CS-MODULES AND GENERALIZATIONS

by

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CONTENTS

STATEMENT

Chapter 1 is a review of existing work apart from Example 1.2.3. The rest of the thesis is original except where stated otherwise. The work included in [32] and [33] arose out of discussions with my supervisor Professor P. F. Smith.

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ABSTRACT

As a generalization of the divisibility of an abelian group, injectivity was defined for modules by Baer in 1940. Since then this concept has attracted much interest.

The starting point of this thesis is that for any torsion-free abelian group A ($\mathbb{Z}$-module) let $B \leq A$ such that A/B is torsion-free, can any homomorphism $\varphi : B \to A$ be lifted to A (i.e. does there exist a homomorphism $\theta : A \to A$ such that $\theta|_B = \varphi$)? Since the answer is no, it is decided to investigate lifting homomorphisms from submodules to M and relationships with one (or two) of the following properties :

(C_1) Every submodule of M is essential in a direct summand of M. Equivalently, every complement submodule of M is a direct summand of M.

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand of M.

(C_3) If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is a direct summand of M.

A module with the property (C_1) is called a CS-module and a CS-module with the property (C_2) ((C_3)) is called continuous (quasi-continuous) module.

In particular Kamal and Muller's result : "M_R satisfies (C_1) if and only if $M = Z_2(M) \oplus N$ and $Z_2(M)$ is N-injective", allows us to consider nonsingular modules.

Special rings are then considered and it is investigated when they are CS-rings for nonsingular cases. In particular, let

$$R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix},$$

where S, T are rings and ${}_SM_T$ bimodule such that ${}_SM$ is faithful. Then the necessary and sufficient conditions for R to be a right nonsingular right

CS-ring are given.

In general, the full matrix ring over a CS-ring needs not be a CS-ring. This thesis contains the equivalent conditions for a full matrix ring being CS over a domain.

Kamal and Muller proved that over a commutative integral domain, any torsion-free reduced CS-module is a finite direct sum of uniform modules. This result is generalized to nonsingular modules over a commutative ring with finitely many minimal prime ideals.

This thesis also deals with the characterization of continuous and quasi-continuous modules in terms of lifting homomorphisms.

Since the direct sum of two CS-modules need not be a CS-module (C_1) is weakened to (C_{11}) as follows:

(C_{11}) Every submodule of M has a complement which is a direct summand of M.

In contrast to CS-modules it is observed that any direct sum of modules with (C_{11}) satisfies (C_{11}). However, it is not possible to determine whether any direct summand of a module with (C_{11}) satisfies (C_{11}) or not. A module M satisfies $(C_{11}{}^{+})$ if any direct summand of M satisfies (C_{11}). Moreover the weaker condition than (C_{11}) is given as follows :

(C_{12}) For every submodule N of M there exists a direct summand K of M and a monomorphism $\alpha : N \to K$ such that $\alpha(N)$ is essential in K.

It is worth knowing whether any direct summand of M is a direct sum of uniform modules whenever M is itself a direct sum uniform modules. It was shown that this is true for modules over $\mathbb{Z}$.

The work was completed by considering conditions on a module M which imply that M is a direct sum of uniform modules and chain conditions with $(C_{11}{}^{+})$.

Chapter 1

PRELIMINARIES

In this chapter we will give basic definitions and some well-known results which will be needed in the following chapters. In particular, we will define CS-modules, continuous modules and quasi-continuous modules and we will give fundamental properties of these modules as well as recent developments.

1.1. Basic definitions and ideas.

Let R be a ring (with identity) and M a right R-module, usually just called a "module". We shall write "$N \leq M$" to indicate that N is a submodule of M. The right R-module R is denoted R_R. Thus "$E \leq R_R$" means that E is a right ideal of R. Let $m \in M$, $N \leq M$. Define

$$m^{-1}N = \{r \in R : mr \in N\}.$$

It is easy to check that $m^{-1}N \leq R_R$.

Definition 1.1.1. A submodule N of M is called an *essential submodule*, or is *essential in M*, provided $N \cap K \neq 0$ for all $0 \neq K \leq M$.

We first give some properties of essential submodules in the following Proposition. For proofs and more information about this section refer to [1], [6], [8], [34], [41].

Proposition 1.1.2. Let M be a module. Then

(i) N is essential in M if and only if $N \cap mR \neq 0$ for all $0 \neq m \in M$.

(ii) Given $K \leq N \leq M$, K is essential in M if and only if K is essential in N and N is essential in M.

(iii) Let N be essential in M and $K \leq M$. Then $N \cap K$ is essential in K.

(iv) For any $t \geq 1$, let N_i be essential in K_i $(1 \leq i \leq t)$. Then $N_1 \cap N_2 \cap \ldots \cap N_t$ is essential in $K_1 \cap K_2 \cap \ldots \cap K_t$.

(v) Given $K \leq N \leq M$, let N/K be essential in M/K. Then N is essential in M.

(vi) Let N be essential in M and $m \in M$. Then $m^{-1}N$ is essential in R_R.

(vii) For any non-empty index set Λ, let N_λ be essential in M_λ $(\lambda \in \Lambda)$. Then $\oplus_\Lambda N_\lambda$ is essential in $\oplus_\Lambda M_\lambda$.

Let N be a submodule of M. By Zorn's Lemma, the collection of submodules L of M such that $N \cap L = 0$ has a maximal member.

Definition 1.1.3. Given $L \leq M$, by a *complement* (*submodule*) *of L in M*, we mean a submodule K of M, maximal with respect to the property $K \cap L = 0$. Thus, K is a complement of L in M if and only if (i) $K \cap L = 0$, and (ii) $N \cap L \neq 0$ for all $K \subset N \leq M$.

Proposition 1.1.4. Let $L, N \leq M$ with $N \cap L = 0$. Then there exists a complement K of L in M such that $N \subseteq K$.

Proposition 1.1.5. Let $L \leq M$ and let K be any complement of L in M. Then $K \oplus L$ is essential in M.

For any module M, the *socle*, Soc M of M is defined to be the sum of all simple submodules of M, or zero if M has no simple submodules.

Corollary 1.1.6. For any module M,

$$\mathrm{Soc}\ M = \bigcap \{N : N \text{ is essential in } M\}.$$

Definition 1.1.7. A submodule K of a module M will be called a *complement* (*submodule*) (in M), provided there exists $L \leq M$ such that K is a complement of L in M. Clearly 0, M are complements in M. Moreover, any direct summand of M is a complement in M.

Proposition 1.1.8. Let $N \leq M$. Then there exists $K \leq M$, containing N such that N is essential in K and K is a complement in M.

Proposition 1.1.9. Let $K \leq M$. Then K is a complement in M if and only if whenever K is essential in L, for any $L \leq M$, then $K = L$.

Proposition 1.1.10. Let K be a complement in N, and N a complement in M. Then K is a complement in M.

Proof. Let K be a complement of K' in N and let N be a complement of N' in M. It is easy to show that $K \cap (K' + N') = 0$. By Proposition 1.1.4, there exists a complement C of $K' + N'$ in M with $K \subseteq C$. Set $D = N \cap (C + N')$. We have $K \subseteq D \subseteq N$ and $D \cap K' = 0$, so that $K = D$. It is now straightforward to show that $(C + N) \cap N' = 0$, giving $C + N = N$, i.e. $C \subseteq N$, so that $C = K$.

Definition 1.1.11. Let M, X be unital right R-modules. Then X is called *M-injective* provided for each submodule N of M, every R-homomorphism $\varphi : N \to X$ can be lifted to an R-homomorphism $\theta : M \to X$ (i.e. $\theta(n) = \varphi(n)$ for all $n \in N$). A module X is called *quasi-injective* (or *self-injective*) provided X is X-injective. A module which contains no non-zero injective submodule will be called *reduced.*

Note. Any injective module is M-injective, for any module M, and any (R_R)-injective module is injective.

Notation. For any module A, E(A) will denote the injective hull of A.

Definition 1.1.12. A submodule U of M is called *uniform*, provided $U \neq 0$ and $X \cap Y \neq 0$ for all $0 \neq X, Y \leq U$. In other words, U is uniform if and only if every non-zero submodule X of U is essential in U.

The module M has *finite uniform (Goldie) dimension* if M does not contain an infinite direct sum of non-zero submodules. It is well known that a module M has finite uniform dimension if and only if there exists a positive integer n and uniform submodules U_i $(1 \leq i \leq n)$ of M such that $U_1 \oplus U_2 \oplus \ldots \oplus U_n$ is an essential submodule of M, and in this case n is an invariant of the module called the *uniform dimension* of M (see, for example, [1, p.294, ex.2]).

Definition 1.1.13. The *singular submodule* Z(M) of a module M is defined by

$$Z(M) = \{m \in M : mE = 0 \text{ for some essential right ideal } E \text{ of } R\}.$$

The *Goldie torsion submodule* (or *second singular submodule*) $Z_2(M)$ of M is that submodule of M, containing Z(M), such that $Z_2(M) / Z(M)$ is the singular submodule of $M / Z(M)$. The module M is called *singular* if $M = Z(M)$ and *nonsingular* if $Z(M) = 0$. Further, for any module M, $M / Z_2(M)$ is a nonsingular module and $Z_2(M)$ is a complement in M (see [34]).

Definition 1.1.14. A right ideal A of R is a *right annihilator* for M provided there exists a non-empty subset X of M such that

$$A = \{r \in R : xr = 0 \text{ for all } x \in X\} = \{r \in R : Xr = 0\}.$$

Note that for $X = \{m\}$ for any $m \in M$, we will use $\underset{\sim}{r}(m)$ to indicate the right annihilator of m in M, i.e, $\underset{\sim}{r}(m) = \{r \in R : mr = 0\}$.

Definition 1.1.15. A ring R is *semiprime right Goldie* provided it satisfies the following.

(i) $I \leq R$, $I^2 = 0$ implies $I = 0$.

(ii) R_R has finite uniform dimension, and

(iii) R_R has ACC (ascending chain condition) on right annihilators.

Definition 1.1.16. An element c of a ring R is *regular* (a *non-zero divisor*) provided $cr \neq 0$ and $rc \neq 0$ for all non-zero $r \in R$. Then a module M is called *divisible* provided $M = Mc = \{mc : m \in M\}$, for every regular element c of R.

Definition 1.1.17. Let R be a ring. Then R is called a *pp-ring* if every principal right ideal of R is projective. Note that for any $x \in R$, xR is projective if and only if $\underset{\sim}{r}(x) = eR$ for some idempotent e. Thus R is right pp-ring if and only if for each $x \in R$ there is an idempotent e such that $\underset{\sim}{r}(x) = eR$.

Definition 1.1.18. A *torsion theory* for Mod-R is a pair $(\underline{\underline{T}}, \underline{\underline{F}})$ of classes of right R-modules such that

(i) $\mathrm{Hom}(T,F) = 0$ for all $T \in \underline{\underline{T}}$, $F \in \underline{\underline{F}}$.

(ii) $\underline{\underline{T}}$ and $\underline{\underline{F}}$ are maximal classes having property (i).

The modules in $\underline{\underline{T}}$ are called *torsion* modules and the modules in $\underline{\underline{F}}$ are *torsion-free*.

Any given non-empty class $\underline{\underline{G}}$ of modules generates a torsion theory in the following way,

$$\underline{\underline{F}} = \{F : \mathrm{Hom}(G,F) = 0 \text{ for all } G \in \underline{\underline{G}}\}$$

$$\underline{\underline{T}} = \{T : \mathrm{Hom}(T,F) = 0 \text{ for all } F \in \underline{\underline{F}}\}.$$

In particular, the torsion theory which is generated by the class

$$\underline{\underline{G}} = \{M / L : L \text{ is essential in } M\}$$

is called the *Goldie torsion theory.*

Definition 1.1.19. An integral domain R is said to be a *right Ore domain* if $aR \cap bR \neq 0$, for all nonzero $a, b \in R$. For example, every commutative integral domain is a right Ore domain.

Definition 1.1.20. [21, Definition 2.15] Let $X = \sum_{\lambda \in \Lambda} X_\lambda$ be a direct sum of submodules X_λ $(\lambda \in \Lambda)$ of a module M. Then X is called a *local summand* of M if $\sum_{\lambda \in \Lambda'} X_\lambda$ is a direct summand of M for each finite subset Λ' of Λ.

Definition 1.1.21. A module M is called *locally Noetherian* provided every finitely generated submodule N of M is Noetherian. Note that M is locally Noetherian implies that N and M / N are both locally Noetherian. Also if N is Noetherian and M / N is locally Noetherian then M is locally Noetherian. But N and M / N are both locally Noetherian (even if M / N is Noetherian) does not imply M is locally Noetherian.

Example 1.1.22. Let K be a field and V an infinite dimensional vector space over K. Let $R = \begin{bmatrix} K & V \\ 0 & K \end{bmatrix} = \left\{\begin{bmatrix} k & v \\ 0 & k \end{bmatrix} : k \in K,\ v \in V\right\}$. Let $I = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$. Then I and R / I are locally Noetherian. But R is not locally Noetherian.

Proof. Clearly I is semisimple (in fact $I = \mathrm{Soc}\, R$). Thus I is locally Noetherian. Define $\varphi : R \to K$ by

$$\varphi\left(\begin{bmatrix} k & v \\ 0 & k \end{bmatrix}\right) = k \quad (k \in K,\ v \in V).$$

Then φ is an epimorphism with kernel I. Therefore $R / I \cong K$ so R / I is a simple R-module. Hence R / I is Noetherian. Now $R = R1$ so R is finitely generated. Since V is infinite dimensional then I is not finitely generated. It follows that R is not Noetherian and hence not locally Noetherian.

1.2. Historical background and recent developments of CS-modules.

Definition 1.2.1. Let M be a right R-module. Then M is called a *CS-module* (*module with* (C_1), *extending module*, etc.) if every complement L in M is a direct summand of M. Equivalently, any submodule N of M is essential in a direct summand K of M (see Proposition 5.1.2).

We shall say that R is a right *CS-ring* whenever R_R is a CS-module, i.e if I is any right ideal of R which is a complement submodule of R_R, then $I = eR$ for some idempotent e.

Among examples of CS-modules, we could point out semisimple modules, uniform modules and injective modules. On the other hand, any free abelian group of finite rank is a CS-module (see Example 2.1.17 or 4.2.2).

To illustrate a usage of Definition 1.2.1, any injective module M is a CS-module. Let K be a complement in M. Then K is essential in its injective hull E(K). Therefore K = E(K) and hence K is injective. Thus K is a direct summand of M.

CS is an abbreviation for "complements are summands". CS-modules were originated by von Neumann [38], [39], [40] in 1930. They were developed by Utumi [35], [36], [37] for rings in the 1960's, were extended to modules by Jeremy [13], [14] in the 1970's and have been investigated by Oshiro [24], [25] and others. The following set-up summarises the origin of CS-modules and the concepts related to it.

$$\begin{tikzcd}
& \text{Quantum mechanics} \arrow[dl] \arrow[dr] & \\
\text{Projective geometry} \arrow[dr] & & \text{Operator theory} \arrow[dl] \\
& \text{Lattices} \arrow[d] & \\
& \text{Rings} \arrow[d] & \\
& \text{Modules} &
\end{tikzcd}$$

<u>Proposition 1.2.2.</u> [21, Proposition 2.7]. Any direct summand of a CS-module is a CS-module.

Proof. By Proposition 1.1.10.

Clearly, any complement submodule of a CS-module is a CS-module, but arbitrary submodules of a CS-module need not be a CS-module. For example, let M be a module which is not CS and let $M' = E(M)$. Then $M \leq M'$ and M' is a CS-module. A direct sum of CS-modules need not be a CS-module either. We shall give an example to illustrate this. More examples can be found in [4] and [5].

<u>Example 1.2.3.</u> [33] Let p be any prime and M the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$. Let $M_1 = (\mathbb{Z}/\mathbb{Z}p) \oplus 0$ and $M_2 = 0 \oplus (\mathbb{Z}/\mathbb{Z}p^3)$. Then

(i) K is a complement in M if and only if $K = 0$, M, M_1, M_2 or $\mathbb{Z}(1 + \mathbb{Z}p, b + \mathbb{Z}p^3)$ for some $b \in \mathbb{Z}$ such that p^3 does not divide b.

(ii) M is not a CS-module.

Proof. (i) First we show $K = \mathbb{Z}(1 + \mathbb{Z}p, b + \mathbb{Z}p^3)$ $(b \notin \mathbb{Z}p^3)$ is a complement in M. Since K is cyclic and $p^3K = 0$, K is uniform. Suppose K is essential in a submodule L of M. Then L is uniform and hence cyclic, because M is finitely generated. Therefore $L = \mathbb{Z}(c + \mathbb{Z}p, d + \mathbb{Z}p^3)$ for some c, $d \in \mathbb{Z}$. Thus there exists $n \in \mathbb{Z}$ such that

$$(1 + \mathbb{Z}p, b + \mathbb{Z}p^3) = n(c + \mathbb{Z}p, d + \mathbb{Z}p^3), \text{ i.e.}$$

$$1 \equiv nc(\text{mod } p),\ b \equiv nd(\text{mod } p^3).$$

Now if p divides n then $1 \equiv 0(\text{mod } p)$, which is a contradiction.

Hence p does not divide n. It follows that $1 = nc + sp$ for some $s \in \mathbb{Z}$. Thus $(1 - nc)^3 = s^3p^3$. Then $1 - nt = s^3p^3$ for some $t \in \mathbb{Z}$. Therefore

$$t(1 + \mathbb{Z}p, b + \mathbb{Z}p^3) = nt(c + \mathbb{Z}p, d + \mathbb{Z}p^3) = (1 - s^3p^3)(c + \mathbb{Z}p, d + \mathbb{Z}p^3)$$
$$= (c + \mathbb{Z}p, d + \mathbb{Z}p^3).$$

Thus $K = L$. It follows that K is a complement in M.

Let N be a complement in M such that $N \neq 0, M, M_1, M_2$. Note that N is a maximal uniform submodule of M. Then $(a + \mathbb{Z}p, b + \mathbb{Z}p^3) \in N$ for some $a \notin \mathbb{Z}p$, $b \notin \mathbb{Z}p^3$. Without loss of generality, we can suppose $a = 1$.

Thus $\mathbb{Z}(1 + \mathbb{Z}p, b + \mathbb{Z}p^3) \subseteq N$ and then $\mathbb{Z}(1 + \mathbb{Z}p, b + \mathbb{Z}p^3)$ is essential in N. Thus $N = \mathbb{Z}(1 + \mathbb{Z}p, b + \mathbb{Z}p^3)$, because $\mathbb{Z}(1 + \mathbb{Z}p, b + \mathbb{Z}p^3)$ is a complement in M.

(ii) Let $N = \mathbb{Z}(1 + \mathbb{Z}p, p + \mathbb{Z}p^3)$. Then N is a complement in M of order p^2. If N were a direct summand of M, then $M = N \oplus N'$ for some $N' \leq M$, and hence N' has order p^2 also, giving $p^2M = 0$, a contradiction. Thus M is not a CS-module.

Theorem 1.2.4. [4, Theorem 2.4 and Corollary] Consider the following conditions.

(i) R is a right CS-ring.

(ii) Every non-zero complement right ideal of R is non-nil.

(iii) Every maximal uniform right ideal of R is generated by an idempotent element.

(a) The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) always hold.

(b) If R has no infinite sets of orthogonal idempotents and every non-zero complement right ideal of R contains a uniform right ideal then (i) and (ii) are equivalent.

(c) If R is left perfect (i.e. R satisfies DCC (descending chain condition) for principal right ideals) then (i), (ii), (iii) are equivalent.

Note that conditions (i) and (iii) of Theorem 1.2.4 are equivalent for a ring with finite uniform dimension, but the following example shows that a Noetherian ring satisfying (ii) of Theorem 1.2.4 need not be a right CS-ring.

Example 1.2.5. [4, Example 6.2] Let

$$R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}.$$

Then R is a Noetherian ring which satisfies (ii) of Theorem 1.2.4. But R is not a right CS-ring.

Proof. The only nilpotent right ideals of R are those of the form $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ where A is an ideal of $\mathbb{Z}$. If $A \neq 0$, then $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ is essential as a proper right R-submodule of $\begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$ if $A \neq \mathbb{Z}$; and, if $A = \mathbb{Z}$, $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$ is essential as a proper right R-submodule of $\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$. Thus every non-zero nil right ideal of R is not a complement, i.e. R satisfies (ii) of Theorem 1.2.4. To show that R is not a right CS-ring, let $u = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, then uR is a uniform right ideal of R. It is easy to see that the identity element of R is the only idempotent element e of R such that $eu = u$, i.e. such that $uR \subseteq eR$. Therefore R is not a right CS-ring.

Proposition 1.2.6. [21, Proposition 2.5] Let M be an indecomposable module which is CS. Then M is uniform.

Proof. Clear.

Lemma 1.2.7. [23, Lemma 3] Let M be a CS-module and suppose that R has ACC on right annihilators for M. Then M is a direct sum of uniform modules.

Theorem 1.2.8. [15, Theorem 1] Let R be a ring and M a right R-module. Then M is a CS-module if and only if $M = Z_2(M) \oplus N$ where $Z_2(M)$ and N are CS-modules and $Z_2(M)$ is N-injective.

Proof. Suppose first that M is a CS-module. Since $Z_2(M)$ is a complement in M, we have $M = Z_2(M) \oplus N$, where N is nonsingular. By Proposition 1.2.2, $Z_2(M)$ and N are CS-modules. Let $\varphi : X \to Z_2(M)$ be a homomorphism where $X \leq N$. Consider $X' = \{x - \varphi(x) : x \in X\}$. By hypothesis, there exists a direct summand L of M such that X' is essential in L. Write $M = L \oplus Y$. Since $X' \cap Z_2(M) = 0$ and X' is essential in L, it follows that L is nonsingular and that $Z_2(M) = Z_2(Y)$. Hence $Z_2(M)$ is a direct summand of Y, say $Y = Y' \oplus Z_2(M)$. Let $\pi : L \oplus Y' \oplus Z_2(M) \to Z_2(M)$ be the canonical projection. It is easy to see that $\pi|_X = \varphi$.

Conversely, let $M = Z_2(M) \oplus N$, where $Z_2(M)$ and N are CS-modules and $Z_2(M)$ is N-injective. Let A be a complement in M. Since $Z_2(A)$ is a complement in A then $Z_2(A)$ is a complement in M. But $Z_2(A) \leq Z_2(M)$, so that $Z_2(A)$ is a complement in $Z_2(M)$. Thus $Z_2(A)$ is a direct summand of $Z_2(M)$, therefore also of A. Write $A = Z_2(A) \oplus B$, where B is a nonsingular submodule of A. Since $B \cap Z_2(M) = 0$ and $Z_2(M)$ is N-injective, there exists a homomorphism $\theta : N \to Z_2(M)$ such that $\theta\pi_2|_B = \pi_1|_B$, where π_1, π_2 are the projections of M onto $Z_2(M)$ and N respectively. Consider $N' = \{n + \theta(n) : n \in N\}$. It follows that B is contained in N'. Since $N' \cong N$ is a CS-module, we have B is a direct summand of N'. It is clear that $M = Z_2(M) \oplus N'$. Therefore A is a direct summand of M.

Theorem 1.2.9. [15, Theorem 5] Let M be a torsion-free reduced CS-module over a commutative integral domain R. Then M is a finite direct sum of uniform modules.

Lemma 1.2.10. [3, Lemma 3] Let M be a CS-module such that $M/\operatorname{soc} M$ has finite uniform dimension. Then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 of M and submodule M_2 with finite uniform dimension.

Proposition 1.2.11. [3, Proposition 5] Let M be a CS-module. Then M has ACC (respectively, DCC) on essential submodules if and only if $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2.

1.3. Continuous and quasi-continuous modules.

Consider the following conditions on a module M :

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand of M.

(C_3) If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$. Then $M_1 \oplus M_2$ is a direct summand of M.

Definition 1.3.1. A module M is called *continuous* if it satisfies (C_1) (i.e CS) and (C_2), *quasi-continuous* if it satisfies (C_1) and (C_3).

Semisimple modules and injective modules are continuous modules. Uniform modules are quasi-continuous. Any continuous module is quasi-continuous as we shall see next.

<u>Proposition 1.3.2.</u> [21, Proposition 2.2] If a module M satisfies (C_2), then it satisfies (C_3).

Proof. Let K, L be direct summands of M with $K \cap L = 0$. Then $M = K \oplus K'$ for some $K' \leq M$. Let $\pi : M \to K'$ denote the canonical projection. Since $K \cap L = 0$, then $\pi(L) \cong L$ and $\pi(L) \leq K'$. But $\pi(L)$ is a direct summand of M by (C_2) and hence $M = \pi(L) \oplus L'$ for some $L' \leq M$. Thus

$$K' = \pi(L) \oplus (K' \cap L')$$

and $M = K \oplus \pi(L) \oplus (K' \cap L')$. Hence $K \oplus \pi(L)$ is a direct summand of M. But $K \oplus L \cong K \oplus \pi(L)$. Thus, $K \oplus L$ is a direct summand of M. Thus M satisfies (C_3).

<u>Proposition 1.3.3.</u> [21, Proposition 2.7] The conditions (C_2) and (C_3) are inherited by direct summands.

The following Proposition gives a necessary condition for $M_1 \oplus M_2$ to be quasi-continuous.

<u>Proposition 1.3.4.</u> [21, Proposition 2.10] If $M_1 \oplus M_2$ is quasi-continuous, then M_1 and M_2 are relatively injective.

Let S denote the endomorphism ring of a module M, J the Jacobson radical of S and $\Delta = \{f \in S : \ker f \text{ is essential in } M\}$.

Proposition 1.3.5. [21, Proposition 3.5] If M is a continuous module, then S/Δ is a (von Neumann) regular ring and Δ equals J.

Note that at chapter 5 the condition (C_1) will be weakened and similar result to Proposition 1.3.5 will be shown.

Chapter 2

RELATIVE INJECTIVITY AND CLS-MODULES

2.1. Lifting submodules

Let R be a ring with identity. Let M, X be unital right R-modules. We define lifting submodules for X in M and obtain basic properties of this class of submodules of M. In particular, we investigate relationships between the class and (C_1), (C_2) and (C_3). Finally, we define CLS-modules as being closed submodules which are direct summands.

Notation.

$\underline{\underline{L}}(M) = \{N : N \text{ is a submodule of } M\}$.

$\underline{\underline{E}}(M) = \{N : N \text{ is an essential submodule of } M\}$.

$\underline{\underline{D}}(M) = \{N : N \text{ is a direct summand of } M\}$.

$\underline{\underline{C}}(M) = \{N : N \text{ is a complement submodule of } M\}$.

Definition 2.1.1. A submodule N of M is called *a lifting submodule for X in M* provided for any $\varphi \in \mathrm{Hom}_R(N,X)$ there exists $\theta \in \mathrm{Hom}_R(M,X)$ such that $\varphi = \theta|_N$. We set

$$\mathrm{Lift}_X(M) = \{N : N \leq M \text{ and } N \text{ is a lifting submodule for } X \text{ in } M\}.$$

Clearly $0 \in \mathrm{Lift}_X(M)$ and $M \in \mathrm{Lift}_X(M)$. More generally, we have:

Lemma 2.1.2. $\underline{\underline{D}}(M) \subseteq \mathrm{Lift}_X(M)$.

Proof. Let $N \in \underline{\underline{D}}(M)$. Then $M = N \oplus N'$ for some submodule N' of M. Suppose

$\varphi \in \mathrm{Hom}_R(N,X)$. Define $\theta : M \to X$ by

$$\theta(n + n') = \varphi(n) \quad (n \in N,\ n' \in N').$$

It is easy to check $\theta \in \mathrm{Hom}_R(M,X)$ and $\varphi = \theta\,|_N$.

<u>Lemma 2.1.3</u>. The following statements are equivalent.

(i) X is M-injective.

(ii) $\mathrm{Lift}_X(M) = \underline{\underline{L}}(M)$.

(iii) $\underline{\underline{E}}(M) \subseteq \mathrm{Lift}_X(M)$.

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (i). Let N be a submodule of M. Let N′ be a complement of N in M. Then $N \oplus N' \in \underline{\underline{E}}(M)$. Let $\varphi \in \mathrm{Hom}_R(N,X)$. By Lemma 2.1.2, there exists $\theta \in \mathrm{Hom}_R(N \oplus N',X)$ such that $\theta\,|_N = \varphi$. By (iii), there exists $\chi \in \mathrm{Hom}_R(M,X)$ such that $\chi\,|_{N \oplus N'} = \theta$. Thus $\chi\,|_N = \varphi$. It follows that X is M-injective.

<u>Lemma 2.1.4</u>. Let $K \le N \le M$. Then

(i) $K \in \mathrm{Lift}_X(N)$, $N \in \mathrm{Lift}_X(M)$ implies that $K \in \mathrm{Lift}_X(M)$.

(ii) $K \in \mathrm{Lift}_X(M)$ implies that $K \in \mathrm{Lift}_X(N)$.

(iii) $N \in \mathrm{Lift}_X(M)$ implies that $N / K \in \mathrm{Lift}_X(M / K)$.

(iv) $K \in \mathrm{Lift}_X(M)$, $N / K \in \mathrm{Lift}_X(M / K)$ implies that $N \in \mathrm{Lift}_X(M)$.

Proof. (i) and (ii) are clear.

(iii). Let $\varphi \in \mathrm{Hom}_R(N / K,X)$. Let $\pi : N \to N / K$ denote the canonical projection. Then $\varphi\pi : N \to X$ is a homomorphism. Since $N \in \mathrm{Lift}_X(M)$, there exists $\theta \in \mathrm{Hom}_R(M,X)$ such that $\theta(n) = \varphi\pi(n) = \varphi(n + K)$ for all $n \in N$.
Define $\overline{\theta} : M / K \to X$ by $\overline{\theta}(m + K) = \theta(m)$ $(m \in M)$.

Suppose $m + K = m' + K$ where $m, m' \in M$. Then $m - m' \in K$ and hence $\varphi\pi(m - m') = 0$. Thus $\theta(m - m') = 0$ so that $\theta(m) = \theta(m')$. Hence $\overline{\theta}$ is well defined. Clearly $\overline{\theta} \in \mathrm{Hom}_R(M / K,X)$. For any $n \in N$, $\overline{\theta}(n + K) = \theta(n) = \varphi(n + K)$. It follows that $N / K \in \mathrm{Lift}_X(M / K)$.

(iv) Let $\varphi \in \mathrm{Hom}_R(N,X)$. Then $\varphi|_K \in \mathrm{Hom}_R(K,X)$. There exists $\theta \in \mathrm{Hom}_R(M,X)$ such that $\varphi|_K = \theta|_K$. Define $\chi : N / K \to X$ by

$$\chi(n + K) = \varphi(n) - \theta(n) \quad (n \in N).$$

Note that χ is well defined and a homomorphism. There exists $\psi \in \mathrm{Hom}_R(M / K,X)$ such that $\psi|_{N/K} = \chi$. Let $\pi : M \to M / K$ denote the canonical projection. Let $\alpha = \psi\pi + \theta \in \mathrm{Hom}_R(M,X)$. For any $n \in N$,

$$\alpha(n) = \psi\pi(n) + \theta(n) = \psi(n + K) + \theta(n) = \chi(n + K) + \theta(n) = \varphi(n).$$

Thus $\alpha|_N = \varphi$.

Corollary 2.1.5. For any $N \in \mathrm{Lift}_X(M)$, $\mathrm{Lift}_X(N) = \{K \leq N : K \in \mathrm{Lift}_X(M)\}$.

Proof. Suppose $K \in \mathrm{Lift}_X(N)$. Then $K \leq N$ and by Lemma 2.1.4(i), $K \in \mathrm{Lift}_X(M)$. Therefore, $\mathrm{Lift}_X(N) \subseteq \{K \leq N : K \in \mathrm{Lift}_X(M)\}$. Conversely, suppose $K \leq N$ and $K \in \mathrm{Lift}_X(M)$. By Lemma 2.1.4(ii), $K \in \mathrm{Lift}_X(N)$.

Let $K \leq N \leq M$. Then $K \in \mathrm{Lift}_X(M)$ does not imply $N \in \mathrm{Lift}_X(M)$, as the following example illustrates.

Example 2.1.6. Let X be a non-injective module. There exists $E \in \underline{\underline{E}}(R_R)$ such that $E \notin \mathrm{Lift}_X(R_R)$. Let $M = R \oplus R$, $K = R \oplus 0$, $N = R \oplus E$. Then $K \in \mathrm{Lift}_X(M)$ by Lemma 2.1.2 but $N \notin \mathrm{Lift}_X(M)$ by Lemma 2.1.4.

Lemma 2.1.7. Let $N, K \leq M$ such that $N + K$ and $N \cap K$ both belong to $\mathrm{Lift}_X(M)$. Then N and K both belong to $\mathrm{Lift}_X(M)$.

Proof. Let $\varphi \in \mathrm{Hom}_R(N,X)$. Then $\varphi|_{N \cap K} \in \mathrm{Hom}_R(N \cap K,X)$. There exists

$\theta_1 \in \mathrm{Hom}_R(M,X)$ such that $\theta_1 |_{N \cap K} = \varphi |_{N \cap K}$. Define $\chi : N + K \to X$ by

$$\chi(n + k) = \varphi(n) + \theta_1(k) \quad (n \in N,\ k \in K).$$

Suppose $n, n' \in N$; $k, k' \in K$ and $n + k = n' + k'$. Then $n - n' = k' - k$, so that $k' - k \in N \cap K$. Hence,

$$\theta_1(k') - \theta_1(k) = \theta_1(k - k') = \varphi(k' - k) = \varphi(n - n') = \varphi(n) - \varphi(n'),$$

which implies $\varphi(n) + \theta_1(k) = \varphi(n') + \theta_1(k')$. Thus χ is well defined. Clearly $\chi \in \mathrm{Hom}_R(N + K,X)$, so, by hypothesis, there exists $\theta \in \mathrm{Hom}_R(M,X)$ such that $\theta |_{N+K} = \chi$. For any $n \in N$,

$$\theta(n) = \chi(n) = \varphi(n).$$

Thus $\theta |_N = \varphi$. It follows that $N \in \mathrm{Lift}_X(M)$. Similarly $K \in \mathrm{Lift}_X(M)$.

<u>Corollary 2.1.8</u>. Let $K, N \leq M$.

(i) If $N \cap K = 0$ and $N \oplus K \in \mathrm{Lift}_X(M)$ then $N, K \in \mathrm{Lift}_X(M)$.

(ii) If $N + K = M$ and $N \cap K \in \mathrm{Lift}_X(M)$ then $N, K \in \mathrm{Lift}_X(M)$.

Proof. Clear by Lemma 2.1.7.

<u>Lemma 2.1.9</u>. Let $K \in \mathrm{Lift}_X(M)$, $N \leq M$. Suppose $N \cap K \in \mathrm{Lift}_X(K)$ and $(N + K) / K \in \mathrm{Lift}_X(M / K)$. Then $N \in \mathrm{Lift}_X(M)$.

Proof. By Lemma 2.1.4 (i) and (iv), we deduce $N \cap K$ and $N + K$ both belong to $\mathrm{Lift}_X(M)$. Apply Lemma 2.1.7.

<u>Corollary 2.1.10</u>. Let $K \leq M$. Then X is M-injective if and only if (i) X is K-injective, (ii) X is (M / K)-injective, and (iii) $K \in \mathrm{Lift}_X(M)$.

Proof. By Lemmas 2.1.4, 2.1.9.

Lemma 2.1.11. Let $X = \Pi_{\lambda \in \Lambda} X_\lambda$. Then $\mathrm{Lift}_X(M) = \bigcap_{\lambda \in \Lambda} \mathrm{Lift}_{X_\lambda}(M)$, for any module M.

Proof. Let $\lambda \in \Lambda$. Let $Y = X_\lambda$. Let $N \in \mathrm{Lift}_X(M)$. Let $\varphi \in \mathrm{Hom}_R(N,Y)$. Let $i : Y \to X$

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\varphi"'] & M \arrow[ddl, dashed, "\theta"] \\
& Y \arrow[d, shift right, "i"'] & \\
& X \arrow[u, shift right, "\pi"'] &
\end{tikzcd}$$

denote the inclusion mapping and $\pi : X \to Y$ the canonical projection. Then $i\varphi \in \mathrm{Hom}_R(N,X)$. By hypothesis there exists $\theta \in \mathrm{Hom}_R(M,X)$ such that $\theta|_N = i\varphi$. Now $\pi\theta \in \mathrm{Hom}_R(M,Y)$ and, for any $n \in N$,

$$\pi\theta(n) = \pi i\varphi(n) = \varphi(n).$$

Thus, $\varphi = \pi\theta|_N$. It follows that $N \in \mathrm{Lift}_Y(M)$. Hence $\mathrm{Lift}_X(M) \subseteq \mathrm{Lift}_Y(M)$. Therefore, $\mathrm{Lift}_X(M) \subseteq \bigcap_{\lambda \in \Lambda} \mathrm{Lift}_{X_\lambda}(M)$.

Conversely, let $K \in \bigcap_{\lambda \in \Lambda} \mathrm{Lift}_{X_\lambda}(M)$. Let $\alpha \in \mathrm{Hom}_R(K,X)$. For each $\lambda \in \Lambda$, let $\pi_\lambda : X \to X_\lambda$ denote the canonical projection. Then $\pi_\lambda\alpha \in \mathrm{Hom}_R(K,X_\lambda)$, $\lambda \in \Lambda$. By hypothesis, for each $\lambda \in \Lambda$, there exists $\beta_\lambda \in \mathrm{Hom}_R(M,X_\lambda)$ such that $\beta_\lambda(k) = \pi_\lambda\alpha(k)$, $k \in K$. Define $\beta : M \to X$ by

$$\beta(m) = \{\beta_\lambda(m)\}_{\lambda \in \Lambda} \quad (m \in M).$$

For each $k \in K$, $\beta(k) = \alpha(k)$. Thus $K \in \mathrm{Lift}_X(M)$.

Corollary 2.1.12. Let $X = \Pi_{\lambda \in \Lambda} X_\lambda$. Then X is M-injective if and only if X_λ is M-injective for all $\lambda \in \Lambda$.

Proof. By Lemmas 2.1.3 and 2.1.11.

Lemma 2.1.13. Let $\underline{\underline{A}}$ be a non-empty collection of submodules of M. Then the following statements are equivalent.

(i) $\underline{\underline{A}} \subseteq \underline{\underline{D}}(M)$.

(ii) $\underline{\underline{A}} \subseteq \mathrm{Lift}_X(M)$ for all right R-modules X.

(iii) $\underline{\underline{A}} \subseteq \mathrm{Lift}_X(M)$ for all $X \in \underline{\underline{A}}$.

Proof. (i) $\Rightarrow$ (ii). By Lemma 2.1.2.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Let $A \in \underline{\underline{A}}$. Consider the identity mapping $1 : A \to A$. Because $A \in \mathrm{Lift}_A(M)$ by (iii), there exists $\theta \in \mathrm{Hom}_R(M,A)$ such that $\theta(a) = a$ $(a \in A)$. It can easily be checked that $M = A \oplus (\ker\theta)$. Thus $A \in \underline{\underline{D}}(M)$. Hence $\underline{\underline{A}} \subseteq \underline{\underline{D}}(M)$.

Corollary 2.1.14. The following statements are equivalent for a module M.

(i) M is semisimple.

(ii) Every right R-module X is M-injective.

(iii) Every submodule of M is M-injective.

Proof. Apply Lemma 2.1.13 to $\underline{\underline{A}} = \underline{\underline{L}}(M)$, and use Lemma 2.1.3. (see [1, Theorem 9.6]).

Corollary 2.1.15. The following statements are equivalent for a module M.

(i) M is a CS-module (i.e. $\underline{\underline{C}}(M) \subseteq \underline{\underline{D}}(M)$).

(ii) $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_X(M)$ for all right R-modules X.

(iii) $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_X(M)$ for all $X \in \underline{\underline{C}}(M)$.

Proof. Apply Lemma 2.1.13 to $\underline{\underline{A}} = \underline{\underline{C}}(M)$.

Example 2.1.16. Let p be any prime integer and let R denote the local ring $\mathbb{Z}_p$. Let M denote the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$. Then

(i) M is an R-module.

(ii) $K \in \underline{\underline{C}}(M)$ if and only if $K \in \underline{\underline{D}}(M)$ or $K = R(1 + \mathbb{Z}p, q)$ for some non-zero element q in $\mathbb{Q}$.

(iii) $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M)$, but M is not a CS-module.

Proof. (i) Let $M_1 = (\mathbb{Z}/\mathbb{Z}p) \oplus 0$ and $M_2 = 0 \oplus \mathbb{Q}$, so that M is the direct sum $M_1 \oplus M_2$ of its submodules M_1, M_2. The ring R is the subring of $\mathbb{Q}$ consisting of all rational numbers s/t such that $s, t \in \mathbb{Z}$, $t \neq 0$ and t is coprime to p. Note first that for any element m in M and any $s, t \in \mathbb{Z}$ such that p does not divide t, there exists a unique element $m' \in M$ such that $tm' = sm$, and we shall denote m' by $(s/t)m$. In this way M is an R-module.

(ii) Let $q \in \mathbb{Q}$ and $K = R(1 + \mathbb{Z}p, q)$. We show first that $K \in \underline{\underline{C}}({}_{\mathbb{Z}}M)$. Note that K is a uniform submodule of M. Suppose that N is a submodule of M such that $K \in \underline{\underline{E}}(N)$. Let $x \in N$. Then $U = \mathbb{Z}x + \mathbb{Z}(1 + \mathbb{Z}p, q)$ is a finitely generated uniform $\mathbb{Z}$-module, and hence U is cyclic (see [7, volume I, Theorem 15.5]). Suppose that $U = \mathbb{Z}(a + \mathbb{Z}p, b)$, where $a \in \mathbb{Z}$, $b \in \mathbb{Q}$. There exists $n \in \mathbb{Z}$ such that $(1 + \mathbb{Z}p, q) = n(a + \mathbb{Z}p, b)$. Note that $1 - na \in \mathbb{Z}p$, and hence n is coprime to p, and $(a + \mathbb{Z}p, b) \in R(1 + \mathbb{Z}p, q) = K$. Thus $x \in K$. It follows that $K = N$. Hence $K \in \underline{\underline{C}}(M)$.

Let $L \in \underline{\underline{C}}(M)$ and suppose $L \neq M$. Note that M has uniform dimension 2 and hence L is uniform (see [6, Lemma 1.9]). We shall show first that L is an R-submodule of M. Let

$$L' = \{m \in M : tm \in L \text{ for some } t \in \mathbb{Z},\ t \text{ coprime to } p\}.$$

Then L' is a submodule of M containing L, in fact $L' = RL$. If $0 \neq m \in L'$ then $tm \in L$ for some $t \in \mathbb{Z}$, coprime to p, and hence $tm \neq 0$. It follows that $L \in \underline{\underline{E}}(L')$. Thus $L = L'$, and L is an R-submodule of M.

Next we show that $L = 0$, M, M_1, M_2 or $R(1 + \mathbb{Z}p,q)$ for some $q \in \mathbb{Q}$. Suppose that $L \neq 0$, M, M_1 or M_2. Note that M_1 and M_2 are both uniform, so that L is not contained in either M_1 or M_2. Thus $(c + \mathbb{Z}p,d) \in L$ for some $c \in \mathbb{Z}$, coprime to p and $0 \neq d \in \mathbb{Q}$. Without loss of generality we can suppose that $c = 1$. Because L is an R-submodule of M, $R(1 + \mathbb{Z}p,d) \subseteq L$. But $R(1 + \mathbb{Z}p,d) \in \underline{\underline{C}}(M)$, and so $L = R(1 + \mathbb{Z}p,d)$. This completes the proof of (ii).

(iii) Note that $K = R(1 + \mathbb{Z}p,1)$ is a complement in M. Suppose $K \in \underline{\underline{D}}(M)$. Then $M = K \oplus L$ for some $L \leq M$. Let $(a + \mathbb{Z}p,b) \in L$ $(a \in \mathbb{Z},\ b = m / n \in \mathbb{Q})$. Thus $p(a + \mathbb{Z}p,b) = (0 + \mathbb{Z}p,pm / n) \in L$. Therefore

$$n(0 + \mathbb{Z}p,pm / n) = (0 + \mathbb{Z}p,pm) = (0 + \mathbb{Z}p,0),$$

because $K \cap L = 0$. Hence $npb = pm = 0$ gives $b = 0$, so that if $x \in L$ then $x = (y + \mathbb{Z}p,0)$ $(y \in \mathbb{Z})$. Thus $L \leq M_1$ which is simple, so $L = M_1$. Now $M = K \oplus M_1$. Hence

$$K \cong M / M_1 \cong M_2 \cong \mathbb{Q} \cong p\mathbb{Q}.$$

Then there exists an element $(c + \mathbb{Z}p,d) \in K$ such that $(1 + \mathbb{Z}p,1) = p(c + \mathbb{Z}p,d) = (0 + \mathbb{Z}p,pd)$ $(c \in \mathbb{Z},\ d \in \mathbb{Q})$. Thus $1 \in \mathbb{Z}p$, a contradiction. Hence $K \notin \underline{\underline{D}}(M)$. Thus M is not a CS-module. To show that $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M)$, it is sufficient to prove that for any non-zero $q \in \mathbb{Q}$ and any homomorphism $\varphi : R(1 + \mathbb{Z}p,q) \to M$, φ can be lifted to an endomorphism θ of M. Let $K = R(1 + \mathbb{Z}p,q)$. Suppose that $\varphi(1 + \mathbb{Z}p,q) = (a + \mathbb{Z}p,b)$, for some $a \in \mathbb{Z}$, $b \in \mathbb{Q}$. Define a mapping $\theta : M \to M$ by

$$\theta(c + \mathbb{Z}p,d) = (ca + \mathbb{Z}p,db / q) \quad (c \in \mathbb{Z},\ d \in \mathbb{Q}).$$

It is clear that θ is well defined. It can be checked that $\theta : M \to M$ is a homomorphism and that φ is the restriction of θ to K. Thus $K \in \mathrm{Lift}_M(M)$.

Hence, if M is a CS-module, then $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M)$, but not conversely. Even if M is a finitely generated module which satisfies $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M)$, M need not be a CS-module. As we see in the next example.

Example 2.1.17. Let $R = \mathbb{Z}$. Any finitely generated torsion-free R-module is a CS-module, but there exists a finitely generated R-module M such that $\underline{\underline{C}}(M)$ is not contained in $\mathrm{Lift}_M(M)$.

Proof. Let M' be any finitely generated torsion-free R-module. Let N be a submodule of M'. Let $N \leq K \leq M'$ such that K/N is the torsion submodule of M'/N. Then M'/K is finitely generated torsion-free, hence free, so $K \in \underline{\underline{D}}(M')$. Also $N \in \underline{\underline{E}}(K)$ because K is torsion-free and K/N torsion. Thus M' is a CS-module.

Now, let p be any prime, $M_1 = \mathbb{Z}/\mathbb{Z}p$, $M_2 = \mathbb{Z}$ and $M = M_1 \oplus M_2$. Let L denote the cyclic submodule $\mathbb{Z}(1 + \mathbb{Z}p, p)$ of M. Since, as an abelian group, L is infinite cyclic, it follows that L is a uniform $\mathbb{Z}$-module. Suppose that K is a submodule of M and $L \in \underline{\underline{E}}(K)$. Then K is uniform, and hence cyclic, because K is a finitely generated abelian group. There exist elements $a, b \in \mathbb{Z}$ such that $K = \mathbb{Z}(a + \mathbb{Z}p, b)$. Now there exists $n \in \mathbb{Z}$ such that

$$(1 + \mathbb{Z}p, p) = n(a + \mathbb{Z}p, b)$$

and hence $1 - na \in \mathbb{Z}p$ and $p = nb$. It follows that $n = 1$ or -1, and hence $L = K$. Thus $L \in \underline{\underline{C}}(M)$.

We claim $L \notin \mathrm{Lift}_M(M)$. Suppose not. Define $\varphi \in \mathrm{Hom}_R(L,M)$ by

$$\varphi(1 + \mathbb{Z}p, p) = (0,1).$$

There exists $\theta \in \mathrm{Hom}_R(M,M)$ such that $\theta|_L = \varphi$.

Suppose $\theta(1 + \mathbb{Z}p, 0) = (a_1 + \mathbb{Z}p, b_1)$, $\theta(0,1) = (a_2 + \mathbb{Z}p, b_2)$, for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. Then $p(a_1 + \mathbb{Z}p, b_1) = 0$ implies $b_1 = 0$. Hence

$$(0,1) = \varphi(1 + \mathbb{Z}p, p) = \theta(1 + \mathbb{Z}p, p) = (a_1 + \mathbb{Z}p, 0) + p(a_2 + \mathbb{Z}p, b_2),$$

and this implies $1 = pb_2$, a contradiction. Thus $L \notin \mathrm{Lift}_M(M)$.

Proposition 2.1.18. Let ${}_{\mathbb{Z}}M = M_1 \oplus M_2$ where M_1 is torsion and M_2 is infinite cyclic. If M satisfies $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M)$ then $M_1 = pM_1$ for each prime p.

Proof. Let $M_2 = \mathbb{Z}m_2 \neq 0$ for some $m_2 \in M_2$. Suppose $M_1 \neq pM_1$ for some prime p. Let $m_1 \in M_1$, $m_1 \notin pM_1$. Let $K = \mathbb{Z}(m_1,pm_2)$. Suppose $K \in \underline{\underline{E}}(L)$ for some $L \leq M$. Then for any $n \in \mathbb{Z}$,

$n(m_1,pm_2) = (nm_1,npm_2) = (0,0) \Leftrightarrow nm_1 = 0,\ npm_2 = 0 \Leftrightarrow n = 0$ (M_2 is infinite cyclic). Therefore K is infinite cyclic, and hence K is a uniform $\mathbb{Z}$-module. Thus L is a uniform $\mathbb{Z}$-module. Let $x \in L$ and $a = (m_1,pm_2)$. Then $K + \mathbb{Z}x = \mathbb{Z}a + \mathbb{Z}x$ is finitely generated, so that $K + \mathbb{Z}x \leq L$, and is a direct sum of cyclic modules. But $K + \mathbb{Z}x$ is uniform, hence $K + \mathbb{Z}x$ is cyclic. Then $\mathbb{Z}a \subseteq K + \mathbb{Z}x = \mathbb{Z}y$ for some $y \in M$. Suppose $y = (m_1',km_2)$ for some $m_1' \in M_1$ and $k \in \mathbb{Z}$. Then $a = sy$ for some $s \in \mathbb{Z}$. Hence

$$(m_1,pm_2) = s(m_1',km_2),$$

which gives $m_1 = sm_1'$, $pm_2 = skm_2$. Since M_2 is infinite cyclic, $s = \pm 1$ or $k = \pm 1$. If $k = \pm 1$ then $s = \pm p$ so that $m_1 = \pm pm_1' \in pM_1$, a contradiction. Thus $s = \pm 1$. Therefore $y \in \mathbb{Z}a$ and hence $x \in \mathbb{Z}y \subseteq \mathbb{Z}a$, i.e. $L \subseteq \mathbb{Z}a = K$. Hence $K = L$ so $K \in \underline{\underline{C}}(M)$.

Now define a homomorphism $\varphi : K \to M$ by

$$\varphi(m_1,pm_2) = (0,m_2).$$

Suppose that φ can be lifted to $\theta : M \to M$. Then $\theta(m_1,0) = (u,0)$ for some $u \in M_1$ and $\theta(0,m_2) = (v,tm_2)$ for some $v \in M_1$, $t \in \mathbb{Z}$. Hence

$$(0,m_2) = \varphi(m_1,pm_2) = \theta(m_1,pm_2) = \theta(m_1,0) + p\theta(0,m_2) = (u,0) + p(v,tm_2).$$

Then we obtain, $0 = u + pv$, $m_2 = ptm_2$, so that $1 = pt$, a contradiction. Therefore φ cannot be lifted. It follows that $M_1 = pM_1$ for each prime p.

2.2. Lifting submodules with (C_2) or (C_3)

Let M be a module and $\underline{\underline{A}}(M)$ a non-empty collection of submodules of M. Let n be a positive integer.

Notation.

$\underline{\underline{A}}'(M) = \{N \lesssim M$: there exists $K \in \underline{\underline{A}}(M)$ such that $K \cong N\}$, and

$\underline{\underline{A}}^{(n)}(M) = \{L_1 + L_2 + \ldots + L_n : L_i \in \underline{\underline{A}}(M)$ for $1 \leq i \leq n$ and $L_1 + L_2 + \ldots + L_n$ is a direct sum$\}$.

Note that, the condition (C_2) (respectively (C_3)), becomes $\underline{\underline{D}}'(M) \subseteq \underline{\underline{D}}(M)$ (respectively $\underline{\underline{D}}^{(2)}(M) \subseteq \underline{\underline{D}}(M)$) with the above notation.

Proposition 2.2.1. The following statements are equivalent for a module M.

(i) M has (C_2).

(ii) $\underline{\underline{D}}'(M) \subseteq \mathrm{Lift}_X(M)$ for all right R-modules X.

(iii) $\underline{\underline{D}}'(M) \subseteq \mathrm{Lift}_X(M)$ for all $X \in \underline{\underline{D}}'(M)$.

(iv) $\underline{\underline{D}}'(M) \subseteq \mathrm{Lift}_M(M)$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). By Lemma 2.1.13.

(iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (i). Let $N' \in \underline{\underline{D}}'(M)$. Then there exist $N \in \underline{\underline{D}}(M)$ and an isomorphism $\varphi : N' \to N$. By Lemma 2.1.11, $\mathrm{Lift}_M(M) \subseteq \mathrm{Lift}_N(M)$. Thus, by (iv), $N' \in \mathrm{Lift}_N(M)$, and there exists $\theta \in \mathrm{Hom}_R(M,N)$ such that $\theta|_{N'} = \varphi$. For any $m \in M$, $\theta(m) \in N$ and hence $\theta(m) = \varphi(n')$ for some $n' \in N'$. Thus $\theta(m) = \theta(n')$ and so $m - n' \in \ker\theta$. It follows that $M = N' + (\ker\theta)$. But $N' \cap (\ker\theta) = \ker\varphi = 0$. Thus $M = N' \oplus (\ker\theta)$. Therefore, $N' \in \underline{\underline{D}}(M)$. It follows that $\underline{\underline{D}}'(M) \subseteq \underline{\underline{D}}(M)$.

Proposition 2.2.2. The following statements are equivalent for a module M.

(i) M has (C_3).

(ii) $\underline{\underline{D}}^{(2)}(M) \subseteq \mathrm{Lift}_X(M)$ for all right R-modules X.

(iii) $\underline{\underline{D}}^{(2)}(M) \subseteq \mathrm{Lift}_X(M)$ for all $X \in \underline{\underline{D}}^{(2)}(M)$.

(iv) $\underline{\underline{D}}^{(2)}(M) \subseteq \mathrm{Lift}_M(M)$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). By Lemma 2.1.13.

(iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (i). Let K, $L \in \underline{\underline{D}}(M)$ with $K \cap L = 0$. Let $\pi : K \oplus L \to K$ denote the canonical projection. By (iv) and Lemma 2.1.11,

$$K \oplus L \in \mathrm{Lift}_M(M) \subseteq \mathrm{Lift}_K(M)$$

and hence there exists $\theta \in \mathrm{Hom}_R(M,K)$ such that $\theta|_{K \oplus L} = \pi$. It follows that $M = K \oplus (\ker\theta)$. Now $\theta(L) = \pi(L) = 0$ implies $L \subseteq \ker\theta$. But $M = L \oplus L'$ for some submodule L' of M. Thus $\ker\theta = L \oplus (\ker\theta \cap L')$, and hence $M = K \oplus L \oplus (\ker\theta \cap L')$. It follows that $K \oplus L \in \underline{\underline{D}}(M)$. Hence M has (C_3).

Note. Suppose M has (C_3), i.e $\underline{\underline{D}}^{(2)}(M) \subseteq \underline{\underline{D}}(M)$. Then $\underline{\underline{D}}^{(n)}(M) \subseteq \underline{\underline{D}}(M)$ for all positive integer n. For, suppose $n \geq 3$ and $L_i \in \underline{\underline{D}}(M)$ $(1 \leq i \leq n)$ with $L_1 + \ldots + L_n$ a direct sum. By induction $L_1 + \ldots + L_{n-1} \in \underline{\underline{D}}(M)$ and hence $L_1 + \ldots + L_n \in \underline{\underline{D}}^{(2)}(M) \subseteq \underline{\underline{D}}(M)$.

Lemma 2.2.3. Let X be any right R-module. Then the following statements are equivalent for a module M.

(i) $\underline{\underline{C}}^{(2)}(M) \subseteq \mathrm{Lift}_X(M)$.

(ii) $\underline{\underline{C}}^{(n)}(M) \subseteq \mathrm{Lift}_X(M)$ for all $n \geq 2$.

Proof. (ii) $\Rightarrow$ (i). Obvious.

(i) $\Rightarrow$ (ii). Suppose (i) holds. Let $k \geq 3$ and $N_i \in \underline{\underline{C}}(M)$ $(1 \leq i \leq k)$ such that $N_1 + \ldots + N_k$ is a direct sum. Let $N = N_1 + \ldots + N_k$ and let $\varphi \in \mathrm{Hom}_R(N,X)$. There exists $N' \in \underline{\underline{C}}(M)$ such that $N_2 + \ldots + N_k \in \underline{\underline{E}}(N')$. By induction, $N_2 + \ldots + N_k \in \mathrm{Lift}_X(M)$ and hence there exists $\alpha \in \mathrm{Hom}_R(M,X)$ such that

$$\alpha(m) = \varphi(m) \quad (m \in N_2 + \ldots + N_k).$$

Now $N_1 \cap N' = 0$, because $N_1 \cap (N_2 + \ldots + N_k) = 0$, so we can define

$\beta \in \mathrm{Hom}_R(N_1 \oplus N', X)$ by

$$\beta(n + n') = \varphi(n) + \alpha(n') \quad (n \in N_1,\ n \in N').$$

Then, by (i), there exists $\delta \in \mathrm{Hom}_R(M,X)$ such that $\delta|_{N_1 \oplus N'} = \beta$.

For any $n_i \in N$ $(1 \le i \le k)$,

$$\delta(n_1 + \dots + n_k) = \beta(n_1 + \dots + n_k) = \varphi(n_1) + \alpha(n_2 + \dots + n_k)$$
$$= \varphi(n_1) + \varphi(n_2 + \dots + n_k) = \varphi(n_1 + \dots + n_k).$$

Thus $\delta|_N = \varphi$. It follows that $N \in \mathrm{Lift}_X(M)$. Hence $\underline{\underline{C}}^{(k)}(M) \subseteq \mathrm{Lift}_X(M)$.

<u>Lemma 2.2.4.</u> Let $K \le M_R$. Then M/K is nonsingular if and only if $m \in M$, $E \in \underline{\underline{E}}(R_R)$ and $mE \le K$ implies $m \in K$.

Proof. Suppose M/K is nonsingular. Let $m \in M$ and $mE \le K$ where $E \in \underline{\underline{E}}(R_R)$. Thus $(m + K)E = 0$ in M/K. Hence $m + K = 0$. It follows that $m \in K$. Conversely, suppose that $mE \le K$ implies $m \in K$. Let $x \in Z(M/K)$. Then $x = y + K$ for some $y \in M$ and $xF = 0$ for some $F \in \underline{\underline{E}}(R_R)$. Thus $(y + K)F = 0$ and hence $yF + K = 0$ in M/K. It follows that $yF \le K$ so that $y \in K$. Therefore $x = y + K = 0$ in M/K.

<u>Lemma 2.2.5.</u> [28, Lemma 2.3] Let M be a nonsingular module and $K \le M$. Then $K \in \underline{\underline{C}}(M)$ if and only if M/K is nonsingular.

Proof. Suppose M/K is nonsingular. Let N be a submodule of M such that $K \in \underline{\underline{E}}(N)$. Then $N/K \le Z(M/K)$ so that $N/K = 0$, and hence $K = N$. Thus $K \in \underline{\underline{C}}(M)$. (This part is true for any module). Conversely, suppose M/K is not nonsingular. There exists $m \in M$, $m \notin K$ such that $mE \le N$ for some $E \in \underline{\underline{E}}(R_R)$. Let $r \in R$, $k \in K$ and consider $mr + k$. Let

$$F = \{s \in R : rs \in E\}.$$

Then $F \in \underline{\underline{E}}(R_R)$ and $(mr + k)F \leq K$. If $mr + k \neq 0$ then $(mr + k)F \neq 0$ and hence $K \cap (mr + k)R \neq 0$. Thus $K \in \underline{\underline{E}}(mR + K)$. Thus $K \notin \underline{\underline{C}}(M)$.

Note that $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M)$ does not imply $\underline{\underline{C}}^{(2)}(M) \subseteq \mathrm{Lift}_M(M)$ (see Example 2.1.16).

<u>Lemma 2.2.6.</u> Consider the following conditions for any submodule N of a module M :

(i) $\theta(M) \leq X$ for any $\theta \in \mathrm{Hom}_R(M,E(X))$ with $\theta(N) \leq X$.

(ii) $N \in \mathrm{Lift}_X(M)$.

(iii) $\theta(M) \leq X$ for any $\theta \in \mathrm{Hom}_R(M,E(X))$ with $\theta(N) \leq X$ and $\theta^{-1}(X) \in \mathrm{Lift}_X(M)$.

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii). Let $\varphi \in \mathrm{Hom}_R(N,X)$. Then there exists $\theta \in \mathrm{Hom}_R(M,E(X))$ such that $\theta|_N = i\varphi$, where $i : X \to E(X)$ is the inclusion mapping. Thus $\theta(N) \leq X$. By hypothesis, $\theta(M) \leq X$ and hence $\theta \in \mathrm{Hom}_R(M,X)$. It follows that $N \in \mathrm{Lift}_X(M)$.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $\theta \in \mathrm{Hom}_R(M,E(X))$ such that $N \leq \theta^{-1}(X) \in \mathrm{Lift}_X(M)$. There exists $\theta' \in \mathrm{Hom}_R(M,X)$ such that $\theta'(k) = \theta(k)$ $(k \in \theta^{-1}(X))$. Consider $\theta - \theta' : M \to E(X)$. If $(\theta - \theta')(M) \neq 0$ then $(\theta - \theta')(M) \cap X \neq 0$, and hence there exists $0 \neq x \in X$, $m \in M$ such that $x = (\theta - \theta')(m) = \theta(m) - \theta'(m)$. Thus $\theta(m) = x + \theta'(m) \in X$ and hence $m \in \theta^{-1}(X)$. In this case, $\theta'(m) = \theta(m)$, so that $x = 0$, a contradiction. Thus $(\theta - \theta')(M) = 0$ and hence $\theta(M) = \theta'(M) \leq X$.

2.3. CLS-modules.

Even though it is not the first aim of this work it will be interesting to work on CLS-modules as a class of modules which contain the class of CS-modules.

Definition 2.3.1. Let M be a right R-module and N be a submodule of M. Then N is a *closed submodule of M* provided M / N is nonsingular. Note that the concept "closed submodule" has been used by some other authors. For example, according to [21], complement and closed submodule are the same. However, in [8], closed submodule is in the sense of complement submodule as in this thesis.

Let $\underline{\underline{CL}}(M) = \{N : N \text{ is a closed submodule of } M\}$. Then the following provides the link between $\underline{\underline{C}}(M)$ and $\underline{\underline{CL}}(M)$ for a module M.

Lemma 2.3.2. Let M be a module. Then

(i) $\underline{\underline{CL}}(M) \subseteq \underline{\underline{C}}(M)$.

(ii) For M nonsingular, $\underline{\underline{CL}}(M) = \underline{\underline{C}}(M)$.

Proof. Obvious by Lemma 2.2.5.

Observe that for a module M, $\underline{\underline{C}}(M) \subseteq \underline{\underline{CL}}(M)$ is not true in general as shown in the following example.

Example 2.3.3. Let K be a field and V be a vector space over K such that $\dim_K V \geq 2$. Let

$$R = \begin{bmatrix} K & V \\ 0 & K \end{bmatrix} = \left\{ \begin{bmatrix} k & v \\ 0 & k \end{bmatrix} : k \in K,\ v \in V \right\}.$$

Hence clearly R is a commutative ring with respect to the usual matrix operations. Then $\underline{\underline{C}}(R_R)$ is not contained in $\underline{\underline{CL}}(R_R)$.

Proof. Let $E = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$. Then $E \in \underline{\underline{E}}(R_R)$. Let $F_v = \begin{bmatrix} 0 & Kv \\ 0 & 0 \end{bmatrix}$ $(v \in V)$.

Suppose that $G \leq R$ such that $F_v \in \underline{\underline{E}}(G)$. Then $F_v \in \underline{\underline{E}}(G \cap E)$ and hence $F_v = G \cap E$. Let $\begin{bmatrix} k & w \\ 0 & k \end{bmatrix} \in G$ for some $w \in V$, $0 \neq k \in K$. Let $x \in V$ such that $x \notin Kv$. Thus

$$\begin{bmatrix} k & w \\ 0 & k \end{bmatrix} \begin{bmatrix} 0 & (1/k)x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \in G \cap E.$$

Therefore $x \in Kv$, a contradiction. Thus $k = 0$. Hence $G \leq E$ so $F_v = G$. It follows that $F_v \in \underline{\underline{C}}(R_R)$ for all $v \in V$. But $E^2 = 0$ so $E^2 \leq F_v$. However E is not contained in F_v. Thus $F_v \notin \underline{\underline{CL}}(R_R)$.

Definition 2.3.4. A module M is called a *CLS-module* if every closed submodule of M is a direct summand of M.

Clearly, over a commutative integral domain R, any torsion module M is a CLS-module. Moreover,

Corollary 2.3.5. (i) Every CS-module is a CLS-module. In particular, any injective module is a CLS-module.

(ii) Every nonsingular CLS-module is a CS-module.

Proof. By Lemma 2.3.2.

Lemma 2.3.6. Any direct summand of a CLS-module is a CLS-module.

Proof. Suppose $M = K \oplus K'$. Let $L \in \underline{\underline{CL}}(K)$. Then $L \oplus K' \in \underline{\underline{CL}}(M)$, because

$$M / (L \oplus K') = (K \oplus K') / (L \oplus K') \cong K / L.$$

Thus, $L \oplus K' \in \underline{\underline{D}}(M)$, so that $L \in \underline{\underline{D}}(M)$ and hence $L \in \underline{\underline{D}}(K)$. Thus K is a CLS-module.

Proposition 2.3.7. A module M is a CLS-module if and only if there exists a submodule M' of M such that $M = Z_2(M) \oplus M'$ and M' is a CS-module.

Proof. Suppose M is a CLS-module. Then $Z_2(M) \in \underline{\underline{D}}(M)$, so that $M = Z_2(M) \oplus M'$ for some submodule M' of M. Note that M' is nonsingular and, by Lemma 2.3.6, a CLS-module. Thus M' is a CS-module by Corollary 2.3.5.

Conversely, suppose $M = Z_2(M) \oplus M'$ for some CS-module M'. Let $K \in \underline{\underline{CL}}(M)$. Then $Z(M) \leq K$ and hence $Z_2(M) \leq K$. Thus $K = Z_2(M) \oplus (K \cap M')$. Now $M / K \cong M' / (K \cap M')$, so that $K \cap M' \in \underline{\underline{CL}}(M')$. Thus by Corollary 2.3.5, $M' = (K \cap M') \oplus K'$ for some submodule K'. Hence $M = K \oplus K'$. It follows that M is a CLS-module.

Note. In Proposition 2.3.7, M' is $Z_2(M)$-injective. For suppose N is a submodule of $Z_2(M)$ and $\varphi : N \to M'$ a homomorphism. If $x \in N \cap Z(M)$ then $xE = 0$ for some $E \in \underline{\underline{E}}(R_R)$, so that $\varphi(x)E = 0$ and hence $\varphi(x) = 0$, because M' is nonsingular. Thus $\varphi(N \cap Z(M)) = 0$. If $y \in N$ then $yF \subseteq N \cap Z(M)$ for some $F \in \underline{\underline{E}}(R_R)$, so that $\varphi(y)F = 0$ and hence $\varphi(y) = 0$. Thus $\varphi(y) = 0$ for all $y \in N$. Thus $\varphi = 0$, and φ can be lifted to $Z_2(M)$.

Theorem 2.3.8. Suppose a right R-module M is a direct sum $M_1 \oplus M_2$ of CLS-modules M_1 and M_2, such that M_1 is M_2-injective. Then M is a CLS-module.

Proof. Let $N \in \underline{\underline{CL}}(M)$. Then M/N is nonsingular. Now, $M_1/(N \cap M_1) \cong (M_1 + N)/N$ implies $N \cap M_1 \in \underline{\underline{CL}}(M_1)$. Thus $N \cap M_1 \in \underline{\underline{D}}(M_1)$ and hence $N \cap M_1 \in \underline{\underline{D}}(M)$. It follows that $N \cap M_1 \in \underline{\underline{D}}(N)$. Hence $N = (N \cap M_1) \oplus K$ for some submodule K of N. Let $\pi_i : M \to M_i$ $(i = 1, 2)$ denote the canonical projections. Consider the diagram

$$\begin{array}{ccccl} 0 & \longrightarrow & K & \xrightarrow{\alpha} & M_2 \quad \text{exact} \\ & & \beta \downarrow & & \\ & & M_1 & & \end{array}$$

where $\alpha = \pi_2|_K$ and $\beta = \pi_1|_K$. Note that α is a monomorphism and M_1 is M_2-injective. Thus there exists a homomorphism $\varphi : M_2 \to M_1$ such that $\varphi\alpha = \beta$. Let $L = \{x + \varphi(x) : x \in M_2\}$. Then it can easily be checked that L is a submodule of M and $L \cong M_2$. Moreover, $M = M_1 \oplus L$. If $k \in K$ then $k = m_1 + m_2$ for some $m_i \in M_i$ $(i = 1, 2)$. Then

$$m_1 = \beta(k) = \varphi\alpha(k) = \varphi(m_2)$$

and this implies $k = \varphi(m_2) + m_2 \in L$. Thus $K \subseteq L$. Since M/N is nonsingular it follows that L/K is nonsingular. Hence $K \in \underline{\underline{CL}}(L)$. But $L \cong M_2$, so that L is a CLS-module and $K \in \underline{\underline{D}}(L)$, and hence $N \in \underline{\underline{D}}(M)$. It follows that M is a CLS-module.

<u>Corollary 2.3.9</u>. Suppose a nonsingular right R-module M is a direct sum $M_1 \oplus M_2$ of CS-modules M_1, M_2, such that M_1 is M_2-injective. Then M is a CS-module.

Proof. By Theorem 2.3.8 and Corollary 2.3.5. (see [15, Theorem 1]).

<u>Corollary 2.3.10</u>. Suppose a right R-module M is a direct sum $M_1 \oplus M_2$ of CS-modules M_1, M_2 such that M_1 is M_2-injective and M_2 is nonsingular. Then M is a CS-module.

Proof. It is clear that $Z_2(M) = Z_2(M_1) \in \underline{\underline{D}}(M_1)$. Thus $M_1 = Z_2(M) \oplus M_1'$ for some nonsingular submodule M_1'. Now

$$M = Z_2(M) \oplus M_1' \oplus M_2.$$

Note that M_1' is M_2-injective, M_1' is a CS-module and $M_1' \oplus M_2$ is nonsingular. By Corollary 2.3.9, $M_1' \oplus M_2$ is a CS-module. But, by [15, Theorem 1], $Z_2(M)$ is M_1'-injective. Thus $Z_2(M)$ is $(M_1' \oplus M_2)$-injective. Again, by [15, Theorem 1], M is a CS-module.

Remark. Suppose $M = M_1 \oplus M_2$, where M_1 and M_2 are CS-modules such that M_1 is M_2-injective. Then M is a CS-module if and only if $Z_2(M)$ is a CS-module.

Proof. The necessity is clear by [15, Theorem 1]. Conversely, suppose that $Z_2(M) = Z_2(M_1) \oplus Z_2(M_2)$ is a CS-module. There exist submodules M_1' of M_1 and M_2' of M_2 such that

$$M_1 = Z_2(M_1) \oplus M_1' \text{ and } M_2 = Z_2(M_2) \oplus M_2'.$$

Then $M = [Z_2(M_1) \oplus Z_2(M_2)] \oplus [M_1' \oplus M_2']$. By [15, Theorem 1] and the fact that M_1 is M_2-injective, we know that $Z_2(M_1) \oplus Z_2(M_2)$ is $(M_1' \oplus M_2')$-injective. Also $M_1' \oplus M_2'$, being nonsingular, is a CS-module by Corollary 2.3.9. By [15, Theorem 1] again, $M = M_1 \oplus M_2$ is a CS-module.

Example 2.3.11. Let M be the $\mathbb{Z}$-module $(\mathbb{Z} / \mathbb{Z}p) \oplus \mathbb{Q}$. Then M is a CLS-module (but not a CS-module). Also M satisfies (C_2).

Proof. Recall that M is not a CS-module (Example 2.1.16). Clearly $(\mathbb{Z} / \mathbb{Z}p) \oplus 0$, $M \in \underline{\underline{CL}}(M)$ and $0 \oplus \mathbb{Q} \notin \underline{\underline{CL}}(M)$. Let $N = R(1 + \mathbb{Z}p, q)$ $(0 \neq q \in \mathbb{Q})$ and R be as in Example 2.1.16. Suppose that $N \in \underline{\underline{CL}}(M)$. Let $m = (1 + \mathbb{Z}p, 0)$. Then $m \notin N$. Therefore $0 \neq m + N \in M / N$. Now

$$p(m + N) = p(1 + \mathbb{Z}p, 0) + N = (0 + \mathbb{Z}p, 0) + N = N.$$

Thus $p(m + N) = 0$ in M/N i.e, $p \in \underset{\sim}{r}(m + N)$. Since $\underset{\sim}{r}(m + N) \in \underline{\underline{E}}({}_{\mathbb{Z}}\mathbb{Z})$, $m + N \in Z(M/N) = 0$, a contradiction. It follows that $N \notin \underline{\underline{CL}}(M)$. Thus, the only closed submodules of M are $(\mathbb{Z}/\mathbb{Z}p) \oplus 0$ and M. Thus M is a CLS-module.

Let $0 \neq L \in \underline{\underline{D}}(M)$. Suppose $L \neq M$. Then L is uniform because M has uniform dimension 2. By Example 2.1.16, $L = (\mathbb{Z}/\mathbb{Z}p) \oplus 0$, $0 \oplus \mathbb{Q}$, M or $L = R(1 + \mathbb{Z}p, q)$ for some $0 \neq q \in \mathbb{Q}$, where R is the local ring $\mathbb{Z}_p$. Now $M = L \oplus L'$ for some submodule L' of M. Suppose $L = R(1 + \mathbb{Z}p, q)$ for some $0 \neq q \in \mathbb{Q}$. Then $pL \cap L' = 0$, so that $R(0, pq) \cap L' = 0$ and hence $L' \cap (0 \oplus \mathbb{Q}) = 0$. Thus L' embeds in $\mathbb{Z}/\mathbb{Z}p$ which is simple. It follows that $L' = (\mathbb{Z}/\mathbb{Z}p) \oplus 0$. Therefore, $M = L \oplus L' = (\mathbb{Z}/\mathbb{Z}p) \oplus Rq$, a contradiction because $\mathbb{Q} \neq Rq$. Thus $L = M$, $(\mathbb{Z}/\mathbb{Z}p) \oplus 0$, or $0 \oplus \mathbb{Q}$. Let $\varphi : L \to M$ be a monomorphism. If $L = (\mathbb{Z}/\mathbb{Z}p) \oplus 0$ then $\varphi(L)$ is simple so that $\varphi(L) = L$. If $L = 0 \oplus \mathbb{Q}$ then $\varphi(L)$ is torsion-free injective. Let $(a + \mathbb{Z}p, b) \in \varphi(L)$ $(a \in \mathbb{Z},\ b \in \mathbb{Q})$. Then $(a + \mathbb{Z}p, b) = p(x + \mathbb{Z}p, y)$ for some $x \in \mathbb{Z}$, $y \in \mathbb{Q}$. Thus $(a + \mathbb{Z}p, b) = (0 + \mathbb{Z}p, py)$. It follows that $a = 0$ so that $\varphi(L) \leq L$. However, L is uniform and $\varphi(L)$ is injective, so $\varphi(L) = L$. If $L = M$ then $\varphi(L) = \varphi((\mathbb{Z}/\mathbb{Z}p) \oplus 0) + \varphi(0 \oplus \mathbb{Q}) = (\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q} = L$. Thus $\varphi(L) = L$ for every $L \in \underline{\underline{D}}(M)$ and monomorphism $\varphi : L \to M$. Thus M satisfies (C_2).

<u>Lemma 2.3.12.</u> The following statements are equivalent for a module M.

(i) M is a CLS-module.

(ii) $\underline{\underline{CL}}(M) \subseteq \mathrm{Lift}_X(M)$ for all right R-modules X.

(iii) $\underline{\underline{CL}}(M) \subseteq \mathrm{Lift}_X(M)$ for all $X \in \underline{\underline{CL}}(M)$.

Proof. Apply Lemma 2.1.13 to $\underline{\underline{A}} = \underline{\underline{CL}}(M)$.

Chapter 3

CERTAIN RIGHT CS-RINGS

This chapter consists of some results on right CS-rings. Some results, related to Kamal-Muller's Theorem (see [15, Theorem 1]), will be given for the nonsingular case.

3.1. Right CS-rings.

Let R be a ring and M a unital right R-module. Recall that, the ring R is a right CS-ring provided R_R is a CS-module.

<u>Notation</u>. Let S, T be rings, M a left S-, right T-bimodule such that ${}_SM$ is faithful. We can think of S as a subring of $\mathrm{End}(M_T)$, because the mapping $\varphi : S \to \mathrm{End}(M_T)$ given by

$$\varphi(s)(m) = sm \ (s \in S, \ m \in M)$$

is a ring monomorphism. Let

$$R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix} = \left\{ \begin{bmatrix} s & m \\ 0 & t \end{bmatrix} : s \in S, \ t \in T, \ m \in M \right\}.$$

Then R is a ring with respect to the usual addition and multiplication of matrices.

<u>Lemma 3.1.1</u>. $I \in \underline{\underline{E}}(R_R)$ if and only if there exist $N \in \underline{\underline{E}}(M_T)$ and $E \in \underline{\underline{E}}(T_T)$ such that $\begin{bmatrix} 0 & N \\ 0 & E \end{bmatrix} \le I$.

Proof. ($\Leftarrow$) Suppose $\begin{bmatrix} 0 & N \\ 0 & E \end{bmatrix} \leq I$. Let $0 \neq r = \begin{bmatrix} s & m \\ 0 & t \end{bmatrix} \in R$ ($s \in S$, $t \in T$, $m \in M$). Suppose $s \neq 0$. Therefore $sM \neq 0$ and hence $sm' \neq 0$ for some $m' \in M$. Consider

$$\begin{bmatrix} s & m \\ 0 & t \end{bmatrix}\begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & sm' \\ 0 & 0 \end{bmatrix} \in rR.$$

But there exists $0 \neq t' \in T$ such that $0 \neq (sm')t' \in N$. Thus $\begin{bmatrix} 0 & sm' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & t' \end{bmatrix} = \begin{bmatrix} 0 & (sm')t' \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & N \\ 0 & 0 \end{bmatrix} \leq I$. It follows that $rR \cap I \neq 0$.

Suppose $s = 0$. Then $r = \begin{bmatrix} 0 & m \\ 0 & t \end{bmatrix}$.

Case 1. $m \neq 0$. Therefore $0 \neq mt'' \in N$ for some $t'' \in T$. Thus

$$\begin{bmatrix} 0 & m \\ 0 & t \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & t'' \end{bmatrix} = \begin{bmatrix} 0 & mt'' \\ 0 & tt'' \end{bmatrix}.$$

Now, if $tt'' = 0$, then $\begin{bmatrix} 0 & mt'' \\ 0 & 0 \end{bmatrix} \in I$ so $rR \cap I \neq 0$.

If $tt'' \neq 0$ then $0 \neq tt''t''' \in E$ for some $t''' \in T$. Thus $\begin{bmatrix} 0 & mt'' \\ 0 & tt'' \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & t''' \end{bmatrix} = \begin{bmatrix} 0 & mt''t''' \\ 0 & tt''t''' \end{bmatrix} \in I$. Again $rR \cap I \neq 0$.

Case 2. $m = 0$. Therefore $r = \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix}$, where $0 \neq t \in T$. Thus there exists $\bar{t} \in T$ such that $0 \neq t\bar{t} \in E$ so that $\begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \bar{t} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & t\bar{t} \end{bmatrix} \in I$.

Hence $rR \cap I \neq 0$. It follows that $I \in \underline{\underline{E}}(R_R)$.

($\Rightarrow$) Suppose $I \in \underline{\underline{E}}(R_R)$. Define

$$N = \{m \in M : \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in I\} \text{ and } E = \{t \in T : \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \in I\}.$$

Then clearly $N \leq M_T$ and $E \leq T_T$. Let $0 \neq A \leq M_T$. Let $0 \neq a \in A$. Then $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} R \leq R$. Since $I \in \underline{\underline{E}}(R_R)$, $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} R \cap I \neq 0$. Therefore there exists

$0 \neq \alpha \in \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} R \cap I$. Now $\alpha = \begin{bmatrix} 0 & at \\ 0 & 0 \end{bmatrix}$ for some $0 \neq t \in T$. Then $at \in A$ and hence $0 \neq at \in A \cap N$. It follows that $N \in \underline{\underline{E}}(M_T)$.

Now let $0 \neq B \leq T_T$. Let $0 \neq b \in B$. Then $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} R \leq R$. Thus $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} R \cap I \neq 0$. Let $0 \neq \beta \in \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} R \cap I$. Then $\beta = \begin{bmatrix} 0 & 0 \\ 0 & bt \end{bmatrix}$ for some $0 \neq t \in T$. Since $bt \in B$, $0 \neq bt \in T \cap E$. Thus $E \in \underline{\underline{E}}(T_T)$.

Lemma 3.1.2. Suppose M_T is nonsingular and $sN = 0$ for some $s \in S$ where $N \leq M_T$. Then $N \in \underline{\underline{E}}(M_T)$ implies that $s = 0$.

Proof. Let $0 \neq m \in M$. Then $mE \leq N$ for some $E \in \underline{\underline{E}}(R)$. Therefore $smE = 0$. Hence $sm = 0$, i.e. $sM = 0$. Thus $s = 0$.

Corollary 3.1.3. R is right nonsingular if and only if M_T and T_T are nonsingular.

Proof. ($\Leftarrow$) Let $\begin{bmatrix} s & m \\ 0 & t \end{bmatrix} \in Z(R_R)$. By Lemma 3.1.1, there exist $N \in \underline{\underline{E}}(M_T)$, $E \in \underline{\underline{E}}(T_T)$ such that $sN = 0$, $mE = 0$, $tE = 0$. Thus $m = t = 0$, and, by Lemma 3.1.2, $s = 0$.

($\Rightarrow$) Let $m \in Z(M_T)$. Let E denote the right annihilator of m in T. Then $E \in \underline{\underline{E}}(T_T)$. Note that $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S & M \\ 0 & E \end{bmatrix} = 0$. Hence $\begin{bmatrix} S & M \\ 0 & E \end{bmatrix}$ is contained in the right annihilator of $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$, so that $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in Z(R_R) = 0$ (Lemma 3.1.1). It follows that $m = 0$, i.e M_T is nonsingular.

Again let $x \in Z(T_T)$ and E denote the right annihilator of x in T. Hence $E \in \underline{\underline{E}}(T_T)$. Note that $\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} S & M \\ 0 & E \end{bmatrix} = 0$. Hence $\begin{bmatrix} S & M \\ 0 & E \end{bmatrix}$ is contained in the right annihilator of $\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}$, so that $\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \in Z(R_R) = 0$.

It follows that $x = 0$. Thus T_T is nonsingular.

<u>Lemma 3.1.4.</u> Suppose R is a right nonsingular right CS-ring. Then

(i) For every $K \in \underline{\underline{C}}(M_T)$ there exists an idempotent e in S such that $K = eM$,

(ii) T is a right CS-ring.

Proof. (i) Let $K \in \underline{\underline{C}}(M_T)$. Let

$$X = \{s \in S : sM \leq K\} \leq S_S.$$

Let $A = \begin{bmatrix} X & K \\ 0 & 0 \end{bmatrix} \leq R_R$. Let $\alpha \in R$, $I \in \underline{\underline{E}}(R_R)$ such that $\alpha I \leq A$. Then $\alpha = \begin{bmatrix} s & m \\ 0 & t \end{bmatrix}$ for some $s \in S$, $m \in M$, $t \in T$. Moreover, by Lemma 3.1.1, there exist $N \in \underline{\underline{E}}(M_T)$, $E \in \underline{\underline{E}}(T_T)$ such that $\begin{bmatrix} 0 & N \\ 0 & E \end{bmatrix} \leq I$. Thus

$$\begin{bmatrix} 0 & sN+mE \\ 0 & tE \end{bmatrix} = \alpha \begin{bmatrix} 0 & N \\ 0 & E \end{bmatrix} \leq \alpha I \leq A.$$

It follows that $sN + mE \leq K$, $tE = 0$. By Corollary 3.1.3, $t = 0$. Also $mE \leq K$ implies $m \in K$ (Lemma 2.2.4 and Corollary 3.1.3). Let $m \in M$. Since $N \in \underline{\underline{E}}(M_T)$ it follows that $mF \leq N$ for some $F \in \underline{\underline{E}}(T_T)$. Thus $smF \leq sN \leq K$, so $sm \in K$ (Lemma 2.2.4). It follows that $sM \leq K$. Hence $\alpha \in A$ and $A \in \underline{\underline{C}}(R_R)$ (Lemma 2.2.4). By hypothesis, $A \in \underline{\underline{D}}(R_R)$. Thus there exists $e \in X$, $k \in K$ such that if $\beta = \begin{bmatrix} e & k \\ 0 & 0 \end{bmatrix}$ then $\beta = \beta^2$ and $A = \beta R$. Now

$$\beta = \beta^2 \Rightarrow e^2 = e,\ k = ek.$$

Also, for every $x \in X$, $y \in K$,

$$\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e & k \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$$

implies $x = ex$, $y = ey$. In particular, $K = eK$. Also, $e \in X$ implies $eM \leq K$. It follows that $K = eM$.

(ii) Let $G \in \underline{\underline{C}}(T_T)$. Let $B = \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} \leq R_R$. Let $\alpha \in R$ and $\alpha I \leq B$ for some $I \in \underline{\underline{E}}(R_R)$. Now $\alpha = \begin{bmatrix} s & m \\ 0 & t \end{bmatrix}$ for some $s \in S$, $m \in M$, $t \in T$ and $\begin{bmatrix} 0 & N \\ 0 & E \end{bmatrix} \leq I$ for some $N \in \underline{\underline{E}}(M_T)$, $E \in \underline{\underline{E}}(T_T)$. Thus $\alpha \begin{bmatrix} 0 & N \\ 0 & E \end{bmatrix} \leq \alpha I \leq B$ implies

$$sN + mE = 0, \ tE \leq G.$$

It follows that $m = 0$, $t \in G$. Also, if $m \in M$ then $mF \leq N$ for some $F \in \underline{\underline{E}}(T_T)$ and $smF \leq sN = 0$; thus $sm = 0$. Hence $sM = 0$ and so $s = 0$. Thus $\alpha \in B$ and $B \in \underline{\underline{C}}(R_R)$. There exists $f \in G$ such that $\gamma = \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$ is idempotent and $B = \gamma R$. Then $f = f^2$ and $G = fT$. It follows that T is a right CS-ring.

<u>Lemma 3.1.5</u>. Let R be a right nonsingular right CS-ring. Then M is an injective right T-module.

Proof. Let A be a right ideal of T and $\varphi : A \to M$ a homomorphism. Let

$$F = \left\{ \begin{bmatrix} 0 & \varphi(a) \\ 0 & a \end{bmatrix} : a \in A \right\}.$$

Then F is a right ideal of R. There exists an idempotent $e \in R$ such that $F \in \underline{\underline{E}}(eR)$. Now $e = \begin{bmatrix} s & m \\ 0 & t \end{bmatrix}$ for some $s \in S$, $m \in M$, $t \in T$. There exists $E \in \underline{\underline{E}}(R)$ such that $eE \leq F$. Now $\begin{bmatrix} 0 & N \\ 0 & B \end{bmatrix} \leq E$ for some $N \in \underline{\underline{E}}(M_T)$, $B \in \underline{\underline{E}}(T_T)$. In particular,

$$\begin{bmatrix} s & m \\ 0 & t \end{bmatrix} \begin{bmatrix} 0 & N \\ 0 & 0 \end{bmatrix} \leq F.$$

Thus $\begin{bmatrix} 0 & sN \\ 0 & 0 \end{bmatrix} \leq F$ and hence $sN = 0$. By Lemma 3.1.2, $s = 0$.

Now for each $a \in A$,

$$\begin{bmatrix} 0 & \varphi(a) \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & m \\ 0 & t \end{bmatrix}\begin{bmatrix} 0 & \varphi(a) \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & ma \\ 0 & ta \end{bmatrix},$$

so that $\varphi(a) = ma$. It follows that M_T is injective.

<u>Lemma 3.1.6.</u> Suppose T is a right nonsingular right CS-ring and M_T is nonsingular such that for every $K \in \underline{\underline{C}}(M_T)$ there exists $e = e^2 \in S$ such that $K = eM$. Let $A \in \underline{\underline{C}}(R_R)$, then there exists $a = a^2 \in A$ such that $A = aR \oplus B$, $B \leq \begin{bmatrix} 0 & M \\ 0 & T \end{bmatrix}$ and $B \cap \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} = 0$ where $B = (1-a)R \cap A$.

Proof. Let $A \in \underline{\underline{C}}(R_R)$. Define

$$K = \{m \in M : \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in A\}.$$

Then $K \leq M_T$. Also define

$$X = \{s \in S : sM \leq K\}.$$

Then $X \leq S_S$. We prove first :

$$\begin{bmatrix} X & K \\ 0 & 0 \end{bmatrix} \leq A \leq \begin{bmatrix} X & M \\ 0 & T \end{bmatrix} \tag{1}$$

Let $s \in X$. Then

$$\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & M \\ 0 & T \end{bmatrix} = \begin{bmatrix} 0 & sM \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} \leq A$$

implies $\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} \in A$ (Lemmas 3.1.1 and 2.2.4). Thus $\begin{bmatrix} X & K \\ 0 & 0 \end{bmatrix} \leq A$. Next let $\alpha = \begin{bmatrix} s & m \\ 0 & t \end{bmatrix} \in A$. Then

$$\begin{bmatrix} 0 & sM \\ 0 & 0 \end{bmatrix} = \alpha\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \leq A$$

implies $sM \leq K$, so $s \in X$. This proves (1).

Next we prove :

$$K \in \underline{\underline{C}}(M_T) \tag{2}$$

Let $m \in M$ such that $mE \leq K$ for some $E \in \underline{\underline{E}}(T_T)$. Then

$$\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & M \\ 0 & E \end{bmatrix} = \begin{bmatrix} 0 & mE \\ 0 & 0 \end{bmatrix} \leq A$$

implies $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in A$ (Lemmas 3.1.1 and 2.2.4), so $m \in K$. By Lemma 2.2.5, $K \in \underline{\underline{C}}(M_T)$.

Next we prove :

$$\text{There exists } e = e^2 \in S \text{ such that } X = eS,\ K = eM \tag{3}$$

By hypothesis there exists $e = e^2 \in S$ such that $K = eM$. Thus $e \in X$. Let $s \in X$. Then $sM \leq K$. Let $m \in M$. Then $sm = em'$ for some $m' \in M$ and hence

$$sm = em' = e^2m' = e(em') = esm,$$

so that $(s - es)m = 0$. Thus $(s - es)M = 0$, and hence $s = es$. It follows that $X = eX \leq eS \leq X$, so that $X = eS$. This proves (3).

Let $a = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \in A$, by (1). Note $a = a^2$. Thus $R = aR \oplus (1 - a)R$ and hence $A = aR \oplus B$ where $B = A \cap (1 - a)R \leq R_R$. Let $\beta \in B$. Then

$$\beta = \begin{bmatrix} 1-e & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s & m \\ 0 & t \end{bmatrix} = \begin{bmatrix} (1-e)s & (1-e)m \\ 0 & t \end{bmatrix}$$

for some $s \in S$, $m \in M$, $t \in T$. Since $\beta \in A$, (1) gives $(1 - e)s \in X$ so that $(1 - e)s = e(1 - e)s = 0$, by (3). Thus $\beta \in \begin{bmatrix} 0 & M \\ 0 & T \end{bmatrix}$. Suppose $\beta \in \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$. Then $t = 0$, and $\beta \in A$ implies $(1 - e)m \in K$, so $(1 - e)m = e(1 - e)m = 0$, by (3). Thus $\beta = 0$. It follows that $B \leq \begin{bmatrix} 0 & M \\ 0 & T \end{bmatrix}$ and $B \cap \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} = 0$. This completes the proof of Lemma 3.1.6.

Theorem 3.1.7. Suppose that ${}_SM$ is faithful. Then R is a right nonsingular right CS-ring if and only if,

(i) the right T-module M is nonsingular, injective,

(ii) T is a right nonsingular right CS-ring, and

(iii) for every $K \in \underline{\underline{C}}(M_T)$ there exists an idempotent e in S such that $K = eM$.

Proof. The necessity follows by Corollary 3.1.3 and Lemmas 3.1.4, 3.1.5.

For the converse suppose (i), (ii), (iii) hold. In view of Lemma 3.1.6, it is sufficient to prove that if B is a right ideal of R such that $B \leq \begin{bmatrix} 0 & M \\ 0 & T \end{bmatrix}$ and $B \cap \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} = 0$ then B is essential in a direct summand of R_R. Let

$$X = \{(m,t) \in M \oplus T : \begin{bmatrix} 0 & m \\ 0 & t \end{bmatrix} \in B\}.$$

Then X is a T-submodule of $M \oplus T$ with $X \cap M = 0$. Let $\pi_1 : M \oplus T \to M$, $\pi_2 : M \oplus T \to T$ denote the canonical projections. Note that $\pi_2|_B : B \to T$ is a monomorphism. Consider

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\pi_2|_B"] \arrow[d, "\pi_1|_B"'] & T \arrow[dl, dashed, "\varphi"] \\ & M & \end{tikzcd}$$

Because M_T is injective, there exists a mapping $\varphi : T \to M$ such that $\varphi\pi_2|_B = \pi_1|_B$. Let

$$E = \left\{ \begin{bmatrix} 0 & \varphi(t) \\ 0 & t \end{bmatrix} : t \in T \right\}.$$

Then E is a right ideal of R. Also, for any $b \in B$,

$$b = \begin{bmatrix} 0 & \pi_1(b) \\ 0 & \pi_2(b) \end{bmatrix} = \begin{bmatrix} 0 & \varphi\pi_2(b) \\ 0 & \varphi\pi_1(b) \end{bmatrix} \in E.$$

Because T is a right CS-ring, there exists an idempotent e in T such that $A = \pi_2(B) \in \underline{\underline{E}}(eT)$. Let

$$f = \begin{bmatrix} 0 & \varphi(e) \\ 0 & e \end{bmatrix}.$$

Then f is an idempotent in R. For any $a \in A$, $a = ea$ and hence

$$\begin{bmatrix} 0 & \varphi(a) \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & \varphi(e) \\ 0 & e \end{bmatrix} \begin{bmatrix} 0 & \varphi(a) \\ 0 & a \end{bmatrix}.$$

Thus $B = fB$. There exists $C \in \underline{\underline{E}}(T)$ such that $eC \leq A$. Let $c \in C$. Then

$$\begin{bmatrix} 0 & \varphi(e) \\ 0 & e \end{bmatrix} \begin{bmatrix} 0 & M \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & \varphi(ec) \\ 0 & ec \end{bmatrix} \in B.$$

It follows that $B \in \underline{\underline{E}}(fR)$. Thus R is a right CS-ring.

Corollary 3.1.8. Let K be a field and V a non-zero vector space over K. Let $S = \mathrm{End}_K(V)$, the ring of K-endomorphisms of V. Let $R = \begin{bmatrix} S & V \\ 0 & K \end{bmatrix}$. Then R is a right nonsingular right CS-ring with right uniform dimension $1 + \dim_K V$.

Proof. Let $U \in \underline{\underline{C}}(V_K)$. Then $V = U \oplus U'$ for some $U' \leq V_K$. Let $p : V \to U$ denote the canonical projection. Then $p = p^2 \in S$ and $U = pV$. The other hypotheses of Theorem 3.1.7 are obviously satisfied. Thus R is a right nonsingular right CS-ring.

Let $I = \begin{bmatrix} 0 & V \\ 0 & K \end{bmatrix}$. Then $I \in \underline{\underline{E}}(R_R)$, $I = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$ and $\text{u-dim}I = \dim_K V + 1$. If $A_1 \oplus A_2 \oplus A_3 \oplus \ldots$, is a direct sum of non-zero right ideals of R then $A_i \cap I \neq 0$ $(i \geq 1)$ and hence

$$(A_1 \cap I) \oplus (A_2 \cap I) \oplus (A_3 \cap I) \oplus \ldots$$

is a direct sum of non-zero submodules of I_R. It follows that

$$\text{u-dim}R = \text{u-dim}I = 1 + \dim_K V.$$

Note. Take K, V, S as in Corollary 3.1.8 and $R = \begin{bmatrix} S & V \\ 0 & K \end{bmatrix}$.

(i) If $\dim_K V = n < \infty$ then R is a right nonsingular right CS-ring with right uniform dimension $n + 1$.

(ii) If $\dim_K V = \infty$ then R is a right nonsingular right CS-ring which does not have finite right uniform dimension.

Example 3.1.9. Let K be a field. Set

$$R = \begin{bmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & K \end{bmatrix}.$$

Then R is a right CS-ring.

Proof. Take $T = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$, $M = [K\ K]$ and $S = K$. Then T is a right nonsingular right CS-ring, M is a right nonsingular, injective, uniform T-module. Thus by Theorem 3.1.7, R is a right CS-ring.

Example 3.1.10. [4, Example 5.5]. Let K be a field and $R = \begin{bmatrix} K & 0 & K \\ 0 & K & K \\ 0 & 0 & K \end{bmatrix}$. Then R does not satisfy (C_2), but it is a right CS-ring.

Proof. Take $S = K$, $T = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ and $M = [0\ K]$. Then T is a right nonsingular right CS-ring and M_T is a simple, injective module. Thus again by Theorem 3.1.7, R is a right CS-ring.

Let $E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} R$. Then clearly $E \in \underline{\underline{D}}(R_R)$. Then the mapping which is defined by $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k \\ 0 & 0 & 0 \end{bmatrix}$, is an isomorphism i.e $E \cong \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & 0 & 0 \end{bmatrix}$.

However $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & 0 & 0 \end{bmatrix} \notin \underline{\underline{D}}(R_R)$.

Example 3.1.11. Let $R = \begin{bmatrix} K & K & K \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix}$. Then R is not a right CS-ring.

Proof. Take $S = K$, $M = [K\ K]$ and $T = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$. Then T is semisimple ring and M_T is a nonsingular, injective module. Suppose R is a right CS-ring. Now

$\underline{\underline{C}}(M_T) = \{[K\ 0], [0\ K], [0\ 0], [K\ K]\}$. Let $L = K \oplus 0$. Then $L = e(K \oplus K)$ for some $e = e^2 \in K$ (Theorem 3.1.7). But $e = 0$ or 1, as K is a field. Hence $L = 0$ or $K \oplus K$, a contradiction.

In the remainder of this section unless otherwise stated R is a general ring and S a subring of R such that $S = eRe$ for some idempotent e of R with $R = ReR$. For example $R = M_n(S)$ for any $1 \leq n$. Let M be a right R-module. Then Me is a right S-module.

<u>Lemma 3.1.12.</u> Let $K, K' \leq M_R$ and $N, N' \leq Me_S$. Then

(i) $K = KeR$ and $N = NRe$,

(ii) $K \cap K' = 0$ if and only if $Ke \cap K'e = 0$, and

(iii) $N \cap N' = 0$ if and only if $NR \cap N'R = 0$.

Proof. (i) $K = KR = KReR = KeR$ and $N = NS = NeRe = NRe$.

(ii) If $K \cap K' = 0$ then $Ke \cap K'e \leq K \cap K'$ gives $Ke \cap K'e = 0$. Conversely, suppose $Ke \cap K'e = 0$. Let $x \in K \cap K'$. Then $xRe \leq Ke \cap K' = 0$, and hence $xReR = 0$. Thus $xR = 0$ and hence $x = 0$. It follows that $K \cap K' = 0$.

(iii) By (i), (ii).

<u>Corollary 3.1.13.</u> Let $L \leq M_R$. Then $L \in \underline{\underline{E}}(M_R)$ if and only if $Le \in \underline{\underline{E}}(Me_S)$.

Proof. Suppose $L \in \underline{\underline{E}}(M_R)$. Let $0 \neq N \leq Me$. Then $L \cap NR \neq 0$ and hence $Le \cap N \neq 0$, by Lemma 3.1.12. Thus $Le \in \underline{\underline{E}}(Me_S)$.

Conversely, suppose $Le \in \underline{\underline{E}}(Me_S)$. Let $0 \neq K \leq M_R$. By Lemma 3.1.12, $K = KeR$, so that $0 \neq Ke \leq Me_S$. Hence $Ke \cap Le \neq 0$. But $Ke \cap Le \leq K \cap L$. Thus $K \cap L \neq 0$. It follows that $L \in \underline{\underline{E}}(M_R)$.

Lemma 3.1.14. Let $L, N \leq Me_S$. Then L is a complement of N in the S-module Me if and only if LR is a complement of NR in the R-module M.

Proof. ($\Rightarrow$) Suppose L is a complement of N in Me. Then $L \cap N = 0$ and hence $LR \cap NR = 0$ (Lemma 3.1.12). Suppose $LR \leq K \leq M_R$ and $K \cap NR = 0$. Therefore Lemma 3.1.12 gives:

$$L = LRe \leq Ke \leq Me \text{ and } Ke \cap N \leq K \cap NR = 0.$$

It follows that $L = Ke$ and hence $LR = KeR = K$ (Lemma 3.1.12). It follows that LR is a complement of NR in M

($\Leftarrow$) Suppose that LR is a complement of NR in M. Then $L \cap N = 0$. Suppose $L \leq H \leq Me_S$ and $H \cap N = 0$. Now

$$LR \leq HR \leq M \text{ and } HR \cap NR = 0,$$

by Lemma 3.1.12. Thus $LR = HR$ and hence $L = LRe = HRe = H$, by Lemma 3.1.12, again. Thus L is a complement of N in Me.

Corollary 3.1.15. $L \in \underline{\underline{C}}(M_R)$ if and only if $Le \in \underline{\underline{C}}(Me_S)$.

Proof. By Lemma 3.1.14.

Lemma 3.1.16. Let $K \leq M_R$. Then $K \in \underline{\underline{D}}(M_R)$ if and only if $Ke \in \underline{\underline{D}}(Me_S)$.

Proof. Suppose $K \in \underline{\underline{D}}(M_R)$. Then $M = K \oplus K'$ for some $K' \leq M_R$. Thus $Me = Ke + K'e$. But $Ke \cap K'e \leq K \cap K' = 0$. Therefore $Me = Ke \oplus K'e$. Conversely suppose that $Me = Ke \oplus L$ for some $L \leq Me_S$. By Lemma 3.1.12, $K \cap LR = 0$, and

$$M = MeR = (Ke + L)R = KeR + LR = K + LR.$$

Thus $M_R = K \oplus LR$.

Theorem 3.1.17. M_R is a CS-module if and only if Me_S is a CS-module.

Proof. ($\Rightarrow$) By Lemma 3.1.12, Corollary 3.1.15, and Lemma 3.1.16.

($\Leftarrow$) By Lemmas 3.1.12 and 3.1.16.

Corollary 3.1.18. The ring R is a right CS-ring if and only if the right eRe-module Re is a CS-module.

Proof. Immediate by Theorem 3.1.17.

Corollary 3.1.19. Let $R = M_n(T)$ where T is a ring and $e = e_{11}$. Then R is a right CS-ring if and only if the right T-module T^n is a CS-module.

Proof. It is clear that $R = Re_{11}R$ and $e_{11}Re_{11} \cong T$, where e_{ij} is the matrix where (i,j)th entry is 1, and all other entries are zero. Moreover, $Re_{11} = Te_{11} + Te_{21} + \ldots + Te_{n1} \cong T^n$ (as right T-modules). Hence by Corollary 3.1.18, the result follows.

Lemma 3.1.20. M_R satisfies (C_3) if and only if Me_S satisfies (C_3).

Proof. Let A, $B \in \underline{\underline{D}}(Me_S)$ with $A \cap B = 0$. Therefore $A = ARe$, $B = BRe$, by Lemma 3.1.12. Hence AR, $BR \in \underline{\underline{D}}(M_R)$ (Lemma 3.1.16). Since $ARe \cap BRe = A \cap B = 0$ then by Lemma 3.1.12, $AR \cap BR = 0$ in M_R. Thus $AR \oplus BR \in \underline{\underline{D}}(M_R)$. By Lemma 3.1.16, $A \oplus B = (AR \oplus BR)e \in \underline{\underline{D}}(Me_S)$. Conversely let K, $L \in \underline{\underline{D}}(M_R)$ with $K \cap L = 0$. Therefore Ke, $Le \in \underline{\underline{D}}(Me_S)$ (Lemma 3.1.16). Then $Ke \cap Le \leq K \cap L = 0$ gives that $(Ke \oplus Le)R = KeR \oplus LeR = K \oplus L \in \underline{\underline{D}}(M_R)$.

Corollary 3.1.21. M_R is quasi-continuous if and only if Me_S is quasi-continuous.

Proof. By Theorem 3.1.17 and Lemma 3.1.20.

Let T be a ring such that $M_2(T)$ is right quasi-continuous. By Corollary 3.1.19 and Lemma 3.1.20 the right T-module T^2 is quasi-continuous and hence, by [21, Proposition 2.10], T is right self-injective. Thus we have the following result.

Corollary 3.1.22. The following statements are equivalent for a ring T.

(i) T is right self-injective.

(ii) $M_2(T)$ is right quasi-continuous.

(iii) $M_n(T)$ is right quasi-continuous for every positive integer n.

(iv) $M_n(T)$ is right self-injective for every positive integer n.

Lemma 3.1.23. M_R satisfies (C_2) if and only if Me_S satisfies (C_2).

Proof. Let K, $L \leq M_R$ such that $K \cong L \in \underline{\underline{D}}(M_R)$. Let $f : K \to L$ be an R-homomorphism. Since $L \in \underline{\underline{D}}(M_R)$ then $Le \in \underline{\underline{D}}(Me_S)$ (Lemma 3.1.16). Now $\varphi = f|_{Ke} : Ke \to Le$ is an isomorphism. Therefore $Ke \in \underline{\underline{D}}(Me)$ and hence $K \in \underline{\underline{D}}(M_R)$. For the converse, let A, $B \leq Me$ such that $A \cong B \in \underline{\underline{D}}(Me_S)$. Thus $BR \in \underline{\underline{D}}(M_R)$. Suppose that $\varphi : A \to B$ is an isomorphism. Define $\theta : AR \to BR$ and $\theta' : BR \to AR$ by

$$\theta(\textstyle\sum_{i=1}^{n} a_i r_i) = \sum_{i=1}^{n} \varphi(a_i) r_i, \quad \theta'(\sum_{i=1}^{n} b_i r_i) = \sum_{i=1}^{n} \varphi^{-1}(b_i) r_i$$

for all $n \geq 1$, $a_i \in A$, $b_i \in B$, $r_i \in R$ $(1 \leq i \leq n)$. Now suppose $\sum_{i=1}^{n} a_i r_i = 0$. Then $\sum_{i=1}^{n} a_i r_i se = 0$ for all $s \in R$. Therefore $\sum_{i=1}^{n} a_i e r_i se = 0$ and hence $\sum_{i=1}^{n} \varphi(a_i) e r_i se = 0$. Thus $\sum_{i=1}^{n} \varphi(a_i) r_i se = 0$. It follows that $(\sum_{i=1}^{n} \varphi(a_i) r_i)Re = 0$, so that $(\sum_{i=1}^{n} \varphi(a_i) r_i)ReR = 0$ i.e, $\sum_{i=1}^{n} \varphi(a_i) r_i = 0$. Therefore θ is a well-defined mapping. It is easy to check that θ is an R-homomorphism. Similarly, θ' is an R-homomorphism. Clearly, $\theta'\theta = 1|_{AR}$

and $\theta\theta' = 1|_{BR}$. Hence θ is an isomorphism. By hypothesis, $AR \in \underline{\underline{D}}(M_R)$ and then by Lemmas 3.1.12, 3.1.16, $A \in \underline{\underline{D}}(M_R)$.

<u>Corollary 3.1.24</u>. M_R is continuous if and only if Me_S is continuous.

Proof. By Theorem 3.1.17 and Lemma 3.1.23.

<u>Proposition 3.1.25</u>. M_R is nonsingular if and only if Me_S is nonsingular.

Proof. ($\Leftarrow$) Let $m \in Z(M_R)$. Let $r \in R$. Therefore $mre \in Z(M_R)$. There exists $F \in \underline{\underline{E}}(R_R)$ such that $mreF = 0$. Now by Corollary 3.1.13, $eR \cap F \in \underline{\underline{E}}(eR)$ and hence $(eR \cap F)e \in \underline{\underline{E}}(eRe_S) = \underline{\underline{E}}(S_S)$. But $mre \in Me$ and $(eR \cap F)e \leq Fe \leq F$. Thus $(mre)[(eR \cap F)e] = 0$. Then $mre = 0$ because Me_S is nonsingular. Hence $mRe = 0$. Therefore $mReR = 0$, so that $mR = 0$ i.e $m = 0$.

($\Rightarrow$) Let $me \in Z(Me_S)$. Then $meG = 0$ for some $G \in \underline{\underline{E}}(S_S)$. By Corollary 3.1.13, $GR \in \underline{\underline{E}}(eR_R)$. Thus $GR \oplus (1-e)R \in \underline{\underline{E}}(R_R)$. Since $me[GR \oplus (1-e)R] = 0$ then $me \in Z(M_R)$ and hence $me = 0$.

Note that if R is a right nonsingular right CS-ring then R is a right pp-ring. However, the following example shows that there exists a ring which is Artinian CS but not pp.

<u>Example 3.1.26</u>. Let K be a field of characteristic $p > 0$. Let $G = <x : x^p = 1>$, the cyclic group of order p. Let R denote the group algebra K[G]. Then R is an Artinian CS-ring which is not pp.

Proof. Let $S = K[X]$ polynomial ring. Define $\varphi : S \to R$ by

$$\varphi(a_0 + a_1X + a_2X^2 + \dots + a_kX^k) = a_0 + a_1x + a_2 + \dots + a_kx^k.$$

Then φ is a epimorphism and $R \cong S / S(X^p - 1)$. Note that R is commutative, Artinian and $\dim_K R = p$. Let $\varepsilon : R \to K$ be the augmentation mapping defined by

$$\varepsilon(a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) = a_0 + a_1 + \dots + a_{p-1}.$$

Therefore $\ker\varepsilon = R(x - 1) = P$ is the augmentation ideal. Hence $R / R(x - 1) \cong K$ and $R(x - 1)$ is maximal in R. Now

$$(x - 1)^p = x^p - \binom{p}{1} x^{p-1} + \binom{p}{2} x^{p-2} + \dots + (-1)^p$$

$$= 1 - 0 + 0 + \dots + (-1)^p = 0.$$

Therefore $[R(x - 1)]^p = R(x - 1)^p = 0$. Let $A < R$. Suppose $0 < A \leq P$. Then there exists $k \geq 1$ such that $A \leq R(x - 1)^k$, $A \leq R(x - 1)^{k+1}$. Thus there exists $a \in A$ such that $a \notin R(x - 1)^{k+1}$. Therefore $a = r(x - 1)$ for some $r \in R$ and $r \notin P$ (if $r \in P$ then $a \in R(x - 1)^{k+1}$). Now P is the unique maximal ideal. Hence r is a unit in R. It follows that $(x - 1)^k = r^{-1} a \in A$ i.e, $A = R(x - 1)^k$. Then the only ideals of R are, $R > P > P^2 > \dots > P^p = 0$. Therefore ${}_R R$ is uniform and hence R is a CS-ring. Since $P^{p-1} \in \underline{\underline{E}}(R)$ then $P = Z(R)$. Now let $I = \underset{\sim}{r}(x - 1) = (1 + x + \dots + x^{p-1})R$. Then

$$I^2 = (1 + x + \dots + x^{p-1})^2 R = p(1 + x + \dots + x^{p-1})R = 0.$$

Therefore $\underset{\sim}{r}(x - 1) \neq Re$ for any $e = e^2 \in R$. That is, R is not a pp-ring.

Recall that over a CS-ring a full matrix ring does not need to be CS (see for example [4, Example 6.9]). On the other hand, we know $M_n(R)$ is a right CS-ring if and only if $(R^n)_R$ is a CS-module (Corollary 3.1.19).

<u>Proposition 3.1.27.</u> Let S be a domain. If $M_2(S)$ is a right CS-ring then S is left and right Ore and right CS.

Proof. Let $R = M_2(S)$. Let $0 \neq x, y \in S$ and suppose $Sx \cap Sy = 0$. Set

$$u = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}.$$

Then $\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}\begin{bmatrix} r & s \\ w & v \end{bmatrix} = \begin{bmatrix} xr & xs \\ yr & ys \end{bmatrix}$, $\begin{bmatrix} r & s \\ w & v \end{bmatrix} \in R$. Then $uR \in \underline{\underline{E}}(eR)$ for some $e^2 = e \in R$. Let say $e = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $(a, b, c, d \in S)$. Thus

$$\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}$$

and hence $\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} = \begin{bmatrix} ax + by & 0 \\ cx + dy & 0 \end{bmatrix}$. Then $x = ax + by$ and $y = cx + dy$. Now $(1 - a)x = by$. Therefore $(1 - a)x = 0$ and hence $a = 1$, $b = 0$. Also $(1 - d)y = cx$ implies that $d = 1$, $c = 0$. Thus $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence $uR \in \underline{\underline{E}}(R)$.

On the other hand,

$$0 \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} xr & xs \\ yr & ys \end{bmatrix} \quad (a', b', c', d' \in R),$$

gives that $\begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} xr & xs \\ yr & ys \end{bmatrix}$. Hence $r = s = 0$. Therefore $a' = 0$, $b' = 0$ a contradiction. It follows that S is left Ore.

Since R is a right CS-ring then $(S \oplus S)_S$ is a CS-module (Corollary 3.1.19). Therefore S is a CS-ring by Proposition 1.2.2.

Let $0 \neq z \in S$. Then $zS \in \underline{\underline{E}}(fS_S)$ for some $f^2 = f \in S$. Thus $f = 1$, so that $zS \in \underline{\underline{E}}(S_S)$. Hence S is right Ore.

By adapting the proof of [4, Proposition 6.8] we can prove the following generalization.

<u>Proposition 3.1.28.</u> Suppose R is a semiprime right and left Goldie ring. Then the following statements are equivalent.

(i) R is a right CS-ring.

(ii) R is a right and left pp-ring.

(iii) R is a left CS-ring.

Proof. (i) $\Rightarrow$ (ii) Because R is a right (left) nonsingular, every right (left) annihilator is a complement in R, so a direct summand. Therefore R is a right (left) pp-ring.

(ii) $\Rightarrow$ (i) Let Q denote the semiprime Artinian classical ring of quotients of R. Note that a right ideal of Q is minimal if and only if it is uniform. Let $u \in R$. Suppose uR is uniform, then uQ is a minimal right ideal of Q. We have $Qu = Qe$ for some idempotent e, so that $\underset{\sim}{r}_Q(u) = (1 - e)Q$. Thus

$$eQ \cong Q / (1 - e)Q = Q / \underset{\sim}{r}(u) \cong uQ,$$

so that eQ is a minimal right ideal of the semiprime Artinian ring Q. Hence Qe is a minimal left ideal of Q. Therefore Qu is minimal, i.e. Ru is uniform.

Now let U be a maximal uniform right ideal of R and let $0 \neq u \in U$. Then the left annihilator of u, $\underset{\sim}{l}(u) = Rf$ for some $f^2 = f \in R$. We have $uR \leq U$, so that uR and Ru are uniform. Also, $Ru \cong R(1 - f)$, so that $R(1 - f)$ is a uniform left ideal. Hence $(1 - f)R$ is a uniform right ideal. But $uR \leq (1 - f)R$, so that $U \cap (1 - f)R \neq 0$. Hence $U + (1 - f)R$ is also uniform (R is nonsingular). Since $(1 - f)R \in \underline{\underline{D}}(U + (1 - f)R)$ we must have $U + (1 - f)R = (1 - f)R$. Thus $U \leq (1 - f)R$, and the maximality of U gives $U = (1 - f)R$. By the Corollary of Theorem 1.2.4, R is a right CS-ring.

(i) $\Leftrightarrow$ (iii) By symmetry.

<u>Theorem 3.1.29</u>. Let R be a domain. Then the following statements are equivalent.

(i) $M_2(R)$ is a right CS-ring.

(ii) R is a right and left Ore and every 2-generator right or left ideal is projective.

(iii) $M_2(R)$ is left CS-ring.

Proof. (i) $\Rightarrow$ (ii). By Proposition 3.1.27, R is a right and left Ore domain.

Also $M_2(R)$ is right nonsingular so right and left pp-ring. Hence by Small's Theorem [6, Theorem 8.17], every 2-generator right or left ideal is projective.

(ii) $\Rightarrow$ (i). Now $M_2(R)$ is right and left pp-ring by [6, Theorem 8.17]. Since $M_2(R)$ is a prime Goldie ring then by Proposition 3.1.28, $M_2(R)$ is a right CS-ring.

(i) $\Leftrightarrow$ (iii). By symmetry.

Note. In the above Theorem, 2 can be replaced by n.

3.2. Nonsingular CS-modules

Let R be a ring and $(\underline{\underline{T}}, \underline{\underline{F}})$ a torsion theory for Mod–R (see [34]). For any R-module M let $\tau(M)$ denote the torsion submodule of M. Recall that a right R-module M is said to be *reduced* provided it contains no non-zero injective submodule.

Lemma 3.2.1. Let R be a ring and $(\underline{\underline{T}}, \underline{\underline{F}})$ a torsion theory for Mod–R. Let M be a torsion-free reduced right R-module. Let I be an ideal of R such that $I \leq \tau(R_R)$. Then $MI = 0$ and M is a reduced right (R / I)-module.

Proof. Let

$$\Gamma = \{E : E \text{ is a right ideal of } R \text{ and } R / E \in \underline{\underline{T}}\}.$$

Thus $\tau(X) = \{x \in X : xE = 0 \text{ for some } E \in \Gamma\}$, for any right R-module X.

Let $m \in M$, $a \in I$. There exists $G \in \Gamma$ such that $aG = 0$ and hence $(ma)G = m(aG) = 0$. Thus $ma = 0$. It follows that $MI = 0$. Define

$$m(r + I) = mr \quad (m \in M,\ r \in R).$$

With this definition, the abelian group M becomes a right (R / I)-module.

Let N be an injective submodule of the right (R / I)-module M. Let A be a right ideal of R and $\theta : A \to N$ an R-homomorphism. Define

$\varphi : (A + I) / I \to N$ by

$$\varphi(a + I) = \theta(a) \ (a \in A).$$

First we show that φ is well-defined. Suppose $a, \ b \in A$ and $a + I = b + I$. Then $a - b \in I \leq \tau(R_R)$. Thus there exists $F \in \Gamma$ such that $(a - b)F = 0$. We have :

$$[\theta(a) - \theta(b)]F = [\theta(a - b)]F = \theta\{(a - b)F\} = \theta(0) = 0.$$

But M is torsion-free. Thus $\theta(a) - \theta(b) = 0$, i.e. $\theta(a) = \theta(b)$. It follows that φ is well-defined.

Let $a, \ b \in A, \ r \in R$. Then

$$\varphi((a + I) + (b + I)) = \varphi((a + b) + I) = \theta(a + b) = \theta(a) + \theta(b)$$
$$= \varphi(a + I) + \varphi(b), \text{ and}$$
$$\varphi((a + I)(r + I)) = \varphi((ar) + I) = \theta(ar) = \theta(a)r = \theta(a)(r + I)$$
$$= \varphi(a + I)(r + I).$$

Thus φ is an (R / I)-homomorphism. Since N is (R / I)-injective it follows that there exists $n \in N$ such that $\varphi(a + I) = n(a + I) \ (a \in A)$. Hence, for all $a \in A$,

$$\theta(a) = \varphi(a + I) = n(a + I) = na.$$

It follows that N is an injective submodule of the reduced R-module M. Thus $N = 0$. Hence M is a reduced (R / I)-module.

Lemma 3.2.2. Let R be a ring, I an ideal of R and M a right R-module such that $MI = 0$. Then M is a right (R / I)-module. Moreover, the R-module M is a CS-module if and only if the (R / I)-module M is a CS-module.

Proof. Define

$$m(r + I) = mr \ (m \in M, \ r \in R).$$

Then the abelian group M is a right (R / I)-module. Let N be a subgroup of M. Let $n \in N$, $r \in R$. Clearly $nr = n(r + I)$ implies $nr \in N$ if and only if $n(r + I) \in N$. Thus

N is an R-submodule of M if and only if N is an (R / I)-submodule of M. (4)

Suppose that the R-module M is a CS-module. Let $N \in \underline{\underline{C}}(M_{R/I})$. Let K be an R-submodule of M such that $N \in \underline{\underline{E}}(K)$. Let L be an (R / I)-submodule of K such that $N \cap L = 0$. By (4), L is an R-submodule of K and hence $L = 0$. Thus $N \in \underline{\underline{E}}(K_{R/I})$ and hence $N = K$. Thus $N \in \underline{\underline{C}}(M_R)$. Since M is a CS-module, there exists an R-submodule N' of M such that $M = N \oplus N'$. By (4), N' is an (R / I)-submodule of M. It follows that $\underline{\underline{C}}(M_{R/I}) \subseteq \underline{\underline{D}}(M_{R/I})$. Hence the (R / I)-module M is a CS-module.

Similarly, if the (R / I)-module M is a CS-module then the R-module M is a CS-module.

Lemma 3.2.3. Let I be an ideal of a ring R and E a right ideal of R such that $I \leq E$ and $E / I \in \underline{\underline{E}}((R / I)_{R/I})$. Then $E \in \underline{\underline{E}}(R_R)$.

Proof. Let $0 \neq r \in R$. If $r \in I$ then $r \in E$ and hence $0 \neq rR \leq rR \cap E$. If $r \notin I$ then $(r + I)(R / I) \cap (E / I) \neq 0$. Thus

$$[(rR \cap E) + I] / I = [(rR + I) / I] \cap (E / I) = (r + I)(R / I) \cap (E / I) \neq 0$$

and hence $rR \cap E \neq 0$. It follows that $E \in \underline{\underline{E}}(R_R)$.

Corollary 3.2.4. Let I be an ideal of a ring R and M a nonsingular right R-module such that $MI = 0$. Then M is a nonsingular right (R / I)-module.

Proof. Again make M into a right (R / I)-module by defining

$$m(r + I) = mr \quad (m \in M,\ r \in R).$$

Suppose $m \in M$ and $m\bar{E} = 0$ for some $\bar{E} \in \underline{\underline{E}}(\bar{R}_{\bar{R}})$ where $\bar{R} = R / I$. There exists a right ideal E of R containing I such that $\bar{E} = E / I$. By Lemma 3.2.3, $E \in \underline{\underline{E}}(R_R)$. Moreover, $mE = 0$. Hence $m = 0$, because M is nonsingular. It follows that M is a nonsingular right (R / I)-module.

For the Goldie torsion theory $(\underline{\underline{T}}, \underline{\underline{F}})$, $\underline{\underline{F}}$ consists precisely of the nonsingular right R-modules.

<u>Proposition 3.2.5.</u> Let R be a ring and M a right R-module such that M is a nonsingular reduced CS-module. Let $I \leq Z_2(R_R)$. Then $MI = 0$ and the right (R / I)-module M is a nonsingular reduced CS-module.

Proof. By Lemmas 3.2.1 and 3.2.2 and Corollary 3.2.4.

<u>Lemma 3.2.6.</u> [29, Proposition 2.6] Let R be a semiprime right Goldie ring and M a nonsingular right R-module. Then M is injective if and only if M is divisible.

Proof. Suppose M is injective. Let $c \in R$, c regular. Let $y \in M$. Define $\theta : cR \to M$ by

$$\theta(cr) = yr \quad (r \in R).$$

Then θ is well-defined (because c is regular) and an R-homomorphism. It follows that there exists $x \in M$ such that $\theta(cr) = xcr$ $(r \in R)$. In particular, $y = \theta(c) = xc \in Mc$. It follows that $M = Mc$. Hence M is divisible. (This part is true for any ring R).

Conversely, suppose M is divisible. Let $E \in \underline{\underline{E}}(R_R)$ and $\varphi : E \to M$ be an R-homomorphism. Since R is semiprime right Goldie, E contains a regular element d. Now $\varphi(d) \in M = Md$ and hence $\varphi(d) = md$ for some $m \in M$. Let $e \in E$. Then $ed' = dr$ for some $r, d' \in R$, d' regular (see [6, Theorem 1.27]).

Thus

$$\varphi(e)d' = \varphi(ed') = \varphi(dr) = \varphi(d)r = mdr = med'$$

and so $(\varphi(e) - me)d' = 0$. But $d'R \in \underline{\underline{E}}(R_R)$ and M is nonsingular. Thus $\varphi(e) - me = 0$. That is, $\varphi(e) = me$ $(e \in E)$. It follows that M is injective.

<u>Lemma 3.2.7.</u> [20, Proposition p.70] Let R be a commutative ring with a finite collection of prime ideals $P_1, P_2, \ldots, P_n$ such that $P_1 \cap P_2 \cap \ldots P_n = 0$. Then R is semiprime Goldie.

Proof. Without loss of generality P_i does not contained in P_j $(1 \le i \ne j \le n)$. Thus

$P_1 \cap P_2 \cap \ldots \cap P_{i-1} \cap P_{i+1} \cap \ldots \cap P_n$ does not contained in P_i $(1 \le i \le n)$. Let $c \in R$, c regular. Suppose $cR \cap I = 0$ for some ideal I of R. Then $cI \le cR \cap I$, so $cI = 0$ and $I = 0$. Thus $cR \in \underline{\underline{E}}(R_R)$.

Conversely, let $E \in \underline{\underline{E}}(R_R)$. Then E is not contained in P_i $(1 \le i \le n)$. Thus $E \cap P_1 \cap \ldots \cap P_{i-1} \cap P_{i+1} \cap \ldots \cap P_n$ is not contained in P_i $(1 \le i \le n)$.

Let $e_i \in E \cap P_1 \cap \ldots \cap P_{i-1} \cap P_{i+1} \cap \ldots \cap P_n$, $e_i \notin P_i$ $(1 \le i \le n)$. Let $e = e_1 + e_2 + \ldots + e_n \in E$. Let $r \in R$ and suppose $er = 0$. Let $1 \le i \le n$. If $e \in P_i$ then $e_i \in P_i$, a contradiction. Thus $e \notin P_i$. Therefore, $er = 0 \in P_i$ implies $r \in P_i$. Hence

$$r \in P_1 \cap P_2 \cap \ldots \cap P_n = 0.$$

It follows that e is regular. Thus R is semiprime Goldie.

The next result extends [15, Theorem 5].

<u>Corollary 3.2.8.</u> Let R be a commutative ring with finitely many minimal prime ideals $P_1, \ldots, P_n$. Let M_R be a nonsingular reduced CS-module. Then M

has finite uniform dimension.

Proof. Let $N = \{r \in R : r^k = 0$ for some $k \geq 1\}$. Then $N = P_1 \cap P_2 \cap \ldots \cap P_n$. Let $r \in N$. Let $E = \{s \in R : rs = 0\}$. Let $0 \neq a \in R$. Then there exists $k \geq 1$ such that $r^{k-1}a \neq 0$ but $r^k a = 0$ (convention : $r^0 = 1$). Thus $0 \neq r^{k-1}a \in Ra \cap E$. It follows that $E \in \underline{\underline{E}}(R_R)$. Hence $r \in Z(R)$. Thus $N \leq Z(R) \leq Z_2(R)$. By Proposition 3.2.5, $MN = 0$ and the (R/N)-module M is a nonsingular CS-module. Thus, without loss of generality, $N = 0$, i.e. $P_1 \cap P_2 \cap \ \ldots \ \cap P_n = 0$.

Then $P_1P_2 \ldots P_n = 0$, so that $MP_1P_2 \ldots P_n = 0$. Let $K = \{m \in M : mP_1 = 0\}$. Then $K \leq M$. Let $L \leq M$ such that $K \in \underline{\underline{E}}(L)$. Let $x \in L$. There exists $I \in \underline{\underline{E}}(R_R)$ such that $xI \leq K$ and hence $xIP_1 = 0$. Thus $(xP_1)I = 0$, so that $xP_1 = 0$. It follows that $x \in K$. Hence $L = K$, and $K \in \underline{\underline{C}}(M)$. By hypothesis, $M = K \oplus K'$ for some $K' \leq M$. Note that $K' \cong M/K$ and $(P_2 \cap \ldots \cap P_n)P_1 \leq P_1 \cap P_2 \cap \ \ldots \ \cap P_n = 0$ gives $M(P_2 \cap \ \ldots \ \cap P_n)P_1 = 0$, so that $M(P_2 \cap \ \ldots \ \cap P_n) \leq K$. Hence $K'(P_2 \cap \ \ldots \ \cap P_n) = 0$. By Lemma 3.2.2 and Corollary 3.2.4, the (R/P_1)-module K is a nonsingular CS-module and the $(R/(P_2 \cap \ \ldots \ \cap P_n))$-module K' is a nonsingular CS-module. Let X be an injective submodule of the (R/P_1)-module K. Let c be a regular element of R. Then $c \notin P_1$ and hence $X = Xc$ (Lemma 3.2.6). Thus X_R is divisible. By Lemmas 3.2.6 and 3.2.7, X is an injective R-module. But M is reduced, so that $X = 0$. Hence the (R/P_1)-module K is reduced. But R/P_1 is a domain, so that by Kamal-Muller's Theorem (see [15, Theorem 5]), K has finite uniform dimension. By the same argument, the $R/(P_2 \cap \ \ldots \ \cap P_n)$-module K' is reduced and K' has finite uniform dimension by induction on n. Thus $M = K \oplus K'$ has finite uniform dimension.

Chapter 4.

A CHARACTERIZATION OF CONTINUOUS AND QUASI-CONTINUOUS MODULES

In this chapter we shall characterize continuous and quasi-continuous modules in terms of lifting homomorphisms from certain submodules of M to M itself. Note that in [19, Theorem 1.3] a different lifting condition is given to characterize continuous modules.

4.1. Modules with $\underline{\underline{C}}^{(n)}(M) \subseteq \mathrm{Lift}_M(M)$ or (Q_n).

Recall that the module M is called *continuous* if it satisfies (C_1) (i.e, CS) and (C_2). Consequently, the module M is called *quasi-continuous* if it satisfies (C_1) and (C_3). For basic results refer to Chapter 1, section 3. Moreover, for a good general account, see the comprehensive study of Mohamed-Muller [21].

Let R be a ring with identity and M a unital right R-module.

<u>Lemma 4.1.1.</u> Let $K \in \underline{\underline{C}}(M)$. Then $K \in \underline{\underline{D}}(M)$ if and only if there exists a complement L of K in M such that $K \oplus L \in \mathrm{Lift}_M(M)$.

Proof. Suppose first that $K \in \underline{\underline{D}}(M)$. Then $M = K \oplus K'$ for some submodule K' of M. Clearly, $L = K'$ will do.

Conversely, suppose that there exists a complement L of K in M with the stated property. Let $\varphi : K \oplus L \to M$ be the homomorphism defined by

$$\varphi(x + y) = x \ (x \in K,\ y \in L).$$

By hypothesis, there exists a homomorphism $\theta : M \to M$ such that

$$\theta(x + y) = x \ (x \in K,\ y \in L).$$

Note that $K \subseteq \operatorname{im}\theta$ and $L \subseteq \ker\theta$.

Let $0 \neq v \in \operatorname{im}\theta$. Then there exists $u \in M$ such that $v = \theta(u)$. Note that $u \notin L$. Thus $K \cap (L + uR) \neq 0$. There exist $x \in K$, $y \in L$ and $r \in R$ such that $0 \neq x = y + ur$. Then $x = \theta(x) = \theta(y + ur) = vr$. It follows that $vR \cap K \neq 0$ for all non-zero $v \in \operatorname{im}\theta$. Thus $K \in \underline{\underline{E}}(\operatorname{im}\theta)$. But $K \in \underline{\underline{C}}(M)$. Hence $K = \operatorname{im}\theta$.

Now it is easy to check that $M = K \oplus (\ker\theta)$. Thus $K \in \underline{\underline{D}}(M)$.

Corollary 4.1.2. A module M satisfies (C_1) if and only if for every $K \in \underline{\underline{C}}(M)$ there exists a complement L of K in M such that $K \oplus L \in \mathrm{Lift}_M(M)$.

Proof. Immediate by Lemma 4.1.1.

Let n be a positive integer. It is clear that if M satisfies $\underline{\underline{C}}^{(n)}(M) \subseteq \mathrm{Lift}_M(M)$ then M satisfies $\underline{\underline{C}}^{(n-1)}(M) \subseteq \mathrm{Lift}_M(M)$, for all $n \geq 2$. Note that modules satisfying $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M)$ have been considered in [27].

Next we establish the connection between $\underline{\underline{C}}^{(n)}(M) \subseteq \mathrm{Lift}_M(M)$ and the quasi-continuity of a module M, as was pointed out at the beginning of this chapter.

Theorem 4.1.3. The following statements are equivalent for a module M.

(i) M is quasi-continuous

(ii) M satisfies $\underline{\underline{C}}^{(n)}(M) \subseteq \mathrm{Lift}_M(M)$ for every positive integer n.

(iii) M satisfies $\underline{\underline{C}}^{(n)}(M) \subseteq \mathrm{Lift}_M(M)$ for some integer $n \geq 2$.

(iv) M satisfies $\underline{\underline{C}}^{(2)}(M) \subseteq \mathrm{Lift}_M(M)$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (i) By Proposition 2.2.2 and Corollary 4.1.2.

We now consider continuous modules. Let us consider the following condition for any positive integer n and a module M.

(Q_n) For every submodule K of M such that K is a direct sum $K_1 \oplus K_2 \oplus \ldots \oplus K_n$ of submodules K_i ($1 \le i \le n$) of M, each isomorphic to a complement in M, $K \in \mathrm{Lift}_M(M)$.

It is clear that if M satisfies (Q_n) then M satisfies (Q_{n-1}) for all $n \geq 2$. Moreover, if M satisfies (Q_n) then M satisfies $\underline{\underline{C}}^{(n)}(M) \subseteq \mathrm{Lift}_M(M)$, for all $n \geq 1$.

Theorem 4.1.4. The following statements are equivalent for a module M.

(i) M is continuous.

(ii) M satisfies (Q_n) for every positive integer n.

(iii) M satisfies (Q_n) for some integer $n \geq 2$.

(iv) M satisfies (Q_2).

(v) M satisfies (Q_1) and (C_1).

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Clear.

(iv) $\Rightarrow$ (i). By Corollary 4.1.2 and Proposition 2.2.1.

(i) $\Rightarrow$ (v). Clear.

(v) $\Rightarrow$ (i). By Proposition 2.2.1.

4.2. An outline and counter examples.

Theorems 4.1.3 and 4.1.4, allow us to construct the following outline.

$$
\begin{tikzcd}
\text{quasi-injective} \arrow[r, Rightarrow] & \text{continuous} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{quasi-continuous} \arrow[d, Leftrightarrow] & \\
(Q_n)\ \forall n \arrow[r, Leftrightarrow] & (Q_2) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \underline{\underline{C}}^{(2)}(M) \subseteq \mathrm{Lift}_M(M) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \underline{\underline{C}}^{(n)}(M) \subseteq \mathrm{Lift}_M(M)\ \forall\, n. \arrow[d, Rightarrow] \\
& (Q_1) \arrow[r, Rightarrow] & \underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M) & (C_1), \arrow[l, Rightarrow]
\end{tikzcd}
$$

for any integer $n \geq 2$. No other implications can be added to this table, in general. To see why this is the case we shall give a number of examples. First of all, note that Utumi [36, Example 3] has given an example of a continuous module which is not quasi-injective. The next example is easy.

Example 4.2.1. Let $\mathbb{Z}$ denote the ring of rational integers and M be the $\mathbb{Z}$-module $\mathbb{Z}$. Then M satisfies $\underline{\underline{C}}^{(2)}(M) \subseteq \mathrm{Lift}_M(M)$ but does not satisfy (Q_1).

Proof. It is clear that M satisfies (C_1) and (C_3), so that M satisfies $\underline{\underline{C}}^{(2)} \subseteq \mathrm{Lift}_M(M)$, by Theorem 4.1.3. Let N denote the submodule $2\mathbb{Z}$ of $\mathbb{Z}$. Then $N \cong M$, but the homomorphism $\varphi : N \to M$ given by

$$\varphi(2n) = n \quad (n \in \mathbb{Z})$$

does not lift to M. Suppose that there exists a homomorphism $\theta : M \to M$ such that $\theta|_N = \varphi$. Then there exists $x \in M$ such that $\theta(m) = xm$ $(m \in M)$. Therefore

$$2xm = \theta(2m) = \varphi(2m) = m,$$

so that $2x = 1$, a contradiction. Thus M does not satisfy (Q_1).

Example 4.2.1 shows that, for a module M, none of the implications quasi-continuous $\Rightarrow$ continuous, $\underline{\underline{C}}^{(2)}(M) \subseteq \mathrm{Lift}_M(M) \Rightarrow (Q_2)$, $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M) \Rightarrow (Q_1)$, is true in general.

Now we shall show that, for $n \geq 2$, $(C_1) \Rightarrow \underline{\underline{C}}^{(n)}(M) \subseteq \mathrm{Lift}_M(M)$ is not true in general. Note that $\underline{\underline{C}}^{(n)}(M) \subseteq \mathrm{Lift}_M(M) \Rightarrow (C_3)$ (see, Proposition 2.2.2).

<u>Example 4.2.2</u>. Let M be any free $\mathbb{Z}$-module of non-zero finite rank k. Then

(i) M satisfies (C_1), and

(ii) M satisfies (C_3) if and only if $k = 1$.

Proof. (i) Let $N \in \underline{\underline{C}}(M)$, $N \neq M$. Then M/N is torsion-free, and hence free. Thus $N \in \underline{\underline{D}}(M)$.

(ii) If $k = 1$ then M is uniform and hence satisfies (C_3). Conversely, suppose that $k \geq 2$. Let $f_1, \ldots, f_k$ be a basis of M. Let $K_1 = \mathbb{Z}f_1$ and $K_2 = \mathbb{Z}(f_1 + 2f_2)$. Clearly, $M = K_1 \oplus L = K_2 \oplus L$, where $L = \mathbb{Z}f_2 + \ldots + \mathbb{Z}f_k$. Also $K_1 \cap K_2 = 0$, but $K_1 \oplus K_2 = \mathbb{Z}f_1 \oplus \mathbb{Z}2f_2$, which is not a direct summand of $\mathbb{Z}f_1 \oplus \mathbb{Z}f_2$, and hence not a direct summand of M. Thus M does not satisfy (C_3).

Note that neither of the implications

$$\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M) \Rightarrow \underline{\underline{C}}^{(2)}(M) \subseteq \mathrm{Lift}_M(M), \quad \underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M) \Rightarrow (C_1),$$

is true for a module M, in general (see Example 2.1.16). Moreover, the module M in Example 2.1.16, does not satisfy (Q_1). To see why this is so, let $K = \mathbb{Z}(1 + \mathbb{Z}p, 1)$. Then $K \in \underline{\underline{C}}(M)$ and $K \cap M_1 = 0$. Let $\pi : M \to M_2$ denote the canonical projection. Let $L = \pi(K)$. Then $L \cong K$. Note that $L = \mathbb{Z}(0 + \mathbb{Z}p, 1)$. Define $\varphi : L \to M$ by

$$\varphi(r(0 + \mathbb{Z}p, 1)) = r(1 + \mathbb{Z}p, 1) \quad (r \in \mathbb{Z}).$$

Then φ is a homomorphism which does not lift to M. For, suppose that φ could be lifted to a homomorphism $\theta : M \to M$. Then

$$(1 + \mathbb{Z}p, 1) = \varphi(0 + \mathbb{Z}p, 1) = \theta(0 + \mathbb{Z}p, 1) = p\theta(0 + \mathbb{Z}p, 1/p),$$

a contradiction. It follows that M does not satisfy (Q_1).

Finally, we turn to the conditions (Q_1) and (Q_2). The next example, which is due to B. L. Osofsky, shows that (Q_1) does not imply (Q_2).

Example 4.2.3. There exists a commutative local ring R such that the R-module R satisfies (Q_1) but does not satisfy (C_1).

Proof. By [18, Remark (i)], there exists a commutative valuation domain S such that every homomorphic image of S is a self-injective ring, but S is not Noetherian. Suppose that every proper image of S has non-zero socle. Then every proper homomorphic image of S is finitely cogenerated (S is a valuation ring!), and hence is Artinian by [29, Theorem 3.21]. Thus every proper homomorphic image of S is Noetherian. By [29, Theorem 3.25 Corollary], S is Noetherian, a contradiction. Thus there exists a non-zero ideal A of S such that the ring S/A has zero socle. Let $T = S/A$ and note that T is a local self-injective ring.

Let J denote the unique maximal ideal of T. Let R denote the subring of the commutative ring $T \oplus T$ defined by

$$R = \{(t,t') : t - t' \in J\}.$$

Then R is the pullback of

$$\begin{array}{ccc} & & T \\ & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow{\pi} & T/J \end{array}$$

where $\pi : T \to T/J$ is the canonical epimorphism. The ring R is a commutative local ring with unique maximal ideal $J \oplus J$. Let M denote the R-module R. Then, being local, M does not have (C_1) and hence M does not have (Q_2) (Theorem 4.1.4).

It remains to show that M satisfies (Q_1). Let t and t′ be any non-zero elements of T. If t has zero annihilator in T then $T = Tt$, because T is self-injective (see, Lemma 3.2.6 or [29, Proposition 2.6]), and hence

$M = R(t,t')$. Similarly, $M = R(t,t')$ if t' has zero annihilator. Now suppose that t and t' both have non-zero annihilator. Then $R(t,t') \cap (J \oplus 0) \neq 0$ and $R(t,t') \cap (0 \oplus J) \neq 0$. Thus $R(t,t') \in \underline{\underline{E}}(M)$. It follows that $\underline{\underline{C}}(M) = \{0, M, J \oplus 0, 0 \oplus J\}$.

Let N be a submodule and $K \in \underline{\underline{C}}(M)$ such that there exists an isomorphism $\alpha : K \to N$. Let $\varphi : N \to M$ be a homomorphism. If $K = 0$ then $N = 0$ and φ can be lifted to M. Now suppose that $K = M$. In this case, $N = R(t,t')$ for some elements t, t' in T such that both t and t' have zero annihilator. As we have just seen, this gives $N = R(t,t') = M$. Again, φ lifts to M trivially.

Now suppose that $K = J \oplus 0$. For any $a \in J$, $\alpha(a,0) = (b,c)$ for some b, $c \in T$. Now $(a,0)(0 \oplus J) = 0$ implies $(b,c)(0 \oplus J) = 0$, and hence $cJ = 0$. Because T has zero socle, we have $c = 0$. Thus $N = \alpha(K) \subseteq J \oplus 0$. It follows that $N = L \oplus 0$ for some proper ideal L of T. Now consider $\varphi : N \to M$. Because $N(0 \oplus J) = 0$, the same argument gives $\varphi(N) \subseteq J \oplus 0$. Thus φ induces a homomorphism $\varphi' : L \to T$. But T is self-injective, and hence φ' can be lifted to T and this allows us to lift φ to M. A similar proof shows that if $K = 0 \oplus J$ then φ can be lifted to M. It follows that M satisfies (Q_1).

Chapter 5.

GENERALIZATIONS OF CS-MODULES

In this chapter two generalizations of CS-modules and the conditions on a module which imply that it is a direct sum of uniform modules will be investigated. Finally we shall consider chain conditions.

5.1. Modules with (C_{11}).

Let R be a ring. In this section we shall establish some properties of R-modules which satisfy (C_{11}). Note that modules which satisfy (C_{11}) are mentioned by Mohamed and Müller in [21, p.106].

Definition 5.1.1. A module M *satisfies* (C_{11}) if every submodule of M has a complement which is a direct summand of M, i.e. for each $N \in \underline{\underline{L}}(M)$ there exists a $K \in \underline{\underline{D}}(M)$ such that K is a complement of N in M.

For purposes of comparision we first prove

Proposition 5.1.2. A module M satisfies (C_1) if and only if for all submodules N and L such that $N \cap L=0$ there exists a $K \in \underline{\underline{D}}(M)$ such that $L \leq K$ and $N \cap K=0$. Moreover, in this case $N \oplus K \in \underline{\underline{E}}(M)$.

Proof. Suppose first that M satisfies (C_1). Suppose that N and L are submodules of M such that $N \cap L=0$. There exists a complement K of N in M such that $L \leq K$. By hypothesis, $K \in \underline{\underline{D}}(M)$.

Conversely, suppose that M satisfies the stated condition. Let $L \in \underline{\underline{C}}(M)$. There exists a submodule N of M such that L is a complement of N in M. By hypothesis, there exists a $K \in \underline{\underline{D}}(M)$ such that $L \leq K$ and $K \cap N = 0$. Thus $L = K$. It follows that every complement in M is a direct summand. Therefore M satisfies (C_1).

For the last part, use Proposition 1.1.5.

Lemma 5.1.3. Let N be a submodule of a module M and let $K \in \underline{\underline{D}}(M)$. Then K is a complement of N in M if and only if $K \cap N = 0$ and $K \oplus N \in \underline{\underline{E}}(M)$.

Proof. The necessity follows by Proposition 1.1.5. Conversely, suppose that K and N have the stated properties. There exists a submodule K' of M such that $M = K \oplus K'$. Suppose that there exists a submodule K_1 of M such that $K \subseteq K_1$ and $K_1 \cap N = 0$. Then $K_1 = K_1 \cap M = K_1 \cap (K \oplus K') = K \oplus (K_1 \cap K')$. Let $0 \neq y \in (K_1 \cap K')$. Therefore $0 \neq yr = n + k$ for some $n \in N$, $k \in K$, $r \in R$ (because $N \oplus K \in \underline{\underline{E}}(M)$). Therefore, $yr - k = n \in K_1 \cap N = 0$, so that $yr = k \in K \cap K' = 0$, a contradiction. Hence $K_1 \cap K' = 0$ and $K = K_1$. That is, K is a complement of N in M.

Compare the next result with Proposition 5.1.2.

Proposition 5.1.4. The following statements are equivalent for a module M.

(i) M has (C_{11}).

(ii) For any $L \in \underline{\underline{C}}(M)$, there exists a $K \in \underline{\underline{D}}(M)$ such that K is a complement of L in M.

(iii) For any submodule N of M, there exists a $K \in \underline{\underline{D}}(M)$ such that $N \cap K = 0$ and $N \oplus K \in \underline{\underline{E}}(M)$.

(iv) For any $L \in \underline{\underline{C}}(M)$, there exists a $K \in \underline{\underline{D}}(M)$ such that $L \cap K = 0$ and

$L \oplus K \in \underline{\underline{E}}(M)$.

Proof. (i)$\Rightarrow$(ii), (iii)$\Rightarrow$(iv) Obvious.

(i)$\Leftrightarrow$(iii), (ii)$\Leftrightarrow$(iv) Clear by Lemma 5.1.3.

(iv)$\Rightarrow$(i). Let A be any submodule of M. Then there exists $B \in \underline{\underline{C}}(M)$ such that $A \in \underline{\underline{E}}(B)$. By hypothesis, there exists a $K \in \underline{\underline{D}}(M)$ such that $B \cap K = 0$ and $B \oplus K \in \underline{\underline{E}}(M)$. Hence, by Lemma 5.1.3, K is a complement of B in M. Note that $K \cap A = 0$. Suppose that K' is a submodule of M which properly contains K. Therefore $K' \cap B \neq 0$ and hence $K' \cap B \cap A \neq 0$, i.e. $K' \cap A \neq 0$. Thus K is a complement of A in M.

It is clear from Proposition 5.1.2 that any module M with (C_1) satisfies (C_{11}), because any complement submodule of M is a direct summand (or see Proposition 5.1.4). In particular uniform modules, semisimple modules and injective modules satisfy (C_{11}). On the other hand, any indecomposable module with (C_{11}) is uniform. Let us justify this statement :

Let N be a non-zero submodule of M. By (C_{11}), there exists a $K \in \underline{\underline{D}}(M)$ such that K is a complement of N in M. But M is indecomposable thus $K = 0$ and hence $N \in \underline{\underline{E}}(M)$ (Proposition 1.1.5). Therefore $N \in \underline{\underline{E}}(M)$. It follows that M is uniform.

We shall show that every module which is a direct sum of uniform modules satisfies (C_{11}). In particular, for any prime p, the $\mathbb{Z}$-module $M = (\mathbb{Z} / \mathbb{Z}p) \oplus (\mathbb{Z} / \mathbb{Z}p^3)$ satisfies (C_{11}). However M does not satisfy (C_1). (see Example 1.2.3). Thus a sum of uniform modules does not have (C_1), in general.

<u>Theorem 5.1.5.</u> Any direct sum of modules with (C_{11}) satisfies (C_{11}).

Proof. Let M_λ $(\lambda \in \Lambda)$ be a non-empty collection of modules, each satisfying (C_{11}). Let $M = \oplus_{\lambda \in \Lambda} M_\lambda$. Let N be any submodule of M. Let $\lambda \in \Lambda$. Note that $N \cap M_\lambda$ is a submodule of M_λ and M_λ satisfies (C_{11}). By Proposition 5.1.4, there exists $K_\lambda \in \underline{\underline{D}}(M_\lambda)$ such that $(N \cap M_\lambda) \cap K_\lambda = 0$ and $(N \cap M_\lambda) \oplus K_\lambda \in \underline{\underline{E}}(M_\lambda)$. Note that $N \cap K_\lambda = 0$, $(N \oplus K_\lambda) \cap M_\lambda = (N \cap M_\lambda) \oplus K_\lambda$ and $(N \oplus K_\lambda) \cap M_\lambda \in \underline{\underline{E}}(M_\lambda)$. Let Λ' be a non-empty subset of Λ containing λ such that there exists a $K' \in \underline{\underline{D}}(M' = \oplus_{\lambda \in \Lambda'} M_\lambda)$, with $N \cap K' = 0$ and with $(N \oplus K') \cap M' \in \underline{\underline{E}}(M')$. Suppose $\Lambda' \neq \Lambda$. Let $\mu \in \Lambda$, $\mu \notin \Lambda'$. Now $L = (N \oplus K') \cap M_\mu$ is a submodule of M_μ, so there exists $K_\mu \in \underline{\underline{D}}(M_\mu)$ such that $L \cap K_\mu = 0$ and $L \oplus K_\mu \in \underline{\underline{E}}(M_\mu)$. Let $\Lambda'' = \Lambda' \cup \{\mu\}$ and $M'' = \oplus_{\lambda \in \Lambda''} M_\lambda = M' \oplus M_\mu$. Note that $K' \cap K_\mu = 0$. Let $K'' = K' \oplus K_\mu$. Note that $K'' \in \underline{\underline{D}}(M'')$ and moreover $N \cap K'' = 0$.

Consider the submodule $N \oplus K''$. Note that $(N \oplus K'') \cap M'$ contains $(N \oplus K') \cap M'$, so that $(N \oplus K'') \cap M' \in \underline{\underline{E}}(M')$. Moreover,

$$(N \oplus K'') \cap M_\mu = (N \oplus K' \oplus K_\mu) \cap M_\mu = [(N \oplus K') \cap M_\mu] \oplus K_\mu = L \oplus K_\mu \in \underline{\underline{E}}(M_\mu).$$

It follows that $(N \oplus K'') \cap M'' \in \underline{\underline{E}}(M'')$. Repeating this argument, there exists $K \in \underline{\underline{D}}(M)$ such that $N \cap K = 0$ and $N \oplus K \in \underline{\underline{E}}(M)$. By Proposition 5.1.4, M satisfies (C_{11}).

<u>Corollary 5.1.6.</u> Any direct sum of modules with (C_1) satisfies (C_{11}).

Proof. Immediate by Theorem 5.1.5.

<u>Corollary 5.1.7.</u> Any direct sum of uniform modules satisfies (C_{11}).

Proof. Immediate by Corollary 5.1.6.

The next result shows that the study of modules with (C_{11}) reduces to the case of Goldie torsion modules and nonsingular modules. It is the analogue of

[15, Theorem 1]. Recall that the *singular submodule* $Z(M)$ of a module M is defined by

$$Z(M) = \{m \in M : mE = 0 \text{ for some } E \in \underline{\underline{E}}(R)\},$$

and the *Goldie torsion submodule* (or *second singular submodule*) $Z_2(M)$ of M is that submodule of M, containing $Z(M)$, such that $Z_2(M) / Z(M)$ is the singular submodule of $M / Z(M)$.

Theorem 5.1.8. A module M satisfies (C_{11}) if and only if $M = Z_2(M) \oplus K$ for some (nonsingular) submodule K of M, and $Z_2(M)$ and K both satisfy (C_{11}).

Proof. The sufficiency is an immediate consequence of Theorem 5.1.5. Conversely, suppose M satisfies (C_{11}). We prove first that $Z_2(M) \in \underline{\underline{D}}(M)$. Let $L = Z_2(M)$. There exist submodules K and K' of M such that $M = K \oplus K'$, $L \cap K = 0$ and $L \oplus K \in \underline{\underline{E}}(M)$ (Proposition 5.1.4). Now $L = Z_2(M) = Z_2(K \oplus K') = Z_2(K) \oplus Z_2(K')$. But it is clear that $Z_2(K) = 0$. Thus $L = Z_2(K') \subseteq K'$. Because $L \oplus K \in \underline{\underline{E}}(M)$, $L \in \underline{\underline{E}}(K')$, and hence K' / L is singular. Thus $L = K'$, and $L \in \underline{\underline{D}}(M)$.

We have proved that $M = L \oplus K$. Now we prove that L satisfies (C_{11}). Let N be any submodule of L. Then $N \oplus K$ is a submodule of M. Because M satisfies (C_{11}), there exist submodules P, P' of M such that $M = P \oplus P'$, $(N \oplus K) \cap P = 0$ and $N \oplus K \oplus P \in \underline{\underline{E}}(M)$. Note that $P \cap K = 0$ and hence P embeds in $M / K \cong L$. Thus $P = Z_2(P)$ and $P \leq L$. It follows that $P \in \underline{\underline{D}}(L)$ (in fact $L = P \oplus (L \cap P')$) and $N \oplus P \in \underline{\underline{E}}(L)$. By Proposition 5.1.4, L satisfies (C_{11}).

Finally we prove that K satisfies (C_{11}). Let $\pi : M \to K$ denote the canonical projection. Let H be any submodule of K. Then $L \cap H = 0$, and there exist submodules Q and Q' of M such that $M = Q \oplus Q'$, $(L \oplus H) \cap Q = 0$ and $L \oplus H \oplus Q \in \underline{\underline{E}}(M)$. Note that $L = Z_2(M) = Z_2(Q) \oplus Z_2(Q') = Z_2(Q')$, because $Q \cap L = 0$ gives $Z_2(Q) = 0$. Hence $L \leq Q'$, and $Q' = L \oplus (Q' \cap K)$. Now $M = Q \oplus Q' = Q \oplus L \oplus (Q' \cap K)$. Thus $L \oplus Q \in \underline{\underline{D}}(M)$. But $L \oplus Q = L \oplus \pi(Q)$. Therefore

$\pi(Q) \in \underline{\underline{D}}(M)$, and hence $\pi(Q) \in \underline{\underline{D}}(K)$. However $H \oplus \pi(Q) \oplus L \in \underline{\underline{E}}(M)$. Thus $H \oplus \pi(Q) \in \underline{\underline{E}}(K)$. By Proposition 5.1.4, K satisfies (C_{11}).

In a similar vein to Theorem 5.1.8 we show that the study of nonsingular modules satisfying (C_{11}) reduces to the case of modules with essential socle and modules with zero socle. For any module M, Soc M will denote the socle of M. First we prove.

Lemma 5.1.9. Let M be a module which satisfies (C_{11}). Then $M = M_1 \oplus M_2$ where M_1 is a submodule of M with Soc $M_1 \in \underline{\underline{E}}(M_1)$ and M_2 a submodule of M with zero socle.

Proof. Let S denote the socle of M. There exist submodules K and K′ of M such that $M = K \oplus K'$, $S \cap K = 0$ and $S \oplus K \in \underline{\underline{E}}(M)$. By [1, Proposition 9.19],

$$S = \text{Soc } M = (\text{Soc } K) \oplus (\text{Soc } K').$$

Clearly Soc $K = 0$ so that $S \leq K'$. Now $S \oplus K \in \underline{\underline{E}}(M)$ implies $S \in \underline{\underline{E}}(K')$, and the result is proved.

Let M be a module and N any submodule of M. Let us form the submodule

$$c(N) = \{m \in M : mE \leq N \text{ for some essential right ideal } E \text{ of } R\},$$

of M. Then

Lemma 5.1.10. Let M be a nonsingular module and N any submodule of M. Then c(N) is a unique complement in M such that $N \in \underline{\underline{E}}(c(N))$.

Proof. Suppose that $N \in \underline{\underline{E}}(K)$ for any $K \leq M$. Let $0 \neq x \in K$. Then $xR \cap N \neq 0$. Thus $E = x^{-1}N \in \underline{\underline{E}}(R_R)$ by Proposition 1.1.2. Therefore $xE \leq N$ and hence $x \in c(N)$ so that $K \leq c(N)$. Now let $0 \neq y \in c(N)$. Thus $yF \leq N$ for some

$F \in \underline{\underline{E}}(R_R)$. Since M is nonsingular, $yF \neq 0$. Therefore $0 \neq yF \leq N \cap yR$. It follows that $N \in \underline{\underline{E}}(c(N))$. The result follows.

The proof of the next result is very similar to that of Theorem 5.1.8 but we include it for completeness.

<u>Theorem 5.1.11</u>. A nonsingular module M satisfies (C_{11}) if and only if $M = M_1 \oplus M_2$ where M_1 is a module satisfying (C_{11}) with Soc $M_1 \in \underline{\underline{E}}(M_1)$ and M_2 is a module satisfying (C_{11}) and having zero socle.

Proof. The sufficiency is clear by Theorem 5.1.5.

Conversely, suppose that M satisfies (C_{11}). By Lemma 5.1.9, $M = M_1 \oplus M_2$ where Soc $M_1 \in \underline{\underline{E}}(M_1)$ and Soc $M_2 = 0$. Let S denote the socle of M. Clearly $M_1 = c(S)$.

We prove next that M_1 satisfies (C_{11}). Let N be any submodule of M_1. By Proposition 5.1.4, there exists $P \in \underline{\underline{D}}(M)$ such that $(N \oplus M_2) \cap P = 0$ and $N \oplus M_2 \oplus P \in E(M)$. Now P embeds in M_1 and hence Soc $P = S \cap P \in \underline{\underline{E}}(P)$ (see [1, Corollary 9.9]). Thus $P = c(S \cap P) \leq c(S) = M_1$. Hence $P \in \underline{\underline{D}}(M_1)$ and $N \oplus P \in \underline{\underline{E}}(M_1)$. By Proposition 5.1.4, M_1 satisfies (C_{11}).

Now consider M_2. Let $\pi : M \to M_2$ denote the canonical projection. Let H be any submodule of M_2. By Proposition 5.1.4, M has submodules Q and Q' such that $M = Q \oplus Q'$, $(M_1 \oplus H) \cap Q = 0$ and $M_1 \oplus H \oplus Q \in \underline{\underline{E}}(M)$. Now $S \cap Q = 0$ implies $S \subseteq Q'$, by [1, Proposition 9.19]. Thus $M_1 = c(S) \subseteq Q'$. It follows that $M_1 \in \underline{\underline{D}}(Q')$ and hence $M_1 \oplus Q \in \underline{\underline{D}}(M)$. This implies that $M_1 \oplus \pi(Q) \in \underline{\underline{D}}(M)$, $\pi(Q) \in \underline{\underline{D}}(M_2)$, and $H \oplus \pi(Q) \in \underline{\underline{E}}(M_2)$. By Proposition 5.1.4, M_2 satisfies (C_{11}).

Theorems 5.1.8 and 5.1.11 raise the following natural question :

Let M be a module which satisfies (C_{11}). Does any direct summand of M

satisfy (C_{11}) ?

We do not know the answer to this question in general. A special case of this question is of interest, namely, if a module M is a direct sum of uniform submodules, does any direct summand of M satisfy (C_{11}) ? The next result deals with a special case (see also Theorem 5.3.3). First we prove.

Lemma 5.1.12. Let $N \in \underline{\underline{D}}(M)$ and let K be an injective submodule of M such that $N \cap K = 0$. Then $N \oplus K \in \underline{\underline{D}}(M)$.

Proof. There exists a submodule N' of M such that $M = N \oplus N'$. Let $\pi : M \to N'$ be the canonical projection. Then $N \cap K = 0$ implies $K \cong \pi(K)$ so that $\pi(K)$ is injective. It follows that $\pi(K) \in \underline{\underline{D}}(N')$. But $N \oplus K = N \oplus \pi(K)$, and hence $N \oplus K \in \underline{\underline{D}}(M)$.

Proposition 5.1.13. Let M be a module which satisfies (C_{11}). Let $N \in \underline{\underline{D}}(M)$ such that M / N is an injective module. Then N satisfies (C_{11}).

Proof. Let L be any submodule of N. There exists an injective submodule N' of M such that $M = N \oplus N'$. Consider the submodule $L \oplus N'$. There exists a $K \in \underline{\underline{D}}(M)$ such that $(L \oplus N') \cap K = 0$ and $L \oplus N' \oplus K \in \underline{\underline{E}}(M)$ (Proposition 5.1.4). By Lemma 5.1.12, $N' \oplus K \in \underline{\underline{D}}(M)$. But

$$N' \oplus K = N' \oplus \pi(K),$$

where $\pi : M \to N$ denotes the canonical projection. Thus $\pi(K) \in \underline{\underline{D}}(N)$. However $L \oplus \pi(K) \oplus N' \in \underline{\underline{E}}(M)$. It follows that $L \oplus \pi(K) \in \underline{\underline{E}}(N)$. By Proposition 5.1.4, N satisfies (C_{11}).

Let U be a torsion uniform $\mathbb{Z}$-module. Then the injective hull E(U) of U is a torsion indecomposable injective $\mathbb{Z}$-module, so that E(U) is a quasicyclic group $\mathbb{Z}(p^{\infty})$ for some prime p (see, for example, [7, Volume I, Theorem 23.1]).

Lemma 5.1.14. Let M be an abelian p-group for some prime p. Suppose that the $\mathbb{Z}$-module M is a direct sum of uniform modules. Then every direct summand of M is a direct sum of uniform modules.

Proof. Our above remarks show that $M = M_1 \oplus M_2$ where M_1 is an injective submodule of M and the submodule M_2 is a direct sum of cyclic groups. Let N be a direct summand of M. There exists a submodule N' of M such that $M = N \oplus N'$. Let $\pi : M \to N$ denote the canonical projection. Then $\pi(M_1)$ is injective ([7, Volume I, p.98 (D)]), and hence $N = \pi(M_1) \oplus L$ for some submodule L of N. Let $L' = \pi(M_1) \oplus N'$. Then $M = L \oplus L'$. Note that $M_1 \subseteq L'$. Let $\pi_1 : M \to L$ denote the canonical projection. Then $\pi_1(M_1) = 0$, and hence $M_1 \leq \ker\pi_1 = L'$. Thus $L' = M_1 \oplus (L' \cap M_2)$. Now $L \oplus M_1 \oplus (L' \cap M_2) = M = M_1 \oplus M_2$, so that $L \oplus (L' \cap M_2) \cong M_2$. By a theorem of Kulikov ([7, Volume I, Theorem 18.1]) L is a direct sum of cyclic groups. It follows that L is a direct sum of uniform submodules. Moreover $\pi(M_1)$ is a direct sum of uniform submodules, by [7, Volume I, Theorem 23.1]. Thus $N = \pi(M_1) \oplus L$ is a direct sum of uniform submodules.

For any abelian group A, let $\tau(A)$ denote its torsion subgroup and, for any prime p, let $\tau_p(A)$ denote the p-component of $\tau(A)$ [7, Volume I, p.43]. We now prove.

Theorem 5.1.15. Let M be a $\mathbb{Z}$-module such that M is a direct sum of uniform modules. Then any direct summand of M is a direct sum of uniform modules.

Proof. Each uniform $\mathbb{Z}$-module is torsion or torsion-free. Thus $M = M_1 \oplus M_2$

where M_1 is a torsion-free module, M_2 is a torsion module and both M_1 and M_2 are direct sums of uniform modules. Let $N \in \underline{\underline{D}}(M)$. Then $M = N \oplus N'$ for some submodule N' of M. Note that $M_2 = \tau(N) \oplus \tau(N')$, so that $\tau(N), \tau(N') \in \underline{\underline{D}}(M)$. It follows that $N = \tau(N) \oplus K$ and $N' = \tau(N') \oplus K'$ for some submodules K of N and K' of N'. Now $M_1 \oplus M_2 = M = N \oplus N' = M_2 \oplus K \oplus K'$, and hence $M_1 \cong K \oplus K'$. By [7, Volume II, Theorem 86.7], K is a direct sum of uniform modules.

Next, every torsion uniform submodule of M_2 is a p-group for some prime p [7, Volume I, Theorem 8.4]. Thus $M_2 = \oplus \tau_p(M_2)$, where the direct sum is over all primes p, and $\tau_p(M_2)$ is a direct sum of uniform modules for each p, again by [7, Volume I, Theorem 8.4]. Moreover, it is clear that $\tau_p(M_2) = \tau_p(N) \oplus \tau_p(N')$, for each prime p. By Lemma 5.1.14, $\tau_p(N)$ is a direct sum of uniform modules for each prime p, and hence so too is $\tau(N) = \oplus_p \tau_p(N)$. Thus $N = \tau(N) \oplus K$ is a direct sum of uniform modules.

Combining Corollary 5.1.7 and Theorem 5.1.15, we conclude that over the ring $\mathbb{Z}$, any direct sum of uniform modules satisfies $(C_{11}{}^{+})$.

Proposition 5.1.16. Let M be a torsion-free module over $\mathbb{Z}$. Then M satisfies (C_2) if and only if M is injective.

Proof. Suppose M satisfies (C_2). Suppose $M = I \oplus R$, where I is injective and R is reduced. Thus R satisfies (C_2). Let p be any prime, then $pR \cong R$ (with $r \to pr$). Therefore $pR \in \underline{\underline{D}}(M)$ and hence $R = pR \oplus X$ for some $X \leq R$. Now $pR \in \underline{\underline{E}}(R)$ and hence $X = 0$. It follows that $R = pR$ for all primes p. Therefore R is injective so that $R = 0$. Hence $M = I$ is injective.

We make one final comment in this section. A free abelian group M is a $\mathbb{Z}$-module with (C_1) if and only if it has finite rank, by [15, Theorem 5].

However, any free abelian group satisfies (C_{11}) by Corollary 5.1.7.

5.2. Modules with (C_{12}).

We show that for any ring R the class of R-modules with (C_{12}) properly contains the class of modules which satisfy (C_{11}), and we establish some properties of modules with (C_{12}). Let us start with the definition of a module which satisfies (C_{12}).

Definition 5.2.1. A module M *satisfies* (C_{12}) if, for every submodule N of M, there exists a $K \in \underline{\underline{D}}(M)$ and a monomorphism $\alpha : N \to K$ such that $\alpha(N) \in \underline{\underline{E}}(K)$.

Lemma 5.2.2. A module M satisfies (C_{12}) if and only if for each $N \in \underline{\underline{C}}(M)$ there exists a $K \in \underline{\underline{D}}(M)$ and a monomorphism α: $N \to K$ such that $\alpha(N) \in \underline{\underline{E}}(K)$.

Proof. The necessity is clear. Conversely, suppose M satisfies the stated property for complements. Let L be any submodule of M. There exists $N \in \underline{\underline{C}}(M)$ such that $L \in \underline{\underline{E}}(N)$. By hypothesis, there exists $K \in \underline{\underline{D}}(M)$ and a monomorphism $\alpha : N \to K$ such that $\alpha(N) \in \underline{\underline{E}}(K)$. But $\alpha(L) \in \underline{\underline{E}}(\alpha(N))$, and this implies that $\alpha(L) \in \underline{\underline{E}}(K)$. It follows that M satisfies (C_{12}).

Lemma 5.2.2 makes it clear that modules with (C_1) satisfy (C_{12}). We show next that modules with (C_{11}) satisfy (C_{12}).

Proposition 5.2.3. If a module M satisfies (C_{11}) then M satisfies (C_{12}).

Proof. Let N be a submodule of M. Then there exist submodules K and K′ of M such that $M = K \oplus K'$, $N \cap K' = 0$ and $N \oplus K' \in \underline{\underline{E}}(M)$ (Proposition 5.1.4). Let

$\pi : M \to K$ be the canonical projection mapping and α the restriction of π to the submodule N. Then $\alpha : N \to K$ is a monomorphism. Let $0 \neq k \in K$. Then there exists $r \in R$ such that $0 \neq kr = x + k'$ for some $x \in N$, $k' \in K'$. Now

$$kr = \pi(kr) = \pi(x + k') = \pi(x) = \alpha(x).$$

Thus $kR \cap \alpha(N) \neq 0$ for all $0 \neq k \in K$. Thus $\alpha(N) \in \underline{\underline{E}}(K)$.

Proposition 5.2.4. Let M be any module. Then M is isomorphic to a direct summand of a module which satisfies (C_{12}).

Proof. For any module X, let E(X) denote the injective hull of X. Let $M' = E(E(M) \oplus E(M) \oplus E(M) \oplus \ldots)$. Note that M' is injective. Let $M'' = M \oplus M'$. So M is isomorphic to $M \oplus 0 \in \underline{\underline{D}}(M'')$. Now we show that M'' satisfies (C_{12}). Note that

$$E(M'') \cong E(M) \oplus M' \cong E(E(M) \oplus E(M) \oplus E(M) \oplus \ldots),$$

which is isomorphic to M' and hence there exists a monomorphism β: $M'' \to M'$. Let N be a submodule of M''. Then $\beta(N)$ is a submodule of M'. But M' is injective thus there exists $K \in \underline{\underline{D}}(M')$ (and hence $K \in \underline{\underline{D}}(M'')$) such that $\beta(N) \in \underline{\underline{E}}(K)$. Thus M'' satisfies (C_{12}).

Our next objective is to give an example of a $\mathbb{Z}$-module which satisfies (C_{12}) but which does not satisfy (C_{11}). First we prove :

Lemma 5.2.5. The Specker group does not satisfy (C_{12}).

Proof. Let M be the Specker group $\Pi_1^\infty \mathbb{Z}$ and let N be the subgroup $\oplus_1^\infty \mathbb{Z}$ of M. Suppose that there exists $K \in \underline{\underline{D}}(M)$ and a monomorphism $\alpha : N \to K$ such that $\alpha(N) \in \underline{\underline{E}}(K)$. Note that N is isomorphic to $\alpha(N)$. By Nunke's Theorem [22, Theorem 5], K is isomorphic to M. This implies K has uncountable rank. But

$\alpha(N)$ has countable rank. Thus $\alpha(N) \notin \underline{\underline{E}}(K)$.

Corollary 5.2.6. There exists a $\mathbb{Z}$-module M satisfying (C_{12}) such that some $K \in \underline{\underline{D}}(M)$ does not satisfy (C_{12}).

Proof. By Proposition 5.2.4 and Lemma 5.2.5.

Proposition 5.2.7. There exists a $\mathbb{Z}$-module M which satisfies (C_{12}) but M does not satisfy (C_{11}).

Proof. By the construction of Proposition 5.2.4, if M is the Specker group $\Pi_1^{\infty}\, \mathbb{Z}$ then there exists an injective $\mathbb{Z}$-module M' such that $M'' = M \oplus M'$ satisfies (C_{12}). By Propositions 5.1.13 and 5.2.3, and Lemma 5.2.5, M'' does not satisfy (C_{11}).

Recall that it is proved in Theorem 4.1.4 that a module M is continuous if and only if M satisfies (Q_1) and (C_1). We shall show that M is continuous if it satisfies (Q_1) and (C_{12}). First we need the following Lemma.

Lemma 5.2.8. Suppose that M satisfies (Q_1). Then, for every $K \in \underline{\underline{C}}'(M)$ and any $L \in \underline{\underline{D}}(M)$ such that $K \cap L = 0$, $K \oplus L \in \mathrm{Lift}_M(M)$.

Proof. Let $\varphi : K \oplus L \to M$ be a homomorphism. Now $M = L \oplus L'$ for some submodule L' of M. There exists a homomorphism $\alpha : M \to M$ such that $\varphi(y) = \alpha(y)$ $(y \in L)$. Let $\chi = \varphi - \alpha$; hence $\chi : K \oplus L \to M$ is a homomorphism and $\chi(L) = 0$. Let $\pi : M \to L'$ denote the canonical projection. Then $\pi(K)$ is isomorphic to K. Define $\beta : \pi(K) \to M$ by

$$\beta(\pi(k)) = \chi(k) \ (k \in K).$$

By hypothesis, there exists a homomorphism $\gamma : M \to M$ such that $\gamma(\pi(k)) = \chi(k)$ $(k \in K)$. Now define a homomorphism $\theta : M \to M$ by

$$\theta(x + x') = \gamma(x') \ (x \in L,\ x' \in L').$$

Let $m \in K \oplus L$, so that $m = y + z$ for some $y \in K$, $z \in L$. Then, $m = \pi(y) + y - \pi(y) + z$. Hence $\theta(m) = \gamma\pi(y) = \chi(y) = \chi(m)$. Thus χ lifts to M. Therefore φ lifts to M. That is $K \oplus L \in \mathrm{Lift}_M(M)$.

The next result generalizes Proposition 2.2.1.

<u>Lemma 5.2.9.</u> Let $K \in \underline{\underline{C}}'(M)$. Then $K \in \underline{\underline{D}}(M)$ if and only if there exists a complement L of K in M such that $K \oplus L \in \mathrm{Lift}_M(M)$.

Proof. Suppose that $K \in \underline{\underline{D}}(M)$. Then $M = K \oplus K'$ for some submodule K' of M. Thus $L = K'$ will do.

Conversely suppose that K is isomorphic to a $K' \in \underline{\underline{C}}(M)$. Hence there exists an isomorphism $\alpha : K \to K'$. Define $\beta : K \oplus L \to K'$ by

$$\beta(x + y) = \alpha(x) \ (x \in K,\ y \in L).$$

By hypothesis, there exists a homomorphism $\theta : M \to M$ such that the restriction of θ to $K \oplus L$ is β. Note that $K' = \beta(K) = \theta(K)$ which is a submodule of $\theta(M)$. Let $0 \neq m \in \theta(M)$. Therefore $m = \theta(m')$ for some $m' \in M$ and $m' \notin \ker\theta$. But $L \subseteq \ker\theta$, so we have $m' \notin L$. This implies that $K \cap (L + m'R) \neq 0$. Let $0 \neq x \in K \cap (L + m'R)$. There exist $y \in L$, $r \in R$ such that $x = y + m'r$. Thus $\alpha(x) = 0 + mr \neq 0$. Hence $K' \cap mR \neq 0$ for all $0 \neq m \in \theta(M)$, i.e. $K' \in \underline{\underline{E}}(\theta(M))$. Because $K' \in \underline{\underline{C}}(M)$, we have $K' = \theta(M)$. Let $m \in M$. Then $\theta(m) = \theta(x)$ for some $x \in K$. Therefore $M = K + (\ker\theta)$. Since $K \cap (\ker\theta) = 0$, $M = K \oplus (\ker\theta)$.

<u>Theorem 5.2.10.</u> A module M is continuous if and only if M satisfies (Q_1) and

(C_{12}).

Proof. The necessity is clear by Theorem 4.1.4.

Conversely, suppose that M satisfies (Q_1) and (C_{12}). Let $K \in \underline{\underline{C}}(M)$. There exist some $L \in \underline{\underline{D}}(M)$ and a monomorphism $\alpha : K \to L$ such that $\alpha(K) \in \underline{\underline{E}}(L)$. There exists a submodule L' of M such that $M = L \oplus L'$.

Consider the submodule $\alpha(K) \oplus L'$. Let $\varphi : \alpha(K) \oplus L' \to M$ be any homomorphism. By Lemma 5.2.8, φ lifts to M. Now apply Lemma 5.2.9 to give that $\alpha(K) \in \underline{\underline{D}}(M)$. By Proposition 2.2.1, $K \in \underline{\underline{D}}(M)$. It follows that M satisfies (C_1). By Proposition 2.2.1, M satisfies (C_2). Thus M is continuous.

<u>Note</u>. Let M be a right R-module. Suppose that M satisfies (C_{12}) (or even (C_{11})) then M does not need to satisfy $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M)$.

For example, let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Z}$ where p is prime. Then $\underline{\underline{C}}(M)$ is not contained in $\mathrm{Lift}_M(M)$ (see, Example 2.1.17). But M satisfies (C_{11}), by Corollary 5.1.7 and (C_{12}) by Proposition 5.2.3.

The converse is not true in general either. For example, let $M = {}_R R$ be as in Example 4.2.3. Then M satisfies (Q_1) and hence $\underline{\underline{C}}(M) \subseteq \mathrm{Lift}_M(M)$. Since M does not satisfy (Q_2) (see, Example 4.2.3), it does not satisfy (C_{12}), by Theorem 5.2.10.

<u>Example 5.2.11</u>. Let K be a field and V be a vector space over K such that $\dim_K V = n$. Let

$$R = \begin{bmatrix} K & V \\ 0 & K \end{bmatrix} = \left\{ \begin{bmatrix} k & v \\ 0 & k \end{bmatrix} : k \in K,\ v \in V \right\}.$$

Then clearly R is a commutative ring with respect to the usual matrix operations. Moreover,

(i) R_R is an indecomposable module.

(ii) If $A \in \underline{\underline{E}}(R_R)$ then $n \leq \dim_K A$.

(iii) R_R satisfies (C_{12}) if and only if $n = 1$.

Proof. (i) Let $e = \begin{bmatrix} a & v \\ 0 & a \end{bmatrix}$ be an idempotent in R for some $a \in K$, $v \in V$.

Then

$$\begin{bmatrix} a & v \\ 0 & a \end{bmatrix} = \begin{bmatrix} a & v \\ 0 & a \end{bmatrix}^2 = \begin{bmatrix} a^2 & 2av \\ 0 & a^2 \end{bmatrix},$$

gives $a = 0$ or $a = 1$. Therefore $e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. It follows that R_R is indecomposable.

(ii) Let $A \in \underline{\underline{E}}(R_R)$. Then for any $0 \neq v \in V$, $0 \neq \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} k & v' \\ 0 & k \end{bmatrix} \in A$ i.e, $\begin{bmatrix} 0 & kv \\ 0 & 0 \end{bmatrix} \in A$ for some $0 \neq k \in K$, $v' \in V$. Thus

$$\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/k & 0 \\ 0 & 1/k \end{bmatrix}\begin{bmatrix} 0 & kv \\ 0 & 0 \end{bmatrix} \in A.$$

Hence $\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} \leq A$, so that $n \leq \dim_K A$.

(iii) Suppose first that $n = 1$. Let $0 \neq \begin{bmatrix} k & v \\ 0 & k \end{bmatrix} \in R$ and $0 \neq \begin{bmatrix} k' & v' \\ 0 & k' \end{bmatrix} \in R$ ($k, k' \in K$ and $v, v' \in V$).

Case 1. $k \neq 0$, $k' \neq 0$.

Since $\begin{bmatrix} k & v \\ 0 & k \end{bmatrix}\begin{bmatrix} k' & v \\ 0 & k' \end{bmatrix} = \begin{bmatrix} kk' & kv \\ 0 & kk' \end{bmatrix} \neq 0$, $\begin{bmatrix} k & v \\ 0 & k \end{bmatrix}R \cap \begin{bmatrix} k' & v' \\ 0 & k' \end{bmatrix}R \neq 0$.

Case 2. $k = 0$, $k' \neq 0$.

Since $\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} k' & v' \\ 0 & k' \end{bmatrix} = \begin{bmatrix} 0 & k'v \\ 0 & 0 \end{bmatrix} \neq 0$, $\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}R \cap \begin{bmatrix} k' & v' \\ 0 & k' \end{bmatrix}R \neq 0$.

Case 3. $k = 0$, $k' = 0$.

Thus $v \neq 0$, $v' \neq 0$. There exists $k'' \in K$ such that $v = k''v'$. Hence

$$0 \neq \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} k'' & 0 \\ 0 & k'' \end{bmatrix}\begin{bmatrix} 0 & v' \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} k & v \\ 0 & k \end{bmatrix}R \cap \begin{bmatrix} k' & v' \\ 0 & k' \end{bmatrix}R.$$

Thus R_R is uniform and hence it satisfies (C_{12}).

For the converse, suppose that $n > 1$. Let $V = Kv_1 \oplus Kv_2 \oplus \ldots \oplus Kv_n$ where $v_i \in V$ $(1 \leq i \leq n)$. Let $N = \begin{bmatrix} 0 & Kv_1 \\ 0 & 0 \end{bmatrix}$. Suppose that there exists a monomorphism $\alpha : N \to R$ such that $\alpha(N) \in \underline{\underline{E}}(R_R)$. Then $\dim_K N = \dim_K \alpha(N) = 1$. Thus $\alpha(N) \notin \underline{\underline{E}}(R_R)$, by (ii). Therefore there is no such a monomorphism i.e, R_R does not satisfy (C_{12}). Then the result follows.

We will finish this section with the following two examples which illustrate neither $(C_{11}) \Rightarrow$ CLS, nor CLS $\Rightarrow (C_{11})$ is true in general.

<u>Example 5.2.12.</u> Let ${}_{\mathbb{Z}}M = \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$. Then M satisfies (C_{11}) but is not a CLS-module.

Proof. Suppose $\varphi : M \to \mathbb{Q}$ is an epimorphism. Let $K = \ker\varphi$. Thus $M / K \cong \mathbb{Q}$ which is nonsingular. Hence $K \in \underline{\underline{CL}}(M)$. If K was a direct summand of M then we would have $M = K \oplus L$ for some $L \leq M$, where $L \cong \mathbb{Q}$ which is a contradiction. It follows that M is not a CLS-module. On the other hand, M satisfies (C_{11}) by Corollary 5.1.7.

<u>Example 5.2.13.</u> Let R denote the ring as in Example 5.2.11. Suppose $\dim_K V = 2$. Then R_R is a CLS-module which does not satisfy (C_{11}).

Proof. By Example 5.2.11, R does not satisfy (C_{11}). Since R_R has no proper closed submodules (Example 2.3.3) then R_R is a CLS-module.

5.3. Modules with (C_{11}^{+})

We mentioned in §5.1 that we do not know if direct summands of modules with (C_{11}) satisfy (C_{11}). We showed in §5.2 that direct summands of modules with (C_{12}) need not satisfy (C_{12}). This is in contrast to the situation for modules which satisfy (C_1) (see [21, Proposition 2.7]).

Definition 5.3.1. Let (P) be some property of modules. Then we shall say that a module M *satisfies* (P^{+}) if every direct summand of M satisfies (P).

For example, an indecomposable module satisfies (P^{+}) if and only if it satisfies (P). A module M satisfies (C_1) if and only if M satisfies (C_1^{+}) (see [21, Proposition 2.7]). We can abbreviate this fact to $(C_1) = (C_1^{+})$. We know that, in general, $(C_{12}) \neq (C_{12}^{+})$ (Corollary 5.2.6), and have been unable to settle whether $(C_{11}) = (C_{11}^{+})$.

Lemma 5.3.2. Let $M = U \oplus V$ be a direct sum of uniform modules U and V. Then M satisfies (C_{11}^{+}).

Proof. Let $0 \neq K \in \underline{\underline{D}}(M)$. If $K = M$ then K satisfies (C_{11}) by Corollary 5.1.7. If $K \neq M$ then K is uniform and hence K satisfies (C_{11}). Thus M satisfies (C_{11}^{+}).

Recall that $(C_2^{+}) = (C_2)$ and $(C_3^{+}) = (C_3)$ (see Proposition 1.3.3). The next result shows that if a module M satisfies (C_{11}) and (C_3) then so too does every direct summand of M.

Theorem 5.3.3. Let M be a module such that M satisfies (C_{11}) and (C_3). Then M satisfies (C_{11}^{+}).

Proof. Suppose M satisfies (C_{11}) and (C_3). Let $N \in \underline{\underline{D}}(M)$. There exists a submodule N' of M such that $M = N \oplus N'$. Let $\pi : M \to N$ denote the canonical projection. Let K be any submodule of N. There exists $L \in \underline{\underline{D}}(M)$ such that $(K \oplus N') \cap L = 0$ and $K \oplus N' \oplus L \in \underline{\underline{E}}(M)$. Because M satisfies (C_3), $N' \oplus L \in \underline{\underline{D}}(M)$. Note that $N' \oplus L = N' \oplus \pi(L)$ and hence $\pi(L) \in \underline{\underline{D}}(N)$. Moreover, $K \oplus N' \oplus L = K \oplus \pi(L) \oplus N' \in \underline{\underline{E}}(M)$ implies $K \oplus \pi(L) \in \underline{\underline{E}}(N)$. It follows that N satisfies (C_{11}) (Proposition 5.1.4). Thus M satisfies $(C_{11}{}^{+})$.

We now consider conditions on a module M which imply that M is a direct sum of uniform modules. First we prove :

Proposition 5.3.4. Let M be a non-zero module with finite uniform dimension. Then the following statements are equivalent.

(i) Every direct summand of M is a (finite) direct sum of uniform modules.

(ii) M satisfies $(C_{11}{}^{+})$.

(iii) M satisfies $(C_{12}{}^{+})$.

Proof. (i)$\Rightarrow$(ii). By Corollary 5.1.7.

(ii)$\Rightarrow$(iii). By Proposition 5.2.3.

(iii)$\Rightarrow$(i). Every direct summand of M satisfies $(C_{12}{}^{+})$ and has finite uniform dimension. Thus it suffices to prove that M is a direct sum of uniform submodules.

Let n denote the uniform dimension of M. If $n = 1$ then M is uniform. Suppose $n > 1$. Let U be any uniform submodule of M. Because M satisfies (C_{12}), there exist submodules K and K' of M such that $M = K \oplus K'$ and a monomorphism $\alpha : U \to K$ such that $\alpha(U) \in \underline{\underline{E}}(K)$. Then $\alpha(U)$, and hence K, is uniform. Because K' has uniform dimension $n - 1$ and satisfies $(C_{12}{}^{+})$ it follows that K' is a direct sum of uniform submodules, by induction on n. Thus M is a direct sum of uniform modules, as required.

Let R be a ring and M a right R-module. For any element m in M, let $\underset{\sim}{r}(m)$ denote the annihilator of m in M, i.e. $\underset{\sim}{r}(m) = \{r \in R : mr = 0\}$.

Definition 5.3.5. We shall say that M *satisfies (A)* if every ascending chain of right ideals of the form $\underset{\sim}{r}(m_1) \subseteq \underset{\sim}{r}(m_2) \subseteq \underset{\sim}{r}(m_3) \subseteq \ldots$, with $m_i \in M$ $(i \geq 1)$, terminates. For example, if R is a right Noetherian ring then every right R-module satisfies (A). More generally, if M is a locally Noetherian module then M satisfies (A). For, let $m \in M$. Then $R / \underset{\sim}{r}(m) \cong mR$, and thus $R / \underset{\sim}{r}(m)$ is a Noetherian right R-module. It follows that M satisfies (A).

The proof of the next result is taken from [21, Proposition 2.18], but it is given for completeness.

Lemma 5.3.6. Let M be a module which satisfies (A). Then every local summand of M is a complement in M.

Proof. Let $X = \sum_{\lambda \in \Lambda} X_\lambda$ be a local summand of M. There exists a $X^* \in \underline{C}(M)$ such that $X \in \underline{\underline{E}}(X^*)$. Suppose $X \neq X^*$. Choose $m \in X^* \setminus X$ with $\underset{\sim}{r}(m)$ maximal. There exists $0 \neq a \in R$ such that $0 \neq ma \in X$. There exists a finite subset Λ' of Λ such that $ma \in \sum_{\lambda \in \Lambda'} X_\lambda$. Now $M = (\sum_{\lambda \in \Lambda'} X_\lambda) \oplus Y$ for some submodule Y of M. Hence $m = x + y$ for some $x \in \sum_{\lambda \in \Lambda'} X_\lambda$ and $y \in Y$. Note that $y = m - x \in X^*$, $y \notin X$. Also if $b \in \underset{\sim}{r}(m)$ then $0 = mb = xb + yb$, which gives $yb = 0$. Therefore $\underset{\sim}{r}(y)$ contains $\underset{\sim}{r}(m)$. Hence $\underset{\sim}{r}(m) = \underset{\sim}{r}(y)$, by the choice of m. But $ya = ma - xa \in (\sum_{\lambda \in \Lambda'} X_\lambda) \cap Y = 0$. Thus $ma = 0$, a contradiction. It follows that $X = X^*$.

Lemma 5.3.7. Let M be a module which satisfies (A). Suppose further that either

(a) M satisfies (C_{11}) and (C_3), or

(b) M satisfies (C_{12}) and (C_2).

Then every local summand of M is a direct summand.

Proof. (a) Let $X = \sum_{\lambda \in \Lambda} X_\lambda$ be a local summand of M. Because M satisfies (C_{11}), there exists $K \in \underline{\underline{D}}(M)$ such that $X \cap K = 0$ and $X \oplus K \in \underline{\underline{E}}(M)$ (Proposition 5.1.4). Consider $X \oplus K$. For any finite subset Λ' of Λ, $Y = \oplus_{\lambda \in \Lambda'} X_\lambda \in \underline{\underline{D}}(M)$ and hence $Y \oplus K \in \underline{\underline{D}}(M)$, because M satisfies (C_3). Thus $X \oplus K$ is a local summand. By Lemma 5.3.6, $X \oplus K \in \underline{\underline{C}}(M)$. But $X \oplus K \in \underline{\underline{E}}(M)$. Thus $M = X \oplus K$.

(b) Let $X = \sum_{\lambda \in \Lambda} X_\lambda$ be a local summand of M. Because M satisfies (C_{12}), there exists $K \in \underline{\underline{D}}(M)$ and a monomorphism $\alpha : X \to K$ such that $\alpha(X) \in \underline{\underline{E}}(K)$. Now $M = K \oplus K'$ for some submodule K' of M. Consider $\alpha(X) \oplus K' = (\sum_{\lambda \in \Lambda} \alpha(X_\lambda)) \oplus K'$. For any finite subset Λ' of Λ, $Y = \oplus_{\lambda \in \Lambda'} \alpha(X_\lambda) \in \underline{\underline{D}}(M)$, because M satisfies (C_2). Thus $\alpha(X) \oplus K'$ is a local summand of M. By Lemma 5.3.6, $\alpha(X) \oplus K' \in \underline{\underline{C}}(M)$. But $\alpha(X) \oplus K' \in \underline{\underline{E}}(M)$. Thus $M = \alpha(X) \oplus K'$. By hypothesis, $X \in \underline{\underline{D}}(M)$.

Theorem 5.3.8. Let M be a module which satisfies (A), (C_{11}) and (C_3). Then M is a direct sum of uniform submodules.

Proof. By Lemma 5.3.7 and [21, Theorem 2.17] M is a direct sum of indecomposable submodules. But by Theorem 5.3.3, every indecomposable direct summand of M satisfies (C_{11}), and hence is uniform. The result follows.

Corollary 5.3.9. Let M be a locally Noetherian module which satisfies (C_3). Then M satisfies (C_{11}) if and only if M is a direct sum of uniform modules.

Proof. By Corollary 5.1.7 and Theorem 5.3.8.

Corollary 5.3.10. Let R be a right Noetherian ring and M a right R-module which satisfies (C_3). Then M satisfies (C_{11}) if and only if M is a direct sum of uniform modules.

Proof. By Corollary 5.3.9.

Corollary 5.3.11. Let M be a nonsingular module such that mR has finite uniform dimension for each $m \in M$. Suppose M satisfies (C_3). Then M satisfies (C_{11}) if and only if M is a direct sum of uniform modules.

Proof. The sufficiency follows by Corollary 5.1.7. Conversely, suppose

$$\underset{\sim}{r}(m_1) \subseteq \underset{\sim}{r}(m_2) \dots, \text{ where } m_i \in M \ (i \geq 1).$$

Now $R / \underset{\sim}{r}(m_1) \cong m_1R$ $(r \to m_1r)$ $(r \in R)$ has finite uniform dimension. Then the right R-module $R / \underset{\sim}{r}(m_1)$ has finite uniform dimension, thus there exists $k \geq 1$ such that $\underset{\sim}{r}(m_i) / \underset{\sim}{r}(m_1) \in \underline{\underline{E}}(\underset{\sim}{r}(m_{i+1}) / \underset{\sim}{r}(m_1)$ and hence $\underset{\sim}{r}(m_i) \in \underline{\underline{E}}(\underset{\sim}{r}(m_{i+1}))$ for all $i \geq k$. Let $i \geq k$ and let $a \in \underset{\sim}{r}(m_{i+1})$. Then $aE \leq \underset{\sim}{r}(m_i)$ for some $E \in \underline{\underline{E}}(R_R)$. Thus $m_iaE = 0$ and hence $m_ia = 0$. It follows that $\underset{\sim}{r}(m_i) = \underset{\sim}{r}(m_{i+1})$ for all $i \geq k$. Thus M satisfies (A). By Theorem 5.3.8, M is a direct sum of uniform modules.

Corollary 5.3.12. Let R be a ring with finite right uniform dimension and M a nonsingular right R-module which satisfies (C_3). Then M satisfies (C_{11}) if and only if M is a direct sum of uniform modules.

Proof. Let R and M be as stated. Let $m \in M$. Then $\underset{\sim}{r}(m)$ is a complement in the right R-module R. Then the R-module $R / \underset{\sim}{r}(m)$ has finite uniform dimension. Since $mR \cong R / \underset{\sim}{r}(m)$ then apply Corollary 5.3.11.

Theorem 5.3.13. Let M be a module which satisfies (A), (C_{12}) and (C_2). Then M is a direct sum of indecomposable modules.

Proof. By Lemma 5.3.7 and [21, Theorem 2.17].

Corollary 5.3.14. Let R be a right Noetherian ring and M a right R-module which satisfies $(C_{12}{}^{+})$ and (C_2). Then M is a direct sum of uniform modules. Moreover, M satisfies $(C_{11}{}^{+})$.

Proof. By Theorem 5.3.13, there exists an index set Λ such that $M = \oplus_{\lambda \in \Lambda} M_\lambda$ where M_λ is an indecomposable submodule of M for each $\lambda \in \Lambda$. Let $\lambda \in \Lambda$. By hypothesis, M_λ satisfies (C_{12}). Suppose $M_\lambda \neq 0$. Let $0 \neq m \in M_\lambda$. Then mR is Noetherian, and hence mR contains a uniform submodule U. There exists a $K \in \underline{\underline{D}}(M_\lambda)$ and a monomorphism $\varphi : U \to K$ such that $\varphi(U) \in \underline{\underline{E}}(K)$. But M_λ indecomposable implies $K = M_\lambda$ and hence M_λ is uniform. Thus M is a direct sum of uniform submodules. For the last part, apply Corollary 5.1.7.

Note. Proposition 2.10 of [21], for a module M which satisfies (C_{11}) and (C_2), fails. For example, let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$. Then M satisfies $(C_{11}{}^{+})$ and (C_2) (see, Example 2.3.11 and Lemma 5.3.2). But $\mathbb{Z}/\mathbb{Z}p$ is not $\mathbb{Q}$-injective.

Proof. Suppose that $\mathbb{Z}/\mathbb{Z}p$ is $\mathbb{Q}$-injective and let $\pi : \mathbb{Z} \to \mathbb{Z}/\mathbb{Z}p$ denote the canonical epimorphism, defined by

$$\pi(n) = n + \mathbb{Z}p \quad (n \in \mathbb{Z}).$$

Then there exists a homomorphism $\alpha : \mathbb{Q} \to \mathbb{Z}/\mathbb{Z}p$ such that $\alpha|_{\mathbb{Z}} = \pi$.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"'] & \mathbb{Q} \arrow[dl, dashed, "\alpha"] \\ & \mathbb{Z}/\mathbb{Z}p & \end{tikzcd}$$

Now $\alpha(1/p) = x + \mathbb{Z}p$ for some $x \in \mathbb{Z}$. Thus

$$p\alpha(1/p) = \alpha(1) = \pi(1) = 1 + \mathbb{Z}p.$$

It follows that $px + \mathbb{Z}p = 1 + \mathbb{Z}p$ and hence $1 \equiv 0 (\mathrm{mod}\ p)$, a contradiction. Thus $\mathbb{Z}/\mathbb{Z}p$ is not $\mathbb{Q}$-injective.

However, Proposition 3.5 of [21] is true for any module M which satisfies (C_{11}) and (C_2), as we shall see in the following Proposition (see Proposition 1.3.5).

<u>Proposition 5.3.15.</u> Let M be a module which satisfies (C_{11}) and (C_2). Then S/Δ is a (von Neumann) regular ring and Δ equals J.

Proof. Let $\alpha \in S$ and $K = \ker\alpha$. By (C_{11}), there exists $L \in \underline{\underline{D}}(M)$ such that L is a complement of K in M. Since $\alpha|_L$ is a monomorphism, $\alpha(L) \in \underline{\underline{D}}(M)$, by (C_2). Hence there exists $\beta \in S$ such that $\beta\alpha = 1|_L$. Then

$$(\alpha - \alpha\beta\alpha)(K \oplus L) = (\alpha - \alpha\beta\alpha)(L) = 0,$$

and so $K \oplus L \leq \ker(\alpha - \alpha\beta\alpha)$. Since $K \oplus L \in \underline{\underline{E}}(M)$, $\alpha - \alpha\beta\alpha \in \Delta$. Therefore S/Δ is a regular ring. This also proves that $J \leq \Delta$.

Let $a \in \Delta$. Since $\ker a \cap \ker(1-a) = 0$ and $\ker a \in \underline{\underline{E}}(M)$, $\ker(1-a) = 0$. Hence $(1-a)M \in \underline{\underline{D}}(M)$ by (C_2). However $(1-a)M \in \underline{\underline{E}}(M)$ since $\ker a \leq (1-a)M$. Thus $(1-a)M = M$, and therefore $1-a$ is a unit in S. It follows that $a \in J$, and hence $\Delta \leq J$.

<u>Lemma 5.3.16.</u> [42, Lemma 1.3] Let M be a nonsingular right R-module.

Then $\Delta = 0$.

Proof. Let $f \in \Delta$ and $N = \ker f$. Then for any $x \in M$, $x^{-1}N \in \underline{\underline{E}}(R_R)$. Now $f(x)x^{-1}N = 0$. Since M is nonsingular, $f(x) = 0$, and since x was arbitrary, $f = 0$.

Corollary 5.3.17. Let M be a nonsingular right R-module which satisfies (C_{11}) and (C_2). Then S is a (von Neumann) regular ring.

Proof. By Lemma 5.3.16 , $\Delta = 0$ and hence by Proposition 5.3.15 the result follows.

5.4. A remark.

Let R be a ring. Let M be a right R-module. We first consider modules M which satisfy the ascending chain condition (ACC) or descending chain condition (DCC) on essential submodules.

Armendariz [2, Proposition 1.1] proved that the module M satisfies DCC on essential submodules if and only if M / (Soc M) is an Artinian module. On the other hand, Goodearl [9, Proposition 3.6] essentially proved that the module M satisfies ACC on essential submodules if and only if M / (Soc M) is a Noetherian module.

It is proved in [30, Theorem 2.1] that the following statements are equivalent for a module M :

(i) M / N has finite uniform dimension for every essential submodule N of M

(ii) every homomorphic image of M / (Soc M) has finite uniform dimension.

Camillo and Yousif [3, Corollary 3] prove that if M is a module which satisfies (C_1) and $M/(\mathrm{Soc}\ M)$ has finite uniform dimension, then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 of M and submodule M_2 with finite uniform dimension, and in this case M is a direct sum of uniform modules. They deduced in [3, Proposition 5] that if M is a module with (C_1), then M has ACC (respectively, DCC) on essential submodules if and only if $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2 of M.

These results were generalized by Smith [31]. A module M is called a *weak CS-module* if, for each semisimple submodule S of M, there exists a direct summand K of M such that S is essential in K. Clearly modules with (C_1) are weak CS-modules. Smith [31, Corollary 2.7, Theorem 2.8] showed that the results of [3] mentioned above can be extended to weak CS-modules.

We now show that these results from [3] also extend to modules which satisfy $(C_{11}{}^{+})$. Note that if a module M satisfies (C_{11}) then Soc M is essential in a direct summand of M (Lemma 5.1.9), but we have been unable to determine whether M is a weak CS-module.

Lemma 5.4.1. Let M be a module such that M satisfies $(C_{11}{}^{+})$ and $M/(\mathrm{Soc}\ M)$ has finite uniform dimension. Suppose that Soc M is contained in a finitely generated submodule of M. Then M has finite uniform dimension.

Proof. Suppose M does not have finite uniform dimension. Then Soc M is not finitely generated. There exist submodules S_1, S_2 of Soc M such that S_i is not finitely generated for $i = 1,2$, and $\mathrm{Soc}\ M = S_1 \oplus S_2$. There exist submodules K, K′ of M such that $M = K \oplus K'$, $S_1 \cap K = 0$ and $S_1 \oplus K \in \underline{\underline{E}}(M)$. Note that, by [1, Propositions 9.7 and 9.19],

$$S_1 \oplus S_2 = \mathrm{Soc}\ M = \mathrm{Soc}\ (S_1 \oplus K) = S_1 \oplus (\mathrm{Soc}\ K).$$

Thus $\mathrm{Soc}\ K \cong S_2$ and hence Soc K is not finitely generated. On the other hand,

$\mathrm{Soc}\ K \oplus \mathrm{Soc}\ K' = \mathrm{Soc}\ M = S_1 \oplus \mathrm{Soc}\ K$, so that $\mathrm{Soc}\ K' \cong S_1$, and hence $\mathrm{Soc}\ K'$ is not finitely generated.

By hypothesis, there exists a finitely generated submodule N of M such that $\mathrm{Soc}\ M \leq N$. Suppose that $K = \mathrm{Soc}\ K$. Then $\mathrm{Soc}\ K \in \underline{\underline{D}}(M)$ and hence also $K \in \underline{\underline{D}}(N)$, and it follows that $\mathrm{Soc}\ K$ is finitely generated, a contradiction. Thus $K \neq \mathrm{Soc}\ K$. Similarly $K' \neq \mathrm{Soc}\ K'$. But, by [1, Proposition 9.19],

$$M / \mathrm{Soc}\ M \cong [K / (\mathrm{Soc}\ K)] \oplus [K' / (\mathrm{Soc}\ K')].$$

It follows that the modules $K / (\mathrm{Soc}\ K)$ and $K' / (\mathrm{Soc}\ K')$ each have smaller uniform dimension than $M / (\mathrm{Soc}\ M)$. By induction on the uniform dimension of $M / (\mathrm{Soc}\ M)$, we conclude that K and K' both have finite uniform dimension, and hence so also does $M = K \oplus K'$, a contradiction. Thus M has finite uniform dimension.

<u>Theorem 5.4.2.</u> Let M be a module such that M satisfies $(C_{11}{}^{+})$ and $M / (\mathrm{Soc}\ M)$ has finite uniform dimension. Then M contains a semisimple submodule M_1 and a submodule M_2 with finite uniform dimension such that $M = M_1 \oplus M_2$. In particular, M is a direct sum of uniform submodules.

Proof. If $M = \mathrm{Soc}\ M$ then there is nothing to prove. Suppose that $M \neq \mathrm{Soc}\ M$. Let $m \in M$, $m \notin \mathrm{Soc}\ M$. There exist submodules K, K' of M such that $M = K \oplus K'$, $mR \cap K = 0$ and $mR \oplus K \in \underline{\underline{E}}(M)$. Let $\pi' : M \to K'$ denote the canonical projection. Then clearly $mR \oplus K = \pi'(m)R \oplus K$. It follows that $\pi'(m)R \in \underline{\underline{E}}(K')$ and hence $\mathrm{Soc}\ K' \leq \pi'(m)R$ (see [1, Proposition 9.7]). By Lemma 5.4.1, K' has finite uniform dimension.

Note that $\pi'(m)R \cong mR$ and hence $\pi'(m) \in K'$, $\pi'(m) \notin \mathrm{Soc}\ K'$. Thus $K' \neq \mathrm{Soc}\ K'$. Now

$$(M / \mathrm{Soc}\ M) \cong [K / (\mathrm{Soc}\ K)] \oplus [K' / (\mathrm{Soc}\ K')]$$

implies that the module $K/(\mathrm{Soc}\ K)$ has smaller uniform dimension than $M/(\mathrm{Soc}\ M)$. By induction on the uniform dimension of $M/(\mathrm{Soc}\ M)$, there exist submodules K_1, K_2 of K such that $K = K_1 \oplus K_2$, K_1 is semisimple and K_2 has finite uniform dimension. Thus M is the direct sum of the semisimple submodule K_1 and the submodule $K_2 \oplus K'$ which has finite uniform dimension. The last part is a consequence of Proposition 5.3.4.

Corollary 5.4.3. Let M be a module which satisfies (C_{11}^{+}) and ACC (respectively, DCC) on essential submodules. Then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2.

Proof. We prove the result in the ACC case; the DCC case is similar. Suppose M satisfies ACC on essential submodules. By [9, Proposition 3.6], $M/(\mathrm{Soc}\ M)$ is Noetherian. Hence, by Theorem 5.4.2, $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and submodule M_2 with finite uniform dimension. Now $\mathrm{Soc}\ M = M_1 \oplus (\mathrm{Soc}\ M_2)$, by [1, Proposition 9.19], and hence $M/(\mathrm{Soc}\ M) \cong M_2/(\mathrm{Soc}\ M_2)$. Thus $M_2/(\mathrm{Soc}\ M_2)$ is Noetherian. But $\mathrm{Soc}\ M_2$ is Noetherian, because M_2 has finite uniform dimension. Therefore M_2 is Noetherian.

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INDEX

$\subseteq$: set inclusion

$\cap$: set intersection

$\leq$: submodule and ideal

$\oplus$: direct sum

Π : direct product