > 0$ then by (17), $v_{r+1} > 0$. But

$$r + 1 = \frac{\ln\left\{\frac{(u_{r+1} + v_{r+1}\sqrt{D})(u_0 - v_0\sqrt{D})}{4N}\right\}}{\ln\left\{\frac{x_0 + y_0\sqrt{D}}{2}\right\}}$$

giving

$$s - 1 = \frac{\ln\left\{\frac{u_{r+s} + v_{r+s}\sqrt{D}}{u_{r+1} + v_{r+1}\sqrt{D}}\right\}}{\ln\left\{\frac{x_0 + y_0\sqrt{D}}{2}\right\}}$$

$$< \frac{2\ln\left\{\frac{2\beta}{\alpha}\right\}}{\ln\left\{\frac{D}{4}\right\}}$$

and

$$s \leqslant \frac{2\ln\left\{\frac{2\beta}{\alpha}\right\}}{\ln\left\{\frac{D}{4}\right\}} + 1 \quad . \tag{22}$$

So

$$M(N,D,\alpha,\beta) \leqslant \left\{\frac{2\ln\left\{\frac{2\beta}{\alpha}\right\}}{\ln\left\{\frac{D}{4}\right\}} + 1\right\} \times \text{(number of classes)}.$$

For $N = p_1^{2a_1} p_2^{2a_2} \dots p_m^{2a_m} q_1^{2b_1+1} \dots q_n^{2b_n+1}$ it follows from results B. Stolt [7] achieves that the number of classes is at most

$$2^n (2a_1+1)(2a_2+1)\dots(2a_m+1)(b_1+1)\dots(b_n+1)$$

$$= (2a_1+1)(2a_2+1)\dots(2a_m+1)(2b_1+2)\dots(2b_n+2)$$

$$= \tau(N).$$

So

$$M(N,D,\alpha,\beta) \leqslant \left\{\frac{2\ln\left\{\frac{2\beta}{\alpha}\right\}}{\ln\left\{\frac{D}{4}\right\}} + 1\right\} \tau(N) \quad . \tag{23}$$

<u>STEP 3</u>: *Number of positive integer solutions of* $x^2 - Dy^2 = 4N$ *for* $\alpha<x\leqslant\beta$, *with* $D>0$, $N<0$.

Here, as in Step 3, denoting the number of positive solutions of $x^2 - Dy^2 = 4N$ for $\alpha<x\leqslant\beta$ where $D>0$ and $N<0$ as $M(N,D,\alpha,\beta)$ we get

$$M(N,D,\alpha,\beta) \leqslant \left\{ \frac{2\ln\left\{\frac{2\beta}{\alpha}\right\}}{\ln\left\{\frac{D}{4}\right\}} + 1 \right\} \tau(N) .$$

The proof goes through almost identically, except that here we define a class of solutions as those solutions given by

$$u_n + v_n\sqrt{D} = (u_0 + v_0\sqrt{D}\,)\left[\frac{x_0 + y_0\sqrt{D}}{2}\right]^n \;;\; n=1,2,\ldots$$

where (u_0,v_0) is defined as the smallest non-negative value of v, rather than of u.

Following from this definition we are able to show

(i) $v_1>0$

(ii) $v_{n+1}>v_n>0$ for all n

(iii) If $u_k>0$ for some k then $u_n>0$ for all $n>k$

(iv) If $u_k<0$ and $u_{k+1}<0$ for some k then $u_n<0$ for all $n>k$

and complete the proof mutatis mutandis.

STEP 4: *The completion of the Lemma.*

In summary, we have so far that the number of integer solutions of $z^2-ady^2=b^2-ac$ for $b^2-ac\equiv 0 \bmod 4$ with $0<\alpha<z\leqslant\beta$ is

$$\leqslant \begin{cases} \tau(b^2-ac) & ;\ ad\leqslant 0 \\ \dfrac{\tau(b^2-ac)}{2} & ;\ ad \text{ a perfect square} \\ \tau\left[\dfrac{b^2-ac}{4}\right]\left\{\dfrac{2\ln\left\{\frac{2\beta}{\alpha}\right\}}{\ln\left\{\frac{ad}{4}\right\}} + 1\right\} & ;\ \text{otherwise} \end{cases}$$

In the cases $ad\leqslant 0$ and ad a perfect square the lemma follows immediately.

If however $ad>0$, and ad is not a perfect square, from (1) we see that

$$x = \frac{-b\pm z}{a} .$$

As we require $0\leqslant x\leqslant A$, if a and b are both positive we can only obtain x within this range if we take

$$x = \frac{-b+z}{a}$$

in which case

$$0 \leq \frac{-b+z}{a} \leq A$$

and $\qquad b \leq z \leq Aa+b.$

The number of positive integer solutions of

$$z^2 - ady^2 = b^2-ac$$

for z in this range is, hence,

$$\ll \quad \tau\left[\frac{b^2-ac}{4}\right]\left\{\frac{2\ln\left\{\frac{2(Aa+b)}{b}\right\}}{\ln\left\{\frac{ad}{4}\right\}} + 1\right\}.$$

Continuing in this way, we find that the number of solutions of $ax^2 + 2bx + c = dy^2$, for $ad>0$ and not a perfect square, and for $0 \leq x \leq A$ is

$$\ll \begin{cases} \tau\left[\frac{b^2-ac}{4}\right]\left\{\frac{2\ln\left\{\frac{2(Aa+b)}{b}\right\}}{\ln\left\{\frac{ad}{4}\right\}} + 1\right\} & ;\ ab>0 \\ \tau\left[\frac{b^2-ac}{4}\right]\left\{\frac{2\ln\left\{\frac{2|b|}{|b|-A|a|}\right\}}{\ln\left\{\frac{ad}{4}\right\}} + 1\right\} & ;\ \begin{matrix} ab<0 \\ A|a|<|b| \end{matrix} \\ \tau\left[\frac{b^2-ac}{4}\right]\left\{\frac{2\ln(2|b|)}{\ln(^{ad}/_{4})} + 1\right\} & ;\ \begin{matrix} ab<0 \\ A|a|=|b| \end{matrix} \\ \tau\left[\frac{b^2-ac}{4}\right]\left\{\frac{2\ln(2|b|)}{\ln(^{ad}/_{4})} + \frac{2\ln(2(A|a|-|b|))}{\ln(^{ad}/_{4})} + 1\right\} & ;\ \begin{matrix} ab<0 \\ |b|<A|a| \end{matrix} \end{cases}$$

These collectively imply that the number of solutions of $ax^2 + 2bx + c = dy^2$ with $0 \leq x \leq A$ is

$$\ll \tau\left[\frac{b^2-ac}{4}\right]\left\{\frac{\ln(A|a|)}{\ln(^{ad}/_{4})}\right\}$$

$$\ll \tau\left[\frac{b^2-ac}{4}\right]\frac{\ln A}{\ln(ad)}$$

$$\leqslant \begin{cases} \tau\left[\frac{b^2-ac}{4}\right]\ln\left\{\frac{A}{ad}\right\} & ; ad<A \\ \tau\left[\frac{b^2-ac}{4}\right] & ; ad \geqslant A \end{cases}$$

which completes the lemma. □

N.B. We have achieved an upper bound on the number of solutions of $b^2-a(c-dy^2)=z^2$ as required. However, recalling the comments following the opening of the proof, we also require that either $-b+z$ or $b+z$ be divisible by a. With regards to this additional restraint we make the following observation.

If the solutions in a class $\mathbb{C}$ *of* $u^2-adv^2=4N$, *for* ad *positive and squarefree, are given by* $(u_0,v_0),(u_1,v_1),\ldots,$ $(u_n,v_n)\ldots$ *then, if* $u_n-b\equiv 0 \bmod a$, *it follows that all the solutions in the series*

$$\ldots,u_{n-2},\ u_n,\ u_{n+2},\ u_{n+4},\ldots$$

also satisfy $u-b\equiv 0 \bmod a$.

Following the comments near the start of the proof we may assume that $(a,4)=1$. From equation (9) we have

$$u_n+v_n\sqrt{ad} = (u_{n-2}+v_{n-2}\sqrt{ad})\left[\frac{x_0+y_0\sqrt{ad}}{4}\right]^2$$

so that

$$4u_n = u_{n-2}(x_0^2+y_0^2ad)+2adv_{n-2}x_0y_0.$$

Hence

$$4u_n \equiv u_{n-2}x_0^2 \bmod a.$$

But (x_0,y_0) satisfies $x_0^2-ady_0^2=4$ giving $x_0^2\equiv 4 \bmod a$. So

$$4u_n\equiv 4u_{n-2} \bmod a.$$

Since $(a,4)=1$,

$$u_n\equiv u_{n-2} \bmod a.$$

Similarly $u_n\equiv u_{n+2} \bmod a$. The rest of the proof follows by induction.

It follows that, for each class of solutions, we need only observe the first two solutions to know whether or not that particular class will yield solutions to $ax^2+2bx+c=dy^2$.

LEMMA 5.1

Let y *and* Q *be large real numbers. Let* α *be a positive real number satisfying* $\alpha>10$ *and* $\exp(Q^{1/\alpha})>\exp(c\ln^2 Q)$. *Let* $\ln^{2\alpha}Q<y<z$. *Then there are at most* $O(Q^{9/\alpha})$ *distinct primitive characters to moduli not exceeding* Q *for which the estimate*

$$\prod_{p<z}\left(1-\frac{\chi(p)}{p}\right)^{-1} = \prod_{p<y}\left(1-\frac{\chi(p)}{p}\right)^{-1}\left\{1+O(\alpha y^{-1/\alpha})+O(Q^{-3/\alpha})\right\}$$

fails.

The O*-constants are absolute, effectively computable, and independent of the value of* α.

PROOF

The lemma is a generalisation of Lemmas 22.5, 22.6 and 22.7 of Elliott [8], and of Lemma 1 of Nair and Perelli [1]. The proof is broken into three steps, Step 1 corresponding to Lemma 22.5 of Elliott, Step 2 to Lemma 22.6 and Step 3 to Lemma 22.6 and Lemma 1 of Nair and Perelli.

The proofs are essentially very similar to their originals. More detail is given where it is considered helpful.

STEP ONE

Let Q and U be large real numbers. Let α be a positive real number, $\alpha>10$. Then, if $\ln^{2\alpha}Q<U<Q^2$, there are at most $O(Q^{8/\alpha+1/160})$ distinct primitive characters to moduli not exceeding Q for which the inequality

$$\left| \sum_{U<p\leq 2U} \frac{\chi(p)}{p} \right| \leq U^{-1/\alpha} \tag{1}$$

fails. The 0-constant is independent of α.

PROOF OF STEP ONE

The proof is very similar to that of Lemma 22.5 and is omitted here.

STEP TWO

Let Q and U be large real numbers with $U>Q^2$. Let α be a positive real number, $\alpha\geq 10$. Then there are at most $O(Q^{8/\alpha})$ distinct primitive characters to moduli not exceeding Q for which the inequality

$$\left| \sum_{U<p\leq 2U} \frac{\chi(p)}{p} \right| \leq Q^{-4/\alpha}$$

fails. The 0-constant is independent of α.

PROOF OF STEP TWO

Again omitted.

STEP THREE : COMPLETION OF THE LEMMA.

We observe that

$$\prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)^{-1} = \prod_{p\leq y} \left(1-\frac{\chi(p)}{p}\right)^{-1} \prod_{y<p<z} \left(1-\frac{\chi(p)}{p}\right)^{-1}$$

$$= \prod_{p\leq y} \left(1-\frac{\chi(p)}{p}\right)^{-1} \prod_{y<p\leq \exp(Q^{1/\alpha})} \left(1-\frac{\chi(p)}{p}\right)^{-1} \prod_{\exp(Q^{1/\alpha})<p<z} \left(1-\frac{\chi(p)}{p}\right)^{-1}$$

if $\exp(Q^{1/\alpha})\leq z$.

The product $\prod_{y<p\leq \exp(Q^{1/\alpha})} \left(1-\frac{\chi(p)}{p}\right)^{-1}$ may be dealt with by firstly showing that

$$\prod_{y<p\leq \exp(Q^{1/\alpha})} \left(1-\frac{\chi(p)}{p}\right)^{-1} = 1 + O\left(\left| \sum_{y<p\leq \exp(Q^{1/\alpha})} \frac{\chi(p)}{p} \right| \right)$$

and then applying Steps One and Two to the error term to give

$$\prod_{y<p\leq\exp(Q^{1/\alpha})} (1-\frac{\chi(p)}{p})^{-1} = 1 + O(\alpha y^{-1/\alpha}) + O(Q^{-3/\alpha})$$

with at most $O(Q^{9/\alpha+1/80})$ exceptions.

On the other hand statement (18) of Nair and Perelli's paper [1] which reads

$$\text{"}\ L(1,\chi) = \prod_{p<w} (1-\frac{\chi(p)}{p})^{-1} \{1+O(\exp(-c(\ln w)^{\frac{1}{2}}))\}$$

holds uniformly for $w\geq\exp(c\ln^2 Q)$ and for all primitive characters to a modulus $q\leq Q$ with at most one exception, χ_1"

may be used to estimate the product $\prod_{\exp(Q^{1/\alpha})<p\leq z} (1-\frac{\chi(p)}{p})^{-1}$.

Assuming that $\exp(Q^{1/\alpha})\geq\exp(c\ln^2 Q)$ we have

$$\prod_{\exp(Q^{1/\alpha})<p\leq z} (1-\frac{\chi(p)}{p})^{-1} = \prod_{p\leq z} (1-\frac{\chi(p)}{p})^{-1} \prod_{p\leq\exp(Q^{1/\alpha})} (1-\frac{\chi(p)}{p})$$

$$= L(1,\chi)\{1 + O(\exp(-c(\ln z)^{\frac{1}{2}}))\}$$

$$\times L(1,\chi)^{-1}\{1 + O(\exp(-cQ^{1/2\alpha}))\}$$

$$= 1 + O(\exp(-Q^{1/2\alpha}))$$

$$= 1 + O(Q^{-3/\alpha}).$$

This completes the lemma. □

Using a version of Lemma 5.1 Elliott [8] extended his results over non-primitive characters. He proved

"LEMMA 22.8 Let x be a real number, $x\geq 9$. Then the estimate

$$L(1,\chi_D) = \{1 + O((\ln x)^{-2})\} \prod_{p\leq\ln^{20}x} (1-\frac{\chi_D(p)}{p})^{-1} \tag{1}$$

holds for all D, $2<D<x$, $-D\equiv 0,1$ mod4, with the possible exception of at most $O(x^{7/8})$ moduli."

It is clear that the number of exceptional moduli, $O(x^{7/8})$, may be decreased if we widen the range of the product on the right hand side of (1), or if we accept weaker error terms. Lemma 5.2 below is an attempt to minimise the number of exceptional moduli if the problem is approached via Elliott's methods. The only major divergence from his mode of argument is in the use of estimates such as Lemma 5.3 below. (It is of course conceivable that other methods of proof than Elliotts may generate better results and so Lemma 5.2 makes no claim to be a best possible result.)

Lemma 5.2 is not used in any of the theorems of the thesis and stands independently of the rest of the work. It is included as it follows a natural line of inquiry from Lemma 5.1.

LEMMA 5.2

Let x *be a large real number. Let* α *be a positive real number satisfying* $\alpha>20$ *and* $\alpha\leqslant(\ln\ln x)^{\frac{1}{4}}$. *Then the estimate*

$$L(1,\chi_D) = \prod_{p<\ln^{2\alpha}x}\left(1-\frac{\chi_D(p)}{p}\right)^{-1}\left(1+O(\exp(-c(\ln\ln x)^{\frac{1}{4}}))\right) \qquad (2)$$

holds for all D, $2<D<x$, $-D\equiv 0,1$ mod4, *with the possible exception of at most*

$$O\left(\left[\frac{x}{\ln\ln x}\right]^{\frac{1}{2}}\right)$$

moduli. The O-constant is absolute, effectively computable, and independent of the value of α.

To prove Lemma 5.2 we firstly require the following lemma:

LEMMA 5.3

$$L(1,\chi) = \prod_{p\leq w} \left(1-\frac{\chi(p)}{p}\right)^{-1} (1 + O(\exp(-c(\ln w)^{\frac{1}{4}}))) \qquad (3)$$

holds uniformly for $Q \leq \exp(c\ln\ln w)$, *and for all primitive characters* χ *to a modulus* $q\leq Q$. *The O-constant is absolute and effectively computable.*

PROOF OF LEMMA 5.3

The proof of Lemma 5.3 follows classical multiplicative number theory arguments and is only briefly outlined. Firstly it may be shown that if

$$\Psi(x,\chi) = \sum_{n\leq x} \Lambda(n)\chi(n)$$

where $\Lambda(n)$ is the von Mangoldt function, then following the arguments of Chapter 20 of Davenport [9],

$$\Psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O\left[\frac{x}{\exp(\ln^{\frac{1}{4}}x)}\right] \qquad (4)$$

uniformly, for all primitive characters χ modq, $q\leq Q$, if

$$Q \leq \exp(c \ln\ln x)$$

and where β_1 is the possible exceptional zero of $L(s,\chi)$ satisfying $\beta_1 > 1-\frac{c}{\ln q}$.

It is well known (see Davenport) that an upper bound on β_1 is given by

$$\beta < 1 - \frac{c}{q^{\frac{1}{2}}\ln q} \leq 1 - \frac{c}{Q^{\frac{1}{2}}\ln Q}.$$

From (4) it follows, via the argument of Lemma 1 of Nair and Perelli [1], that

$$\Psi(x,\chi) = O(\exp(-c(\ln x)^{\frac{1}{4}})) \qquad (5)$$

uniformly, for all primitive characters χ modq, $q\leq Q$.

Further, from (5),

$$\sum_{p \leq x} \chi(p) = O(x\exp(-c(\ln x)^{\frac{1}{4}})) \quad . \tag{6}$$

Since

$$\prod_{p>w} \left(1-\frac{\chi(p)}{p}\right)^{-1} = 1 + O\left[\frac{1}{w}\right] + O\left[\left| \sum_{p>w} \frac{\chi(p)}{p} \right| \right]$$

the result follows by partial summation. □

We cannot extend the method of proof of Lemma 5.3 to include Q much larger without there being a possible exceptional modulus. This case was covered by Nair and Perelli [1] where they proved

" $$L(1,\chi) = \prod_{p \leq w} \left(1-\frac{\chi(p)}{p}\right)^{-1} (1 + O(\exp(-c(\ln w)^{\frac{1}{2}}))) \tag{6}$$

holds uniformly for $w \geq \exp(c\ln^2 Q)$ and for all primitive characters χ to a modulus $q \leq Q$ with at most one exception, χ_1."

We are now in a position to prove Lemma 5.2.

PROOF OF LEMMA 5.2

Each discriminant $-D$ may be written in the unique form $-D=\ell^2 d$ where, if s is defined to be 4d or d as $d \not\equiv 1 \bmod 4$ and $d \equiv 1 \bmod 4$ respectively, s is the disciminant of the quadratic field $\mathbb{Q}(\sqrt{-D})$. Further $\chi_d(p)$ defined by the Kronecker symbol $\left[\frac{d}{p}\right]$ is a real primitive character, mod $|d|$.

We have

$$L(1,\chi_D) = \prod_{p} \left(1-\frac{\chi_D(p)}{p}\right)^{-1}$$

and

$$\chi_D(p) = \begin{cases} \chi_d(p) & ; \ (p,\ell)=1 \\ 0 & ; \ (p,\ell)>1 \end{cases}$$

giving

$$L(1,\chi_D) = \prod_{p \nmid \ell} \left(1-\frac{\chi_d(p)}{p}\right)^{-1}$$

$$= L(1,\chi_d) \prod_{p \mid \ell} \left(1-\frac{\chi_d(p)}{p}\right) \quad . \qquad (7)$$

Now $|-D|=|\ell^2 d|\leq x$ implies that $|d|\leq\frac{x}{\ell^2}$. Define $Q=\frac{x}{\ell^2}$.

Writing $\exp((\ln\ln x)^{\frac{1}{2}})$ as $F(x)$, for $\left[\frac{x}{F(x)}\right]^{\frac{1}{2}}\leq\ell\leq x^{\frac{1}{2}}$ we have $Q\leq F(x)$ and we may apply equation (6) to (7) to give

$$L(1,\chi_d) = \prod_{p\leq \ln^{2\alpha}x} \left(1-\frac{\chi_d(p)}{p}\right)^{-1} \{1 + O(\exp(-c(\ln\ln x)^{\frac{1}{2}}))\} \qquad (8)$$

with at most one exception. Let this exceptional modulus, if it exists, be denoted χ_1, having modulus $|\bar{d}|$.

If $|\bar{d}|\leq\ln\ln x$ then from Lemma 5.3,

$$L(1,\chi_{\bar{d}}) = \prod_{p\leq \ln^{2\alpha}x} \left(1-\frac{\chi_{\bar{d}}(p)}{p}\right)^{-1} \{1 + O(\exp(-c(\ln\ln x)^{\frac{1}{4}}))\}. \qquad (9)$$

Together (9) and (8) give, for $\left[\frac{x}{F(x)}\right]^{\frac{1}{2}}<\ell\leq x^{\frac{1}{2}}$,

$$L(1,\chi_d) = \prod_{p\leq \ln^{2\alpha}x} \left(1-\frac{\chi_d(p)}{p}\right)^{-1} \{1 + O(\exp(-c(\ln\ln x)^{\frac{1}{4}}))\} \qquad (10)$$

with at most one exception, χ_1, say with modulus $|\bar{d}|\geq\ln\ln x$.

If on the other hand $\ell<\left[\frac{x}{F(x)}\right]^{\frac{1}{2}}$ then we have $F(x)<Q$ and applying Lemma 5.1 gives

$$L(1,\chi_d) = \prod_{p\leq \ln^{2\alpha}x} \left(1-\frac{\chi_d(p)}{p}\right)^{-1} \left\{1 + O\left[\frac{\alpha}{\ln^2 x}\right] + O(F(x)^{-3/\alpha})\right\} \qquad (11)$$

with at most $O(Q^{9/\alpha})$ exceptions.

For non-exceptional moduli (10) and (11) give

$$L(1,\chi_D) = \prod_{p\mid\ell} \left(1-\frac{\chi_d(p)}{p}\right) \prod_{p\leq \ln^{2\alpha}x} \left(1-\frac{\chi_d(p)}{p}\right)^{-1} \{1 + O(\exp(-c(\ln\ln x)^{\frac{1}{4}}))\} \qquad (12)$$

if $\alpha\ll(\ln\ln x)^{\frac{1}{4}}$.

However

$$\prod_{p|\ell}\left(1-\frac{\chi_d(p)}{p}\right)\prod_{p<\ln^{2\alpha}x}\left(1-\frac{\chi_d(p)}{p}\right)^{-1} = \prod_{\substack{p|\ell\\ p<\ln^{2\alpha}x}}\left(1-\frac{\chi_d(p)}{p}\right)\prod_{p<\ln^{2\alpha}x}\left(1-\frac{\chi_d(p)}{p}\right)^{-1}$$

$$\times \prod_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\left(1-\frac{\chi_d(p)}{p}\right)$$

$$= \prod_{\substack{p\nmid\ell\\ p<\ln^{2\alpha}x}}\left(1-\frac{\chi_d(p)}{p}\right)\prod_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\left(1-\frac{\chi_d(p)}{p}\right)$$

$$= \prod_{p<\ln^{2\alpha}x}\left(1-\frac{\chi_D(p)}{p}\right)\prod_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\left(1-\frac{\chi_d(p)}{p}\right) . \tag{13}$$

So (12) and (13) will complete the proof of the lemma for non-exceptional moduli if an appropriate estimate of

$$\prod_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\left(1-\frac{\chi_d(p)}{p}\right)$$

can be found.

But

$$\prod_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\left(1-\frac{\chi_d(p)}{p}\right) = \exp\left[\sum_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\ln\left[1-\frac{\chi_d(p)}{p}\right]\right]$$

and

$$\sum_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\left|\ln\left[1-\frac{\chi_d(p)}{p}\right]+\frac{\chi_d(p)}{p}\right| < \sum_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\frac{2}{p^2} .$$

So

$$\sum_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\ln\left[1-\frac{\chi_d(p)}{p}\right] < 2\sum_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\frac{1}{p^2} + \left|\sum_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\frac{\chi(p)}{p}\right|$$

$$< 3\sum_{\substack{p|\ell\\ p>\ln^{2\alpha}x}}\frac{1}{p}$$

$$< \frac{3\ \omega(\ell)}{\ln^{2\alpha} x}$$

where $\omega(\ell)$ denotes the number of prime divisors of ℓ.

Since $\ell \leqslant x^{\frac{1}{2}}$, $\omega(\ell) \ll \ln x$, and

$$\sum_{\substack{p|\ell \\ p>\ln^{2\alpha}x}} \ln\left[1 - \frac{\chi_d(p)}{p}\right] \ll \frac{1}{\ln^{2\alpha-1}x}$$

we have

$$\prod_{\substack{p|\ell \\ p>\ln^{2\alpha}x}} (1-\frac{\chi_d(p)}{p}) < \exp\left[\frac{c}{\ln^{2\alpha-1}x}\right] = 1 + O\left[\frac{1}{\ln^{2\alpha-1}x}\right] .$$

As

$$\prod_{\substack{p|\ell \\ p>\ln^{2\alpha}x}} (1-\frac{\chi_d(p)}{p}) = 1 + (\text{extra terms})$$

it follows that

$$\prod_{\substack{p|\ell \\ p>\ln^{2\alpha}x}} (1-\frac{\chi_d(p)}{p}) = 1 + O\left[\frac{1}{\ln^{2\alpha-1}x}\right]$$

and for non-exceptional characters we have, as required,

$$L(1,\chi_D) = \prod_{p\leqslant\ln^{2\alpha}x} (1-\frac{\chi_D(p)}{p})^{-1}\{1 + O(\exp(-c(\ln\ln x)^{\frac{1}{4}}))\} . \qquad (14)$$

Turning now to the exceptional moduli, according to (10) for $\frac{x^{\frac{1}{2}}}{F(x)^{\frac{1}{2}}} < \ell \leqslant x^{\frac{1}{2}}$ there is at most one exceptional modulus for d, namely $|\bar{d}|$. This satisfies $|\bar{d}| \geqslant \ln\ln x$. Since $|-D| = \ell^2|d| \leqslant x$ it follows that only those ℓ within the range $\frac{x^{\frac{1}{2}}}{F(x)^{\frac{1}{2}}} < \ell \leqslant \frac{x^{\frac{1}{2}}}{\ln^{\frac{1}{2}}\ln x}$ generate a value of $-D$. Hence there are at most

$$O\left[\left[\frac{x}{\ln\ln x}\right]^{\frac{1}{2}}\right]$$

exceptional moduli for $\frac{x^{\frac{1}{2}}}{F(x)^{\frac{1}{2}}} < \ell \leqslant x^{\frac{1}{2}}$.

If however $\ell < \frac{x^{\frac{1}{2}}}{F(x)^{\frac{1}{2}}}$ then, from (12) it follows that for each value of ℓ there are at most $0\left(\left[\frac{x}{\ell^2}\right]^{9/\alpha}\right)$ exceptions. Varying ℓ over the range $\ell < \frac{x^{\frac{1}{2}}}{F(x)^{\frac{1}{2}}}$ we deduce that the number of additional exceptional moduli is of the order

$$\sum_{\ell < \frac{x^{\frac{1}{2}}}{F(x)^{\frac{1}{2}}}} \left[\frac{x}{\ell^2}\right]^{9/\alpha} \leqslant x^{9/\alpha} \sum_{\ell < \frac{x^{\frac{1}{2}}}{F(x)^{\frac{1}{2}}}} \frac{1}{\ell^{18/\alpha}} \ll x^{9/\alpha} \left[\frac{x}{F(x)}\right]^{\frac{1}{2}(1-18/\alpha)}$$

$$= \frac{x^{\frac{1}{2}}}{F(x)^{\frac{1}{2}-9/\alpha}}.$$

So the total number of exceptional moduli is

$$0\left(\left[\frac{x}{\ln\ln x}\right]^{\frac{1}{2}} + \frac{x^{\frac{1}{2}}}{F(x)^{\frac{1}{2}-9/\alpha}}\right) = 0\left(\left[\frac{x}{\ln\ln x}\right]^{\frac{1}{2}}\right)$$

which completes the lemma. □

Recall our earlier comment that we cannot extend Lemma 5.3 to include Q much larger without having to introduce the possibility of there being an exceptional modulus. Since the proof of Lemma 5.2 is dependent upon Lemma 5.3 it follows that Lemma 5.2 is effectively best possible for this method of proof.

<u>LEMMA 6</u>

Let $f(m) = \alpha m^2 + 2\beta m + \gamma$ *be a polynomial with integer coefficients, with* $\alpha \neq 0$, *and write*

$$F(y) := \max_{0<m\leqslant y} |f(m)|.$$

Let M *be a constant defined by*

$$M = \max(\,|\alpha+2\beta+\gamma|, |-\beta^2-\alpha\gamma|\,)$$

and define

$$M_1 = \beta^2-\alpha\gamma+|\alpha|M.$$

If

$$y > N = \begin{cases} \dfrac{|\beta| + (M_1)^{\frac{1}{2}}}{|\alpha|} & ; \quad M_1 \geq 0 \\ 0 & ; \quad M_1 < 0 \end{cases}$$

then

(i) $F(y) = |\alpha y^2+2\beta y+\gamma|$.

Assuming in addition that

$$y > \max\left\{ \frac{5|\beta|}{|\alpha|} , \frac{3|\gamma|}{|\alpha|} \right\}$$

gives

(ii) $\frac{y^2}{2} \leq F(y) \leq 4|\alpha|y^2$.

PROOF

Assume firstly that $\alpha>0$.

The polynomial $|\alpha m^2+2\beta m+\gamma|$ has a local maximum at $m=-\beta/\alpha$, so over the range $0<m\leq y$, it will be greatest at one of the following three points:

(a) $m=y$; $|f(m)| = |\alpha y^2+2\beta y+\gamma|$

(b) $m=1$; $|f(m)| = |\alpha+2\beta+\gamma|$

(c) $m=-\beta/\alpha$; $|f(m)| = |-\beta^2/\alpha + \gamma|$.

Now $|\alpha y^2+2\beta y+\gamma| = \alpha y^2+2\beta y+\gamma$.

This is easily seen to be the case, as

$$\alpha y^2+2\beta y+\gamma = 0$$

has roots at

$$y = \frac{-\beta\pm(\beta^2-\alpha\gamma)^{\frac{1}{2}}}{\alpha} .$$

So certainly, as $\beta^2-\alpha\gamma \leq M_1$, assuming that $y\geq N$ ensures that

$$y > \frac{-\beta+(\beta^2-\alpha\gamma)^{\frac{1}{2}}}{\alpha}$$

and that

$$\alpha y^2+2\beta y+\gamma\geq 0.$$

For F(y) to equal $|\alpha y^2+2\beta y+\gamma|$ then, it is enough to show both

(1) $\alpha y^2+2\beta y+\gamma \geq |\alpha+2\beta+\gamma|$

and

(2) $\alpha y^2+2\beta y+\gamma \geq |-\beta^2/_\alpha + \gamma|$.

Now (1) holds whenever

$$\alpha y^2+2\beta y+\gamma-|\alpha+2\beta+\gamma| \geq 0.$$

By the same reasoning as above we see that this occurs whenever

$$y \geq \begin{cases} \dfrac{-\beta + (\beta^2-\alpha(\gamma-|\alpha+2\beta+\gamma|))^{\frac{1}{2}}}{\alpha} & ;\ \beta^2-\alpha(\gamma+|\alpha+2\beta+\gamma|)\geq 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Certainly it occurs if $y \geq N$.

Similarly (2) holds whenever

$$y \geq \begin{cases} \dfrac{-\beta+(\beta^2-\alpha(\gamma-|-\beta^2/_\alpha+\gamma|))^{\frac{1}{2}}}{\alpha} & ;\ \beta^2-\alpha(\gamma-|-\beta^2/_\alpha+\gamma|) \geq 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

i.e. whenever

$$y \geq \begin{cases} \dfrac{-\beta+(\beta^2-(\alpha\gamma-|-\beta^2+\alpha\gamma|))^{\frac{1}{2}}}{\alpha} & ;\ \beta^2-(\alpha\gamma-|-\beta^2-\alpha\gamma|) \geq 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Again, this occurs if $y \geq N$.

This concludes the proof of (i) for $\alpha>0$. The proof for $\alpha<0$ is very similar.

For (ii) we have

$$F(y) = |\alpha y^2+2\beta y+\gamma|$$
$$\leq |\alpha|y^2+2|\beta|y+|\gamma|$$

and

$$|\alpha|y^2+2|\beta|y+|\gamma| \leq 4|\alpha|y^2$$

whenever

$$\frac{2|\beta|}{3|\alpha|} + \frac{|\gamma|}{3|\alpha|y} < y.$$

Certainly this is the case if $y > \max\left\{ \frac{5|\beta|}{|\alpha|}, \frac{3|\gamma|}{|\alpha|} \right\}$.

On the other hand

$$F(y) = |\alpha y^2+2\beta y+\gamma| > |\alpha|y^2-2|\beta|y-|\gamma|$$

and

$$|\alpha|y^2-2|\beta|y-|\gamma| > y^2/2$$

whenever

$$y > \frac{4|\beta|}{2|\alpha|-1} + \frac{2|\gamma|}{(2|\alpha|-1)y} .$$

This holds if

$$y > \frac{4|\beta|}{|\alpha|} + \frac{2|\gamma|}{|\alpha|y}$$

and it consequently holds if $y > \max\left\{ \frac{5|\beta|}{|\alpha|}, \frac{3|\gamma|}{|\alpha|} \right\}$.

This completes the lemma.

□

CHAPTER TWO

INTRODUCTION

In Chapter Two we prove our first major theorem. Define the function S(x,y,z) to be

$$S(x,y,z) = \left|\{(n,m);\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leqslant y,\ m\equiv\ell_2 \bmod k_2,\ ((an^2+bn+c)m+(dn^2+en+f),\prod_{p<z}p)=1\}\right|$$

where the product Π is over all primes up to z.

In Theorem One we give an asymptotic formula for S(x,y,z) when an^2+bn+c and dn^2+en+f have no common factors either constant, linear or quadratic. We assume that a and d are not both zero. The proof hinges on the observation, first exploited by Nair and Perelli [1] when estimating the simpler function

$$S(x,y,z) = \left|\{(n,m):\ n\leqslant x,\ m\leqslant y,\ (n^2+m,\prod_{p<z}p)=1\}\right|,$$

that S(x,y,z) may be written as two different sums. Namely

$$\sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv\ell_1 \bmod k_1}} \left|\{m:\ 0<m\leqslant y,\ m\equiv\ell_2 \bmod k_2,\ ((an^2+bn+c)m+(dn^2+en+f),\prod_{p<z}p)=1\}\right|$$

$$= S(x,y,z) =$$

$$\sum_{\substack{0<m\leqslant y \\ m\equiv\ell_2 \bmod k_2}} \left|\{n:\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ ((an^2+bn+c)m+(dn^2+en+f),\prod_{p<z}p)=1\}\right|.$$

Since, whenever $z\leqslant {}^{y}/k_2$, the function within the first sum

$$\left|\{m:\ 0<m\leqslant y,\ m\equiv\ell_2 \bmod k_2,\ ((an^2+bn+c)m+(dn^2+en+f),\prod_{p<z}p)=1\}\right|$$

is relatively easy to estimate this gives us a starting point from which to estimate the second sum whenever ${}^{y}/k_2\leqslant z\leqslant {}^{x}/k_1$.

The proof draws on various areas of, what is now considered to be, classical Number Theory, such as Sieve Theory, Dirichlet

L-functions and Ideal Theory, as well as on more recent work. One of the most recent papers to be referred to in the proof of Theorem One is that by Gross and Zagier [11] in which it is proved that, for any $\delta>0$, there is an effectively computable constant $c_\epsilon>0$ such that for any imaginary quadratic field F, $h_F>c_\delta(\ln|d_F|)^{1-\epsilon}$, where h_F and d_F are the class-number and discriminant of F respectively.

As noted in the proof, were this paper not available we would be forced to make use of Siegal's Theorem. Although in this circumstance the error terms in the estimate of S(x,y,z) would be sharper it would unfortunately mean that the associated O-constants were, with current knowledge, non-computable.

It will be noted that, although in S(x,y,z) we have taken n to lie in the range $\alpha<n\leqslant\alpha+x$ for any α, we have not taken m to lie in an arbitrary range of length y. Furthermore the estimate of S(x,y,z) for ${}^x/k_1 \geqslant {}^y/k_2$ in Theorem One is independent of α. The reasoning behind this will be explained at the end of the proof of Theorem One.

Before we state Theorem One we give some definitions.

Firstly we define a function H_Z as

$$H_Z = \sum_{\substack{w \\ w=1 \text{ or } \gamma(w)>z}} \Gamma_Z(w)$$

where $\gamma(w)$ denotes the smallest prime factor of w, and where $\Gamma_Z(w)$ is defined to be

$$\Gamma_Z(w) = \sum_{\substack{\alpha_i \bmod Fk_2 \\ \alpha_i \equiv \ell_1 \bmod (k_1, Fk_2)}} \prod_{p<z} \left(1 + \frac{\rho(p)}{p(p-1)}\right) .$$

The notation of $\Gamma_Z(w)$ is as follows:

(i) $F = (cd-fa)^2-(bd-ea)(ce-fb)$

(ii) $\alpha_1, \ldots, \alpha_\mu$ denote the integers, n, in the interval $1 \leq n \leq Fk_2$ for which both

$$(an^2+bn+c, dn^2+en+f) = w$$

and

$$\left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right], \prod_{\substack{p<z \\ p \mid k_2}} p\right)=1$$

hold.

(iii) the unique solution, mod$[k_1,Fk_2]$, of the two congruences $n \equiv \ell_1 \bmod k_1$ and $n \equiv \alpha_i \bmod Fk_2$ is denoted, if it exists, by $\beta_i = \beta_i(\ell_1, \alpha_i)$. Letting $h=(a,b,c)$; $a=a_1h$, $b=b_1h$, $c=c_1h$, then

$$\rho(p) = \begin{cases} \left|\{t \bmod p;\ a_1([k_1,Fk_2]t+\beta_i)^2+b_1([k_1,Fk_2]t+\beta_i)+c_1 \equiv 0 \bmod p\}\right| & ;\ p \nmid k_2h \\ p & ;\ p \mid k_2h \end{cases}.$$

If D denotes the discriminant of the polynomial an^2+bn+c then define $G(x,\alpha)$ as

$$G(x,\alpha) = \begin{cases} \max\limits_{\alpha<n\leq\alpha+x} |a_1n^2+b_1n+c_1| & ;\ D\neq 0 \\ \max\limits_{\alpha<n\leq\alpha+x} |a_1n^2+b_1n+c_1|^{\frac{1}{2}} & ;\ D=0 \end{cases}$$

and λ by

$$\ln\lambda = \begin{cases} \left[\left[\frac{\ln(2.D^2)}{\ln 2}\right]+1\right]\ln 2 & ;\ D\neq 0 \\ 0 & ;\ D=0 \end{cases}.$$

Define $A := \max(\ln\ln G(x,\alpha)\ln^\lambda G(x,\alpha),\ \ln^2 z)$.

Finally define $\zeta=b^2-4ac$, $\eta=be-2cd-2fa$, and $\theta=e^2-4fd$.

With these definitions we have:

<u>THEOREM ONE</u>

Let an^2+bn+c *and* dn^2+en+f *be polynomials with integer coefficients and with* a *and* d *not both zero. Assume that these polynomials have no common factors. Let* $x,y \in Z$ *and* ℓ_1, ℓ_2, k_1, k_2

$\in\mathbb{N}$ *with*

$$\exp((\ln {}^{y}/k_2)^{\frac{1}{2}}) > \max(|a|,|b|,|c|,|d|,|e|,|f|,k_1,k_2).$$

If z *satisfies* $2 \leq z \leq \max({}^{x}/k_1, {}^{y}/k_2)$, *then if* ${}^{y}/k_2 > {}^{x}/k_2$,

$$S(x,y,z) = \frac{xy}{k_2[k_1,Fk_2]} \prod_{p<z} (1-\frac{1}{p}) H_z \Big\{1 + O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))$$

$$+ O(\exp(-u(\ln u - \ln\ln 3u - 2))) + O\left[\frac{a_1[k_1,Fk_2]^{3A}}{x}\right]$$

$$+ O\left[a_1[k_1 Fk_2]^2 \frac{\ln\ln G(x,\alpha)\ln^{\lambda}G(x,\alpha)}{G(x,\alpha)}\right] \Big\}$$

and if ${}^{x}/k_1 \geq {}^{y}/k_2$, *for any* $\epsilon>0$,

$$S(x,y,z) = \frac{xy}{k_2[k_1,Fk_2]} \prod_{p<z} (1-\frac{1}{p}) H_z \Big\{1 + O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}}))$$

$$+ O(\exp(-v(\ln v - \ln\ln 3v - \ln 2 - 2)))$$

$$+ O_{\epsilon}\left[|\zeta|^{\frac{1}{2}}.\ln\ln^2|a\zeta|.\tau\left[\frac{\eta^2-\zeta\theta}{4}\right].\frac{k_1k_2[k_1,Fk_2]}{\varphi(k_1)^2}.\ \frac{\ln\ln^2 y}{(\ln y)^{1-\epsilon}}\right] \Big\}$$

where $u = \dfrac{\ln {}^{y}/k_2}{\ln z}$; $v = \dfrac{\ln {}^{x}/k_1}{\ln z}$; *and where* $\tau(n)$ *denotes the number of divisors of* n. *The* O*-constants are absolute, effectively computable, and independent of* $a,b,c,d,e,f,\ell_1,\ell_2,$ k_1 *and* k_2. *In the case of* ${}^{x}/k_1 \geq {}^{y}/k_2$ *the* O*-constants may however depend on* ϵ.

PROOF OF THEOREM ONE

Owing to the length of the proof of Theorem One we split it into thirteen steps.

STEP ONE *An asymptotic formula for*

$$S(x,y,z) = \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\ell_1 \bmod k_1}} \left|\{ m: 0<m\leq y,\ m\equiv\ell_2 \bmod k_2,\ ((an^2+bn+c)m+(dn^2+en+f), \prod_{p<z} p)=1\}\right|$$

whenever $z \leq {}^{y}/k_2$.

Define M(y,z,n) as

$$M(y,z,n) = \left|\{m : 0<m\leq y,\ m\equiv \ell_2 \bmod k_2,\ ((an^2+bn+c)m+(dn^2+en+f), \prod_{p<z} p) = 1\}\right|$$

so that

$$S(x,y,z) = \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv \ell_1 \bmod k_1}} M(y,z,n) \tag{1}$$

Define r_n to be the highest common factor of the two polynomials an^2+bn+c and dn^2+en+f. It is apparent that if

$$(r_n, \prod_{p<z} p) > 1$$

then $M(y,z,n) = 0$.

Assuming, however, that $(r_n, \prod_{p<z} p)=1$ we have

$$M(y,z,n) = \left|\{m : 0<m\leq y,\ m\equiv \ell_2 \bmod k_2,\ \left(\frac{(an^2+bn+c)}{r_n}m+\frac{(dn^2+en+f)}{r_n}, \prod_{p<z} p\right) = 1\}\right|$$

which, on applying Lemma 1.1, gives

M(y,z,n)

$$= \begin{cases} \dfrac{y}{k_2} \displaystyle\prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho(p)}{p}\right)\left\{1+O(\exp(-u(\ln u-\ln\ln 3u-2))) + O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))\right\} & ;\ \left(\left[\dfrac{an^2+bn+c}{r_n}\right]\ell_2+\left[\dfrac{dn^2+en+f}{r_n}\right], \displaystyle\prod_{\substack{p<z \\ p\mid k_2}} p\right)=1 \\ 0 & ;\ \text{otherwise} \end{cases}$$

where

$$u = \frac{\ln {}^{y}/k_2}{\ln z}$$

and

$$\rho(p) = \left|\{ m \bmod p : \left[\frac{an^2+bn+c}{r_n}\right]m + \left[\frac{dn^2+en+f}{r_n}\right] \equiv 0 \bmod p \}\right|$$

provided, of course, that

$\rho(p)<p$ for all primes, p.

But in fact

$$\rho(p) = \begin{cases} 1 & ; p \nmid \left[\frac{an^2+bn+c}{r_n}\right] \\ 0 & ; p \mid \left[\frac{an^2+bn+c}{r_n}\right] . \end{cases}$$

We note that, under the condition $(r_n, \prod_{p<z} p)=1$, whenever $p<z$

$$\rho(p) = \begin{cases} 1 & ; p \nmid an^2+bn+c \\ 0 & ; p \mid an^2+bn+c . \end{cases}$$

Summing $M(y,z,n)$ over n gives

$$S(x,y,z) = \frac{y}{k_2} \sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (r_n, \prod_{p<z} p)=1 \\ \left(\left[\frac{an^2+bn+c}{r_n}\right]\ell_2+\left[\frac{dn^2+en+f}{r_n}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\nmid(an^2+bn+c)k_2}} \left(1-\frac{1}{p}\right) \Big\{ 1 + O(\exp(-u(\ln u-\ln\ln 3u-2))) + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) \Big\} \qquad (2)$$

Taking the product $\prod_{p<z} \left(1-\frac{1}{p}\right)$ out to the front of the sum gives

$$S(x,y,z) = \frac{y}{k_2} \prod_{p<z}\left(1-\frac{1}{p}\right) \sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (r_n, \prod_{p<z} p)=1 \\ \left(\left[\frac{an^2+bn+c}{r_n}\right]\ell_2+\left[\frac{dn^2+en+f}{r_n}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\mid(an^2+bn+c)k_2}} \left(1-\frac{1}{p}\right)^{-1} \Big\{ 1 + O(\exp(-u(\ln u-\ln\ln 3u-2))) + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) \Big\} \qquad (3)$$

Now the sum

$$\sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (r_n, \prod_{p<z} p)=1 \\ \left(\left[\frac{an^2+bn+c}{r_n}\right]\ell_2+\left[\frac{dn^2+en+f}{r_n}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\mid(an^2+bn+c)k_2}} \left(1-\frac{1}{p}\right)^{-1}$$

$$= \sum_{\substack{w=1 \text{ or} \\ \gamma(w)>z}} \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (an^2+bn+c,\ dn^2+en+f)=w \\ \left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{p<z,\ p\mid k_2} p\right)=1}} \prod_{\substack{p<z \\ p\mid(an^2+bn+c)k_2}} (1-\frac{1}{p})^{-1}$$

where $\gamma(w)$ denotes the smallest prime factor of w.

By Lemma 2.9 this sum is equal to

$$\frac{x}{[k_1,Fk_2]} \sum_{w=1 \text{ or } \gamma(w)>z} \Gamma_z(w) \left\{ 1 + 0\left[\frac{a_1[k_1,Fk_2]^2 \ln\ln G(x,\alpha)\ \ln^{\lambda}G(x,\alpha)}{G(x,\alpha)} \right] + 0\left[\frac{a_1[k_1,Fk_2]^3 A}{x} \right] \right\} \qquad (4)$$

where

(i) $F = |(cd-fa)^2-(bd-ea)(ce-fb)|$

(ii)
$$\Gamma_z(w) = \sum_{\substack{\alpha_i \bmod Fk_2 \\ \alpha_i\equiv\ell_1 \bmod (k_1,Fk_2)}} \prod_{p<z} (1+\frac{\rho(p)}{p(p-1)})$$

where $\alpha_1,\ldots,\alpha_\mu$ denote the integers, n, in the interval $1\leq n\leq Fk_2$ for which both

$$(an^2+bn+c,\ dn^2+en+f)=w$$

and
$$\left\{ \left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{p<z,\ p\mid k_2} p \right\}=1$$

hold.

(iii) the unique solution, mod$[k_1,Fk_2]$, of the congruences

$$n\equiv\ell_1 \bmod k_1 \text{ and } n\equiv\alpha_i \bmod Fk_2$$

is denoted, if it exists, by $\beta_i=\beta_i(\ell_1,\alpha_i)$. Letting

$$h=(a,\ b,\ c);\ a=a_1h,\ b=b_1h,\ c=c_1h,$$

then,

$$\rho(p) = \begin{cases} \left|\{ t \bmod p;\ a_1([k_1,Fk_2]t+\beta_i)^2+b_1([k_1,Fk_2]t+\beta_i)+c_1 \equiv 0 \bmod p\}\right| & ;\ p\nmid k_2h \\ p & ;\ p\mid k_2h \end{cases}$$

If D denotes the discriminant of the polynomial an^2+bn+c then

(iv)
$$G(x,\alpha)=\begin{cases}\max\limits_{\alpha<n\leq\alpha+x}|a_1n^2+b_1n+c_1| & ; D\neq 0\\ \max\limits_{\alpha<n\leq\alpha+x}|a_1n^2+b_1n+c_1|^{\frac{1}{2}} & ; D=0\end{cases}$$

(v)
$$\ln\lambda=\begin{cases}\left(\left[\frac{\ln 2D^2}{\ln 2}\right]+1\right)\ln 2 & ; D\neq 0\\ 0 & ; D=0.\end{cases}$$

and finally

(vi) $A = \max(\ln\ln G(x,\alpha)\ln^{\lambda}G(x,\alpha),\ \ln^2 z)$.

Let

$$H_z = \sum_{\substack{w\\ w=1 \text{ or } \gamma(w)>z}} \Gamma_z(w) \quad .$$

By Lemma 2.1 we have that, if $(an^2+bn+c,dn^2+en+f)=w$ for some integer n, then $F\equiv 0 \bmod w$. So the number of possible w is at most $\tau(F)$ where $\tau(F)$ denotes the number of divisors of F. We may however ascertain exactly the number of possible w, for $(an^2+bn+c,dn^2+en+f)=w$ if and only if $(a(n+F)^2+b(n+F)+c,d(n+F)^2+e(n+F)+f)=w$, and consequently, the smallest integer n, if it exists, for which $(an^2+bn+c,dn^2+en+f)=w$ will be less than or equal to F.

Let these possible w be denoted $1,w_1,w_2,\ldots,w_r$. Then

$$H_z = \Gamma_z(1)+\Gamma_z(w_1)+\ldots+\Gamma_z(w_r).$$

We note here that if $z\geq F$ then $H_z=\Gamma_z(1)$.

Substituting (4) into (3) gives

$$S(x,y,z)=\frac{xy}{k_2[k_1,Fk_2]}\prod_{p<z}\left(1-\frac{1}{p}\right) H_z\left\{1 + O\left[\frac{a_1[k_1,Fk_2]^2\ \ln\ln G(x,\alpha)\ \ln^{\lambda}G(x,\alpha)}{G(x,\alpha)}\right] + O\left[\frac{a_1[k_1,Fk_2]^3\ A}{x}\right] + O(\exp(-u(\ln u-\ln\ln 3u-2))) + O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))\right\}. \qquad (5)$$

Equation (5) completes the theorem for the case ${}^{y}/k_2>{}^{x}/k_1$, so we may assume henceforth that

$^x/k_1 = \max\{^x/k_1, ^y/k_2\}$.

STEP TWO

An asymptotic formula for

$$S(x,y,z) = \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2}} \left| \{ n: \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1 \quad (n^2(am+d)+n(bm+e)+(cm+f),\ \prod_{p<z} p)=1 \} \right|$$

whenever $z\leq {}^x/k_1$.

Define N(x,z,m) as

$$N(x,z,m) = \left| \{ n: \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1, \quad (n^2(am+d)+n(bm+e)+(cm+f), \prod_{p<z} p)=1 \} \right|$$

so that

$$S(x,y,z) = \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2}} N(x,z,m) \qquad (6)$$

To find an asymptotic formula for N(x,z,m) we follow the argument of Step One, and firstly remove from (6) any N(x,y,m) obviously zero.

Define s_m to be the highest common factor of (am+d), (bm+e), and (cm+f). Then if

$$(s_m, \prod_{p<z} p) > 1$$

it follows that N(x,z,m)=0.

Further if

$$(am+d+bm+e, cm+f) \equiv 0 \bmod 2$$

then

$$n^2(am+d)+n(bm+e)+(cm+f) \equiv 0 \bmod 2$$

giving, again, N(x,z,m)=0.

Assuming, then, that

$$(s_m, \prod_{p<z} p) = 1 \tag{7}$$

and

$$(am+d+bm+e, cm+f) \equiv 1 \bmod 2 \tag{8}$$

Lemma 1.1 gives

$N(x,z,m) =$

$$\left| \left\{ n:\ \alpha < n \leq \alpha + x,\ n \equiv \ell_1 (\bmod k_1),\ \left(n^2\left[\frac{am+d}{s_m}\right] + n\left[\frac{bm+e}{s_m}\right] + \left[\frac{cm+f}{s_m}\right],\ \prod_{p<z} p\right) = 1 \right\} \right|$$

$$= \begin{cases} \dfrac{x}{k_1} \displaystyle\prod_{\substack{p<z \\ p \nmid k_1}} \left(1 - \frac{\rho_m(p)}{p} \right) \{ 1 + O(\exp(-v(\ln v - \ln\ln 3v - \ln 2 - 2)) + O(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}})) \} & ;\ \left(\ell_1^2\left[\frac{am+d}{s_m}\right] + \ell_1\left[\frac{bm+e}{s_m}\right] + \left[\frac{cm+f}{s_m}\right], \prod_{\substack{p<z \\ p \mid k_1}}\right) = 1 \\ 0 & ;\ \text{otherwise} \end{cases} \tag{9}$$

where

$$v = \frac{\ln {}^x/k_1}{\ln z}$$

and

$$\rho_m(p) = \left| \left\{ n \bmod p:\ n^2\left[\frac{am+d}{s_m}\right] + n\left[\frac{bm+e}{s_m}\right] + \left[\frac{cm+f}{s_m}\right] \equiv 0 \bmod p \right\} \right|$$

provided that

$$\rho_m(p) < p \quad \text{for all primes } p. \tag{10}$$

To verify (10), as it is certainly the case that $\rho_m(p) \leq p$ we have only to show that $\rho_m(p) \neq p$, for any prime p. If $p>2$, $\rho_m(p) = p$ if and only if $\left(\frac{am+d}{s_m}, \frac{bm+e}{s_m}, \frac{cm+f}{s_m} \right) \equiv 0 \bmod p$ which, by the definition of s_m, cannot occur.

If $p=2$ then $\rho_m(2) = 2$ if and only if $\frac{cm+f}{s_m} \equiv 0 \bmod 2$ and $\frac{am+d+bm+e}{s_m} \equiv 0 \bmod 2$. But this would imply $cm+f \equiv 0 \bmod 2$ and $am+d+bm+e \equiv 0 \bmod 2$ contradicting (8).

Hence (10) is satisfied, as required.

Summing (9) over m gives

$$S(x,y,z) = \frac{x}{k_1} \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (s_m, \prod_{p<z} p)=1 \\ (am+d+bm+e,\, cm+f)\equiv 1 \bmod 2 \\ \left(\ell_1^2\left[\frac{am+d}{s_m}\right]+\ell_1\left[\frac{bm+e}{s_m}\right]+\left[\frac{cm+f}{s_m}\right],\, \prod_{\substack{p<z \\ p\mid k_1}} p\right)=1}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \Big\{ 1 + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2))) + O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}})) \Big\}$$

(11)

for $z < {}^{x}/k_1$.

To simplify the expression under the summation sign we write "(m,z) appropriate" or "(m,z) app" for those m satisfying the conditions

(i) $(s_m, \prod_{p<z} p)=1$

(ii) $(am+d+bm+e, cm+f)\equiv 1 \bmod 2$

(iii) $\left(\ell_1^2\left[\frac{am+d}{s_m}\right]+\ell_1\left[\frac{bm+e}{s_m}\right]+\left[\frac{cm+f}{s_m}\right],\ \prod_{\substack{p<z \\ p\mid k_1}} p\right)=1$.

Any m satisfying conditions (i), (ii), and (iii) will be said to be "z appropriate".

(11) becomes

$$S(x,y,z) = \frac{x}{k_1} \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \text{app}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \Big\{ 1 + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2))) + O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}})) \Big\}.$$

(12)

Recalling from (5), that for $z \leq {}^{y}/k_2$, and with $A=\max(\ln\ln G(x,\alpha)\ln^{\lambda}G(x,\alpha),\ \ln^2 z)$

$$S(x,y,z)=\frac{xy}{k_2[k_1,Fk_2]}\prod_{p<z}\left(1-\frac{1}{p}\right)H_z\Big\{1+$$

$$O\left[\frac{a_1[k_1,Fk_2]^2\ \ln\ln G(x,\alpha)\ \ln^{\lambda}G(x,\alpha)}{G(x,\alpha)}\right]+O\left[\frac{a_1[k_1,Fk_2]^3\ A}{x}\right]$$

$$+\ O(\exp(-u(\ln u-\ln\ln 3u-2)))+O(\exp(-(\ln^{y}/k_2)^{\frac{1}{2}}))\Big\}. \quad (5)$$

If we further restrain z to $z \leqslant \exp(27(\ln^{y}/k_2)^{\frac{1}{2}})$ the error term $O(\exp(-u(\ln u-\ln\ln 3u-2)))$ is absorbed into the final error term to give,

$$S(x,y,z)=\frac{xy}{k_2[k_1,Fk_2]}\prod_{p<z}\left(1-\frac{1}{p}\right)H_z\Big\{1+$$

$$O\left[\frac{a_1[k_1,Fk_2]^2\ \ln\ln G(x,\alpha)\ \ln^{\lambda}G(x,\alpha)}{G(x,\alpha)}\right]+O\left[\frac{a_1[k_1,Fk_2]^3\ A}{x}\right]$$

$$+\ O(\exp(-(\ln^{y}/k_2)^{\frac{1}{2}}))\Big\}. \quad (13)$$

A comparison of (12) and (13) for $z \leqslant \exp(27(\ln^{y}/k_2)^{\frac{1}{2}})$ taking $A=\ln^2 z$, gives (by Lemma 3),

$$\sum_{\substack{0<m\leqslant y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \mathrm{app}}}\ \prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{\rho_m(p)}{p}\right)=\frac{k_1 y}{k_2[k_1,Fk_2]}\prod_{p<z}\left(1-\frac{1}{p}\right)H_z\Big\{1+$$

$$+\ O\left[\frac{a_1[k_1,Fk_2]^2\ln\ln G(x,\alpha)\ln^{\lambda}G(x,\alpha)}{G(x,\alpha)}\right]$$

$$+\ O\left[\frac{a_1[k_1,Fk_2]^3\ln^2 z}{x}\right]+O(\exp(-(\ln^{y}/k_2)^{\frac{1}{2}})))$$

$$+\ O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))$$

$$+\ O(\exp(-(\ln^{x}/k_1)^{\frac{1}{2}})))\Big\}. \quad (14)$$

As the left hand side of (14) is independent of x, we may let

$x \to \infty$, thus ensuring that $x/k_1 > z$ is satisfied and that $v \to \infty$. Hence

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \frac{k_1 y}{k_2[k_1,Fk_2]} \prod_{p<z}\left(1-\frac{1}{p}\right) H_z \left\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\right\} \qquad (15)$$

for

$$2 \leq z \leq \exp(27(\ln y/k_2)^{\frac{1}{2}}) . \qquad (16)$$

To complete the theorem we need to find a result similar to (15) for $z>\exp(27(\ln y/k_2)^{\frac{1}{2}})$.

<u>STEP THREE</u> *Determination of $\rho_m(p)$ in terms of the Legendre symbol.*

It is clear that $\rho_m(p)$ is closely related to the Kronecker symbol, $\chi(p)$, and that the product $\prod_{p<z}(1-\frac{\rho_m(p)}{p})$ is related to the product $\prod_{p<z}(1-\frac{\chi(p)}{p})$. The translation into this latter product will in later steps enable us to make use of results on Dirichlet's L-function.

However, to begin with, we relate $\rho_m(p)$ to the less general Legendre symbol.

Recall that

$$\rho_m(p) = \left|\left\{n \bmod p:\ n^2\left[\frac{am+d}{s_m}\right]+n\left[\frac{bm+e}{s_m}\right]+\left[\frac{cm+f}{s_m}\right]\equiv 0 \bmod p\right\}\right| .$$

For $p<z$, and assuming $(s_m, \prod_{p<z} p)=1$,

$$\rho_m(p) = \left|\{n \bmod p:\ n^2(am+b)+n(bm+e)+(cm+f)\equiv 0 \bmod p\}\right| .$$

If $p\nmid 2(am+d)$ then

$$n^2(am+d)+n(bm+e)+(cm+f)\equiv 0 \bmod p$$

if and only if

$$4n^2(am+d)^2+4n(bm+e)(am+d)+4(cm+f)(am+d)\equiv 0 \bmod p$$

i.e. if and only if

$$(2n(am+d)+(bm+e))^2-(bm+e)^2+4(cm+f)(am+d)\equiv 0 \bmod p.$$

But the integers

$$\{\ 2n(am+d)+(bm+e);\ 1\leq n\leq p\ \}$$

form an incongruent set of residues (mod p) and it follows that

$$\rho_m(p) = \left|\{s \bmod p:\ s^2\equiv(bm+e)^2-4(cm+f)(am+d) \bmod p\ \}\right|$$

$$= \left|\{s \bmod p:\ s^2\equiv(b^2-4ac)m^2+2(be-2cd-2fa)m+(e^2-4fd) \bmod p\ \}\right|. \tag{17}$$

If however $p|(am+d)$, $p<z$ and assuming that m is "z appropriate" we have

$$\rho_m(p) = \begin{cases} 1 & ;\ p\nmid(bm+e) \\ 0 & ;\ p|(bm+e) \end{cases} \tag{18}$$

and if $p=2$,

$$\rho_m(p) = \begin{cases} 1 & ;\ 2\nmid(am+d+bm+e) \\ 0 & ;\ 2|(am+d+bm+e) \end{cases} \tag{19}$$

Defining g_m as

$$g_m = (b^2-4ac)m^2+2(be-2cd-2fa)m+(e^2-4fd),$$

(17), (18), and (19) give for m "z appropriate", and $p<z$,

$$\rho_m(p) = \begin{cases} (g_m/_p) + 1 & ;\ p\nmid 2(am+d) \\ 1 & ;\ p|2(am+d)\ \&\ p\nmid(am+d+bm+e) \\ 0 & ;\ p|2(am+d)\ \&\ p|(am+d+bm+e) \end{cases} \tag{20}$$

where $(\cdot/_p)$ denotes the Legendre symbol.

Elaborating on the comments at the start of Step Three, and writing $g_m = r^2s$, where s is square-free and $s\neq 1$, the Legendre symbol, $(g_m/_p)$, may be reduced to the Kronecker symbol

$(^s/_p)$ or $(^{4s}/_p)$, and consequently the product $\prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{\rho_m(p)}{p})$ may be related to the Dirichlet L-function, $L(1,\chi)$, as will be demonstrated in Steps Five onwards.

However, we firstly deal with the case of g_m a square. ie where s=1.

<u>STEP FOUR</u> g_m *a square.*

We firstly show that g_m cannot be a complete square given our assumption that an^2+bn+c and dn^2+en+f have no common linear factor with integer coefficients.

If it were the case then, as

$$g_m = (b^2-4ac)m^2+2(be-2cd-2fa)m+(e^2-4fd)$$

it would follow that both b^2-4ac and e^2-4fd were squares and that

$$(be-2cd-2fa)^2 = (b^2-4ac)(e^2-4fd). \qquad (21)$$

Now, for b^2-4ac a square, an^2+bn+c may be written as the product of two linear polynomials with integer coefficients

$$an^2+bn+c = (An+B)(Cn+D) \qquad (22)$$

say.

[It is not immediately apparent that A,B,C,D are all integers but it is clear that if $an^2+bn+c=h(a_1n^2+b_1n+c_1)$ with $(a_1,b_1,c_1)=1$, and $a_1n^2+b_1n+c_1$ the product of two linear polynomials, then we may write

$$a_1n^2+b_1n+c_1 = (\alpha n+\beta)\left[\frac{\gamma_1}{\gamma_2}n + \frac{\delta_1}{\delta_2}\right]$$

say, with $\alpha,\beta,\gamma_1,\gamma_2,\delta_1,\delta_2$ integers, $(\alpha,\beta)=1$, $(\gamma_1,\gamma_2)=1$ and $(\delta_1,\delta_2)=1$. Assuming that γ_2 and δ_2 are not both 1, we have

$$a_1 = \alpha\frac{\gamma_1}{\gamma_2}, \quad b_1 = \alpha\frac{\delta_1}{\delta_2} + \beta\frac{\gamma_1}{\gamma_2}, \quad c_1 = \beta\frac{\delta_1}{\delta_2}$$

and certainly $\gamma_2|\alpha$ and $\delta_2|\beta$.

For b_1 an integer either $\delta_2|\alpha$ and $\gamma_2|\beta$ or $(\gamma_2,\delta_2)>1$. However if $\delta_2|\alpha$ and $\gamma_2|\beta$ then $(\alpha,\beta)\neq 1$ contradicting our assumption above. This leaves the possibility that $(\gamma_2,\delta_2)>1$, for which we have the same objection.

So γ_2 and δ_2 are both 1 as required.]

For e^2-4fd a square, dn^2+en+f may also be written as the product of two linear factors with integer coefficients,

$$dn^2+en+f = (En+F)(Gn+H) \tag{23}$$

say.

(22) and (23) substituted into (21) give

$$(AD+BC)(EH+FG)-2(BDEG+FHAC) = (AD-BC)(EH-FG)$$

$$\text{i.e.} \quad AF(DG-HC) = BE(DG-HC).$$

For this to occur, either we must have $DG=HC$ implying $D/H = C/G$ or $AF=BE$ implying $A/E = B/F$. Either case would contradict our assumption that an^2+bn+c and dn^2+en+f have no common linear factor with integer coefficients.

For g_m a square, from (20),

$$\rho_m(p) = \begin{cases} 2 \;; & p\nmid 2(am+d)g_m \\ 1 \;; & p\nmid 2(am+d) \;\&\; p|g_m \;\; \underline{\text{or}} \\ & p|2(am+d) \;\&\; p\nmid(am+d+bm+e) \\ 0 \;; & \text{otherwise} \end{cases} \tag{24}$$

and

$$\prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \prod_{\substack{p<z \\ p\nmid k_1 \\ p\nmid 2(am+d)g_m}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z \\ p\nmid k_1 \\ p\nmid 2(am+d) \\ p|g_m}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_1 \\ p|2(am+d) \\ p\nmid(am+d+bm+e)}} \left(1-\frac{1}{p}\right)$$

$$\leqslant \prod_{\substack{p<z \\ p\nmid k_1 \\ p\nmid(am+d)g_m}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_1 \\ p\nmid(am+d) \\ p|g_m}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_1 \\ p|(am+d) \\ p\nmid(am+d+bm+e)}} \left(1-\frac{1}{p}\right)$$

$$= \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_1 \\ p|(am+d) \\ p|(bm+e)}} \left(1-\frac{1}{p}\right)^{-1}$$

$$\ll \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p|(am+d)}} \left(1-\frac{1}{p}\right)^{-1} .$$

It follows that

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \mathrm{app} \\ g_m\ \mathrm{a\ square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \ll \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \mathrm{app} \\ g_m\ \mathrm{a\ square}}} \ln\ln(|am+d|).$$

Now $\max_{0<m\leq y} |am+d| \leq |a|y+|d|$ and $|a|y+|d| \leq 2|a|y$ if $|d|/|a| \leq y$.

So assuming $|d|/|a| \leq y$, we have

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \mathrm{app} \\ g_m\ \mathrm{a\ square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \ll \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) \ln\ln(|a|y) \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \mathrm{app} \\ g_m\ \mathrm{a\ square}}} 1 .$$

(The assumption $|d|/|a|\leq y$ will be clarified in Step 13 as will any subsequent assumption on the size of y.)

From Lemma 4 we have

$$\sum_{\substack{0<m\leq y \\ g_m\ \mathrm{a\ square}}} 1 \ll \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{|\zeta|}\right]$$

where $\zeta=b^2-4ac$, $\eta=be-2cd-2fa$, and $\theta=e^2-4fd$, assuming that $y\geq|\zeta|$.

(It is clear that $\eta^2-\zeta\theta\equiv 0 \bmod 4$ as is required for the application of the lemma.)

So

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\text{ app} \\ g_m \text{ a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \ll \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\ln(|a|y)\ln\left[\frac{y}{|\zeta|}\right] . \tag{25}$$

Note that we have nowhere made any assumption about the size of z, so (25) holds for all z.

Substitution of (25) into (15) gives

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\text{ app} \\ g_m \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \frac{k_1 y}{k_2[k_1,Fk_2]} \prod_{p<z} \left(1-\frac{1}{p}\right) H_z \left\{ 1 \right.$$

$$+ \; O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))$$

$$\left. + \; O\left[\tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_2[k_1,Fk_2]}{\varphi(k_1)} \frac{\ln\ln(|a|y)\ \ln({}^{y}/{}_{|\zeta|})}{y} \right] \right\} \tag{26}$$

for $2 \leq z \leq \exp(27(\ln {}^{y}/k_2)^{\frac{1}{2}})$.

This completes Step Four.

<u>STEP FIVE</u> *Reduction of* $\prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$.

Let $(\cdot/_p)$ denote the Legendre symbol. Recalling (20), we have for m "z appropriate",

$$\rho_m(p) = \begin{cases} ({}^{g_m}/_p) + 1 & ;\ p\nmid 2(am+d) \\ 1 & ;\ p\,|\,2(am+d)\ \&\ p\nmid(am+d+bm+e) \\ 0 & ;\ p\,|\,2(am+d)\ \&\ p\,|\,(am+d+bm+e) \end{cases}$$

giving

$$\prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \prod_{\substack{p<z \\ p\nmid 2(am+d)k_1}} \left(1-\frac{({}^{g_m}/_p)+1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_1 \\ p|2(am+d) \\ p\nmid(am+d+bm+e)}} \left(1-\frac{1}{p}\right) . \tag{27}$$

The aim of this step is to rewrite this product to involve the products $\prod_{\substack{p<z\\ p\nmid k_1}}(1-\frac{1}{p})$ and $\prod_{p<z}(1-\frac{\chi(p)}{p})$ where $\chi(p)$ denotes the Kronecker symbol as described below.

Firstly we note that

$$\prod_{\substack{p<z\\ p\nmid 2(am+d)k_1}} \left(1-\frac{(g_m/p)+1}{p}\right)$$

$$= \prod_{\substack{p<z\\ p\nmid 2(am+d)k_1}} \left(\frac{p-1}{p}\right) \prod_{\substack{p<z\\ p\nmid 2(am+d)k_1}} \left(\frac{p-(g_m/p)}{p}\right) \prod_{\substack{p<z\\ p\nmid 2(am+d)k_1}} \left[\frac{p^2-((g_m/p)+1)p)}{p^2-((g_m/p)+1)p+(g_m/p)}\right]$$

$$= \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z\\ p\nmid k_1\\ p\mid 2(am+d)}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z\\ p\nmid 2(am+d)k_1}} \left(1-\frac{(g_m/p)}{p}\right) \times$$

$$\prod_{\substack{p<z\\ p\nmid 2(am+d)k_1}} \left[\frac{p^2-((g_m/p)+1)p}{p^2-((g_m/p)+1)p+(g_m/p)}\right]. \tag{28}$$

Let $\chi(n) = \chi_D(n)$ denote the Kronecker symbol $(^D/_n)$, where if $g_m = r^2s$, for s square-free and not equal to 1, D=4s or s as $s\not\equiv 1$ mod 4 and $s\equiv 1$ mod 4 respectively.

For $g_m = r^2s$ the Legendre symbol $(^{g_m}/_p)=(^s/_p)$ if $p\nmid r$. If in addition $p\nmid 2s$ then the Legendre symbol $(^s/_p)$ is the Kronecker symbol $(^s/_p)$ and further $(^s/_p)=(^{4s}/_p)$.

So for $p\nmid 2g_m$

'the Legendre symbol $(^{g_m}/_p)=(^D/_p)$ the Kronecker symbol'.

Applying this to (28) gives

$$\prod_{\substack{p<z\\ p\nmid 2(am+d)k_1}} \left(1-\frac{(g_m/p)+1}{p}\right) = \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z\\ p\nmid k_1\\ p\mid 2(am+d)}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z\\ p\nmid 2(am+d)g_m k_1}} \left(1-\frac{\chi(p)}{p}\right)$$

$$\times \prod_{\substack{p<z\\ p\nmid 2(am+d)g_m k_1}} \left[1-\frac{\chi(p)}{p^2-(\chi(p)+1)p+\chi(p)}\right]$$

$$= \prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{1}{p}) \prod_{p<z} (1-\frac{\chi(p)}{p}) \prod_{\substack{p<z \\ p\nmid k_1 \\ p|2(am+d)}} (1-\frac{1}{p})^{-1} \prod_{\substack{p<z \\ p|2(am+d)g_m k_1}} (1-\frac{\chi(p)}{p})^{-1}$$

$$\times \prod_{\substack{p<z \\ p\nmid 2(am+d)g_m k_1}} \left[1-\frac{\chi(p)}{p^2-(\chi(p)+1)p+\chi(p)} \right] .$$

This on substitution into (27) gives

$$\prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{\rho_m(p)}{p}) = \prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{1}{p}) \prod_{p<z} (1-\frac{\chi(p)}{p}) \quad c(g_m,z)$$

where

$$c(g_m,z) = \prod_{\substack{p<z \\ p\nmid k_1 \\ p|2(am+d) \\ p\nmid(am+d+bm+e)}} (1-\frac{1}{p}) \prod_{\substack{p<z \\ p\nmid k_1 \\ p|2(am+d)}} (1-\frac{1}{p})^{-1} \prod_{\substack{p<z \\ p|2(am+d)g_m k_1}} (1-\frac{\chi(p)}{p})^{-1}$$

$$\times \prod_{\substack{p<z \\ p\nmid 2(am+d)g_m k_1}} \left[1-\frac{\chi(p)}{p^2-(\chi(p)+1)p+\chi(p)}\right] .$$

So

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z) \text{ app} \\ g_m \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{\rho_m(p)}{p}) = \prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{1}{p})$$

$$\times \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z) \text{ app} \\ g_m \text{ not a square}}} \prod_{p<z} (1-\frac{\chi(p)}{p}) \quad c(g_m,z) . \tag{29}$$

Equation (26) together with (29) gives

$$\prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{1}{p}) \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \mathrm{app} \\ g_m\ \text{not a square}}} \prod_{p<z} (1-\frac{\chi(p)}{p})\ c(g_m,z) = \frac{k_1 y}{k_2[k_1,Fk_2]} \prod_{p<z} (1-\frac{1}{p})$$

$$\times H_z \left\{1 + O(\exp(-\ln y/k_2)^{\frac{1}{2}}) + O\left[\tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_2[k_1,Fk_2]}{\varphi(k_1)} \frac{\ln\ln(|a|y)\ \ln(y/|\zeta|)}{y}\right]\right\}$$

for $z \leq \exp(27(\ln y/k_2)^{\frac{1}{2}})$

or

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \mathrm{app} \\ g_m\ \text{not a square}}} \prod_{p<z} (1-\frac{\chi(p)}{p})\ c(g_m,z) = \frac{k_1 y}{k_2[k_1,Fk_2]} \prod_{\substack{p<z \\ p\mid k_1}} (1-\frac{1}{p})$$

$$\times H_z \left\{1 + O(\exp(-\ln y/k_2)^{\frac{1}{2}}) + O\left[\tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_2[k_1,Fk_2]}{\varphi(k_1)} \frac{\ln\ln(|a|y)\ \ln(y/|\zeta|)}{y}\right]\right\} \tag{30}$$

for $z \leq \exp(27(\ln y/k_2)^{\frac{1}{2}})$.

In particular, writing $z_0 = \exp(27(\ln y/k_2)^{\frac{1}{2}})$, we have

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z_0)\ \mathrm{app} \\ g_m\ \text{not a square}}} \prod_{p<z_0} (1-\frac{\chi(p)}{p})\ c(g_m,z_0) = \frac{k_1 y}{k_2[k_1,Fk_2]} \prod_{\substack{p<z_0 \\ p\mid k_1}} (1-\frac{1}{p})$$

$$\times H_{z_0} \left\{1 + O(\exp(-\ln y/k_2)^{\frac{1}{2}}) + O\left[\tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_2[k_1,Fk_2]}{\varphi(k_1)} \frac{\ln\ln(|a|y)\ \ln(y/|\zeta|)}{y}\right]\right\}. \tag{31}$$

To obtain an asymptotic formula for

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\text{ app} \\ g_m \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right).$$

for all z, it is clearly sufficient for us to show that, whenever $z\geq z_0$,

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\text{ app} \\ g_m \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m,z)$$

$$= \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z_0)\text{ app} \\ g_m \text{ not a square}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m,z_0) \left\{ 1 + 0\left[\frac{1}{T(y)}\right] \right\} \tag{32}$$

for some function $T(y)>1$.

Our first step in this direction is to remove the dependence on z of the conditions under the summation sign of (31). We recall that the condition, (m,z) app, applies to those m for which

(i) $(s_m, \prod_{p<z} p)=1$

(ii) $(am+d+bm+e, cm+f)\equiv 1 \bmod 2$

(iii) $\left(\ell_1^2\left[\frac{am+d}{s_m}\right]+\ell_1\left[\frac{bm+e}{s_m}\right]+\left[\frac{cm+f}{s_m}\right], \prod_{\substack{p<z \\ p\mid k_1}} p \right)=1$

where $s_m = (am+d, bm+e, cm+f)$.

For $z>\exp(27(\ln y/k_2)^{\frac{1}{2}})$, we may assume that y is large enough to ensure $z>k_1$. (See Step 13.) So condition (iii) is satisfied if and only if

$$\left(\ell_1^2\left[\frac{am+d}{s_m}\right]+\ell_1\left[\frac{bm+e}{s_m}\right]+\left[\frac{cm+f}{s_m}\right], k_1 \right)=1.$$

For condition (i) to be satisfied either s_m must be 1 or s_m must have smallest prime factor greater than or equal to z. If we assume that y is large, with z consequently large, and satisfying

$$z > \max\{ |d|,|e|,|f|,|bd-ea|,|dc-fa|,|ec-fb| \}$$

then it is a simple matter to show that the latter case cannot arise.

For if either a, b or c equal zero then s_m must divide either $|d|$, $|e|$, or $|f|$ respectively. If neither a, b, or c equal zero then

$$(am+d)b-(bm+e)a \equiv 0 \mod s_m$$

i.e. $db-ea \equiv 0 \mod s_m$.

If $db-ea=0$ we have instead

$$(am+d)c-(cm+f)a \equiv 0 \mod s_m$$

i.e. $dc-fa \equiv 0 \mod s_m$

and if both $db-ea$ and $dc-fa$ are equal to zero then we have

$$(bm+e)c-(cm+f)b \equiv 0 \mod s_m$$

i.e. $ec-fb \equiv 0 \mod s_m$.

The situation $db-ea = dc-fa = ec-fb = 0$ cannot arise for otherwise $a/b = d/e$, $a/c = d/f$ and $c/b = f/e$ a situation implying that $am+d$, $bm+e$, and $cm+f$ are constant multiples of each other, a position contradictory to our assumptions about $S(x,y,z)$.

Hence assuming that

$$z > \max\{ |d|,|e|,|f|,|bd-ea|,|dc-fa|,|ec-fb| \}$$

ensures that condition (i) is satisfied whenever $(am+d,bm+e,cm+f)=1$, and that the only possible s_m is $s_m=1$.

Consistent with our previous notation we term the integers m satisfying the conditions

(i) $(am+d,bm+e,cm+f)=1$

(ii) $(am+d+bm+e,cm+f) \equiv 1 \mod 2$

(iii) $(\ell_1^2(am+d)+\ell_1(bm+e)+(cm+f),\ k_1)=1$

as "m appropriate".

We may now write

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \text{app}\\ g_m \text{ not a square}}} \prod_{p<z}\left(1-\frac{\chi(p)}{p}\right) c(g_m,z) = \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m\ \text{app}\\ g_m \text{ not a square}}} \prod_{p<z}\left(1-\frac{\chi(p)}{p}\right) c(g_m,z) \tag{33}$$

for

$$z \geq \exp(27(\ln y/k_2)^{\frac{1}{2}}) . \tag{34}$$

Having dealt with m "z appropriate" we now turn to the product $c(g_m,z)$.

Define $c(g_m)$ as

$$c(g_m) = \prod_{\substack{p\nmid k_1\\ p\mid 2(am+d)\\ p\nmid (am+d+bm+e)}}\left(1-\frac{1}{p}\right) \prod_{\substack{p\nmid k_1\\ p\mid 2(am+d)}}\left(1-\frac{1}{p}\right)^{-1} \prod_{p\mid 2(am+d)g_m k_1}\left(1-\frac{\chi(p)}{p}\right)^{-1}$$

$$\times \prod_{p\nmid 2(am+d)g_m k_1}\left[1-\frac{\chi(p)}{p^2-(\chi(p)+1)p+\chi(p)}\right] .$$

Then, for $z\geq\exp(27(\ln y/k_2)^{\frac{1}{2}})$, we have

$$c(g_m,z) = c(g_m)\{ 1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) \} . \tag{35}$$

Owing to the length of the proof of (35), we write it as a seperate step.

<u>STEP SIX</u> *Proof of statement* (35).

Clearly, for $z>k_1$,

$$\frac{c(g_m,z)}{c(g_m)} = \prod_{\substack{p\geq z\\ p\mid (am+d)\\ p\nmid (bm+e)}}\left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p\geq z\\ p\mid (am+d)}}\left(1-\frac{1}{p}\right) \prod_{\substack{p\geq z\\ p\mid (am+d)g_m}}\left(1-\frac{\chi(p)}{p}\right)$$

$$\times \prod_{\substack{p\geq z\\ p\mid (am+d)g_m}}\left[1-\frac{\chi(p)}{p^2-(\chi(p)+1)p+\chi(p)}\right]^{-1}$$

$$= \prod_{\substack{p>z \\ p|(am+d) \\ p|(bm+e)}} (1-\frac{1}{p}) \prod_{\substack{p>z \\ p|(am+d)g_m}} (1-\frac{\chi(p)}{p}) \prod_{\substack{p>z \\ p|(am+d)g_m}} \left[1-\frac{\chi(p)}{p^2-(\chi(p)+1)p+\chi(p)}\right]^{-1}$$

$= T_1T_2T_3$ say.

We deal with each product T_i $(i=1,2,3)$ in turn and show in each case that

$$T_i = 1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) \qquad (36)$$

whenever $z > \exp(27(\ln y/k_2)^{\frac{1}{2}})$, and y is large.

(I) $T_1 = \prod_{\substack{p>z \\ p|(am+d) \\ p|(bm+e)}} (1-\frac{1}{p})$

If either a or b are zero then it is apparent that $T_1=1$, for $z > \max\{ |d|,|e|,|f|,|bd-ea|,|dc-fa|,|ec-fb| \}$, a condition stipulated in Step Five.

If neither a or b are zero then $(am+d)\equiv 0 \bmod p$ and $(bm+e)\equiv 0 \bmod p$ together imply

$$(am+d)b-(bm+e)a\equiv 0 \bmod p$$

i.e. $\qquad db-ea\equiv 0 \bmod p.$

If $db-ea\neq 0$ it again follows that $T_1=1$.

This leaves only the possibility that $db-ea=0$. If, however, $db=ea$ then $b/a = e/d = \gamma$ say and T_1 becomes

$$T_1 = \prod_{\substack{p>z \\ p|(am+d,\gamma(am+d))}} (1-\frac{1}{p}) = \prod_{\substack{p>z \\ p|(am+d)}} (1-\frac{1}{p})$$

Taking logarithms of both sides we have

$$\ln T_1 = \sum_{\substack{p>z \\ p|(am+d)}} \ln (1-\frac{1}{p})$$

and consequently

$$- \sum_{\substack{p>z \\ p|(am+d)}} \frac{1}{p-1} < \ln T_1 < - \sum_{\substack{p>z \\ p|(am+d)}} \frac{1}{p}$$

giving

$$\frac{-\ \omega(am+d)}{z-1} < \ln T_1 < 0$$

where $\omega(A)$ denotes the number of prime divisors of A.

Now, for any integer A, $\omega(A) \leqslant 2\ln|A|$. So

$$- \frac{2\ln|am+d|}{z-1} < \ln T_1 < 0$$

and

$$\exp\left\{- \frac{2\ln|am+d|}{z-1}\right\} < T_1 < 1.$$

But

$$\exp\left\{\frac{2\ln|am+d|}{z-1}\right\} < \exp\left\{\frac{2\ln|am+d|}{\exp(27(\ln y/k_2)^{\frac{1}{2}})-1}\right\}$$

$$< \exp\left\{\frac{1}{\exp((\ln y/k_2)^{\frac{1}{2}})}\right\}$$

whenever $\ln|am+d| < \exp(26(\ln y/k_2)^{\frac{1}{2}})$, which we take to be the case.

As $\exp\left\{\frac{1}{\exp((\ln y/k_2)^{\frac{1}{2}})}\right\} = 1 + 0(\exp(-\ln y/k_2)^{\frac{1}{2}})$

we have

$$T_1 = 1 + 0(\exp(-\ln y/k_2)^{\frac{1}{2}}) \qquad (37)$$

as required.

(II) $T_2 = \prod_{\substack{p>z \\ p|(am+d)g_m}} (1-\frac{\chi(p)}{p})$

Clearly

$$\prod_{\substack{p>z \\ p|(am+d)g_m}} (1-\frac{1}{p}) < T_2 < \prod_{\substack{p>z \\ p|(am+d)g_m}} (1+\frac{1}{p})$$

Following the argument for T_1 we have

$$-\sum_{\substack{p>z \\ p|(am+d)g_m}} \frac{1}{p-1} < \ln T_2 < \sum_{\substack{p>z \\ p|(am+d)g_m}} \frac{1}{p}$$

Again, assuming that y is large enough for

$$\ln (|am+d||g_m|) < \exp(26(\ln y/k_2)^{\frac{1}{2}})$$

to hold, gives

$$T_2 = 1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})). \qquad (38)$$

(III) $$T_3 = \prod_{\substack{p>z \\ p|(am+d)g_m}} \left[1 - \frac{\chi(p)}{p^2-(\chi(p)+1)p+\chi(p)} \right]^{-1}$$

It is a simple matter to show that

$$\prod_{\substack{p>z \\ p|(am+d)g_m}} (1-\frac{1}{p^2}) < T_3 < \prod_{\substack{p>z \\ p|(am+d)g_m}} (1 + \frac{1}{p(p-2)}) .$$

or

$$\prod_{p>z} (1 - \frac{1}{p^2}) < T_3 < \prod_{p>z} (1 + \frac{1}{p(p-2)}) .$$

Following the argument for T_1 again we get

$$-\sum_{p>z} \frac{1}{p^2-1} < \ln T_3 < \sum_{p>z} \frac{1}{p(p-2)}$$

and

$$-\frac{1}{2} \sum_{n>z} \frac{1}{n^2} < \ln T_3 < 2 \sum_{n>z} \frac{1}{n^2} .$$

As $\sum_{n>z} \frac{1}{n^2} = O\left[\frac{1}{z}\right]$

we have

$$T_3 = 1 + O(^1/_z)$$

$$= 1 + O(\exp(-27(\ln {}^y/k_2)^{\frac{1}{2}})$$

$$= 1 + O(\exp(-(\ln {}^y/k_2)^{\frac{1}{2}}) \qquad (39)$$

A combination of (37), (38) and (39) give, as required,

$$c(g_m,z) = c(g_m)(1 + O(\exp(-(\ln {}^y/k_2)^{\frac{1}{2}})))$$

whenever

$$z > \exp(27(\ln {}^y/k_2)^{\frac{1}{2}})$$

for y large.

In passing we note that, as

$$\prod_{p\nmid 2(am+d)g_m k_1} \left[1 - \frac{\chi(p)}{p^2-(\chi(p)+1)p+\chi(p)}\right]$$

is absolutely convergent, there exist constants c_1 and c_2 such that

$$c_1 \prod_{\substack{p\nmid k_1 \\ p\mid 2(am+d) \\ p\mid (am+d+bm+e)}} (1 - \frac{1}{p})^{-1} \prod_{p\mid 2(am+d)g_m k_1} (1 - \frac{\chi(p)}{p})^{-1} < c(g_m) <$$

$$c_2 \prod_{\substack{p\nmid k_1 \\ p\mid 2(am+d) \\ p\mid (am+d+bm+e)}} (1 - \frac{1}{p})^{-1} \prod_{p\mid 2(am+d)g_m k_1} (1 - \frac{\chi(p)}{p})^{-1} . \qquad (40)$$

This completes Step Six.

<u>STEP SEVEN</u> *Continuation of Step Five.*

Recalling (33) and having now proved (35), we have

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \text{app} \\ g_m\ \text{not a square}}} \prod_{p<z} \left(1 - \frac{\chi(p)}{p}\right) c(g_m,z)$$

$$= \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m\ \text{app} \\ g_m\ \text{not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) \left\{1+0(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))\right\} \qquad (41)$$

for $z > \exp(27(\ln^{y}/k_2)^{\frac{1}{2}})$.

We have now reduced the problem such that to complete the theorem, we have only to derive an asymptotic equation of the form

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m\ \text{app} \\ g_m\ \text{not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m)$$

$$= \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m\ \text{app} \\ g_m\ \text{not a square}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m)\left\{ 1 + 0\left[\frac{1}{T(y)}\right] \right\} \qquad (42)$$

for $z>z_0$, and for some function $T(y)\geq 1$.

Now Lemma 5.1 allows us to write $\prod_{p<z} (1-\frac{\chi(p)}{p})$ in terms of $\prod_{p<z_0} (1-\frac{\chi(p)}{p})$ for at least some of the primitive characters if indeed our $\chi(p)$ are primitive characters. Recall from Step Five that $\chi(p)$ denotes the Kronecker symbol $({}^{D}/_{n})$, where if $g_m=r^2s$, $D=4s$ or s as $s\not\equiv 1 \bmod 4$ and $s\equiv 1 \bmod 4$ respectively. It is well known (see for example Davenport [9]) that the quadratic field $\mathbb{Q}(\sqrt{(g_m)})$ has discriminant D and that $\chi(n)$ is a primitive character mod$|D|$.

In line with the results of Lemma 5.1 we split the discriminants, D, into two groups; those that are exceptions in the sense of Lemma 5.1 we denote "bad" D, the rest "good" D. Further, an integer g_m will be called "bad" if it gives rise to a "bad" D as explained above. Otherwise g_m will be called "good".

Clearly $|D| \leq 4 \max_{0<m\leq y} |g_m|$.

Taking $\alpha = 27$ in Lemma 5.1 and writing $Q = 4 \max_{0<m\leq y} |g_m|$, we have, for $s \geq \ln^{54} Q$, and $z \geq s$

$$\prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)^{-1} = \prod_{p\leq s} \left(1-\frac{\chi(p)}{p}\right)^{-1} \left\{1 + O(s^{-1/27}) + O\left(\left(\max_{0<m\leq y} |g_m|\right)^{-1/9}\right)\right\}$$

with at most $O((\max_{0<m\leq y} |g_m|)^{1/3})$ exceptions.

From Lemma 6, assuming that y is large enough to satisfy the condition

$$y > \begin{cases} \dfrac{-|\eta| + M_1^{\frac{1}{2}}}{|\zeta|} & ; M_1 \geq 0 \\ 0 & ; M_1 < 0 \end{cases}$$

where $g_m = \zeta m^2 + 2\eta m + \theta$ with $\zeta = b^2 - 4ac$, $\eta = be - 2cd - 2fa$ and $\theta = e^2 - 4fd$; where $M_1 = \eta^2 - \zeta\theta + |\zeta| M$; and where $M = \max(|\zeta + 2\eta + \theta|, |-\eta^2 - \zeta\theta|)$; and the condition

$$y > \max\left\{ \frac{5|\eta|}{|\zeta|}, \frac{3|\theta|}{|\zeta|} \right\}$$

we have

$$\frac{y^2}{2} \leq \max_{0<m\leq y} |g_m| \leq 4|\zeta| y^2.$$

So

$$\prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)^{-1} = \prod_{p<s} \left(1-\frac{\chi(p)}{p}\right)^{-1} \left\{ 1 + O(s^{-1/27}) + O(y^{-2/9}) \right\}$$

with at most $O(|\zeta|^{1/3} y^{2/3})$ exceptions.

Taking $s = z_0 = \exp(27(\ln y/k_2)^{\frac{1}{2}})$, and assuming

$$\exp(27(\ln y/k_2)^{\frac{1}{2}}) > \ln^{54}(\max_{0<m\leq y}|g_m|)$$

we have

$$\prod_{p<z}\left(1-\frac{\chi(p)}{p}\right)^{-1} = \prod_{p<z_0}\left(1-\frac{\chi(p)}{p}\right)^{-1}\left\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) + O(y^{-2/9})\right\}$$

$$= \prod_{p<z_0}\left(1-\frac{\chi(p)}{p}\right)^{-1}\left\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\right\}$$

with at most $O(|\zeta|^{1/3}y^{2/3})$ exceptions, and, (by Lemma 3),

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ good}}} \prod_{p<z}\left(1-\frac{\chi(p)}{p}\right)c(g_m) = \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ good}}} \prod_{p<z_0}\left(1-\frac{\chi(p)}{p}\right)c(g_m)\left\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\right\}. \qquad (43)$$

This is some way towards the asymptotic formula required.

Further

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square}}} \prod_{p<z_0}\left(1-\frac{\chi(p)}{p}\right)c(g_m) = \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ good}}} \prod_{p<z_0}\left(1-\frac{\chi(p)}{p}\right)c(g_m) + \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{p<z_0}\left(1-\frac{\chi(p)}{p}\right)c(g_m)$$

To find $\displaystyle\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ good}}} \prod_{p<z_0}\left(1-\frac{\chi(p)}{p}\right)c(g_m)$ in terms of

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m\ \mathrm{app}\\ g_m\ \text{not a square}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right)^{c(g_m)}$$ which we have previously gained

information on, we require an upper bound on the sum

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m\ \mathrm{app}\\ g_m\ \text{not a square}\\ g_m\ \text{bad}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right)^{c(g_m)} .$$

In Step Five we had

$$\prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{\rho_m(p)}{p}\right) = \prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{1}{p}\right) \prod_{p<z}\left(1-\frac{\chi(p)}{p}\right)^{c(g_m)}\{1+O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\}$$

for $z\geq z_0$.

By Lemma 3 this implies

$$\prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{1}{p}\right) \prod_{p<z}\left(1-\frac{\chi(p)}{p}\right)^{c(g_m)} = \prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{\rho_m(p)}{p}\right)\{1+O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\}$$

and

$$\prod_{p<z}\left(1-\frac{\chi(p)}{p}\right)^{c(g_m)} = \prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{\rho_m(p)}{p}\right)\{1+ O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\} \quad (44)$$

for $z\geq z_0$.

Hence

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m\ \mathrm{app}\\ g_m\ \text{not a square}\\ g_m\ \text{bad}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right)^{c(g_m)} \ll \prod_{\substack{p<z_0\\ p\nmid k_1}}\left(1-\frac{1}{p}\right)^{-1} \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m\ \mathrm{app}\\ g_m\ \text{not a square}\\ g_m\ \text{bad}}} \prod_{\substack{p<z_0\\ p\nmid k_1}}\left(1-\frac{\rho_m(p)}{p}\right). \quad (45)$$

The next four steps are devoted to finding an upper bound for the sum

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$$

for any $z>z_0$.

STEP EIGHT *The sum* $\displaystyle\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$ *for*

$$\exp\left(27(\ln y/k_2)^{\frac{1}{2}}\right) \leq z \leq \exp\left(y^{1/7}\right).$$

As $\rho_m(p)$ is always greater than or equal to zero we have the rather crude upper bound

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \leq \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} 1 .$$

But

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} 1 \ll \sum_{\substack{0<m\leq y \\ g_m \text{ not a square} \\ g_m \text{ bad}}} 1 \ll \sum_{\substack{s \\ |s|\ll (|\zeta|^{1/3}y^{2/3})}} \sum_{\substack{0<m\leq y \\ g_m=r^2 s}} 1$$

From Lemma 4,

$$\sum_{\substack{0<m\leqslant y \\ g_m=r^2s}} 1 \ll \begin{cases} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{\zeta s}\right] & ; \ |\zeta s|<A,\ \zeta s>0 \text{ and } \zeta s \text{ not a perfect square} \\ \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] & ; \text{ otherwise} \end{cases}$$

where $\zeta=b^2-4ac$, $\eta=bd-2cd-2fa$, and $\theta=e^2-4fd$.

We certainly have

$$\sum_{\substack{0<m\leqslant y \\ g_m=r^2s}} 1 \ll \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{|\zeta|}\right]$$

and consequently

$$\sum_{\substack{0<m\leqslant y \\ m\equiv\ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} 1 \ll \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \sum_{|s|\leqslant|\zeta|^{1/3}y^{2/3}} \ln\left[\frac{y}{|\zeta|}\right]$$

$$\ll \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{|\zeta|}\right] |\zeta|^{1/3}y^{2/3}.$$

So

$$\sum_{\substack{0<m\leqslant y \\ m\equiv\ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \ll \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{|\zeta|}\right] |\zeta|^{1/3}y^{2/3}$$

and

$$\sum_{\substack{0<m\leqslant y \\ m\equiv\ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)c(g_m) \ll \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right)^{-1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{|\zeta|}\right] |\zeta|^{1/3}y^{2/3}$$

$$\ll \frac{\varphi(k_1)}{k_1}(\ln z)\tau\left[\frac{\eta^2-\zeta\theta}{4}\right](\ln y)|\zeta|^{1/3}y^{2/3}.$$

For $\exp(27(\ln y/k_2)^{\frac{1}{2}}) < z < \exp(y^{1/7})$ we note that this gives

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) \ll \frac{\varphi(k_1)}{k_1}\, \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{5/6}. \tag{46}$$

This gives us, for $\exp(27(\ln y/k_2)^{\frac{1}{2}}) \leq z \leq \exp(y^{1/7})$,

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) = \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ good}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) \tag{47}$$

$$+ O\left\{ \frac{\varphi(k_1)}{k_1}\, \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{5/6}\right\}.$$

By (43) this gives ,

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) = \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ good}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m) \left\{ 1\right.$$

$$\left. + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\right\} + O\left\{ \frac{\varphi(k_1)}{k_1}\, \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{5/6}\right\} \tag{48}$$

for $z_0 < z \leq \exp(y^{1/7})$.

But, by similar reasoning,

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ good}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m) = \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m)$$

$$+ O\left\{ \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ bad}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m) \right\}$$

$$= \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m) + O\left\{ \frac{\varphi(k_1)}{k_1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{5/6} \right\}.$$

Comparing this with (31) which reads, in the light of (33) and (35),

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} H_{z_0}\Big\{1 + O(\exp(-\ln y/k_2)^{\frac{1}{2}}))$$

$$+ O\left\{ \frac{k_2[k_1,Fk_2]}{\varphi(k_1)} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{\ln\ln(|a|y)\ \ln y}{y} \right\} \Big\}$$

we have

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ good}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} H_{z_0}\Big\{1 + O(\exp(-\ln y/k_2)^{\frac{1}{2}}))$$

$$+ O\left\{ \frac{k_2[k_1,Fk_2]}{\varphi(k_1)} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{\ln\ln(|a|y)\ \ln y}{y} \right\}$$

$$+ O\left\{ \frac{k_2[k_1,Fk_2]}{k_1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{-1/6} \right\} \Big\}$$

$$= \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} H_{z_0}\Big\{1 + O(\exp(-\ln y/k_2)^{\frac{1}{2}}))$$

$$+ O\left\{ \frac{k_2[k_1,Fk_2]}{k_1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{-1/6} \right\} \Big\} \tag{49}$$

as the second error term is absorbed into the third.

Substitution back into (48) gives

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} H_{z_0}\Big\{1 + O(\exp(-\ln y/k_2)^{\frac{1}{2}}))$$

$$+ O\left\{ \frac{k_2[k_1,Fk_2]}{k_1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3}y^{-1/6}\right\} \Big\} \times$$

$$\Big\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) \Big\}$$

$$+ O\left\{ \frac{\varphi(k_1)}{k_1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3}y^{5/6} \right\}$$

$$= \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} H_{z_0} \Big\{ 1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))$$

$$+ O\left\{ \frac{k_2[k_1,Fk_2]}{k_1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3}y^{-1/6}\right\} \Big\}. \tag{50}$$

Given (44) we conclude

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) H_{z_0} \Big\{ 1 +$$

$$O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) + O\left\{ \frac{k_2[k_1,Fk_2]}{k_1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3}y^{-1/6} \right\} \Big\} \tag{51}$$

for

$$\exp(27(\ln y/k_2)^{\frac{1}{2}}) \leq z \leq \exp(y^{1/7}) . \tag{52}$$

We now, in Step Nine, turn to the case $z > \exp(y^{1/7})$.

<u>STEP NINE</u> *The sum* $\sum\limits_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod\limits_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$ *for* $z > \exp(y^{1/7})$

We have already seen that

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \ll \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) \tag{53}$$

and that for $\exp(27(\ln y/k_2)^{\frac{1}{2}}) \leq z \leq \exp(y^{1/7})$,

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) \ll \frac{\varphi(k_1)}{k_1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{5/6} . \tag{54}$$

Now statement (18) of Nair & Perelli [1] reads

$$\text{" } L(1,\chi) = \prod_{p<w} \left(1-\frac{\chi(p)}{p}\right)^{-1} \left\{ 1 + O(\exp(-c(\ln w)^{\frac{1}{2}})) \right\}$$

holds uniformly for $w > \exp(c\ln^2 Q)$ and for all primitive characters χ to a modulus $q \leq Q$ with at most one exception χ_1. "

Writing $z_1 = \exp(y^{1/7})$ and $Q = 4 \max\limits_{0<m\leq y} |g_m|$, and recalling that $\max\limits_{0<m\leq y} |g_m| \leq 4|\zeta| y^2$ for y large, it is apparent that $z_1 > \exp(c\ln^2 Q)$ and hence that

$$L(1,\chi) = \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)^{-1} \left\{1 + O(\exp(-c(\ln z_1)^{\frac{1}{2}}))\right\} \tag{55}$$

for any $z > z_1$, with at most one exception.

It follows that

$$\prod_{z_1\le p<z}\left(1-\frac{\chi(p)}{p}\right) = \prod_{p<z}\left(1-\frac{\chi(p)}{p}\right)\prod_{p<z_1}\left(1-\frac{\chi(p)}{p}\right)^{-1}$$

$$= L(1,\chi)^{-1}\,\{1+O(\exp(-c(\ln z_1)^{\frac{1}{2}}))\}$$

$$\times\; L(1,\chi)\,\{1+O(\exp(-c(\ln z_1)^{\frac{1}{2}}))\}$$

$$= 1 + O(\exp(-c(\ln z_1)^{\frac{1}{2}}))$$

with at most one exception.

More generally

$$\prod_{z_1\le p<z}\left(1-\frac{\chi(p)}{p}\right) \ll 1$$

with at most one exception.

Consequently

$$\sum_{\substack{0<m\le y\\ m\equiv \ell_2 \bmod k_2\\ m\ \text{app}\\ g_m\ \text{not a square}\\ g_m\ \text{bad}}}\prod_{p<z}\left(1-\frac{\chi(p)}{p}\right)c(g_m) = \sum_{\substack{0<m\le y\\ m\equiv \ell_2 \bmod k_2\\ m\ \text{app}\\ g_m\ \text{not a square}\\ g_m\ \text{bad}}}\prod_{p<z_1}\left(1-\frac{\chi(p)}{p}\right)\prod_{z_1\le p<z}\left(1-\frac{\chi(p)}{p}\right)c(g_m)$$

$$\ll \sum_{\substack{0<m\le y\\ m\equiv \ell_2 \bmod k_2\\ m\ \text{app}\\ g_m\ \text{not a square}\\ g_m\ \text{bad}}}\prod_{p<z_1}\left(1-\frac{\chi(p)}{p}\right)c(g_m)$$

$$\ll \frac{\varphi(k_1)}{k_1}\,\tau\left[\frac{\eta^2-\zeta\theta}{4}\right]\,|\zeta|^{1/3}y^{5/6} \qquad (56)$$

with at most one exception, whenever $z\ge\exp(y^{1/7})$.

Steps Ten and Eleven are devoted to the possible exceptional modulus of (56).

STEP TEN *Translation of* $\prod_{\substack{p<z \\ p \nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$ *into a product involving the norms of prime ideals in* $\mathbb{Q}(\sqrt{g_m})$.

By, for example, Theorem 90 and Theorem 81 of Hecke [10], for (p) the principal ideal in $\mathbb{Q}(\sqrt{g_m})$ generated by the prime number p, we have that

$$(p) = \begin{cases} \beta_1\beta_2 & ; \ N\beta_1 = N\beta_2 = p \quad \text{if } (D/p)=1 \\ \beta^2 & ; \ N\beta = p \quad \text{if } (D/p)=0 \\ \beta & ; \ N\beta = p^2 \quad \text{if } (D/p)=-1 \end{cases}$$

where β, β_i are prime ideals and $N\beta$ is the norm of β. It follows that

$$\prod_{N\beta<z} \left(1 - \frac{1}{N\beta}\right) = \prod_{\substack{p<z \\ (D/p)=1}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z \\ p \mid D}} \left(1-\frac{1}{p}\right) \prod_{\substack{p^2<z \\ (D/p)=-1}} \left(1-\frac{1}{p^2}\right).$$

Now, we have

$$\prod_{\substack{p<z \\ p \nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \prod_{\substack{p<z \\ p \nmid k_1}} \left(1-\frac{1}{p}\right) \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m, z)$$

and for $z \geq z_0$

$$\prod_{\substack{p<z \\ p \nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \ll \prod_{\substack{p<z \\ p \nmid k_1}} \left(1-\frac{1}{p}\right) \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m)$$

$$= \frac{k_1}{\varphi(k_1)} \prod_{p<z} \left(1-\frac{1}{p}\right) \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m).$$

But

$$\prod_{p<z} \left(1-\frac{1}{p}\right) \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)$$

$$= \prod_{\substack{p<z \\ p \mid D}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ (D/p)=1}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ (D/p)=-1}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ (D/p)=1}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ (D/p)=-1}} \left(1+\frac{1}{p}\right)$$

$$= \prod_{\substack{p<z \\ p|D}} (1-\frac{1}{p}) \prod_{\substack{p<z \\ (D/p)=1}} (1-\frac{1}{p})^2 \prod_{\substack{p<z \\ (D/p)=-1}} (1-\frac{1}{p^2})$$

$$\ll \prod_{\substack{p<z \\ p|D}} (1-\frac{1}{p}) \prod_{\substack{p<z \\ (D/p)=1}} (1-\frac{1}{p})^2 \prod_{\substack{p^2<z \\ (D/p)=-1}} (1-\frac{1}{p^2})$$

$$= \prod_{N\beta<z} (1-\frac{1}{N\beta}) .$$

So

$$\prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{\rho_m(p)}{p}) \ll \frac{k_1}{\varphi(k_1)} \prod_{N\beta<z} (1-\frac{1}{N\beta}) \quad c(g_m) \tag{57}$$

and we immediately deduce

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{\rho_m(p)}{p}) \ll \frac{k_1}{\varphi(k_1)} \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{N\beta<z} (1-\frac{1}{N\beta}) \; c(g_m). \tag{58}$$

Nair and Perelli [1] have shown that

$$\prod_{N\beta<z} (1-\frac{1}{N\beta}) \ll \frac{1}{L(1,\chi_D)\ln z}$$

whenever $z \gg D^6$, and g_m is negative. The proof is also applicable in the case g_m positive.

Consequently we have

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{\rho_m(p)}{p}) \ll \frac{k_1}{\varphi(k_1)\ln z} \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} L(1,\chi)^{-1} \; c(g_m) \tag{59}$$

for $z \gg (\max_{0<m\leq y} |g_m|)^6$, and as $(\max_{0<m\leq y} |g_m|)^6 > z_0$, so that

$\prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{1}{p}) = \frac{k_1}{\varphi(k_1)\ln z}$, we further have

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) \ll \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ bad}}} L(1,\chi)^{-1}\, c(g_m) \qquad (60)$$

for $z > (\max_{0<m\leq y} |g_m|)^6$.

Certainly if $z>\exp(y^{1/7})$ then $z>(\max_{0<m\leq y}|g_m|)^6$, and we may use

(60) to estimate the sum

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m)$$

over the possible exceptional modulus of (56).

<u>STEP ELEVEN</u> *The possible exceptional modulus of* (56).

We require an upper bound on the sum

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ bad}}}{}' L(1,\chi)^{-1}\, c(g_m)$$

where the sum $\sum'$ is over g_m which give rise to the possible exceptional modulus of (56).

We firstly find an upper bound on $L(1,\chi)^{-1}$.

Dirichlets class number formula states

$$h(d) = \begin{cases} \dfrac{d^{\frac{1}{2}}\, L(1,\chi)}{\ln \epsilon} & \text{for } d>0 \\[2ex] \dfrac{\omega\, |d|^{\frac{1}{2}}\, L(1,\chi)}{2\pi} & \text{for } d<0 \end{cases} \qquad (61)$$

where h(d) is the class number of the quadratic field with discriminant d, and where

$$\omega = \begin{cases} 2 & \text{if } d<-4 \\ 4 & \text{if } d=-4 \\ 6 & \text{if } d=-3 \end{cases} .$$

(N.B. The range of d for which ω is defined is complete for it is not possible for d to be either -1 or -2. If it was then we would have either s=-1 or s=-2 respectively, with s≡1 mod 4, clearly a contradiction.)

and where $\epsilon = \frac{1}{2}(t_0+u_0\sqrt{d})$ with (t_0,u_0), $t_0>0$, $u_0>0$ denoting the fundamental solution of the Pellian equation $t^2-du^2=4$.

Certainly $\epsilon>\frac{1}{2}d^{\frac{1}{2}}$ when d>0.

(61) gives

$$L(1,\chi)^{-1} = \begin{cases} \dfrac{d^{\frac{1}{2}}}{\ln \epsilon .\ h(d)} & \text{for } d>0 \\ \\ \dfrac{\omega\ |d|^{\frac{1}{2}}}{2\pi .\ h(d)} & \text{for } d<0 \end{cases}$$

Clearly $h(d)\geqslant 1$ always, and $\ln \epsilon \gg \ln d$.

So

$$L(1,\chi)^{-1} \ll \frac{d^{\frac{1}{2}}}{\ln d} \quad \text{for } d>0. \tag{62}$$

If d<0, however, from the recent paper of Gross-Zagier [11] we have that, for every $\epsilon>0$, there exists an effectively computable constant $c_\epsilon>0$ such that

$$h(d) > c_\epsilon(\ln |d|)^{1-\epsilon} .$$

Hence

$$L(1,\chi)^{-1} \ll_\epsilon \frac{|d|^{\frac{1}{2}}}{(\ln |d|)^{1-\epsilon}} \quad \text{for } d<0. \tag{63}$$

(62) and (63) give

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}}' L(1,\chi)^{-1}\, c(g_m) \ll_\epsilon \frac{|\bar{s}|^{\frac{1}{2}}}{(\ln |\bar{s}|)^{1-\epsilon}} \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}}' c(g_m) \tag{64}$$

where, as before, the sum $\sum'$ is over g_m which give rise to the possible exceptional modulus $\bar{D}$ of (58); and where $\bar{s}=\bar{D}$ or $\bar{s}=\bar{D}/4$ according to either $\bar{D} \not\equiv 0 \bmod 4$ or $\bar{D} \equiv 0 \bmod 4$.

We note in passing that were the Gross-Zagier theorem not available then to estimate $\sum'$ effectively we would be forced to make use of Siegel's theorem which states that "for any $\epsilon>0$ there exists a positive number $c(\epsilon)$ such that $h(d)>c(\epsilon)|d|^{\frac{1}{2}-\epsilon}$ for $d<0$." Although the use of this theorem would improve any bound we may reach the constant $c(\epsilon)$ is unfortunately non-computable with current knowledge.

Now from (40) back in Step Six we have

$$c(g_m) \ll \prod_{\substack{p \nmid k_1 \\ p | 2(am+d) \\ p | (am+d+bm+e)}} \left(1-\frac{1}{p}\right)^{-1} \prod_{p | 2(am+d) g_m k_1} \left(1-\frac{\chi(p)}{p}\right)^{-1}.$$

Less strongly

$$c(g_m) \ll \prod_{\substack{p \nmid k_1 \\ p | (am+d) \\ p | (bm+e)}} \left(1-\frac{1}{p}\right)^{-1} \prod_{p | (am+d) g_m k_1} \left(1-\frac{1}{p}\right)^{-1}$$

$$\ll \prod_{\substack{p | (am+d) \\ p | (bm+e)}} \left(1-\frac{1}{p}\right)^{-1} \prod_{p | (am+d) g_m} \left(1-\frac{1}{p}\right)^{-1} \prod_{p | k_1} \left(1-\frac{1}{p}\right)^{-1}.$$

As $am+d\equiv 0 \bmod p$ and $bm+e\equiv 0 \bmod p$ together imply that $am+d\equiv 0 \bmod p$ and $g_m\equiv 0 \bmod p$ we have

$$c(g_m) \ll \frac{k_1}{\varphi(k_1)} \prod_{p|(am+d)g_m} (1-\frac{1}{p})^{-2}$$

$$\ll \frac{k_1}{\varphi(k_1)} \ln\ln^2 (\max_{0<m\leq y} |(am+d)g_m|).$$

Assuming, as we have done previously in Steps Four and Seven, that y is large enough to ensure

(i) $\max_{0<m\leq y} |am+d| \leq 2|a|y$

and

(ii) $\max_{0<m\leq y} |g_m| \leq 4|\zeta|y^2$

where $\zeta=b^2-4ac$, gives

$$c(g_m) \ll \frac{k_1}{\varphi(k_1)} \ln\ln^2 (|a||\zeta|y^3)$$

$$\ll \frac{k_1}{\varphi(k_1)} \ln\ln^2 (|a||\zeta|) \ln\ln^2 y.$$

Substitution into (64) gives

$$\sum'_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} L(1,\chi)^{-1} c(g_m) \ll_\epsilon$$

$$\frac{|\bar{s}|^{\frac{1}{2}}}{(\ln|\bar{s}|)^{1-\epsilon}} \frac{k_1}{\varphi(k_1)} \ln\ln^2|a\zeta| \ln\ln^2 y \sum'_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} 1 . \tag{65}$$

Now

$$\sum'_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} 1 \ll \sum_{\substack{0<m\leq y \\ g_m=r^2\bar{s}}} 1 .$$

But, from Lemma 4,

$$\sum_{\substack{0<m\leqslant y\\ g_m=r^2\bar{s}}} 1 \ll \begin{cases} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{|\bar{s}\zeta|}\right] & ;\ \text{if } |\bar{s}\zeta|\leqslant y,\ \bar{s}\zeta>0 \text{ and } \bar{s}\zeta \text{ not a perfect square} \\ \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] & ;\ \text{otherwise} \end{cases}$$

and certainly

$$\sum_{\substack{0<m\leqslant y\\ g_m=r^2\bar{s}}} 1 \ll \begin{cases} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln y & \text{if } |\bar{s}|\leqslant y \\ \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] & \text{if } |\bar{s}|\geqslant y \end{cases} \quad (66)$$

Assuming firstly that $|\bar{s}|\leqslant y$, (66) together with (65), gives

$$\sum_{\substack{0<m\leqslant y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ bad}}}{}' L(1,\chi)^{-1} c(g_m) \ll_\epsilon$$

$$\frac{y^{\frac{1}{2}}}{(\ln y)^{1-\epsilon}} \frac{k_1}{\varphi(k_1)} \ln\ln^2|a\zeta| \ln\ln^2 y\ \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln y$$

$$\ll y^{2/3} \frac{k_1}{\varphi(k_1)} \ln\ln^2|a\zeta|\ \tau\left[\frac{\eta^2-\zeta\theta}{4}\right]. \quad (67)$$

If, on the contrary, $|\bar{s}|\geqslant y$ then

$$\sum_{\substack{0<m\leqslant y\\ m\equiv \ell_2 \bmod k_2\\ m \text{ app}\\ g_m \text{ not a square}\\ g_m \text{ bad}}}{}' L(1,\chi)^{-1} c(g_m) \ll_\epsilon$$

$$\frac{|\bar{s}|^{\frac{1}{2}}}{(\ln|\bar{s}|)^{1-\epsilon}} \frac{k_1}{\varphi(k_1)} \ln\ln^2|a\zeta| \ln\ln^2 y\ \tau\left[\frac{\eta^2-\zeta\theta}{4}\right].$$

However, we certainly have $|\bar{s}|\leqslant \max_{0<m\leqslant y}|g_m|$, and we have

previously assumed that y is large enough to ensure

$$\max_{0<m\leq y} |g_m| < 4|\zeta|y^2.$$

So for $|\bar{s}|>y$,

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}}{}' L(1,\chi)^{-1} c(g_m) \ll_\epsilon |\zeta|^{\frac{1}{2}} \frac{k_1}{\varphi(k_1)} \ln \ln^2|a\zeta| \; \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{y \ln \ln^2 y}{(\ln y)^{1-\epsilon}}. \tag{68}$$

Incorporating the results for $|\bar{s}|<y$ and $|\bar{s}|>y$ gives

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}}{}' L(1,\chi)^{-1} c(g_m) \ll_\epsilon |\zeta|^{\frac{1}{2}} \frac{k_1}{\varphi(k_1)} \ln \ln^2|a\zeta| \; \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{y \ln \ln^2 y}{(\ln y)^{1-\epsilon}} \tag{69}$$

for any $\bar{s}$.

(69) together with (60) and (56) gives the general result for all g_m bad,

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) \ll_\epsilon |\zeta|^{\frac{1}{2}} \frac{k_1}{\varphi(k_1)} \ln \ln^2|a\zeta| \; \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{y \ln \ln^2 y}{(\ln y)^{1-\epsilon}} \tag{70}$$

for $z>\exp(y^{1/7})$.

This completes Step Eleven.

<u>STEP TWELVE</u> *The completion of the Theorem.*

The reasoning of Step Twelve largely follows that of Step Eight. Equation (43) gave us

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ good}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) = \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ good}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) c(g_m) \left\{ 1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) \right\}$$

for all $z\geq\exp(27(\ln y/k_2)^{\frac{1}{2}})$.

But equation (49) gave us further that the right hand side of this equation is

$$\frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} H_{z_0} \left\{ 1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) + O\left\{ \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_2[k_1,Fk_2]}{k_1} |\zeta|^{1/3} y^{-1/6} \right\} \right\}$$

so that

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ good}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} H_{z_0} \left\{1 + O(\exp(-(\ln y/k_2))) + O\left\{ \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_2[k_1,Fk_2]}{k_1} |\zeta|^{1/3} y^{-1/6} \right\} \right\}. \tag{71}$$

Now, if $z>\exp(y^{1/7})$, from (70) we have

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ good}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) = \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m)$$

$$+ O_\epsilon\left\{ |\zeta|^{\frac{1}{2}} \frac{k_1}{\varphi(k_1)} \ln \ln^2 |a\zeta| \; \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{y \ln \ln^2 y}{(\ln y)^{1-\epsilon}} \right\}$$

and this combined with (71) gives

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_m) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} H_{z_0}\left\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\right.$$

$$+ O\left\{ \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_2[k_1,Fk_2]}{k_1} |\zeta|^{1/3} y^{-1/6} \right\}$$

$$\left. + O_\epsilon\left\{ |\zeta|^{\frac{1}{2}} \ln \ln^2 |a\zeta| \; \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_1 k_2[k_1,Fk_2]}{\varphi(k_1)^2} \frac{\ln \ln^2 y}{(\ln y)^{1-\epsilon}} \right\} \right\}$$

$$= \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} H_{z_0} \left\{ 1 + \right.$$

$$\left. + O_\epsilon\left\{ |\zeta|^{\frac{1}{2}} \ln \ln^2 |a\zeta| \; \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_1 k_2[k_1,Fk_2]}{\varphi(k_1)^2} \frac{\ln \ln^2 y}{(\ln y)^{1-\epsilon}} \right\} \right\} \tag{72}$$

for $z>\exp(y^{1/7})$, the first and second error terms being absorbed into the third.

Further, given (44),

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) H_{z_0}\left\{1 + \right.$$

$$\left. + O_\epsilon\left\{ |\zeta|^{\frac{1}{2}} \ln \ln^2 |a\zeta| \; \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_1 k_2[k_1,Fk_2]}{\varphi(k_1)^2} \frac{\ln \ln^2 y}{(\ln y)^{1-\epsilon}} \right\} \right\} \tag{72}$$

for $z>\exp(y^{1/7})$.

Combining (72) with (51), a similar result but for

$\exp(27(\ln y/k_2)^{\frac{1}{2}}) \leq z \leq \exp(y^{1/17})$, gives

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) H_{z_0}\Big\{1 +$$

$$+ O_\epsilon\Big\{ |\zeta|^{\frac{1}{2}} \ln\ln^2 |a\zeta|\ \tau\Big[\frac{\eta^2-\zeta\theta}{4}\Big] \frac{k_1k_2[k_1,Fk_2]}{\varphi(k_1)^2} \frac{\ln\ln^2 y}{(\ln y)^{1-\epsilon}} \Big\}\Big\} \tag{73}$$

for $z \geq \exp(27(\ln y/k_2)^{\frac{1}{2}})$.

To complete the theorem we require an asymptotic formula for the sum

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z) \text{ app}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$$

for $z \geq \exp(27(\ln y/k_2)^{\frac{1}{2}})$ which we may then substitute into (12). To re-introduce the condition "(m,z) app" simply recall from Step 5, (33), that

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z) \text{ app} \\ g_m \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$$

for $z \geq \exp(27(\ln y/k_2)^{\frac{1}{2}})$, and (73) becomes

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z) \text{ app} \\ g_m \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) H_{z_0}\Big\{1 +$$

$$+ O_\epsilon\Big\{ |\zeta|^{\frac{1}{2}} \ln\ln^2 |a\zeta|\ \tau\Big[\frac{\eta^2-\zeta\theta}{4}\Big] \frac{k_1k_2[k_1,Fk_2]}{\varphi(k_1)^2} \frac{\ln\ln^2 y}{(\ln y)^{1-\epsilon}} \Big\}\Big\} \tag{74}$$

for $z \geq \exp(27(\ln y/k_2)^{\frac{1}{2}})$.

To extend the sum of (74) to include g_m a square we return to (25) which reads

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\text{ app} \\ g_m \text{ a square}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \ll \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\ln(|a|y) \ln\left[\frac{y}{|\zeta|}\right]$$

for any z.

This together with (74) gives

$$\sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\text{ app}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{1}{p}\right) H_{z_0}\Big\{1 +$$

$$+ O_\epsilon\Big\{ |\zeta|^{\frac{1}{2}}\ln \ln^2|a\zeta| \; \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{k_1k_2[k_1,Fk_2]}{\varphi(k_1)^2} \frac{\ln \ln^2 y}{(\ln y)^{1-\epsilon}} \Big\} \Big\} \tag{75}$$

for $z>\exp(27(\ln^y/k_2)^{\frac{1}{2}})$.

Before we conclude the theorem we remove the dependence on z_0 of the right hand side of (75). This dependence occurs only in the term H_{z_0} and we will show that

$$H_{z_0} = H_z \{1 + O(\exp(-(\ln {}^y/k_2)^{\frac{1}{2}}))\} \tag{76}$$

for $z>z_0$.

We recall from Step One that

$$H_z = \sum_{\substack{w \\ w=1 \text{ or } \gamma(w)>z}} \Gamma_z(w)$$

where $\gamma(w)$ denotes the smallest prime factor of w, and

$$\Gamma_z(w) = \sum_{\substack{\alpha_1 \bmod Fk_2 \\ \alpha_1\equiv \ell_1 \bmod (k_1,Fk_2)}} \prod_{p<z} \left(1 + \frac{\rho(p)}{p(p-1)}\right) .$$

The notation of $\Gamma_z(w)$ is defined as follows:

(i) $\alpha_1,\ldots,\alpha_\mu$ denote the integers, n, in the interval $1\leq n\leq Fk_2$ for which both

$$(an^2+bn+c, dn^2+en+f) = w$$

and

$$\left(\left[\frac{an^2+bn+c}{w}\right]\ell_2 + \left[\frac{dn^2+en+f}{w}\right] , \prod_{\substack{p<z \\ p\mid k_2}} p \right) = 1$$

hold.

(ii) the unique solution $\bmod[k_1,Fk_2]$ of the two congruences $n\equiv\ell_1 \bmod k_1$ and $n\equiv\alpha_i \bmod Fk_2$ is denoted, if it exists, by $\beta_i=\beta_i(\ell_1,\alpha_i)$. Letting $h=(a,b,c)$; $a=a_1h$, $b=b_1h$, $c=c_1h$ we have

$$\rho(p) = \begin{cases} \left|\{t:\ t \bmod p;\ a_1([k_1,Fk_2]t+\beta_i)^2 + b_1([k_1,Fk_2]t+\beta_i) + c_1 \equiv 0 \bmod p\ \}\right| & ;\ p\nmid k_2h \\ p & ;\ p\mid k_2h \end{cases}$$

Now, firstly, from Lemma 2.1 it follows that if w with $\gamma(w)>z$ exists such that $(an^2+bn+c, dn^2+en+f)=w$ then w must divide F. So assuming that $z_0>F$ rules out this possibility and H_{z_0} becomes $\Gamma_{z_0}(1)$. ie w=1 only.

Secondly, assuming that $z_0>k_2$, and that $z>z_0$,

$$\left((an^2+bn+c)\ell_2 + (dn^2+en+f), \prod_{\substack{p<z_0 \\ p\mid k_2}} p \right)=1$$

$$\Leftrightarrow \left((an^2+bn+c)\ell_2 + (dn^2+en+f), \prod_{\substack{p<z \\ p\mid k_2}} p \right)=1 .$$

This leaves only the term $\prod_{p<z_0} \left(1 + \frac{\rho(p)}{p(p-1)}\right)$ dependent on z_0.

But

$$\rho(p) = \begin{cases} \left|\{t:\ t \bmod p;\ a_1[k_1,Fk_2]^2t^2 + [k_1,Fk_2](2\beta_i a_1+b_1)t + (a_1\beta_i^2 + b_1\beta_i + c_1) \equiv 0 \bmod p\ \}\right| & ;\ p\nmid k_2h \\ p & ;\ p\mid k_2h \end{cases}$$

and so for $p\geq z_0$, as $\beta_i\leq[k_1,Fk_2]$, and assuming

$z_0 > \max\{ a_1[k_1,Fk_2]^2,\ [k_1,Fk_2](2[k_1,Fk_2]a_1+b_1),$

$a_1[k_1,Fk_2]^2+b_1[k_1,Fk_2]+c_1 \}$

we have $\rho(p) \leq 2$ if $p \nmid k_2h$.

Consequently

$$\prod_{p<z}\left(1+\frac{\rho(p)}{p(p-1)}\right) = \prod_{p<z_0}\left(1+\frac{\rho(p)}{p(p-1)}\right)\prod_{z_0\leq p<z}\left(1+\frac{\rho(p)}{p(p-1)}\right)$$

and

$$\prod_{\substack{z_0\leq p<z\\ p\mid k_2h}}\left(1+\frac{1}{p-1}\right) < \prod_{z_0\leq p<z}\left(1+\frac{\rho(p)}{p(p-1)}\right) < \prod_{\substack{z_0\leq p<z\\ p\mid k_2h}}\left(1+\frac{1}{p-1}\right)\prod_{\substack{z_0\leq p<z\\ p\nmid k_2h}}\left(1+\frac{2}{p(p-1)}\right).$$

Assuming further that $z_0 \geq h$ gives

$$1 < \prod_{z_0\leq p<z}\left(1+\frac{\rho(p)}{p(p-1)}\right) < \prod_{\substack{z_0\leq p<z\\ p\nmid k_2h}}\left(1+\frac{2}{p(p-1)}\right).$$

Arguing as in Step Six we get

$$1 < \prod_{z_0\leq p<z}\left(1+\frac{\rho(p)}{p(p-1)}\right) \leq 1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))$$

giving

$$\prod_{z_0\leq p<z}\left(1+\frac{\rho(p)}{p(p-1)}\right) = 1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})).$$

Hence, (by Lemma 3),

$$\prod_{p<z_0}\left(1+\frac{\rho(p)}{p(p-1)}\right) = \prod_{p<z}\left(1+\frac{\rho(p)}{p(p-1)}\right)(1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})))$$

and $H_{z_0} = H_z(1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})))$ as required.

Equation (75) becomes

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \frac{\varphi(k_1)y}{k_2[k_1,Fk_2]} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{1}{p}\right) H_z \Big\{1 +$$

$$+ O_\epsilon\Big\{ |\zeta|^{\frac{1}{2}}\ln\ln^2|a\zeta|\ \tau\Big[\frac{\eta^2-\zeta\theta}{4}\Big] \frac{k_1k_2[k_1,Fk_2]}{\varphi(k_1)^2} \frac{\ln\ln^2 y}{(\ln y)^{1-\epsilon}} \Big\} \Big\}$$

$$= \frac{k_1 y}{k_2[k_1,Fk_2]} \prod_{p<z} \left(1-\frac{1}{p}\right) H_z \Big\{1 +$$

$$+ O_\epsilon\Big\{ |\zeta|^{\frac{1}{2}}\ln\ln^2|a\zeta|\ \tau\Big[\frac{\eta^2-\zeta\theta}{4}\Big] \frac{k_1k_2[k_1,Fk_2]}{\varphi(k_1)^2} \frac{\ln\ln^2 y}{(\ln y)^{1-\epsilon}} \Big\} \Big\} \tag{76}$$

for $z\geq\exp(27(\ln {}^{y}/k_2)^{\frac{1}{2}})$.

Equation (15) covers the case for $2\leq z\leq\exp(27(\ln {}^{y}/k_2)^{\frac{1}{2}})$; a combination of (76) and (15) gives

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \frac{k_1 y}{k_2[k_1,Fk_2]} \prod_{p<z} \left(1-\frac{1}{p}\right) H_z \Big\{1 +$$

$$+ O_\epsilon\Big\{ |\zeta|^{\frac{1}{2}}\ln\ln^2|a\zeta|\ \tau\Big[\frac{\eta^2-\zeta\theta}{4}\Big] \frac{k_1k_2[k_1,Fk_2]}{\varphi(k_1)^2} \frac{\ln\ln^2 y}{(\ln y)^{1-\epsilon}} \Big\} \Big\} \tag{77}$$

for $z\geq2$, and (77) substituted into (12) gives

$$S(x,y,z) = \frac{x\ y}{k_2[k_1,Fk_2]} \prod_{p<z} \left(1-\frac{1}{p}\right) H_z \Big\{1 +$$

$$+ O_\epsilon\Big\{ |\zeta|^{\frac{1}{2}}\ln\ln^2|a\zeta|\ \tau\Big[\frac{\eta^2-\zeta\theta}{4}\Big] \frac{k_1k_2[k_1,Fk_2]}{\varphi(k_1)^2} \frac{\ln\ln^2 y}{(\ln y)^{1-\epsilon}} \Big\}$$

$$+ O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2))) + O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}})) \Big\} \tag{78}$$

for $2 \leq z \leq {}^{x}/k_1$ thus effectively completing the theorem for ${}^{y}/k_2\leq{}^{x}/k_1$. Recalling that (5) completed the theorem for ${}^{x}/k_1<{}^{y}/k_2$ the theorem is essentially complete.

STEP THIRTEEN *Some mopping up.*

Scattered throughout the proof of Theorem One (from Step Four onwards) are various assumptions about the size of y. In Step Thirteen we aim to show that making the single assumption

$$\exp((\ln {}^{y}/k_2)^{\frac{1}{2}}) > \max(|a|,|b|,|c|,|d|,|e|,|f|,k_1,k_2) \tag{79}$$

for $^{y}/k_2$ large is enough to cover them all.

We deal firstly with the recurring assumptions, namely

(I) $^{|d|}/_{|a|} < y$.

Assuming this allowed us to make use of the inequality $|am+d| \leq 2|a|y$. It is clear that it is satisfied if (79)is satisfied.

(II)
$$y > \begin{cases} \dfrac{-|\eta| + M_1^{\frac{1}{2}}}{|\zeta|} & ; M_1 > 0 \\ 0 & ; M_1 < 0 \end{cases}$$

with $\zeta = b^2-4ac$, $\eta = be-2cd-2fa$, $\theta = e^2-4fd$, $M_1 = \eta^2 - \zeta\theta + |\zeta|M$, and $M = \max(|\zeta + 2\eta + \theta|, |-\eta^2 - \zeta\theta|)$. Assuming this allowed us to make use of the double inequality $\frac{y^2}{2} < \max_{0<m\leq y} |g_m| < 4|\zeta|y^2$. Certainly $M \leq \max(|\zeta|+2|\eta|+|\theta|, |\eta^2|+|\zeta||\theta|)$. But, assuming (79),

$$\begin{aligned} |\zeta|+2|\eta|+|\theta| &\leq b^2+4|a||c|+2|b||e|+2|c||d|+2|f||a|+e^2+4|f||d| \\ &\leq 16\max(|a|,|b|,|c|,|d|,|e|,|f|)^2 \\ &\leq 16\exp(2(\ln {}^{y}/k_2)^{\frac{1}{2}}). \end{aligned}$$

Similarly

$$\eta^2 + |\zeta||\theta| \leq 35\exp(4(\ln {}^{y}/k_2)^{\frac{1}{2}}).$$

So $M \leq 35\exp(4(\ln {}^{y}/k_2)^{\frac{1}{2}})$ and

$$\begin{aligned} M_1 &\leq \eta^2+|\zeta||\theta|+|\zeta|35\exp(4(\ln {}^{y}/k_2)^{\frac{1}{2}}) \\ &\leq 71\exp(4(\ln {}^{y}/k_2)^{\frac{1}{2}}) \end{aligned}$$

and finally

$$\frac{-|\eta|+M_1^{\frac{1}{2}}}{|\zeta|} \leq |\eta|+M_1^{\frac{1}{2}} \leq 72\exp(4(\ln {}^{y}/k_2)^{\frac{1}{2}}) \leq y$$

as required. So assuming (79) we have

$$\frac{y^2}{2} < |g_m| < 4|\alpha|y^2$$

as required.

For the rest of the assumptions in the proof we deal with each in the order they appear. We take each step seperately for ease of reference. In brackets below each stated assumption we briefly show that assuming (79) is sufficient. The first of the assumptions, as stated previously, occurs in Step Four and so we may assume in all the following that $z \geq \exp(27(\ln y/k_2)^{\frac{1}{2}})$.

STEP FOUR

(i) $y > |\zeta|$ where $\zeta = b^2-4ac$

($|b^2-4ac| \leq b^2+4|a||c| \leq 5\exp^2((\ln y/k_2)^{\frac{1}{2}}) \leq y$)

STEP FIVE

(ii) $z > k_1$

($k_1 < \exp((\ln y/k_2)^{\frac{1}{2}}) < \exp(27(\ln y/k_2)^{\frac{1}{2}}) \leq z$)

(iii) $z > \max\{|d|,|e|,|f|,|bd-ea|,|dc-fa|,|ec-fb|\}$

($\max\{|d|,|e|,|f|,|bd-ea|,|dc-fa|,|ec-fb|\}$

$\leq \max\{|b||d|+|e||a|,\ |d||c|+|f||a|,\ |e||c|+|f||b|\}$

$\leq 2\exp^2((\ln y/k_2)^{\frac{1}{2}})$

$\leq \exp(27(\ln y/k_2)^{\frac{1}{2}})$

$\leq z$)

STEP SIX

(iv) $\ln(|am+d||g_m|) \leq \exp(26(\ln y/k_2)^{\frac{1}{2}})$

($\ln(|am+d||g_m|) \leq \ln(8|a||\zeta|y^3)$ from (I) and (II), and

$\ln(8|a||\zeta|y^3) \leq \ln(8y^5) \leq \ln(8(y/k_2)^{10}) \leq \exp(26(\ln y/k_2)^{\frac{1}{2}})$)

STEP SEVEN

(v) $\ln^{54}(\max_{0<m\leq y}|g_m|) \leq \exp(27(\ln y/k_2)^{\frac{1}{2}})$

(From (II), $\ln^{54}(\max_{0<m\leq y}|g_m|) \leq \ln^{54}(4|\zeta|y^2)$

$\leq \ln^{54}(4y^3) \leq \ln^{54}(4(y/k_2)^6) \leq \exp(27(\ln y/k_2)^{\frac{1}{2}})$)

STEP NINE

(vi) $\exp(y^{1/7}) > \exp(c\ln^2(\max_{0<m\leq y}|g_m|))$

($\ln^2(\max_{0<m\leq y}|g_m|) \leq \ln^2(4|\zeta|y^2) \leq \ln^2(4y^3) < y^{1/7}$)

STEP TEN

(vii) $(\max_{0<m\leq y}|g_m|)^6 \geq \exp(27(\ln y/k_2)^{\frac{1}{2}})$

(From (I), $(\max_{0<m\leq y}|g_m|)^6 \geq \left[\frac{y^2}{2}\right] \geq \left[\frac{1}{2}\left[\frac{y}{k_2}\right]^2\right]^6 \geq$

$\exp(27(\ln y/k_2)^{\frac{1}{2}})$)

(viii) $\exp(y^{1/7}) \geq (\max_{0<m\leq y}|g_m|)^6$

($(\max_{0<m\leq y}|g_m|)^6 \leq (4|\zeta|y^2)^6 \leq (4y^3)^6 \leq \exp(y^{1/7})$)

STEP TWELVE

(ix) $\exp(27(\ln y/k_2)^{\frac{1}{2}}) > F$ where

$$F = \begin{cases} |ce-fb| & \text{if } a=0,\ d=0 \\ |(cd-fa)^2-(bd-ea)(ce-fb)| & \text{otherwise} \end{cases}$$

($$F \leq \begin{cases} |c||e|+|f||b| & \text{if } a=0,\ d=0 \\ (|c||d|+|f||a|)^2+(|b||d|+|e||a|)(|c||e|+|f||b|) & \text{otherwise} \end{cases}$$

$\leq 4\max^2\{|a|,|b|,|c|,|d|,|e|,|f|\}$

$\leq 4\exp(2(\ln y/k_2)^{\frac{1}{2}}) < \exp(27(\ln y/k_2)^{\frac{1}{2}})$)

(x) $\exp(27(\ln y/k_2)^{\frac{1}{2}}) > k_2$

(Obvious)

(xi) $\exp(27(\ln y/k_2)^{\frac{1}{2}}) > \max\{ a_1[k_1,Fk_2]^2,$

$[k_1,Fk_2]^2(2[k_1,Fk_2]a_1+b_1),\ a_1[k_1,Fk_2]^2+b_1[k_1,Fk_2]+c_1 \}$

where $a_1={}^a/_h$, $b_1={}^b/_h$, $c_1={}^c/_h$ with $h=(a,b,c)$.

(Firstly,

$a_1[k_1,Fk_2]^2 \leq |a|k_1{}^2F^2k_2{}^2 \leq 16\exp(9(\ln y/k_2)^{\frac{1}{2}})$

$\leq \exp(27(\ln y/k_2)^{\frac{1}{2}})$.

Secondly,

$[k_1,Fk_2]^2(2[k_1,Fk_2]a_1+b_1) \leq 64\exp(13(\ln y/k_2)^{\frac{1}{2}})$

$\leq \exp(27(\ln y/k_2)^{\frac{1}{2}})$

and lastly,

$$a_1[k_1,Fk_2]^2+b_1[k_1,Fk_2]+c_1 \leqslant 48\exp(9(\ln {}^{y}/k_2)^{\frac{1}{2}})$$

$$\leqslant \exp(27(\ln {}^{y}/k_2)^{\frac{1}{2}}) \quad).$$

(xii) $\exp(27(\ln {}^{y}/k_2)^{\frac{1}{2}}) > h.$

(Obvious)

This completes the proof of Theorem One.

AFTERWORD

It will be noticed from the statement of Theorem One that if ${}^{y}/k_2 > {}^{x}/k_1$ then the error terms in the estimate of $S(x,y,z)$ are not independent of α, whereas when ${}^{x}/k_1 \geqslant {}^{y}/k_2$ the error terms are independent of α.

On the other hand an examination of Step One of the proof of Theorem One will reveal that, for ${}^{y}/k_2 > {}^{x}/k_1$ and $z \leqslant {}^{y}/k_2$, had we taken $S(x,y,z)$ to be

$$S_1(x,y,z) = \Big|\{(n,m);\alpha<n\leqslant\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ \beta<m\leqslant\beta+y,\ m\equiv\ell_2 \bmod k_2,$$
$$((an^2+bn+c)m+(dn^2+en+f),\prod_{p<z}p)=1\}\Big|$$

then we could have found an asymptotic formula for $S_1(x,y,z)$ independent of β. The obvious course of enquiry is to examine whether or not an estimate of $S_1(x,y,z)$ can be found with all terms independent of β when ${}^{x}/k_1 \geqslant {}^{y}/k_2$. Our method of proof does not allow us to answer this conclusively.

The main stumbling block occurs when we try to extend the function

$$\prod_{p<z_0}\left(1-\frac{\chi(p)}{p}\right) \qquad \text{for } z_0 \leqslant {}^{y}/k_2$$

to $\prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)$ for $z \geq z_0$.

Whereas for m in the range $0<m\leq y$ with $Q=\max_{0<m\leq y}|g_m|$ we were able to apply Lemma 5.1 this lemma becomes inapplicable for β arbitrarily large, since in this instance we would be forced to take $Q=\max_{\beta<m\leq\beta+y}|g_m|$ and we could not ensure that $\ln^{2\alpha}Q\leq z$ is satisfied, a condition of the lemma.

So in summary we have, writing $S_1(x,y,z)$ to be

$$S_1(x,y,z) = \Big|\{(n,m);\alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ \beta<m\leq\beta+y,\ m\equiv\ell_2 \bmod k_2,\ ((an^2+bn+c)m+(dn^2+en+f),\prod_{p<z}p)=1\}\Big|,$$

that an estimate of $S_1(x,y,z)$ may be found independently of α if $x/k_1 \geq y/k_2$ and independently of β if $y/k_2 \geq x/k_1$.

With reference to our assumption in Theorem One that a and d are not both zero, were the contrary true then we would require an estimate of the function

$$S(x,y,z) = \Big|\{(n,m);\alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2,\ ((bn+c)m+(en+f),\prod_{p<z}p)=1\}\Big|.$$

Essentially the method of argument of Step One of the proof repeated twice would suffice to give such an estimate. We omit the details.

We conclude Chapter Two with an examination of the other case concerning the function

$$S(x,y,z) = \left|\{ (n,m): \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2, ((an^2+bn+c)m+(dn^2+en+f), \prod_{p<z} p)=1 \}\right|$$

excluded by Theorem One, namely where an^2+bn+c and dn^2+en+f have a common factor.

The case when an^2+bn+c and dn^2+bn+c have a constant integer in common is essentially trivial and is not examined. We assume in what follows that $(a,b,c,d,e,f)=1$.

We assume firstly that an^2+bn+c and dn^2+en+f are constant multiples of each other so that $S(x,y,z)$ may be written

$$S(x,y,z) = \left|\{ (n,m): \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2, ((An^2+Bn+C)(Dm+E), \prod_{p<z} p)=1 \}\right|$$

for some integers A,B,C,D and E.

We prove the following:

<u>THEOREM TWO</u>

For $x,y,z\in Z$ *let* $M=\min(x/k_1, y/k_2)$ *and assume that* z *satisfies* $2\leq z\leq M$. *Let* $(\cdot/p)$ *denote the Legendre symbol and let* $\delta=B^2-4AC$. *Then*

$$S(x,y,z) = \frac{xy}{k_1k_2} \prod_{\substack{p<z \\ p\nmid k_2D}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid 2Ak_1}} \left(1-\frac{(\delta/p)+1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_1 \\ p\mid 2A \\ p\nmid A+B}} \left(1-\frac{1}{p}\right) \Big\{1 + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) + O(\exp(-(\ln x/k_1)^{\frac{1}{2}})) + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\Big\}$$

for $u = \dfrac{\ln M}{\ln z}$ *if the conditions*

(i) $(D\ell_2+E, \prod_{\substack{p<z\\ p|k_2}} p)=1$

(ii) $(A+B,C)\equiv 1 \bmod 2$

(iii) $(A\ell_1^2+B\ell_1+C, \prod_{\substack{p<z\\ p|k_1}} p)=1$

are satisfied.

Otherwise $S(x,y,z)=0$.

Further, if $x/k_1 \leq z \leq y/k_2$, *then under conditions* (i), (ii) and (iii),

$\prod \ (1-\frac{1}{}) \quad \prod \ (1- \frac{(\delta/p)+1}{}$

$$S(x,y,z) < \frac{xy}{k_1k_2} \prod_{\substack{p<z\\ p\nmid k_2D}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<x/k_1\\ p\nmid 2Ak_1}} \left(1-\frac{(\delta/p)+1}{p}\right) \prod_{\substack{p<x/k_1\\ p\nmid k_1\\ p|2A\\ p\nmid A+B}} \left(1-\frac{1}{p}\right) \Big\{1 + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2))) + O(\exp(-(\ln x/k_1)^{\frac{1}{2}}))\Big\}$$

where $v = \frac{\ln x/k_1}{\ln z}$.

The O*-constants are absolute, effectively computable, and independent of* A,B,C,D,E,k_1 and k_2

PROOF OF THEOREM TWO

Assuming firstly that $z \leq y/k_2$, define the function $M(y,z)$ to be

$$M(y,z) = \left|\{m:\ 0<m\leq y,\ m\equiv \ell_2 \bmod k_2,\ (Dm+E, \prod_{p<z} p)=1\}\right|$$

so that

$$S(x,y,z) = \sum_{\substack{\alpha<n\leq \alpha+x\\ n\equiv \ell_1 \bmod k_1\\ (An^2+Bn+C, \prod_{p<z} p)=1}} M(y,z) \quad .$$

An application of Lemma 1.1 gives

$$M(y,z) = \begin{cases} \frac{y}{k_2} \prod_{\substack{p<z \\ p\nmid k_2 D}} (1-\frac{1}{p}) \left\{1 + O(\exp(-u(\ln u - \ln\ln 3u - \ln 2 - 2))) \right. \\ \qquad \left. + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\right\} \\ \qquad ; (D\ell_2+E, \prod_{\substack{p<z \\ p\mid k_2}} p)=1 \\ 0 \qquad ; \text{otherwise} \end{cases}$$

where $u = \frac{\ln y/k_2}{\ln z}$.

So if $(D\ell_2 + E, \prod_{\substack{p<z \\ p\mid k_2}} p)>1$ then $S(x,y,z)=0$ whenever $z\leqslant y/k_2$.

Otherwise

$$S(x,y,z) = \frac{y}{k_2} \prod_{\substack{p<z \\ p\nmid k_2 D}} (1-\frac{1}{p}) \sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (An^2+Bn+C, \prod_{p<z} p)=1}} 1 \left\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) \right.$$

$$\left. + O(\exp(-u(\ln u - \ln\ln u - \ln 2 - 2)))\right\} \tag{1}$$

Write the sum

$$\sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (An^2+Bn+C, \prod_{p<z} p)=1}} 1 = \left|\{n: \alpha<n\leqslant\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ (An^2+Bn+C, \prod_{p<z} p)=1\}\right|$$

as $N(x,z)$. Then $N(x,z)$ may be estimated if we assume in addition that $z\leqslant x/k_1$.

If $(A+B,C)\equiv 0 \bmod 2$ then $N(x,z)=0$.

Assuming that $(A+B,C)\equiv 1 \bmod 2$, a second application of Lemma 1.1 gives

$$N(x,z) = \begin{cases} \frac{x}{k_1} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho(p)}{p}\right) \Big\{1 + O(\exp(-v(\ln v - \ln\ln 3v - \ln 2 - 2))) \\ \qquad + O(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}}))\Big\} \\ \qquad ; \ (A\ell_1^2 + B\ell_1 + C, \prod_{\substack{p<z \\ p\mid k_2}} p)=1 \\ 0 \qquad ; \ \text{otherwise} \end{cases}$$

where $v = \frac{\ln {}^x/k_1}{\ln z}$ and where

$$\rho(p) = \left|\{n \bmod p: An^2 + Bn + C \equiv 0 \bmod p\}\right|.$$

Consequently, if $(A\ell_1^2 + B\ell_1 + C, \prod_{\substack{p<z \\ p\mid k_2}} p\ )>1$, $S(x,y,z)=0$.

Otherwise

$$S(x,y,z) = \frac{x\ y}{k_1 k_2} \prod_{\substack{p<z \\ p\nmid k_2 D}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho(p)}{p}\right) \Big\{1 + O(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}}))$$

$$+ O(\exp(-(\ln {}^y/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u - \ln\ln 3u - \ln 2 - 2)))\Big\}$$

where $u = \frac{\ln(\min({}^x/k_1, {}^y/k_2))}{\ln z}$.

Now if $p\nmid 2A$,

$$\rho(p) = \left|\{n \bmod p: n^2 \equiv B^2 - 4AC \bmod p\}\right|$$

$$= \left(\frac{B^2-4AC}{p}\right) + 1$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

On the other hand if $p \mid 2A$ then

$$\rho(p) = \begin{cases} 1 \ ; \ p\nmid(A+B) \\ 0 \ ; \ p\mid(A+B) \ . \end{cases}$$

So

$$\prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{\rho(p)}{p}\right) = \prod_{\substack{p<z\\ p\nmid 2Ak_1}}\left(1-\frac{(\delta/p)+1}{p}\right)\prod_{\substack{p<z\\ p\nmid k_1\\ p\mid 2A\\ p\nmid (A+B)}}\left(1-\frac{1}{p}\right)$$

where $\delta=B^2-4AC$ which completes the theorem for

$z\leqslant\min(x/k_1, y/k_2)$.

If $x/k_1\leqslant z\leqslant y/k_2$ then from (1)

$$S(x,y,z) = \frac{y}{k_2}\prod_{\substack{p<z\\ p\nmid k_2D}}\left(1-\frac{1}{p}\right)\sum_{\substack{\alpha<n\leqslant\alpha+x\\ n\equiv\ell_1 \bmod k_1\\ (An^2+Bn+C,\prod_{p<z}p)=1}} 1\ \Big\{1+O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))$$

$$+\ O(\exp(-u(\ln u-\ln\ln u-\ln 2-2)))\Big\}$$

Writing x/k_1 as x_1 for convenience it follows that

$$S(x,y,z) \leqslant \frac{y}{k_2}\prod_{\substack{p<z\\ p\nmid k_2D}}\left(1-\frac{1}{p}\right)\sum_{\substack{\alpha<n\leqslant\alpha+x\\ n\equiv\ell_1 \bmod k_1\\ (An^2+Bn+C,\prod_{p<x_1}p)=1}} 1\ \Big\{1+O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))$$

$$+\ O(\exp(-u(\ln u-\ln\ln u-\ln 2-2)))\Big\}$$

$$= \frac{xy}{k_1k_2}\prod_{\substack{p<z\\ p\nmid k_2D}}\left(1-\frac{1}{p}\right)\prod_{\substack{p<x_1\\ p\nmid 2Ak_1}}\left(1-\frac{(\delta/p)+1}{p}\right)\prod_{\substack{p<x_1\\ p\nmid k_1\\ p\mid 2A\\ p\nmid A+B}}\left(1-\frac{1}{p}\right)\Big\{1+$$

$$O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))+O(\exp(-(\ln x/k_1)^{\frac{1}{2}}))\Big\}$$

which completes the theorem. □

N.B. A quick examination of the proof will reveal that Theorem Two holds for $z\leqslant\min(x/k_1, y/k_2)$ or $x/k_1\leqslant z\leqslant y/k_2$ even if we take

$$S(x,y,z) = \left|\{(n,m):\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ \beta<m\leq\beta+y,\ m\equiv\ell_2 \bmod k_2,\ ((An^2+Bn+C)(Dm+E), \prod_{p<z} p)=1\}\right|.$$

However, consistent with Theorem One, we leave $S(x,y,z)$ in its original form.

We may also extend the proof to cover z in the range $^y/k_2\leq z\leq {}^x/k_1$, to give

$$S(x,y,z) \leq \frac{xy}{k_1k_2} \prod_{\substack{p<^y/k_2\\ p\nmid k_2D}} (1-\frac{1}{p}) \prod_{\substack{p<z\\ p\nmid 2Ak_1}} (1-\frac{(^\delta/p)+1}{p}) \prod_{\substack{p<z\\ p\nmid k_1\\ p\mid 2A\\ p\nmid A+B}} (1-\frac{1}{p}) \Big\{1 + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) + O(\exp(-(\ln {}^y/k_2)^{\frac{1}{2}}))\Big\}$$

for $u = \frac{\ln {}^y/k_2}{\ln z}$.

We now turn to the case where an^2+bn+c and dn^2+en+f have a linear factor only in common. In this instance $S(x,y,z)$ may be written

$$S(x,y,z) = \left|\{\ (n,m):\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2,\ ((An+B)((Cn+D)m+(En+F)), \prod_{p<z} p)=1\}\right|$$

for some integers A,B,C,D,E and F.

Firstly we give some definitions:

(i) $R=DE-FC$

(ii) $$\Gamma_z(w) = \sum_{\substack{w\\ w=1 \text{ or } \gamma(w)>z}} \frac{A}{\varphi(A)} \prod_{\substack{p<z\\ p\nmid k_2\\ p\mid h}} (1-\frac{1}{p})^{-1} \frac{[k_1,Rk_2]^{\frac{1}{2}}}{\varphi([k_1,Rk_2])^{\frac{1}{2}}}\ k_1 \times \prod_{\substack{p<z\\ p\nmid k_2h}} (1+\frac{p^{\frac{1}{2}}}{(p-1)^2})\ \Psi_z(w)$$

where $h=(C,D)$ and $C_1=C/h$, $D_1=D/h$;

and where $\Psi_z(w)$ denotes the number of integers n in the interval $1\leq n\leq Rk_2$ for which both

$$(Cn+D,En+F)=w$$

and

$$\left(\left[\frac{Cn+D}{w}\right]\ell_2+\left[\frac{En+F}{w}\right],\ \prod_{\substack{p<z\\ p\nmid k_2}} p\right)=1$$

hold.

With these definitions we have the following:

THEOREM THREE

For $x,y\in Z$ *let* $M=\max(y/k_2,x/k_1)$. *Define* $z_1=\min(z,x/k_1)$ *and assume that* z *satisfies* $2\leq z\leq M$.

Then, whenever $\exp((\ln M)^{\frac{1}{2}})\geq\max\{|a|,|b|,|c|,|d|,|e|,|f|,k_1,k_2\}$,

$$S(x,y,z)\leq \frac{x\ y}{k_1k_2}\prod_{\substack{p<z\\ p\nmid k_2}}\left(1-\frac{1}{p}\right)\prod_{p<z_1}\left(1-\frac{1}{p}\right)\Gamma_z(w)\Big\{1+ O(\exp(-\ln y/k_2)^{\frac{1}{2}}))+O(\exp(-(\ln x/k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\Big\}$$

for $v=\dfrac{\ln x/k_1}{\ln z}$. *The O-constants are absolute, effectively computable, and independent of* A,B,C,D,E,k_1 and k_2.

PROOF OF THEOREM THREE

Assume to begin with that $y/k_2\leq x/k_1$.

We follow the procedure of Theorem One.

Define

$$M(y,z,n)=\Big|\{m:\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2,\ \ ((Cn+D)m+(En+F),\prod_{p<z}p)=1\}\Big|$$

so that

$$S(x,y,z) = \sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (An+B,\prod_{p<z}p)=1}} M(y,z,n) \qquad (1)$$

Define r_n to be the highest common factor of the two polynomials Cn+D and En+F. It is apparent that if $(r_n, \prod_{p<z}p)>1$ then $M(y,z,n)=0$. Assuming that $(r_n, \prod_{p<z}p)=1$ we have

$$M(y,z,n) = \left|\{m:\ 0<m\leqslant y,\ m\equiv\ell_2 \bmod k_2,\ \left(\left[\frac{Cn+D}{r_n}\right]m+\left[\frac{En+F}{r_n}\right], \prod_{p<z}p\right)=1\}\right|$$

and an application of Lemma 1.1 for $z\leqslant y/k_2$ gives

$$M(y,z,n) = \begin{cases} \dfrac{y}{k_2} \prod\limits_{\substack{p<z \\ p\nmid k_2 \\ p\nmid(Cn+D)}} (1-\frac{1}{p}) \Big\{1 + O(\exp(-u(\ln u-\ln\ln m3u-\ln 2-2))) \\ \qquad + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\Big\} \\ \qquad ;\ \left(\left[\frac{Cn+D}{r_n}\right]\ell_2+\left[\frac{En+F}{r_n}\right], \prod\limits_{\substack{p<z \\ p\mid k_2}} p\right)=1 \\ 0 \qquad ;\ \text{otherwise} \end{cases}$$

where $u = \dfrac{\ln y/k_2}{\ln z}$.

Summing M(y,z,n) over n gives

$$S(x,y,z) = \frac{y}{k_2} \sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (r_n,\prod_{p<z}p)=1 \\ \left(\left[\frac{Cn+D}{r_n}\right]\ell_2+\left[\frac{En+F}{r_n}\right], \prod_{p<z,\ p\mid k_2} p\right)=1 \\ (An+B,\prod_{p<z}p)=1}} \prod_{\substack{p<z \\ p\nmid(Cn+D)k_2}} (1-\frac{1}{p}) \Big\{1 +$$

$$O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\Big\}. \qquad (2)$$

Taking the product $\prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{1}{p})$ out to the left of the sum gives

$$S(x,y,z) = \frac{y}{k_2} \prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{1}{p}) \sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (r_n, \prod_{p<z} p)=1 \\ ([\frac{Cn+D}{r_n}]\ell_2+[\frac{En+F}{r_n}], \prod_{p<z, p|k_2} p)=1 \\ (An+B, \prod_{p<z} p)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p|(Cn+D)}} (1-\frac{1}{p})^{-1} \Big\{1 +$$

$$O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\Big\}.$$

Now the sum

$$\sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (r_n, \prod_{p<z} p)=1 \\ ([\frac{Cn+D}{r_n}]\ell_2+[\frac{En+F}{r_n}], \prod_{p<z, p|k_2} p)=1 \\ (An+B, \prod_{p<z} p)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p|(Cn+D)}} (1-\frac{1}{p})^{-1}$$

$$= \sum_{\substack{w \\ w=1 \text{ or } \gamma(w)>z}} \sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (Cn+D, En+F)=w \\ ([\frac{Cn+D}{w}]\ell_2+[\frac{En+F}{w}], \prod_{p<z, p|k_2} p)=1 \\ (An+B, \prod_{p<z} p)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p|(Cn+D)}} (1-\frac{1}{p})^{-1}$$

where $\gamma(w)$ denotes the smallest prime factor of w.

Assuming, in addition to $z\leqslant {}^{y}/k_2$, that $z\leqslant\exp(10(\ln x)^{\frac{1}{2}})$ we may apply Lemma 2.12 to this sum to give

$$S(x,y,z) \leqslant \frac{x\ y}{k_1 k_2} \prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{1}{p}) \prod_{p<z} (1-\frac{1}{p})\ \Gamma_z(w) \Big\{1 +$$

$$O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))$$

$$+ O(\exp(-(\ln x)^{\frac{1}{2}})) + O\Big[\varphi([k_1, Fk_2])\ \frac{\ln\ln x.\ln^{3/2}x}{x^{\frac{1}{2}}}\Big]\Big\} \qquad (3)$$

where R and $\Gamma_z(w)$ are as described in the introduction to the Theorem.

We now estimate $S(x,y,z)$ in a different way.

Define

$$N(x,z,m) = \left|\{n: \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ ((An+B)((Cm+E)n+(Dm+F)), \prod_{p<z} p)=1\}\right|$$

so that

$$S(x,y,z) = \sum_{\substack{0<m\leq y \\ m\equiv\ell_2 \bmod k_2}} N(x,z,m).$$

Define s_m to be the highest common factor of $(Cm+E)$ and $(Dm+F)$. Then if $(s_m, \prod_{p<z} p)>1$, $N(x,z,m)=0$. Further if $((A+B)((Cm+E)+(Dm+F)), B(Dm+F))\equiv 0 \bmod 2$ then $S(x,y,z)=0$. Assuming that $(s_m, \prod_{p<z} p)=1$ and $((A+B)((Cm+E)+(Dm+F)), B(Dm+F))\equiv 1 \bmod 2$ we may write

$$N(x,z,m) = \left|\{n: \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ ((An+B)(\left[\frac{Cm+E}{s_m}\right]n+\left[\frac{Dm+F}{s_m}\right]), \prod_{p<z} p)=1\}\right|$$

and applying Lemma 1.1 again (this time for $z\leq x/k_1$) gives

$$N(x,z,m) = \begin{cases} \dfrac{x}{k_1} \displaystyle\prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \Big\{1 + O(\exp(-(\ln x/k_1)^{\frac{1}{2}})) \\ \qquad\qquad + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\Big\} \\ \qquad ;\ ((A\ell_1+B)(\left[\frac{Cm+E}{s_m}\right]\ell_1+\left[\frac{Dm+F}{s_m}\right]), \prod_{\substack{p<z \\ p\mid k_1}} p)=1 \\ 0 \qquad ;\ \text{otherwise} \end{cases} \tag{4}$$

where $v = \dfrac{\ln x/k_1}{\ln z}$ and

$$\rho_m(p) = \left|\{n \bmod p: (An+B)(\left[\frac{Cm+E}{s_m}\right]n+\left[\frac{Dm+F}{s_m}\right])\equiv 0 \bmod p\}\right|$$

provided that $\rho_m(p)<p$ for all primes p, a condition which is easily seen to be satisfied under the conditions $(s_m, \prod_{p<z} p)=1$

and $((A+B)((Cm+E)+(Dm+F)),B(Dm+F))\equiv 1 \bmod 2$. Summing (4) over m gives

$$S(x,y,z) = \frac{x}{k_1} \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (s_m,\prod_{p<z}p)=1 \\ ((A+B)((Cm+E)+(Dm+F)),B(Dm+F))\equiv 1 \bmod 2 \\ ((A\ell_1+B)(\left[\frac{Cm+E}{s_m}\right]\ell_1+\left[\frac{Dm+F}{s_m}\right]),\prod_{\substack{p<z\\ p|k_1}} p)=1}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \Big\{1 + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2))) + O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}}))\Big\} \tag{5}$$

for $z\leq {}^{x}/k_1$.

To simplify this expression we use the notation "(m,z) app" to denote those m which satisfy the conditions

(i) $(s_m,\prod_{p<z}p)=1$

(ii) $((A+B)((Cm+E)+(Dm+F)),B(Dm+F))\equiv 1 \bmod 2$

(iii) $((A\ell_1+B)(\left[\frac{Cm+E}{s_m}\right]\ell_1+\left[\frac{Dm+F}{s_m}\right]),\prod_{\substack{p<z\\ p|k_1}} p)=1$

so that (5) becomes

$$S(x,y,z) = \frac{x}{k_1} \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \Big\{1 + O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\Big\}. \tag{6}$$

Recalling our assumption that ${}^{y}/k_2\leq {}^{x}/k_1$, if $z\leq \exp(10(\ln y)^{\frac{1}{2}})$, (6) and (3) give

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \text{app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) < \frac{y}{k_2} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \prod_{p<z} \left(1-\frac{1}{p}\right) \Gamma_z(w) \Big\{1 +$$

$$O(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}})) + O(\exp(-(\ln {}^y/k_2)^{\frac{1}{2}}))$$

$$+ O\left[\frac{\varphi([k_1,Rk_2]).\ln\ln x.\ln^{3/2}x}{x^{\frac{1}{2}}}\right]$$

$$+ O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\Big\}.$$

Letting $x\to\infty$ we get,

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \text{app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) < \frac{y}{k_2} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \prod_{p<z} \left(1-\frac{1}{p}\right) \Gamma_z(w) \Big\{1 +$$

$$O(\exp(-(\ln {}^y/k_2)^{\frac{1}{2}}))\Big\} \tag{7}$$

for $z\leq\exp(10(\ln y)^{\frac{1}{2}})$.

In particular writing $z_0=\exp(10(\ln y)^{\frac{1}{2}})$ we get

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z_0)\ \text{app}}} \prod_{\substack{p<z_0\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) < \frac{y}{\varphi(k_2)} \prod_{p<z_0} \left(1-\frac{1}{p}\right)^2 \Gamma_{z_0}(w) \Big\{1 +$$

$$O(\exp(-(\ln {}^y/k_2)^{\frac{1}{2}}))\Big\}. \tag{8}$$

We now determine the nature of $\displaystyle\prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$.

For $p<z$ and $(s_m,\prod_{p<z} p)=1$,

$$\rho_m(p) = \Big|\{n \bmod p:\ (An+B)((Cm+E)n+(Dm+F))\equiv 0 \bmod p\}\Big|.$$

The linear congruence

(i) $An+B\equiv 0 \bmod p$

has one solution if $p\nmid A$ and no solution if $p\mid A$. Similarly the linear congruence

(ii) $(Cm+E)n+(Dm+F)\equiv 0 \bmod p$

has one solution if $p \nmid (Cm+E)$ and no solution otherwise. So certainly $\rho_m(p) \leq 2$ for all primes p.

Suppose $p \nmid A(Cm+E)$ so that both (i) and (ii) have exactly one solution. Then $\rho_m(p)=2$ unless (i) and (ii) have the same solution. If this is the case multiplying (i) by Cm+E and (ii) by A it follows that

(iii) $A(Bm+F)-B(Cm+E) \equiv 0 \bmod p$.

So if $A(Dm+F)-B(Cm+E) \not\equiv 0 \bmod p$ then $\rho_m(p)=2$. Otherwise $\rho_m(p)=1$ or 2 and we leave $\rho_m(p)$ undetermined in this instance.

We have shown that

$$\prod_{\substack{p<z \\ p \nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \prod_{\substack{p<z \\ p \nmid k_1 \\ p \nmid A(Cm+E) \\ p \nmid A(Dm+F)-B(Cm+E)}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z \\ p \nmid k_1 \\ p \nmid A(Cm+E) \\ p \mid A(Dm+F)-B(Cm+E)}} \left(1-\frac{\rho_m(p)}{p}\right)$$

$$\times \prod_{\substack{p<z \\ p \nmid k_1 \\ p \mid A \\ p \nmid (Cm+E)}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p \nmid k_1 \\ p \nmid A \\ p \mid (Cm+E)}} \left(1-\frac{1}{p}\right) . \qquad (9)$$

This may be reduced to read

$$\prod_{\substack{p<z \\ p \nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \prod_{\substack{p<z \\ p \nmid k_1}} \left(1-\frac{1}{p}\right)^2 c(m,z) \qquad (10)$$

where

$$c(m,z) = \prod_{\substack{p<z \\ p \nmid k_1 \\ p \nmid A(Cm+E) \\ p \nmid A(Dm+F)-B(Cm+E)}} \left(1-\frac{1}{(p-1)^2}\right) \prod_{\substack{p<z \\ p \nmid k_1 \\ p \nmid A(Cm+E) \\ p \mid A(Dm+F)-B(Cm+E)}} \left(1+\frac{p(2-\rho_m(p))-1}{(p-1)^2}\right)$$

$$\times \prod_{\substack{p<z \\ p \nmid k_1 \\ p \mid A \\ p \nmid (Cm+E)}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z \\ p \nmid k_1 \\ p \nmid A \\ p \mid (Cm+E)}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z \\ p \nmid k_1 \\ p \mid A \\ p \mid (Cm+E)}} \left(1-\frac{1}{p}\right)^{-2} .$$

Only straightforward arguments are necessary, following Step Six of Theorem One, to show that, whenever $z>z_0$,

$$c(m,z) = c(m,z_0)\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\}. \tag{11}$$

Substituting (10) into (8) gives

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z_0)\text{ app}}} \prod_{\substack{p<z_0\\ p\nmid k_1}} (1-\frac{1}{p})^2 c(m,z_0) < \frac{y}{\varphi(k_2)} \prod_{p<z_0} (1-\frac{1}{p})^2 \Gamma_{z_0}(w) \Big\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\Big\}$$

so that

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z_0)\text{ app}}} c(m,z_0) < \frac{y}{\varphi(k_2)} \cdot \frac{\varphi^2(k_1)}{k_1^{\,2}} \cdot \Gamma_{z_0}(w)\Big\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\Big\} \tag{12}$$

To complete the Theorem for $y/k_2 \leq x/k_1$ we require an upper bound on the sum

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\text{ app}}} \prod_{\substack{p<z\\ p\nmid k_1}} (1-\frac{\rho_m(p)}{p}) \quad \text{for } z>z_0.$$

But from (10) and (11), for $z>z_0$,

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\text{ app}}} \prod_{\substack{p<z\\ p\nmid k_1}} (1-\frac{\rho_m(p)}{p}) = \prod_{\substack{p<z\\ p\nmid k_1}} (1-\frac{1}{p})^2 \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\text{ app}}} c(m,z)$$

$$= \prod_{\substack{p<z\\ p\nmid k_1}} (1-\frac{1}{p})^2 \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\text{ app}}} c(m,z_0)\ \{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\}\ .$$

Since, for $z>z_0$ and for y large, those m in the range $0<m\leq y$ which satisfy "(m,z) app" are exactly those satisfying "(m,z_0)

app" we have

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{1}{p}\right)^2 \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z_0)\ \mathrm{app}}} c(m,z_0)\{1 + O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))\}.$$

From (12) we now have, for $z\geq z_0$,

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \leq \frac{y}{\varphi(k_2)} \prod_{p<z} \left(1-\frac{1}{p}\right)^2 \Gamma_{z_0}(w) \left\{1 + O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))\right\}.$$

Further for $z\geq z_0$, $\Gamma_{z_0}(w) = \Gamma_z(w)\{1 + O(\exp(-\ln {}^{y}/k_2)^{\frac{1}{2}}))\}$.

So for $z\geq z_0$,

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \leq \frac{y}{k_2} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \prod_{p<z} \left(1-\frac{1}{p}\right) \Gamma_z(w) \left\{1 + O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))\right\}. \tag{13}$$

(13) and (7) substituted into (6) give

$$S(x,y,z) \leq \frac{x\,y}{k_1k_2} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \prod_{p<z} \left(1-\frac{1}{p}\right) \Gamma_z(w) \left\{1 + O(\exp(-\ln {}^{y}/k_2)^{\frac{1}{2}})) + O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\right\} \tag{14}$$

which completes the theorem for ${}^{y}/k_2\leq {}^{x}/k_1$.

We now turn to the second case, namely where $z\leq {}^{x}/k_1\leq {}^{y}/k_2$.

Equation (6) gave us, for $z\leq {}^{x}/k_1$,

$$\left|\{(n,m):\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1\bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2\bmod k_2,\ ((An+B)((Cn+D)m+(En+F)),\prod_{p<z}p)=1\}\right|$$

$$=\frac{x}{k_1}\sum_{\substack{0<m\leq y\\ m\equiv\ell_2\bmod k_2\\ (m,z)\ \mathrm{app}}}\ \prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{\rho_m(p)}{p}\right)\left\{1+O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}}))+O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\right\}.$$

An exactly parallel argument gives

$$\left|\{(n,m):\ \alpha+x<n\leq\alpha+2x,\ n\equiv\ell_1\bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2\bmod k_2,\ ((An+B)((Cn+D)m+(En+F)),\prod_{p<z}p)=1\}\right|$$

$$=\frac{x}{k_1}\sum_{\substack{0<m\leq y\\ m\equiv\ell_2\bmod k_2\\ (m,z)\ \mathrm{app}}}\ \prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{\rho_m(p)}{p}\right)\left\{1+O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}}))+O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\right\}.$$

Consequently

$$\left|\{(n,m):\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1\bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2\bmod k_2,\ ((An+B)((Cn+D)m+(En+F)),\prod_{p<z}p)=1\}\right|$$

$$=\tfrac{1}{2}\left|\{(n,m):\ \alpha<n\leq\alpha+2x,\ n\equiv\ell_1\bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2\bmod k_2,\ ((An+B)((Cn+D)m+(En+F)),\prod_{p<z}p)=1\}\right|$$

$$\times\{1+O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}}))+O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\}.$$

Writing $(\lambda-1)^{x}/k_1\leq {}^{y}/k_2\leq\lambda^{x}/k_1$, λ repetitions of the above argument gives

$$\left|\{(n,m):\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1\bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2\bmod k_2,\ ((An+B)((Cn+D)m+(En+F)),\prod_{p<z}p)=1\}\right|$$

$$= \frac{1}{\lambda} \Big| \{(n,m):\ \alpha<n\leq\alpha+\lambda x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2,$$

$$((An+B)((Cn+D)m+(En+F)), \prod_{p<z} p)=1\} \Big|$$

$$\times \{1 + O(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\}. \quad (15)$$

Since ${}^y/k_2 \leq \lambda {}^x/k_1$ we may apply (14) to the function

$$\Big| \{(n,m):\ \alpha<n\leq\alpha+\lambda x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2,$$

$$((An+B)((Cn+D)m+(En+F)), \prod_{p<z} p)=1\} \Big|$$

to give

$$\Big| \{(n,m):\ \alpha<n\leq\alpha+\lambda x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2,$$

$$((An+B)((Cn+D)m+(En+F)), \prod_{p<z} p)=1\} \Big|$$

$$\leq \lambda \frac{x}{k_1}\frac{y}{k_2} \prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{1}{p}) \prod_{p<z} (1-\frac{1}{p})\ \Gamma_z(w) \Big\{1 +$$

$$O(\exp(-\ln {}^y/k_2)^{\frac{1}{2}})) + O(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}}))$$

$$+ O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\Big\}. \quad (16)$$

(16) together with (15) gives

$$\Big| \{(n,m):\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv\ell_2 \bmod k_2,$$

$$((An+B)((Cn+D)m+(En+F)), \prod_{p<z} p)=1\} \Big|$$

$$\leq \frac{x}{k_1}\frac{y}{k_2} \prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{1}{p}) \prod_{p<z} (1-\frac{1}{p})\ \Gamma_z(w) \Big\{1 +$$

$$O(\exp(-\ln {}^y/k_2)^{\frac{1}{2}})) + O(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}}))$$

$$+ O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\Big\} \quad (17)$$

which completes the theorem for the case $z\leq {}^x/k_1 \leq {}^y/k_2$.

Finally we look at the case ${}^x/k_1 \leq z \leq {}^y/k_2$.

Equation (2) gave us for $z \leqslant y/k_2$,

$$S(x,y,z) = \frac{y}{k_2} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (r_n, \prod_{p<z} p)=1 \\ \left(\left[\frac{Cn+D}{r_n}\right]\ell_2+\left[\frac{En+F}{r_n}\right], \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1 \\ (An+B, \prod_{p<z} p)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid (Cn+D)}} \left(1-\frac{1}{p}\right)^{-1} \Big\{1 +$$

$$O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\Big\}.$$

Writing for convenience $x/k_1=x_1$, (2) and (17) give for $z=x_1$,

$$\sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (r_n, \prod_{p<x_1} p)=1 \\ \left(\left[\frac{Cn+D}{r_n}\right]\ell_2+\left[\frac{En+F}{r_n}\right], \prod_{\substack{p<x_1 \\ p\mid k_2}} p\right)=1 \\ (An+B, \prod_{p<x_1} p)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid (Cn+D)}} \left(1-\frac{1}{p}\right)^{-1} < \frac{x}{k_1} \prod_{p<x_1} \left(1-\frac{1}{p}\right) \Gamma_{x_1}(w) \Big\{1 +$$

$$O(\exp(-(\ln x/k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\Big\}. \tag{18}$$

Now for $z \geqslant x_1$,

$$\sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (r_n, \prod_{p<z} p)=1 \\ \left(\left[\frac{Cn+D}{r_n}\right]\ell_2+\left[\frac{En+F}{r_n}\right], \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1 \\ (An+B, \prod_{p<z} p)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid (Cn+D)}} \left(1-\frac{1}{p}\right)^{-1} < \sum_{\substack{\alpha<n\leqslant\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (r_n, \prod_{p<x_1} p)=1 \\ \left(\left[\frac{Cn+D}{r_n}\right]\ell_2+\left[\frac{En+F}{r_n}\right], \prod_{\substack{p<x_1 \\ p\mid k_2}} p\right)=1 \\ (An+B, \prod_{p<x_1} p)=1}} \prod_{\substack{p<x_1 \\ p\nmid k_2 \\ p\mid (Cn+D)}} \left(1-\frac{1}{p}\right)^{-1}$$

$$\times \{1 + O(\exp(-(\ln x/k_1)^{\frac{1}{2}}))\} \tag{19}$$

which follows easily from the observation that

$$\prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid (Cn+D)}} \left(1-\frac{1}{p}\right)^{-1} = \prod_{\substack{p<x_1 \\ p\nmid k_2 \\ p\mid (Cn+D)}} \left(1-\frac{1}{p}\right)^{-1} \{1 + O(\exp(-(\ln x/k_1)^{\frac{1}{2}}))\}.$$

Substitution of (19) and (18) into (2) gives

$$S(x,y,z) \leq \frac{x\,y}{k_1 k_2} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \prod_{p<x/k_1} \left(1-\frac{1}{p}\right) \Gamma_{x_1}(w) \Big\{1 +$$

$$O(\exp(-(\ln x/k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v - \ln\ln 3v - \ln 2 - 2)))\Big\}. \tag{20}$$

Since $\Gamma_{x_1}(w) = \Gamma_z(w)\{1 + O(\exp(-(\ln x/k_1)^{\frac{1}{2}}))\}$ for $z \geq x_1$ the Theorem is complete. □

CHAPTER THREE

INTRODUCTION

The major theorem of Chapter Three, Theorem Four, is an upper bound on the function

$$P(x,y,z) = \left|\{(q,r);\ \alpha<q\leq\alpha+x,\ q\equiv\ell_1 \bmod k_1,\ \beta<r\leq\beta+y,\ r\equiv\ell_2 \bmod k_2,\ ((aq^2+bq+c)r+(dq^2+eq+f), \prod_{p<z} p)=1\}\right|$$

where the q's and r's are primes. We assume as in Theorem One that a and d are not both zero and that the polynomials aq^2+bq+c and dq^2+eq+f have no common factors. (A study of the case where aq^2+bq+c and dq^2+eq+f have common factors may be undertaken without introducing any new methods of argument and is consequently omitted here.)

If $y/k_2 \geq x/k_1$ we find an upper bound on $P(x,y,z)$ with $\alpha=0$, and if $x/k_1 \geq y/k_2$ we find an upper bound on $P(x,y,z)$ with $\beta=0$. To emphasise the different approaches we define

$$P_1(x,y,z) = \left|\{(q,r);\ 0<q\leq x,\ q\equiv\ell_1 \bmod k_1,\ \beta<r\leq\beta+y,\ r\equiv\ell_2 \bmod k_2,\ ((aq^2+bq+c)r+(dq^2+eq+f), \prod_{p<z} p)=1\}\right|$$

and

$$P_2(x,y,z) = \left|\{(q,r);\ \alpha<q\leq\alpha+x,\ q\equiv\ell_1 \bmod k_1,\ 0<r\leq y,\ r\equiv\ell_2 \bmod k_2,\ ((aq^2+bq+c)r+(dq^2+eq+f), \prod_{p<z} p)=1\}\right|.$$

As would be expected an upper bound on $P_1(x,y,z)$ may be found independently of β if $y/k_2 \geq x/k_1$, and an upper bound on $P_2(x,y,z)$ may be found independently of α if $x/k_1 \geq y/k_2$.

There are at least two possible approaches to the problem of finding such upper bounds. Firstly we might, following the method of argument of Theorem One, write

$$P(x,y,z) = \sum_{\substack{\alpha<q\leq\alpha+x \\ q\equiv\ell_1 \bmod k_1}} \Big|\{r:\ \beta<r\leq\beta+y,\ r\equiv\ell_2 \bmod k_2,\ ((aq^2+bq+c)r+(dq^2+eq+f),\ \prod_{p<z} p)=1\}\Big|$$

and then apply Lemma 1.4 to the function within the summation sign. However, as the remarks after the statement of Lemma 1.4 indicate, any asymptotic formula or upper bound derived in this way would have associated error terms with non-computable 0-constants. Furthermore we would be forced to take both α and β to be zero.

An alternative approach, and the one that is adopted here, is to firstly study the functions

$$T_1(x,y,z) = \Big|\{(q,m);\ 0<q\leq x,\ q\equiv\ell_1 \bmod k_1,\ \beta<m\leq\beta+y,\ m\equiv\ell_2 \bmod k_2,\ (((aq^2+bq+c)m+(dq^2+eq+f))m,\ \prod_{p<z} p)=1\}\Big|$$

and

$$T_2(x,y,z) = \Big|\{(n,q);\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ 0<q\leq y,\ q\equiv\ell_2 \bmod k_2,\ (((an^2+bn+c)q+(dn^2+en+f))n,\ \prod_{p<z} p)=1\}\Big|,$$

the former when ${}^x/k_1\leq{}^y/k_2$, the latter when ${}^y/k_2\leq{}^x/k_1$. Upper bounds may be found for both of these functions with the associated error terms having computable 0-constants. In Step Thirteen of the proof of Theorem Four we demonstrate how these upper bounds can be used to give upper bounds on the functions $P_1(x,y,z)$ and $P_2(x,y,z)$.

It is worthy of note that were we to take the former route i.e. via Lemma 1.4, the main term of the subsequent asymptotic formula for $P(x,y,z)$ would be

$$c_1 \frac{x.y}{k_1.k_2.\ln{}^x/k_1.\ln{}^y/k_2} \prod_{p<z} (1-\frac{1}{p}) \qquad \text{(i)}$$

for some constant c_1 depending only on the constants a,b,c,d,e,

f, k_1, k_2. In contrast the leading term of the upper bound for $P_2(x,y,z)$ when $x/k_1 \geq y/k_2$ for $a+b+c \not\equiv d+e+f$ mod2 is of the order

$$c_2 \frac{x.y}{k_1.k_2} \prod_{p<z} (1-\frac{1}{p})^2 \prod_{p<z_2} (1-\frac{1}{p}) \qquad \text{(ii)}$$

where $z_2 = \min(z, \exp((\ln y/k_2)^{1-\epsilon})$ for some ϵ, $0<\epsilon<\frac{1}{2}$.

Assuming, for example, that z is approximately $x^{\frac{1}{2}}$, (ii) is weaker than (i) by a factor of $(\ln y)^{\epsilon}$. So it seems we pay rather a high penalty for computable 0-constants when $x/k_1 \geq y/k_2$. On the other hand when $y/k_2 \geq x/k_1$ and z is approximately $y^{\frac{1}{2}}$ then the two approaches give similar leading terms.

Before stating Theorem Four we give some definitions. Firstly we define the functions J_Z and G_Z as

$$J_Z = \frac{[k_1, 2Fk_2]^{\frac{1}{2}}}{\varphi([k_1, 2Fk_2])} \ln[k_1, 2Fk_2] \prod_{\substack{p<z \\ p \nmid 2k_2 h}} (1 + \frac{4p^{2/3}}{(p-1)^2}) \sum_{\substack{w \\ w=1 \text{ or } \gamma(w)>z}} \Upsilon_Z(w)$$

and

$$G_Z = \frac{[k_1, Fk_2]^{\frac{1}{2}}}{\varphi([k_1, Fk_2])} \prod_{\substack{p<z \\ p \nmid k_2 h}} (1 + \frac{4p^{2/3}}{(p-1)^2}) \sum_{\substack{w \\ w=1 \text{ or } \gamma(w)>z}} \Upsilon'_Z(w) .$$

The notation of J_Z and G_Z is as follows:

(i) $\gamma(w)$ denotes the smallest prime factor of w

(ii) $F=(cd-fa)^2-(bd-ea)(ce-fb)$

(iii) $h=(ad, ae+bd, af+be+cd, bf+ce, cf)$ and $A=ad/h$, $B=(ae+bd)/h$, $C=(af+be+cd)/h$, $D=(bf+ce)/h$, $E=cf/h$.

(iv) If ℓ_3 denotes the solution of the congruences $m \equiv 1 \bmod 2$ and $m \equiv \ell_2 \bmod k_2$, then $\Upsilon_Z(w)$ denotes the number of integers n in the interval $1 \leq n \leq 2Fk_2$ for which both

$$(an^2+bn+c, dn^2+en+f)=w$$

and

$$\left(\left[\frac{an^2+bn+c}{w}\right]\ell_3+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{\substack{p<z\\ p\nmid 2k_2}} p\right)=1$$

hold,

and finally

(v) $\Upsilon_z'(w)$ denotes the number of integers n in the interval $1\leq n\leq 2Fk_2$ for which both

$$(an^2+bn+c, dn^2+en+f)=w$$

and

$$\left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{\substack{p<z\\ p\nmid k_2}} p\right)=1$$

hold.

Let Δ denote the discriminant of the polynomial $(an^2+bn+c)(dn^2+en+f)$ if neither an^2+bn+c nor dn^2+en+f have repeated factors. If an^2+bn+c has a repeated factor, say $an^2+bn+c=\theta(\gamma n+\delta)^2$ and dn^2+en+f does not have a repeated factor, then let Δ denote the discriminant of $\theta(\gamma n+\delta)(dn^2+en+f)$. Similarly if dn^2+en+f has a repeated factor.

With this definition of Δ define

$$G(x) := \max_{0<q\leq x} |Aq^4+Bq^3+Cq^2+Dq+E|$$

for q prime, and

$$\ln \lambda = \left\{ \left[\frac{\ln(2.\Delta^2)}{\ln 2}\right]+1\right\}\ln 2.$$

Finally define, as in Theorem One,

$$\zeta=b^2-4ac,\ \eta=be-2cd-2fa, \text{ and } \theta=e^2-4fd.$$

With these definitions we have:

<u>THEOREM FOUR</u> *Let* an^2+bn+c *and* dn^2+en+f *be polynomials with integer coefficients, and with* a *and* d *not both zero. Assume that the polynomials have no common factors. Let* $x,y\in\mathbb{Z}$ *and* $\ell_1,\ell_2,k_1,k_2\in\mathbb{N}$ *with* $(\ell_1,k_1)=1$, $(\ell_2,k_2)=1$ *and*

$\exp((\ln{}^y/k_2)^{\frac{1}{2}}) > \max(|a|,|b|,|c|,|d|,|e|,|f|,k_1,k_2)$.

Let ϵ *be any constant satisfying* $0<\epsilon<\frac{1}{2}-\frac{4}{\ln\ln y}$.

Then we have the following upper bounds on $P_1(x,y,z)$ *and* $P_2(x,y,z)$.

(I) *If* ${}^y/k_2 > {}^x/k_1$ *and* z *satisfies* $3 \leq z \leq {}^y/k_2$ *then for* $a+b+c \not\equiv d+e+f \bmod 2$,

$$P_1(x,y,z) < \frac{2xy}{[2,k_2]\ln x} \prod_{\substack{p<z\\ p\nmid 2k_2}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z\\ p\nmid 2k_2\\ p\mid h}} \left(1-\frac{1}{p}\right)^{-1} J_z \left\{1 + O\left[\frac{\varphi([k_1,2Fk_2])}{[k_1,2Fk_2]^{\frac{1}{2}}} \frac{\ln\ln G(x)\ \ln^{\lambda}G(x)\ \ln x}{x}\right] + O\left[\frac{\varphi([k_1,2Fk_2])}{[k_1,2Fk_2]^{\frac{1}{2}}} \frac{k_2}{\varphi(k_1)} \frac{z \ln^2 z}{y}\right] + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) + O(\exp(-(\ln{}^y/k_2)^{\frac{1}{2}}))\right\}$$

where $u = \frac{\ln{}^y/k_2}{\ln z}$.

(II) *If* ${}^y/k_2 > {}^x/k_1$ *and* $a+b+c \equiv d+e+f \bmod 2$ *and if the conditions*

(i) $2 \equiv \ell_1 \bmod k_1$

(ii) $c \not\equiv f \bmod 2$

(iii) $(r, \prod_{p<z} p)=1$

(iv) $\left(\left[\frac{4a+2b+c}{r}\right]\ell_3+\left[\frac{4d+2e+f}{r}\right], \prod_{\substack{p<z\\ p\mid 2k_2}} p\right)=1$

are satisfied where $r=(4a+2b+c,4d+2e+f)$ *then*

$$P_1(x,y,z) < \frac{y}{[2,k_2]} \prod_{\substack{p<z\\ p\nmid 2k_2\\ p\nmid (4a+2b+c)(4d+2e+f)}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z\\ p\nmid 2k_2\\ p\mid (4a+2b+c)(4d+2e+f)}} \left(1-\frac{1}{p}\right)$$

$$\times \left\{1 + O(\exp(-(\ln{}^y/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) + O\left[\frac{k_2}{\varphi(k_1)} \frac{x.z.\ln^2 z}{y.\ln x}\right]\right\}.$$

If on the other hand at least one of the conditions (i) *to* (iv) *is not satisfied then* $P_1(x,y,z)=0$.

(III) *If* $x/k_1 > y/k_2$ *and* z *satisfies* $3 \le z \le x/k_1$ *with* $a+b+c \not\equiv d+e+f$ mod2 *then*

$$P_2(x,y,z) \le \frac{y.x}{k_2} \prod_{p<z}\left(1-\frac{1}{p}\right)^2 \prod_{p<z_2}\left(1-\frac{1}{p}\right) \prod_{z_2 \le p<z}\left(1-\frac{1}{(p-1)^2}\right)$$

$$\times \prod_{\substack{p<z_2\\ p|k_2}}\left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z_2\\ p|k_2 h}}\left(1-\frac{1}{p}\right)^{-1} \quad G_z \left\{1 + O(\exp(-(\ln^x/k_1)^{\frac{1}{2}}))\right.$$

$$+ O_\epsilon\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}}\, \sigma\left[\frac{\eta^2-\zeta\theta}{4}\right] \varphi(k_2)\ |\zeta|^{\frac{1}{2}} \ln\ln^3|a\zeta c|\ \frac{\ln\ln^3 y}{(\ln y)^{\epsilon/2}}\right]$$

$$+ O\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}}\ \frac{k_2}{\varphi(k_2)}\ \frac{z.\ln^2 z}{x}\right]$$

$$\left. + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))\right\}$$

where $v = \frac{\ln^x/k_1}{\ln z}$ *and* $z_2 = \min(z,\ \exp((\ln^y/k_2)^{1-\epsilon}))$.

(IV) *If* $x/k_1 > y/k_2$ *and* z *satisfies* $3 \le z \le x/k_1$ *with* $a+b+c \equiv d+e+f$ mod2 *then*

$$P_2(x,y,z) \le \frac{x}{k_1} \prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{1}{p}\right) \left\{1 + O(\exp(-(\ln^x/k_1)^{\frac{1}{2}}))\right.$$

$$\left. + O(\exp(-v(\ln v-\ln\ln 3v-2))) + O\left[\frac{k_1}{\varphi(k_1)}\ \frac{z.\ln z}{x}\ \frac{y}{\ln y}\right] \right\}$$

where $v = \frac{\ln^x/k_1}{\ln z}$.

The O*-constants are absolute, effectively computable, and independent of* a,b,c,d,e,f,k_1, *and* k_2.

PROOF OF THEOREM FOUR

As in Theorem One we split the proof into steps. Since the proof of Theorem Four is in many respects similar to that of Theorem One wherever possible the steps are kept parallel. The proof of Theorem Four is not given in as much detail as that of Theorem One, except where wholly new material and arguments are employed.

STEP ONE *An upper bound for*

$$T_1(x,y,z) = \Big|\{(q,m):\ 0<q\le x,\ q\equiv\ell_1 \bmod k_1,\ \beta<m\le\beta+y,\ m\equiv\ell_2 \bmod k_2,\ (((aq^2+bq+c)m+(dq^2+eq+f))m,\ \prod_{p<z} p\ )=1\ \}\Big|$$

with $x/k_1 \le y/k_2$ *and* $z \le y/k_2$.

Define for q fixed the function

$$M(y,z,q) = \Big|\{m:\ \beta<m\le\beta+y,\ m\equiv\ell_2 \bmod k_2,\ (((aq^2+bq+c)m+(dq^2+eq+f))m,\prod_{p<z} p\ )=1\}\Big|$$

so that $T_1(x,y,z)$ may be written

$$T_1(x,y,z) = \sum_{\substack{0<q\le x \\ q\equiv\ell_1 \bmod k_1}} M(y,z,q)\ . \tag{1}$$

It is clear that we may rewrite $M(y,z,q)$ as

$$M(y,z,q) = \Big|\{m:\ \beta<m\le\beta+y,\ m\equiv 1 \bmod 2,\ m\equiv\ell_2 \bmod k_2,\ (((aq^2+bq+c)m+(dq^2+eq+f))m,\prod_{p<z} p\ )=1\}\Big|.$$

The congruences $m\equiv 1 \bmod 2$ and $m\equiv\ell_2 \bmod k_2$ have a common solution $\bmod[2,k_2]$ if and only if $(2,k_2)\mid(\ell_2-1)$. If $(2,k_2)=2$ then, for $(\ell_2,k_2)=1$, it follows that $\ell_2\equiv 1 \bmod 2$ and consequently that $(2,k_2)\mid(\ell_2-1)$. Hence there always exists a constant ℓ_3 with $(\ell_3,[2,k_2])=1$ such that

$$M(y,z,q) = \Big|\{m:\ \beta<m\leq\beta+y,\ m\equiv\ell_3 \bmod[2,k_2],$$
$$(((aq^2+bq+c)m+(dq^2+eq+f))m, \prod_{p<z} p\ )=1\}\Big|. \qquad (2)$$

Suppose firstly that $(a+b+c)\equiv(d+e+f) \bmod 2$. Then, for $q>2$ and $m\equiv 1 \bmod 2$, it follows that $(aq^2+bq+c)m+(dq^2+eq+f)\equiv 0 \bmod 2$ and consequently that $M(y,z,q)=0$. Under these circumstances (1) becomes

$$T_1(x,y,z) = \begin{cases} 0 & ;\ \text{if } 2\not\equiv\ell_1 \bmod k_1 \\ M(y,z,2) & ;\ \text{otherwise} \end{cases} \qquad (3)$$

If, in addition to $a+b+c\equiv d+e+f \bmod 2$, we have $c\equiv f \bmod 2$ then for $q=2$ and $m\equiv 1 \bmod 2$ it again follows that $(aq^2+bq+c)m+(dq^2+eq+f)\equiv 0 \bmod 2$ and that $M(y,z,q)=0$. So we may further adapt (1) to read

$$T_1(x,y,z) = \begin{cases} 0 & ;\ \text{if } 2\not\equiv\ell_1 \bmod k_1 \ \underline{\text{or}}\ c\equiv f \bmod 2 \\ M(y,z,2) & ;\ \text{otherwise} \end{cases} \qquad (4)$$

So in the case $a+b+c\equiv d+e+f \bmod 2$ we are left with the problem of estimating $M(y,z,2)$ when $c\not\equiv f \bmod 2$.

From (2), for $q=2$, we have

$$M(y,z,2) = \Big|\{m:\ \beta<m\leq\beta+y,\ m\equiv\ell_3 \bmod[2,k_2],$$
$$(((4a+2b+c)m+(4d+2e+f))m, \prod_{p<z} p\ )=1\}\Big|.$$

Define r_2 to be the highest common factor of $4a+2b+c$ and $4d+2e+f$. It is apparent that if $(r_2, \prod_{p<z} p)>1$ then $M(y,z,q)=0$. Assuming that $(r_2, \prod_{p<z} p)=1$, $M(y,z,2)$ becomes

$$M(y,z,2) = \Big|\{m:\ \beta<m\leq\beta+y,\ m\equiv\ell_3 \bmod[2,k_2],$$
$$((\left[\frac{4a+2b+c}{r_2}\right]m+\left[\frac{4d+2e+f}{r_2}\right])m, \prod_{p<z} p\ )=1\}\Big|.$$

An application of Lemma 1.2 gives

$$M(y,z,2) = \begin{cases} \frac{y}{[2,k_2]} \prod_{\substack{p<z \\ p\nmid 2k_2}} \left(1-\frac{\rho'(p)}{p}\right) \{1+O(\exp(-(\ln {}^{y}/2k_2)^{\frac{1}{2}})) \\ \qquad +O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\} \\ \qquad ; \quad \left(\left[\frac{4a+2b+c}{r_2}\right]\ell_3+\left[\frac{4d+2e+f}{r_2}\right], \prod_{\substack{p<z \\ p\mid 2k_2}} p\right)=1 \\ 0 \qquad ; \quad \text{otherwise} \end{cases}$$

where $u = \frac{\ln {}^{y}/2k_2}{\ln z}$, and where

$$\rho'(p) = \begin{cases} \rho(p)+1 \ ; & p\nmid \frac{4d+2e+f}{r_2} \\ \rho(p) \ ; & p\mid \frac{4d+2e+f}{r_2} \end{cases}$$

with

$$\rho(p) = \left|\{m \bmod p:\ \left[\frac{4a+2b+c}{r_2}\right]m+\left[\frac{4d+2e+f}{r_2}\right] \equiv 0 \bmod p\}\right|$$

provided that

(i) $\rho(p)<p$ for all primes p; and

(ii) $\rho(p)<p-1$ if $p\nmid \frac{4d+2e+f}{r_2}$.

But

$$\rho(p) = \begin{cases} 1 \ ; & p\nmid \frac{4a+2b+c}{r_2} \\ 0 \ ; & p\mid \frac{4a+2b+c}{r_2} \end{cases}$$

so that (i) is easily seen to be satisfied, and (ii) is satisfied for all $p>2$. If $p=2$ then for $\frac{4d+2e+f}{r_2} \not\equiv 0 \bmod 2$ we require $\rho(2)=0$. But under our assumption $(r_2, \prod_{p<z} p)=1$ we may assume that $2\nmid r_2$ so that $\frac{4d+2e+f}{r_2} \not\equiv 0 \bmod 2 \Leftrightarrow 4d+2e+f\not\equiv 0 \bmod 2 \Leftrightarrow$ $f\not\equiv 0 \bmod 2$. For $c\not\equiv f \bmod 2$ this implies $c\equiv 0 \bmod 2$ and consequently that $\frac{4a+2b+c}{r_2} \equiv 0 \bmod 2$ giving, as required, $\rho(2)=0$.

We are now in a position to state

$$T_1(x,y,z) = \begin{cases} 0\ ; & \text{if } 2\not\equiv \ell_1 \bmod k_1 \ \underline{\text{or}}\ c\equiv f \bmod 2\ \underline{\text{or}} \\ & (r_2, \prod_{p<z} p)>1\ \underline{\text{or}}\ \left(\left[\frac{4a+2b+c}{r_2}\right]\ell_3+\left[\frac{4d+2e+f}{r_2}\right], \prod_{\substack{p<z\\ p\mid 2k_2}} p\right)>1 \\ \frac{y}{[2,k_2]} \prod_{\substack{p<z\\ p\nmid 2k_2}} \left(1-\frac{\rho'(p)}{p}\right) \{1+O(\exp(-(\ln y/2k_2)^{\frac{1}{2}})) \\ \qquad +O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\} & ; \text{ otherwise} \end{cases} \tag{5}$$

where $r_2=(4a+2b+c,4d+2e+f)$; ℓ_3 denotes the solution of the congruences $m\equiv 1 \bmod 2$ and $m\equiv \ell_2 \bmod k_2$; $u=\frac{\ln y/2k_2}{\ln z}$;

$$\rho'(p) = \begin{cases} 2\ ; & p\nmid \left[\frac{4a+2b+c}{r_2}\right]\left[\frac{4d+2e+f}{r_2}\right] \\ 1\ ; & p\mid \left[\frac{4a+2b+c}{r_2}\right]\left[\frac{4d+2e+f}{r_2}\right] \end{cases}$$

if $a+b+c\equiv d+e+f \bmod 2$.

We now turn to the second case, namely where $a+b+c\not\equiv d+e+f \bmod 2$. Define r_q to be the highest common factor of the two polynomials aq^2+bq+c and dq^2+eq+f. It is apparent that if $(r_q, \prod_{p<z} p)>1$ then $M(y,z,q)=0$. Assuming that $(r_q, \prod_{p<z} p)=1$, $M(y,z,q)$ becomes

$$M(y,z,q) = \left|\{m:\ \beta<m\leq\beta+y,\ m\equiv\ell_3 \bmod [2,k_2],\ \left(\left(\left[\frac{aq^2+bq+c}{r_q}\right]m+\left[\frac{dq^2+eq+f}{r_q}\right]\right)m, \prod_{p<z} p\right)=1\}\right|$$

We have already dealt with $M(y,z,2)$ so we may assume here that $q\geq 3$. A second application of Lemma 1.2 gives

$$M(y,z,q) = \begin{cases} \frac{y}{[2,k_2]} \prod_{\substack{p<z\\ p\nmid 2k_2}} \left(1-\frac{\rho'(p)}{p}\right) \{1+O(\exp(-(\ln y/2k_2)^{\frac{1}{2}})) \\ \qquad +O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\} \\ \qquad ;\ \left(\left[\frac{aq^2+bq+c}{r_q}\right]\ell_3+\left[\frac{dq^2+eq+f}{r_q}\right], \prod_{\substack{p<z\\ p\mid 2k_2}} p\right)=1 \\ 0 \qquad ;\ \text{otherwise} \end{cases} \tag{6}$$

where $u = \frac{\ln y/2k_2}{\ln z}$, and where

$$\rho'(p) = \begin{cases} \rho(p)+1 \;;\; p \nmid \frac{dq^2+eq+f}{r_q} \\ \rho(p) \;;\; p \mid \frac{dq^2+eq+f}{r_q} \end{cases}$$

with

$$\rho(p) = \left| \{ m \bmod p : \left(\frac{aq^2+bq+f}{r_q}\right)m + \left(\frac{dq^2+eq+f}{r_q}\right) \equiv 0 \bmod p \} \right|$$

provided that

(i) $\rho(p)<p$ for all primes p; and

(ii) $\rho(p)<p-1$ if $p \nmid \frac{dq^2+eq+f}{r_q}$.

But

$$\rho(p) = \begin{cases} 1 \;;\; p \nmid \frac{aq^2+bq+c}{r_q} \\ 0 \;;\; p \mid \frac{aq^2+bq+c}{r_q} \end{cases}$$

so that to satisfy (i) and (ii) we have only to show that $\rho(2)=0$ when $\frac{dq^2+eq+f}{r_q} \not\equiv 0 \bmod 2$. But, as $(r_q, \prod_{p<z} p)=1$, we have

$$\rho(2) = \left| \{ m \bmod 2 : (aq^2+bq+c)m+(dq^2+eq+f) \equiv 0 \bmod 2 \} \right|$$

$$= \left| \{ m \bmod 2 : (a+b+c)m+(d+e+f) \equiv 0 \bmod 2 \} \right|$$

for $q \geq 3$. As $a+b+c \not\equiv d+e+f \bmod 2$ it is easily seen that we have $\rho(2)=0$ as required.

Recalling that

$$M(y,z,2) = \begin{cases} \left[\frac{y}{2,k_2}\right] \prod_{\substack{p<z \\ p \nmid 2k_2}} \left(1-\frac{\rho'(p)}{p}\right) \{1+O(\exp(-(\ln y/2k_2)^{\frac{1}{2}})) \\ \qquad +O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\} \\ \qquad ;\; \left(\left[\frac{4a+2b+c}{r_2}\right]\ell_3+\left[\frac{4d+2e+f}{r_2}\right], \prod_{\substack{p<z \\ p \mid 2k_2}} p\right)=1 \\ \qquad \text{and } c \not\equiv f \bmod 2,\; (r_2, \prod_{p<z} p)=1 \\ 0 \;;\; \text{otherwise} \end{cases} \tag{7}$$

summing (6) and (7) over q gives, for $a+b+c\not\equiv d+e+f \bmod 2$,

$$T_1(x,y,z) = \frac{y}{[2,k_2]} \sum_{\substack{\alpha<q\leq x \\ q\equiv \ell_1 \bmod k_1 \\ (r_q,\prod_{p<z} p)=1 \\ \left(\left[\frac{aq^2+bq+c}{r_q}\right]\ell_3+\left[\frac{dq^2+eq+f}{r_q}\right],\ \prod_{\substack{p<z \\ p\mid 2k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\nmid 2k_2}} \left(1-\frac{\rho'(p)}{p}\right) \Big\{ 1 +$$

$$O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) + O(\exp(-(\ln y/2k_2)^{\frac{1}{2}})) \Big\} \tag{8}$$

where

$$\alpha = \begin{cases} 0 & \text{if } c\not\equiv f \bmod 2 \\ 2 & \text{if } c\equiv f \bmod 2 \end{cases} .$$

Now, for $(r_q,\prod_{p<z} p)=1$,

$$\prod_{\substack{p<z \\ p\nmid 2k_2}} \left(1-\frac{\rho'(p)}{p}\right) = \prod_{\substack{p<z \\ p\nmid 2k_2 \\ p\nmid (aq^2+bq+c)(dq^2+eq+f)}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z \\ p\nmid 2k_2 \\ p\mid (aq^2+bq+c)(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)$$

$$\leq \prod_{\substack{p<z \\ p\nmid 2k_2 \\ p\nmid (aq^2+bq+c)(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z \\ p\nmid 2k_2 \\ p\mid (aq^2+bq+c)(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)$$

$$= \prod_{\substack{p<z \\ p\nmid 2k_2}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid 2k_2 \\ p\nmid (aq^2+bq+c)(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)$$

$$= \prod_{\substack{p<z \\ p\nmid 2k_2}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z \\ p\nmid 2k_2 \\ p\mid (aq^2+bq+c)(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)^{-1}$$

and (8) becomes

$$T_1(x,y,z) < \frac{y}{[2,k_2]} \prod_{\substack{p<z \\ p\nmid 2k_2}} \left(1-\frac{1}{p}\right)^2 \sum_{\substack{\alpha<q\leq x \\ q\equiv \ell_1 \bmod k_1 \\ (r_q, \prod_{p<z} p)=1 \\ \left(\left[\frac{aq^2+bq+c}{r_q}\right]\ell_3+\left[\frac{dq^2+eq+f}{r_q}\right],\ \prod_{\substack{p<z \\ p\mid 2k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\nmid 2k_2 \\ p\mid (aq^2+bq+f)(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)^{-1} \Big\{ 1$$

$$+ O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) + O(\exp(-(\ln {}^{y}/{}_{2k_2})^{\frac{1}{2}})) \Big\}. \tag{9}$$

Now the sum

$$\sum_{\substack{\alpha<q\leq x \\ q\equiv \ell_1 \bmod k_1 \\ (r_q, \prod_{p<z} p)=1 \\ \left(\left[\frac{aq^2+bq+c}{r_q}\right]\ell_3+\left[\frac{dq^2+eq+f}{r_q}\right],\ \prod_{\substack{p<z \\ p\mid 2k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\nmid 2k_2 \\ p\mid (aq^2+bq+f)(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)^{-1}$$

$$= \sum_{\substack{w \\ w=1 \text{ or } \gamma(w)>z}} \ \sum_{\substack{\alpha<q\leq x \\ q\equiv \ell_1 \bmod k_1 \\ (aq^2+bq+c,\, dq^2+eq+f)=w \\ \left(\left[\frac{aq^2+bq+c}{w}\right]\ell_3+\left[\frac{dq^2+eq+f}{w}\right],\ \prod_{\substack{p<z \\ p\mid 2k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\nmid 2k_2 \\ p\mid (aq^2+bq+f)(dq^2+eq+f)}} \left(1-\frac{1}{p}\right)^{-1}$$

where $\gamma(w)$ denotes the smallest prime factor of w.

By Lemma 2.10 this sum is less than

$$\frac{2x}{\ln x} \prod_{\substack{p<z \\ p\nmid 2k_2 \\ p\mid h}} \left(1-\frac{1}{p}\right)^{-1} \frac{[k_1,2Fk_2]^{\frac{1}{2}} \ln[k_1,2Fk_2]}{\varphi([k_1,2Fk_2])} \prod_{\substack{p<z \\ p\nmid 2k_2 h}} \left(1+\frac{4p^{2/3}}{(p-1)^2}\right)$$

$$\times \sum_{\substack{w \\ w=1 \text{ or } \gamma(w)>z}} \Upsilon_z(w) \left\{ 1 + O\left[\frac{\varphi([k_1,2Fk_2])}{[k_1,2Fk_2]^{\frac{1}{2}}} \cdot \frac{\ln\ln G(x).\ln^{\lambda} G(x) \ln x}{x}\right]\right\} \tag{10}$$

where

(i) $F = |(cd-fa)^2-(bd-ea)(ce-fb)|$

(ii) $h = (ad, ae+bd, af+be+cd, bf+ce, cf)$

and $A={}^{ad}/{}_{h}$, $B={}^{(ae+bd)}/{}_{h}$, $C={}^{(af+be+cd)}/{}_{h}$, $D={}^{(bf+ce)}/{}_{h}$

$E={}^{cf}/{}_{h}$.

(iii) $G(x) = \max\limits_{0<q\leq x} |Aq^4+Bq^3+Cq^2+Dq+E|$

and $\ln\lambda = \left\{ \left[\frac{\ln(2.\Delta^2)}{\ln 2}\right]+1 \right\} \ln 2$

where Δ denotes the discriminant of $(aq^2+bq+c)(dq^2+eq+f)$ if neither aq^2+bq+c nor dq^2+eq+f have repeated factors. If aq^2+bq+c has a repeated factor, say $aq^2+bq+c=\theta(\gamma q+\delta)^2$ and dq^2+eq+f does not have a repeated factor then Δ is the discriminant of $\theta(\gamma q+\delta)(dq^2+eq+f)$. Similarly if dq^2+eq+f has a repeated factor. Clearly with this definition $\Delta\neq 0$.

and where

(iv) $\Upsilon_z(w)$ denotes the number of integers n in the interval $1<n\leqslant 2Fk_2$ for which both

$$(an^2+bn+c, dn^2+en+f)=w$$

and $\left(\left[\frac{an^2+bn+c}{w}\right]\ell_3+\left[\frac{dn^2+en+f}{w}\right], \prod_{\substack{p<z\\ p\mid 2k_2}} p \right)=1$.

Let $$J_z = \frac{[k_1,2Fk_2]^{\frac{1}{2}}.\ln[k_1,2Fk_2]}{\varphi([k_1,2Fk_2])} \prod_{\substack{p<z\\ p\nmid 2k_2h}} \left(1+\frac{4p^{2/3}}{(p-1)^2}\right) \sum_{\substack{w\\ w=1 \text{ or } \gamma(w)>z}} \Upsilon_z(w).$$

Substituting (10) into (9) gives

$$T_1(x,y,z)<\frac{2xy}{[2,k_2].\ln x} \prod_{\substack{p<z\\ p\nmid 2k_2}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z\\ p\nmid 2k_2\\ p\mid h}} \left(1-\frac{1}{p}\right)^{-1} J_z \Bigg\{1 + O\left[\frac{\varphi([k_1,2Fk_2])}{[k_1,2Fk_2]^{\frac{1}{2}}} . \frac{\ln\ln G(x).\ln^{\lambda}G(x)}{x}\right]$$
$$+ O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) + O(\exp(-(\ln {}^{y}/2k_2)^{\frac{1}{2}}))\Bigg\}. \quad (11)$$

This completes Step One.

<u>STEP TWO</u> *An upper bound for*

$$T_2(x,y,z) = \sum_{\substack{\alpha<n\leqslant\alpha+x\\ n\equiv\ell_1 \bmod k_1}} \Big|\{q:\ 0<q\leqslant y,\ q\equiv\ell_2 \bmod k_2, (((an^2+bn+c)q+(dn^2+en+f))n, \prod_{p<z} p)=1\}\Big|$$

with $^{x}/k_1 \geq {}^{y}/k_2$ *and* $z \leq \exp((\ln {}^{y}/k_2)^{1-\epsilon})$, $0<\epsilon<\frac{1}{2}$.

The reasoning of Step Two follows very closely that of Step One. Define for n fixed

$$N(y,z,n) = \left|\{q:\ 0<q\leq y,\ q\equiv \ell_2 \bmod k_2,\ ((an^2+bn+c)q+(dn^2+en+f),\ \prod_{p<z} p)=1\ \}\right|$$

so that $T_2(x,y,z)$ may be written

$$T_2(x,y,z) = \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (n,\prod_{p<z}p)=1}} N(y,z,n) \qquad (12)$$

Certainly within this sum $n\equiv 1 \bmod 2$ for otherwise $(n,\prod_{p<z}p)>1$. Now if $(a+b+c)\equiv(d+e+f) \bmod 2$ then for $q>2$, q prime, $(an^2+bn+c)q+(dn^2+en+f)\equiv 0 \bmod 2$. Consequently, if $(a+b+c)\equiv(d+e+f) \bmod 2$, $N(y,z,n)\leq 1$ and

$$T_2(x,y,z) \leq \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (n,\prod_{p<z}p)=1}} 1 .$$

But by Lemma 1.1 we have

$$\sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (n,\prod_{p<z}p)=1}} 1 = \frac{x}{k_1} \prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{1}{p}) \{1+O(\exp(-v(\ln v-\ln\ln 3v-2))) + O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}}))\} \qquad (13)$$

where $v=\dfrac{\ln {}^{x}/k_1}{\ln z}$.

Hence if $a+b+c\equiv d+e+f \bmod 2$ we have

$$T_2(x,y,z) \leq \frac{x}{k_1} \prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{1}{p}) \{1+O(\exp(-v(\ln v-\ln\ln 3v-2))) + O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}}))\} \qquad (14)$$

and this completes our estimate of $T_2(x,y,z)$ in this instance.

Assuming instead that $a+b+c \not\equiv d+e+f \bmod 2$, define r_n to be the highest common factor of an^2+bn+c and dn^2+en+f. Clearly if $(r_n, \prod_{p<z} p)>1$ then $N(y,z,n)=0$. Assuming then that both $a+b+c \not\equiv d+e+f \bmod 2$ and $(r_n, \prod_{p<z} p)=1$, $N(y,z,n)$ becomes

$$N(y,z,n) = \left| \{ q:\ 0<q\leq y,\ q\equiv \ell_2 \bmod k_2,\ \left(\left[\frac{an^2+bn+c}{r_n}\right]q+\left[\frac{dn^2+en+f}{r_n}\right],\ \prod_{p<z} p\right)=1 \} \right|$$

which on applying Lemma 1.3 gives

$$N(y,z,n) \leq \begin{cases} \dfrac{y}{k_2} \prod\limits_{\substack{p<z \\ p\nmid k_2}} \left(1-\dfrac{\rho'(p)}{p}\right) \Big\{ 1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) \\ \qquad + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) \Big\} \\ \qquad + O(A) & ;\ \left(\left[\dfrac{an^2+bn+c}{r_n}\right]\ell_2+\left[\dfrac{dn^2+en+f}{r_n}\right],\ \prod\limits_{\substack{p<z \\ p\mid k_2}} p\right)=1 \\ 0 & ;\ \text{otherwise} \end{cases} \qquad (15)$$

where $u= \dfrac{\ln y/k_2}{\ln z}{}^2$,

$$A = \begin{cases} \dfrac{z}{\varphi(k_2)\ln z/k_2} & ;\ z>k_2 \\ 1 & ;\ z\leq k_2 \end{cases}$$

and

$$\rho'(p) = \begin{cases} \rho(p)+1 & ;\ p\nmid \dfrac{dn^2+en+f}{r_n} \\ \rho(p) & ;\ p\mid \dfrac{dn^2+en+f}{r_n} \end{cases}$$

with

$$\rho(p) = \left| \{ m \bmod p:\ \left[\frac{an^2+bn+c}{r_n}\right]m+\left[\frac{dn^2+en+f}{r_n}\right]\equiv 0 \bmod p \} \right|$$

provided that (i) $\rho(p)<p$ for all primes p, and

(ii) $\rho(p)<p-1$ if $p\nmid \dfrac{dn^2+en+f}{r_n}$.

But

$$\rho(p) = \begin{cases} 1 & ; p \nmid \frac{an^2+bn+c}{r_n} \\ 0 & ; p \mid \frac{an^2+bn+c}{r_n} \end{cases}$$

and (i) and (ii) are seen to hold by reasoning similar to that used in Step One.

Taking the O(A) term into the main term gives, for

$$\left(\left[\frac{an^2+bn+c}{r_n}\right]\ell_2+\left[\frac{dn^2+en+f}{r_n}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1\ ,$$

$$N(y,z,n) < \frac{y}{k_2} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho'(p)}{p}\right) \Big\{ 1 + O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))$$

$$+ O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) + O\left[\frac{Ak_2}{y} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho'(p)}{p}\right)^{-1}\right] \Big\}. \tag{16}$$

But clearly

$$\prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho'(p)}{p}\right)^{-1} < \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{2}{p}\right)^{-1} \ll \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right)^{-2} \ll \begin{cases} \ln^2 z . \frac{\varphi^2(k_2)}{k_2{}^2} & ; z>k_2 \\ \ln^2 z & ; z\leqslant k_2 \end{cases}$$

For $z\leqslant\exp((\ln {}^{y}/k_2)^{1-\epsilon})$ this gives

$$\prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho'(p)}{p}\right)^{-1} \ll \begin{cases} (\ln {}^{y}/k_2)^{2(1-\epsilon)} . \frac{\varphi^2(k_2)}{k_2^2} & ; z>k_2 \\ (\ln {}^{y}/k_2)^{2(1-\epsilon)} & ; z\leqslant k_2 \end{cases} .$$

Further

$$A \ll \begin{cases} \frac{1}{\varphi(k_2)} . \exp((\ln {}^{y}/k_2)^{1-\epsilon}) & ; z>k_2 \\ 1 & ; z\leqslant k_2 \end{cases}$$

and from here it is a simple matter to show that

$$\frac{Ak_2}{y} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho'(p)}{p}\right)^{-1} \ll \exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}})$$

and the third error term of (16) becomes absorbed into the second.

Summing (15) over n gives

$$T_2(x,y,z) < \frac{y}{k_2} \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (n,\prod_{p<z} p)=1 \\ (r_n,\prod_{p<z} p)=1 \\ \left(\left[\frac{an^2+bn+c}{r_n}\right]\ell_2+\left[\frac{dn^2+en+f}{r_n}\right],\ \prod_{\substack{p<z\\ p\mid k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho'(p)}{p}\right) \Big\{1 +$$

$$O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\Big\}. \tag{17}$$

From the definition of $\rho'(p)$ we have

$$\rho'(p) = \begin{cases} 2 \ ; \ p\nmid(an^2+bn+c)(dn^2+en+f) \\ 1 \ ; \ p\mid(an^2+bn+c)(dn^2+n+f) \end{cases}$$

for $(r_n,\prod_{p<z} p)=1$ and $p<z$. Hence the product $\prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho'(p)}{p}\right)$ in

(17) becomes

$$\prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho'(p)}{p}\right) = \prod_{\substack{p<z \\ p\nmid k_2 \\ p\nmid(an^2+bn+c)(dn^2+en+f)}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid(an^2+bn+c)(dn^2+en+f)}} \left(1-\frac{1}{p}\right)$$

$$< \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid(an^2+bn+c)(dn^2+en+f)}} \left(1-\frac{1}{p}\right)^{-1}$$

and (17) becomes

$$T_2(x,y,z) < \frac{y}{k_2} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right)^2 \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv \ell_1 \bmod k_1 \\ (n,\prod_{p<z} p)=1 \\ (r_n,\prod_{p<z} p)=1 \\ \left(\left[\frac{an^2+bn+c}{r_n}\right]\ell_2+\left[\frac{dn^2+en+f}{r_n}\right],\ \prod_{\substack{p<z\\ p\mid k_2}} p\right)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid(an^2+bn+c)(dn^2+en+f)}} \left(1-\frac{1}{p}\right)^{-1} \Big\{1 +$$

$$O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\Big\}. \tag{18}$$

Now the sum

$$\sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (n,\prod_{p<z}p)=1 \\ (r_n,\prod_{p<z}p)=1 \\ \left(\left[\frac{an^2+bn+c}{r_n}\right]\ell_2+\left[\frac{dn^2+en+f}{r_n}\right],\ \prod_{\substack{p<z \\ p|k_2}}p\right)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p|(an^2+bn+c)(dn^2+en+f)}} (1-\tfrac{1}{p})^{-1}$$

$$= \sum_{\substack{w \\ w=1 \text{ or } \gamma(w)>z}} \sum_{\substack{\alpha<n\leq\alpha+x \\ n\equiv\ell_1 \bmod k_1 \\ (n,\prod_{p<z}p)=1 \\ (an^2+bn+c,\,dn^2+en+f)=w \\ \left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right],\ \prod_{\substack{p<z \\ p|k_2}}p\right)=1}} \prod_{\substack{p<z \\ p\nmid k_2 \\ p|(an^2+bn+c)(dn^2+en+f)}} (1-\tfrac{1}{p})^{-1}$$

where $\gamma(w)$ denotes the smallest prime factor of w.

By Lemma 2.11 this sum is less than or equal to

$$x \prod_{\substack{p<z \\ p|k_2}} (1-\tfrac{1}{p}) \prod_{\substack{p<z \\ p\nmid k_2 h}} (1-\tfrac{1}{p}) \sum_{\substack{w \\ w=1 \text{ or } \gamma(w)>z}} \Lambda_z(w) \left\{ 1 + \right.$$

$$\left. O(\exp(-(\ln x)^{\epsilon})) + O\left[\frac{\varphi([k_1,Fk_2])\ \ln\ln x\ \ln^{\lambda+1}x}{x^{\frac{1}{2}}}\right]\right\} \tag{19}$$

where $\Lambda_z(w) = \frac{[k_1,Fk_2]^{\frac{1}{2}}}{\varphi([k_1,Fk_2])} \prod_{\substack{p<z \\ p\nmid k_2 h}} \left(1 + \frac{4p^{2/3}}{(p-1)^2}\right) \Upsilon_z'(w)$

and where

(i) $F = |(cd-fa)^2-(bd-ea)(ce-fb)|$

(ii) $h = (ad,ae+bd,af+be+cd,bf+ce,cf)$

and $A={}^{ad}/_{h}$, $B={}^{(ae+bd)}/_{h}$, $C={}^{(af+be+cd)}/_{h}$, $D={}^{(bf+ce)}/_{h}$

$E={}^{cf}/_{h}$.

(iii) $G(x,\alpha) = \max_{\alpha<n\leq\alpha+x} |An^4+Bn^3+Cn^2+Dn+E|$

and $\ln\lambda = \left\{ \left[\frac{\ln(2.\Delta^2)}{\ln 2}\right]+1 \right\} \ln 2$

where Δ denotes the discriminant of $(an^2+bn+c)(dn^2+en+f)$ if neither an^2+bn+c nor dn^2+en+f have repeated factors.

If an^2+bn+c has a repeated factor, say $an^2+bn+c=\theta(\gamma n+\delta)^2$ and dn^2+en+f does not have a repeated factor then Δ is the discriminant of $\theta(\gamma n+\delta)(dn^2+en+f)$. Similarly if dn^2+en+f has a repeated factor.

and where

(iv) $\Upsilon_z^1(w)$ denotes the number of integers n in the interval $1\leq n\leq Fk_2$ for which both

$$(an^2+bn+c, dn^2+en+f)=w$$

and $$\left(\left[\frac{an^2+bn+c}{w}\right]\ell_2+\left[\frac{dn^2+en+f}{w}\right] \ , \prod_{\substack{p<z\\ p\nmid k_2}} p \right)=1 .$$

Let $$G_z = \sum_{\substack{w\\ w=1 \text{ or } \gamma(w)>z}} \Lambda_z(w) .$$

(As in Theorem One we have that the number of possible w is at most $\tau(F)$ where $\tau(F)$ denotes the number of divisors of F.)

Substituting (19) into (18) gives

$$T_2(x,y,z) \leq \frac{yx}{k_2} \prod_{\substack{p<z\\ p\nmid k_2}} (1-\frac{1}{p})^2 \prod_{\substack{p<z\\ p\mid k_2}} (1-\frac{1}{p}) \prod_{\substack{p<z\\ p\nmid k_2 h}} (1-\frac{1}{p})\, G_z \Big\{ 1 + O(\exp(-(\ln x)^{\epsilon})) + O\left[\frac{\varphi([k_1,Fk_2])\ \ln\ln x\ \ln^{\lambda+1}x}{x^{\frac{1}{2}}}\right]$$
$$+ O(\exp(-(\ln{}^{y}/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\Big\}. \tag{20}$$

This completes Step Two.

<u>STEP THREE</u> *An asymptotic formula for*

$$T_2(x,y,z) = \sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2}} \Big|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ (((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p)=1\}\Big|$$

with $3<z\leq {}^{x}/k_1$

The initial stages of Step Three are very similar to those of Steps One and Two. Define for q fixed

$$R(x,z,q) = \Big|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ (((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p)=1\ \}\Big|$$

so that the sum under consideration becomes

$$T_2(x,y,z) = \sum_{\substack{0<q\leq y \\ q\equiv\ell_2 \bmod k_2}} R(x,z,q)\ . \tag{21}$$

To find an asymptotic formula for $R(x,z,q)$ we continue as we have done previously and remove from (21) any $R(x,z,q)$ obviously zero. Certainly for $z>3$

$$R(x,z,q) = \Big|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv 1 \bmod 2,\ n\not\equiv 0 \bmod 3,\ n\equiv\ell_1 \bmod k_1,\ (((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p)=1\ \}\Big|$$

$$= \Big|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv 1 \bmod 6,\ n\equiv\ell_1 \bmod k_1,\ (((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p)=1\ \}\Big|$$

$$+ \Big|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv 5 \bmod 6,\ n\equiv\ell_1 \bmod k_1,\ (((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p)=1\ \}\Big|.$$

Now there exists a constant ℓ_3 with $(\ell_3,6,k_1)=1$ such that

$$\Big|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv 1 \bmod 6,\ n\equiv\ell_1 \bmod k_1,\ (((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p)=1\ \}\Big|$$

$$= \begin{cases} \Big|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell_3 \bmod [6,k_1],\ (((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p)=1\ \}\Big| & ;\ \text{if } (6,k_1)\mid(\ell_1-1) \\ 0 & ;\ \text{if } (6,k_1)\nmid(\ell_1-1) \end{cases}$$

and there exists a constant ℓ_4 with $(\ell_4,6,k_1)=1$ such that

$$\Big|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv 5 \bmod 6,\ n\equiv\ell_1 \bmod k_1,\ (((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p)=1\ \}\Big|$$

$$= \begin{cases} \left|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell_4 \bmod[6,k_1],\ \left(((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p\right)=1\ \}\right| & ;\ \text{if } (6,k_1)\mid(\ell_1-5) \\ 0 & ;\ \text{if } (6,k_1)\nmid(\ell_1-5)\ . \end{cases}$$

If $(6,k_1)=3$ or $(6,k_1)=6$ with $(\ell_1,k_1)=1$ then $(6,k_1)\mid(\ell_1-1)$ or $(6,k_1)\mid(\ell_1-5)$ but not both in which case there exists a constant ℓ_5 with $(\ell_5,6,k_1)=1$ such that

$$R(x,z,q) = \left|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell_5 \bmod[6,k_1],\ \left(((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p\right)=1\ \}\right| . \tag{22}$$

If on the other hand $(6,k_1)=1$ or $(6,k_1)=2$ then both $(6,k_1)\mid(\ell_1-1)$ and $(6,k_1)\mid(\ell_1-5)$ and

$$R(x,z,q) = \left|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell_3 \bmod[6,k_1],\ \left(((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p\right)=1\ \}\right|$$
$$+ \left|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell_4 \bmod[6,k_1],\ \left(((aq+d)n^2+(bq+e)n+(cq+f))n, \prod_{p<z} p\right)=1\ \}\right| . \tag{23}$$

Define s_q to be the highest common factor of $aq+d$, $bq+e$, and $cq+f$. If $(s_q, \prod_{p<z} p)>1$ then $R(x,z,q)=0$ so assume the contrary. If $aq+d+bq+e+cq+f\equiv 0 \bmod 2$ then $(aq+d)n^2+(bq+e)n+(cq+f)\equiv 0 \bmod 2$ for $n\equiv 1 \bmod 2$ so again assume the contrary.

If $(aq+d+bq+e+cq+f, 4(aq+d)+2(bq+e)+(cq+f))\equiv 0 \bmod 3$ then $(aq+d)n^2+(bq+e)n+(cq+f)\equiv 0 \bmod 3$ for $n\not\equiv 0 \bmod 3$. Again assume the contrary.

Assuming then that

(i) $(s_q, \prod_{p<z} p)=1$

(ii) $(aq+d+bq+e+cq+f)\equiv 1 \bmod 2$

(iii) $(aq+d+bq+e+cq+f, 4(aq+d)+2(bq+e)+(cq+f))\not\equiv 0 \bmod 3$,

an application of Lemma 1.2 to (22) gives, for $(6,k_1)=3$ or 6,

$$R(x,z,q) = \begin{cases} \frac{x}{[6,k_1]} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) \{1+O(\exp(-(\ln {}^x/6k_1)^{\frac{1}{2}})) \\ \qquad +O(\exp(-v(\ln v - \ln\ln 3v - \ln 3 - 2)))\} \\ \qquad ; \left(\left[\frac{aq+d}{s_q}\right]\ell_5^2+\left[\frac{bq+e}{s_q}\right]\ell_5+\left[\frac{cq+f}{s_q}\right], \prod_{\substack{p<z \\ p\mid 6k_1}} p\right)=1 \\ 0 \qquad ; \text{otherwise} \end{cases}$$

with $v = \frac{\ln {}^x/6k_1}{\ln z}$ and

$$\rho_q'(p) = \begin{cases} \rho_q(p)+1 & ; \ p\nmid \frac{cq+f}{s_q} \\ \rho_q(p) & ; \ p\mid \frac{cq+f}{s_q} \end{cases}$$

with

$$\rho_q(p) = \left|\left\{n \bmod p: \left[\frac{aq+d}{s_q}\right]n^2+\left[\frac{bq+e}{s_q}\right]n+\left[\frac{cq+f}{s_q}\right] \equiv 0 \bmod p\right\}\right|$$

provided that

(a) $\rho_q(p)<p$ for all primes p

(b) $\rho_q(p)<p-1$ if $p\nmid \frac{cq+f}{s_q}$.

Our conditions (i), (ii) and (iii) are enough to ensure that (a) and (b) hold.

Summing R(x,z,q) over q gives

$$T_2(x,y,z) = \frac{x}{[6,k_1]} \sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \text{app}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right)\{1+ O(\exp(-(\ln {}^x/6k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))\} \tag{23}$$

when $(6,k_1)=3$ or 6, where the term "(q,z) appropriate" or "(q,z) app" denotes those primes q for which the following conditions hold:

(i) $(s_q, \prod_{p<z} p)=1$

(ii) $aq+d+bq+e+cq+f\equiv 1 \bmod 2$

(iii) $(aq+d+bq+e+cq+f, 4(aq+d)+2(bq+e)+(cq+f)) \not\equiv 0 \bmod 3$

(iv) $\left(\left[\frac{aq+d}{s_q}\right]\ell_5^2+\left[\frac{bq+e}{s_q}\right]\ell_5+\left[\frac{cq+f}{s_q}\right], \prod_{\substack{p<z\\ p|6k_1}} p\right)=1$.

Any q satisfying conditions (i) to (iv) is said to be "z appropriate".

If we extend the range of definition for "(q,z) app" so that when $(6,k_1)=1$ or 2 it becomes the set of conditions

(i) $(s_q, \prod_{p<z} p)=1$

(ii) $aq+d+bq+e+cq+f \equiv 1 \bmod 2$

(iii) $(aq+d+bq+e+cq+f, 4(aq+d)+2(bq+e)+(cq+f)) \not\equiv 0 \bmod 3$

(iv) $\left(\left[\frac{aq+d}{s_q}\right]\ell_3^2+\left[\frac{bq+e}{s_q}\right]\ell_3+\left[\frac{cq+f}{s_q}\right], \prod_{\substack{p<z\\ p|6k_1}} p\right)=1$ <u>or</u>

$\left(\left[\frac{aq+d}{s_q}\right]\ell_4^2+\left[\frac{bq+e}{s_q}\right]\ell_4+\left[\frac{cq+f}{s_q}\right], \prod_{\substack{p<z\\ p|6k_1}} p\right)=1$

then a similar argument gives

$$T_2(x,y,z) = \frac{x}{[6,k_1]} \sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ (q,z)\ \text{app}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) \{1+$$

$$O(\exp(-(\ln x/6k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))\ \} \tag{24}$$

for any value of $(6,k_1)$.

Recalling from (20), that for $z \leq \exp((\ln y/k_2)^{1-\epsilon})$,

$$T_2(x,y,z) \leq \frac{yx}{k_2} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z\\ p|k_2}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z\\ p\nmid k_2 h}} \left(1-\frac{1}{p}\right) G_z \left\{ 1 + \right.$$

$$O(\exp(-(\ln x)^{\epsilon})) + O\left[\frac{\varphi([k_1,Fk_2])\ \ln\ln x\ \ln^{\lambda+1} x}{x^{\frac{1}{2}}}\right]$$

$$\left. + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) \right\}$$

a comparison with (24) gives

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \text{app}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) < \frac{y[6,k_1]}{k_2} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z \\ p\mid k_2}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_2 h}} \left(1-\frac{1}{p}\right)$$

$$\times\ G_z \left\{ 1 + O(\exp(-(\ln x)^{\epsilon})) + O\left[\frac{\varphi([k_1,Fk_2])\ \ln\ln x\ \ln^{\lambda+1}x}{x^{\frac{1}{2}}}\right] \right.$$

$$+ O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))$$

$$\left. + O(\exp(-(\ln {}^{x}/6k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))\right\}. \tag{25}$$

As the left hand side of (25) is independent of x we may let $x\to\infty$ and (25) becomes

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \text{app}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) < \frac{y[6,k_1]}{k_2} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z \\ p\mid k_2}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid k_2 h}} \left(1-\frac{1}{p}\right)$$

$$\times\ G_z \left\{1 + O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\right\} \tag{26}$$

for $3<z\leq\exp((\ln {}^{y}/k_2)^{1-\epsilon})$. (27)

<u>**STEP FOUR**</u> *Determination of* $\rho_q'(p)$ *in terms of the Legendre symbol.*

Arguing as in Theorem One we have, for p<z and q "z appropriate",

$$\rho_q(p) = \begin{cases} ({}^{g}q/_{p})+1 & ;\ p\nmid 2(aq+d) \\ 1 & ;\ p\mid 2(aq+d),\ p\nmid (aq+d+bq+e) \\ 0 & ;\ p\mid 2(aq+d),\ p\mid (aq+d+bq+e) \end{cases} \tag{28}$$

where $g_q = (b^2-4ac)q^2+2(be-2cd-2fa)q+(e^2-4fd)$, and where $(\cdot/_p)$ denotes the Legendre symbol.

So, for 3<p<z and q "z appropriate", since

$$\rho_q'(p) = \begin{cases} \rho_q(p)+1 & ; p \nmid (cq+f) \\ \rho_q(p) & ; p \mid (cq+f) \end{cases}$$

we have

$$\rho_q'(p) = \begin{cases} (g_q/p)+2 & ; p \nmid (aq+d)(cq+f) \\ (g_q/p)+1 & ; p \nmid (aq+d),\ p \mid (cq+f) \\ 2 & ; p \mid (aq+d),\ p \nmid (bq+e)(cq+f) \\ 1 & ; p \mid (aq+d),\ p \mid (bq+e)(cq+f) \end{cases} \qquad (29)$$

<u>STEP FIVE</u> g_q *a square*

As in Theorem One we may assume that if aq^2+bq+c and dq^2+eq+f have no common factors then g_q is not a complete square.

For g_q a square, from (29), we have

$$\rho_q'(p) = \begin{cases} 3 & ; \quad p \nmid (aq+d)(cq+f)g_q \\ 2 & ; \quad p \nmid (aq+d)(cq+f)\ \&\ p \mid g_q \\ & \quad \underline{\text{or}}\ p \nmid (aq+d)g_q\ \&\ p \mid (cq+f) \\ & \quad \underline{\text{or}}\ p \mid (aq+d)\ \&\ p \nmid (bq+e)(cq+f) \\ 1 & ; \quad p \nmid (aq+d)\ \&\ p \mid (cq+f)\ \&\ p \mid g_q \\ & \quad \underline{\text{or}}\ p \mid (aq+d)\ \&\ p \mid (bq+e)(cq+f) \end{cases} \qquad (30)$$

and

$$\prod_{\substack{p<z \\ p \nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) = \prod_{\substack{p<z \\ p \nmid 6k_1 \\ p \nmid (aq+d)(cq+f)g_q}} \left(1-\frac{3}{p}\right) \prod_{\substack{p<z \\ p \nmid 6k_1 \\ p \nmid (aq+d)(cq+f) \\ p \mid g_q}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z \\ p \nmid 6k_1 \\ p \nmid (aq+d)g_q \\ p \mid (cq+f)}} \left(1-\frac{2}{p}\right)$$

$$\times \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\mid(aq+d)\\ p\nmid(bq+e)(cq+f)}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\nmid(aq+d)\\ p\mid(cq+f)\\ p\mid g_q}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\mid(aq+d)\\ p\mid(bq+e)(cq+f)}} \left(1-\frac{1}{p}\right).$$

If $p\mid(aq+d)$ then $p\mid(bq+e) \Leftrightarrow p\mid g_q$. So

$$\prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) \ll \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\nmid(aq+d)(cq+f)g_q}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\nmid(aq+d)(cq+f)\\ p\mid g_q}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\nmid(aq+d)g_q\\ p\mid(cq+f)}} \left(1-\frac{2}{p}\right)$$

$$\times \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\nmid(aq+d)\\ p\mid(cq+f)\\ p\mid g_q}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\mid(aq+d)\\ p\nmid(cq+f)g_q}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\mid(aq+d)\\ p\mid(cq+f)g_q}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\nmid(aq+d)\\ p\mid(cq+f)\\ p\mid g_q}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\mid(aq+d)\\ p\mid(cq+f)g_q}} \left(1-\frac{1}{p}\right)^{-1}$$

$$= \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\nmid(aq+d)\\ p\mid(cq+f)\\ p\mid g_q}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\mid(aq+d)\\ p\mid(cq+f)g_q}} \left(1-\frac{1}{p}\right)^{-1}$$

$$\ll \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z\\ p\mid(cq+f)g_q}} \left(1-\frac{1}{p}\right)^{-1}$$

$$\ll \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) \ln\ln(|(cq+f)g_q|) .$$

Now in Theorem One (Step Thirteen) we saw that assuming

$$\exp((\ln y/k_2)^{\frac{1}{2}}) > \max\{ |a|,|b|,|c|,|d|,|e|,|f|,k_1,k_2 \}$$

was enough to ensure that

$$\frac{y^2}{2} < \max_{0<q\leq y} |g_q| < 4|\zeta|y^2$$

with $\zeta=b^2-4ac$.

It is also clear that under the same assumption we have

$$|cq+f| < 2|c|y.$$

So

$$\prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) \ll \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) \ln\ln(8|\zeta c|y^3)$$

and it follows that

$$\sum_{\substack{0<q\leqslant y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \text{app} \\ g_q\ \text{a square}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) \ll \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) \ln\ln(|\zeta c|y^3) \sum_{\substack{0<q\leqslant y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \text{app} \\ g_q\ \text{a square}}} 1$$

$$\ll \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) \ln\ln(|\zeta c|y^3) \sum_{\substack{0<m\leqslant y \\ m\equiv \ell_2 \bmod k_2 \\ g_m\ \text{a square}}} 1 .$$

Following the argument of Step 4, Theorem One, we have

$$\sum_{\substack{0<q\leqslant y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \text{app} \\ g_q\ \text{a square}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) \ll \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\ln(|\zeta c|y^3) \times \ln\left[\frac{y}{|\zeta|}\right] \qquad (31)$$

where $\zeta=b^2-4ac$, $\eta=be-2cd-2fa$, $\theta=e^2-4fd$.

We note here that (31) holds for all z.

Substitution of (31) into (26) gives

$$\sum_{\substack{0<q\leqslant y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \text{app} \\ g_q\ \text{not a square}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) < \frac{y\ [6,k_1]}{k_2} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z \\ p\mid k_2}} \left(1-\frac{1}{p}\right)$$

$$\times \prod_{\substack{p<z \\ p\nmid k_2 h}} \left(1-\frac{1}{p}\right) G_z \Big\{1 + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))$$

$$O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))$$

$$+ O\left[\frac{\varphi([k_1,Fk_2])k_2}{[k_1,Fk_2]^{\frac{1}{2}}} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{\ln\ln(|\zeta c|y)}{y} \ln\left[\frac{y}{|\zeta|}\right] \ln z\right] \Big\} \qquad (32)$$

for $3 < z < \exp((\ln {}^{y}/k_2)^{1-\epsilon})$.

This completes Step Five.

STEP SIX *Reduction of* $\prod_{\substack{p<z \\ p\nmid 6k_1}} (1-\frac{\rho_q'(p)}{p})$.

Let $(\cdot/_p)$ denote the Legendre symbol. Recalling (29) we have for q "z appropriate",

$$\prod_{\substack{p<z \\ p\nmid 6k_1}} (1-\frac{\rho_q'(p)}{p}) = \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d)(cq+f)}} (1-\frac{(gq/_p)+2}{p}) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d) \\ p\mid (cq+f)}} (1-\frac{(gq/_p)+1}{p})$$

$$\times \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\mid (aq+d) \\ p\nmid (bq+e)(cq+f)}} (1-\frac{2}{p}) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\mid (aq+d) \\ p\mid (bq+e)(cq+f)}} (1-\frac{1}{p}) .$$

This may be rewritten as

$$\prod_{\substack{p<z \\ p\nmid 6k_1}} (1-\frac{\rho_q'(p)}{p}) = \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d)}} (1-\frac{(gq/_p)}{p}) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d)(cq+f)}} (1-\frac{2}{p}) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d) \\ p\mid (cq+f)}} (1-\frac{1}{p})$$

$$\times \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\mid (aq+d) \\ p\nmid (bq+e)(cq+f)}} (1-\frac{2}{p}) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d)(cq+f)}} (1-\frac{\theta_1(p)}{p^2}) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d) \\ p\mid (cq+f)}} (1-\frac{\theta_2(p)}{p^2}) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\mid (aq+d) \\ p\mid (bq+e)(cq+f)}} (1-\frac{1}{p}) \qquad (33)$$

where $\theta_1 = \frac{2p^2.(gq/_p)}{(p-2)(p-(gq/_p))}$ and $\theta_2 = \frac{p^2.(gq/_p)}{(p-1)(p-(gq/_p))}$.

Let $\chi(p) = \chi_D(p)$ denote the Kronecker symbol $(^D/_p)$ where if $g_q = r^2 s$ for s square-free and not equal to 1, D=4s or s as $s \not\equiv 1 \bmod 4$ and $s \equiv 1 \bmod 4$ respectively. Then, as in Theorem One, the Legendre symbol $(^{g_q}/_p)$ is equivalent to the Kronecker symbol $(^D/_p)$ whenever $p \nmid 2g_q$. Applying this to (33) gives

$$\prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) = \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d)g_q}} \left(1-\frac{\chi(p)}{p}\right) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d)(cq+f)}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d) \\ p\mid (cq+f)}} \left(1-\frac{1}{p}\right)$$

$$\times \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\mid (aq+d) \\ p\nmid (bq+e)(cq+f)}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d)(cq+f)g_q}} \left(1-\frac{\theta_1(p)}{p^2}\right) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d)g_q \\ p\mid (cq+f)}} \left(1-\frac{\theta_2(p)}{p^2}\right) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\mid (aq+d) \\ p\mid (bq+e)(cq+f)}} \left(1-\frac{1}{p}\right)$$

where $\theta_1 = \dfrac{2p^2\chi(p)}{(p-2)(p-\chi(p))}$ and $\theta_2 = \dfrac{p^2\chi(p)}{(p-1)(p-\chi(p))}$.

This may also be written

$$\prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) = \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q,z) \qquad (34)$$

where

$$f(g_q,z) = \prod_{\substack{p<z \\ p\mid 6k_1(aq+d)g_q}} \left(1-\frac{\chi(p)}{p}\right)^{-1} \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d)(cq+f)g_q}} \left(1-\frac{\theta_1(p)}{p^2}\right) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d)g_q \\ p\mid (cq+f)}} \left(1-\frac{\theta_2(p)}{p}\right)$$

$$\times \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\nmid (aq+d) \\ p\mid (cq+f)}} \left(1+\frac{1}{p-2}\right) \prod_{\substack{p<z \\ p\nmid 6k_1 \\ p\mid (aq+d) \\ p\mid (bq+e)(cq+f)}} \left(1+\frac{1}{p-2}\right) . \qquad (35)$$

Equation (34) together with (32) gives

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\text{ app} \\ g_q \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q,z) < \frac{y[6,k_1]}{k_2} \prod_{p<z}\left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1+\frac{1}{p^2-2p}\right)$$

$$\times \prod_{\substack{p<z \\ p\mid k_2}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z \\ p\mid k_2h}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z \\ p\mid 6k_1}} \left(1-\frac{1}{p}\right)^{2} G_z \Big\{ 1 +$$

$$O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))) + O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))$$

$$+O\left[\frac{\varphi([k_1,Fk_2])k_2}{[k_1,Fk_2]^{\frac{1}{2}}} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{\ln\ln(|\zeta c|y)}{y} \ln\left[\frac{y}{|\zeta|}\right] \ln z\right] \Big\} \qquad (36)$$

for $3<z\leq\exp((\ln {}^{y}/k_2)^{1-\epsilon})$.

In particular writing $z_0=\exp((\ln {}^{y}/k_2)^{1-\epsilon})$, since we may assume that $z_0>\max(k_2,k_1,h)$, we have

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z_0)\text{ app} \\ g_q \text{ not a square}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) f(g_q,z_0) < \frac{y[6,k_1]}{\varphi(k_2)} \cdot \frac{k_2h}{\varphi(k_2h)} \cdot \frac{\varphi^2(6k_1)}{36k_1^{\,2}} \prod_{p<z_0}\left(1-\frac{1}{p}\right)$$

$$\times \prod_{\substack{p<z_0 \\ p\nmid 6k_1}} \left(1+\frac{1}{p^2-2p}\right) G_{z_0}\Big\{ 1 + O(\exp(-(\ln {}^{y}/k_2)^{\epsilon}))$$

$$+O\left[\frac{\varphi([k_1,Fk_2])k_2}{[k_1,Fk_2]^{\frac{1}{2}}} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{\ln\ln(|\zeta c|y)}{y} \ln\left[\frac{y}{|\zeta|}\right] \ln y\right] \Big\}. \qquad (37)$$

By a proof identical in most respects to that of Theorem One we are able to show that the primes q that are "z_0 appropriate" are exactly those primes q that are "appropriate". i.e. those primes q satisfying the conditions

(i) $(aq+d,bq+e,cq+f)=1$

(ii) $aq+d+bq+e+cq+f\equiv 1 \bmod 2$

(iii) $(aq+d+bq+e+cq+f,4(aq+d)+2(bq+e)+(cq+f))\not\equiv 0 \bmod 3$

(iv) $((aq+d)\ell_5^2+(bq+e)\ell_5+(cq+e),\ 6k_1)=1$

if $(6,k_1)=3$ or 6.

Similarly for $(6,k_1)=1$ or 6.

Further we may show for $z>z_0$ that

$$f(g_q,z) = f(g_q)\{1 + O(\exp(-(\ln {}^{y}/k_2)^{\frac{1}{2}}))\} \tag{38}$$

where

$$f(g_q) = \prod_{p\mid 6k_1(aq+d)g_q} \left(1-\frac{\chi(p)}{p}\right)^{-1} \prod_{\substack{p\nmid 6k_1 \\ p\nmid(aq+d)(cq+f)g_q}} \left(1-\frac{\theta_1(p)}{p^2}\right) \prod_{\substack{p\nmid 6k_1 \\ p\nmid(aq+d)g_q \\ p\mid(cq+f)}} \left(1-\frac{\theta_2(p)}{p}\right)$$

$$\times \prod_{\substack{p\nmid 6k_1 \\ p\nmid(aq+d) \\ p\mid(cq+f)}} \left(1+\frac{1}{p-2}\right) \prod_{\substack{p\nmid 6k_1 \\ p\mid(aq+d) \\ p\mid(bq+e)(cq+f)}} \left(1-\frac{2}{p}\right)^{-1} . \tag{39}$$

The proof of (38) follows the arguments used in Step Six of Theorem One and it is not repeated here.

We finally note that as both $\prod_{\substack{p\nmid 6k_1 \\ p\nmid(aq+d)(cq+f)g_q}} \left(1-\frac{\theta_1(p)}{p^2}\right)$ and $\prod_{\substack{p\nmid 6k_1 \\ p\nmid(aq+d)g_q \\ p\mid(cq+f)}} \left(1-\frac{\theta_2(p)}{p^2}\right)$ are absolutely convergent there exist constants c_1 and c_2 such that

$$c_1 \prod_{p\mid 6k_1(aq+d)g_q} \left(1-\frac{\chi(p)}{p}\right)^{-1} \prod_{\substack{p\nmid 6k_1 \\ p\nmid(aq+d) \\ p\mid(cq+f)}} \left(1+\frac{1}{p-2}\right) \prod_{\substack{p\nmid 6k_1 \\ p\mid(aq+d) \\ p\mid(bq+e)(cq+f)}} \left(1+\frac{1}{p-2}\right) \leqslant f(g_q) \leqslant$$

$$c_2 \prod_{p\mid 6k_1(aq+d)g_q} \left(1-\frac{\chi(p)}{p}\right)^{-1} \prod_{\substack{p\nmid 6k_1 \\ p\nmid(aq+d) \\ p\mid(cq+f)}} \left(1+\frac{1}{p-2}\right) \prod_{\substack{p\nmid 6k_1 \\ p\mid(aq+d) \\ p\mid(bq+e)(cq+f)}} \left(1+\frac{1}{p-2}\right) . \tag{40}$$

In line with Theorem One we end Step Six here.

STEP SEVEN *Continuation of Step Six.*

We split g_q, as we split g_m, into "good" and "bad" corresponding to whether or not the related D is exceptional in the sense of Lemma 5.1.

Clearly $|D| \leq 4 \max_{0<q\leq y} |g_q| \leq 16|\zeta| y^2$.

Taking $\alpha=27$ in Lemma 5.1, and writing $Q = 4 \max_{0<q\leq y} |g_q|$, we derive, for $z \geq z_0$,

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ good}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q) = \sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ good}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) f(g_q) \left\{ 1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) \right\}. \tag{41}$$

The next four steps are devoted to finding an upper bound for the sum

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ bad}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right)$$

for $z \geq z_0$.

STEP EIGHT *The sum* $\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ bad}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right)$ *for*

$\exp((\ln y/k_2)^{1-\epsilon}) \leq z \leq \exp(y^{1/7})$.

As $\rho_q'(p)$ is always greater than or equal to one we have

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ bad}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) < \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ bad}}} 1$$

$$< \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ g_m \text{ bad}}} 1$$

$$\ll \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{1}{p}\right) \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{|\zeta|}\right] |\zeta|^{1/3} y^{2/3}$$

from Step Eight of Theorem One.

So

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ bad}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) \ll \frac{k_1}{\varphi(k_1)} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{|\zeta|}\right] \frac{|\zeta|^{1/3} y^{2/3}}{\ln z}$$

and

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right)^{f(g_q)} \ll \frac{k_1}{\varphi(k_1)} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{|\zeta|}\right] \frac{|\zeta|^{1/3} y^{2/3}}{\ln z} \times \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right)^{-1}$$

$$\ll \frac{\varphi(k_1)}{k_1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \ln\left[\frac{y}{|\zeta|}\right] |\zeta|^{1/3} y^{2/3} \ln z .$$

For $\exp((\ln {}^{y}/k_2)^{1-\epsilon}) \leq z \leq \exp(y^{1/7})$ this gives

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q) \ll \frac{\varphi(k_1)}{k_1}\, \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{5/6} . \qquad (42)$$

So, for $\exp((\ln {}^{y}/k_2)^{1-\epsilon}) \leq z \leq \exp(y^{1/7})$,

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q) = \sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ good}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q)$$

$$+ 0\left[\frac{\varphi(k_1)}{k_1}\, \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{5/6}\right]$$

which, by (41), gives

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q) = \sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ good}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) f(g_q) \{1 +$$

$$0(\exp(\ln {}^{y}/k_2)^{\frac{1}{2}}))\}$$

$$+ 0\left[\frac{\varphi(k_1)}{k_1}\, \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{5/6}\right].$$

But from (37),

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ good}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) f(g_q) \leq \frac{y[6,k_1]}{\varphi(k_2)} \cdot \frac{k_2 h}{\varphi(k_2 h)} \cdot \frac{\varphi^2(6k_1)}{36k_1^2} \cdot \prod_{p<z_0} \left(1-\frac{1}{p}\right)$$

$$\times \prod_{\substack{p<z_0 \\ p\nmid 6k_1}} \left(1+\frac{1}{p^2-2p}\right) G_{z_0} \left\{ 1 + O(\exp(-(\ln y/k_2)^\epsilon))\right.$$

$$+ O\left[\frac{\varphi([k_1,Fk_2])k_2}{[k_1,Fk_2]^{\frac{1}{2}}} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \frac{\ln\ln(|\zeta c|y)}{y} \ln\left[\frac{y}{|\zeta|}\right] \ln y\right]$$

$$\left. + O\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \frac{\varphi(k_2)\varphi(k_2 h)}{\varphi(k_1)\ k_2 h} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{-1/6} \ln y\right]\right\}. \tag{43}$$

Hence

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q) \leq \frac{y[6,k_1]}{\varphi(k_2)} \cdot \frac{k_2 h}{\varphi(k_2 h)} \cdot \frac{\varphi^2(6k_1)}{36k_1^2} \cdot \prod_{p<z_0} \left(1-\frac{1}{p}\right)$$

$$\times \prod_{\substack{p<z_0 \\ p\nmid 6k_1}} \left(1+\frac{1}{p^2-2p}\right) G_{z_0} \left\{ 1 + O(\exp(-(\ln y/k_2)^\epsilon))\right.$$

$$\left. + O\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \frac{\varphi(k_2)\varphi(k_2 h)}{\varphi(k_1)\ k_2 h} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{-1/6} \ln y\right] \right\}$$

and

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) \leq \frac{y[6,k_1]}{\varphi(k_2)} \cdot \frac{k_2 h}{\varphi(k_2 h)} \cdot \frac{\varphi^2(6k_1)}{36k_1^2} \cdot \prod_{p<z_0} \left(1-\frac{1}{p}\right)$$

$$\times \prod_{\substack{p<z_0 \\ p\nmid 6k_1}} \left(1+\frac{1}{p^2-2p}\right) \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) G_{z_0} \left\{ 1 + O(\exp(-(\ln y/k_2)^\epsilon))\right.$$

$$\left. + O\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \frac{\varphi(k_2)\varphi(k_2 h)}{\varphi(k_1)\ k_2 h} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{-1/6} \ln y\right] \right\} \tag{44}$$

for $\exp((\ln {}^{y}/k_2)^{1-\epsilon}) \leqslant z \leqslant \exp(y^{1/7})$.

In Step Nine we turn to the case $y > \exp(y^{1/7})$.

<u>STEP NINE</u> *The sum* $\displaystyle\sum_{\substack{0<q\leqslant y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ bad}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right)$ *for* $z > \exp(y^{1/7})$.

Writing $z_1 = \exp(y^{1/7})$ we saw in Step Nine of Theorem One that for $z > z_1$,

$$\prod_{z_1 \leqslant p<z} \left(1-\frac{\chi(p)}{p}\right) \ll 1$$

with at most one exception.

So, for $z > \exp(y^{1/7})$, by (42) we have

$$\sum_{\substack{0<q\leqslant y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q) \ll \sum_{\substack{0<q\leqslant y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ bad}}} \prod_{p<z_1} \left(1-\frac{\chi(p)}{p}\right) f(g_q)$$

$$\ll \frac{\varphi(k_1)}{k_1} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] |\zeta|^{1/3} y^{5/6} \qquad (45)$$

with at most one exception.

<u>STEP TEN</u> *Translation of* $\displaystyle\prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right)$ *into a product involving the norms of prime ideals in* $\mathbb{Q}(\sqrt{g_q})$.

As in Theorem One we have

$$\prod_{N\beta<z} \left(1-\frac{1}{N\beta}\right) = \prod_{\substack{p<z \\ (D/p)=1}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z \\ p\mid D}} \left(1-\frac{1}{p}\right) \prod_{\substack{p^2<z \\ (D/p)=-1}} \left(1-\frac{1}{p^2}\right)$$

where β is a prime ideal in $\mathbb{Q}(\sqrt{g_q})$, and $N\beta$ is the norm of β. We were consequently able to show that

$$\prod_{p<z}\left(1-\frac{1}{p}\right)\prod_{p<z}\left(1-\frac{\chi(p)}{p}\right) \ll \prod_{N\beta<z}\left(1-\frac{1}{N\beta}\right).$$

So, for $z>\exp((\ln y/k_2)^{1-\epsilon})$,

$$\prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{\rho_q'(p)}{p}\right) \ll \prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{2}{p}\right)\prod_{p<z}\left(1-\frac{\chi(p)}{p}\right) f(g_q)$$

$$\ll \frac{k_1^2}{\varphi^2(k_1)}\prod_{p<z}\left(1-\frac{1}{p}\right)^2\prod_{p<z}\left(1-\frac{\chi(p)}{p}\right) f(g_q)$$

$$\ll \frac{k_1^2}{\varphi^2(k_1)}\prod_{p<z}\left(1-\frac{1}{p}\right)\prod_{N\beta<z}\left(1-\frac{1}{N\beta}\right) f(g_q).$$

Further, as $\prod_{N\beta<z}\left(1-\frac{1}{N\beta}\right) \ll \frac{1}{L(1,\chi)\ln z}$ for $z\gg D^6$, and

$$\exp(y^{1/7}) \gg \left(\max_{0<q\leq y}|g_q|\right)^6 \gg \exp((\ln y/k_2)^{1-\epsilon}),$$

we have

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ bad}}}\prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{\rho_q'(p)}{p}\right) \ll \frac{k_1^2}{\varphi^2(k_1)}\cdot\frac{1}{\ln^2 z}\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ bad}}} L(1,\chi)^{-1} f(g_q) \tag{46}$$

for $z>\exp(y^{1/7})$, and

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ bad}}}\prod_{p<z}\left(1-\frac{\chi(p)}{p}\right) f(g_q) \ll \sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ q \text{ app}\\ g_q \text{ not a square}\\ g_q \text{ bad}}} L(1,\chi)^{-1} f(g_q). \tag{47}$$

In Step Eleven we estimate (47) over the possible exceptional modulus of (45).

STEP ELEVEN *The possible exceptional modulus of* (45).

From (62) and (63) of Theorem One we have that for every $\epsilon>0$ there exists an effectively computable constant $c>0$ such that

$$L(1,\chi)^{-1} < c_\epsilon \frac{|\bar{s}|^{\frac{1}{2}}}{(\ln|\bar{s}|)^{1-\epsilon/2}}$$

where $\bar{D}$ is the possible exceptional modulus of (45) and where $\bar{s}=\bar{D}$ or $\bar{s}=\bar{D}/4$ according to either $\bar{D}\not\equiv 0 \bmod 4$ or $\bar{D}\equiv 0 \bmod 4$. We will take the ϵ above and the ϵ appearing in $z_0=\exp((\ln y/k_2)^{1-\epsilon})$ to be identical.

Now, from (40) in Step Six, we have

$$f(g_q) \ll \prod_{p|6k_1(aq+d)g_q} \left(1-\frac{\chi(p)}{p}\right)^{-1} \prod_{\substack{p\nmid 6k_1 \\ p\nmid(aq+d) \\ p|(cq+f)}} \left(1+\frac{1}{p-2}\right) \prod_{\substack{p\nmid 6k_1 \\ p|(aq+d) \\ p|(bq+e)(cq+f)}} \left(1+\frac{1}{p-2}\right).$$

Less strongly,

$$f(g_q) \ll \prod_{p|k_1(aq+d)g_q} \left(1-\frac{1}{p}\right)^{-1} \prod_{p|(cq+f)} \left(1+\frac{1}{p-2}\right) \prod_{p|(aq+d)} \left(1+\frac{1}{p-2}\right)$$

$$\ll \frac{k_1}{\varphi(k_1)} \ln\ln|(aq+d)g_q| . \ln\ln|cq+f| . \ln\ln|aq+d|$$

$$\ll \frac{k_1}{\varphi(k_1)} \ln\ln^3|a\bar{c}c| . \ln\ln^3 y.$$

Consequently

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ bad}}}' L(1,\chi)^{-1} f(g_q) \ll_\epsilon \frac{|\bar{s}|^{\frac{1}{2}}}{(\ln|\bar{s}|)^{1-\epsilon/2}} \cdot \frac{k_1}{\varphi(k_1)} \ln\ln^3|a\zeta c| .$$

$$\times \ln\ln^3 y \sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ bad}}}' 1$$

where the sum $\sum'$ is over g_q which give rise to the possible exceptional modulus of (45).

Clearly

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ bad}}}' L(1,\chi)^{-1} f(g_q) \ll_\epsilon \frac{|\bar{s}|^{\frac{1}{2}}}{(\ln|\bar{s}|)^{1-\epsilon/2}} \cdot \frac{k_1}{\varphi(k_1)} \ln\ln^3|a\zeta c| .$$

$$\times \ln\ln^3 y \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ g_m \text{ not a square} \\ g_m \text{ bad}}}' 1$$

so that following Theorem One we have

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ bad}}}' L(1,\chi)^{-1} f(g_q) \ll_\epsilon |\zeta|^{\frac{1}{2}} . \frac{k_1}{\varphi(k_1)} . \ln\ln^3|a\zeta c| . \tau\left[\frac{\eta^2-\zeta\theta}{4}\right]$$

$$\times \frac{y \ln\ln^3 y}{(\ln y)^{1-\epsilon/2}} . \qquad (48)$$

(48) together with (47) and (45) gives

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ bad}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q) \ll_\epsilon |\zeta|^{\frac{1}{2}}.\frac{k_1}{\varphi(k_1)}.\ \ln\ln^3|a\zeta c|.\tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \times \frac{y\ln\ln^3 y}{(\ln y)^{1-\epsilon/2}} \qquad (49)$$

for $z>\exp(y^{1/7})$.

This completes Step Eleven.

<u>STEP TWELVE</u> *Completion of the estimate of* $T_2(x,y,z)$ *for* $x/k_1 > y/k_2$.

Recall that equation (41) gave

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ good}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q) = \sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ good}}} \prod_{p<z_0} \left(1-\frac{\chi(p)}{p}\right) f(g_q) \left\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\right\}$$

for all $z>\exp((\ln y/k_2)^{1-\epsilon})=z_0$.

But equation (43) gave us that the right hand side of this equation is less than or equal to

$$\frac{y[6,k_1]}{\varphi(k_2)}.\frac{k_2h}{\varphi(k_2h)}\frac{\varphi^2(6k_1)}{36k_1^2}.\prod_{p<z_0}\left(1-\frac{1}{p}\right)\prod_{\substack{p<z_0 \\ p\nmid 6k_1}}\left(1+\frac{1}{p^2-2p}\right) G_{z_0}\left\{1 + O(\exp(-(\ln y/k_2)^{\epsilon}))\right.$$

$$\left. + O\left[\frac{\varphi(k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}}.\tau\left[\frac{\eta^2-\zeta\theta}{4}\right].\frac{\varphi(k_2)\varphi(k_2h)}{\varphi(k_1).k_2h}.|\zeta|^{1/3}y^{-1/6}\ln y\right]\right\}$$

so that

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square} \\ g_q \text{ good}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q) \leq \frac{y[6,k_1]}{\varphi(k_2)} \cdot \frac{k_2 h}{\varphi(k_2 h)} \frac{\varphi^2(6k_1)}{36k_1^2} \cdot \prod_{p<z_0} \left(1-\frac{1}{p}\right)$$

$$\times \prod_{\substack{p<z_0 \\ p\nmid 6k_1}} \left(1+\frac{1}{p^2-2p}\right) G_{z_0} \left\{1 + O(\exp(-(\ln y/k_2)^{\epsilon}))\right.$$

$$\left. + O\left[\frac{\varphi(k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \cdot \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \cdot \frac{\varphi(k_2)\varphi(k_2 h)}{\varphi(k_1) \cdot k_2 h} \cdot |\zeta|^{1/3} y^{-1/6} \ln y\right] \right\} \tag{50}$$

for $z \geq z_0$.

But (50) together with (49) and (44) gives

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ q \text{ app} \\ g_q \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) f(g_q) \leq \frac{y[6,k_1]}{\varphi(k_2)} \cdot \frac{k_2 h}{\varphi(k_2 h)} \frac{\varphi^2(6k_1)}{36k_1^2} \cdot \prod_{p<z_0} \left(1-\frac{1}{p}\right)$$

$$\times \prod_{\substack{p<z_0 \\ p\nmid 6k_1}} \left(1+\frac{1}{p^2-2p}\right) G_{z_0} \left\{1 + \right.$$

$$\left. O_{\epsilon}\left[\frac{\varphi(k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \cdot \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \cdot \varphi(k_2) \cdot |\zeta|^{\frac{1}{2}} . \ln\ln^3 |a\zeta c| \cdot \frac{\ln\ln^3 y}{(\ln y)^{\epsilon/2}}\right] \right\} \tag{51}$$

for $z \geq z_0$.

In a similar fashion to Theorem One we are able to show that (51) can be extended over q "z appropriate" and g_q a square, and that we may write

$$G_{z_0} = G_z \left\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\right\}$$

for $z \geq z_0$.

Hence

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) < \frac{y[6,k_1]}{\varphi(k_2)} \cdot \frac{k_2 h}{\varphi(k_2 h)} \frac{\varphi^2(6k_1)}{36k_1^{\,2}} \cdot \prod_{p<z_0} \left(1-\frac{1}{p}\right)$$

$$\times \prod_{\substack{p<z_0 \\ p\nmid 6k_1}} \left(1+\frac{1}{p^2-2p}\right) \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{2}{p}\right) G_z \Bigg\{1 +$$

$$O_\epsilon\left[\frac{\varphi(k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \cdot \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] . \varphi(k_2) . |\zeta|^{\frac{1}{2}} . \ln\ln^3|a\zeta c| . \frac{\ln\ln^3 y}{(\ln y)^{\epsilon/2}}\right] \Bigg\} \tag{52}$$

for $z\geq z_0$.

Equation (26) covers the case for $z\leq z_0$, and a combination of the two gives

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) < \frac{y[6,k_1]}{k_2} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) \prod_{p<z_2} \left(1-\frac{1}{p}\right)$$

$$\times \prod_{\substack{p<z_2 \\ p\mid 6k_1}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z_2 \\ p\mid k_2}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z_2 \\ p\mid k_2 h}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z_2 \\ p\nmid 6k_1}} \left(1+\frac{1}{p^2-2p}\right) G_z \Bigg\{1$$

$$+ \quad O_\epsilon\left[\frac{\varphi(k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \cdot \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] . \varphi(k_2) . |\zeta|^{\frac{1}{2}} . \ln\ln^3|a\zeta c| . \frac{\ln\ln^3 y}{(\ln y)^{\epsilon/2}}\right] \Bigg\} \tag{53}$$

for $z>3$ with $z_2 = \min(\exp((\ln{}^{y}/k_2)^{1-\epsilon}),z)$,

and (53) substituted into (24) gives

$$T_2(x,y,z) < \frac{yx}{k_2} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{2}{p}\right) \prod_{p<z_2} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z_2 \\ p\mid 6k_1}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p<z_2 \\ p\mid k_2}} \left(1-\frac{1}{p}\right)^{-1}$$

$$\times \prod_{\substack{p<z_2 \\ p\mid k_2 h}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z_2 \\ p\nmid 6k_1}} \left(1+\frac{1}{p^2-2p}\right) G_z \Bigg\{1 +$$

$$O_\epsilon\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \cdot \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] . \varphi(k_2) . |\zeta|^{\frac{1}{2}} . \ln\ln^3|a\zeta c| . \frac{\ln\ln^3 y}{(\ln y)^{\epsilon/2}}\right]$$

$$+ O(\exp(-(\ln {}^{x}/k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v - \ln\ln 3v - \ln 3 - 2))) \Bigg\}.$$

This may be restated as

$$T_2(x,y,z) \leq \frac{yx}{k_2} \prod_{p<z}\left(1-\frac{1}{p}\right)^2 \prod_{p<z_2}\left(1-\frac{1}{p}\right) \prod_{z_2\leq p<z}\left(1-\frac{1}{(p-1)^2}\right)$$

$$\times \prod_{\substack{p<z_2 \\ p|k_2}}\left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p<z_2 \\ p|k_2h}}\left(1-\frac{1}{p}\right)^{-1} G_z \Big\{1 +$$

$$0_\epsilon\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}}\right].\tau\left[\frac{\eta^2-\zeta\theta}{4}\right].\varphi(k_2).|\zeta|^{\frac{1}{2}}.\ln\ln^3|a\zeta c|.\frac{\ln\ln^3 y}{(\ln y)^{\epsilon/2}}\right]$$

$$+ 0(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}})) + 0(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))\Big\}. \qquad (54)$$

which concludes Step Twelve.

STEP THIRTEEN *The completion of the Theorem.*

To complete the theorem we require an upper bound on the function

$$P_2(x,y,z) = \Big|\{(q,r);\ \alpha<q\leq\alpha+x,\ q\equiv\ell_1 \bmod k_1,\ 0<r\leq y,\ r\equiv\ell_2 \bmod k_2,$$
$$((aq^2+bq+c)r+(dq^2+eq+f), \prod_{p<z} p)=1\}\Big|$$

whenever ${}^x/k_1 \geq {}^y/k_2$; and an upper bound on the function

$$P_1(x,y,z) = \Big|\{(q,r);\ 0<q\leq x,\ q\equiv\ell_1 \bmod k_1,\ \beta<r\leq\beta+y,\ r\equiv\ell_2 \bmod k_2,$$
$$((aq^2+bq+c)r+(dq^2+eq+f), \prod_{p<z} p)=1\}\Big|$$

whenever ${}^y/k_2 \geq {}^x/k_1$. The variables q and r denote primes throughout.

Dealing firstly with the case ${}^x/k_1 \geq {}^y/k_2$ we observe that the function

$$\Big|\{q:\ \alpha<q\leq\alpha+x,\ q\equiv\ell_1 \bmod k_1,\ ((aq^2+bq+c)r+(dq^2+eq+f), \prod_{p<z} p)=1\}\Big|$$

counts the integers n satisfying $\alpha<n\leq\alpha+x$, $n\equiv\ell_1 \bmod k_1$, for which n is a prime and $((an^2+bn+c)r+(dn^2+en+f), \prod_{p<z} p)=1$. If in addition $n\geq z$ then n is counted in

$$\left|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ (((an^2+bn+c)r+(dn^2+en+f))n,\prod_{p<z}p)=1\}\right|.$$

Otherwise $n\leq z$. Since there are $O\left[\frac{z}{\varphi(k_1)\ln^{z}/k_1}\right]+O(1)$ primes less than or equal to z that are congruent to $\ell_1 \bmod k_1$ it follows that

$$\left|\{q:\ \alpha<q\leq\alpha+x,\ q\equiv\ell_1 \bmod k_1,\ ((aq^2+bq+c)r+(dq^2+eq+f),\prod_{p<z}p)=1\}\right|$$

$$\leq \left|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ (((an^2+bn+c)r+(dn^2+en+f))n,\prod_{p<z}p)=1\}\right|$$

$$+\ O\left[\frac{z}{\varphi(k_1)\ln^{z}/k_1}\right] + O(1).$$

Consequently

$$P_2(x,y,z) \leq \sum_{\substack{0<r\leq y\\ r\equiv\ell_2 \bmod k_2}} \left|\{n:\ \alpha<n\leq\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ (((an^2+bn+c)r+(dn^2+en+f))n,\prod_{p<z}p)=1\}\right|$$

$$+\ O\left[\frac{y}{\varphi(k_2)\ln^{y}/k_2}\ \frac{z}{\varphi(k_1)\ln^{z}/k_1}\right] + O\left[\frac{y}{\varphi(k_2)\ln^{y}/k_2}\right]$$

$$= T_2(x,y,z) + O\left[\frac{y}{\varphi(k_2)\ln y}\ \frac{z}{\varphi(k_1)\ln^{z}/k_1}\right] + O\left[\frac{y}{\varphi(k_2)\ln y}\right]. \qquad (55)$$

Assuming that, in addition to ${}^x/k_1\geq{}^y/k_2$, we have $a+b+c\not\equiv d+e+f \bmod 2$ then from (54) of Step Twelve we have

$$T_2(x,y,z) \leq \frac{yx}{k_2} \prod_{p<z}\left(1-\frac{1}{p}\right)^2 \prod_{p<z_2}\left(1-\frac{1}{p}\right) \prod_{z_2\leq p<z}\left(1-\frac{1}{(p-1)^2}\right) \prod_{\substack{p<z_2\\ p|k_2}}\left(1-\frac{1}{p}\right)^{-1}$$

$$\times \prod_{\substack{p<z_2\\ p|k_2h}}\left(1-\frac{1}{p}\right)^{-1} G_z \Big\{1 + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))$$

$$+\ O_\epsilon\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac12}}\ \tau\left[\frac{\eta^2-\zeta\theta}{4}\right]\ \varphi(k_2)\ |\zeta|^{\frac12}\ \ln\ln^3|a\zeta c|\ \frac{\ln\ln^3 y}{(\ln y)^{\epsilon/2}}\right]$$

$$+\ O(\exp(-(\ln^{x}/k_1)^{\frac12}))\Big\}.$$

Substituting this into (55) gives

$$P_2(x,y,z) < \frac{yx}{k_2} \prod_{p<z}\left(1-\frac{1}{p}\right)^2 \prod_{p<z_2}\left(1-\frac{1}{p}\right) \prod_{z_2\leq p<z}\left(1-\frac{1}{(p-1)^2}\right) \prod_{\substack{p<z_2\\ p|k_2}}\left(1-\frac{1}{p}\right)^{-1}$$

$$\times \prod_{\substack{p<z_2\\ p|k_2h}}\left(1-\frac{1}{p}\right)^{-1} G_z \Big\{1 + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))$$

$$+ O_\epsilon\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \tau\left[\frac{\eta^2-\zeta\theta}{4}\right] \varphi(k_2)\ |\zeta|^{\frac{1}{2}} \ln\ln^3|a\zeta c| \frac{\ln\ln^3 y}{(\ln y)^{\epsilon/2}}\right]$$

$$+ O(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}})) + O\left[\frac{\varphi([k_1,Fk_2])}{[k_1,Fk_2]^{\frac{1}{2}}} \frac{k_2}{\varphi(k_2)} \frac{z\ln^2 z}{x}\right]\Big\}. \tag{56}$$

If ${}^x/k_1 > {}^y/k_2$ but $a+b+c \equiv d+e+f$ mod2 from equation (14) of Step Two we have

$$P_2(x,y,z) < \frac{x}{k_1} \prod_{\substack{p<z\\ p\nmid k_1}}\left(1-\frac{1}{p}\right) \Big\{1 + O(\exp(-v(\ln v-\ln\ln 3v-2)))$$

$$+ O(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}})) + O\left[\frac{k_1}{\varphi(k_2)} \frac{z\ln z}{x} \frac{y}{\ln y}\right]\Big\}. \tag{57}$$

We may similarly show that whenever ${}^y/k_2 > {}^x/k_1$,

$$P_1(x,y,z) < T_1(x,y,z) + O\left[\frac{x}{\varphi(k_1)\ln {}^x/k_1}\left[\frac{z}{\varphi(k_2)\ln {}^z/k_2} + 1\right]\right]. \tag{58}$$

Part (I) of the Theorem follows on applying equation (11) of Step One to (58). Part (II) follows on applying equation (5) of Step One when the conditions

(i) $2 \equiv \ell_1 \bmod k_1$

(ii) $c \not\equiv f$ mod2

(iii) $(r_2, \prod_{p<z} p) = 1$

(iv) $\left(\left[\frac{4a+2b+c}{r_2}\right]\ell_3 + \left[\frac{4d+2e+f}{r_2}\right], \prod_{\substack{p<z\\ p|2k_2}} p\right) = 1$

are satisfied. If at least one of these conditions is not satisfied then a repetition of the argument leading to equation

(5) but applied to the function $P_1(x,y,z)$ rather than $T_2(x,y,z)$ gives

$$P_1(x,y,z)=0.$$

This completes the theorem. □

CHAPTER FOUR

INTRODUCTION

In this chapter we attempt to extend the method of argument of Theorem One to the evaluation of the more general function

$$F(x,y,z) = \Big|\{(n,m):\ 0<n\leq x,\ n\equiv \ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv \ell_2 \bmod k_2,$$
$$((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i),\prod_{p<z} p\ )=1\}\Big|.$$

We will assume in what follows that $x/k_1 \geq y/k_2$. The argument when $y/k_2 \geq x/k_1$ is very similar. Beginning as we did in Theorem One we write F(x,y,z) in two differant ways for $z\leq y$.

Firstly for $z\leq y$,

$$F(x,y,z) = \frac{y}{k_2} \sum_{\substack{0<n\leq x \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))$$
$$+ O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\} \qquad (1)$$

where $u = \dfrac{\ln y/k_2}{\ln z}$;

where, if $r_n=(an^2+bn+c, dn^2+en+f, gn^2+hn+i)$, then for $p<z$ and $(r_n,\prod_{p<z} p)=1$,

$$\rho_n(p) = \Big|\{m \bmod p:\ (an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i)\equiv 0 \bmod p\}\Big|;$$

and where "(n,z) app" denotes those integers n satisfying the conditions

(i) $(r_n,\prod_{p<z} p)=1$

(ii) $(an^2+bn+c+dn^2+en+f, gn^2+hn+i)\equiv 1 \bmod 2$

(iii) $\left(\left[\dfrac{an^2+bn+c}{r_n}\right]\ell_2^{\,2}+\left[\dfrac{bn^2+en+f}{r_n}\right]\ell_2+\left[\dfrac{gn^2+hn+i}{r_n}\right],\ \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1.$

Similarly for $z\leq x$,

$$F(x,y,z) = \frac{x}{k_1} \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \text{app}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) (1 + O(\exp(-(\ln {}^x/k_1)^{\frac{1}{2}}))$$

$$+ O(\exp(-v(\ln v - \ln\ln 3v - \ln 2 - 2)))) \tag{2}$$

where $v = \frac{\ln {}^x/k_1}{\ln z}$;

where, if $s_m=(am^2+dm+g, bm^2+em+h, cm^2+fm+i)$, then for $p<z$ and $(s_m, \prod_{p<z} p)=1$,

$$\rho_m(p) = \left|\{n \bmod p: (am^2+dm+g)n^2+(bm^2+em+h)n+(cm^2+fm+i)\equiv 0 \bmod p\}\right|;$$

and where "(m,z) app" denotes those integers m satisfying the conditions

(i) $(s_m, \prod_{p<z} p)=1$

(ii) $(am^2+dm+g+bm^2+em+h, cm^2+fm+i)\equiv 1 \bmod 2$

(iii) $\left(\left[\frac{am^2+dm+g}{s_m}\right]\ell_1^{\,2}+\left[\frac{bm^2+em+h}{s_m}\right]\ell_1+\left[\frac{cm^2+fm+i}{s_m}\right], \prod_{\substack{p<z \\ p\mid k_1}} p\right)=1$.

Recalling the method of argument of Theorem One we were firstly (Step 1) able to find a relatively simple expression for the function S(x,y,z) for $z\leq y$, and then compare this with the more complicated expression we derived for S(x,y,z) with $z\leq x$. This gave us a starting point from which to develop the argument.

In order to follow the same method here we would compare expressions (1) and (2) for $z\leq y$. However before this can be done in a meaningful way we require at least an upper bound on the function

$$\sum_{\substack{0<n\leq x \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\ \text{app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \tag{3}$$

for $z\leq y$.

Assuming that an^2+bn+c and dn^2+en+f have no common factors, and that a and d are not both zero, define F to be

$F = |(cd-fa)^2-(bd-ea)(ce-fb)|$

(The assumption that an^2+bn+c and dn^2+en+f have no common factors and that a and d are not both zero is only an artificial restriction. Similar results to what follows may still be derived. To include all possible cases leads only to unnecessary complication.)

If $z \geq \max(F,k_2)$ then, by Lemma 2.1,

$$\sum_{\substack{0<n\leq x \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \sum_{\substack{0<n\leq x \\ n\equiv \ell_1 \bmod k_1 \\ n\text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$$

where "n app" denotes those integers n satisfying the conditions

(i) $(an^2+bn+c, dn^2+en+f, gn^2+hn+i)=1$

(ii) $(an^2+bn+c+dn^2+en+f, gn^2+hn+i)\equiv 1 \bmod 2$

(iii) $((an^2+bn+c)\ell_2{}^2+(dn^2+en+f)\ell_2+(gn^2+hn+i),\ k_2)=1$.

If however $z \leq \max(F,k_2)$ then

$$\sum_{\substack{0<n\leq x \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \leq \sum_{\substack{0<n\leq x \\ n\equiv \ell_1 \bmod k_1}} 1 \leq \frac{x}{k_1} + 1 \qquad (4)$$

which is possibly very weak but will suffice.

For $z>\max(F,k_2)$ we work with the sum

$$\sum_{\substack{0<n\leq x \\ n\equiv \ell_1 \bmod k_1 \\ n\text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) .$$

Firstly we give some definitions.

(I) For $g_n'=(d^2-4ag)n^4+2(de-2ah-2gb)n^3+(2df+e^2-4ai-4bh-4cg)n^2+2(ef-2bi-2ch)n+(f^2-4ci)$

define

$$g_n := \begin{cases} g_n' & \text{if } g_n' \text{ has no squared linear factor} \\ \dfrac{g_n'}{(\xi n+\eta)^2} & \text{if } g_n' \text{ has a squared linear factor } (\xi n+\eta)^2; \\ & (\xi,\eta)=1 \end{cases}$$

(II) Define $T(y,s)$ to be the number of integer solutions, (n,r), in the range $0<n\leq y$ of the equation $g_n=r^2s$.

(III) Define

$$S(y) := \max_{0<s\leq y^5} |T(y,s)|.$$

and finally

(IV) λ = maximum coefficient of g_n in modulus,

$\zeta = \max\{|a|,|b|,|c|,|d|,|e|,|f|,|g|,|h|,|i|\}$.

Before stating Lemma One which will give an upper bound on the sum

$$\sum_{\substack{0<n\leq x \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$$

as required, we will attempt to give upper bounds on $T(y,s)$ and $S(y)$.

In the case where $g_n=\frac{g_n'}{(\xi n+\eta)^2}$ ie where g_n is a quadratic, writing $g_n=An^2+2Bn+C$ it follows from Lemma 4 that

$$T(y,s) \ll \tau\left[\frac{B^2-AC}{4}\right]\ln y \tag{7}$$

and consequently that

$$S(y) \ll \tau\left[\frac{B^2-AC}{4}\right]\ln y\ . \tag{8}$$

For the case $g_n=g_n'$ ie where g_n' has no squared linear factor, a bound on $T(y,s)$ follows from Theorem One of Evertse and Silverman's paper "Uniform Bounds for the number of solutions to $Y^n=f(x)$." [12] which states, for $n=2$,

"THEOREM 1: *Set the following notation*:

K *an algebraic number field of degree* m

S *a finite set of places of* K, *containing the infinite places*

s = #S

R_S *the ring of* S-*integers of* K

f(X) $\epsilon R_S[X]$, *a polynomial of degree* d *with discriminant* $\mathrm{disc}(f)\epsilon R_S^*$

L/K *an extension of degree* M

$k_2(L)$ *the* 2-*rank of the ideal class group of* L.

Let $V(R_S,f) = \{x\epsilon R_S : f(x)\epsilon K^{*2}\}$.

Let d≥3 *and assume that* L *contains at least three zeros of* f.

Then

$$\# V(R_S,f) \leq 7^{d^3(4m+9s)}4^{k_2(L)}."$$

I am indebted to J.H.Evertse for outlining the application of this theorem to the integral case. In detail, we take K to be the set of rationals and $p_1,\ldots,p_r$ to be the prime divisors of disc(f). Let S be the set of p-adic valuations for $p_1\ldots p_r$ together with the valuation corresponding to the unique infinite prime divisor.

ie $S=\{v_{p_1},v_{p_2},\ldots,v_{p_r},v_\infty\}$.

Then

$s=\#S=\omega(\mathrm{disc}(f))+1$.

By definition R_S, the ring of S-integers of $\mathbb{Q}$ is the set

$\{a\epsilon\mathbb{Q} \mid v_p(a)\geq 0\ \forall v_p\notin S\}$.

Since, for a an integer, $v_p(a)\geq 0$ it follows that $f(x)\epsilon R_S[x]$.

R_S^*, the unit ring of S-integers is defined as

$R_S^*=\{a\epsilon\mathbb{Q} \mid v_p(a)=0\ \forall v_p\notin S\}$.

If a divides disc(f) then clearly $v_p(a)=0$ for all v_p not in S. It follows that $\mathrm{disc}(f)\epsilon R_S^*$.

Let L be an algebraic number field of degree less than or equal to d^3 containing at least three zeros of f. Define $k_2(L)$ to be the 2-rank of the ideal class group of L. Then by Theorem 1 above the number of solutions in integers of $f(x)=y^2$ is at most

$$7^{d^3(4+9s)}4^{k_2(L)}$$

or

$$7^{d^3(13+9t)}4^{k_2(L)}$$

where $t=\omega(disc(f))$ and d is the degree of f(x).

To extend this result to find an upper bound on the number of integer solutions of

$$\ell y^2=f(x) \qquad (9)$$

we write $F(x)=\ell f(x)$ so that $(\ell y)^2=F(x)$.

Since $disc(f)=\ell^{(2d-2)}disc(f)$ we have

$$\omega(disc(F)) \leq \omega(\ell)+\omega(disc(f)).$$

It follows that the number of integer solutions of (9) is at most

$$7^{d^3(13+9(\omega(\ell)+\omega(disc(f)))}4^{k_2(L)}.$$

So we have

$$T(y,s) \leq 7^{4^3(13+9(\omega(s)+\omega(disc(g_n)))}4^{k_2(L)}$$

where L is an algebraic number field of degree less than or equal to 4^3, containing at least three zeros of g_n, and where $k_2(L)$ is the 2-rank of the ideal class group of L.

Writing G as

$$G:= 7^{4^3(9\omega(disc(g_n))+13)}4^{k_2(L)}$$

we have

$$T(y,s)\leq 7^{4^3.9.\omega(s)}.G.$$

Since $2^{\omega(s)}\leq\tau(s)$, where $\tau(s)$ denotes the number of prime divisors of s, we have

$$T(y,s) \leq G.(2^{\omega(s)})^{1618} \leq G.(\tau(s))^{1618} \leq G.s^{1/300} \qquad (10)$$

say.

We further have, recalling the definition of $S(y)$,

$$S(y) \ll G.y^{1/60}.$$

With the above definitions of G, g_n, λ and ζ we may prove the following:

LEMMA ONE

Suppose $z<y<x$. *Then there exists an absolute and effectively computable constant* c_1, *independent of* a,b,c,d,e,f, g,h,i,k_1, *and* k_2, *for which it follows that*

$$\sum_{\substack{0<n<x \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) < c_1 \frac{x}{k_1} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \prod_{(35\lambda)^{100}} \left(1-\frac{1}{p}\right)^{-1} \times$$

$$\left\{1 + \frac{\ln(\zeta.k_2.\lambda)}{(7\lambda)^{99}} + \frac{1}{(7\lambda)^2} + k_1.G.\ln\ln\zeta\right\}$$

if $z>\max(F,k_2)$.

If on the other hand $z<\max(F,k_2)$ *then*

$$\sum_{\substack{0<n<x \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) < \frac{x}{k_1} + O(1).$$

PROOF OF LEMMA ONE

The second part of the lemma ie where $z<\max(F,k_2)$ has been dealt with previously. (see (4)).

We assume for the time being that $g_n=g_n'$ ie g_n' has no squared linear factor, and that g_n is of degree 3 or 4 so that we may apply the bounds on $T(y,s)$ and $S(y)$ derived above.

Many of the methods of argument of Lemma One will be by now familiar and are not given in detail. We will assume throughout the proof that $z>\ln^{100}(y^5)$. If z is smaller than this then a shortened version of the proof will suffice.

Firstly we more clearly define $\rho_n(p)$.

We have, for "n app" and $p<z$,

$$\rho_n(p) = \left|\{m \bmod p:\ (an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i)\equiv 0 \bmod p\}\right|.$$

For $p\nmid 2(an^2+bn+c)$ this becomes

$$\rho_n(p) = \left|\{m \bmod p:\ m^2\equiv (dn^2+en+f)^2-4(an^2+bn+c)(gn^2+hn+i) \bmod p\}\right|.$$

Writing

$$\begin{aligned} g_n' &= (dn^2+en+f)^2-4(an^2+bn+c)(gn^2+hn+i) \\ &= (d^2-4ag)n^4+2(de-2ah-2gb)n^3+(2df+e^2-4ai-4bh-4cg)n^2 \\ &\qquad +2(ef-2bi-2ch)n+(f^2-4ci) \end{aligned}$$

we have

$$\rho_n(p) = \begin{cases} (g_n/p)+1 & ;\ p\nmid 2(an^2+bn+c) \\ 1 & ;\ p\mid 2(an^2+bn+c)\ \&\ p\nmid (an^2+bn+c+dn^2+en+f) \\ 0 & ;\ p\mid 2(an^2+bn+c)\ \&\ p\mid (an^2+bn+c+dn^2+en+f) \end{cases}$$

where

$$g_n = \begin{cases} g_n' & \text{if } g_n' \text{ has no squared linear factor} \\ \dfrac{g_n'}{(\xi n+\eta)^2} & \text{if } g_n' \text{ has a squared linear factor } (\xi n+\eta)^2; \\ & (\xi,\eta)=1. \end{cases}$$

For g_n not a square, define $\chi(n)=\chi_D(n)$ to be the Kronecker symbol $(^D/_p)$, where if $g_n=r^2s$, for s squarefree and not equal to 1, $D=4s$ or s as $s\not\equiv 1 \bmod 4$ and $s\equiv 1 \bmod 4$ respectively. This enables us to write

$$\prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_n,z) \qquad (12)$$

for all z, where

$$c(g_n,z) = \prod_{\substack{p<z \\ p\nmid k_2 \\ p\mid 2(an^2+bn+c) \\ p\mid(an^2+bn+c+dn^2+en+f)}} (1-\frac{1}{p})^{-1} \prod_{\substack{p<z \\ p\mid 2(an^2+bn+c)g_n k_2}} (1-\frac{\chi(p)}{p})^{-1}$$

$$\times \prod_{\substack{p<z \\ p\nmid 2(an^2+bn+c)g_n k_2}} (1-\frac{\chi(p)}{p^2-(\chi(p)+1)p+\chi(p)}) .$$

It will be of advantage to us later to note that, for any $z>z_0$,

$$c(g_n,z) = c(g_n,z_0)\left\{1 + O\left[\frac{\ln(\zeta.k_2.\lambda).\ln n}{z_0}\right]\right\} \tag{13}$$

where $\zeta=\max\{|a|,|b|,|c|,|d|,|e|,|f|,|g|,|h|,|i|\}$ and λ=maximum coefficient of g_n in modulus.

To find an upper bound on the sum

$$\sum_{\substack{0<n\leq x \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{\rho_n(p)}{p})$$

it seems clear that we will need an upper bound on the product $\prod_{p<z} (1-\frac{\chi(p)}{p})$ when often D is relatively large in comparison to z, a seemingly difficult problem. However we are able to avoid the problem by reducing the sum to a form wherin D becomes "small" in comparison to z using the following observation:

" *If* $\max(F,k_2) \leq z \leq f(y)/k_1 \leq y/k_2$, *then for any* $A\in\mathbb{N}$,

$$\sum_{\substack{Af(y)<n\leq(A+1)f(y) \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{\rho_n(p)}{p}) = \sum_{\substack{0<n\leq f(y) \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{\rho_n(p)}{p})\left\{1 + O(E(v)) + O(\exp(-(\ln f(y)/k_1)^{\frac{1}{2}}))\right\} \tag{14}$$

where $v = \frac{\ln f(y)/k_1}{\ln z}$, $E(v) = \exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))$. "

The proof of (14) follows from an examination of the two functions

$$M(x,y,z) = \Big|\{(n,m):\ 0<n\leq f(y),\ n\equiv \ell_1 \bmod k_1,\ 0<m\leq y,\ m\equiv \ell_2 \bmod k_2,$$
$$((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i), \prod_{p<z} p)=1\Big|$$

and

$$N(x,y,z) = \Big|\{(n,m): Af(y)<n\leq (A+1)f(y), n\equiv \ell_1 \bmod k_1,\ 0<m\leq y, m\equiv \ell_2 \bmod k_2$$
$$((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i), \prod_{p<z} p)=1\Big|.$$

If $\max(F,k_2)\leq z\leq f(y)/k_1\leq y/k_2$, following the by now well-used arguments we have

$$M(x,y,z) = \frac{y}{k_2} \sum_{\substack{0<n\leq f(y)\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \{1 + O(E(u))$$
$$+ O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\}$$

where $u = \dfrac{\ln y/k_2}{\ln z}$, or alternatively

$$M(x,y,z) = \frac{f(y)}{k_1} \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z) \text{ app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \{1 + O(E(v))$$
$$+ O(\exp(-(\ln f(y)/k_1)^{\frac{1}{2}}))\}$$

where $v = \dfrac{\ln f(y)/k_1}{\ln z}$.

Comparing these two gives

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z) \text{ app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) = \frac{y.k_1}{f(y).k_2} \sum_{\substack{0<n\leq f(y)\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \{1 +$$
$$+ O(E(v)) + O(\exp(-(\ln f(y)/k_1)^{\frac{1}{2}}))\}. \tag{15}$$

Similarly a calculation of $N(x,y,z)$ in two different ways gives

$$\sum_{\substack{Af(y)<n\leq(A+1)f(y)\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \frac{f(y).k_2}{y.k_1} \sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z) \text{ app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$$

$$\times \{1 + O(E(v)) + O(\exp(-(\ln {}^{f(y)}/k_1)^{\frac{1}{2}}))\}. \tag{16}$$

(16) together with (15) gives (14) as required.

Consequently, for $z\leq y$,

$$\sum_{\substack{0<n\leq x\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \left(\frac{x}{y} + O(1)\right) \sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \{1$$

$$+ O(E(v)) + O(k_1^{\frac{1}{2}} \exp(-(\ln y)^{\frac{1}{2}}))\} \tag{17}$$

with $v=\frac{\ln {}^{y}/k_2}{\ln z}$, and we need only find an upper bound on the sum

$$\sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right). \tag{18}$$

We firstly deal with the sum over g_n square ie

$$\sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}\\ g_n \text{ a square}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right).$$

For g_n a square we have

$$\prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \prod_{\substack{p<z\\ p\nmid k_2\\ p\nmid 2(an^2+bn+c)g_n}} \left(1-\frac{2}{p}\right) \prod_{\substack{p<z\\ p\nmid k_2\\ p\nmid 2(an^2+bn+c)\\ p\mid g_n}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z\\ p\nmid k_2\\ p\mid 2(an^2+bn+c)\\ p\nmid (an^2+bn+c+dn^2+en+f)}} \left(1-\frac{1}{p}\right).$$

Following the argument of Theorem One this gives

$$\prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \ll \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z\\ p\mid (an^2+bn+c)}} \left(1-\frac{1}{p}\right)^{-1}$$

so that

$$\sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ a square}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \ll \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \ln\ln\zeta . \ln\ln y \sum_{\substack{0<n\leq y \\ g_n \text{ a square}}} 1 .$$

Since by definition, and (10),

$$\sum_{\substack{0<n\leq y \\ g_n \text{ a square}}} 1 = T(y,1) = O(G)$$

we have

$$\sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ a square}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \ll \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) . \ln\ln\zeta . \ln\ln y . G. \tag{19}$$

Consequently,

$$\sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$$

$$+ O\left[\prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) . \ln\ln\zeta . \ln\ln y . G\right] \tag{20}$$

and our task is reduced to finding an upper bound on the sum

$$\sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) . \tag{21}$$

From (12) this sum is equal to

$$\prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ not a square}}} \prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) c(g_n,z) \tag{22}$$

where $\chi(p)=(^D/_p)$.

Our intention is to change the dependence of (22) on z to one of dependence on z_1 for $z_1<z$.

Clearly $|D|\leq 4 \max_{0<n\leq y} |g_n|$.

Writing $g_n=a_4n^4+a_3n^3+a_2n^2+a_1n+a_0$ it is also clear that if $t\geq 20\max\{|a_4|,|a_3|,|a_2|,|a_1|,|a_0|\}$ then

$$4 \max_{0<n\leq t} |g_n| \leq t^5. \tag{23}$$

So we certainly have

$$|D| \leq 4 \max_{0<n\leq y} |g_n| \leq y^5.$$

Writing $Q=y^5$ and putting $\alpha=50$ in Lemma 5.1 gives

$$\prod_{p<z} \left(1-\frac{\chi(p)}{p}\right) = \prod_{p<z_1} \left(1-\frac{\chi(p)}{p}\right) \{1 + O(z_1^{-1/50}) + O(y^{-3/10})\}$$

for any real number z_1 satisfying $z\geq z_1\geq \ln^{100}(y^5)=5^{100}\ln^{100}y$, with at most $O(y^{9/10})$ exceptions.

With the usual understanding of "good" and "bad" g_n we have, taking $z_1=5^{100}\ln^{100}y$,

$$\sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ not a square} \\ g_n \text{ good}}} \prod_{p<z_1} \left(1-\frac{\chi(p)}{p}\right) c(g_n,z)$$

$$\times \{1 + O(\ln^{-2}y)\}$$

$$+ \sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ not a square} \\ g_n \text{ bad}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) .$$

But, by (11),

$$\sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ not a square} \\ g_n \text{ bad}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) < \sum_{\substack{0<n\leq y \\ g_n \text{ bad}}} 1 \ll \sum_{\substack{s \\ |s|\leq y^{9/10}}} \sum_{\substack{0<n\leq y \\ r^2 s = g_n}} 1$$

$$\ll S(y)y^{9/10} \ll G.y^{11/12}$$

where $S(y)$ is as defined in the introduction.

Secondly, from (13), for $n\leq y$,

$$c(g_n,z) = c(g_n,z_1)\left\{1 + 0\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99}y}\right]\right\}.$$

So

$$\sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ not a square}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \prod_{\substack{z_1\leq p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ not a square} \\ g_n \text{ good}}} \prod_{\substack{p<z_1 \\ p\nmid k_2}} \left(1-\frac{1}{p}\right)$$

$$\times \prod_{p<z_1} \left(1-\frac{\chi(p)}{p}\right) c(g_n,z_1)\left\{1 + 0\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99}y}\right] + 0\left[\frac{1}{\ln^2 y}\right]\right\}$$

$$+ 0(G.y^{11/12})$$

$$= \prod_{\substack{z_1\leq p<z \\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leq y \\ n\equiv \ell_1 \bmod k_1 \\ n \text{ app} \\ g_n \text{ not a square} \\ g_n \text{ good}}} \prod_{\substack{p<z_1 \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \left\{1 + 0\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99}y}\right]\right.$$

$$\left. + 0\left[\frac{1}{\ln^2 y}\right] \right\} + 0(G.y^{11/12}).$$

But, going backwards through the argument, we may write this final sum as

$$\sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n\ \mathrm{app}\\ g_n\ \text{not a square}\\ g_n\ \text{good}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n\ \mathrm{app}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) + O(G.y^{11/12})$$

$$+ O\left[\prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right).\ln\ln\zeta.\ln\ln y.G\right]$$

so that

$$\sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n\ \mathrm{app}\\ g_n\ \text{not a square}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \prod_{\substack{z_1\leq p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n\ \mathrm{app}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)\Bigg\{1 +$$

$$O\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99}y}\right] + O\left[\frac{1}{\ln^2 y}\right]\Bigg\} + O(G.y^{11/12})$$

$$+ O\left[\prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right).\ln\ln\zeta.\ln\ln y.G\right]. \tag{24}$$

Substitution of (24) back into (20) gives

$$\sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \prod_{\substack{z_1\leq p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n\ \mathrm{app}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)\Bigg\{1 +$$

$$O\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99}y}\right] + O\left[\frac{1}{\ln^2 y}\right]\Bigg\} + O(G.y^{11/12})$$

$$+ O\left[\prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right).\ln\ln\zeta.\ln\ln y.G\right] \tag{25}$$

and substitution of (25) into (17) gives

$$\sum_{\substack{0<n\leq x\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \frac{x}{y} \prod_{\substack{z_1\leq p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \left\{1\right.$$

$$+ 0\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99}y}\right] + 0\left[\frac{1}{\ln^2 y}\right] + 0(E(v)) + 0(k_1^{\frac{1}{2}}.\exp(-(\ln y)^{\frac{1}{2}}))$$

$$\left. + 0\left[\frac{y}{x}\right]\right\}$$

$$+ 0(x.G.y^{-1/12}) + 0\left[\frac{x}{y}.\prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right).\ln\ln\zeta.\ln\ln y.G\right]. \qquad (26)$$

thus returning us to our original sum.

We have however now reduced the problem to one of finding an upper bound on the sum

$$\sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right).$$

We repeat the process now writing this sum in terms of a sum dependent on z_2 with $z_2<z_1$ rather than on z_1 and so reducing the problem further.

Firstly from (14) we have

$$\sum_{\substack{0<n\leq y\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \left[\frac{y}{\exp^{\frac{1}{2}} y} + 0(1)\right] \sum_{\substack{0<n\leq \exp(\ln^{\frac{1}{2}} y)\\ n\equiv \ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$$

$$\times (1 + 0(k_1^{\frac{1}{2}}.\exp(-(\ln y)^{\frac{1}{4}}))) \qquad (27)$$

and

$$\sum_{\substack{0<n\leq\exp(\ln^{\frac12}y)\\ n\equiv\ell_1 \bmod k_1\\ n\ \mathrm{app}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \sum_{\substack{0<n\leq\exp(\ln^{\frac12}y)\\ n\equiv\ell_1 \bmod k_1\\ n\ \mathrm{app}\\ g_n\ \text{not a square}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$$

$$+ O\left[\prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right).\ln\ln\zeta.\ln(\ln y)^{\frac12}.G\right]. \tag{28}$$

Further

$$\sum_{\substack{0<n\leq\exp(\ln^{\frac12}y)\\ n\equiv\ell_1 \bmod k_1\\ n\ \mathrm{app}\\ g_n\ \text{not a square}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right)$$

$$\times \sum_{\substack{0<n\leq\exp(\ln^{\frac12}y)\\ n\equiv\ell_1 \bmod k_1\\ n\ \mathrm{app}\\ g_n\ \text{not a square}}} \prod_{p<z_1} \left(1-\frac{\chi(p)}{p}\right) c(g_n,z_1)$$

with $|D| \leq 4 \max_{0<n\leq\exp(\ln^{\frac12}y)} |g_n| \leq \exp(5(\ln^{\frac12}y))$ from (23).

Writing $Q=\exp(5(\ln^{\frac12}y))$ we have from Lemma 5.1 again, with $\alpha=50$,

$$\prod_{p<z_1} \left(1-\frac{\chi(p)}{p}\right) = \prod_{p<z_2} \left(1-\frac{\chi(p)}{p}\right) (1 + O(\ln^{-1}y))$$

where $z_2=5^{100}(\ln y)^{50}$, with at most $O(\exp({}^9/_{10}\ln^{\frac12}y))$ exceptions.

We also have, from (13), for $n\leq\exp(\ln^{\frac12}y)$,

$$c(g_n,z_1) = c(g_n,z_2)\left\{1 + O\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99/2}y}\right]\right\}.$$

So

$$\sum_{\substack{0<n\leqslant\exp(\ln^{\frac12}y)\\ n\equiv \ell_1 \bmod k_1\\ n\ \text{app}\\ g_n\ \text{not a square}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \prod_{\substack{z_2\leqslant p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leqslant\exp(\ln^{\frac12}y)\\ n\equiv \ell_1 \bmod k_1\\ n\ \text{app}\\ g_n\ \text{not a square}\\ g_n\ \text{good}}} \prod_{\substack{p<z_2\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$$

$$\times\left\{1+0\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99/2}y}\right]+0\left[\frac{1}{\ln y}\right]\right\}$$

$$+\sum_{\substack{0<n\leqslant\exp(\ln^{\frac12}y)\\ n\equiv \ell_1 \bmod k_1\\ n\ \text{app}\\ g_n\ \text{not a square}\\ g_n\ \text{bad}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$$

$$= \prod_{\substack{z_2\leqslant p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leqslant\exp(\ln^{\frac12}y)\\ n\equiv \ell_1 \bmod k_1\\ n\ \text{app}\\ g_n\ \text{not a square}\\ g_n\ \text{good}}} \prod_{\substack{p<z_2\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \left\{1+0\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99/2}y}\right]\right.$$

$$\left.+0\left[\frac{1}{\ln y}\right]\right\} + 0(G.\exp({}^{11}/_{12}\ln^{\frac12}y)).$$

Thus we have, arguing backwards again,

$$\sum_{\substack{0<n\leqslant\exp(\ln^{\frac12}y)\\ n\equiv \ell_1 \bmod k_1\\ n\ \text{app}\\ g_n\ \text{not a square}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \prod_{\substack{z_2\leqslant p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leqslant\exp(\ln^{\frac12}y)\\ n\equiv \ell_1 \bmod k_1\\ n\ \text{app}}} \prod_{\substack{p<z_2\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$$

$$\times\left\{1+0\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99/2}y}\right]+0\left[\frac{1}{\ln y}\right]\right\}$$

$$+ 0(G.\exp({}^{11}/_{12}\ln^{\frac12} y))$$

$$+ 0\left[\prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \ln\ln\zeta.\ln(\ln y)^{\frac12}.G\right]. \qquad (29)$$

Substituting (29) into (28) gives

$$\sum_{\substack{0<n\leq\exp(\ln^{\frac{1}{2}}y)\\ n\equiv\ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \prod_{\substack{z_2\leq p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<n\leq\exp(\ln^{\frac{1}{2}}y)\\ n\equiv\ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z_2\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$$

$$\times \left\{1 + 0\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99/2}y}\right] + 0\left[\frac{1}{\ln y}\right]\right\}$$

$$+ 0(G.\exp({}^{11}/_{12}\ln^{\frac{1}{2}}y))$$

$$+ 0\left[\prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \ln\ln\zeta.\ln(\ln y)^{\frac{1}{2}}.G\right] \qquad (30)$$

and (30) substituted into (27) gives

$$\sum_{\substack{0<n\leq y\\ n\equiv\ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \frac{y}{\exp(\ln^{\frac{1}{2}}y)} \prod_{\substack{z_2\leq p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right)$$

$$\times \sum_{\substack{0<n\leq\exp(\ln^{\frac{1}{2}}y)\\ n\equiv\ell_1 \bmod k_1\\ n \text{ app}}} \prod_{\substack{p<z_2\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) \left\{1 + 0\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99/2}y}\right] + 0\left[\frac{1}{\ln y}\right]\right\}$$

$$+ 0(k_1^{\frac{1}{2}}.\exp(-\ln^{\frac{1}{4}}y))\Big\} + 0(G.y.\exp(-{}^{1}/_{12}\ln^{\frac{1}{2}}y))$$

$$+ 0\left[\frac{y}{\exp(\ln^{\frac{1}{2}}y)} . \prod_{\substack{p<z_1\\ p\nmid k_2}} \left(1-\frac{1}{p}\right) \ln\ln\zeta.\ln(\ln y)^{\frac{1}{2}}.G\right]. \qquad (31)$$

Finally substituting (31) into (26), and writing $z=z_0$, gives

$$\sum_{\substack{0<n\leq x\\ n\equiv \ell_1 \bmod k_1\\ n\ \text{app}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \frac{x}{\exp((\ln y)^{\frac{1}{2}})} \prod_{\substack{z_2\leq p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right)$$

$$\times \sum_{\substack{0<n\leq \exp((\ln y)^{\frac{1}{2}})\\ n\equiv \ell_1 \bmod k_1\\ n\ \text{app}}} \prod_{\substack{p<z_2\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)\left\{1 + O\left[\ln(\zeta.k_2.\lambda).\sum_{0\leq t\leq 1}\frac{1}{\ln^{99/2^t}y}\right]\right.$$

$$+ O\left[\sum_{0\leq t\leq 1}\frac{1}{\ln^{2/2^t}y}\right] + O\left[k_1^{\frac{1}{2}}\sum_{0\leq t\leq 1}\exp(-(\ln y)^{1/2^{t+1}})\right] + O(E(v))$$

$$\left. + O\left[\frac{y}{x}\right]\right\}$$

$$+ O\left[x.G.\sum_{0\leq t\leq 1}\prod_{\substack{z_t\leq p<z\\ p\nmid k_2}}\left(1-\frac{1}{p}\right).\exp(-{}^1/_{12}(\ln y)^{1/2^t})\right]$$

$$+ O\left[x.\frac{\ln\ln\zeta}{\ln z}.\frac{k_2}{\varphi(k_2)}.G.\sum_{0\leq t\leq 1}\frac{\ln(\ln y)^{1/2^t}}{\exp((\ln y)^{1/2^t})}\right]. \tag{32}$$

Although this estimate has such unpleasant looking error terms we will shortly see that these can be greatly simplified if we accept some loss of strength. We have also now reduced the problem to one of finding an upper bound on the sum

$$\sum_{\substack{0<n\leq \exp(\ln^{\frac{1}{2}}y)\\ n\equiv \ell_1 \bmod k_1\\ n\ \text{app}}} \prod_{\substack{p<z_2\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right).$$

It is clear that we may approach this in the same way, splitting into the smaller sums

$$\sum_{\substack{0<n\leq \exp(\ln^{\frac{1}{4}}y)\\ n\equiv \ell_1 \bmod k_1\\ n\ \text{app}}} \prod_{\substack{p<z_2\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right).$$

We may continue the process until we reach the sum

$$\sum_{\substack{0<n\leq\exp(\ln^{1/2^A}y)\\ n\equiv \ell_1 \bmod k_1\\ n\ \mathrm{app}}} \prod_{\substack{p<z_{A+1}\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) .$$

where $z_{A+1}= 5^{100}(\ln y)^{100/2^{A+1}}$, and where

$$\ln^{1/2^A}y > 7\max\{|a_4|,|a_3|,|a_2|,|a_1|,|a_0|\} > \ln^{1/2^{A+1}}y$$

where $a_4,\ldots a_0$ are the coefficients of g_n.

Recalling that $\lambda=\max\{|a_4|,|a_3|,|a_2|,|a_1|,|a_0|\}$ we will then have

$$\sum_{\substack{0<n\leq x\\ n\equiv \ell_1 \bmod k_1\\ n\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) = \frac{x}{\exp((\ln y)^{1/2^t})} \prod_{\substack{z_{A+1}\leq p<z\\ p\nmid k_2}} \left(1-\frac{1}{p}\right)$$

$$\times \sum_{\substack{0<n\leq\exp((\ln y)^{1/2^A})\\ n\equiv \ell_1 \bmod k_1\\ n\ \mathrm{app}}} \prod_{\substack{p<z_{A+1}\\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)\left\{1 + O\left[\ln(\zeta.k_2.\lambda).\sum_{0\leq t\leq A}\frac{1}{\ln^{99/2^t}y}\right]\right.$$

$$+ O\left\{\sum_{0\leq t\leq A}\frac{1}{\ln^{2/2^t}y}\right\} + O\left[k_1^{\frac{1}{2}}\sum_{0\leq t\leq A}\exp(-(\ln y)^{1/2^{t+1}})\right] + O(E(v))$$

$$\left. + O\left[\frac{y}{x}\right]\right\}$$

$$+ O\left[x.G.\sum_{0\leq t\leq A}\prod_{\substack{z_t\leq p<z\\ p\nmid k_2}}\left(1-\frac{1}{p}\right).\exp(-{}^1/_{12}(\ln y)^{1/2^t})\right]$$

$$+ O\left[x.\frac{\ln\ln\zeta}{\ln z}.\frac{k_2}{\varphi(k_2)}G.\sum_{0\leq t\leq A}\frac{\ln(\ln y)^{1/2^t}}{\exp((\ln y)^{1/2^t})}\right]. \qquad (33)$$

As stated previously we may tidy this sum up somewhat by noting that the first error term satisfies

$$O\left[\ln(\zeta.k_2.\lambda)\sum_{0\leq t\leq A}\frac{1}{\ln^{99/2^t}y}\right] = O\left[\ln(\zeta.k_2.\lambda).\frac{1}{\ln^{99/2^A}y}\right] ;$$

the second error term satisfies

$$0\left[\sum_{0\leq t\leq A}\frac{1}{\ln^{2/2^t}y}\right]=0\left[\frac{1}{\ln^{2/2^A}y}\right];$$

and the third error term satisfies

$$0\left[k_1^{\frac{1}{2}}.\sum_{0\leq t\leq A}\exp(-(\ln y)^{1/2^{t+1}})\right]=0(k_1^{\frac{1}{2}}).$$

Furthermore the sum from the leading term satisfies

$$\sum_{\substack{0<n\leq\exp(\ln y)^{1/2^A})\\ n\equiv\ell_1\bmod k_1\\ n\ \mathrm{app}}}\prod_{\substack{p<z_{A+1}\\ p\nmid k_2}}(1-\frac{\rho_n(p)}{p})\leq\frac{\exp((\ln y)^{1/2^A})}{k_1}+0(1).$$

So

$$\sum_{\substack{0<n\leq x\\ n\equiv\ell_1\bmod k_1\\ n\ \mathrm{app}}}\prod_{\substack{p<z\\ p\nmid k_2}}(1-\frac{\rho_n(p)}{p})\leq\frac{x}{k_1}\prod_{\substack{p<z\\ p\nmid k_2}}(1-\frac{1}{p})\prod_{\substack{p<z_{A+1}\\ p\nmid k_2}}(1-\frac{1}{p})^{-1}\left\{1+\right.$$

$$\left.0\left[\frac{\ln(\zeta.k_2.\lambda)}{\ln^{99/2^A}y}\right]+0(\ln^{-2/2^A}y)+0(k_1^{\frac{1}{2}})\right\}$$

$$+0\left[x.G.\sum_{0\leq t\leq A}\prod_{\substack{z_t\leq p<z\\ p\nmid k_2}}(1-\frac{1}{p}).\exp(-1/_{12}(\ln y)^{1/2^t})\right]$$

$$+0\left[x.\frac{\ln\ln\zeta}{\ln z}.\frac{k_2}{\varphi(k_2)}G.\sum_{0\leq t\leq A}\frac{\ln(\ln y)^{1/2^t}}{\exp((\ln y)^{1/2^t})}\right].\qquad(34)$$

From the definition of A straightforward arguments give

$$\sum_{\substack{0<n\leq x\\ n\equiv\ell_1\bmod k_1\\ n\ \mathrm{app}}}\prod_{\substack{p<z\\ p\nmid k_2}}(1-\frac{\rho_n(p)}{p})\leq\frac{x}{k_1}\prod_{\substack{p<z\\ p\nmid k_2}}(1-\frac{1}{p})\prod_{p<(35\lambda)^{100}}(1-\frac{1}{p})^{-1}\left\{1+\right.$$

$$\left.0\left[\frac{\ln(\zeta.k_2.\lambda)}{(7\lambda)^{99}}\right]+0((7\lambda)^{-2})+0(k_1^{\frac{1}{2}})\right\}$$

$$+ O\left[x.G.\sum_{0<t\leq A}\prod_{\substack{z_t\leq p<z\\ p\nmid k_2}}\left(1-\frac{1}{p}\right).\exp\left(-{}^1/_{12}(\ln y)^{1/2^t}\right)\right]$$

$$+ O\left[x.\frac{\ln\ln\zeta}{\ln z}.\frac{k_2}{\varphi(k_2)}G.\sum_{0<t\leq A}\frac{\ln(\ln y)^{1/2^t}}{\exp((\ln y)^{1/2^t})}\right]. \tag{35}$$

Finally we may give upper bounds on the remaining error terms

$$O\left[x.G.\sum_{0<t\leq A}\prod_{\substack{z_t\leq p<z\\ p\nmid k_2}}\left(1-\frac{1}{p}\right).\exp\left(-{}^1/_{12}(\ln y)^{1/2^t}\right)\right]$$

and

$$O\left[x.\frac{\ln\ln\zeta}{\ln z}.\frac{k_2}{\varphi(k_2)}G.\sum_{0<t\leq A}\frac{\ln(\ln y)^{1/2^t}}{\exp((\ln y)^{1/2^t})}\right].$$

Firstly

$$\prod_{\substack{z_t\leq p<z\\ p\nmid k_2}}\left(1-\frac{1}{p}\right) = \prod_{\substack{p<z\\ p\nmid k_2}}\left(1-\frac{1}{p}\right)\prod_{\substack{p<z_t\\ p\nmid k_2}}\left(1-\frac{1}{p}\right)^{-1} \ll \prod_{\substack{p<z\\ p\nmid k_2}}\left(1-\frac{1}{p}\right).\ln z_t.$$

Since $z_t=5^{100}(\ln y)^{100/2^t}$ it follows that $\ln z_t \ll \ln(\ln y)^{1/2^t}$ and that

$$x.G.\sum_{0<t\leq A}\prod_{\substack{z_t\leq p<z\\ p\nmid k_2}}\left(1-\frac{1}{p}\right).\exp\left(-{}^1/_{12}(\ln y)^{1/2^t}\right)$$

$$\ll x.G.\prod_{\substack{p<z\\ p\nmid k_2}}\left(1-\frac{1}{p}\right)\sum_{0<t\leq A}\frac{\ln(\ln y)^{1/2^t}}{\exp({}^1/_{12}(\ln y)^{1/2^t})}.$$

For any t satisfying $0<t\leq A$,

$$\frac{\ln(\ln y)^{1/2^t}}{\exp(1/12(\ln y)^{1/2^t})} > 2.\frac{\ln(\ln y)^{1/2^{t-1}}}{\exp(1/12(\ln y)^{1/2^{t-1}})}$$

holds.

So the sum

$$\sum_{0<t<A} \frac{\ln(\ln y)^{1/2t}}{\exp(1/12(\ln y)^{1/2t})}.$$

is convergent.

Hence

$$0\left[x.G.\sum_{0<t<A} \prod_{\substack{z_t<p<z \\ p\nmid k_2}} (1-\frac{1}{p}) .\exp(-1/12(\ln y)^{1/2t})\right]$$

$$= 0\left[x.G.\prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{1}{p})\right].$$

Similarly

$$0\left[x.\frac{\ln\ln\zeta}{\ln z}.\frac{k_2}{\varphi(k_2)}G.\sum_{0<t<A} \frac{\ln(\ln y)^{1/2t}}{\exp((\ln y)^{1/2t})}\right].$$

$$= 0\left[x.\ln\ln\zeta.G.\prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{1}{p})\right].$$

It follows that there exists an absolute and effectively computable constant c_1 independent of a,b,c,d,e,f,g,h,i,k_1 and k_2 for which

$$\sum_{\substack{0<n<x \\ n\equiv\ell_1 \bmod k_1 \\ n \text{ app}}} \prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{\rho_n(p)}{p}) < c_1.\frac{x}{k_1} \prod_{\substack{p<z \\ p\nmid k_1}} (1-\frac{1}{p}) \prod_{p<(35\lambda)^{100}} (1-\frac{1}{p})^{-1} \left\{1 + \frac{\ln(\zeta.k_2.\lambda)}{(7\lambda)^{99}} + \frac{1}{(7\lambda)^2} + k_1.G.\ln\ln\zeta\right\}.$$

This completes the lemma in the case $g_n=g_n'$ ie where g_n' has no squared linear factor and is of degree 3 or 4.

In the alternative case ie where $g_n=\frac{g_n'}{(\xi n+\eta)^2}$ or $g_n=g_n'$ is of degree 2 a very similar proof may be constructed using the observation that here, writing $g_n=An^2+2Bn+C$,

$$T(y,s) \ll \tau\left[\frac{B^2-AC}{4}\right]\ln y$$

and

$$S(y) \ll \tau\left[\frac{B^2-AC}{4}\right]\ln y$$

from Lemma 4, and further that

$$\tau\left[\frac{B^2-AC}{4}\right]\ln y \ll G.y^{1/60}.$$

This completes the lemma.

Lemma One may now be applied to equation (1), our initial estimate of F(x,y,z) and, following the arguments of Theorem One, an upper bound on F(x,y,z) may be constructed.

Write, for convenience, the function

$$\prod_{p<(35\lambda)^{100}} (1-\frac{1}{p})^{-1} \left\{1 + \frac{\ln(\zeta.k_2.\lambda)}{(7\lambda)^{99}} + \frac{1}{(7\lambda)^2} + k_1.G.\ln\ln\zeta\right\}$$

as Γ.

Before stating Theorem Five we make some definitions and observations.

Let $g_m' := (b^2-4ac)m^4+2(be-2af-2cd)m^3+(2bh+e^2-4ai-4df-4gi)m^2$
$+2(eh-2di-2gf)m+(h^2-4gi)$

and write

$$\mu = b^2-4ac.$$

Define

$$g_m := \begin{cases} g_m' & \text{if } g_m' \text{ has no squared linear factor} \\ \dfrac{g_m'}{(\xi m+\eta)^2} & \text{if } g_m' \text{ has a squared linear factor } (\xi m+\eta)^2; \\ & (\xi,\eta)=1 \end{cases}$$

Define U(y,s) to be the number of integer solutions, (m,r), for m in the range $0<m\leq y$ of the equation $g_m=r^2s$; and V(y) to be

$$V(y) := \max_{0<s\leq 4|\mu|y^4} |U(y,s)|.$$

Whenever $g_m = g_m'$ define H as

$$H := 7^{4^3(9\omega(\mathrm{disc}(g_m))+13)}{}_4 k_2(L')$$

where L' is an algebraic number field of degree less than or equal to 4^3 containing at least three zeros of g_m, and where $k_2(L')$ is the 2-rank of the ideal class of L. With this definition we conclude, as previously, that

$$V(y) = O(H.|\mu|^{1/60}.y^{1/60}).$$

From here we derive

THEOREM FIVE.

Let an^2+bn+c, dn^2+en+f, *and* gn^2+hn+i *be polynomials with integer coefficients,* an^2+bn+c *and* dn^2+en+f *having no common factors. Let* $x,y\in\mathbb{Z}$ *and* $\ell_1,k_1,\ell_2,k_2\in\mathbb{N}$ *with* $\exp((\ln{}^{y}/k_2)^{\frac{1}{2}}) > \max\{|a|,|b|,|c|,|d|,|e|,|f|,|g|,|h|,|i|,k_1,k_2\}$. *Then, for* $2\leq z\leq x$, *and* ${}^{x}/k_1 \geq {}^{y}/k_2$,

$$F(x,y,z) < c_1.\frac{x.y}{k_1k_2} \prod_{\substack{p<z \\ p\nmid k_2}} (1-\frac{1}{p})\ \Gamma \Big\{1 + O(\exp(-(\ln{}^{y}/k_2)^{\frac{1}{2}}))$$

$$+ O(\varphi(k_2).|\mu|^{1/5}.y^{-1/7}.H) + O(\exp(-v(\ln v - \ln\ln 3v - \ln 2 - 2)))\Big\}$$

where c_1 *is the constant appearing in Lemma One, and where*

$$v = \frac{\ln{}^{x}/k_1}{\ln z}.$$

The O*-constants are absolute, and independent of* a,b,c,d,e, f,g,h,i,ℓ_1,ℓ_2,k_1, *and* k_2. *(They are however non-computable with current knowledge.)*

The proof of Theorem Five is not given as it is essentially the same as that of Theorem One. It really differs in only one respect. Recall Step Eleven of Theorem One where we found an upper bound on the function

$$\sum_{\substack{0<m\leqslant y \\ m\equiv \ell_2 \bmod k_2 \\ m \text{ app} \\ g_m \text{ not a square} \\ g_m \text{ bad}}}' L(1,\chi)^{-1}.c(g_m)$$

where the sum $\sum'$ was over g_m giving rise to a possible exceptional modulus. An equivalent sum occurs in the proof of Theorem Five. Since g_m may be of degree 4 in this instance (see the definition before the statement of the theorem) our previous estimate

$$L(1,\chi)^{-1} \ll \frac{|\bar{s}|^{\frac{1}{2}}}{\ln|\bar{s}|}$$

merely gives

$$L(1,\chi)^{-1} \ll \frac{y^2}{\ln y}$$

which is too large for our purposes. To avoid this difficulty we use Siegel's Theorem. Unfortunately this leads to non-computable error terms being introduced into the upper bound.

In line with the results of Theorem One we would expect that an upper bound for the function

$$F_1(x,y,z) = \Big|\{(n,m);\ \alpha<n\leqslant\alpha+x,\ n\equiv\ell_1 \bmod k_1,\ 0<m\leqslant y,\ m\equiv\ell_2 \bmod k_2,$$
$$((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i),\ \prod_{p<z} p)=1\}\Big|$$

could be found independently of α whenever $z\leqslant x$, and similarly an upper bound on the function

$$F_2(x,y,z) = \Big|\{(n,m);\ 0<n\leqslant x,\ n\equiv \ell_1 \bmod k_1,\ \beta<m\leqslant \beta+y,\ m\equiv \ell_2 \bmod k_2,$$
$$((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i), \prod_{p<z} p)=1\}\Big|$$

independently of β whenever $z\leqslant y$. This is indeed the case.

Looking firstly at $F_1(x,y,z)$ for $z\leqslant x$ we may rewrite the function as

$$F_1(x,y,z) = \Big|\{(s,m):\ 0<s\leqslant x,\ s\equiv(\ell_1-\alpha) \bmod k_1,\ 0<m\leqslant y,\ m\equiv \ell_2 \bmod k_2,$$
$$((a(s+\alpha)^2+b(s+\alpha)+c)m^2+(d(s+\alpha)^2+e(s+\alpha)+f)m+(g(s+\alpha)^2+h(s+\alpha)+$$
$$i), \prod_{p<z} p)=1\}\Big|.$$

Following previous arguments we get

$$F_1(x,y,z) = \frac{x}{k_1} \sum_{\substack{0<m\leqslant y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)\Big\{1 + O(\exp(-(\ln^x/k_1)^{\frac{1}{2}}))$$
$$+ O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\Big\}$$

where if s_m is the highest common factor of am^2+dm+g, $2a\alpha m^2+bm^2+2d\alpha m+em+2g\alpha+h$, and $a\alpha^2m^2+b\alpha m^2+cm^2+d\alpha^2m+e\alpha m+fm+g\alpha^2+h\alpha +i$ then "(m,z) app" denotes those integers satisfying the conditions

(i) $(s_m, \prod_{p<z} p)=1$

(ii) $(am^2+dm+g+2a\alpha m^2+dm^2+2d\alpha m+em+2g\alpha+h,$
$a\alpha^2m^2+b\alpha m^2+cm^2+d\alpha^2m+e\alpha m+fm+g\alpha^2+h\alpha+i)\equiv 1 \bmod 2$

(iii) $$\left(\left[\frac{am^2+bm+c}{s_m}\right]\ell_1^2+\left[\frac{2a\alpha m^2+bm^2+2d\alpha m+em+2g\alpha+h}{s_m}\right]\ell_1+\right.$$
$$\left.+\left[\frac{a\alpha^2m^2+b\alpha m^2+cm^2+d\alpha^2m+e\alpha m+fm+g\alpha^2+h\alpha+i}{s_m}\right], \prod_{\substack{p<z \\ p\mid k_2}} p\right)=1.$$

Under these conditions $\rho_m(p)$ is defined by

$$\rho_m(p) = \Big|\{n(\bmod p):\ (am^2+dm+g)n^2+(2a\alpha m^2+bm^2+2d\alpha m+em+2g\alpha+h)n$$
$$+(a\alpha^2m^2+b\alpha m^2+cm^2+d\alpha m^2+e\alpha m+fm+g\alpha^2+h\alpha+i) \equiv 0 \bmod p\}\Big|.$$

Clearly s_m may be rewritten as the highest common factor of am^2+dm+g, bm^2+em+h, and cm^2+fm+i so that the condition

$(s_m, \prod_{p<z} p)=1$ is satisfied if and only if the condition $(am^2+dm+g, bm^2+em+h, cm^2+fm+i, \prod_{p<z} p)=1$ is satisfied. Further, condition (ii) is satisfied if and only if

$$(am^2+dm+g+bm^2+em+h, cm^2+fm+i) \not\equiv 1 \bmod 2,$$

and condition (iii) is satisfied if and only if

$$\left(\left[\frac{am^2+dm+g}{s_m}\right](\ell_1+\alpha)^2+\left[\frac{bm^2+em+h}{s_m}\right](\ell_1+\alpha)+\left[\frac{cm^2+fm+i}{s_m}\right], \prod_{\substack{p<z \\ p \mid k_1}} p\right)=1.$$

Finally

$$\rho_m(p) = \begin{cases} (g_m/p)+1 & ; p \nmid 2(am^2+dm+g) \\ 1 & ; p \mid 2(am^2+dm+g) \ \& \ p \nmid (am^2+dm+g+2a\alpha m^2+bm^2+em+2g\alpha+h) \\ 0 & ; \text{otherwise} \end{cases}$$

where

$$g_m = (2a\alpha m^2+bm^2+2d\alpha m+em+2g\alpha+h)^2-4(am^2+dm+g)(a\alpha^2m^2+b\alpha m^2+cm^2 +d\alpha^2m+e\alpha m+fm+g\alpha^2+h\alpha+i).$$

g_m may be simplified to read

$$g_m = (bm^2+em+h)^2-4(am^2+dm+g)(cm^2+fm+i)$$

and

$$\rho_m(p) = \begin{cases} (g_m/p)+1 & ; p \nmid 2(am^2+dm+g) \\ 1 & ; p \mid 2(am^2+dm+g) \ \& \ p \nmid (am^2+dm+g+bm^2+em+h) \\ 0 & ; \text{otherwise} \end{cases}$$

or

$$\rho_m(p) = \left|\{n(\bmod p) : (am^2+dm+g)n^2+(bm^2+em+h)n+(cm^2+fm+i) \equiv 0 \bmod p\}\right|.$$

So $F_1(x,y,z)$ may be rewritten

$$F_1(x,y,z) = \frac{x}{k_1} \sum_{\substack{0<m\leq y \\ m\equiv \ell_2 \bmod k_2 \\ (m,z) \text{ app}}} \prod_{\substack{p<z \\ p \nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right) \left\{1 + O(\exp(-(\ln^x/k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 2-2)))\right\}$$

where if s_m is the highest common factor of am^2+dm+g, bm^2+em+h, and cm^2+fm+i then "(m,z) app" denotes those integers m satisfying the conditions

(i) $(s_m, \prod_{p<z} p)=1$

(ii) $(am^2+dm+g+bm^2+em+h, cm^2+fm+i)\equiv 1 \bmod 2$

(iii) $\left(\left[\frac{am^2+dm+g}{s_m}\right](\ell_1+\alpha)^2+\left[\frac{bm^2+em+h}{s_m}\right](\ell_1+\alpha)+\left[\frac{cm^2+fm+i}{s_m}\right], \prod_{\substack{p<z\\ p|k_1}} p\right)=1$.

But the sum

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$$

for these definitions occurs in the estimate of the function

$$\left|\{(n,m):\ 0<n\leq x, n\equiv(\ell_1-\alpha)\bmod k_1,\ 0<m\leq y,\ m\equiv \ell_2 \bmod k_2,\right.$$
$$\left.((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i), \prod_{p<z} p)=1\}\right|;$$

the function covered in Theorem Five. Since the upper bound on $F(x,y,z)$ in Theorem Five is independent of the value of ℓ_1 it follows that an upper bound on

$$\sum_{\substack{0<m\leq y\\ m\equiv \ell_2 \bmod k_2\\ (m,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid k_1}} \left(1-\frac{\rho_m(p)}{p}\right)$$

may be found independently of the value of α when $z\leq x$ as required. The proof when $z\leq y$ is very similar.

CHAPTER FIVE

To apply the reasoning of Chapter Four to the general prime function

$$P(x,y,z) = \Big|\{(q,r):\ q\leq x,\ q\equiv \ell_1 \bmod k_1,\ r\leq y,\ r\equiv \ell_2 \bmod k_2,\ ((aq^2+bq+c)r^2+(dq^2+eq+f)r+(gq^2+hq+i),\prod_{p<z} p)=1\}\Big|$$

for q and r primes, we would, following the argument of Theorem Four, firstly find an upper bound on the function

$$R(x,y,z) = \Big|\{(n,q):\ n\leq x,\ n\equiv \ell_1 \bmod k_1,\ q\leq y,\ q\equiv \ell_2 \bmod k_2,\ (((an^2+bn+c)q^2+(dn^2+en+f)q+(gn^2+hn+i))n,\prod_{p<z} p)=1\}\Big|$$

for $z\leq x$ and $x/k_1 \geq y/k_2$.

Progressing as in Steps 2 and 3 of Theorem Four we see that for $z\leq \exp((\ln y/k_2)^{\frac{1}{2}})$, $R(x,y,z)$ may be written

$$R(x,y,z) \leq \frac{y}{k_2} \sum_{\substack{0<n\leq x \\ n\equiv \ell_1 \bmod k_1 \\ (n,\prod_{p<z} p)=1 \\ (n,z)\ \text{app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n'(p)}{p}\right) \Big\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))\Big\} \tag{1}$$

where "(n,z) app" represents a series of conditions on n, the exact evaluation of which need not concern us here: and where

$$\rho_n'(p) = \begin{cases} \rho_n(p)+1 & ;\ p\nmid(gn^2+hn+i) \\ \rho_n(p) & ;\ p\mid(gn^2+hn+i) \end{cases}$$

for

$$\rho_n(p) = \Big|\{m(\bmod p): (an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i)\equiv 0 \bmod p\}\Big|$$

whenever $p<z$.

On the other hand, for $z\leq x$, defining s_q to be the highest common factor of aq^2+dq+e, bq^2+eq+h, and cq^2+fq+i, $R(x,y,z)$ may be written

$$R(x,y,z) = \frac{x}{[6,k_1]} \sum_{\substack{0<q\leqslant y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\text{ app}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right)\Big\{1 +$$

$$O(\exp(-(\ln x/6k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))\Big\}$$

(2)

where for some integers ℓ_3, ℓ_4 and ℓ_5 all satisfying $(\ell_i,6,k_1)=1$; $i=1,2,3$, "(q,z) app" denotes the primes satisfying

(i) $(s_q, \prod_{p<z} p)=1$

(ii) $aq^2+dq+g+bq^2+eq+h+cq^2+fq+i\equiv 1 \bmod 2$

(iii) $(aq^2+dq+g+bq^2+eq+h+cq^2+fq+i,$

$4(aq^2+dq+g)+2(bq^2+eq+h)+(cq^2+fq+i))\not\equiv 0 \bmod 3$

(iv) $\left(\left[\frac{aq^2+dq+g}{s_q}\right]\ell_5^2+\left[\frac{dq^2+eq+h}{s_q}\right]\ell_5+\left[\frac{cq^2+fq+i}{s_q}\right], \prod_{\substack{p<z \\ p\mid 6k_1}} p\right)=1$

if $(6,k_1)=3$ or 6; and

(i) $(s_q, \prod_{p<z} p)=1$

(ii) $aq^2+dq+g+bq^2+eq+h+cq^2+fq+i\equiv 1 \bmod 2$

(iii) $(aq^2+dq+g+bq^2+eq+h+cq^2+fq+i,$

$4(aq^2+dq+g)+2(bq^2+eq+h)+(cq^2+fq+i))\not\equiv 0 \bmod 3$

(iv) $\left(\left[\frac{aq^2+dq+g}{s_q}\right]\ell_3^2+\left[\frac{dq^2+eq+h}{s_q}\right]\ell_3+\left[\frac{cq^2+fq+i}{s_q}\right], \prod_{\substack{p<z \\ p\mid 6k_1}} p\right)=1$ <u>or</u>

$\left(\left[\frac{aq^2+dq+g}{s_q}\right]\ell_4^2+\left[\frac{dq^2+eq+h}{s_q}\right]\ell_4+\left[\frac{cq^2+fq+i}{s_q}\right], \prod_{\substack{p<z \\ p\mid 6k_1}} p\right)=1$

otherwise. In what follows we will assume that $(6,k_1)=3$ or 6. When $(6,k_1)=1$ or 2 a similar argument may be applied.

Further

$$\rho_q'(p) = \begin{cases} \rho_q(p)+1 & ; \ p\nmid(cq^2+fq+i) \\ \rho_q(p) & ; \ p\mid(cq^2+fq+i) \end{cases}$$

where

$$\rho_q(p) = \left|\{n(\bmod p):(aq^2+dq+g)n^2+(bq^2+eq+h)n+(cq^2+fq+i)\equiv 0 \bmod p\}\right|$$

whenever $p<z$.

The obvious way forward is now to find an upper bound on the function

$$\sum_{\substack{0<n\leqslant x \\ n\equiv \ell_1 \bmod k_1 \\ (n,\prod_{p<z} p)=1 \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n'(p)}{p}\right) . \tag{3}$$

which is a problem similar to that tackled in Lemma One, where the sum we required an upper bound on was

$$\sum_{\substack{0<n\leqslant x \\ n\equiv \ell_1 \bmod k_1 \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right) .$$

(NB "(n,z) app" may be defined differantly in the two cases.) In that instance we wrote the product $\prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$ in terms of the product $\prod_{\substack{p<z_1 \\ p\nmid k_2}} \left(1-\frac{\rho_n(p)}{p}\right)$ for $z_1<z$, thus reducing the sum to one effectively dependent only on z_1.

Although as in Lemma One the product of (3), $\prod_{\substack{p<z \\ p\nmid k_2}} \left(1-\frac{\rho_n'(p)}{p}\right)$ may be written in terms of the product $\prod_{\substack{p<z_1 \\ p\nmid k_2}} \left(1-\frac{\rho_n'(p)}{p}\right)$ for some $z_1<z$ it is not clear how the condition $(n,\prod_{p<z} p)=1$ appearing under the summation sign may be reduced to $(n,\prod_{p<z_1} p)=1$.

To avoid this difficulty we take a different approach. Rather than work with the function

$$R(x,y,z) = \Big|\{(n,q):\ n\leqslant x,\ n\equiv \ell_1 \bmod k_1,\ q\leqslant y,\ q\equiv \ell_2 \bmod k_2,$$
$$(((an^2+bn+c)q^2+(dn^2+en+f)q+(gn^2+hn+i))n,\prod_{p<z} p)=1\}\Big|$$

we look instead at

$$T(x,y,z) = \Big|\{(n,q):\ n\leqslant x,\ n\equiv \ell_5 \bmod [6,k_1],\ q\leqslant y,\ q\equiv \alpha \bmod \beta,$$
$$((an^2+bn+c)q^2+(dn^2+en+f)q+(gn^2+hn+i),\prod_{p<z} p)=1\}\Big|$$

where (α,β) is not necessarily equal to 1.

For $z\leqslant \exp(54(\ln y/k_2)^{\frac{1}{2}}))$, an upper bound may be found on $T(x,y,z)$ following the usual argument ie

$$T(x,y,z) < \frac{y}{2\beta} \sum_{\substack{0<n\leq x \\ n\equiv \ell_5 \bmod [6,k_1] \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid 2\beta}} \left(1-\frac{\rho_n'(p)}{p}\right)\Big\{1 + O(\exp(-(\ln^y/_\beta)^{\frac{1}{2}}))$$

$$+ O(\exp(-u(\ln u-\ln\ln 3u-\ln 3-2)))\Big\} \tag{4}$$

for some set of conditions "(n,z) app" and where

$$\rho_n'(p) = \begin{cases} \rho_n(p)+1 & ;\ p\nmid gn^2+hn+i \\ \rho_n(p) & ;\ p\mid gn^2+hn+i \end{cases}$$

with

$$\rho_n(p) = \left|\{m(\bmod p):(an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i)\equiv 0 \bmod p\}\right|$$

for $p<z$.

On the other hand for $z\leq x$, defining, as before, s_q to be the highest common factor of aq^2+dq+g, bq^2+eq+h, and cq^2+fq+i we have the alternative estimate of $T(x,y,z)$,

$$T(x,y,z) = \frac{x}{[6,k_1]} \sum_{\substack{0<q\leq y \\ q\equiv\alpha \bmod \beta \\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} \left(1-\frac{\rho_q(p)}{p}\right)\Big\{1 +$$

$$O(\exp(-(\ln^x/6k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))\Big\} \tag{5}$$

where, for $p<z$,

$$\rho_q(p) = \left|\{n(\bmod p):(aq^2+dq+g)n^2+(bq^2+eq+h)n+(cq^2+fq+i)\equiv 0 \bmod p\}\right|$$

and where "(q,z) app" is the set of conditions

(i) $(s_q, \prod_{p<z} p)=1$

(ii) $aq^2+dq+g+bq^2+eq+h+cq^2+fq+i\equiv 1 \bmod 2$

(iii) $(aq^2+dq+g+bq^2+eq+h+cq^2+fq+i,$

$$4(aq^2+dq+g)+2(bq^2+eq+h)+(cq^2+fq+i))\not\equiv 0 \bmod 3$$

(iv) $\left(\left[\frac{aq^2+dq+g}{s_q}\right]\ell_5^2+\left[\frac{dq^2+eq+h}{s_q}\right]\ell_5+\left[\frac{cq^2+fq+i}{s_q}\right], \prod_{\substack{p<z \\ p\mid 6k_1}} p\right)=1$

a set of conditions identical to those defining "(q,z) app" in (2).

Now, applying the method of argument of Lemma One to the sum of (5) instead of (1) we attain

LEMMA TWO

Suppose $2<z<y<x$. *Define* $F:=|(cd-fa)^2-(bd-ea)(ce-fb)|$. *Then there exists an absolute and effectively computable constant* c_1, *independent of* $a,b,c,d,e,f,g,h,i,k_1,\ell_5,\alpha$ *and* β *for which it follows that*

$$\sum_{\substack{0<n\leq x \\ n\equiv \ell_5 \bmod [6,k_1] \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid 2\beta}} \left(1-\frac{\rho_n'(p)}{p}\right) < c_1 . \frac{x}{k_1} \prod_{\substack{p<z \\ p\nmid 2\beta}} \left(1-\frac{2}{p}\right) \prod_{p<(35\lambda)^{100}} \left(1+\frac{1}{p-2}\right)$$

$$\times \left\{1 + \frac{\ln(\zeta.\beta.\lambda)}{(7\lambda)^{99}} + \frac{1}{(7\lambda)^2} + k_1.G.\ln\ln\zeta\right\}$$

if $z>\max(F,2\beta)$.

If on the other hand $z\leq\max(F,2\beta)$ *then*

$$\sum_{\substack{0<n\leq x \\ n\equiv \ell_5 \bmod [6,k_1] \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid 2\beta}} \left(1-\frac{\rho_n'(p)}{p}\right) < \left[\frac{x}{k_1} + O(1)\right] \prod_{\substack{p<z \\ p\nmid 2\beta}} \left(1-\frac{1}{p}\right) .$$

The notation is as described in Lemma One.

Writing Γ as

$$\Gamma = c_1 . \prod_{\substack{p<(35\lambda)^{100} \\ p\neq 2}} \left(1+\frac{1}{p-2}\right) \left\{1 + \frac{\ln(\zeta.\lambda)}{(7\lambda)^{99}} + \frac{1}{(7\lambda)^2} + k_1.G.\ln\ln\zeta\right\}$$

it follows that

$$\sum_{\substack{0<n\leq x \\ n\equiv \ell_5 \bmod [6,k_1] \\ (n,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid 2\beta}} \left(1-\frac{\rho_n'(p)}{p}\right) < \frac{x}{k_1} . \Gamma . \prod_{\substack{p<z \\ p\nmid 2\beta}} \left(1-\frac{2}{p}\right) . \ln\beta \qquad (6)$$

for $z<y\leq x$ with Γ independent of β if $z>\max(F,2\beta)$. Furthermore (6) holds for $z\leq\max(F,2\beta)$ as well.

Substitution of (6) into (4) gives

$$T(x,y,z) < \frac{x.y}{2\beta k_1}\Gamma.\ln\beta.\prod_{\substack{p<z\\ p\nmid 2\beta}}\left(1-\frac{2}{p}\right)\left\{1+O(\exp(-(\ln y/k_2)^{\frac{1}{2}}))\right\} \qquad (7)$$

for $z<\exp(54(\ln y/k_2)^{\frac{1}{2}})$, and a comparison with the alternative estimate of $T(x,y,z)$, (5), gives

$$\sum_{\substack{0<q\leq y\\ q\equiv\alpha\bmod\beta\\ (q,z)\ \mathrm{app}}}\ \prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{\rho_q(p)}{p}\right) < \frac{y.[6,k_1]}{2\beta k_1}\Gamma.\ln\beta.\prod_{\substack{p<z\\ p\nmid 2\beta}}\left(1-\frac{2}{p}\right)\left\{1+O(\exp(-(\ln y/\beta)^{\frac{1}{2}}))\right\} \qquad (8)$$

for $z<\exp(54(\ln y/k_2)^{\frac{1}{2}})$. We emphasise here that Γ is independent of β.

Given this upper bound we will demonstrate how this information may be used to find an upper bound on the function

$$\sum_{\substack{0<q\leq y\\ q\equiv\ell_2\bmod k_2\\ (q,z)\ \mathrm{app}}}\ \prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{\rho_q'(p)}{p}\right)$$

which appears in (2), the estimate for $R(x,y,z)$, whenever $z<\exp(54(\ln y/k_2)^{\frac{1}{2}})$. In doing so we sidestep the difficulty of having to find an upper bound on

$$\sum_{\substack{0<n\leq x\\ n\equiv\ell_1\bmod k_1\\ (n,\prod_{p<z}p)=1\\ (n,z)\ \mathrm{app}}}\ \prod_{\substack{p<z\\ p\nmid k_2}}\left(1-\frac{\rho_n'(p)}{p}\right).$$

Recall that

$$\rho_q'(p)=\begin{cases}\rho_q(p)+1 & ;\ p\nmid(cq^2+fq+i)\\ \rho_q(p) & ;\ p\mid(cq^2+fq+i)\end{cases}$$

where

$$\rho_q(p) = \left|\{n(\bmod p):(aq^2+dq+g)n^2+(bq^2+eq+h)n+(cq^2+fq+i)\equiv 0 \bmod p\}\right|.$$

It follows that

$$\prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{\rho_q'(p)}{p}\right) = \prod_{\substack{p<z\\ p\nmid 6k_1\\ p\nmid cq^2+fq+i}}\left(1-\frac{\rho_q(p)+1}{p}\right)\prod_{\substack{p<z\\ p\nmid 6k_1\\ p\mid cq^2+fq+i}}\left(1-\frac{\rho_q(p)}{p}\right)$$

$$= \prod_{p<z}\left(1-\frac{1}{p}\right)\prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{\rho_q(p)}{p}\right)\prod_{\substack{p<z\\ p\mid 6k_1(cq^2+fq+i)}}\left(1-\frac{1}{p}\right)^{-1}$$

$$\times\prod_{\substack{p<z\\ p\nmid 6k_1\\ p\nmid cq^2+fq+i}}\left(1-\frac{\rho_q(p)}{(p-1)(p-\rho_q(p))}\right).$$

Since $\rho_q(p)\geq 0$,

$$\prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{\rho_q'(p)}{p}\right) \leq \prod_{p<z}\left(1-\frac{1}{p}\right)\prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{\rho_q(p)}{p}\right)\prod_{\substack{p<z\\ p\mid 6k_1(cq^2+fq+i)}}\left(1-\frac{1}{p}\right)^{-1}$$

and we have

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ (q,z)\text{ app}}}\prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{\rho_q'(p)}{p}\right) \leq \prod_{p<z}\left(1-\frac{1}{p}\right)\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ (q,z)\text{ app}}}\prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{\rho_q(p)}{p}\right)$$

$$\times\prod_{\substack{p<z\\ p\mid 6k_1(cq^2+fq+i)}}\left(1-\frac{1}{p}\right)^{-1}.$$

The second sum is clearly similar to that of equation (8).

The reasoning from here is along the lines of Lemmas 2.9 and 2.11.

Let θ denote the highest common factor of c, f, and i and write $c_1={}^{c}/_{\theta}$, $f_1={}^{f}/_{\theta}$ and $i_1={}^{i}/_{\theta}$. Then

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) < \prod_{\substack{p<z\\ p\nmid 6k_1\theta}} \left(1-\frac{1}{p}\right) \sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q(p)}{p}\right) \times \prod_{\substack{p<z\\ p\mid (c_1q^2+f_1q+i_1)\\ p\nmid 6k_1\theta}} \left(1-\frac{1}{p}\right)^{-1} .$$

Write the sum

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q(p)}{p}\right) \prod_{\substack{p<z\\ p\mid c_1q^2+f_1q+i_1\\ p\nmid 6k_1}} \left(1-\frac{1}{p}\right)^{-1}$$

as S so that the sum we require satisfies

$$\sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q'(p)}{p}\right) < \prod_{\substack{p<z\\ p\nmid 6k_1\theta}} \left(1-\frac{1}{p}\right) S . \tag{9}$$

Now the second product of S, $\prod_{\substack{p<z\\ p\mid c_1q^2+f_1q+i_1\\ p\nmid 6k_1\theta}} \left(1-\frac{1}{p}\right)^{-1}$, is equal to

$$\sum_{\substack{c_1q^2+f_1q+i_1\equiv 0 \bmod m\\ (6k_1\theta,m)=1\\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)}$$

where P(m) denotes the largest prime factor of m.

Consequently

$$S = \sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q(p)}{p}\right) \sum_{\substack{c_1q^2+f_1q+i_1\equiv 0 \bmod m\\ (6k_1\theta,m)=1\\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)}$$

which on changing the order of summation gives

$$S = \sum_{\substack{1\leq m\leq G(y)\\ (6k_1,\theta,m)=1\\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{0<q\leq y\\ q\equiv \ell_2 \bmod k_2\\ (q,z)\ \mathrm{app}\\ c_1q^2+f_1q+i_1\equiv 0 \bmod m}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q(p)}{p}\right)$$

where $G(y):=\max_{0<q\leq y}|c_1q^2+f_1q+i_1|$.

Let $\gamma_1(m),\ldots,\gamma_r(m)$ be the $\rho(m)$ solutions of

$$c_1n^2+f_1n+i_1\equiv 0 \bmod m$$

and let $\delta_i=\delta_i(\ell_2,\gamma_i(m))$ be the unique solution, $\bmod[k_2,m]$, if it exists of the pair of congruences $n\equiv\ell_2 \bmod k_2$ and $n\equiv\gamma_i(m) \bmod m$. Then

$$S = \sum_{\substack{1\leq m\leq G(y)\\ (6k_1,\theta,m)=1\\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\gamma_i(m) \bmod m\\ \gamma_i(m)\equiv\ell_2 \bmod (k_2,m)}} \sum_{\substack{0<q\leq y\\ q\equiv\delta_i \bmod [k_2,m]\\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q(p)}{p}\right).$$

We divide the sum S into two to read

$$S = \sum_{\substack{1\leq m\leq \exp((\ln y)^{\frac12})\\ (6k_1,\theta,m)=1\\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\gamma_i(m) \bmod m\\ \gamma_i(m)\equiv\ell_2 \bmod (k_2,m)}} \sum_{\substack{0<q\leq y\\ q\equiv\delta_i \bmod [k_2,m]\\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q(p)}{p}\right)$$

$$+ \sum_{\substack{\exp((\ln y)^{\frac12})\leq m\leq G(y)\\ (6k_1,\theta,m)=1\\ P(m)<z}} \frac{\mu^2(m)}{\varphi(m)} \sum_{\substack{\gamma_i(m) \bmod m\\ \gamma_i(m)\equiv\ell_2 \bmod (k_2,m)}} \sum_{\substack{0<q\leq y\\ q\equiv\delta_i \bmod [k_2,m]\\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q(p)}{p}\right). \qquad (10)$$

Now, if $m\leq\exp((\ln y)^{\frac12})$ then from (8) we have that the first innermost sum satisfies

$$\sum_{\substack{0<q\leq y\\ q\equiv\delta_i \bmod [k_2,m]\\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z\\ p\nmid 6k_1}} \left(1-\frac{\rho_q(p)}{p}\right) < \frac{y[6,k_1]}{2k_1}.\Gamma.\frac{\ln[k_2,m]}{[k_2,m]} \prod_{\substack{p<z\\ p\nmid 2[k_2,m]}} \left(1-\frac{2}{p}\right) \Big\{1 + O(\exp(-(\ln y/[k_2,m])^{\frac12}))\Big\}$$

$$< 2.\frac{y}{k_1}.[6,k_1].\Gamma.\ln[k_2,m].\frac{[k_2,m]}{\varphi^2([k_2,m])} \prod_{p<z} (1-\frac{1}{p})^2 \Big\{1 + O(\exp(-(\ln y)^{\frac{1}{2}}))\Big\} \qquad (11)$$

with Γ independent of m.

(We recall here that the derivation of this upper bound stemmed from an analysis of the function T(x,y,z) introduced on page 228.)

If however $m \geq \exp((\ln y)^{\frac{1}{2}})$ then the second innermost sum satisfies

$$\sum_{\substack{0<q\leq y \\ q\equiv\delta_i \bmod [k_2,m] \\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} (1-\frac{\rho_q(p)}{p}) \leq \sum_{\substack{0<q\leq y \\ q\equiv\delta_i \bmod [k_2,m]}} 1 \leq \frac{2y}{\ln y}.\frac{\ln[k_2,m]}{\varphi([k_2,m])} + 1. \qquad (12)$$

Substitution of (11) and (12) back into (10) gives

$$S \leq 2.\frac{y}{k_1}.[6,k_1].\Gamma.\prod_{p<z} (1-\frac{1}{p})^2 \sum_{\substack{1\leq m\leq \exp((\ln y)^{\frac{1}{2}}) \\ (6k_1\theta,m)=1 \\ P(m)<z}} \rho(m)\frac{\mu^2(m)}{\varphi(m)}\frac{\ln[k_2,m].[k_2,m]}{\varphi^2([k_2,m])}$$

$$\times\ (1 + O(\exp(-(\ln y)^{\frac{1}{2}})))$$

$$+ O\Big[\frac{y}{\ln y} \sum_{\substack{\exp((\ln y)^{\frac{1}{2}})<m\leq G(y) \\ (6k_1\theta,m)=1 \\ P(m)<z}} \frac{\mu^2(m).\rho(m)}{\varphi(m)} \frac{\ln[k_2,m]}{\varphi([k_2,m])}\Big]$$

$$+ O\Big[\sum_{\substack{\exp((\ln y)^{\frac{1}{2}})<m\leq G(y) \\ (6k_1\theta,m)=1 \\ P(m)<z}} \frac{\mu^2(m).\rho(m)}{\varphi(m)}\Big]. \qquad (13)$$

But the first sum of (13) satisfies

$$\sum_{\substack{1\leq m\leq \exp((\ln y)^{\frac{1}{2}}) \\ (6k_1\theta,m)=1 \\ P(m)<z}} \rho(m).\frac{\mu^2(m)}{\varphi(m)}.\frac{\ln[k_2,m].[k_2,m]}{\varphi^2([k_2,m])} \leq k_2 \sum_{\substack{(6k_1\theta,m)=1 \\ P(m)<z}} \frac{\mu^2(m)\rho(m)m^{5/4}}{\varphi^3(m)}$$

$$< k_2 \prod_{\substack{p<z \\ p\nmid 6k_1\theta}} \left(1+ \frac{2p^{5/4}}{(p-1)^3}\right)$$

$$< 2k_2.\zeta(3/2) \qquad (14)$$

where $\zeta(\ )$ denotes the Riemann zeta function.

The second part of (13) satisfies

$$\sum_{\substack{\exp((\ln y)^{\frac12})<m\leq G(y) \\ (6k_1\theta,m)=1 \\ P(m)<z}} \frac{\mu^2(m).\rho(m)}{\varphi(m)} \frac{\ln[k_2,m]}{\varphi([k_2,m])} < k_2 \sum_{m>\exp((\ln y)^{\frac12})} \frac{\rho(m).m^{\frac14}}{\varphi^2(m)}$$

$$\ll k_2 \sum_{m>\exp((\ln y)^{\frac12})} \frac{\rho(m).\ln\ln m}{\varphi(m).m^{\frac34}}$$

$$\ll k_2 \frac{\ln\ln y}{\exp(\frac34(\ln y)^{\frac12})} \sum_{m>\exp((\ln y)^{\frac12})} \frac{\rho(m)}{\varphi(m)}$$

$$\ll k_2.\frac{(\ln\ln y)^2}{\exp(\frac34(\ln y)^{\frac12})}.(\ln y)^{M/2} \qquad (15)$$

by Lemma 2.7 with $\ln M=\left\{\left[\frac{\ln(2.D^2)}{\ln 2}\right]+1\right\}\ln 2$ for D the discriminant of $c_1n^2+f_1n+i_1$.

The third sum of (13) satisfies

$$\sum_{\substack{\exp((\ln y)^{\frac12})<m\leq G(y) \\ (6k_1\theta,m)=1 \\ P(m)<z}} \frac{\mu^2(m).\rho(m)}{\varphi(m)} < \prod_{\substack{p<z \\ p\nmid 6k_1\theta}} \left(1+\frac{2}{p-1}\right). \qquad (16)$$

So

$$S < 4.\frac{y}{k_1}.[6,k_1].k_2.\Gamma.\zeta(3/2)\prod_{p<z}\left(1-\frac{1}{p}\right)^2.\left\{1+O(\exp(-(\ln y)^{\frac12}))\right\}$$

$$+ O\left[\frac{y}{\ln y}.k_2.\frac{(\ln\ln y)^2.(\ln y)^{M/2}}{\exp(\frac34(\ln y)^{\frac12})}\right] + O\left[\prod_{\substack{p<z \\ p\nmid 6k_1\theta}}\left(1+\frac{2}{p-1}\right)\right].$$

For $z\leq\exp(54(\ln y/k_2)^{\frac12})$ this yields

$$S < 4.\frac{y}{k_1}.[6,k_1].k_2.\Gamma.\zeta(3/2).\prod_{p<z}(1-\frac{1}{p})^2.\left\{1 + O(\exp(-(\ln y)^{\frac{1}{2}}))\right\}$$

for $M \ll \ln\ln y$.

From (9) we now have

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} (1-\frac{\rho_q'(p)}{p}) \leq 4.\frac{y}{k_1}.[6,k_1].k_2.\Gamma.\zeta(3/2).\prod_{\substack{p<z \\ p\nmid 6k_1\theta}} (1-\frac{1}{p})$$

$$\times \prod_{p<z} (1-\frac{1}{p})^2 \left\{1 + O(\exp(-(\ln y)^{\frac{1}{2}}))\right\} \qquad (17)$$

for $z \leq \exp(54(\ln^y/k_2))$. This is the upper bound we required. (It appears in our estimate of $R(x,y,z)$.)

Given this starting point we may proceed as in Theorem Four to reach

$$\sum_{\substack{0<q\leq y \\ q\equiv \ell_2 \bmod k_2 \\ (q,z)\ \mathrm{app}}} \prod_{\substack{p<z \\ p\nmid 6k_1}} (1-\frac{\rho_q'(p)}{p}) \leq 4.\frac{y}{k_1}.[6,k_1].k_2.\Gamma.\zeta(3/2).\prod_{\substack{p<z \\ p\nmid 6k_1}} (1-\frac{2}{p})$$

$$\times \prod_{\substack{p<z \\ p\nmid 6k_1\theta}} (1-\frac{1}{p}) \prod_{p<z_1} (1-\frac{1}{p})^2 \prod_{\substack{p<z_1 \\ p\nmid 6k_1}} (1-\frac{2}{p})^{-1} \left\{1 + \right.$$

$$\left. O(\exp(-(\ln^y/k_2)^{\frac{1}{2}})) + O(|\mu|^{1/5}.H.y^{-1/7}.\ln y)\right\} \qquad (18)$$

for any $z \leq x$, where

$$z_1 = \begin{cases} \exp(54(\ln^y/k_2)^{\frac{1}{2}}) & ;\ z>\exp(54(\ln^y/k_2)^{\frac{1}{2}}) \\ z & ;\ z\leq \exp(54(\ln^y/k_2)^{\frac{1}{2}}) \end{cases} ;$$

where, if we define g_m to be

$$g_m=(b^2-4ac)m^4+2(be-2af-2cd)m^3+(2bh+e^2-4ai-4df-4gi)m^2$$
$$+ 2(eh-2di-2gf)m+(h^2-4gi)$$

then μ:=largest coefficient of g_m in modulus;

and where H is as defined for Theorem Five.

This upper bound may now be substituted into our initial estimate of $F(x,y,z)$, (2).

Substitution of (18) into (2) gives

$$R(x,y,z) \leq 4.\frac{k_2}{k_1}.\Gamma.\zeta(^3/_2).x.y.\prod_{\substack{p<z\\ p\nmid 6k_1}}(1-\frac{2}{p})\prod_{\substack{p<z_1\\ p\nmid 6k_1\theta}}(1-\frac{1}{p})\prod_{p<z_1}(1-\frac{1}{p})^2$$

$$\times \prod_{\substack{p<z_1\\ p\nmid 6k_1}}(1-\frac{2}{p})^{-1}\Big\{1 + O(\exp(-(\ln^y/k_2)^{\frac{1}{2}})) + O(|\mu|^{1/5}.H.y^{1/7}\ln y)$$

$$+ O(\exp(-(\ln^x/6k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))\Big\} \qquad (19)$$

which concludes the case for $(6,k_1)=3$ or 6.

If $(6,k_1)=1$ or 2 then an almost identical proof gives

$$R(x,y,z) \leq 8.\frac{k_2}{k_1}.\Gamma.\zeta(^3/_2).x.y.\prod_{\substack{p<z\\ p\nmid 6k_1}}(1-\frac{2}{p})\prod_{\substack{p<z_1\\ p\nmid 6k_1\theta}}(1-\frac{1}{p})\prod_{p<z_1}(1-\frac{1}{p})^2$$

$$\times \prod_{\substack{p<z_1\\ p\nmid 6k_1}}(1-\frac{2}{p})^{-1}\Big\{1 + O(\exp(-(\ln^y/k_2)^{\frac{1}{2}})) + O(|\mu|^{1/5}.H.y^{1/7}\ln y)$$

$$+ O(\exp(-(\ln^x/6k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))\Big\}. \qquad (20)$$

A combination gives

$$R(x,y,z) \leq \frac{8}{\varphi((6,k_1))}\Gamma.\frac{k_2}{k_1}.\zeta(^3/_2).x.y.\prod_{\substack{p<z\\ p\nmid 6k_1}}(1-\frac{2}{p})\prod_{\substack{p<z_1\\ p\nmid 6k_1}}(1-\frac{1}{p})\prod_{p<z_1}(1-\frac{1}{p})^2$$

$$\times \prod_{\substack{p<z_1\\ p\nmid 6k_1}}(1-\frac{2}{p})^{-1}\Big\{1 + O(\exp(-(\ln^y/k_2)^{\frac{1}{2}})) + O(|\mu|^{1/5}.H.y^{1/7}\ln y)$$

$$+ O(\exp(-(\ln^x/6k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))\Big\} \qquad (21)$$

for any value of $(6,k_1)$.

Returning to the start of the chapter we recall that the function we were really interested in was $P(x,y,z)$. Now

$$P(x,y,z) \leq R(x,y,z) + O\left[\frac{y.z}{\varphi(k_2)\varphi(k_1)}\right]$$

so we have the following theorem:

THEOREM SIX

Let an^2+bn+c, dn^2+en+f *and* gn^2+hn+i *be polynomials with integer coefficients*, an^2+bn+c *and* dn^2+en+f *having no common factors. Let* $x,y\epsilon Z$ *and* $\ell_1,k_1,\ell_2,k_2\epsilon N$.

Then for $3\leqslant z\leqslant x$, *and* $x/k_1 \geqslant y/k_2$,

$$P(x,y,z) \leqslant \frac{8}{\varphi((6,k_1))}\Gamma.\frac{k_2}{k_1}.\zeta(3/2).x.y.\prod_{\substack{p<z\\ p\nmid 6k_1}}\left(1-\frac{2}{p}\right)\prod_{\substack{p<z_1\\ p\nmid 6k_1\theta}}\left(1-\frac{1}{p}\right)\prod_{p<z_1}\left(1-\frac{1}{p}\right)^2$$

$$\times \prod_{\substack{p<z_1\\ p\nmid 6k_1}}\left(1-\frac{2}{p}\right)^{-1}\Big\{1 + O(\exp(-(\ln y/k_2)^{\frac{1}{2}})) + O(|\mu|^{1/5}.H.y^{1/7}\ln y)$$

$$+ O(\exp(-(\ln x/6k_1)^{\frac{1}{2}})) + O(\exp(-v(\ln v-\ln\ln 3v-\ln 3-2)))$$

$$+ O\left[\frac{z.\ln^3 z}{x.\varphi(k_1).\varphi(k_2)}\right]\Big\}$$

where $v=\frac{\ln x/k_1}{\ln z}$.

The O-constants are absolute and independent of a,b,c,d,e, f,g,h,i,ℓ_1,ℓ_2,k_1 *and* k_2.

CHAPTER 6

As a final note we include the observation that the methods developed throughout the previous chapters may be used to estimate functions of the form

$$\Phi_k(x,y) = \left|\{(n,m):\ n\leq x,\ m\leq y,\ ((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i),\ k)=1\}\right|.$$

Defining

$$\Phi_k(x,y,z) = \left|\{(n,m):\ n\leq x,\ m\leq y,\ ((an^2+bn+c)m^2+(dn^2+en+f)m+(gn^2+hn+i),\ \prod_{\substack{p<z\\ p|k}} p\ )=1\}\right|$$

we proceed in a manner similar to that adopted previously. By a simple adaptation of Lemma 1.1 we have, for $z\leq y$,

$$\Phi_k(x,y,z) = y \sum_{\substack{0<n\leq x\\ (n,z)\ \text{app}}} \prod_{\substack{p<z\\ p|k}} \left(1-\frac{\rho_n(p)}{p}\right) \{1 + O(E(u)) + O(\exp(-(\ln y)^{\frac{1}{2}}))\} \tag{1}$$

where $E(u)=\exp(-u(\ln u-\ln\ln 3u-\ln 2-2))$ and $u=\frac{\ln y}{\ln z}$;

and for $z\leq x$,

$$\Phi_k(x,y,z) = x \sum_{\substack{0<m\leq y\\ (m,z)\ \text{app}}} \prod_{\substack{p<z\\ p|k}} \left(1-\frac{\rho_m(p)}{p}\right) \{1 + O(E(v)) + O(\exp(-(\ln x)^{\frac{1}{2}}))\} \tag{2}$$

where $v=\frac{\ln x}{\ln z}$.

Following the arguments of Lemma 5.1 we may show that for y and Q large real numbers, $\alpha\geq 10$, and $\ln^{2\alpha}Q\leq y\leq z$,

$$\prod_{\substack{p<z\\ p|k}} \left(1-\frac{\chi(p)}{p}\right)^{-1} = \prod_{\substack{p\leq y\\ p|k}} \left(1-\frac{\chi(p)}{p}\right)^{-1} \left\{1 + O(\alpha y^{-1/\alpha}) + O(Q^{-3/\alpha}) + O\left[\frac{P(k)}{\exp(Q^{1/\alpha})}\right]\right\} \tag{3}$$

with at most $O(Q^{9}/\alpha)$ exceptions, where P(k) denotes the largest prime factor of k.

Given (1), (2) and (3) an asymptotic formula, or upper bound, for $\Phi_k(x,y,z)$ may be derived. No major changes occur in the argument; in many instances the arguments are simpler.

The details of a general theorem have not been derived but we give as an example (without proof) the following relatively simple case:

THEOREM 7

For $x,y\in\mathbb{N}$, *let* $M=\max(x,y)$. *Then defining the function*

$$\Phi_k(x,y,z) = \left|\{(n,m): 0<n\leq x,\ 0<m\leq y,\ (n^2+m, \prod_{\substack{p<z\\ p\mid k}} p)=1\}\right|$$

we have

$$\Phi_k(x,y,z) = xy.\prod_{\substack{p<z\\ p\mid k}}\left(1-\frac{1}{p}\right)\Big\{1 + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))$$

$$+ O(\exp(-(\ln y)^{\frac{1}{2}})) + O(\exp(-(\ln x)^{\frac{1}{2}})) + O\left[\frac{P(k)}{\exp(y^{1/10})}\right]$$

$$+ O\left[\frac{k}{\varphi(k)}.y^{-1/32}\right]\Big\} \tag{4}$$

where $u=\frac{\ln M}{\ln z}$, *and* z *satisfies* $2\leq z\leq M$. *All the implicit constants are absolute and effectively computable.*

Since $\frac{k}{\varphi(k)} \leq \prod_{p<P(k)}\left(1-\frac{1}{p}\right)^{-1} \ll \ln P(k)$ it follows that, for $P(k)\leq\exp(y^{1/64})$, (4) may be rewritten

$$\varphi_k(x,y,z) = xy.\prod_{\substack{p<z\\ p\mid k}}\left(1-\frac{1}{p}\right)\Big\{1 + O(\exp(-u(\ln u-\ln\ln 3u-\ln 2-2)))$$

$$+ O(\exp(-(\ln y)^{\frac{1}{2}})) + O(\exp(-(\ln x)^{\frac{1}{2}}))\Big\}.$$

For $\Phi_k(x,y) = \left|\{(n,m): 0<n\leq x,\ 0<m\leq y,\ (n^2+m,k)=1\}\right|$ clearly

$\Phi_k(x,y) \leq \Phi_k(x,y,z)$. So we have the following:

COROLLARY

For all $x,y\in\mathbb{N}$, *and for* $k\in\mathbb{N}$ *satisfying*

$$P(k) \leq \min(x^{\frac{1}{2}}, \exp(y^{1/64})),$$

there exists an absolute and effectively computable constant c_1 *such that*

$$\Phi_k(x,y) \leq c_1 xy \frac{\varphi(k)}{k}.$$

To judge the effectiveness of the Corollary we note that Theorem 3.5 of Halberstam–Richert [2], gives

$$\sum_{\substack{m\leq y\\(m,k)=1}} 1 < 7.\frac{\varphi(k)}{k}.y \qquad \text{if } y>e^6 \text{ and } P(k)\leq y.$$

An almost identical proof yields

$$\sum_{\substack{\alpha<m\leq\alpha+y\\(m,k)=1}} 1 < 7.\frac{\varphi(k)}{k}.y \qquad \text{if } y>e^6 \text{ and } P(k)\leq y.$$

This implies that

$$\Phi_k(x,y) = \sum_{\substack{n\leq x,\ m\leq y\\(n^2+m,k)=1}}\sum 1 < 7.\frac{\varphi(k)}{k}.xy \qquad \text{if } y>e^6 \text{ and } P(k)\leq y.$$

Our corollary does not of course improve on this result except that it allows for a much wider range of k whenever $y \ll \sqrt{x}$.

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