A thesis submitted for the degree of Doctor

of Philosophy in the University of Glasgow

SEMILINEAR AND QUASILINEAR ELLIPTIC EQUATIONS

by

ZONGMING GUO

Department of Mathematics, University of Glasgow

July, 1992

To

my wife and daughter

SEMILINEAR AND QUASILINEAR ELLIPTIC EQUATIONS

CONTENTS

PREFACE

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow. It presents the results of research undertaken by the author between October 1989 and September 1992.

All the results of this thesis are the original work of the author, except for the instances indicated within the text.

I would like to express my deep gratitude to my supervisor, Professor J. R. L. Webb, for his unswerving assistance and encouragement, without which this thesis would not have been possible; and to Professor R. W. Ogden, who provided me with the opportunity to study at the University of Glasgow. I should also like to thank the University of Glasgow and Committee of Vice-Chancellors and Principals of the Universities of the United Kingdom for financing this research through a scholarship and an ORS Award from October 1990 to September 1992.

ZONGMING GUO

SUMMARY

We are concerned with the solvability of boundary value problems and related general properties of solutions of semilinear equations

$$\Delta u + f(x, u\, Du) = 0, \text{ in } \Omega \tag{*}$$

and of quasilinear elliptic equations

$$\operatorname{div}(|Du|^{p-2} Du) + \lambda f(u) = 0, \text{ in } \Omega \tag{**}$$

with a variety of domains.

In Chapter 2, we concentrate on the study of existence and uniqueness of positive radially symmetric solutions of the equation (*) with a variety of Dirichlet and Neumann boundary conditions in annular domains. Using Leray–Schauder degree theory, we establish some new existence results.

In Chapter 3, we shall give a new description of the generalized degree theory. Using this new generalized degree, we establish the existence of solutions to the periodic boundary value problem

$$(|u'|^{p-2} u') + f(t, u, u') = y(t), \ u(0) = u(1), \ u'(0) = u'(1),$$

under various conditions on the function $y : [0,1] \to \mathbb{R}$ and the function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$.

The aim of Chapters 4, 5 is to consider the existence and multiplicity of positive solutions of the eigenvalue problem

$$\left.\begin{aligned} -\operatorname{div}(|Du|^{p-2} Du) &= \lambda f(u) && \text{in } \Omega \subset \mathbb{R}^N \\ u &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{***}$$

for $\lambda > 0$, $N \geq 2$ and $p > 1$. The domain Ω is assumed to be bounded, connected and to have a smooth boundary.

In Chapter 4, we prove that there is a strong maximum principle for $A+\theta$ when $1<p\leq 2$, where $A\cdot = -\mathrm{div}(|D\cdot|^{p-2}D\cdot)$, $\theta>0$. We show that the positive solutions of (***) occur in pairs when f satisfies either (F_1) or (F_2):

(F_1) f is strictly increasing on $\mathbb{R}^+$, $f(0)=0$ and $\lim_{s\to 0} f(s)/s^{p-1}=0$; there exist α_1, $\alpha_2>0$ such that $f(s)\leq \alpha_1+\alpha_2|s|^{\tau}$, $0<\tau<p-1$.

(F_2) There is $\beta>0$ such that

$$f(0)=f(\beta)=0,\ f>0 \text{ in } (0,\beta) \text{ and } f<0 \text{ in } (\beta,\infty) \text{ and } \lim_{s\to 0} f(s)/s^{p-1}=0.$$

We also give a necessary and sufficient condition for (***) to possess a positive solution when f satisfies (F_2).

In Chapter 5, the existence and uniqueness of positive radial solutions of the problem (***) on $\Omega=B_R$ with Dirichlet condition are proved for λ large enough and f satisfying a condition

(F_1') $f\in C^1(0,\infty)\cap C^{\gamma}([0,\infty))$ is non-decreasing on $\mathbb{R}^+$, $0<\gamma\leq 1$, $f(0)=0$; $\lim_{s\to\infty} f(s)/s^{\beta}=1$, $0<\beta<p-1$; $\lim_{s\to 0} f(s)/s^{p-1}=\infty$ and $(f(s)/s^{p-1})'\leq 0$, for $s>0$.

It is also proved that all the positive solutions of (***) are radially symmetric solutions for f satisfying (F_1') and λ large enough.

INTRODUCTION

The principal objective of this work is some new developments of the general theory of second order elliptic equations and the theory of quasilinear elliptic equations. We shall be concerned with the solvability of boundary value problems (primarily the Dirichlet problem) and related general properties of solutions of semilinear equations

$$\Delta u + f(x, u, Du) = 0, \quad \text{in } \Omega \tag{0.1}$$

and of quasilinear elliptic equations

$$\operatorname{div}(|Du|^{p-2} Du) + \lambda f(u) = 0, \quad \text{in } \Omega \tag{0.2}$$

with a variety of domains. Here $\lambda > 0$, $p > 1$, $Du = (D_1 u, \cdots, D_N u)$, where $D_i = \frac{\partial u}{\partial x_i}$. We are especially interested in the existence and multiplicity of positive radial solutions of the above equations in symmetric domains.

Recently, the problem of the existence and multiplicity of the solutions of equations (0.1) and (0.2) has been studied by many authors. These equations arise in many branches of mathematics and applied mathematics, see, for example, [50, 76]. Problem (0.1) has been treated, if not exhaustively, at least with reasonable completness and the fundamental questions have received rather simple answers. To do so, the authors always use the *nice* properties of the operator Δ, for example, maximum principle, strong maximum principle, comparison principle, Serrin's sweeping principle and so on. Meanwhile, the problem of the existence and multiplicity of the positive radial solutions of (0.1) has attracted much interenst, see [4, 15, 24-25, 27, 43, 45-46, 50, 54-56]. In 1979, Gidas, Ni and Nirenberg proved some very interesting facts, for example, they showed that all positive solutions in $C^2(\overline{\Omega})$ of the problem

$$\left.\begin{aligned} \Delta u + f(u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{0.3}$$

are radially symmetric solutions provided that Ω is a N-ball (see [28]). They also proved that no such result can automatically apply to the annulus (see also [46]). On the other hand, when $f(u)$ is superlinear ($\lim_{t\to\infty} f(t)/t = \infty$) the existence of positive solutions of problem (0.3) with a general Ω has been proved under various sets of assumptions, always including a restriction on the growth of f at infinity (see [3, 9, 48]). It is known that such a growth condition is, in general, necessary for starlike domains (see [63]). In the case of the annulus, such a growth condition is not necessary (see [4]). Therefore, the problem of existence of solutions of second order elliptic equations in symmetric domains is of much interest.

Motivated by the study of equation (0.1), many results have been obtained for the equation (0.2) (see, for example, [30, 33, 65, 73]). Unfortunately, there are many differences between these two equations. The major stumbling block in the case of $p \neq 2$ is the fact that certain 'nice' features inherent to semilinear problems seem to be lost or at least difficult to verify. In the case of (0.1), the solutions are classical (that is, smooth), but in the case of (0.2) they are generally weak solutions, since the pseudo-Laplacian is degenerate elliptic. Precisely, it was shown in [73-74] that the bounded solutions to the problem

$$\left.\begin{aligned} -\mathrm{div}(|Du|^{p-2} Du) &= f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{0.4}$$

belong to $C^{1+\alpha}(\overline{\Omega})$ for some α $(0<\alpha<1)$ but not always to $C^2(\overline{\Omega})$. For example, when the domain Ω is a ball centred at the origin 0, the function $u(x)$ defined by

$$u(x) = a\,|x|^{p/(p-1)} + b,$$

with constants a and b ($a<0$, $b>0$), is a solution to the problem

$$\left.\begin{aligned} -\operatorname{div}(|Du|^{p-2}Du) &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\}. \tag{0.5}$$

Moreover, the solutions of (0.4) on a symmetric domain are not necessarily radially symmetric. Tolksdorf [73] showed that there exist solutions of the problem

$$-\operatorname{div}(|Du|^{p-2}Du) = 0, \quad \text{in } \mathbb{R}^2 \tag{0.6}$$

that are of the form

$$u(x) = r^{\lambda}\,\phi(\theta), \tag{0.7}$$

where $r=|x|$. When $\Omega=B_\rho(0)=\{x : |x|<\rho\}$, Kichenassary and Smoller [41] proved the following theorem.

<u>Theorem A</u> *Suppose f satisfies:*

(C) *If $f : \mathbb{R}\to\mathbb{R}$ is a compactly supported C^1 function satisfying $f(0)=0=f(1)$; $f'(0)>0>f'(1)$, $\int_{s_0}^{1} f(t)dt>0$ for some $s_0\in(0,1)$, $\int_0^1 f(t)dt=0$ and $\int_0^a f(t)dt>0$ for some $a\in(0,1)$ and $f(a)=0$, then, for $p>2$, the problem*

$$\left.\begin{aligned} -\operatorname{div}(|Du|^{p-2}Du) &= f(u) \qquad |x|\le\rho \\ u &= 0 \qquad\quad\ |x|=\rho \end{aligned}\right\} \tag{0.8}$$

has, for sufficiently large ρ (depending on p), a compactly supported solution.

From Theorem A, they easily obtained the consequence that (0.8) has non-negative non-radial solutions.

In the same paper, they also proved the following theorem.

Theorem B *Let $p>2$ and f be as in Theorem A. For $a>0$ and every $\varepsilon>0$ small enough, the solution of the problem*

$$\left.\begin{array}{ll} -\mathrm{div}(|Du|^{p-2}Du)=f(u) & |x|\le\rho \\ u=a-\varepsilon & |x|=\rho \end{array}\right\}, \tag{0.9}$$

satisfies, with some $L<\rho/2$, independent of ρ,

$$u\equiv a \quad \text{for } |x|\le(\rho/2)-L.$$

Theorem B implies that there is no strong maximum principle for the equation (0.9) when $p>2$. We know that Equation (0.9) is solved by looking for the minimum in $W^{1,p}\cap\{u=a \text{ for } |x|=\rho\}$ of

$$\int_{|x|\le\rho}\left(\frac{1}{p}|Du|^p-\bar{F}(u)\right)dx,$$

where $\bar{F}(s) := \int_0^s \bar{f}(t)dt$ and $\bar{f}$ is strictly decreasing, $\bar{f}\equiv f$ for $s\in[a-\varepsilon, a]$. As the problem is monotone, there is exactly one minimum, u. The maximum principle shows $a-\varepsilon\le u\le a$. Theorem B says $\max u=a$. But there is strong maximum principle for $1<p<2$ (see Proposition 4.14 of Chapter 4). Now, we can raise the question whether some existence and multiplicity results of semilinear elliptic equations are true or not for the above quasilinear elliptic equations.

In Chapter 2, we concentrate on the study of existence and uniqueness of positive radially symmetric solutions of the equation (0.1) with a variety of Dirichlet and Neumann boundary conditions in annular domains. The function f is assumed to be decreasing in u and u' and is allowed to be singular when either $u=0$ and $u'=0$. To study this problem, we use the transformations

$$t=[(N-2)r^{N-2}]^{-1}, \tag{0.10}$$

$$\phi(t)=[(N-2)t]^{-k},\ k=\frac{2N-2}{N-2}, \tag{0.11}$$

$$t_i=[(N-2)R_i^{N-2}]^{-1},\ i=0,\ 1 \tag{0.12}$$

to discuss the two-point boundary value problem of an ordinary differential equation

$$u''(t) + \phi(t)\, g(t, u(t), u'(t)) = 0, \tag{0.13}$$

where g is decreasing in u and u', g also has singularities at $u=0$ and $u'=0$. Using topological transversality theorem (see Theorem 1.3.5 of Chapter 1), Bobisud, O'Regan and Royalty have obtained many results on this problem (see, for example, [5, 57]), but they only treated some special cases. For example, when $g(t, u, u') = u^{-\alpha}$, $\alpha > 0$, they only discussed the case $0 < \alpha \leq 1$. In this Chapter, we shall construct a new homotopy and use Leray-Schauder degree theory to show that some technical restrictions imposed in their papers are unnecessary. Meanwhile, we also give a sufficient condition for solvability when $\alpha > 1$. There still exist many interesting problems on the more general cases, we conjecture that there exists a necessary and sufficient condition for the solvability with a general function g.

Part of this chapter has been accepted for publication (see [35-36]).

In Chapter 3, we establish the existence of solutions to the Periodic Boundary Value Problem

$$(|u'|^{p-2} u')' + f(t, u, u') = y(t), \quad u(0) = u(1), \; u'(0) = u'(1) \tag{0.14}$$

under various conditions on the function $y : [0, 1] \to \mathbb{R}$ and the function $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$. We also consider the problem

$$\left.\begin{aligned} Au - f(t, u) &= 0 \quad \text{in} \quad (0, 1) \\ u'(0) = u'(1) &= 0 \end{aligned}\right\}, \tag{0.15}$$

where $Au = -(a(|u'|^2 u')'$, $a : \mathbb{R} \to \mathbb{R}$ is a continuous mapping such that $h(t^2) = \int_0^{t^2} a(\tau) d\tau$ is a strictly convex function on $\mathbb{R}$. (0.14) and (0.15) may be written in the general form

$$Au - Nu = y. \tag{0.16}$$

When A is a linear operator and is Fredholm of index 0, Eq (0.16) has been studied by many authors (see [22-23, 40, 51-52, 58, 60-61]) using various theories of topological degree. When A is not a Fredholm map or is not linear, these ideas are not directly applicable. In a recent paper [62], the problem

$$(|u'|^{p-2}u')' + f(t, u) = 0, \quad u(0)=u(1)=0, \tag{0.17}$$

was considered. Under homogeneous Dirichlet boundary condition, $G_p = A^{-1}$ is compact from $C^0[0, 1]$ to $C^0[0, 1]$ and Eq (0.17) is equivalent to

$$u - G_p(f(t, u)) = 0, \tag{0.18}$$

and Leray-Schauder degree theory can be used. But, under the periodic boundary condition or Neumann boundary condition, A is not invertible. In this Chapter, we shall give a new description of the generalized degree, the main ideas are similar to those of [58]. Then, we use this new generalized degree to obtain existence of problem (0.14) and (0.15).

Part of this Chapter has been accepted for publication in *Journal of Differential and Integral Equations.*

The aim of Chapter 4 and Chapter 5 is to consider the existence and multiplicity of positive solutions of the following eigenvalue problem

$$\left.\begin{aligned} -\mathrm{div}(|Du|^{p-2}Du) &= \lambda f(u) \quad \text{in } \Omega\subset\mathbb{R}^N \\ u &= 0 \qquad\quad \text{on } \partial\Omega \end{aligned}\right\}, \tag{0.19}$$

for $\lambda>0$, $N\geq 2$ and $p>1$. The domain Ω is assumed to be bounded, connected and to have a smooth boundary $\partial\Omega$ which is connected.

In Chapter 4, we shall consider the problem when f satisfies either (F_1) or (F_2):

(F_1) f is strictly increasing on $\mathbb{R}^+$, $f(0)=0$ and $\lim_{s\to 0} f(s)/s^{p-1}=0$; there exist α_1, $\alpha_2>0$ such that $f(s)\leq\alpha_1+\alpha_2|s|^{\gamma}$, $0<\gamma<p-1$.

(F_2) There is $\beta>0$, such that

$f(0)=f(\beta)=0$, $f>0$ in $(0, \beta)$ and $f<0$ in (β, ∞) and $\lim_{s\to 0} f(s)/s^{p-1}=0$.

For $p=2$, the problem has been studied by many authors. Rabinowitz [64] first established the existence of a subsolution and a supersolution to the problem

$$\left.\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{in } \Omega\subset\mathbb{R}^{N} \\ u &= 0 \qquad\quad \text{on } \partial\Omega \end{aligned}\right\}. \tag{0.20}$$

Using the strong maximum principle and Leray-Schauder degree theory, he found that positive solutions of (0.20) occur in pairs when f satisfies $f(0)=f(\beta)=0$, $f>0$ in $(0, \beta)$ and $f'(0)=0$. Furthermore, Dancer [19] and De Figueiredo [20] extended the results of [64] to allow more general f but they assumed that Ω possesses some special properties, such as, starshapedness. Clement and Sweers [16] gave a necessary and sufficient condition for problem (0.20) possessing a positive solution by using the sweeping principle of Serrin [66, 70], and the fact that positive solutions of (0.20) in N-ball are radial.

Our problem is that all the good properties as above are not readily available for problem (0.19). In Chapter 4, we prove that there is a strong maximum principle for $A + \theta$ when $1<p\leq 2$, where $A\cdot=-\mathrm{div}(|D\cdot|^{p-2}D\cdot)$, $\theta>0$. We show that the positive solutions of (0.19) when f satisfies (F_1) or (F_2) occur in pairs using the degree of mappings of class $(S)_+$. We also give a necessary and sufficient condition for problem (0.19) to possess a positive solution when f satisfies (F_2).

Part of this chapter has been accepted for publication (see [34]).

In Chapter 5, the existence and uniqueness of positive radial solutions of the problem (0.19) on $\Omega=B_R(0)$ with Dirichlet condition are proved for λ large enough and f satisfying a condition

(F_1') $\quad f\in C^1(0, \infty)\cap C^{\gamma}([0, \infty))$ is non-decreasing on $\mathbb{R}^+$, $0<\gamma\leq 1$, $f(0)=0$; $\lim_{s\to\infty} f(s)/s^{\beta}=1$, $0<\beta<p-1$; $\lim_{s\to 0} f(s)/s^{p-1}=\infty$ and $(f(s)/s^{p-1})'\leq 0$, for $s>0$.

A *Generalized Serrin's Sweeping Principle* in the radially symmetric case is given in this Chapter. For the general case, it is still not known whether there is such a Sweeping Principle or not. It is also proved that all the positive solutions in $C_0^1(B_R)$ of the problem (0.19) are radially symmetric solutions for f satisfying (F_1') and λ large enough. This is interesting because under some conditions of f, Kichenassary and Smoller [41] showed that there do exist positive nonradial solutions for problem (0.19).

Part of this chapter has been accepted for publication (see [37]).

CHAPTER ONE

PRELIMINARIES

1.1 NOTATION AND GENERAL CONCEPTS

We shall write $\mathbb{Z}$ for the set of all integers, $\mathbb{N}$ for the set of all positive integers, 1, 2, $\cdots$, $\mathbb{R}$ for the set of real numbers, $\mathbb{R}^+$ for the set of positive real numbers and $\mathbb{R}^-$ for the set of negative real numbers.

$\mathbb{R}^N$ denotes N-dimensional Euclidean space; $x=(x_1, \cdots, x_N)$ denotes an arbitrary point in it, with norm $\|x\|=(\sum|x_i|^2)^{1/2}$. Throughout, N is assumed to be 2 or greater. Ω denotes a bounded domain in $\mathbb{R}^N$, that is, an arbitrary open connected set contained in some sphere of sufficiently great radius; $\overline{\Omega}$ denotes the closure of Ω, $\partial\Omega$ denotes the boundary of Ω, so that $\overline{\Omega}=\Omega\cup\partial\Omega$.

Unless otherwise stated X, Y, Z and E will denote Banach spaces with norms denoted $\|\cdot\|_X$, etc., or simply $\|\cdot\|$ when the underlying space is clear.

If G is a subset of a Banach space Z and $z\in Z$ an arbitrary point, $\mathrm{dist}(z,G):=\inf\{\|z-g\|: g\in G\}$ denotes the distance of z from the set G. The closure and boundary of a set G will be denoted, respectively, by $\overline{G}$ and ∂G.

$B_\rho(z):=\{x\in Z : \|x-z\|<\rho\}$ denotes the open ball in Z, centre z and radius ρ.

In what follows, we shall encounter various constants defined by quantities that are known to us from the conditions. We shall denote these constants by a upper-case letter C with various subscripts. Where there is no danger of confusion and where the value of constant in question is of no significance, we shall drop the subscripts on the C, so that even in a single proof the letter C with the same subscript or even with no subscript at all may be used to denote different constants. In other cases, when we need to emphasize the dependence of a constant on some quantity or other, this dependence will be shown explicitly.

If L is an operator in a Banach space, we shall denote its closure by $\overline{L}$. We shall denote the domain of definition of L by $\mathrm{D}(L)$ and its range of values by $\mathrm{R}(L)$.

All the functions and quantities considered in this thesis will be real. If $u(x)$ is some differentiable function of $x \in \mathbb{R}^N$, then its derivative is written $Du(x)=u_x(x)=(u_{x_1}(x), \cdots, u_{x_N}(x))$; $|Du|=(\sum_1^N (u_{x_i})^2)^{1/2}$.

n denotes an outwardly directed unit vector (relative to Ω) normal to $\partial\Omega$ at any point on $\partial\Omega$. $\dfrac{\partial}{\partial n}$ denotes differentiation in the direction n.

We shall use the following notations for the function spaces that we shall encounter.

$L^p(\Omega)$ denotes the Banach space of all functions on Ω that are measurable and p-summable with respect to Ω with $p \geq 1$. The norm in this space is defined by

$$\|u\|_p = \left(\int_\Omega |u|^p dx \right)^{1/p}.$$

Measurability and summability are always understood in the sense of Lebesgue. The elements $L^p(\Omega)$ are the classes of equivalent functions on Ω.

Generalized derivatives are understood in the way that is now customary in the majority of works on differential equations. Different but equivalent definitions of them and their fundamental properties can be found, for example, in [68-69].

$W^{m,p}(\Omega)$ denotes the Banach space of all elements of $L^p(\Omega)$ that have generalized derivatives of the first m orders that are p-summable over Ω. The norm in $W^{m,p}(\Omega)$ is defined by

$$\|u\|_{m,p}=\left[\int_{\Omega}(|u|^p+\sum_{k=1}^{m}\sum_{(k)}|D^{(k)}u|^p)dx\right]^{1/p}, \qquad (1.1.1)$$

where the symbol $D^{(k)}$ denotes an arbitrary k-th derivative of $u(x)$ with respect to x and $\sum_{(k)}$ denotes summation over all possible k-th derivatives of u.

For $\rho>0$ we write $\Omega_\rho=B_\rho\cap\Omega$, B_ρ denotes an arbitrary sphere of radius ρ in $\mathbb{R}^N$. We shall say that the function $u(x)$ satisfies a Hölder condition with exponent α, where $\alpha\in(0,1]$, and with Hölder constant M in the region $\overline{\Omega}$ if

$$\|u\|_{(\alpha),\Omega}=\sup\rho^{-\alpha}\operatorname{osc}\{u;\ \Omega^i_\rho\}=M. \qquad (1.1.2)$$

Where $\operatorname{osc}\{u(x);\ \Omega\}$ is the oscillation of $u(x)$ on x, that is, the difference between essential $\max_\Omega u(x)$ and $\min_\Omega u(x)$; the supremum is over all components Ω^i_ρ of all Ω_ρ such that $\rho\leq\rho_0$, $\rho_0>0$ is a constant.

$C^{0,\alpha}(\overline{\Omega})$ is the Banach space the elements in which are all functions $u(x)$ that are continuous in Ω having finite norm $\|u\|_{\alpha,\Omega}$, where the norm in $C^{0,\alpha}(\overline{\Omega})$

is defined by the equation

$$\|u\|_{\alpha,\Omega}=\max |u| + \|u\|_{(\alpha),\Omega} . \tag{1.1.3}$$

$C^{m,\alpha}(\overline{\Omega})$ is a Banach space the elements of which are functions that are continuous in Ω and have continuous derivatives in Ω of the first m orders endowed with the norm

$$\|u\|_{m,\alpha,\Omega}= \sum_{k=0}^{m} \sum_{(k)} \max_{\Omega} |D^{(k)}u(x)| + \sum_{(k)} |D^{(k)}u|_{(\alpha),\Omega} . \tag{1.1.4}$$

The Banach space $C^{m}(\overline{\Omega})$ and the linear set $C^{m}(\Omega)$ are defined analogously. The norm in $C^{m}(\overline{\Omega})$ or, what amounts to the same thing in $C^{m,0}(\overline{\Omega})$ is defined by the equation

$$\|u\|_{m,\Omega}=\|u\|_{m,0,\Omega}= \sum_{k=0}^{m} \sum_{(k)} \max_{\Omega} |D^{(k)}u| . \tag{1.1.5}$$

In brief, $C^{m}(\overline{\Omega})$ consists of all functions that are m times continuously differentiable in $\overline{\Omega}$.

Definition 1.1.1 A continuous mapping $f: Z \to E$, which is one-to-one (injective), onto (surjective) and whose inverse mapping $f^{-1}: E \to Z$ is also continuous, is called a homeomorphism.

If $D \subset Z$ is a linear subspace, then $\dim D$ will be written for the dimension of D, which may be infinite. If $A \subset \mathbb{R}^N$ is an open set, then the dimension of A, denoted by $\dim A$, is defined in the same way as in [39] or as the definition of a manifold.

Remark 1.1.2 (Hurewicz, Wallman [39]) The dimension is a topological invariant. That is, if $f: Z \to E$ is a homeomorphism, Z_n is a n-dimension subspace of Z, then

$$\dim (f(Z_n)) = \dim (Z_n) = n. \tag{1.1.6}$$

Where $f(Z_n) \subset E$ is an open set, $\dim (f(Z_n))$ is understood as above.

Definition 1.1.3 A mapping $f: Z \to E$ is said to be compact if it is continuous and $\overline{f(D)}$ is compact in E whenever D is a bounded subset in Z.

Definition 1.1.4 A function $f: Z \to \mathbb{R}$ is said to be Fréchet differentiable at the point $z_0 \in Z$, if there exists a bounded, linear map $f'(z_0): Z \to \mathbb{R}$ such that

$$f(z_0+h) - f(z_0) - f'(z_0)h = R(z_0, h),$$

where $R: Z \times Z \to \mathbb{R}$ is such that $|R(z_0,h)|/\|h\| \to 0$ as $\|h\| \to 0$. We call $f'(z_0)$ the Fréchet derivative of f at the point z_0.

Definition 1.1.5 We call z_0 a critical point of f if $f'(z_0)=0$.

1.2 GENERALIZED TOPOLOGICAL DEGREES

Some of the main results in this thesis are based on the generalized topological degree. So, we first discuss A-proper maps and the maps of class $(S)_+$. Throughout the text we shall assume that the reader is familiar with the definition and properties of the classical Brouwer degree, which we denote by deg, the classical Leray-Schauder topological degree, denoted by $\deg_{LS}$, for infinite dimensional map of the form identity minus compact and Mawhin's Coincidence degree, denoted by $\deg_M$. These concepts may be found in the book of Lloyd [18], and in [42, 49] and Mawhin [51].

A-proper maps were first named as such by Browder and Petryshyn [12], although Petryshyn had used them earlier in [59], where he referred to them as mappings satisfying condition (H). To define A-proper mappings, we need the following definition.

Definition 1.2.1 (Petryshyn, [60]) $\Gamma=\{X_n, Y_n, Q_n\}$ is said to be an admissible scheme for maps from X into Y provided that:

(*i*) $\{X_n\}\subset X$ and $\{Y_n\}\subset Y$ are sequences or oriented finite dimensional subspaces with $\dim X_n=\dim Y_n$, for each $n\in\mathbb{N}$;

(*ii*) $\{Q_n\}$ is a sequence of linear, continuous projections, with $Q_n: Y\to Y_n$ for each $n\in\mathbb{N}$, and $Q_n y\to y$ as $n\to\infty$, for each $y\in Y$;

(*iii*) $\mathrm{dist}(x, X_n)\to 0$ as $n\to\infty$ for each $x\in X$.

In definition 1.2.1 by 'oriented' finite dimensional spaces X_n, Y_n, we mean that bases have been chosen for X_n and Y_n such that the bounded, linear operator L: $X_n\to Y_n$ which maps the basis in X_n onto the basis in Y_n, is such that the determinant of the matrix of L is positive.

Remark 1.2.2 In this thesis, we use the following admissible scheme $\Gamma'=\{A_n, Y_n, Q_n\}$ for maps from X into Y provided that

(*i*) $\{X_n\}\subset X$ and $\{Y_n\}\subset Y$ are oriented finite dimensional subspaces with $\dim X_n=\dim Y_n$ for each $n\in\mathbb{N}$, $A_n\subset X_n$ is an open set with $\dim A_n=\dim X_n$ for each $n\in\mathbb{N}$;

(*ii*) $\{Q_n\}$ is a sequence of linear, continuous projections with $Q_n: Y\to Y_n$

for each $n\in\mathbb{N}$, and $Q_n y \to y$ as $n\to\infty$, for each $y\in Y$;

(*iii*) $\mathrm{dist}(x, A_n) \to 0$ as $n\to\infty$ for each $x\in X$.

Let $G\subset X$ be an open, bounded set, and write $G_n = G\cap A_n \subset X_n$, $\overline{G}_n = \overline{G}\cap A_n$ and $\partial G_n = \partial G\cap A_n$.

Definition 1.2.3 A map $T\colon \overline{G}\subset X\to Y$ is said to be A-proper with respect to Γ' if and only if

(*i*) $T_n = Q_n T|_{\overline{G}_n} : \overline{G}_n \subset A_n \subset X_n \to Y_n$ is continuous; and

(*ii*) if $\{x_{n_j} : x_{n_j} \in G_{n_j}\}$ is any bounded sequence such that $T_{n_j}(x_{n_j}) \to g$ for some g in Y, then there exists a subsequence $\{x_{n_{j(k)}}\}$ and $x\in\overline{G}$ such that $x_{n_{j(k)}} \to x$ as $k\to\infty$ and $Tx = g$.

Thus, in the class of A-proper maps, the problem of finding solutions to an infinite dimensional problem $f(x)=y$ may be reduced to that of solving the associated finite dimensional problems $Q_m f(x_m) = Q_m y$. The required solution is then the strong limit of some subsequence of $\{x_m\}$, provided the sequence $\{x_m\}$ is bounded.

Definition 1.2.4 Let $T\colon \overline{G}\to Y$ be A-proper with respect to $\Gamma' = \{A_n, Y_n, Q_n\}$ and $f\notin T(\partial G)$. We define $\deg_A(T, G, f)$ *the degree of* T *on* G *over* f *with respect to* Γ', to be a subset of $\mathbb{Z}' = \mathbb{Z}\cup\{\pm\infty\}$, given by:

(*a*) An integer $m\in\deg_A(T, G, f)$ provided there is a sequence $\{n_j\}$ of positive integers such that $\deg(T_{n_j}, G_{n_j}, Q_{n_j} f) = m$, for all $j \geq 1$.

(*b*) ∞ (or $-\infty$)$\in \deg_A(T, G, f)$ provided there exists a sequence $\{n_j\}$ such that $\lim_j \deg(T_{n_j}, G_{n_j}, Q_{n_j} f) = \infty$ (or $-\infty$).

<u>*Remark* 1.2.5</u> (1) The degree $\deg(T_n, G_n, Q_n f)$ used in definition 1.2.4 is the classical Brouwer degree for continuous maps acting between oriented finite dimension spaces of the same finite dimension.

(2) Note that if $f \notin T(\partial G)$, then it follows from the A-properness of $T : \overline{G} \to Y$ that there exists an integer $n_0 \geq 1$ such that $Q_n f \notin Q_n T(\partial G_n)$ for all $n \geq n_0$. Consequently, the above definition makes sense and the statement of Definition 1.2.4 implies that $\deg_A(T, G, f) \neq \emptyset$.

(3) The degree $\deg_A(T, G, f)$ of an A-proper T according to scheme Γ' is the same as the degree defined in [60].

Using the properties of Brouwer degree and of A-proper maps, it was shown in [11, 49] that, although in general $\deg_A(T, G, f)$ is *multivalued* it has most of the useful properties of the Brouwer degree as the following indicates.

(P_1) If $\deg_A(T, G, f) \neq \{0\}$, then there is $x \in G$ such that $Tx = f$.

(P_2) If $G \subset G_1 \cap G_2$, $\overline{G} = \overline{G}_1 \cup \overline{G}_2$, with G_1 and G_2 open and bounded, $G_1 \cap G_2 = \emptyset$ and $f \notin (T(\partial G_1) \cup T(\partial G_2))$, then

$$\deg_A(T, G, f) \subseteq \deg_A(T, G_1, f) + \deg_A(T, G_2, f)$$

with equality holding if either $\deg_A(T, G_1, f)$ or $\deg_A(T, G_2, f)$ is a singleton, where we use the convention that $\infty+(-\infty)=\mathbb{Z}'$.

(P_3) If $H : [0,1]\times\overline{G}\to Y$ is an A-proper homotopy (see the definition below) such that $H(t,x)\neq f$ for $t\in[0,1]$ and $x\in\partial G$, then

$$\deg_A(H(0,\cdot), G, f)=\deg_A(H(1,\cdot), G, f).$$

(P_4) If G is symmetric about $0\in G$, and $T : \overline{G}\to Y$ is A-proper and odd, and $0\notin T(\partial G)$, then $\deg_A(T, G, 0)$ is odd, that is $2m\notin\deg_A(T, G, 0)$ for any m, so that, in particular, $0\notin\deg_A(T, G, 0)$.

We recall that, for any set V in X, the map $H : [0,1]\times V\to Y$ is called an A-proper (with respect to Γ') homotopy provided $H_n : [0,1]\times V_n\to Y_n$ is continuous and if $\{x_{n_j} : x_{n_j}\in V_{n_j}\}$ is bounded and $\{t_{n_j}\}\subseteq[0,1]$ are such that $H_{n_j}(t_{n_j}, x_{n_j})\to g$ for some g in Y, then there exist subsequences $\{x_{n_{j(k)}}\}$ and $\{t_{n_{j(k)}}\}$ and $x_0\in V$ and $t_0\in[0,1]$ such that $x_{n_{j(k)}}\to x_0$ in X, $t_{n_{j(k)}}\to t_0$ and $H(t_0, x_0)=g$.

Let X^* be the dual space of X. We write $<x^*, x>$ for the value of $x^*\in X^*$ at $x\in X$.

<u>Definition 1.2.6</u> (Browder [10]) A mapping $T : G\subseteq X\to X^*$ is said to be of class $(S)_+$ if for any sequence $\{x_n\}$ of X which converges weakly to x and $\limsup_{n\to\infty}<Tx_n, x_n-x>\leq 0$, then x_n converges strongly to x.

<u>Theorem 1.2.7</u> (Browder [10] *Let* X *be a reflexive Banach space, and consider the family F of maps* $f:\overline{G}\to X^*$, *where* G *is a bounded open subset of* X, f *is a*

mapping of class $(S)_+$ *with* f *demicontinuous (i.e. continuous from the strong topology of* X *to the weak topology of* X^**). Let* H *be the class of affine homotopies in* F*, and let* J *be the duality mapping from* X *to* X^* *corresponding to an equivalent norm on* X *in which both* X *and* X^* *are locally uniformly convex.*

Then there exists one and only one degree function on F *which is invariant under* H *and normalized by the map* J.

We denote by $\deg_S(T, U, p)$ the degree of mapping $T : X \to X^*$ of class $(S)_+$ at p relative to U.

Theorem 1.2.8 (Hirano [38]) *Let* $Df : X \to X^*$ *be the gradient of a functional* f *such that* Df *is of class* $(S)_+$ *and* Df *maps bounded sets of* X *to bounded sets of* X^*. *Suppose that, for some* β*, the set* $V = f^{-1}(-\infty, \beta)$ *is bounded. Moreover suppose the following condition*:

There exist numbers $\alpha < \beta$ *and* $r > 0$ *and an element* x_0 *of* X *such that* $f^{-1}(-\infty, \alpha) \subset B_r(x_0) \subset V$ *and* $Df \neq 0$ *for all* $x \in f^{-1}[\alpha, \beta]$.

Then $\deg_S(Df, V, 0) = 1$.

This extends a result of Amann [2] who worked in Hilbert space and assumed Df had the form Identity–Compact.

1.3 EMBEDDINGS, INEQUALITIES AND MAXIMUM PRINCIPLES

Theorem 1.3.1 (Gilbarg, Trudinger [29]) **(Sobolev inequality)** *For* $\Omega \subset \mathbb{R}^N$, $p > 1$,

$$W_0^{1,p}(\Omega) \subset \begin{cases} L^{Np/(N-p)}(\Omega), & p < N, \\ C^0(\overline{\Omega}), & p \geq N. \end{cases}$$

Furthermore, there exists a constant $C = C(N, p)$ *such that for any* $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{Np/(N-p)} \leq C\|Du\|_p, \qquad p < N, \tag{1.3.1}$$

$$\sup_{\Omega} |u| \leq C|\Omega|^{1/N-1/p}\|Du\|_p, \quad p \geq N. \tag{1.3.2}$$

Theorem 1.3.2 (Gilbarg, Trudinger [29]) *For* $p > 1$,

$$W_0^{k,p}(\Omega) \subset \begin{cases} L^{Np/(N-kp)}(\Omega), & kp < N, \\ C^m(\overline{\Omega}), & 0 \leq m < k - N/p. \end{cases}$$

Definition 1.3.3 (Granas [31-32]) Let U be an open subset of a convex set $K \subset E$. A compact map $f : \overline{U} \to K$ which is fixed point free on ∂U is called essential if every compact map $g : \overline{U} \to K$ which agrees with f on ∂U has a fixed point in U.

Definition 1.3.4 (Granas [31-32]) Two compact maps f, g : $\overline{U} \to K$ which are fixed point free on ∂U are called homotopic if there is a compact homotopy $H: \overline{U} \times [0,1] \to K$ such that $H_t(u) = H(u, t)$ is fixed point free on ∂U for each t in $[0,1]$, $f = H_0$, and $g = H_1$.

Theorem 1.3.5 (Granas [32]) **(Topological transversality theorem)** *If f and g are homotopic, then f is essential if and only if g is essential.*

We need the following inequalities in the following proofs.

(*a*) Young's inequality:

$$ab \le a^p/p + b^q/q, \ p, \ q > 1$$

this holds for positive real numbers a, b, p, q satisfying

$$1/p + 1/q = 1.$$

The case $p = q = 2$ of the inequality is known as Cauchy's inequality.

(*b*) Hölder's inequality:

$$\int_\Omega u \, v \, dx \le \|u\|_p \, \|v\|_q;$$

this holds for functions $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, $1/p + 1/q = 1$. When $p = q = 2$, Hölder's inequality reduces to the well-known Schwarz inequality.

In the following we give some important properties of the elliptic operators.

Definition 1.3.6 (Gilbarg, Trudinger [29]) Let $Lu = a^{ij}(x) D_{ij} u + b^i(x) D_i u + c(x) u$, $a^{ij} = a^{ji}$. L is called elliptic at x if for all $\xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N - \{0\}$,

$$0 < \lambda(x) |\xi|^2 \le a^{ij}(x) \xi_i \xi_j \le \Lambda(x) |\xi|^2.$$

Here $\lambda(x)$, $\Lambda(x)$ denote respectively the minimum and maximum eigenvalues of the coefficient matrix $[a^{ij}(x)]$. L is called elliptic in Ω if $\lambda > 0$ in Ω; L is called uniformly elliptic in Ω if Λ/λ is bounded in Ω.

Theorem 1.3.7 (Gilbarg, Trudinger [29]) (Maximum principle) *Let L be elliptic in the bounded domain Ω. Suppose that*

$$Lu \ge 0 \ (\le 0) \ \text{in} \ \Omega, \ c = 0 \ \text{in} \ \Omega.$$

with $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$. *Then the maximum (minimum) of* u *in* $\overline{\Omega}$ *is achieved on* $\partial\Omega$, *that is,*

$$\sup_{\Omega} u=\sup_{\partial\Omega} u \quad (\inf_{\Omega} u=\inf_{\partial\Omega} u).$$

Corollary 1.3.8 (Gilbarg, Trudinger [29]) *Let* L *be elliptic in* Ω. *Suppose that in* Ω, $Lu\geq 0$ (≤ 0), $c\leq 0$, *with* $u\in C^0(\overline{\Omega})$. *Then writing* $u^+=\max(u,0)$, $u^-=\min(u,0)$,

$$\sup_{\Omega} u\leq\sup_{\partial\Omega} u^+ \quad (\inf_{\Omega} u\geq\inf_{\partial\Omega} u^-).$$

If $Lu=0$ *in* Ω, *then*

$$\sup_{\Omega} |u|=\sup_{\partial\Omega} |u|.$$

Definition 1.3.9 Ω is said to satisfy the interior sphere condition if for any $x_0\in\partial\Omega$ there exists a ball $B\subset\Omega$ such that $x_0\in\partial B$.

Theorem 1.3.10 (Gilbarg, Trudinger [29]) **(Strong maximum principle)** *Let* L *be uniformly elliptic,* $c\equiv 0$ *and* $Lu\geq 0$ (≤ 0) *in a domain* Ω *which satisfies the interior sphere condition. Then if* u *achieves its maximum (minimum) in the interior of* Ω, u *is a constant. If* $c\leq 0$ *and* c/λ *is bounded, then* u *cannot achieve a non-negative maximum (non-positive minimum) in the interior of* Ω *unless it is a constant.*

Theorem 1.3.11 (Gilbarg, Trudinger [29]) **(The comparison principle)** *Let* L *be elliptic in* Ω. *Suppose* u, $v\in C^0(\overline{\Omega})\cap C^2(\Omega)$ *satisfy* $Lu\geq Lv$ *in* Ω, $u\leq v$ *on* $\partial\Omega$. *It then follows that* $u\leq v$ *in* Ω. *Furthermore, if* $Lu>Lv$ *in* Ω, $u\leq v$ *on* $\partial\Omega$, *we have the* *strict inequality* $u<v$ *in* Ω.

Definition 1.3.12 We say $v\in C^1(\Omega)$ is a subsolution (supersolution) of the problem

$$\left.\begin{aligned}\Delta u+h(u)&=0 \quad \text{in } \Omega\\ u&=g \quad \text{on } \partial\Omega\end{aligned}\right\}, \tag{1.3.3}$$

if $v\le g$ $(\ge g)$ on $\partial\Omega$ and for any $\phi\in C_0^{\infty}(\Omega)$, $\int_\Omega Dv\, D\phi dx\le(\ge)\int_\Omega h(v)\phi dx$.

Theorem 1.3.13 (Clement, Sweers [16]) (Serrin's sweeping principle) *Let* u *be a solution of the problem* (1.3.3). *Let* $\mathrm{A}=\{v_\tau;\ \tau\in[0,1]\}$ $(\mathrm{B}=\{w_\tau;\ \tau\in[0,1]\})$ *be a family of subsolutions (supersolutions) of* (1.3.3) *satisfying* $v_\tau\le g$ $(w_\tau\ge g)$ *on* $\partial\Omega$, *for all* $\tau\in[0,1]$. *If*

(1) $\tau\to v_\tau$ $(w_\tau)\in C^1(\overline{\Omega})$ *is continuous with respect to the* $\|\cdot\|_0$*-norm,*

(2) $u\ge v_0$ $(\le w_0)$ *in* $\overline{\Omega}$, *and*

(3) $u\not\equiv v_\tau$ (w_τ), *for all* $\tau\in[0,1]$.

Then, $u>v_1$ $(<w_1)$.

In the following we give some results for quasilinear elliptic equations. We consider the equation

$$\frac{d}{dx_i}a_i(x,u,u_x)+a(x,u,u_x)=0. \tag{1.3.4}$$

Let us assume that the functions $a_i(x,u,y)$ and $a(x,u,y)$ are defined for $x\in\overline{\Omega}$

and arbitrary u and y, that they are measurable, and the functions satisfy the conditions

$$\sum_{1}^{N} a_i(x,u,y)\, y_i \geq \upsilon(|u|)|y|^m - \mu(|u|), \tag{1.3.5}$$

$$\sum_{1}^{N} |a_i(x,u,y)|(1+|y|)+|a(x,u,y)| \leq \mu(|u|)(1+|y|)^m, \tag{1.3.6}$$

where $|y|^2=\sum_{1}^{N} y_i^2$; $\upsilon(t)$ and $\mu(t)$ are positive functions, and the constant $m>1$.

We shall refer to a function $u(x)$ in $W^{1,m}(\Omega)$ such that $\|u\|_{\infty,\Omega}<\infty$ and

$$\mathrm{I}(u,\eta)\equiv\int_{\Omega}[a_i(x,u,u_x)\eta_{x_i}-a(x,u,u_x)\eta]dx=0, \tag{1.3.7}$$

for an arbitrary bounded function $\eta(x)$ in $W_0^{1,m}(\Omega)$ as a bounded generalized solution of Eq (1.3.4).

Definition 1.3.14 (Ladyzhenskaya, Ural'tseva [44], p. 6) We shall say that the boundary $\partial\Omega$ of a region Ω satisfies condition (A) if there exist two positive numbers a_0 and θ_0 such that, for an arbitrary sphere B_ρ with center on $\partial\Omega$ of radius $\rho\leq a_0$ and for an arbitrary component $\overline{O}$ of $B_\rho\cap\Omega$, the inequality

$$\text{meas}\,\overline{O}\leq(1-\theta_0)\,\text{meas}\,B_\rho$$

holds.

Theorem 1.3.15 (Ladyzhenskaya, Ural'tseva [44], p. 251) *Suppose that conditions* (1.3.5), (1.3.6) *are satisfied. Then, an arbitrary bounded generalized solution* $u(x)$ *of* Eq (1.3.4) *belongs to the class* $C^{0,\alpha}(\Omega)$ *with* *exponent* $\alpha > 0$ *depending on* $M = \|u\|_{\infty,\Omega}$ *and the constant* m, $\upsilon(M)$ *and* $\mu(M)$ *in* (1.3.5) *and* (1.3.6). *For arbitrary* $\Omega' \subset \Omega$, *the norm* $\|u\|_{\alpha,\Omega'}$ *is bounded above by an expression involving* M, m, $\upsilon(M)$, $\mu(M)$ *and the distance from* Ω' *to* $\partial\Omega$.

If the boundary $\partial\Omega$ *satisfies condition* (A) *and if* $u|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega)$, *then the quantity* $\|u\|_{\alpha,\Omega}$ *(where* $\alpha \le \beta$*) is bounded above by an expression determined by* M, $\upsilon(M)$, $\mu(M)$, m, *the constants* α_0 *and* θ_0 *in the definition of condition* (A), β *and the norm* $\|u\|_{\beta,\partial\Omega}$.

Suppose that the a_i and a satisfy the inequalities

$$a_i(x,u,y)\, y_i \ge \upsilon_1(|u|)\,|y|^m - (1+|u|^{\alpha_1})\varphi_1(x), \tag{1.3.8}$$

$$\operatorname{sign} u \cdot a(x,u,y) \le (1+|u|^{\alpha_2})\varphi_2(x) + (1+|u|^{\alpha_3})\varphi_3(x)\,|y|^{m-\varepsilon}, \tag{1.3.9}$$

and α_i, the φ_i and ε satisfy the conditions

(1) $\dfrac{N}{N+q} \le \varepsilon \le m$, where $q \ge \dfrac{Nm}{N-m}$;

(2) $\varphi_i \in L_{r_i}(\Omega)$, i=1, 2, 3,

$$r_1,\ r_2 > \frac{N}{m};\quad r_3 > \begin{cases} N/\varepsilon & \text{for } \varepsilon \ge 1, \\ \dfrac{Nq}{q\varepsilon + N(\varepsilon - 1)} > N/\varepsilon & \text{for } \varepsilon < 1; \end{cases}$$

(3) $0 \le \alpha_1 < m\dfrac{N+q}{N} - \dfrac{q}{r_1}$,

$0 \le \alpha_2 < m\dfrac{N+q}{N} - \dfrac{q}{r_1} - 1$,

$0 \le \alpha_1 < m\dfrac{N+q}{N} - \dfrac{q}{r_3} - 1$.

Then the following theorem holds.

Theorem 1.3.16 (Ladyzhenskaya, Ural'tseva [44], p. 286) *Suppose that* $u(x)$ *is a generalized solution in* $W^{1,m}(\Omega)\cap L^{q}(\Omega)$, $N\geq m>1$, $q\geq q^{*}=\dfrac{Nm}{N-m}$, *of* Eq (1.3.4) *and suppose that* $\|u\|_{\infty,\partial\Omega}=M_0<\infty$. *Suppose that inequalities* (1.3.8), (1.3.9) *are satisfied for the* $a_i(x,u,y)$ *and* $a(x,u,y)$ *and that, in these inequalities, the parameters* ε *and the* α_i (i=1, 2, 3) *and the functions* φ_i *(for* i=1, 2, 3) *satisfy conditions* (1)-(3). *Then,* $\|u\|_{\infty,\Omega}$ *is bounded by an expression in terms of* $\|u\|_q$, M_0, υ_1, ε, α_i, $\|\varphi\|_{r_i}$, i=1, 2, 3, u *and* meas Ω.

Now we consider the following special equation

$$\left.\begin{aligned} -\operatorname{div}(a(|Du|^2 Du)=f, &\quad \text{in } \Omega\cap B_{2R}(0) \\ u=0, &\quad \text{on } \partial\Omega\cap B_{2R}(0) \end{aligned}\right\} \tag{1.3.10}$$

and we suppose that a satisfies the ellipticity and growth conditions

$$\gamma\,(k+t)^{p-2}\leq a(t^2)\leq\Gamma\,(k+t)^{p-2}, \tag{1.3.11}$$

$$(\gamma-1/2)\,a(t)\leq a'(t)t\leq\Gamma\,a(t), \tag{1.3.12}$$

for some $\gamma>0$, some $\Gamma>0$, some $k\in[0,1]$ and all $t>0$. Moreover, we need the limit condition

$$\lim_{t\to\infty}\frac{a'(t)\,t}{a(t)}=\frac{p-2}{2}. \tag{1.3.13}$$

The following regularity result is known.

Theorem 1.3.17 (Tolksdorf [74]) *Let* $B_R(0)$ *be a ball with radius* R *such that, either* $\partial\Omega\cap B_{2R}(0)$ *is empty or that* $\partial\Omega\cap B_{2R}(0)$ *is a regular* C^{∞}*-surface. Suppose* *that* $u\in W^{1,m}(\Omega)\cap L^{\infty}(\Omega\cap B_{2R}(0))$ *is a solution of* (1.3.10), *where* $f\in L^{\infty}(\Omega\cap B_{2R}(0))$ *and where* a *satisfies the ellipticity and growth conditions* (1.3.11), (1.3.12). *Then,*

$$\|u\|_{1,\alpha,\Omega\cap B_R}\le K, \tag{1.3.14}$$

for some $\alpha\in(0,1)$ *and some* $K>0$ *depending only on* N, m, γ, Γ, $\Omega\cap B_{2R}$ *and a bound for* $\|u\|_{\infty,\Omega\cap B_{2R}}$ *and* $\|f\|_{\infty,\Omega\cap B_{2R}}$.

Remark 1.3.18 It is clear that the function $\xi_i(x,u,y)=|y|^{p-2}y_i$, $p>1$, satisfies the conditions of $a_i(x,u,y)$ in the above theorems. Therefore, the above theorems hold for the equation $\operatorname{div}(|Du|^{p-2}Du)+a(x,u,Du)=0$, where a is as above.

Theorem 1.3.19 (Sakaguchi [65]) (**Hopf's lemma**) *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ $(N\ge 2)$ *with smooth boundary* $\partial\Omega$. *Let* $u\in C^1(\overline{\Omega})$ *satisfy*

$$-\operatorname{div}(|Du|^{p-2}Du)\ge 0 \quad \text{in } \Omega \text{ (in the weak sense)}$$
$$u>0 \quad \text{in } \Omega, \text{ and } u=0 \text{ on } \partial\Omega.$$

Then $\dfrac{\partial u}{\partial n}<0$ *on* $\partial\Omega$.

CHAPTER TWO

SOLVABILITY OF SOME SINGULAR NONLINEAR BOUNDARY VALUE PROBLEMS AND EXISTENCE OF POSITIVE RADIAL SOLUTIONS OF SOME NONLINEAR ELLIPTIC PROBLEMS IN ANNULAR DOMAINS

INTRODUCTION

In this chapter we study the existence and uniqueness of positive solutions of singular second order differential equations of the form

$$y'' + f(t, y, y') = 0, \qquad t \in (t_1, t_2) \tag{2.0.1}$$

satisfying a mixture of Dirichlet conditions and Neumann conditions, where $0 \le t_1 < t_2 < +\infty$. f is allowed to have suitable singularities at $t = t_1$, $t = t_2$ and when $y = 0$ or $y' = 0$.

Equations of this type arise in diffusion and osmotic flow theory, one particular example being the generalized Emden-Fowler equation (see [50, 76]).

They also arise from the study of the existence of positive radially symmetric solutions of the equation

$$\Delta y + f(r, y, y'(r)) = 0, \qquad \text{in } R_2 < r < R_1 \tag{2.0.2}$$

subject to one of the following sets of boundary conditions

$$y = 0 \quad \text{on } r = R_2 \quad \text{and} \quad y = 0 \quad \text{on } r = R_1, \tag{2.0.3a}$$

$$y=0 \quad \text{on } r=R_2 \quad \text{and} \quad \frac{\partial y}{\partial r}=0 \quad \text{on } r=R_1, \tag{2.0.3b}$$

$$\frac{\partial y}{\partial r}=0 \quad \text{on } r=R_2 \quad \text{and} \quad y=0 \quad \text{on } r=R_1. \tag{2.0.3c}$$

Here $r=|x|$, $x\in\mathbb{R}^N$, $N\geq 3$; $\frac{\partial}{\partial r}$ denotes differentiation in the radial direction; $0<R_2<R_1\leq\infty$; f is continuous on $(R_2, R_1)\times(0, \infty)\times(-\infty, \infty)$. The equation is singular because f is allowed to be singular at $y=0$, $y'=0$ and $r=R_2$, R_1.

When f is nondecreasing and $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a symmetric domain, the problem of the existence of positive radially symmetric solutions of the problem

$$\left.\begin{aligned} \Delta y+f(y)&=0 \quad \text{in } \Omega \\ y&=0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{2.0.4}$$

has been treated by many authors (see [17, 24, 27, 55-56]). In this chapter, we concentrate on the solvability of the problem (2.0.2) with f decreasing.

Putting

$$t=[(N-2)r^{N-2}]^{-1},$$

$$\phi(t)=[(N-2)t]^{-k}, \quad k=(2N-2)/(N-2), \tag{2.0.5}$$

$$t_i=[(N-2)\,R_i^{N-2}]^{-1} \quad i=1,\ 2,$$

radial solutions of (2.0.2) are solutions of

$$y''(t) + \phi(t)f(t, y, y'(t))= 0, \qquad t_1<t<t_2 \tag{2.0.1$'$}$$

(see [4]). Now the boundary conditions become

$$y(t_2)=0 \quad \text{and} \quad y(t_1)=0, \tag{2.0.3a$'$}$$

$$y'(t_2)=0 \quad \text{and} \quad y(t_1)=0, \tag{2.0.3b$'$}$$

$$y(t_2)=0 \quad \text{and} \quad y'(t_1)=0 \tag{2.0.3c$'$}$$

(when $R_1=\infty$, $t_1=0$).

By a solution of (2.0.1) we mean a function $y\in C^1[t_1, t_2]\cap C^2(t_1, t_2)$ that satisfies (2.0.1).

Existence of solutions for Problem (2.0.1) has been considered by Callegari and Nachman [15], who effectively consider (2.0.1) with $f(t, y, y')=\phi(t)\, g(y)$, $\phi(t)=[\min(t, 1-t)]$, $g(y)=y^{-1}$ and $t_1=0$, $t_2=1$; by Luning and Perry [50], who establish constructive results for $g(y)=y^{-\alpha}$ when $0<\alpha\leq 1$ and $t_1=0$, $t_2=1$; and by Taliaferro [71], who proves, in particular, that a necessary and sufficient condition for existence of solutions with first derivative continuous on [0,1] is that

$$\int_0^1 t^{-\alpha}(1-t)^{-\alpha}\phi(t)dt<+\infty \tag{2.0.6}$$

Using topological transversality theorem (see Theorem 1.3.5 of Chapter 1), Bobisud, O'Regan and Royalty extended the above results by allowing a more general $f(t, y, y')$ and obtained the following theorms:

<u>Theorem 2.0.1</u> (Bobisud, O'Regan and Royalty [5]) *Suppose that:*

(*a*) f *is continuous on* $(0, 1)\times(0, \infty)\times(-\infty, \infty)$;

(*b*) $0<f(t, y, z)\leq\phi(t)\, g(y)$ *on* $(0, 1)\times(0, \infty)\times(-\infty, \infty)$, *where*

(i) $g(y)$ *is continuous and nonincreasing on* $(0, \infty)$,

(ii) $y\, g(y)$ *is nondecreasing on* $(0, \infty)$,

(iii) $\phi>0$ *is continuous on* $(0, 1)$,

(iv) $\dfrac{f(t, y, z)}{\phi(t)\, g(kt\,(1-t))}$ *is continuous on* $[0, 1]\times(0, \infty)\times(-\infty, \infty)$

for each constant $k>0$,

(v) $\int_0^1 g(kt\,(1-t))\,\phi(t)dt<\infty$ *for any constant* $k>0$;

(*c*) *for each constant* $M_0 > 0$ *there exists* $\psi(t)$ *continuous and positive on* $(0,1)$ *such that* $f(t, y, z) \geq \psi(t)$ *on* $(0, 1) \times (0, M_0] \times (-\infty, \infty)$ *and*

$$\int_0^1 t\,(1-t)\,\psi(t)dt < \infty.$$

Then problem (2.0.1) *possesses a solution.*

<u>Theorem 2.0.2</u> (Bobisud, O'Regan and Royalty [5]) *Suppose* f *is independent of* z *and satisfies*

(*a*) $f(t, y)$ *is continuous and positive on* $(0, 1) \times (0, \infty)$;

(*b*) $f(t, y)$ *is strictly decreasing in* y *for* $y > 0$ *and* $t \in (0, 1)$;

(*c*) *for some constant* k,

$$\lambda f(t, \lambda y) \leq k f(t, y)$$

for $0 < \lambda \leq 1$, $0 < t < 1$ *and* $y > 0$;

(*d*) *there exists a nonnegative* $\alpha(t)$ *satisfying*

(i) $\alpha''(t) + f(t, \alpha(t)) > 0$ *on* $(0, 1)$, $\alpha(0) = \alpha(1) = 0$,

(ii) $\displaystyle\int_0^1 f(t, \alpha(t))dt < \infty$,

(iii) $f(t, y) / f(t, \alpha(t))$ *is continuous on* $[0, 1] \times (0, \infty)$.

Then problem (2.0.1) *possesses a solution.*

Using the same methods as in [5], O'Regan [57] discussed the above equations with Neumann boundary value conditions. In this case, he assumed that $g(y)$ satisfies $\int_0^1 g(y)\,dy < \infty$.

The above results give no information when $\int_0^1 g(y)dy = \infty$, for example,

$g(y)=y^{-\alpha}$, $\alpha \geq 1$. In this chapter, we prove some further results which cover this case.

In section 2.1, we use Leray-Schauder degree to seek positive solutions of the problem

$$y'' + \phi(t)\, g(y) = 0 \quad \text{in } (t_1, t_2), \qquad y(t_1)=y(t_2)=0. \tag{2.0.7}$$

We shall extend some of the results of [5, 57, 71, 76] by allowing a more general $g(y)$ and at the same time generalize the sufficient condition of [71] to problems with a more general $g(y)$. In section 2.2, we discuss problem (2.0.1) with nonlinear term $f(t, y, y')$ and Neumann conditions. In section 2.3, we apply the results obtained in sections 2.1, 2.2 to the equation (2.0.2) and obtain the existence of a positive radial solution of (2.0.2).

Part of this chapter has been published (see [35–36]).

2.1 SOLVABILITY OF DIRICHLET PROBLEMS WITH FIRST ORDER DERIVATIVES ABSENT

In this section we discuss the existence of positive solutions of a special problem

$$y'' + \phi(t)\, g(y) = 0 \quad \text{in } (t_1, t_2), \qquad y(t_1)=y(t_2)=0. \tag{2.0.7}$$

We first prove the following theorem.

Theorem 2.1.1 *Suppose that* $\phi(t) \in C^0[t_1, t_2]$, $\phi(t)>0$ *in* (t_1, t_2), *where* $0<t_1<t_2<\infty$; g *satisfies:*

(*a*) g *is continuous and nonincreasing on* $(0, \infty)$,

(*b*) $g(y)>0$ *on* $(0, \infty)$,

(c) $\int_{t_1}^{t_2} \phi(s)\, g(k\,(s-t_1)\,(t_2-s))ds < +\infty$, *for* $0<k<1$,

(d) $\lim_{t\to\infty} g(t) \int_0^t [g(s)]^{-1} ds = \infty$.

Then Problem (2.0.7) *has a positive solution.*

To prove this theorem, we consider two cases:

(i) g has singularity at 0;

(ii) g has no singularity at 0.

We only prove Theorem 2.1.1 in case (i), the proof of case (ii) is similar to case (i).

In order to avoid the possible singularity of g at 0, we consider, for each $n \in \mathbb{N}$ and $\lambda \in [0,1]$, the family of problems:

$$y'' + (1-\lambda)\delta\, \phi_1(t) + \lambda\phi(t)\, g(y) = 0, \qquad y(t_1)=y(t_2)=1/n, \qquad (2.1.1)_{\lambda,n}$$

where $0<\delta<1$ is a positive real number which is determined below, $\phi_1 \in C^0[t_1, t_2]$ and $\phi_1 > 0$ on (t_0, t_1).

We shall use Leray-Schauder degree to show that the existence of a solution for $\lambda=0$ implies the existence of a solution for $\lambda=1$; passage to the limit as $n\to\infty$ will yield the existence for (2.0.7).

We let

$$g_n(y) = \begin{cases} g(|y|) & |y| \geq 1/n \\ g(1/n) & |y| < 1/n \end{cases} \qquad (2.1.2)$$

and consider

$$y'' + (1-\lambda)\delta\, \phi_1(t) + \lambda\phi(t)\, g_n(y) = 0, \qquad y(t_1)=y(t_2)=1/n. \qquad (2.1.3)_{\lambda,n}$$

From $g_n > 0$ and $\phi_1 > 0$, it follows that $y \geq 1/n$ for any solution y of $(2.1.3)_{\lambda,n}$, and hence any solution of $(2.1.3)_{\lambda,n}$ is a solution of $(2.1.1)_{\lambda,n}$. We set

$$v(t)=y(t)-1/n \tag{2.1.4}$$

to get that

$$v'' + (1-\lambda)\delta\,\phi_1(t) + \lambda g_n(v+1/n)\phi(t)=0,\ v(t_1)=v(t_2)=0. \qquad (2.1.5)_{\lambda,n}$$

Therefore, any solution of $(2.1.5)_{\lambda,n}$ satisfies $v\geq 0$. Now we establish the *a priori* bounds necessary for application of Leray-Schauder degree.

Lemma 2.1.2 *There exists a constant M_0 independent of $\lambda\in[0,1]$, $0<\delta<1$ and $n\in\mathbb{N}$ such that*

$$|y(t)|\leq M_0, \tag{2.1.6}$$

for any solution y of $(2.1.5)_{\lambda,n}$.

Proof. Similar to the proof of Lemma 1 of [5]. Let y_{max} be the maximum of $y(t)$ on $[t_1,t_2]$ and suppose it occurs at t_3. Then $y'(t_3)=0$. Integrate the inequality

$$y'' = -(1-\lambda)\delta\,\phi_1(t)-\lambda\phi(t)\,g_n(y(t)+1/n)\geq -(1-\lambda)\delta\,\phi_1(t)-\lambda\phi(t)\,g(y(t))$$

from t_3 to $t>t_3$ to obtain

$$y'(t)\geq -(1-\lambda)\delta\int_{t_1}^{t_2}\phi_1(t)dt - \lambda\int_{t_3}^{t}\phi(s)\,g(y(s))ds$$

$$\geq -(1-\lambda)\delta\int_{t_1}^{t_2}\phi_1(t)dt - \lambda\,g(y(t))\int_{t_3}^{t}\phi(s)ds,$$

since $\phi>0$, $y\geq 0$ and $g(y)$ is nonincreasing. Divide by $g(y(t))$ and integrate from t_3 to t_2 to get

$$\int_{t_3}^{t_2}[g(y(t))]^{-1}y'dt\geq -(1-\lambda)\delta\left[\int_{t_1}^{t_2}\phi_1(t)dt\right]\left[\int_{t_3}^{t_2}[g(y(t))]^{-1}dt\right]$$

$$-\lambda \int_{t_3}^{t_2} (t_2-t)\, \phi(t)dt,$$

that is

$$\int_0^{y_{\max}} [g(u)]^{-1} du \leq \delta \left(t_2-t_1\right) \left(\int_{t_1}^{t_2} \phi_1(t)dt\right) \left([g(y_{\max})]^{-1}\right)$$

$$+(t_3-t_1)^{-1} \int_{t_3}^{t_2} (t-t_1)(t_2-t)\, \phi(t)dt.$$

A similar argument on $[t_1,t_3]$ yields

$$\int_0^{y_{\max}} [g(u)]^{-1} du \leq \delta \left(t_2-t_1\right) \left(\int_{t_1}^{t_2} \phi_1(t)dt\right) \left([g(y_{\max})]^{-1}\right)$$

$$+(t_2-t_3)^{-1} \int_{t_1}^{t_3} (t-t_1)(t_2-t)\, \phi(t)dt.$$

From these two inequalities it follows that

$$\int_0^{y_{\max}} [g(u)]^{-1} du \leq \delta \left(t_2-t_1\right) \left(\int_{t_1}^{t_2} \phi_1(t)dt\right) \left([g(y_{\max})]^{-1}\right)$$

$$+2(t_2-t_1)^{-1} \max \left\{ \int_{\beta}^{t_2} (t-t_1)(t_2-t)\phi(t)dt,\ \int_{t_1}^{\beta} (t-t_1)(t_2-t)\phi(t)dt \right\}$$

$$\leq \delta \left(t_2-t_1\right) \left(\int_{t_1}^{t_2} \phi_1(t)dt\right) \left([g(y_{\max})]^{-1}\right) + 2\left(t_2-t_1\right)^{-1} C_1. \tag{2.1.7}$$

Here $\beta=(t_1+t_2)/2$, $\quad C_1 = \max \left\{ \int_{\beta}^{t_2} (t-t_1)(t_2-t)\phi(t)dt,\ \int_{t_1}^{\beta} (t-t_1)(t_2-t)\phi(t)dt \right\}.$

From (d) it follows that there exists a M_0, such that

$$y_{\max} \leq M_0. \tag{2.1.8}$$

Lemma 2.1.3 *There exist constants* $K>0$, $K_1>0$, *such that*

$$y(t) \geq (t_2-t_1)^{-1} [(1-\lambda)\delta K_1+\lambda K]\,(t-t_1)(t_2-t), \tag{2.1.9}$$

for any solution of $(2.1.5)_{\lambda,n}$.

Proof. From Lemma 2.1.2, we know that $y \leq M_0$. Therefore,

$$y'' = -(1-\lambda)\delta\,\phi_1(t) - \lambda\phi(t)\,g(y+1/n) \leq -(1-\lambda)\delta\,\phi_1(t) - \lambda\phi(t)\,g(M_0+1).$$

We deduce by integration that

$$y(t) \geq (t_2-t_1)^{-1} [(1-\lambda)\delta\,\theta_1(t) + \lambda\,g(M_0+1)\theta(t)], \tag{2.1.10}$$

where $\theta(t)=(t-t_1)\int_t^{t_2}(t_2-s)\,\phi(s)ds + (t_2-t)\int_{t_1}^{t}(s-t_1)\phi(s)ds,\ t\in[t_1,t_2];$

$$\theta_1(t)=(t-t_1)\int_t^{t_2}(t_2-s)\,\phi_1(s)ds + (t_2-t)\int_{t_1}^{t}(s-t_1)\phi_1(s)ds,\ t\in[t_1,t_2].$$

Then

$$\theta'(t)=\int_t^{t_2}(t_2-s)\,\phi(s)ds - \int_{t_1}^{t}(s-t_1)\,\phi(s)ds,$$

for $t\in(t_1,t_2)$. Let $k_0=\int_{t_1}^{t_2}(t_2-s)\,\phi(s)ds$, then $\theta'(t_1)=k_0$ (In the following theorems, we suppose ϕ has singularities at t_1 or t_2, $\theta'(t_1)$ may become ∞, but in either case, $\theta'(t_1)\geq k_0$). Hence there is an $\varepsilon>0$ such that $\theta(t)\geq \frac{1}{2}k_0(t-t_1)\geq \frac{1}{2}k_0(t-t_1)(t_2-t)$ on $[t_1, t_1+\varepsilon]$. By a similar argument, $\theta(t)\geq \frac{1}{2}k_1(t-t_1)(t_2-t)$ on $[t_2-\delta, t_2]$ for some k_1 and $\delta>0$. Since $\theta(t)/[(t-t_1)(t_2-t)]$ is bounded on $[t_1+\varepsilon, t_2-\delta]$, there is a constant C_1 such that

$$\theta(t) \geq C_2(t-t_1)(t_2-t), \text{ for } t\in[t_1, t_2]. \tag{2.1.11}$$

Similar methods shows that

$$\theta_1(t) \geq C_3(t-t_1)(t_2-t), \text{ for } t\in[t_1, t_2]. \tag{2.1.12}$$

Let $K=g(M_0+1)C_2$, $K_1=C_3$, then we get (2.1.9).

<u>Lemma 2.1.4</u> *There exist positive constants* M_1, M_2 *such that, when*

$$0<\delta< \min\{K/K_1, (t_2-t_1)/K_1, 1\},$$

for any solution $y(t)$ *of* $(2.1.5)_{\lambda,n}$ *we have*

$$|y'(t)| \leq M_1, \tag{2.1.13}$$

$$|\xi(t)y''(t)| \leq M_2, \tag{2.1.14}$$

where M_1, M_2 *are independent of* λ *and* n; $\xi(t)=[g(\delta K_1(t_2-t_1)^{-1}(t-t_1)(t_2-t))]^{-1}$.

Proof. We have from Lemma 2.1.3 and the fact that g is nonincreasing that, for any solution $y(t)$ of $(2.1.5)_{\lambda,n}$,

$$|y''(t)| \leq (1-\lambda)\delta\, \phi_1(t)+\lambda\phi(t)\, g(\delta K_1(t_2-t_1)^{-1}(t-t_1)(t_2-t)).$$

By conditions $g\in C^0(0,\infty)$, $g>0$ and g has singularity at 0, we know that $\xi(t)\in C^0[t_1,t_2]$ and

$$|\xi(t)\, y''(t)| \leq (1-\lambda)\delta\, \xi(t)\, \phi_1(t)+ \lambda\phi(t) \leq M_2,$$

where M_2 is independent of λ and n.

Let $y(t_3)= \max\limits_{[t_1,t_2]} y(t)$; then

$$|y'(t)| = \left|\int_{t_3}^{t} y''(s)ds\right| \leq (1-\lambda)\, \delta \int_{t_1}^{t_2} \phi_1(t)dt$$

$$+\lambda \int_{t_1}^{t_2} \phi(t)\, g((t_2-t_1)^{-1}((1-\lambda)\delta K_1+\lambda K)(t-t_1)(t_2-t))dt$$

$$\le \delta \int_{t_1}^{t_2} \phi_1(t)dt + \int_{t_1}^{t_2} \phi(t)\, g((t_2-t_1)^{-1}\delta K_1(t-t_1)(t_2-t))dt.$$

Let $M_1=\delta \int_{t_1}^{t_2} \phi_1(t)dt + \int_{t_1}^{t_2} \phi(t)\, g(\delta K_1(t_2-t_1)^{-1}(t-t_1)(t_2-t))dt.$ By (c), we get (2.1.13).

For $u\in C^2(t_1, t_2)\cap C^1[t_1, t_2]$, define

$$\|u\|_0= \sup_{[t_1,t_2]} |u(t)|,$$

$$\|u\|_1= \max (\|u\|_0, \|u'\|_0),$$

$$\|u\|_2= \max (\|u\|_0, \|u'\|_0, \sup_{[t_1,t_2]} |\xi(t)\, u''(t)|).$$

Here $\xi(t)=[g(\delta K_1(t_2-t_1)^{-1}(t-t_1)(t_2-t))]^{-1}$. Set

$$\mathbb{K}= \{ u\in C^2(t_1, t_2)\cap C^1[t_1, t_2]:\ u(t_1)=u(t_2)=0 \text{ and } \|u\|_2<\infty\} ,$$

with norm $\|.\|_2$ and

$$\mathbb{C}=\{ u\in C^0(t_1, t_2) :\ \sup_{(t_1,t_2)} |u(t)|<\infty\} ,$$

with the norm $\|u\|_0= \sup_{(t_1,t_2)} |u(t)|$;

$L :\ \mathbb{K}\longrightarrow \mathbb{C},\qquad Lu(t)=\xi(t)\, u''(t);$

$j :\ \mathbb{K}\longrightarrow C^1[t_1, t_2],\qquad ju=u;$

$F_{\lambda,n}\ :\ C^1[t_1, t_2]\longrightarrow \mathbb{C},$

$$F_{\lambda,n}(u)(t)=\lambda\xi(t)\,\phi(t)\,g_n(u(t)+1/n)\,+\,(1-\lambda)\delta\,\xi(t)\,\phi_1(t).$$

We claim that $\mathbb{K}$ and $\mathbb{C}$ are Banach spaces. Only completeness of $\mathbb{K}$ requires effort. Let $\{u_n\}$ be Cauchy sequence in $\mathbb{K}$. Then there exists $u\in C^1[t_0, t_1]$ such that $\|u_n-u\|_1\to 0$ since $C^1[t_0, t_1]$ is a Banach space. Because $\xi(t)$ is bounded below on $[t_0+\delta, t_1-\delta]$ for any $\delta>0$, $u_n''\to u''$ uniformly on $[t_0+\delta, t_1-\delta]$, and thus $u\in C^2(t_0, t_1)$. Since a Cauchy sequence is bounded, $u\in\mathbb{K}$. To conclude that $\|u_n-u\|_2\to 0$, observe that $\|u_m-u_n\|_2<\varepsilon$ for large m, n and let $m\to\infty$.

Consider the triangle of maps in Fig.1. We have the following lemma.

<u>Lemma 2.1.5</u> (i) *j is completely continuous;*

(ii) *F is continuous;*

(iii) *L^{-1} is continuous.*

$$\begin{tikzcd}
 & C^1[t_0,t_1] \arrow[d, "F"] \\
\mathbb{K} \arrow[ur, "j"] \arrow[r, "L"] & \mathbb{C}
\end{tikzcd}$$

Fig. 1

Proof. (i) Let $\Omega\subset\mathbb{K}$ be bounded; that is, there is a constant C such that $\|u\|_2\le C$ for all $u\in\Omega$. Then clearly $j\Omega=\{ju\,:\,u\in\Omega\}$ is bounded in $\|\cdot\|_1$; we therefore have to show only that $j\Omega$ is equicontinuous in order to apply the Ascoli–Arzela theorem (see [53]) and conclude that j is completely continuous. For $0\le s\le t\le 1$ and $u\in\Omega$ we have

$$|u'(t)-u'(s)|\le\int_s^t|u''(\eta)|\,d\eta\le M_2\int_s^t\frac{d\eta}{\xi(\eta)}=M_2\int_s^t\phi(\eta)\,g(k(\eta-t_1)(t_2-\eta))d\eta$$

and the final integral can be made small by choosing $(t-s)$ small because $g(k(t-t_1)(t_2-t))\,\phi(t)$ is integrable. Similarly

$$|u(t)-u(s)| \leq \int_s^t |u'(\eta)|\,d\eta \leq M_1\,|t-s|.$$

(ii) $F_{\lambda,n}$ is continuous since

$$\xi(t)\,g_n(u+1/n) = g_n(u+1/n)\,/\,[\phi(t)\,g(k(t-t_1)(t_2-t))]$$

is uniformly continuous on $[t_1, t_2] \times [-C, C]$, where $C>0$.

(iii) L is linear, and L is continuous since

$$\sup_{(t_1,t_2)} |Lu(t)| = \sup_{(t_1,t_2)} |\xi(t)\,u''(t)| \leq \|u\|_2.$$

L is one-to-one because $Lu=0$ together with the boundary conditions $u(t_1)=u(t_2)=0$ imply $u\equiv 0$. To see that L is onto, let $\rho \in \mathbb{C}$, then the solution of the boundary value problem

$$\xi(t)\,u''(t)=\rho(t),\ \ u(t_1)=u(t_2)=0$$

is given by

$$u(t)=(t-t_1)\int_t^{t_2}(t_2-s)\,\phi(s)\,g(k(s-t_1)(t_2-s))\rho(s)ds$$

$$+(t_2-t)\int_{t_1}^{t}(s-t_1)\,\phi(s)\,g(k(s-t_1)(t_2-s))\,\rho(s)ds,\ \ t\in[t_1, t_2].$$

From this it follows directly that $\|u\|_1 < \infty$.

Since $\sup_{[t_1,t_2]} |\xi(t)u''(t)| = \sup_{[t_1,t_2]} |\rho(t)| < \infty$, $u\in\mathbb{K}$. Therefore L^{-1} is continuous by the bounded inverse theorem.

Proof of Theorem 2.1.1

Problem $(2.1.5)_{\lambda,n}$ is equivalent to

$$(I + L^{-1}F_{\lambda,n}j)\, y=0,$$

for $\lambda\in[0,1]$. It follows from above that $L^{-1}F_{\lambda,n}j:\ \mathbb{K}\longrightarrow\mathbb{K}$ is a compact homotopy.

Let $M=\max\{M_0, M_1, M_2, (\delta \max_{[t_1,t_2]} \theta_1(t))\}+2$ and $\Omega=\{u\in\mathbb{K} \mid \|u\|_2\le M\}$, where M_0, M_1, M_2 are as in (2.1.6), (2.1.13), (2.1.14). From above we know that

$$(I + L^{-1}F_{\lambda,n}j)\, y \neq 0, \qquad y\in\partial\Omega.$$

Therefore, properties of Leray-Schauder degree give

$$\deg_{LS}(I+L^{-1}(\delta\xi(t)\,\phi_1(t)),\ \Omega,\ 0)=\deg_{LS}(I+L^{-1}Gj,\ \Omega,\ 0). \tag{2.1.15}$$

Here $G(y)(t)=\xi(t)\,\phi(t)\,g_n(y(t)+1/n)$. Using the fact that $L^{-1}(\delta\,\xi(t)\,\phi_1(t)) =-\delta\,\theta_1(t)$ and $\delta\,\theta_1(t)\in\Omega$, we have

$$\deg_{LS}(I+L^{-1}(\delta\xi(t)\,\phi_1(t)),\ \Omega,\ 0)=\deg_{LS}(I,\ \Omega,\ \delta\theta_1(t))=1. \tag{2.1.16}$$

Therefore,

$$\deg_{LS}(I + L^{-1}Gj,\ \Omega,\ 0)= 1. \tag{2.1.17}$$

From this, we see that

$$y'' + \phi(t)\, g_n(y+1/n)=0, \qquad y(t_1)=y(t_2)=0 \tag{2.1.18}$$

has at least one solution in Ω, call it υ_n. As we have noted before, $\upsilon_n\ge0$. Thus the problem

$$y'' + \phi(t)\, g_n(y)=0, \qquad y(t_1)=y(t_2)=1/n$$

has a solution $y_n=\upsilon_n+1/n$ satisfying $y_n\ge 1/n$, and therefore y_n is a solution of

$$y_n'' + \phi(t)\, g(y_n)=0, \qquad y_n(t_1)=y_n(t_2)=1/n.$$

Moreover, $\|y_n\|_2\le M+2$ for all $n\in\mathbb{N}$.

By the Ascoli-Arzela theorem (see [53]), there is a subsequence (still call it $\{y_n\}$) such that $\|y_n-y\|_1\to0$ for some $y\in C^1[0, 1]$. Now, for any fixed

$t_0 \in (0, 1)$, y_n satisfies the integral equation

$$y_n(t) = y_n(t_0) + (t-t_0)\, y_n'(t_0) + \int_{t_0}^{t} (s-t)\, g(y_n(s))ds$$

and for $t \in (0, 1)$ this converges as $n \to \infty$ to

$$y(t) = y(t_0) + (t-t_0)\, y'(t_0) + \int_{t_0}^{t} (s-t)\, g(y(s))ds,$$

because g is uniformly continuous on $[-M-2, M+2]$. Thus $y \in C^2(t_1, t_2) \cap C^1[t_1, t_2]$ and y satisfies (2.0.7).

Remark a For the proof of Theorem 2.1.1 it is sufficient to take $\phi_1(t) \equiv 1$. However, in the proof of Theorem 2.1.10 below we will need a general $\phi_1(t)$.

If g has no singularity at 0, we deduce the following theorem.

Theorem 2.1.6 *Let ϕ be as in Theorem 2.1.1, $0 < t_1 < t_2 < \infty$, g satisfy the conditions (a), (b), (d) of Theorem 2.1.1 and $g(0) \neq \infty$. Then Problem (2.1.1) has a positive solution.*

Proof. Similar to the proof of Theorem 2.1.1. In this case, we can directly consider the problem

$$y'' + (1-\lambda)\delta + \lambda\phi(t)\, g(y) = 0, \qquad y(t_1) = y(t_2) = 0,$$

and condition (c) of Theorem 2.1.1 automatically holds.

Now we consider the special case $g(y) = y^{-\alpha}$.

From [71] we know that when $0 < \alpha \leq 1$, Problem

$$y'' + \phi(t)\, y^{-\alpha} = 0, \qquad y(0) = y(1) = 0, \tag{2.1.19}$$

has a solution in $C^1[0, 1]$ if and only if (2.0.6) holds. When $\alpha > 1$, we have

Theorem 2.1.7 *Let $\phi > 0$ be continuous on* (0,1) *and*

$$\int_0^1 (1-t)^{-\alpha}\, t^{-\alpha}\, \phi(t)\, dt < \infty. \tag{2.1.20}$$

Then problem (2.1.19) *has a positive solution.*

Proof. By (2.1.20), we know that $\phi(t)$ has no singularities at $t=0$ and $t=1$ and condition (c) of Theorem 2.1.1 holds. Following the steps of the proof of Theorem 2.1.1 we prove this theorem.

Remark b When $\phi(t)$ has no singularities at $t=0$ and $t=1$, then,

(i) If $g(y)$ has a singularity at $y=0$, then Theorem 2.1.1 holds for $t_1=0$, $t_2=1$;

(ii) If $g(y)$ has no singularity at $y=0$, then Theorem 2.1.6 holds for $t_1=0$, $t_2=1$.

Remark c If $\phi(t) \in C^0(t_1, t_2)$, ϕ has singularities at $t=t_1$, $t=t_2$, $\phi(t) > 0$ on (t_1, t_2) and $\int_{t_1}^{t_2} (t-t_1)(t_2-t)\, \phi(t) dt < \infty$; g satisfies all the conditions of Theorem 2.1.1, g is singular at 0. Then, the result of Theorem 2.1.1 is still true. In this case, we let $\phi_1(t) \equiv 1$ and $\xi(t) = [\phi(t)\, g(2^{-1}\delta(t-t_1)(t_2-t))]^{-1}$. Now K , δ in the proof of Theorem 2.1.1 are $K=(t_2-t_1)^{-1}\, g(M_0+1)\, C_2$ and $0 < \delta = \min\{K, 1\}$.

If $\phi(t)$ has a singularity at $t=0$ and g has no singularity at $y=0$, then using the remark c above we obtain

Theorem 2.1.8 *Let* $\phi(t) \in C^0(0,1]$, $\phi(t) > 0$ *in* $(0,1]$ *and* $\phi(t)$ *satisfy* $\int_0^1 t\phi(t)dt < \infty$;

g satisfy:

(*a*) *g is continuous and nonincreasing on* $[0, \infty)$,

(*b*) $g(y) > 0$, *on* $[0, \infty)$,

(*c*) $\lim_{t \to \infty} g(t) \int_0^t [g(s)]^{-1} ds = \infty$.

Then the problem

$$y'' + \phi(t)\, g(y) = 0, \qquad y(0) = y(1) = 0, \tag{2.1.21}$$

has a positive solution.

Theorem 2.1.9 *Let* $\phi(t) \in C^0(0, 1)$, $\phi(t) > 0$ *in* $(0,1)$ *and* $\phi(t)$ *satisfy*

$$\int_0^1 t\,(1-t)\,\phi(t)dt < \infty; \tag{2.1.22}$$

g satisfy the conditions of Theorem 2.1.1 *with* $t_0 = 0$, $t_1 = 1$. *Then problem* (2.1.21) *has a positive solution.*

Now, we show that some technical restrictions on g in Theorem 2.0.2 proved by Bobisud, O'Regan and Royalty [5] are unnecessary.

Theorem 2.1.10 *Suppose that*

(*a*) $g(t, y)$ *is continuous and positive on* $(0, 1) \times (0, \infty)$,

(*b*) $g(t, y)$ *is strictly decreasing in* y, *for* $y > 0$ *and* $t \in (0, 1)$,

(*c*) *there exists a nonnegative* $\alpha(t) \in C^2\,[0, 1]$ *satisfying:*

(*i*) $\alpha''(t) + g(t, \alpha(t)) > 0$, *on* $(0, 1)$, $\alpha(0) = \alpha(1) = 0$,

(*ii*) $\int_0^1 g(t, \alpha(t))dt < \infty$, *(iii)* $g(t, y) / g(t, \alpha(t))$ *is continuous on* $[0, 1]\times(0, \infty)$.

Then the problem

$$y'' + g(t, y) = 0, \qquad y(0)=y(1)=0, \qquad (2.1.23)$$

has a positive solution.

Proof. Define

$$g_n(t, y)= \begin{cases} g(t, |y|) & |y| \geq 1/n \\ g(t, 1/n)+1/n-|y| & |y| < 1/n \end{cases},$$

then g_n is strictly decreasing for $y\geq 0$. As in the preceding argument, we consider the problem

$$y'' + (1-\lambda)\, g(t, \alpha(t)+1/n) + \lambda\, g_n(t, y+1/n)= 0, \quad y(0)=y(1)=0, \qquad (2.1.24)_{\lambda,n}$$

and prove the following lemmas which replace Lemmas 2.1.2, 2.1.4 in the proof of Theorem 2.1.1. The remainder of the proof goes through with only minor alterations of Theorem 2.1.1, and so will be omitted.

Lemma 2.1.11 *Let the hypotheses of Theorem* 2.1.10 *hold, and* $y_\lambda=y_{\lambda,n}$ *denote a solution of* $(2.1.24)_{\lambda,n}$. *Then*

$$y_\lambda(t)\geq \alpha(t) \qquad \textit{for } \lambda\in[0,1],\ t\in[0,1]. \qquad (2.1.25)$$

Proof. For n large enough, we claim that $\alpha''(t)+g(t, \alpha(t)+1/n)\geq 0$, for $t\in(0,1)$. Since $\alpha''(t)$ is bounded on $[0,1]$, $g(t, \alpha(t))$ is unbounded at $t=0$ and $t=1$, then there exist $\beta>0$, N, such that $\alpha''(t) + g(t, \alpha(t) + 1/n) \geq 0$ for $n>N$, $t\in(0, \beta)\cup(1-\beta, 1)$; On $[\beta, 1-\beta]$, $\alpha''(t) + g(t, \alpha(t))$ is bounded below by a positive constant and hence for n large enough, $\alpha''(t) + g(t, \alpha(t)+1/n) \geq 0$. To see that $y_\lambda\geq\alpha(t)$, on $[0,1]$, assume the contrary and let $\alpha(t)-y_\lambda(t)$ achieve its maximum positive value at t_0. Then

$$\alpha''(t_0) - y''_\lambda(t_0) \le 0. \tag{2.1.26}$$

But, $\quad \alpha''(t_0) - y''_\lambda(t_0) \ge -g(t_0, \alpha(t_0)+1/n)$

$$+(1-\lambda)\, g(t_0, \alpha(t_0)+1/n) + \lambda\, g(t_0, y_\lambda(t_0)+1/n)$$

$$=\lambda\, [g(t_0, y_\lambda(t_0)+1/n)-g(t_0, \alpha(t_0)+1/n)].$$

By $\alpha(t_0) > y_\lambda(t_0)$, we know $\alpha''(t_0)-y''_\lambda(t_0) > 0$. This contradicts (2.1.26).

<u>Lemma 2.1.12</u> *There exist $M_3 > 0$, $M_4 > 0$ independent of λ, n, such that*

$$|\xi(t)\, y''_\lambda(t)| \le M_3,$$

where $\xi(t)=[g(\alpha(t))]^{-1}$, and

$$\|y'_\lambda\|_0 \le M_4. \tag{2.1.27}$$

Proof. From Lemma 2.1.11 we know that $y_\lambda(t) \ge \alpha(t)$. Hence, $|y''_\lambda(t)| \le 2\, g(t, \alpha(t))$ and $M_3=2$. Since each y_λ has a zero derivative somewhere in (0,1), we have

$$|y'_\lambda(t)| \le (1-\lambda)\int_0^1 g(t, \alpha(t)+1/n)dt + \lambda \int_0^1 g_n(t, y_\lambda+1/n)dt$$

$$\le \int_0^1 g(t, \alpha(t))dt + \int_0^1 g_n(t, \alpha(t)+1/n)dt \le 2\int_0^1 g(t, \alpha(t))dt \le M_4,$$

where we use condition *(ii)* and the fact that g is strictly decreasing in y.

From (2.1.27) and $y_\lambda(0)=0$, we also have

$$\|y_\lambda\|_0 \le M_4.$$

<u>Theorem 2.1.13</u> *For each $t\in(t_1,t_2)$ let $f(t,\cdot)$ be strictly decreasing on $(0,\infty)$. Then the solution of the problem*

$$y'' + f(t, y) = 0, \text{ on } (t_1, t_2) \qquad y(t_1) = y(t_2) = 0, \ y > 0 \text{ on } (t_1, t_2),$$

is unique.

Proof. Let y and z be two distinct solutions, and suppose there is a $t_0 \in (t_1, t_2)$ such that $y(t_0) > z(t_0)$. Then there is a $t_3 \in (t_1, t_2)$ such that $y(t) - z(t)$ has a positive maximum at t_3. Therefore $y''(t_3) - z''(t_3) \le 0$, $y'(t_3) = z'(t_3)$, and $y(t_3) > z(t_3)$. But then

$$y''(t_3) - z''(t_3) = -f(t_3, y(t_3)) + f(t_3, z(t_3)) > 0,$$

a contradiction.

Remark d The solution of Theorem 2.1.7 and Theorems 2.1.9–10 is unique. Also uniqueness holds in Theorems 2.1.1, 2.1.6 and 2.1.8, if $g(y)$ is strictly decreasing.

2.2 SOLVABILITY OF NEUMANN AND DIRICHLET PROBLEMS WITH FIRST ORDER DERIVATIVES PRESENT

In this section we establish the existence of positive solutions on $[t_1, t_2]$ of

$$\begin{cases} y'' + \phi(t)\, g(t, y, y'(t)) = 0, \\ y(t_1) = 0, \qquad y'(t_2) = b \ge 0, \end{cases} \tag{2.2.1}$$

where $t_1 > 0$, $\phi(t)$ is as in (2.0.5).

Theorem 2.2.1 *Suppose that:*

(*i*) *g is continuous on $[t_1, t_2] \times (0, \infty) \times (-\infty, \infty)$;*

(ii) $0 < g(t, y, z) \leq \psi(t)\, h(y)$ *on* $(t_1, t_2) \times (0, \infty) \times (-\infty, \infty)$, *where*

(*a*) $h(y) > 0$ *is continuous and nonincreasing on* $(0, \infty)$,

(*b*) $\psi > 0$ *is continuous on* $[t_1, t_2]$,

(*c*) $1/h(k(t-t_1))$ *is continuous on* $[t_1, t_2]$ *for each constant* k, $0 < k < 1$.

(*d*) $\int_{t_1}^{t_2} h(k\,(t-t_1))\, \psi(t) dt < \infty$ *for any constant* k, $0 < k < 1$,

(*e*) $\lim_{t \to \infty} h(t) \int_1^t [h(s)]^{-1} ds = \infty$.

(iii) *For each constant* $M_0 > 0$ *there exists* $\xi(t)$ *continuous on* $[t_1, t_2]$ *and positive on* (t_1, t_2) *such that* $g(t, y, z) \geq \xi(t)$ *on* $[t_1, t_2] \times (0, M_0] \times (-\infty, \infty)$.

Then problem (2.0.1) *has a positive solution.*

Before giving the proof we illustrate applications of the theorem.

Example 2.2.2 Let $g(t, y, z) = t^{-2} y^{-1/2} (1 + 3y^{1/2})(2 + z^2)(1 + z^2)^{-1}$ and $\xi(t) = 3t^{-2}$. We let $h(y) = y^{-1/2}(1 + 3y^{1/2})$, $\psi(t) = 2t^{-2}$. Easy calculations show that $g(t, y, z)$, $h(y)$ and $\xi(t)$ satisfy all the conditions of Theorem 2.2.1. Therefore, the problem

$$\left.\begin{aligned} y'' + \phi(t) t^{-2} y^{-1/2}(1 + 3y^{1/2})[2 + (y')^2][1 + (y')^2]^{-1} &= 0 \\ y(t_1) = 0, \; y'(t_2) &= 0 \end{aligned}\right\}$$

has at least one positive solution.

<u>Example 2.2.3</u> Let $g(t,y,z)=(t-t_1)^2 y^{-5/2}(3+z^2)(1+z^2)^{-1}$, $\xi(t)=C^{5/2}(t-t_1)^2$, where C is as in *(iii)*. Let $h(y)=y^{-5/2}$, $\psi(t)=3(t-t_1)^2$, we obtain from Theorem 2.2.1 that the equation

$$\Delta y+(N-2)^2[r^{-(N-2)}-R_1^{-(N-2)}]^2 y^{-5/2}[3+r^{2(N-1)}(y'(r))^2][1+r^{2(N-1)}(y'(r))^2]^{-1}=0$$

for $r\in[R_2,R_1]$ with the boundary condition (2.0.3*c*) has a positive radially symmetric solution.

Proof of Theorem 2.2.1

The main ideas of the proof of this theorem are same as that of the proof of Theorem 2.1.1.

We consider the problem

$$\begin{cases} y''+\phi(t)\,g(t,y,y')=0, \\ y(t_1)=1/\mathrm{n}, \quad y'(t_2)=b\geq 0, \end{cases} \tag{2.2.2}$$

where $\mathrm{n}\in\mathbb{N}$, to avoid the possible singularity of g at $y=0$. If y is any solution to (2.2.2), then $y''<0$ on (t_1,t_2). So, $y'>b\geq 0$ on (t_1,t_2) which implies y is strictly increasing on (t_1,t_2). Accordingly, we may remove the singularity at $y=0$ by defining

$$g_{\mathrm{n}}(t,y,z)=\begin{cases} g(t,\,|y|,\,z), & t_1<t<t_2, \quad |y|\geq 1/\mathrm{n}, \\ g(t,\,1/\mathrm{n},\,z), & t_1<t<t_2, \quad |y|<1/\mathrm{n}. \end{cases}$$

So, every solution υ of

$$\begin{cases} \upsilon''+\phi(t)\,g_{\mathrm{n}}(t,\upsilon,\upsilon')=0, \\ \upsilon(t_1)=1/\mathrm{n}, \quad \upsilon'(t_2)=b, \end{cases} \tag{2.2.3}$$

is a solution to (2.2.2). We now consider the family of problems

$$\begin{cases} y'' + (1-\lambda)\delta+\lambda\ \phi(t) g_n(t, y, y') = 0 \\ y(t_1)=1/n, \quad y'(t_2)=b \end{cases}, \tag{2.2.4$^n_\lambda$}$$

where $0<\delta<1$ is a positive real number which is determined below and $\lambda\in[0,1]$. Let $y(t)$ be a solution of $(2.2.4)^n_\lambda$. Then $y(t)\geq 1/n$, $y'(t)\geq b$ for $t\in[t_1, t_2]$. We also have

$$y'' + (1-\lambda)\delta + \lambda\phi(t)\,\psi(t)\,h(y) \geq y'' + (1-\lambda)\delta + \lambda\phi(t)\,g_n(t, y, y')=0$$

and this implies $-y'' \leq (1-\lambda)\delta + \lambda\,\phi(t)\,\psi(t)\,h(y)$. Integrate from t to t_2 to obtain that

$$y'(t)-b \leq (1-\lambda)\,\delta\,(t_2-t_1) + \lambda \int_t^{t_2} \phi(s)\,\psi(s)\,h(y(s))ds$$

$$\leq \delta\,(t_2-t_1) + h(y(t)) \int_{t_1}^{t_2} \phi(s)\,\psi(s)ds,$$

since h is nonincreasing. Thus,

$$y'(t) \leq \delta\,(t_2-t_1) + h(y(t)) \int_{t_1}^{t_2} \phi(s)\,\psi(s)ds + b. \tag{2.2.5}$$

Divide by $h(y(t))$ and integrate from t_1 to t to obtain

$$\int_{1/n}^{y(t)} \frac{dv}{h(v)} \leq [\delta(t_2-t_1)+b]\,\frac{(t_2-t_1)}{h(y(t))} + (t_2-t_1)\int_{t_1}^{t_2} \phi(t)\,\psi(t)\,dt. \tag{2.2.6}$$

It follows from (e) that there exists $C_0>0$ independently of λ, δ, n, such that

$$y(t)\leq C_0 \qquad t\in[t_1,t_2], \tag{2.2.7}$$

therefore $1/n\leq y(t)\leq C_0$. On the other hand, from $y''(t)\leq-(1-\lambda)\delta-\lambda\phi(t)\zeta(t)$, we obtain $y(t)\geq\theta(t)+1/n$, where

$$\theta(t)=b(t-t_1)+2^{-1}(1-\lambda)\,\delta\,(2t_2-t_1-t)(t-t_1)+\lambda\int_{t_1}^{t}\int_s^{t_2}\phi(v)\zeta(v)dvds\ . \tag{2.2.8}$$

Let $\zeta(t)=\int_{t_1}^{t}\int_{s}^{t_2}\phi(v)\,\xi(v)dvds$ and $k_0=\int_{t_1}^{t_2}\phi(s)\,\xi(s)ds$, then $\zeta'(t_1)=k_0>0$. Hence there is an $\varepsilon>0$ such that $\zeta(t)\geq 2^{-1}k_0(t-t_1)$ on $[t_1, t_1+\varepsilon]$. Since $\zeta(t)/(t-t_1)$ is bounded below on $[t_1+\varepsilon, t_2]$, there is a constant $k_1>0$ such that $\zeta(t)\geq k_1(t-t_1)$ on $[t_1+\varepsilon, t_2]$. Let $\bar{k}=\min\{k_0/2, k_1\}$, therefore,

$$1/\mathrm{n}+[b+2^{-1}(1-\lambda)\delta(t_2-t_1)+\lambda\bar{k}](t-t_1)\leq y(t)\leq C_0. \tag{2.2.9}$$

Let $0<\delta<\min\{(t_2-t_1)^{-1}, \bar{k}(t_2-t_1)^{-1}, \ 1\}$. We obtain

$$1/\mathrm{n}+[2^{-1}\delta(t_2-t_1)](t-t_1)\ \leq y(t)\leq C_0. \tag{2.2.10}$$

Using $|y''(t)|\leq 1+\phi(t)\,\psi(t)\,h(y(t))$, we know

$$|y''|\leq 1+\phi(t)\,\psi(t)\,h(\,[2^{-1}\delta(t_2-t_1)](t-t_1)), \tag{2.2.11}$$

and then

$$|y'(t)|\leq b+(t_2-t_1)+\int_{t_1}^{t_2}\phi(t)\,\psi(t)\,h([2^{-1}\delta(t_2-t_1)](t-t_1))dt=C_1, \tag{2.2.12}$$

where $C_1>0$ is independent of λ, n. Let $x(t)=1/h([2^{-1}\delta(t_2-t_1)](t-t_1))$, then

$$|x(t)\,y''(t)|\leq x(t)+\phi(t)\,\psi(t)\leq C_2. \tag{2.2.13}$$

For $u\in C^2(t_1, t_2)\cap C^1[t_1, t_2]$ define

$$\mathbb{K}_{a,b}=\{\,u\in C^2(t_1, t_2)\cap C^1[t_1, t_2]\ :\ u(t_1)=a\geq 0,\ u'(t_2)=b\geq 0 \text{ and } \|u\|_2<\infty\},$$

where $\|\cdot\|_2$, $\|\cdot\|_1$ are the same as in the proof of Theorem 2.1.1. We also let $\mathbb{C}$ be same as in the proof of Theorem 2.1.1. So, $\mathbb{K}_{a,b}$ and $\mathbb{C}$ are Banach spaces.

Define mappings $G_{\lambda,n} : C^1[t_1, t_2] \longrightarrow \mathbb{C}$, $j : \mathbb{K}_{1/n,\, b} \longrightarrow C^1[t_1, t_2]$ and $L : \mathbb{K}_{1/n,\, b} \longrightarrow \mathbb{C}$ by $G_{\lambda,n}(u)(t) = x(t)\,[(1-\lambda)\delta + \lambda\phi(t)\, g_n(t, u, u')]$, $ju = u$, and $Lu = x(t)\, u''(t)$. Clearly $G_{\lambda,n}$ is continuous by the continuity of $x(t)\, g_n$. By the same idea as in the proof of Theorem 2.1.1, we know that j is completely continuous and L^{-1} exists and is continuous. Now, $(2.2.4)^n_\lambda$ is equivalent to

$$(I + L^{-1} G_{\lambda,n} j)(y) = 0. \tag{2.2.14}$$

Let

$$C = \max\left\{ C_0, C_1, C_2, 1 + b\,(t_2 - t_1) + \delta(t_2 - t_1)^2, \right.$$

$$\left. \delta \max_{[t_1, t_2]} (x(t)),\ b + \delta(t_2 - t_1) \right\}. \tag{2.2.15}$$

and define

$$U = \{ u \in \mathbb{K}_{1/n,\, b},\ \|u\|_2 < C + 1 \}, \tag{2.2.16}$$

then $(I + L^{-1} G_{\lambda,n} j)(y) \neq 0$ on ∂U. Finally, by $1/n + b(t - t_1) + 2^{-1}\delta(2t_2 - t_1 - t)(t - t_1) \in U$ and properties of Leray-Schauder degree, we have

$$\deg_{LS}(I + G_{1,n} j, U, 0) = 1, \tag{2.2.17}$$

thus, (2.2.3) has a solution in U. The remainder of the proof is similar to the proof of Theorem 2.1.1.

Remark e The result of Theorem 2.2.1 extend the results of Theorem 2.0.1 and the results of [57]. In [57], one of the condition on $h(y)$ is $\int_0^c h(s)ds < \infty$, for all $c \in [0, \infty)$. So, for example, no result of [57] applies to Example 2.2.3 above.

Theorem 2.2.4 *Suppose that*

(*i*) g *is continuous on* $(t_1, t_2)\times(0, \infty)\times(0, \infty)$,

(*ii*) $\xi(t) \le g(t, y, z) \le \psi(t)\, h(y)\, p(z)$ *on* $(t_1, t_2) \times (0, \infty) \times (0, \infty)$,

where

(*a*) $h(y)>0$, $p(z)>0$ *are continuous and nonincreasing on* $(0, \infty)$,

(*b*) $\psi(t)>0$ *and* $\xi(t)>0$ *are continuous on* (t_1, t_2); $\xi(t_2)>0$ *and* $\int_{t_1}^{t_2} \xi(t)dt<\infty$,

(*c*) $\int_{t_1}^{t_2} \psi(t)\, h(\alpha(t-t_1))\, p(\beta(t_2-t))dt<\infty$, *for each pair* (α, β), $0<\alpha, \beta<1$,

(*d*) $g(t,y,z) / [\psi(t)\, h(\alpha(t-t_1))\, p(\beta(t_2-t))]$, $[\psi(t)h(\alpha(t-t_1))p(\beta(t_2-t))]^{-1}$ *are continuous on* $[t_1, t_2]\times(0, \infty)\times(0, \infty)$ *and* $[t_1, t_2]$ *respectively for* α, β, $0<\alpha,\beta<1$. *Then problem* (2.2.1) *has a positive solution.*

Example 2.2.5 Let $g(t, y, z)=\psi(t)\, h(y)\, p(z)$, here $\psi(t)=t$, $h(y)=y^{-3/4}(1+y^{3/4})$ and

$$p(z)=\begin{cases} z^{-1/2}, & 0<z\le 1 \\ \frac{1}{2}(1+z^{-4/5}), & z>1 \end{cases}.$$

Let $\xi(t)=t/2$. A calculation shows that the functions satisfy the conditions of Theorem 2.2.4.

Proof of Theorem 2.2.4

We only discuss the case when $y'(t_2)=b=0$, for $b\neq 0$ this theorem follows easily from the proof of Theorem 2.2.1. We consider the family of problems

$$\begin{cases} y'' + (1-\lambda)\delta + \lambda\,\phi(t)\; g_n(t, y, y') = 0 \\ y(t_1) = 1/n, \quad y'(t_2) = 1/n \end{cases}, \tag{2.2.18$^n_\lambda$}$$

where

$$g_n(t, y, z) = \begin{cases} g(t, |y|, |z|), & t_1 < t < t_2,\ |y| \geq 1/n,\ |z| \geq 1/n \\ g(t, 1/n, 1/n), & t_1 < t < t_2,\ |y| < 1/n,\ |z| < 1/n \end{cases},$$

$0 < \delta < 1$ is determined as in the proof of Theorem 2.2.1 and $\lambda \in [0,1]$. Let y be a solution of $(2.2.18)^n_\lambda$, then $y(t) \geq 1/n$, $y'(t) \geq 1/n$, for $t \in [t_1, t_2]$. It follows from (*ii*) that $y(t) \geq \theta(t) + 1/n$ and $y'(t) \geq \theta'(t) + 1/n$, where

$$\theta(t) = 2^{-1}(1-\lambda)\delta(2t_2 - t_1 - t)(t - t_1) + \lambda \int_{t_1}^{t} \int_{s}^{t_2} \phi(v)\, \xi(v)\, dv\, ds, \tag{2.2.19}$$

$$\theta'(t) = (1-\lambda)\delta(t_2 - t) + \lambda \int_{t}^{t_2} \phi(s)\, \xi(s)\, ds, \tag{2.2.20}$$

$$\theta''(t_2) = -(1-\lambda)\delta - \lambda\, \phi(t_2)\, \xi(t_2). \tag{2.2.21}$$

Then, there exists a $C_3 > 0$ such that $\int_{t_1}^{t_2} \phi(s)\, \xi(s)\, ds > C_3$ and

$$\theta'(t_1) \geq (1-\lambda)\delta(t_2 - t_1) + \lambda C_3. \tag{2.2.22}$$

Using the same methods as in the proof of Theorem 2.2.1, we have that there exists k_2, $0 < k_2 < 1$ such that

$$y(t) \geq k_2(t - t_1) \quad \text{for } t \in (t_1, t_2). \tag{2.2.23}$$

Whether $\xi(t_2) = \infty$ or not, there exists $C_4 > 0$ such that $\phi(t_2)\, \xi(t_2) > C_4$, and

$$\theta''(t_2) \leq -(1-\lambda)\delta - \lambda C_4. \tag{2.2.24}$$

Using the same idea as above, we have that there exists k_3, $0 < k_3 < 1$ such that

$$\theta'(t) \geq k_3(t_2 - t), \quad \text{for } t \in (t_1, t_2). \tag{2.2.25}$$

Therefore,

$$y'(t) \geq k_3(t_2-t)+1/n \qquad \text{and} \qquad |y''(t)| \leq 1 + \phi(t)\,\psi(t)\,h(k_2(t-t_1))\,p(k_3(t_2-t)).$$

Hence, by condition (c),

$$|y'(t)| \leq (t_2-t_1) + \int_{t_1}^{t_2} \phi(t)\,\psi(t)\,h(k_2(t-t_1))\,p(k_3(t_2-t))dt = C_5,$$

and

$$|y(t)| \leq C_6 \quad \text{for } t \in [t_1, t_2]. \tag{2.2.26}$$

where $C_6 = C_5(t_2-t_1)$. Let $x(t) = [\,\psi(t)\,h(k_2(t-t_1))\,p(k_3(t_2-t))]^{-1}$, so that,

$$|x(t)\,y''(t)| \leq x(t) + \phi(t) \leq C_7.$$

Here C_5, C_6 and C_7 are positive constants which are independent of λ, n. The remainder of the proof is similar to the proofs of Theorem 2.2.1.

If $g(t, y, z)$ has singularities at $t=t_1$, $t=t_2$ and $\lim\limits_{y\to\infty} g(t, y, z)=0$ for $(t, z) \in (t_1, t_2) \times (-\infty, \infty)$, then the condition (ii) of Theorem 2.2.4 does not hold. To cover this case, we have the following theorem.

Theorem 2.2.6 *Suppose that*

(i) g *is continuous on* $(t_1, t_2) \times (0, \infty) \times (-\infty, \infty)$*;*

(ii) $0 < g(t, y, z) \leq \psi(t)\,h(y)$ *on* $(t_1, t_2) \times (0, \infty) \times (-\infty, \infty)$*, where*

(a) $h > 0$ *is continuous and nonincreasing on* $(0, \infty)$*,*

(b) $\psi > 0$ *is continuous on* (t_1, t_2) *and* $\int_{t_1}^{t_2} (t-t_1)\,\psi(t)dt < \infty$*,*

(c) $g(t, y, z) / [\psi(t)\,h(k(t-t_1))]$ *and* $1 / [\psi(t)\,h(k(t-t_1))]$ *are continuous on* $[t_1, t_2] \times (0, \infty) \times (-\infty, \infty)$ *and* $[t_1, t_2]$ *respectively, for each*

constant k, $0<k<1$,

$$(d)\quad \int_{t_1}^{t_2} h(k(t-t_1))\,\psi(t)\,\phi(t)dt<\infty \text{ for any constant } k,\ 0<k<1,$$

$$(e)\quad \lim_{t\to\infty} h(t)\int_1^t [h(s)]^{-1}ds=\infty.$$

(*iii*) *For each constant* $M_0>0$ *there exists* $\xi(t)>0$ *continuous on* (t_1, t_2) *and* $\int_{t_1}^{t_2}(t-t_1)\,\xi(t)dt<\infty$, *such that* $g(t, y, z)\geq\xi(t)$ *on* $(t_1, t_2)\times(0, M_0]\times(-\infty,\infty)$.

Then problem (2.2.1) *has a positive solution.*

Proof. Let $x(t)=\{\psi(t)\,h([2^{-1}\delta(t_2-t_1)](t-t_1))\}^{-1}$ be as in Theorem 2.2.1, then the result of this theorem follows from a slight modification of the proof of Theorem 2.1.1 by changing the order of integration.

If $g(t, y, z)$ has singularities at $t=t_1$, $t=t_2$ and $\lim_{z\to\infty} g(t, y, z)=0$ for $(t, y)\in(t_1, t_2)\times(0, \infty)$, then we have the following theorem.

Theorem 2.2.7 *Suppose that*

(*i*) g *is continuous on* $(t_1, t_2)\times[0, \infty)\times(0, \infty)$,

(*ii*) $0<g(t, y, z)\leq\psi(t)\,p(z)$ *on* $(t_1, t_2)\times(0, \infty)\times(0, \infty)$, *where*

(*a*) $p(z)>0$ *is continuous and nonincreasing on* $(0, \infty)$ *and* $z\,p(z)$ *is nondecreasing on* $(0,\infty)$,

(*b*) $\psi(t)>0$ *is continuous on* (t_1, t_2) *and* $\int_{t_1}^{t_2}\psi(t)dt<\infty$,

(*c*) $g(t, y, z)\,/\,[\psi(t)\,p(k(t_2-t))]$ *is continuous on*

$[t_1, t_2] \times [0, \infty) \times (0, \infty)$ *for each constant* $k > 0$.

(iii) For some constants M_1, $M_2 > 0$ *there exists* $\xi(t)$ *continuous, positive on* (t_1, t_2) *and* $\int_{t_1}^{t_2} \xi(t)dt < \infty$ *such that* $g(t, y, z) \geq \xi(t)$ *on* $(t_1, t_2) \times (0, M_1] \times (0, M_2]$.

Then problem (2.0.7) *has a positive solution.*

Proof. We only discuss the case when $y'(t_2) = b = 0$. We consider the family of problems

$$\begin{cases} y'' + \lambda\ \phi(t)\, g_n(t,\ y\ ,\ y') = 0 \\ y(t_1) = 1/n,\ \ y'(t_2) = 1/n \end{cases}, \qquad (2.2.27)_\lambda^n$$

here $g_n(t, y, y')$, λ are as in the proof of Theorem 2.2.4. Let y be a solution of $(2.2.27)_\lambda^n$, then $y(t) \geq 1/n$ and $y'(t) \geq 1/n$, for $t \in [t_1, t_2]$. From *(ii)* we know that $(1\,/\,p(y'(t)))\, y''(t) + \lambda\, \phi(t)\, \psi(t) \geq 0$. Let $f(z) = \int_0^z \frac{dz}{p(z)}$, $f(z)$ is increasing since $p(z)$ is decreasing and $(f(y'(t)))' + \lambda\phi(t)\, \psi(t) \geq 0$. Therefore,

$$f(y'(t)) \leq f(1/n) + \int_{t_1}^{t_2} \phi(t)\, \psi(t)dt.$$

The fact that $f(z)$ is increasing and condition *(b)* imply that $|y'(t)| \leq C_8$ and $|y(t)| \leq C_8(t_2 - t_1)$. Let $C_9 = C_8(t_2 - t_1)$, then, C_8 and C_9 are independent of λ, n. By condition *(iii)* and the same idea as in the proof of Theorem 2.2.4, we have that there exists $k_4 > 0$ such that $y'(t) \geq \lambda k_4(t_2 - t)$ and

$$|y''(t)| \leq \lambda\phi(t)\, \psi(t)\, p(\lambda k_4(t_2 - t)) \leq \phi(t)\, \psi(t)\, p(k_4(t_2 - t)). \qquad (2.2.28)$$

Let $x(t) = 1/\,[\psi(t)p(k_4(t_2 - t))]$. Define $\mathbb{K}_{1/n,1/n}$, L, j, $G_{\lambda,n}$ as in Theorem 2.2.1 for $\delta = 0$. The proof is then a consequence of Leray–Schauder degree theory as in the proof of Theorem 2.2.1.

Remark f By the same methods we can discuss the following problem

$$\begin{cases} y'' + \phi(t)g(t, y, y') = 0 \\ y'(t_1) = 0, \quad y(t_2) = 0 \end{cases}, \tag{2.2.29}$$

and obtain results for this problem similar to the above theorems.

In the following, we use the above methods to show the existence of positive solutions on (t_1, t_2) $(t_1 > 0)$ to the problem

$$\begin{cases} y'' + \phi(t)\, g(t, y, y') = 0, \\ y(t_1) = y(t_2) = 0. \end{cases} \tag{2.2.30}$$

Theorem 2.2.8 *Suppose that*

(*i*) *g is continuous on $(t_1, t_2) \times (0, \infty) \times (-\infty, \infty)$;*

(*ii*) *$0 < g(t, y, z) \le \psi(t)\, h(y)$ on $(t_1, t_2) \times (0, \infty) \times (-\infty, \infty)$, where*

(*a*) *$h(y)$ is continuous and nonincreasing on $(0, \infty)$,*

(*b*) *$\psi > 0$ is continuous on (t_1, t_2) and*

$$\int_{t_1}^{t_2} (t_2 - t)(t - t_1)\, \psi(t)\,dt < \infty,$$

(*c*) $$g(t,y,z)/[\psi(t)h(k(t-t_1)(t_2-t))] \in C^0([t_1,t_2] \times (0,\infty) \times (-\infty,\infty)),$$

for each constant k, $0 < k < 1$,

(*d*) $\int_{t_1}^{t_2} \psi(t)\, h[k(t-t_1)(t_2-t)]dt < \infty$ *for any constant k, $0 < k < 1$,*

(*e*) $\lim_{t \to \infty} h(t) \int_1^t [h(s)]^{-1} ds = \infty.$

(*iii*) *For each constant $M_0 > 0$ there exists $\xi(t)$ continuous and positive* on (t_1, t_2), $\int_{t_1}^{t_2} (t-t_1)(t_2-t)\,\xi(t)dt < \infty$ such that

$$g(t, y, y') \geq \xi(t) \text{ on } (t_1, t_2) \times (0, M_0] \times (-\infty, \infty).$$

Then problem (2.2.30) *has a positive solution.*

Proof. Using the same methods as in the proof of Theorem 2.1.1.

Theorem 2.2.9 *Suppose that*

(*i*) $g(t, y, z)$ *is continuous on* $[t_1, t_2] \times (0, \infty) \times (0, \infty)$;

(*ii*) $\xi(t) \leq g(t, y, z) \leq \psi(t)\,h(y)\,p(z)$ *on* $[t_1, t_2] \times (0, \infty) \times (0, \infty)$, *where*

(*a*) $h > 0$, $p > 0$ *are continuous and nonincreasing on* $(0, \infty)$, $\lim_{z \to 0} p(z) = \infty$,

(*b*) $\xi(t) > 0$, $\psi(t) > 0$ *are continuous on* $[t_1, t_2]$, $\xi > 0$ *at* $t = t_1, t_2$,

(c) $\int_{t_1}^{t_1+\varepsilon} \psi(t)\,h(k(t-t_1))dt < \infty$; $\int_{t_2-\varepsilon}^{t_2} \psi(t)\,h(k(t_2-t))dt < \infty$; $\int_1^{\varepsilon} p(t)dt < \infty$,

for all constants ε, k, *with* $0 < \varepsilon$, $k < 1$,

(*d*) $1/h(k(t_2-t))$ and $1/h(k(t-t_1))$ *are continuous on* $[t_1, t_2]$.

Then the problem

$$\begin{cases} y'' + \phi(t)\, g(t, y, |y'|) = 0 \\ y(t_1) = y(t_2) = 0 \end{cases}, \tag{2.2.31}$$

has a positive solution in $C^2(t_1, t_2) \cap C^1[t_1, t_2]$.

Proof. We consider the family of problems

$$\begin{cases} y'' + (1-\lambda)\delta + \lambda\phi(t)\, g_n(t, y+1/n, y') = 0 \\ y(t_1) = y(t_2) = 0 \end{cases} . \qquad (2.2.32)_\lambda^n$$

Here $g_n(t, y, y')$ is as in the proof of Theorem 2.2.4. Let y be a solution of $(2.2.32)_\lambda^n$, then $y \geq 0$ for $t \in [t_1, t_2]$. Let $t_3 \in (t_1, t_2)$, $y(t_3) = \max_{t_1 < t < t_2} y(t)$. By the proof of Theorem 2.2.4, we know that there exist k_5, k_6 satisfying $0 < k_5, k_6 < 1$ such that $y(t) \geq k_5(t-t_1)$, $y'(t) \geq k_6(t_3-t)$ for $t \in [t_1, t_3]$. A similar argument on $[t_3, t_2]$ yields $y(t) \geq k_7(t_2-t)$, $|y'(t)| \geq k_8(t-t_3)$ for $t \in [t_3, t_2]$. Here, $0 < k_7, k_8 < 1$. Then,

$$|y''(t)| \leq (1-\lambda)\,\delta + \lambda\,\phi(t)\,\psi(t)\,h(y)\,p(|y'|)$$

$$\leq \begin{cases} 1 + \phi(t)\,\psi(t)\,h(k_5(t-t_1))\,p(k_6(t_3-t)), & t \in (t_1, t_3) \\ 1 + \phi(t)\,\psi(t)\,h(k_7(t_2-t))\,p(k_8(t-t_3)), & t \in (t_3, t_2) \end{cases} . \tag{2.2.33}$$

Let $x(t) = \begin{cases} 1 / h(k_5(t-t_1))\,p(k_6(t_3-t)), & t \in (t_1, t_3) \\ 1 / h(k_7(t_2-t))\,p(k_8(t-t_3)), & t \in (t_3, t_2) \end{cases}$. By condition (d), $x \in C^0[t_1, t_2]$. Therefore,

$$|x(t)\,y''(t)| \leq C_{10}. \tag{2.2.34}$$

Condition (c) and (2.2.33) imply

$$|y'(t)| \leq C_{11}\ , \ |y(t)| \leq C_{12}. \tag{2.2.35}$$

Here C_{10}, C_{11} and C_{12} are positive constants which are independent of λ, n and δ. The remainder of the proofs follows from a slight modification of the proof of Theorem 2.1.1.

Remark g It follows easily that the results of Theorems 2.2.6 and 2.2.9 still hold for $t_1 = 0$ if the function $m(t) = \phi(t)\,\psi(t)$ satisfies the conditions imposed on $\psi(t)$, $n(t) = \phi(t)\,\xi(t)$ satisfies the conditions imposed on $\xi(t)$. The result of Theorem 2.2.4 holds for $t_1=0$, if $m(t)$ satisfies the conditions imposed on $\psi(t)$ and $\int_0^{t_2} \phi(t)\,\xi(t)dt<\infty$.

Remark h Directly consider the problem:

$$\begin{cases} y'' + \lambda\ \phi(t)\ \ g_n(t, y+1/n, y')=0 \\ y(t_1)=y(t_2)=0 \end{cases} \tag{2.2.36}$$

Here $g_n(t, y, y')$ is as in Theorem 2.2.9. Using the same idea as in the proof of Theorem 2.2.9, we can obtain a existence result for problem (2.2.31) if $g(t,y,z)$, $\psi(t)$ and $p(z)$ satisfy all the conditions of Theorem 2.2.7 where also $\lim_{z\to 0} p(z)=\infty$ but with (c) replaced by

$(c)'$ $\quad g(t, y, z)\,/\,\psi(t)$ is continuous on $[t_1, t_2] \times [0, \infty)\times(0,\infty)$.

2.3 APPLICATIONS TO EXISTENCE OF RADIAL SOLUTIONS

In this section, we use the results obtained in section 2.1 to obtain the existence of positive radial solutions of problem (2.0.4) in annular domains. We only state the results, the proofs are immediate consequences of the theorems of section 2.1. We use g to replace f in problem (2.0.4).

Theorem 2.3.1 *Let g satisfy*

(*a*) *g is continuous and strictly decreasing on* $(0, \infty)$,

(*b*) $g(y)>0$ *on* $(0, \infty)$ *and* $g(0)\neq\infty$,

(*c*) $\lim_{t\to\infty} g(t)\int_0^t [g(s)]^{-1}ds=\infty$.

Then problem (2.0.4) *has a unique positive radial solution* $y(r)\in C^2(R_2, R_1)\cap C^1[R_2, R_1]$, *where* $0<R_2<R_1<\infty$.

Theorem 2.3.2 *Let* $\alpha > 0$; $R_3 > 0$, $\xi(r) \in C^0[R_3, \infty)$; $\xi(r) > 0$ *on* $[R_3, \infty)$ *and*

$$\int_0^{t_3} (t_3 - t)^{-\alpha} t^{-\alpha} \xi_1(t) [(N-2)t]^{-(2N-2)/(N-2)} dt < \infty.$$

Then, the problem

$$\begin{cases} \Delta y + \xi(r) y^{-\alpha} = 0 & r > R_3, \quad N \geq 3 \\ y = 0 & on \quad r = R_3 \end{cases} \tag{2.3.1}$$

has a unique positive radial solution $y(r) \in C^2(R_3, \infty) \cap C^1[R_3, \infty)$ *that tends to* 0 *as* $r \to \infty$, *where*

$$t = [(N-2)\, r^{N-2}]^{-1};\ r = |x|\ ;\ t_3 = [(N-2)\, R_3^{N-2}]^{-1};\quad \xi_1(t) = \xi(((N-2)t)^{-1/(N-2)}).$$

Theorem 2.3.3 *Let* R_3, r, $\xi(r)$, t_3 *be as in Theorem* 2.3.2; $\xi_1(t)$ *satisfy*

$$\int_0^1 \xi_1(t) [(N-2)t]^{-(2N-2)/(N-2)} dt < \infty; \tag{2.3.2}$$

let g satisfy:

(*a*) *g is continuous and strictly decreasing on* $(0, \infty)$,

(*b*) $g(y) > 0$ *on* $(0, \infty)$; $g(0) \neq \infty$,

(*c*) $\lim_{t \to \infty} g(t) \int_0^t [g(s)]^{-1} ds = \infty$.

Then, the problem

$$\begin{cases} \Delta y + \xi(r) g(y) = 0 & r > R_3, \quad N \geq 3 \\ y = 0 & on \quad r = R_3 \end{cases} \tag{2.3.3}$$

has a unique positive radial solution in $C^2(R_3, \infty) \cap C^1[R_3, \infty)$ *that tends to* 0 *as* $r \to \infty$.

Theorem 2.3.4 *Let* R_3, t, $\xi_1(t)$, t_3 *be as in Theorem* 2.3.2; $\xi(r) \in C^0(R_3, \infty)$ *and* $\xi(r) > 0$ *on* $(0, \infty)$. *Let* $f(t) = \xi_1(t) [(N-2)t]^{-(2N-2)/(N-2)}$ *and* $f(t)$ *satisfy* *the conditions imposed on* $\phi(t)$ *in Theorem* 2.2.7*;* $g(y)$ *satisfy the conditions of Theorem* 2.2.7. *Then, problem* (2.3.3) *has a unique positive radial solution in* $C^2(R_3, \infty) \cap C^1[R_3, \infty)$ *that tends to* 0 *as* $r \to \infty$.

CHAPTER THREE

BOUNDARY VALUE PROBLEMS FOR A CLASS OF QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS

INTRODUCTION

In this chapter, we establish the existence of solutions to the Periodic Boundary Value Problem (BVP)

$$(|u'|^{p-2}u')' + f(t, u, u') = y(t), \quad u(0)=u(1),\ u'(0)=u'(1) \tag{3.0.1}$$

under various conditions on the function $y : [0, 1] \longrightarrow \mathbb{R}$ and the function $f : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$.

We also consider the problem of the following form

$$\left.\begin{aligned} Au - f(t, u) &= 0 \quad \text{in } (0,1) \\ u'(0) = u'(1) &= 0 \end{aligned}\right\}, \tag{3.0.2}$$

where $Au = -(a(|u'|^2)u')'$, $a : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous mapping such that $h(t^{h(t^2)} = \int_0^{2} \imath(\tau)d\tau$ is a strictly convex function on $\mathbb{R}$. That is, the equation (3.0.2) coincides with the equation $g'(u)=0$, where

$$g(u) = \frac{1}{2}\int_0^1 h(|u'|^2)dt - \int_0^1\int_0^{u(t)} f(\tau, u(\tau))d\tau dt. \tag{3.0.3}$$

By a C^1–solution of problem (3.0.1) we mean that $u \in C^1([0, 1])$, $u(0)=u(1)$,

$u'(0)=u'(1)$ and u satisfies:

$$|u'(t)|^{p-2}u'(t)-|u'(0)|^{p-2}u'(0)$$

$$=-\int_0^t\left[f(s,u(s),u'(s))-y(s)\right]ds.$$

By a C^1-solution of problem (3.0.2) we mean $u\in C^1([0,1])$ satisfying $u'(0)=u'(1)=0$ and

$$-(a(|u'|^2)u')'-f(t,u)=0,\quad a.e.\text{ in }[0,1]. \tag{3.0.4}$$

In a recent paper [62], the problem

$$(|u'|^{p-2}u')'+f(t,u)=0,\qquad u(0)=u(1)=0, \tag{3.0.5}$$

was considered. Under homogeneous Dirichlet boundary condition, $G_p=A^{-1}$ is compact from $C^0[0,1]$ to $C^0[0,1]$ and Eq (3.0.5) is equivalent to

$$u-G_p(f(t,u))=0, \tag{3.0.6}$$

so, Leray-Schauder degree theory can be used. But under the periodic boundary condition or the Neumann boundary condition, A is not invertible, and the approach of [62] is not applicable. Problems of this kind were also studied in [6, 10, 38, 58, 75]. In these papers, all the authors considered the problems in the Sobolev spaces $W_0^{1,p}$ or $W^{1,p}$ and proved the existence of solutions in these spaces by generalized degree theory (see [10, 38, 58]).

It is clear that these methods do not work with our boundary conditions, we need solutions belonging to $C^1([0,1])$. We note that the special case of problem (3.0.4) with Dirichlet boundary conditions can be easily dealt with in $H_0^1(0,1)$, but we do not know how to deal with the problem with our boundary conditions. Let $Au=-(b(u)u')'$, for $b\in C^0(\mathbb{R},\mathbb{R}^+)$, and consider the problem

$$\left.\begin{array}{ll} Au-f(t,u)=0 & \text{in }(0,1)\\ u(0)=u(1)=0 & \end{array}\right\}, \tag{3.0.7}$$

for $f \in C^0([0,1] \times \mathbb{R})$. In this case, the equation (3.0.7) does not have a variational structure. But it can be easily shown that $B: H_0^1(0,1) \to (H_0^1(0,1))^*$, $Bu = Au - f(t,u)$ is of type $(S)_+$, where $(H_0^1(0,1))^*$ is the dual space of $H_0^1(0,1)$. So, using the degree theory of the mapping of type $(S)_+$, we can obtain some existence results for (3.0.7) by using the same ideas as in section 3.3 below. But it is not clear how the existence of solutions of the equation in (3.0.7) with periodic boundary condition or Neumann boundary condition can be treated.

In sections 3.1–2, we study the existence of solutions of problem (3.0.1) and the problem:

$$-(|u'|^{p-2}u')' + f(t,u,u') = y(t), \quad u(0)=u(1),\ u'(0)=u'(1). \tag{3.0.8}$$

Our methods are closely related to [58], but we consider the problems in $C^2_{per}(0,1)$, where $C^2_{per}(0,1) = \{ u \in C^2(0,1) \mid u(0)=u(1),\ u'(0)=u'(1)\}$. In this case, we do not know whether $A : C^2_{per}(0,1) \longrightarrow C^0[0,1]$ belongs to a class of mappings with a degree theory such as A–proper maps because we cannot prove that A maps $C^2_{per}(0,1)$ onto a linear subspace of $C^0[0,1]$. In order to overcome this difficulty, we replace A by $A_\varepsilon : C^2_{per}(0,1) \longrightarrow C^0[0,1]$ where $A_\varepsilon u = -\varepsilon u'' - (|u'|^{p-2}u')'$ and $\varepsilon > 0$. Our results are related to those of the recent papers of [60–61]. In section 3.3, we directly use Leray–Schauder degree theory to give an existence result for problem (3.0.2).

3.1 AN ABSTRACT EXISTENCE THEOREM

In this section we give an abstract existence result similar to that of [60–61] where it was assumed that $p=2$. To do this, we first show that A_ε

possesses properties similar to those of a Fredholm map of index 0.

For convenience, we let $\phi_p(s)=|s|^{p-2}s$, where $p>2$ is fixed and let $g_\varepsilon(s)=\varepsilon s+\phi_p(s)$.

<u>Lemma 3.1.1</u> *For any* $h\in C^0[0,1]$ *with* $\int_0^1 h(s)ds=0$, *there exists a unique* $u\in C^2(0,1)$ *such that* u *satisfies*

$$-\varepsilon u'' - (|u'|^{p-2}u')' = h, \quad u(0)=u(1)=0,\ u'(0)=u'(1). \tag{3.1.1}$$

Proof. For a given $h\in C^0[0,1]$ with $\int_0^1 h(t)dt=0$, we look for a function $u\in C_0^1([0,1])$, such that $g_\varepsilon(u')$ is absolutely continuous and

$$-\varepsilon u'' - (|u'|^{p-2}u')' = h, \quad a.e. \text{ on } [0,1]. \tag{3.1.2}$$

First we find a solution $u\in W_0^{1,p}(0,1)$ of (3.1.2). It is well-known that searching for $u\in W_0^{1,p}(0,1)$ satisfying (3.1.2) is equivalent to finding critical points of the functional $\psi_h(w)=\frac{\varepsilon}{2}\int_0^1|w'|^2+\frac{1}{p}\int_0^1|w'|^p-\int_0^1 hw$. We find that ψ_h is a continuous functional such that $\psi_h\longrightarrow\infty$ as $\|w\|_{1,p}\longrightarrow\infty$. Hence (see [22]) it possesses a critical point $u\in W_0^{1,p}(0,1)$ at which it reaches its minimum. So, u satisfies $u(0)=u(1)=0$ and

$$\int_0^1\left[\varepsilon u'+\phi_p(u')\right]v'=\int_0^1 h\,v, \tag{3.1.3}$$

for all $v\in W_0^{1,p}(0,1)$. Then, it follows that $\varepsilon u'+\phi_p(u')$ belongs to $L^q(0,1)$ and satisfies (3.1.3) for all $v\in C_0^\infty(0,1)$. Here $q=p/(p-1)$, so $q<p$ for $p>2$. Therefore, $\varepsilon u'+\phi_p(u')\in W^{1,q}(0,1)$. From this and theorem VIII of [7] we can see that $g_\varepsilon(u')$ is an absolutely continuous function which satisfies (3.1.2). Since g_ε is invertible and $g_\varepsilon^{-1}\in C^1(\mathbb{R})$, using Remark 6 of [7] we find that $u\in C^1$. (3.1.2) means that $u''=-h/[\varepsilon+(p-1)|u'|^{p-2}]$ *a.e.* in $[0,1]$ and $u'(0)=u'(1)$. The latter equality follows from the fact that the function $g_\varepsilon(s)$ is strictly increasing, (3.1.2) and $\int_0^1 h(s)\,ds=0$. Now we can redefine u'' on a set of measure 0 so that $u''=-h/[\varepsilon+(p-1)|u'|^{p-2}]$, for all $t\in[0,1]$, then $u\in C_0^2(0,1)$ and $u'(0)=u'(1)$.

Now we prove u is unique. Suppose that there is $u_1\in C^2(0,1)$ such that $-\varepsilon u_1'' - (|u_1'|^{p-2}u_1')'=h$ and $u_1(0)=u_1(1)=0$, $u_1'(0)=u_1'(1)$, then

$$\varepsilon(u''-u_1'') + [(|u'|^{p-2}u')'-(|u_1'|^{p-2}u_1')']=0.$$

Multiplying by $(u-u_1)$ and integrating from 0 to 1, we obtain

$$\varepsilon\int_0^1 (u'-u_1')^2dt + \int_0^1 (\phi_p(u')-\phi_p(u_1'))\,(u'-u_1')dt=0.$$

By the monotonicity of ϕ_p we get $u'=u_1'$ on $[0, 1]$ and hence $u\equiv u_1$ on $[0, 1]$.

<u>*Remark a*</u> If $\varepsilon=0$, then $g_0(s)=\phi_p(s)$. It follows directly that ϕ_p^{-1} does not belong to $C^1(\mathbb{R})$. So, we cannot prove that $u\in C^2(0,1)$ from (3.1.2).

Let $C^2_{0,\text{per}}(0,1)=\left\{u\in C^2_{\text{per}}(0,1) \mid u(0)=u(1)=0\right\}$, then

$$C^2_{\text{per}}(0,1) = \mathbb{R}\oplus C^2_{0,\text{per}}(0,1).$$

From above we know $u_0\in C^2_{0,\text{per}}(0,1)$. For $h\in C^0(0,1)$, we write $h=h_0+h_1$, $h_0=\int_0^1 h$, and $\int_0^1 h_1=0$, then $C^0=\mathbb{R}\oplus Z$, here $Z=\left\{h\in C^0(0,1) \mid \int_0^1 h=0\right\}$. Let $A_{\varepsilon,1}$ be defined by $A_{\varepsilon,1}: C^2_{0,\text{per}}(0,1)\longrightarrow Z$, $A_{\varepsilon,1}(\bar{u})=A_\varepsilon(u)$, where $u=u(0)+\bar{u}$. Then $A_{\varepsilon,1}$ is invertible.

<u>Lemma 3.1.2</u> $A_\varepsilon : C^2_{\text{per}}(0,1)\longrightarrow C^0[0,1]$ *is a continuous map.*

Proof. Note that the function $T : (0,1)\times\mathbb{R}^2 \longrightarrow \mathbb{R}$, $T(t,x,y)=(\varepsilon+|x|^{p-2})\,y$ is continuous, the proof of this lemma is then routine.

Lemma 3.1.3 $A^{-1}_{\varepsilon,1} : Z \longrightarrow C^1_{0,\mathrm{per}}$ *is compact.*

Proof. For a bounded sequence $h_n\in Z$, $A^{-1}_{\varepsilon,1}(h_n)=u_n$, then $u_n\in C^2_{0,\mathrm{per}}(0,1)$ and

$$\|[\varepsilon u_n' + \phi_p(u_n')]'\|_0 \le \|h_n\|_0 \le M. \tag{3.1.4}$$

As $u_n(0)=u_n(1)=0$, we have that for any n, there exists a $t_n\in(0,1)$ such that $u_n'(t_n)=0$ and $\phi_p(u_n'(t_n))=0$. From this and $A_{\varepsilon,1}u_n=h_n$, it clearly follows that $-\varepsilon u_n'(t)-\phi_p(u_n'(t))=\int_{t_n}^{t} h_n(s)ds$. Therefore,

$$\|\varepsilon u_n' + \phi_p(u_n'(t))\|_0 \le \|h_n\|_0 \le M. \tag{3.1.5}$$

(3.1.4)-(3.1.5) imply

$$\|\varepsilon u_n' + \phi_p(u_n')\|_1 \le M.$$

Using the fact that $i : C^1([0,1]) \longrightarrow C^0[0,1]$ is compact, we obtain that there exists a convergent subsequence (still denoted by $\{\varepsilon u_n' + \phi_p(u_n')\}$) such that $\varepsilon u_n' + \phi_p(u_n') \longrightarrow v$ in $C^0[0,1]$. As $\varepsilon+\phi_p : \mathbb{R}\longrightarrow\mathbb{R}$ is strictly increasing and continuous, we have $u_n' \to (\varepsilon+\phi_p)^{-1}(v)$ in $C^0[0,1]$. Let $\bar{u}(t)=\int_0^t \left[\varepsilon+\phi_p\right]^{-1} v(s)ds$, then, it follows from $u_n' \to (\varepsilon+\phi_p)^{-1}(v)$ in $C^0[0,1]$ that $\bar{u}(1)=0$ and $\bar{u}'(0)=\bar{u}'(1)$. Therefore, $\bar{u}\in C^1_{0,\mathrm{per}}([0,1])$ and $u_n(t)$ tends to $\int_0^t \left[\varepsilon+\phi_p\right]^{-1} v(s)ds$ in $C^1_{0,\mathrm{per}}([0,1])$.

Lemma 3.1.4 $A^{-1}_{\varepsilon,1} : Z \longrightarrow C^2_{0,\mathrm{per}}$ *is continuous.*

Proof. Let $h_n \in Z$ and $u_n \in C^2_{0,\mathrm{per}}$ be such that $h_n \to h$ in C^0 and $A_{\varepsilon,1}(u_n)=h_n$. Then from Lemma 3.1.3 we have $u_n \to u$ in $C^1_{0,\mathrm{per}}$. As $u''_n = h_n / (\varepsilon + (p-1)|u'_n|^{p-2})$ we also have that $u_n \to u$ in $C^2(0, 1)$ and $u'' = h / (\varepsilon + (p-1)|u'|^{p-2})$ on $(0, 1)$.

In the following, we prove that for some N, $A_\varepsilon - \lambda N$ is A-proper with respect to some Γ', where Γ' is as in Definition 1.2.4 of Chapter 1, and give a existence result for problem (3.1.1).

Let $Z_n \subset Z$ (Z is as in lemma 3.1.3) be sequences of oriented finite dimensional spaces, $Y_n = \mathbb{R} \oplus Z_n$ and $Q_n : Y \to Y_n$ be a linear projection of Y onto Y_n for each $n \geq 1$ such that for any $y \in C^0[0, 1]$, $Q_n y \to y$. Let $E_n = \mathbb{R} \oplus (A^{-1}_{\varepsilon,1}(Z_n))$. Then E_n are sequences of oriented finite dimensional open sets in $C^2_{\mathrm{per}}(0, 1)$ and from the properties of Y_n, we have that for any $u \in C^2_{\mathrm{per}}(0, 1)$, $\mathrm{dist}\,(u, E_n) \to 0$ as $n \to \infty$.

Lemma 3.1.5 *The Scheme* $\Gamma' \equiv \left\{ E_n, Y_n, Q_n \right\}$ *is admissible.*

Proof. For any n, $A_{\varepsilon,1} : A^{-1}_{\varepsilon,1}(Z_n) \longrightarrow Z_n$ is a homeomorphism. By the fact that dimension is a topological invariant (see Remark 1.1.2 of Chapter 1), we know

$$\dim (A^{-1}_{\varepsilon,1}(Z_n)) = \dim (Z_n),$$

and so, $\dim (E_n) = \dim (Y_n)$. Here $\dim (A^{-1}_{\varepsilon,1}(Z_n))$ is as in Chapter 1.

Lemma 3.1.6 *Suppose* A_ε *is as in Lemma* 3.1.1, Γ' *is as in Lemma* 3.1.5. *Let* $G \subset C^2_{\mathrm{per}}(0,1)$ *be an open, bounded set. Then* $A_\varepsilon : \overline{G} \subset C^2_{\mathrm{per}}(0, 1) \to C^0(0, 1)$ *is*

A-proper with respect to Γ'.

Proof. Let $\left\{u_{n_j} \mid u_{n_j} \in G_{n_j}\right\}$ be any sequence, bounded in $C^2_{per}(0, 1)$, such that $g_{n_j} \equiv Q_{n_j} A_\varepsilon(u_{n_j}) \longrightarrow g$ for some g in C^0. Since $u_{n_j}(t) = u_{n_j}(0) + \bar{u}_{n_j}(t)$, $\bar{u}_{n_j} \in E_{1,n_j}$, here $E_{1,n_j} \subset C^2_{0,per}$, $A_\varepsilon(u_{n_j}) = A_{\varepsilon,1}(\bar{u}_{n_j}) \subset Z_{n_j}$, then $Q_{n_j} A_\varepsilon(u_{n_j}) = A_\varepsilon(u_{n_j})$. We see that $g_{n_j} \equiv A_\varepsilon(u_{n_j}) = A_{\varepsilon,1}(\bar{u}_{n_j}) \longrightarrow g$ in C^0. By the continuity of $A^{-1}_{\varepsilon,1}$, we know that $\bar{u}_{n_j} \longrightarrow A^{-1}_{\varepsilon,1} g$. As $\{u_{n_j}(0)\}$ is bounded, we also have $u_{n_j}(0) \longrightarrow C$ and so, u_{n_j} converges to $C + A^{-1}_{\varepsilon,1} g$. Let $u = C + A^{-1}_{\varepsilon,1} g$, then $u \in C^2_{per}(0, 1)$ and $A_\varepsilon u = g$. Since $\overline{G}$ is a closed set, $u \in \overline{G}$.

Now, using the A-proper property of A_ε and using generalized degree theory as in Chapter 1, we give an existence theorem similar to Corollary 2 of [60], but in [60] the map A is linear and Fredholm of index 0.

<u>Theorem 3.1.7</u> (Existence Theorem) *Let $y \in Z$, A_ε be as in Lemma 3.1.1, let $G \subset C^2_{per}$ be an open bounded set with $0 \in G$ and $N: C^2_{per}(0, 1) \longrightarrow C^0[0, 1]$ be a bounded continuous nonlinear map such that*

(*a*) $A_\varepsilon - \lambda N : \overline{G} \to C^0[0, 1]$ *is A-proper w. r. t. Γ', for each $\lambda \in (0, 1]$,*

(*b*) $A_\varepsilon u \neq \lambda N u + \lambda y$ *for $u \in \partial G$ and $\lambda \in (0,1]$,*

(*c*) $QNu \neq 0$ *for $u \in \mathbb{R} \cap \partial G$, where Q is a linear projection of C^0 onto $\mathbb{R}$ with $C^0 = \mathbb{R} \oplus Z$,*

(*d*) $\deg_A(A_\varepsilon - QN, G, 0) \neq \{0\}$.

Then there exist $u \in \overline{G}$ such that $A_\varepsilon u - Nu = y$.

Proof. Let $H(\lambda,u)=A_\varepsilon u-(1-\lambda)QNu-\lambda Nu-\lambda y$ for $u\in\overline{G}$ and $\lambda\in[0,1]$. Since QN is compact, $A_\varepsilon-\lambda N:\overline{G}\to C^0[0,1]$ is A-proper with respect to Γ' for each $\lambda\in[0,1]$ and the additional condition on $N(\partial G)$, it follows that $H(\lambda,u)$ is an A-proper (with respect to Γ') homotopy. Moreover, in virtue of our conditions, we may assume that $H(\lambda,u)\neq 0$ for $\lambda\in[0,1]$ and $u\in\partial G$. Indeed, if this were not the case, then there would exist $\lambda_0\in[0,1]$ and $u_0\in\partial G$ such that $H(\lambda_0,u_0)=0$. Now if $\lambda_0=1$, then $0=H(1,u_0)=A_\varepsilon u_0-Nu_0-y$ with $u_0\in\partial G$, that is, u_0 is a solution and so we finish the proof. Hence, we may exclude this case from further considerations. If $\lambda_0=0$, then $0=H(0,u_0)=A_\varepsilon u_0-QNu_0$. Since $A_\varepsilon u_0\in Z$ and $QNu_0\in\mathbb{R}$, it follows that $A_\varepsilon u_0=0$ and $QNu_0=0$. This means $u_0\in\mathbb{R}\cap\partial G$ such that $QNu_0=0$. This contradicts (c). If $\lambda_0\in(0,1)$, then $A_\varepsilon u_0-\lambda_0Nu_0-\lambda_0y=(1-\lambda_0)QNu_0\neq 0$ by (b). Hence $QNu_0\neq 0$ and, since $A_\varepsilon u_0-\lambda_0y\in Z$, we get from the last equality the contradictory relation

$$-\lambda_0QNu_0=Q[A_\varepsilon u_0-\lambda_0Nu_0-\lambda_0y]=(1-\lambda_0)QNu_0,\ 0<\lambda_0<1.$$

The above discussion shows that $H(\lambda,u)\neq 0$ for $\lambda\in[0,1]$ and $u\in\partial G$. Consequently,

$$\deg_A(A_\varepsilon-N,G,0)=\deg_A(A_\varepsilon-QN,G,0)\neq\{0\},\ \text{by } (d).$$

Hence, there exists $u\in G$ such that $A_\varepsilon u-Nu=y$.

Corollary 3.1.8 *Let $y\in Z$ and assume that*

(*a*) $A_\varepsilon-\lambda N:\overline{G}\to C^0[0,1]$ *is A-proper with respect to Γ' for each $\lambda\in(0,1]$ with $N(\partial G)$ bounded,*

(*b*) $A_\varepsilon u\neq\lambda Nu+\lambda y$ *for $u\in\partial G$ and $\lambda\in(0,1)$,*

(*c*) $QNu\neq 0$ *for $u\in\mathbb{R}\cap\partial G$,*

(*e*) *For $u\in\mathbb{R}\cap\partial G$, either* (e_1): $[QNu,u]\geq 0$ *or*

(e_2): $[QNu,u]\leq 0$.

Here $[QNu, u]=QNu \times u$. *Then there exists* $u\in G$ *such that* $A_\varepsilon u-Nu=y$.

Proof. By Theorem 3.1.7, We prove that (*c*) and (*e*) imply

$$\deg_A(A_\varepsilon-QN, G, 0)\neq\{0\}.$$

Indeed, suppose first that (e_1) holds and consider the homotopy

$$H : [0,1]\times\overline{G}\to C^0,\ H(\lambda,u)=A_\varepsilon u-(1-\lambda)Bu-\lambda QNu$$

for $u\in\overline{G}$ and $\lambda\in[0,1]$ with $B=P$, P is a projection of $C^2_{per}(0,1)$ onto $\mathbb{R}$ if (e_1) holds and $B=-P$ if (e_2) holds. Since B and QN are compact, it follows that H is an A-proper homotopy and the fact that $(A_\varepsilon-B)\,u=0$, $u\in C^2_{per}(0,1)$ if and only if $u\equiv 0$. Moreover, $H(\lambda,u)\neq 0$ for $u\in\partial G$ and $\lambda\in[0,1]$. Indeed, if this were not the case, then there would exist $\lambda_0\in[0,1]$ and $u_0\in\partial G$ such that $H(\lambda_0,u_0)=0$. Now, if $\lambda_0=0$, then $0=H(0,u_0)=A_\varepsilon u_0-Bu_0$ with $u_0\neq 0$. From above, we know $u_0\equiv 0$. This is a contradiction. If $\lambda_0=1$, then $H(1,u_0)=A_\varepsilon u_0-QNu_0=0$. Since $A_\varepsilon u_0=QNu_0$ with $A_\varepsilon u_0\in Z$ and $QNu_0\in\mathbb{R}$, it follows that $u_0\in\mathbb{R}\cap\partial G$ and $QNu_0=0$, in contradiction to (*c*). Thus $\lambda_0\in(0,1)$ and $A_\varepsilon u_0=(1-\lambda_0)Bu_0+\lambda_0 QNu_0$. This again implies that $A_\varepsilon u_0=0$, that is, $u_0\in\mathbb{R}$ and

$$\pm(1-\lambda_0)Pu_0+\lambda_0 QNu_0=0.$$

Since $Pu_0=u_0$, $\pm(1-\lambda_0)u_0+\lambda_0 QNu_0=0$, Then,

$$\pm(1-\lambda_0)u_0^2+\lambda_0\ QNu_0\times u_0=0.$$

Thus, in both cases, $H(\lambda,u)\neq 0$ for all $u\in\partial G$ and all $\lambda\in[0,1]$. Consequently,

$$\deg_A(A_\varepsilon-QN, G, 0)=\deg_A(A_\varepsilon-P, G, 0).$$

Since $A_\varepsilon-P$ is one-to-one and $0\in G$, we have from the properties of the Brouwer degree of one-to-one map and the definition of $\deg_A$, that

$$\deg_A(A_\varepsilon-P, G, 0)\subset\{1,-1\},\ \text{and,}$$

in particular, $0\notin\deg_A(A_\varepsilon-N, G, 0)$.

Corollary 3.1.9 *Let* $y\in C^0[0, 1]$, A_ε *be as in Lemma* 3.1.1, *let* $G\subset C^2_{per}$ *be an open bounded set with* $0\in G$ *and* $N : C^2_{per}(0, 1)\to C^0[0, 1]$ *be a bounded continuous nonlinear map such that*

(*a*) $A_\varepsilon-\lambda N : \overline{G}\to C^0[0, 1]$ *is A-proper w. r. t.* Γ', *for each* $\lambda\in(0, 1]$ *with* $N(\partial G)$ *bounded,*

(*b*) $A_\varepsilon u \neq \lambda Nu+\lambda y$ *for* $u\in\partial G$ *and* $\lambda\in(0,1]$,

(*c*) $QNu+Qy \neq 0$ *for* $u\in\mathbb{R}\cap\partial G$, *where* Q *is a linear projection of* C^0 *onto* $\mathbb{R}$ *with* $C^0=\mathbb{R}\oplus Z$,

(*d*) *Either* (*i*) $(QNu+Qy)\times u\geq 0$ *or*

(*ii*) $(QNu+Qy)\times u\leq 0$

for $u\in\mathbb{R}\cap\partial G$. *Then there exists* $u\in\overline{G}$ *such that* $A_\varepsilon u-Nu=y$.

Proof. Replacing N in Corollary 3.1.8 by $N+y$, this corollary follows from Corollary 3.1.8.

3.2 EXISTENCE RESULTS FOR PERIODIC BOUNDARY VALUE PROBLEMS

In this section we use Corollary 3.1.9 to establish the existence of solutions to the problems:

$$\varepsilon u'' + (|u'|^{p-2}u')'+f(t, u, u')= y(t), \quad u(0)=u(1),\ u'(0)=u'(1), \qquad (3.2.1)_\varepsilon$$

where $p>2$, $f : [0, 1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous, $y\in C^0[0, 1]$. Then we shall let ε tend to 0 and obtain the existence of a solution to (3.0.1). We assume that:

(H_1) $f : [0, 1]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is a continuous function and there are positive constants A, B, C such that $B+C<\pi_p$ and

$$|f(t, q, r)|\leq A+B|q|^{p-1}+C|r|^{p-1} \qquad (3.2.2)$$

for $t\in[0, 1]$ and q, $r\in\mathbb{R}$, where $\pi_p>0$ is the first eigenvalue of the problem

$$(|u'|^{p-2}u')' + \lambda|u|^{p-2}u=0, \quad u(0)=u(1)=0.$$

$(\mathrm{H}_1)'$ $f : [0, 1]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function and satisfies

(*i*) there exists a continuous function $f_1 : [0, 1]\times\mathbb{R}\longrightarrow\mathbb{R}$ and a constant C such that $|f(t, q, r)| \leq f_1(t, q)+ C|r|^p$,

(*ii*) there are constants α, β, $\gamma\geq 0$ and σ, $\tau<p$ such that

$$q f(t, q, r)\geq -\alpha|q|^{\sigma}-\beta|r|^{\tau}-\gamma \tag{3.2.3}$$

for $t\in[0,1]$, $q,r\in\mathbb{R}$.

Now we prove the following theorem.

Theorem 3.2.1 *Suppose that in addition to* (H_1) *we assume that*

(H_2) *To a given* $y\in C^0[0,1]$ *there exists* $M>0$ *(depending on* y*) such that* $\int_0^1\left\{f(t, u, u') - y(t)\right\}dt\neq 0$, *for* $u\in C^2(0, 1)$ *with* $|u|>M$ *for* $t\in[0, 1]$.

(H_3) *There are* $M_1\geq M$ *and* a, $b\in\mathbb{R}$ *such that either (i) or (ii) holds, where*

(*i*) $a\geq b$, $u\in\mathbb{R}$ *and* $u\geq M_1\Rightarrow f(t, u, 0)\geq a;$

$u\leq -M\Rightarrow f(t, u, 0)\leq b$ *for* $t\in[0, 1]$ *and* $b\leq y_1\leq a$ *with* $y_1=\int_0^1 y\,dt$,

(*ii*) $a\leq b$, $u\in\mathbb{R}$ *and* $u\geq M_1\Rightarrow f(t, u, 0)\leq a;$

$u\leq -M\Rightarrow f(t, u, 0)\geq b$ *for* $t\in[0, 1]$ *and* $a\leq y_1\leq b$.

Then the periodic BVP $(3.2.1)_\varepsilon$ *has a solution in* $C^2(0, 1)$.

Proof. Let $Nu=f(t, u, u')$, then the map $N : C^2(0, 1)\longrightarrow C^0[0, 1]$ is compact since $C^2(0, 1)$ is compactly embedded in $C^1([0, 1])$ and, by Theorem 3.1.6, $A_\varepsilon-\lambda N$ is A-proper w.r.t. Γ' for each $\lambda\in(0, 1]$. Condition (*a*) in Corollary 3.1.9 holds. Next we show that if (H_1) holds, then there exists $r>0$ such that if we let $G=\{x\in C^2(0,1) : \|x\|_2\le r\}$, then (b) of Corollary 3.1.9 holds. For that it suffices to show that if $u\in C^2(0,1)$ is a solution of

$$-\varepsilon u'' - (|u'|^{p-2}u')' = \lambda f(t,u,u') - \lambda y, \quad u(0)=u(1), \quad u'(0)=u'(1), \tag{3.2.4}$$

for some $\lambda\in(0,1]$, then $\|u\|_2\le M_2$ for some $M_2>0$ independent of u and λ.

So, let $u\in C^2(0,1)$ be a solution of (3.2.4) and integrate from 0 to 1 to obtain

$$\int_0^1 \Big\{ f(t,u(t),u'(t)) - y(t) \Big\} dt = 0. \tag{3.2.5}$$

It follows from (3.2.5) and (H_2) that there exists $t_0\in[0,1]$ such that $|u(t_0)|\le M$. We write $u(t)=u(t_0)+\int_{t_0}^t u'(s)ds$, and so

$$|u(t)|\le M+\|u'\|_p, \quad \text{where} \quad \|v\|_p=\left(\int_0^1 |v|^p dt\right)^{1/p}. \tag{3.2.6}$$

For $u\in C^2_{per}(0,1)$, we write $u(t)=u(0)+h$ with $h\in C^2_{0,per}$, then $u'=h'$ and $\|u'\|_p=\|h'\|_p$. Therefore, $\|u(t)\|_p\le M+\|h'\|_p$. From the equality (3.2.4), we obtain

$$\varepsilon\int_0^1 h'^2 dt + \int_0^1 |h'(t)|^p dt = \lambda\int_0^1 \Big\{ f(t,u,u') - y(t) \Big\} h(t)dt, \tag{3.2.7}$$

then,

$$\begin{aligned}\int_0^1 |h'|^p dt &\le \|(|f(t,u,u')| + |y|)\|_q \|h\|_p \\ &\le (A + B\|u\|_p^{p-1} + C\|h'\|_p^{p-1} + \|y\|_0)\|h\|_p.\end{aligned} \tag{3.2.8}$$

In view of the fact that $h(0)=h(1)=0$ and $\|h\|_p\le\pi_p^{-1}\|h'\|_p$ (see [62]), one easily derives from (3.2.6) and (H_1) that there exist $A_1>0$ independent of λ, ε such that $\|h'\|_p\le A_1$ and so, $|u|\le M+A_1$ for $t\in[0,1]$ and $\|u\|_p\le M+A_1$.

It follows from (3.2.4) that

$$\varepsilon h'(t) + |h'(t)|^{p-2}h'(t) = \lambda\int_{t_1}^t \Big\{ f(s,u,h') - y(s) \Big\} ds, \tag{3.2.9}$$

where $t_1\in(0,1)$ is such that $h'(t_1)=0$. Then, we get from (3.2.9) that

$$|\varepsilon h'(t)+|h'(t)|^{p-2}h'(t)|\leq A_2, \tag{3.2.10}$$

where $A_2=A_2(M, A_1, A, B, C, \|y\|_0)$. Using the facts that ϕ_p is strictly increasing and A_2 is independent of ε and (3.2.10), we easily obtain that there exist A_3, such that $|h'|\leq A_3$ for $t\in(0, 1)$, A_3 is also independent of λ and ε. Hence, $\|u\|_1\leq A_3+M$. From (3.2.4) we also obtain

$$-(\varepsilon+|u'|^{p-2})u''=\lambda f(t, u, u')-\lambda y,$$

therefore,

$$|u''|\leq \varepsilon^{-1}(|f(t, u, u')|+|y|)\leq \varepsilon^{-1}A_4, \tag{3.2.11}$$

where $A_4>0$ is independent of ε, λ. Now let $r>\max\{A_3+M, \varepsilon^{-1}A_4\}$ and $G=\{u\in C^2_{per}(0, 1) : \|u\|_2\leq r\}$, then for $\lambda\in(0, 1]$ and $u\in\partial G$,

$$-\varepsilon u''-(|u'|^{p-2}u')'\neq\lambda f(t, u, u')-\lambda y, \tag{3.2.12}$$

that is, condition (*b*) of Corollary 3.1.9 holds. Note that Corollary 3.1.9

(*c*) holds, for $QNx-Qy=\int_0^1\left\{f(t, x, 0)-y(t)\right\}dt\neq 0,\quad x\in\mathbb{R}\cap\partial G$. This follows from the fact that if $x\in\mathbb{R}\cap\partial G$, then $|x|=r>M$ and by using (H_2), Corollary 3.1.9 (*d*) follows directly from (H_3) (see [60]). Hence, the conclusion of Theorem 3.2.1 follows from Corollary 3.1.9.

<u>Corollary 3.2.2</u> *Suppose that* (H_1)-(H_3) *of Theorem* 3.2.1 *hold,* $y\in C^0[0, 1]$. *Then* BVP (3.0.1) *has at least one solution in* $C^1([0, 1])$.

Proof. From the proof of Theorem 3.2.1, we know that for any $\varepsilon>0$, there exist at least one $u_\varepsilon\in C^2_{per}(0, 1)$ which satisfies

$$\varepsilon u''_\varepsilon+(|u'_\varepsilon|^{p-2}u'_\varepsilon)'+f(t, u_\varepsilon, u'_\varepsilon)=y(t), \tag{3.2.13}$$

and $\|u_\varepsilon\|_1\leq A_3+M$ and A_3+M is independent of ε. Let $u_\varepsilon=u_\varepsilon(0)+\bar{u}_\varepsilon$, then

$\bar{u}_\varepsilon$ satisfies $A_{\varepsilon,1}(\bar{u}_\varepsilon)=f(t, u_\varepsilon, u'_\varepsilon)- y(t)$ and $\|f(t, u_\varepsilon, u'_\varepsilon)- y\|_0 \le A_5$, here $A_5>0$ is independent of ε. Lemma 3.1.3 implies that there exists $\bar{u} \in C^1_{0,\mathrm{per}}([0, 1])$ such that $\bar{u}_\varepsilon \to \bar{u}$ in $C^1([0, 1])$ as $\varepsilon \to 0$. The boundedness of $\|u_\varepsilon\|_0$ implies that $u_\varepsilon(0) \longrightarrow C$ as $\varepsilon \to 0$, here $C \in \mathbb{R}$. Hence $u_\varepsilon \longrightarrow u$ in $C^1([0, 1])$, $u=C+\bar{u}$. Integrating (3.2.13) from 0 to t and letting $\varepsilon \to 0$, we obtain that $u \in C^1_{\mathrm{per}}([0, 1])$ is a solution of (3.0.1).

Remark b If the function f in problem (3.0.1) is independent of u', then $N(u)=f(t, u)$ is also compact as a map from $C^2_{\mathrm{per}}(0, 1)$ to $C^0\,[0, 1]$ and so Corollary 3.2.2 yields existence results for the periodic BVP

$$(|u'|^{p-2}u')' + f(t, u) = y, \quad u(0)=u(1), \; u'(0)=u'(1). \qquad (3.2.14)$$

Combining the facts in [61] and Theorem 3.2.1, we have the following theorem.

Theorem 3.2.3 *Let* $y_1=0$ *and suppose* $f : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is continuous and satisfies the conditions*

(H_4) *There is* $M>0$ *such that* $f(t, u, u')\,u<0$ *for* $u \in C^2_{\mathrm{per}}(0, 1)$ *with* $|u(t)| \ge M$ *for* $t \in [0, 1]$.

(H_5) *There exists a continuous function* $f_1 : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *and a constant* $C \in \mathbb{R}^+$ *with* $C<\pi_p$ *such that* $|f(t, q, r)| \le f_1(t, q) + C|r|^{p-1}$ *for* $t \in [0,1]$ *and* $q, r \in \mathbb{R}$.

Then the periodic BVP (3.0.1) *has a solution in* $C^1([0, 1])$.

Now we give an existence theorem for a different version of the BVP $(3.2.1)_\varepsilon$, namely,

$$-\varepsilon u'' - (|u'|^{p-2}u')' + f(t, u, u') = y(t), \quad u(0)=u(1), \quad u'(0)=u'(1). \quad (3.2.1)'_\varepsilon$$

Theorem 3.2.4 *Suppose that in addition to* $(H_1)'$ *we assume that* (H_2) *and* (H_3) *in Theorem* 3.2.1 *hold. Then the periodic* BVP $(3.2.1)'_\varepsilon$ *has a solution in* $C^2(0, 1)$.

Proof. Let $Nu=-f(t, u, u')$, then condition (*a*) of Corollary 3.1.9 holds. To prove this theorem, it suffices to show that $u\in C^2(0, 1)$ is a solution of

$$-\varepsilon u'' - (|u'|^{p-2}u')' = -\lambda f(t, u, u') + \lambda y, \quad u(0)=u(1), \ u'(0)=u'(1), \quad (3.2.15)$$

for some $\lambda\in(0,1]$, then $\|u\|_2 \le M_3$ for some $M_3>0$ independent of u and λ.

Following the same ideas as in the proof of Theorem 3.2.1, we obtain that

$$\|u\|_p \le M + \|u'\|_p$$

and

$$\varepsilon\int_0^1 u'^2 dt + \int_0^1 |u'|^p\, dt + \lambda\int_0^1 f(t, u, u')\, u\, dt = \lambda\int_0^1 y\, u\, dt.$$

In view of (3.2.3), one easily derives that

$$\int_0^1 f(t, u, u')\, u\, dt \ge -\alpha\int_0^1 |u|^\sigma - \beta\int_0^1 |u'|^\tau - \gamma \ge -\alpha\,\|u\|_p^\sigma - \beta\|u'\|_p^\tau - \gamma.$$

Then, $\|u'\|_p^p \le \alpha\,\|u\|_p^\sigma + \beta\,\|u'\|_p^\tau + \|y\|_0\|u\|_p + \gamma$. So, there exists M_4, which may depend on M, α, β, γ, σ, τ, and $\|y\|_0$, but is independent of λ and ε such that $\|u'\|_p \le M_4$. Therefore, $\|u\|_{1,p} \le M+2M_4$. Since $W^{1,p}(0,1)$ is embedded in $C^0(0,1)$, $\|u\|_0 \le M_5$, where M_5 is independent of ε and λ. Let

$$M_6 = \sup\{\ |f(t, q)|\ :\ 0\le t\le 1,\ |q| \le M_5\},$$

then M_6 is also independent of λ, ε. From this, $(H_1)'$ and (3.2.15), it easily follows that

$$|\varepsilon u' + (|u'|^{p-2}u')| \le \int_0^1 |f(t, u, u')|\, dt + \|y\|_0 \le M_6 + C(M+2M_4)^p + \|y\|_0. \quad (3.2.16)$$

Let $M_7=M_6+C(M+2M_4)^p+\|y\|_0$, we know that M_7 is independent of λ and ε. Therefore, $|u'|_0\le M_7^{1/(p-1)}$. From (3.2.15) we also get that there exist $M_8>0$ independent of λ, ε such that $|u''|\le\varepsilon^{-1}M_8$. Now let $r>\max\left\{M_5,\ M_7^{1/(p-1)},\ \varepsilon^{-1}M_8\right\}$ and $G=\{u\in C^2_{per}(0,1):\|u\|_2\le r\}$, then for $\lambda\in(0,1]$ and $u\in\partial G$,

$$-\varepsilon u''-(|u'|^{p-2}u')'\neq-\lambda f(t,u,u')+\lambda y. \tag{3.2.17}$$

Condition (b) of Corollary 3.1.9 holds.

<u>Corollary 3.2.5</u> *Suppose that the conditions of Theorem* 3.2.4 *hold,* $y\in C^0[0,1]$. *Then* BVP

$$-(|u'|^{p-2}u')'+f(t,u,u')=y(t),\quad u(0)=u(1),\ u'(0)=u'(1),$$

has at least one solution in $C^1([0,1])$.

Proof. Similar to the proof of Corollary 3.2.2.

If the function f in $(3.0.1)'_\varepsilon$ is independent of u', then we have the following corollary.

<u>Corollary 3.2.6</u> *Suppose that* $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ *is a continuous function and there exist constants* α, $\beta\ge0$ *and* $\sigma<p$ *such that*

$$q\,f(t,q)\ge-\alpha|q|^{\sigma}-\beta, \tag{3.2.18}$$

for $t\in[0,1]$ *and* $q\in\mathbb{R}$. *We also assume that:*

$(\mathrm{H}_2)'$ *To a given* $y\in C^0[0,1]$ *there exist* $M>0$ *(depending on* y*) such that* $\int_0^1\{f(t,u)-y(t)\}\,dt\neq0$, *for* $u\in C^2_{per}(0,1)$ *with* $|u|\ge M$ *for* $t\in[0,1]$.

$(\mathrm{H}_3)'$ *There are* $M_1\ge M$ *and* $a,b\in\mathbb{R}$ *such that either* (i) *or* (ii) *holds, where*

(i) $a\ge b$, $u\in\mathbb{R}$ *and* $u\ge M_1\Rightarrow f(t,u)\ge a$;

$$u \leq -M \Rightarrow f(t,u) \leq b \text{ for } t \in [0, 1] \text{ and } b \leq y_1 \leq a \text{ with } y_1 = \int_0^1 y dt.$$

(*ii*) $a \leq b$, $u \in \mathbb{R}$ and $u \geq M_1 \Rightarrow f(t, u) \leq a$;

$$u \leq -M \Rightarrow f(t,u) \geq b \text{ for } t \in [0, 1] \text{ and } a \leq y_1 \leq b.$$

Then the periodic BVP

$$-(|u'|^{p-2} u')' + f(t, u) = y, \quad u(0) = u(1), \ u'(0) = u'(1),$$

has a solution in $C^1([0,1])$.

Proof. This corollary follows from Theorem 3.2.4 and Corollary 3.2.5.

3.3 EXISTENCE RESULTS FOR NEUMANN BOUNDARY VALUE PROBLEMS

In this section, we directly use Leray-Schauder degree theory [51] to discuss the problem

$$\left.\begin{aligned} &Au - f(t, u) = 0 \quad \text{in } (0,1) \\ &u'(0) = u'(1) = 0 \end{aligned}\right\} \tag{3.3.1}$$

where $Au = -(a(|u'|^2) u')'$, $a : \mathbb{R} \to \mathbb{R}$ is a continuous mapping that satisfies the following conditions:

(a_1) the mapping $h(t^2)$ is strictly convex, where

$$h(t^2) = \int_0^{t^2} a(\tau) d\tau;$$

(a_2) there exist $p > 1$, c_0, $c_3 > 0$ and c_1, $c_2 \geq 0$ such that

$$c_0 |t|^{p-2} + c_1 \leq a(t^2) \leq c_2 + c_3 |t|^{p-2} \tag{3.3.2}$$

for all $t \in \mathbb{R}$.

Using the same ideas as above, we can obtain some existence results for problem (3.3.1). But in this section, we use a simpler method.

The condition (a_2) holds if $a(t)$ is a polynomial with $a(t)\geq c_1$. Other examples are $a(t^2)=1+(1+t^2)^{-2}$ and for any $p>1$, $a(t^2)=(t^2)^{(p-2)/2}$. We first establish the following lemma.

Lemma 3.3.1 *Let* $Au=-(a(|u'|^2)\,u')'$, $S(u)=a(|u|^2)$ *and* $C^1_*([0,1])=\left\{u\in C^1([0,1]):\ u'(0)=u'(1)=0\right\}$. *For any* $\delta>0$, *let* $J_\delta(u)=Au+\delta\, S(u)\,u$. *Suppose the function* a *satisfies conditions* (a_1) *and* (a_2). *Then,* J_δ *is invertible and*

$$J_\delta^{-1}\ :\ L^q(0,\,1)\to C^1_*([0,\,1]) \text{ is compact.}$$

Here $q=p/(p-1)$, p *is as in* (3.3.2).

Proof. For a given $x\in L^q(0,\,1)$ with $q=p/(p-1)$, we look for a function $u\in C^1([0,\,1])$ satisfying

$$\left.\begin{aligned} &Au+\delta\ S(u)u=x \quad a.e. \text{ on } [0,1]\\ &u'(0)=u'(1)=0\end{aligned}\right\} \tag{3.3.3}$$

with $a(|u'|^2)\,u'$ an absolutely continuous function on [0,1]. Clearly, if u is such a solution, then it satisfies

$$\int_0^1 a(|u'|^2)\,u'\,v' + \delta\int_0^1 a(|u|^2)\,u\,v=\int_0^1 x\,v, \tag{3.3.4}$$

for all $v\in W^{1,p}(0,\,1)$. Conversely, if $u\in W^{1,p}(0,\,1)$ satisfies (3.3.4) for all $v\in W^{1,p}(0,\,1)$, then by condition (a_2), $a(|u'|^2)\,u'\in L^q(0,\,1)$, $a(|u|^2)\,u\in L^q(0,\,1)$ and satisfies (3.3.4) for all $v\in C_0^\infty(0,\,1)$. Hence, $a(|u'|^2)\,u'\in W_0^{1,p}(0,\,1)$; $a(|u'|^2)\,u'$ is an absolutely continuous function on [0, 1]. The embedding of $W_0^{1,p}(0,1)$ to $C^0[0,\,1]$ implies that $a(|u'|)u'\in C^0(0,1)$ and $a(|u'(0)|^2)u'(0)=0$ $=a(|u'(1)|^2)\,u'(1)$. Since the function $\phi(t)=a(t^2)t$ is strictly increasing, we conclude that $u \in C^1([0, 1])$ and $u'(0)=u'(1)=0$. This means $u \in C^1_*([0, 1])$.

Now, we prove that there exists $u \in W^{1,p}(0, 1)$ such that (3.3.4) holds. Consider

$$I(u)=\frac{1}{2}\int_0^1 h(|u'|^2)+\frac{\delta}{2}\int_0^1 h(|u|^2)-\int_0^1 xu \ . \tag{3.3.5}$$

By the properties of $h(t)$, we know that

$$I(u)\geq C\|u\|^p_{1,p} \ \text{, (with some } C>0). \tag{3.3.6}$$

We also find I is a continuous convex functional on $W^{1,p}(0, 1)$. Hence, it possesses a critical point $u \in W^{1,p}(0, 1)$ at which it reaches its minimum. We also know that at u, (3.3.4) holds for all $v \in W^{1,p}(0, 1)$.

For $x \in L^q(0,1)$, there is only one $u \in W^{1,p}(0, 1)$ satisfying (3.3.4). Indeed, suppose not, there are $u_1, u_2 \in C^1_*([0, 1])$, $u_1 \neq u_2$ such that

$$\int_0^1 a(|u_1'|^2)u_1'v'+\delta\int_0^1 a(|u_1|^2)u_1v=\int_0^1 xv \tag{3.3.7}$$

$$\int_0^1 a(|u_2'|^2)u_2'v'+\delta\int_0^1 a(|u_2|^2)u_2v=\int_0^1 xv \tag{3.3.8}$$

for all $v \in W^{1,p}(0, 1)$. Then,

$$0=(J_\delta(u_1)-J_\delta(u_2),u_1-u_2)=\int_0^1 (a(|u_1'|^2)u_1'-a(|u_2'|^2)u_2')(u_1'-u_2')$$

$$+\delta\int_0^1 (a(|u_1|^2)u_1-a(|u_2|^2)u_2)(u_1-u_2)$$

$$\geq\int_0^1 (a(|u_1'|^2)|u_1'|-a(|u_2'|^2)|u_2'|)(|u_1'|-|u_2'|)$$

$$+\delta\int_0^1 (a(|u_1|^2)|u_1|-a(|u_2|^2)|u_2|)(|u_1|-|u_2|)>0.$$

This is a contradiction. Therefore, J_δ is invertible.

To end the proof of this lemma, we have to prove that J_δ^{-1} is compact. We first show that J_δ^{-1} : $L^q(0,1)\to C^1_*([0, 1])$ is continuous. Let $\{x_n\}$ be a sequence in $L^q(0,1)$ such that $x_n \to x$ as $n \to \infty$. Suppose that $J_\delta^{-1}(x_n)$ does not converge to $J_\delta^{-1}(x)$ as $n \to \infty$. Hence, there exists an $\varepsilon > 0$ and a subsequence of $\{x_n\}$ which we will call again $\{x_n\}$ such that

$$\|J_\delta^{-1}(x_n) - J_\delta^{-1}(x)\|_1 \ge \varepsilon \tag{3.3.9}$$

for all $n \in \mathbb{N}$. Based on the definition of the mappings ϕ, J_δ^{-1} and the fact that for a solution u of problem (3.3.3), $\phi(u')$ is an absolutely continuous function on $[0,1]$, setting

$$u_n = J_\delta^{-1}(x_n) \tag{3.3.10}$$

$$u = J_\delta^{-1}(x) \tag{3.3.11}$$

we find that

$$-(\phi(u_n'))' + \delta\ S(u)u = x_n \tag{3.3.12}$$

for each fixed $n \in \mathbb{N}$, $u_n \in C_*^1([0,1])$. Equation (3.3.12) and the boundedness of $\{x_n\}$ in $L^q(0,1)$ tell us that $\|u_n\|_{1,p}$ is bounded. There exists a subsequence of $\{u_n\}$ (still call it $\{u_n\}$) such that there exists $w \in C^0[0,1]$,

$$u_n \to w \quad \text{in } C^0[0,1]. \tag{3.3.13}$$

Here we use the compactness of the embedding of $W^{1,p}(0,1)$ in $C^0[0,1]$. From (3.3.12) we also know that the sequence $\{\phi(u_n')\}$ meets the requirements of Ascoli–Arzela's theorem in $C^0[0,1]$. Therefore, there exists a subsequence of $\{\phi(u_n')\}$ (still call it $\{\phi(u_n')\}$) which is convergent in C^0. This and the fact that the function ϕ has a continuous inverse imply that $\{u_n\}$ contains a convergent subsequence in $C^1([0,1])$ and $u_n \to w$ in $C^1([0,1])$. From the equality

$$\int_0^1 [\phi(u_n')\, v' + \delta\ S(u_n)\, u_n v] = \int_0^1 x_n\, v, \tag{3.3.14}$$

for all $v \in W^{1,p}(0,1)$ and $n \in \mathbb{N}$, and recalling that ϕ is continuous, we can let n go to infinity in (3.3.14) to obtain

$$\int_0^1 [\phi(w')\, v' + \delta\; S(w)\, v] = \int_0^1 x\, v, \tag{3.3.15}$$

for all $v \in W^{1,p}$. This and the argument above imply that $w=u$, which is a contradiction in light of (3.3.9). Following the same ideas as above, we can show that J_δ^{-1} is compact.

<u>Theorem 3.3.2</u> *Let $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function, i.e. $f(\cdot,u)$ is measurable for every $u \in \mathbb{R}$ and $f(t,\cdot)$ is continuous for almost every $t \in (0,1)$. Mereover, assume that for every $\mathrm{R}>0$, there is a $k_{\mathrm{R}} \in L^1(0,1)$ such that*

$$|f(t,u)| \le k_{\mathrm{R}}(t)$$

for all $|u| \le \mathrm{R}$ and almost every $t \in (0,1)$. Let $\alpha \in L^1(0,1)$ be such that

(A_1) *for any $\varepsilon > 0$, there exist $\beta_\varepsilon \in L^p(0,1)$, $\gamma_\varepsilon \in L^1(0,1)$ such that*

$$f(t,u)\, u \le (\alpha(t)+\varepsilon)|u|^p + \beta_\varepsilon(t)|u|^{p-1} + \gamma_\varepsilon(t);$$

(B_1) *for any $u \in W^{1,p}(0,1)$, one has $\int_0^1 (|u'|^p - \frac{\alpha}{c_0}|u|^p)dt > 0.$*

Then problem (3.3.1) has a solution in C^1_*, where c_0 is as in (3.3.2).

<u>*Remark c*</u> It is easy to show that assumption (A_1) is true if for some $\alpha \in L^1(0,1)$,

$$\limsup_{|u| \to \infty} \left(f(t,u)/|u|^{p-1} \right) \le \alpha(t)$$

for almost every $t \in (0,1)$.

<u>*Remark d*</u> The condition (B_1) is equivalent to

(B_2) there exists $\bar{\varepsilon} > 0$ such that for any $u \in W^{1,p}(0,1)$ one has

$$\int_0^1 \left[|u'|^p - \frac{\alpha}{c_0}|u|^p \right] dt \ge \bar{\varepsilon}\|u\|^p_{1,p}\,. \tag{3.3.16}$$

Suppose (B_2) is false, we can find a sequence $\{u_n\}$ in $W^{1,p}(0,1)$ such that $\|u_n\|_{1,p}=1$ and $\int_0^1 \left(|u_n'|^p - \frac{\alpha}{c_0}|u_n|^p\right) \to 0$. Taking a subsequence, we can assume $u_n \to u$ weakly in $W^{1,p}(0,1)$. Then $\{u_n\}$ converges to u in $L^p(0,1)$ and the weak semicontinuity of the L^p-norm of u_n' implies $\int_0^1 \left(|u'|^p - \frac{\alpha}{c_0}|u|^p\right) \le 0$. By (B_1), $u=0$ and the above implies that $\{u_n\}$ converges to 0 in $L^p(0,1)$. Since $\int_0^1 \left(|u_n'|^p - \frac{\alpha}{c_0}|u_n|^p\right) \to 0$, it follows that $\|u_n\|_{1,p} \to 0$, which is impossible.

Lemma 3.3.3 *Suppose (B_1) holds. Then for any $\delta>0$ and $\alpha \in L^1(0,1)$, the equation*

$$-(a(|u'|^2)u')' + \delta\ S(u)u - \alpha(t)|u|^{p-2}u = 0 \tag{3.3.17}$$

has only the trivial solution in $C^1_*([0,1])$.

Proof. Suppose $u \in C^1_*([0,1])$ is a solution of (3.3.17), then

$$\int_0^1 [a(|u'|^2)|u'|^2 + \delta\ S(u)u^2 - \alpha(t)|u|^p]dt = 0. \tag{3.3.18}$$

So,

$$0 \ge c_0 \int_0^1 [|u'|^p + \delta|u|^p - \frac{\alpha(t)}{c_0}|u|^p]dt > 0.$$

This is a contradiction.

Proof of Theorem 3.3.2

From Lemma 3.3.1 we know that for any $\delta>0$, problem (3.3.1) is equivalent to

$$u - J_\delta^{-1}\left[\delta\ S(u)u + f(t,u)\right] = 0 \tag{3.3.19}$$

and $J_\delta^{-1} : L^q(0,1) \to C^1_*([0,1])$ is compact. Now we establish *the priori* bounds necessary for application of coincidence degree.

We consider the family of equations

$$u - J_\delta^{-1}\left[\lambda\delta\, S(u)u + \lambda f(t,u) - (1-\lambda)\alpha(t)|u|^{p-2}u\right] = 0 \qquad (3.3.20)$$

for $\lambda\in[0,1]$. Let $u\in C^1_*([0,1])$ be a solution of (3.3.20), then u satisfies

$$Au + \delta\, S(u)\, u - \lambda\left[\delta\, S(u)\, u + f(t,u)\right] - (1-\lambda)\alpha(t)|u|^{p-2}u$$

$$= 0 \ a.e. \text{ on } [0,1], \qquad (3.3.21)$$

$$\int_0^1 \left[a(|u'|^2)|u'|^2 + \delta\, S(u)\, u^2 - \lambda\left[\delta\, S(u)\, u^2 + f(t,u)\, u\right] - (1-\lambda)\alpha(t)|u|^p\right]dt = 0.$$

Fix $\varepsilon < c_0\bar{\varepsilon}$. Condition (A_1) and the above Remark d imply that

$$0 \ge \int_0^1 \left\{a(|u'|^2)|u'|^2 - \lambda\left[(a(t)+\varepsilon)|u|^p + \beta_\varepsilon(t)|u|^{p-1} + \gamma_\varepsilon(t)\right] - (1-\lambda)\alpha(t)|u|^p\right\}$$

$$\ge c_0\int_0^1 \left[\left(|u'|^p - \frac{\alpha}{c_0}|u|^p\right) - \varepsilon|u|^p - \lambda\beta_\varepsilon(t)|u|^{p-1} - \gamma_\varepsilon(t)\right] dt$$

$$\ge (c_0\bar{\varepsilon} - \varepsilon)\|u\|^p_{1,p} - \int_0^1 \left[\beta_\varepsilon(t)|u|^{p-1} + \gamma_\varepsilon(t)\right] dt.$$

From this we obtain that there exists $M_9 > 0$ independent of λ and u such that

$$\|u\|_{1,p} \le M_9.$$

The embedding of $W^{1,p}(0,1)$ to $C^0[0,1]$ imply that there exists $M_{10} > 0$ independent of λ and u such that

$$\|u\|_0 \le M_{10}.$$

Directly integrating in (3.3.21), we obtain that there exists $M_{11} > 0$ such that

$$\|u\|_1 \le M_{11}.$$

Let

$$O = \{\, u\in C^1_*([0,1]) \mid \|u\|_1 \le M_{11} + 1\},$$

$$G(t,u) = \delta\, S(u)u + f(t,u) \text{ and } R(t,u) = -\alpha(t)|u|^{p-2}u\ .$$

Then,

$$u - J_\delta^{-1}\left[\lambda\delta\, S(u)u + \lambda f(t,u) - (1-\lambda)\alpha(t)\,|u|^{p-2}u\right] \neq 0, \text{ for } u \in \partial O.$$

Using Lemma 3.3.1 and the properties of Leray-Schauder degree, we have

$$\deg_{\mathrm{LS}}(I - J_\delta^{-1} G, O, 0) = \deg_{\mathrm{LS}}(I - J_\delta^{-1} R, O, 0).$$

From Lemma 3.3.3, we know that

$$\deg_{\mathrm{LS}}(I - J_\delta^{-1} R, O, 0) = 1,$$

then Problem (3.3.19) has a solution in O.

CHAPTER FOUR

SOME EXISTENCE AND MULTIPLICITY RESULTS FOR A CLASS OF QUASILINEAR ELLIPTIC EIGENVALUE PROBLEMS

INTRODUCTION

In this chapter, we consider the existence of positive solutions of the following eigenvalue problems:

$$\left.\begin{array}{ll} -\operatorname{div}(|Du|^{p-2}Du)=\lambda f(u) & \text{in } \Omega\subset\mathbb{R}^N \\ u=0 & \text{on } \partial\Omega \end{array}\right\}, \tag{P}$$

for $\lambda>0$, $N\geq 2$ and $p>1$. The domain Ω is assumed to be open bounded, connected and to have a smooth boundary $\partial\Omega$ which is connected.

A positive solution of (P) will be a pair (λ,u) in $\mathbb{R}^+\times C^1(\overline{\Omega})$ satisfying (P) in a weak sense with $u>0$ in Ω. That is, u is positive in Ω and satisfies

$$\int_\Omega\left[|Du|^{p-2}Du\,D\phi\right]=\lambda\int_\Omega\left[f(u)\,\phi\right],$$

for every $\phi\in\mathcal{D}^+(\Omega)$, where $\mathcal{D}^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$.

The function $f\in C^1(\mathbb{R}^+)$ and we shall suppose that it satisfies either (F_1) or (F_2):

(F_1) f is strictly increasing on $\mathbb{R}^+$, $f(0)=0$ and $\lim_{s\to 0} f(s)/s^{p-1}=0$; there exist α_1, $\alpha_2>0$ such that $f(s)\leq\alpha_1+\alpha_2|s|^\mu$, $0<\mu<p-1$.

(F_2) There is $\beta>0$, such that

$$f(0)=f(\beta)=0,\ f>0 \text{ in } (0,\beta) \text{ and } f<0 \text{ in } (\beta,\infty) \text{ and } \lim_{s\to 0} f(s)/s^{p-1}=0.$$

We shall obtain some existence and multiplicity results for Problem (P). When $p=2$, such problems have been considered by many authors (see [16, 19–20, 64, 66, 70]), but when $p\neq 2$, many of the 'nice' properties of Δ are lost and the methods used for Δ are not applicable.

In section 4.1, we prove that a strong maximum principle is available for $A+\vartheta$ when $1<p\leq 2$, $\vartheta>0$, where $A\cdot = -\operatorname{div}(|D\cdot|^{p-2}D\cdot)$. In section 4.2, we show that the positive solutions of (P) when f satisfies (F_1) or (F_2) occur in pairs, using the theory of degree of mappings of class $(S)_+$. In section 4.3, we give a necessary and sufficient condition for the existence of solutions of (P) when $1<p\leq 2$ and a necessary condition for $p>2$.

4.1 SOME PROPERTIES OF THE p-LAPLACIAN

In this section we obtain some results which will be useful in the coming proofs. We say that $u_1,\ u_2\in C^1(\overline{\Omega})$ satisfy

$$-\operatorname{div}(|Du_1|^{p-2}Du_1)+\vartheta u_1\geq(\leq)-\operatorname{div}(|Du_2|^{p-2}Du_2)+\vartheta u_2 \quad \text{in } \Omega,$$

in the weak sense if for every $\phi\in\mathscr{D}^+(\Omega)$,

$$\int_\Omega(|Du_1|^{p-2}Du_1D\phi+\vartheta u_1\phi)\geq(\leq)\int_\Omega(|Du_2|^{p-2}Du_2D\phi+\vartheta u_2\phi),$$

where $\mathscr{D}^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$.

Lemma 4.1.1 **(Weak comparison principle)** *Let Ω be a bounded domain in* $\mathbb{R}^N$ $(N\geq 2)$ *with smooth boundary $\partial\Omega$, and let $\vartheta\geq 0$. Let $u_1,\ u_2\in W^{1,p}(\Omega)$ satisfy*

$$\int_\Omega|Du_1|^{p-2}Du_1\cdot D\psi dx+\int_\Omega\vartheta u_1\psi dx\leq\int_\Omega|Du_2|^{p-2}u_2\cdot D\psi dx+\int_\Omega\vartheta u_2\psi dx, \tag{4.1.1}$$

for all non-negative $\psi\in W^{1,p}(\Omega)$. Then the inequality

$$u_1 \le u_2 \quad on\ \partial\Omega$$

implies that

$$u_1 \le u_2 \quad in\ \Omega. \tag{4.1.2}$$

Proof. Let $\eta = Du_1$, $\eta' = Du_2$ and $\psi = \max\{u_1 - u_2, 0\}$. Since $u_1 \le u_2$ on $\partial\Omega$, ψ belongs to $W_0^{1,p}(\Omega)$. Inserting this function ψ into (4.1.1), we have

$$\int_{\{u_1 > u_2\}} (|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta')(\eta - \eta')dx \le 0. \tag{4.1.3}$$

We suppose that $\mathscr{A}(\mathscr{B}) := \{x : |\eta(x)| \le (>) |\eta'(x)|\}$, then

$$\frac{1}{4}|\eta - \eta'| \le |\eta' + t(\eta - \eta')| \le 1 + |\eta| + |\eta'| \quad \text{in } \mathscr{A}(\mathscr{B}),$$

for all $t \in [0, 1/4]$ ([3/4, 1]). Therefore, writing $a^i(\eta) = |\eta|^{p-2}\eta_i$,

$$\int_{\{u_1 > u_2\}} (|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta')(\eta - \eta')dx$$

$$= \int_{\{u_1 > u_2\}} \int_0^1 \sum_1^N \frac{\partial a^i}{\partial \eta_j}(\eta' + t(\eta - \eta'))(\eta_i - \eta_i')(\eta_j - \eta_j')dt\, dx$$

$$\ge \int_{\{u_1 > u_2\}} \left[\int_0^{1/4} + \int_{3/4}^1\right] \sum_1^N \frac{\partial a^i}{\partial \eta_j}(\eta' + t(\eta - \eta'))(\eta_i - \eta_i')(\eta_j - \eta_j')dt\, dx$$

$$\ge \gamma_0 \begin{cases} \displaystyle\int_{\{u_1 > u_2\}} (1 + |\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2 dx & \text{if } p < 2, \\ \displaystyle\int_{\{u_1 > u_2\}} |\eta - \eta'|^p\, dx & \text{if } p \ge 2, \end{cases}$$

where $\gamma_0 > 0$. From this and (4.1.3), we see that (4.1.2) is true.

<u>Lemma 4.1.2</u> **(Hopf type maximum principle)** *Let* B *be a ball contained in* Ω *and assume that* $\vartheta > 0$, $1 < p \le 2$ *and* $u \in C^1(\overline{\mathrm{B}})$ *satisfies*

$$-\mathrm{div}(|Du|^{p-2}Du) + \vartheta u \ge 0 \qquad in\ \mathrm{B}\ \ (in\ the\ weak\ sense), \tag{4.1.4}$$

$$u > 0 \text{ in } B, \text{ and } u(x_0) = 0, \text{ for some } x_0 \in \partial B. \tag{4.1.5}$$

Then

$$Du(x_0) \neq 0. \tag{4.1.6}$$

The conclusion is still true for all $p > 1$ when $\vartheta = 0$.

Proof. For $p = 2$, it is well-known that the result is true. Now we only consider the case of $1 < p < 2$. Without loss of generality, we may suppose that $B = B_1$ is the unit ball, centred at 0. For $k > 0$ and $\alpha > 0$, we set

$$b(x) = k(e^{-\alpha|x|^2} - e^{-\alpha}).$$

Then, writing $r = |x|$,

$$-\mathrm{div}(|Db|^{p-2}Db) + \vartheta b = -r^{-(N-1)}(r^{N-1}|b_r|^{p-2}b_r)_r + \vartheta b$$

$$= (2\alpha k)^{p-1} r^{p-2}((N-2+p) - 2\alpha(p-1)r^2)e^{-\alpha(p-1)r^2} + \vartheta k(e^{-\alpha r^2} - e^{-\alpha}).$$

As $1 < p < 2$, we can choose k and α in such a way that

$$b \le u, \qquad \text{on } \partial(B_1 - B_{1/2}) \tag{4.1.7}$$

and

$$-\mathrm{div}(|Db|^{p-2}Db) + \vartheta b \le 0 \qquad \text{in } B_1 - B_{1/2}. \tag{4.1.8}$$

The weak comparison principle (Lemma 4.1.1), (4.1.5), (4.1.7) and (4.1.8) imply that

$$b \le u \qquad \text{in } B_1 - B_{1/2}. \tag{4.1.9}$$

From (4.1.7) and (4.1.9) we know that $\frac{\partial u}{\partial n}\Big|_{\partial B_1} \le \frac{\partial b}{\partial n}\Big|_{\partial B_1} < 0$. This implies that (4.1.6) is true. Following the same ideas as above, we can obtain the result for all $p > 1$, when $\vartheta = 0$.

Lemma 4.1.3 (Strong Maximum Principle) *Assume that $1 < p \le 2$, $\vartheta > 0$, Ω is connected. Moreover, suppose that $u \in C^1(\Omega)$ satisfies* (4.1.4) *in Ω, u is* *nonnegative and that it does not vanish identically. Then,*

$$u > 0, \text{ in } \Omega.$$

The conclusion is still true for all $p > 1$ *when* $\vartheta = 0$.

Proof. Suppose there is a $x_0 \in \Omega$ such that $u(x_0) = 0$, then we can find $x_1 \in \Omega$ such that there is a ball B contained in Ω, $x_1 \in \partial B$, $u(x_1) = 0$, $u(x) > 0$ in B and $Du(x_1) = 0$. This contradicts the conclusion of Lemma 4.1.2.

Now, we prove the following proposition.

Proposition 4.1.4 *Let* f *satisfy* (F_2), $1 < p \le 2$. *Suppose* $u \in C^1(\overline{\Omega})$ *is a solution of the problem*

$$\left.\begin{aligned} -\mathrm{div}(|Du|^{p-2}Du) = \lambda f(u) \quad & \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned}\right\}, \tag{P}$$

with $u \ge 0$, $\max u \le \beta$. *Then* $\max u < \beta$.

Proof. By (F_2), we have that there exists $M > 0$ such that $f(s) + Ms$ is monotone increasing in s for $s \in [0, \beta]$. Since

$$(A + \lambda M)(\beta - u) = \lambda[(f(\beta) + M\beta) - (f(u) + Mu)] \ge 0, \qquad \text{in } \Omega. \tag{4.1.10}$$

This proposition follows from Lemma 4.1.3.

For $\vartheta \ge 0$ and a given $h \in C^0(\Omega)$, we are going to look for a function $u \in C^1(\overline{\Omega})$ satisfying

$$\left.\begin{aligned} -\mathrm{div}(|Du|^{p-2}Du) + \vartheta u = h \quad & a.e \text{ in } \Omega \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned}\right\}. \tag{4.1.11}$$

To prove that there is a solution, we first prove that there is $u \in W_0^{1,p}(\Omega)$

such that u satisfies (4.1.11) in a weak sense, then applying some of the theorems in Chapter 1, we obtain that $u \in C^1(\overline{\Omega})$.

We observe that searching for $u \in W_0^{1,p}(\Omega)$ satisfying (4.1.11) is equivalent to finding critical points of the functional

$\Psi_h : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\Psi_h(v) = (1/p)\int_\Omega |Dv|^p + (\vartheta/2)\int_\Omega v^2 - \int_\Omega hv. \tag{4.1.12}$$

We find that Ψ_h is a continuous strictly convex functional such that $\Psi_h(v) \longrightarrow \infty$, as $\|v\|_{1,p} \to \infty$. By Proposition 3.1 of [30], Ψ_h is differentiable. So, it possesses a unique critical point $w \in W_0^{1,p}(\Omega)$ at which it reaches its global minimum. It also follows from [30] that w satisfies:

$$\int_\Omega |Dw|^p + \vartheta\int_\Omega w^2 = \int_\Omega hw. \tag{4.1.13}$$

Therefore,

$$\int_\Omega |Dw|^p \leq \|h\|_0 (\text{meas}\Omega)^{1/q} C_1 \|Dw\|_p. \tag{4.1.14}$$

Here $1/q + 1/p = 1$, and C_1 is the Sobolev embedding constant. So, $\|w\|_{1,p} \leq C_2 \|h\|_0^{(p-1)/p}$. When $1 < p < \mathrm{N}$, the embedding of $W_0^{1,p}(\Omega)$ in $L^{\mathrm{N}p/(\mathrm{N}-p)}(\Omega)$ implies that $w \in L^{\mathrm{N}p/(\mathrm{N}-p)}(\Omega)$. Applying Theorem 1.3.16 of Chapter 1, we obtain the estimate:

$$\sup\{|w|;\ x \in \Omega\} \leq C_3, \tag{4.1.15}$$

here $C_3 = C_3(\|h\|_0)$. If $p \geq \mathrm{N}$, we get (4.1.15) from the Sobolev embedding theorem. Using Theorem 1.3.15 of Chapter 1, we see that w belongs to $C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$, and

$$\|w\|_{C^\alpha} \leq C_4, \tag{4.1.16}$$

here C_4 is determined by C_3. By Theorem 1.3.17 of Chapter 1, we also know that w belongs to $C^{1,\alpha}(\overline{\Omega})$ and

$$\|w\|_{C^{1,\alpha}} \leq C_5. \tag{4.1.17}$$

Here C_5 is determined by C_4.

From the previous arguments we conclude that (4.1.11) has a unique solution $u \in C^1(\overline{\Omega})$. Thus we can define a mapping $G_p : C^0(\Omega) \to C^1(\overline{\Omega})$ by $G_p(h) = w$.

Proposition 4.1.5 *For $p > 1$, the mapping $G_p : C^0(\Omega) \longrightarrow C^1(\overline{\Omega})$ is compact.*

Proof. Let $G_p(h) = u$. From the arguments above we see that there exists $C_5 > 0$ such that $\|u\|_{C^{1+\alpha}} \leq C_5$. The compact embedding of $C^{1+\alpha}(\overline{\Omega})$ in $C^1(\overline{\Omega})$ implies that the mapping $G_p : C^0(\Omega) \longrightarrow C^1(\overline{\Omega})$ is compact.

4.2 EXISTENCE RESULTS

In this section we first give the following existence theorem.

Theorem 4.2.1 *Let $f \in C^1(\mathbb{R}^+)$ satisfy (F_1), $p > 1$. Then there exists $\underline{\lambda} > 0$ such that for all $\lambda > \underline{\lambda}$, (P) possesses at least 2 distinct positive solutions $u_1(\lambda)$, $u_2(\lambda)$.*

To prove this theorem, we first prove the following lemmas.

Lemma 4.2.2 *Suppose f satisfies (F_1) or (F_2), $p > 1$. Then for any $\lambda_1 > 0$, there exists $\rho = \rho(\lambda_1)$ such that for $\lambda \in (0, \lambda_1]$, $u \equiv 0$ is the unique nonnegative solution of (P) in $B_\rho(0) \subset C_0^1(\overline{\Omega})$. Here $B_\rho(0)$ is a ball with centre 0 and radius ρ.*

Proof. Suppose not, then (P) possesses solutions (λ_m, u_m) with $u_m \geq 0$ and $u_m \not\equiv 0$, and $\lim_{m \to \infty} u_m \to 0$ in $C_0^1(\overline{\Omega})$; $\lambda_m > 0$ and some $\lambda_m \in (0, \lambda_1]$. From the equation (P) we have

$$\int_\Omega |Du_m|^p dx = \lambda_m \int_\Omega f(u_m)u_m dx. \tag{4.2.1}$$

Then, using $\lim_{s\to 0} f(s)/s^{p-1} = 0$ and Hölder inequality, we get

$$\int_\Omega |Du_m|^p dx \le \lambda_1 C_6(m) \int_\Omega (u_m)^p \le \lambda_1 C_7(m) \int_\Omega |Du_m|^p dx,$$

where $C_7(m) \to 0$ as $m \to \infty$. Thus in both cases, we obtain a contradiction.

<u>*Remark a*</u> By the proof of Lemma 4.2.2, we see that if f satisfies (F_2), then for any $\lambda_1 > 0$, there exists $\rho = \rho(\lambda_1) > 0$ small enough such that for $\lambda \in (0, \lambda_1]$, $u \equiv 0$ is the unique nonnegative solution of Problem (P) in $B_\rho(0) \subset W_0^{1,p}(\Omega)$.

<u>Lemma 4.2.3</u> *Let f satisfy* (F_1), $p > 1$ *and* λ *be large enough. Then problem* (P) *possesses a positive solution* (λ, u) *with* $u \in C^1(\overline{\Omega})$.

Proof. First modify the function f by setting $f(s) = -f(-s)$ for $s < 0$. Now we want to minimize

$$I(\lambda,u) = 1/p \int_\Omega |Du|^p - \lambda \int_\Omega F(u), \quad \text{in } W_0^{1,p}(\Omega), \tag{4.2.2}$$

where $F(u) = \int_0^u f(s)ds$.

For $\lambda > 0$, by (F_1) we know that $I(\lambda,u)$ is bounded below in $W_0^{1,p}(\Omega)$. Let u_n be a minimizing sequence of $I(\lambda,u)$ for a fixed λ, then

$$I(\lambda, |u_n|) = 1/p \int_\Omega |D|u_n||^p - \lambda \int_\Omega F(|u_n|)$$

$$\le 1/p \int_\Omega |Du_n|^p - \lambda \int_\Omega \int_0^{u_n} f(s)ds \le I(\lambda, u_n).$$

Since $I(\lambda,\cdot)$ is sequentially weakly lower semicontinuous and convex in $W_0^{1,p}(\Omega)$, $I(\lambda,\cdot)$ possesses a nonnegative minimizer, which we denote by u_λ. It follows from the equation (P) and condition (F_1) that

$$\int_\Omega |Du_\lambda|^p \le \lambda \int_\Omega f(u_\lambda)u_\lambda \le \lambda\alpha_1 \int_\Omega (u_\lambda)^{\mu+1} + \lambda\alpha_2 \int_\Omega u_\lambda$$

$$\le \lambda T_1 \left[\int_\Omega |Du_\lambda|^p \right]^{(\mu+1)/p} + \lambda T_2 \left[\int_\Omega |Du_\lambda|^p \right]^{1/p}.$$

From this we get that $\|u_\lambda\|_{1,p}$ is bounded. Then following the same steps as in the proof of Proposition 4.1.5 we get $u_\lambda \in C^1(\overline{\Omega})$.

To prove $u_\lambda > 0$ in Ω, by Lemma 4.1.3 we only prove $u_\lambda \not\equiv 0$. If $u_\lambda \equiv 0$, choose $v \in C_0^\infty(\Omega)$, $C > 0$ such that $0 \le v \le C$ in $\Omega_\delta = \{ x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$ and $v = C$ in $\Omega \setminus \Omega_\delta$. Then,

$$I(\lambda,v) - I(\lambda,u_\lambda) = 1/p \int_\Omega |Dv|^p dx - \lambda \int_\Omega F(v) dx$$

$$\le 1/p \int_\Omega |Dv|^p dx - \lambda \left[\int_\Omega F(C)dx + \int_{\Omega_\delta} (F(v) - F(C))dx \right]$$

$$\le 1/p \int_\Omega |Dv|^p dx - \lambda \int_\Omega \int_0^C f(s)ds + \lambda MC \,\mathrm{meas}(\Omega_\delta),$$

where $M = \sup_{[0,C]} f(s)$. For $\mathrm{meas}(\Omega_\delta)$ small enough and λ large enough, then

$$I(\lambda,v) - I(\lambda,u_\lambda) < 0.$$

This contradicts the fact that $I(\lambda,u_\lambda) = \min I(\lambda,u)$ for all $u \in W_0^{1,p}(\Omega)$.

From Lemma 4.2.3 we can easily obtain a subsolution y_λ to problem (P). In fact, let $g(s) = (1/2)f(s)$, then $g(s)$ satisfies the same hypotheses as f. Following the ideas of Lemma 4.2.3 we get a positive solution y_λ to the problem

$$\left.\begin{aligned} -\mathrm{div}(|Du|^{p-2}Du) &= (\lambda/2)f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\}. \tag{4.2.3}$$

Therefore, y_λ satisfies:

$$-\mathrm{div}(|Dy_\lambda|^{p-2}Dy_\lambda) \le \lambda f(y_\lambda) \quad \text{and } \|y_\lambda\|_1 \text{ is bounded by } C(\lambda). \tag{4.2.4}$$

Moreover, $y_\lambda > 0$ in Ω. This follows from Lemma 4.1.3. For convenience, we denote $C(\lambda)$ by C.

From Lemma 4.2.3 we also can obtain a supersolution to problem (P). Let $g(s)=f(s+C)$, then $g(0)>0$. Writing $g(s)=2g(0)-g(-s)$ when $s<0$, and following the same steps as the proof of Lemma 4.2.3, we get a positive solution z_λ to the problem

$$\left.\begin{aligned} -\mathrm{div}(|Du|^{p-2}Du)&=\lambda f(u+C) && \text{in } \Omega \\ u&=0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{4.2.5}$$

in the weak sense. Setting $w_\lambda = z_\lambda + C$, w_λ is a supersolution of (P). It is clear that $w_\lambda \in C^1(\overline{\Omega})$ and $w_\lambda > y_\lambda$, $w_\lambda > z_\lambda$ in Ω.

Now, we will show that $z_\lambda > y_\lambda$ in Ω. Since $w_\lambda > y_\lambda$ in Ω and f is strictly increasing, then

$$-\mathrm{div}(|Dz_\lambda|^{p-2}Dz_\lambda) > -\mathrm{div}(|Dy_\lambda|^{p-2}Dy_\lambda). \tag{4.2.6}$$

It follows from Lemma 4.1.1 that $y_\lambda \le z_\lambda$ in Ω. From this, we can prove the following lemma.

<u>Lemma 4.2.4</u> *Let y_λ and z_λ be as above. Then $y_\lambda < z_\lambda$ in Ω.*

Proof. By the fact that Ω satisfies the interior sphere condition, applying Lemma 4.1.2 to y_λ and z_λ , we obtain, for n the normal direction,

$$\frac{\partial z_\lambda}{\partial n} < 0 \quad \text{and} \quad \frac{\partial y_\lambda}{\partial n} < 0 \quad \text{on } \partial\Omega \tag{4.2.7}$$

Since $\frac{\partial z_\lambda}{\partial n}$ and $\frac{\partial y_\lambda}{\partial n}$ are continuous, then we get from (4.2.7) that

$$\frac{\partial z_\lambda}{\partial n} < -\delta \quad \text{and} \quad \frac{\partial y_\lambda}{\partial n} < -\delta \quad \text{on } \overline{G} \tag{4.2.8}$$

where δ is a positive constant and G is an open one-side connected neighbourhood of $\partial\Omega$ in Ω. Suppose there exist a $x_0 \in \Omega$ such that $z_\lambda(x_0)=y_\lambda(x_0)$, we show that there exists a $x_1 \in G$ such that $z_\lambda - y_\lambda$ vanishes.

Choose a bounded subdomain Ω_1 of Ω with smooth boundary $\partial\Omega_1$ which satisfies

$$\overline{\Omega}_1 \subset \Omega, \ \partial\Omega_1 \subset G \ \text{ and } x_0 \in \Omega_1.$$

Then we have a point $x_1 \in \partial\Omega_1$ where $z_\lambda - y_\lambda$ vanishes. Indeed, suppose $z_\lambda - y_\lambda > 0$ on $\partial\Omega_1$. By the continuity we have

$$z_\lambda - y_\lambda \geq \tau > 0 \qquad \text{on } \partial\Omega_1 \tag{4.2.9}$$

for some $\tau > 0$. Since the function $v_\lambda = y_\lambda + \tau$ satisfies

$$-\mathrm{div}(|Dv_\lambda|^{p-2} Dv_\lambda) \leq \lambda f(y_\lambda) \quad \text{in } \Omega_1 \tag{4.2.10}$$

in the weak sense, we have

$$-\mathrm{div}(|Dz_\lambda|^{p-2} Dz_\lambda) > -\mathrm{div}(|Dv_\lambda|^{p-2} Dv_\lambda) \quad \text{in } \Omega_1 \tag{4.2.11}$$

and $z_\lambda \geq v_\lambda$ on $\partial\Omega_1$. Then it follows from Lemma 4.1.1 that

$$z_\lambda \geq v_\lambda \qquad \text{in } \Omega_1. \tag{4.2.12}$$

Since $x_0 \in \Omega_1$, we have $z_\lambda(x_0) \geq v_\lambda(x_0) = y_\lambda(x_0) + \tau$. This contradicts $z_\lambda(x_0) - y_\lambda(x_0) = 0$. Thus we have a point $x_1 \in \partial\Omega_1 \subset G$ where $z_\lambda - y_\lambda$ vanishes.

We now use the maximum principle to obtain a contradiction. We have

$$0 < -\mathrm{div}(|Dz_\lambda|^{p-2} Dz_\lambda) - \left\{ -\mathrm{div}\left[|Dy_\lambda|^{p-2} Dy_\lambda \right] \right\}$$

$$= -\sum_{i,j} [a_{ij}(x)(z_\lambda - y_\lambda)_{x_j}]_{x_i}, \quad \text{in } G, \tag{4.2.13}$$

where $a_{ij}(x) = \int_0^1 [a_i(tDz_\lambda + (1-t)Dy_\lambda)]_{\rho_j} dt$, and $a_i(\rho) = |\rho|^{p-2}\rho_i$ $(i = 1, 2, \ldots, N)$, for $\rho = (\rho_1, \ldots, \rho_N) \in \mathbb{R}^N$. Put $L \cdot = \sum_{i,j} [a_{ij}(x) \frac{\partial}{\partial x_j}]_{x_i}$. From (4.2.8) we see that L is a uniformly elliptic operator on G. Consequently, we have

$$\left.\begin{aligned} -L(z_\lambda - y_\lambda) &> 0 \qquad \text{in } G \\ z_\lambda - y_\lambda &\geq 0 \qquad \text{on } \partial G \end{aligned}\right\} \tag{4.2.14}$$

and $x_1 \in G$, $z_\lambda(x_1) - y_\lambda(x_1) = 0$. Then, by using Hopf's strong maximum principle for uniformly elliptic operator (see Theorem 1.3.12 of Chapter 1), we get

$$z_\lambda \equiv y_\lambda \qquad \text{in } G.$$

This contradicts (4.2.14).

Proof of Theorem 4.2.1.

We note from the proof of Lemma 4.2.2 that if $u_\lambda \in C^1(\overline{\Omega})$ is a solution of problem (P), there exists $q(\lambda) > 0$ such that $\|u_\lambda\|_1 < q(\lambda)$.

Let $O_\lambda = \left\{ \phi \in C_0^1(\overline{\Omega}) \colon y_\lambda < \phi < z_\lambda, \text{ in } \Omega,\ \|\phi\|_1 < q(\lambda) \right\}$ where y_λ and z_λ were defined by (4.2.3) and (4.2.5). Let $T_\lambda u = G_p(\lambda f)(u)$ (here $\vartheta = 0$), by Proposition 4.1.5 T_λ is a completely continuous mapping from $C^1(\overline{\Omega})$ to $C^1(\overline{\Omega})$. The result of Lemma 4.2.4 and $z_\lambda < w_\lambda$ shows that if $\phi \in \overline{O}_\lambda$,

$$y_\lambda \;<\; T_\lambda(y_\lambda) \;\le\; T_\lambda(\phi) \;\le\; T_\lambda(z_\lambda) \;<\; z_\lambda \text{ in } \Omega. \tag{4.2.15}$$

Let $\Phi(\lambda, u) = u - T_\lambda(u)$, By (4.2.15), $0 \notin \Phi(\lambda, \partial O_\lambda)$. Hence the Leray-Schauder degree of $\Phi(\lambda, \cdot)$ relative to the set O_λ and the point 0 is defined and will be denoted by $\mathrm{d}_{\mathrm{LS}}(\Phi(\lambda, \cdot), O_\lambda, 0)$. Since $T_\lambda \colon \overline{O}_\lambda \to O_\lambda$, then $\mathrm{d}_{\mathrm{LS}}(\Phi(\lambda, \cdot), O_\lambda, 0) = 1$. To verify this, let $a \in O_\lambda$, $S_t(\lambda, u) = t T_\lambda(u) + (1-t)a$, and $\Psi_t(\lambda, u) = u - S_t(\lambda, u)$ for $t \in [0,1]$. Then $S_t \colon \overline{O}_\lambda \to O_\lambda$, so, $0 \notin \Psi_t(\lambda, \partial O_\lambda)$ for $t \in [0,1]$ and therefore by the homotopy invariance of degree,

$$\mathrm{d}_{\mathrm{LS}}(\Psi_t(\lambda, \cdot), O_\lambda, 0) \equiv \text{constant} = \omega_1,\ t \in [0,1].$$

Since $\Psi_0(\lambda, u) = u - a$, the definition of degree implies $\omega_1 = 1$.

Let $\mathrm{B}_\gamma(\psi) = \left\{ \varphi \in C^1(\overline{\Omega}) : \|\varphi - \psi\|_1 < \rho \right\}$, $\mathrm{B}_\gamma = \mathrm{B}_\gamma(0)$ and $\mathrm{Q}^+ = \left\{ \varphi \in C^1(\overline{\Omega}),\ \varphi > 0 \text{ in } \Omega \right\}$.

By Lemma 4.2.2, if $\gamma = \gamma(\lambda)$ is sufficiently small, $u \equiv 0$ is the unique solution of (P) in $\overline{\mathrm{B}}_{\gamma(\lambda)}$ and γ can be chosen independently of λ for λ in a bounded interval. Fix $\lambda > \underline{\lambda}$, choose such a γ for $[0,\lambda]$ and let

$$\mathscr{C}(\lambda) = \left\{ (\mu, \varphi) \in \mathbb{R} \times \mathrm{Q}^+ \ :\ \mu \in [0,\lambda],\ \gamma < \|\varphi\|_1 < q(\lambda) \right\}.$$

Since $q(\lambda)$ is continuous, $\mathscr{G}(\lambda)$ is a bounded open set in $[0,\lambda]\times C^1(\Omega)$ with no zeros of Φ on $\partial\mathscr{G}(\lambda)$ (relative to $[0,\lambda]\times C^1(\overline{\Omega})$). By the homotopy invariance of degree,

$$d_{LS}(\Phi(\mu,\cdot),\ \mathscr{G}_\mu(\lambda),\ 0)\equiv \text{const.}=\omega_2,\ \mu\in[0,\lambda],$$

where $\mathscr{G}_\mu(\lambda)=\{\varphi\in C^1(\Omega):\ (\mu,\varphi)\in\mathscr{G}(\lambda)\}$. Since $\Phi(0,\cdot)$ has no zeros in $\mathscr{G}_0(\lambda)$, $\omega_2=0$. By the additivity of degree,

$$\begin{aligned} d_{LS}(\Phi(\lambda,\cdot),\ \mathscr{G}_\lambda(\lambda),\ 0) &= d_{LS}(\Phi(\lambda,\cdot),\ O_\lambda,\ 0) \\ &\quad + d_{LS}(\Phi(\lambda,\cdot),\ \mathscr{G}_\lambda(\lambda)-\overline{O}_\lambda,\ 0) \\ &= 1+d_{LS}(\Phi(\lambda,\cdot),\ \mathscr{G}_\lambda(\lambda)-\overline{O}_\lambda,\ 0). \end{aligned}$$

Hence, $d_{LS}(\Phi(\lambda,\cdot),\ \mathscr{G}_\lambda(\lambda)-\overline{O}_\lambda,\ 0)=-1$ and therefore, there exists a solution of (P) belonging to $\mathscr{G}_\lambda(\lambda)-\overline{O}_\lambda\subset Q^+$. This completes the proof.

We now give a result assuming f satisfies (F_2).

Theorem 4.2.5 *Let $f\in C^1(\mathbb{R}^+)$ satisfy (F_2), $1<p\le 2$. Then there exists $\underline{\lambda}>0$ such that for all $\lambda>\underline{\lambda}$, (P) possesses at least 2 distinct positive solutions.*

To prove this theorem, we give the following lemmas.

Lemma 4.2.6 *Let f satisfy (F_2), $1<p\le 2$. Then problem (P) possesses a positive solution (λ, u) with $u\in C^1(\overline{\Omega})$ and* $\max u\in(0,\beta)$.

Proof. First modify the function f by setting $f(s)=-f(-s)$ for $s<0$ and $f(s)=0$ for $s>\beta$. It follows from Lemma 4.2.3 that there exist a solution $u_\lambda\in C^1(\overline{\Omega})$ such that $u>0$. Suppose that $\max u>\beta$. By the continuity we have that there exist a connected domain $\Omega_1\subset\Omega$ such that $u>\beta$ in Ω_1 and $u=\beta$ on $\partial\Omega_1$. Hence, u satisfies

$$\left.\begin{aligned} -\mathrm{div}(|D(u-\beta)|^{p-2}D(u-\beta)) \equiv 0 \ \text{ in } \Omega_1 \\ u-\beta=0 \ \text{ on } \partial\Omega_1 \end{aligned}\right\}, \tag{4.2.16}$$

Thus, $u\equiv\beta$ in Ω_1. This is a contradiction. By Proposition 4.1.4 this proves the lemma.

Proof of Theorem 4.2.5.

Using the same arguments as after Lemma 4.2.3, we obtain a positive subsolution y_λ of (P) which satisfies

$$\left.\begin{aligned} -\mathrm{div}(|Dy_\lambda|^{p-2}Dy_\lambda)=(\lambda/2)f(y_\lambda) \quad & \text{in } \Omega \ \text{(in the weak sense)} \\ y_\lambda=0 \quad & \text{on } \partial\Omega \end{aligned}\right\}. \tag{4.2.17}$$

Therefore, y_λ satisfies

$$-\mathrm{div}(|Dy_\lambda|^{p-2}Dy_\lambda)\le\lambda f(y_\lambda) \text{ and } \|y_\lambda\|_0<\beta.$$

So, y_λ also satisfies:

$$-\mathrm{div}(|Dy_\lambda|^{p-2}Dy_\lambda)+\lambda My_\lambda\le\lambda(f(y_\lambda)+My_\lambda) \quad \text{in } \Omega, \tag{4.2.18}$$

where $M>0$ is a constant such that $g(s)=f(s)+Ms$ is strictly increasing for $s>0$.

Let $z_\lambda\in C^1(\overline{\Omega})$ be the positive solution of

$$\left.\begin{aligned} -\mathrm{div}(|Du|^{p-2}Du)+\lambda Mu=\lambda g(\beta) \quad & \text{in } \Omega \ \text{(in the weak sense)} \\ u=0 \quad & \text{on } \partial\Omega \end{aligned}\right\}. \tag{4.2.19}$$

We will show that $\beta>z_\lambda$ in Ω and $z_\lambda>y_\lambda$ in Ω. In fact $z_\lambda<\beta$ is a consequence of Lemma 4.2.6.

Now, we prove that $z_\lambda>y_\lambda$ in Ω. The idea of proof is the same as that of Lemma 4.2.4. Since $y_\lambda<\beta$ in Ω and g is monotone increasing in $s>0$, there exists $\alpha>0$ such that $g(\beta)\ge g(y_\lambda)+\alpha$ in Ω. So, it follows from Lemma 4.1.1, that $z_\lambda\ge y_\lambda$ in Ω. To prove $z_\lambda>y_\lambda$ in Ω, it suffices to prove that if there exists a $x_0\in\Omega$ such that $z_\lambda(x_0)=y_\lambda(x_0)$, then there exists $x_1\in G$ such that $z_\lambda-y_\lambda$ vanishes. Here G is as in the proof of Theorem 4.2.1.

Choose a bounded subdomain $\Omega_1 \subset \Omega$ as in the proof of Lemma 4.2.4. Then we have a point $x_1 \in \partial\Omega_1$ where $z_\lambda - y_\lambda$ vanishes. Suppose $z_\lambda - y_\lambda > 0$ on $\partial\Omega_1$. By the continuity we have

$$z_\lambda - y_\lambda \geq \tau \qquad \text{on } \partial\Omega_1, \tag{4.2.20}$$

where $0 < \tau < \alpha/M$. Since the function $v_\lambda = y_\lambda + \tau$ satisfies

$$-\mathrm{div}(|Dv_\lambda|^{p-2}Dv_\lambda) + \lambda M v_\lambda \leq \lambda(g(y_\lambda) + M\tau) \leq \lambda g(\beta) \quad \text{in } \Omega_1 \tag{4.2.21}$$

in the weak sense, we have

$$-\mathrm{div}(|Dz_\lambda|^{p-2}Dz_\lambda) + \lambda M z_\lambda \geq -\mathrm{div}(|Dv_\lambda|^{p-2}Dv_\lambda) + \lambda M v_\lambda \quad \text{in } \Omega_1 \tag{4.2.22}$$

and $z_\lambda \geq v_\lambda$ on $\partial\Omega_1$. Then it follows from Lemma 4.1.1 that

$$z_\lambda \geq v_\lambda \quad \text{in } \Omega_1.$$

This contradicts $z_\lambda(x_0) - y_\lambda(x_0) = 0$. The remainder of the proof is same as the proof of Lemma 4.2.4.

For $p > 2$, we have the following theorem:

<u>Theorem 4.2.7</u> *Suppose* $p > 2$, f *satisfies* (F_2) *and* $f''(x) < 0$, *for* $x \in (0, \beta)$. *Then there exists* $\underline{\lambda} > 0$ *such that for all* $\lambda > \underline{\lambda}$, (P) *possesses at least* 2 *distinct positive solutions.*

To prove this theorem, we let

$$f_1(s) = \begin{cases} f(s) & 0 \leq s \leq \beta \ , \\ 0 & \text{otherwise}, \end{cases}$$

and consider the problem

$$\left.\begin{aligned} \mathrm{div}(|Du|^{p-2}Du) + \lambda f_1(u) = 0, &\qquad \text{on } \Omega \\ u = 0, &\qquad \text{on } \partial\Omega \end{aligned}\right\} . \tag{4.2.23}$$

By the maximum principle, we know that if $u \in C_0^1(\Omega)$ is a nontrival solution of

(4.2.23), then $0 \le u \le \beta$. So, from Lemma 4.1.3 we know that u is a positive solution of (P). For convenience, we still write f rather than f_1.

Let $W = W_0^{1,p}(\Omega)$, $W^* = W^{-1,q}(\Omega)$, $1/p + 1/q = 1$. Now we give some lemmas.

Lemma 4.2.8 *For each $t \in (0, 1)$, the mapping $A(t) = A - t\lambda f : W \to W^*$ is of type $(S)_+$. Here $(S)_+$ is as in Definition* 1.2.6 *of Chapter* 1.

Proof. Let $\{u_n\} \subset W$ be a sequence such that u_n converges weakly to a point $u \in W$ and $\limsup < Au_n - t\lambda f(u_n), u_n - u > \le 0$. Then from the Sobolev compact embedding theorem, we have that u_n converges strongly to u in $L^p(\Omega)$. Therefore,

$$\lim < f(u_n), u_n - u > = 0.$$

Thus we find

$$\limsup < Au_n, u_n - u > = \limsup \sum_1^N < |Du_n|^{p-2} \frac{\partial u_n}{\partial x_i}, \frac{\partial (u_n - u)}{\partial x_i} > \le 0. \quad (4.2.24)$$

On the other hand, we have that there exists $C > 0$ such that

$$\| |Du_n|^{p-2} |Du_n| \|_q \le C, \text{ for all } n \ge 1.$$

We also have that $(|t|^{p-2}t - |s|^{p-2}s)(t-s) > 0$ for s, $t \in \mathbb{R}$ and $s \ne t$, since the mapping $s \to |s|^{p-2}s$ is strictly increasing. Then for each $x \in \Omega$ with $|Du_n(x)| \ne |Du(x)|$, we have

$$|Du_n(x)|^{p-2}(|Du_n(x)|^2 - \sum_1^N \frac{\partial}{\partial x_i}(u_n) \frac{\partial u}{\partial x_i}) \ge |Du_n|^{p-2}(|Du_n|^2 - |Du_n|\,|Du|)$$

$$> |Du(x)|^{p-2}(|Du_n|\,|Du| - |Du|^2) \ge \sum_1^N < |Du|^{p-2} \frac{\partial u}{\partial x_i}, \frac{\partial}{\partial x_i}(u_n - u) > . \quad (4.2.25)$$

Since u_n converges weakly to u in W, we have

$$\lim \sum_1^N < |Du|^{p-2} \frac{\partial u}{\partial x_i}, \frac{\partial}{\partial x_i}(u_n - u) > = 0.$$

Then combining (4.2.24) and (4.2.25) with the equality above, we find that $|Du_n|$ converges to $|Du|$ *a. e.* on Ω. Suppose that u_n does not converge to u strongly in W. Then we may assume by extracting a subsequence that there exists $\varepsilon > 0$ such that

$$\lim_{n\to\infty} \int_\Omega |Du_n - Du|^p dx = \varepsilon.$$

By using Fatou's lemma, we can see that for each $m > 0$,

$$\lim_{n\to\infty} \int_{\{x:\ |Du_n(x)| \leq m\}} |Du_n - Du|^p dx$$

$$\leq \int \limsup \chi_{\{x:\ |Du_n(x)| \leq m\}} |Du_n - Du|^p dx = 0.$$

Here χ_A denotes the characteristic function of the set A. Then we have

$$\limsup \sum_1^N \int_{\{x:\ |Du_n(x)| \leq m\}} |Du_n|^{p-2} \frac{\partial}{\partial x_i}(u_n) \cdot \frac{\partial}{\partial x_i}(u_n - u) dx$$

$$\leq \limsup \left\{ \int_\Omega \left[|Du_n|^{p-2} |Du_n| \right]^q \right\}^{1/q} \left\{ \int_{\{x:\ |Du_n(x)| \leq m\}} |Du_n - Du|^p dx \right\}^{1/p}$$

$$\leq \limsup C \left\{ \int_{\{x:\ |Du_n(x)| \leq m\}} |Du_n - Du|^p \right\}^{1/p} = 0. \tag{4.2.26}$$

Noting that meas $\{x \in \Omega:\ |Du_n(x)| > m\} \to 0$ as $m \to \infty$, uniformly in n, we find that $\int_{\{x:\ |Du_n(x)| > m\}} |Du|^p dx \to 0$ as $m \to \infty$, uniformly in n. Therefore, we obtain that there exists $m_0 > 0$ such that for each $m \geq m_0$,

$$\liminf \int_{\{x:\ |Du_n(x)| > m\}} |Du_n|^p dx \geq \varepsilon/2 \tag{4.2.27}$$

and

$$\limsup \left\{ \int (|Du_n|^{p-2} |Du_n|)^q dx \right\}^{1/q} \left\{ \int_{\{x:\ |Du_n(x)| > m\}} |Du|^p dx \right\}^{1/p} \leq \varepsilon/4.$$

Here we obtain from (4.2.26), (4.2.27) and the above equality that

$$0 \ge \limsup \sum_1^N < |Du_n|^{p-2} \frac{\partial}{\partial x_i} u_n, \frac{\partial}{\partial x_i}(u_n - u) >$$

$$= \limsup \sum_1^N \int_{\{x:\ |Du_n| > m\}} |Du_n|^{p-2} \frac{\partial}{\partial x_i} u_n \cdot \frac{\partial}{\partial x_i}(u_n - u) dx$$

$$= \limsup \sum_1^N \int_{\{x:\ |Du_n| > m\}} |Du_n|^{p-2} \frac{\partial}{\partial x_i} u_n \cdot \frac{\partial}{\partial x_i} u_n dx - \varepsilon/4$$

$$\ge \int_{\{x:\ |Du_n| > m\}} |Du_n|^p dx - \varepsilon/4 \ge \varepsilon/2.$$

This is a contradiction, and the proof is completed.

Lemma 4.2.9 *For any $\lambda > 0$, there exists $M_1 = M_1(\lambda) > 0$ which is independent of $t \in [0, 1]$ such that if $u \in W$ is a weak solution of the problem*

$$\left.\begin{aligned} \operatorname{div}(|Du|^{p-2}Du) + t\lambda f(u) = 0 \quad & in\ \Omega \\ u = 0 \quad & on\ \partial\Omega \end{aligned}\right\},$$

then, $\|u\|_{1,p} < M_1$.

Proof. From the equation we have

$$\int_\Omega |Du|^p dx = \lambda t \int_\Omega f(u)u\ dx \le \lambda M_2 \int_\Omega u\ dx \le \lambda M_2\ [\mathrm{meas}(\Omega)]^{1/q} \|u\|_p^{1/p}$$

$$\le \lambda M_3 \|u\|_{1,p}^{1/p}.$$

From this we obtain the result.

Lemma 4.2.10 *For any $r > M(\lambda)$,*

$$\deg_S(A - \lambda f, B_r(0), 0) = \deg_S(A - \lambda f, B_\rho(0), 0),$$

where ρ is as in Remark a (p 87) and is independent of $t \in [0, 1]$.

Proof. For any $\lambda > 0$, we consider the mapping $A(t) = A - t\lambda f$, $t \in [0,1]$. It follows from Remark a that $u\equiv 0$ is the unique solution of $A(t)$ in $B_\rho(0)$ (ρ only depending on λ). Therefore, using Lemma 4.2.9, we obtain

$$\deg_S(A-\lambda f,\ B_\rho(0),\ 0)=\deg_S(A,\ B_\rho(0),\ 0),$$

$$\deg_S(A-\lambda f,\ B_r(0),\ 0)=\deg_S(A,\ B_r(0),\ 0).$$

By the fact that $Au\neq 0$ for any $u\in B_r(0)\setminus B_\rho(0)$, we obtain

$$\deg_S(A,\ B_r(0),\ 0)=\deg_S(A,\ B_\rho(0),\ 0).$$

From this we get the result.

We recall the following result from Chapter 1 (Theorem 1.2.8).

Lemma 4.2.11 *Let $Dg : W\to W^*$ be the gradient of a functional g such that Dg is of class $(S)_+$ and Dg maps bounded sets of W to bounded sets of W^*. Suppose that, for some b, the set $V=g^{-1}(-\infty,b)$ is bounded. Moreover, suppose the following condition holds:*

There exist numbers $a<b$ and $r>0$ and an element u_0 of W such that

$$g^{-1}(-\infty,a)\subset B_r(u_0)\subset V$$

and $Dg\,(x)\neq 0$ for all $x\in g^{-1}[a,b]$. Then, $\deg_S(Dg,\ V,\ 0)=1$.

Proof of Theorem 4.2.7

Let

$$g(u)=1/p\int_\Omega |Du|^p dx-\lambda\int_\Omega\int_0^{u(x)} f(s)dsdx. \tag{4.2.28}$$

First, we prove that

$$\inf\{g(u):\ u\in W\}<0. \tag{4.2.29}$$

Let ϕ be the eigenfunction corresponding to the first eigenvalue $\mu_1>0$ of

$$\left.\begin{aligned} \operatorname{div}(|Du|^{p-2}Du)+\mu|u|^{p-2}u=0 &\quad \text{in } \Omega\\ u=0 &\quad \text{on } \partial\Omega \end{aligned}\right\}$$

with $\|\phi\|_p=1$. From [65], we know that $\phi\in C^1(\overline{\Omega})$ and $\phi>0$ in Ω. Let $\gamma=\max_{[0,\beta]}\{f(u)\}$ and $\beta_1\in(0,\beta)$ be the unique maximum point of f. Then $f(s)\geq(\gamma/\beta_1)s$ for $s\in(0,\beta_1)$ as $f''(x)<0$ on $(0,\beta)$. Let $\varepsilon>0$ be so small that $\varepsilon\phi\leq\beta_1$ for $x\in\Omega$. Then,

$$g(\varepsilon\phi)=(1/p)\varepsilon^p\mu_1-\lambda\int_\Omega\int_0^{\varepsilon\phi}f(s)dsdx\leq(1/p)\varepsilon^p\mu_1-(\gamma/2\beta_1)\varepsilon^2\lambda\int_\Omega\phi^2dx<0, \tag{4.2.30}$$

for $\lambda>\underline{\lambda}>0$. Here we use the fact that $p>2$.

Since $g(u)\to\infty$ as $\|u\|_p\to\infty$, we find that there exists $u_0\in\mathrm{W}$ ($u_0\neq0$) such that $g(u_0)=\inf\{g(u)\colon u\in\mathrm{W}\}$. It is obvious that u_0 is a solution of (P). If g attains its minimum at a point $v\neq u_0$, v is also a solution of (P) and therefore the assertion of Theorem 4.2.7 holds. Now suppose that u_0 is the unique minimum point of g. Then, using the inequality

$$< Au-\lambda f(u), u > \geq C\|u\|_p^p \ , \text{ for all } u\in\mathrm{W},$$

with $\|u\|_p\geq r_0$, $r_0>0$ and $C>0$, we can choose a, $b<0$, $a<b$, such that $\mathrm{V}=\{u\in\mathrm{W} : g(u)<b\}$ is bounded and

$$g^{-1}(-\infty,a)\subset\overline{\mathrm{B}}_r(u_0)\subset\mathrm{V} \quad \text{for some } r>0.$$

If the set $g^{-1}[a,b]$ contains a point u with $Dg(u)=0$, then u is a nontrivial solution of (P) different from u_0.

On the other hand, if

$$Dg(u)\neq0 \quad \text{for all } u\in g^{-1}[a,b],$$

then by Lemma 4.2.11, we have $\deg_S(A-\lambda f,\mathrm{V},0)=1$.

Here we choose positive numbers $r\geq M(\lambda)$ and $s\leq\rho$ such that

$$\overline{\mathrm{B}}_s(0)\cap\overline{\mathrm{V}}=\emptyset,\ \overline{\mathrm{V}}\subset\overline{\mathrm{B}}_r(0),$$

where $M(\lambda)$ and ρ are as in Lemma 4.2.10. Then we have from Lemma 4.2.10 and Lemma 4.2.11 that

$$\deg_S(A-\lambda f,\mathrm{B}_r(0)\setminus(\overline{\mathrm{B}}_s(0)\cup\mathrm{V}))$$

$$=\deg_S(A-\lambda f, B_r(0),0) - \deg_S(A-\lambda f, \overline{B}_s(0),0) - \deg_S(A-\lambda f,V,0)$$

$$=-1\neq 0.$$

Thus we obtain that there exists $u\in B_r(0)\setminus(\overline{B}_s(0)\cup V)$ such that $Au-\lambda f(u)=0$. This completes the proof.

The proof as above shows that there exist two nontrivial solutions in W. Using the same ideas as after 4.1.14, we know that the solutions belong to $C^1(\Omega)$. The Maximum principle implies the solutions are positive.

4.3 A NECESSARY AND SUFFICIENT CONDITION

In this section we give a necessary and sufficient condition for the existence of a solution of the problem (P) for $\lambda>(p-1)/p$ large enough.

Theorem 4.3.1 *Let $1<p\leq 2$, $f\in C^1(\mathbb{R})$ satisfy*

(F_3) *f is bounded on $\mathbb{R}$; there are two numbers β_1 and β_2 such that $0<\beta_1<\beta_2$,*

$$f(\beta_1)=f(\beta_2)=0,\ f>0 \text{ in } (\beta_1,\beta_2) \text{ and } f<0 \text{ in } (\beta_2,\infty).$$

Then for $\lambda>(p-1)/p$ sufficiently large, problem (P) *possesses a solution* (λ, u) *with $u\in C_0^1(\overline{\Omega})$*, $\max u\in(\beta_1,\beta_2)$ *if and only if*

$$J(s)=\int_s^{\beta_2} f(t)dt>0 \quad \text{for every } s\in[0,\beta_1). \tag{4.3.1}$$

To prove this theorem we first prove the following lemma, a generalization of Serrin's Sweeping Principle.

Lemma 4.3.2 *Let $\{v_t\in C^1(\overline{\Omega}),\ t\in[0,1]\}$ be a family of functions satisfying $v_t>0$ on $\partial\Omega$ for all $t\in[0,1]$, and for some $c>0$,*

$$-\mathrm{div}(|Dv_t|^{p-2}Dv_t)\geq\lambda(f(v_t)+c)\quad in\ \Omega,\ for\ all\ t\in[0,1].$$

Let $u\in C_0^1(\overline{\Omega})$ *be a solution of* (P) *with* f *satisfying* (F_3). *If*

(1) $t\to v_t\in C^0(\overline{\Omega})$ *is continuous with respect to the* $\|\cdot\|_0$*-norm,*

(2) $u\leq v_0$ *in* $\overline{\Omega}$, *and*

(3) $u\not\equiv v_t$, *for all* $t\in[0,1]$.

Then for all $t\in[0,1]$, $u<v_t$ *in* $\overline{\Omega}$.

Proof. Set $E=\{t\in[0,1]\ :\ u\leq v_t\ \text{in}\ \overline{\Omega}\}$. By (2) E is not empty and moreover E is closed. For $t\in E$, v_t satisfies

$$-\mathrm{div}(|Dv_t|^{p-2}Dv_t)+\lambda Mv_t\geq\lambda g(v_t)+\lambda c>\lambda g(u)=-\mathrm{div}(|Du|^{p-2}Du)+\lambda Mu$$

in Ω, where $g(s)=f(s)+Ms$ is a strictly increasing function. Since $u<v_t$ on $\partial\Omega$, it follows that there exists a one-side neighbourhood G of $\partial\Omega$ such that $G\subset\Omega$ and $u<v_t$ in G. Let $\Omega_1\subset\Omega$ be a subdomain of Ω with $\partial\Omega_1\subset G$. We show that $u<v_t$ in Ω_1. In fact, $u<v_t$ on $\partial\Omega_1$, so, there exists τ, $0<\tau<c/M$ such that $u+\tau<v_t$ on $\partial\Omega_1$. Let $w=u+\tau$. Then, w satisfies

$$-\mathrm{div}(|Dw|^{p-2}Dw)+\lambda Mw=\lambda g(u)+\lambda M\tau\leq\lambda g(v_t)+\lambda c\leq-\mathrm{div}(|Dv_t|^{p-2}Dv_t)+\lambda Mv_t$$

in Ω_1. By Lemma 4.1.1, we obtain $w\leq v_t$ in Ω_1. This shows $u<v_t$ in $\overline{\Omega}$. The continuity of u and v_t in Ω and the continuity of $t\to v_t$ imply that E is also open. So, $E=[0,1]$. This proves the lemma.

Proof of Theorem 4.3.1

NECESSITY: Write $J(s)=\int_s^{\beta_2}f(t)dt$, and define

$$J^*:=\min\{J(s)\ :\ s\in[0,\beta_1]\},\quad J^*=\int_{s*}^{\beta_2}f(t)dt,\ s*\in[0,\beta_1].$$

Suppose condition (4.3.1) is not satisfied, that is, $J^*\leq 0$. Let (λ,u) be a solution of (P) satisfying $\lambda > (p-1)/p$ large enough, $u \in C_0^1(\overline{\Omega})$ and $\max u \in (\beta_1, \beta_2)$, we will obtain a contradiction.

First, if $J^*=0$, modify f to f^* in C^1 such that $f > f^* > 0$ in $(\max u,\ \beta_2)$ and $f = f^*$ elsewhere. Still u is a solution of (P), but now $J^* < 0$. Hence, we assume without loss of generality that $J^* < 0$.

Consider the initial value problem

$$\left.\begin{aligned} (|v'|^{p-2}v')' &= \lambda(f(v)+k) := \lambda h(v) \quad \text{for } t>0 \\ v(0) &= \beta_2,\ v'(0) = -(-J^*)^{1/p} \end{aligned}\right\}. \tag{4.3.2}$$

Here k satisfies $0 < k < (-J^*)(\lambda-(p-1)/p)/(\lambda\beta_2)$. Since $f < 0$ for $s > \beta_2$, we have that there is $\beta_3 > \beta_2$ such that $h(\beta_3)=0$, $h(s)<0$ for $s>\beta_3$ and $\beta_3 - \beta_2$ is small enough when k is small enough. Let $w = |v'|^{p-2}v'$, then $v' = \phi(w)$ and $\phi \in C^1(\mathbb{R}\backslash\{0\})$. Now,Problem (4.3.2) is equivalent to

$$\left\{\begin{aligned} w' &= \lambda h(v), \\ v' &= \phi(w), \end{aligned}\right. \quad (w,v)(0) = (-(-J^*)^{(p-1)/p},\ \beta_2). \tag{4.3.3}$$

So, Problem (4.3.3) has a unique solution in C^1 for $w \neq 0$. Therefore, for a solution of (4.3.2), there are two possibilities:

(*i*) $v' \neq 0$ for all $t > 0$.

(*ii*) there exists t_0 such that $v'(t_0)=0$ and $v'(t) \neq 0$ for $0 < t < t_0$.

For case (*i*), one has

$$(1-1/p)|v'(t)|^p = \lambda\int_{v(t)}^{\beta_2} h(s)ds - (1-1/p)J^* \tag{4.3.4}$$

for all $t > 0$. For case (*ii*), one has (4.3.4) for $0 < t < t_0$. Next, we only give some properties of v in case (*ii*). Using the same idea, we know that v has similar properties in case (*i*).

Property (1) $v(t_0) < \beta_1$.

Suppose not, $v(t_0) \geq \beta_1$. Then $v(t) \geq \beta_1$ in $(0, t_0)$. From (4.3.4) we know

$$|v'(t)| > (-J^*)^{1/p} \text{ for } t\in(0,t_0). \tag{4.3.5}$$

This is impossible.

Property (2) $v > s*$ in $(0, t_0)$.

Suppose not, there exists a $t^*\in(0, t_0)$ such that $v(t^*)=s*$ and since (4.3.4) holds, we obtain

$$(1-1/p)|v'(t^*)|^p=(\lambda-(p-1)/p)J^*+\lambda k(\beta_2-s*)<0. \tag{4.3.6}$$

This is impossible.

So, either $v(t)\downarrow s_0\in(s*, \beta_1)$ as $t\to\infty$, or, v has a first positive minimum t_0. In the first case define

$$\bar{v}(t)=\begin{cases} v_1(t_1), & \text{for } t\le t_1, \\ v_1(t), & \text{for } t_1 \le t< 0, \\ v(t), & \text{for } t\ge 0, \end{cases} \tag{4.3.7}$$

where $v_1(t)=\beta_2+p_1(t)$, $p_1(t)=[(p-1)/(\lambda kp)]\left\{(-J^*)-\left[\lambda kt+(-J^*)^{(p-1)/p}\right]^{p/(p-1)}\right\}$

and $t_1=-\lambda k/(-J^*)^{(p-1)/p}$, $v_1(t_1)=\beta_2+[(p-1)/(\lambda kp)](-J^*)$. A direct calculation shows that $v_1>\beta_2$ for $t_1\le t<0$. The structures of $v_1(t)$ and $v(t)$ guarantee that $\bar{v}'(t)$ is continuous at t_1 and 0, which implies $\bar{v}\in C^1(\mathbb{R})$. Furthermore, $\bar{v}(t)$ satisfies

$$-(|\bar{v}'|^{p-2}\bar{v}')'\ge\lambda(f(\bar{v})+k) \text{ for all } t\in\mathbb{R} \text{ and some } k\in(0, [(-J^*)(\lambda-(p-1)/p)/\lambda\beta_2]),$$

since $f(\bar{v})<0$ for $t\in(t_1,0)$ and $f(\bar{v})+k<0$ for $t\le t_1$ when k is small enough. In the second case, let

$$\bar{v}(t)=\begin{cases} v_1(t_1) & \text{for } t \le t_1, \\ v_1(t) & \text{for } t\in(t_1, 0), \\ v(t) & \text{for } t\in[0, t_0), \\ \lim_{t\to t_0} v(t) & \text{for } t=t_0, \\ v(2t_0-t) & \text{for } t\in(t_0, 2t_0], \\ v_2(t) & \text{for } t\in(2t_0, t_2), \\ v_2(t_2) & \text{for } t\ge t_2. \end{cases}$$

Here v_1 and t_1 are as above, $v_2=\beta_2+p_2(t)$, $t_2=2t_0+(1/\lambda k)(-J^*)^{(p-1)/p}$ and $p_2(t)=(p-1)/(\lambda kp)\left\{(-J^*)-\left[(-J^*)^{(p-1)/p}+2\lambda kt_0-\lambda kt\right]^{p/(p-1)}\right\}$. A direct calculation shows that $v_2(t)\ge\beta_2$ for $t\in(2t_0, t_2]$. Furthermore, $v_2(t)$ satisfies

$$-(|v_2'|^{p-2}v_2')'\ge\lambda(f(v_2)+k) \quad \text{in } (2t_0, \infty).$$

Therefore, $\bar{v}\in C^1(\mathbb{R})$, $(|\bar{v}'|^{p-2}\bar{v}')'\ge\lambda(f(\bar{v})+k)$ for all $t\in\mathbb{R}$ and some possibly different k, as above.

Set $y(t, x)=\bar{v}(x_1-t)$, where $x=(x_1, x_2, \cdots, x_N)$. Then, $\{y(t,x)\in C^1(\bar{\Omega}), t\in\mathbb{R}\}$ is a family of supersolutions of

$$\left.\begin{aligned} -\mathrm{div}(|Du|^{p-2}Du)=\lambda f(u)+\lambda k \quad & \text{in } \Omega \\ u=0 \quad & \text{on } \partial\Omega \end{aligned}\right\} \tag{4.3.8}$$

and for all t large enough $y(t,\cdot)>\beta_2$ in Ω. From Lemma 4.3.2 and the structure of $\bar{v}(t)$, we obtain

$$u(x)<y(t,x) \quad \text{in } \Omega \text{ for all } t. \tag{4.3.9}$$

Hence,

$$u(x)\le\inf\{y(t,x) : t\in\mathbb{R}\}=\inf \bar{v}<\beta_1. \tag{4.3.10}$$

This is a contradiction.

SUFFICIENCY: Note that f is bounded on $\mathbb{R}$. As above we want to minimize

$$I(u,\lambda)=1/p\int_\Omega |Du|^p - \lambda\int_\Omega F(u), \quad \text{in } W_0^{1,p}(\Omega),$$

where $F(u)=\int_0^u f(s)ds$.

For $\lambda>0$, $I(u,\lambda)$ is bounded below. Since $I(\cdot,\lambda)$ is sequentially weakly lower semicontinuous and convex in $W_0^{1,p}(\Omega)$, $I(\cdot,\lambda)$ possesses a minimizer, which we denote by u_λ. As in Lemma 4.2.3, we can prove that $u_\lambda\in C_0^1(\overline{\Omega})$.

Suppose $\max u_\lambda>\beta_2$, then there is a subdomain $\Omega_1\subset\Omega$ such that $u_\lambda>\beta_2$ in Ω_1 and $u_\lambda=\beta_2$ on $\partial\Omega_1$. So,

$$\left.\begin{aligned} -\mathrm{div}(|D(u_\lambda-\beta_2)|^{p-2}D(u_\lambda-\beta_2))=\lambda f(u_\lambda)<0 \quad &\text{in } \Omega_1 \\ u_\lambda-\beta_2=0 \quad &\text{on } \partial\Omega_1 \end{aligned}\right\}. \tag{4.3.11}$$

From (4.3.11) we obtain $u_\lambda\equiv\beta_2$ in Ω_1. This contradicts $\max u_\lambda>\beta_2$. Therefore, $\max u_\lambda\le\beta_2$. Note that Proposition 4.1.4 is still true for u_λ, then, $\max u_\lambda<\beta_2$.

Now we prove that $\max u_\lambda>\beta_1$.

Set

$$\alpha=\min\left\{\int_s^{\beta_2} f(t)dt;\ 0\le s\le \beta_1\right\},$$

$$\beta=\max\left\{\int_s^{\beta_2} f(t)dt;\ 0\le s\le\beta_2\right\}.$$

Suppose that for λ large enough, $\|u_\lambda\|_\infty\le\beta_1$, then we will obtain a contradiction.

We choose $\delta>0$ such that $2|\Omega_\delta|\ \beta<|\Omega|\ \alpha$, with Ω_δ as in Lemma 4.2.3 and $|\Omega|$ denoting the Lebesgue-measure of Ω. This is possible since $\alpha>0$ and $\lim_{\delta\to 0}|\Omega_\delta|=0$.

Next we choose $w\in C_0^\infty(\Omega)$, satisfying $0\le w\le\beta_2$ in Ω_δ and $w=\beta_2$ in $\Omega\setminus\Omega_\delta$; then

$I(w,\lambda)-I(u_\lambda,\lambda)$

$$=(1/p)\int_{\Omega}(|Dw|^p-|Du_\lambda|^p)dx-\lambda\int_{\Omega}(F(w)-F(u_\lambda))dx$$

$$\le(1/p)\int_{\Omega}|Dw|^p dx-\lambda\left[\int_{\Omega}F(\beta_2)dx+\int_{\Omega_\delta}(F(w)-F(\beta_2))dx-\int_{\Omega}F(u_\lambda)dx\right]$$

$$\le(1/p)\int_{\Omega}|Dw|^p dx+2\lambda|\Omega_\delta|\,\beta-\lambda\int_{\Omega}(F(\beta_2)-F(u_\lambda))dx$$

$$=(1/p)\int_{\Omega}|Dw|^p dx+2\lambda|\Omega_\delta|\,\beta-\lambda\int_{\Omega}\int_{u_\lambda}^{\beta_2}f(s)ds$$

$$\le(1/p)\int_{\Omega}|Dw|^p dx+\lambda(2|\Omega_\delta|\,\beta-|\Omega|\,\alpha)<0,$$

for λ large enough, since $2|\Omega_\delta|\,\beta-|\Omega|\,\alpha<0$.

Then $I(w,\lambda)<I(u_\lambda,\lambda)$, contradicting the fact that u_λ is a minimizer. This completes the proof of the sufficiency.

Remark b From the proof above, we know that (4.3.1) is a necessary condition for the existence of a solution u_λ with $\max u_\lambda\in(\beta_1,\beta_2)$ for all $p>1$ and λ large enough.

CHAPTER FIVE

EXISTENCE AND UNIQUENESS OF POSITIVE RADIAL SOLUTIONS FOR A CLASS OF DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS

INTRODUCTION

In this chapter we discuss the following problem

$$\left.\begin{aligned} \operatorname{div}(|Du|^{p-2}Du)+\lambda f(u)=0 \quad & x\in\Omega \\ u=0 \quad & x\in\partial\Omega \end{aligned}\right\}, \tag{5.0.1}$$

where $f(u)>0$ for $u>0$, $\lambda>0$, $p>1$ and $\Omega=B_R=\{x\in\mathbb{R}^N;\ \|x\|<R\}$.

Problem (5.0.1) has been treated by many authors when $p=2$, see, for example, [1, 14, 47–48]. One of the major stumbling blocks in the case that $p\neq 2$ is the fact that certain "nice" features inherent to semilinear problems seem to be lost or at least difficult to verify. For example, when $p\neq 2$, we do not know whether Serrin's sweeping principle is true or not in a general domain Ω, but it plays an important role in the proof of the existence of solutions of problem (5.0.1) when $p=2$.

When $p>1$, problem (5.0.1) has been studied in chapter 4. Under the hypotheses:

(F_1) f is strictly increasing on $\mathbb{R}^+$, $f(0)=0$ and $\lim_{s\to 0} f(s)/s^{p-1}=0$; there exist $\alpha_1,\ \alpha_2>0$ such that $f(s)\le\alpha_1+\alpha_2 s^{\sigma}$, $0<\sigma<p-1$, we showed that, when λ is large enough, there exist at least two positive solutions for (5.0.1) on an

arbitrary bounded domain $\Omega\subset\mathbb{R}^N$, $N\geq 2$.

In this chapter, we are interested in the case that f satisfies

$$(F_1')\quad f\in C^1(0,\infty)\cap C^\gamma([0,\infty)) \text{ is non-decreasing on } \mathbb{R}^+,\ 0<\gamma\leq 1,\ f(0)=0;$$

$$\lim_{s\to\infty} f(s)/s^\beta=1,\ 0<\beta<p-1;\ \lim_{s\to 0} f(s)/s^{p-1}=\infty \text{ and } (f(s)/s^{p-1})'\leq 0, \text{ for } s>0.$$

We shall study the existence and uniqueness of solutions of (5.0.1) when f satisfies (F_1'). By constructing some appropriate super- and sub-solutions and applying a generalized Serrin's sweeping principle, we show that for λ large enough, the positive radial solution of (5.0.1) exists and is unique. Meanwhile, we prove that when f satisfies (F_1'), nonradial positive solutions of (5.0.1) do not exist. That is, all the positive solutions of (5.0.1) are radial solutions. For $p=2$, this is a well-known result (see [28]), but when $p>2$ and f satisfies some other conditions, there do exist nonradial positive solutions to problem (5.0.1) (see [41]). We still do not know whether the above results are true or not for general Ω. In this case, it seems difficult to give the generalized Serrin's sweeping principle.

By a solution u of (5.0.1) we mean $u\in C_0^1(\Omega)$ which satisfies

$$\int_\Omega |Du|^{p-2}Du\ Dv = \lambda\int_\Omega f(u)v$$

for any $v\in C_0^\infty(\Omega)$.

We say that u is a subsolution (supersolution) of problem (5.0.1) if $u\in C^1(\Omega)$, $u(\partial\Omega)\leq 0$ (≥ 0) and satisfies

$$\int_\Omega \left[\,|Du|^{p-2}Du\ D\psi - \lambda f(u)\psi\right]\leq 0\ (\geq 0),$$

for every $\psi\in D^+(\Omega)$, where $D^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$.

In section 5.1, we give existence and uniqueness results for some special equations. We also give a generalized Serrin's sweeping principle to problem (5.0.1) in the radially symmetric case. In section 5.2, we construct some super– and sub–solutions to problem (5.0.1). Using degree theory, we show that positive radial solutions of problem (5.0.1) with f satisfying (F_1') exist. In section 5.3, we establish the asymptotic behaviour of the solutions of problem (5.0.1) when λ is large enough. In section 5.4, we prove that the solution is unique. In section 5.5, we show that all the positive solutions of problem (5.0.1) are radial solutions when λ is large enough.

5.1 GENERALIZED SERRIN'S SWEEPING PRINCIPLE

In this section, we obtain some basic results which will be useful in the coming proofs.

Lemma 5.1.1 *For $p>1$, the problem*

$$\left.\begin{aligned} -\operatorname{div}(|Dv|^{p-2}Dv)&=1 \quad \text{for } x\in B_R \\ v&=0 \quad \text{for } x\in \partial B_R \end{aligned}\right\} \tag{5.1.1}$$

has a unique solution $v_0(r)\in C^1([0,R])$,

$$v_0(r)=\left(\frac{1}{N}\right)^{1/(p-1)}\frac{(p-1)}{p}\left(R^{p/(p-1)}-r^{p/(p-1)}\right),$$

where $r=|x|$.

Proof. The uniqueness of the solution of (5.1.1) has been proved in [65]. We can easily check that $v_0(r)$ is a solution of (5.1.1).

Lemma 5.1.2 *Let $p>1$ and $\rho(x)=\rho(r)\in C^0(0,R)$, $\rho\geq 0$, $\rho(r)\not\equiv 0$. Then the problem*

$$\left.\begin{aligned} -\operatorname{div}(|Du|^{p-2}Du)&=\rho(x) && \text{for } x\in B_R \\ u&=0 && \text{for } x\in\partial B_R \end{aligned}\right\} \tag{5.1.2}$$

has a unique solution $u\in C^1([0,R])$,

$$u(r)=\int_r^R\left[s^{1-N}\int_0^s t^{N-1}\rho(t)dt\right]^{1/(p-1)}ds, \tag{5.1.3}$$

which is such that there exist constants $l>k>0$ with

$$k(R-r)<u(r)<l(R-r). \tag{5.1.4}$$

For $\delta>0$, define the function ρ_δ by

$$\rho_\delta(r)=\begin{cases} 1 & \text{for } R-\delta\leq r<R \\ 0 & \text{for } 0<r<R-\delta \end{cases}.$$

Then the solution $u_\delta(r)$ of (5.1.2) with $\rho(r)=\rho_\delta(r)$ satisfies

$$u_\delta(r)\leq C_p\ \delta^{1/(p-1)}(R-r),\ \text{for } 0<r<R, \tag{5.1.5}$$

where C_p is a positive constant dependent of p and R.

Proof. The uniqueness of solutions of (5.1.2) can be obtained from [65]. But $u(r)$ is a radial solution of (5.1.2), so, $u(r)$ is the unique solution of (5.1.2).

Now we prove that $u(r)$ in (5.1.3) satisfies (5.1.4), (5.1.5).

Since $\rho(r)\geq 0$ and $\rho\not\equiv 0$, by (5.1.3), $u\in C_0^1([0,R])$, $u\geq 0$, $u\not\equiv 0$. Using Lemmas 4.1.2 and 4.1.3, we have that $u(r)>0$ for $0\leq r<R$ and $\frac{\partial u}{\partial r}(R)<0$. Since ∂B_R is compact, there exists $\delta'>0$ and $l'\geq k'>0$ such that

$$-l'\leq\frac{\partial u}{\partial r}\leq -k'<0$$

for any $r\in[R-\delta', R]$. By writing

$$u(r)=-(\mathrm{R}-r)\int_0^1 \frac{\partial u}{\partial \tau}(r+\tau(\mathrm{R}-r))d\tau$$

for any $r\in[\mathrm{R}-\delta',\mathrm{R}]$ and choosing $l\geq k>0$ appropriately for $r\in[0,\mathrm{R}-\delta')$, we can obtain (5.1.4). It is clear that k and l depend on u and ρ.

From $u_\delta(r)=\int_r^{\mathrm{R}}\left[s^{1-\mathrm{N}}\int_0^s t^{\mathrm{N}-1}\rho_\delta(t)dt\right]^{1/(p-1)} ds,$

for $0<r\leq\mathrm{R}-\delta$, we obtain

$$u_\delta(r)=\int_{\mathrm{R}-\delta}^{\mathrm{R}}\left[s^{1-\mathrm{N}}\int_{\mathrm{R}-\delta}^{s} t^{\mathrm{N}-1}dt\right]^{1/(p-1)} ds$$

$$\leq\left(\frac{1}{\mathrm{N}}\right)^{1/(p-1)}\int_{\mathrm{R}-\delta}^{\mathrm{R}} s^{1/(p-1)}(\mathrm{R}^{\mathrm{N}}-(\mathrm{R}-\delta)^{\mathrm{N}})^{1/(p-1)}ds$$

$$\leq\mathrm{R}^{\mathrm{N}/(p-1)}\delta^{1/(p-1)}\delta\leq\mathrm{R}^{\mathrm{N}/(p-1)}\delta^{1/(p-1)}(\mathrm{R}-r).$$

For $\mathrm{R}-\delta\leq r\leq\mathrm{R}$, we can obtain the same estimate.

Lemma 5.1.3 **(Generalized Serrin's sweeping principle)** *Suppose $f\in C^1(0,\infty)$, $f>0$ for $s>0$ and f is a nondecreasing function. Let $\{\phi_\tau : 0\leq\tau\leq1\}$ be a family of positive radial subsolutions of (5.0.1), ϕ_τ satisfies*

(*a*) $\mathrm{div}(|D\phi_\tau|^{p-2}D\phi_\tau)+\lambda f(\phi_\tau)\geq0$ *for* $x\in\mathrm{B}_\mathrm{R}$,

(*b*) $\phi_\tau=0$ *for* $x\in\partial\mathrm{B}_\mathrm{R}$,

(*c*) $\dfrac{\partial\phi_\tau}{\partial r}<0$ *for* $\tau\in[0,1]$ *and* $0<r\leq\mathrm{R}$,

(*d*) *for* $\tau\in[0,1]$, ϕ_τ *is not a solution of the problem*

$$\left.\begin{aligned}\mathrm{div}(|Du|^{p-2}Du)+\lambda f(u)=0 \quad &for\quad x\in\mathrm{B}_\mathrm{R}\\ u=0\quad &for\quad x\in\partial\mathrm{B}_\mathrm{R}\end{aligned}\right\}, \tag{5.1.6}$$

(*e*) $\tau \to \phi_\tau \in C^1([0,\mathrm{R}])$ *is continuous with respect to the* $\|\cdot\|_0$*-norm.*

Suppose that $u \in C^1([0,\mathrm{R}])$ *is a positive radial solution of* (5.1.6) *with*

$$u \geq \phi_1 \quad on\ [0,\mathrm{R}]. \tag{5.1.7}$$

Then $u > \phi_0(r)$ *for* $r \in [0,\mathrm{R})$.

Proof. Set $E = \{\tau \in [0,1] : u \geq \phi_\tau \text{ in } \overline{\mathrm{B}}_\mathrm{R}\}$. By (5.1.7), E is not empty. Moreover, E is closed by (*e*). For $\tau \in E$, ϕ_τ satisfies

$$\mathrm{div}(|D\phi_\tau|^{p-2}D\phi_\tau) + \lambda f(\phi_\tau) \geq 0 \quad \text{in } \mathrm{B}_\mathrm{R};$$

u satisfies

$$\mathrm{div}(|Du|^{p-2}Du) + \lambda f(u) = 0 \quad \text{in } \mathrm{B}_\mathrm{R}.$$

Since u is a positive radial solution, u satisfies

$$(r^{\mathrm{N}-1}|u_r|^{p-2}u_r)_r + \lambda r^{\mathrm{N}-1} f(u) = 0, \tag{5.1.8}$$

and $u'(0) = 0$. Integrating (5.1.8) from 0 to r, we obtain

$$u'(r) < 0, \text{ for } r \in (0,\mathrm{R}]. \tag{5.1.9}$$

Therefore,

$$-\mathrm{div}(|Du|^{p-2}Du) - \{-\mathrm{div}(|D\phi_\tau|^{p-2}D\phi_\tau\} \geq \lambda(f(u) - f(\phi_\tau)) \geq 0. \tag{5.1.10}$$

Here we use the fact that f is non-decreasing.

In view of (5.1.9), (*c*) and the continuity of the derivatives $\frac{\partial u}{\partial r}$ and $\frac{\partial \phi_\tau}{\partial r}$, we get

$$\frac{\partial u}{\partial r} < -\mu \quad \text{and} \quad \frac{\partial \phi_\tau}{\partial r} < -\mu \quad \text{on } \overline{G}, \tag{5.1.11}$$

where μ is a positive constant and G is an open one-side connected neighbourhood of $\partial\mathrm{B}_\mathrm{R}$ in B_R (since $\partial\mathrm{B}_\mathrm{R}$ is connected, we can choose G to be connected). Furthermore we have from (5.1.11)

$$t\frac{\partial u}{\partial r} + (1-t)\frac{\partial \phi_\tau}{\partial r} \leq -\mu \quad \text{on } G, \tag{5.1.12}$$

for all numbers t $(0 \leq t \leq 1)$. Thus we obtain from (5.1.10) and the mean value theorem

$$0\leq -\mathrm{div}(|Du|^{p-2}Du)-\{-\mathrm{div}(|D\phi_\tau|^{p-2}D\phi_\tau\}=-\sum_{i,j}\frac{\partial}{\partial x_i}[a^{ij}(x)\frac{\partial}{\partial x_j}(u-\phi_\tau)] \text{ in } G.$$

Where $a^{ij}(x)=\int_0^1\frac{\partial a^i}{\partial q_j}[tDu+(1-t)D\phi_\tau]dt$ and $a^i(q)=|q|^{p-2}q_i$ (i=1, 2,$\cdots$, N) for $q=(q_1, q_2,\cdots, q_N)\in\mathbb{R}^N$. Put $L\cdot=\sum_{i,j}\frac{\partial}{\partial x_i}[a^{ij}(x)\frac{\partial}{\partial x_j}\cdot]$, from (5.1.12) we see that L is a uniformly elliptic operator on G. Consequently, we have

$$-L(u-\phi_\tau)\geq 0 \text{ in } G, \tag{5.1.13}$$

$$u\geq\phi_\tau \text{ in } G \text{ and } u-\phi_\tau=0 \text{ on } \partial B_R. \tag{5.1.14}$$

Now we prove that

$$u>\phi_\tau \text{ in } B_R.$$

Suppose there exists $z_1\in B_R$ such that $u(z_1)=\phi_\tau(z_1)$. Then we will show that there exists $z\in G$ such that $u(z)=\phi_\tau(z)$. Suppose $u>\phi_\tau$ in G, then there exists a ball $B_{R'}$, $\partial B_{R'}\subset G$ such that $z_1\in B_{R'}$ and $u\geq\phi_\tau+\beta$ on $\partial B_{R'}$, here $\beta>0$. Let $w=\phi_\tau+\beta$, then

$$-\mathrm{div}(|Du|^{p-2}Du)-\{-\mathrm{div}(|Dw|^{p-2}Dw\}\geq\lambda(f(u)-f(\phi_\tau))\geq 0 \text{ in } B_{R'},$$

$$u(\partial B_{R'})\geq w(\partial B_{R'}).$$

The weak comparison principle (see Lemma 4.1.1 of Chapter 4) implies that $u\geq w$ in $B_{R'}$, that is, $u\geq\phi_\tau+\beta$ in $B_{R'}$. This contradicts $u(z_1)=\phi_\tau(z_1)$.

Let $z\in G$ be such that $u(z)=\phi_\tau(z)$. By (5.1.14), $u-\phi_\tau$ attains its minimum in G. It follows from (5.1.13) and the maximum principle of uniformly elliptic operator that $u\equiv\phi_\tau$ in G. Since $u'(r)<0$, $\phi_\tau'(r)<0$, we can repeat the above steps and obtain $u\equiv\phi_\tau$ in [0,R]. This contradicts the assumption (*d*).

By Hopf's boundary point lemma for uniformly elliptic operators, we also obtain

$$\frac{\partial}{\partial r}(u-\phi_\tau)<0 \quad \text{on } \partial B_R. \tag{5.1.15}$$

As in Lemma 5.1.2, there exists $\theta>0$ such that $u-\phi_\tau \geq \theta(\mathrm{R}-r)$, for $r\in[0,\mathrm{R}]$. This and (*e*) imply that E is open. Therefore, $E=[0,1]$ and $u>\phi_0(r)$ on $[0,\mathrm{R})$.

Remark 5.1.4 The generalized Serrin's sweeping principle remains true for a family of positive radial supersolutions of problem (5.0.1).

5.2 EXISTENCE RESULTS

In this section we shall show that when λ is large enough, problem (5.0.1) has at least one positive radial solution.

Consider the eigenvalue problem

$$\left.\begin{aligned} \operatorname{div}(|Du|^{p-2}Du)+\lambda|u|^{p-2}u=0 \quad &\text{for} \quad x\in \mathrm{B}_\mathrm{R} \\ u=0 \quad &\text{for} \quad x\in\partial \mathrm{B}_\mathrm{R} \end{aligned}\right\}. \tag{5.2.1}$$

Thelin [72] showed that there is a unique positive radial eigenfunction ψ of norm 1 to (5.2.1) corresponding to the smallest eigenvalue λ_1. We also know $\psi\in C^1([0,\mathrm{R}])$.

Let $\lambda_1(>0)$ be the least eigenvalue and $\psi=\psi(r)>0$ be the corresponding eigenfunction with normalization $\|\psi\|_\infty=1$. Then, we easily obtain

$$\frac{\partial\psi}{\partial r}<0, \text{ for } r\in(0,\mathrm{R}].$$

We first prove the following lemma.

Lemma 5.2.1 *Suppose f satisfies*

(*a*) $f\in C^1((0,\infty))\cap C^\gamma([0,\infty))$ *for some* $\gamma\in(0,1]$,

(*b*) $f(u)>0$, *for* $u>0$,

(c) $f(u) > \sigma u^{p-1}$ *in* $(0,\mu]$, *for some* σ, $\mu > 0$.

Then the positive radial solution u_λ *of* (5.0.1) *satisfies*

$$u_\lambda \geq \mu\psi \quad in\ [0,R],\ if\ \lambda \geq \lambda_1/\sigma.$$

Proof. For $\tau \in (0,\mu]$ and $\lambda \geq \lambda_1/\sigma$, we have

$$\operatorname{div}(|D(\tau\psi)|^{p-2}D(\tau\psi)) + \lambda f(\tau\psi) > -\tau^{p-1}\lambda_1|\psi|^{p-2}\psi + \lambda\sigma\tau^{p-1}\psi^{p-1}$$

$$= \tau^{p-1}\psi^{p-1}(\lambda\sigma - \lambda_1) \geq 0.$$

Therefore, $\{\tau\psi,\ \tau \in (0,\mu]\}$ is a family of subsolutions of (5.0.1). It follows from Lemma 5.1.3 that $u_\lambda \geq \mu\psi$.

<u>Theorem 5.2.2</u> *Suppose f satisfies (a), (b), (c) in Lemma* 5.2.1 *and*

(d) $\lim_{s \to \infty} f(s)/s^\beta = 1,\ 0 < \beta < p-1.$

Then problem (5.0.1) *has at least one positive radial solution when* $\lambda \geq \lambda_1/\sigma$.

Proof. (d) implies that there exists $B > 1$ and a constant $A \geq 0$ such that

$f(u) \leq A + Bu^\beta$, for $u \geq 0$.

We first prove that $\left\{\lambda^{1/(p-1-\beta)}\delta v_0(r)\right\}$ is a family of supersolutions of (5.0.1) if δ is large enough. Here v_0 is as in (5.1.1). In fact,

$$\operatorname{div}(|D(\lambda^{1/(p-1-\beta)}\delta v_0|^{p-2}D(\lambda^{1/(p-1-\beta)}\delta v_0)) + \lambda f(\lambda^{1/(p-1-\beta)}\delta v_0)$$

$$\leq -(\lambda^{1/(p-1-\beta)}\delta)^{p-1} + A\lambda + B\lambda\lambda^{\beta/(p-1-\beta)}\delta^\beta v_0^\beta$$

$$= -\lambda^{(p-1)/(p-1-\beta)}[\delta^{p-1} - A\lambda^{-\beta/(p-1-\beta)} - B\delta^\beta v_0^\beta] \leq 0,$$

if $\delta^{p-1} \geq A\lambda^{-\beta/(p-1-\beta)} + B\delta^{\beta}\|v_0\|_\infty^{\beta}$. It is easy to see this is true if

$$\delta \geq 2 \max \{(B\|v_0\|_\infty^{\beta}+1)^{1/(p-1-\beta)},\ A^{1/(p-1)}\lambda^{-\beta/(p-1-\beta)(p-1)}\}.$$

For $\lambda \geq 1$, we can choose

$$\delta \geq M_\beta = 2 \max \left\{ \left[B\|v_0\|_\infty^{\beta}+1 \right]^{1/(p-1-\beta)},\ A^{1/(p-1)} \right\} + 1.$$

Therefore, $\lambda^{1/(p-1-\beta)}\delta v_0$ is not a solution of (5.0.1) when $\delta \geq M_\beta$ is large enough. Hence, by Remark 5.1.4,

$$u_\lambda \leq \lambda^{1/(p-1-\beta)} M_\beta v_0 \quad \text{in } B_R,$$

for $\lambda \geq \lambda_0$ and some $\lambda_0 > 0$. Let $\xi_\lambda = \lambda^{1/(p-1-\beta)} M_\beta v_0$. By Lemma 5.1.2, there exist c_1, $c_2 > 0$ such that $\psi \leq c_1(R-r)$ and $v_0 \geq c_2(R-r)$. Then when λ is large enough,

$$\mu\psi < \xi_\lambda \quad \text{in } [0,R).$$

and

$$\frac{\partial(\mu\psi)}{\partial r}(R) > \frac{\partial(\xi_\lambda)}{\partial r}(R).$$

Let $E = \{\phi \in C^1([0,R]);\ \phi(R)=0\}$;

$$O_\lambda = \left\{ \phi \in E;\ \mu\psi < \phi < \lambda^{1/(p-1-\beta)} M_\beta v_0 \text{ in } [0,R),\ \frac{\partial(\mu\psi)}{\partial r}(R) > \frac{\partial\phi}{\partial r}(R) > \frac{\partial(\xi_\lambda)}{\partial r}(R) \right\}$$

and

$$(Au)(r) = -(r^{N-1}|u'|^{p-2}u')'.$$

Then O_λ is a bounded, open, convex set. Using the ideas of the proof of Proposition 4.1.5 of Chapter 4, we also can prove that for any $h \in C^0([0,R])$, there exists only one solution for the problem

$$\left.\begin{aligned} (Au)(r) &= r^{N-1}h(r) \text{ for } 0 \leq r \leq R \\ u(R) &= 0,\ u'(0) = 0 \end{aligned}\right\}, \tag{5.2.2}$$

and let G_p be the inverse of A, then $G_p : C^0([0,R]) \to C^1([0,R])$ is compact.

Set $(T_\lambda u)(r)=G_p(\lambda r^{N-1}f(u))$. Since f is increasing, we can use the same ideas as in Lemma 5.1.3 to obtain that if $\phi\in\overline{O}_\lambda$, $\mu\psi< T_\lambda(\mu\psi)\le T_\lambda(\phi)\le T_\lambda(\xi_\lambda)<\xi_\lambda$ in [0,R) with reversed inequalities for $\partial/\partial r$ on ∂B_R. Hence $T_\lambda:\overline{O}_\lambda\to O_\lambda$. Let $e\in O_\lambda$, $S_t(\lambda,u)=tT(\lambda,u)+(1-t)e$ and $\Psi_t(\lambda,u)=u-S_t(\lambda,u)$ for $t\in[0,1]$. Then $S_t:\overline{O}_\lambda\to O_\lambda$, so, $0\notin\Psi_t(\lambda,\partial O_\lambda)$ for $t\in[0,1]$ and therefore by the homotopy invariance of degree,

$$\deg_{LS}(\Psi_t(\lambda,\cdot),O_\lambda,0)=1,\ t\in[0,1]. \tag{5.2.3}$$

Hence, $\deg_{LS}(\Psi_1(\lambda,\cdot),O_\lambda,0)=1.$ (5.2.4)

(5.2.4) shows that (5.0.1) has a radial solution u_λ in O_λ.

<u>Corollary 5.2.3</u> *For $p>1$ and $0<\beta<p-1$, there exists a positive radial solution v_β of the problem*

$$\left.\begin{aligned}-\mathrm{div}(|Du|^{p-2}Du)&=u^\beta\quad x\in B_R\\ u&=0\quad x\in\partial B_R\end{aligned}\right\}. \tag{5.2.5}$$

<u>Lemma 5.2.4</u> *Suppose that there exist two positive C^1 radial functions $u_0(r)\le v_1(r)$ satisfying $u_0'(r)<0$, $v_1'<0$ in (0,R] such that $u_0(r)$ is a subsolution of (5.0.1) and $v_1(r)$ is a supersolution of (5.0.1). Then (5.0.1) has a minimum positive radial solution $\underline{u}(r)$ and a maximum positive radial solution $\overline{u}(r)$ such that if u is a radial solution of (5.0.1) satisfying $u_0(r)\le u(r)\le v_1(r)$, then, $\underline{u}(r)\le u(r)\le\overline{u}(r)$ and $\underline{u}'(r)\ge u'(r)\ge\overline{u}'(r)$ in [0,R].*

Proof. Let the operators A and T_λ be as above. Assume that

$$\mathrm{div}(|Du_0|^{p-2}Du_0)+\lambda f(u_0)\ne 0;$$

$$\mathrm{div}(|Dv_1|^{p-2}Dv_1)+\lambda f(v_1)\ne 0.$$

Define

$$u_1 = T_\lambda u_0 \text{ and } w_1 = T_\lambda v_1.$$

Let us show that $u_1 > u_0$ and $w_1 < v_1$ (strict inequalities in B_R). We have

$$\left.\begin{aligned} \operatorname{div}(|Du_1|^{p-2}Du_1) = -\lambda f(u_0) \text{ in } B_R \\ u_1 = 0 \text{ on } \partial B_R \end{aligned}\right\},$$

so,

$$\left.\begin{aligned} -\operatorname{div}(|Du_1|^{p-2}Du_1) \geq -\operatorname{div}(|Du_0|^{p-2}Du_0) \text{ in } B_R \\ u_1 = u_0 = 0 \text{ on } \partial B_R \end{aligned}\right\}.$$

By the weak comparison principle (see Lemma 4.1.1 of Chapter 4), we know that

$$u_0 \leq u_1 \text{ in } B_R.$$

Since $u_1'(r) < 0$, $u_0'(r) < 0$ in (0,R], using the same idea as in Lemma 5.1.3, we obtain that $u_0 < u_1$ in B_R. A similar argument shows that $w_1 < v_1$ in B_R. Thus the sequence defined inductively by $w_1 = T_\lambda v_1$, $w_n = T_\lambda w_{n-1}$ is monotone decreasing. Similarly, $u_n = T_\lambda u_{n-1}$, $u_1 = T_\lambda u_0$ defines a monotone increasing sequence. Furthermore, we have

$$u_n < w_n \text{ for all n:}$$

$$u_0 < u_1 < u_2 < \cdots < u_n < \cdots < w_n < \cdots < w_1 < v_1.$$

In fact, $u_0 < v_1$; suppose $u_{n-1} < w_{n-1}$. Then,

$$w_n = T_\lambda w_{n-1} > T_\lambda u_{n-1} = u_n,$$

so, the proof follows by induction.

Since the sequences $\{u_k\}$ and $\{w_k\}$ are monotone, the pointwise limits

$$\underline{u}(r) = \lim_{k\to\infty} u_k(r) \text{ and } \bar{u}(r) = \lim_{k\to\infty} w_k(r)$$

both exist. The operator T_λ is a composition of the nonlinear operation $u \to \lambda f(u)$ with the inversion of the nonlinear boundary value problem $\phi \to v$

defined by

$$\left.\begin{aligned} -\mathrm{div}(|Dv|^{p-2}Dv)=\phi \quad &\text{in } \mathrm{B_R} \\ v=0 \quad &\text{on } \partial\mathrm{B_R} \end{aligned}\right\}.$$

For u bounded and $f(u)$ bounded on the range of u, the first operation takes bounded pointwise convergent into pointwise convergent sequences, The operation $\phi\to v$ maps $C^0(\mathrm{B_R})$ compactly into the space $C^1(\mathrm{B_R})$ (see Proposition 4.1.5 of Chapter 4). Thus, since $u_k=T_\lambda u_{k-1}$ and since $\{u_k\}$ is a bounded, pointwise convergent sequence, it converges also in $C^1(\mathrm{B_R})$. We thus have

$$\underline{u}=\lim_{k\to\infty}u_k=\lim_{k\to\infty}T_\lambda u_{k-1}=T_\lambda\lim_{k\to\infty}u_{k-1}=T_\lambda\underline{u}$$

and similarly for $\bar{u}$, by the continuity of T_λ. Thus $\underline{u}$ and $\bar{u}$ are fixed points of T_λ, and furthermore, they are of class $C^1(\mathrm{B_R})$. This idea comes from [67]. Since f is increasing and $u'(r)$, $\underline{u}'(r)\le 0$ in (0,R], it is clear that

$$(r^{N-1}|u'|^{p-2}u')'\le(r^{N-1}|\underline{u}'|^{p-2}\underline{u}')' \text{ for } r\in(0,\mathrm{R}]. \tag{5.2.6}$$

Integrating (5.2.6) from 0 to r to obtain $u'(r)\le\underline{u}'(r)$ in [0,R]. Using the same steps, we obtain $u'(r)\ge\bar{u}'(r)$ in [0,R].

In the following we give the following asymptotic theorem for the solution u_λ of problem (5.0.1) when λ is large enough.

Theorem 5.2.5 *Suppose f satisfies the conditions (a)–(d) of Theorem* 5.2.2. *Then*

$$\lim_{\lambda\to\infty} u_\lambda(r)\,/\,(\lambda^{1/(p-1-\beta)}\,v_\beta(r)) = 1 \textit{ uniformly on } [0,\mathrm{R}], \tag{5.2.7}$$

where $u_\lambda\in C^1([0,\mathrm{R}])$ is a positive radial solution of (5.0.1) *and v_β is as in Corollary* 5.2.3.

Proof. We prove this theorem by using an idea from Lin [47]. The proof will be divided into three steps.

1. There exists $\lambda_2 > 0$, such that if $\lambda > \lambda_2$ then

$$u_\lambda \geq \lambda^{1/(p-1)} v_\beta \text{ in } [0,\mathrm{R}], \tag{5.2.8}$$

where u_λ is a positive radial solution of Problem (5.0.1) and v_β is as in Corollary 5.2.3.

2. For any $\varepsilon > 0$ and $\varepsilon_1 > 0$, there exists $\Lambda_\varepsilon = \Lambda(\varepsilon,\varepsilon_1) > 0$, if $\lambda > \Lambda_\varepsilon$, then

$$u_\lambda \geq \lambda^{1/(p-1-\beta)}(1-\varepsilon)^{1/(p-1-\beta)}(1-\varepsilon_1)^{(p-1)/(p-1-\beta)} v_\beta. \tag{5.2.9}$$

3. For any $\varepsilon > 0$ and $\varepsilon_1 > 0$, there exists $\Lambda'_\varepsilon = \Lambda'(\varepsilon,\varepsilon_1) > 0$, if $\lambda > \Lambda'_\varepsilon$, then

$$u_\lambda \leq \lambda^{1/(p-1-\beta)}(1+\varepsilon)^{1/(p-1-\beta)}(1+\varepsilon_1)^{(p-1)/(p-1-\beta)} v_\beta. \tag{5.2.10}$$

Step 1. We shall first prove that there exists $m > 0$ such that $u_\lambda \geq \lambda^{1/(p-1)} m v_0$, if $\lambda \geq \lambda_1/\sigma$. Here λ_1, σ are as in Lemma 5.1.1.

By Lemma 5.1.2, there exists $k_1 > 0$ such that $\psi \geq k_1 v_0$. Hence, From Lemma 5.2.1 we have $u_\lambda \geq \mu_1 v_0$, if $\lambda \geq \lambda_1/\sigma$, where $\mu_1 = \mu k_1 > 0$.

Let $\mu' \in (0,\mu_1)$ and $m' = \min\{f(u);\ u \in [\mu',\infty)\}$. Then $m' > 0$. Denote by $[\phi \geq \mathrm{U}] = \{r \in [0,\mathrm{R}];\ \phi(r) \geq \mathrm{U}\}$. Since $(1-r)^{p/(p-1)} \leq 1 - r^{p/(p-1)}$ in (0,1), then

$$(\mathrm{R}-r)^{p/(p-1)} \leq \mathrm{R}^{p/(p-1)} - r^{p/(p-1)} \text{ in } (0,\mathrm{R}).$$

$v_0(r) \leq \mu'/\mu_1$ implies that $(\mathrm{R}-r)^{p/(p-1)} \leq (\mu'/\mu_1)\, \mathrm{N}^{1/(p-1)} p/(p-1)$. If $\lambda \geq \lambda_1/\sigma$, then using (5.1.4), (5.1.5), we obtain

$$u_\lambda(r) = \int_r^{\mathrm{R}} \left[\lambda s^{1-\mathrm{N}} \int_0^s t^{\mathrm{N}-1} f(u_\lambda(t))dt \right]^{1/(p-1)} ds$$

$$\geq (\lambda m')^{1/(p-1)} \int_{[\mu_1 v_0 \geq \mu']} \left[s^{1-N} \int_0^s t^{N-1} dt \right]^{1/(p-1)} ds$$

$$= (\lambda m')^{1/(p-1)} \left\{ v_0(r) - \int_{[v_0 \leq \mu'/\mu_1]} \left[s^{1-N} \int_0^s t^{N-1} dt \right]^{1/(p-1)} ds \right\}$$

$$= (\lambda m')^{1/(p-1)} \left\{ v_0(r) - C_p(\mu'/\mu_1)^{1/p}\, v_0(r) \right\}.$$

Choosing μ' such that $1-C_p(\mu'/\mu_1)^{1/p}=1/2$, we obtain

$$u_\lambda \geq \lambda^{1/(p-1)}(m'^{1/(p-1)}/2)v_0(r).$$

Next, we prove that there exists $\lambda_2 \geq \lambda_1/\sigma$ such that when $\lambda \geq \lambda_2$, $u_\lambda \geq \lambda^{1/(p-1)} v_\beta$. By (5.1.4) of Lemma 5.1.2, there exists $M > 0$ such that

$$(M^{1/(p-1)}/2)v_0 \geq v_\beta.$$

For this M, there exists $\mathrm{U}=\mathrm{U}(M)>0$, such that if $u \geq \mathrm{U}$, then $f(u) \geq M$. Therefore,

$$u_\lambda(r) \geq (\lambda M)^{1/(p-1)} \int_{[\lambda^{1/(p-1)} m v_0 \geq \mathrm{U}]} \left[s^{1-N} \int_0^s t^{N-1} dt \right]^{1/(p-1)} ds$$

$$\geq (\lambda M)^{1/(p-1)}(v_0(r)-(\mathrm{U}/\lambda m)^{1/p} C_p v_0(r)) \geq \lambda^{1/(p-1)} v_\beta(r).$$

For $\lambda \geq \lambda_2$, λ_2 is chosen such that $C_p(\mathrm{U}/\lambda_2 m)^{1/p}=1/2$. The proof of step 1 is completed.

Step 2. By (5.1.4) and (5.1.5) with $\rho = v_\beta(r)$,

$$u_\delta(r) = \int_{[\mathrm{R}-r \leq \delta]} \left[s^{1-N} \int_0^s t^{N-1}\, v_\beta^\beta dt \right]^{1/(p-1)} ds$$

$$\leq C_\beta \delta^{1/(p-1)} v_\beta(r),$$

where C_β depends on υ_β and R. By (*d*), for any $\varepsilon>0$, there exists $\mathrm{U}=\mathrm{U}(\varepsilon)>0$ such that $u\geq \mathrm{U}$ implies $f(u)\geq(1-\varepsilon)u^{\beta}$. Therefore,

$$u_\lambda(r)\geq(\lambda(1-\varepsilon))^{1/(p-1)}\int_{[\lambda^{1/(p-1)}\upsilon_\beta\geq \mathrm{U}]}\left[s^{1-\mathrm{N}}\int_0^s t^{\mathrm{N}-1}\lambda^{\beta/(p-1)}\upsilon_\beta^\beta dt\right]^{1/(p-1)}ds$$

$$\geq\lambda^{1/(p-1)+\beta/(p-1)^2}(1-\varepsilon)^{1/(p-1)}\left[1-C_\beta(\mathrm{U}/\lambda^{1/(p-1)})^{1/(p-1)}\right]\upsilon_\beta$$

$$\geq\lambda^{1/(p-1)+\beta/(p-1)^2}(1-\varepsilon)^{1/(p-1)}(1-\varepsilon_1)\upsilon_\beta.$$

Where $\lambda\geq\max\{\lambda_2,\lambda_3\}$ and $\lambda_3=(C_\beta/\varepsilon_1)^{(p-1)^2}\mathrm{U}^{p-1}$.

Repeating this argument, we have that for $\mathrm{n}\geq 4$,

$$u_\lambda\geq\lambda^{1/(p-1)+\beta/(p-1)^2+\cdots+\beta^{\mathrm{n}-1}/(p-1)^{\mathrm{n}}}(1-\varepsilon)^{1/(p-1)+\beta/(p-1)^2+\cdots+\beta^{\mathrm{n}-2}/(p-1)^{\mathrm{n}-1}}$$

$$\times(1-\varepsilon_1)^{1+\beta/(p-1)+\cdots+\beta^{\mathrm{n}-3}/(p-1)^{\mathrm{n}-3}}\upsilon_\beta$$

and

$$\lambda_{\mathrm{n}}=\left\{(C_\beta^{p-1}\mathrm{U})/\varepsilon_1^{p-1}\times\left(1/\left[(1-\varepsilon)^{\beta/(p-1)+\cdots+\beta^{\mathrm{n}-4}/(p-1)^{\mathrm{n}-3}}\times\right.\right.\right.$$

$$\left.\left.\left.(1-\varepsilon_1)^{1+\beta/(p-1)+\cdots+\beta^{\mathrm{n}-4}/(p-1)^{\mathrm{n}-4}}\right]\right)\right\}^{1/[1/(p-1)+\beta/(p-1)+\cdots+\beta^{\mathrm{n}-3}/(p-1)^{\mathrm{n}-3}]}$$

Since $0<\beta/(p-1)<1$, $\lambda_{\mathrm{n}}\to\lambda_\infty$ as $\mathrm{n}\to\infty$, here

$$\lambda_\infty=\left\{(C_\beta^{p-1}\mathrm{U})/\varepsilon_1^{p-1}\times\left(1/\left[(1-\varepsilon)^{\beta/(p-1-\beta)}(1-\varepsilon_1)^{(\beta-1)/(p-1-\beta)}\right]\right)\right\}^{(p-1-\beta)}.$$

Let

$$\Lambda_\varepsilon=\max\{\lambda_2,\lambda_3,\ \cdots\}=\Lambda(\varepsilon,\varepsilon_1)<\infty.$$

If $\lambda\geq\Lambda_\varepsilon$, then

$$u_\lambda(r)\geq\lambda^{1/(p-1-\beta)}(1-\varepsilon)^{1/(p-1-\beta)}(1-\varepsilon_1)^{(p-1)/(p-1-\beta)}\upsilon_\beta(r)\text{ in }[0,\mathrm{R}].$$

This completes the proof of step 2.

The proof of step 3 is exactly similar to the proof of step 2.

5.3 UNIQUENESS RESULTS

In this section we shall give the following uniqueness result for problem (5.0.1) when λ is large enough.

Theorem 5.3.1 *Let $p>1$, f satisfy all the conditions in Theorem 5.2.2, $f'(s)\geq 0$ for $s>0$ and $(f(s)/s^{p-1})'\leq 0$ for $s>0$. Then problem (5.0.1) has a unique positive radial solution in $C^1[0,R]$ for λ large enough.*

Proof. In fact, it follows from the proof of Theorem 5.2.2 that when λ is large enough, there exist a positive radial solution of problem (5.0.1). Suppose there are two positive radial solutions $u_1(r)$, $u_2(r)\in C^1([0,R])$ to problem (5.0.1). Since $u_1'(r)$, $u_2'(r)<0$ in $(0,R]$, by Lemma 5.1.2, there exist k_1, k_2, l_1, $l_2>0$ such that

$$k_1(R-r)\leq u_1(r)\leq l_1(R-r),\ k_2(R-r)\leq u_2(r)\leq l_2(R-r). \tag{5.3.1}$$

We define β_1, β_2 by

$$\beta_1=\sup\left\{\mu\in\mathbb{R};\ u_1-\mu u_2>0 \text{ in } [0,R)\right\}, \tag{5.3.2}$$

$$\beta_2=\sup\left\{\mu\in\mathbb{R};\ u_2-\mu u_1>0 \text{ in } [0,R)\right\}. \tag{5.3.3}$$

It follows from (5.3.1) that β_1, $\beta_2>0$. We suppose $\beta_1\leq 1$ (otherwise $\beta_2\leq 1$). Obviously, $u_1-\beta_1u_2\geq 0$ in B_R. Furthermore, we can show that there exists a point $z\in B_R$ where $u_1-\beta_1u_2$ vanishes. Indeed, suppose $u_1-\beta_1u_2>0$ in B_R. Since $f'(s)\geq 0$ for $s>0$ and $(f(s)/s^{p-1})'\leq 0$ for $s>0$, then

$$f(\beta_1 u_2)/(\beta_1 u_2)^{p-1} \geq f(u_2)/u_2^{p-1}. \tag{5.3.4}$$

Therefore, $f(u_1) \geq f(\beta_1 u_2) \geq \beta_1^{p-1} f(u_2)$. So,

$$-\operatorname{div}(|Du_1|^{p-2}Du_1) \geq -\operatorname{div}(|D(\beta_1 u_2)|^{p-2}D(\beta_1 u_2)) \text{ in } \mathrm{B_R}. \tag{5.3.5}$$

Following the same ideas as in the proof of Lemma 5.1.3, we have

$$(u_1 - \beta_1 u_2)'(\mathrm{R}) < 0. \tag{5.3.6}$$

Therefore, in view of the continuity, combining this and the assumption $u_1 - \beta_1 u_2 > 0$ in $\mathrm{B_R}$, we have

$$u_1 - (\beta_1 + \theta)u_2 > 0 \text{ in } \mathrm{B_R}, \tag{5.3.7}$$

for some positive number θ. This contradicts the definition of the number β_1. Thus we see that there exists a point $z \in \mathrm{B_R}$ where $u_1 - \beta_1 u_2$ vanishes. The proof of Lemma 5.1.3 gives that there exists $\bar{z} \in G$ where $u_1 - \beta_1 u_2$ vanishes. Here G is as in the proof of Lemma 5.1.3.

Observing that

$$\begin{cases} -L(u_1 - \beta_1 u_2) \geq 0 & \text{in } G \\ u_1 - \beta_1 u_2 \geq 0 \text{ in } G \text{ and } u_1 - \beta_1 u_2 = 0 \text{ at } z \in G \end{cases}, \tag{5.3.8}$$

where L is as in Lemma 5.1.3, we obtain by the strong maximum principle for uniformly elliptic operators (see Theorem 1.3.10 of Chapter 1),

$$u_1 - \beta_1 u_2 \equiv 0 \text{ in } G. \tag{5.3.9}$$

Since $u_1'(r)$ and $u_2'(r) < 0$ for $r \in (0,\mathrm{R}]$, following the same ideas as above, we obtain

$$u_1(r) \equiv \beta_1 u_2(r) \text{ in } [0,\mathrm{R}]. \tag{5.3.10}$$

Since u_1, u_2 are solutions of problem (5.0.1), (5.3.10) implies

$$f(\beta_1 u_2) = \beta_1^{p-1} f(u_2). \tag{5.3.11}$$

By (5.2.7), u_2 is large when λ is large enough. From the condition (d), we know from (5.3.11) that $\beta_1(\lambda)\equiv 1$ when λ is large enough. This shows $u_1 \equiv u_2$.

5.4 THE SHOOTING METHOD

In this section we prove that when f satisfies the conditions of Theorem 5.3.1, all the positive solutions of (5.0.1) are radial solutions. This implies that when f satisfies the conditions of Theorem 5.3.1, problem (5.0.1) only has a positive radial solution when λ is large enough. We first prove the following lemmas.

Lemma 5.4.1 *Suppose f satisfies* (F_1'). *Then there exists $\Lambda_1>0$ such that* (5.0.1) *has arbitrarily small nontrivial radially symmetric subsolutions for all $\lambda>\Lambda_1$.*

Proof. Let λ_1, $\psi(r)$ be as in (5.2.1). By (F_1'), for $\lambda\geq\lambda_1$,

$$\tau(\lambda)=\sup\{\, t\in[0,1];\ \lambda f(u) > \lambda_1 u^{p-1} \text{ for } 0\leq u\leq t\} > 0.$$

Suppose $\lambda>\lambda_1$ and $0<\varepsilon<\tau(\lambda)$. Then

$$\operatorname{div}(|D(\varepsilon\psi)|^{p-2}D(\varepsilon\psi))+\lambda f(\varepsilon\psi)$$

$$\geq -\lambda_1\varepsilon^{p-1}(|\psi|^{p-2}\psi)+\lambda f(\varepsilon\psi)\geq 0,$$

for all $r\in[0,R]$ and so the function $\underline{u}_\varepsilon(r)=\varepsilon\psi(r)$ is a subsolution of (5.0.1).

Lemma 5.4.2 *Suppose f satisfies all the conditions in Theorem 5.2.2. Then for any nonradial positive solution u_λ of* (5.0.1), *there exists $\Lambda_2>0$ such that when $\lambda>\Lambda_2$, there exist nontrivial positive radial supersolutions ξ_λ' of* (5.0.1) *satisfying $u_\lambda\leq\xi_\lambda'$.*

Proof. Let $u_\lambda = \lambda^{1/(p-1-\beta)} \upsilon$. Then $\upsilon > 0$ in B_R and υ satisfies

$$\left.\begin{aligned} \operatorname{div}(|D\upsilon|^{p-2} D\upsilon) + f(\lambda^{1/(p-1-\beta)}\upsilon)/(\lambda^{1/(p-1-\beta)}\upsilon)^{\beta} = 0 \text{ in } B_R \\ \upsilon = 0 \text{ on } \partial B_R \end{aligned}\right\}. \tag{5.4.1}$$

As $\lim\limits_{\lambda \to \infty} f(\lambda^{1/(p-1-\beta)}\upsilon)/(\lambda^{1/(p-1-\beta)}\upsilon)^{\beta} = 1$, we obtain

$$-\operatorname{div}(|D\upsilon|^{p-2} D\upsilon) \geq 0 \quad \text{in } B_R, \tag{5.4.2}$$

$$\upsilon > 0 \text{ in } B_R \text{ and } \upsilon = 0 \text{ on } \partial B_R. \tag{5.4.3}$$

Using Theorem 1.3.19 of Chapter 1, we know that $\frac{\partial \upsilon}{\partial r}(R) < 0$ on ∂B_R. Since ∂B_R is compact, there exists $\delta' > 0$ and $M_1 \geq M_2 > 0$ such that

$$-M_1 \leq \frac{\partial \upsilon}{\partial n_{r(x)}}(x) \leq -M_2 < 0, \tag{5.4.4}$$

for any $x \in \Omega_{\delta'} = \{x \in B_R;\ 0 < R - r < \delta'\}$, where $\frac{\partial}{\partial n_{r(x)}}$ is the directional derivative along the direction of r. Then by

$$\upsilon(x) = -(R-r)\int_0^1 \frac{\partial u}{\partial n_{r(x)}}(x + \tau(R-r)n_{r(x)})d\tau, \text{ for any } x \in \Omega_{\delta'},$$

and by choosing $M_3 > 0$ appropriately for $x \in B_R - \Omega_{\delta'}$, we have $\upsilon \leq M_3(R-r)$ in B_R. Therefore, $u_\lambda \leq \lambda^{1/(p-1-\beta)} M_4 \upsilon_0(r)$. Here $M_4 > 0$ and $\upsilon_0(r)$ is as in Lemma 5.1.1. From Theorem 5.2.2, we get the existence of positive radial supersolutions.

<u>Theorem 5.4.3</u> *Suppose f satisfies all the conditions in Theorem 5.2.2, $f'(s) \geq 0$ for $s > 0$ and $(f(s)/s^{p-1})' < 0$ for $s > 0$; $\vartheta \geq 0$. Then the boundary value problem*

$$\left.\begin{aligned} \operatorname{div}(|Du|^{p-2} Du) + \lambda f(u) = 0 \ \ in\ B_R \\ u > \vartheta \ \ in\ B_R,\ u = \vartheta \quad on\ \partial B_R \end{aligned}\right\} \tag{5.4.5}$$

has at most one positive radial solution.

Proof. Let $\upsilon = u - \vartheta$, then υ satisfies

$$\left.\begin{aligned} \operatorname{div}(|D\upsilon|^{p-2}D\upsilon) + \lambda f(\upsilon + \vartheta) = 0 \quad \text{in } B_R \\ \upsilon > 0 \ \text{ in } B_R, \ \upsilon = 0 \quad \text{on } \partial B_R \end{aligned}\right\}. \tag{5.4.6}$$

Then (5.4.5) has two positive radial solutions if and only if (5.4.6) has two positive radial solutions. Let υ_1, υ_2 be two positive radial solutions of (5.4.6). The proof of Theorem 5.3.1 shows that we can choose β_1, $0 < \beta_1 \le 1$ such that $\upsilon_1 \ge \beta_1 \upsilon_2$ in [0,R] and if $\beta_1 < 1$,

$$f(\beta_1\upsilon_2 + \vartheta)/(\beta_1\upsilon_2 + \vartheta)^{\beta} > f(\upsilon_2 + \vartheta)/(\upsilon_2 + \vartheta)^{\beta}. \tag{5.4.7}$$

So,

$$f(\upsilon_1 + \vartheta) \ge f(\beta_1\upsilon_2 + \vartheta) > \beta_1^{p-1} f(\upsilon_2 + \vartheta). \tag{5.4.8}$$

Following the same ideas as in the proof of Theorem 5.3.1, we get

$$\upsilon_1 \equiv \beta_1 \upsilon_2 \ \text{ in } [0,R]. \tag{5.4.9}$$

(5.4.9) implies that $f(\upsilon_1 + \vartheta) = \beta_1^{p-1} f(\upsilon_2 + \vartheta)$, which contradicts (5.4.8). Therefore, $\beta_1 = 1$ and $\upsilon_1 \equiv \upsilon_2$. The proof is completed.

Consider the initial value problem

$$\left.\begin{aligned} (r^{N-1}|u'|^{p-2}u')' + \lambda r^{N-1} f(u) = 0 \quad \text{for } r > 0 \\ u'(0) = 0\,,\ u(0) = \zeta, \quad \text{where } \zeta > 0 \end{aligned}\right\}. \tag{5.4.10}$$

We denote a solution of (5.4.10) by $u(\cdot,\zeta,\lambda)$. It follows from standard theorems on the continuous dependence of solutions on parameters and on initial data that $(\zeta,\lambda) \to u(\cdot,\zeta,\lambda)$ is a continuous function from $[0,\infty) \times [0,\infty)$ to $C[0,R_1]$ for any $R_1 > 0$.

Let $A(\lambda) = \{\zeta \in \mathbb{R}^+;$ there exists $R_2 > 0$ such that $0 < u(r,\zeta,\lambda)$ for $0 \le r < R_2$ and

$u(R_2,\zeta,\lambda)=0\}$ and $B(\lambda)=\{\zeta\in\mathbb{R}^+;\ u(r,\zeta,\lambda)>0$ for $0\le r\le R\}$. It is straightforward to prove from the continuous dependence of solutions on parameters that $A(\lambda)$ and $B(\lambda)$ are open disjoint subsets of $\mathbb{R}^+$. Thus we can prove the following results.

<u>Theorem 5.4.4</u> (*i*) *If* $A(\lambda)\neq\emptyset$, *then there exists* $\zeta(\lambda)$, $0<\zeta(\lambda)<\infty$, *such that* $A(\lambda)=(0,\ \zeta(\lambda))$.

(*ii*) *Suppose the problem*

$$\left.\begin{array}{r}(r^{N-1}|u'|^{p-2}u')'+\lambda r^{N-1}f(u)=0 \quad \textit{for } r>0\\ u'(0)=0,\ u(R)=0\end{array}\right\}. \tag{5.4.11}$$

has a solution. Then there exists $\zeta(\lambda)$, $0<\zeta(\lambda)<\infty$ *such that* $A(\lambda)=(0,\ \zeta(\lambda))$, $B(\lambda)=(\zeta(\lambda),\ \infty)$ *and* $u(\cdot,\ \zeta(\lambda),\ \lambda)$ *is the solution of* (5.4.11).

Proof. If $\zeta\in A(\lambda)$, there exists $R_2>0$ such that $0<u(r,\zeta,\lambda)$ for $0<r<R_2$ and $u(R_2,\zeta,\lambda)=0$. Assume that $q<\zeta$ and $q\notin A(\lambda)$. Then either $u(R,q,\lambda)=0$ or $u(r,q,\lambda)>0$ for $0<r\le R$. In either case, there exists r_1, $0<r_1<R$ such that

$$u(r_1,q,\lambda)=u(r_1,\zeta,\lambda)\ (=\vartheta,\ \text{say}).$$

By $u'(\cdot,q,\lambda)<0$ and $u'(\cdot,\zeta,\lambda)<0$, $u(\cdot,q,\lambda)$ and $u(\cdot,\zeta,\lambda)$ correspond to distinct radial solutions of (5.4.5), which is impossible. Part (*ii*) follows from the fact that under our hypotheses on f, (5.0.1) has at most one positive radial solution.

<u>Theorem 5.4.5</u> *Suppose* f *satisfies the conditions of Theorem* 5.3.1. *Then every positive solution of* (5.0.1) *is radially symmetric.*

Proof. Suppose u is a positive nonradially symmetric solution of (5.0.1). Then by Lemmas 5.4.1 and 5.4.2, there exist a radially symmetric subsolution $\underline{u}$ and a radially symmetric supersolution $\overline{u}$ such that $\underline{u}(|x|)\le u(x)\le\overline{u}(|x|)$, for $x\in B_R$.

Lemma 5.2.4 implies the existence of radially symmetric solutions u_1 and u_2 such that $\underline{u}(|x|) \le u_1(|x|) \le u(x) \le u_2(|x|) \le \bar{u}(|x|)$. Since u is nonradially symmetric, $u_1 \ne u$ and $u_2 \ne u$. So, $u_1 \ne u_2$. But this is impossible because of Theorem 5.3.1, and so every positive solution of (5.0.1) is radially symmetric when λ is large enough.

REFERENCES

1. H. Amann, On the existence of positive solutions of nonlinear elliptic boundary value problems, *Indiana Univ. Math. J.* 21 (1971), 125–146.

2. ———, A note on degree theory for gradient mappings, *Proc. Amer. Math. Soc.* 85 (1982), 591–595.

3. A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* 14 (1973), 349–381.

4. C. Bandle, C. V. Coffman and M. Marcus, Nonlinear ellptic problems in annular domains, *J. Diff. Eqns.* 69 (1987), 322–345.

5. L. E. Bobisud, D. O'Regan and W. D. Royalty, Solvability of some nonlinear boundary value problems, *Nonlinear Analysis T. M. A.* 12 (9) (1988), 855–869.

6. L. Boccardo, P. Drabek and M. Kucera, Landesman–Lazer conditions for strongly nonlinear boundary value problems, *Comment. Math. Univ. Carolinae* 30, (1989), 411–427.

7. H. Brézis, Analyse fonctionnelle, *Masson, Paris,* 1983.

8. H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* 34 (1983).

9. H. Brézis and R. E. L. Turner, On a class of superlinear elliptic problems, *Comm. Partial Diff. Eqns.* 2 (1977), 601–614.

10. F. E. Browder, Fixed point theory and nonlinear problems, *Bull. Amer. Math. Soc.* 9(1), (1983).

11. F. E. Browder and W. V. Petryshyn, The topological degree and the Galerkin approximation for noncompact operators in Banach spaces, *Bull. Amer. Math. Soc.* 74 (1968), 641–646.

12. ——————————, Approximation methods and the

generalized topological degree for nonlinear mappings in Banach spaces, *J. Funct. Anal.* 3 (1968), 217–245.

13. ———————————————, Approximation methods and the generalized topological degree for nonlinear mappings in locally convex topological vector spaces, *J. Diff. Eqns.* 12 (1972), 610–636.

14. K. J. Brown and A. Tertikas, On the bifurcation of radially symmetric steady-state solutions arising in population genetics, *SIAM J. Math. Anal.* 22 (2) (1991), 400–413.

15. A. Callegari and A. Nachman, Some singular nonlinear differential equations arising in boundary layer theory, *J. Math. Anal. Appl.* 64 (1978), 96–105.

16 Ph. Clement and G. Sweers, Existence and multiplicity results for a semilinear elliptic eigenvalue problem, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 14 (1987), 97–121.

17. C. V. Coffman, On the positive solutions of boundary value problems for a class of nonlinear differential equations, *J. Diff. Eqns.* 3 (1967), 92–111.

18. J. Cronin, Fixed points and topological degree in nonlinear analysis, Amer. Math. Soc. Providence, R. I., 1964.

19. E. N. Dancer, Uniqueness for elliptic equations when a parameter is large, *Nonlinear Analysis T. M. A.* 8 (1984), 835–836.

20. D. G. de Figueiredo, On the uniqueness of positive solutions of the Dirichlet problem $-\Delta u=\lambda \sin u$, *Nonlinear P.D.E. and Appl. College de France seminar vol* 7, *Pitman* 1985, 80–83.

21. A. Fonda and J. Mawhin, Quadratic forms, weighted eigenfunctions and boundary value problems for non-linear second order ordinary differential equations, *Proc. Royal. Soc. Edin.* 112A (1989) 145–153.

22. S. Fucik and A. Kufner, Nonlinear differential equations, Academic press, New York 1980.

23. R. E. Gaines and J. Mawhin, Coincidence degree and nonlinear differential equations, Lect. Notes in Math. 568 (Berlin: Springer 1977).

24. X. Garaizer, Existence of positive radial solutions for semilinear elliptic equations in the annulus, *J. Diff. Eqns.* 70 (1987), 69–92.

25. J. A. Gatica, V. Oliker and P. Waltman, Singular nonlinear boundary value problems for second order ordinary differential equations, *J. Diff. Eqns.* 79 (1989), 62-78.

26. I. M. Gelfand, Some problems in the theory of quasilinear equations, *Amer. Math. Soc. Transl.* 1(2) 29 (1963), 295–381.

27. B. Gidas, W. M. Ni and L. Nirenberg, Symmetry and related properties in almost spherically symmetric domains, *Arch. Rat. Mech. Anal.* 96 (1986), 167–197.

28. B. Gidas, W. M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* 68 (1979), 209–243.

29. D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer–Verlag, 1977.

30. J. P. Gossez, Strongly nonlinear elliptic problems near resonance: A Variational Approach, *to appear in Comm. P.D.E.*

31. A. Granas, Sur la méthode de continuité de Poincaré, *Comptus Rendus Acad. Sci. Paris* 282 1976, 983–985.

32. A. Granas, R. B. Guenther and J. W. Lee, Nonlinear boundary value problems for ordinary differential equations, Dissertationes Mathematcae, Warsaw, 1985.

33. M. Guedda and L. Veron, Local and global properties of solutions of qusilinear elliptic equations, *J. Diff. Eqns.* 76 (1988), 159–189.

34. Z. M. Guo, Some existence and multiplicity results for a class of quasilinear elliptic eigenvalue problems, *Nonlinear Analysis T.M.A.* in press, 1991.

35. ————, Solvability of some singular nonlinear boundary value problems

and existence of positive radial solutions of some nonlinear elliptic problems, *Nonlinear Analysis T. M. A.* 16(9) (1991), 781–790.

36. ———, Existence of positive radial solutions for a class of nonlinear singular elliptic problems in annular domains, *Proc. Edin. Math. Soc.* in press.

37. ———, Existence and uniqueness of positive radial solutions for a class of quasilinear elliptic equations, *Applicable Analysis,* in press.

38. N. Hirano, Multiple solutions for quasilinear elliptic equations, *Nonlinear Analysis T. M. A.* 15 (7) (1990), 625–638.

39. W. Hurewicz and H. Wallman, Dimension Theory, Princeton University Press 1948.

40. R. Iannacci and M. N. Nkashama, Unbounded perturbations of forced second order ordinary differential equations at resonance, *J. Diff. Eqns.* 69 (1987) 289–309.

41. S. Kichenassamy and J. Smoller, On the existence of radial solutions of quasi-linear elliptic equations, *Nonlinearity,* 3 (1990), 677-694.

42. M. A. Krasnosel'skii, Topological methods in the theory of nonlinear integral equations, Pergamon, New York, 1964.

43. J. C. Kurtz, Weighted Sobolev spaces with applications to singular nonlinear boundary value problems, *J. Diff. Eqns.* 49 (1983), 105–123.

44. O. A. Ladyzhenskaya and N.N. Ural'tseva, Linear and quasilinear elliptic equations, Academic Press, New York and London, 1968.

45. S. S. Lin, On the existence of positive radial solutions for nonlinear elliptic equations in annular domains, *J. Diff. Eqns.* 81 (1989), 221-223.

46. ————, Positive radial solutions and non-radial bifurcation for semilinear elliptic equations in annular domains, *J. Diff. Eqns.* 86 (1990), 367–391.

47. ————, On the number of positive solutions for nonlinear elliptic

equations when a parameter is large, *Nonlinear Analysis T. M. A.* 16(3) (1990), 283–297.

48. P. L. Lions, On the existence of positive solutions of semilinear elliptic equations, *SIAM. Rev.* 24 (1982), 441–467.

49. N.G. Lloyd, Degree theory, Cambridge University Press, London, 1978.

50. C. D. Luning and W. L. Perry, Positive solutions of negative exponent generalized Emden–Fowler boundary value problems, *SIAM J. Math. Anal.* 12 (1981), 874–879.

51. J. Mawhin, Topological degree methods in nonlinear boundary value problems, *Regional conference series in mathematics* no 40, 1979.

52. J. Mawhin and J. R. Ward, Periodic solutions of some forced Liénard differential equations at resonance, *Arch. Math.* 41 (1983), 337–351.

53. E. J. Mcshane and T. A. Botts, Real analysis, D. Van Nostrand Company, INC. Princeton, New Jersey, Toronto, London, New York, 1959.

54. Z. Nehari, On a class of second order differential equations, *Trans. Amer. Math. Soc.* 95 (1960), 101–123.

55. W. M. Ni, Uniqueness of solutions of nonlinear Dirichlet problems, *J. Diff. Eqns.* 50 (1983), 289–304.

56. W. M. Ni and R. Nussbaum, Uniqueness and non-uniqueness for positive radial solutions of $\Delta u+f(u,r)=0$, *Comm. Pure Appl. Math.* 38 (1985), 67–108.

57. D. O'Regan, Positive solutions to singular and non-singular second-order boundary value problems, *J. Math. Anal. Appl.* 142 (1989), 40–52.

58. W. V. Petryshyn, On the approximation solvability of equations involving A–proper and pseudo A–proper mappings, *Bull. Amer. Math. Soc.* 81 (1975) 223–312.

59. ——————, Iterative construction of fixed points of contractive type mappings in Banach spaces, *Numerical Analysis of Partical Differential Equations* (CIME 2nd Ciclo, Ispra, 1967), Edizioni Cremonese, Rome, 1968,

307–339.

60. ——————, Using degree theory for densely defined A–proper maps in the solvability of semilinear equations with unbounded and noninverttible linear part, *Nonlinear Analysis T. M. A.* 4(2) (1983), 259–281.

61. W. V. Petryshyn and Z. S. Yu, Periodic solutions of nonlinear second–order differential equations which are not solvable for the highest derivative, *J. Math. Anal. Appl.* 89 (1982) 462–488.

62. M. D. Pino, M. Elgueta and R. Manasevich, A homotopic deformation along p of a Leray–Schauder degree result and existence for $(|u'|^{p-2}u')'+f(t,u)=0$, $u(0)=u(T)=0$, $p>1$, *J. Diff. Eqns.* 80 (1989), 1–13.

63. S. I. Pohozaev, Eigenfunctions of the equation $\Delta u+\lambda f(u)=0$, *Soviet Math. Dokl.* 5 (1965), 1408-1411.

64. P. H. Rabinowitz, Pairs of positive solutions of nonlinear elliptic partial differential equations, *Indiana Univ. Math. J.* 23 (2) (1973), 172-185.

65. S. Sakaguchi, Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems, *Ann. Scuola Norm. Sup. Pisa. Cl. Sci.* 4 (14) (1987), 403-421.

66. D. H. Sattinger, Topics in stability and bifurcation theory, *Lect. Notes in Math.* 309, *Springer, Berlin Heidelberg, New York*, 1973.

67. ——————, Monotone methods in nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J.*, 21 (1972), 979–1000.

68. V. I. Smirnov, A course of higher mathematics, Vol. V, Reading, Mass., Addison-Wesley, 1964.

69. S. L. Sobolev, Applications of functional analysis in mathematical physics, Providence, Rhode Island, American Mathematical Society, 1963.

70. J. Serrin, Nonlinear equations of second order, *A.M.S. Sympos. Partial Diff. Eqns., Berkeley*, August 1971.

71. S. D. Taliaferro, A nonlinear singular boundary value problem, *Nonlinear Analysis T. M. A.* 3 (1979), 897–904.

72. F. de Thelin, Sur l'espace propre associé à la première valeur propre du pseudo–Laplacien, *C. R. Acad. Sc. Paris, Ser.* A 303 (1980), 355–358.

73. P. Tolksdorf, Regularity for a more general class of qusilinear elliptic equations, *J. Diff. Eqns.* 51 (1984), 126–150.

74. ——————, On the Dirichlet problem for quasilinear equations in domains with conical boundary points, *Comm. Partial Diff. Eqns.*, 8(7) (1983), 773–817.

75. J. R. L. Webb, Boundary value problems for strongly nonlinear elliptic equations, *J. London Math. Soc.* (2) 21 (1980) 123–132.

76. J. S. W. Wong, On the generalized Emden–Fowler equations, *SIAM Review* 17 (1975), 339–360.