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NOETHERIAN MODULES OVER HYPERFINITE GROUPS

by

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To

my wife and son

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NOETHERIAN MODULES OVER HYPERFINITE GROUPS

CONTENTS

SUMMARY

Let G be a group and A a $\mathbb{Z}G$-module. If $A = A^{f} \oplus A^{\bar{f}}$, where A^{f} is a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A^{f} is finite and the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of A has no nonzero finite $\mathbb{Z}G$-factors, then A is said to have an f–decomposition. If G is a hyperfinite locally soluble group, then it is known that any artinian $\mathbb{Z}G$-module A has an f–decomposition. In this thesis, especially by investigating the properties of the torsion-free noetherian $\mathbb{Z}G$-modules, we prove that any noetherian $\mathbb{Z}G$-module A over a hyperfinite locally soluble group G has an f–decomposition, too. Further, the structure of the noetherian $\mathbb{Z}G$-submodule A^{f} is well described and the structure of the noetherian $\mathbb{Z}G$-submodule $A^{\bar{f}}$ is discussed in detail.

If G is a Černikov group (not necessarily locally soluble) or, more generally, if G is a finite extension of a periodic abelian group with $|\pi(G)| < \infty$, where $\pi(G) = \{\text{prime } p;\ G \text{ has an element of order } p\}$, then, for any noetherian $\mathbb{Z}G$-module A, we have that:

(1) A has an f–decomposition;

(2) A^{f} is finitely generated as an abelian group and $G/C_G(A^{f})$ is finite; and

(3) $A^{\bar{f}}$ is torsion as a group and has a finite $\mathbb{Z}G$-composition series as well as a finite exponent.

Moreover, we have generalized Zaĭcev's results about modules over hyperfinite locally soluble groups to modules over hyper–(cyclic or finite) groups. In fact, we have got the following results:

Theorem C: Any periodic artinian $\mathbb{Z}G$-module A over a hyper–(cyclic or finite)

locally soluble group G has an f–decomposition.

Theorem **D**: Let E be an extension of a periodic abelian group A by a hyper–(cyclic or finite) locally soluble group G. If A is an artinian $\mathbb{Z}G$-module, then E splits conjugately over A modulo A^{f}. And

Theorem **E**: Let E be an extension of an abelian group A by a hyper–(cyclic or finite) locally soluble group G. If A is a noetherian $\mathbb{Z}G$-module with $A = A^{\bar{f}}$, then E splits conjugately over A.

A number of questions are given at the end of the work.

INTRODUCTION

A group G is a hyperfinite group if G has an ascending normal series $1 = G_0 \le G_1 \le \cdots \le G_\alpha = G$ in which each factor $G_{\beta+1}/G_\beta$ is finite, where $\beta < \alpha$. The class of hyperfinite groups forms a subclass of the class of locally finite groups. In our work, we mainly consider a hyperfinite group G acting on an abelian group A and, by the action of G on A, we consider A as a $\mathbb{Z}G$-module.

In 1986, D. I. Zaĭcev proved that: if G is a hyperfinite locally soluble group, then any artinian $\mathbb{Z}G$-module A has an f–decomposition. That is, $A = A^f \oplus A^{\bar{f}}$, where A^f is a $\mathbb{Z}G$-submodule of A such that the irreducible $\mathbb{Z}G$-factors of A^f are all finite and the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of A has no nonzero finite $\mathbb{Z}G$-factors. Using this result, he proved a splitting theorem that: let E be an extension of an abelian group A by a hyperfinite locally soluble group G and assume that A is an artinian $\mathbb{Z}E$-module, then E splits conjugately over A modulo A^f (we will explain this result later). In 1988, he used a strong condition for proving a splitting theorem dual to the above. That is, he proved the result that: let E be an extension of an abelian group A by a hyperfinite locally soluble group G and assume that A is a noetherian $\mathbb{Z}G$-module with $A = A^{\bar{f}}$, then E splits conjugately over A. Can we remove the condition $A = A^{\bar{f}}$ and get exactly the form that E splits conjugately over A modulo A^f? This leads us to consider whether any noetherian $\mathbb{Z}G$-module A over a hyperfinite locally soluble group G has an f–decomposition. After investigating the properties of the torsion-free noetherian $\mathbb{Z}G$-modules, we have now successfully proved the required result. That is, we have:

<u>Theorem A</u>: Any noetherian $\mathbb{Z}G$-module A over a hyperfinite locally soluble

group G has an f–decomposition, too.

In our proof, we proceed in the following steps.

Step 1: The important lemmas.

Lemma 1.2.5: (Wilson, [17]) Let G be a group, H a normal subgroup of finite index in G, and A a $\mathbb{Z}G$-module. Then A is a noetherian (resp. artinian) $\mathbb{Z}G$-module if and only if A is a noetherian (resp. artinian) $\mathbb{Z}H$-module.

Lemma 1.2.14: (Zaĭcev, [22]) Let H be a hyperfinitely embedded subgroup of a group G and A a noetherian $\mathbb{Z}G$-module. If $C_A(H) = 0$, then H contains a subgroup K and A contains a nonzero $\mathbb{Z}G$-submodule B such that K is normal in G, $C_B(K) = 0$, and $|K/C_K(B)| < \infty$.

Lemma 2.1.4: Let G be a locally finite group and A a torsion-free noetherian $\mathbb{Z}G$-module. Then $pA < A$ and $\bigcap_{i=1}^{\infty} p^iA = 0$ for any prime p.

Lemma 2.4.5: Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p. If all irreducible $\mathbb{Z}G$-factors of A are finite, then A is finite.

Lemma 2.4.7: Let G be a group, A a $\mathbb{Z}G$-module, and M a $\mathbb{Z}G$-submodule of A such that the factor module A/M is a p-group for some prime p. If $H = C_G(A/M)$ contains a nontrivial finite subgroup K being a q-group for some prime $q \neq p$, then $A = C_A(x)+M$ for any $x \in K$. Further, $A = C_A(K)+M$.

Step 2: Reducing A to be either torsion-free or an elementary abelian p-group for some prime p.

This step is very important in our proof and it much depends on the following result.

Corollary **3.3**: Let G be a hyperfinite locally soluble group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of B is finite (resp. infinite) and A/B contains no finite (resp. infinite) irreducible $\mathbb{Z}G$-factors. Then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

Step 3: Reducing A to be torsion-free with all finite irreducible $\mathbb{Z}G$-factors being p-groups for some fixed prime p.

This has been achieved in Proposition **3.10**.

Proposition **3.10**: Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module. If A has no f–decomposition, then A has a nonzero $\mathbb{Z}G$-image $\overline{A}$ satisfying:

(*a*) $\overline{A}$ has no f–decomposition;

(*b*) for every nonzero $\mathbb{Z}G$-submodule $\overline{C}$ of $\overline{A}$, $\overline{A}/\overline{C}$ has an f–decomposition;

(*c*) $\overline{A}$ has no nonzero $\mathbb{Z}G$-submodules with an f–decomposition;

(*d*) $\overline{A}$ is torsion-free; and

(*e*) the finite irreducible $\mathbb{Z}G$-factors of $\overline{A}$ are all p-groups for some fixed prime p.

Step 4: Discussing the properties of the torsion-free noetherian $\mathbb{Z}G$-modules.

Specially, for a fixed prime p, we have got a descending series of $\mathbb{Z}G$-submodules

$$A_{00} > A_{01} > A_{02} > \cdots > \bigcap_i A_{0i} = A_{0\infty},$$

in which, for any $\mathbb{Z}G$-submodule A_{0i}, $pA_{0i} < A_{0,i+1}$, the irreducible $\mathbb{Z}G$-factors of $A_{0i}/A_{0,i+1}$ are all finite, and the $\mathbb{Z}G$-submodule $A_{0\infty}$ has no nonzero finite $\mathbb{Z}G$-factors being p-groups.

Step 5: Completing the proof.

The key result in completing our proof is the following:

Proposition 3.14: Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module. If all finite irreducible $\mathbb{Z}G$-factors of A are p-groups for some fixed prime p, then A has an f–decomposition.

Furthermore, in Chapter **4**, we have well described the structure of the noetherian $\mathbb{Z}G$-submodule A^{f} of A and, under some conditions, that of the noetherian $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of A. More exactly, we have:

Theorem **B**: Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module. Then A^{f} is finitely generated as an abelian group and $G/C_G(A^{f})$ is finite.

Proposition **4.4.6**: Let G be a periodic abelian group with $|\pi(G)| < \infty$, where $\pi(G) = \{\text{prime } p;\ G \text{ has an element of order } p\}$, and let A be a noetherian $\mathbb{Z}G$-module. Then $A^{\bar{f}}$ is torsion and $A^{\bar{f}}$ has a finite $\mathbb{Z}G$-composition series as well as a finite exponent.

In all the above results, G has been assumed to be locally soluble. However, it is not a necessary condition as we can take G to be a Černikov group (not necessarily locally soluble). More generally, we have:

Theorem: If G is a finite extension of a periodic abelian group with $\pi(G)$ finite, then

(1) any noetherian $\mathbb{Z}G$-module A has an f–decomposition:

$$A = A^{f} \oplus A^{\bar{f}},$$

where A^{f} is a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A^{f} is finite and the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ has no nonzero finite $\mathbb{Z}G$-factors;

(2) A^{f} as a group is finitely generated and $G/C_G(A^{f})$ is finite; and

(3) $A^{\bar{f}}$ is torsion and has a finite $\mathbb{Z}G$-composition series as well as a finite exponent.

In Chapter **5**, we have generalized Zaĭcev's results about modules over hyperfinite locally soluble groups to modules over hyper-(cyclic or finite) locally soluble groups. Especially, the obtained splitting theorems are new in the splitting theory, for which we will give a short review below.

A group E is said to split over its subgroup A modulo C for some subgroup C of E if there is a subgroup B of E such that $E = AB$ and $A \cap B = C$. The subgroup B is called a supplement to A in E or a complement to A in E modulo C. If all complements to A in E modulo C are conjugate in E modulo C, then E is said to split conjugately over A modulo C. If $C = 1$, then we naturally have the concepts of a complement to A in E and E splitting conjugately over A.

It is well-known that if A is a subgroup of a finite group E such that $(|A|, |E/A|) = 1$ then E splits conjugately over A (The Schur–Zassenhaus Theorem, **[15]**). After this splitting theorem, many splitting results sprang up in the later years. Among these, we quote a few to stand as a background for our results.

a (M. L. Newell, 1975, **[10]**): Let A be an abelian normal subgroup of a group E such that E/A is locally supersoluble and $[E', A] = 1$. If A is noetherian as $\mathbb{Z}E$-module and has no nonzero cyclic $\mathbb{Z}E$-images, then E splits conjugately over A.

b (M. L. Newell, 1975, **[10]**): Let A be an abelian minimal normal subgroup of a group E such that E/A is hypercyclic. If A is not cyclic, then E splits conjugately over A.

c (M. J. Tomkinson, 1978, **[16]**): Let G be a hypercyclic group and A a finite $\mathbb{Z}$G-module. If A has no nonzero cyclic $\mathbb{Z}$G-images, then any extension E of A by G splits conjugately over A.

d (D. I. Zaǐcev, 1979 and 1980, **[19]** and **[20]**): Let A be an abelian normal subgroup of a group E such that E/A is hypercyclic. If A is artinian (resp. noetherian) as a $\mathbb{Z}$E-module and has no nonzero cyclic $\mathbb{Z}$E-submodules (resp. $\mathbb{Z}$E-images), then E splits conjugately over A.

e (D. I. Zaǐcev, 1986 and 1988, **[21]** and **[22]**): Let A be an abelian normal subgroup of a group E such that E/A is a hyperfinite locally soluble group. If A is artinian (resp. noetherian) as a $\mathbb{Z}$E-module and has no nonzero finite $\mathbb{Z}$E-submodules (resp. $\mathbb{Z}$E-images), then E splits conjugately over A.

Now we have proved the following results.

<u>Theorem D</u>: Let A be a periodic abelian normal subgroup of a group E such that E/A is a hyper-(cyclic or finite) locally soluble group. If A is artinian as a $\mathbb{Z}$E-module and has no nonzero finite $\mathbb{Z}$E-submodules, then E splits conjugately over A.

<u>Theorem E</u>: Let A be an abelian normal subgroup of a group E such that E/A is a hyper-(cyclic or finite) locally soluble group. If A is noetherian as a $\mathbb{Z}$E-module and has no nonzero finite $\mathbb{Z}$E-images, then E splits conjugately over

A.

There are lots of questions still remaining open. We list out some of these in Chapter **6**.

1 PRELIMINARIES

In this chapter, as a beginning, we use two sections to recall some definitions and some well–known results, which will be quoted at least once in the later work. The indicated source of most of the results does not mean the original one. Some of the results have been literally rewritten to make them easily quoted. For the sake of convenience, we have also given a proof for some evident results. The terminology used in the work is standard as used in [15].

§1.1 NOTATION AND TERMINOLOGY

Throughout, we let G denote a group, $\mathbb{Z}$ the ring of integers, $\mathbb{Z}_p$ the prime field of characteristic p, and $\mathbb{Z}G$ (resp. $\mathbb{Z}_pG$) the group ring with G as a basis and the coefficients in $\mathbb{Z}$ (resp. $\mathbb{Z}_p$).

Definition 1.1.1: An ascending series of a group G is a set of subgroups $\{G_\beta;\ \beta \le \alpha\}$ indexed by ordinals less than or equal to an ordinal α such that

(a) $H_{\beta_1} \le H_{\beta_2}$ if $\beta_1 \le \beta_2$,

(b) $H_0 = 1$ and $H_\alpha = G$,

(c) H_β is normal in $H_{\beta+1}$, and

(d) $H_\lambda = \bigcup_{\beta<\lambda} H_\beta$ if λ is a limit ordinal.

It is convenient to write the ascending series in the form

$$1 = G_0 \le G_1 \le \cdots \le G_\alpha = G.$$

If each G_β $(\beta \le \alpha)$ is normal in G, then we say the ascending series $\{G_\beta;\ \beta \le \alpha\}$ of G is an ascending normal series of G.

Definition 1.1.2: Let $\mathscr{P}$ denote a group property. A group is said to be a hyper-$\mathscr{P}$ group if G has an ascending series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G,$$

in which each factor $G_{\beta+1}/G_\beta$ has $\mathscr{P}$ and each G_β is normal in G, where $\beta < \alpha$. In particularly, if $\mathscr{P}$ is the group property of finiteness, then we call G a hyperfinite group. If α is finite and G_β need not to be normal in G, $\beta < \alpha$, then we call G a poly-$\mathscr{P}$ group. Thus a poly-cyclic group G is that G has an ascending series $1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G$, in which each factor $G_{\beta+1}/G_\beta$ is cyclic, $\beta < \alpha$, and $\alpha < \infty$.

Definition 1.1.3: A normal subgroup H of G is said to be hyper-$\mathscr{P}$ embedded in G if H itself is a hyper-$\mathscr{P}$ group and in the corresponding ascending normal series $1 = H_0 \leq H_1 \leq \cdots \leq H_\alpha = H$ each H_β $(\beta \leq \alpha)$ is normal in G.

For a $\mathbb{Z}G$-module A, being similar with the ascending series of groups, we define the ascending (resp. descending) $\mathbb{Z}G$-composition series of A as a set of $\mathbb{Z}G$-submodules $\{A_\gamma;\ \gamma \leq \delta\}$ indexed by ordinals less than or equal to an ordinal δ such that

(a) $A_{\gamma_1} \leq A_{\gamma_2}$ (resp. $A_{\gamma_1} \geq A_{\gamma_2}$) if $\gamma_1 \leq \gamma_2$,

(b) $A_0 = 0$ and $A_\delta = A$ (resp. $A_0 = A$ and $A_\delta = 0$),

(c) the $\mathbb{Z}G$-module $A_{\gamma+1}/A_\gamma$ (resp. $A_\gamma/A_{\gamma+1}$) is irreducible for $\gamma < \delta$,

(d) $A_\lambda = \bigcup_{\gamma<\lambda} A_\gamma$ (resp. $A_\lambda = \bigcap_{\gamma<\lambda} A_\gamma$) if λ is a limit ordinal.

If δ is finite, then we say that A has a finite $\mathbb{Z}G$-composition series and usually write in the form

$$0 = A_0 < A_1 < \cdots < A_\delta = A. \quad (\text{resp. } A = A_0 > A_1 > \cdots > A_\delta = 0.)$$

Definition 1.1.4: A $\mathbb{Z}G$-module A is said to be completely reducible if A is a direct sum of some irreducible $\mathbb{Z}G$-submodules. Furthermore, A is said to be semisimple if A is a direct sum of finitely many irreducible $\mathbb{Z}G$-submodules.

The important concept is that:

Definition 1.1.5: A $\mathbb{Z}G$-module A is said to have an f–decomposition if

$$A = A^{f} \oplus A^{\bar{f}},$$

where A^f is a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A^f is finite and the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ has no nonzero finite $\mathbb{Z}G$-factors. Sometimes, we call A^f the f–component of A and $A^{\bar{f}}$ the $\bar{f}$–component of A.

Definition 1.1.6: Let G be a group, H a normal subgroup of G, and A a $\mathbb{Z}G$-module. Then A is said to be H–perfect if $A = [A, H]$.

Some other definitions are:

Definition 1.1.7: Let E, A, and G be groups. If A is normal in E and the factor group $E/A \cong G$, then E is called an extension of A by G.

Definition 1.1.8: A group E is said to split over its subgroup A modulo C for some subgroup C of E if there is a subgroup B of E such that $E = AB$ and $A \cap B = C$. The subgroup B is called a supplement to A in E or a complement to A in E modulo C. If all complements to A in E modulo C are conjugate in E modulo C, then E is said to split conjugately over A modulo C. If $C = 1$, then we naturally have the concepts of a complement to A in E and E splitting conjugately over A.

A ring R is called (right) semisimple if each right R–module is semisimple or equivalently R is a direct sum of finitely many simple (right) artinian rings. Also, a ring R is called regular if each finitely generated right ideal of R is generated by a single element e with $e^2 = e$ (such e is called an idempotent of R).

Other notations are: $\pi(G) = \{\text{prime } p;\ G \text{ has an element of order } p\}$, $A]K$ denotes the semidirect product of A by K, and $A > B$ means B is properly contained in A.

§1.2 THE WELL–KNOWN RESULTS

The following two results are the module version of the according results in **[15]**.

Lemma 1.2.1: (Thm 3.3.11 in **[15]**) If a $\mathbb{Z}G$-module A is a sum of a set of its irreducible $\mathbb{Z}G$-submodules, then it is the direct sum of certain of these $\mathbb{Z}G$-submodules. Thus A is a completely reducible $\mathbb{Z}G$-module.

Lemma 1.2.2: (Remark, Thm 3.3.12 in **[15]**) Let $A = \mathrm{Dr}_{\lambda \in \Lambda} A_\lambda$, where A_λ is an irreducible $\mathbb{Z}G$-submodule of A. Suppose that B is a $\mathbb{Z}G$-submodule of A. Then $A = B \oplus \mathrm{Dr}_{\upsilon \in M} A_\upsilon$ for some $M \leq \Lambda$. Also, B is completely reducible.

If G is a locally finite group, then the group structure of an irreducible $\mathbb{Z}G$-module has been well described. That is, we have:

Lemma 1.2.3: (Baer, Lemma 5.26 in **[14]**) If G is a locally finite group, then any irreducible $\mathbb{Z}G$-module A (as a group) is an elementary abelian p-group for some prime p.

Two useful fundamental lemmas are:

Lemma 1.2.4: (Fitting's Lemma, Thm 5.2.3 in [3]) Let p be a prime, A an abelian p-group, and $H \leq \mathrm{Aut}(A)$. If H is a finite p'-group, then

$$A = C_A(H) \oplus [A, H].$$

Here A is written additively.

Lemma 1.2.5: (Wilson, [17]) Let G be a group, H a normal subgroup of finite index in G, and A a $\mathbb{Z}G$-module. Then A is a noetherian (resp. artinian) $\mathbb{Z}G$-module if and only if A is a noetherian (resp. artinian) $\mathbb{Z}H$-module.

We mention that:

Lemma 1.2.6: Let G be a group, A a $\mathbb{Z}G$-module, and H a normal subgroup of G. Then both $C_A(H)$ and $[A, H]$ are $\mathbb{Z}G$-submodules of A.

For convenience, we prove:

Lemma 1.2.7: Let G be a group, A a $\mathbb{Z}G$-module, and K a finite p-subgroup of G, where p is a prime. If A (as a group) is also a p-group, then $C_A(K) \neq 0$.

Proof: We note firstly that $A_1 = \langle a \rangle K$ is a finite $\mathbb{Z}K$-module for any fixed $0 \neq a \in A$. Since A is a p-group, so A_1 is a finite p-group and then the semidirect product $L = A_1]K$ is a finite p-group. Thus the normal subgroup A_1 of L contains a nontrivial central element of L, say a_0. Therefore $a_0 \neq 0$ and $a_0 \in C_A(K)$. That is, $C_A(K) \neq 0$. The lemma is true.

Corollary 1.2.8: Under the hypotheses of Lemma 1.2.7, if A is further irreducible, then $A = C_A(K)$.

About hyperfinite groups, it is worth mentioning some results here.

Lemma 1.2.9: If G is a hyperfinite group, then the subgroups and the homomorphic images of G are all hyperfinite.

Proof: Suppose G has an ascending normal series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G$$

in which each G_β is normal in G and $G_{\beta+1}/G_\beta$ is finite, $\beta < \alpha$.

For a subgroup H of G, it is clear that

$$1 = H \cap G_0 \leq H \cap G_1 \leq \cdots \leq H \cap G_\alpha = H$$

is an ascending normal series of H, and also each factor $(H \cap G_{\beta+1})/(H \cap G_\beta)$ $\{\cong (H \cap G_{\beta+1})G_\beta/G_\beta \leq G_{\beta+1}/G_\beta\}$ is finite, $\beta < \alpha$. So H is hyperfinite.

Now, for a homomorphic image $\overline{G}$ of G, we may assume $\overline{G} = G/N$, where N is a normal subgroup of G. Then

$$\overline{1} = NG_0/N \leq NG_1/N \leq \cdots \leq NG_\alpha/N = \overline{G}$$

is an ascending normal series of $\overline{G}$, and the factor $(NG_{\beta+1}/N)/(NG_\beta/N)$ $\{\cong NG_{\beta+1}/NG_\beta \cong G_{\beta+1}/((G_{\beta+1} \cap N)G_\beta)\}$ is finite, $\beta < \alpha$. So $\overline{G}$ is hyperfinite.

Lemma 1.2.10: If G is a hyperfinite group, then any nontrivial normal subgroup of G contains nontrivial finite subgroups being (minimal) normal in G.

Proof: Since G is hyperfinite, there is an ascending normal series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G$$

in which each factor $G_{\beta+1}/G_\beta$ is finite for all $\beta < \alpha$.

Suppose H is a nontrivial normal subgroup of G, then $H \cap G_\beta$ is normal in G, $\forall\, \beta < \alpha$. Since $H \cap G_\alpha = H \neq 1$, there exists $\alpha_0 \leq \alpha$ such that $H \cap G_{\alpha_0} \neq 1$ but $H \cap G_\beta = 1$ for all $\beta < \alpha_0$. If α_0-1 does not exist, then $G_{\alpha_0} = \bigcup_{\beta < \alpha_0} G_\beta$. By $H \cap G_{\alpha_0} \neq 1$, there is $1 \neq g \in H \cap G_{\alpha_0}$ and then $1 \neq g \in H \cap G_\beta$ for some $\beta < \alpha_0$, contrary to $H \cap G_\beta = 1$. So α_0-1 exists. By $H \cap G_{\alpha_0 -1} = 1$ we have

$$1 \neq H \cap G_{\alpha_0} \cong \{G_{\alpha_0 -1}(H \cap G_{\alpha_0})\}/G_{\alpha_0 -1} \leq G_{\alpha_0}/G_{\alpha_0 -1},$$

and then $|H \cap G_{\alpha_0}| \leq |G_{\alpha_0}/G_{\alpha_0 -1}| < \infty$. That is, $H \cap G_{\alpha_0}$ is a nontrivial finite subgroup of H and is normal in G. Furthermore, by $H \cap G_{\alpha_0}$ being finite, we can find a nontrivial finite subgroup N of H such that N is minimal with respect to N being normal in G.

<u>**Lemma 1.2.11**</u>: A hyperfinite group is locally finite.

<u>Proof</u>: Suppose G is a hyperfinite group, then G has an ascending normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G$, in which each G_β is normal in G and $G_{\beta+1}/G_\beta$ is finite, $\beta < \alpha$.

Let $\gamma \leq \alpha$, and suppose that G_β is a locally finite group for all $\beta < \gamma$, we prove that G_γ is also a locally finite group.

If γ-1 exists, then G_γ is a (locally finite)–by–finite group and then is a locally finite group **[15]**; and on the other hand, if γ-1 does not exist, then for any $x_1, \cdots, x_n \in G_\gamma = \bigcup_{\beta < \gamma} G_\beta$ there exists β_0 such that $x_1, \cdots, x_n \in G_{\beta_0}$

and then $<x_1, \cdots, x_n>$ is a finite subgroup of G_{β_0} (and of G_γ). So G_γ is a locally finite group. That is, from G_β being locally finite groups for all $\beta < \gamma$ we have proved that G_γ is also a locally finite group. The lemma holds.

Lemma 1.2.12: A hyperfinite p-group G is a locally nilpotent group. Therefore, G has a nontrivial centre and then is a hypercentral group.

Before giving the proof for Lemma **1.2.12**, we point out another well-known result:

Lemma 1.2.13: (Mal'cev, McLain, Thm 12.1.6 in **[15]**) A principal factor of a locally nilpotent group G is central.

Proof of Lemma 1.2.12: By Lemma **1.2.10**, G is locally finite, then since G is a p-group we have G is a locally nilpotent group. By Lemma **1.2.13** and G having a minimal normal subgroup, G has a nontrivial centre. Let h(G) be the maximal hypercentrally embedded subgroup of G, i.e., h(G) is the hypercentre of G. If $h(G) < G$, then G/h(G) as a nontrivial hyperfinite p-group has a nontrivial centre, say H/h(G). It is clear that H is a hypercentrally embedded subgroup of G and $h(G) < H$ contrary to the maximality of h(G). So $h(G) = G$ and then G is hypercentral.

In our work, we will quote a lot of results proved by D. I. Zaičev in the series of papers **[19—22]**. Among these, the most important one is:

Lemma 1.2.14: (Zaičev, **[22]**) Let H be a hyperfinitely embedded subgroup of G, and A a nonzero noetherian $\mathbb{Z}G$-module. If $C_A(H) = 0$, then H contains a subgroup K and A contains a nonzero $\mathbb{Z}G$-submodule B such that K is normal in G, $C_B(K) = 0$, and $|K/C_K(B)| < \infty$.

The above lemma has a generalization for H being a hyper–(cyclically or finitely) embedded subgroup of G. That is, it is a special case of our result—Proposition **5.1.2** (see p.100).

The other results proved by D. I. Zaičev or appearing in D. I. Zaičev's papers are:

<u>Lemma **1.2.15**</u>: (Zaičev, [19]) If G is a hypercentral group, then any artinian $\mathbb{Z}G$-module A has a Z–decomposition, i.e., $A = A^{Z} \oplus A^{\bar{Z}}$, in which A^{Z} is a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A^{Z} has G as its centralizer in G and the $\mathbb{Z}G$-submodule $A^{\bar{Z}}$ of A has no such irreducible $\mathbb{Z}G$-factors.

<u>Lemma **1.2.16**</u>: (Frattini argument, [19]) Let A be an abelian normal subgroup of the group G. Let $x \in G$ such that $A = [A, x]$ and $A<x>$ is normal in G. Then $G = AN_G(<x>)$.

<u>Lemma **1.2.17**</u>: ([19]) If A is a $\mathbb{Z}G$-module and $<x>$ is a normal cyclic subgroup of G, then A(x–1) and $C_A(x)$ are $\mathbb{Z}G$-submodules of A.

<u>Lemma **1.2.18**</u>: ([20]) Let A be a $\mathbb{Z}G$-module and $<x>$ a normal cyclic subgroup of G. The map $a \longmapsto a(x-1)$ induces an isomorphism of the groups $A/C_A(x)$ and A(x–1) under which $\mathbb{Z}G$-submodules of $A/C_A(x)$ correspond to $\mathbb{Z}G$-submodules of A(x–1). (If, further, x is in the centre of G, then the map $a \longmapsto a(x-1)$ induces a $\mathbb{Z}G$-isomorphism between the $\mathbb{Z}G$-modules $A/C_A(x)$ and A(x–1).)

<u>Lemma **1.2.19**</u>: ([20]) Let A be a noetherian $\mathbb{Z}G$-module and $<x>$ a normal cyclic subgroup of G. Then there is an integer n such that $A(x-1)^n \cap C_A(x) = 0$.

Lemma 1.2.20: ([21]) Let A be an abelian normal p-subgroup of E and K/A a finite normal p-subgroup of the factor group E/A such that $A = [A, K]$. Then A contains a proper subgroup D such that D is normal in E and (1) A has a complement in E modulo D; (2) if A has a complement in E, then any two complements are conjugate modulo D.

Lemma 1.2.21: ([22]) Let A be an abelian normal subgroup of E and K/A a finite normal subgroup of E/A such that $|K/A| = p^k$, $C_A(K) = 1$. Then (1) if A has a complement in E modulo A^{p^k} then A has a complement in E; (2) if A has a complement in E and the complements are conjugate modulo A^{p^k} then they are conjugate in E.

Lemma 1.2.22: (Zaičev, [22]) Let G be a hyperfinite group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that B (resp. A/B) is finite. If each irreducible $\mathbb{Z}G$-factor of A/B (resp. that of B) is infinite, then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

There are some simple results we should mention:

Lemma 1.2.23: Let G be a group, and A a $\mathbb{Z}G$-module. If A has an f–decomposition, then each $\mathbb{Z}G$-submodule of A and each $\mathbb{Z}G$-image of A has an f–decomposition, too. Furthermore, if $A = A^{f} \oplus A^{\bar{f}}$, where A^f is the f–component of A and $A^{\bar{f}}$ is the $\bar{f}$–component of A, then (1) for any $\mathbb{Z}G$-submodule B of A, $B = (B \cap A^f) \oplus (B \cap A^{\bar{f}})$, in which $(B \cap A^f) = B^f$ and $(B \cap A^{\bar{f}}) = B^{\bar{f}}$; and (2) for any $\mathbb{Z}G$-image A/D of A, $A/D = (A^f + D)/D \oplus (A^{\bar{f}} + D)/D$, in which $(A^f + D)/D = (A/D)^f$ and $(A^{\bar{f}} + D)/D = (A/D)^{\bar{f}}$.

Proof: Let $A = A^f \oplus A^{\bar{f}}$, where A^f is a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A^f is finite and the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of A has no nonzero finite $\mathbb{Z}G$-factors. Let B be a $\mathbb{Z}G$-submodule of A. It is clear that $B \geq (B\cap A^f) \oplus (B\cap A^{\bar{f}})$, each irreducible $\mathbb{Z}G$-factor of $B\cap A^f$ is finite and $B\cap A^{\bar{f}}$ contains no nonzero finite $\mathbb{Z}G$-factors. Also, by $B/B\cap A^f \cong_{\mathbb{Z}G} (B+A^f)/A^f$ and $B/B\cap A^{\bar{f}} \cong_{\mathbb{Z}G} (B+A^{\bar{f}})/A^{\bar{f}}$, we have each nonzero irreducible $\mathbb{Z}G$-factor of $B/B\cap A^f$ is infinite and all irreducible $\mathbb{Z}G$-factors of $B/B\cap A^{\bar{f}}$ are finite. Thus $B/\{(B\cap A^f) \oplus (B\cap A^{\bar{f}})\}$ has no nonzero irreducible $\mathbb{Z}G$-factors and then $B = (B\cap A^f) \oplus (B\cap A^{\bar{f}})$, where $(B\cap A^f) = B^f$ and $(B\cap A^{\bar{f}}) = B^{\bar{f}}$. That is, B has an f–decomposition.

For the $\mathbb{Z}G$-image, say A/D, it is clear that $A/D = (A^f+D)/D + (A^{\bar{f}}+D)/D$. Since $(A^f+D)/D \cong_{\mathbb{Z}G} A^f/(A^f\cap D)$ and $(A^{\bar{f}}+D)/D \cong_{\mathbb{Z}G} A^{\bar{f}}/(A^{\bar{f}}\cap D)$, we have each irreducible $\mathbb{Z}G$-factor of $(A^f+D)/D$ is finite and the $\mathbb{Z}G$-submodule $(A^{\bar{f}}+D)/D$ has no nonzero finite irreducible $\mathbb{Z}G$-factors. Thus $(A^f+D)/D \cap (A^{\bar{f}}+D)/D = 0$. It follows that $A/D = (A^f+D)/D \oplus (A^{\bar{f}}+D)/D$, $(A^f+D)/D = (A/D)^f$ and $(A^{\bar{f}}+D)/D = (A/D)^{\bar{f}}$. That is, A/D has an f–decomposition. The lemma is proved.

<u>**Lemma 1.2.24**</u>: Let $A = \sum_{i\in I}A_i$, where A is a $\mathbb{Z}G$-module and A_i are $\mathbb{Z}G$-submodules of A. If each A_i has an f–decomposition: $A_i = A_i^f \oplus A_i^{\bar{f}}$, then A has an f–decomposition with $A^f = \sum_{i\in I}A_i^f$ and $A_i^{\bar{f}} = \sum_{i\in I}A_i^{\bar{f}}$.

<u>Proof:</u> It is clear that $A = \sum_{i\in I}A_i = \sum_{i\in I}(A_i^f \oplus A_i^{\bar{f}}) = (\sum_{i\in I}A_i^f)+(\sum_{i\in I}A_i^{\bar{f}})$.

For $\sum_{i\in I}A_i^f$, if it contains an infinite irreducible $\mathbb{Z}G$-factor, say B_0/C_0, then for $x \in B_0\backslash C_0$, we have $x = x_1+\cdots+x_r$ for some $x_j \in A_{i_j}$, $j = 1,\cdots,r$. Thus $B_0\cap\sum_{j=1}^r A_{i_j}^f > C_0\cap\sum_{j=1}^r A_{i_j}^f$. By B_0/C_0 being irreducible and

$$0 \neq (B_0\cap\sum_{j=1}^r A_{i_j}^f)/(C_0\cap\sum_{j=1}^r A_{i_j}^f) \cong_{\mathbb{Z}G} [C_0+(B_0\cap\sum_{j=1}^r A_{i_j}^f)]/C_0,$$

we have $\sum_{j=1}^{r} A_{i_j}^{f}$ contains infinite irreducible $\mathbb{Z}G$-factors.

Let r_0 be minimal such that $\sum_{j=1}^{r_0} A_{i_j}^{f}$ contains infinite irreducible $\mathbb{Z}G$-factors. It is clear that $r_0 > 1$. Let B/C be an infinite irreducible $\mathbb{Z}G$-factor of $\sum_{j=1}^{r_0} A_{i_j}^{f}$, then by the minimality of r_0 we must have

$$B\cap\sum_{j=1}^{r_0-1} A_{i_j}^{f} = C\cap\sum_{j=1}^{r_0-1} A_{i_j}^{f} \quad \text{and then} \quad B+\sum_{j=1}^{r_0-1} A_{i_j}^{f} > C+\sum_{j=1}^{r_0-1} A_{i_j}^{f}.$$

By B/C being irreducible and

$$0 \neq (B+\sum_{j=1}^{r_0-1} A_{i_j}^{f})/(C+\sum_{j=1}^{r_0-1} A_{i_j}^{f}) \cong_{\mathbb{Z}G} B/[B\cap(C+\sum_{j=1}^{r_0-1} A_{i_j}^{f})],$$

we have $(B+\sum_{j=1}^{r_0-1} A_{i_j}^{f})/(C+\sum_{j=1}^{r_0-1} A_{i_j}^{f})$ is infinite and irreducible. Also

$$(B+\sum_{j=1}^{r_0-1} A_{i_j}^{f})/(C+\sum_{j=1}^{r_0-1} A_{i_j}^{f}) \leq \sum_{j=1}^{r_0} A_{i_j}^{f}/(C+\sum_{j=1}^{r_0-1} A_{i_j}^{f}) \cong_{\mathbb{Z}G} A_{i_{r_0}}^{f}/[A_{i_{r_0}}^{f}\cap(C+\sum_{j=1}^{r_0-1} A_{i_j}^{f})],$$

which shows that $A_{i_{r_0}}^{f}$ has an infinite irreducible $\mathbb{Z}G$-factor, a contradiction. Therefore $\sum_{i\in I} A_i^{f}$ contains no infinite irreducible $\mathbb{Z}G$-factors.

Similarly, $\sum_{i\in I} A_i^{\bar{f}}$ contains no nonzero finite irreducible $\mathbb{Z}G$-factors. Thus $(\sum_{i\in I} A_i^{f})\cap(\sum_{i\in I} A_i^{\bar{f}}) = 0$ and then $A = (\sum_{i\in I} A_i^{f})\oplus(\sum_{i\in I} A_i^{\bar{f}})$ with $A^{f} = \sum_{i\in I} A_i^{f}$ and $A^{\bar{f}} = \sum_{i\in I} A_i^{\bar{f}}$.

<u>Lemma **1.2.25**</u>: Let $A = B + A_1$ with $B\cap A_1 = B_1$, where A is a $\mathbb{Z}G$-module, B, A_1, and B_1 are $\mathbb{Z}G$-submodules. If $A_1 = B_1 \oplus C_1$ for some $\mathbb{Z}G$-submodule C_1, then $A = B \oplus C_1$.

<u>Proof</u>: Since $A = B + A_1 = B + (B_1 + C_1) = (B + B_1) + C_1 = B + C_1$,

and $B \cap C_1 = B \cap (A_1 \cap C_1) = (B \cap A_1) \cap C_1 = B_1 \cap C_1 = 0$, so $A = B \oplus C_1$.

The following form of Maschke's Theorem will be used in a later proof.

Lemma 1.2.26: (Maschke's Theorem) Let V be a torsion-free $\mathbb{Z}G$-module with G being finite, and W a $\mathbb{Z}G$-submodule of V. If $V = W \oplus V_1$ for some subgroup V_1 of V, then there exists a $\mathbb{Z}G$-submodule U such that $|G| V \leq W \oplus U$.

Proof: It is a special case of Theorem **4.1** in **[12]**.

Finally, we end this chapter with three results related to semisimple rings.

Lemma 1.2.27: (Corollary **2.16**, **[2**, p.21**]**) Any noetherian regular ring is semisimple.

Lemma 1.2.28: If F is a finite p'-group, then the group ring $\mathbb{Z}_p F$ is semisimple.

Proof: Since $\mathbb{Z}_p F$ is finite, so the result is a consequence of Theorem **4.2** in **[11]** and Theorem **0.1.11** in **[9]**.

Lemma 1.2.29: Any right ideal of a semisimple ring is generated by a single idempotent.

Proof: Using Lemma 1.2.2 in this section and Lemma **6** in **[13]**, we get the result.

2 THE BASIC LEMMAS

This chapter consists of basic lemmas, of which most will be necessarily used in the later proof of our main results. We deal with them in four sections.

§2.1 ON TORSION-FREE MODULES

Among the $\mathbb{Z}G$-factors of a torsion-free $\mathbb{Z}G$-module A, there are some factors which have a very nice relation between them. We list some of these as

Lemma **2.1.1**: Let G be a group, A a torsion-free $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A. Then $p^i A/p^i B \cong_{\mathbb{Z}G} A/B$ for any integer $p > 0$ and any integer $i \geq 0$.

Proof: Let φ: $a \longmapsto p^i a + p^i B$, where $a \in A$. Since

a. $\varphi(a + b) = p^i(a + b) + p^i B$

$= (p^i a + p^i B) + (p^i b + p^i B) = \varphi(a) + \varphi(b),$ and

b. $\varphi(ag) = p^i(ag) + p^i B$

$= (p^i a)g + p^i B = (p^i a + p^i B)g = [\varphi(a)]g,$

where $a, b \in A$ and $g \in G$. So φ is a $\mathbb{Z}G$-homomorphism from A to $p^i A/p^i B$, thus $A/\ker\varphi \cong_{\mathbb{Z}G} p^i A/p^i B$.

Since $B \leq \ker\varphi$ is clear; and, on the other hand, $\alpha \in \ker\varphi$ implies that $p^i\alpha \in p^i B$. By A being torsion-free, $\alpha \in B$ and so $\ker\varphi \leq B$. Thus $\ker\varphi = B$ and then $p^i A/p^i B \cong_{\mathbb{Z}G} A/B$ for any integer $p > 0$ and any integer $i \geq 0$.

From Lemma **2.1.1**, we have:

Corollary 2.1.2: Let G be a group, A a torsion-free $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A. Then $p^iA/p^iB \cong_{\mathbb{Z}G} p^jA/p^jB$ for any integer $p > 0$ and any integers $i, j \geq 0$.

If we let $B = p^tA$, where t is an integer not less than zero, then we have:

Corollary 2.1.3: Let G be a group and A a torsion-free $\mathbb{Z}G$-module. Then

$$k^iA/k^{i+t}A \cong_{\mathbb{Z}G} k^jA/k^{j+t}A$$

for any integer $k > 0$ and any integers $i, j, t \geq 0$.

In the torsion-free case, we often need A contains some nonzero $\mathbb{Z}G$-factors being p-groups for some fixed prime p. The following lemma indicates that conditions for the purpose.

Lemma 2.1.4: Let G be a locally finite group and A a torsion-free noetherian $\mathbb{Z}G$-module. Then $pA < A$ and $\bigcap_{i=1}^{\infty} p^iA = 0$ for any integer $p > 0$.

Proof: Firstly, we prove $pA < A$ for any integer $p > 0$.

Since A is noetherian, there exist $a_1, \cdots, a_n$ such that

$$A = \langle a_1, \cdots, a_n \rangle^G.$$

Suppose that $pA = A$ for some integer $p > 0$, then

$$a_i = p(a_{i_1} m_{i_1} g_{i_1} + \cdots + a_{i_t} m_{i_t} g_{i_t})$$

for some integer t, where $m_{i_j} \in \mathbb{Z}$, $g_{i_j} \in G$, $j = 1, 2, \cdots, t$, and $i = 1, 2, \cdots, n$.

Let $F = \langle g_{i_j} \in G;\ j = 1, \cdots, t,\ i = 1, \cdots, n \rangle$. By G being locally finite, F is a finite group. Also, $a_i \in p\langle a_1, \cdots, a_n\rangle^F$ for all i. Thus $\langle a_1, \cdots, a_n\rangle \le p\langle a_1, \cdots, a_n\rangle^F$ and then

$$\langle a_1 \cdots, a_n\rangle^F = p\langle a_1, \cdots, a_n\rangle^F.$$

But $\langle a_1, \cdots, a_n\rangle^F$ is a finitely generated torsion-free abelian group and so $p\langle a_1, \cdots, a_n\rangle^F$ is a proper subgroup of $\langle a_1, \cdots, a_n\rangle^F$ for $p > 0$, a contradiction. Thus we have proved that $pA < A$ for any integer $p > 0$.

Secondly, if $B = \bigcap_{i=1}^{\infty} p^i A \neq 0$, then since B is also a torsion-free noetherian $\mathbb{Z}G$-module, thus $pB < B$ for the integer $p > 0$. But by A being torsion-free we can easily get $B = pB$, a contradiction. Thus $\bigcap_{i=1}^{\infty} p^i A = 0$. The lemma holds.

Related to Lemma **2.1.4**, we have:

Lemma **2.1.5**: For any noetherian $\mathbb{Z}G$-module A, $pA < A$ if and only if A has a nonzero $\mathbb{Z}G$-factor being a p-group, where p is a prime.

Proof: The necessity is evident.

We prove the sufficiency. Let U/V be a nonzero $\mathbb{Z}G$-factor of A such that U/V is a p-group. By the noetherian condition, we may assume that U/V is an elementary abelian p-group and A/U has no nonzero $\mathbb{Z}G$-factors being p-groups. If $pA = A$, then $A/V = (pA+V)/V = p(A/V)$. For $(V \neq)\ u+V \in U/V \le p(A/V)$, there is $a_0 \in A$ such that $u+V = p(a_0+V)$. Thus $a_0 \notin U$. But $p^2(a_0+V) = p(u+V) = V$, which implies that $(A/U)[p] = \{a+U \in A/U;\ p(a+U) = U\}$ is a nonzero $\mathbb{Z}G$-submodule, contrary to A/U having no nonzero $\mathbb{Z}G$-factors being p-groups. So $pA < A$, the result holds.

The following lemmas are useful and interesting.

Lemma 2.1.6: Let G be a locally finite group, A a torsion-free noetherian $\mathbb{Z}G$-module, and $C_G(A) = 1$. Then

(1) $H = C_G(A/pA) = 1$, if p is an odd prime; and

(2) $H = C_G(A/pA)$ is an elementary abelian 2-group, if $p = 2$.

Proof: We prove first that $H = C_G(A/pA)$ is a p-group for the prime p.

Suppose $x \in H$ and x is of order q for some prime q other than p. Since $x \notin C_H(A) \le C_G(A) = 1$, there exists $a_0 \in A$ such that $a_0x \ne a_0$. Let $A_i = p^iA$, then, by applying Lemma 2.1.4, we have $\bigcap_i A_i = 0$. Therefore, there exists i_0 such that $a_0 \in A_{i_0} \backslash A_{i_0+1}$. By $A/A_1 \cong_{\mathbb{Z}G} A_i/A_{i+1}$ for any integer i, we have $x \in H = C_G(A_i/A_{i+1})$ for any integer i. So $a_0x = a_0+a_1$, where $a_1 \ne 0$ and $a_1 \in A_{i_1} \backslash A_{i_1+1}$ for some integer $i_1 > i_0$. If $a_1x = a_1$, then $a_0 = a_0x^q = a_0+qa_1$. That is, $qa_1 = 0$ and then $a_1 = 0$, a contradiction. So $a_1x \ne a_1$.

Suppose $a_0x^r = \sum_{j=0}^r \binom{r}{j}a_j$, where for any j, $a_j \in A_{i_j} \backslash A_{i_j+1}$, $a_jx \ne a_j$, and $a_{j-1}x = a_{j-1}+a_j$, $i_r > i_{r-1} > \cdots > i_1 > i_0 \ge 0$. By $a_rx \ne a_r$ and $x \in H = C_G(A_{i_r}/A_{i_r+1})$, we have $a_rx = a_r+a_{r+1}$, where $a_{r+1} \in A_{i_{r+1}} \backslash A_{i_{r+1}+1}$ and $i_{r+1} > i_r$. Also we easily obtain $a_{r+1}x \ne a_{r+1}$. Now we have

$$\begin{aligned} a_0x^{r+1} &= (a_0x^r)x = [\textstyle\sum_{j=0}^r \binom{r}{j}a_j]x = \sum_{j=0}^r \binom{r}{j}(a_jx) \\ &= \textstyle\sum_{j=0}^r \binom{r}{j}(a_j+a_{j+1}) = a_0+\sum_{j=1}^r [\binom{r}{j-1}+\binom{r}{j}]a_j+a_{r+1} \\ &= a_0+\textstyle\sum_{j=1}^r \binom{r+1}{j}a_j+a_{r+1} = \sum_{j=0}^{r+1}\binom{r+1}{j}a_j. \end{aligned}$$

Hence $a_0 = a_0x^q = \sum_{j=0}^q \binom{q}{j}a_j = a_0+\sum_{j=1}^q \binom{q}{j}a_j$. That is,

$$0 = \sum_{j=1}^q \binom{q}{j}a_j = q[a_1+\sum_{j=2}^{q-1} \frac{1}{q}\binom{q}{j}a_j]+a_q.$$

Since $q \neq p$, so $q = \mathrm{k}p + \mathrm{t}$ for some t with $0 < \mathrm{t} < p$. Thus we have $\mathrm{ta}_1 = -p(\mathrm{ka}_1) - \sum_{j=2}^{q}\binom{q}{j}\mathrm{a}_j \in (\mathrm{A}_{i_1+1} + \mathrm{A}_{i_2}) \leq \mathrm{A}_{i_1+1}$ and then $\mathrm{a}_1 \in \mathrm{A}_{i_1+1}$, a contradiction. Hence we have in fact proved that $\mathrm{H} = \mathrm{C_G}(\mathrm{A}/p\mathrm{A})$ contains only p-elements.

Now we begin to prove the required results.

(1) If $p > 2$ and $\mathrm{H} \neq 1$, then, as above, we have the equation

$$0 = p[\mathrm{a}_1 + \sum_{j=2}^{p-1} \frac{1}{p}\binom{p}{j}\mathrm{a}_j] + \mathrm{a}_p,$$

where $\mathrm{a}_{j-1}\mathrm{x} = \mathrm{a}_{j-1} + \mathrm{a}_j$ for some $\mathrm{x} \in \mathrm{H}$, $\mathrm{a}_j \in \mathrm{A}_{i_j} \backslash \mathrm{A}_{i_j+1}$, and $i_p > i_{p-1} > \cdots > i_1 > i_0 \geq 0$. Since $i_p - 1 \geq i_3 - 1 \geq i_2 \geq i_1 + 1$, therefore $\mathrm{a}_1 + \sum_{j=2}^{p-1} \frac{1}{p}\binom{p}{j}\mathrm{a}_j \in \mathrm{A}_{i_p-1} \leq \mathrm{A}_{i_2}$ and then $\mathrm{a}_1 \in \mathrm{A}_{i_2} \leq \mathrm{A}_{i_1+1}$, contrary to $\mathrm{a}_1 \in \mathrm{A}_{i_1} \backslash \mathrm{A}_{i_1+1}$. So $p > 2$ implies that $\mathrm{H} = \mathrm{C_G}(\mathrm{A}/p\mathrm{A}) = 1$.

(2) For the 2-group $\mathrm{H} = \mathrm{C_G}(\mathrm{A}/2\mathrm{A})$, if H has an element, say x, with order 4, then, since $\mathrm{x}^2 \neq 1$ and $\mathrm{x}^2 \notin \mathrm{C_G}(\mathrm{A}) = 1$, there exists $0 \neq \mathrm{a}_0 \in \mathrm{A}$ such that $\mathrm{a}_0\mathrm{x}^2 \neq \mathrm{a}_0$. Let $\mathrm{A}_i = 2^i\mathrm{A}$, then, by Lemma **2.1.4**, $\mathrm{A}_i < \mathrm{A}_{i+1}$ and $\bigcap_i \mathrm{A}_i = 0$. Thus $\mathrm{a}_0 \in \mathrm{A}_{i_0} \backslash \mathrm{A}_{i_0+1}$ for some $i_0 \geq 0$. Since $\mathrm{x} \in \mathrm{H} = \mathrm{C_G}(\mathrm{A}_0/\mathrm{A}_1) = \mathrm{C_G}(\mathrm{A}_i/\mathrm{A}_{i+1})$ for any integer $i \geq 0$, so $\mathrm{a}_0\mathrm{x} = \mathrm{a}_0 + \mathrm{a}_1$ for some $\mathrm{a}_1 \in \mathrm{A}_{i_0+1}$. Evidently, $\mathrm{a}_1\mathrm{x} \neq \mathrm{a}_1$, for otherwise $\mathrm{a}_0 = \mathrm{a}_0\mathrm{x}^4 = \mathrm{a}_0 + 4\mathrm{a}_1$, which implies that $4\mathrm{a}_1 = 0$ and then $\mathrm{a}_1 = 0$, contrary to $\mathrm{a}_0\mathrm{x}^2 \neq \mathrm{a}_0$. So we have $\mathrm{a}_1\mathrm{x} \neq \mathrm{a}_1$. Let $\mathrm{a}_1 \in \mathrm{A}_{i_1} \backslash \mathrm{A}_{i_1+1}$, where $i_1 > i_0$, then, similarly, we have $\mathrm{a}_1\mathrm{x} = \mathrm{a}_1 + \mathrm{a}_2$ for some $\mathrm{a}_2 \in \mathrm{A}_{i_2} \backslash \mathrm{A}_{i_2+1}$, where $i_2 > i_1$ and $\mathrm{a}_2\mathrm{x} \neq \mathrm{a}_2$. Generally, we have

$a_{j-1}x = a_{j-1}+a_j$, with $a_j \in A_{i_j} \setminus A_{i_j+1}$, $a_jx \neq a_j$, $i_j > i_{j-1}$, $j=1,2,3,4$. Therefore $a_0 = a_0x^4 = a_0 + 4a_1 + 6a_2 + 4a_3 + a_4$. That is, $4a_1+6a_2+4a_3+a_4 = 0$. Since $i_4 \geq i_3+1 \geq i_2+2 > 2$, so we have $a_4 = 2a_4^*$, where $a_4^* \in A_{i_4-1} \setminus A_{i_4}$. Thus $2a_1 + 3a_2 + 2a_3 + a_4^* = 0$ and then $a_2 = -2a_1 - (2a_2+2a_3+a_4^*) = -2a_1 + b_2$, where $b_2 = -(2a_2+2a_3+a_4^*) \in A_{i_2+1}$. By $a_0x^2 \neq a_0$ and $a_0x^2 = (a_0+a_1)x$ $= a_0 + 2a_1 + a_2 = a_0 + b_2$, we have $b_2 \neq 0$, thus $b_2 \in A_{j_2} \setminus A_{j_2+1}$ for some $j_2 > i_2$. If $b_2x = b_2$, then $a_0 = a_0x^4 = (a_0x^2)x^2 = (a_0+b_2)x^2 = a_0+2b_2$, which implies that $2b_2 = 0$ contrary to $b_2 \neq 0$. So $b_2x \neq b_2$. As above, we may have $b_{k-1}x = b_{k-1}+b_k$, where $b_k \in A_{j_k} \setminus A_{j_k+1}$, $b_kx \neq b_k$, $j_k > j_{k-1}$, and $k = 3, 4$. Therefore,

$$\begin{aligned} a_0 &= a_0x^4 = (a_0x^2)x^2 = (a_0+b_2)x^2 \\ &= a_2x^2+b_2x^2 = (a_2+b_2)+(b_2+b_3)x \\ &= (a_0+b_2)+(b_2+2b_3+b_4) \\ &= a_0+(2b_2+2b_3+b_4). \end{aligned}$$

That is, $2b_2+2b_3+b_4 = 0$ and then $b_2+b_3 \in A_{j_4-1} \leq A_{j_3}$. Hence we get the contradiction that $b_2 \in A_{j_3} \leq A_{j_2+1}$.

By the above contradiction, we have H contains no elements of order 4 and then, since H is a 2-group, we must have H is an elementary abelian 2-group. The result is proved.

A consequence of Lemma **2.1.6** is that:

Corollary 2.1.7: Let G be a locally finite group and A a torsion-free noetherian $\mathbb{Z}G$-module. Then $C_G(A/pA) = C_G(A)$ for any prime $p > 2$.

The following possibly well–known result follows immediately from Corollary **2.1.7**.

Corollary 2.1.8: If a locally finite group G is contained in the group $GL_n(\mathbb{Z})$, then G is isomorphic with some subgroup of $GL_n(\mathbb{Z}_p)$ for any prime $p>2$. Further, G is finite with order at most $3^{\frac{n(n-1)}{2}} \Pi_{i=1}^{n}(3^i-1)$.

As an interesting aside, we prove a lemma, even though it has not been used anywhere in the later discussion, but it has a very close relationship with Lemma **2.1.6**.

Lemma 2.1.9: Let G be a locally finite group, A a torsion-free noetherian $\mathbb{Z}G$-module, and $C_G(A) = 1$. Then $H = C_G(A/4A) = 1$.

Proof: Since $H = C_G(A/4A) \leq C_G(A/2A)$, so, by Lemma **2.1.6**, H is an elementary abelian 2-group. If $H \neq 1$, then H contains an element x, say, of order 2. By $C_G(A) = 1$, there exists $a_0 \in A$ such that $a_0x \neq a_0$. Let $A_i = 4^iA$, then, by Lemma **2.1.4**, $A_{i+1} < A_i$ and $\bigcap_i A_i = 0$. Thus there exists $i_0 \geq 0$ such that $a_0 \in A_{i_0} \backslash A_{i_0+1}$. Using Corollary **2.1.3**, $A_0/A_1 \cong_{\mathbb{Z}G} A_i/A_{i+1}$, so $H = C_G(A_i/A_{i+1})$ for any $i \geq 0$. Since $x \in H = C_G(A_{i_0}/A_{i_0+1})$ and $a_0 \in A_{i_0}$, so $a_0x = a_0+a_1$ for some $0 \neq a_1 \in A_{i_0+1}$. Let $a_1 \in A_{i_1} \backslash A_{i_1+1}$, where $i_1 > i_0$, then by $x \in H = C_G(A_{i_1}/A_{i_1+1})$ we have $a_1x = a_1+a_2$ for some $a_2 \in A_{i_1+1}$. On the other hand, $a_0 = a_0x^2 = (a_0+a_1)x = a_0+a_1+a_1x$. That is, $a_1x = -a_1$. So $a_2 = -2a_1$ and then $2a_1 = -a_2$. Similarly, if we let $a_2 \in A_{i_2} \setminus A_{i_2+1}$, where $i_2 > i_1$, then $a_2x = a_2+a_3$ for some $a_3 \in A_{i_2+1}$ and $2a_2 = -a_3$. Thus $4a_1 = -2a_2 = a_3 \in A_{i_2+1}$. Since A is torsion-free, $a_1 \in A_{i_2} \le A_{i_1+1}$ a contradiction. Hence we must have $H = C_G(A/4A)$ is a trivial group. That is, $H = 1$. The result is proved.

§2.2 RELATED TO THE NORMAL SUBGROUPS OF FINITE INDEX

From Wilson's lemma (Lemma **1.2.5**), we know that some important properties are inherited by the normal subgroups of finite index. In this section, we give some more properties of this kind.

Lemma **2.2.1**: Let H be a normal subgroup of finite index in a group G, and A a $\mathbb{Z}G$-module. Then A has a finite $\mathbb{Z}G$-composition series if and only if A has a finite $\mathbb{Z}H$-composition series.

Proof: Since A has a finite composition series if and only if A is both artinian and noetherian, thus the lemma follows from Lemma **1.2.5**.

Corollary **2.2.2**: If H is a normal subgroup of finite index in a group G, then any finite (resp. infinite) irreducible $\mathbb{Z}G$-module A contains a finite (resp. infinite) irreducible $\mathbb{Z}H$-submodule.

Proof: By Lemma **2.2.1**, A contains an irreducible $\mathbb{Z}H$-submodule, say V. Let T be a left transversal to H in G, then $\sum_{t\in T} Vt$ is a nonzero $\mathbb{Z}G$-submodule of A and then $A = \sum_{t\in T} Vt$ by the irreducibility of A. Since T is finite, so A is finite if and only if V is finite. Thus V is finite (resp. infinite) if A is finite (resp. infinite).

<u>Corollary **2.2.3**</u>: If H is a normal subgroup of finite index in a group G, and if a $\mathbb{Z}G$-module A contains an irreducible $\mathbb{Z}G$-factor B/C, then A contains an irreducible $\mathbb{Z}H$-factor U/V such that: if B/C is finite, infinite, or a p-group for some prime p, then so is U/V.

<u>Proof</u>: Consider the $\mathbb{Z}G$-module B/C, then the result follows from Corollary **2.2.2** and its proof.

Let G be a group, H a normal subgroup of G, and A a RG-module, where R is a ring with 1. Let $\tilde{A}^H = \{U/V;\ U/V \text{ is a RH-factor of } A\}$, then Ug and Vg are RH-submodules of A and so G acts naturally on $\tilde{A}^H$ by the action

$$(U/V)g = Ug/Vg \quad \text{for every } g \in G.$$

<u>Lemma **2.2.4**</u>: If H is a normal subgroup of finite index in a group G, and if a $\mathbb{Z}G$-module A contains a nonzero (irreducible) $\mathbb{Z}H$-factor U/V, then

(1) Ug/Vg is also a nonzero (irreducible) $\mathbb{Z}H$-factor of A, $\forall\, g \in G$;

(2) as groups, $U/V \cong Ug/Vg$, $\forall\, g \in G$;

(3) $C_H(Ug/Vg) = g^{-1}C_H(U/V)g$, $\forall\, g \in G$;

(4) if $V = 0$ and U is irreducible, then A contains an irreducible $\mathbb{Z}G$-submodule B such that B has an irreducible $\mathbb{Z}H$ -submodule W with $W \cong_{\mathbb{Z}G} Ug$ for some $g \in G$;

(5) for B and W in the above (4), $B = \mathrm{Dr}_{s\in S} Ws$, where S is a subset of

a transversal T to H in G; and

(6) if U/V is irreducible, then A contains an irreducible $\mathbb{Z}G$-factor B/C such that: if U/V is finite, infinite, or a p-group for some prime p, then so is B/C.

<u>Proof</u>: (1) For any $g \in G$, we show firstly that: (*a*) U is a $\mathbb{Z}H$-submodule of A if and only if Ug is also, and (*b*) for any $\mathbb{Z}H$-submodules U and V of A, $U \geq V$ if and only if $Ug \geq Vg$.

(*a*) "$\Longrightarrow$": For any $a_1g, a_2g \in Ug$ and any $h \in H$,

$$a_1g - a_2g = (a_1 - a_2)g \in Ug, \quad \text{and}$$
$$(a_1g)h = a_1(gh) = a_1(h'g) = (a_1h')g \in Ug$$

since U is a $\mathbb{Z}H$-submodule and H is normal in G. So Ug is a $\mathbb{Z}H$-submodule.

"$\Longleftarrow$": Since Ug is a $\mathbb{Z}H$-submodule of A, by the arbitrarity of g and the above proof, we have $U = (Ug)g^{-1}$ is also a $\mathbb{Z}H$-submodule of A.

(*b*) "$\Longrightarrow$": For any $ag \in Vg$, since $a \in V \leq U$ so $ag \in Ug$, thus

$$Vg \leq Ug;$$

"$\Longleftarrow$": By the arbitrarity of g and the above proof, we have

$$V = (Vg)g^{-1} \leq (Ug)g^{-1} = U.$$

We secondly note that: (*c*) if W is a $\mathbb{Z}H$-submodule of Ug for some $\mathbb{Z}H$-submodule U of A, then there exists a $\mathbb{Z}H$-submodule V of U such that $W = Vg$, namely $V = Wg^{-1}$.

By (*a*), (*b*) and (*c*), it is clear that Ug/Vg is also a nonzero (irreducible) $\mathbb{Z}H$-factor of A for any nonzero (irreducible) $\mathbb{Z}H$-factor U/V of A, where $g \in G$.

(2) Let φ: $u \longmapsto ug + Vg$, where $u \in U$. Then it is clear that φ is a group homomorphism from U to Ug/Vg, thus $U/\ker\varphi \cong Ug/Vg$. Obviously,

$V \leq \ker\varphi$. On the other hand, $u \in \ker\varphi$ implies that $ug \in Vg$, thus there exists $v \in V$ such that $ug = vg$ and then $u = (ug)g^{-1} = (vg)g^{-1} = v$, so $\ker\varphi \leq V$ and so $V = \ker\varphi$. That is, as groups, $U/V \cong Ug/Vg$.

(3) Let $h \in C_H(Ug/Vg)$; then $ghg^{-1} \in H$ and

$$(u + V)ghg^{-1} = (ug + Vg)hg^{-1} = (ug + Vg)g^{-1} = u + V,$$

so $ghg^{-1} \in C_H(U/V)$ and then $h \in g^{-1}C_H(U/V)g$. That is,

$$C_H(Ug/Vg) \leq g^{-1}C_H(U/V)g.$$

On the other hand, $h \in g^{-1}C_H(U/V)g$ implies that $h = g^{-1}h^*g \in H$, where $h^* \in C_H(U/V)$. Since

$$(ug + Vg)h = (ug + Vg)g^{-1}h^*g = (u + V)h^*g = (u + V)g = ug + Vg,$$

so $h \in C_H(Ug/Vg)$ and thus $g^{-1}C_H(U/V)g \leq C_H(Ug/Vg)$.

Therefore $C_H(Ug/Vg) = g^{-1}C_H(U/V)g$.

(4) Let T be a left transversal to H in G, then T is a finite set. Let $D = \sum_{t \in T} Ut$, then D is a $\mathbb{Z}G$-submodule of A and, by (1), D is a sum of finitely many irreducible $\mathbb{Z}H$-submodules thus D is completely reducible (Lemma **1.2.1**). Since D has a finite $\mathbb{Z}H$-composition series, so D has a finite $\mathbb{Z}G$-composition series (Lemma **2.2.1**); thus D (and then A) contains an irreducible $\mathbb{Z}G$-submodule, say B. By Lemma **1.2.2**, $D = B \oplus \mathrm{Dr}_{s \in S} Us$ and $B \cong_{\mathbb{Z}H} \mathrm{Dr}_{s \in S'} Us$, where S is a subset of T and $S' = T \setminus S$. Thus, it is clear that B contains an irreducible $\mathbb{Z}H$-submodule W such that $W \cong_{\mathbb{Z}H} Ug$ for some $g \in S' \subseteq G$.

(5) For the above B and W, $\sum_{t \in T} Wt$ is a nonzero $\mathbb{Z}G$-submodule of B and then, by the irreducibility of B, $B = \sum_{t \in T} Wt$. Thus by Lemma **1.2.1**, $B = \mathrm{Dr}_{s \in S} Ws$ for some subset $S \subseteq T$.

(6) Let U/V be an irreducible $\mathbb{Z}H$-factor of A, then by Zorn's lemma,

there is a $\mathbb{Z}H$-submodule M of A maximal with respect to $U \cap M = V$. This implies that $V \leq M$ and so $(U+M)/M \cong_{\mathbb{Z}H} U/V$ is irreducible. If $M < M_1$ then $U+M \leq M_1$ otherwise $(U+M)\cap M_1 = M$ so $U\cap M_1 = U\cap(U+M)\cap M_1 = U\cap M = V$, contrary to the choice of M. If M is not a $\mathbb{Z}G$-submodule of A, then there is a $\mathbb{Z}H$-submodule W of A of the form $W = \bigcap_{i=1}^{n} Mg_i$ such that $M\cap W$ is a $\mathbb{Z}G$-submodule properly contained in W; thus M is properly contained in the $\mathbb{Z}H$-submodule $M+W$ and $U+M = (U+M)\cap(M+W) = M+[(U+M)\cap W]$, so $(U+M)/M \cong_{\mathbb{Z}H}[(U+M)\cap W]/(M\cap W)$. That is $[(U+M)\cap W]/(M\cap W)$ is irreducible and $M\cap W$ is a $\mathbb{Z}G$-submodule of A. Also, by

$$[(U+M)\cap W]/(M\cap W) \cong_{\mathbb{Z}H} (U+M)/M \cong_{\mathbb{Z}H} U/V$$

we have that: if U/V is finite, infinite, or a p-group, then so is $[(U+M)\cap W]/(M\cap W)$. Now by (4) and (5) above, we have the required $\mathbb{Z}G$-factor B/C.

The lemma is proved.

From Lemma **2.2.4**, we have the following consequence:

Lemma **2.2.5**: Let H be a normal subgroup of finite index in a group G, and let the $\mathbb{Z}G$-module A have a nonzero $\mathbb{Z}H$-submodule W. Then there is a one-to-one correspondence between the $\mathbb{Z}H$-factors of W and those of Wg for any $g \in G$, and this correspondence preserves the finiteness and the irreducibility of these $\mathbb{Z}H$-factors.

Proof: From the proof of (1) in Lemma **2.2.4**, it is clear that the mapping φ: $U/V \longmapsto Ug/Vg$ is a one-to-one correspondence between the $\mathbb{Z}H$-factors of W and those of Wg, and from (1) and (2) in Lemma **2.2.4**, φ preserves the finiteness and the irreducibility of the $\mathbb{Z}H$-factors. So the lemma is true.

Another consequence of Lemma **2.2.4** is:

Lemma 2.2.6: Let H be a normal subgroup of finite index in a group G and let A be a $\mathbb{Z}G$-module. Then A contains an irreducible $\mathbb{Z}G$-factor being finite, infinite, or a p-group for some prime p if and only if A as a $\mathbb{Z}H$-module contains an irreducible $\mathbb{Z}H$-factor being finite, infinite, or a p-group for the prime p, respectively.

Proof: It follows from Corollary **2.2.3** and (6) in Lemma **2.2.4**.

Furthermore, we have:

Lemma 2.2.7: If H is a normal subgroup of finite index in a group G, and if D is a $\mathbb{Z}H$-submodule of a $\mathbb{Z}G$-module A, then the $\mathbb{Z}G$-submodule D^G $(=\sum_{g\in G} Dg)$ of A has a finite (resp. infinite) irreducible $\mathbb{Z}G$-factor if and only if D has a finite (resp. infinite) irreducible $\mathbb{Z}H$-factor.

Proof: By (6) in Lemma **2.2.4**, the sufficiency is evident. So we prove the necessity.

Suppose $T = \{t_1 = 1, t_2, \cdots, t_n\}$ is a transversal to H in G. If D^G has a finite (resp. infinite) irreducible $\mathbb{Z}G$-factor then, by Corollary **2.2.3**, D^G has a finite (resp. infinite) irreducible $\mathbb{Z}H$-factor, say B_0/C_0. Since $D^G = \sum_{i=1}^{n} Dt_i$, $t_i \in T$, we may choose an integer n_0 such that n_0 is minimal with respect to $\sum_{i=1}^{n_0} Dt_i$ has a finite (resp. infinite) irreducible $\mathbb{Z}H$-factor B/C. We show that $n_0 = 1$. If not, we should have $(B+\sum_{i=1}^{n_0-1} Dt_i) > (C+\sum_{i=1}^{n_0-1} Dt_i)$ for otherwise by B/C being irreducible we have

$$B/C \cong_{\mathbb{Z}H} (B\cap\sum_{i=1}^{n_0-1} Dt_i)/(C\cap\sum_{i=1}^{n_0-1} Dt_i)$$

contrary to the minimality of n_0. So

$$B/C \cong_{\mathbb{Z}H} (B+\sum_{i=1}^{n_0-1} Dt_i)/(C+\sum_{i=1}^{n_0-1} Dt_i)$$

$$\leq \sum_{i=1}^{n_0} Dt_i/(C+\sum_{i=1}^{n_0-1} Dt_i)$$

$$\cong_{\mathbb{Z}H} Dt_{n_0}/[Dt_{n_0} \cap (C+\sum_{i=1}^{n_0-1} Dt_i)].$$

That is, Dt_{n_0} has a finite (resp. infinite) irreducible $\mathbb{Z}H$-factor. By Lemma **2.2.5**, D $(=Dt_1)$ has a finite (resp. infinite) irreducible $\mathbb{Z}H$-factor, contrary to the minimality of n_0 again. So $n_0=1$. That is, B/C is a finite (resp. infinite) irreducible $\mathbb{Z}H$-factor of D. The lemma is true.

Using Lemma **2.2.7**, we have:

Corollary **2.2.8**: If H is a normal subgroup of finite index in a group G, then the $\mathbb{Z}G$-module A contains no nonzero $\mathbb{Z}G$-submodules with an f–($\mathbb{Z}G$)–decomposition if and only if A as a $\mathbb{Z}H$-module contains no nonzero $\mathbb{Z}H$–submodules with an f–($\mathbb{Z}H$)–decomposition.

Proof: For a $\mathbb{Z}G$-submodule C of A, if C has an f–($\mathbb{Z}G$)–decomposition

$$C = C^f \oplus C^{\bar{f}},$$

then it follows from Lemma **2.2.6** that this is also an f–($\mathbb{Z}H$)–decomposition.

Suppose A as a $\mathbb{Z}H$-module contains a nonzero $\mathbb{Z}H$-submodule D with an f–($\mathbb{Z}H$)–decomposition, i.e., $D = D^f \oplus D^{\bar{f}}$, where D^f is the f–component of D and $D^{\bar{f}}$ is the $\bar{f}$–component of D. Since $D^G = (D^f)^G + (D^{\bar{f}})^G$, and by Lemma **2.2.7** $(D^f)^G$ has only finite irreducible $\mathbb{Z}G$-factors and $(D^{\bar{f}})^G$ has only infinite irreducible $\mathbb{Z}G$-factors, it follows that $D^G = (D^f)^G \oplus (D^{\bar{f}})^G$ is the f–($\mathbb{Z}$G)–decomposition of D^G. That is, the nonzero $\mathbb{Z}$G-submodule D^G of A has an f–($\mathbb{Z}$G)–decomposition, contrary to A having no such $\mathbb{Z}$G-submodules. So the corollary is true.

From the proof of Corollary **2.2.8**, we have the following two corollaries.

Corollary **2.2.9**: Let H be a normal subgroup of finite index in a group G and let D be a $\mathbb{Z}$H-submodule of the $\mathbb{Z}$G-module A. Then D has an f–($\mathbb{Z}$H)–decomposition if and only if D^G has an f–($\mathbb{Z}$G)–decomposition.

Corollary **2.2.10**: If H is a normal subgroup of finite index in a group G, then a $\mathbb{Z}$G-module A has an f–($\mathbb{Z}$G)–decomposition if and only if A has an f–($\mathbb{Z}$H)–decomposition.

§**2.3** RELATED TO f–DECOMPOSITION

At the end of the last section, we have noted some results related to the f–decomposition of the $\mathbb{Z}$G-modules. Here, we prove some more results, which will play an important role in the proof of our main results.

Lemma **2.3.1**: Let G be a group, F a nontrivial finite normal subgroup of G, A a $\mathbb{Z}$G-module, and B a $\mathbb{Z}$G-submodule of A. If A/B has an f–decomposition and $F \leq C_G(B)$ but F is not contained in $C_G(A)$, then A has a nonzero $\mathbb{Z}$G-submodule D with an f–decomposition, too. Furthermore, D can be chosen such that:

(1) if $(A/B)^f = 0$ then $D^f = 0$, and

(2) if $(A/B)^{\bar{f}} = 0$ then $D^{\bar{f}} = 0$.

Proof: Let $H = C_G(F)$, then $|G/H| < \infty$. By Corollary **2.2.10**, A/B as a $\mathbb{Z}H$-module has an f–($\mathbb{Z}H$)–decomposition, and then each $\mathbb{Z}H$-image of A/B has an f–($\mathbb{Z}H$)–decomposition (Lemma **1.2.23**). For $x \in F$, we have $B \leq C_A(x) \leq A$. Since F is not contained in $C_G(A)$, there exists $x_0 \in F$ such that $B \leq C_A(x_0) < A$. Thus the nonzero $\mathbb{Z}H$-submodule $C = A(x_0-1)$ $(\cong_{\mathbb{Z}H} A/C_A(x_0))$ has an f–($\mathbb{Z}H$)–decomposition. By (6) in Lemma **2.2.4**, the $\mathbb{Z}H$-submodule C has the properties that: (*i*) if $(A/B)^f = 0$, then $C^f = 0$; and (*ii*) if $(A/B)^{\bar{f}} = 0$, then $C^{\bar{f}} = 0$. Let $D = C^G = \sum_{g \in G} Cg$, then D is a nonzero $\mathbb{Z}G$-submodule of A, and, by Corollary **2.2.9**, D has an f–($\mathbb{Z}G$)–decomposition and, further, satisfies (1) if $(A/B)^f = 0$, then $D^f = 0$; and (2) if $(A/B)^{\bar{f}} = 0$, then $D^{\bar{f}} = 0$. The lemma is proved.

Corollary **2.3.2**: Let G be a hyperfinite group, A a $\mathbb{Z}G$-module, and B a nonzero $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A/B is finite (resp. infinite). If A has no nonzero $\mathbb{Z}G$-submodules with all irreducible $\mathbb{Z}G$-factors being finite (resp. infinite), then $C_G(B) = C_G(A)$.

Proof: We assume that G acts faithfully on A, i.e., $C_G(A) = 1$. If $C_G(B) \neq 1$, then since G is hyperfinite, $C_G(B)$ contains a nontrivial finite subgroup F being normal in G. By Lemma **2.3.1**, A has a nonzero $\mathbb{Z}G$-submodule with all irreducible $\mathbb{Z}G$-factors being finite (resp. infinite), a contradiction. So $C_G(B) = 1$. That is, $C_G(B) = C_G(A)$ as required.

The following lemma will be important in our proof for the main decomposition theorem in Chapter **3** as it allows us to assume that G acts faithfully when we pass to certain submodules.

Lemma **2.3.3**: Let G be a hyperfinite group which contains a normal locally soluble subgroup H, A a noetherian $\mathbb{Z}G$-module, and B a nonzero $\mathbb{Z}G$-submodule of A satisfying $B = B^{f}$ and $A/B = (A/B)^{\bar{f}}$ (resp. $B = B^{\bar{f}}$ and $(A/B)^{f} = A/B$). If $C_B(H) = 0$, A/C has an f–decomposition for any nonzero $\mathbb{Z}G$-submodule C of B, and A has no nonzero $\mathbb{Z}G$-submodule D with $D = D^{\bar{f}}$ (resp. $D = D^{f}$), then there is a $K \le H$ and a nonzero $\mathbb{Z}G$-submodule $B^* \le B$ such that K is normal in G, $A/B^* = B/B^* \oplus A^*/B^*$ for some $\mathbb{Z}G$-submodule A^* of A, $C_{B^*}(KC_G(A^*)/C_G(A^*)) = 0$, and $KC_G(A^*)/C_G(A^*)$ is a finite elementary abelian q-subgroup of $G/C_G(A^*)$ for some prime q.

<u>Proof</u>: By Lemma **1.2.14**, there is a $K \le H$ and a nonzero $\mathbb{Z}G$-submodule $B_1 \le B$ such that K is normal in G, $C_{B_1}(K) = 0$, and $|K/C_K(B_1)| < \infty$.

Let A_1 be the $\mathbb{Z}G$-submodule of A such that $A/B_1 = B/B_1 \oplus A_1/B_1$, then by Corollary **2.3.2**, $C_G(B_1) = C_G(A_1)$. Since $|K/C_K(B_1)| < \infty$ and $KC_G(A_1)/C_G(A_1) = KC_G(B_1)/C_G(B_1) \cong K/C_K(B_1)$, $KC_G(A_1)/C_G(A_1)$ is a finite subgroup of $G/C_G(A_1)$.

Choose K such that $KC_G(A_1)/C_G(A_1)$ is minimal with respect to $C_{B_1}(K) = 0$. Let K_0 be a normal subgroup of G such that $KC_G(A_1) > K_0 \ge C_G(A_1)$ and $KC_G(A_1)/K_0$ is a minimal normal subgroup of G/K_0. By the minimality of $KC_G(A_1)/C_G(A_1)$, we have $C_{B_1}(K_0) \ne 0$. Let $B_2 = C_{B_1}(K_0)$, then $A_1/B_2 = B_1/B_2 \oplus A_2/B_2$. Since $C_{B_2}(K) \le C_{B_1}(K) = 0$ and $C_G(A_1) \le C_G(A_2)$, so K is not contained in $C_G(A_2)$ and $KC_G(A_1)C_G(A_2) = KC_G(A_2)$. That is, $KC_G(A_2)/C_G(A_2)$ is a nontrivial subgroup of $G/C_G(A_2)$. Also, by Corollary **2.3.2** again, $C_G(B_2) = C_G(A_2)$. Since $K_0 \le C_G(B_2)$, so $K_0 \le C_G(A_2)$, and then $KC_G(A_1) \ge C_G(A_2) \cap (KC_G(A_1)) \ge K_0$. By

$$1 \neq KC_G(A_2)/C_G(A_2) = KC_G(A_1)C_G(A_2)/C_G(A_2) \cong KC_G(A_1)/[C_G(A_2)\cap(KC_G(A_1))]$$

and $KC_G(A_1)/K_0$ being a finite characteristically simple group, we must have $KC_G(A_2)/C_G(A_2)$ is a finite characteristically simple group. Since H is locally soluble and $K \leq H$, $KC_G(A_2)/C_G(A_2)$ is a finite characteristically simple subgroup of the locally soluble group $HC_G(A_2)/C_G(A_2)$. So $KC_G(A_2)/C_G(A_2)$ is a finite elementary abelian q-group for some prime q. Since $C_{B_2}(K) = 0$, we have $C_{B_2}(KC_G(A_2)/C_G(A_2)) = 0$.

Now, since $A = B+A_1 = B+B_1+A_2 = B+A_2$ and $B\cap A_2 = B\cap(A_1\cap A_2) = (B\cap A_1)\cap A_2 = B_1\cap A_2 = B_2$, so $A/B_2 = B/B_2 \oplus A_2/B_2$. Let $B^* = B_2$ and $A^* = A_2$. The required results follow.

§2.4 SOME OTHERS

This section comprises generalizations of one of Zaĭcev's results as well as other important results.

Lemma 2.4.1: Let G be a hyperfinite group, A a noetherian $\mathbb{Z}$G-module, and B a $\mathbb{Z}$G-submodule of A such that each irreducible $\mathbb{Z}$G-factor of B is not a finite p-group for some fixed prime p. If A/B is a finite p-group, then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}$G-submodule C of A.

Proof: Suppose B does not have a complement in A, then by the noetherian condition we may assume that for every nonzero $\mathbb{Z}$G-submodule C of B, B/C has a complement in A/C. Let D_0 be a $\mathbb{Z}$G-submodule of A maximal subject to $B\cap D_0 = 0$.

Since $A \neq B \oplus D_0$, by replacing A by A/D_0 we may assume that for each nonzero $\mathbb{Z}G$-submodule D of A, $B \cap D \neq 0$.

We show first that A is not torsion-free. For otherwise, since A/B is a finite p-group, there is a $\mathbb{Z}G$-submodule A^* of A such that A^*/B is a nontrivial elementary abelian p-group and, by A being torsion-free, A^* is also torsion-free, thus $pA^* \neq 0$ and then $A^* \cong_{\mathbb{Z}G} pA^* \leq B$ contrary to B containing no irreducible $\mathbb{Z}G$-factors being finite p-groups. So A is not torsion-free. Let T(A) be the torsion part of A, then T(A) is a nonzero $\mathbb{Z}G$-submodule of A. Thus $T(B) = T(A) \cap B \neq 0$. Let B_1 be the nonzero $\mathbb{Z}G$-submodule of B generated by all the elements of order q for some prime q, then $A/B_1 = B/B_1 \oplus A_1/B_1$ for some $\mathbb{Z}G$-submodule A_1 of A. Since $A_1/B_1 \cong_{\mathbb{Z}G} A/B$, A_1/B_1 is also a finite p-group. If $q \neq p$, then it is clear that $A_1 = B_1 \oplus A_2$ for some $\mathbb{Z}G$-submodule A_2 of A_1 (and hence of A), thus $A = B \oplus A_2$ contrary to B having no complement in A. Therefore, $q = p$ and then A_1 is a p-group. Thus B_1 is a $\mathbb{Z}G$-submodule of A_1 such that A_1/B_1 is finite while B_1 has no nonzero finite irreducible $\mathbb{Z}G$-factors. By Lemma **1.2.22**, B_1 has a complement in A_1, and then B has a complement in A, a contradiction.

Lemma **2.4.2**: Let G be a hyperfinite group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that B is a finite p-group for some prime p. If A/B has no nonzero finite $\mathbb{Z}G$-factors being p-groups, then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

Proof: Suppose B does not have a complement in A, then by the noetherian condition we may assume that for every nonzero $\mathbb{Z}G$-submodule C of B, B/C has a complement in A/C. Let D_0 be a $\mathbb{Z}G$-submodule of A maximal with respect to $B \cap D_0 = 0$. Since $A \neq B \oplus D_0$, by replacing A by A/D_0 we may assume that for each nonzero $\mathbb{Z}G$-submodule D of A, $B \cap D \neq 0$, i.e., each nonzero $\mathbb{Z}G$-submodule of A contains nonzero $\mathbb{Z}G$-submodules being finite p-groups.

Let B_1 be the nonzero $\mathbb{Z}G$-submodule of B generated by all the elements of order p, then $A/B_1 = B/B_1 \oplus A_1/B_1$ for some $\mathbb{Z}G$-submodule A_1 of A. Let $\varphi: A_1 \longmapsto pA_1$ defined by $\varphi: a \longmapsto pa$ ($\forall$ a $\in A_1$), then φ is a $\mathbb{Z}G$-homomorphism from A_1 to pA_1. It is clear that $B_1 \leq \ker\varphi$. Since $pA_1 \cong_{\mathbb{Z}G} A_1/\mathrm{Ker}\varphi$ and A_1/B_1 ($\cong_{\mathbb{Z}G}$ A/B) has no nonzero $\mathbb{Z}G$-factors being finite p-groups, pA_1 has no nonzero $\mathbb{Z}G$-factors being finite p-groups. But each nonzero $\mathbb{Z}G$-submodule of A (and hence of A_1) contains nonzero $\mathbb{Z}G$-submodules being finite p-groups, therefore we must have $pA_1 = 0$ and then A_1 is a $\mathbb{Z}G$-module with the $\mathbb{Z}G$-submodule B_1 such that B_1 is finite and the factor-module A_1/B_1 has no nonzero finite $\mathbb{Z}G$-factors. By Lemma **1.2.22**, B_1 has a complement in A_1 and then B has a complement in A, a contradiction.

Combining the above two lemmas, we have:

<u>Corollary **2.4.3**</u>: Let G be a hyperfinite group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that B (resp. A/B) is a finite p-group for some prime p. If A/B (resp. B) contains no nonzero finite $\mathbb{Z}G$-factors being p-groups, then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

From Corollary **2.4.3**, we have:

<u>Corollary **2.4.4**</u>: Let G be a hyperfinite group and A a noetherian $\mathbb{Z}G$-module. Then some irreducible $\mathbb{Z}G$-image of A is a finite p-group for some prime p if and only if so is some irreducible $\mathbb{Z}G$-factor of A.

<u>Proof</u>: If A has an irreducible $\mathbb{Z}G$-factor being a finite p-group for the

prime p, then by the noetherian condition A has an irreducible $\mathbb{Z}G$-factor, say B/C, such that the $\mathbb{Z}G$-submodule B/C of A/C is a finite p-group and the factor-module $A/C/B/C$ $(\cong_{\mathbb{Z}G} A/B)$ contains no irreducible $\mathbb{Z}G$-factors being finite p-groups. By Corollary **2.4.3**, A/C and then A has an irreducible $\mathbb{Z}G$-image being a finite p-group.

The above are the generalizations of Zaĭcev's result (see Lemma **1.2.22**). Now we prove some other important lemmas.

<u>Lemma 2.4.5</u>: Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p. If all irreducible $\mathbb{Z}G$-factors of A are finite, then A is finite.

<u>Proof</u>: Suppose A is not finite, then by the noetherian condition we may assume that for every nonzero $\mathbb{Z}G$-submodule C of A, A/C is finite.

We have $A = \langle a_1, \cdots, a_n \rangle^G$ with n being an integer and the order of a_i is p for each i. Also we may assume that G acts faithfully on A and, since A is infinite, G is infinite.

Let M be a maximal $\mathbb{Z}G$-submodule of A, then A/M is finite and hence for $H = C_G(A/M)$ we have $|G/H| < \infty$. Since H is nontrivial, so H contains a nontrivial finite subgroup, say K, being minimal normal in G (Lemma **1.2.10**). By G being a locally soluble group, K is an elementary abelian subgroup of G. Let $G_1 = C_G(K)$, then $K < G_1$ and $|G/G_1| < \infty$. By Lemma **1.2.5** and (6) in Lemma **2.2.4**, A is an infinite noetherian $\mathbb{Z}G_1$-module and all irreducible $\mathbb{Z}G_1$-factors of A are finite. Since the $\mathbb{Z}G_1$-image A/M is finite, using the noetherian condition , we may have a $\mathbb{Z}G_1$-image A^* $(= A/D)$ of A such that $D < M$ and A^* is infinite but for every nonzero $\mathbb{Z}G_1$-submodule C^* of A^*, A^*/C^* is finite. By G being faithful on A, we have the $\mathbb{Z}G$-submodule $[A, K] \neq 0$ and then the $\mathbb{Z}G$-image (and so $\mathbb{Z}G_1$-image) $A/[A, K]$ is finite. Thus $[A, K]$ is not contained in D and then K is not contained in $C_{G_1}(A/D) = C_{G_1}(A^*)$.

Replacing G_1 by $G_1/C_{G_1}(A^*)$ and K by $(KC_{G_1}(A^*))/C_{G_1}(A^*)$ we may assume that G_1 acts faithfully on A^*, then K is a nontrivial finite central subgroup of G_1. Let $1 \neq x \in K$ such that x is of order q for some prime q. If $q \neq p$, then, by Lemma **1.2.4**, $A^* = C_{A^*}(<x>) \oplus [A^*, <x>]$. Since $C_{G_1}(A^*) = 1$, so the $\mathbb{Z}G_1$-submodule $A^*(x-1) = [A^*, x] = [A^*, <x>] \neq 0$. Also, since $M^* = M/D$ and $A^*/M^* \cong_{\mathbb{Z}G_1} A/M$, so $<x> \leq K \leq C_{G_1}(A^*/M^*)$, therefore $A^*(x-1) \leq M^* < A^*$. Thus $C_{A^*}(x) = C_{A^*}(<x>) \neq 0$. But $C_{A^*}(x)$ $(\cong_{\mathbb{Z}G_1} A^*/A^*(x-1))$ and $A^*(x-1)$ $(\cong_{\mathbb{Z}G_1} A^*/C_{A^*}(x))$ are both finite and then A^* is finite, a contradiction. So $q = p$. Consider the finite $\mathbb{Z}<x>$-module $A_1^* = <a>^{<x>}$, where $0 \neq a \in A^*$. Since A_1^* is a finite p-group, there exists $0 \neq a_0 \in A_1^*$ such that $a_0 \in C_{A^*}(x)$. By $C_{G_1}(A^*) = 1$ we have $A^* \neq C_{A^*}(x)$ and then $A^*(x-1)$ $(\cong_{\mathbb{Z}G_1} A^*/C_{A^*}(x))$ is a nonzero finite $\mathbb{Z}G_1$-module. Therefore $A^*/A^*(x-1)$ is finite and then A^* is finite, a contradiction again. The result holds.

For a general hyperfinite group G (that is, G need not to be locally soluble), we have:

Lemma 2.4.6: Let G be a hyperfinite group and A a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p. If G is a p'-group and all irreducible $\mathbb{Z}G$-factors of A are finite, then A is finite.

Proof: Suppose A is infinite, then using the noetherian condition we may assume that for any nonzero $\mathbb{Z}G$-submodule C of A, A/C is finite.

We have $A = \langle a_1, \cdots, a_n \rangle^G$ with n being an integer and the order of a_i is p for each i. Also we may assume that G acts faithfully on A and, since A is infinite, G is infinite.

Let M be a maximal $\mathbb{Z}G$-submodule of A, then A/M is finite and hence for $H = C_G(A/M)$ we have $|G/H| < \infty$. Since H contains nontrivial finite subgroups being normal in G (Lemma **1.2.10**), so we may let $K \leq H$ and K is a finite normal subgroup of G. By G being a p'-group, we have K is a finite p'-group, thus, by Lemma **1.2.4**, $A = C_A(K) \oplus [A, K]$. Since $K \leq H = C_G(A/M)$, so $[A, K] \leq M$ and then $C_A(K) \neq 0$. By G being faithful on A, we have $C_A(K) \neq A$ and then $[A, K] \neq 0$. Thus $C_A(K)$ $(\cong_{\mathbb{Z}G} A/[A, K])$ and $[A, K]$ $(\cong_{\mathbb{Z}G} A/C_A(K))$ are both finite and then A is finite, a contradiction. Hence the result holds.

The final result of this chapter has a very special but simple proof. We mention that it will play an important role in our work similar with that of Fitting's lemma.

<u>Lemma 2.4.7</u>: Let G be a group, A a $\mathbb{Z}G$-module, and M a $\mathbb{Z}G$-submodule of A such that A/M is a p-group for some prime p. If $H = C_G(A/M)$ contains a nontrivial subgroup K such that K is a finite q-group for some prime q other than p, then $A = C_A(x) + M$ for any $x \in K$. Further, $A = C_A(K) + M$. (We note that the subgroups $C_A(x)$ and $C_A(K)$ may not be $\mathbb{Z}G$-submodules of A.)

<u>Proof</u>: Let $x \in K$, then $x^{q^n} = 1$ for some integer n. Since

$$(x^{q^n-1} + x^{q^n-2} + \cdots + x + 1)(x-1) = x^{q^n} - 1 = 0, \qquad \text{and}$$

$$q^n = (x^{q^n-1} + x^{q^n-2} + \cdots + x + 1) - [x^{q^n-2} + 2x^{q^n-3} + \cdots + (q^n-2)x + (q^n-1)](x-1). \qquad (*)$$

So $q^nA \leq A(x^{q^n-1} + x^{q^n-2} + \cdots + x + 1) + A(x-1) \leq C_A(x) + M$. That is, $A/(C_A(x) + M)$ is a q-group. But A/M is a p-group and $p \neq q$, so we must have $A = C_A(x) + M$.

Let $K = \{x_1=1, x_2, \cdots, x_t\}$ and let $C_m = C_A(x_1, \cdots, x_m)$, where $m = 1, 2, \cdots, t$. Suppose that $A = C_m + M$, we prove that $A = C_{m+1} + M$. By the equations in (*) above, we have

$$q^nC_m \leq C_m(x_{m+1}^{q^n-1} + x_{m+1}^{q^n-2} + \cdots + x_{m+1} + 1) + C_m(x_{m+1}-1)$$

$$\leq C_{C_m}(x_{m+1}) + (M\cap C_m) \leq C_{m+1} + (M\cap C_m).$$

Since $A = C_m + M$ so, as groups, $A/M \cong C_m/(M\cap C_m)$. That is, $C_m/(M\cap C_m)$ is also a p-group. By $q^nC_m \leq C_{m+1} + (M\cap C_m)$ and $q \neq p$, we have $C_m = C_{m+1} + (M\cap C_m)$, and then

$$A = C_m + M = C_{m+1} + (M\cap C_m) + M = C_{m+1} + M$$

as required. So $A = C_m + M$ for all m. In particular, put $m = t$, then $C_t = C_A(K)$ and $A = C_A(K) + M$. The lemma is proved.

3 THE f–DECOMPOSITION OF THE NOETHERIAN MODULES

In this chapter, we aim to prove the main result—Theorem A.

If a noetherian $\mathbb{Z}G$-module A over a hyperfinite group G contains a $\mathbb{Z}G$-submodule B such that B (resp. A/B) is finite and A/B (resp. B) contains no nonzero finite irreducible $\mathbb{Z}G$-factors, then B has a complement in A (Lemma **1.2.22**). In §**2.4**, we have given a generalization of this result. Now we will prove a number of other generalizations and corollaries under the condition that G is hyperfinite and locally soluble.

The following is in fact the beginning of the main proof for Theorem A.

Proposition 3.1: Let G be a hyperfinite locally soluble group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of B is finite while A/B has no finite irreducible $\mathbb{Z}G$-factors. Then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

Proof: Suppose that B does not have a complement in A. By considering an appropriate factor-module of A we may assume that for every nonzero $\mathbb{Z}G$-submodule $\tilde{B}$ of B, $B/\tilde{B}$ has a complement in $A/\tilde{B}$. Also, let C be a $\mathbb{Z}G$-submodule of A maximal with respect to $B \cap C = 0$. Then $A \neq B \oplus C$. By replacing A by A/C we may assume that for every nonzero $\mathbb{Z}G$-submodule D of A, $B \cap D \neq 0$. That is, A contains no nonzero $\mathbb{Z}G$-submodules with all irreducible $\mathbb{Z}G$-factors being infinite.

If A is not torsion-free, let T(A) be the torsion part of A, then T(A) is a nonzero $\mathbb{Z}G$-submodule of A and then $T(B) = B \cap T(A) \neq 0$. Let B^* be the nonzero $\mathbb{Z}G$-submodule of B generated by all the elements of order p for some prime p, then $A/B^* = B/B^* \oplus A^*/B^*$ for some $\mathbb{Z}G$-submodule A^* of A. Let φ: $A^* \longrightarrow pA^*$

be defined by φ: $a \longmapsto pa$ for all $a \in A^*$, then φ is a $\mathbb{Z}G$-homomorphism from A^* to pA^* and $B^* \leq \ker\varphi$. If $pA^* \neq 0$, then since $pA^* \cong_{\mathbb{Z}G} A^*/\ker\varphi$ and A^*/B^* has no finite irreducible $\mathbb{Z}G$-factors (as $A^*/B^* \cong_{\mathbb{Z}G} A/B$), the irreducible $\mathbb{Z}G$-factors of pA^* are all infinite, a contradiction. So $pA^* = 0$. That is, A^* is an elementary abelian p-group for the prime p. If B^* has a complement in A^*, i.e., $A^* = B^* \oplus C^*$ for some $\mathbb{Z}G$-submodule C^*, then $A = B \oplus C^*$ (Lemma **1.2.25**), a contradiction. So B^* has no complement in A^*, and A^* satisfies all the conditions that are satisfied by A. Hence we may replace A by A^* (when A is not torsion-free) and then we may assume that A is either torsion-free or an elementary abelian p-group for some prime p.

Let $L = C_G(A/B)$, where we have B is a nonzero proper $\mathbb{Z}G$-submodule of A such that all irreducible $\mathbb{Z}G$-factors of B are finite and A/B has no finite irreducible $\mathbb{Z}G$-factors. We consider the following two cases: (*a*) $L=1$, or (*b*) $L \neq 1$.

(*a*) $L = C_G(A/B) = 1$. In this case, let N_1 and N_2 be two maximal $\mathbb{Z}G$-submodules of B such that B/N_1 is a finite p-group and B/N_2 is a finite q-group, where p and q are primes and, if A is torsion-free, we can assume $p \neq q$ (by using Lemma **2.1.4**). Let M_1 and M_2 be two $\mathbb{Z}G$-submodules of A such that $A/M_1 \cong_{\mathbb{Z}G} B/N_1$ and $A/M_2 \cong_{\mathbb{Z}G} B/N_2$ (such M_1 and M_2 exist as we can take $A/N_1 = B/N_1 \oplus M_1/N_1$ and $A/N_2 = B/N_2 \oplus M_2/N_2$). Let $M = M_1 \cap M_2$ and let $H = C_G(A/M)$, then $|G/H| < \infty$.

Let $W = C_B(H)$. If $W \neq 0$, then let $0 \neq a \in W$, we have $U = \langle a \rangle^G$ is a nonzero $\mathbb{Z}G$-submodule of B and then $A/U = B/U \oplus V/U$ for some $\mathbb{Z}G$-submodule V of A. By Corollary **2.3.2**, we have $C_G(U) = C_G(V)$. But $H \leq C_G(W) \leq C_G(U)$ and $C_G(V) \leq C_G(V/U) = C_G(A/B) = L = 1$, then it follows from $|G/H| < \infty$ that G is a finite group, contrary to A having infinite irreducible $\mathbb{Z}G$-factors. So

$W = C_B(H) = 0$. Now, by Lemma **2.3.3**, there exists a $K \le H$ and a nonzero $\mathbb{Z}G$-submodule B_1 of B such that K is normal in G, $A/B_1 = B/B_1 \oplus A_1/B_1$ for some $\mathbb{Z}G$-submodule A_1 of A, $C_{B_1}(KC_G(A_1)/C_G(A_1)) = 0$, and $KC_G(A_1)/C_G(A_1)$ is a finite elementary abelian r-subgroup of $G/C_G(A_1)$ for some prime r. Since $C_G(A_1) \le C_G(A_1/B_1) = C_G(A/B) = L = 1$, so $C_G(A_1) = 1$ and then K is a nontrivial finite elementary abelian r-subgroup of G for some prime r.

If $C_A(K) \ne 0$, then since $B_1 \cap C_A(K) = C_{B_1}(K) = 0$, we have $C_A(K) \cong_{\mathbb{Z}G} (C_A(K) \oplus B_1)/B_1 \le A/B_1 = B/B_1 \oplus A_1/B_1$. Using Lemma **1.2.23**, we get $C_A(K)$ has an f–decomposition and, by the fact that A has no nonzero $\mathbb{Z}G$-submodules with all irreducible $\mathbb{Z}G$-factors being infinite, we have $C_A(K) < B$. Let $B_3 = C_A(K)$, then $A/B_3 = B/B_3 \oplus A_3/B_3$ for some $\mathbb{Z}G$-submodule A_3 and $K \le C_G(B_3) = C_G(A_3)$ (Corollary **2.3.2**). But

$$C_G(A_3) \le C_G(A_3/B_3) = C_G(A/B) = L=1,$$

thus $K = 1$, a contradiction. So $C_A(K) = 0$.

If A is an elementary abelian p-group, then by $C_A(K) = 0$ we have $r \ne p$; also, on the other hand, if A is torsion-free, then we have $r \ne p$ or $r \ne q$. By Lemma **2.4.7** and $C_A(K) = 0$ we have the contradiction that

$$A = C_A(K) + M_i = M_i < A,$$

where $i = 1$ or 2 if $r \ne p$ or $r \ne q$.

Case (a) is proved.

(b) $L = C_G(A/B) \ne 1$. In this case, let $B^* = C_B(L)$, and we consider the following two subcases: (i) $B^* = 0$, or (ii) $B^* \ne 0$.

(i) $B^* = C_B(L) = 0$. By Lemma **2.3.3**, there exists a $K \le L$ and a nonzero $\mathbb{Z}G$-submodule B_1 of B such that K is normal in G, $A/B_1 = B/B_1 \oplus A_1/B_1$

for some $\mathbb{Z}G$-submodule A_1 of A, $C_{B_1}(KC_G(A_1)/C_G(A_1)) = 0$, and $KC_G(A_1)/C_G(A_1)$ is a finite elementary abelian q-subgroup of $G/C_G(A_1)$ for some prime q. Consider A_1 as a $\mathbb{Z}\overline{G}$-module, where $\overline{G} = G/C_G(A_1)$. Then it is evident that $\overline{1} \neq \overline{K} \leq \overline{L} = C_{\overline{G}}(A_1/B_1)$ and $C_{B_1}(\overline{K}) = 0$. Also it is clear that all the irreducible $\mathbb{Z}\overline{G}$-factors of B_1 are finite, the factor-module A_1/B_1 has no finite irreducible $\mathbb{Z}\overline{G}$-factors, and A_1 has no nonzero $\mathbb{Z}\overline{G}$-submodules with all irreducible $\mathbb{Z}\overline{G}$-factors being infinite. Thus, since $B_1 \cap C_{A_1}(\overline{K}) = C_{B_1}(\overline{K}) = 0$, we have $C_{A_1}(\overline{K}) = 0$.

If A_1 is an elementary abelian p-group, then by $C_{A_1}(\overline{K}) = 0$ we have $q \neq p$. Also, if A_1 is torsion-free but A_1/B_1 is not torsion-free, then we may assume that A_1 has a $\mathbb{Z}\overline{G}$-submodule A_2 such that A_2/B_1 ($\leq A_1/B_1$) is a nontrivial elementary abelian r-group for some prime r, and then $A_2 \cong_{\mathbb{Z}\overline{G}} rA_2 \leq B_1$ (by A_2 being torsion-free), contrary to B_1 having no infinite irreducible $\mathbb{Z}\overline{G}$-factors. So A_1 is torsion-free implies that A_1/B_1 is torsion-free, too. By Lemma **2.1.4**, we have $A_1/B_1 > p(A_1/B_1)$ for any prime p. Therefore A_1 always contains a proper $\mathbb{Z}\overline{G}$-submodule M such that $B_1 \leq M$ and A_1/M is a p-group for some prime p other than q. Since $\overline{K} \leq \overline{L} = C_{\overline{G}}(A_1/B_1) \leq C_{\overline{G}}(A_1/M)$, by Lemma **2.4.7**, $A_1 = C_{A_1}(\overline{K}) + M = M < A_1$, a contradiction.

(*ii*) $B^* = C_B(L) \neq 0$. We write A as a sum $A = B + A^*$ with $B \cap A^* = B^*$. For any $C \leq B^*$ ($C \neq 0$), since $A^*/C = B^*/C \oplus \overline{A}/C$ for some $\mathbb{Z}G$-submodule $\overline{A}$ of A, and since L centralizes both B^*/C and $\overline{A}/C$ (as $L \leq C_G(B^*)$ and $L = C_G(A/B)$

$= C_G(A^*/B^*) = C_G(\overline{A}/C))$, then $L \le C_G(A^*/C)$. Let $C^* = \bigcap C$, where $0 \neq C \le B^*$, then either $C^* = 0$ or C^* is a finite irreducible $\mathbb{Z}G$-submodule of A. If $C^* \neq 0$, then $A/C^* = B/C^* \oplus \tilde{A}/C^*$ for some $\mathbb{Z}G$-submodule $\tilde{A}$ of A and then, by Lemma **1.2.22**, $\tilde{A} = C^* \oplus \tilde{\tilde{A}}$ for some $\mathbb{Z}G$-submodule $\tilde{\tilde{A}}$. It is clear that $\tilde{\tilde{A}}$ is a nonzero $\mathbb{Z}G$-submodule with all irreducible $\mathbb{Z}G$-factors being infinite, contrary to A having no such $\mathbb{Z}G$-submodules. So we must have $C^* = 0$. As $[A^*,L] \le C$ for all $0 \neq C \le B^*$, so $[A^*,L] \le \bigcap C = C^* = 0$; that is, $L \le C_G(A^*)$. Also since $C_G(A^*) \le C_G(A^*/B^*) = L$, so $L = C_G(A^*)$. Now consider A^* as a $\mathbb{Z}\overline{G}$-module, where $\overline{G} = G/C_G(A^*)$, and let $\overline{L} = C_{\overline{G}}(A^*/B^*)$, then it is clear that $\overline{L} = \overline{1}$. Thus, by the proof of the above for the case (*a*), we get a contradiction. So we have in fact finished the proof.

Dual to Proposition **3.1**, we have

<u>Proposition **3.2**</u>: Let G be a hyperfinite locally soluble group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A/B is finite while B has no finite irreducible $\mathbb{Z}G$-factors. Then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

<u>Proof:</u> Suppose B does not have a complement in A. By considering an appropriate factor-module of A we may assume that for every nonzero $\mathbb{Z}G$-submodule $\tilde{B}$ of B, $B/\tilde{B}$ has a complement in $A/\tilde{B}$. Also, let C be a $\mathbb{Z}G$-submodule of A maximal subject to $B \cap C = 0$. Then $A \neq B \oplus C$. By replacing A by A/C we may assume that for every nonzero $\mathbb{Z}G$-submodule D of A, $B \cap D \neq 0$. That is, A contains no nonzero $\mathbb{Z}G$-submodules with all irreducible $\mathbb{Z}G$-factors

being finite.

If A is torsion-free but A/B is not, then let A* be a $\mathbb{Z}G$-submodule of A such that A*/B is a nontrivial elementary abelian p-group for some prime p. Thus $A^* \cong_{\mathbb{Z}G} pA^* \leq B$ by A* being torsion-free, but this is contrary to B containing no finite irreducible $\mathbb{Z}G$-factors. So A being torsion-free implies that A/B is also torsion-free.

If A is not torsion-free, let T(A) be the torsion part of A, then T(A) is a nonzero $\mathbb{Z}G$-submodule of A and then $T(B) = T(A) \cap B \neq 0$. Let B_1 be the nonzero $\mathbb{Z}G$-submodule of B generated by all the elements of order p for some prime p, then $A/B_1 = B/B_1 \oplus A_1/B_1$ for some $\mathbb{Z}G$-submodule A_1 of A. Let φ: $A_1 \longrightarrow pA_1$ be defined by φ: $a \longmapsto pa$ for all $a \in A_1$; then φ is a $\mathbb{Z}G$-homomorphism from A_1 to pA_1 and $B_1 \leq \ker\varphi$. If $pA_1 \neq 0$, then since $pA_1 \cong_{\mathbb{Z}G} A_1/\ker\varphi$ and A_1/B_1 has no infinite irreducible $\mathbb{Z}G$-factors, pA_1 is a nonzero $\mathbb{Z}G$-submodule with all irreducible $\mathbb{Z}G$-factors being finite, contrary to A having no such $\mathbb{Z}G$-submodules. So $pA_1 = 0$. That is, A contains a nonzero $\mathbb{Z}G$-submodule A_1, which is an elementary abelian p-group for some prime p, such that $A = B + A_1$.

As before, we can replace A by A_1 (if necessary), so we may assume that A is either torsion-free or an elementary abelian p-group for some prime p. For the nonzero $\mathbb{Z}G$-submodule B of A with $B = B^{\bar{f}}$ and $A/B = (A/B)^{f}$, since A/B is accordingly either torsion-free or an elementary abelian p-group, we can assume that A contains two maximal $\mathbb{Z}G$-submodules M_1 and M_2, both containing B, such that A/M_1 is a finite p-group and A/M_2 is a finite q-group, where p and q are primes, and in the case that A/B is torsion-free, by Lemma **2.1.4**, we may assume that $p \neq q$. Let $M = M_1 \cap M_2$ and let $H = C_G(A/M)$, then $|G/H| < \infty$.

If $a \in C_B(H)$, then the $\mathbb{Z}G$-submodule $<a>^G$ as a group is finitely generated and so each irreducible $\mathbb{Z}G$-factor of $<a>^G$ is finite. Since B has no nonzero finite irreducible $\mathbb{Z}G$-factors, we must have $\langle a \rangle^G = 0$ and so $a = 0$; that is, $C_B(H) = 0$. Now, by Lemma **2.3.3**, there is a $K \leq H$ and a nonzero $\mathbb{Z}G$-submodule $B^* \leq B$ such that K is normal in G, $A/B^* = B/B^* \oplus A^*/B^*$ for some $\mathbb{Z}G$-submodule A^* of A, $C_{B^*}\big(KC_G(A^*)/C_G(A^*)\big) = 0$, and $KC_G(A^*)/C_G(A^*)$ is a finite elementary abelian r-subgroup of $G/C_G(A^*)$ for some prime r.

Since $A^*/B^* \cong_{\mathbb{Z}G} A/B$, there exist $\mathbb{Z}G$-submodules M^*, M_1^*, and M_2^* such that $M^* = M_1^* \cap M_2^*$, $A^*/M_1^* \cong_{\mathbb{Z}G} A/M_1$, $A^*/M_2^* \cong_{\mathbb{Z}G} A/M_2$, $A^*/M^* \cong_{\mathbb{Z}G} A/M$, and $B^* \leq M^*$. Let $\overline{G} = G/C_G(A^*)$, then, since $C_G(A^*) \leq C_G(A^*/M^*) = C_G(A/M) = H$, $\overline{H} = H/C_G(A^*)$. It is clear that $\overline{H} = C_{\overline{G}}(A^*/M^*)$. Also $\overline{1} \neq \overline{K} \leq \overline{H}$ and $\overline{K}$ is a finite elementary abelian r-group for the prime r. Since $C_{B^*}(\overline{K}) = 0$, if A^* is a p-group, then $r \neq p$, and if A^*/B^* is torsion-free, then $r \neq p$ or $r \neq q$. By Lemma **2.4.7**, we have $A^* = C_{A^*}(\overline{K}) + M_i^*$, where $M_i^* = M_1^*$ or M_2^* if $r \neq p$ or $r \neq q$. Since $B^* \cap C_{A^*}(\overline{K}) = C_{B^*}(\overline{K}) = 0$ and A^* has no nonzero $\mathbb{Z}\overline{G}$-submodules with all irreducible $\mathbb{Z}\overline{G}$-factors being finite, so $C_{A^*}(\overline{K}) = 0$ and then $A^* = M_i^* < A^*$, a contradiction.

The result is proved.

Joining these two propositions, we get the following corollary, which generalizes Lemma **1.2.22** in the case that G is hyperfinite and locally soluble.

<u>Corollary 3.3</u>: Let G be a hyperfinite locally soluble group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of B is finite (resp. infinite) and A/B contains no finite (resp. infinite) irreducible $\mathbb{Z}G$-factors. Then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

From Corollary 3.3, we have

Corollary 3.4: Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module. Then A has a finite (resp. infinite) irreducible $\mathbb{Z}G$-image if and only if A has a finite (resp. infinite) irreducible $\mathbb{Z}G$-factor.

Proof: Suppose A has a finite (resp. infinite) irreducible $\mathbb{Z}G$-factor, then by the noetherian condition we have A contains an irreducible $\mathbb{Z}G$-factor, say B/C, such that the irreducible $\mathbb{Z}G$-submodule B/C of A/C is finite (resp. infinite) and the factor-module $A/C / B/C$ ($\cong_{\mathbb{Z}G}$ A/B) contains no finite (resp. infinite) irreducible $\mathbb{Z}G$-factors. By Corollary 3.3, we have A/C and hence A has a finite (resp. infinite) irreducible $\mathbb{Z}G$-image. The corollary is true.

Another consequence of Corollary 3.3 is that

Corollary 3.5: Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module. If A has a $\mathbb{Z}G$-composition series in which the finite (resp. infinite) irreducible $\mathbb{Z}G$-factors of A are only finitely many, then A has an f–decomposition.

Proof: It follows from Corollary 3.3 and induction on the finite number of the finite (resp. infinite) irreducible $\mathbb{Z}G$-factors in a $\mathbb{Z}G$-composition series.

As in the hyperfinite case, we can generalize Corollary 3.3 in the following forms, and these results will be seen to be useful in the later discussions.

Proposition 3.6: Let G be a hyperfinite locally soluble group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A/B is a finite (resp. infinite) p-group and B has no irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) p-groups, where p is a fixed prime. Then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

Proof: Suppose B does not have a complement in A, then by the noetherian condition we may assume that for every nonzero $\mathbb{Z}G$-submodule C of B, B/C has a complement in A/C. Let D_0 be a $\mathbb{Z}G$-submodule of A maximal with respect to $B \cap D_0 = 0$. Since $A \neq B \oplus D_0$, by replacing A by A/D_0 we may assume that for any nonzero $\mathbb{Z}G$-submodule D of A, $B \cap D \neq 0$.

Since each irreducible $\mathbb{Z}G$-factor of A/B is a finite (resp. infinite) p-group for the fixed prime p, by Lemma **2.1.4**, A/B is not torsion-free, and further A/B is a p-group. We claim that A is also not torsion-free. For otherwise, let A^* be a $\mathbb{Z}G$-submodule of A such that A^*/B is a nontrivial elementary abelian p-group, then $pA^* \neq 0$ and then $A^* \cong_{\mathbb{Z}G} pA^* \leq B$, contrary to B having no irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) p-groups. So A is not torsion-free. Let T(A) be the torsion part of A, then T(A) is a nonzero $\mathbb{Z}G$-submodule of A and then $T(B) = B \cap T(A) \neq 0$. Let B_1 be the nonzero $\mathbb{Z}G$-submodule of B generated by all the elements of order q for some prime q, then $A/B_1 = B/B_1 \oplus A_1/B_1$ for some $\mathbb{Z}G$-submodule A_1 of A. By $A_1/B_1 \cong_{\mathbb{Z}G} A/B$, we have A_1/B_1 is a p-group. If $q \neq p$, then it is clear that $A_1 = B_1 \oplus A_2$ for some $\mathbb{Z}G$-submodule A_2 and then $A = B \oplus A_2$, a contradiction. Thus $q = p$, and then A_1 is a p-group. Now each irreducible $\mathbb{Z}G$-factor of A_1/B_1 is finite (resp. infinite) and the $\mathbb{Z}G$-submodule B_1 has no finite (resp. infinite) irreducible $\mathbb{Z}G$-factors, by Corollary **3.3**, we have $A_1 = B_1 \oplus C_1$ and then $A = B \oplus C_1$, a contradiction again.

Corollary 3.7: Let G be a hyperfinite locally soluble group, A a noetherian $\mathbb{Z}G$-module and p a prime. Then A has an irreducible $\mathbb{Z}G$-image being not a finite (resp. infinite) p-group if and only if the same is true for some irreducible $\mathbb{Z}G$-factor of A.

Proof: If A has an irreducible $\mathbb{Z}G$-factor being not a finite (resp. infinite) p-group, then by the noetherian condition we have A contains an irreducible $\mathbb{Z}G$-factor, say B/C, such that the irreducible $\mathbb{Z}G$-submodule B/C of A/C is not a finite (resp. infinite) p-group but the irreducible $\mathbb{Z}G$-factors of the factor-module $A/C/B/C$ ($\cong_{\mathbb{Z}G}$ A/B) are all finite (resp. infinite) p-groups. By Proposition **3.6**, we have A/C and then A has a nonzero irreducible $\mathbb{Z}G$-image being not a finite (resp. infinite) p-group.

The dual of proposition **3.6** is:

Proposition **3.8**: Let G be a hyperfinite locally soluble group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of B is a finite (resp. infinite) p-group and A/B has no irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) p-groups, where p is a fixed prime. Then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

Proof: Suppose B does not have a complement in A, then by the noetherian condition we may assume that for every nonzero $\mathbb{Z}G$-submodule C of B, B/C has a complement in A/C. Let D_0 be a $\mathbb{Z}G$-submodule of A maximal subject to $B \cap D_0 = 0$. Since $A \neq B \oplus D_0$, by replacing A by A/D_0 we may assume that for each nonzero $\mathbb{Z}G$-submodule D of A, $B \cap D \neq 0$, i.e., each nonzero $\mathbb{Z}G$-submodule of A contains irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) p-groups for the fixed prime p.

By Lemma **2.1.4**, B is not torsion-free, and further B is a p-group for the prime p. Let B_1 be the nonzero $\mathbb{Z}G$-submodule of B generated by all the elements of order p, then $A/B_1 = B/B_1 \oplus A_1/B_1$ for some $\mathbb{Z}G$-submodule A_1 of A. Let $\varphi: A_1 \longrightarrow pA_1$ be defined by $\varphi: a \longmapsto pa$ for all $a \in A$; then φ is a $\mathbb{Z}G$-homomorphism from A_1 to pA_1, and it is clear that $B_1 \leq \ker\varphi$. Since $pA_1 \cong_{\mathbb{Z}G} A_1/\ker\varphi$ and A_1/B_1 ($\cong_{\mathbb{Z}G}$ A/B) has no irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) p-groups, so pA_1 has no irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) p-groups. But each nonzero $\mathbb{Z}G$-submodule of A (and hence of A_1) contains irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) p-groups; therefore we must have $pA_1 = 0$ and then A_1 is a $\mathbb{Z}G$-module with the $\mathbb{Z}G$-submodule B_1 such that each irreducible $\mathbb{Z}G$-factor of B_1 is finite (resp. infinite) and the factor-module A_1/B_1 has no finite (resp. infinite) irreducible $\mathbb{Z}G$-factors. Thus, by Corollary **3.3**, we have $A_1 = B_1 \oplus C_1$ and then $A = B \oplus C_1$, a contradiction.

<u>Corollary 3.9</u>: Let G be a hyperfinite locally soluble group, A a noetherian $\mathbb{Z}G$-module and p a prime. Then A has an irreducible $\mathbb{Z}G$-image being a finite (resp. infinite) p-group if and only if the same is true for some irreducible $\mathbb{Z}G$-factor of A.

<u>Proof</u>: If A has an irreducible $\mathbb{Z}G$-factor being a finite (resp. infinite) p-group, then by the noetherian condition we have A contains an irreducible $\mathbb{Z}G$-factor, say B/C, such that the irreducible $\mathbb{Z}G$-submodule B/C of A/C is a finite (resp. infinite) p-group and the factor-module A/C/B/C ($\cong_{\mathbb{Z}G}$ A/B) has no irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) p-groups. By Proposition **3.8**, we have A/C and hence A contains the required $\mathbb{Z}G$-images.

Comparing with Corollary **2.4.3** and Corollary **2.4.4**, we see that Proposition **3.8** and Corollary **3.9** are generalizations of these results in the locally soluble case.

An important step in proving Theorem **A** is the following reduction result.

Proposition **3.10**: Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module. If A has no f–decomposition, then A has a nonzero $\mathbb{Z}G$-image $\overline{A}$ satisfying:

(*a*) $\overline{A}$ has no f–decomposition;

(*b*) for every nonzero $\mathbb{Z}G$-submodule $\overline{C}$ of $\overline{A}$, $\overline{A}/\overline{C}$ has an f–decomposition;

(*c*) $\overline{A}$ has no nonzero $\mathbb{Z}G$-submodules with an f–decomposition;

(*d*) $\overline{A}$ is torsion-free; and

(*e*) the finite irreducible $\mathbb{Z}G$-factors of $\overline{A}$ are all *p*-groups for some fixed prime *p*.

Proof: Since A has no f–decomposition, then by the noetherian condition there is a nonzero $\mathbb{Z}G$-image $\overline{A}$ satisfying the conditions (*a*) and (*b*).

For $\overline{A}$, suppose $\overline{B} \leq \overline{A}$ and $\overline{B}$ has an f–decomposition, i.e., $\overline{B} = \overline{B}^f \oplus \overline{B}^{\bar{f}}$. If $\overline{B}^f \neq 0$, then since $\overline{B}^f$ is a nonzero $\mathbb{Z}G$-submodule of $\overline{A}$, by (*b*), $\overline{A}/\overline{B}^f$ has an f–decomposition. Let $\overline{A}/\overline{B}^f = \overline{A}_1/\overline{B}^f \oplus \overline{A}_2/\overline{B}^f$, where $\overline{A}_1$ is a $\mathbb{Z}G$-submodule of $\overline{A}$ such that $\overline{A}_1 \geq \overline{B}^f$ and $\overline{A}_1/\overline{B}^f = (\overline{A}/\overline{B}^f)^f$ and the $\mathbb{Z}G$-submodule $\overline{A}_2$ ($\geq \overline{B}^f$) is such that $\overline{A}_2/\overline{B}^f = (\overline{A}/\overline{B}^f)^{\bar{f}}$. By Proposition **3.1**, $\overline{A}_2 = \overline{B}^f \oplus \overline{C}$ for some $\mathbb{Z}G$-submodule $\overline{C}$ whose irreducible $\mathbb{Z}G$-factors are all infinite. And then it is clear that $\overline{A} = \overline{A}_1 \oplus \overline{C}$, where evidently each irreducible $\mathbb{Z}G$-factor of $\overline{A}_1$ is finite. That is, $\overline{A}$ has an f–decomposition, contrary to (*a*). So $\overline{B}^f = 0$. Similarly, by applying Proposition **3.2**, we can prove $\overline{B}^{\bar{f}} = 0$. So $\overline{B} = 0$. Therefore the condition (*c*) is satisfied by $\overline{A}$.

If $\overline{A}$ does not satisfy the condition (*d*) or the condition (*e*), then by (*c*), we may assume that $\overline{A}$ is either an elementary abelian *p*-group for some prime *p* or a torsion-free group which contains at least two nonzero finite irreducible $\mathbb{Z}G$-factors, one being a *p*-group and the other a *q*-group, where *p* and *q* are two distinct primes.

We may also assume that G acts faithfully on $\overline{A}$, i.e., $C_G(\overline{A})=1$. By Corollary **3.9**, we may assume that $\overline{A}$ contains two maximal $\mathbb{Z}G$-submodules $\overline{M}_1$ and $\overline{M}_2$ such that $\overline{A}/\overline{M}_1$ is a nontrivial finite *p*-group and $\overline{A}/\overline{M}_2$ is a nontrivial finite *r*-group, where $r = p$ or q according to $\overline{A}$ being torsion or torsion-free. Let $\overline{M} = \overline{M}_1 \cap \overline{M}_2$ and let $H = C_G(\overline{A}/\overline{M})$, then $|G/H| < \infty$. For $C = C_{\overline{M}}(H)$, if $C \neq 0$, let $0 \neq a \in C$, then the $\mathbb{Z}G$-submodule $<a>^G$ is nonzero and each of its irreducible $\mathbb{Z}G$-factors is finitely generated as an abelian group and hence is finite, contrary to the condition (*c*). So $C = 0$. By Lemma **1.2.14**, there exists a $K \leq H$ and a nonzero $\mathbb{Z}G$-submodule $\overline{M}^* \leq \overline{M}$ such that K is normal in G, $C_{\overline{M}^*}(K) = 0$, and $|K/C_K(\overline{M}^*)| < \infty$.

We show that $C_K(\overline{M}^*) = 1$. If not, then $C_K(\overline{M}^*)$ contains a nontrivial finite subgroup F being normal in G. Since F is not contained in $C_G(\overline{A}) = 1$ and $\overline{A}/\overline{M}^*$ has an f–decomposition, by Lemma **2.3.1**, $\overline{A}$ has a nonzero $\mathbb{Z}G$-submodule with an f–decomposition, a contradiction. So $C_K(\overline{M}^*) = 1$ and then K is finite.

Choose K to be minimal with respect to $C_{\overline{M}^*}(K) = 0$. If K is not a minimal normal subgroup of G, then K contains a nontrivial proper subgroup K_1 being normal in G. By the minimality of K, we have $C_{\overline{M}^*}(K_1) \neq 0$. Let $\overline{M}^{**} = C_{\overline{M}^*}(K_1)$, then $1 \neq K_1 \leq C_K(\overline{M}^{**})$. As in the last paragraph, we obtain a contradiction. So K must be a minimal normal subgroup of G. Since G is locally soluble, K is an elementary abelian *k*-group for some prime *k*.

In order to apply Lemma **2.4.7**, we should consider that $k \neq p$ or $k \neq q$

in the case that $\overline{A}$ is torsion-free; or we have $k \neq p$ by using $C_{\overline{M^*}}(K) = 0$ in the case that $\overline{A}$ is a p-group (Lemma **1.2.7**). Now from Lemma **2.4.7**we have $\overline{A} = C_{\overline{A}}(K)+\overline{M}_i$, where $\overline{M}_i = \overline{M}_1$ or $\overline{M}_2$ according to $k \neq p$ or $k \neq q$. Since $\overline{M}_i < \overline{A}$, so $C_{\overline{A}}(K) \neq 0$. But, since $\overline{M}^* \cap C_{\overline{A}}(K) = C_{\overline{M^*}}(K) = 0$, we have $C_{\overline{A}}(K) \cong_{\mathbb{Z}G} (C_{\overline{A}}(K) \oplus \overline{M}^*)/\overline{M}^* \leq \overline{A}/\overline{M}^*$ and then, since $\overline{A}/\overline{M}^*$ has an f–decomposition, each $\mathbb{Z}G$-submodule of $\overline{A}/\overline{M}^*$ has an f–decomposition (Lemma **1.2.23**) and hence so does $C_{\overline{A}}(K)$, a contradiction. So we have, in fact, proved that $\overline{A}$ satisfies the conditions (*d*) and (*e*).

The result is proved.

From Proposition **3.10** and its proof, we can get a number of results which assert that the noetherian $\mathbb{Z}G$-module A has an f–decomposition under certain conditions. Among these, the simplest and most useful one is:

<u>Corollary **3.11**</u>: If G is a hyperfinite locally soluble group, then any periodic noetherian $\mathbb{Z}G$-module A has an f–decomposition.

By Proposition **3.10**, the task for proving Theorem **A** is now reduced to considering torsion-free noetherian $\mathbb{Z}G$-modules. For such modules, we need to prove the following

<u>Proposition **3.12**</u>: Let G be a hyperfinite locally soluble group, A a torsion-free noetherian $\mathbb{Z}G$-module, and $A_{ii} = p^i A$, where p is a prime and $i=0,1,2,\cdots$. Then

(1) for any $0 \leq j < i$, A_{jj}/A_{ii} has an f–decomposition

$$A_{jj}/A_{ii} = A_{ij}/A_{ii} \oplus A_{ji}/A_{ii},$$

where A_{ij} is the $\mathbb{Z}G$-submodule of A_{jj} such that $A_{ij} \geq A_{ii}$ and $A_{ij}/A_{ii} = (A_{jj}/A_{ii})^f$

and the $\mathbb{Z}G$-submodule $A_{ji}(\geq A_{ii})$ is such that $A_{ji}/A_{ii} = (A_{jj}/A_{ii})^{\bar{f}}$;

(2) $A_{ij} \leq A_{ik}$ and $A_{ij} \leq A_{sj}$, $\forall k \leq j$ and $\forall s \leq i$;

(3) $A_{ii} = A_{ik} \cap A_{si}$, where $k \leq i$, $s \leq i$, and $i = 0,1,2,\cdots$;

(4) $A_{ij} = A_{ik} \cap A_{sj}$, where $k \leq j$, $s \leq i$, and $i,j = 0,1,2,\cdots$;

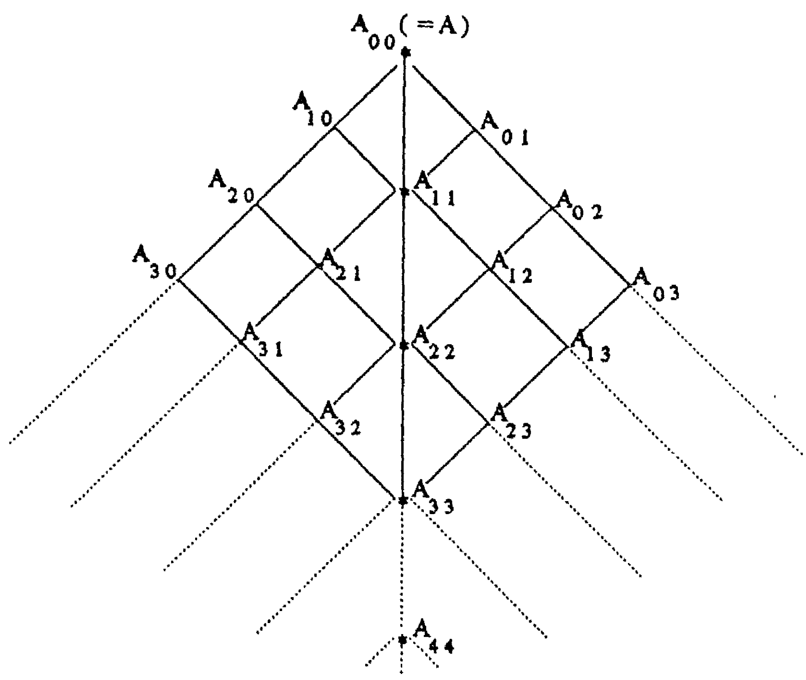
(5) $A_{ij}/A_{kk} = A_{kj}/A_{kk} \oplus A_{ik}/A_{kk}$, $\quad A_{kj}/A_{kk} = (A_{ij}/A_{kk})^{f}$, and $A_{ik}/A_{kk} = (A_{ij}/A_{kk})^{\bar{f}}$, where $k \geq i$, j;

(6) $A_{ij}/A_{sk} = A_{sj}/A_{sk} \oplus A_{ik}/A_{sk}$, $\quad A_{sj}/A_{sk} = (A_{ij}/A_{sk})^{f}$, and $A_{ik}/A_{sk} = (A_{ij}/A_{sk})^{\bar{f}}$, where $k \geq j$, $s \geq i$, and i, $j = 0,1,2,\cdots$;

(7) $A_{ij}/A_{i,j+t} \cong_{\mathbb{Z}G} A_{ks}/A_{k,s+t}$ and $A_{ij}/A_{i+t,j} \cong_{\mathbb{Z}G} A_{ks}/A_{k+t,s}$

where i, j, k, s, $t = 0,1,2,\cdots$;

(8) $p^k A_{ij} = A_{i+k,j+k}$, $\quad i$, j, $k = 0,1,2,\cdots$.



<u>Proof</u>: (1) By Lemma 2.1.4, $A_{11} = pA < A$, and then $A_{ii} < A_{jj}$ for any $0 \leq j < i$. Since

A is noetherian, so A_{jj}/A_{ii} is noetherian and is also periodic. Thus, by Corollary **3.11**, A_{jj}/A_{ii} has an f–decomposition, i.e.,

$$A_{jj}/A_{ii} = A_{ij}/A_{ii} \oplus A_{ji}/A_{ii},$$

where A_{ij} is the $\mathbb{Z}G$-submodule of A_{jj} such that $A_{ij} \geq A_{ii}$ and $A_{ij}/A_{ii} = (A_{jj}/A_{ii})^{f}$ and the $\mathbb{Z}G$-submodule $A_{ji}(\geq A_{ii})$ is such that $A_{ji}/A_{ii} = (A_{jj}/A_{ii})^{\bar{f}}$.

(2) We prove $A_{ij} \leq A_{ik}$ for any $k \leq j$ and $i,j=0,1,2,\cdots$.

If $j \leq i$, then $A_{ii} \leq A_{jj} \leq A_{kk}$ for any $k \leq j$. Thus $A_{jj}/A_{ii} \leq A_{kk}/A_{ii}$ and then $A_{ij}/A_{ii} \leq A_{ik}/A_{ii}$. That is, $A_{ij} \leq A_{ik}$.

If $j > i$, then if $k \leq i$, we have $A_{ik} \geq A_{ii} \geq A_{ij}$; so we may assume that $i < k \leq j$. Thus $A_{ii} \geq A_{ik} \geq A_{kk} \geq A_{jj}$. Since A_{ii}/A_{jj} has an f–decomposition, we have A_{ik}/A_{jj} has an f–decomposition (Lemma **1.2.23**). Let B be the $\mathbb{Z}G$-submodule of A_{ik} such that $B \geq A_{jj}$ and $B/A_{jj} = (A_{ik}/A_{jj})^{\bar{f}}$, then $B \leq A_{ii}$ and every irreducible $\mathbb{Z}G$-factor of A_{ii}/B is isomorphic either with one of the irreducible $\mathbb{Z}G$-factors of A_{ii}/A_{ik} or with one of those of A_{ik}/B and so is finite. So we will have $B = A_{ij}$ and then $A_{ik} \geq A_{ij}$. Thus, we have proved that $A_{ij} \leq A_{ik}$ for any $k \leq j$.

Similarly, we have $A_{ij} \leq A_{sj}$, $\forall\ s \leq i$.

(3) By (2), $A_{ii} \leq A_{ik} \cap A_{si}$ for any $k \leq i$ and any $s \leq i$.

Since $(A_{ik} \cap A_{si})/A_{ii} \leq A_{ik}/A_{ii} = (A_{kk}/A_{ii})^{f}$ and $(A_{ik} \cap A_{si})/A_{ii} \leq A_{si}/A_{ii} = (A_{ss}/A_{ii})^{\bar{f}}$, so $(A_{ik} \cap A_{si})/A_{ii}$ is trivial. That is, $A_{ii} = A_{ik} \cap A_{si}$

for any $k,s \le i$, and $i=0,1,2,\cdots$.

(4) By (2), $A_{ij} \le A_{ik} \cap A_{sj}$ for any $k \le j$ and any $s \le i$.

On the other hand, we let $j \le i$, then by

$$(A_{ik} \cap A_{jj})/A_{ii} \le A_{ik}/A_{ii} \cap A_{jj}/A_{ii} \le (A_{jj}/A_{ii})^{f} = A_{ij}/A_{ii},$$

we have $A_{ik} \cap A_{jj} \le A_{ij}$. For $s \le i$, if $s<j$, then

$$A_{ik} \cap A_{sj} = (A_{ik} \cap A_{jk}) \cap A_{sj} = A_{ik} \cap (A_{jk} \cap A_{sj}) = A_{ik} \cap A_{jj} \le A_{ij};$$

and if $j \le s \le i$, then $A_{ik} \cap A_{sj} \le A_{ik} \cap A_{jj} \le A_{ij}$. That is, $A_{ij} \ge A_{ik} \cap A_{sj}$ for $j \le i$, $k \le j$, and $s \le i$. Similarly, we have $A_{ij} \ge A_{ik} \cap A_{sj}$ for $j>i$, $k \le j$, and $s \le i$. Thus $A_{ij} = A_{ik} \cap A_{sj}$ for any $k \le j$ and any $s \le i$.

(5) Suppose $i \ge j$, then $A_{jj} \ge A_{ij} \ge A_{ii} \ge A_{kk}$, where $k \ge i$. Since A_{jj}/A_{kk} has an f–decomposition, we have A_{ij}/A_{kk} has an f–decomposition. Let $A_{ij}/A_{kk} = B/A_{kk} \oplus C/A_{kk}$, in which B is the $\mathbb{Z}G$-submodule of A_{ij} such that $B \ge A_{kk}$ and $B/A_{kk} = (A_{ij}/A_{kk})^{f}$ and the $\mathbb{Z}G$-submodule C ($\ge A_{kk}$) is such that $C/A_{kk} = (A_{ij}/A_{kk})^{\bar{f}}$. Then, by Lemma **1.2.23**, $B \le A_{kj}$ and $C \le A_{ik}$. Since $A_{ij}/A_{kk} \ge (A_{jj}/A_{kk})^{f}$ and A_{ij}/B has no finite irreducible $\mathbb{Z}G$-factors, so $B = A_{kj}$. Meanwhile, by $A_{ij}/A_{kk} \ge A_{ik}/A_{kk} = (A_{ii}/A_{kk})^{\bar{f}}$ and A_{ij}/C having no infinite irreducible $\mathbb{Z}G$-factors, we have $C = A_{ik}$. Thus

$$A_{ij}/A_{kk} = A_{kj}/A_{kk} \oplus A_{ik}/A_{kk} \qquad \text{for any } k \ge i \ge j.$$

Similarly, the result is true for $k \ge j > i$.

(6) Suppose $i \leq j$, then, by (2), $A_{ij} \geq A_{sj}$ and $A_{ij} \geq A_{ik}$ for any $s \geq i$ and any $k \geq j$. Thus $A_{ij} \geq A_{sj} + A_{ik}$. By (5), if $s \geq k$, then $A_{ij} = A_{sj} + A_{is} \leq A_{sj} + A_{ik}$, and if $s < k$, then $A_{ij} = A_{kj} + A_{ik} \leq A_{sj} + A_{ik}$. So $A_{ij} = A_{sj} + A_{ik}$.

By (4), $A_{sk} = A_{sj} \cap A_{ik}$, so $A_{ij}/A_{sk} = A_{sj}/A_{sk} \oplus A_{ik}/A_{sk}$, where $k \geq j$, $s \geq i$, and $j \geq i = 0,1,2,\cdots$.

For A_{sj}/A_{sk}: *(i)* if $s \geq k(\geq j)$, then $A_{sj} \geq A_{sk} \geq A_{ss}$ and, since each irreducible $\mathbb{Z}G$-factor of A_{sj}/A_{ss} is finite, we have each irreducible $\mathbb{Z}G$-factor of A_{sj}/A_{sk} is finite; *(ii)* if $k \geq s \geq j$, then $A_{sj} \geq A_{ss} \geq A_{sk}$ and, by each irreducible $\mathbb{Z}G$-factor of A_{sj}/A_{sk} is isomorphic to one of the irreducible $\mathbb{Z}G$-factors of A_{sj}/A_{ss} or one of that of A_{ss}/A_{sk}, we have A_{sj}/A_{sk} contains only finite irreducible $\mathbb{Z}G$-factors; and *(iii)* if $(k \geq)j \geq s$, then $A_{ss} \geq A_{sj} \geq A_{sk}$ and, since each irreducible $\mathbb{Z}G$-factor of A_{ss}/A_{sk} is finite, we have any irreducible $\mathbb{Z}G$-factor of A_{sj}/A_{sk} is finite. Thus $A_{sj}/A_{sk} \leq (A_{ij}/A_{sk})^{f}$. Similarly, we have $A_{ik}/A_{sk} \leq (A_{ij}/A_{sk}^{\bar{f}})$. Therefore, $A_{sj}/A_{sk} = (A_{ij}/A_{sk}^{f})$ and $A_{ik}/A_{sk} = (A_{ij}/A_{sk})^{\bar{f}}$.

For $i > j$, the proof is similar.

(7) We only consider the case in which $i \leq j$, $k \leq s$, and $k \leq i$ (as we can similarly prove the other cases).

By (5), we have

$$A_{ij}/A_{j+t,j+t} = A_{j+t,j}/A_{j+t,j+t} \oplus A_{i,j+t}/A_{j+t,j+t}, \qquad \text{and}$$

$$A_{ks}/A_{s+t,s+t} = A_{s+t,s}/A_{s+t,s+t} \oplus A_{k,s+t}/A_{s+t,s+t}.$$

Thus $A_{ij}/A_{i,j+t} \cong_{\mathbb{Z}G} A_{j+t,j}/A_{j+t,j+t}$ and $A_{ks}/A_{k,s+t} \cong_{\mathbb{Z}G} A_{s+t,s}/A_{s+t,s+t}$.

By $A_{jj}/A_{j+t,j+t} \cong_{\mathbb{Z}G} A_{ss}/A_{s+t,s+t}$ (Corollary **2.1.3**), we have

$$A_{j+t,j}/A_{j+t,j+t} = (A_{jj}/A_{j+t,j+t})^f \cong_{\mathbb{Z}G} (A_{ss}/A_{s+t,s+t})^f = A_{s+t,s}/A_{s+t,s+t}.$$

Thus $A_{ij}/A_{i,j+t} \cong_{\mathbb{Z}G} A_{ks}/A_{k,s+t}$.

Similarly, we have $A_{ij}/A_{i+t,j} \cong_{\mathbb{Z}G} A_{ks}/A_{k+t,s}$.

(8) By induction, we only need to prove $pA_{ij}=A_{i+1,j+1}$ for any i, $j \geq 0$.

Let $i \geq j$; by (6), $A_{ij}/A_{i+1,j+1} = A_{i+1,j}/A_{i+1,j+1} \oplus A_{i,j+1}/A_{i+1,j+1}$,

and by (7), $A_{i+1,j}/A_{i+1,j+1} \cong_{\mathbb{Z}G} A_{00}/A_{01}$ and $A_{i,j+1}/A_{i+1,j+1} \cong_{\mathbb{Z}G} A_{00}/A_{10}$. So

$$\begin{aligned} p(A_{ij}/A_{i+1,j+1}) &= p(A_{i+1,j}/A_{i+1,j+1}) \oplus p(A_{i,j+1}/A_{i+1,j+1}) \\ &\cong_{\mathbb{Z}G} p(A_{00}/A_{01}) \oplus p(A_{00}/A_{10}) = 0. \end{aligned}$$

That is, $pA_{ij} \leq A_{i+1,j+1}$.

On the other hand, let $a \in A_{i+1,j+1}$, then $a \in A_{j+1,j+1} \setminus A_{i+1,i+1}$. Thus $a=pb$ for some $b \in A_{jj} \setminus A_{ii}$. If $b \notin A_{ij}$, then $(<b>^G+A_{ii})/A_{ii}$ is not contained in A_{ij}/A_{ii} and then $(<b>^G+A_{ii})/A_{ii}$ contains infinite irreducible $\mathbb{Z}G$-factors. Since

$$\begin{aligned} A_{i+1,j+1}/A_{i+1,i+1} &\geq (<a>^G+A_{i+1,i+1}) / A_{i+1,i+1} \\ &= (p<b>^G+A_{i+1,i+1}) / A_{i+1,i+1} \\ &\cong_{\mathbb{Z}G} p<b>^G / (p<b>^G \cap A_{i+1,i+1}) \\ &= p<b>^G / p(<b>^G \cap A_{ii}) && \text{(as A is torsion-free)} \\ &\cong_{\mathbb{Z}G} <b>^G / (<b>^G \cap A_{ii}) && \text{(Lemma 2.1.1)} \\ &\cong_{\mathbb{Z}G} (<b>^G+A_{ii}) / A_{ii}, \end{aligned}$$

we have $A_{i+1,j+1}/A_{i+1,i+1}$ contains infinite irreducible $\mathbb{Z}G$-factors, a contradiction. So $b\in A_{ij}$ and then $a\in pA_{ij}$. Thus $A_{i+1,j+1}\leq pA_{ij}$. Therefore, $pA_{ij}=A_{i+1,j+1}$ for $i\geq j$.

Similarly, we have $pA_{ij}=A_{i+1,j+1}$ for $i<j$. Thus $pA_{ij}=A_{i+1,j+1}$, for any $i,j\geq 0$.

Furthermore, let $A_{\infty i} = \bigcap_j A_{ji}$ and $A_{i\infty} = \bigcap_j A_{ij}$ for $i=0,1,2,\cdots$, then by applying Proposition **3.12**, we can prove the result which will be very important in the following critical proof for Theorem **A**.

<u>Proposition 3.13</u>: Under the hypothesis of Proposition **3.12** and the notation above, one has:

(*a*) $p^k A_{\infty i} = A_{\infty,i+k}$ and $p^k A_{i\infty} = A_{i+k,\infty}$, $i,k=0,1,2,\cdots$;

(*b*) $A_{\infty k} = A_{\infty j}\cap A_{ik}$ and $A_{k\infty} = A_{j\infty}\cap A_{ki}$, $k\geq j$, and $i=0,1,2,\cdots$;

(*c*) $A_{\infty j}/A_{\infty k} \cong_{\mathbb{Z}G} (A_{\infty j}+A_{kk})/A_{kk} \leq A_{kj}/A_{kk}$ and

$A_{j\infty}/A_{k\infty} \cong_{\mathbb{Z}G} (A_{j\infty}+A_{kk})/A_{kk} \leq A_{jk}/A_{kk}$, $k\geq j=0,1,2,\cdots$; and

(*d*) $A_{i\infty}$ (resp. $A_{\infty i}$) has no finite (resp. infinite) irreducible $\mathbb{Z}G$-factors being p-groups, $i=0,1,2,\cdots$.

<u>Proof</u>: (*a*) We only prove $pA_{\infty i}=A_{\infty,i+1}$ for $i=0,1,\cdots$.

By (8) in Proposition **3.12**, $pA_{ji} = A_{j+i,i+1}$, so

$$pA_{\infty i} = p(\bigcap_j A_{ji}) = \bigcap_j (pA_{ji}) \qquad \text{(as A is torsion-free)}$$

$$= \bigcap_j A_{j+1,i+1} = \bigcap_j A_{j,i+1} = A_{\infty,i+1}.$$

(*b*) For $k \geq j \geq 0$, using (6) in Proposition **3.12**, we have

$$A_{\infty j} \cap A_{ik} = (\bigcap_{s>i} A_{sj}) \cap A_{ik} = \bigcap_{s>i} (A_{sj} \cap A_{ik}) = \bigcap_{s} A_{sk} = A_{\infty k}.$$

Similarly, we have $A_{k\infty} = A_{j\infty} \cap A_{ki}$ $k \geq j \geq 0$, $i=0,1,\cdots$.

(*c*) By (*b*), we have $A_{\infty k} = A_{\infty j} \cap A_{kk}$ for any $k \geq j \geq 0$. Also, it is clear that $A_{\infty j} + A_{kk} \leq A_{kj}$, thus

$$A_{\infty j}/A_{\infty k} = A_{\infty j}/(A_{\infty j} \cap A_{kk}) \cong_{\mathbb{Z}G} (A_{\infty j} + A_{kk})/A_{kk} \leq A_{kj}/A_{kk}, \qquad k \geq j \geq 0.$$

Similarly, $A_{j\infty}/A_{k\infty} \cong_{\mathbb{Z}G} (A_{j\infty} + A_{kk})/A_{kk} \leq A_{jk}/A_{kk}, \qquad k \geq j \geq 0.$

(*d*) By (*a*) and (*c*), we have

$$A_{i\infty}/pA_{i\infty} = A_{i\infty}/A_{i+1,\infty} \cong_{\mathbb{Z}G} (A_{i\infty} + A_{i+1,i+1})/A_{i+1,i+1} \leq A_{i,i+1}/A_{i+1,i+1}.$$

Since $A_{i,i+1}/A_{i+1,i+1}$ has no finite irreducible $\mathbb{Z}G$-factors, we have $A_{i\infty}/pA_{i\infty}$ has no finite irreducible $\mathbb{Z}G$-factors. Since $A_{i\infty}$ is also a noetherian $\mathbb{Z}G$-module, by Corollary **3.9**, $A_{i\infty}$ has no finite irreducible $\mathbb{Z}G$-factors being p-groups ($i=0,1,2,\cdots$).

Similarly, we have $A_{\infty i}$ has no infinite irreducible $\mathbb{Z}G$-factors being p-groups, $i=0,1,2,\cdots$.

The Proposition **3.13** is proved.

Now the critical proof for Theorem **A** is coming. It enables us to deal with those modules which remain after Proposition **3.10**.

<u>Proposition **3.14**</u>: Let G be a hyperfinite locally soluble group and A a

noetherian $\mathbb{Z}G$-module. If all finite irreducible $\mathbb{Z}G$-factors of A are p-groups for some fixed prime p, then A has an f–decomposition.

Proof: Suppose that A does not have an f–decomposition, then by Proposition **3.10**, we may further assume that A satisfies the following conditions:

(*a*) for every nonzero $\mathbb{Z}G$-submodule C of A, A/C has an f–decomposition;

(*b*) A is torsion-free; and

(*c*) A has no nonzero $\mathbb{Z}G$-submodules with an f–decomposition.

Furthermore, we assume that G acts faithfully on A, i.e., $C_G(A)=1$.

For the prime p, by Lemma **2.1.4**, $pA<A$ (and then $p^{i+1}A<p^iA$ for any integer i) and $\bigcap_i p^iA=0$. Applying Corollary **3.11**, we have p^jA/p^iA has an f–decomposition for any integers $0\leq j<i$. Let $A_{kk}=p^kA$ for any integer $k\geq 0$ and, for any $0\leq j<i$, let $A_{jj}/A_{ii} = A_{ij}/A_{ii} \oplus A_{ji}/A_{ii}$, where A_{ij} is the $\mathbb{Z}G$-submodule of A_{jj} such that $A_{ij}/A_{ii} = (A_{jj}/A_{ii})^f$ and the $\mathbb{Z}G$-submodule $A_{ji}(\geq A_{ii})$ is such that $A_{ji}/A_{ii} = (A_{jj}/A_{ii})^{\bar{f}}$. Since A does not have an f–decomposition, it does have finite irreducible $\mathbb{Z}G$–factors. Together with the hypothesis of the proposition, this shows that A contains finite irreducible $\mathbb{Z}G$-factors being p-groups for the prime p; then by Corollary **3.9**, A contains irreducible $\mathbb{Z}G$-images being finite p-groups. Thus A_{00}/A_{01} ($\cong_{\mathbb{Z}G} A_{10}/A_{11}$) is nonzero and then, by Lemma **2.4.5**, $|A_{00}/A_{01}|<\infty$.

Let $H = C_G(A_{00}/A_{01})$, then $|G/H|<\infty$. Consider A as a $\mathbb{Z}H$-module and then, by Lemma **1.2.5**, A is noetherian and, by Corollary **2.2.10**, A has no f–($\mathbb{Z}$H)–decomposition. By Proposition **3.10**, there is a $\mathbb{Z}$H–image A* (=A/C, where C is a $\mathbb{Z}$H–submodule) of A such that

(*a*) A* has no f–($\mathbb{Z}$H)–decomposition;

(*b*) for every nonzero $\mathbb{Z}$H–submodule D* of A*, A*/D* has an f–($\mathbb{Z}$H)–decomposition;

(*c*) A* has no nonzero $\mathbb{Z}$H–submodules with an f–($\mathbb{Z}$H)–decomposition; and

(*d*) A* is torsion-free.

Since the finite irreducible $\mathbb{Z}$G–factors of A are all p-groups for the prime p, applying (6) in Lemma **2.2.4**, the finite irreducible $\mathbb{Z}$H–factors of A, and hence those of A* too, are all p-groups for the prime p.

As above, for A*, we have $pA^* < A^*$ (and then $p^{i+1}A^* < p^iA^*$ for any integer i) and $\bigcap_i p^iA^*=0$. For integers $k\geq 0$ and $0\leq j<i$, let $A^*_{kk}=p^kA^*$ and let $A^*_{jj}/A^*_{ii} = A^*_{ij}/A^*_{ii} \oplus A^*_{ji}/A^*_{ii}$, where A^*_{ij} is the $\mathbb{Z}$H–submodule of A^*_{jj} such that $A^*_{ij}/A^*_{ii} = (A^*_{jj}/A^*_{ii})^f$ and the $\mathbb{Z}$H–submodule $A^*_{ji}(\geq A^*_{ii})$ is such that $A^*_{ji}/A^*_{ii} = (A^*_{jj}/A^*_{ii})^{\bar{f}}$. Then A^*_{00}/A^*_{01} ($\cong_{\mathbb{Z}H} A^*_{10}/A^*_{11}$) is nonzero and $|A^*_{00}/A^*_{01}| < \infty$.

Since
$$A^*_{00} = A^* = A/C = A_{00}/C,$$
$$A^*_{11} = pA^* = p(A/C) = (pA+C)/C = (A_{11}+C)/C,$$
$$(A_{10}+C)/(A_{11}+C) \cong_{\mathbb{Z}H} A_{10}/(A_{11}+(A_{10}\cap C)), \qquad \text{and}$$
$$(A_{01}+C)/(A_{11}+C) \cong_{\mathbb{Z}H} A_{01}/(A_{11}+(A_{01}\cap C)),$$
so $A^*_{00}/A^*_{11} \cong_{\mathbb{Z}H} A_{00}/(A_{11}+C) = (A_{10}+C)/(A_{11}+C) \oplus (A_{01}+C)/(A_{11}+C)$. And also

$$A^*_{00} = A/C = (A+C)/C = (A_{10}+C)/C + (A_{01}+C)/C, \quad \text{and}$$

$$(A_{10}+C)/C \cap (A_{01}+C)/C = \big((A_{10}+C)\cap(A_{01}+C)\big)/C = (A_{11}+C)/C = A^*_{11}.$$

Thus $A^*_{01} = (A_{01}+C)/C$, and then

$$H = C_G(A_{00}/A_{01}) = C_H(A_{00}/A_{01}) \leq C_H(A_{00}/A_{01}+C) = C_H(A^*_{00}/A^*_{01}).$$

That is, $H = C_H(A^*_{00}/A^*_{01})$. Also, by Proposition **3.12**, $A^*_{00}/A^*_{01} \cong_{\mathbb{Z}H} A^*_{ij}/A^*_{i,j+1}$, so $H = C_H(A^*_{00}/A^*_{01}) = C_H(A^*_{ij}/A^*_{i,j+1})$ for any $i, j \geq 0$. By Proposition **3.13**, the $\mathbb{Z}H$-submodule $A^*_{0\infty}$ has no finite irreducible $\mathbb{Z}H$-factors being p-groups and then has no finite irreducible $\mathbb{Z}H$-factors. Since A^* has no nonzero $\mathbb{Z}H$-submodules with an f–decomposition, we must have $A^*_{0\infty} = 0$.

Replacing H by $H/C_H(A^*)$ we may assume that H acts faithfully on A^*. Now we will obtain a contradiction by proceeding in the following four steps:

(*i*) $H = C_H(A^*_{00}/A^*_{01})$ is a p–group for the prime p.

Suppose $x \in H$ and x has order q for some prime q. Since $x \notin C_H(A^*_{00}) = 1$, there exists $a_0 \in A^*_{00}$ such that $a_0x \neq a_0$. By $\bigcap_i A^*_{0i} = A^*_{0\infty} = 0$, we have $a_0 \in A^*_{0,i_0} \setminus A^*_{0,i_0+1}$ for some integer $i_0 \geq 0$. Since $A^*_{00}/A^*_{01} \cong_{\mathbb{Z}H} A^*_{0,i_0}/A^*_{0,i_0+1}$, $x \in H = C_H(A^*_{0,i}/A^*_{0,i+1})$ for any i, so $a_0x = a_0 + a_1 = \sum_{j=0}^{1} \binom{1}{j} a_j$, where $a_1 \neq 0$ and $a_1 \in A^*_{0,i_1} \setminus A^*_{0,i_1+1}$ for some integer $i_1 > i_0$. If $a_1x = a_1$, then $a_0 = a_0x^q = a_0 + qa_1$. That is, $qa_1 = 0$ and then $a_1 = 0$, a contradiction. So $a_1x \neq a_1$. Suppose $a_0x^r = \sum_{j=0}^{r} \binom{r}{j} a_j$, where for any j, $a_j \in A^*_{0,i_j} \setminus A^*_{0,i_j+1}$, $a_jx \neq a_j$, and $a_{j-1}x = a_{j-1} + a_j$, $i_r > i_{r-1} > \cdots > i_1 > i_0 \geq 0$. By $a_rx \neq a_r$ and

$x \in H = C_H(A^*_{0,i_r}/A^*_{0,i_r+1})$, we have $a_r x = a_r + a_{r+1}$, where $a_{r+1} \in A^*_{0,i_{r+1}} \backslash A^*_{0,i_{r+1}+1}$ and $i_{r+1} > i_r$. As above we also have $a_{r+1}x \neq a_{r+1}$.

Now we have:

$$\begin{aligned} a_0 x^{r+1} &= (a_0 x^r) = [\textstyle\sum_{j=0}^{r} \binom{r}{j} a_j]\, x = \sum_{j=0}^{r} \binom{r}{j}(a_j x) \\ &= \textstyle\sum_{j=0}^{r} \binom{r}{j}(a_j + a_{j+1}) = a_0 + \sum_{j=1}^{r} [\binom{r}{j-1} + \binom{r}{j}] a_j + a_{r+1} \\ &= a_0 + \textstyle\sum_{j=1}^{r} \binom{r+1}{j} a_j + a_{r+1} = \sum_{j=0}^{r+1} \binom{r+1}{j} a_j . \end{aligned}$$

Hence $a_0 = a_0 x^q = \sum_{j=0}^{q} \binom{q}{j} a_j = a_0 + \sum_{j=1}^{q} \binom{q}{j} a_j$. That is,

$$0 = \sum_{j=1}^{q} \binom{q}{j} a_j = q\, [a_1 + \sum_{j=2}^{q-1} \frac{1}{q} \binom{q}{j} a_j] + a_q . \qquad (*)$$

If $q \neq p$, by $q = kp + t$ for some t with $0 < t < p$, we have $ta_1 = -p(ka_1) - \sum_{j=2}^{q} \binom{q}{j} a_j \in (A^*_{1,i_1+1} + A^*_{0,i_2}) \leq A^*_{0,i_1+1}$, contrary to the abelian p-group $A^*_{0,i_1}/A^*_{0,i_1+1}$ being elementary. So $p = q$ and then H must be a p-group.

(*ii*) $p = 2$.

Now from (*), we have

$$p[a_1 + \sum_{j=2}^{p-1} \frac{1}{p} \binom{p}{j} a_j] \in pA^*_{00} \cap A^*_{0,i_p} = A^*_{11} \cap A^*_{0,i_p} = A^*_{1,i_p} = pA^*_{0,i_p-1}.$$

Since A^* is torsion-free, $a_1 + \sum_{j=2}^{p-1} \frac{1}{p} \binom{p}{j} a_j \in A^*_{0,i_p-1}$ If $p > 2$, then $i_p - 1 \geq i_3 - 1 \geq i_2 \geq i_1 + 1$. Thus $a_1 + \sum_{j=2}^{p-1} \frac{1}{p} \binom{p}{j} a_j \in A^*_{0,i_2}$ and then $a_1 \in A^*_{0,i_2} \leq A^*_{0,i_1+1}$, contrary to $a_1 \in A^*_{0,i_1} \backslash A^*_{0,i_1+1}$. Thus we must have $p = 2$.

(*iii*) Z(H) contains only one element with order 2.

Since H is a hyperfinite 2–group, by Lemma **1.2.12**, we have Z(H) is nontrivial and so contains at least one element x_0, say, with the order of x_0 being 2.

For $1 \neq x \in Z(H)$, if $C_{A*}(x) \neq 0$, then, by $A^* \neq C_{A*}(x)$, we have the $\mathbb{Z}H$-submodule $A^*(x\text{-}1)$ $(\cong_{\mathbb{Z}H}$ $A^*/C_{A*}(x))$ is nonzero and has an f–($\mathbb{Z}$H)–decomposition, contrary to A^* having no such $\mathbb{Z}$H–submodules. So $C_{A*}(x) = 0$ for any $x \in Z(H)$ with $x \neq 1$. In particular, $C_{A*}(x_0) = 0$, where $x_0 \in Z(H)$ and the order of x_0 is 2. By $A^*(x_0+1) \leq C_{A*}(x_0)$, we have $A^*(x_0+1) = 0$ and then $ax_0 = -a$ for any $a \in A^*$.

If $x \in H$ with the order of x being 2 and $C_{A*}(x) = 0$, then, since $A^*(x+1) \leq C_{A*}(x) = 0$ we have $ax = -a$ for any $a \in A^*$ and then

$$a(xx_0) = (ax)x_0 = (-a)x_0 = -ax_0 = a,$$

for any $a \in A^*$. Thus $xx_0 \in C_H(A^*) = 1$ and so $xx_0 = 1$, i.e., $x = x_0$. So it follows that Z(H) contains only one element with order 2.

(*iv*) H has no elements of order 4.

In fact, from the proof of the above in (*iii*), we have: if $x \in H$ with $x^2 = 1$ and $x \neq x_0$, then $C_{A*}(x) \neq 0$.

Let $y \in H$ and the order of y be 4, then $(y^2x_0)^2 = 1$ and $y^2x_0 \neq x_0$. Thus $C_{A*}(y^2x_0) \neq 0$. Let $0 \neq a \in C_{A*}(y^2x_0)$, since $a \in A^*_{00}$ and $\bigcap_i A^*_{0i} = A^*_{0\infty} = 0$, there exists j_1 such that $a \in A^*_{0,j_1} \backslash A^*_{0,j_1+1}$. By $y \in H = C_H(A^*_{0i}/A^*_{0,i+1})$ for any integer i, we have $ay = a + b$, where $b \in A^*_{0,j_1+1}$. Let

$b \in A^*_{0,j_2} \backslash A^*_{0,j_2+1}$ for some integer $j_2 > j_1$, then $by = b + c$, where $c \in A^*_{0,j_2+1}$. Thus, $-a = ax_0 = a(y^2x_0)y^2 = ay^2 = (a+b)y = a+2b+c$. Therefore, $2(a+b) = -c \in \left(2A^*_{00} \cap A^*_{0,j_2+1}\right) = \left(A^*_{11} \cap A^*_{0,j_2+1}\right) = A^*_{1,j_2+1} = 2A^*_{0,j_2}$. Since A^* is torsion-free, we have $a+b \in A^*_{0,j_2}$ and then $a \in A^*_{0,j_2} \leq A^*_{0,j_1+1}$, contrary to $a \in A^*_{0,j_1} \backslash A^*_{0,j_1+1}$. So H has no elements of order 4.

Now by (*i*), (*ii*) and (*iv*), H is an elementary abelian 2-group and, by (*iii*), $|H| = 2$. But, G is infinite and $|G/H| < \infty$, we must have H is infinite, a contradiction. So the result is true.

Now Theorem **A** is followed.

Theorem A: If G is a hyperfinite locally soluble group, then any noetherian $\mathbb{Z}G$-module A has an f–decomposition.

Proof: Suppose A does not have an f–decomposition, then, by applying Propositions **3.10.** and **3.14**, we will get a contradiction, So the theorem is true.

In our proof of Theorem **A**, the locally soluble condition is necessary. However, it is not a necessary condition for the result as we can see from the following results:

Corollary **A1**: If G is a hyperfinite almost locally soluble group, then any noetherian $\mathbb{Z}G$-module A has an f–decomposition. (Here almost locally soluble means (locally soluble)–by–finite.)

Proof: It follows from Lemma **1.2.5**, Corollary **2.2.10** and Theorem **A**.

A special and very important case of Corollary **A1** is that:

Corollary **A2**: If G is a Černikov group, then any noetherian $\mathbb{Z}G$-module A has an f–decomposition.

Another special case worthy of mention is:

Corollary **A3**: If G is a locally finite group satisfying the minimal condition on subgroups, then any noetherian $\mathbb{Z}G$-module A has an f–decomposition.

Proof: Since a locally finite group G satisfying the minimal condition on subgroups is almost abelian **[8]** and therefore is a Černikov group, so the result follows from Corollary **A2**.

4 THE STRUCTURE OF THE SUBMODULES

From Theorem A, we know that any noetherian $\mathbb{Z}G$-module A over a hyperfinite locally soluble group G has an f–decomposition: $A = A^{f} \oplus A^{\bar{f}}$. In this chapter, we are going to discuss the details of the structure of the submodules A^{f} and $A^{\bar{f}}$.

Because of the complicated structure of $A^{\bar{f}}$, we need first in §**4.1** to recall some knowledge of injective hull and this yields, in §**4.2**, examples of $A^{\bar{f}}$ with exponent n for any integer $n>0$. §**4.3** contains the complete results about the structure of A^{f}. In §**4.4**, we focus our attention on $A^{\bar{f}}$ again and have proved some results which look interesting. Especially, in some important cases we can prove that $A^{\bar{f}}$ must be torsion and so have finite exponent. The general question of whether $A^{\bar{f}}$ must be torsion remains open.

§4.1 INJECTIVE HULL

We follow the treatment given by B.Hartley and D.McDougall in their paper **[6]**.

Let R be a ring with 1. An R–module X is called injective if whenever $U \leq W$ are R–submodules then every R–homomorphism of U into X can be extended to W. This is equivalent (but not immediately) to the requirement that X be a direct summand of every R–module which contains it. A well-known result is that:

Proposition **4.1.1**: (Hartley, **[5]**) Let K be a field of characteristic $p \geq 0$ and H a countable group. Every irreducible KH-module is injective if and only if H is a periodic almost abelian p'-group.

If V is an arbitrary R–module then an injective hull of V (in the category of R–modules) is an R–module $\overline{V}$ satisfying:

(i) $\overline{V}$ is injective, and either

(ii) no proper R–submodule of $\overline{V}$ containing V is injective, or

(ii)′ $\overline{V}$ is an essential extension of V.

Here an R–module W is said to be an essential extension of an R–submodule U if every nonzero R–submodule of W has a nonzero intersection with U. It was shown by Eckmann and Schopf [1] that every R–module V has an injective hull $\overline{V}$ which is unique in the sense that if V* is another injective hull of V then there is an isomorphism from $\overline{V}$ to V* extending the identity map on V.

The following simple fact was proved by B.Hartley and D.McDougall.

Proposition **4.1.2**: Let R be a ring with 1, let V be an R–module and let $\overline{V}$ be an injective hull of V. Suppose $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$, where each V_λ is an R–submodule of V. If either *(i)* Λ is finite, or *(ii)* R satisfies the maximal condition on right ideals, then $\overline{V} = \bigoplus_{\lambda \in \Lambda} \overline{V}_\lambda$, where $\overline{V}_\lambda$ is an injective hull of V_λ.

B.Hartley and D.McDougall had pointed out that every injective R–module U is divisible in the sense that $Ud = U$ for every element d of R which is not a zero-divisor, and they call an R–module V $\mathbb{Z}$-divisible if the additive group V^+ of V is a divisible group. Then, immediately, they have

Proposition **4.1.3**: Every injective $\mathbb{Z}$G-module is $\mathbb{Z}$-divisible.

For a prime p and an abelian group V, let $V[p^k]$ denote the set of elements $v \in V$ satisfying $p^k v = 0$ (where $k \geq 0$ is an integer). If V is in addition an R–module then evidently $V[p^k]$ is an R–submodule of V. B.Hartley and D.McDougall have proved that:

Proposition **4.1.4**: Let G be a centre–by–finite p'-group and V a $\mathbb{Z}G$-module such that, as an additive group, V is a p-group (where p is a prime). Let $\overline{V}$ be an injective hull of V. Suppose that either (*i*) G is finite, or (*ii*) V is an artinian $\mathbb{Z}G$-module. Then

(*a*) $\overline{V}$ (as an additive group) is a p-group and $\overline{V}[p] = V[p]$,

(*b*) V is injective if and only if V is $\mathbb{Z}$-divisible.

§4.2 EXAMPLES OF $A^{\bar{f}}$

First of all, from Čarin's group (cf. **[14]** p.152), there follows a construction of an infinite irreducible $\mathbb{Z}G$-module, which as a group is a p-group, over the group $G = C_{q^\infty}$ for any two distinct primes p and q. By applying Proposition **4.1.4**, we have that:

Proposition **4.2.1**: For any finite integer $n>0$, there exists a noetherian $\mathbb{Z}G$-module A over a periodic abelian group G such that $A^{\bar{f}}$ is of exponent n.

Proof: Suppose $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, where $p_1, \cdots, p_r$ are different primes and $\alpha_1, \cdots, \alpha_r$ are positive integers. Let q be a prime satisfying $q \nmid n$. Let G be the quasicyclic group C_{q^∞} and let V_i, which is an infinite elementary abelian p_i-group, be the irreducible $\mathbb{Z}G$-module arising from the Čarin group $V_i]G$, where $i = 1,2,\cdots,r$. Let $\overline{V}_i$ be an injective hull of V_i then, since V_i is not $\mathbb{Z}$–divisible, $\overline{V}_i > V_i$ and $\overline{V}_i[p_i] = V_i[p_i] = V_i$ (Proposition **4.1.4**).

Let $V_{ij} = \overline{V}_i[p_i^j]$. Since $V_{i,j}/V_{i,j-1} \cong_{\mathbb{Z}G} V_{i1} = V_i$, so the $\mathbb{Z}G$-submodule V_{ij} (and then $V_{i\alpha_i}$) has a finite $\mathbb{Z}G$-composition series with all nonzero $\mathbb{Z}G$-factors being infinite. Put $A_{\alpha_i} = V_{i,\alpha_i}$, then the noetherian $\mathbb{Z}G$-submodule A_{α_i} is of exponent $p_i^{\alpha_i}$ and $A_{\alpha_i} = A_{\alpha_i}^{\overline{f}}$.

Let $A = A_{\alpha_1} \oplus A_{\alpha_2} \oplus \cdots \oplus A_{\alpha_r}$, then the noetherian $\mathbb{Z}G$-module A is of exponent n and $A = A^{\overline{f}}$. That is, A is the required $\mathbb{Z}G$-module.

In order to get some more general examples, we investigate the relations between the RG-modules and the R(G/N)-modules, where N is some normal subgroup of G and R is a ring with 1.

As B.Hartley and D.McDougall [6] have noted: if G is a periodic abelian group, then all irreducible $\mathbb{Z}_p G$-modules can be obtained (up to isomorphism) by the following:

Let G be a periodic abelian group and K an algebraic closure of $\mathbb{Z}_p$. Suppose δ is a homomorphism of G into the multiplicative group K^* of nonzero elements of K. Then since the elements of $\delta(G)$ are all roots of unity, it follows that the additive group L_δ generated by $\delta(G)$ is in fact a field. Let K_δ be the $\mathbb{Z}_p G$-module whose underlying vector space is L_δ with the G-action given by

$$vg = v\cdot\delta(g) \qquad (v \in K_\delta,\ g \in G).$$

Since $\delta(G)$ generates L_δ additively any G-submodule of K_δ is invariant under multiplication by any element of L_δ; consequently K_δ is irreducible.

Lemma 4.2.2: (B.Hartley and D.McDougall) With the above notation

(*i*) every irreducible $\mathbb{Z}_p G$-module is isomorphic to some K_θ;

(*ii*) $K_\theta \cong K_\phi$ if and only if $L_\theta = L_\phi$ and $\theta = \phi\rho$ for some element ρ of the Galois group of L_θ over $\mathbb{Z}_p$.

For a general (irreducible) RG-module V, using a natural method, we can always view V as an (irreducible) R(G/N)-module for some normal subgroup N $(\leq C_G(V))$ of G and in this case we denote the (irreducible) R(G/N)-module V by V*. That is, for any (irreducible) RG-module V, there exists N $(\leq C_G(V))$, being normal in G, and $\theta \in \mathrm{Hom}(G, G/N)$ with $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$ such that on the set V an (irreducible) R(G/N)-module structure (denote this R(G/N)-module by $V^{\vec{\theta}}$) can be given by

$$v\cdot(r\bar{g}) = v(r\tilde{g}),$$

where $v \in V$, $r \in R$, and $\tilde{g} \in G$ is such that $\theta(\tilde{g}) = \bar{g} \in G/N$. If $N = C_G(V)$, then we denote the faithful (irreducible) R(G/N)-module $V^{\vec{\theta}}$ by V^{θ^*}.

On the other hand, if W is an (irreducible) R(G/N)-module for some normal subgroup N of G, then for any $\theta \in \mathrm{Hom}(G, G/N)$ satisfying $\mathrm{Im}\theta = G/N$ and $\ker\theta = N$ we have an (irreducible) RG-module (denoted by $W^{\overleftarrow{\theta}}$) defined by the following:

(1) the underlying vector space of $W^{\overleftarrow{\theta}}$ is W, and

(2) the RG-action $\odot$ is given by

$$w \odot (rg) = w(rg^\theta) \qquad (w \in W, \ r \in R, \ \text{and} \ g \in G).$$

It is clear that the above is well-defined and then $W^{\overleftarrow{\theta}}$ is an (irreducible) RG-module with $C_G(W^{\overleftarrow{\theta}}) \geq N$. Evidently, $C_G(W^{\overleftarrow{\theta}}) = N$ if and only if W is faithful on G/N, and in this case we denote $W^{\overleftarrow{\theta}}$ by $W^{\theta^\dagger}$. Since there exists $\theta \in \mathrm{Hom}(G, G/N)$ satisfying $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$, so for any (irreducible) R(G/N)-module W there is at least one (irreducible) RG-module V satisfying $C_G(V) \geq N$.

From the above definitions, we immediately have:

<u>Lemma 4.2.3</u>: (*a*) If V is an (irreducible) RG-module with $C_G(V) = N$, then for any $\theta \in \mathrm{Hom}(G, G/N)$ with $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$,

$$V = (V^{\vec{\theta}})^{\overleftarrow{\theta}};$$

(*b*) If W is an (irreducible) R(G/N)-module and $\theta \in \mathrm{Hom}(G, G/N)$ with $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$, then

$$W = (W^{\overleftarrow{\theta}})^{\vec{\theta}}.$$

<u>Lemma 4.2.4</u>: (*a*) Let V_1 and V_2 be two RG-modules with N being their centralizer in G for some normal subgroup N of G, and let $\theta \in \mathrm{Hom}(G, G/N)$ with $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$. If V_1 is RG-isomorphic with V_2, i.e., $V_1 \overset{\varphi}{\cong}_{RG} V_2$, then $V_1^{\vec{\theta}} \overset{\psi}{\cong}_{R(G/N)} V_2^{\vec{\theta}}$ for some R(G/N)-isomorphism ψ.

(*b*) Let W_1 and W_2 be two R(G/N)-isomorphic R(G/N)-modules, i.e., $W_1 \overset{\varphi}{\cong}_{R(G/N)} W_2$. Let $\theta \in \mathrm{Hom}(G, G/N)$ with $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$. Then $W_1^{\overleftarrow{\theta}} \overset{\psi}{\cong}_{RG} W_2^{\overleftarrow{\theta}}$ for some RG-isomorphism ψ.

<u>Proof</u>: (*a*) Let ψ: $a \longmapsto \varphi(a)$ for any $a \in V_1^{\vec{\theta}}$, then ψ is a group-isomorphism from $V_1^{\vec{\theta}}$ to $V_2^{\vec{\theta}}$. Now for any $a \in V_1^{\vec{\theta}}$, any $r \in R$, and any $\bar{g} \in G/N$, since $\bar{g} = \theta(\tilde{g})$ for some $\tilde{g} \in G$, and since

$$\begin{aligned}\psi[a\cdot(r\bar{g})] &= \psi[a(r\tilde{g})] = \varphi[a(r\tilde{g})] \\ &= [\varphi(a)](r\tilde{g}) = [\psi(a)](r\tilde{g}) \\ &= \psi(a)\cdot(r\bar{g}),\end{aligned}$$

so ψ is a R(G/N)-isomorphism from $V_1^{\vec{\theta}}$ to $V_2^{\vec{\theta}}$. That is, $V_1^{\vec{\theta}} \cong_{R(G/N)} V_2^{\vec{\theta}}$.

(*b*) The proof is almost as same as that of (*a*).

In fact, let ψ: $a \longmapsto \varphi(a)$ for any $a \in W_1^{\overleftarrow{\theta}}$, then ψ is a group-isomorphism from $W_1^{\overleftarrow{\theta}}$ to $W_2^{\overleftarrow{\theta}}$. Now for any $a \in W_1^{\overleftarrow{\theta}}$, any $r \in R$, and any $g \in G$, since

$$\begin{aligned}\psi[a \odot (rg)] &= \psi[a(rg^{\theta})] = \varphi[a(rg^{\theta})] \\ &= [\varphi(a)](rg^{\theta}) = [\varphi(a)] \odot (rg^{\theta}) \\ &= [\psi(a)] \odot (rg^{\theta}),\end{aligned}$$

so ψ is an RG-isomorphism from $W_1^{\overleftarrow{\theta}}$ to $W_2^{\overleftarrow{\theta}}$. That is, $W_1^{\overleftarrow{\theta}} \cong_{RG} W_2^{\overleftarrow{\theta}}$.

Also, from the definition, we immediately have:

<u>Lemma 4.2.5</u>: (*a*) Let V_1 and V_2 be two RG-modules with N being their centralizer in G for some normal subgroup N of G, and let $\theta \in \mathrm{Hom}(G, G/N)$ satisfying $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$. Then

$$(V_1 \cap V_2)^{\overrightarrow{\theta}} = V_1^{\overrightarrow{\theta}} \cap V_2^{\overrightarrow{\theta}}.$$

(*b*) Let W_1 and W_2 be two R(G/N)-modules and let $\theta \in \mathrm{Hom}(G, G/N)$ with $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$. Then

$$(W_1 \cap W_2)^{\overleftarrow{\theta}} = W_1^{\overleftarrow{\theta}} \cap W_2^{\overleftarrow{\theta}}.$$

<u>Lemma 4.2.6</u>: Let G be a periodic abelian p'-group for some prime p and R a ring with 1.

(*a*) Let V be an irreducible RG-module such that $pV = 0$, $\overline{V}$ an injective hull of V, and $N = C_G(\overline{V})$. Let $\theta \in \mathrm{Hom}(G, G/N)$ satisfying $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$. If the R(G/N)-module $\overline{(V^{\overrightarrow{\theta}})}$ is an injective hull of $V^{\overrightarrow{\theta}}$, then

$$\overline{(V^{\overrightarrow{\theta}})} \cong_{R(G/N)} (\overline{V})^{\overrightarrow{\theta}}.$$

(*b*) Let N be a normal subgroup of G, W an irreducible R(G/N)-module such that $pW = 0$, and let $\theta \in \mathrm{Hom}(G, G/N)$ with $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$. If $\overline{W}$ is an injective hull of W and if the RG-module $\overline{(W^{\overleftarrow{\theta}})}$ is an injective hull of $W^{\overleftarrow{\theta}}$, then

$$\overline{(W^{\overleftarrow{\theta}})} \cong_{RG} (\overline{W})^{\overleftarrow{\theta}}.$$

Proof: (*a*) By Proposition **4.1.4**, it is easy to know that $(\overline{V})^{\overrightarrow{\theta}}$ is injective.

Since $V \le \overline{V}$, so $V^{\overrightarrow{\theta}} \le (\overline{V})^{\overrightarrow{\theta}}$; also for any nonzero R(G/N)-submodule U of $(\overline{V})^{\overrightarrow{\theta}}$, since $U^{\overleftarrow{\theta}} \le ((\overline{V})^{\overrightarrow{\theta}})^{\overleftarrow{\theta}} = \overline{V}$, so $V \cap U^{\overleftarrow{\theta}} \neq 0$ and then

$$V^{\overrightarrow{\theta}} \cap U = V^{\overrightarrow{\theta}} \cap (U^{\overleftarrow{\theta}})^{\overrightarrow{\theta}} = (V \cap U^{\overleftarrow{\theta}})^{\overrightarrow{\theta}} \neq 0.$$

That is, $(\overline{V})^{\overrightarrow{\theta}}$ is an injective hull of $V^{\overrightarrow{\theta}}$ and then $\overline{(V^{\overrightarrow{\theta}})} \cong_{R(G/N)} (\overline{V})^{\overrightarrow{\theta}}$.

(*b*) By Proposition **4.1.4**, we may easily have $(\overline{W})^{\overleftarrow{\theta}}$ is injective.

Since $W \le \overline{W}$, so $W^{\overleftarrow{\theta}} \le (\overline{W})^{\overleftarrow{\theta}}$; also for any nonzero RG-submodule U of $(\overline{W})^{\overleftarrow{\theta}}$, since $U^{\overrightarrow{\theta}} \le ((\overline{W})^{\overleftarrow{\theta}})^{\overrightarrow{\theta}} = \overline{W}$, so $W \cap U^{\overrightarrow{\theta}} \neq 0$ and then

$$W^{\overleftarrow{\theta}} \cap U = W^{\overleftarrow{\theta}} \cap (U^{\overrightarrow{\theta}})^{\overleftarrow{\theta}} = (W \cap U^{\overrightarrow{\theta}})^{\overleftarrow{\theta}} \neq 0.$$

That is, $(\overline{W})^{\overleftarrow{\theta}}$ is an injective hull of $W^{\overleftarrow{\theta}}$ and then $\overline{(W^{\overleftarrow{\theta}})} \cong_{RG} (\overline{W})^{\overleftarrow{\theta}}$.

A special case is $N = 1$, thus the homomorphism θ is actually an automorphism of G and then we use V^{θ} to denote either $V^{\overleftarrow{\theta}}$ or $V^{\overrightarrow{\theta}}$. Thus we have:

Lemma **4.2.7**: Let V be an RG-module and let $\varphi \in \mathrm{Aut}(G)$. If $\overline{V}$ and $\overline{(V^{\varphi})}$ are the injective hull of V and V^{φ}, respectively, then

$$\overline{(V^{\varphi})} \cong_{RG} (\overline{V})^{\varphi}.$$

Now we continue to consider the examples of a noetherian $\mathbb{Z}G$-module A with $A^{\bar{f}}$ having a finite $\mathbb{Z}G$-composition series and being of finite exponent n. In fact, we will get a complete description for the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of A in the case that (1) $G \cong C_{q^\infty}$ $(q \nmid n)$; or (2) G is Černikov such that its finite residual H satisfies that: if $q \in \pi(H)$, then $q \nmid n$.

As B.Hartley and D.McDougall pointed out: if G is a nontrivial locally cyclic p'-group then there is, up to automorphism conjugacy, exactly one faithful irreducible $\mathbb{Z}_pG$-module. Here automorphism conjugacy is defined as: for a ring R with 1, two RG-modules V_1 and V_2 are automorphism conjugate iff there is an automorphism $\varphi \in \mathrm{Aut}(G)$ such that $V_1 \cong_{RG} V_2^{\varphi}$. Using the above discussion, we have: if G is a nontrivial periodic abelian group then there is, up to quotient-automorphism conjugacy, exactly one irreducible $\mathbb{Z}_pG$-module such that N is its centralizer in G for some normal subgroup N of G. Here quotient-automorphism conjugacy is defined as: for a ring R with 1, if W_1 and W_2 are two RG-modules with N being their centralizer in G for some normal subgroup N of G, then the RG-modules W_1 and W_2 are said to be quotient-automorphism conjugate iff there is an automorphism $\varphi \in \mathrm{Aut}(G/N)$ and a homomorphism $\theta \in \mathrm{Hom}(G, G/N)$ with $\mathrm{Im}\theta = G/N$ and $\mathrm{Ker}\theta = N$ such that

$$W_1 \cong_{RG} [(W_2^{\theta^*})^{\varphi}]^{\theta^\dagger}.$$

If $G = C_{q^\infty} = \langle x_0, x_1, x_2, \cdots;\ x_0 = 1,\ x_{i+1}^q = x_i,\ i=0,1,2,\cdots \rangle$ and if A is a noetherian $\mathbb{Z}G$-module such that $A^{\bar{f}}$ has a finite $\mathbb{Z}G$-composition series and is of exponent n $(=p_1^{\alpha_1}\cdots p_r^{\alpha_r})$ with $q \nmid n$, for each $i \leq r$, using Proposition **4.1.1**, we may suppose

$$A^{\bar{f}}[p_i] = \mathrm{D}\Sigma_{k=1}^{t_i} [(V_{ij_k}^{\theta^*_{ij_k}})^{\varphi_{ij_k}}]^{\theta^\dagger_{ij_k}},$$

where V_{ij_k} is the "unique" irreducible $\mathbb{Z}_{p_i}G$-module with $C_G(V_{ij_k}) = \langle x_{j_k} \rangle$, $\varphi_{ij_k} \in \mathrm{Aut}(G/\langle x_{j_k} \rangle)$, $\theta_{ij_k} \in \mathrm{Hom}(G, G/\langle x_{j_k} \rangle)$ satisfying $\mathrm{Im}\theta_{ij_k} = G/\langle x_{j_k} \rangle$ and $\mathrm{Ker}\theta_{ij_k} = \langle x_{j_k} \rangle$, $j_k \geq 0$, $k = 1,2,\cdots t_i$, and $t_i \geq 1$.

We claim that: up to isomorphism, $A^{\bar{f}}$ is a $\mathbb{Z}G$-submodule of

$$\mathrm{Dr}_{i=1}^{r}\ \mathrm{Dr}_{k=1}^{t_i}\ [(A_{\alpha_i j_k})^{\varphi_{ij_k}}]^{\theta^{\dagger}_{ij_k}},$$

where $A_{\alpha_i j_k}$ is obtained from $V_{ij_k}^{\theta^{*}_{ij_k}}$ over the group ring $\mathbb{Z}(G/\langle x_{j_k} \rangle)$ as in the proof of Proposition **4.2.1**. In fact, let $\overline{A^{\bar{f}}}$ be an injective hull of $A^{\bar{f}}$ and let $\overline{A^{\bar{f}}[p_i]}$ be an injective hull of $A^{\bar{f}}[p_i]$, $i=1,2,\cdots,r$. Since $\mathrm{Dr}_{i=1}^{r}\ A^{\bar{f}}[p_i] \leq A^{\bar{f}} \leq \overline{A^{\bar{f}}}$, so the injective module $\overline{A^{\bar{f}}}$ is an extension of $\mathrm{Dr}_{i=1}^{r}\ A^{\bar{f}}[p_i]$. For any nonzero $\mathbb{Z}G$-submodule B of $\overline{A^{\bar{f}}}$, since $B \cap A^{\bar{f}} \neq 0$, so it is clear that $B \cap (\mathrm{Dr}_{i=1}^{r}\ A^{\bar{f}}[p_i]) \neq 0$. That is, $\overline{A^{\bar{f}}}$ is an essential extension of $\mathrm{Dr}_{i=1}^{r}\ A^{\bar{f}}[p_i]$ and then, by definition, $\overline{A^{\bar{f}}}$ is an injective hull of $\mathrm{Dr}_{i=1}^{r}\ A^{\bar{f}}[p_i]$. Thus $\overline{A^{\bar{f}}} \cong_{\mathbb{Z}G} \overline{\mathrm{Dr}_{i=1}^{r}\ A^{\bar{f}}[p_i]} = \mathrm{Dr}_{i=1}^{r}\ \overline{A^{\bar{f}}[p_i]}$. If B is a $\mathbb{Z}G$-submodule of $\mathrm{Dr}_{i=1}^{r}\ \overline{A^{\bar{f}}[p_i]}$ such that B is of exponent n $(=p_1^{\alpha_1}\cdots p_r^{\alpha_r})$, then, by using Proposition **4.1.2**, Lemma **4.2.6**, Lemma **4.2.7** and Lemma **4.2.4**, we have:

$$\mathrm{Dr}_{i=1}^{r}\ \overline{A^{\bar{f}}[p_i]} = \mathrm{Dr}_{i=1}^{r}\ \mathrm{Dr}_{k=1}^{t_i}\ \overline{[(V_{ij_k}^{\theta^{*}_{ij_k}})^{\varphi_{ij_k}}]^{\theta^{\dagger}_{ij_k}}}$$

$$\cong_{\mathbb{Z}G} \mathrm{Dr}_{i=1}^{r}\ \mathrm{Dr}_{k=1}^{t_i}\ [\overline{(V_{ij_k}^{\theta^{*}_{ij_k}})^{\varphi_{ij_k}}}]^{\theta^{\dagger}_{ij_k}}$$

$$\cong_{\mathbb{Z}G} \mathrm{Dr}_{i=1}^{r} \mathrm{Dr}_{k=1}^{t_i} [\overline{(V_{ij_k}^{\theta^*_{ij_k}})}^{\varphi_{ij_k}}]^{\theta^\dagger_{ij_k}},$$

and then

$$\mathrm{Dr}_{i=1}^{r} (\overline{A^{\bar{f}}[p_i]})[p_i^{\alpha_i}] \cong_{\mathbb{Z}G} \mathrm{Dr}_{i=1}^{r} \{\mathrm{Dr}_{k=1}^{t_i} [\overline{(V_{ij_k}^{\theta^*_{ij_k}})}^{\varphi_{ij_k}}]^{\theta^\dagger_{ij_k}}\}[p_i^{\alpha_i}]$$

$$= \mathrm{Dr}_{i=1}^{r} \mathrm{Dr}_{k=1}^{t_i} [\{\overline{(V_{ij_k}^{\theta^*_{ij_k}})[p_i^{\alpha_i}]}\}^{\varphi_{ij_k}}]^{\theta^\dagger_{ij_k}}$$

$$= \mathrm{Dr}_{i=1}^{r} \mathrm{Dr}_{k=1}^{t_i} [(A_{\alpha_i j_k})^{\varphi_{ij_k}}]^{\theta^\dagger_{ij_k}}.$$

Thus, up to isomorphism, B is a $\mathbb{Z}G$-submodule of $\mathrm{Dr}_{i=1}^{r} \mathrm{Dr}_{k=1}^{t_i} [(A_{\alpha_i j_k})^{\varphi_{ij_k}}]^{\theta^\dagger_{ij_k}}$.

So, up to isomorphism, $A^{\bar{f}}$ is a $\mathbb{Z}G$-submodule of $\mathrm{Dr}_{i=1}^{r} \mathrm{Dr}_{k=1}^{t_i} [(A_{\alpha_i j_k})^{\varphi_{ij_k}}]^{\theta^\dagger_{ij_k}}$.

A similar result is also true for any periodic abelian group G (especially, for an abelian Černikov group). That is, we could give a similar description of the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of a noetherian $\mathbb{Z}G$-module A satisfying that $A^{\bar{f}}$ is of finite exponent and has a finite $\mathbb{Z}G$-composition series, where G is a periodic abelian group satisfying that the intersection of the sets $\pi(G)$ and $\{p;\ p \mid \exp A^{\bar{f}}\}$ is empty.

Furthermore, if G is a Černikov group, then the finite residual H of G is a direct product of finitely many quasicyclic groups. If A is an irreducible $\mathbb{Z}G$-module then, as a $\mathbb{Z}H$-module, A is a direct sum of finitely many irreducible $\mathbb{Z}H$-submodules (Lemma **2.2.4**). On the other hand, if V is an (infinite) irreducible $\mathbb{Z}H$-module (it is clear that such a V always exists), then the following method guarantees the existence of the (infinite) irreducible $\mathbb{Z}G$-modules. And, from the "uniqueness" of the infinite irreducible $\mathbb{Z}H$-module V (up to quotient-automorphism conjugacy), it follows that the infinite irreducible $\mathbb{Z}G$-modules are "almost" unique. Here we mean that: if B is an infinite irreducible $\mathbb{Z}G$-module over a Černikov group G, which has H as its finite residual, then

$$B = \mathrm{Dr}_{i=1}^{t} [(V_i^{\theta_i^*})^{\varphi_i}]^{\theta_i^\dagger},$$

where V_i is the "unique" infinite irreducible $\mathbb{Z}H$-module with N_i being its centralizer in H for some normal subgroup N_i of H, $\varphi_i \in \mathrm{Aut}(H/N_i)$, and $\theta_i \in \mathrm{Hom}(H, H/N_i)$ with $\mathrm{Im}\theta_i = H/N_i$ and $\mathrm{Ker}\theta_i = N_i$.

Let G be a Černikov group, H the finite residual of G, and V an (infinite) irreducible $\mathbb{Z}H$-module. Let $T = \{t_1, t_2, \cdots, t_n\}$ be a transversal to H in G. Consider the induced $\mathbb{Z}G$-module $V \otimes_{\mathbb{Z}H} \mathbb{Z}G = (V \otimes_{\mathbb{Z}H} t_1) \oplus \cdots \oplus (V \otimes_{\mathbb{Z}H} t_n)$ defined by

$$(v \otimes t_i)t_j = vh \otimes t_k, \qquad \text{where } t_i t_j = ht_k \text{ with } h \in H,$$

$$(v \otimes t_i)h = vh^{t_i^{-1}} \otimes t_i.$$

Here $\oplus$ is the direct sum of $\mathbb{Z}H$-submodules. It is easy to show that the above $\mathbb{Z}G$-module $V \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is well-defined. Now, since $V \otimes_{\mathbb{Z}H} t_i \cong_{\mathbb{Z}H} V^{\varphi}$ for the automorphism φ of H induced by t_i^{-1} acting on H by conjugation, and since V is irreducible, so $V \otimes_{\mathbb{Z}H} \mathbb{Z}G$ as a $\mathbb{Z}H$-module has a finite $\mathbb{Z}H$-composition series and then, by Lemma **2.2.1**, $V \otimes_{\mathbb{Z}H} \mathbb{Z}G$ has a finite $\mathbb{Z}G$-composition series. Therefore $V \otimes_{\mathbb{Z}H} \mathbb{Z}G$ contains an irreducible $\mathbb{Z}G$-submodule, say B. As in the proof of (4) and (5) in Lemma **2.2.4**, **B** is a direct sum of finitely many irreducible $\mathbb{Z}H$-submodules. Using the "uniqueness" of the infinite irreducible $\mathbb{Z}H$-modules (up to quotient-automorphism conjugacy), we have the fact that B is "almost" uniquely determined.

For a given Černikov group G, let H be its finite residual and let

$$B = \mathrm{Dr}_{i=1}^{t} [(V_i^{\theta_i^*})^{\varphi_i}]^{\theta_i^\dagger},$$

where B is an infinite irreducible $\mathbb{Z}G$-module, V_i is a fixed infinite irreducible $\mathbb{Z}H$-module with N_i as its centralizer in H for some normal subgroup N_i of H, $\varphi_i \in \mathrm{Aut}(H/N_i)$, and $\theta_i \in \mathrm{Hom}(H, H/N_i)$ satisfying $\mathrm{Im}\theta_i = H/N_i$ and $\mathrm{Ker}\theta_i = N_i$. It is clear that $pB = 0$ for some prime $p \notin \pi(H)$. Consider B as a $\mathbb{Z}H$-module and let $\overline{B}$ be an injective hull of B (here we note that it can be shown

$$\overline{B} \cong_{\mathbb{Z}H} \mathrm{Dr}_{i=1}^{t} [(\overline{V_i^{\theta_i^*}})^{\varphi_i}]^{\theta^\dagger}.$$

where $\overline{V_i^{\theta_i^*}}$ is an injective hull of $V_i^{\theta_i^*}$). By Proposition 4.1.4, $B < \overline{B}$ and $\overline{B}[p] = B[p] = B$. Let $B_j = \overline{B}[p^j]$, then $B_j/B_{j-1} \cong_{\mathbb{Z}H} B_1 = B$, so B_j has a finite $\mathbb{Z}H$-composition series in which each factor is infinite. Evidently, B_j is of exponent p^j. From B_j, we consider the induced $\mathbb{Z}G$-module $A_j = B_j \otimes_{\mathbb{Z}H} \mathbb{Z}G$ defined as above, then as A_j is a direct sum of finitely many $\mathbb{Z}H$-submodules, A_j has a finite $\mathbb{Z}H$-composition series with all factors being infinite. Thus A_j has a finite $\mathbb{Z}G$-composition series in which each factor is infinite (Lemma **2.2.1** and Corollary **2.2.3**) and also A_j is of exponent p^j.

We claim that:

$$A_j \cong_{\mathbb{Z}G} \mathrm{Dr}_{i=1}^{t} U_{ij}, \qquad \text{where } U_{ij} = \{[(\{\overline{V_i^{\theta_i^*}}\}[p^j])^{\varphi_i}]^{\theta_i^\dagger}\} \otimes_{\mathbb{Z}H} \mathbb{Z}G.$$

In order to find a $\mathbb{Z}G$-isomorphism from A_j to $\mathrm{Dr}_{i=1}^{t} U_{ij}$, we must first show that (which has been mentioned in the above)

$$\overline{B} \cong_{\mathbb{Z}H} \mathrm{Dr}_{i=1}^{t} [(\overline{V_i^{\theta_i^*}})^{\varphi_i}]^{\theta_i^\dagger} .$$

Since $V_i^{\theta_i^*} \le \overline{V_i^{\theta_i^*}}$, so $[(V_i^{\theta_i^*})^{\varphi_i}]^{\theta_i^\dagger} \le [(\overline{V_i^{\theta_i^*}})^{\varphi_i}]^{\theta_i^\dagger}$ and then $B \le \mathrm{Dr}_{i=1}^{t} [(\overline{V_i^{\theta_i^*}})^{\varphi_i}]^{\theta_i^\dagger}$; also for any nonzero $\mathbb{Z}H$-submodule C of $\mathrm{Dr}_{i=1}^{t} [(\overline{V_i^{\theta_i^*}})^{\varphi_i}]^{\theta_i^\dagger}$, using Proposition **4.1.4**, we can easily get C is also a p-group, so

$$\begin{aligned} 0 \neq C[p] &\le \{\mathrm{Dr}_{i=1}^{t} [(\overline{V_i^{\theta_i^*}})^{\varphi_i}]^{\theta_i^\dagger}\}[p] \\ &= \mathrm{Dr}_{i=1}^{t} [(\overline{\{V_i^{\theta_i^*}\}[p]})^{\varphi_i}]^{\theta_i^\dagger} \\ &= \mathrm{Dr}_{i=1}^{t} [(\{V_i^{\theta_i^*}\}[p])^{\varphi_i}]^{\theta_i^\dagger} \\ &= \{\mathrm{Dr}_{i=1}^{t} [(V_i^{\theta_i^*})^{\varphi_i}]^{\theta_i^\dagger}\}[p] \\ &= B[p] = B, \end{aligned}$$

and then $B \cap C \neq 0$. Therefore, $\mathrm{Dr}_{i=1}^{t} [(\overline{V_i^{\theta_i^*}})^{\varphi_i}]^{\theta_i^\dagger}$ is an injective hull of B and so

$$\overline{B} \cong_{\mathbb{Z}H} \mathrm{Dr}_{i=1}^{t} [(\overline{V_i^{\theta_i^*}})^{\varphi_i}]^{\theta_i^\dagger}.$$

Thus $B_j = \overline{B}[p^j] \overset{\alpha}{\cong}_{\mathbb{Z}H} \{\mathrm{Dr}_{i=1}^{t} [(\overline{V_i^{\theta_i^*}})^{\varphi_i}]^{\theta_i^\dagger}\}[p^j] = \mathrm{Dr}_{i=1}^{t} [(\overline{\{V_i^{\theta_i^*}\}[p^j]})^{\varphi_i}]^{\theta_i^\dagger} = \mathrm{Dr}_{i=1}^{t} W_{ij}$ for some $\mathbb{Z}H$-isomorphism α, where $W_{ij} = [(\overline{\{V_i^{\theta_i^*}\}[p^j]})^{\varphi_i}]^{\theta_i^\dagger}$. Since

$$A_j = B_j \otimes_{\mathbb{Z}H} \mathbb{Z}G = (B_j \otimes_{\mathbb{Z}H} t_1) \oplus \cdots \oplus (B_j \otimes_{\mathbb{Z}H} t_n) \qquad \text{and}$$

$$\mathrm{Dr}_{i=1}^{t} U_{ij} = \mathrm{Dr}_{i=1}^{t} (W_{ij} \otimes_{\mathbb{Z}H} \mathbb{Z}G) = (\mathrm{Dr}_{i=1}^{t} W_{ij}) \otimes_{\mathbb{Z}H} \mathbb{Z}G$$
$$= [(\mathrm{Dr}_{i=1}^{t} W_{ij}) \otimes_{\mathbb{Z}H} t_1] \oplus \cdots \oplus [(\mathrm{Dr}_{i=1}^{t} W_{ij}) \otimes_{\mathbb{Z}H} t_n],$$

so for $a \in A_j$, let $a = (b_1 \otimes t_1) + \cdots + (b_n \otimes t_n)$ with $b_1, \cdots, b_n \in B_j$, and let β: $a \longmapsto (b_1^{\alpha} \otimes t_1) + \cdots + (b_n^{\alpha} \otimes t_n) \in \mathrm{Dr}_{i=1}^{t} U_{ij}$, then it is routine to check that β is a $\mathbb{Z}G$-isomorphism from A_j to $\mathrm{Dr}_{i=1}^{t} U_{ij}$. So we get the required isomorhpism.

For any integer $n > 0$, let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $p_1, \cdots, p_r$ are distinct primes and $\alpha_1, \cdots, \alpha_r$ are positive integers. Suppose G is a Černikov group with $p_i \neq \pi(H)$ for all $1 \leq i \leq r$, where H is the finite residual of G. As above, there exists a $\mathbb{Z}G$-module A_i such that A_i has a finite $\mathbb{Z}G$-composition series in which each factor is infinite and A_i is of exponent $p_i^{\alpha_i}$, where $i = 1, 2, \cdots, r$. Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_r$, then A has a finite $\mathbb{Z}G$-composition series (and so is noetherian), $A = A^{\bar{f}}$, and A is of exponent n.

Let the noetherian $\mathbb{Z}G$-module A over a Černikov group G satisfy the condition that $A^{\bar{f}}$ has a finite $\mathbb{Z}G$-composition series and is of exponent n $(=p_1^{\alpha_1} \cdots p_r^{\alpha_r})$ with $q \nmid n$ for any $q \in \pi(H)$, where H is the finite residual of G. Suppose

$$A^{\bar{f}}[p_i] = \mathrm{Dr}_{j=1}^{t_i} \mathrm{Dr}_{k=1}^{s_j} [(V_{ijk}^{\theta^*_{ijk}})^{\varphi_{ijk}}]^{\theta^{\dagger}_{ijk}},$$

where $\mathrm{Dr}_{k=1}^{s_j} [(V_{ijk}^{\theta^*_{ijk}})^{\varphi_{ijk}}]^{\theta^{\dagger}_{ijk}}$ is an infinite irreducible $\mathbb{Z}G$-submodule, V_{ijk} is the "unique" infinite irreducible $\mathbb{Z}_{p_i}H$-module (viewed as a $\mathbb{Z}H$-module) with N_{ijk} being its centralizer in H for some normal subgroup N_{ijk} of H, $\varphi_{ijk} \in \mathrm{Aut}(H/N_{ijk})$, $\theta_{ijk} \in \mathrm{Hom}(H, H/N_{ijk})$ satisfying $\mathrm{Im}\theta_{ijk} = H/N_{ijk}$ and

$\mathrm{Ker}\theta_{ijk} = N_{ijk}$, and $t_i \geq 1$. Then, up to a $\mathbb{Z}G$-isomorphism, $A^{\bar{f}}$ is a $\mathbb{Z}G$-submodule of the $\mathbb{Z}H$-module

$$M = \mathrm{Dr}_{i=1}^{r}\, \mathrm{Dr}_{j=1}^{t_i}\, \mathrm{Dr}_{k=1}^{s_j}\, [(\overline{\{V_{ijk}^{\theta^*_{ijk}}\}[p_i^{\alpha}i])^{\varphi_{ijk}}}]^{\theta^{\dagger}_{ijk}},$$

where the $\mathbb{Z}H$-module $\overline{V_{ijk}^{\theta^*_{ijk}}}$ is an injective hull of the $\mathbb{Z}H$-module $V_{ijk}^{\theta^*_{ijk}}$. In fact, consider $A^{\bar{f}}$ as a $\mathbb{Z}H$-module, then, up to a $\mathbb{Z}H$-isomorphism, $A^{\bar{f}}$ is a $\mathbb{Z}H$-submodule of M (this claim can be proved by almost just quoting that of quasicyclic case). Since $A^{\bar{f}}$ is a $\mathbb{Z}G$-module, let $U = \psi(A^{\bar{f}}) \leq M$, where ψ is a $\mathbb{Z}H$-isomorphism from $A^{\bar{f}}$ to U, then we may define a $\mathbb{Z}G$-module U^{ψ}, which as a $\mathbb{Z}H$-module is contained in M and is $\mathbb{Z}G$-isomorphic with $A^{\bar{f}}$.

Let the underlying vector space of U^{ψ} be U, and let the G-action $\circ$ on U^{ψ} be given by

$$u \circ g = \psi([\psi^{-1}(u)]g) \qquad (u \in U^{\psi},\ g \in G).$$

It is clear that the above is well-defined and, as a $\mathbb{Z}H$-module, U^{ψ} is contained in M. Now we prove U^{ψ} is $\mathbb{Z}G$-isomorphic with $A^{\bar{f}}$. For $\varphi\colon\ a \longmapsto \psi(a)$, where $a \in A^{\bar{f}}$, it is evident that φ is a group-isomorphism from $A^{\bar{f}}$ to U^{ψ}; also since $\varphi(ag) = \psi(ag) = \psi[(\psi^{-1}[\psi(a)])g] = \psi(a)\circ g = \varphi(a)\circ g$, for any $g \in G$ and any $a \in A^{\bar{f}}$, so we have φ is actually a $\mathbb{Z}G$-isomorphism from $A^{\bar{f}}$ to M. Thus, up to a $\mathbb{Z}G$-isomorphism, $A^{\bar{f}}$ is a $\mathbb{Z}G$-submodule of the $\mathbb{Z}H$-module M.

§4.3 THE STRUCTURE OF A^f

This section is short, however, the results have completely shown the structure of A^f without further restriction on G.

Proposition **4.3.1**: Let G be a hyperfinite locally soluble group and A a torsion-free noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being finite. Then A is finitely generated as an abelian group and $G/C_G(A)$ is finite.

Proof: We may assume that G acts faithfully on A, i.e., $C_G(A) = 1$. In order to apply Corollary **2.1.7**, we let $H = C_G(A/pA)$ for some prime $p > 2$. By Lemma **2.4.5**, A/pA is finite, and then $|G/H| < \infty$. Applying Corollary **2.1.7**, we get $H = C_G(A) = 1$. Thus G is finite. By the noetherian condition, we have $A = \langle a_1, \cdots, a_n\rangle^G$ for some elements $a_1, \cdots, a_n$. Hence, since G is finite, A is finitely generated as an abelian group. The result holds.

A generalization of Proposition **4.3.1** is that:

Corollary **4.3.2**: Let G be a hyperfinite almost locally soluble group and A a torsion-free noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being finite. Then A is finitely generated as an abelian group and $G/C_G(A)$ is finite.

Proof: Let H be a normal subgroup of G such that H is locally soluble and G/H is finite. Consider A as a $\mathbb{Z}H$-module then, by Lemma **1.2.5** and Lemma **2.2.6**, the torsion-free $\mathbb{Z}H$-module A is also noetherian and has all irreducible $\mathbb{Z}H$-factors being finite. Since H is also hyperfinite thus, by Proposition **4.3.1**, A is finitely generated as an abelian group and $H/C_H(A)$ is finite. For $G/C_G(A)$, since $|G/C_H(A)| = |G/H|\cdot|H/C_H(A)| < \infty$ and $C_H(A) \le C_G(A)$, so $|G/C_G(A)| < \infty$, the result is proved.

An important consequence is that:

Corollary **4.3.3**: Let G be a Černikov group and A a torsion-free noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being finite. Then A is finitely generated as an abelian group and $G/C_G(A)$ is finite.

In particular, we have:

<u>Corollary **4.3.4**</u>: If G is a locally finite group satisfying the minimal condition on subgroups, and if A is a torsion-free noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being finite. Then A is finitely generated as an abelian group and $G/C_G(A)$ is finite.

For a general $\mathbb{Z}G$-module A over a hyperfinite locally soluble group G, we have:

<u>Theorem **B**</u>: Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being finite. Then A is finitely generated as an abelian group and $G/C_G(A)$ is finite.

<u>Proof</u>: Let T(A) be the torsion part of A, then, by Proposition **4.3.1**, A/T(A) is finitely generated as an abelian group. Since T(A) is also a noetherian $\mathbb{Z}G$-module, so T(A) has a finite exponent, say n. Let $n = p_1p_2\cdots p_m$, where p_i are primes, $i = 1, 2, \cdots, m$. By Lemma **2.4.5**, $p_1\cdots p_{j-1}T(A)/p_1\cdots p_jT(A)$ is finite for any $j \in \{1, 2, \cdots, m\}$, where $p_0 = 1$. Thus

$$|T(A)| = |T(A)/p_1T(A)|\cdot|p_1T(A)/p_1p_2T(A)|\cdots|p_1\cdots p_{m-1}T(A)| < \infty.$$

Therefore the group A is finite-by-(finitely generated) and then is finitely generated as an abelian group. The other conclusion that $G/C_G(A)$ is finite follows immediately from the following two simple results.

<u>Proposition **4.3.5**</u>: Let G be a group, A a $\mathbb{Z}G$-module, and B a finite $\mathbb{Z}G$-submodule of A such that the $\mathbb{Z}G$-module A/B as a group is finitely

generated. If $C_G(B) = G = C_G(A/B)$, and if $C_G(A) = 1$, then G is finite.

Proof: Let $A/B = <a_1+B, a_2+B, \cdots, a_n+B>$. Since $G = C_G(A/B)$, so for $g \in G$, we have $a_ig+B = (a_i+B)g = a_i+B$, where $1 \le i \le n$. Thus $a_ig = a_i+b_i$ for some $b_i \in B$. Since B is finite and n is finite, there are only finitely many maps $g^*: a_i \longmapsto a_i+b_i$, where $b_i \in B$ and $1 \le i \le n$. If G is infinite, then there exist two elements $g_1 \neq g_2 \in G$ such that $a_ig_1 = a_ig_2$ for all i. Therefore $a_i(g_1g_2^{-1}) = a_i$, $i = 1, 2, \cdots, n$. Also $g_1g_2^{-1} \in G = C_G(B)$, so it is clear that $g_1g_2^{-1} \in C_G(A) = 1$ and then $g_1 = g_2$, a contradiction. So G is finite.

Proposition 4.3.6: If all irreducible $\mathbb{Z}G$-factors of a noetherian $\mathbb{Z}G$-module A over a hyperfinite locally soluble group G are finite, and if $C_G(A) = 1$, then G is a finite group.

Proof: Let T(A) be the torsion part of A, then from the above proof of Theorem **B** we know that the $\mathbb{Z}G$-submodule T(A) is finite, so $|G/C_G(T(A))| < \infty$. Also, using Proposition **4.3.1**, we have $|G/C_G(A/T(A))| < \infty$. Let $H = C_G(T(A)) \cap C_G(A/T(A))$, then $|G/H| < \infty$. Consider A as a $\mathbb{Z}H$-module, then T(A) is a finite $\mathbb{Z}H$-submodule of A and the $\mathbb{Z}H$-module A/T(A) as a group is finitely generated. It is clear that $C_H(T(A)) = H = C_H(A/T(A))$. Also, since $C_G(A) = 1$, so $C_H(A) = 1$. Thus, by Proposition **4.3.5**, H is finite and then G is finite. The result holds.

As before, from Theorem **B**, we have

Corollary B1: Let G be a hyperfinite almost locally soluble group and A a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being finite. Then A is finitely generated as an abelian group and $G/C_G(A)$ is finite.

Corollary B2: Let G be a Černikov group and A a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being finite. Then A is finitely generated as an abelian group and $G/C_G(A)$ is finite.

Corollary B3: Let G be a locally finite group satisfying the minimal condition on subgroups, and let A be a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being finite. Then A is finitely generated as an abelian group and $G/C_G(A)$ is finite.

§4.4 THE STRUCTURE OF $A^{\bar{f}}$

In §4.2, we have saw that for any integer $n > 0$, there exists a noetherian $\mathbb{Z}G$-module A over a periodic abelian (and hence hyperfinite and locally soluble) group G such that $A^{\bar{f}}$ is of exponent n. Must the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of any noetherian $\mathbb{Z}G$-module A over a hyperfinite locally soluble group G necessarily be torsion? Further, if $pA = 0$ for some prime p, does A always have a finite $\mathbb{Z}G$-composition series? Should these two questions both have a positive answer, the structure of the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of a noetherian $\mathbb{Z}G$-module A over a hyperfinite locally soluble group G would become much clearer, and the examples given in §4.2 would be the typical models for other modules.

Conjecture A: If G is a hyperfinite locally soluble group and if A is a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p, then A has a finite $\mathbb{Z}G$-composition series.

Conjecture B: If G is a hyperfinite locally soluble group, then any noetherian $\mathbb{Z}G$-module A with all irreducible $\mathbb{Z}G$-factors being infinite is torsion and so has finite exponent.

We now prove that these two conjectures are positive if G satisfies some further condition, and then the conjectures are also true even for Černikov groups (which need not be locally soluble).

First, we consider Conjecture A.

Proposition **4.4.1**: Let G be a periodic abelian group and A a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p. If G is a p'-group, then A has a finite $\mathbb{Z}G$-composition series.

Proof: Suppose A does not have a finite $\mathbb{Z}G$-composition series, then by the noetherian condition we may assume that for any nonzero $\mathbb{Z}G$-submodule C of A, A/C has a finite $\mathbb{Z}G$-composition series. It is clear that every nonzero $\mathbb{Z}G$-submodule of A does not have a finite $\mathbb{Z}G$-composition series but any proper $\mathbb{Z}G$-image of the nonzero $\mathbb{Z}G$-submodules of A has one, so we may assume that $A = \langle a \rangle^G$. Also we may assume G acts faithfully on A.

Since $pA = 0$, so we may consider A as a $\mathbb{Z}_pG$-module instead of $\mathbb{Z}G$-module. Let L denote the annihilator ideal $\mathrm{Ann}_{\mathbb{Z}_pG}(a) = \{r \in \mathbb{Z}_pG;\ ar = 0\}$, then $\mathbb{Z}_pG/L \cong_{\mathbb{Z}_pG} \langle a \rangle^G = A$. Thus the ring $\mathbb{Z}_pG/L$ is noetherian by A being a noetherian $\mathbb{Z}_pG$-module. If the ring $\mathbb{Z}_pG$ is regular, i.e., every finitely generated (right) ideal is generated by a single idempotent, then so is $\mathbb{Z}_pG/L$ a regular ring. And then, by Lemma **1.2.27**, $\mathbb{Z}_pG/L$ is semisimple and so has only finitely many (right) ideals. But this is not true as $\mathbb{Z}_pG/L \cong_{\mathbb{Z}_pG} A$ and A has no finite $\mathbb{Z}_pG$-composition series.

The remainder is to prove $\mathbb{Z}_pG$ is regular. In fact, let I be a finitely generated ideal of $\mathbb{Z}_pG$, then $I = \sum_{i=1}^{n}\alpha_i\mathbb{Z}_pG$. Since G is a periodic abelian group, so G is locally finite, and then there is a finite subgroup F such that $\alpha_i \in \mathbb{Z}_pF$ for all $i = 1, 2, \cdots, n$. Since G is a p'-group, so is F, and then $\mathbb{Z}_pF$ is semisimple (Lemma **1.2.28**). Thus any right ideal of $\mathbb{Z}_pF$ is generated by a single idempotent (Lemma **1.2.29**). Therefore $\sum_{i=1}^{n}\alpha_i\mathbb{Z}_pF = v\mathbb{Z}_pF$ for some idempotent $v \in \mathbb{Z}_pF$. Hence

$$\begin{aligned} v\mathbb{Z}_pG &= (v\mathbb{Z}_pF)\mathbb{Z}_pG = (\textstyle\sum_{i=1}^{n}\alpha_i\mathbb{Z}_pF)\mathbb{Z}_pG \le \sum_{i=1}^{n}\alpha_i\mathbb{Z}_pG \\ &= I = \textstyle\sum_{i=1}^{n}\alpha_i\mathbb{Z}_pG \le \sum_{i=1}^{n}(v\mathbb{Z}_pF)\mathbb{Z}_pG \\ &= v\textstyle\sum_{i=1}^{n}(\mathbb{Z}_pF)(\mathbb{Z}_pG) \le v\mathbb{Z}_pG. \end{aligned}$$

That is, $I = v\mathbb{Z}_pG$ with $v = v^2$. So $\mathbb{Z}_pG$ is regular, the result is proved.

<u>Proposition **4.4.2**</u>: Let G be a periodic abelian group and let A be a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p. Then A has a finite $\mathbb{Z}G$-composition series.

<u>Proof</u>: Suppose A does not have a finite $\mathbb{Z}G$-composition series, then by the noetherian condition we may assume that for every nonzero $\mathbb{Z}G$-submodule C of A, A/C has a finite $\mathbb{Z}G$-composition series. Also we may assume that G acts faithfully on A, i.e., $C_G(A) = 1$.

Since G is abelian, every subgroup is normal in G. If G contains an element, say x, with the order of x being p for the prime p, then since A is a p-group, we have $C_A(x) \neq 0$. Also by $C_G(A) = 1$, we have $C_A(x) \neq A$. So $A(x-1)$ $(\cong_{\mathbb{Z}G} A/C_A(x))$ is nonzero and has a finite $\mathbb{Z}G$-composition series. Also $A/A(x-1)$ has a finite $\mathbb{Z}G$-composition series and hence so does A. This is contrary to the choice of A. Therefore G contains no elements with order being the prime p, i.e., G is a p'-group. Thus, by Proposition **4.4.1**, A has a finite $\mathbb{Z}G$-composition series, a contradiction again. Hence we have proved the result.

From Proposition **4.4.2**, using Lemma **1.2.5** and Lemma **2.2.1**, we immediately have:

<u>Corollary 4.4.3</u>: Let G be a periodic almost abelian group and let A be a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p. Then A has a finite $\mathbb{Z}G$-composition series.

As before, we have the following results:

<u>Corollary 4.4.4</u>: If G is a Černikov group and if A is a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p, then A has a finite $\mathbb{Z}G$-composition series.

<u>Corollary 4.4.5</u>: If G is a locally finite group satisfying the minimal condition on subgroups and if A is a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p, then A has a finite $\mathbb{Z}G$-composition series.

Now, for Conjecture **B**, we have:

<u>Proposition 4.4.6</u>: Let G be a periodic abelian group with $|\pi(G)| < \infty$, where $\pi(G) = \{\text{prime } p;\ \text{G has an element of order } p\}$, and let A be a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being infinite. Then A is torsion

and so has a finite $\mathbb{Z}G$-composition series as well as a finite exponent.

Proof: We only need to prove A is torsion. Suppose A is not torsion, then by the noetherian condition we may assume that for every nonzero $\mathbb{Z}G$-submodule C of A, A/C is torsion. Certainly, A is a torsion-free $\mathbb{Z}G$-module. Also, we may assume that G acts faithfully on A, i.e., $C_G(A) = 1$.

For any element $1 \neq x \in G$, since G is abelian, so $A(x-1)$ and $C_A(x)$ are both $\mathbb{Z}G$-submodules of A and $A(x-1) \cong_{\mathbb{Z}G} A/C_A(x)$. If $C_A(x) \neq 0$, then $A(x-1) \neq 0$ by $C_G(A) = 1$, thus $A/A(x-1)$ and $A(x-1)$ being torsion implies that A is torsion, a contradiction. So we must have $C_A(x) = 0$ for any $1 \neq x \in G$. If G has two elements x and y satisfying $\langle x \rangle \cap \langle y \rangle = 1$ and both x and y are of the same order p for some prime p, by

$$\begin{aligned} &a[1 + (x^i y^j) + (x^i y^j)^2 + \cdots + (x^i y^j)^{p-1}](x^i y^j - 1) \\ &= a[(x^i y^j)^p - 1] = a0 = 0 \end{aligned}$$

for any $a \in A$, we have

$$a[1 + (x^i y^j) + (x^i y^j)^2 + \cdots + (x^i y^j)^{p-1}] \in C_A(x^i y^j) = 0.$$

Thus

$$\begin{aligned} pa = a[&(1 + 1 + 1 + \cdots + 1) \\ &+(y + xy + x^2y + \cdots + x^{p-1}y) \\ &+(y^2 + x^2y^2 + x^4y^2 + \cdots + x^{2(p-1)}y^2) \\ &\quad \vdots \\ &+(y^{p-1} + x^{p-1}y^{p-1} + x^{2(p-1)}y^{p-1} + \cdots + x^{(p-1)^2}y^{(p-1)})] \\ = a[&(1 + y + y^2 + \cdots + y^{p-1}) \\ &+(1 + xy + (xy)^2 + \cdots + (xy)^{p-1}) \\ &+(1 + x^2y + (x^2y)^2 + \cdots + (x^2y)^{p-1}) \\ &\quad \vdots \\ &+(1 + x^{p-1}y + (x^{p-1}y)^2 + \cdots + (x^{p-1}y)^{p-1})] = 0, \quad \text{where } 0 \neq a \in A. \end{aligned}$$

This is contrary to A being torsion-free. Hence, if $p \in \pi(G)$, then G has only one subgroup of order p, and then the Sylow p-subgroup of G is locally cyclic. Thus the Sylow p-subgroup S_p of G is either finite cyclic or is isomorphic to C_{p^∞}. By A having infinite irreducible $\mathbb{Z}$G-factors, we have G is infinite. Since $\pi(G)$ is a finite set and G is abelian, at least one Sylow subgroup of G is infinite, so there is at least one $p \in \pi(G)$ such that the Sylow p-subgroup S_p is a quasicyclic group. Let

$$S_p = \langle x_1, x_2, x_3, \cdots; \ x_1^p = 1, \ x_{i+1}^p = x_i, \ i = 1, 2, \cdots \rangle.$$

For the prime p, using Lemma **2.1.4**, we have A/pA is nonzero and $\bigcap_i p^i A = 0$. By Proposition **4.4.2**, A has a descending series of $\mathbb{Z}$G-submodules:

$$A = M_0 > M_1 > M_2 > \cdots > \bigcap_i M_i = 0,$$

in which each $\mathbb{Z}$G-factor M_i/M_{i+1} is irreducible and, as a group, is an infinite elementary abelian p-group, $i = 0, 1, 2, \cdots$. For any $x \in S_p$, since x is of order a power of p, by Lemma **1.2.8**, we have $x \in C_G(M_i/M_{i+1})$, and then S_p is contained in $C_G(M_i/M_{i+1})$ for any $i \geq 0$.

Now for any $a \in M_i \backslash M_{i+1}$ and any $x \in S_p$, $ax = a+b$ for some $b \in M_{i+1}$, thus $a(1+ x +\cdots+ x^{p-1}) = pa + b[(p-1) + (p-2)x + \cdots + x^{p-2}] \in M_{i+1}$ and then $M_i(1+ x + \cdots+ x^{p-1}) \leq M_{i+1}$. For $0 \neq a \in A$, by $\bigcap_i M_i = 0$, there exists i_0 such that $a \in M_{i_0} \backslash M_{i_0+1}$. For $x_j \in S_p$, let $ax_j = a+b_j$, where $b_j \in M_{i_0+1}$, $j = 1, 2, \cdots$. Since

$$\begin{aligned} a+b_j = ax_j = ax_{j+1}^p &= (a+b_{j+1})x_{j+1}^{p-1} \\ &= a+b_{j+1}(1+ x_{j+1}+\cdots+ x_{j+1}^{p-1}), \end{aligned}$$

then $b_j = b_{j+1}(1+ x_{j+1}+\cdots+ x_{j+1}^{p-1}) \in M_{i_0+2}$, $j = 1, 2, \cdots$. We suppose that

$b_j \in M_\alpha$, $j = 1, 2, \cdots$. Then by $a+b_j = a+b_{j+1}(1+ x_{j+1}+\cdots+ x_{j+1}^{p-1})$ again, we have $b_j = b_{j+1}(1+ x_{j+1}+\cdots+ x_{j+1}^{p-1}) \in M_{\alpha+1}$. So, by induction, we have $b_j \in M_i$ for all $i = 1, 2, \cdots$. Then $b_j \in \bigcap_i M_i = 0$, i.e., $b_j = 0$ for all j. That is, $a = ax_j$ for all j. Therefore $a \in C_A(S_p)$ and then, by the arbitrarity of a, we have $1 \neq S_p \leq C_G(A)$ contrary to G being faithful on A. By this contradiction, we have proved the result.

Consequently, by using Lemma **1.2.5** and Lemma **2.2.6**, we have:

Corollary **4.4.7**: Let G be a periodic almost abelian group with $\pi(G)$ being finite and let A be a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being infinite. Then A is torsion and so has finite exponent.

Corollary **4.4.8**: If G is a Černikov group and if A is a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being infinite, then A is torsion and so has finite exponent.

Corollary **4.4.9**: If G is a locally finite group satisfying the minimal condition on subgroups, and if A is a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being infinite, then A is torsion and so has finite exponent.

For the general case, we have finally proved neither Conjecture A nor Conjecture **B**. But the following results are worth mentioning here.

Proposition **4.4.10**: Let G be a hyperfinite p-group and A a noetherian

$\mathbb{Z}G$-module with $pA = 0$, where p is a prime. Then A is finite.

Proof: Suppose A is not finite, then by the noetherian condition we may assume that for any nonzero $\mathbb{Z}G$-submodule C of A, A/C is finite. Also, we may assume that $A = \langle a_1, \cdots, a_n \rangle^G$, where a_i is of order p for all i. So it follows that $G \neq C_G(A)$ and then, by replacing G by $G/C_G(A)$, we may assume that G acts faithfully on A.

Since G is a hyperfinite p-group, so $Z(G) \neq 1$ (Lemma **1.2.12**). Let $x \in Z(G)$ with x being of order p, then $A_1 = \langle a_1 \rangle^{\langle x \rangle}$ is a finite $\mathbb{Z}\langle x \rangle$-module. Thus there exists $0 \neq a_0 \in A_1$ such that $a_0 \in C_A(x)$. By G being faithful on A, we have $A \neq C_A(x)$, and so $A(x-1)$ $(\cong_{\mathbb{Z}G} A/C_A(x))$ is a nonzero finite $\mathbb{Z}G$-module. Also $A/A(x-1)$ is finite, which implies that A is finite, a contradiction. So the result is true.

Proposition **4.4.11**: Let G be a hyperfinite p-group and A a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being infinite. Then A is a torsion p'-group of finite exponent.

Proof: Let T(A) be the torsion part of A. If $T(A) < A$, then $A/T(A)$ is a torsion-free noetherian $\mathbb{Z}G$-module. By Lemma **2.1.4**, $p(A/T(A)) < A/T(A)$, and by Proposition **4.4.10**, $(A/T(A))/p(A/T(A))$ is finite. But A and hence $A/T(A)$ has no nonzero finite $\mathbb{Z}G$-factors, a contradiction. Thus $T(A) = A$. Since A is noetherian, so is of finite exponent and then, by Proposition **4.4.10** again, we have A must be a p'-group. So the result holds.

Combining all the main results in the two chapters above, we have:

Theorem: If G is a finite extension of a periodic abelian group with $\pi(G)$

finite, then

(1) any noetherian $\mathbb{Z}G$-module A has an f–decomposition

$$A = A^{f} \oplus A^{\bar{f}}$$

where A^{f} is a $\mathbb{Z}G$-submodule of A such that each nonzero irreducible $\mathbb{Z}G$-factor of A^{f} is finite while the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ has no nonzero finite $\mathbb{Z}G$-factors;

(2) A^{f} as a group is finitely generated and $G/C_G(A^{f})$ is finite; and

(3) $A^{\bar{f}}$ is torsion and has a finite exponent as well as a finite $\mathbb{Z}G$-composition series.

5 MODULES OVER HYPER–(CYCLIC OR FINITE) GROUPS

D.I.Zaǐcev has proved a number of results about modules over hypercyclic groups **[19, 20]** and modules over hyperfinite groups **[21, 22]**. In this chapter, we consider modules over hyper–(cyclic or finite) groups and get a lot of results which generalize all the Zaǐcev's results about modules over hyperfinite groups.

As we shall mention in §**6.1**, there exist torsion-free irreducible $\mathbb{Z}G$-modules over hypercyclic abelian groups (such irreducible $\mathbb{Z}G$-modules do not occur in the previous discussion). Due to the existence of such a module, we meet some difficulties in the research for the structure of the $\mathbb{Z}G$-modules over hyper–(cyclic or finite) groups. However, if we restrict ourselves only to the periodic case for artinian $\mathbb{Z}G$-modules and to the generalization of the Zaǐcev's results for noetherian $\mathbb{Z}G$-modules, we successfully get the required results. But we have not been able to generalize completely our decomposition theorem for hyperfinite groups.

§5.1 THE f–DECOMPOSITION

We have seen that: if A is an artinian (or noetherian) $\mathbb{Z}G$-module over a hyperfinite locally soluble group G, then A always has an f–decomposition

$$A = A^{f} \oplus A^{\bar{f}},$$

where A^f is a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A^f is finite and the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ contains no nonzero finite $\mathbb{Z}G$-factors. Now we consider G to be hyper–(cyclic or finite) instead of G being just hyperfinite and prove the following results (we note that these results are essentially generalizations of those of Zaǐcev).

Artinian Case:

Theorem C: If G is a hyper–(cyclic or finite) locally soluble group, then any periodic artinian $\mathbb{Z}G$-module A has an f–decomposition.

Proof: We may assume that G acts faithfully on A.

Suppose that the $\mathbb{Z}G$-module A does not have an f–decomposition, then one can find a $\mathbb{Z}G$-submodule not having an f–decomposition but each of its proper $\mathbb{Z}G$-submodules does have. We may suppose that A satisfies this condition. It follows that A is not a sum of proper $\mathbb{Z}G$-submodules (Lemma **1.2.24**) and so A has a unique maximal $\mathbb{Z}G$-submodule M, containing every proper $\mathbb{Z}G$-submodule of A. For each $a \in A\backslash M$, certainly $\langle a\rangle^G = A$. If G were finite, then A would be finitely generated as an abelian group and therefore finite, contrary to the choice of A. So G is infinite.

It is clear that A is a p-group for some prime p (since a periodic abelian group is the direct sum of its components). Let $M = M^f \oplus M^{\bar{f}}$ be the f–decomposition of M, we consider the following two cases: (1) A/M is finite, in this case, we may suppose that $M^f = 0$ by considering A/M^f; (2) A/M is infinite, similarly we suppose that $M^{\bar{f}} = 0$.

(1) A/M is finite and $M = M^{\bar{f}}$.

Now, for $H = C_G(A/M)$, since G is infinite, $H \neq 1$. Thus H contains a nontrivial normal subgroup of G being infinite cyclic or finite.

Suppose firstly that $1 \neq \langle x\rangle \leq H$ and $\langle x\rangle$ is normal in G, where x is of infinite order, then the $\mathbb{Z}G$-submodule $A(x-1) \leq M$. Since

$$\varphi:\quad a+M \longmapsto a(x-1) + M(x-1)$$

is clearly a homomorphism from the group A/M to the group $A(x-1)/M(x-1)$, so the $\mathbb{Z}G$-factor $A(x-1)/M(x-1)$ is finite. By M $(= M^{\bar{f}})$ having no nonzero finite $\mathbb{Z}G$-factors, we have $A(x-1) = M(x-1)$. Thus, for $a \in A$, there exists $m \in M$ such that $a(x-1) = m(x-1)$, i.e., $(a-m)(x-1) = 0$. Then $A = M+C_A(x)$. But this is contrary to G acting faithfully on A and all proper $\mathbb{Z}G$-submodules being contained in M. So H contains a nontrivial finite minimal normal subgroup, say N, of G. Since G is locally soluble, N is an elementary abelian q-group for some prime q. We show that $q \neq p$ by showing that $O_p(G) = 1$.

In fact, if $O_p(G) \neq 1$, then G has a finite normal p-subgroup K, $K \neq 1$, and we put $L = C_G(K)$. Since A is an artinian $\mathbb{Z}G$-module and $|G/L| < \infty$, A is also an artinian $\mathbb{Z}L$-module (Lemma **1.2.5**). Thus A has a least $\mathbb{Z}L$-submodule A_1 such that A_1 is not contained in M. Then $A_1/(A_1 \cap M)$ $\left(\cong_{\mathbb{Z}L} (A_1+M)/M\right)$ is a finite irreducible $\mathbb{Z}L$-module. Since A/M is an irreducible $\mathbb{Z}G$-module (and is a p-group), K acts trivially on A/M. Therefore $A_1(x-1) \leq M$ for each $x \in K$. Since $|G/L| < \infty$, M has no nonzero finite $\mathbb{Z}L$-factors (Lemma **2.2.6**) and so $A_1(x-1)$ is a $\mathbb{Z}L$-module with no nonzero finite $\mathbb{Z}L$-factors. Thus $A_1/C_{A_1}(x) \cong_{\mathbb{Z}L} A_1(x-1)$ has no nonzero finite $\mathbb{Z}L$-factors. But $A_1/(A_1 \cap M)$ is finite and so $C_{A_1}(x)$ is not contained in M. By the choice of A_1 we have $C_{A_1}(x) = A_1$ for each $x \in K$. Thus $A_1 \leq C_A(K)$ and so $C_A(K)$ is not contained in M. Since M contains all proper $\mathbb{Z}G$-submodules of A, we must have $C_A(K) = A$, contrary to G being faithful on A. Thus $O_p(G) = 1$ and then $q \neq p$.

By Lemma **1.2.4**, $A = [A, N] \oplus C_A(N)$. Since $N \leq H = C_G(A/M)$, $[A, N] \leq M$ and then $C_A(N) \neq 0$. Since M contains all proper $\mathbb{Z}G$-submodules of A, so we have $C_A(N) = A$ and so $[A, N] = 0$ contrary to G acting faithfully on A. We have proved case (1).

(2) A/M is infinite and $M^{\bar{f}} = 0$.

In this case, we choose a finite $\mathbb{Z}G$-submodule D, say, of M and let $H = C_G(D)$, then $|G/H| < \infty$ and so H contains either a nontrivial finite normal subgroup of G or an infinite cyclic subgroup being normal in G. If H contains a nontrivial finite subgroup N being minimal normal in G, then it is easy to know that N is a p'-group (by almost using the method used in case (1) to show that $O_p(G) = 1$). Hence, by Lemma **1.2.4**, $A = [A, N] \oplus C_A(N)$. The $\mathbb{Z}G$-submodule M does not contain both factors of this decomposition and so one of them is A (and the other is zero). But $[A, N] \neq 0$ by G being faithful on A. On the other hand, $D \leq C_A(N)$ and so $C_A(N) \neq 0$. This contradiction shows that H does not contain any nontrivial finite subgroups being normal in G. So, we may suppose that $1 \neq \langle x \rangle \leq H$, $\langle x \rangle$ is normal in G and x is of infinite order. Let $G_1 = C_G(x)$, then $|G/G_1| \leq 2$ and $x \in Z(G_1)$. Since A/M is infinite and irreducible, A has a least $\mathbb{Z}G_1$-submodule A_1 such that A_1/M is an infinite irreducible $\mathbb{Z}G_1$-module. If A_1 has an f–($\mathbb{Z}G_1$)–decomposition, i.e., $A_1 = B \oplus M$, where B is an infinite irreducible $\mathbb{Z}G_1$-submodule of A_1, then the nonzero $\mathbb{Z}G$-submodule B^G of A has no nonzero finite $\mathbb{Z}G$-factors (Lemma **2.2.7**). Thus $B^G \cap M = 0$ and then $A = B^G \oplus M$. That is, A has an f–($\mathbb{Z}G$)–decomposition with $A^f = M$ and $A^{\bar{f}} = B^G$, a contradiction. So A_1 has no f–($\mathbb{Z}G_1$)–decomposition. By passing from G to G_1 and A to A_1, we may assume that $1 \neq x \in H \cap Z(G)$.

(*a*) If $A(x-1) \neq A$, then $A(x-1) \leq M$. For φ: $a+M \longmapsto a(x-1)+M(x-1)$ $(a \in A)$, we have $A/M \overset{\varphi}{\cong}_{\mathbb{Z}G} A(x-1)/M(x-1)$ and $\operatorname{Ker}\varphi = 0$ or A/M. If $\operatorname{Ker}\varphi = 0$, then $A(x-1)/M(x-1)$ is an infinite irreducible factor of M, a contradiction. So $\operatorname{Ker}\varphi = A/M$. That is, $A(x-1) = M(x-1)$, and then $A = M + C_A(x)$, a contradiction again.

(*b*) $A(x-1) = A$. Then, for $a \in A\backslash M$, there exists $a_0 \in A$ such that $a = a_0(x-1)$ and $A = <a>^G$. Choose a finitely generated subgroup K of G such that $a_0 \in <a>^K$, $D \leq <a>^K$, and $x \in K$. Let $A_1 = <a>^K$, then A_1 is a finitely generated $\mathbb{Z}K$-module and K is a finitely generated hyper–(cyclic or finite) soluble group. If K is a supersoluble–by–finite group, then K is a polycyclic group (since supersoluble groups and finite soluble groups are both polycyclic). Thus A_1 has a $\mathbb{Z}K$-submodule B_1 of finite index such that $D \cap B_1 \neq D$ by the residual finiteness of finitely generated abelian–by-polycyclic groups **[7]**. Consider the finite $\mathbb{Z}K$-module A_1/B_1. Since $x \in Z(K)$, A_1/B_1 can be viewed as a $\mathbb{Z}<x>$-module. Then, by **[19]**, we can get $A_1/B_1 = B/B_1 \oplus C/B_1$, where the $\mathbb{Z}<x>$-submodule B/B_1 has a $\mathbb{Z}<x>$-composition series in which each $\mathbb{Z}<x>$-factor is $<x>$-trivial and the $\mathbb{Z}<x>$-submodule C/B_1 has no nonzero $\mathbb{Z}<x>$-factors which are $<x>$-trivial. Since $(D+B_1)/B_1$ is an $<x>$-trivial $\mathbb{Z}<x>$-submodule of A_1/B_1, so $B/B_1 \neq 0$. Thus A_1/C is a nonzero finite $\mathbb{Z}<x>$-module and $A_1/C \cong_{\mathbb{Z}<x>} (A_1/B_1)/(C/B_1) \cong_{\mathbb{Z}<x>} B/B_1$ shows that A_1/C has a finite $\mathbb{Z}<x>$-composition series in which each $\mathbb{Z}<x>$-factor is $<x>$-trivial. Hence $(A_1(x-1)+C)/C = \overline{A}_1(x-1) < \overline{A}_1$, where $\overline{A}_1 = A_1/C$. That is, $A_1(x-1) < A_1$. But, on the other hand, since $A_1(x-1)$ is a $\mathbb{Z}K$-module and $a_0 \in A_1$, so

$$A_1 = <a>^K = <a_0(x-1)>^K = (<a_0>(x-1))^K \leq (A_1(x-1))^K = A_1(x-1),$$

a contradiction.

The remainder is to prove that K is a supersoluble–by–finite group. However, it follows from the following result.

<u>Lemma **5.1.1**</u>: Any finitely generated hyper–(cyclic or finite) soluble group is a supersoluble–by-finite group.

Proof: In fact, let

$$G = G_\alpha \rhd \cdots \rhd G_1 \rhd G_0 = 1$$

be an ascending normal series of subgroups of a finitely generated soluble group G in which each factor $G_{\beta+1}/G_\beta$ is cyclic or finite. Since $G/G_\alpha = 1$ is clearly a supersoluble–by-finite group, we may assume that there exists $\beta \le \alpha$ such that G/G_β is supersoluble–by–finite but G/G_γ is not for all $\gamma < \beta$. We claim that $\beta = 0$. Otherwise, if $\beta-1$ exists, then, (*i*) $G_\beta/G_{\beta-1}$ is cyclic would imply that $G/G_{\beta-1}$ is supersoluble–by–finite, a contradiction; and (*ii*) $G_\beta/G_{\beta-1}$ is finite implies that $G/G_{\beta-1}$ is polycyclic and so is residually finite, therefore there is an N with G/N finite and $N \cap G_\beta = G_{\beta-1}$, thus $N/G_{\beta-1} \cong NG_\beta/G_\beta$ is supersoluble–by–finite and so is $G/G_{\beta-1}$, a contradiction. Thus, $\beta-1$ does not exist, i.e., β is a limit ordinal. Since G/G_β is finitely generated, by **[15,** p.403**]**, G_β is finitely generated as a G–operator group. Thus, let $G_\beta = \langle x_1, \cdots, x_n \rangle^G$, since $G_\beta = \bigcup_{\gamma<\beta} G_\gamma$ so there exist $\gamma_1, \cdots, \gamma_n$ such that $x_i \in G_{\gamma_i}$. Let $\gamma_0 < \beta$ such that $\gamma_i < \gamma_0$ for all $i = 1, \cdots, n$, then $x_i \in G_{\gamma_0}$ for all i. Since $G \rhd G_{\gamma_0}$, so

$$G_\beta = \langle x_1, \cdots, x_n \rangle^G \le (G_{\gamma_0})^G = G_{\gamma_0}.$$

Thus $G/G_\beta = G/G_{\gamma_0}$, contrary to the hypothesis for β. Hence $\beta = 0$ and then the result is proved.

Noetherian Case:

We have not yet got the complete f–decomposition theory for a noetherian $\mathbb{Z}G$-module over a hyper–(cyclic or finite) group, however, the following results look like a good start.

<u>Proposition 5.1.2</u>: Let H be a normal hyper–(cyclically or finitely) embedded subgroup of a group G, and let A be a nonzero noetherian $\mathbb{Z}G$-module. If $C_A(H) = 0$, then there is a subgroup K of H and a nonzero $\mathbb{Z}G$-submodule B of A such that K is normal in G, $C_B(K) = 0$, and K induces in B a cyclic or finite group of automorphisms.

<u>Proof</u>: Suppose the lemma is false. Using the noetherian condition we may assume that the lemma is true in all proper $\mathbb{Z}G$-images of the $\mathbb{Z}G$-module A. We may also assume that G acts faithfully on A.

There is a cyclic or finite subgroup $F \leq H$ with F being normal in G. If $C_A(F) = 0$ then the lemma is true taking F, A for K,B.

Consider the second possibility $C_A(F) \neq 0$. We let A_1 be the $\mathbb{Z}G$-submodule $C_A(F)$ and let $H_1 = C_H(F)$. Then H_1 is normal in G and $|H/H_1| < \infty$.

(1) Suppose that the centralizer $A_2/A_1 = C_{A/A_1}(H_1)$ is nonzero, i.e., $A_2 \neq A_1$. Consider the $\mathbb{Z}H_1$-isomorphism $A_2/C_{A_2}(f) \cong_{\mathbb{Z}H_1} A_2(f-1)$, where $f \in F$. Since $A_1 \leq C_{A_2}(f)$ and A_2/A_1 is H_1-trivial, we have that $A_2(f-1)$ is H_1-trivial for any $f \in F$. It follows that $[A_2, F] = \sum_{f \in F} A_2(f-1)$ is H_1-trivial and so H induces a finite group of automorphisms on $[A_2, F]$. Since $A_2 \neq A_1$ the $\mathbb{Z}G$-submodule $[A_2, F] \neq 0$ and $C_{[A_2,F]}(H) = 0$ since $C_A(H) = 0$. Therefore the lemma is true with $K = H$, $B = [A_2, F]$.

(2) Suppose now that $A_2 = A_1$, i.e., $C_{A/A_1}(H_1) = 0$. Then the $\mathbb{Z}G$-module A/A_1 and the normal subgroup H_1 satisfy the hypotheses of the lemma and so

there is a subgroup K_1 of H_1 and a nonzero $\mathbb{Z}G$-submodule B_1/A_1 of A/A_1 such that K_1 is normal in G, $C_{B_1/A_1}(K_1) = 0$, and K_1 induces in B_1/A_1 a cyclic or finite group of automorphisms.

Put $G_1 = C_G(F)$; clearly $H_1 = H \cap G_1$, $|G/G_1| < \infty$.

(*a*) We consider firstly the case that $K_1/C_{K_1}(B_1/A_1)$ is cyclic.

Let $B_2 = [B_1, F]$ and let $K_0 = C_{K_1}(B_1/A_1)$. Since $A_1 = C_A(F)$, so

$$[K_0, B_1, F] = [[K_0, B_1], F] \leq [A_1, F] = 0;$$

also by $K_0 \leq K_1 \leq H_1 = C_H(F)$, we have

$$[F, K_0, B_1] = [[F, K_0], B_1] = [1, B_1] = 0.$$

Thus, by three subgroup lemma,

$$[B_2, K_0] = [[B_1, F], K_0] = [B_1, F, K_0] = 0.$$

Therefore $B_2 \leq C_A(K_0)$ and we then can view the noetherian $\mathbb{Z}G$-module B_2 as a noetherian $\mathbb{Z}(G/K_0)$-module. Applying Lemma **1.2.19** to the cyclic normal subgroup K_1/K_0 of G/K_0, there is an integer m such that

$$B_2(k-1)^m \cap C_{B_2}(k) = 0,$$

where k is an element such that $K_1 = K_0\langle k\rangle$.

If $B_2(k-1)^m = 0$, then

$$\begin{aligned} 0 = B_2(k-1)^m &= \Big(\sum_{f\in F} B_1(f-1)\Big)(k-1)^m \\ &= \sum_{f\in F} B_1\big((f-1)(k-1)^m\big) = \sum_{f\in F} B_1\big((k-1)^m(f-1)\big) \\ &= \sum_{f\in F} \big(B_1(k-1)^m\big)(f-1). \end{aligned}$$

That is, $B_1(k-1)^m \le C_A(F) = A_1$. But this is contrary to

$$C_{B_1/A_1}(k) = C_{B_1/A_1}(K_1) = 0.$$

So we have $B_2(k-1)^m \ne 0$ and then the lemma is true by taking $B = B_2(k-1)^m$ and $K = K_1$.

(*b*) Secondly, we consider the case that $K_1/C_{K_1}(B_1/A_1)$ is finite.

Choose in F a least set of elements $\{x_1, \cdots, x_n\}$ satisfying

$$A_1 = C_{B_1}(F) = C_{B_1}(x_1) \cap \cdots \cap C_{B_1}(x_n)$$

and put $B_2 = C_{B_1}(x_1) \cap \cdots \cap C_{B_1}(x_{n-1})$ if $n > 1$ and $B_2 = B_1$ if $n = 1$. Then

$$B_2 \ne A_1 \tag{1}$$

and $C_{B_2}(x_n) = C_{B_1}(x_1) \cap \cdots \cap C_{B_1}(x_n) = A_1$. Consider the $\mathbb{Z}G_1$-isomorphism

$$B_2/A_1 = B_2/\, C_{B_2}(x_n) \cong_{\mathbb{Z}G_1} B_2(x_n-1). \tag{2}$$

Since $K_1 \le G_1$, $B_2 \le B_1$, and K_1 induces a finite group of automorphisms on B_1/A_1, so K_1 induces a finite group of automorphisms on B_2/A_1 and hence on $B_2(x_n-1)$. Since $C_{B_1/A_1}(K_1) = 0$ we also have $C_{B_2(x_n-1)}(K_1) = 0$.

Let $D = B_2(x_n-1)$. Then D is a $\mathbb{Z}G_1$-submodule of B_1, $C_D(K_1) = 0$, and $|K_1/C_{K_1}(D)| < \infty$. Let $\overline{D}$ be the $\mathbb{Z}G$-module generated by D, then $\overline{D} = \sum_{g \in T} Dg$ is a finite sum of $\mathbb{Z}G_1$-submodules Dg, where T is a transversal to G_1 in G.

Note that since K_1 is normal in G, $C_{Dg}(K_1) = C_D(K_1)g = 0$, and $C_{K_1}(Dg) = g^{-1}C_{K_1}(D)g$, it follows that $|K_1/\bigcap_{g \in T} C_{K_1}(Dg)| < \infty$ and so K_1 induces a finite group of automorphisms in $\overline{D}$.

Now consider two cases.

(A) D contains an element of finite order.

Then D contains a maximal elementary abelian p-subgroup D_1 ($\neq 0$) and we let $\overline{D}_1 = \sum_{g\in T} D_1 g$. Let S be the K_1-socle of the $\mathbb{Z}G_1$-submodule D_1, i.e., sum of all irreducible $\mathbb{Z}K_1$-submodules (these irreducible $\mathbb{Z}K_1$-submodules are all finite since K_1 induces a finite group of automorphisms in D). Since D_1 is a $\mathbb{Z}G_1$-submodule and K_1 is normal in G so S is a $\mathbb{Z}G_1$-submodule and $\overline{S} = \sum_{g\in T} Sg$ is a $\mathbb{Z}G$-submodule. Now Sg is a sum of irreducible $\mathbb{Z}K_1$-submodules and so $\overline{S}$ is a sum of irreducible $\mathbb{Z}K_1$-submodules each being contained in some Sg. Since $C_{Dg}(K_1) = 0$ it follows that $C_{\overline{S}}(K_1) = 0$. Thus we can take K_1 and $\overline{S}$ satisfying the conclusion of the lemma.

(B) The group D is torsion-free.

Let $T(\overline{D})$ be the torsion part of $\overline{D}$. Since $\overline{D}$ is a noetherian $\mathbb{Z}G$-module, $T(\overline{D})$ has a finite exponent. Therefore $n\overline{D} \cap T(\overline{D}) = 0$ for some n and $n\overline{D}$ is torsion-free.

We put $m = |K_1/C_{K_1}(\overline{D})|$, $C = C_{\overline{D}}(K_1)$ and show that

$$[mn\overline{D}, K_1] \cap C = 0 \qquad (3)$$

In fact, if $a \in [mn\overline{D}, K_1] \cap C$, then $a \in [mn\tilde{D}, K_1]$ for some finitely generated K_1-admissible subgroup $\tilde{D}$ of $\overline{D}$. Since $n\tilde{D} \cap C = C_{n\tilde{D}}(K_1)$, $\tilde{D} \leq \overline{D}$, and $n\overline{D}$ is torsion-free, so $n\tilde{D}/(n\tilde{D} \cap C)$ is torsion-free and then $n\tilde{D} = (n\tilde{D} \cap C) \oplus V$, where V is a free abelian subgroup. Using Lemma **1.2.26**, there is in $n\tilde{D}$ a

K_1-admissible subgroup W such that $(n\tilde{D} \cap C) \cap W = 0$ and the factor group $n\tilde{D}/[(n\tilde{D} \cap C) \oplus W]$ has a finite exponent, dividing m. Thus $mn\tilde{D} \le (n\tilde{D} \cap C) \oplus W$. It follows that $[mn\tilde{D}, K_1] \le W$ and so $[mn\tilde{D}, K_1] \cap C = 0$. Hence $a = 0$ and (3) is proved.

Note now that $[mn\overline{D}, K_1] \neq 0$. In fact, if $[mn\overline{D}, K_1] = 0$, then $mn\overline{D} \le C_{\overline{D}}(K_1) = C$. Therefore $mnD \le C$ and since D is torsion-free, $D \le C$. This shows that D is a K_1-trivial $\mathbb{Z}G_1$-module and since $D = B_2(x_n - 1)$ and is G_1-isomorphic to B_2/A_1 (2) we have B_2/A_1 is also K_1-trivial. But $C_{B_1/A_1}(K_1) = 0$ and so $B_2 = A_1$ contrary to (1). Thus $[mn\overline{D}, K_1] \neq 0$. Since $[mn\overline{D}, K_1]$ is a $\mathbb{Z}G$-submodule and K_1 induces in it (as in $\overline{D}$) a finite group of automorphisms then it follows from (3) that the conditions of the lemma are satisfied by K_1 and $[mn\overline{D}, K_1]$. This completes the proof of the proposition.

<u>Proposition 5.1.3</u>: Let G be a hyper-(cyclic or finite) group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that B is of finite index in A and B has no nonzero finite $\mathbb{Z}G$-factors, then B has a complement in A, i.e., $A = B \oplus C$ for some finite $\mathbb{Z}G$-submodule C of A.

<u>Proof</u>: Suppose that B does not have a complement in A. By considering an appropriate factor-module of A we may assume that for every $\mathbb{Z}G$-submodule D of B with $D \neq 0$, B/D has a complement in A/D.

Put $H = C_G(A/B)$, then, since G/H is finite and the irreducible $\mathbb{Z}G$-factors of B are all infinite, we have $C_B(H) = 0$ so we can apply Proposition **5.1.2** to the subgroup H and the $\mathbb{Z}G$-module B. So there is a subgroup K of H and a nonzero $\mathbb{Z}G$-submodule D of B such that K is normal in G,

$C_D(K) = 0$ and K induces on D a cyclic or finite group of automorphisms, i.e., $K/C_K(D)$ is cyclic or finite.

We write A as a sum $A = B+A_1$ with $B \cap A_1 = D$ and we will consider the $\mathbb{Z}G$-submodule A_1 as a faithful $\mathbb{Z}G_0$-module, where $G_0 = G/C_G(A_1)$. It is clear that D is a $\mathbb{Z}G_0$-submodule of A_1 such that D is of finite index in A_1 and D has no nonzero finite $\mathbb{Z}G_0$-factors. Also D has no complements in A_1 for otherwise if $A_1 = D \oplus C_1$ for some $\mathbb{Z}G_0$-submodule C_1 of A_1 then C_1 can be viewed as a $\mathbb{Z}G$-submodule of A by $G_0 = G/C_G(A_1)$ and then $A = B+A_1 = B \oplus C_1$ (Lemma **1,2,25**), a contradiction.

Since $C_D(K) = 0$ and $D \leq A_1$, so K is not contained in $C_G(A_1)$. Let $K_0 = (KC_G(A_1))/C_G(A_1)$, then $K_0 \neq 1$. Also, it is clear that $C_D(K_0) = 0$ and K_0 induces on the $\mathbb{Z}G_0$-submodule D of A_1 a cyclic or finite group of automorphisms. We prove that $C_{K_0}(D) = 1$. For suppose $C_{K_0}(D) \neq 1$ and let F_0 be a nontrivial cyclic or finite normal subgroup of G_0 contained in $C_{K_0}(D)$. If $x \in F_0$, then $D \leq C_{A_1}(x)$. Since $|A_1/D| = |A/B| < \infty$ and, as groups, $A_1/C_{A_1}(x) \cong A_1(x-1)$, we see that $A_1(x-1)$ is finite. Thus the $\mathbb{Z}G_0$-submodule $[A_1, F_0]$ is finite. Also $F_0 \leq C_{K_0}(D) \leq K_0 = (KC_G(A_1))/C_G(A_1) \leq (HC_G(A_1))/C_G(A_1) = (C_G(A/B)C_G(A_1))/C_G(A_1)$, thus $[A_1, F_0] \leq B$, and then $[A_1, F_0] \leq D$. By D having no nonzero finite $\mathbb{Z}G_0$-factors, we have $[A_1, F_0] = 0$ contrary to G_0 acting faithfully on A_1. So $C_{K_0}(D) = 1$ and hence K_0 is cyclic or finite.

Now put $G_1 = C_{G_0}(K_0)$, $K_0 = \langle x_1 = 1, x_2, \cdots, x_m \rangle$, $C_n = C_{A_1}(\langle x_1, \cdots, x_n \rangle)$, $n = 1, 2, \cdots, m$. We prove that $A_1 = D + C_n$, $n = 1, 2, \cdots, m$.

It is clear that $A_1 = D+C_1$. Suppose $A_1 = D+C_n$ we prove $A_1 = D+C_{n+1}$. Consider the isomorphism of $\mathbb{Z}G_1$-modules

$$C_n/C_{n+1} = C_n/C_{C_n}(x_{n+1}) \cong_{\mathbb{Z}G_1} C_n(x_{n+1}-1),$$

where $C_n(x_{n+1}-1)$ may not be contained in C_n if K_0 is nonabelian. Since $x_{n+1} \in K_0 = (KC_G(A_1))/C_G(A_1) \leq (HC_G(A_1))/C_G(A_1)$ $= (C_G(A/B)C_G(A_1))/C_G(A_1)$, the $\mathbb{Z}G_1$-module $C_n(x_{n+1}-1)$ of A_1 is contained in B and then in D. Since $|G_0/G_1| < \infty$ it follows from Lemma **2.2.6** that the irreducible $\mathbb{Z}G_1$-factors of D are all infinite, hence so are the factors of C_n/C_{n+1}. But $C_n/(C_{n+1}+(D\cap C_n)) \cong_{\mathbb{Z}G_1} (C_n+D)/(C_{n+1}+D)$, a factor module of the finite module A_1/D. Hence $C_n+D = C_{n+1}+D$ and so $A_1 = C_{n+1}+D$. Thus $A_1 = C_n+D$ for all $n = 1, 2, \cdots, m$. In particular, put $n = m$, $C_m = C_{A_1}(K_0)$ and $A_1 = D+C_{A_1}(K_0)$. BY Lemma **1.2.6**, $C_{A_1}(K_0)$ is a $\mathbb{Z}G_0$-submodule of A_1. Since $D\cap C_{A_1}(K) = C_D(K) = 0$ we have $A_1 = D\oplus C_{A_1}(K)$, contrary to D having no complements in A_1. The proof is completed.

Using almost the same proof as above, we immediately have:

Proposition **5.1.4**: Let G be a hyper-(cyclic or finite) group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that, as group, A/B is a finite p-group for some prime p and the $\mathbb{Z}G$-submodule B contains no nonzero $\mathbb{Z}G$-factors being finite p-groups. Then B has a complement in A, i.e., $A = B\oplus C$ for some $\mathbb{Z}G$-submodule C of A.

Dual to Proposition **5.1.3**, we have:

<u>Proposition **5.1.5**</u>: Let G be a hyper–(cyclic or finite) group, A a $\mathbb{Z}G$-module, and B a finite $\mathbb{Z}G$-submodule of A such that all irreducible $\mathbb{Z}G$-factors of A/B are infinite. Then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

<u>Proof:</u> By Zorn's Lemma, A has a $\mathbb{Z}G$-submodule D maximal with respect to $B \cap D = 0$. We show that $A = B \oplus D$. Suppose not, then by replacing A by A/D we may assume that for any nonzero $\mathbb{Z}G$-submodule C of A, $B \cap C \neq 0$. We also assume that G acts faithfully on A.

Put $H = C_G(B)$, $|G/H| < \infty$, so there is a normal subgroup K of G contained in H such that K is either cyclic or finite. Put $H_1 = C_H(K)$. Since H_1 is normal in G and $|G/H_1| < \infty$ it follows from Lemma **2.2.6** that the irreducible $\mathbb{Z}H_1$-factors of A/B are infinite. If $x \in K$, then $B \leq C_A(x)$ and so the irreducible $\mathbb{Z}H_1$-factors of $A/C_A(x)$ and hence $A(x-1)$ are infinite.

We prove that $[A, K] \cap B = 0$. If not, then there is a minimal set of elements $x_1, \cdots, x_n$ such that $B_1 = B \cap \sum_{i=1}^{n} A(x_i-1) \neq 0$. Then

$$\begin{aligned} B_1 &\cong_{\mathbb{Z}H_1} \left(B_1 \oplus \textstyle\sum_{i=1}^{n-1} A(x_i-1)\right) / \left(\textstyle\sum_{i=1}^{n-1} A(x_i-1)\right) \\ &\leq \left(\textstyle\sum_{i=1}^{n} A(x_i-1)\right) / \left(\textstyle\sum_{i=1}^{n-1} A(x_i-1)\right) \\ &\cong_{\mathbb{Z}H_1} A(x_n-1) / \left(A(x_n-1) \cap \textstyle\sum_{i=1}^{n-1} A(x_i-1)\right). \end{aligned}$$

This shows that $A(x_n-1)$ has a nonzero finite $\mathbb{Z}H_1$-factor contrary to the fact that the irreducible $\mathbb{Z}H_1$-factors of $A(x-1)$ are all infinite. Thus $[A, K] \cap B = 0$ and hence $[A, K] = 0$, contrary to G acting faithfully on A. So the result is true.

From Proposition **5.1.5**, we have:

Corollary **5.1.6**: Let G be a hyper–(cyclic or finite) group, and A a noetherian $\mathbb{Z}$G-module. Then A has a nonzero finite $\mathbb{Z}$G-factor if and only if A has a nonzero finite $\mathbb{Z}$G-image.

Proof: We only need to suppose that A has a finite $\mathbb{Z}$G-factor B/C, then using the noetherian condition we may assume that every irreducible $\mathbb{Z}$G-factor of A/B is infinite. Then applying Proposition **5.1.5** to A/C with the finite $\mathbb{Z}$G-submodule B/C we obtain a finite $\mathbb{Z}$G-image of A.

Also, almost follow the proof of Proposition **5.1.5**, we have:

Proposition **5.1.7**: Let G be a hyper–(cyclic or finite) group, A a $\mathbb{Z}$G-module and B a $\mathbb{Z}$G-submodule of A. If as a group B is a finite p–group for some prime p, and if the factor module A/B contains no nonzero finite $\mathbb{Z}$G-factors being p-groups, then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}$G-submodule C of A.

By a similar proof to Corollary **5.1.6**, we have:

Corollary **5.1.8**: Let G be a hyper–(cyclic or finite) group, and A a noetherian $\mathbb{Z}$G-module. Then A has a nonzero $\mathbb{Z}$G-image being a finite p-group for some prime p if and only if A has such a nonzero $\mathbb{Z}$G-factor.

§5.2 THE SPLITTING THEORY

Depending on the corresponding decomposition of modules, D. I. Zaǐcev has proved the splitting results for hypercyclic (hypercentral) extensions ([19], [20]) and hyperfinite extensions ([21], [22]) of an abelian normal subgroup. Similarly, we will prove splitting results for hyper–(cyclic or finite) extensions of abelian groups in this section. We divide the discussion in two parts: one is for the artinian case, and another for the noetherian case. The results proved in this section can all be viewed as a generalization of Zaĭcev's results.

Artinian Case:

Before proving the main result of this part, we prove some lemmas.

Lemma 5.2.1: Let A be a nonzero artinian $\mathbb{Z}G$-module and H a normal hyper–(cyclically or finitely) embedded subgroup of G. If A is an H–perfect module then H has a subgroup K such that K is normal in G and A has a nonzero K–perfect $\mathbb{Z}G$-image on which K induces a cyclic or finite group of automorphisms.

Proof: (It is similar to the corresponding one in [21].)

We assume that G acts faithfully on A and take a nontrivial cyclic or finite normal subgroup F of G with $F \le H$. If $A = [A, F]$, then we can take F to be the required subgroup and A the F–perfect $\mathbb{Z}G$-image.

So we may suppose that $A_1 = [A, F] \neq A$. Put $H_1 = C_H(F)$, then H_1 is normal in G and $|H/H_1|$ is finite.

(1) Suppose $A_1 \neq [A_1, H_1]$. Consider the $\mathbb{Z}G$-module $\overline{A} = A/[A_1, H_1]$ and its $\mathbb{Z}G$-submodule $\overline{A}_1 = A_1/[A_1, H_1]$. Since $\overline{A}_1$ is an H_1–trivial $\mathbb{Z}H_1$-module and, for each $x \in F$, $\overline{A}(x-1) \le [\overline{A}, F] \le \overline{A}_1$, so $\overline{A}(x-1)$ is an H_1–trivial $\mathbb{Z}H_1$-module. Therefore the $\mathbb{Z}H_1$-isomorphism $\overline{A}/C_{\overline{A}}(x) \cong \overline{A}(x-1)$ shows that

$\overline{A}/C_{\overline{A}}(x)$ is an H_1–trivial $\mathbb{Z}H_1$-module. It follows that $\overline{A}/C_{\overline{A}}(F) = \overline{A}/\bigcap_{x\in F}C_{\overline{A}}(x)$ (or $\overline{A}/C_{\overline{A}}(F) = \overline{A}/C_{\overline{A}}(x_0)$, where $F = \langle x_0\rangle$ if F is cyclic) is an H_1–trivial $\mathbb{Z}H_1$-module.

The factor $\overline{A}/C_{\overline{A}}(F)$ is nonzero, for if $\overline{A} = C_{\overline{A}}(F)$, then $[A, F] \le [A_1, H_1]$ and hence $A_1 = [A, F] = [A_1, H_1]$ contrary to assumption. Furthermore, the $\mathbb{Z}G$-module $\overline{A}/C_{\overline{A}}(F)$ is H–perfect, since A is H–perfect, and H induces in $\overline{A}/C_{\overline{A}}(F)$ a finite group of automorphisms (since it is H_1–trivial and H/H_1 is finite). Thus we can take H to be the required subgroup and $\overline{A}/C_{\overline{A}}(F)$ as the H–perfect $\mathbb{Z}G$-image.

<u>(2) Suppose $A_1 = [A_1, H_1]$.</u> Suppose that the lemma is true for the pair (A_1, H_1), i.e., H_1 has a subgroup K_1 with K_1 being normal in G and A_1 has a nonzero K_1–perfect $\mathbb{Z}G$-image $\hat{A}_1$ on which K_1 induces a cyclic or finite group of automorphisms. We show that this implies the result for the pair (A, H).

Let $B_i = A(x_i-1)$, where x_i $(1 \le i \le m)$ are elements of the group F with $m = 1$ and $F = \langle x_1\rangle$ if F is infinite or $m = |F|$ if F is finite. Then $A_1 = [A, F] = B_1+\cdots+B_m$ and $\hat{A}_1 = \hat{B}_1+\cdots+\hat{B}_m$, where the $\hat{B}_i$ are $\mathbb{Z}G_1$-modules, $G_1 = C_G(F)$. Let k be the largest integer such that $\hat{A}_1 = \hat{B}_k+\cdots+\hat{B}_m$. Then $\hat{A}_1/(\hat{B}_{k+1}+\cdots+\hat{B}_m) \cong_{\mathbb{Z}G_1} \hat{B}_k/(\hat{B}_k\cap(\hat{B}_{k+1}+\cdots+\hat{B}_m))$ is a K_1–perfect $\mathbb{Z}G_1$-module, since $\hat{A}_1$ is K_1–perfect, and is nonzero by the choice of k. Thus $\hat{B}_k$, and hence B_k, has a nonzero K_1–perfect $\mathbb{Z}G_1$-image. By the $\mathbb{Z}G_1$-isomorphism $A/C_A(x_k) \cong A(x_k-1) = B_k$, A has a nonzero K_1–perfect $\mathbb{Z}G_1$-image A/D, say.

Since D is a $\mathbb{Z}G_1$-module and $|G/G_1| < \infty$, there are only finitely many $\mathbb{Z}G_1$-modules of the form Dg $(g \in G)$. Put $C = [A, K_1]\cap D$, C is a $\mathbb{Z}G_1$-module and $C_0 = \sum_{g\in G}([A, K_1]\cap Dg)$ is the $\mathbb{Z}G$-submodule of A generated by C.

<u>(*a*) Suppose $A = C_0$.</u> Since $C_0 \le [A, K_1]$ we have $A = [A, K_1]$, i.e., A

is a K_1–perfect $\mathbb{Z}G$-module. By hypothesis, K_1 induces a cyclic or finite group of automorphisms on the $\mathbb{Z}G$-image $\hat{A}_1$, $C_{K_1}(\hat{A}_1)$ is normal in G and $K_1/C_{K_1}(\hat{A}_1)$ is cyclic or finite. Elements of $C_{K_1}(\hat{A}_1)$ act trivially on $\hat{A}_1$ and hence on $\hat{B}_k$ and so on A/D. Therefore $[A, C_{K_1}(\hat{A}_1)] \leq D$ and $A/[A, C_{K_1}(\hat{A}_1)]$ is a nonzero K_1–perfect $\mathbb{Z}G$-image of A on which K_1 induces a cyclic or finite group of automorphisms.

(*b*) <u>Suppose $A \neq C_0$.</u> Since A/D is a K_1–perfect $\mathbb{Z}G_1$-image we have $A = [A, K_1] + D$ and, since $C = [A, K_1] \cap D \leq C_0$, $A/C_0 = ([A, K_1]/C_0) \oplus ((D+C_0)/C_0)$. By considering the $\mathbb{Z}G$-factor-module A/C_0, we may assume that $C_0 = 0$ and hence $A = [A, K_1] \oplus D$. Put $D_0 = \sum_{g\in G} Dg$. Since $|G/G_1| < \infty$ this is a sum of finitely many $\mathbb{Z}G_1$-modules Dg. Also $[Dg, K_1] \leq [A, K_1]\cap Dg \leq C_0 = 0$ so that each Dg is K_1–trivial. It follows that D_0 has a finite series of $\mathbb{Z}G_1$-submodules in which the factors are K_1–trivial. Therefore, since A/D is K_1–perfect, we must have $D_0 \neq A$. Thus A/D_0 is a nonzero K_1–perfect $\mathbb{Z}G$-image of A. Also K_1 induces on A/D and hence on A/D_0 a cyclic or finite group of automorphisms. Thus we can take K_1 to be the required subgroup and A/D_0 the required $\mathbb{Z}G$-image.

In Case (2), we have now shown that if the lemma fails to hold for the pair (A, H) where A is an H–perfect $\mathbb{Z}G$-module and H is a normal subgroup of G, then there is an H_1–perfect proper $\mathbb{Z}G$-submodule A_1 with H_1 being normal in G and contained in H such that the lemma is false for the pair (A_1, H_1). Hence

A_1 has a proper H_1–perfect $\mathbb{Z}G$-submodule A_2 such that the lemma is false for (A_2, H_2). This process leads to an infinite properly descending chain of $\mathbb{Z}G$-submodules $A > A_1 > A_2 > \cdots$, contrary to the artinian condition, and so the lemma is proved.

<u>Lemma 5.2.2</u>: Let G be a hyper–(cyclic or finite) locally soluble group, B a $\mathbb{Z}G$-module, and A a periodic artinian $\mathbb{Z}G$-submodule of B such that all irreducible $\mathbb{Z}G$-factors of A are infinite. If B/A as an abelian group is finitely generated and is G–trivial, then $B = A \oplus B_1$ for some $\mathbb{Z}G$-submodule B_1 of B.

<u>Proof</u>: Since A is an artinian $\mathbb{Z}G$-submodule of B, it is possible to choose a $\mathbb{Z}G$-submodule B_1 such that $B = A+B_1$ and for each $U \le B_1$ with $B = A+U$, the intersections $A \cap U$ and $A \cap B_1$ are equal. We prove that $A \cap B_1 = 0$.

Suppose $A_1 = A \cap B_1 \neq 0$. Since B/A is finitely generated and $B/A \cong_{\mathbb{Z}G} B_1/A_1$, we have $B_1 = A_1 + \langle b_1, \cdots, b_n \rangle$ for some $b_1, \cdots, b_n \in B_1$. Firstly, if $Z(G) = 1$, since G is a hyper–(cyclic or finite) locally soluble group, G has a nontrivial normal subgroup H, which is either cyclic or abelian and finite. Let $G_1 = C_G(H)$ and consider B as a $\mathbb{Z}G_1$-module. Since $|G/G_1| < \infty$, the conditions of the lemma are clearly satisfied by G_1, B and A. So we may assume that $Z(G) \neq 1$. Secondly, if $C_G(B_1) \neq 1$, then $C_G(B_1) \lhd G$ by the nonzero $\mathbb{Z}G$-submodule A_1 being contained in B_1 and A having no nonzero finite $\mathbb{Z}G$-factors. Let $\overline{G} = G/C_G(B_1)$, then $\overline{G}$ is a hyper–(cyclic or finite) locally soluble group. We regard B_1 as a $\mathbb{Z}\overline{G}$-module, and then, as above, we may assume that $Z(\overline{G}) \neq 1$. Let $1 \neq \overline{g} \in Z(\overline{G})$, then $\langle \overline{g} \rangle$ is a central cyclic subgroup of $\overline{G}$ and $B_1(\overline{g}-1) = A_1(\overline{g}-1) + \langle b_1(\overline{g}-1), \cdots, b_n(\overline{g}-1) \rangle$. Since B_1/A_1 $(\cong_{\mathbb{Z}G} B/A)$ is G–trivial and therefore is $\overline{G}$–trivial, we have $B_1(\overline{g}-1)/A_1(\overline{g}-1)$ is $\overline{G}$–trivial and $B_1(\bar{g}-1) \le A_1$. Also, since A_1 is periodic and has no nonzero finite $\mathbb{Z}G$-factors (therefore has no nonzero finite $\mathbb{Z}\bar{G}$-factors), we have $B_1(\bar{g}-1) = A_1(\bar{g}-1)$. Thus, there exist $a_i \in A_1$ such that $b_i(\bar{g}-1) = a_i(\bar{g}-1)$ for all $1 \le i \le n$, that is, $b_i-a_i \in C_{B_1}(\bar{g})$. So $B_1 = A_1+C_{B_1}(\bar{g})$, where $C_{B_1}(\bar{g})$ is certainly a proper $\mathbb{Z}\bar{G}$-submodule of B_1. Any $\mathbb{Z}\bar{G}$-submodule of B_1 is a $\mathbb{Z}G$-submodule and so $C_{B_1}(\bar{g})$ is a proper $\mathbb{Z}G$-submodule of B_1. But, since $A+C_{B_1}(\bar{g}) = A+B_1 = B$, then $A\cap C_{B_1}(\bar{g}) < A\cap B_1$, which is contrary to the choice of B_1. So $A\cap B_1 = 0$, the lemma is proved.

<u>Corollary 5.2.3</u>: Let G, B, and A be as in Lemma **5.2.2** with one exception that B/A is just cyclic, then the same is true for B, i.e., B has a $\mathbb{Z}G$-submodule B_1 such that $B = A\oplus B_1$.

<u>Proof</u>: Let $G_1 = C_G(B/A)$, then $|G/G_1| \le 2$. Regarding B as a $\mathbb{Z}G_1$-module, then, by Lemma **5.2.2**, $B = A\oplus B_1$ for some $\mathbb{Z}G_1$-submodule B_1 of B. For $g \in G$, if $B_1g \ne B_1$, then $0 \ne B_1g/(B_1\cap B_1g) \cong_{\mathbb{Z}G_1} (B_1+B_1g)/B_1 \le B/B_1 \cong_{\mathbb{Z}G_1} A$. That is, A has a cyclic (and hence finite by A being periodic) $\mathbb{Z}G_1$-factor. By Lemma **2.2.6**, A has a nonzero finite $\mathbb{Z}G$-factor, a contradiction. So $B_1g = B_1$ for all $g \in G$. That is, B_1 is a $\mathbb{Z}G$-submodule of B. Thus the result holds.

<u>Lemma 5.2.4</u>: Let E be an extension of a periodic abelian subgroup A by a hyper–(cyclic or finite) locally soluble group G such that A is an artinian $\mathbb{Z}G$-module without any nonzero finite $\mathbb{Z}G$-factors. If N/A is normal in E/A and $N \le C_E(A)$, then $N = A\times M$, where M is normal in E and is contained in any supplement to A in E.

<u>Proof</u>: Let M be a normal subgroup of E contained in N and maximal subject to

$M \cap A = 1$. By considering the factor group E/M, we may assume that $M = 1$. Then E satisfies the following property (*): if S is normal in E, $S \le N$, and $S \ne 1$, then $A \cap S \ne 1$. We show that in this situation $A = N$.

Suppose that $A \ne N$. Since the factor group E/A is hyper-(cyclic or finite), there is a nontrivial finite subgroup $K/A \le N/A$ such that K is normal in E or an infinite cyclic subgroup $L/A \le N/A$ such that L is normal in E.

For K, by the hypothesis of the lemma, $K \le C_E(A)$ and so K is a finite extension of its central subgroup A. Hence K' is finite (Schur, Thm **10.1.4** in **[15]**). It follows that $A \cap K'$ is finite and by A having no nonzero finite $\mathbb{Z}G$-factors, $A \cap K' = 1$, and so, by (*), $K' = 1$. That is, K is abelian. By Theorem **C**, $K = A \times K^f$, where K^f is a finite normal subgroup of E. By (*) again, $K^f = 1$, that is, $K = A$, contrary to K/A being nontrivial.

For L, by the hypothesis of the lemma, $L \le C_E(A)$ and so L is a cyclic extension of its central subgroup A. Thus L is abelian. By Corollary **5.2.3**, $L = A \times L_1$, where L_1 is a cyclic normal subgroup of E. By (*), $L_1 = 1$, that is, $L = A$, a contradiction.

Thus, we have got $N = A \times M$ with M being normal in E.

If E_1 is a supplement to A in E, then $AE_1 = E$ and so $N = A(N \cap E_1)$ with $N \cap E_1$ being normal in $AE_1 = E$. Since M is hyper-(cyclically or finitely) embedded in E, and so is $M(N \cap E_1)/(N \cap E_1)$ in $E/(N \cap E_1)$. But, since A is periodic and has no nontrivial finite E-factors (otherwise A would have a nonzero finite $\mathbb{Z}G$-factor, a contradiction), by

$$M(N \cap E_1)/(N \cap E_1) \le N/(N \cap E_1) = A(N \cap E_1)/(N \cap E_1),$$

we have $M(N \cap E_1) = N \cap E_1$, i.e., $M \le E_1$. The result is proved.

Comparing with Lemma **5.2.1**, we have the following partial dual result:

Lemma **5.2.5**: Let E be an extension of a periodic abelian group A by a hyper–(cyclic or finite) locally soluble group G and suppose that A is an artinian $\mathbb{Z}G$-module without any nonzero finite $\mathbb{Z}G$-factors. If also G has no nontrivial finite normal subgroups and E has no nontrivial cyclic normal subgroups, then there is a normal subgroup K of E such that $[A\cap K, K] = A\cap K$ and $K/(A\cap K)$ is a nontrivial cyclic group. If, furthermore, A has a complement L in E, then K can be chosen so that $K \leq (A\cap K)L$.

Proof: By G being hyper–(cyclic or finite) and having no nontrivial finite normal subgroups, we have E contains a normal subgroup K_0 such that $A < K_0$ and K_0/A is cyclic. By Lemma **5.2.4** and E having no nontrivial normal cyclic subgroups, we have $C_E(A) = A$. Thus, K_0 is nonabelian. Let $A_0 = [A, K_0]$. If $A = A_0$, then K_0 is the required subgroup. If $A \neq A_0$, consider the $\mathbb{Z}G$-module $K_0/A_0 = \overline{K}_0$. By Corollary **5.2.3**, we have $\overline{K}_0 = \overline{A} \oplus \overline{K}_1$ for some $\mathbb{Z}G$-submodule $\overline{K}_1$, where $\overline{A} = A/A_0$. Let K_1 be the preimage of $\overline{K}_1$, then K_1 is a normal subgroup of E, $K_1\cap A = A_0$, and $K_1/A_0 \cong K_0/A$. Furthermore, if $A_1 = [A_0, K_1] \neq A_1$, then similarly we can find a subgroup K_2 being normal in E such that $K_2 \leq K_1$, $K_2\cap A = A_1$, and $K_2/A_1 \cong K_0/A$. If $A_2 = [A_1, K_2] \neq A_1$, then there exists a subgroup K_3. The chain of submodules $A > A_0 > A_1 > \cdots$ must terminate at A_n, say, and there is a normal subgroup K_{n+1} such that $[A_n, K_{n+1}] = A_n$, $A\cap K_{n+1} = A_n$, and $K_{n+1}/A_n \cong K_0/A$. Thus K_{n+1} has the required properties.

Now suppose that A has a complement L in E. Then $K_0 \leq E = AL = (A\cap K_0)L$ so that the second part is also proved if $K = K_0$. Since $E = A]L$, we have $K_0/A_0 = \overline{K}_0 = \overline{A}\times(\overline{K}_0\cap\overline{L})$. By $\overline{A}$ having no nonzero finite (and hence cyclic) $\mathbb{Z}G$-factors, we clearly have that the direct factor $\overline{A}$ of $\overline{K}_0$ has a unique complement in $\overline{K}_0$. Therefore, it follows that $\overline{K}_0\cap\overline{L} = \overline{K}_1$ and then $K_1 \leq A_0L$

$= (K_1 \cap A)L$. Using a similar argument and induction on n, we immediately have $K_{n+1} \leq A_n L = (A \cap K_{n+1})L$. The result holds.

Now we prove the main result of this part.

Theorem D: Let G be a hyper–(cyclic or finite) locally soluble group and A a periodic artinian $\mathbb{Z}G$-module. If A has no nonzero finite $\mathbb{Z}G$-submodules, then any extension E of A by G splits conjugately over A and A has no nonzero finite $\mathbb{Z}G$-factors. Also, every complement to A in E is self–normalizing.

Proof: By Theorem **C**, A has no nonzero finite $\mathbb{Z}G$-factors.

Existence of self–normalizing complements: Since A is artinian, E has a subgroup E_1 minimal with respect to $E = AE_1$. We prove that $A \cap E_1 = 1$.

Suppose that $A_1 = A \cap E_1 \neq 1$. The group E_1 and its subgroup A_1, considered as a $\mathbb{Z}G$-module, satisfy the conditions of the theorem (since $E/A \cong E_1/A_1$). We may therefore assume that $A = A_1$ and $E = E_1$, then A has no proper supplement in E and $A \neq 1$.

Since A is periodic, by passing from E to its factor group $E/A_{p'}$, we may assume that A is a p-group for some prime p. By A having no nonzero finite $\mathbb{Z}G$-factors, we have $A = [A, G]$. So, by Lemma **5.2.1**, there is a nonzero $\mathbb{Z}G$-image A/B and a nontrivial normal subgroup K/A of E/A such that $A/B = [A/B, K]$ and $K/C_K(A/B)$ is cyclic or finite. Passing to the factor group E/B, we can take $B = 1$ so that $A = [A, K]$ and $K/C_K(A)$ is cyclic or finite.

(1) Suppose $K/C_K(A)$ is cyclic. By Lemma **5.2.4**, $C_E(A) = A \times M$, where M is a normal subgroup of E. Passing to E/M, we may assume that $C_E(A) = A$. So $C_K(A) = K \cap C_E(A) = A$, i.e., K/A is cyclic. Passing from E to E/C, where C is

the maximal hypercyclically embedded normal subgroup of E, we may assume that E has no nontrivial cyclic normal subgroups. Since $A = [A, K]$ and K/A is a nontrivial cyclic normal subgroup of E/A, there is an element x such that $K = A\langle x\rangle$. Then, by Lemma **1.2.16**, we have $E = AN_E(\langle x\rangle)$. By A having no proper supplement in E, we have $E = N_E(\langle x\rangle)$, i.e., $\langle x\rangle$ is a cyclic normal subgroup of E, a contradiction.

(2) Suppose $K/C_K(A)$ is finite. In this case, we need only consider the situation in which $K/C_K(A)$ is a q-group. For, let

$$K = K_0 > K_1 > \cdots > K_s = C_K(A)$$

be a series of normal subgroups of E, the factors of which are p_i-groups. Let i be the largest such that $A = [A, K_i]$. Then $[A, K_{i+1}] < A$, $[A, K_{i+1}]$ is normal in E, and we can pass to the factor group $E/[A, K_{i+1}]$. So we may assume that $[A, K_{i+1}] = 1$, $K_{i+1} \leq C_{K_i}(A)$, and $K_i/C_{K_i}(A)$ is a q-group ($q = p_i$). We can replace K by K_i and so assume that $K/C_K(A)$ is a q-group. By Lemma **5.2.4**, we may further assume that $C_E(A) = A$, and so $C_K(A) = A$. Thus K/A is a finite q-group for some prime q.

Consequently, we need only consider the following situation: E is an extension of the p-group A, where E/A has a nontrivial finite normal q-subgroup K/A such that $A = [A, K]$.

Let $q = p$. By Lemma **1.2.20**, A has a proper supplement in E, a contradiction;

Let $q \neq p$. If Q is a Sylow q-subgroup of K, then $K = AQ$ is abelian–by–finite and so is locally finite, which implies all Sylow q-subgroups of K are conjugate in K **[15]**. Therefore, by the Frattini argument, $E = AN_E(Q)$. Also $A \cap N_E(Q) < A$ for otherwise $[A, K] = [A, Q] = 1$. Hence

$N_E(Q)$ is a proper supplement to A in E, a contradiction again.

So, we have proved the existence of the complements.

If, finally, a complement K is properly contained in its normalizer $N_E(K)$, then consider $K < K\langle x\rangle \leq N_E(K)$. Since $K\langle x\rangle \cap A$ is normal in $AK = E$ and $K\langle x\rangle \cap A \cong ((K\langle x\rangle \cap A)K)/K \leq K\langle x\rangle/K$, we see that $K\langle x\rangle \cap A$ is a cyclic (and hence finite) $\mathbb{Z}G$-submodule of A, contrary to A having no nonzero finite $\mathbb{Z}G$-factors. So all the complements to A in E are self–normalizing.

<u>Conjugacy of complements</u>: Let S_1 and S_2 be two complements to A in E and take an E–invariant subgroup A_0 of A such that S_1 and S_2 are conjugate modulo A_0 but are not conjugate modulo any proper E–invariant subgroup of A_0. If $A_0 \neq 1$, we may clearly assume that $A = A_0$. By A being artinian, E has a subgroup E_1 minimal with respect to $E = AE_1$ and E_1 is generated by $S_1^{g_1}$ and $S_2^{g_2}$ —— conjugates of S_1 and S_2. We prove that $A \cap E_1 = 1$ so that $E_1 = S_1^{g_1} = S_2^{g_2}$, a contradiction.

Suppose that $A_1 = A \cap E_1 \neq 1$ and note that $E_1 = A_1]S_i^{g_i}$. Apply Lemma **5.2.1** to E_1, we obtain a series of normal subgroups

$$B < A_1 \leq C_K(A_1/B) \leq K \leq E_1$$

with $A_1/B = [A_1/B, K]$ and $K/C_K(A_1/B)$ is cyclic or finite. Passing to the factor group E_1/B, we can take $B = 1$ so that $A_1 = [A_1, K]$ and $K/C_K(A_1)$ is cyclic or finite.

(1) Suppose $K/C_K(A_1)$ is cyclic. In this case, we pass from E_1 to the factor group $\overline{E}_1 = E_1/C$, where C is the maximal hypercyclically embedded normal subgroup of E_1. Certainly, by A_1 having no nonzero finite $\mathbb{Z}G$-factors, we have K is not contained in C and, since a complement to A_1 in E_1 is a maximal hyper–(cyclic or finite) subgroup of E_1, C is contained in any

complement to A_1 in E_1. Also, by Lemma **5.2.4**, we may assume that $C_{\overline{K}}(\overline{A}_1) = \overline{A}_1$. Hence, we have $\overline{E}_1 = \overline{A}_1]\overline{S_i^{g_i}}$ and $\overline{K}/\overline{A}_1$ is cyclic. It is clear that $\overline{K} = \overline{A}_1]\overline{K}_i$, where $\overline{K}_i$ is the cyclic subgroup $\overline{K}_i = \overline{K}\cap\overline{S_i^{g_i}}$. By $\overline{A}_1 = [\overline{A}_1, \overline{K}]$, we have $\overline{K}_i$ are conjugate in $\overline{K}$, i.e., $\overline{K}_1 = \overline{K_2^a}$ for some $a \in A_1$. By Lemma **1.2.16**, $\overline{E}_1 = \overline{A}_1N_{\overline{E}_1}(\overline{K}_1)$. By $\overline{E}_1 = E_1/C$, we have $\overline{E}_1 > N_{\overline{E}_1}(\overline{K}_1)$. Let E_2 be the preimage of $N_{\overline{E}_1}(\overline{K}_1)$ in E_1; since $C \le S_1^{g_1}$, $\overline{S_1^{g_1}} \le N_{\overline{E}_1}(\overline{K}_1)$, and $\overline{E}_1 = \overline{A}_1N_{\overline{E}_1}(\overline{K}_1)$, we have $E_2 < E_1$ and $E_1 = A_1E_2$. Thus, $E = AE_1 = AE_2$. But $\overline{K}_1 = \overline{K_2^a}$ so $S_1^{g_1}$ and $S_2^{g_2a}$ are both contained in E_2 and hence the subgroup of E_1 generated by $S_1^{g_1}$ and $S_2^{g_2a}$ should be the group E_1 by the minimality of E_1. That is, $E_2 = E_1$ contrary to $E_2 < E_1$.

(2) Suppose $K/C_K(A_1)$ is finite. As before, we may assume that $K/C_K(A_1)$ is a q-group and, further, we may assume that $C_K(A_1) = A_1$, i.e., K/A_1 is a finite q-group.

If $p = q$, then by Lemma **1.2.20**, $S_1^{g_1}$ and $S_2^{g_2}$ are conjugate modulo some proper E_1-invariant subgroup of A_1 and hence S_1 and S_2 are conjugate modulo some proper E-invariant subgroup of A, a contradiction;

If $p \neq q$, then by Lemma **1.2.4**, $A_1 = C_{A_1}(K) \oplus [A_1, K]$, and it follows from $A_1 = [A_1, K]$ that $C_{A_1}(K) = 1$. The intersection $K\cap S_1^{g_1}$ is a Sylow q-subgroup of K and $K\cap S_1^{g_1}$ is normal in $S_1^{g_1}$ so that $S_1^{g_1} \le N_{E_1}(K\cap S_1^{g_1})$. But $N_{A_1}(K\cap S_1^{g_1}) = C_{A_1}(K\cap S_1^{g_1}) = C_{A_1}(K) = 1$ so that $S_1^{g_1} = N_{E_1}(K\cap S_1^{g_1})$. Similarly, $S_2^{g_2} = N_{E_1}(K\cap S_2^{g_2})$ and so $S_1^{g_1}$ and $S_2^{g_2}$ are conjugate in E_1 (since

$K \cap S_1^{g_1}$ and $K \cap S_2^{g_2}$ are conjugate in K and so in E_1). Thus, S_1 and S_2 are conjugate in E, a contradiction.

Theorem **D** is proved.

Noetherian Case:

Similar with the artinian case, we have the following lemmas:

Lemma 5.2.6: Let G be a hyper–(cyclic or finite) group, B a $\mathbb{Z}G$-module, and A a noetherian $\mathbb{Z}G$-submodule of B such that all irreducible $\mathbb{Z}G$-factors of A are infinite. If B/A is torsion-free and G–trivial, then $B = A \oplus B_1$ for some $\mathbb{Z}G$-submodule B_1 of B.

Proof: Suppose that A has no complements in B. Since A is noetherian , we may assume that for each nonzero $\mathbb{Z}G$-submodule C of A, A/C has a complement in B/C.

In B, we choose a $\mathbb{Z}G$-submodule M maximal with respect to $A \cap M = 0$. We show that if S is any $\mathbb{Z}G$-submodule such that $B = A+S$ then $M \leq S$.

Since $B/A \geq (A \oplus M)/A \cong_{\mathbb{Z}G} M$, we have M is a G–trivial $\mathbb{Z}G$-module and hence all of its irreducible $\mathbb{Z}G$-factors are finite. Also

$$A/(A \cap S) \cong_{\mathbb{Z}G} (A+S)/S = B/S \geq (M+S)/S \cong_{\mathbb{Z}G} M/(M \cap S),$$

by A being noetherian and having no nonzero finite $\mathbb{Z}G$-factors, we must have $M = M \cap S$, i.e., $M \leq S$.

Consider the factor-module B/M. Every nonzero $\mathbb{Z}G$-submodule of B/M has nonzero intersection with $(A \oplus M)/M$. In particular, $(A \oplus M)/M$ has no complements in B/M. If V/M is a nonzero $\mathbb{Z}G$-submodule of $(A \oplus M)/M$ then $V = C \oplus M$, where $C = A \cap V$ is nonzero and so $B/C = A/C \oplus S_1/C$ for some

$\mathbb{Z}G$-submodule S_1 of B. As above, $M \leq S_1$ and so $(A \oplus M) \cap S_1 = (A \cap S_1) \oplus M = C \oplus M = V$. Thus S_1/V is a complement to $(A \oplus M)/V$ in B/V.

By passing to the factor-module B/M we may assume that $M = 1$ so that: (a) A has no complements in B but for any nonzero $\mathbb{Z}G$-submodule C of A, A/C has a complement in B/C; (b) if N is a nonzero $\mathbb{Z}G$-submodule of B then $A \cap N \neq 0$.

We may assume that A is torsion-free. For otherwise, we may let $A[p]$ be the nonzero $\mathbb{Z}G$-submodule generated by all the elements of order p, where p is a prime. By (a), $B/A[p] = A/A[p] \oplus B_1/A[p]$. Since $B_1/A[p]$ $(\cong_{\mathbb{Z}G} B/A)$ is torsion-free, $pB_1 \neq 0$, then, by (b), $0 \neq A \cap pB_1 \leq A[p] \cap pB_1$. That is, B_1 has elements of order p^2, contrary to $B_1/A[p]$ being torsion-free. So A is torsion-free and then B is torsion-free. Since A has no nonzero finite $\mathbb{Z}G$-factors, we have $C_A(G) = 0$. By Proposition **5.1.2**, G has a normal subgroup K and A has a nonzero $\mathbb{Z}G$-submodule A_1 such that $C_{A_1}(K) = 0$ and $K/C_K(A_1)$ is cyclic or finite. By (a), $B/A_1 = A/A_1 \oplus B_1/A_1$. Consider the $\mathbb{Z}G$-module B_1 and we prove that $B_1 = A_1 \oplus B_2$ for some $\mathbb{Z}G$-submodule B_2 (and hence we get $B = A \oplus B_2$ as required).

Suppose $B_1 \neq A_1 \oplus B_2$ for any $\mathbb{Z}G$-submodule B_2 and suppose that G acts faithfully on B_1, i.e., $C_G(B_1) = 1$. It is clear that we still have that K is normal in G, $C_{A_1}(K) = 0$, and $K/C_K(A_1)$ is cyclic or finite. If $C_K(A_1) \neq 1$, then, since $C_K(A_1) = K \cap C_G(A_1)$ is a normal subgroup of G, $C_K(A_1)$ contains a nontrivial cyclic or finite subgroup F being normal in G. Let $F = \langle f_1, \cdots, f_n \rangle$ and let $G_1 = C_G(F)$, then $|G/G_1| < \infty$. By Lemma **2.2.6**, the irreducible $\mathbb{Z}G_1$-factors of A_1 are infinite. Since B_1/A_1 is G–trivial, it is also G_1–trivial. By $B_1/C_{B_1}(f_i) \cong_{\mathbb{Z}G_1} B_1(f_i-1) \leq A_1$ and $A_1 \leq C_{B_1}(f_i)$, we must have

$B_1(f_i-1) = 0$, for all i. That is, $1 \neq F \leq C_G(B_1)$, contrary to G acting faithfully on B_1. So $C_K(A_1) = 1$ and so K is a nontrivial cyclic or finite normal subgroup of G. Let $K = <k_1, \cdots, k_t>$. Being similar with the above, we have $B_1/C_{B_1}(k_i) \cong_{\mathbb{Z}G_2} B_1(k_i-1) \leq A_1$ for all i, where $G_2 = C_G(K)$. Thus $B_1/(A_1+C_{B_1}(k_i))$ must be zero for all i. That is, $B_1 = A_1+C_{B_1}(k_i)$ for any i. Let $C_m = C_{B_1}(<k_1, \cdots, k_m>)$, $m = 1, \cdots, t$. Then we have $B_1 = A_1+C_1$. Suppose that $B_1 = A_1+C_m$; we prove that $B_1 = A_1+C_{m+1}$.

Consider the $\mathbb{Z}G_2$-modules $C_m/C_{m+1} = C_m/C_{C_m}(k_{m+1}) \cong_{\mathbb{Z}G_2} C_m(k_{m+1}-1)$. Since B_1/A_1 is G–trivial, $C_m(k_{m+1}-1) \leq A_1$ and so $C_m(k_{m+1}-1)$ has no nonzero finite $\mathbb{Z}G_2$-factors; hence the irreducible $\mathbb{Z}G_2$-factors of C_m/C_{m+1} are all infinite. But $C_m/(C_{m+1}+(A_1 \cap C_m)) \cong_{\mathbb{Z}G_2} (C_m+A_1)/(C_{m+1}+A_1)$ a factor module of the G_2–trivial $\mathbb{Z}G_2$-module B_1/A_1. Hence $A_1+C_m = A_1+C_{m+1}$. That is, $B_1 = A_1+C_{m+1}$. Therefore $B_1 = A_1+C_m$ for all m. Put $m = n$, then $C_n = C_{B_1}(K)$ and $B_1 = A_1+C_{B_1}(K)$, which implies that $C_{B_1}(K) \neq 0$. Hence, by (b) and $B/A_1 = A/A_1 \oplus B_1/A_1$, we have $C_{A_1}(K) = A_1 \cap C_{B_1}(K) = A \cap C_{B_1}(K) = 0$, a contradiction. So $B_1 = A_1 \oplus B_2$ for some $\mathbb{Z}G$-submodule B_2 and hence the lemma is proved.

<u>Corollary 5.2.7</u>: Let G be a hyper–(cyclic or finite) group, B a $\mathbb{Z}G$-module, and A a noetherian $\mathbb{Z}G$-submodule of B such that all irreducible $\mathbb{Z}G$-factors of A are infinite. If B/A is an infinite cyclic group, then $B = A \oplus B_1$ for some $\mathbb{Z}G$-submodule B_1 of B.

<u>Proof</u>: Let $G_1 = C_G(B/A)$, then $|G/G_1| \leq 2$ and B/A is torsion-free and

G_1–trivial. By Lemma **5.2.6**, $B = A \oplus B_1$ for some G_1–trivial $\mathbb{Z}G_1$-submodule B_1 of B. For $g \in G$, if $B_1 g \neq B_1$, then $B_1 g$ is G_1–trivial and

$$0 \neq B_1 g/(B_1 \cap B_1 g) \cong_{\mathbb{Z}G_1} (B_1 + B_1 g)/B_1 \leq B/B_1 \cong_{\mathbb{Z}G_1} A.$$

That is, A has a nonzero G_1–trivial $\mathbb{Z}G_1$-factor and then a nonzero finite irreducible $\mathbb{Z}G_1$-factor, which will imply that A has a nonzero finite irreducible $\mathbb{Z}G$-factor, a contradiction. So $B_1 g = B_1$ for all $g \in G$. That is, B_1 is a $\mathbb{Z}G$-submodule of B. The result is proved.

<u>Lemma **5.2.8**</u>: Let E be an extension of the abelian group A by a hyper–(cyclic or finite) group G such that A is a noetherian $\mathbb{Z}G$-module and all irreducible $\mathbb{Z}G$-factors of A are infinite. Then if C/A is a normal subgroup of E/A and $C \leq C_E(A)$, then $C = A \times N$, where N is a normal subgroup of E and is contained in every supplement to A in E.

<u>Proof</u>: Let N be a normal subgroup of E contained in C and maximal subject to $N \cap A = 1$. By considering the factor group E/N we may suppose that $N = 1$. Then E satisfies the following condition: if S is normal in E, $S \leq C$, and $S \neq 1$, then $S \cap A \neq 1$. We show that this implies that $A = C$.

Suppose that $A \neq C$. Since E/A is hyper–(cyclic or finite), there is a nontrivial finite subgroup $K/A \leq C/A$ such that K is normal in E or an infinite cyclic subgroup $L/A \leq C/A$ such that L is normal in E.

For K, by the hypothesis of the lemma, $K \leq C_E(A)$ and so K is a finite extension of its central subgroup A. Hence K' is finite. It follows that $A \cap K'$ is finite and so $A \cap K' = 1$ by A having no nonzero finite $\mathbb{Z}G$-factors. By the condition above, we have $K' = 1$ and so K is abelian. Apply Proposition **5.1.3** to the $\mathbb{Z}(E/K)$-module K and its submodule A, then $A = A \times K_1$ for some normal subgroup K_1 of E, contrary to the condition above.

For L, by the hypothesis of the lemma, $L \leq C_E(A)$ and so L is a cyclic extension of its central subgroup A. Thus L is abelian. By Corollary **5.2.7**, $L = A \times L_1$ for some normal subgroup L_1 of E, contrary to the condition above.

Thus we have proved that $C = A \times N$, where N is normal in E.

Now let E_1 be a supplement to A in E so that $E = AE_1$, $C = A(C \cap E_1)$ and $C \cap E_1$ is normal in $AE_1 = E$. We have

$$N(C \cap E_1)/(C \cap E_1) \leq C/(C \cap E_1) = A(C \cap E_1)/(C \cap E_1).$$

Since N is hyper-(cyclically or finitely) embedded in E and the irreducible $\mathbb{Z}G$-factors of A are all infinite, we must have $N(C \cap E_1)/(C \cap E_1) = 1$, i.e., $N(C \cap E_1) = C = E_1$. Hence $N \leq E_1$ as required.

Now, we prove the last main result of this part.

Theorem E: Let G be a hyper-(cyclic or finite) locally soluble group and A a noetherian $\mathbb{Z}G$-module. If A has no nonzero finite $\mathbb{Z}G$-images, then the extension E of A by G splits conjugately over A and A has no nonzero finite $\mathbb{Z}G$-factors.

Proof: By Corollary **5.1.6**, A has no nonzero finite $\mathbb{Z}G$-factors.

Suppose the theorem is false, then using the fact that A is a noetherian $\mathbb{Z}G$-module we may assume that: A has conjugate complements in E modulo any nontrivial E-invariant subgroup of A.

Since A has no nonzero finite $\mathbb{Z}G$-factors, $C_A(E) = 1$. By Proposition **5.1.2**, E/A has a normal subgroup K/A and A has a nontrivial E-invariant subgroup A_0 such that $C_{A_0}(K) = 1$ and $K/C_K(A_0)$ is cyclic or finite.

(1) If $K/C_K(A_0)$ is finite, then we may choose K and A_0 such that $K/C_K(A_0)$ is minimal and so $K/C_K(A_0)$ is a chief factor of E. (For if L is

normal in E and $C_K(A_0) < L < K$ then if $C_{A_0}(L) = 1$ we have L, A_0 contrary to minimality of $|K/C_K(A_0)|$ and if $C_{A_0}(L) \neq 1$ then K, $C_{A_0}(L)$ is contrary to minimality of $|K/C_K(A_0)|$.) Hence $K/C_K(A_0)$ has order p^k for some prime p and integer $k \geq 1$. From $C_{A_0}(K) = 1$ it follows that $A_0[p] = 1$ and so $A_0^{p^k} \neq 1$.

By the assumption on A, we have E splits conjugately over A modulo $A_0^{p^k}$.

Let E_1 be a complement to A in E modulo $A_0^{p^k}$: $E = AE_1$, $A \cap E_1 = A_0^{p^k}$; put $E_0 = A_0E_1$, $K_0 = K \cap E_0$, and $C_0 = C_{K_0}(A_0)$. By Lemma **5.2.8**, $C_0 = A_0 \times N$, where N is normal in E_0 and is contained in E_1. Consider the factor group $\overline{E}_0 = E_0/N$ and the subgroups $\overline{K}_0, \overline{A}_0$. Since

$$\overline{K}_0/\overline{A}_0 = \overline{K}_0/\overline{C}_0 \cong K_0/C_0 \cong K/C_K(A_0),$$

we have $|\overline{K}_0/\overline{A}_0| = p^k$. Corresponding to $C_{A_0}(K) = 1$ we have $C_{\overline{A}_0}(\overline{K}_0) = \overline{1}$ and also $\overline{A}_0 \cap \overline{E}_1 = \overline{A}_0^{p^k}$. It follows, by applying Lemma **1.2.21** to $\overline{E}_0$ and its subgroups $\overline{K}_0$, $\overline{A}_0$, that $\overline{E}_0$ splits over $\overline{A}_0$: $\overline{E}_0 = \overline{A}_0\overline{E}_2$, $\overline{A}_0 \cap \overline{E}_2 = \overline{1}$. The complete preimage E_2 of $\overline{E}_2$ in E_0 gives $E_0 = A_0E_2$ and $A_0 \cap E_2 = 1$. So that E_2 is a complement to A in E. Let S_1, S_2 be any two complements to A in E. Then, since E splits conjugately over A modulo $A_0^{p^k}$, we have S_1 and S_2 are conjugate modulo $A_0^{p^k}$ and we may assume that $A_0^{p^k}S_1 = A_0^{p^k}S_2$. Put $E_0 = A_0S_1 = A_0S_2$, $K_0 = K \cap E_0$, and $C_0 = C_{K_0}(A_0)$. By Lemma **5.2.8**, $C_0 = A_0 \times N$, where N is normal in E_0 and is contained in every supplement to A_0 in E_0; in particular, $N \leq S_1 \cap S_2$. Consider the factor group $\overline{E}_0 = E_0/N$ and its subgroups $\overline{K}_0$, $\overline{A}_0$. Since $\overline{K}_0/\overline{A}_0 \cong K/C_K(A_0)$, so $\overline{K}_0/\overline{A}_0$ is a group of order p^k, and also $C_{\overline{A}_0}(\overline{K}_0) = \overline{1}$ by $C_{A_0}(K) = 1$. From $A_0^{p^k}S_1 = A_0^{p^k}S_2$ it follows that $\overline{S}_1$ and $\overline{S}_2$ are complements to $\overline{A}_0$ in $\overline{E}_0$ which coincide modulo $\overline{A}_0^{p^k}$. Apply Lemma **1.2.21** to the group $\overline{E}_0$ and its subgroups $\overline{K}_0$, $\overline{A}_0$, we have the conjugacy of the complements: $\overline{S}_1^{\bar{a}} = \overline{S}_2$, $a \in A_0$. Since $\overline{S}_1 = S_1/N$, $\overline{S}_2 = S_2/N$, and N is normal in E_0 it follows that $S_1^a = S_2$, i.e., E splits conjugately over A, a contradiction.

(2) Now we may suppose that $K/C_K(A_0)$ is cyclic.

In this case, we let $A_1 = [A_0, K] \leq A_0$, then, by $C_{A_0}(K) = 1$, we have $A_1 \neq 1$. Thus E splits conjugately over A modulo A_1, i.e., $E = AE_1$, $A \cap E_1 = A_1$. Let $K_1 = K \cap E_1$ and $C_1 = C_{K_1}(A_0)$. It is clear that $A_1 \leq C_1 \leq C_{K_1}(A_1) \leq C_{E_1}(A_1)$. By Lemma **5.2.8**, $C_1 = A_1 \times N$ for some normal subgroup N of E_1. Since $K_1/C_1 \cong K/C_K(A_0)$, we have $K_1 = C_1\langle x\rangle$ for some $x \in K_1$. Let $M = N\langle x\rangle$, then $K_1 = C_1\langle x\rangle = A_1M$. Since

$$
\begin{aligned}
[A_1 \cap M, K] &= [A_1 \cap M, C_K(A_0)\langle x\rangle] \\
&= [A_1 \cap M, \langle x\rangle] = [A_1 \cap M, x] \\
&\leq [A_1, x] \cap [M, x] \\
&\leq A_1 \cap N = 1,
\end{aligned}
$$

we have $A_1 \cap M \leq C_{A_0}(K) = 1$. Thus $K_1 = A_1M$, i.e., M is a complement to A_1 in K_1.

Suppose that M_0 is also a complement to A_1 in K_1 with $N \leq M_0$; we show that M and M_0 are conjugate by an element of A_0. We can write $x = a_1x_0$ with $a_1 \in A_1$ and $x_0 \in M_0$. Since

$$A_1 = [A_0, K] = [A_0, C_K(A_0)<x>]$$
$$= [A_0, <x>] = [A_0, x^{-1}],$$

so $a_1 = [a_0^{-1}, x^{-1}]$ for some $a_0 \in A_0$, and therefore

$$x = a_1x_0 = [a_0^{-1}, x^{-1}]x_0 = a_0(a_0^{-1})^{x^{-1}}x_0$$
$$= (a_0^{-1})^{x^{-1}}a_0x_0 = x(x^{-1})^{a_0}x_0,$$

i.e., $x_0 = x^{a_0}$. Since $N \leq M_0$ and $N \leq C_1 = C_{K_1}(A_0)$, we have

$$M^{a_0} = (N<x>)^{a_0} = N<x^{a_0}> = N<x_0> \leq M_0.$$

As $C_K(A_0) = AC_{K_1}(A_0)$ and $K = K_1C_K(A_0)$, so

$$AM_0 = A(A_1M_0) = AK_1 = AC_{K_1}(A_0)K_1 = C_K(A_0)K_1$$
$$= K = K^{a_0} = (AM)^{a_0} = AM^{a_0},$$

also $A \cap M_0 = A_1 \cap M_0 = 1$ and $A \cap M = 1$ implies that $A \cap M^{a_0} = 1$. Thus $M_0 = M^{a_0}$.

We now prove that A has conjugate complements in E and that the complements are of the form $L = N_{E_0}(M)$, where $E_0 = A_0E_1$ and M is , as above, a complement to A_1 in K_1 containing N.

If $g \in E_1$, then since N and K_1 are both normal in E_1 and the subgroup M^g is a complement to A_1 in K_1 containing N, thus $M^g = M^{a_0}$ for some $a_0 \in A_0$ and so $ga_0^{-1} \in N_{E_0}(M) = L$, hence $E = AE_1 = AL$. We show that L is a complement to A in E. That is, we need to prove that $A \cap L = 1$.

Since $L \leq E_0 = A_0E_1$ and $A \cap E_1 = A_1$, so

$$A \cap L = A \cap (E_0 \cap L) = (A \cap E_0) \cap L = (A \cap A_0E_1) \cap L$$
$$= A_0(A \cap E_1) \cap L = A_0A_1 \cap L = A_0 \cap L;$$

also A_0 is normal in E and $L = N_{E_0}(M)$, hence $[A_0 \cap L, M] \leq A_0 \cap M$.

Since $A_0 \cap M = A_0 \cap (E_1 \cap M) = (A_0 \cap E_1) \cap M = A_1 \cap M = 1$, so $A \cap L \leq C_{A_0}(M)$. Therefore, by $K = AM$ and $C_{A_0}(K) = 1$, we have $A \cap L \leq C_{A_0}(M) = C_{A_0}(K) = 1$. That is, $A \cap L = 1$ and so L is a complement to A in E.

Now let S be any complement to A in E. Thus S and L are conjugate modulo A_1 and we may assume that $A_1L = A_1S$. Therefore, we have

$$E_0 = E_0 \cap E = E_0 \cap AL = (E_0 \cap A)L = (A_0E_1 \cap A)L$$
$$= A_0L = A_0A_1L = A_0A_1S = A_0S.$$

Since $K_1 = A_1M \leq A_1L = A_1S$, so $K_1 = A_1M_1$, $A_1 \cap M_1 = 1$, where $M_1 = K_1 \cap S$; thus M and M_1 are complements to A_1 in K_1. We show that $N \leq M_1$. By $K_1 \leq A_1S$ and $C_1 = C_{K_1}(A_0) \leq K_1$ we have $C_1 = C_1 \cap A_1S = A_1(C_1 \cap S)$, thus $C_1 = A_1 \times N_1$, where $N_1 = C_1 \cap S \leq M_1$ and N_1 is normal in $A_0S = E_0$ since $C_1 = C_{K_1}(A_0)$ is normal in $A_0E_1 = E_0$. In particular, N_1 is normal in $E_1 \leq E_0$ and, since E_1/A_1 is hyper–(cyclic or finite), N_1 is hyper–(cyclically or finitely) embedded in E_1. Consider the product NN_1, if $NN_1 \neq N_1$ then, by $C_1 = A_1 \times N = A_1 \times N_1$, $NN_1 \cap A_1 \neq 1$ and so A_1 contains a nontrivial cyclic or finite subgroup normal in E_1. By $A_1 \le A$ and $E_1/A_1 \cong E/A \cong G$, we have A has a nonzero cyclic or finite $\mathbb{Z}G$-submodule and hence contains a nonzero finite $\mathbb{Z}G$-factor, a contradiction. Thus $NN_1 = N_1$, $N \le N_1$ and so $N \le M_1$.

This shows that M and M_1 are conjugate by an element $a_0 \in A_0$, i.e., $M^{a_0} = M_1$, and hence $L^{a_0} = N_{E_0}(M)^{a_0} = N_{E_0}(M^{a_0}) = N_{E_0}(M_1)$. From $K_1 = A_1M$ and M is normal in L it follows that K_1 is normal in A_1L. Therefore, by $A_1L = A_1S$, we have K_1 is normal in A_1S, and so $M_1 = K_1 \cap S$ is normal in S and $S \le N_{E_0}(M_1)$. By $L^{a_0} = N_{E_0}(M_1)$, we have $S \le L^{a_0}$ and so

$$L^{a_0} = AS \cap L^{a_0} = (A \cap L^{a_0})S = S.$$

That is, S and L are conjugate in E, i.e., E splits conjugately over A, a contradiction again.

Thus, we have finished the proof of the theorem.

6 SOME REMARKS

After Theorem **A** is proved, the similar results about the modules over some special groups are expected. However, these expected results are not true in most cases, which can be seen from the examples given in the following §**6.1**. §**6.2** contains questions arising from our work which are still open.

§6.1 EXAMPLES OF SPECIAL MODULES

a There exists a torsion-free irreducible $\mathbb{Z}G$-module A over a hypercyclic group G.

As P. Hall has shown: there exists a 3–generator torsion-free soluble group E with derived length 3 and having a minimal normal subgroup A isomorphic with the direct product of a countable infinity of copies of the additive group of rational numbers **[4]**. That is, in the category of $\mathbb{Z}G$-modules, A is a torsion-free irreducible $\mathbb{Z}(E/A)$-module over the soluble group E/A. From the example given by him, we can have a torsion-free irreducible $\mathbb{Z}G$-module A over a hypercyclic group G. In order to get such a module, we recall firstly P. Hall's example.

Let V be a vector space of dimension $\aleph_0$ over the field of rational numbers $\mathbb{Q}$ and let $\{v_i;\ i = 0, \pm1, \pm2, \cdots\}$ be a basis for V. For each integer i we select a prime number p_i in such a way that $p_i \neq p_j$ if $i \neq j$, and every prime occurs among the p_i.

Let ξ and η be the linear transformations of V defined by

$$v_i\xi = v_{i+1} \quad \text{and} \quad v_i\eta = p_i v_i, \qquad (i = 0, \pm1, \pm2, \cdots).$$

Let $H = <\xi, \eta>$, and writing $\eta^{\xi^j} = \eta_j$, then

$$v_i\eta_j = p_{i-j}v_i.$$

Thus the η_j commute with each other and η^H is a normal abelian subgroup of H. Therefore H is metabelian, and indeed, H is isomorphic with the standard wreath product of two infinite cyclic groups (H has η^H as its base subgroup).

It was shown that V contains no nonzero proper H–admissible additive subgroups. Let A be the additive group of V and let $E = A]H$, the semidirect product of A by H, then E is a 3-generator torsion-free soluble group with derived length 3 and has A as its minimal normal subgroup. View A as a $\mathbb{Z}H$-module, then A is a torsion-free irreducible $\mathbb{Z}H$-module.

Now let $G = \eta^H$ and let A be the additive group of the 1-dimensional vector subspace, say spanned by v_0, of V. Then G is a torsion-free hypercyclic abelian group and A is a torsion-free $\mathbb{Z}G$-module. Let B be a nonzero $\mathbb{Z}G$-submodule of A and let $0 \neq w \in B$, then $w = rv_0$ for some nonzero rational number $r \in \mathbb{Q}$. For any prime p and any integer n, since $p = p_i$ for some i and $v_0\eta_{-i} = p_{0-(-i)}v_0 = p_iv_0 = pv_0$, so

$$rp^nv_0 = rp_i^nv_0 = (rv_0)\eta_{-i}^n \in B.$$

Thus, it follows that B contains each rational multiple of v_0 and then $B = A$. That is, A is irreducible. (If we take A to be the additive group of all rationals and G the multiplicative group of all positive rationals and assume that G acts on A by the natural multiplication, then we get the same required example by taking $\eta_j = p_{-j} \in G$, where p_i runs over all primes when i runs over all integers.)

b There is a noetherian $\mathbb{Z}G$-module A over a hypercyclic group G such that

A has no C-decomposition.

A C–decomposition of a $\mathbb{Z}G$-module A over a group G is that:

$$A = A^{c} \oplus A^{\bar{c}},$$

where A^{c} is a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A^{c} is a cyclic group and the $\mathbb{Z}G$-submodule $A^{\bar{c}}$ has no that kind of irreducible $\mathbb{Z}G$-factors.

D. I. Zaǐcev proved that: any artinian $\mathbb{Z}G$-module A over a hypercyclic group G has a C-decomposition **[19]**. For noetherian modules, we have the following counterexample.

Let $A = \mathbb{Z} \oplus \mathbb{Z}$ $(= \langle a, b;\ ab = ba \rangle)$, the free abelian group of rank 2. Let G $(= \langle x \rangle)$ be a cyclic group of order 3. Define a G-action on A by

$$x: \begin{array}{l} a \longmapsto b \ (\longmapsto -(a+b) \longmapsto a) \\ b \longmapsto -(a+b) \ (\longmapsto a \longmapsto b). \end{array}$$

Then we can see that A is a noetherian $\mathbb{Z}G$-module over the hypercyclic group G.

Since A/3A is finite and is of order 3^2, so the 3-group G acts trivially on the irreducible $\mathbb{Z}G$-factors of A/3A (Corollary **1.2.8**) and then the irreducible $\mathbb{Z}G$-factors of A/3A are cyclic. Thus, the $\mathbb{Z}G$-module A contains irreducible $\mathbb{Z}G$-factors being cyclic groups. On the other hand, it is clear that A/2A is an irreducible $\mathbb{Z}G$-factor of A and is an elementary abelian group of order 4. That is, A contains irreducible $\mathbb{Z}G$-factors being not cyclic groups.

Suppose A has a C–decomposition, i.e., $A = A^{c} \oplus A^{\bar{c}}$, then $A^{c} \neq 0$ and $A^{\bar{c}} \neq 0$. Since A is a free abelian group of rank 2, so both A^{c} and $A^{\bar{c}}$ must be infinite cyclic groups and then $A^{\bar{c}}/3A^{\bar{c}}$ is irreducible and is of order 3, which contrary to $A^{\bar{c}}$ having no irreducible $\mathbb{Z}G$-factors being cyclic groups. So A has no C–decomposition.

c There is a noetherian $\mathbb{Z}G$-module A over a hypercentral group G such that A has no Z–decomposition.

A Z-decomposition of a $\mathbb{Z}G$-module A over a group G is that:

$$A = A^{z} \oplus A^{\bar{z}},$$

where A^{z} is a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A^{z} is G–trivial, i.e., the centralizer in G is the whole group G, and the $\mathbb{Z}G$-submodule $A^{\bar{z}}$ has no nonzero G–trivial $\mathbb{Z}G$-factors.

In the same paper [19], D. I. Zaǐcev pointed out that: any artinian $\mathbb{Z}G$-module A over a hypercentral group G has a Z–decomposition. But, the result does not hold again in the noetherian case as we can see from the following simple example.

Let $A = \langle a \rangle$, the infinite cyclic group, and let $G = \langle x \rangle$, the cyclic group of order 2. Define the G–action on A by

$$a^{x} = a^{-1}.$$

It is clear that the above is, in fact, the definition of an infinite dihedral group, thus we have got a noetherian $\mathbb{Z}G$-module $A\ (= \langle a \rangle)$ over the hypercentral group $G\ (= \langle x \rangle)$. Since A is indecomposable as a group, so A has no nontrivial Z–decomposition. But each irreducible $\mathbb{Z}G$-factor $2^{i}A/2^{i+1}A$ is clearly G–trivial and each irreducible $\mathbb{Z}G$-factor $p^{j}A/p^{j+1}A$ with $p \neq 2$ is clearly not G–trivial, so A does not have a Z–decomposition.

§6.2 UNSOLVED QUESTIONS

Through our work, we often assume that G is locally soluble. However, from the corollaries of our main results in Chapter 3 and Chapter 4, we have noted that this condition is not necessary. Therefore, the general question arises (as D. I. Zaǐcev has mentioned for artinian case).

Question 1: Let G be a hyperfinite group, does any noetherian $\mathbb{Z}G$-module A have an f–decomposition?

From our proof for the main result —— Theorem **A**, we see that the above question may have a positive answer if the following three questions all have a positive answer.

Question 2: Let G be a hyperfinite group and A a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p. If all irreducible $\mathbb{Z}G$-factors of A are finite, should A be finite? (It is almost true, see Lemma **2.4.6**)

Question 3: Let G be a hyperfinite group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A. If all irreducible $\mathbb{Z}G$-factors of B are finite and A/B has no nonzero finite $\mathbb{Z}G$-factors, does B have a complement in A?

Question 4: Let G be a hyperfinite group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A. If all irreducible $\mathbb{Z}G$-factors of A/B are finite while B has no nonzero finite $\mathbb{Z}G$-factors, does B have a complement in A?

In §**4.4**, we have proved that: if A is a noetherian $\mathbb{Z}G$-module over a periodic abelian group G with $\pi(G)$ being finite, then $A^{\bar{f}}$ is torsion and has a finite $\mathbb{Z}G$-composition series as well as a finite exponent. Now the general question is:

<u>Question 5</u>: For any noetherian $\mathbb{Z}G$-module A over a hyperfinite locally soluble group G, must $A^{\bar{f}}$ always be torsion?

Specially, we still have:

<u>Question 6</u>: If A is a noetherian $\mathbb{Z}G$-module over a periodic abelian group G, must $A^{\bar{f}}$ be torsion?

The other challenges rising from Chapter **5** are the following:

<u>Question 7</u>: Let G be a hyper–(cyclic or finite) locally soluble group, does any (torsion-free) artinian $\mathbb{Z}G$-module A have an f–decomposition?

<u>Question 8</u>: Let G be a hyper–(cyclic or finite) locally soluble group, does any noetherian $\mathbb{Z}G$-module have an f–decomposition?

Finally, we have:

<u>Question 9</u>: Let G be a hyperfinite locally soluble group, A a $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A. If B is an artinian (resp. noetherian) $\mathbb{Z}G$-module and A/B is a noetherian (resp. artinian) $\mathbb{Z}G$-module, does A have an f–decomposition?

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