Alternative Compactification of Superstring related Theories

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This is dedicated to Elizabeth and Murray

Contents

Declaration

Except where stated otherwise within the text Chapters 3 to 8 are claimed as original. This work was partly done in collaboration with B.P.Dolan (mainly Chapters 5,6 and 8).

The contents of chapters 3 and 4 have appeared, at the time of writting, in preprint form. Specifically in the preprint entitled

'Compactifying Solutions to an Extended Chaline-Manton Lagrangian'

A brief summary of some of content of Chapters 5 and 6 will appear in the proceedings of the Nato ARW at Simon Fraser University Summer 1986. and will also appear in preprint form in

'Compactification of 10 dimensional Superstring theories on Non-Symmetric Coset Spaces with Torsion'

Acknowledgements

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It is part of being a research student to be sometimes depressed, sometimes ecstatic but always poor and as a consequence I would like to thank all my friends for tolerating my ecstatic moments and supporting me in my black spells between which I have swung all too frequently in the past three years.

Abstact

The aim of this work is to consider the recently introduced ten dimensional Superstring theories and, by considering the low energy field theory limit, consider possible compactification schemes where the original ten dimensions split up into four observed space time dimensions and six ,highly curved, compactified dimensions. We shall attempt to find solutions which satisfy the classical equations of motion and then, using these solutions, we shall try to obtain schemes which give a spectrum of particles which is compatible with the observed spectrum.

We shall, by considering situations where we allow non-zero torsion on the compactified 6-D manifold, investigate possibilities other than the Calabi-Yau spaces which are usually considered.

In Chapter 0 we give a (very biased) review of particle physics and in Chapter 1 we give a little Superstring formalism. In Chapter 2 we discuss the low energy limit of Superstring theories and decide upon the lagrangian which we shall subsequently use. The two types of internal manifold which we shall consider are group manifolds and Coset spaces. We consider these because they provide a natural ansatz for a non-zero torsion. In Chapter 3 we attempt to find solutions to the equations of motion when the internal manifold is a group space and in Chapter 4 we discuss the consequence of any such

solutions. In Chapters 5 and 6 we do the same for Non-Symmetric Coset Spaces and in Chapter 7 we look at Symmetric Coset Spaces. In Chapter 8 we return to the issue of what the low energy field theory should be.

Chapter 0 A review

For many years now it has been the goal of many theoretical physicists to find a theory which would describe nature by a single Force (and a small number of elementary particles) of which the known forces are just different aspects . The recent "Superstring" theories are interesting candidates for such a theory. In this chapter we shall review the known forces and give some of the arguments leading to Superstrings.

In nature there appears to be four forces - Electromagnetism, the Strong interaction, the Weak interaction and finally Gravity. Of these there exists a well known ,experimentally solid ,theory describing the first three which is known as the 'Standard Model' [1] (the introduction of which in 1967 won the Nobel prize in physics for Glashow,Salam and Weinberg). We shall give a brief summary of this model here. The Standard Model is a 'Gauge Theory'. What do we mean by this ? If we take the Dirac lagrangian (describing a free ,spin $\frac{1}{2}$ massless fermion eg an electron)

$$L = -\frac{1}{2}\int \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi \, d^4x \tag{0.1}$$

Then this lagrangian is invariant under the following transformation of the fermion field ψ.

$$\psi \rightarrow e^{i\alpha}\psi \qquad (\ \alpha \text{ a constant }) \tag{0.2}$$

This rather simple observation leads us to a physical conservation law - conservation of electric charge . (from a theorem due to Noether every symmetry of a lagrangian leads to a conservation law). However the conservation law is a global law whereas physically we have local conservation (ie charge is conserved at every point rather than just the total charge of the Universe remaining fixed) This suggests we should try to construct a theory which would be invariant under transformations (0.2) where now α is a function of the coordinates x ie $\alpha(x)$. However if we do this with the lagrangian (0.2) we find that this is not invariant (due to $\partial_\mu \alpha$ terms). We can get around this problem by introducing another field $A_\mu(x)$ which we call a Gauge boson and changing the lagrangian to

$$L = -\frac{1}{2}\int \bar{\Psi}\, \gamma^\mu (\partial_\mu + igA_\mu(x))\psi \, d^4x \qquad (0.3)$$

(g is a constant called the coupling constant)

Then we find this lagrangian is invariant under the following generalisation of (0.2)

$$\psi \rightarrow e^{i\alpha(x)}\psi$$
$$A_\mu \rightarrow A_\mu - g^{-1}\partial_\mu \alpha \qquad (0.4)$$

We can also give the $A_\mu(x)$ field a life of its own by introducing its kinetic term.

$$L = -\frac{1}{4}\int (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \qquad (0.5)$$

This kinetic term is invariant under the gauge transformation (0.4) in its own right. If we identify $A_\mu(x)$ with the potential for the electromagnetic field then the resulting theory proves to be a very sucessful one for describing the interaction of a spin $\frac{1}{2}$ fermion (eg the electon) with the electromagnetic field. It is also about the simplest type of gauge theory. We can extend the concept further, suppose we have a set of fermions ψ_i or $\underset{\sim}{\psi}$, and these have a group of symmetries acting upon them. Suppose this is a Lie group G with generators T^a (the $\underset{\sim}{\psi}$ and T^a must be in some representation of G) Then the lagrangian will be

$$L = -\frac{1}{2}\int \bar{\underset{\sim}{\psi}}\, \gamma^{\mu} \partial_{\mu} \underset{\sim}{\psi}\; d^4x \tag{0.6}$$

This must be invariant under the following global transformation

$$\underset{\sim}{\psi} \longrightarrow e^{i\alpha^a T^a}\, \underset{\sim}{\psi} \tag{0.7}$$

(double a-indices implies summation)

We wish to make the α^a local. To do this we must make the following change in (0.6)

$$\partial_{\mu} \longrightarrow (\partial_{\mu} + igA^a_{\mu}(x)T^a) \tag{0.8}$$

We must also add the bosonic lagrangian which is a generalisation of (0.5)

$$L = -\frac{1}{4}\int F^a_{\mu\nu} F^{a\mu\nu}\, d^4x \tag{0.9}$$

where $F^a_{\mu\nu} = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} + C^a{}_{bc}\, A^b_{\mu}\, A^c_{\nu}$

($C^a{}_{bc}$ are the structure constants of G)

We must also generalise (0.4)

Our simple example had U(1) for its group G. The group G for the standard model is SU(3)xSU(2)xU(1) . The strong interaction being described by the SU(3) and the electro-weak by the SU(2)xU(1).

What are the fermions present ? . These split into two groups - leptons and quarks . The leptons are not affected by the strong force and consist of- the electon e , the muon μ, the tau muon τ and three neutrinos (one for each of e μ τ) ν_e, ν_{μ} ν_{τ} . There are six quarks to match the six leptons - u,d,c,s,t,b (the existance of the t-quark is a little suspect at present) [2] . These fit into three families the first of which is

e , ν_e , u , d

The other two families are just matching sets with the same quantum numbers only with a higher mass - these are

$$\mu\ ,\ \nu_\mu\ ,\ c\ ,\ s$$
$$\tau\ ,\ \nu_\tau\ ,\ t\ ,\ b$$

Each of the above particles (except possibly the neutrinos) has two 'chiralities'. What do we mean by this ? - a fermion field ψ can be split into two parts (called chiralities) thus

$$\psi = \tfrac{1}{2}(1+\gamma_5)\psi + \tfrac{1}{2}(1-\gamma_5)\psi = \psi_L + \psi_R \qquad (0.10)$$

The Lagrangian also splits up into two parts if we do this ie

$$\bar{\psi}\gamma^\mu\partial_\mu\psi = \bar{\psi}_L\gamma^\mu\partial_\mu\psi_L + \bar{\psi}_R\gamma^\mu\partial_\mu\psi_R \qquad (0.11)$$

The ψ_R and ψ_L also transform under Poincare transformations independently. So ψ_L and ψ_R can be taken as different objects, usually refered to as the left and right handed chiralities. It is one of the most intriguing aspects of the standard model that the left and right chiralities appear differently within the theory. The left chiralities fit into SU(2) doublets as follows

$$\begin{pmatrix} e_L \\ \nu_{e\,L} \end{pmatrix},\ \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$ (plus the same for the other families)

Whereas the right handed chiralities appear as SU(2) singlets

u_R , d_R , e_R (and possibly $\nu_{e\,R}$)

This aspect of the standard model is difficult to explain and will appear again. The quarks (both left and right) are SU(3) triplets and the leptons are SU(3) singlets. As we have three families of chiral fermions we often say the number of chiral fermions is three. It is uncertain whether there are more families as yet undiscovered but cosmological evidence suggests the no of families is $\leqslant$ 4 [3] (this is only a good argument if the neutrinos are massless)

The standard model has been very sucessful in describing the Electro-Weak-Strong interactions. It has successfully described the known Electro-Weak phenomena to high quantitative agreement and predicted the existance of the $W^{\pm}$ and Z^{o} bosons which were discovered about 1982 at CERN [2] (the gauge bosons A^{a} of SU(2)xU(1) are identified with the photon,W^{+} ,W^{-} and the Z^{o}) Everywhere that the standard model has been able to make predictions has been a source of excellent agreement (in the SU(3) sector there have been major difficulties in calculating the predictions of the theory but things are getting better via Lattice Gauge Theory [4]). Gauge theories also have the very important property of being consistent quantum theories (ie renormalisable).

There are however criticisms of this model

1. There are a lot of "free parameters" within the theory by this we mean there are a lot ($\sim$ 23) of constants which appear in the Lagrangian without any good theoretical reason why they should have the values they do (from working backward from experiments)

2. The Unification group is hardly in some ways much of a Unification -it is just three groups pasted together.

In (0.8) we have written a single g but for the standard model since we have SU(3)xSU(2)xU(1) , a direct product of three groups , we could have three different g's one for each of the groups and in fact if we fit experiment to theory we must take different values to start with - in a genuine unification we would like to see only one independent coupling constant for one force.)

3. Although not previously mentioned if we wish give the $W^{\pm}$, Z^{0} and the fermions a mass we must have spin-0 Higgs particles present. There is no principle why they must be present. (Most of the free parameters mentioned in 1. are in the Higgs sector.)

4. Charge conservation is a result of the model but charge quantisation is not.

5. Gravity is not incorporated within this model.

The next stage in our Unification scheme was the introduction of 'Grand Unified Theories' [5] (the story of nomenclature in particle physics is rather dramatic). Whose idea was as follows- we should have some grand unification group G which is a single group with one coupling constant, the vacuum state however does not possess the full G symmetry so at energies less than the scale of the vacuum solution we will observe a smaller symmetry than G , G' say, $G' \subset G$. We are trying to obtain G' =SU(3)xSU(2)xU(1) from some G. To make sense the scale of the solution must be of the order 10^{15} GeV. (The coupling constants are a function of energy scale and upon extrapolating one finds they have the same value at about this energy.) Grand unified theories contain more gauge bosons than the standard model but the unseen ones have a mass of the order of the vacuum scale. These, although massive, do have in principle observable effects eg they can mediate proton decay. The proton lifetime is rather large however ~ 10^{30} years [6]. There are various candidates for G -SU(5) was originally a popular candidate for G others are SO(10) and E_6. Of the problems 1.-5. above Grand unification gives good progress on 1.,2. and 4. However 3. and 5. are still problems and an additional problem appears. Why should the scale of Grand Unification and that of the weak interactions be so different ? . We need to 'fine tune' the parameters in the original G theory very carefully to make the scales so different. This is the famous Hierarchy problem.

A solution to this problem was provided by the introduction of 'Supersymmetry' [7] (which also has many other interesting points). Supersymmetry is a symmetry between bosons and fermions and as such this is quite a leap forward in unification - one can regard bosons and leptons as just two aspects of the one particle. Supersymmetry solves the hierarchy problem because the mass of the $W^{\pm}, Z^{0}$ and of the fermions are suppressed , to much less than the unification scale, because their mass term is not supersymmetric and cannot appear if supersymmetry exists. It also answers the tantalising question of why do we need fermions at all -

recall the gauge boson kinetic term was invariant by itself so a gauge theory of bosons alone is perfectly acceptable. Supersymmetry predicts a matching of bosons and fermions. The supersymmetry generators S transform fermions into bosons and vice-versa . So there should be a matching up of fermions and bosons with the same quantum number. Unfortunately this is not observed amongst the known particles ! . So we must be in a similar situation to the Grand unification schemes where the symmetries of the lagrangian are not observed in nature - so Supersymmetry must be broken. This breaking must occur somewhere above the weak interaction scale (100 GeV) but if we are still to solve the hierarchy problem it cannot be too far above, certainly well below the unification scale. It is not really known at what scale it is broken. (Hence the excitment amongst supersymmetry phenomenologists whenever any hint of experimental deviation from the standard model is suggested!).

So far we have been talking of global supersymmetry. It is when we allow the supersymmetry transformation parameters to become local that supersymmetry really starts proving its worth. We obtain (amongst other locally supersymmetric theories) the so-called supergravity theories which contain spin-2 particles which we identify as the graviton, the particle which mediates gravity. So for the first time in our journey we find the fourth force finding a place. In 4-D there are various types of supergravity depending on how much supersymmetry is present ie

N = 1 supergravity has a 1-D group of S's

N = 2 supergravity has a 2-D group of S's etc

We will not consider supergravities (or supersymmetries) with N > 8 since these must contain spin > 2 particles and it is not known how to deal with such objects [8] and it is thought there exists no consistent way of including them.

Our next step forward is the idea, originaly due to Kaluza and Klein in the 20's [9], that we should take the possibility that we live in dimensions > 4 seriously. If we had a 4+k dimensional theory and a solution which was of the form

(4-D flat space)x(k-D compact space with length scale L)

Then for lengths very much larger than L this would appear to be a 4-Dimensional space. So is the Universe really 4-Dimensional or does it only appear to be ? . We shall look briefly at the original Kaluza-Klein model to illustrate the ideas. This model had a space-time which was 4+1 dimensional and the theory was five dimensional gravity. If we take simple 5-D gravity and take (4-D Minkowski) x (1-D torus) as our solution, then the resultant low energy 4-D theory will look rather more complicated than simple gravity. If we take indices A,B to be 1-5 and μ, ν to be 1-4 and carry out a redefinition of our 5-D metric field G_{AB}

$$G_{\mu\nu} = g_{\mu\nu} + A_\mu A_\nu$$

$$G_{\mu 5} = A_\mu$$

$$G_{55} = \phi \qquad (0.12)$$

(these are a definition of $g_{\mu\nu}$, A_μ and ϕ)

Then we find at lengths much larger than that of the torus then the lagrangian will approximate to that for 4-Dimensional gravity ($g_{\mu\nu}$), an Abelian Yang-Mills field (A_μ) and a scalar particle (ϕ). This is a very simple model which was originally introduced to unify gravity and electromagnetism (in the 1920s). The coupling constant for the Yang-Mills field is related to the size of the torus. We shall note a few of the features of this model.

1. Gravity and electomagnetism in 4-D are just different aspects of 5-D gravity

2. Charge conservation arises from this model as a direct analogue of momentum conservation. For momentum conservation since for a wave function $\sim e^{ipx}$ we can have any value of p and so we do not have quantisation. However since the fifth dimension is a torus a wavefunction $\sim e^{ip_5 x_5}$ is not single valued unless p_5 obeys a quantisation condition ie is a constant multiple of some fundamental unit (which is proportional to 1/(length scale))

3. The scalar field arises in a natural manner

4. In fact for our redefined field , $A_\mu(X)$ say ,this would appear in four dimensions as a infinite set of particles since we can expand ($X = x , x_5$)

$$A_\mu(X) = \sum_{n=0}^{\infty} A_\mu{}^n(x)e^{in\frac{1}{2\pi a}x_5} \qquad (0.13)$$

(a is the radius of torus)

Each of the $A_\mu{}^n(x)$ is a valid 4-D field which will have a mass^2 which will be of the order $n^2 a^{-2}$ As the natural length scale for gravity is the planck length then we would expect $a \sim$ (planck length) this results in mass$^2 \sim n^2 \times (10^{19}$ GeV $)^2$ so only the n=0 fields would appear at low energies However we cannot forget about the other fields completely eg they effect the Quantum properties of the theory.

We can generalise this procedure to a more general situation. Starting with the Einstein-Hilbert action in 4+k dimensions

$$\int \frac{1}{2} R_{AB} \wedge {}^*E^{AB} \tag{0.14}$$

Then if we take for our solution

(4-D minkowski)x(k-D compact K)

Then we will find in 4-D gravity, scalars and Yang-Mills fields The Yang-mills fields will have Gauge group G which is related to the symmetries of K [10](in particular if K has no symmetries there will be no (massless) Yang-Mills fields). It would be very nice to obtain Yang-Mills fields with gauge group SU(3)xSU(2)xU(1) or one of the unification groups in this way. If we wish to obtain SU(3)xSU(2)xU(1) in this manner then sheer size arguments imply we must start with k $\geqslant$ 7. Of course any other fields present in the original theory will also appear in various ways in 4-D. Eg an initial spin-$\frac{3}{2}$ Rarita-Schwinger fermion field will split into spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fields in 4-D.

Supergravity theories in dimensions greater than four dimensions are in some ways natural candidates for complete theories. If we take N=1 supergravity in 4+k dimensions then, on the simplest compactification we will find $N = 2^{[\frac{k}{2}]}$ ([ ] denotes integer part) supergravity in 4-D. for k > 7 we will have N > 8 and so we must obtain spins > 2. These particles are very undesirable. If we do not wish to have these particles we must restrict ourselves to $k \leqslant 7$. The two conditions $k \geqslant 7$ and $k \leqslant 7$ which apply if we want our gauge bosons to arise from the metric and not obtain spins > 2 seem strongly to suggest looking carefully at k=7, if a N=1 supergravity exists. In fact such a N=1 theory does exist for dimension 11 [11] and is in fact a very simple supergravity theory containing only three fields- the metric g_{AB}, a spin-$\frac{3}{2}$ fermion field ψ and a three form A (or field A_{ABC} which is antisymmetric in ABC). This theory is very attractive and has been studied very carefully over the last few years However it has been largely discredited due to several problems mainly

1. Although is is possible to obtain SU(3)xSU(2)xU(1) as the gauge group it proved very difficult to obtain the fermions in the correct representation [12]

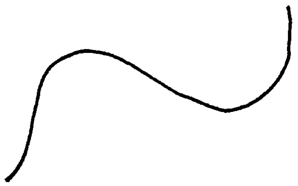
2. The 4-D observed fermions are "chiral" and it is a theorem due to Witten that (with some assumptions) that chiral fermions can only be obtained if (a) the original space-time is even dimensional and (b) Yang-Mills fields exist in the original theory [13] (these are necessary conditions not sufficient) The N=1 D=11 Supergravity theory fails on both counts !.

3. We are really interested in Quantum Mechanical theories. The standard model and the Unified theories are consistent quantum field theories. However when we introduce gravity the theory is no longer able to be quantised consistently. It was hoped that the supergravity theories would due to their high symmetry be able to be quantised. Calculating " loop diagrams" ,which for a theory to be renormalisable must be well behaved, we find for pure gravity that one-loop diagrams are fine but the two-loop and higher are not. For supergravities it is thought that the two-loop diagrams are well-behaved but three or more loops will lead to problems [14]. So it appears that the supergravity theories are not quantum-mechanical consistent.

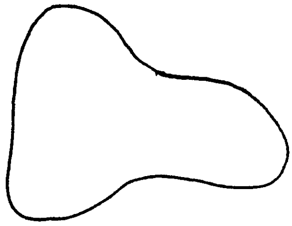
There exist supergravity theories in dimensions less that 11 but greater than 4 however if we wish to obtain SU(3)xSU(2)xU(1) in 4-D we must couple these to Yang-Mills theories (not always possible) these could then solve 1. and 2. but 3. still remains and introducing Yang-Mills without a good reason could just be done in 4-D !. So without some fundamental reason for including the Yang-Mills fields these are unsatisfactory.

The major difficulty of producing a consistent quantum field theory of gravity has led to the introduction of Superstring theories [15] which are not point particle field theories but have fundamental objects which are "strings" rather than point particles. The difficulty with Quantum gravity occurs in the regime where two point particles are very close together. At short distances string theories are radically different from point particle theories so we might hope for a different behaviour.

The fundamental object in a superstring theory is a "string". A string is a one-dimensional object which can either be open or closed ie



OPEN STRING



CLOSED STRING

Bosonic string theories have been around since the early seventies [16] but it has been the introduction of the Supersymmetric Superstring theories which has provoked the recent interest in superstrings. These string theories incorporate both fermions and bosons.

There are various types of string theory.

If we have open and closed strings we say we have a type I superstring theory

If we have only closed strings we say we have a type II superstring theory

There is a third type of string theory -the heterotic string which only has closed strings but the closed strings are rather strange in that the vibrational modes appear differently depending on which way they travel around the string[48].

(N.B. we cannot have only open strings since an open string can join ends to form a closed)

A very crucial point of Superstring theories is that for a few very restictive conditions being satisfied then they are consistent Quantum Theories. For consistency we must have:

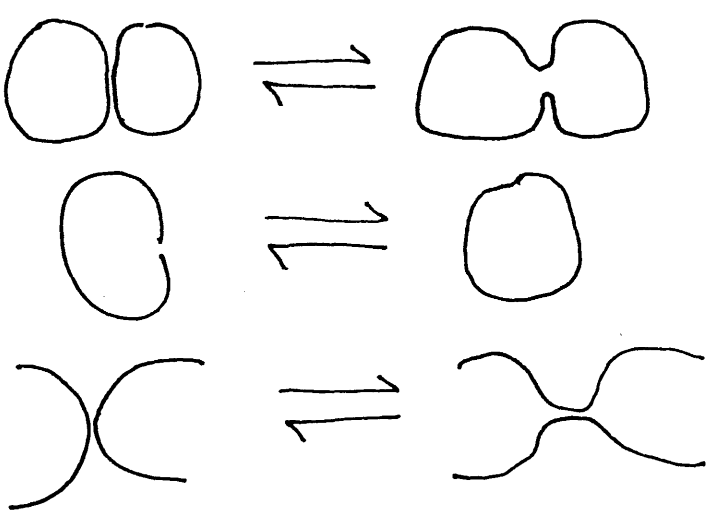
1. The superstring theories are only consistent if they are written down in ten-dimensions

2. For type I and heterotic superstrings there is a Yang-Mills type index associated the strings. The are strong restrictions on what the gauge group may be ie

For Type I we may only have SO(32)

For the Heterotic we may have $SO(32)/Z_2$ or $E_8 x E_8$

The strings may interact in various ways eg.



Of course at experimental energies (at present) we do not see strings we see point particles. To explain this the length scale of the strings must be very small- as string theories have only one length scale it is natural that this must be the planck length which is 10^{-34} m which is much less than experimentally investigated distances. Strings have an infinite number of vibrational modes most of which will have energies/masses planck energy. At low energies the only modes which would be excited are those which are massless. These massless modes would interact amongst each other like point particles. So at low energies the string would simulate a point particle theory. In fact the massless modes of Superstring theories form ten-dimensional Supergravity theories the form of which depends on the string theory.

Type II strings form a N=2 10-D supergravity

Type I and Heterotic strings form a N=1 10-D supergravity which is also coupled to Yang-Mills supermultiplets the gauge group being that of the initial string. (Although we have Yang-Mills fields these are not ad hoc but are specified by the string theory.)

This work will be concerned with the analysis of the effective 10-D point particle theory for Type I and Heterotic Superstrings and the process by which six of the ten dimensions compactify leaving four dimensions. We shall be examining alternate compactification schemes to the popular one where the internal six dimensions are a "Calabi-Yau" space [17]. (These spaces are rather interesting objects- being Ricci-flat and having no symmetries). In the next chapter we shall look at a little Superstring formalism -just (?) enough for our purposes and in Chapter 2 we shall examine closely the the N=1 D=10 supergravity theory which we shall be working with.

Chapter 1 A look at Superstrings

In this brief chapter we shall take a short look at superstring formalism and give a justification of the statement that the zero mass modes of a string form supergravity multiplets. For a more detailed exposure see for example references [15] and [18].

We shall present a little of the superstring formalism, The original bosonic superstring theories were based in 26 dimensions as this was the only dimension where they could be written down consistently. The Superstring theories can only be consistently written down in ten-dimensions for quantum mechanical reasons.

We shall start by describing a little of the bosonic string. A string, which since it is a one-dimensional object , will sweep out a two dimensional 'world sheet' in space-time as it develops. (as opposed to the world line swept out by a point particle.) For the bosonic string the string kinematics are completely given by $X^{\mu}(\sigma, \tau)$, $\sigma \in (0, \pi)$ where X^{μ} are space-time coordinates and σ and τ are world sheet coordinates. σ is space-like and τ is time-like. At fixed τ, σ describes position along the string. We can have two types of string - open strings where the endpoints do not neccessarily coincide and closed strings where we must have

$$X^{\mu}(0, \tau) = X^{\mu}(\pi, \tau) \qquad (1.1)$$

Closed strings can also be orientated or unorientated. If the string is invariant under $\sigma \to \pi - \sigma$ then we call it unorientated otherwise it is orientated. The string is described by the action

$$S = -\frac{1}{2\pi} \int d\sigma\, d\tau \sqrt{-g}\; g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \qquad (1.2)$$

(α, β are world sheet indices 1-2 refering to σ and τ, $g^{\alpha\beta}$ is the world sheet metric.)

Associated with string theories there is only one free parameterthe so called string tension which has is in units of inverse mass^2. The inverse of the string tension should appear premultiplying the action (1.2). It is natural and usual to take this to be the inverse (planck mass)^2. We shall usually work in units where the string tension is one and we shall not explicitly mention it again.

For the world sheet metric $g^{\alpha\beta}$ we can solve its algebraic equations of motion and substitute back into (1.2) (this is only valid for D=26) We have reparameterisation invariance of (1.2) so we can also choose $g^{\alpha\beta}$ such that

$$\sqrt{-g}\;\; g^{\alpha\beta} = \eta^{\alpha\beta} \qquad \eta = \mathrm{diag}(-1,1) \qquad (1.3)$$

The $g^{\alpha\beta}$ equations of motion will then manifest themselves as constraints

$$(\partial_\tau X^\mu \pm \partial_\sigma X^\mu)^2 = 0 \qquad (1.4)$$

These constraints are important. If we started with $\eta^{\alpha\beta}$ in the action instead of $g^{\alpha\beta}$ we would not obtain them. Without these constraints we would not obtain a physical Hilbert space of states when the theory is quantised.

The equations of motion arising from (1.2) are

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(\sigma,\tau) = 0 \qquad (1.5)$$

with additional boundary conditions for open strings

$$\frac{\partial}{\partial\sigma} X^\mu(\sigma,\tau) = 0 \quad \text{at} \quad \sigma = 0 \;\&\; \pi \qquad (1.6)$$

The solution to (1.5) and (1.6) is

$$X^\mu(\sigma,\tau) = X^\mu + p^\mu \tau + i \sum_{n\neq 0} \frac{1}{n} (a^\mu_n e^{-in\tau}) \cos n\sigma \qquad (1.7)$$

This is for a open string. For a closed string we apply (1.5) and (1.1) but not (1.6) to give for the general solution

$$X^{\mu}(\sigma,\tau) = X^{\mu} + p^{\mu}\tau + i\sum_{n\neq 0}\frac{1}{n}\left(b_n^{\mu} e^{-2in(\sigma-\tau)} + \tilde{b}_n^{\mu} e^{-2in(\sigma+\tau)}\right) \qquad (1.8)$$

Notice that closed strings have double the modes that open strings do. As in field theory upon quantisation we let a^{μ}_n, b^{μ}_n and $\tilde{b}^{\mu}_n$ be creation/annihilation operators satisfying appropriate commutation relations. For the closed string the b^{μ}_n and $\tilde{b}^{\mu}_n$ are operators for modes travelling in opposite directions around the string.

In superstring theory we have both bosonic and fermionic coordinates which are functions of the world sheet parameters . We fully describe the string by $X^{\mu}(\sigma,\tau)$ and $\chi^{A}(\sigma,\tau)$ A=1,2 these are interpreted as 10-D superspace coordinates χ^{A} are D=2 (world sheet) scalars but D=10 Majorana-Weyl spinors. We have N=2 superspace/supersymmetry.

We must generalise our bosonic action to a supersymmetric extension. The appropriate form for a non-interacting string is

$$S = S_1 + S_2 \qquad (1.9)$$

Where S_1 is the naive extension

$$S_1 = -\frac{1}{2\pi}\int d\sigma\, d\tau \sqrt{-g}\, g^{\alpha\beta}\, \pi^{\mu}_{\alpha}\, \pi_{\beta\mu} \qquad (1.10)$$

Where $\pi^{\mu}_{\alpha} = \partial_{\alpha} X^{\mu} - i\sum_{A} \bar{\chi}^{A}\gamma^{\mu}\partial_{\alpha}\chi^{A}$

and γ^{μ} are the 10-D gamma matrices

S_1 in fact is not a free string action so we add S_2 as an additional term to make S non-interacting

$$S_2 = \frac{1}{\pi}\int d\sigma\, d\tau\, \varepsilon^{\alpha\beta}\{ -i\partial_{\alpha}X^{\mu}(\bar{\chi}^{1}\gamma_{\mu}\partial_{\beta}\chi^{1} - \bar{\chi}^{2}\gamma_{\mu}\partial_{\beta}\chi^{2})$$

$$+\bar{\chi}^1\gamma^\mu\partial_\alpha\chi^1\;\bar{\chi}^2\gamma_\mu\partial_\beta\chi^2 \quad \} \qquad (1.11)$$

($\varepsilon^{\alpha\beta}$ is the antisymmetric tensor in α and β)

We can again use the equations of motion for $g^{\alpha\beta}$ to solve, substitute back and set equal to $\eta^{\alpha\beta}$ as before again yielding vital constraints. However the constraints are complicated. We will continue this discussion in a particular gauge - the 'Light Cone Gauge'. We change from coordinates X^μ $\mu=0,9$ to X^I $I=1,8$ and $X^\pm$ where

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^9) \qquad (1.12)$$

In this gauge we can use the constraints to solve for $X^\pm$ in terms of X^I so all the physical degrees of freedom will reside within the X^I. Local fermionic symmetries of (1.9) also allow us to impose

$$\gamma^+\chi^1 = \gamma^+\chi^2 = 0 \qquad \text{where } \gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^9) \qquad (1.13)$$

(1.13) truncates χ^1, χ^2 to 8 component SO(8) representations

$$\chi^1 \rightarrow \chi \;,\quad \chi^2 \rightarrow \tilde{\chi} \qquad (1.14)$$

Which spinor representations $\underline{8}_s$ or $\underline{8}_c$ depends on the original D=10 chirality of χ. X^I is in the vector representation $\underline{8}_v$ of SO(8).

In the light cone-gauge the equations of motion become

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)X^I = 0 \qquad (1.15)$$

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma}\right)\chi = 0 \qquad (1.16)$$

$$\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma}\right)\tilde{\chi} = 0 \qquad (1.17)$$

We also obtain boundary conditions for open strings

$$\chi(\sigma,\tau) = \tilde{\chi}(\sigma,\tau) \qquad \sigma = 0 \;\&\; \pi \qquad (1.18)$$

$$\frac{\partial}{\partial\sigma}X^I(\sigma,\tau) = 0, \qquad \sigma = 0 \;\&\; \pi \qquad (1.19)$$

(1.18) requires χ and $\tilde{\chi}$ to have the same chirality and reduces the supersymmetry to N=1. We obtain, by solving (1.16)-(1.17) the mode expansions for χ and $\tilde{\chi}$. That for X^I is identical to (1.7) (with μ replaced by I)

$$\chi(\sigma,\tau) = \sum_{n=-\infty}^{\infty} \chi_n e^{-in(\tau-\sigma)} \tag{1.20}$$

$$\tilde{\chi}(\sigma,\tau) = \sum_{n=-\infty}^{\infty} \tilde{\chi}_n e^{-in(\tau+\sigma)} \tag{1.21}$$

When we quantise the theory we obtain the mass formula

$$M^2 = \sum (\alpha^I_{-n} \alpha^I_n + n\, \chi^a_{-n} \chi^a_n) \tag{1.22}$$

(a is a fermionic index previously omitted)

So the zero mass state of the string will consist of the ground state of the Fock space of the α_n and χ_n oscillators tensored with a general function of the superspace coordinates

$$|0\rangle f(x , \text{fermionic coordinates}) \tag{1.23}$$

The fermionic coordinates are not quite the same as the χ (see [15]). Expansion of the f(x , χ') in powers of the fermionic coordinates will yield 16 functions of x , 8 of which will be bosonic and 8 will be fermionic. We find the massless modes of an open string will be

$$\underline{8}_v \oplus \underline{8}_c \text{ of } SO(8) \tag{1.24}$$

Looking back to the bosonic string we notice that for a closed string we have two sets of operators , each set the same as the open string set of operators , one for each direction of motion around the string. We get a doubling of modes. It can be shown that the open string states fall within multiplets R_n with $\text{mass}^2=n$ and the closed string states are in multiplets with $\text{mass}^2=4n$ formed by $R_n \otimes R_n$. So to find the massless states for a closed string we need just need to product two open string massless multiplets. We obtain type II superstring theories in this manner. SO(8) has the property of having three 8 -dimensional representations $\underline{8}_v$ the vector bosonic representation and two spinor representations $\underline{8}_c$ and $\underline{8}_s$. We can take either spinor rep for the open string. When we product two open strings we have two possibilities. We can either take two open strings with different spinor $\underline{8}$s or two with the same type of spinor $\underline{8}$. The first possibility corresponds to type IIA superstrings and the second to type IIB. We obtain for the massless modes

$$\text{IIA} \quad (\underline{8}_v \oplus \underline{8}_s) \otimes (\underline{8}_v \oplus \underline{8}_c) \tag{1.25}$$

$$\text{IIB} \quad (\underline{8}_v \oplus \underline{8}_s) \otimes (\underline{8}_v \oplus \underline{8}_s) \tag{1.26}$$

These produce the following states (bosonic states first)

$$\text{IIA} \quad \underline{1} \oplus \underline{28} \oplus \underline{35}_v \oplus 8_v \oplus \underline{56}_v \;,\; \underline{8}_c \oplus \underline{56}_c \oplus 8_s \oplus \underline{56}_s \tag{1.27}$$

$$\text{IIB} \quad \underline{1} \oplus \underline{28} \oplus \underline{35}_v \oplus \underline{1} \oplus \underline{28} \oplus \underline{35}_s \;,\; \underline{8}_c \oplus \underline{56}_c \oplus \underline{8}_s \oplus \underline{56}_s \tag{1.28}$$

(1.27) and (1.28) are just the field contents of Type IIA and Type IIB N=2 D=10 supergravities as given in Table 1.1. These string are orientated strings .

The only known theory based on open and closed stings involves unorientated closed strings based on type IIB. Imposing the condition of invariance under $\sigma \rightarrow \pi - \sigma$ eliminates half the states in the IIB theory leaving the same states as the Type I N=1 D=10 Supergravity as given in Table 1.2. The massless content of this theory (known as Type I superstring theory) is that of type I supergravity plus the massless content of the open strings ($\underline{8}_v \oplus \underline{8}_s$). The open strings can (must) have a

Yang-Mills index associated with the free end. If the Yang-Mills group is G then the massless states will be

$$(\underline{8}_v \oplus \underline{8}_s , \underline{1}) \otimes (\underline{1} , \text{adjoint of } G) \qquad (1.29)$$

This is the same content as a D=10 super Yang-Mills multiplet.

For a consistent quantum theory we must be very restrictive in our choice of G . To enable anomalies to cancel when we quantise we have only one choice of G namely SO(32). At this time this appears to give a finite consistent theory.

Apart from the three type of string theory I, IIA and IIB there is one further type of string theory the 'Heterotic Superstring' which is based on closed strings only.

The Heterotic superstring is a very strange object- it is a closed string theory for which the modes moving around the string in the two directions are very different objects[48]. Mathematically one set are superstring modes in 10-D and the other set are bosonic string modes in 26-D. The 26-D is compactified to 10-D. The net result (the analysis is complicated and not really necessary here) for the zero modes is to find Type I supergravity plus a super Yang-Mills multiplet where the Gauge group, if we require anomaly cancelation, can have only two possible choices- $E_8 \times E_8$ or $SO(32)/Z_2$. These are both rank 16 groups.

Here we have shown (or indicated) how we obtain the same 'fields' as D=10 supergravities when we look at the massless modes of superstring theories. However this is not the same as showing that they form a supergravity theory. Detailed analysis of the interaction between the zero modes does confirm that they do form these theories [18].

The Behaviour of the string is determined by its entire (infinite) spectrum of states. However at low energies we would hope that we may approximate the behaviour by analysing the behaviour of the zero mass modes. There must be more to the low energy limit than just the supergravity lagrangians since these suffer from anomalies whereas the full string theories do not. We shall in this work attempt to analyse phenomenological aspects of some of the superstring theories by examining the 10-D field theory lagrangians which are based on the 10-D supergravity lagrangians with appropriate corrections due to the higher mass modes. These Lagrangians are only approximations to the superstring but we should be able to learn something from them. (After all physics is a very good approximation to a point particle world- there is no direct experimental evidence at the moment for matter being extended objects)

We shall be interested mainly in the compactification of the ten dimensions into four flat plus six highly curved dimensions. If we wish to explain our manifestly 4-D universe by a 10-D theory this must certainly happen. We shall attempt to find compactifying solutions to the classical equations of motion arising from the lagrangians which result in physical particle spectra and Yang-Mills symmetries. We regard these classical solutions as the background solutions for when we quantise the theory.

Type II supergravities have no (apart from a U(1) in IIA) fundamental Yang-Mills fields hence it is very difficult to see how after compactification we can find SU(3)xSU(2)xU(1) Yang-Mills fields. (Recall from chapter 0 that we cannot obtain enough Yang-Mills fields from the metric for dimensions less than 11.) Hence we shall only deal with the type I supergravity ,which is derived from both Type I and Heterotic superstrings, coupled to various Yang-Mills. In the next chapter we shall introduce this supergravity lagrangian which is known as the Chapline-Manton lagrangian and discuss the possible alterations to it.

Table 1.1 Field content of the type II Supergravities

N=2 D=10 Type IIA Supergravity

Field	Symbol	Rep of SO(8)
scalar	ϕ	$\underline{1}$
metric	$g_{\mu\nu}$	$\underline{35}_v$
U(1) Yang-Mills	A_μ	$\underline{8}_v$
Two form	$A_{\mu\nu}$	$\underline{28}$
Three form	$A_{\mu\nu\rho}$	$\underline{56}_v$
Gravitino (Majorana)	ψ	$\underline{56}_s \oplus \underline{56}_c$
Spinor (Majorana)	λ	$\underline{8}_s \oplus \underline{8}_c$

N=2 D=10 Type IIB Supergravity

Field	Symbol	Rep of SO(8)
Complex scalar	B	$\underline{1} \oplus \underline{1}$
graviton	$g_{\mu\nu}$	$\underline{35}_v$
complex two form	$A_{\mu\nu}$	$\underline{28} \oplus \underline{28}$
Four form with self dual field strength	$A_{\mu\nu\rho\sigma}$	$\underline{35}_s$
Gravitino (Weyl)	ψ	$\underline{56}_s \oplus \underline{56}_s$
Spinor(Weyl)	λ	$\underline{8}_s \oplus 8_s$

Table 1.2 Field content of type I D=10 N=1 Supergravity

Field	Symbol	Rep of SO(8)
scalar	ϕ	$\underline{1}$
graviton	g	$\underline{35}_v$
two form	$A_{\mu\nu}$	$\underline{28}$
gravitino (Majorana/Weyl)	ψ	$\underline{56}_s$
Spinor (Majorana/Weyl)	λ	$\underline{8}_c$

This can also be coupled to a super Yang-Mills multiplet which will have content (in reps of SO(8)xYang-Mills group G)

Yang-Mills	A_μ	($\underline{8}_v$, adjoint of G)
Spinor	χ	(8 , adjoint of G)

Chapter 2 The Extended Chapline Manton Lagrangian

In this chapter we shall introduce the Lagrangian which we will be working with in Chapters 3-7.

As discussed in chapter 1 at the low energy limit of Type I and Heterotic superstrings the zero modes of the superstring behave as point particles of the d=10,N=1 Chapline-Manton supergravity coupled to specific Yang-Mills fields. For type I superstring theory the Yang-Mills are SO(32) fields and for the heterotic string theory they are $E_8 x E_8$ or $SO(32)/Z_2$. This is the lowest order Lagrangian. There are various reasons to suppose it is not sufficient to consider only the zeroth order approximation and we must consider additional terms from the next order in perturbation theory. The d=10,N=1 supergravity contains the following fields (we are not coupling to Yang-Mills yet).

E^A - the gravitational orthonormal one forms describing a spin 2 particle, A=0,9

$\omega^A{}_B$ - the lorentz connection related to the E^A via the torsion T^A

$$dE^A + \omega^A{}_B \wedge E^B = T^A$$

B - a two form (or B_{AB} a two index field)

ν - a scalar spin 0 field

ψ - a spin $\frac{3}{2}$ fermion field

λ - a spin $\frac{1}{2}$ fermion field

The lagrangian also features field strengths defined from some of these fields

$$R^A{}_B = dw^A{}_B + w^A{}_C \wedge w^C{}_B$$

$$H = dB$$

The appropriate lagrangian is [19]

$$=\frac{1}{2}R_{AB}\wedge *E^{AB} \quad -\frac{1}{4}e^{2\nu}H\wedge *H \quad -d\nu\wedge *d\nu$$
$$-\frac{1}{12}\bar{\psi}\gamma^{ABCD}D\psi \quad *E_{ABCD} -\frac{1}{2}\bar{\lambda}\gamma^{A}D\lambda\wedge *E_{A}$$
$$-\frac{\sqrt{3}}{2}(\bar{\psi}\gamma^{A}\gamma^{B}\lambda\wedge *E_{B})\wedge *(d\phi \quad *E_{A})$$
$$+\frac{1}{8}e^{\nu}H\wedge(K(\lambda,\psi)) \tag{2.1}$$
$$+ \text{ four fermion couplings}$$

Where $K(\lambda, \psi)$ is defined as

$$*(\bar{\psi}\gamma^{A}E_{A}\psi) -\frac{1}{5!}\bar{\psi}\gamma^{ABCDE}\psi *E_{ABCDE} +\frac{\sqrt{2}}{6}E_{ABC}*(\bar{\psi}\gamma^{ABC}\gamma^{D}\lambda *E_{D}) \tag{2.2}$$

Where we have used the following notation/definitions

d is the exterior derivative which takes p-forms to (p+1)-forms d acting twice on any form gives zero ie d(d(a form))=0.

$\wedge$ is the interior product operation which acts on a p-form and a q-form to yield a (p+q)-form.

$E^{A_1..A_p}$ means $E^{A_1}\wedge E^{A_2}\wedge E^{A_3}\wedge.. \wedge E^{A_p}$

$*$ is the operation of Hodge dual which take p-forms to 10-p forms. Its action on the $E^{A_1..A_p}$ is defined by

$$*E^{A_1...A_p} = \frac{1}{(10-p)!}\varepsilon^{A_1..A_p}{}_{B_1...B_{n-p}} E^{B_1...B_{n-p}}$$

($\varepsilon^{A_1...A_{10}}$ is the antisymmetric tensor in ten indices)

Another operation which we will use is that of interior derivation i^A, i^A:p-forms $\rightarrow$ (p-1)-forms and is defined on E^B by $i^A E^B = \eta^{AB}$ where η =diag(-1,+1,+1.....,+1).

This supergravity theory can be coupled to a Yang-Mills supermultiplet by adding the following fields [20]

A - Yang-Mills potential

χ - spin $\frac{1}{2}$ supersymmetric partner to A

These fields have a group index (which we suppress).

We find we must add to the Lagrangian the bosonic term

$$+\frac{1}{2}e^{\phi}\,\mathrm{tr}(\,F\wedge *F\,) \qquad (2.3)$$

where F is the field strength of A ,F=dA +A$\wedge$A. We also must add the fermionic terms

$$-\frac{1}{2}\mathrm{tr}\{\bar{\chi}\,\hat{D}\,\chi\}*1 + M_{ABC}\,\mathrm{tr}\{\bar{\chi}\gamma^{ABC}\chi\}\,*1$$
$$-\frac{1}{8}e^{\frac{1}{2}\phi}\,\mathrm{tr}\{\bar{\chi}\gamma^{A}\gamma^{BC}(\,F_{BC}+\hat{F}_{BC}\,)\}\{i^{A}\psi+\frac{\sqrt{2}}{12}\gamma^{A}\lambda\} \qquad (2.4)$$

where we have introduced M_{ABC} which is defined by

$$M_{ABC} = -\frac{1}{48}i_A i_B i_C H \quad -\frac{1}{512}\bar{\lambda}\gamma_{ABC}\lambda$$
$$-\frac{\sqrt{2}}{1536}i^{D}\bar{\psi}(\,4\gamma_{ABC}\gamma_{D}+3\gamma_{D}\gamma_{ABC})\,\psi \qquad (2.5)$$

(circumflexed quantities are super covariantised)

We find we must also alter our definition of H to $H = dB - \Omega_{YM}$ where Ω_{YM} is the Yang-Mills Chern-Simons term which is defined by

$$\Omega_{YM} = \mathrm{tr}\{\,F\wedge A\;-\frac{1}{3}A\wedge A\wedge A\,\} \qquad (2.6)$$

With these changes we now have the standard Chapline-Manton lagrangian.

When we use our gauge groups arising from superstrings we must be careful what we mean by the trace tr( ). If we have the $E_8 x E_8$ gauge group then the Yang-Mills lie in the adjoint representation the generators of which if taken as anti-hermitian (ie $Q_i^{+} = -Q_i$) are usually normalised to $\mathrm{Tr}(\,Q_i Q_j) = -30\delta_{ij}$.In this case we find we must replace tr( ) be $\frac{1}{30}$Tr( ) whereever this occurs. For SO(32) we have the Yang-Mills in the fundamental representation which is normalised to $\mathrm{Tr}(\,Q_i Q_j) = -\delta_{ij}$. For SO(32) we replace tr( ) by Tr( ).

Throughout this work we shall be setting the fermion fields to zero in our ansatzes. Observationally non-zero fermion fields are not ruled out but there is no positive evidence for them. So we shall only consider the bosonic lagrangian. (As the fermion fields occur in pairs when we set them equal to zero there will be no residual contributions to the equation of motion from the fermionic terms in the Lagrangian when we do this).

$$L_{BOSE} = \tfrac{1}{2}R_{AB}\wedge{*E}^{AB} - \tfrac{1}{4}e^{2\nu}H\wedge{*H} \quad -d\nu\wedge{*d\nu}$$
$$+\tfrac{1}{2}e^{\nu}\,\mathrm{tr}(\,F\wedge{*F}\,) \qquad (2.7)$$

Although this is the lowest order lagrangian there are reasons why it alone cannot describe physics (and hence we must consider higher order terms) One reason is the argument due to Freedman et al [21]) "ten into four won't go" which states that given certain assumptions then there are no solutions when space-time is 4-D maximally symmetric and there are six compactified dimensions. To see this we look at the scalar equation of motion arising from (2.7)

$$2d{*d\nu} - \tfrac{1}{2}e^{2\nu}H\wedge{*H} + \tfrac{1}{2}e^{\nu}\,\mathrm{tr}(\,F\wedge{*F}\,) = 0 \qquad (2.8)$$

$$\therefore 4\Box\nu\,{*1} = \{e^{2\nu}g(\,H,H\,) - e^{\nu}\mathrm{tr}(\,g(F,F)\,)\,\}{*1} \qquad (2.9)$$

now $\mathrm{tr}(\,g(F,F)\,) = -\chi^2 g(F_i,F_i)$ so we find (χ^2 is the normalisation of the generators)

$$4\Box\nu\,{*1} = \{e^{2\nu}g(H,H) + e^{\nu}\chi^2 g(F_i,F_i)\,\}{*1} \qquad (2.10)$$

If H and F have no time components then the RHS of (2.10) is positive, However $\Box\nu$ only has negative or zero modes so the LHS must be negative or zero. The only solution hence is $\Box\nu = 0$ and H = F = 0. This also implies the curvature scalar is zero. So we do not have

any possibilities of interesting compactification. Another problem , of a rather different nature and perhaps more significant, is that if we attempted to quantise this theory we would have both gravitational and Yang-Mills anomalies [22]. Whereas in the original string theories these have been shown to vanish. So the higher order terms must be significant as they must contain elements which will yield a cancelation of the anomalies present in the Chapline-Manton theory.
So it appears we must consider some of the alterations to the lagrangian due to higher order string effects.

The 'full' point particle field theory lagrangian which would simulate string theories would contain an infinite number of fields and terms; however in certain circumstances perhaps we need only consider some of them. If the typical momentum of a field is k then all the terms will have a certain power of k associated with them. For dimensional reasons we will have a factor of M_s with each k where M_s is the mass scale of the string (inverse string tension) so we can regard infinite lagrangian as a infinite sum of terms whose 'size' is powers of (k/M_s). IF this parameter is small (< 1) then we can regard this as a perturbation expansion and for some purposes just consider the first few terms. This is what we are doing ,we are taking the lowest order terms which correspond to the zero mode terms and adding some of the terms in higher powers of (k/M_s). The terms in higher powers of (k/M_s) come from the correspondingly higher modes of the string theory. (These are sometimes refered to as higher derivative terms.) It is an unresolved problem as to whether it is valid to assume this parameter is small. However we shall assume that it sensible to do so, Some justification being that M_s is usually expected to be the planck scale and in standard Kaluza-Klein theories when we have compactifying solutions in is usual to find the typical momentum to be less (0.01-0.1 of) than the planck scale.

We should include the higher order terms which make the lagrangian anomaly free. (Since the string theory is anomaly free then in a perturbation expansion there must be a cancellation of anomalies). It was found that be redefining H to be

$$H = dB - \Omega_{YM} + \Omega_{LOR} \tag{2.11}$$

where Ω_{LOR} is the Lorentz Chern-Simons term

$$\Omega_{LOR} = \mathrm{tr}(R \wedge w - \tfrac{1}{3} w \wedge w \wedge w) \tag{2.12}$$

(where the trace is a SO(10) trace)

then we find we have cancelation of anomalies in the lagrangian [22]. The next order in an expansion would be expected to include curvature squared terms In [17] it was proposed that the curvature squared term take the form

$$-\frac{1}{2} e^{\nu} \mathrm{tr}(R \wedge {*R}) \tag{2.13}$$

however such a term would lead to gravitational ghosts and Zwiebach [23] has proposed the following ghost free alternative.

$$+\frac{1}{4} e^{\nu} R_{AB} \wedge R_{CD} \wedge {*E}^{ABCD} \tag{2.14}$$

This differs from (2.13) by terms involving the (Ricci tensor $)^2$ and (curvature scalar $)^2$ (and so for Calabi-Yau spaces will not be different). In 4-D this term is just the Euler density and locally is a total derivative thus not affecting the dynamics. However in 10-D this is not the case. The addition of (2.14) to the Chapline-Manton lagrangian invalidates the No-go theorem.
We shall attempt to find solutions to the case where we have made these two alterations to the bosonic Chapline-Manton lagrangian. It should be understood that this is not all the alterations neccesary to form a consistent truncation of the perturbation expansion to second order. We shall return to this issue in Chapter 8.

Before presenting the equations of motion we must decide whether to use first or second order formalism (ie do we regard the Lorentz connection as an independent field to be varied giving an equation or not). In the original Chapline-Manton theory , when the fermion fields are zero the role of torsion is clear- the torsion cannot be an independent field and must be set to zero, (supersymmetry fixes it to be zero). When we consider higher order terms arising from string theory however it is not so clear what the situation is - higher order terms could conceivably manifest themselves as degrees of freedom for the connection $\omega^a{}_b$. We should be able to decide this from string arguments. Certainly the string modifications do seem to imply a symmetry between F and R, or A and ω, (we added the Lorentz

Chern-Simons term to match the Yang-Mills Chern-Simons term and we added the Zwiebach term which is closely related to tr($R\wedge{}^*R$) which would match tr($F\wedge{}^*F$)) so it is possible that the similarities may extended to $\omega^A{}_B$ being an independent field in analogy with A. Since we are looking at a modified lagrangian we are including the effects of the massive modes, primarily those in the first massive level, so it is possible that these modes could manifest themselves by giving degees of freedom which appear as those for the Lorentz connection. Hence we will take a look at the first massive level of states to see if a $\underline{9}$ of SO(9) is present. If no such representation is present then it seems we should not regard the connection as an independent field (to this order in the expansion). This is a one-way argument if a $\underline{9}$ of SO(9) does exist in the first massless level then we really can say nothing as to whether it could be a connection without proper analysis.

If we look at the spectra of Type I strings we recall from Chapter 1 that the open strings had massive multiplets of $mass^2=n^2$ and the closed strings had multiplets of $mass^2=4n^2$ (this is in units of (string tension $)^{-1}$). So we look at the first massive level of the open string. This turns out to be fairly simple and for the bosonic modes is a $\underline{45}$ of SO(9) this is a two index symmetric field. The second mass level is $\underline{36}\oplus\underline{115}$ (a two index antisymmetric tensor and a three index symmetric tensor).

For the Heterotic string where we only have closed strings we must look at the modes in the first massive level of the string which arise from

$$R_1^L \otimes R_1^R \qquad (2.15)$$

Where R_1^L, R_1^R are the n=1 modes from the appropriate open string theory, These are representations of SO(9)x(Yang-Mills Gauge group). We find [15]

$$R_1^L = (\underline{44}, \underline{1}) \oplus (\underline{84}, \underline{1}) \oplus (\underline{128}, \underline{1})$$
$$R_1^R = (\underline{44}, \underline{1}) + (\underline{9}, \underline{496}) + (\underline{1}, \underline{69256}) \qquad (2.16)$$

Where the $\underline{69256}$ depends on the gauge group

For $E_8 x E_8$ $\quad \underline{69256} = (\underline{248},\underline{248}) + 2(\underline{1},\underline{1}) + (\underline{3875},1)$

For SO(32) $\quad \underline{69256} = \underline{2}^{15} + \underline{35960} + \underline{527} + \underline{1}$

We wish to look for representations of the form ($\underline{9}$, $\underline{1}$) these can only come from $\{(\underline{44},\underline{1}) \oplus (\underline{84},\underline{1}) \oplus (\underline{128},\underline{1})\} \otimes (\underline{44},\underline{1})$ carrying out the expansions gives (dropping the $\underline{1}$)

$$\underline{44} \otimes \underline{44} = \underline{1} \oplus \underline{36} \oplus \underline{44} \oplus \underline{450} \oplus \underline{495} \oplus \underline{910}$$
$$\underline{84} \otimes \underline{44} = \underline{84} \oplus \underline{231} \oplus \underline{924} \oplus \underline{2457}$$
$$\underline{128} \otimes \underline{44} = \underline{16} \oplus \underline{128} \oplus \underline{432} \oplus \underline{576} \oplus \underline{1920} \oplus \underline{2560} \qquad (2.17)$$

So for both cases we do not find a $\underline{9}$ of SO(9) so we should certainly take second order formalism and not regard ω as an independent field

There is still the question of whether even if the connection is not a free field the torsion is zero or not. In the original Chapline -Manton theory the torsion can only be non-zero if the fermion fields (and in particular certain fermion bilinears see [20]) are non-zero. However for the low energy limit of superstrings this may not be the case.

When approaching compactification ,via the β-function approach [25], some authors [26],[27] have seen the need to have the torsion tensor T_{abc} ($T^a = T^a{}_{bc} E^{bc}$) set equal to the three index tensor H_{abc} ($H = H_{abc} E^{abc}$) to within a factor of $\sqrt{3}$. This is closely related to what is done in Chapter 8 where we are mainly interested in further modifications to the lagrangian. In this work however we shall mainly be interested in the effects of allowing variable torsion, that is not having H=T but allowing more freedom. In actual fact the ansatzes we consider still have H_{abc} proportional to T_{abc} but the proportionality in non fixed. What is the source of this torsion ?. It could easily come from higher order terms arising from the string theory although we have shown ,at least to first order, that the torsion cannot be propagating. Even at the Chapline-Manton lagrangian level we can have ,via non-zero fermion bilinears, torsion in the system and when considering the extra fermion terms which are the supersymetric partners to the Lorentz Chern-Simmons and Zwiebach [28] terms then we have extra possibilities. Of course to be completely rigourous here we would have to produce the field and show they satisfied the (complicated) fermionic equations. Even if the background fields are zero then Quantum fluctuations of the fermion fields could produce a torsion.This was discussed in [29] where the authors also considered variable torsion. So we will take the viewpoint that even if not a free field there is still the possibility that the torsion may be non-zero and we shall attempt to solve the equations of motion with this crucial difference. Whenever we have solutions we shall investige where ,if at all, H=T.

We can now calculate the equations of motion. First we give the Einstein equation(s)

$$\begin{aligned}\tfrac{1}{2}R_{AB}\wedge i^{E}*E^{AB} \quad & -\tfrac{1}{4}e^{2\nu}\{\, i^{E}(H\wedge *H)-2(i^{E}H)\wedge *H \,\} \\ & -\{\, i^{E}(d\nu\wedge *d\nu) -2(i^{E}d\nu)\wedge *d\nu \,\} \\ & +\tfrac{1}{2}e^{\nu}\mathrm{tr}\{\, i^{E}(F\wedge *F) -2(i^{E}F)\wedge *F \,\} \\ & +\tfrac{1}{2}e^{\nu}R_{AB}\wedge R_{CD}\wedge i^{E}*E^{ABCD} = 0 \qquad (2.18)\end{aligned}$$

For the Yang-Mills we will have

$$D^{YM}(\, e^{\nu} *F \,) \; +e^{2\nu}F\wedge *H = 0 \qquad (2.19)$$

(The $F\wedge *H$ arises from the Yang-Mills Chern-Simons term.)

D^{YM} is the Yang-Mills covariant derivative.

Variation wrt B yields

$$d(\, e^{2\nu} *H \,) = 0 \qquad (2.20)$$

Variation of the scalar field gives us

$$\begin{aligned}2d*d\nu - \tfrac{1}{2}e^{2\nu}H\wedge *H \quad & +\tfrac{1}{2}e^{\nu}\mathrm{tr}(F\wedge *F) \\ & +\tfrac{1}{4}e^{\nu}R_{AB}\wedge R_{CD}\wedge *E^{ABCD} = 0 \qquad (2.21)\end{aligned}$$

The field strengths F and H must satisfy Bianchi identities arising from their definitions

$$dH = \mathrm{tr}(\, R\wedge R \,) -\mathrm{tr}(\, F\wedge F \,) \qquad (2.22)$$

$$D^{YM}F = 0 \quad = dF +A\wedge F -F\wedge A \qquad (2.23)$$

We shall be attempting to find solutions to these classical equations of motion. We are trying to determine the 'background solution' for the quantum theory about which quantum fluctuations (particles) propagate. The very obvious solution to the equations of motion is 10-D Minkowski space-time with F, H and all the fermion fields zero (ν any constant). However if this were the solution we would observe a ten-dimmensional world with gauge fields SO(32) or $E_8 x E_8$ which is rather different from the observed 4-D world with SU(3)xSU(2)xU(1) fields. We shall look for solutions which are of the form

(4-D space-time) x (6-D internal space)

If we are not to 'see' the internal six dimensions they must be highly compactified. This high curvature endangers the expansion in terms of (k/M_s) however as mentioned previously 'typical' Kaluza-Klein theories give $k < M_s$. Even when such solutions exist it is unresolved why such a solution should be preferred to M_{10} or even solutions with split ups into dimensions other than 4 x 6 (3 x 7 , 5 x 5 etc).

If the torsion on the internal manifold is zero then there are arguments which suggest that the three form H must be zero, the 6-space should be one of the now celebrated Ricci-flat Calabi-Yau, and the Yang-Mills should be set equal to the curvature (regarding the curvature as a SO(6) field). These have been extensively studied in the literature [17]. We shall investigate the alternative possibility of finding solutions where the torsion is non-zero (and H possibly non-zero also). We shall consider 6-D spaces which are Group manifolds or Coset spaces. These have the advantage that a natural ansatz exists for the torsion with these spaces [30]. (Not all coset spaces however only 'non-symmetric' ones).

If we take the Einstein equations (2.18) and take the product of E_E with it we obtain the equation

$$4R_{AB}\wedge *E^{AB} - e^{2\nu} H\wedge *H - 8d\nu\wedge *d\nu$$
$$+3e^{\nu}\mathrm{tr}(F\wedge *F) + \tfrac{3}{2}e^{\nu} R_{AB}\wedge R_{CD}\wedge *E^{ABCD} = 0 \qquad (2.24)$$

If ν is a constant we can take this minus 6 x scalar equation (2.21) to obtain

$$4R_{AB}\wedge *E^{AB} + 2e^{2\nu} H \wedge *H = 0 \qquad (2.25)$$

Using (defining) $R_{AB}\wedge *E^{AB} = \mathcal{R} *1$, $H\wedge *H = g(H,H)*1$ this becomes

$$\mathcal{R} = -\tfrac{1}{2}g(H,H) \qquad (2.26)$$

This is a fairly simple equation which will be very useful later. If for the moment we specialise to space time being 4-D Minkowski with H zero on space-time then g(H,H) will be positive and hence the internal curvature must be negative (unless both zero). If our compact internal manifolds have zero torsion their curvature would be positive. Hence our torsion must be large enough to change the sign of the curvature.(This is not a valid argument when space-time is for RxS^3/HS^3 or deSitter/AntideSitter space-time). When the torsion is equal to zero we must have H = 0 and the curvature = 0.(This is one of the properties of the Ricci-flat Calabi-Yau spaces).

Having decided to investigate letting the torsion be non-zero we must decide what it must be !. For Group manifolds and non-symmetric Coset Spaces there is a very natural ansatz for the torsion [30]. An investigation of these two types of space (with a brief mention for symmetric coset spaces) will be the main aim of this work. In Chapters 3-4 we analyse group spaces and in Chapters 5-7 we look at coset spaces. These spaces also have their geometrical structures given more explicitly than for Calabi-Yau spaces.

We shall also mention a useful property of the Einstein equations namely that , under certain conditions , the scalar equation in contained within them.

If H and F have no E^0 component ie $i^0 H = i^0 F = 0$, $R^{A0} = 0$ for all A and $d\mu = 0$ then the 0-th Einstein equation will be

$$1/2 R_{AB} \wedge i^0 *E^{AB} - 1/4 i^0 (H \wedge *H) + 1/2 i^0 (tr(F \wedge *F)) + 1/4 R_{AB} \wedge R_{CD} \wedge i^0 *E^{ABCD} = 0 \qquad (2.27)$$

$$i^0 \{ 1/2 R_{AB} *E^{AB} - 1/4 H \wedge *H + 1/2 tr(F \wedge *F) + 1/4 R_{AB} \wedge R_{CD} \wedge *E^{ABCD} \} = 0 \qquad (2.28)$$

so $$1/2 R_{AB} \wedge *E^{AB} - 1/4 H \wedge *H + 1/2 tr(F \wedge *F) + 1/4 R_{AB} \wedge R_{CD} \wedge *E^{ABCD} = 0 \qquad (2.29)$$

We also have equn (2.24) which always holds. (This was obtained by multiplying (2.18) with E_E). (2.22) minus eight times (2.27) yields

$$H \wedge *H - tr(F \wedge *F) - 1/2 R_{AB} \wedge R_{CD} \wedge *E^{ABCD} \qquad (2.30)$$

This is just the scalar equation (2.19) (to within a factor) So with fairly modest assumptions ($d\mu = 0$, $i^0 F = 0$, $i^0 H = 0$ and $R^{A0} = 0$) we have that the scalar equation is not an independent equation but is contained in the Einstein equations.

The condition on R^{AO} will be satisfied by 4-D Minkowski and Rx(three-sphere or three-hypersphere) but not for deSitter/Anti-deSitter.

This fact will prove quite useful in the following chapters and we will refer to it again.

Chapter 3 Group Manifolds

In this chapter we shall attempt to find solutions to the equations of motion (2.18)-(2.21) of the extended Chapline-Manton lagrangian ,which was discussed in Chapter 2, which take the form

(4-d space-time)x(6-d group manifold)

Why are group manifolds interesting ?. - As discussed in chapter 2 we wish to be able to define a non-zero torsion and for a group manifold G we have a natural ansatz

$$T^a \sim C^a{}_{bc} E^{bc} \tag{3.1}$$

where $C^a{}_{bc}$ are the structure constants for G [30].

How many 6-D Lie groups are there? We find only three SU(2)xSU(2), $SU(2)xU(1)^3$ & $U(1)^6$.

($SO(4) \approx SU(2)xSU(2)$)

The SU(2)xSU(2) case is particularly interesting since by taking the three form field H to be the sum of the volume elements for the two 3-dimensional manifolds we have a natural ansatz which will give compactification in an analogous manner to the Freund-Rubin mechanism [31] (which was introduced for 11-D supergravity). SU(2) is isomorphic to the three sphere.

We shall take the case of SU(2)xSU(2) first. We shall use indices 4-6 for the first SU(2) and 7-9 for the second.

We shall take the following ansatz for the fields :-

$$\mu = \text{constant} \tag{3.2}$$

For the three form H a natural ansatz for H is the sum of the volume elements of the two SU(2)s (SU(2) is three dimensional)

$$H = e^{-\frac{3}{2}\nu}(h_1 E^{456} + h_2 E^{789}) \qquad (3.3)$$

For the torsion we have

$$T^a = \begin{cases} 0 & \text{for } a=0-3 \\ t_1 \pi^a{}_{bc} E^{bc} & \text{for } a,b,c=4-6 \\ t_2 \Gamma^a{}_{bc} E^{bc} & \text{for } a,b,c=7-9 \end{cases} \qquad (3.4)$$

where $\pi^a{}_{bc}$ is the totally antisymmetric tensor in a,b,c=4,5,6 with $\pi_{456}=1$ and similarly for $\Gamma^a{}_{bc}$ (these are the structure constants for SU(2))

The internal curvatures are given by

$$R^{ab} = \frac{1}{6} r_1 e^{-\nu} E^{ab} \qquad a,b=4-6 \qquad (3.5)$$

$$R^{ab} = \frac{1}{6} r_2 e^{-\nu} E^{ab} \qquad a,b=7-9 \qquad (3.6)$$

(r_1 and r_2 are strictly positive if the torsion is zero. However if the torsion is non-zero they may be negative.)

For the space-time curvature we work with two different cases

(A) $$R^{ab} = \frac{1}{12} R_4 e^{-\nu} E^{ab} \qquad a,b=0-3 \qquad (3.7)$$

this corresponds to 4-D Minkowski (M_4), deSitter (dS) or Anti-deSitter (AdS) according to the value of R_4 (=0 , > 0 , < 0 respectively).

(B) $$R^{i0} = 0 \qquad i=1-3 \qquad (3.8)$$

$$R^{ij} = \frac{1}{6} R_3 e^{-\nu} E^{ij} \qquad i,j=1-3 \qquad (3.9)$$

this is a time-independent spacelike 3-sphere (S^3) or hypersphere (HS^3) depending on the sign of R_3 (> 0 , < 0 respectively).

In case (B) we may also add to H the extra term

$$e^{-\frac{3}{2}\nu} h_0 E^{123} \qquad (3.10)$$

(This is the volume element of a three sphere/hypersphere.)

The vielbiens obey the following for the internal dimensions

$$dE^{a} = \frac{1}{2}R_1^{-1}\pi^{a}{}_{bc}E^{bc} \qquad a,b,c = 4,5,6 \qquad (3.11)$$

$$dE = \frac{1}{2}R_2^{-1}\rho^{a}{}_{bc}E^{bc} \qquad a,b,c = 7,8,9 \qquad (3.12)$$

R_1 and R_2 are the length scales of the SU(2)s and are strictly positive.

The Yang-Mills field strength F is a SU(2)xSU(2) field. If we label the first SU(2) by 4,5,6 and the second by 7,8,9 then our ansatz for F is given by

$$F^{a} = \frac{1}{2}e^{-\nu}f_1\pi^{a}{}_{bc}E^{bc} \qquad a,b,c=4,5,6 \qquad (3.13)$$

$$F^{a} = \frac{1}{2}e^{-\nu}f_2\rho^{a}{}_{bc}E^{bc} \qquad a,b,c=7,8,9 \qquad (3.14)$$

the $\pi^{a}{}_{bc}$ and $\rho^{a}{}_{bc}$ are as defined previously. There exists a well defined Yang-Mills potential corresponding to this F ($A^{a}=kE^{a}$) and so the Bianchi identity (2.23) will be satisfied automatically. Since F is topologically trivial f_1 and f_2 are free parameters. ie they are not subject to a quantisation condition.

We have inserted appropriate powers of e^{ν} in our ansatz so that when we look at the resulting equations e^{ν} has vanished. ie we have scaled e^{ν} out of the problem. We shall do this in chapters 5 & 7 also.

This means any single solution will be in fact a one-parameter family (and choosing the value of ν will fix the scale).

With this ansatz the equations of motion reduce to a system of non-linear algebraic equations. For simplicity we shall present here the case where the two SU(2)s and the fields on them are identical ie

$$h_1 = h_2 = h$$
$$t_1 = t_2 = t$$
$$r_1 = r_2 = r$$
$$R_1 = R_2 = R$$
$$\& \quad f_1 = f_2 = f \tag{3.15}$$

We shall take case (A) first For this case we find the ten Einstein equations reduce to two separate equations

$$\{\tfrac{1}{2}R_4 + 2r \} - h^2 - 6\chi^2 f^2 + \{ R_4 . r + r^2 \} = 0 \tag{3.16}$$
$$\{ R_4 + \tfrac{4}{3}r \} - 2\chi^2 f^2 + \{\tfrac{4}{3}R_4 . r + \tfrac{1}{3}r^2 + \tfrac{1}{12}R_4^2 \} = 0 \tag{3.17}$$

where χ^2 is a normalisation factor arising from the generators of SU(2)xSU(2) obeying $tr(O_i O_j) = -\chi^2 \delta_{ij}$

We also have the scalar equation

$$2h^2 + 6\chi^2 f^2 - \{ 2R_4 . r + r^2 + \tfrac{1}{12}R_4^2 \} = 0 \tag{3.18}$$

We also have (contained in these three, see page 40 (2.26))

$$g(H,H) = -2 (R_4 + 2r) \tag{3.19}$$
$$\text{or} \qquad h^2 = -(R_4 + 2r) \tag{3.20}$$

We can use the scalar equation to define f^2

$$6\chi^2 f^2 = \{ 2R_4 . r + r^2 + \tfrac{1}{12}R_4^2 \} - 2h^2 \tag{3.21}$$

Eliminating f^2 from (3.16) and using (3.20) to eliminate h^2 we find the following equation

$$-\tfrac{1}{2} R_4 . \{ 1+2r + \tfrac{1}{6}R_4 \} = 0 \tag{3.22}$$

so we have two possibilities

(1) $R_4 = 0$ (2) $R_4 = -6-12r$

In case (1) we find

$$h^2 = -2r \tag{3.23}$$
$$6\chi^2 f^2 = r(r+4) \tag{3.24}$$

The requirement that $f^2 \geqslant 0$ and $h^2 \geqslant 0$ is only satisfied for $r \leqslant -4$

In case (2) we find

$$R_4 = -6(1 +2r) \tag{3.25}$$

$$h^2 = 10r +6 \tag{3.26}$$

$$6\chi^2 f^2 = -9r^2 -20r -9 \tag{3.27}$$

The function on the rhs of (3.27) is negative for all values of r so we do not find any consistent solutions in this case. So only Minkowski solutions exist

We shall now look at case (B) ie space-time being $Rx(S^3/HS^3)$. Here we find three Einstein equations which are

$$\{ R_3 +2r \} -\tfrac{1}{2}h_0^2 -h^2-6\chi^2 f^2 +\{ 2R_4.r +r^2 \} = 0 \tag{3.28}$$

$$\{\tfrac{1}{3}R_3 +2r \} +\tfrac{1}{2}h_0^2 -h^2-6\chi^2 f^2 +\{ \tfrac{4}{3}R_3.r +r^2 \} = 0 \tag{3.29}$$

$$\{ R_3 +\tfrac{4}{3}r \} -\tfrac{1}{2}h_0^2 -2\chi^2 f^2 +\{ \tfrac{4}{3}R_3.r +\tfrac{1}{3}r^2\} = 0 \tag{3.30}$$

With a little algebraic manipulation we obtain for the hs

$$h_0^2 = \tfrac{2}{3}\{ 2r.R_3 +R_3 \} \tag{3.31}$$

$$h^2 = \tfrac{2}{3}\{ r.R_3 +r \} -4\chi^2 f^2 \tag{3.32}$$

and for f^2

$$4\chi^2 f^2 = \tfrac{2}{3}\{ r(2R_3 +r) +2(2r+ R_3) \} \tag{3.33}$$

substituting back into (3.32) we obtain

$$h^2 = -2r - \tfrac{4}{3}R_3 - \tfrac{2}{3}r.R_3 \tag{3.34}$$

We will only have valid solutions whenever h_0^2, h^2 & f^2 are all positive. Requiring $h_0^2 > 0$ gives us a restiction on the values of r and R_3

$$R_3 \geqslant 0 \quad \text{and} \quad r \geqslant -\tfrac{1}{2} \tag{3.35}$$

$$\text{or} \quad R_3 \leqslant 0 \quad \text{and} \quad r \leqslant -\tfrac{1}{2} \tag{3.36}$$

Requiring $h^2 \geqslant 0$ gives us the restriction

$$R_3 \geqslant -3r/(r+2) \tag{3.37}$$

finally requiring $f^2 \geqslant 0$ gives

$$R_3 \leqslant - (r^2+4r)/(r+1) \tag{3.38}$$

There is a non-zero region where all three conditions are satisfied. This is illustrated on diagram (3.1). The region where valid solutions exist is bounded by two curves. Along the upper curve $f^2=0$ and along the lower $h^2=0$. Note that solutions only exist for both r and R_3 negative. R_3 being negative means that space-time is a hypersphere (HS^3) rather than a sphere. For r it means that the torsion must be large enough to make r negative. We have the relation between the torsion t, the length scale R and r being

$$r = e^{-\mu}\{3/(2R^2) - 6t^2\} \qquad (3.39)$$

As we are using second order formalism t is an arbitrary parameter and we can choose t to satisfy this relation. (As r is negative we are always guaranteed $t^2 \geqslant 0$).

We should note that the Yang-Mills equation is not satisfied trivially but reduces to a constraint. Note that if $A^a = kE^a$ then

$$f = k(\tfrac{1}{2}k + R^{-1}) \qquad (3.40)$$

Both $D{*}F^a$ and $F^a \wedge {*}H$ are non-zero and proportional to ${*}E^a$. We find the resulting constraint is

$$R^{-1} + k = h \qquad (3.41)$$

or

$$R^{-1} = h - k \qquad (3.42)$$

There is no immediate reason why the RHS of (3.42) should be positive (which it must be) however since only h^2 and f^2 are fixed in terms of r (and R_3) we can choose the sign of h and f to ensure R_3 is positive. We can show that (3.24) in terms of k (with some manipulation and (3.41)) becomes

$$k = h \pm \sqrt{h^2 - f} \quad \text{so} \quad \frac{1}{R} = \mp\sqrt{h^2 - f} \qquad (3.43)$$

by choosing the negative root (and the -ve root for f) we can ensure R is positive.

Another possible ansatz for the Yang-Mills field is a $U(1)^3 \times U(1)^3$ field

$$F^4 = f_1 E^{56}, F^5 = f_2 E^{64}, F^6 = f_1 E^{45}$$
$$F^7 = f_2 E^{89}, F^8 = f_2 E^{97}, F^9 = f_2 E^{78} \qquad (3.44)$$

(The coefficients are the same on each SU(2) to ensure that the energy momentum tensor is a product of a unit matrix.)

We find solutions in a very similar manner over the same range of parameters as for F being a SU(2)xSU(2) field.

If we take $R_3 = 0$ then the $R \times HS^3$ case reduces (as one would hope !) to the minkowski case. If we take r =-4 then we find we reduce to the case of Dolan et al who [32] studied the case where the Yang-Mills fields were set to zero.

We shall now look at the case where the internal manifold is $U(1)^6$

This is just a six-torus, We can have a coordinate system with coordinates Θ^i i=1..6, with vielbiens $E^i = \frac{1}{R_i} d\Theta_i$, since $dE^i = 0$ we find the (torsion-free) curvature to be zero. We note immediately that solutions to the equations of motion with T = F = H = R = 0 must exist. We shall attempt to find other solutions where some of the fields are non-zero !. (we shall only give for simplicity the case where all the R are identical.) The natural torsion defined in (3.1) is zero here and there is no obvious alternative so we shall have $T = 0$ and hence $R_{ab} = 0$ (for the internal dimensions) What should the F field be ? If we take

$$A = \frac{a}{2R_1^2}(\theta_1 d\theta_2 - \theta_2 d\theta_1) \tag{3.45}$$

then we find

$$F = a E^{12} \tag{3.46}$$

This is non-zero and obeys $D{*}F = d{*}F = 0$, generalising we can take for our F field a U(1) ansatz ie

$$F^1 = fE^{45}, \quad F^2 = fE^{67}, \quad F^3 = fE^{78} \tag{3.47}$$

What do we take for H ? Almost anything will obey $dH = d{*}H = 0$ but we must also have $F \wedge {*}H = 0$ if the Yang-Mills equation is to be obeyed. An ansatz which satisfies this is

$$H = h_1(E^{135} + E^{246}) \tag{3.48}$$

(Notice that with this F and H the U(1) cannot be regarded as $U(1)^3 \times U(1)^3$ with no mixed fields.)

With this ansatz we find the equations of motion follow in a very similar manner to those for SU(2)xSU(2) and we can obtain them from the SU(2)xSU(2) case by setting the internal curvature r equal to zero. Letting $r = 0$ for the case where space-time is AdS or dS we find (from equations (3.23) and (3.24))

$$h_1^2 = f^2 = 0 \tag{3.48}$$

So we find no non-trivial solutions. If we take space-time to be $R \times (S^3/HS^3)$ we find the equations are (from (3.31),(3.32) and (3.34))

$$h_0^2 = \tfrac{2}{3} R_3 \tag{3.49}$$

$$h^2 = -\tfrac{2}{3} R_3 \tag{3.50}$$

$$4\chi^2 f^2 = 2R_3 \tag{3.51}$$

The positivity conditions on the three objects cannot simultaneously be satisfied so we will obtain no valid solutions.

So the case for $U(1)^6$ is simple. We find no solutions other than the case where $F = H = 0$.

We lastly consider the case of the internal dimensions being $SU(2)xU(1)^3$, We shall deal first with the case where space-time is AdS or dS. We shall use indices 0-3 for space-time, 4-6 for the SU(2) and 7-9 for the $U(1)^3$. We shall take the F field to be a SU(2) field living on the SU(2) manifold (as in (3.13)) and on the $U(1)^3$ manifold we take the F field as a U(1) field

$$F = f(E^{78} + E^{89} + E^{97}) \tag{3.52}$$

For the H field we will take the volume elements of the SU(2) and the $U(1)^3$ ie

$$H = h_1 E^{456} + h_2 E^{789} \tag{3.53}$$

We consider case (A) first. R_4 and r shall be the 4-d and internal curvatures respectively. (As in (3.5) and (3.7)) With this choice of ansatz we find the three Einstein equations become

$$\frac{1}{2}\{ \frac{1}{2}R_4 + r \} + \frac{1}{4}\{ R_4.r \} - \frac{1}{4}\{ h_1^2 + h_2^2 \} - \chi^2\{ f_1^2 + f_2^2 \} = 0 \tag{3.54}$$

$$\{ R_4 + r\frac{1}{3} \} + \frac{1}{4}\{ \frac{2R_4}{3}.r \} - \frac{1}{4}\{ -h_1^2 + h_2^2 \} - \chi^2\{ -\frac{1}{3}f_1^2 + f_2^2 \} = 0 \tag{3.55}$$

$$\frac{1}{2}\{ R_4 + r \} + \frac{1}{4}\{ 2R_4.r \} - \frac{1}{4}\{ h_1^2 - h_2^2 \} - \chi^2\{ f_1^2 - \frac{1}{3}f_2^2 \} = 0 \tag{3.56}$$

When we look at the Yang-Mills equation we find for the SU(2) part neither $D{*}F$ nor $F \wedge {*}H$ is zero and we obtain a similar constraint to that for the SU(2)xSU(2) case. For the U(1) field we find $D{*}F=d{*}F=0$ but $F \wedge {*}H \neq 0$. This means either h_2 or f_2 must be zero. We take the case of $h_2=0$ first. We find on using the equations to find h_1^2, f_1^2 and f_2^2 in terms of R_4 and r that

$$\tfrac{1}{2}h_1^2 = -\{ 5R_4 +2r \} -\tfrac{3}{2}R_4.r \tag{3.57}$$

$$\chi^2 f_2^2 = \tfrac{1}{4}\{ -\tfrac{3}{2}R_4 \} - \tfrac{3}{8}R_4.r \tag{3.58}$$

$$\chi^2 f_1^2 = \tfrac{1}{8}\{ 27R_4+16r \} + \tfrac{19}{8}R_4.r \tag{3.59}$$

The scalar equation places one constraint on R_4 and r. We find

$$R_4(1 +r) = 0 \tag{3.60}$$

so either $R_4 = 0$ or $r = -1$. In either of these two cases it is impossible to find any values or R_4 and r such that all of h_0^2, f_1^2 and f_2^2 are simultaneously non-negative so we find no solutions. We now can consider $f_2 = 0$ and $h_2 \neq 0$ We find our three Einstein equations can be solved giving

$$h_2^2 = -\tfrac{1}{2}R_4 - \tfrac{1}{2}R_4.r \tag{3.61}$$

$$h_1^2 = \{ -\tfrac{9}{2}R_4 -2r \} - \tfrac{5}{2}R_4.r \tag{3.62}$$

$$\chi^2 f_1^2 = \{ 3R_4 +2r \} + 2R_4.r \tag{3.63}$$

The scalar equation again reduces to

$$R_4(1+r) = 0 \tag{3.64}$$

So again $R_4 = 0$ or $r = -1$. If we substitute either possibility into (3.61)-(3.63) then we find we do not simultaneously have all of h_1^2, h_2^2 and f_1^2 positive at any point.

We consider our last possibility - space-time being $Rx(S^3/HS^3)$. We find four Einstein equations. (We can forget the scalar see P41-42). These are

$$\tfrac{1}{2}\{ R_3 +r \} + \tfrac{1}{4}\{ 2R_3.r \} - \tfrac{1}{2}\{ f_1^2 +f_2^2 \}$$

$$-\tfrac{1}{4}\{\ h_0^2 + h_1^2 + h_2^2\} = 0 \quad (3.65)$$

$$\tfrac{1}{2}\{\tfrac{1}{3}R_3 + r\} + \tfrac{1}{4}\{\tfrac{2}{3}R_3.r\} - \tfrac{1}{2}\{f_1^2 + f_2^2\}$$
$$-\tfrac{1}{4}\{-h_0^2 + h_1^2 + h_2^2\} = 0 \quad (3.66)$$

$$\tfrac{1}{2}\{R_3 + r\} + \tfrac{1}{4}\{2R_3.r\} - \tfrac{1}{2}\{-\tfrac{1}{3}f_1^2 + f_2^2\}$$
$$-\tfrac{1}{4}\{h_0^2 - h_1^2 + h_2^2\} = 0 \quad (3.67)$$

$$\tfrac{1}{2}\{R_3 + r\} + \tfrac{1}{4}\{2R_3.r\} - \tfrac{1}{2}\{f_1^2 - \tfrac{1}{3}f_2^2\}$$
$$-\tfrac{1}{4}\{h_0^2 + h_1^2 - h_2^2\} = 0 \quad (3.68)$$

Again we find the Yang-mills equation for the U(1) field on the U(1) manifold reduces to $h_2 . f_2 = 0$ so either $f_2=0$ or $h_2= 0$. So again we have two possibilities. However we find if we take either case then we can deduce that $h_2 = f_2 = 0$ so we have only one case. We can solve for everything in terms of R_3 and r. We find

$$\tfrac{1}{4}h_0^2 = \tfrac{1}{2}\{\tfrac{1}{3}R_3\} + \tfrac{1}{4}\{\tfrac{2}{3}R_3.r\} \quad (3.69)$$

$$\tfrac{1}{4}h_2^2 = 0 \quad (3.70)$$

$$\tfrac{1}{4}h_1^2 = \tfrac{1}{2}\{-\tfrac{4}{3}R_3 - r\} + \tfrac{1}{4}\{-\tfrac{2}{3}R_3.r\} \quad (3.71)$$

$$\tfrac{1}{2}f_2^2 = 0 \quad (3.72)$$

$$\tfrac{1}{2}f_1^2 = \tfrac{1}{2}\{2R_3 + 2r\} + \tfrac{1}{4}\{2R_3.r\} \quad (3.73)$$

We must ask whether we can all five (coefficients) non-negative. On Diagram 3.2 We show the region where all the positivity constraints are satisfied As can be seen solutions only exist for both R_3 and r being negative . As for the SU(2)xSU(2) case this means space-time must be a hypersphere (HS^3) and on the internal space the torsion (on the SU(2)) must be large enough to change the sign of the curvature. For our solutions both H and F are zero on the $U(1)^3$ so there are no fields being non-zero on the tori. As a difference between the SU(2)xSU(2) and $SU(2)xU(1)^3$ notice that Minkowski space-time is not a limiting case for $SU(2)xU(1)^3$ whereas it was for SU(2)xSU(2).

So in summary we find solutions to the equations of motion in several instances

for SU(2)xSU(2) both Minkowki and $RxHS^3$

for $U(1)^3$ xSU(2) $RxHS^3$

for $U(1)^6$ we find only the rather trivial case

$$F=H=R=0$$

We shall examine the consequences of these solutions in the following Chapter.

We should mention that our solutions in many cases can be expanded to form a larger class as follows - Suppose we have a F field which is a solution with gauge group G. This means for the (large) gauge group in the theory certain of the component fields are non-zero. If the gauge group of the theory is large enough to contain GxG then it is possible to have a field FxF as a solution. We must change the coefficient by $\frac{1}{\sqrt{2}}$ but once this is done the algebraic equations will be almost identical.(Note we are only free to do this since the coefficient of F was a free parameter). So if F is a solution we can have fields FxFxF.. with as many Fs nonzero as we can fit into the gauge group of the theory. This has a major effect on the possible resultant 4-D gauge groups predicted after compactification. This will be dealt with in the next chapter.

We now turn our attention to the possibility, as discussed in Chapter 2 of having T=H for our solutions. The precise statement of T=H is

$$T_{abc} = \sqrt{3}\, H_{abc} \tag{3.74}$$

(see Chapter 2 p37, Chapter 8 and [33].)

For our SU(2)xSU(2) ansatz this reduces to

$$h^2 = \frac{1}{108} t^2 \tag{3.75}$$

t^2 is defined by (3.39) ie

$$6t^2 = \frac{3}{2\Lambda^2} - r \tag{3.76}$$

R is specified by the Yang-Mills equation (3.42). For our ansatz with $RxHS^3$ 4-D space-time we find (3.76) becomes

$$6t^2 = \frac{3}{2}\left(-\frac{8}{3}r - \frac{4}{3}R_3 - \frac{2}{3}r.R_3\right) \mp \frac{3}{4|\chi|}\sqrt{\frac{2}{3}r(2R_3+r) + 4r + 2R_3} \tag{3.77}$$

Applying the constraint and substituting for h^2 in terms of r and R_3 we obtain an equation

$$aR_3 + b\sqrt{cR_3 + d} + e = 0 \tag{3.78}$$

where

$$a = -\frac{4}{3} - \frac{2}{3}r + \frac{1}{108}(r+2)$$

$$b = \mp \frac{3}{4.108|\chi|}$$

$$c = \left(\frac{4}{3}r + 2\right)$$

$$d = \frac{2}{3}r^2 + 4r$$

$$e = -\frac{4}{108}r + 2r \tag{3.79}$$

When we solve (3.78) for $R_3(r)$ we find a curve which lies within the allowed region on Diagram 3.1 and very close to the boundary given by $h^2=0$ (so close as to be indistinguishable at the scale of the Diagram). This is not surprising since (3.75) is only going to be satisfied when h^2 is small. We have one more possibility. When f =0 we find in actual fact that the

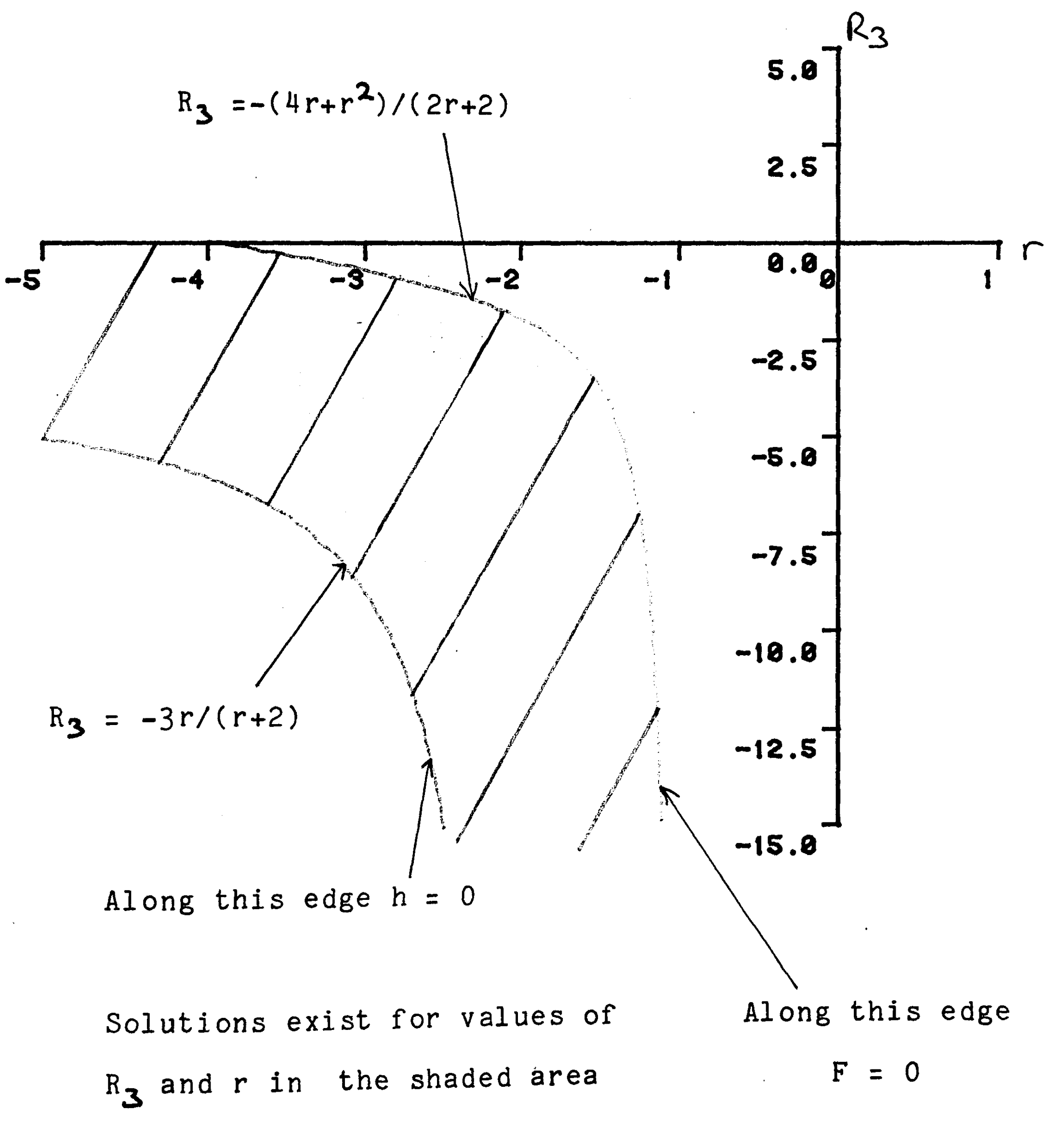
Yang-Mills equation does not yield any constraint so R is not fixed by the Yang-Mills equation. Instead we can fix it via

$$\frac{3}{2R^2} = 6t^2 + r = 648h^2 + r$$

(Although r is -ve the total RHS is +ve so $\frac{1}{R}$ is well defined.)

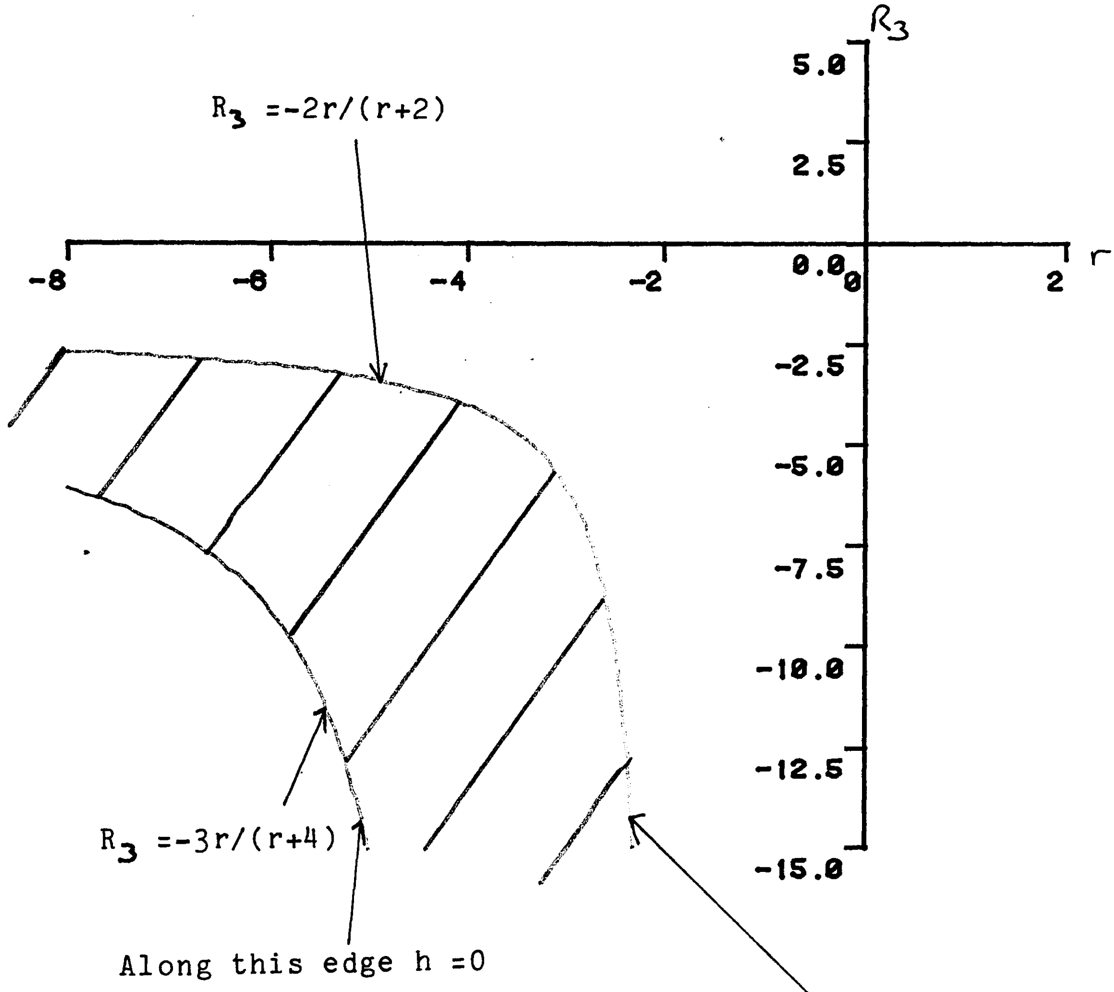
So along the boundary given by F=0 we can satisfy H=T. Since we have H=0 on 4-D space-time we must ask is H=T for the 4-D space-time components. In our original ansatz we took the torsion as zero on space-time. There are two possible viewpoints. One would be to say we only need H=T on the internal space and the other would be to introduce torsion on 4-D space time. Since we have F=0 on 4-D space-time this poses no difficulties for our ansatz (we are in the same situation as for the F=0 boundary) but may give observational difficulties. So in conclusion ,for the SU(2)xSU(2) case ,we can have two distinct one parameter families where T=H. For $SU(2)xU(1)^3$ we find a similar pattern.

Diagram 3.1 Area of R_3-r plane for which solutions exist for manifolds of the form $(\underline{R \times S^3}/\underline{HS^3}) \times (\underline{SU(2) \times SU(2)})$



Solutions exist for values of R_3 and r in the shaded area

Diagram 3.2 Area of R_3-r plane for which solutions exist for manifolds of the form $(R \times S^3/HS^3) \times (SU(2) \times U(1)^3)$



Solutions exist for values of R_3 and r in the shaded area

Chapter 4 Physics from Group Manifolds

We shall now investigate the consequences of taking our solutions from Chapter 3 seriously. We shall investigate several consequences namely

1. Efffective 4-D Yang-Mills fields
2. 4-D Fermions
3. Cosmological aspects

1. Effective 4-D Yang-Mills fields

If we have an extended Chapline-Manton Lagrangian with Yang-Mills group G - $E_8 x E_8$, SO(32) or SO(32)/Z_2 then at high energies this will be the observed gauge group (and space-time would appear ten dimensional). However if we have a compactifying solution of the form (4-D space-time)x(some six dimensional compact manifold) then at energies much less than the compactification scale space-time will appear four dimensional and the gauge group will be different from G. For the gauge bosons in G some of them will become massive ,with masses the order of the compactification scale, and so we will not observe them directly at energies much less than the compactification scale (which if we assume is roughly the planck scale) since

$$100\text{GeV} \;/\; \text{compactification scale} \sim 10^{-17}$$

The present experimental energies available are roughly 100GeV. So if we have gauge group G at high energies then of these bosons we will only see a smaller group G' at low energy ($G' \subset G$). If the compactifying solution has Yang-Mills fields F set equal to zero then G will be unbroken and we will have $G' = G$. If the Yang-Mills field is non-zero ,F_0 say , then when we expand $F \wedge *F$ about F_0 then some of the fields will aquire a mass from this term. Only those fields whose generators have zero commutator with those in F_0 will be massless (on the compactification scale). These will form the resultant gauge group. This is the "usual case" when bosons aquire a mass from $F \wedge *F$. Notice that a U(1) solution for F_0 will not break the symmetry group G. However for the extended Chapline-Manton lagrangian the presence of the Yang-Mills Chern-Simons term gives a mass to these U(1) fields also [34]. We find in total that if the compactifying solution has F_0 a H group and if $G' \times H$ ($\subset$ G) is maximal in the sense that we can't expand G' at all then the resulting low energy symmetry group will be G'.

We can also have 4-D Yang-Mills fields arising from the Einstein part of the lagrangian $R_{AB} \wedge *E^{AB}$. If we split up our 10-D metric g^{AB} as follows

$$g^{AB} = \begin{bmatrix} g^{\mu\nu} - A^{\mu}_{\alpha} A^{\nu}_{\alpha} , & A^{\mu}_{\alpha} K^{b}_{\alpha} \\ A^{\nu}_{\alpha} K^{a}_{\alpha} , & g^{ab} \end{bmatrix} \tag{4.1}$$

Where K^{a}_{α} are the Killing vectors for the internal manifold. α is a label for the isometries of the internal manifold (which form a Lie group). For the internal manifold we then find the 10-D Einstein-Hilbert action $R_{AB} \wedge *E^{AB}$ splits up into the 4-D Einstein-Hilbert action + the Yang-Mills lagrangian for the F field. The index α is the group index of these Yang-

Mills fields. If the internal manifold is a gauge group G then we find these "Kaluza-Klein" fields are GxG. If we have a coset space G/H ($H \neq 1$) then these fields are a G field (+) [30]. If the internal manifold has no isometries (eg as is the case for Calabi-Yau spaces) then there is no massless Yang-Mills fields.

This is the usual case for Einstein gravity with lagrangian density $R_{AB} \wedge {*E}^{AB}$, however for our theory we also have present the Zwiebach form $R_{AB} \wedge R_{CD} \wedge {*E}^{ABCD}$ to consider (and also the Lorentz Chern-Simons term in H) Although we speak of this as a curvature squared term it is more than just the curvature scalar squared in fact (in index notation) it is [23]

$$R_{ABCD}R^{CDAB} + 4R_{ABCB}R^{CDAD} + R_{ABAB}R^{CDCD} \qquad (4.2)$$

(where $R_{AB} = \frac{1}{2}R_{ABCD}\,E^{CD}$)

The curvature scalar is just R_{ABAB} so as we can see we have terms other than the square of the curvature scalar. It seems quite possible that these extra terms may upon compactification yield mass terms for some ,or all ,of the Kaluza-Klein bosons. So whether we would expect to see any bosons from the metric is at present unclear.(For the Calabi-Yau spaces the problem does not exist since these spaces do not have any symmetries and hence no Kaluza-Klein bosons).

The total gauge symmetry is the product of the "Kaluza-Klein" group and the remnant of the original gauge symmetry.

(+) This is the case when H is maximal, otherwise we obtain $G \times N'(H)$ where $N'(H)$ is st $H \times N'(H)$ is maximal

What are the large symmetry groups broken to ?. The groups which we start with are are $SO(32)$, $SO(32)/Z_2$ and $E_8 x E_8$.

We shall first deal with the case where the internal manifold is $SU(2)xSU(2)$. We have two Yang-Mills ansatzes - $SU(2)^2$ and $U(1)^6$. (We could also 'mix' and have $SU(2)xU(1)^3$). We shall look first at the possibilities of breaking E_8 via $SU(2)$s. (Recall we could have multiple $SU(2)$s in our solution see p54). If $H'xSU(2)$ is a maximal subgroup of E_8 then if this $SU(2)$ is non-zero the gauge group will break down to H'. If H'' $xSU(2)$ is a maximal subgroup of H' then we can let this $SU(2)$ be non-zero and be left with H'' etc . So we have a large number of possibilities for the resultant group. Diagram 4.1 indicates the possible groups left over from breaking E_8 via maximal $SU(2)$s. (This is not exhaustive of the possible imbeddings of $SU(2)^n$ within E_8). The resultant from $E_8 x E_8$ will just be the direct product of two of the possibilities. If one E_8 is unbroken then the fields from this E_8 will only interact with the other fields gravitationally and so will appear as 'dark matter' . The existence of which is not inconsistent with Cosmological evidence. It is noticable that none of the interesting groups E_6, $SO(10)$ or $SU(5)$ appear in Diagram 4.1.

If we try to break $E_8 x E_8$ via $U(1)^6$ then we have different possibilities depending on how many $U(1)$s go into each E_8. If all six are imbedded within one E_8 then we will be left with E_8x(a rank 2 group). As

$SU(3)xSU(2)xU(1)$ has rank 4 this is obviously not going to give us a physical gauge symmetry. Since the U(1) fall into two sets of 3 it seems natural to keep these U(1)s together within the same E_8 so the only other possibility is imbedding $U(1)^3$ within each E_8 this will lead to the product of two rank 5 groups one possible 'physical' route would be

$E_8 \supset U(1) \times E_7$ breaking the U(1) gives E_7

$E_7 \supset U(1) \times E_6$ breaking this U(1) gives E_6

$E_6 \supset U(1) \times SO(10)$ breaking this U(1) gives SO(10)

So it is possible to obtain a 'physical' group , SO(10), via this ansatz. Multiple imbeddings of $U(1)^6$ are not very interesting since they break the E_8 too far.

For SO(32) there are even more possibilities than for E_8. SO(32) has rank 16 so imbedding SU(2)xSU(2) would leave us with a rank 14 group. We shall not try to categorise the possibilities but mention a few possibilities - as ($SU(2) \simeq SO(3)$ as algebras we might expect to be able to break SO(32) down to SO(32-3n)xsome U(1)s with ease. However SU(2) and SO(3) are not quite the same groups and there are subtleties involved). $SU(2)xSU(2) \simeq SO(4)$ so we would expext to be able to break SO(32) down to SO(32-4n). This is indeed possible. However we obtain SO(12) and SO(8) (amongst others) in this way but not the desirable SO(10).

Imbedding $U(1)^6$ within SO(32) will give us a rank 10 group which is too big. A double imbedding will yield a rank 4 group which can be SU(5) via the following pathway.

$SO(32) \supset SO(22)xSO(10)$

Since SO(22) is rank 11 we can break this via eleven U(1)s leaving SO(10).
Now SO(10) $\supset$ U(1)xSU(5) so imbedding one U(1) within the U(1) will leave SU(5) as the low energy gauge group. This is a fairly attractive scheme since this is the most $U(1)^6$s we can imbed within SO(32).

For our manifold being $SU(2)xU(1)^3$ we had a gauge group of SU(2) (Recall that we started with a ansatz of SU(2)xU(1) but found the U(1) part to be zero.) So we can obtain the same groups as for SU(2)xSU(2).
It is difficult to take this manifold seriously, however, when no Minkowski space-time solution exists.

For the case where the internal manifold is $U(1)^6$ we have no solutions to consider other than the trivial case F = 0 which would not lead to any symmetry breaking.

2. 4-D fermions

Although the background field for the fermions are zero we will still have different looking fermions in 4-D at low energy from those which appear in the 10-D lagrangian. The original fermions lay in the adjoint of the original Yang-Mills gauge group G when this symmetry is broken to G' then this representation will split up into various representations of G'. Eg if we have $E_8 x E_8$ to start with then we will have a $(\underline{248},\underline{248})$ as our fermion represention. Then if, for example, the symmetry was broken to $E_8 x E_7$ via a SU(2) being non-zero ie

$$E_8 \times E_8 \supset E_8 \times E_7 \times SU(2)$$

then we find the $(\underline{248},\underline{248})$ splits into the following representations

$$(\underline{248},\underline{248}) \rightarrow (\underline{248},\underline{1},\underline{3}) \oplus (\underline{248},\underline{133},\underline{1}) \oplus (\underline{248},\underline{56},\underline{2})$$

The most popular physical groups are SU(5), SO(10) and E_6. The representations which we would like to obtain are, respectively, the $10 + \bar{5}$, the 16 and the 27. As we have three (or four) families of chiral fermions we would like to obtain three (or four) of these with no matching chiral partners.

We now look into the possibility of obtaining chiral fermions in four dimensions. If we were setting F = R à la Calabi-Yau then since the Euler characteristic for our 6-D manifolds is zero we would obtain no chiral fermions. Since our F field is different from R we must look further at the Index theorem.

Suppose we have a solution with Yang-Mills field F set $= F_0$ F_0 has gauge group G_0 which has centraliser H_0 within the overall gauge group. An original fermion representation A will split into (B,C) plus possibly $(\bar{B},\bar{C})$ plus others of $G_0 \times H_0$($\bar{B}$ is the opposite chirality to B). The imbalance upon compactification of massless $\bar{C}$s of H_0 over Cs in 4-D will be given by the imbalance of $\bar{B}$s of G_0 over Bs in the background field F_0. This number is given by the index theorem for a six dimensional manifold which is [35]

$$n_+ - n_- = \frac{1}{6}\int (3c_3 - 3c_2 c_1 + c_1^3) - \frac{1}{24}\int p_1 c_1 \qquad (4.3)$$

Where the c_i are the i-th Chern classes for the manifold

$$c_1 = \frac{i}{2\pi}\mathrm{Tr}(F)$$

$$c_2 = \frac{1}{8\pi^2}\mathrm{Tr}(F \wedge F) - \frac{1}{8\pi^2}\mathrm{Tr}(F) \wedge \mathrm{Tr}(F)$$

$$c_3 = \frac{-i}{24\pi^3}Tr(F\wedge F\wedge F) + \frac{i}{16\pi^3}Tr(F\wedge F \wedge Tr(F)) - \frac{i}{48\pi^3}Tr(F)\wedge Tr(F)\wedge Tr(F) \quad (4.4)$$

and p_1 is the first Pontrjagin class

$$p_1 = \frac{1}{4\pi^2} Tr(R\wedge R)$$

On substitititing the c_is into the first integral in (4.3) it reduces to

$$-\frac{i}{24\pi^2}\int Tr(F\wedge F\wedge F)$$

This is ,for the SU(2), more useful. For our SU(2) fields we have

$$Tr(F\wedge F\wedge F) = 0$$

(Since tr( ) will not give any terms mixing the SU(2)s this will yield six forms on the SU(2)s which will reduce to zero.) So the first term will be zero. The first Chern class c_1 is zero for non-U(1) fields so we will find $n_+ - n_- = 0$ so for our SU(2) fields we cannot obtain Chiral fermions.

For our U(1) fields we have that c_2 and c_3 are zero however we must also look at $c_1c_1c_1$ and $p_1 c_1$. $c_1c_1c_1$ is

$$tr\ F\wedge tr\ F\wedge tr\ F \quad (4.5)$$

tr F is not zero for our U(1) fields however (4.5) must be (a sum of) a (four form in one SU(2)) $\wedge$ (a two form in the other). Since a four form must be zero $c_1 c_1 c_1$ must be zero.The class p_1 must also be zero (since $SU(2)xSU(2) \cong S^3xS^3$). So for our U(1) ansatz we also obtain that $n_+ - n_- = 0$ and so no chiral fermions. (The above analsis will follow through for any 6-D manifold which is of the form (3-D)x(3-D) with no mixed Yang-Mills fields.)

So we do not obtain any chiral fermions for our solutions. This is a major difficulty if we wish to regard our solutions as physical. It is possible that some mechanism , operating at energies intermediate between the compactification scale and 100 GeV may give a mass to one chirality of fermion but we have no concrete suggestions to make for such a mechanism.

3. Cosmological Aspects

Observationally we live in a four dimensional universe whose three spatial dimensions seem to be (at large enough scale) homogeneous ,isotropic and expanding. This can be described by the Robertson-Walker solutions where the Universe is of the form $R \times S^3/HS^3$ with a time dependent scale for the S^3/HS^3. At the present moment the curvature of the universe is very small compared to the planck scale ($\sim 10^{-46}$). If we are really in a 10-D world with six dimensions curled up then the curvature of the internal six dimensions must be reasonably large ($\gtrsim 10^{-4}$ of planck scale) otherwise they would be observed directly The large difference in the curvatures is something which hopefully a sucessful theory would explain. Experimentally it seem's that the universe initially started with an initial state which was highly curved in 4-D also (big bang model)[36]

We have been trying to find solutions of the form

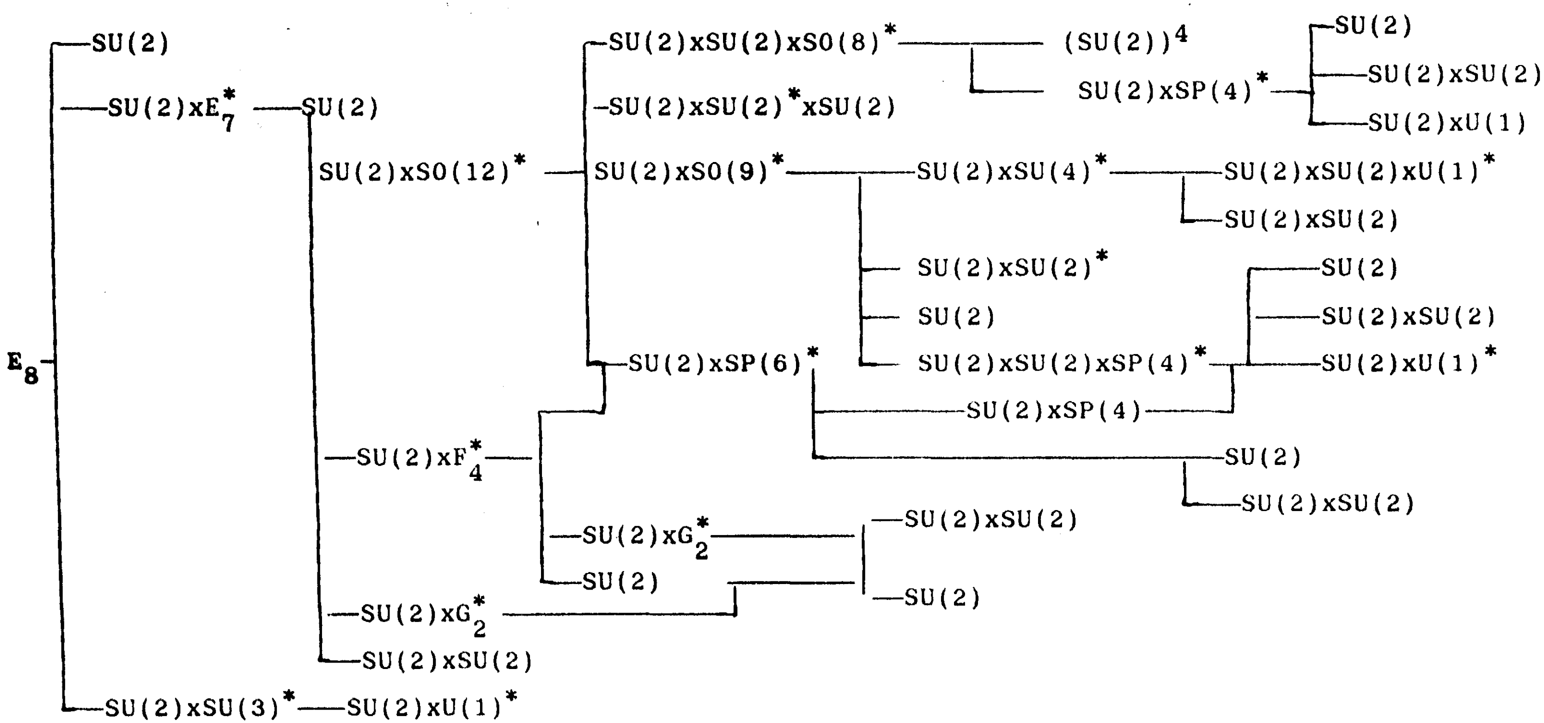
(4-D space-time)x(6-D internal space)

One of the possibilities we have considered is space -time being flat ie Minkowski M_4 obviously this does not fit in with the initial state of the Big-Bang model however M_4 seems to be the $t \to \infty$ limit of the universe so M_4 x compact space should be a limiting solution of a fundamental theory. Of the two cases SU(2)xSU(2) and SU(2)xU(1)^3 M_4 is only a solution for SU(2)xSU(2). It is very interesting to note that for this case the solution space extends from the case of M_4x(curved 6-D) to the case (curved 4-D)x(curved 6-D) since a slow variation with time between these two cases is compatible with the big-bang model.

We do not discuss the possibility of resultant 4-D supersymmetry, although this is an important question , because the lagrangian we are using is an extended Chapline-Manton lagrangian and hence the Chapline-Manton supersymmetry transformations will no longer be valid. At present we do not know which changes ,to the transformations, are neccessary to restore supersymmetry. It may be true that we must add more terms before we can reach a supersymmetric lagrangian.

In conclusion with have great difficulty in matching our solutions for the internal space being a group manifold to the physical world and none can be described as remotely realistic. In particular the non-chiral nature of the fermions is a huge stumbling block. The existance of a family of solutions linking (R$\times S^3$/HS^3)x(compact 6-D) to M_4x(compact 6-D) is interesting.

Diagram 4.1 Possible 4-D Symmetries arising from breaking E via multiple SU(2)s



This is an illustration of possible breakdown schemes of the E_8 symmetry group by letting a series of SU(2) subgroups be non-zero. We consider only breaking via maximal subgroups of the form SU(2)xG', letting the SU(2) be non-zero breaks the symmetry down to G'. (We then consider maximal subgroups of G' etc). The asterixed groups are possible unification groups. (For tables of maximal subgroups see reference [37]).

Chapter 5 Non-Symmetric Coset Spaces

We shall attempt to solve the equations of motion for the case of the internal manifold being a Non-Symmetric Coset Space (N.S.C.S).

First we present a summary of our definitions and notation.

If we have a Lie-group S which has a Lie-subgroup R then we can give the left cosets of R a differential structure in the standard manner [30] a summary of which we shall present here. Let $\hat{a}=1..\dim S$, $\bar{a}=1..\dim R$, $a=1..\dim S-\dim R$. If $Q_{\hat{a}}$ are a suitable choice of the generators for S then they will split into two sets $Q_{\bar{a}}$ which are the generators for R and the remaining Q_a. We have -(since S is a Lie group)

$$[\ Q_{\hat{b}},Q_{\hat{c}}\ ] = \lambda C^{\hat{a}}{}_{\hat{b}\hat{c}}Q_{\hat{a}} \qquad (5.1)$$

The $\lambda C^{\hat{a}}{}_{\hat{b}\hat{c}}$ are the structure constants for S. (We have introduced λ so we can normalise the $C^{\hat{a}}{}_{\hat{b}\hat{c}}$s and then will give the scale).

We can set up a co-ordinate system $\underline{y}$ (at least locally) on S/R. Each independent value of $\underline{y}$ will label distinct left cosets of R within S. For each value of $\underline{y}$ we can choose an element $L(\underline{y})$ of S from the appropriate coset. Since S/R is a differentiable manifold $L(\underline{y})$ is a differentiable function wrt the coordinate system $\underline{y}$.

Hence we can define the S-lie algebra valued one-form

$$E(\underline{y}) = -L^{-1}(\underline{y})dL(\underline{y}) \qquad (5.2)$$

This can be expanded in terms of the generators of S

$$E(\underline{y}) = E^{\hat{a}}(\underline{y})Q_{\hat{a}} = E^{\bar{a}}(\underline{y})Q_{\bar{a}} + E^{a}(\underline{y})Q_{a} \qquad (5.3)$$

where $E^{\bar{a}}(y)$ and $E^{a}(y)$ are one forms on S/R

Since $d^2=0$ we will have

$$dE(\underline{y}) = E(\underline{y}) \wedge E(\underline{y}) \tag{5.4}$$

Using (5.1) this will lead us to

$$dE^{\hat{a}} = -\frac{\lambda}{2} C^{\hat{a}}{}_{\hat{b}\hat{c}} E^{\hat{b}\hat{c}} \tag{5.5}$$

A metric on S/R can be constructed from the E^{a} by using them as orthonormal one-forms. Considering the metric as a rank two symmetric co-variant tensor we define it to be

$$g = \eta_{AB} E^{A} \otimes E^{B} \tag{5.6}$$

Thus we now have our Coset space with its metric. We now look at the structure constants $C^{\hat{a}}{}_{\hat{b}\hat{c}}$.

Since R is a Lie-subgroup we will have

$$C^{a}{}_{\bar{b}\bar{c}} = 0 \tag{5.7}$$

We always have

$$C^{\hat{a}}{}_{\hat{b}\hat{c}} = -C^{\hat{a}}{}_{\hat{c}\hat{b}} \tag{5.8}$$

However we can further choose our generators such that the $C^{\hat{a}}{}_{\hat{b}\hat{c}}$ are cyclic ie

$$C_{\hat{a}\hat{b}\hat{c}} = C_{\hat{b}\hat{c}\hat{a}} = C_{\hat{c}\hat{a}\hat{b}} \quad (C_{\hat{a}\hat{b}\hat{c}} = \eta_{\hat{a}\hat{a}'} C^{\hat{a}'}{}_{\hat{b}\hat{c}}) \tag{5.9}$$

We can also choose to normalise the C ,s so that

$$C^{\hat{a}}{}_{\hat{b}\hat{c}} C^{\hat{b}\hat{c}\hat{d}} = \delta^{\hat{a}\hat{d}} \tag{5.10}$$

It is also possible that some or all of the following will be obeyed

$$C^{a}{}_{b\bar{c}} C^{b\bar{c}d} = n_1 \delta^{ad} \tag{5.11}$$

$$C^{a}{}_{bc} C^{bcd} = n_2 \delta^{ad} \tag{5.12}$$

$$C^{\bar{a}}{}_{bc} C^{\bar{d}bc} = n_3 \delta^{\bar{a}\bar{d}} \tag{5.13}$$

$$C^{\bar{a}}{}_{\bar{b}\bar{c}} C^{\bar{b}\bar{c}\bar{d}} = n_4 \delta^{\bar{a}\bar{d}} \tag{5.14}$$

If these are obeyed then we will have (from (5.10))

$$\dim R . n_3 = (\dim S/R) . n_1 \tag{5.15}$$

$$2n_1 + n_2 = 1 \tag{5.16}$$

$$n_2 + n_3 = 1 \tag{5.17}$$

If we find that all the $C^a{}_{bc}$ are zero then we define this to be a 'Symmetric Coset Space' if there exists a nonzero $C^a{}_{bc}$ then we define this to be a 'Non-Symmetric Coset Space' (N.S.C.S) As previously discussed on P37 we are interested in cases where we can define a non-zero torsion. For the non-symmetric case we have a natural ansatz for the torsion- (see ref [30])

$$T^a = (1-\beta)\frac{\lambda}{2} C^a{}_{bc} E^{bc} \tag{5.18}$$

β being a free parameter (For symmetric coset spaces this ansatz is zero.)

With this choice for the torsion we find the connection $\omega^a{}_b$ to be

$$\omega^a{}_b = \frac{\lambda}{2}(1-\beta) C^a{}_{bc} E^{bc} + \lambda C^a{}_{b\bar{c}} E^{\bar{c}} \tag{5.19}$$

and the curvature two forms to be

$$\begin{aligned} R^a{}_b = \; & \frac{\lambda^2}{2} C^a{}_{b\bar{c}} \; C^{\bar{c}}{}_{de} \; E^{de} \\ & + \beta \frac{\lambda^2}{4} C^a{}_{bc} \; C^c{}_{de} \; E^{de} \\ & + \beta^2 \frac{\lambda^2}{4} C^a{}_{dc} \; C^c{}_{be} \; E^{de} \end{aligned} \tag{5.20}$$

We find the Ricci one-forms are (this involves knowledge of the structure constants)

$$R^a = \frac{\lambda^2}{12}(4 + 2\beta - \beta^2) E^a$$

So our coset spaces are Einstein spaces.

We note the two special cases $\beta = 1, 0$ which are refered to in the mathematical literature as [38] canonical connections of the first, second type

For $\beta = 0$ we find $R^a{}_b = \frac{\lambda^2}{2} C^a{}_{b\bar{c}} C^{\bar{c}}{}_{de} E^{de}$ (5.21)

For $\beta = 1$ we have the torsion free case.

For $\beta = 0$ $R^a{}_b$ will have holonomy group R (for coset space S/R).

For $\beta \neq 0$ the holonomy group is SO(6).

For non-symmetric coset spaces we can with a suitable choice of β ($= 1 \pm \sqrt{5}$) obtain a Ricci-flat space; this was noted by Lust [26] who also noted that with H = F = 0 he had a solution to the equations of motion for the unextended Chapline-Manton lagrangian. However with this choice of β the Zwiebach form $R_{AB} \wedge R_{CD} \wedge {}^*E^{ABCD}$ is not zero and we do not find solutions to the extended lagrangian (as we will see.) Also in [27] Lust noted that for a specific value of β there was a cancelation of the conformal anomaly.

We are interested in finding solutions to the equations of motion of the extended 10-D supergravity lagrangian which are of the form

(4-D space-time)x(6-D non-symmetric coset space)

So we are interested in six dimensional N.S.C.Ss. There are only three of these, they are

SU(3)/{ U(1)xU(1) }, Sp(4)/{ SU(2)xU(1) }, G2/SU(3)

The root diagrams of SU(3), Sp(4) & G2 are shown in Appendix 1. As can be seen there exist two distinct imbedings of SU(2)xU(1) within Sp(4) only one of which yields a N.S.C.S. Also in Appendix 1 we give the structure constants and the explicit form of the curvature two forms $R^a{}_b$.

As in chapter 3 we shall take the scalar field to be a constant and also as in chapter 3 p45 we can rescale our fields so that the scalar field does not appear in the resultant equations of motion. Hereafter we will assume this has been done.

What shall we take as our ansatz for the three form field H?. The following is a natural possibility

$$H = h_1 C^{abc} E_{abc} \tag{5.22}$$

This choice of H has several important properties

1. d*H=0 automatically so if the scalar field is a constant the equation of motion (2.20) will be automatically satisfied

2. the energy momentum tensor is the product of block diagonal matrices ie

$$i^{\varepsilon}(H \wedge *H) - 2i^{\varepsilon}(H) \wedge *H = \begin{array}{ll} 36 n_2 h_1^2 \; *E^{\varepsilon} & \varepsilon = 0-3 \\ 0 \quad *E^{\varepsilon} & \varepsilon = 4-9 \end{array} \tag{5.23}$$

3. In most cases (see later) dH is proportional to both tr(F$\wedge$ F) and tr(R$\wedge$R) hence leaving the Bianchi identity as a single constraint

Another possibility for the H field would be

$$H \cong C_{ad}{}^{e} C_{bf}{}^{d} C_{ce}{}^{f} E^{abc} \tag{5.24}$$

This also satisfies (at least) properties 1. and 2. However explicit calculation of this term for the particular coset spaces analysed revealed it to be zero

For the case where space-time is RxS^3/HS^3 we add to H the extra term (where S^3 denotes a three sphere and HS^3 denotes a three hypersphere.)

$$h_0 E^{123} \tag{5.25}$$

(This is the volume element of S^3/HS^3)

We have several possibilities open to us as to what the Yang-Mills field could be. The first is to simply take F to be zero ie

$$\text{F1} \quad F = 0 \tag{5.26}$$

secondly we could have A (the Yang-Mills potential) imbedded within the large gauge group as an R field (for coset spaces S/R) taking

$$A^{\bar{a}} = -\lambda E^{\bar{a}} \tag{5.27}$$

then we find

$$F2 \qquad F^{\bar{a}} = -\frac{\lambda^2}{2} C^{\bar{a}}{}_{bc} \, E^{bc} \tag{5.28}$$

For $\beta=0$ this F is identical to having F a SO(6) field equal to the curvature (recall that for $\beta=0$ $R^a{}_b$ had holonomy R).

as our third choice we can imbed A as a S-field

$$A^{\bar{a}} = -\lambda E^{\bar{a}} \quad , A^{a} = \lambda E^{a} \tag{5.29}$$

this leads to

$$F3 \qquad F^{\bar{a}} = 0$$

$$F^{a} = -\frac{\lambda^2}{2} C^{a}{}_{bc} \, E^{bc} \tag{5.30}$$

A possible fourth choice of F would be (as in Calabi-Yau) to take $F^a{}_b = R^a{}_b$ ie imbed F as a SO(6) field. If we do this however $\mathrm{Tr}\, F^2 = \mathrm{Tr}\, R^2$ and so we must have dH=0. For our ansatz this means H=0. It then follows that the Yang-Mills equation will reduce to

$$D^{ym} {*F} = D^{LOR} {*R^a{}_b} = 0 \tag{5.31}$$

However $D^{LOR} {*R^a{}_b} \neq 0$ unless $\beta=0$ (or $\lambda^2=0$!). So F=R is only any good if $\beta=0$. However for $\beta=0$ our ansatz F2 is exactly that !. So we shall not consider F=R further.

In all cases we shall assume F has no components in or functional dependence on the 4-D space-time. Having $F\neq0$ on Space-time would probably destroy 4-D Lorentz invariance.

Do these F fields satisfy the Yang-Mills equation of Motion (2.19) ?. F1 obviously does. For F2 we find that both $D{*}F^{a}$ and $F^{a}\wedge{*}H$ are zero so the equation is satisfied leaving no constraint. For F3 we find that both $D{*}F^{a}$ and $F^{a}\wedge{*}H$ are proportional to ${*}E^{a}$ and we are left with the single constraint

$$\lambda + 3h_1 = 0 \tag{5.32}$$

This extra constraint makes the existence of solutions unlikely for space-time being M_4 or AdS/dS.

We note that for the cases SU(3)/{ U(1)xU(1) } and G2/SU(3) all of (5.11) to (5.14) are satisfied

for SU(3)/{U(1)xU(1)} $n_1 = \frac{1}{3}, n_2 = \frac{1}{3}, n_3 = 1$ & $n_4 = 0$

for G2/SU(3) $n_1 = \frac{1}{3}, n_2 = \frac{1}{3}, n_3 = \frac{1}{4}$ & $n_4 = \frac{3}{4}$

However for SP(4)/{SU(2)xU(1)} we find (5.11) and (5.12) are satisfied with $n_1 = n_2 = \frac{1}{3}$ but (5.13) and (5.14) are not. This has important consequences for our ansatz it means $R_{AB}\wedge R_{CD}\wedge i_{E}{*}E^{ABCD}$ is not a constant multiple of ${*}E^{E}$ for E =4-9. Also we find tr($R\wedge R$) is not proportional to dH.

It is possible ,by having a U(1) field ,to 'cancel' the problem part of $R^{a}{}_{b}$. This however can only be done for the case of the Yang-Mills field being a SU(2)xU(1) field ie Case F2. It is detailed in Appendix 2 how this may be done. If we do this it is possible to treat the case Sp(4)/SU(2)xU(1) along with the other two (provided we use the n_i's and normalisations appopriate for the SU(2) alone). From now on we shall assume implicitly that this has been done. However this can only be done for case F2 so in the remainder of this

Chapter we shall only look at case F2 for the Coset space Sp(4)/SU(2)xU(1).

We have three possible cases for 4-D space-time M_4, AdS or dS & RxS^3/HS^3 we shall consider these in turn. First we introduce a few definitions for the curvatures/fields on the internal 6-D space :-

$$R_{mn} \wedge {}^*E^{mn} \;(m,n=4-9) = r(\beta,\lambda){}^*1 \tag{5.33}$$

$$R_{mn} \wedge R_{pq} \wedge {}^*E^{mnpq} \;(m,n,p,q=4-9) = z(\beta,\lambda){}^*1 \tag{5.34}$$

$$tr(F \wedge {}^*F) = F(\beta,\lambda){}^*1 \tag{5.35}$$

$$H \wedge {}^*H = g(H,H){}^*1 \tag{5.36}$$

Minkowski space-time

We now look at the case of Minkowski space-time. As mentioned on P41 we can take the scalar equation as a consequence of the Einstein equations. so we are left with two independent equations -the internal Einstein equations reducing to one algebraic equation and the 4-D equations reducing to one. We find

$$\tfrac{1}{2}r(\beta,\lambda)+\tfrac{1}{4}z(\beta,\lambda)-\tfrac{1}{4}g(H,H)+\tfrac{1}{2}F(\beta,\lambda) = 0 \tag{5.37}$$

$$\tfrac{1}{3}r(\beta,\lambda)+\tfrac{1}{12}z(\beta,\lambda) \qquad\qquad +\tfrac{1}{6}F(\beta,\lambda) = 0 \tag{5.38}$$

These contain the equation (see (2.26))

$$g(H,H) = -r(\beta,\lambda) \tag{5.39}$$

this we can use to define the coefficient of H in terms of β & λ . $g(H,H) = 36n_2h_1^2$. so we have

$$18n_2h_1^2 = -r(\beta,\lambda) \tag{5.40}$$

We have left one remaining Einstein equation which is a constraint on β & λ

$$4r(\beta,\lambda)+ z(\beta,\lambda)+2F(\beta,\lambda) = 0 \tag{5.41}$$

We still have the dH equation to consider ,for both $SU(3)/\{U(1)xU(1)\}$ and $G2/SU(3)$, $tr(R\wedge R)$ and $tr(F\wedge F)$ are proportional to dH. If $H = h_1 H_0$ then

$$tr(R\wedge R) = k_0(\beta,\lambda)dH_0 \tag{5.42}$$

$$tr(F\wedge F) = l_i(\beta,\lambda)dH \tag{5.43}$$

(i refers to which F field is considered (5.26), (5.28) or (5.30).)

so we can write down the H Bianchi identity

$$h_1 = k_0(\beta,\lambda)-l_i(\beta,\lambda) \tag{5.44}$$

squaring $$h_1^2 = (k_0(\beta,\lambda)-l_i(\beta,\lambda))^2 \tag{5.45}$$

substituting in h_1^2 from (5.40) gives us

$$-\frac{1}{18n_2}r(\beta,\lambda) = (k_0(\beta,\lambda)-l_i(\beta,\lambda))^2 \tag{5.46}$$

This is another constraint on β and λ so we now have with (5.41) two constraints. To proceed further we must evaluate the form of the functions we have introduced.

For all three coset spaces

$$r(\beta,\lambda)= \lambda^2\{ 2+\beta-\tfrac{1}{2}\beta^2) \tag{5.47}$$

For $z(\beta,\lambda)$ we obtain slight differences for the different coset spaces. We find

$$z(\beta,\lambda)= \tfrac{1}{6}\lambda^4(\beta^4-2\beta^3-3\beta^2+8\beta+20) \quad \text{for } SU(3)/U(1)xU(1)$$

$$= \tfrac{1}{6}\lambda^4(\beta^4-2\beta^3-3\beta^2+8\beta+12) \quad \text{for } Sp(4)/SU(2)xU(1)$$

$$= \tfrac{1}{6}\lambda^4(\beta^4-2\beta^3-3\beta^2+8\beta+11) \quad \text{for } G2/SU(3) \tag{5.48}$$

For the Yang-Mills fields we obtain

$$F(\beta,\lambda)= 0 \quad \text{for case F1} \tag{5.49}$$

$$F(\beta,\lambda)= -3\chi^2 n_2 = -\chi^2 \quad \text{for case F2} \tag{5.50}$$

$$F(,)= -3\chi^2 n_1 = -\chi^2 \quad \text{for case F3} \tag{5.51}$$

(χ^2 is the normalisation factor from the generators ie $tr(O_i O_j)=-\chi^2\delta_{ij}$)

so $z(\beta,\lambda)+2F(\beta,\lambda) =$

$$\frac{\lambda^4}{6}(\beta^4-2\beta^3-3\beta^2+8\beta+K_1) \qquad (5.52)$$

where $K_1 = 20-12\chi^2$ for $SU(3)/\{U(1)xU(1)\}$

$= 12-12\chi^2$ for $Sp(4)/\{SU(2)xU(1)\}$

$= 11-12\chi^2$ for $G2/SU(3)$ (5.53)

so we can write down our first constraint (5.41)

$$2\lambda^2(\beta^2-2\beta-4) = \frac{1}{6}\lambda^4(\beta^4-2\beta^3-3\beta^2+8\beta+K_1) \qquad (5.54)$$

now we find by explicit calculation

$$k_0(\beta,\lambda) = -\frac{\lambda^3}{24}(\beta^2-4) \quad \text{for } SU(3)/\{U(1)xU(1)\}$$

$$= -\frac{\lambda^3}{24}(\beta^2-4/3) \quad \text{for } Sp(4)/\{SU(2)xU(1)\}$$

$$= -\frac{\lambda^3}{24}(\beta^2-1) \quad \text{for } G2/SU(3) \qquad (5.55)$$

$$l_1(\beta,\lambda) = 0 \quad (\text{case F1})$$

$$l_2(\beta,\lambda) = \frac{1}{6}\lambda^3\chi^2 \quad (\text{case F2})$$

$$l_3(\beta,\lambda) = -\frac{1}{6}\lambda^3\chi^2 \quad (\text{case F3}) \qquad (5.56)$$

We can hence write (5.46) as

$$\lambda^2(\beta^2-2\beta-4) = \frac{1}{48}\lambda^6(\beta^2-K_2)^2 \qquad (5.57)$$

Where $K_2 = 4$ for $SU(3)/\{U(1)xU(1)\}$ case F1

$= 4-4\chi^2$ for $SU(3)/\{U(1)xU(1)\}$ case F2

$= 4+4\chi^2$ for $SU(3)/\{U(1)xU(1)\}$ case F3 (5.58)

For $G2/SU(3)$ we have $1, 1-4\chi^2, 1+4\chi^2$ respectively and for $Sp(4)/\{SU(2)xU(1)\}$ we have only $4/3-4\chi^2$ for the case F2.

We can rearrange our system of two constraints (5.54) and (5.57) thus

$$\lambda^2(\beta^2-2\beta-4) = \frac{1}{12}\lambda^4(\beta^4-2\beta^3-3\beta^2+8\beta+K_1) \qquad (5.59)$$

$$4\lambda^4(\beta^4-2\beta^3-3\beta^2+8\beta+K_1) = \lambda^6(\beta^2-K_2)^2 \qquad (5.60)$$

We find we can solve for λ^2 thus

$$\lambda^2 = 12(\beta^2-2\beta-4)/(\beta^4-2\beta^3-3\beta^2+8\beta+K_1) \qquad (5.61)$$

Leaving a single constraint on β

$$(\beta^4-2\beta^3-3\beta^2+8\beta+K_1)$$
$$-3(\beta^2-K_2)^2(\beta^2-2\beta-4) = 0 \qquad (5.62)$$

This we write as $G(\beta) = 0$.

So we must determine whether (5.62) is satisfied for any value of β for the given values of K_1 and K_2.

Before we can do this we must decide what our χ^2 must be. This is the normalisation of our Yang-Mills generators. We must decide what the appropriate values are. This we do in Appendix 3 with the following results

	F2	F3
SU(3)/{U(1)xU(1)}	1.0	1/4
Sp(4)/{SU(2)xU(1)}	1/3	not appropriate
G_2/SU(3)	1/4	1/4 (5.63)

(We obtain case F1 ie F=0 by setting $\chi^2=0$.)

We also have some ambiguity left over in the case SU(3)/U(1)xU(1) to see how this arises we look at how the factor in front of the $F^{\bar{a}}$ field is fixed (to be $-\frac{\lambda^2}{2}$) .If we let $A^{\bar{a}} = aE^{\bar{a}}$ then

$$
\begin{aligned}
F^{\bar{a}} &= dA^{\bar{a}} + \tfrac{1}{2}C^{\bar{a}}{}_{\hat{b}\hat{c}}A^{\hat{b}}\wedge A^{\hat{c}} \\
&= \tfrac{1}{2}a\lambda\, C^{\bar{a}}{}_{\hat{b}\hat{c}}E^{\hat{b}\hat{c}} + \tfrac{1}{2}a^2 C^{\bar{a}}{}_{\bar{b}\bar{c}}E^{\bar{b}\bar{c}} \\
&= \tfrac{1}{2}(a\lambda + a^2)C^{\bar{a}}{}_{\bar{b}\bar{c}}E^{\bar{b}\bar{c}} + \tfrac{1}{2}a\lambda C^{\bar{a}}{}_{bc}E^{bc} \qquad (5.64)
\end{aligned}
$$

In general unless $a = -\lambda$ the first term will not vanish. This term is undesirable since it involves $E^{\bar{a}}$. Since the $E^{\bar{a}}$ are involved when we look at the energy-momentum tensor

$$i^{\varepsilon}(F\wedge *F) - 2(i^{\varepsilon}F)\wedge *F$$

This will not simply be a constx$*E^{\varepsilon}$ but will be more complicated with a functional dependence on the coordinates $\underline{y}$ of the coset space. Since our coset spaces are Einstein spaces this will give great difficulty in find finding solutions to the Einstein equations. However for the U(1)xU(1) case this term is trivially zero so we would not have problems with the F field if we took a to be arbitrary. This is a local consideration we must be careful how we deal with the global properties. When we consider these the coefficient will be subject to a quantisation constraint and will be integer multiples of the minimal value (eg as in the Dirac monopole which is a U(1) bundle over S^3 - here we have a $U(1)^2$ bundle over SU(3)/U(1)xU(1)) which is $-\frac{\lambda^2}{2}$ for the normalisation we have. So we have an infinite possibilities for this case - but we are not really introducing another free parameter. For convenience we keep the coefficient as $-\frac{\lambda^2}{2}$ but shall allow ourselves the possibility of letting λ^2 be an integer multiple of the minimum value (1). This is also discussed in Appendix 3 where we deal with the normalisations.

We à priori have seven cases to consider-two coset spaces each with F1,F2 and F3 and Sp(4)/SU(2)xU(1) with F2. However we see that in all three cases of F2 we find K_1 =8 and K_2 =0 so we only have five separate cases to consider.

Why do these reduce to the same case ? as we can see in Appendix 1 the S-R structure constants the $C^a{}_{bc}$ are the same (upon relabeling) for the three coset spaces so any property depending solely upon these will be the same for both. With F=0 however the part of $R^a{}_b$ not dependant on these will be important and so we have different properties. For $F\neq 0$ case F2 however we have a cancelation between the F and R^a_b parts. This is because this choice of F is equivalent, in the special case $\beta=0$, to setting $F=R^a_b$ So the part of R^a_b which is not proportional to β or β^2 cancels with F in both TrR^2-TrF^2 and in $R_{AB}\wedge R_{CD}\wedge *E^{ABCD}+TrF\wedge *F$. This is the only part of R^a_b which depends on the $\bar{C}^a_{bc}$. So for the case F2 we only have single case to consider.

For the case where F is zero (F1) we plot the function $G(\beta)$ for the two cases in Diagram 5.1. As we can see there is no root in any of the two cases so for F=0 we have no solutions.

For case F2 we have the single case. $G(\beta)$ for this case is given on diagram 5.2 . For this case we find we have two roots both for negative.

For the case F3 we find a similar pattern to F2. However we do not find (5.32) satisfied at the roots hence we do not find solutions for the case F3.

We still have several positivity conditions to satisfy for our F2 roots to be valid solutions. We need $\lambda^2 \geq 0$ and $h^2 \geq 0$ now

$$h_1^2 \sim (\beta^2-2\beta-4) = (\beta-b_+).(\beta-b_-) \tag{5.65}$$

where $b_\pm = 1\pm\sqrt{5}$

So if $\beta \geq 1+\sqrt{5}$ or $\beta \leq 1-\sqrt{5}$ we will find h^2 positive . As can be seen (?) from diagram 5.2 this is the case for one root but not the other so we will have positive h^2 at one of the roots.

Turning to

$$\lambda^2 = 12(\beta^2 - 2\beta - 4)/(\beta^4 - 2\beta^3 - 3\beta^2 + 8\beta + K_1) \qquad (5.66)$$

as $(\beta^2 - 2\beta - 4) > 0$ at the one remaining interesting root we are reduced to evaluating $(\beta^4 - 2\beta^3 - 3\beta^2 + 8\beta + K_1)$ at that root. We indeed find this function to be > 0 in the appropriate region.

In summary then we find for the case where space-time is Minkowski that we find solutions , for all three coset spaces, when the Yang-Mills field is non-zero and of type F2 but not for Yang-Mills fields of type F1 or F3. The consequences of these solutions will be analysed later. We shall now turn to our next case for the 4-D space-time.

<u>deSitter or Anti-deSitter space-time</u>

We now turn our attention to the case where space-time is deSitter (dS) or Anti deSitter (AdS) The curvature on 4-D space-time is given by

$$R^{\mu\nu} = \frac{1}{12} R_4 E^{\mu\nu} \quad \mu,\nu = 0\text{-}3 \qquad (5.67)$$

The F and H fields will be as before ,however there will be changes in the R and R^2 terms. As discussed on P41 the scalar equation must now be treated as an independant equation. With the definitions of $r(\beta,\lambda)$, $z(\beta,\lambda)$ and $F(\beta,\lambda)$ as before we find the two Einstein equations become

$$\frac{1}{2}\{ r(\beta,\lambda) + \frac{1}{2}R_4 \} + \frac{1}{4}\{ z(\beta,\lambda) + R_4 . r(\beta,\lambda) \}$$
$$+ \frac{1}{2}F(\beta,\lambda) - \frac{1}{4}g(H,H) = 0 \qquad (5.68)$$

$$\frac{1}{2}\{ \frac{2}{3}r(\beta,\lambda) + R \} + \frac{1}{4}\{\frac{1}{3}z(\ , \ ) + R_4 . r(\ , \ ).\frac{4}{3} + \frac{1}{6}R_4^2 \}$$

$$+\frac{1}{6}F(\beta,\lambda) \qquad = 0 \qquad (5.69)$$

The scalar equation is

$$\frac{1}{4}\{\ z(\beta,\lambda) + 2R_4.r(\beta,\lambda) + \frac{1}{6}R_4^2\ \}$$
$$-\frac{1}{2}g(\ H,H\ ) + \frac{1}{2}F(\beta,\lambda) = 0 \qquad (5.70)$$

Again we have (P40)

$$g(\ H,H\ ) = -2\{\ r(\beta,\lambda) + R_4\} \qquad (5.71)$$

This enables us to remove g(H,H) from the system leaving the two equations

$$r(\beta,\lambda) + \frac{3}{4}R_4 + \frac{1}{4}\{\ z(\beta,\lambda) + 2F(\beta,\lambda)$$
$$+R_4.r(\beta,\lambda)\ \} = 0 \qquad (5.72)$$

$$r(\beta,\lambda) + R_4 + \frac{1}{4}\{\ z(\beta,\lambda) + 2F(\beta,\lambda)$$
$$+2R_4.r(\beta,\lambda) + \frac{1}{6}R_4^2\} = 0 \qquad (5.73)$$

subtracting gives us

$$\frac{1}{4}R_4 + \frac{1}{4}\{\ R_4.r(\beta,\lambda) + \frac{1}{6}R_4^2\} = 0 \qquad (5.74)$$

or $\quad R_4\{\ 1 + r(\beta,\lambda) + \frac{1}{6}R_4\ \} = 0 \qquad (5.75)$

so either $R_4 = 0$ or

$$R_4 = -6(\ 1 + r(\beta,\lambda)\ ) \qquad (5.76)$$

$R_4 = 0$ is just the minkowski case considered previously so we shall look at the other case ,substituting back into our one remaining Einstein/scalar equation we find the following constraint on β and λ.

$$\frac{1}{4}\{\ z(\beta,\lambda) + 2F(\beta,\lambda) - 6r^2(\beta,\lambda) - 6r(\beta,\lambda)\ \}$$
$$-\frac{9}{2} - \frac{7}{2}r(\beta,\lambda) = 0 \qquad (5.77)$$

The remaining constaint arising from the $dH = trR^2 - trF^2$ is unaltered

$$h_1^2 = \{\ k_0(\beta,\lambda) - 1_i(\beta,\lambda)\ \}^2 \qquad (5.78)$$

substituting in h_1^2

$$-\frac{1}{18n_2}\{\ r(\beta,\lambda) + R_4\ \} = \{\ k_0(\beta,\lambda) - 1_i(\beta,\lambda)\ \}^2 \qquad (5.79)$$

substituting in R_4 from (5.76)

$$-\frac{1}{18n_2}\{\ -6 - 5r(\beta,\lambda)\ \} = \{\ k_0(\beta,\lambda) - 1_i(\beta,\lambda)\ \}^2 \qquad (5.80)$$

(5.77) and (5.80) now form a system of two constraints in β and λ. Substituting in the exact form of $r(\beta,\lambda)$ etc leads us to

$$\lambda^4\{\tfrac{1}{24}(\beta^4-2\beta^3-3\beta^2+8\beta+K_1) - \tfrac{3}{8}(\beta^2-2\beta-4)^2\} + \tfrac{5}{2}\lambda^2(\beta^2-2\beta-4) - \tfrac{9}{2} = 0 \quad (5.81)$$

$$36 - 15\lambda^2(\beta^2-2\beta-4) - \tfrac{1}{16}\lambda^6(\beta^2-K_2) = 0 \quad (5.82)$$

We shall use equation (5.81) to solve for λ^2 obtaining

$$\lambda^2(\beta) = \frac{-\tfrac{5}{2}(\beta^2-2\beta-4) \pm \sqrt{\{\tfrac{3}{4}(\beta^4-2\beta^3-3\beta^2+8\beta+K_1) - (\beta^2-2\beta-4)^2\}}}{\{\tfrac{1}{12}(\beta^4-2\beta^3-3\beta^2+8\beta+K_1) - \tfrac{3}{4}(\beta^2-2\beta-4)^2\}} \quad (5.83)$$

the remaining constraint (5.82) is

$$Q(\beta) = \tfrac{1}{16}\lambda^2(\beta)(\beta^2-K_2)^2 - 36 + 15\lambda^2(\beta)(\beta^2-2\beta-4) = 0 \quad (5.84)$$

So we are left with finding the roots of $Q(\beta)$. Notice that in (5.83) we have a choice of solutions depending on whether we take the +ve or -ve sign in (5.83). The function $Q(\beta)$ is plotted for the case of F=0 on SU(3)/{U(1)xU(1)}, for both +ve and -ve choices for λ^2, on diagrams 5.3 and 5.4 respectively. As can be seen we find roots in both cases. However we must also check on whether $\lambda^2 \geqslant 0$ and $h^2 \geqslant 0$. When we do this we find no roots for the -ve choice which have both these satisfied however for the +ve case we do. This pattern is repeated for the other coset space with F=0 and for the case F2 (the same for all three). The function $Q(\beta)$ with details of the roots for these cases is given on Diagrams 5.5 and 5.6. As we can see R_4 and $r(\beta,\lambda)$ have opposite signs at the solutions. We find solutions both when space-time is deSitter and Anti-deSitter. We find we have two types of root described by

(i) $R_4 > 0$, $r < 0$ ie 4-D space-time is dS. For this type of root $|R_4|$ almost equals $|r|$ and so $g(H,H)$ is small (relative to $|R_4|$ and $|r|$).

(ii) $R_4 < 0$, $r > 0$ ie space-time is AdS. For this type of root we find $|r| \ll |R_4|$ and $g(H,H)$ is of the same scale as $|R_4|$.

Again the Yang-Mills fields being case F3 we find a similarish pattern to the case F2 but the extra constraint (5.32) is not satisfied so we find no solutions.

RxS^3/HS^3 4-D space-time

We now turn to our remaining case where space-time is RxS^3/HS^3. We find we have three Einstein equations (the scalar equation is a consequence of the Einstein equations). We add to H the extra term $H_0 = h_0 E^{123}$ otherwise the H and F fields are as previously The space-time curvature will be given by

$$R^{i0} = 0 \qquad i=1-3$$

$$R^{ij} = \frac{1}{6} R_3 E^{ij} \qquad i,j=1-3 \tag{5.85}$$

We have for the H field $H = H_0 + H_1$ where

$$H_0 = h_0 E_{123} \ , \ H_1 = h_1 C_{abc} E^{abc}$$

and we find $g(H,H) = g(H_0,H_0) + g(H_1,H_1)$.

The Einstein equations are

$$\frac{1}{2}\{ R_3 + r(\rho,\lambda) \} + \frac{1}{4}\{ z(\rho,\lambda) + 2r(\rho,\lambda).R_4 \} + \frac{1}{2}F(\rho,\lambda) - \frac{1}{4}\{ g(H_0,H_0) + g(H_1,H_1) \} = 0 \tag{5.86}$$

$$\frac{1}{2}\{ \frac{1}{3}R_3 + r(\rho,\lambda) \} + \frac{1}{4}\{ z(\rho,\lambda) + \frac{2}{3}r(\rho,\lambda).R_4 \} + \frac{1}{2}F(\rho,\lambda) - \frac{1}{4}\{ -g(H_0,H_0) + g(H_1,H_1) \} = 0 \tag{5.87}$$

$$\frac{1}{2}\{ R_3 + \frac{2}{3}r(\rho,\lambda)\} + \frac{1}{4}\{\frac{1}{3}z(\rho,\lambda) + \frac{4}{3}r(\rho,\lambda).R_4 \} + \frac{1}{6}F(\rho,\lambda)$$

$$-\frac{1}{4}\{\ g(\ H_0,H_0\ )\qquad\qquad\qquad\qquad\}\quad=\quad 0 \tag{5.88}$$

Manipulation of these equations allows us to solve for $g(\ H_0,H_0\ )$ and $g(\ H_1,H_1\ )$, we find

$$\frac{1}{4}g(\ H_0,H_0\ ) = \frac{1}{2}\{\ R_3+\frac{2}{3}r(\rho,\lambda)\ \} + \frac{1}{6}F(\rho,\lambda) + \frac{1}{4}\{\frac{1}{3}z(\rho,\lambda) + \frac{4}{3}r(\rho,\lambda).R_3\ \} \tag{5.89}$$

$$\frac{1}{4}g(\ H_1,H_1\ ) = \frac{1}{2}\{\quad \frac{1}{3}r(\rho,\lambda)\ \} + \frac{1}{3}F(\rho,\lambda) + \frac{1}{4}\{\frac{2}{3}z(\rho,\lambda) + \frac{2}{3}r(\rho,\lambda).R_3\ \} \tag{5.90}$$

We are left with one independent equation which for convenience we take to be (5.86)-(5.87). We find after substituting in the values of the $g(\ H,H\ )$s that this becomes

$$\frac{2}{3}R_3+\frac{1}{3}r(\rho,\lambda).R_3 + \frac{2}{3}r(\rho,\lambda) + \frac{1}{6}z(\rho,\lambda) + \frac{1}{3}F(\rho,\lambda) = 0 \tag{5.91}$$

we can solve for R_3

$$R_3 = -\{\ 2r(\rho,\lambda)+\frac{1}{2}z(\rho,\lambda) + F(\rho,\lambda)\ \}/\{\ r(\rho,\lambda)+2\ \} \tag{5.92}$$

The remaining equation is $dH = trR^2 - trF^2$ which is (as usual)

$$h_1^2 = \{\ k_0(\rho,\lambda) - l_i(\rho,\lambda)\ \}^2 \tag{5.93}$$

We now substituting away h_1^2 ($h_1^2 = (1/36n_2)g(\ H_1\ ,H_1)$) using (5.90) giving us

$$\frac{1}{3}r(\rho,\lambda) + \frac{1}{3}\{\ z(\rho,\lambda) + 2F(\rho,\lambda)\ \} + \frac{1}{3}r(\rho,\lambda).R_3(\rho,\lambda) = 18n_2\{\ k_0(\rho,\lambda) - l_i(\rho,\lambda)\ \}^2 \tag{5.94}$$

Now eliminating R_3 using (5.92) and multiplying by $\{\ r(\rho,\lambda) + 2\ \}$ will give us (with a little rearranging)

$$2r(\rho,\lambda) - r^2(\rho,\lambda) + 2\{\ z(\rho,\lambda) + 2F(\rho,\lambda)\ \} + \frac{1}{2}r(\rho,\lambda).z(\rho,\lambda) = 18n_2\{\ r(\rho,\lambda) + 2\ \}.\{k_0(\rho,\lambda) - l_i(\rho,\lambda)\ \} \tag{5.95}$$

Now we can substitute in the explicit form of all the functions and after dividing by λ^2 be left with a cubic polynomial in λ^2 which is of the form

$$A(\rho)\lambda^6 + B(\rho)\lambda^4 + C(\rho)\lambda^2 + D(\rho) = 0 \tag{5.96}$$

where the coefficients are given by

$$A(\beta) = 9/(24^2).(\beta^2-K_2)^2(\beta^2-2\beta-4) \qquad (5.97)$$

$$B(\beta) = -1/24.(\beta^2-2\beta-4).(\beta^4-2\beta^3-3\beta^2+8\beta+K_1) - 36/(24^2).(\beta^2-K_2)^2 \qquad (5.98)$$

$$C(\beta) = \tfrac{1}{3}(\beta^4-2\beta^3-3\beta^2+8\beta+K_1) - \tfrac{1}{4}(\beta^2-2\beta-4) \qquad (5.99)$$

$$D(\beta) = -(\beta^2-2\beta-4) \qquad (5.100)$$

The cubic polynomial will have for each value of β at least one root and possibly up to three. Defining the "radicant" (we are using the prescription specified in [39]) by

$$\text{RADICANT} = \tfrac{1}{4}\{ \tfrac{2}{27}(B/A)^3-(B.C)/A^2 +D/A \}^2 + \tfrac{1}{27}\{ C/A- (B/A)^2 \}^3 \qquad (5.101)$$

We find one,two,three roots if this is positive, zero, negative respectively. When it is positive our one root is given by

$$\lambda^2(\beta) = \sqrt[3]{\{ -\tfrac{1}{27}(B/A)^3 - \tfrac{1}{6}(B.C)/A^2 - \tfrac{1}{2}D/A + \sqrt{\text{RADICANT}} \}} - \tfrac{1}{3}B/A \qquad (5.102)$$

When the radicant is negative we find three roots-if we first define r and ϕ by

$$r = \sqrt{\{ -\tfrac{1}{27}(C/A - \tfrac{1}{3}(B/A)^2)^3 \}} \qquad (5.103)$$

$$r\cos(\phi) = -\tfrac{1}{2}(\tfrac{2}{27}(B/A)^3 -(B.C)/A^2+D/A) \qquad (5.104)$$

then our three solutions are given by

$$\lambda_1^2(\beta) = 2\ r^{\frac{1}{3}}.\cos(\tfrac{1}{3}\phi) - \tfrac{1}{3}B/A$$

$$\lambda_2^2(\beta) = 2\ r^{\frac{1}{3}}.\cos(\tfrac{1}{3}\phi + \tfrac{2\pi}{3}) - \tfrac{1}{3}B/A$$

$$\lambda_3^2(\beta) = 2\ r^{\frac{1}{3}}.\cos(\tfrac{1}{3}\phi + \tfrac{4\pi}{3}) - \tfrac{1}{3}B/A \qquad (5.105)$$

The solutions of λ^2 as a function of β are shown on Diagram 5.7 for the case F=0 on SU(3)/{U(1)xU(1)}. As can be seen it is quite complicated with many branches. We require that λ^2, h_0^2 and h_1^2 be positive. Requiring $\lambda^2 > 0$ rules out a few branches of solution. Requiring the hs be positive rules out a large number leaving only those two branches shown in red on Diagram 5.7. We present in Diagrams 5.8 and 5.9 a more detailed desciption of the behaviour of the functions in these regions . We also graph $R_3(\beta)$ and $r(\beta)$. This is only for SU(3)/{U(1)xU(1)} with F=0. For the other cases we find a similar pattern. In diagrams 5.10 and 5.11 we give the solutions for $G_2/SU(3)$ with F=0 and in 5.12 and 5.13 we present the solutions for the case F2 on the three spaces. As can be seen we have two branches for each solution one for the positive region and one in the negative. In 5.13 we have a point where $h_0 = R_3 = 0$ this is the special case of our Minkowski solution. For most of the RxS^3/HS^3 solutions we have $R_3>0$ this means we are dealing with a three-sphere in space-time. In these cases we find $r<0$ and $|(R_3+r)| \ll |r|$ or $|R|$. So g(H,H) will be small. The exception is given in Diagram 5.12 ,Which is for the F2 cases, where both R_3 and r are negative (so space-time will be $RxHS^3$) and we do not find g(H,H) small. This unusual solution has M_4xCoset space as a limiting case. (This is neccesary since we found M_4 as a solution for this case earlier). For the case of our Yang-Mills fields taking the form F3 we find a similar pattern but to find solutions we must apply (5.32). We can rewrite this as

$$H(\beta) = 9h_1^2/\lambda^2 = 1 \qquad (5.106)$$

When we examine $H(\beta)$ for SU(3)/U1)xU(1) we find that $H(\beta) \neq 1$ for any value of β for which the other positivity constraints are satisfied so we do not have any valid solutions. For $G_2/SU(3)$ however we find that for one of the branches of $\lambda^2(\beta)$,where all the positivity constraints are satisfied, that there exists a value of β for which $H(\beta)=1$. On Diagram 5.14 we give λ^2, h_0^2, h_1^2, R_3, and r, on this region and in Diagram 5.15 we give $H(\beta)$ and the details of the root.

In chapter 3 we discussed if we had a solution F then we could also have solutions FxFxF.. Can this also occur for non-symmetric coset spaces ? The answer is not clear immediately - the F fields we are dealing with are not,as for the groups, topologically trivial so the coefficients are not arbitrary so we are not allowed to change the coefficient by $\frac{1}{\sqrt{2}}$ as we did in chapter 3 (p54). Making our field FxFxF... (n-Fs) would have the effect of introducing n in front of $F(\beta,\lambda)$ everywhere ,We could incorporate this into the normalisation factor χ^2. Explicit analysis of the effect on increasing in this manner shows very little difference. Solutions still exist (although with different values) whereever they existed before. So in actual fact we can have multiple factors of a given F field just as in chapter 3.

In the next chapter it will be of interest to take Yang-Mills fields ,on SU(3)/U(1)xU(1), where we have a Yang-Mills field (U(1)xU(1)) but with imbedding such that (effectively) $\chi^2 = 8$. For this special case we present $G(\beta)$ on Diagram 5.16 (recall that to have Minkowski 4-D space-time as a solution we needed $G(\beta)$ to have roots where h^2 and λ^2 were positive). Since this has roots where the positivity conditions are satisfied Minkowski 4-D space-time is definately a valid solution for this special Yang-Mills field.

In summary we find a large class of solutions to the equations of motion for our non-symmetric coset spaces. A summary is given on Table 5.1. In the next chapter we shall try to analyse the consequences of these solutions.

We shall now determine whether H=T at any of our solutions. The condition

$$T_{abc} = \pm\sqrt{3} \quad H_{abc} \tag{5.107}$$

reduces ,for our ansatz, to

$$(1-\rho)\frac{\lambda}{2} = \pm\sqrt{3} \; h \tag{5.108}$$

or

$$(1-\rho)^2 \frac{\lambda^2}{4} = 3h^2 \tag{5.109}$$

Dividing the RHS by the LHS and substituting in h^2 from (5.45) we find

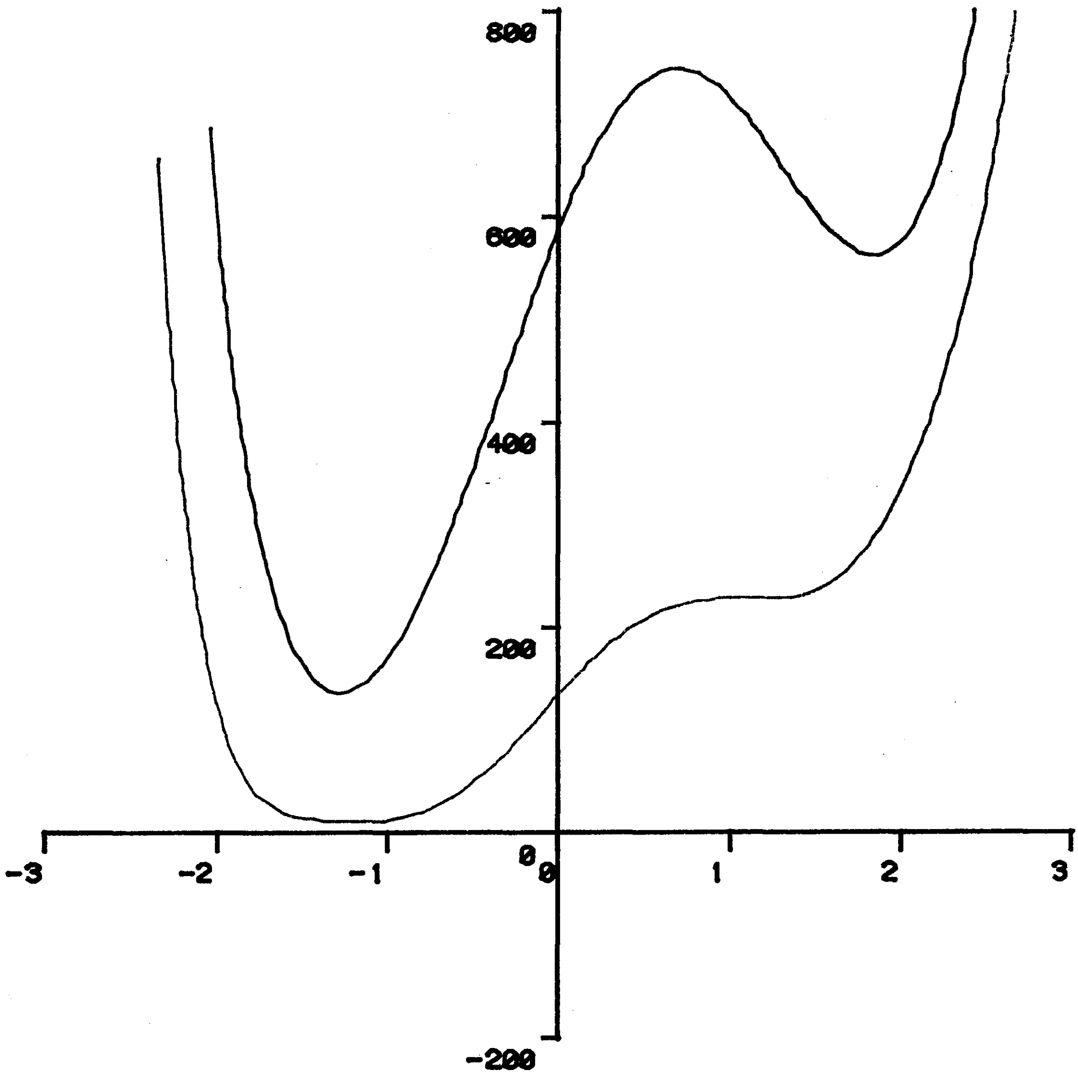
$$\frac{\lambda^4(\rho)(\rho^2-K_2)^2}{48(1-\rho)^2} = 1 = M(\rho) \tag{5.110}$$

So we must determine whether $M(\rho)=1$ at any of our solutions. For M_4 and AdS/dS 4-D space-time it is unlikely that we will find (5.110) satisfied at our single points and indeed by inspection of the solution this is the case. When we look at the case of RxS^3/HS^3 4-D space-time we have three cases to consider. Namely F=0 for SU(3)/U(1)xU(1) and G_2/SU(3) and F being case F2 for all three cosets. We have two branches of solution in each case. We plot $M(\rho)$ for the three -ve branches on Diagram 5.17 and for the three +ve branches on Diagram 5.18. As we can see we have only one place where $M(\rho)=1$. This is for the F2 case. So for all three coset spaces we have a single point where H=T.

Table 5.1 Summary of Solutions

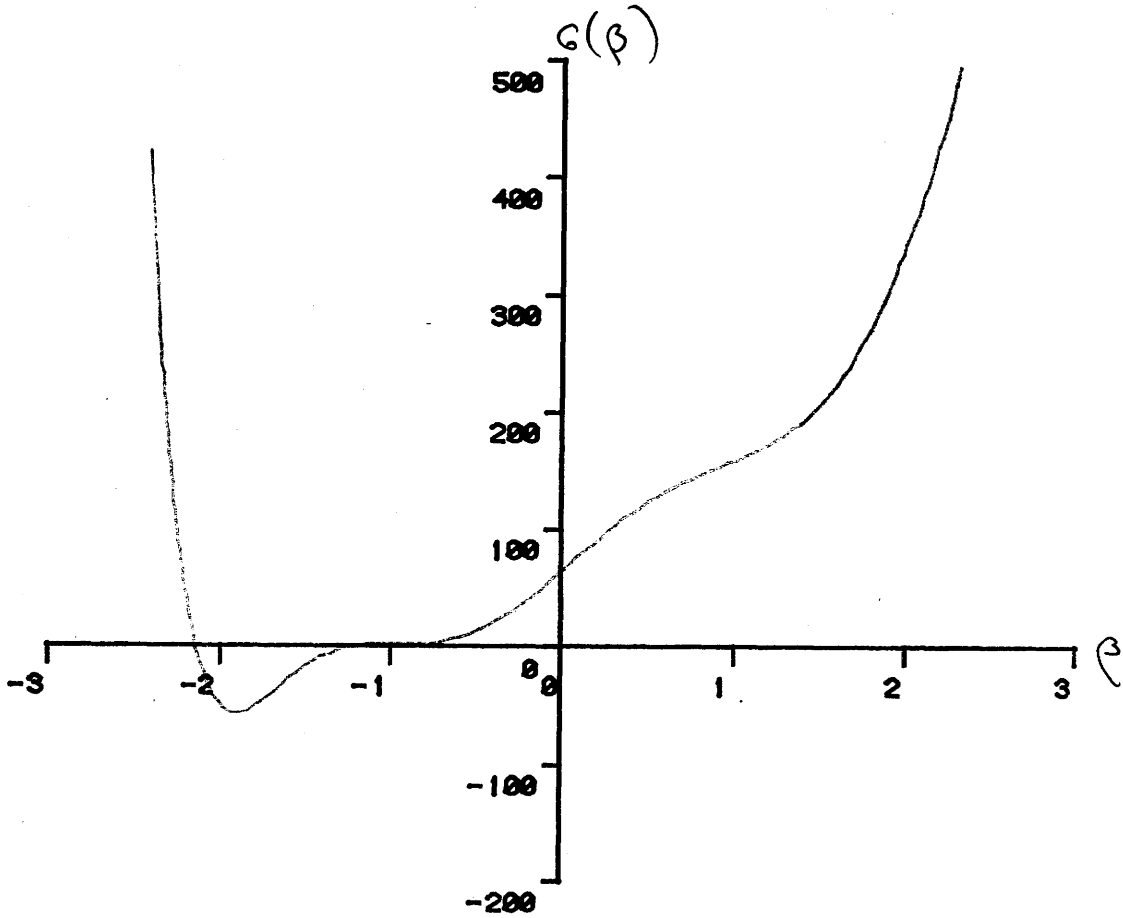
Case	Existence of solutions
Minkowki space time F1,F3 $G_2/SU(3)$ $SU(3)/U(1)xU(1)$	No solutions exist
Minkowski space-time F2 All three spaces	A single solution for β =-2.13
AdS/dS F1 $SU(3)/\{U(1)xU(1)\}$ $G_2/SU(3)$	Four solutions space-time both AdS & dS Three solutions space-time both Ads & dS
Ads/dS F2 All three spaces	Two solutions space-time both Ads & dS
Ads/dS F3 $G_2/SU(3)$ $SU(3)/U(1)xU(1)$	No solutions exist
R x three hypersphere F1 $G_2/SU(3)$ $SU(3)/U(1)xU(1)$	Solutions exist in in one parameter families which we take as In each case solutions exist for β in two small regions one -ve and one +ve eg for $SU(3)/U(1)xU(1)$ solns exist for $-1{\cdot}7 < \beta < -1{\cdot}27$ and $3{\cdot}3 < \beta < 4{\cdot}9$
R x three hypersphere F2 all three spaces	Solutions exist as above for $-2{\cdot}13 < \beta < -1{\cdot}26$ and $3{\cdot}26 < \beta < 4{\cdot}8$
R x three hyprsphere F3 $SU(3)/U(1)xU(1)$ $G_2/SU(3)$	no solutions exist a single solution exists

Diagram 5.1 G(β) For F=0 Both Coset Spaces



We can see that there are no solutions to the equation G(β)=0 for either of the relevent coset spaces

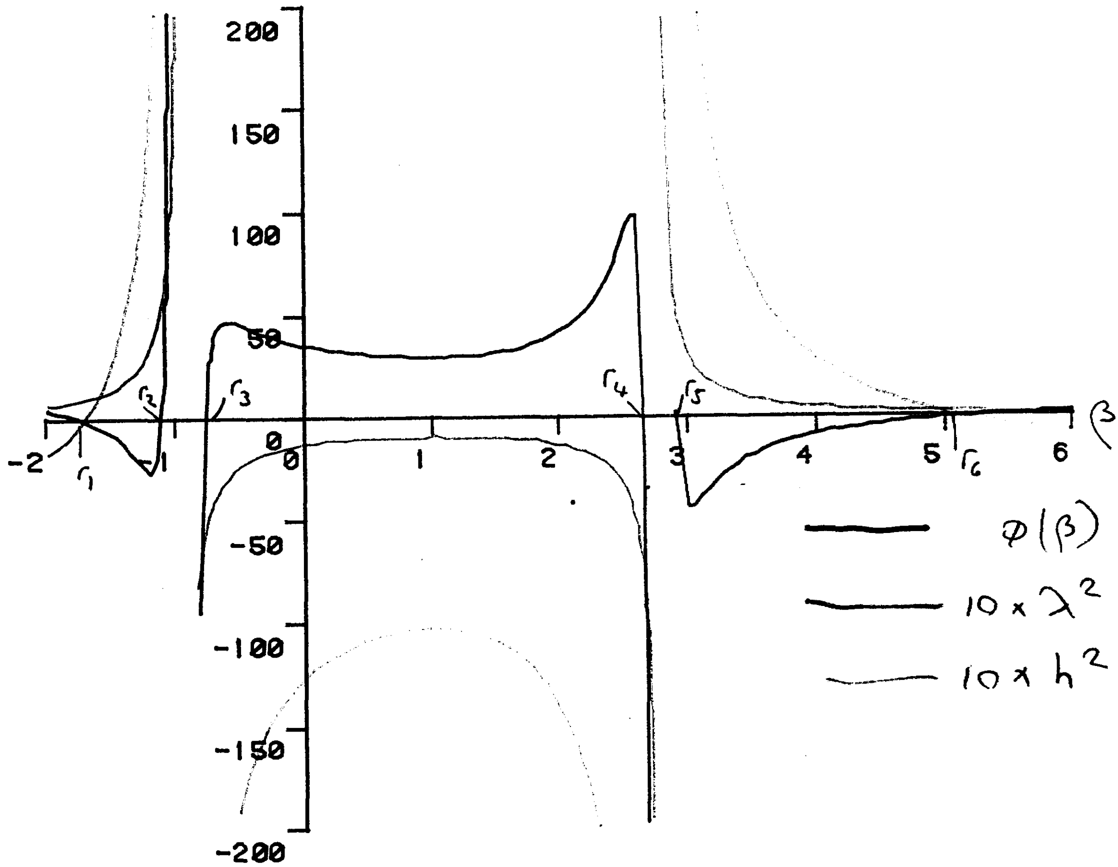
Diagram 5.2 $G(\beta)$ For the Yang-Mills field being case F2



As can be seen we have two roots to the equation $G(\beta)=0$ at $\beta=-1.3$ and at $\beta=-2.13$

At -1.3 we do not find $\lambda^2 \geqslant 0$ and $h^2 \geqslant 0$ so we do not have a valid solution. At -2.13 however we find $\lambda^2=3.3$ and $h^2=16.0$ so we have a single valid solution at $\beta=-2.13$

Diagram 5.3 AdS/dS Solutions for SU(3)/U(1)xU(1) With F=0 taking the positive sign in Equation (5.83)



As can be seen there are six roots r1-r6

At r3 & r4 both λ^2 and h^2 are -ve

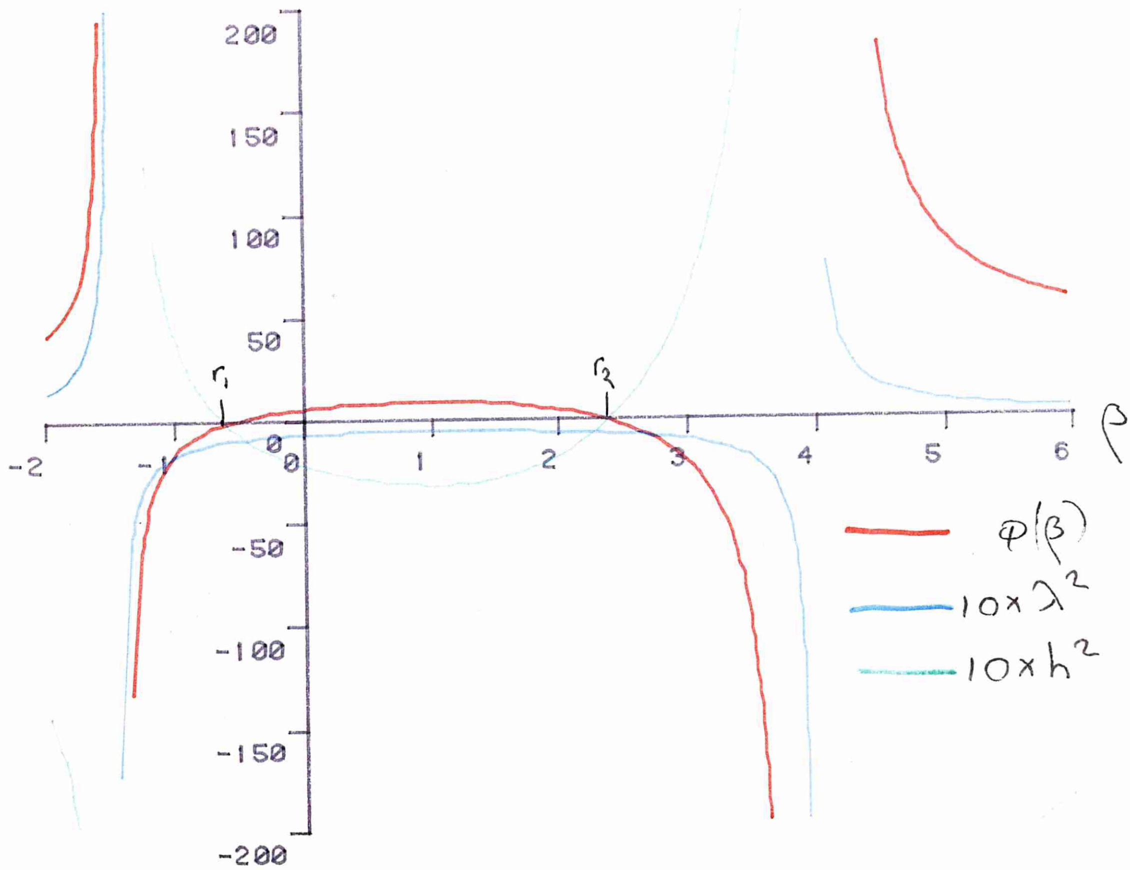
At r2 & r5 both λ^2 and h^2 are +ve

At r1 and r6 (not clear from graph) λ^2 & h^2 are +ve

So we have four good roots to the equn $Q(\beta)=0$. At the roots the functions take the following values

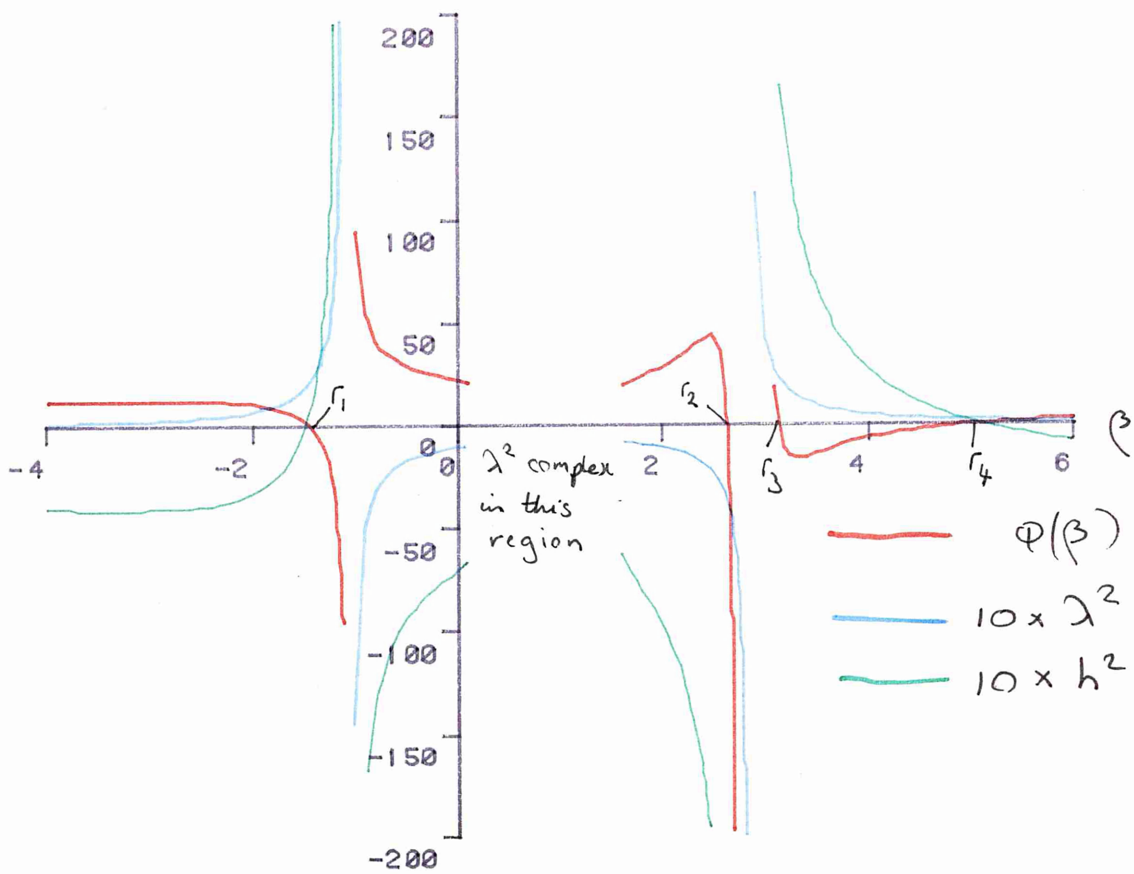
	β	λ^2	g(H,H)	R_4	r
r1	-1.73	1.1	0.019	1.188	-1.198
r2	-1.12	5.5	26.1	-14.6	1.43
r5	2.895	2.89	4.88	-26.6	3.43
r6	5.09	0.205	0.084	1.150	-1.192

Diagram 5.4 Ads/dS solutions for SU(3)/U(1)xU(1) for F=0 and choosing the -ve sign in Equation (5.83)



There are two roots, r1 and r2, at both of these $\lambda^2 < 0$ so there are no consistent solutions

Diagram 5.5 AdS/dS solutions for G2/SU(3) case F=0



There are four roots r1-r4

At r2 λ^2 & h^2 are both +ve

At r3 λ^2 & h^2 are both -ve

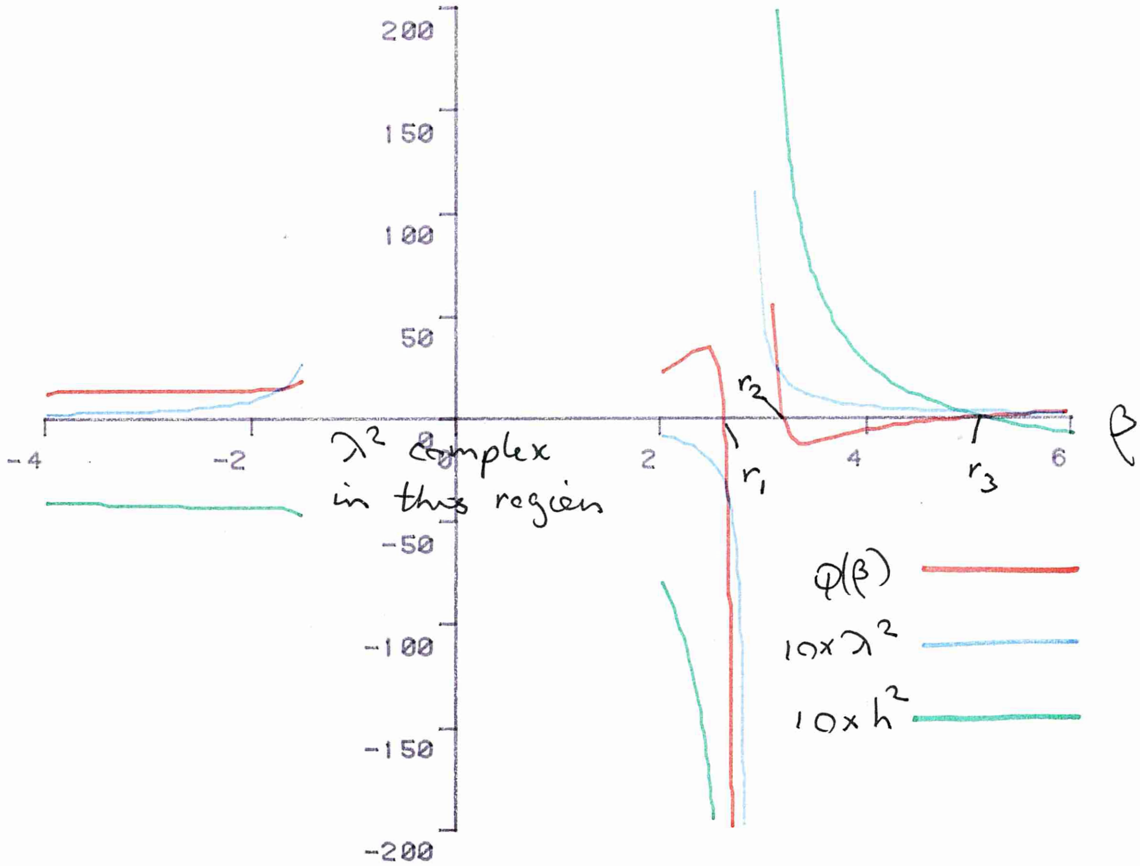
At r1 and r4 (not clear from diagram) λ^2 & h^2 are +ve

So we have three good roots to the equn $Q(\beta)=0$. At the roots the functions take the following values

	β	λ^2	g(H,H)	R_4	r
r1	-1.474	2.095	0.265	1.042	-1.173
r3	3.093	2.32	19.17	-10.31	0.718
r4	5.016	0.2134	0.1184	1.129	-1.188

Diagram 5.6 Ads/dS Solutions for Case F2

(all three spaces)



We have three roots to $Q(\beta)=0$

At r1 λ^2 & h^2 are both +ve

At r2 λ^2 & h^2 are both -ve

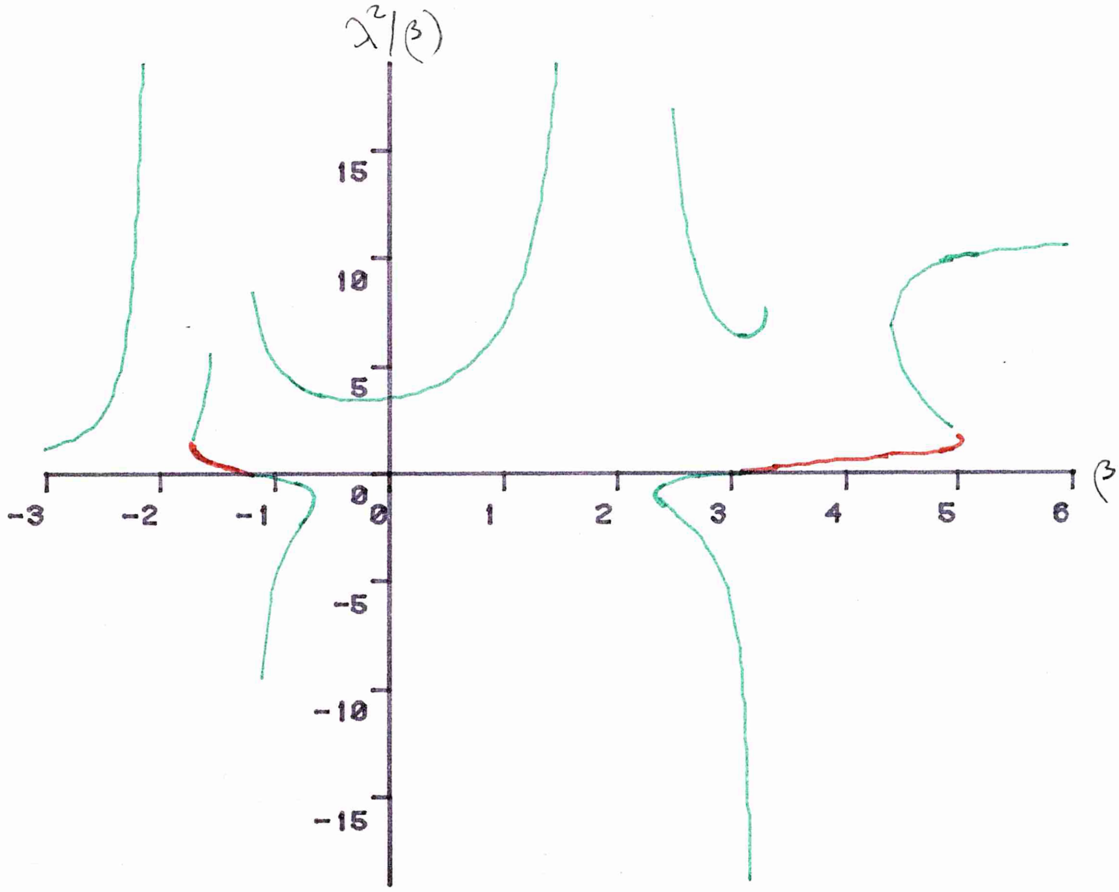
At r3 (not clear from diagram) λ^2 & h^2 are both +ve

So we have two valid solutions. At these the functions take the following values

	β	λ^2	g(H,H)	R_4	r
r2	3.16	1.945	15.25	-7.950	0.0325
r3	4.989	0.2176	0.1324	1.120	-1.1865

Diagram 5.7 $\lambda^2(\beta)$ For $R\times S^3/HS^3$, $SU(3)/U(1)\times U(1)$

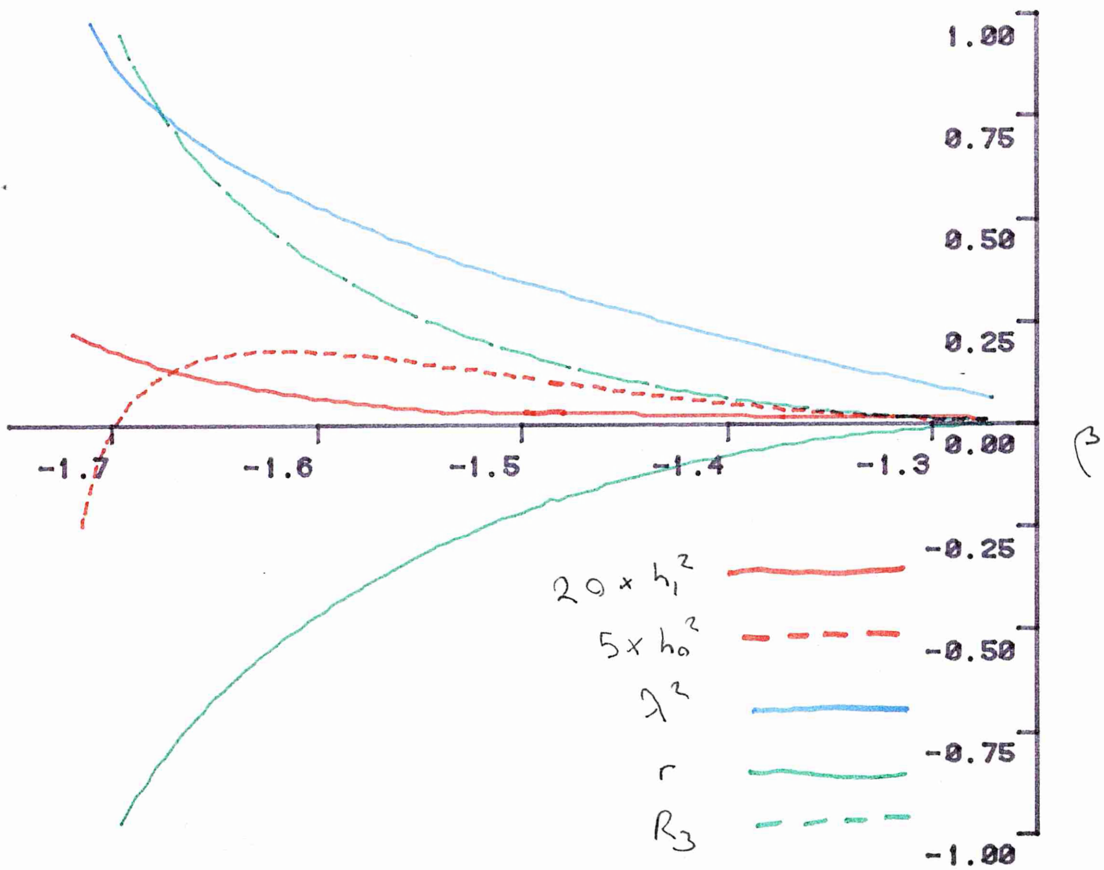
F = 0



The branches in red are those where the positivity conditions on λ^2, h_1^2, and h_0^2 are satisfied. Some of the branches have been scaled up/down to enable them to fit on the same diagram.

Diagram 5.8 Solutions For RxS^3/HS^3,F=0 Region A

For SU(3)/U(1)xU(1) case

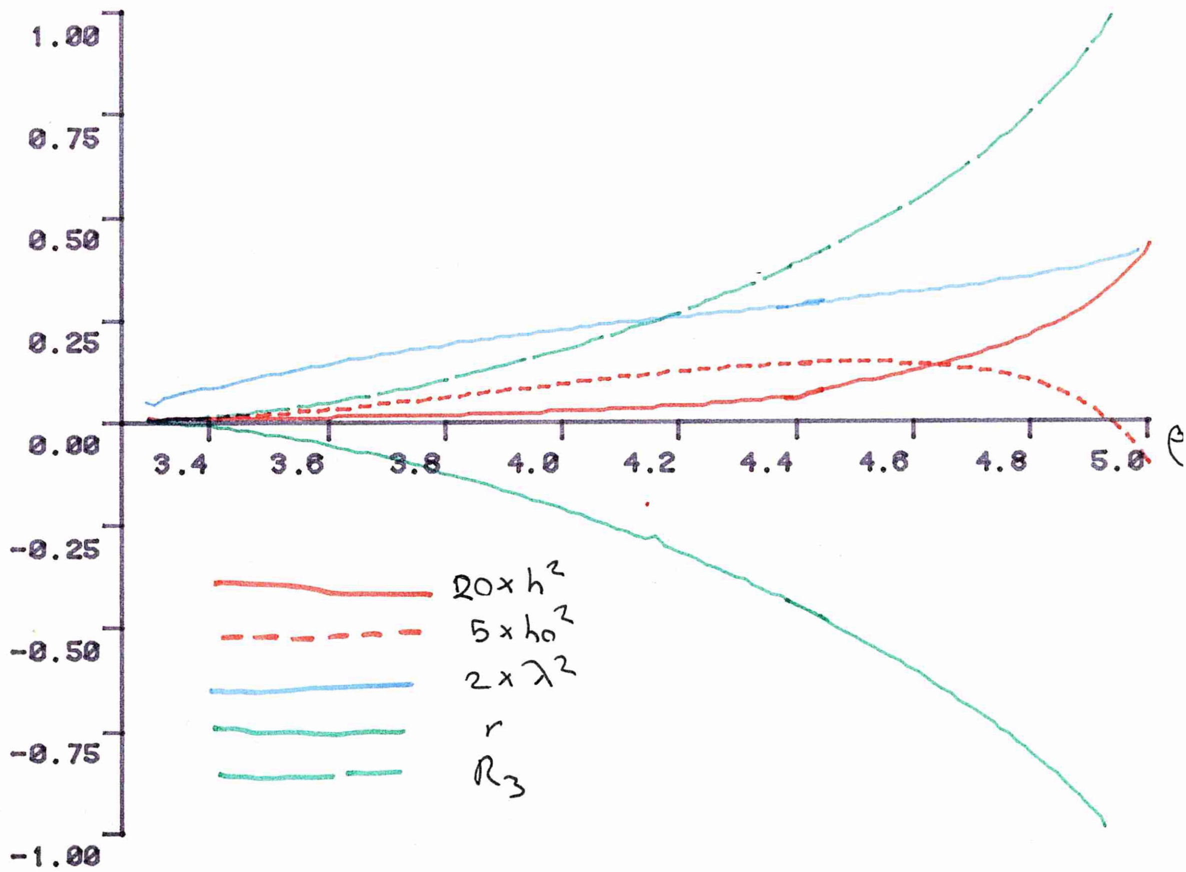


The solutions are a one-parameter family which we take as β. Solutions exist for $-1.7 < \beta < -1.27$. The variables h_0^2, h^2, λ^2, R_3, and r are given on this graph as as function of β.

Since $R_3 < 0$ our solutions have 4-D space-time as (3-D Hypersphere)xR rather than (3-D sphere)xR.

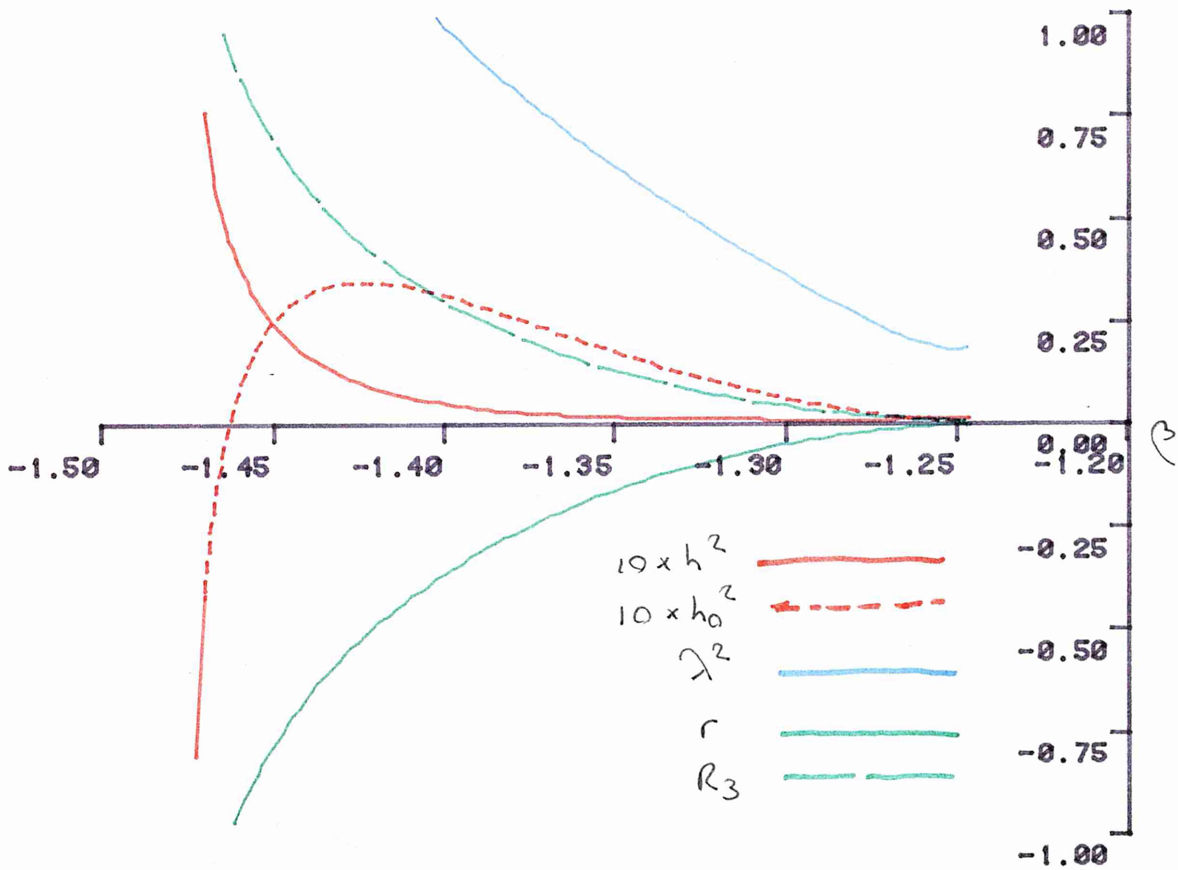
Diagram 5.9 Solutions For RxS^3/HS^3, F=0 Region B

For SU(3)/U(1)xU(1) case



The solutions are a one-parameter family which we take as β. Solutions exist for $3{\cdot}3 < \beta < 4{\cdot}9$. The variables h_0^2, h^2, λ^2, R_3, and r are given on this graph as as function of β.

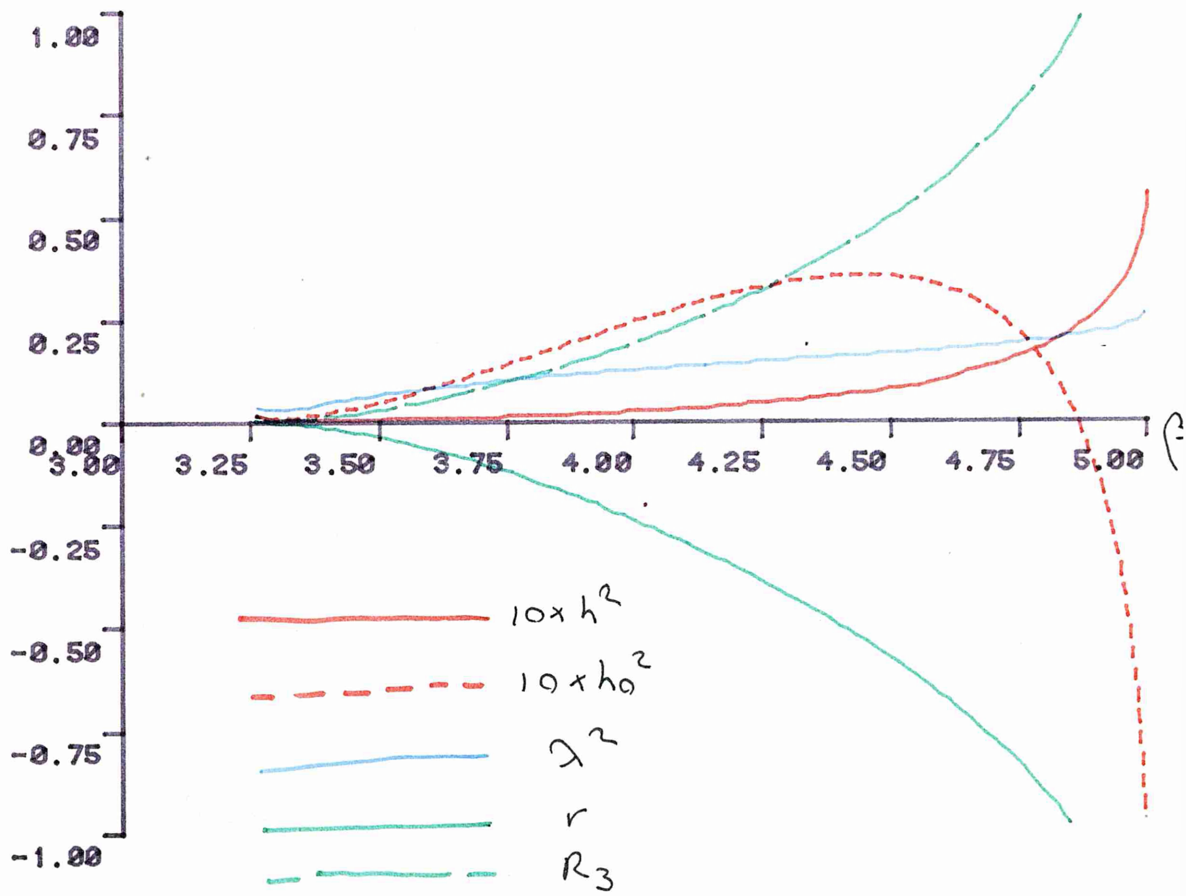
Diagram 5.10 Solutions For $RxS^3/HS^3, F=0$ Region A For $G_2/SU(3)$ case



The solutions are a one-parameter family which we take as β. Solutions exist for $-1.46 < \beta < -1.24$. The variables h_0^2, h^2, λ^2, R_3, and r are given on this graph as as function of β.

Diagram 5.11 Solutions For RxS^3/HS^3, $F=0$ Region B

For $G_2/SU(3)$ case

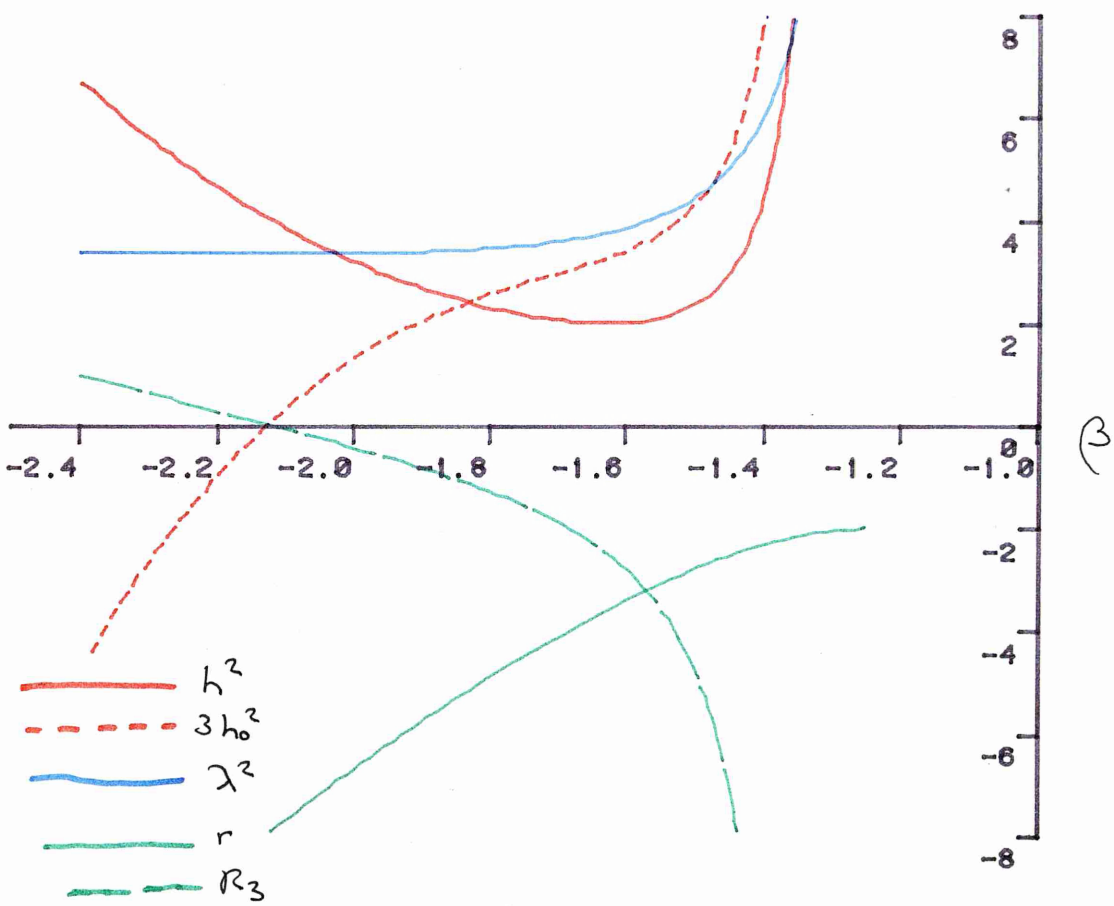


The solutions are a one-parameter family which we take as β. Solutions exist for $3{\cdot}26 < \beta < 4{\cdot}9$. The variables h_0^2, h^2, λ^2, R_3, and r are given on this graph as as function of β.

Diagram 5.12 Solutions For RxS^3/HS^3,F a F2 field

For all three Coset spaces

Region A



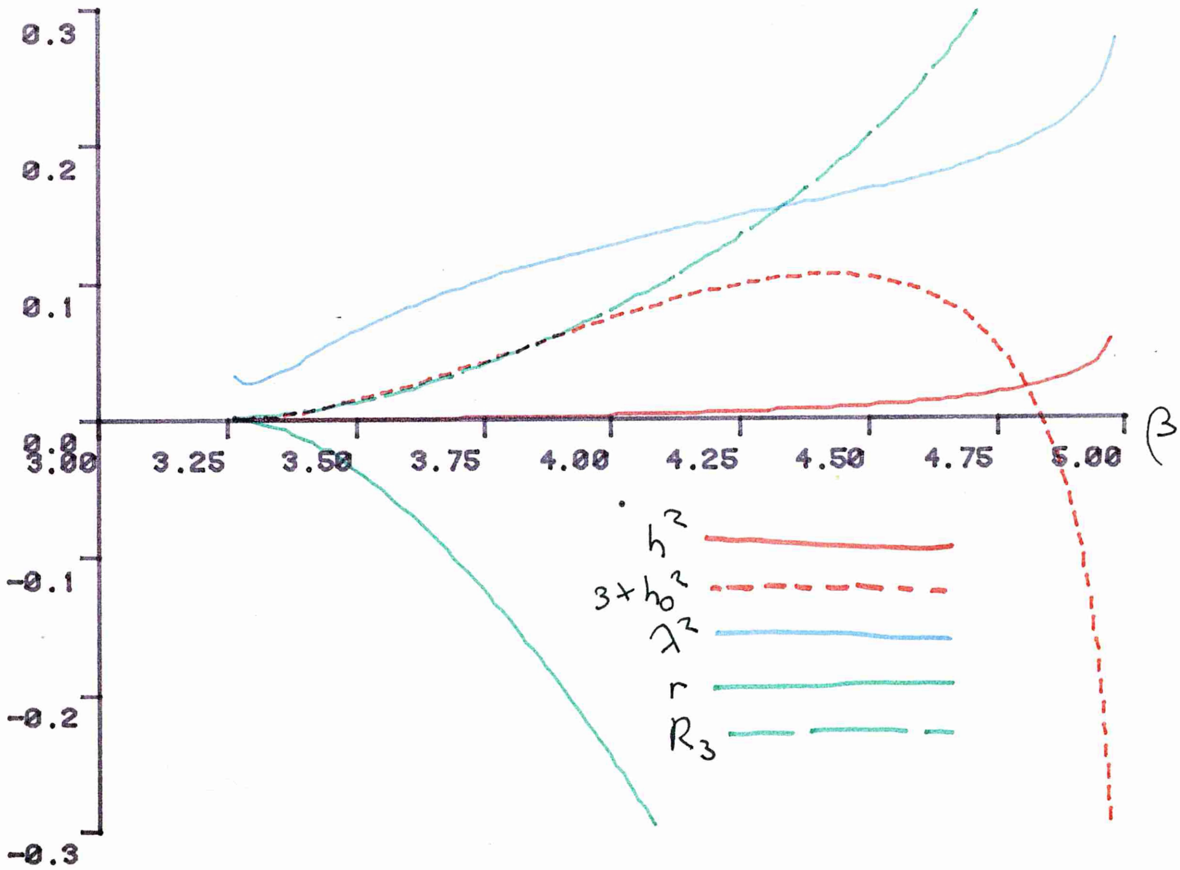
The solutions are a one-parameter family which we take as β. Solutions exist for $-2.15 < \beta < -1.24$. The variables h_0, h^2, λ^2, R_3, and r are given on this graph as as function of β.

Notice that there is a point where $h_0 = R_3 = 0$ and we reduce to our Minkowski space-time solution

Diagram 5.13 Solutions For $R \times S^3/HS^3$, F a F2 field

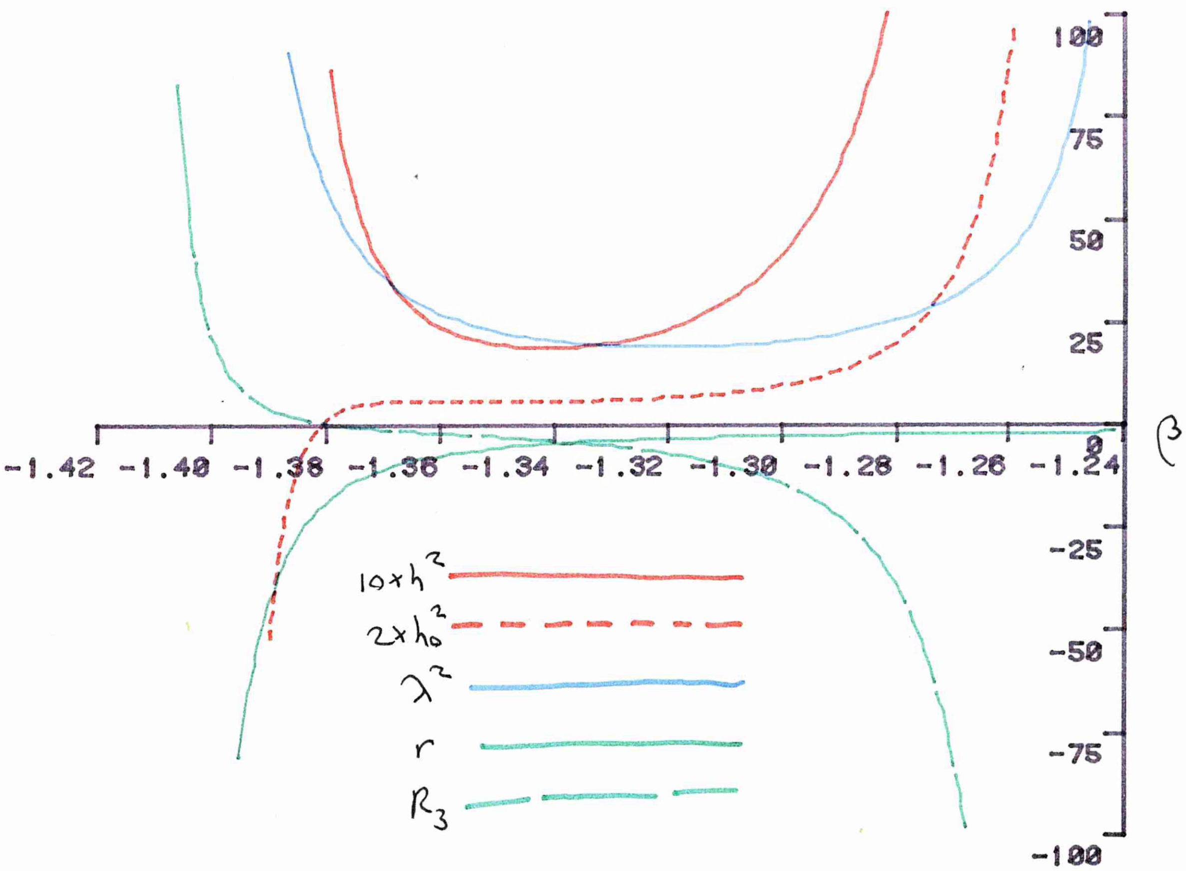
For all three Coset spaces

Region B



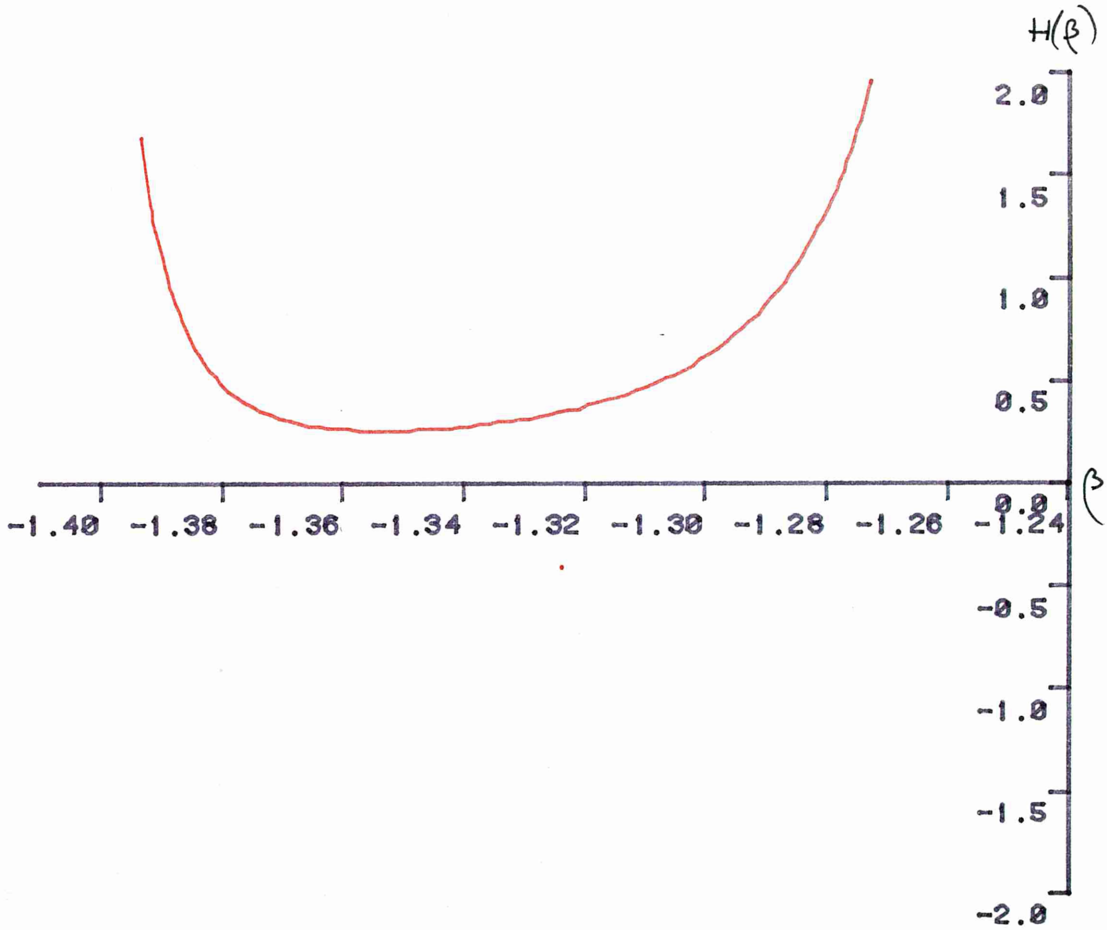
The solutions are a one-parameter family which we take as β. Solutions exist for $3{\cdot}26 < \beta < 4{\cdot}8$. The variables h_0, h^2, λ^2, R_3, and r are given on this graph as as function of β.

Diagram 5.14 Solutions For RxS^3/HS^3, F a F3 field For $G_2/SU(3)$



This is a graph of the functional dependance of h_0^2, h^2 λ^2, R_3, and r on β for the Yang-Mills being a F3 field (ie for coset spaces S/R F is a S field) We have still got the constraint (5.106) to be satisfied so these are not solutions. In Diagram 5.15 we show this constraint is satisfied at $\beta = -1.285$.

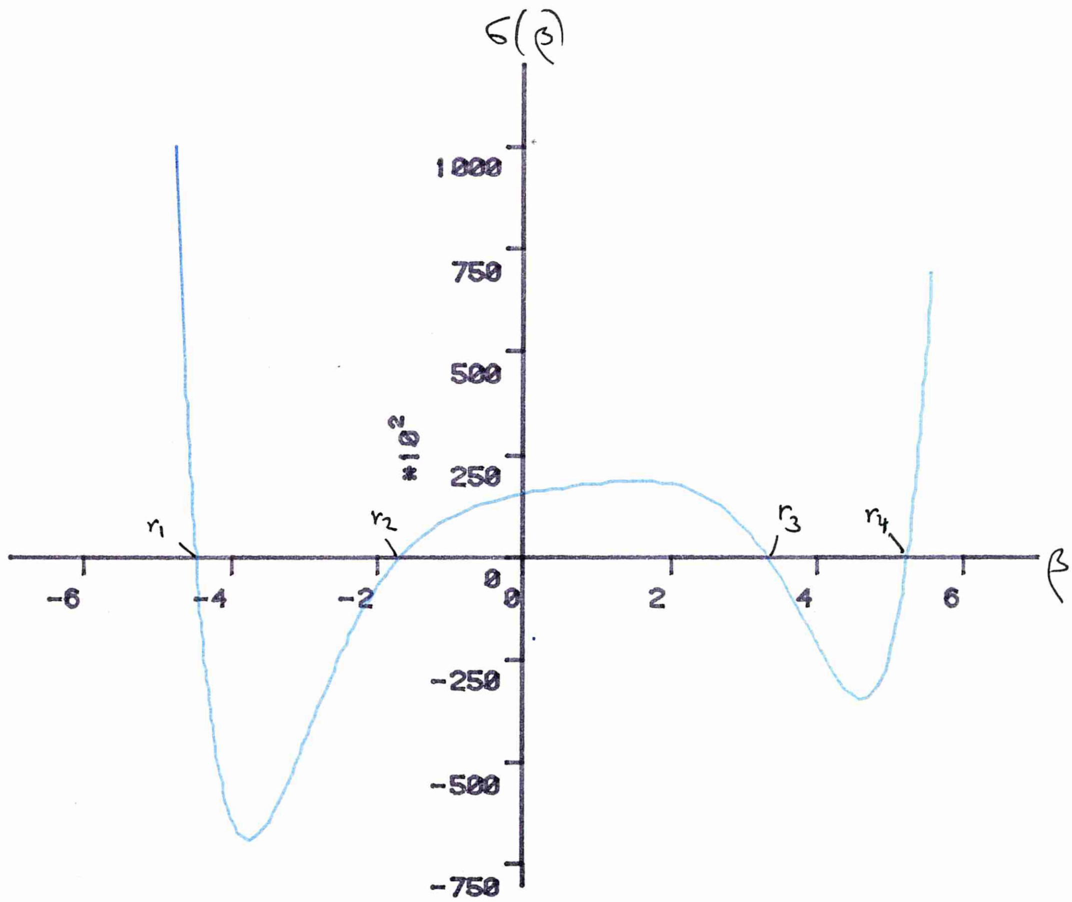
Diagram 5.15 $H(\beta)$ for $G_2/SU(3)$ for the region shown in Diagram 5.14



As we can see there are two values where $H(\beta)=1$. At one of these the positivity condition on h^2 is not satified so we have no solution. At the remaining point $\beta = -1.285$ all the positivity conditions are satisfied and we will have a valid solution.

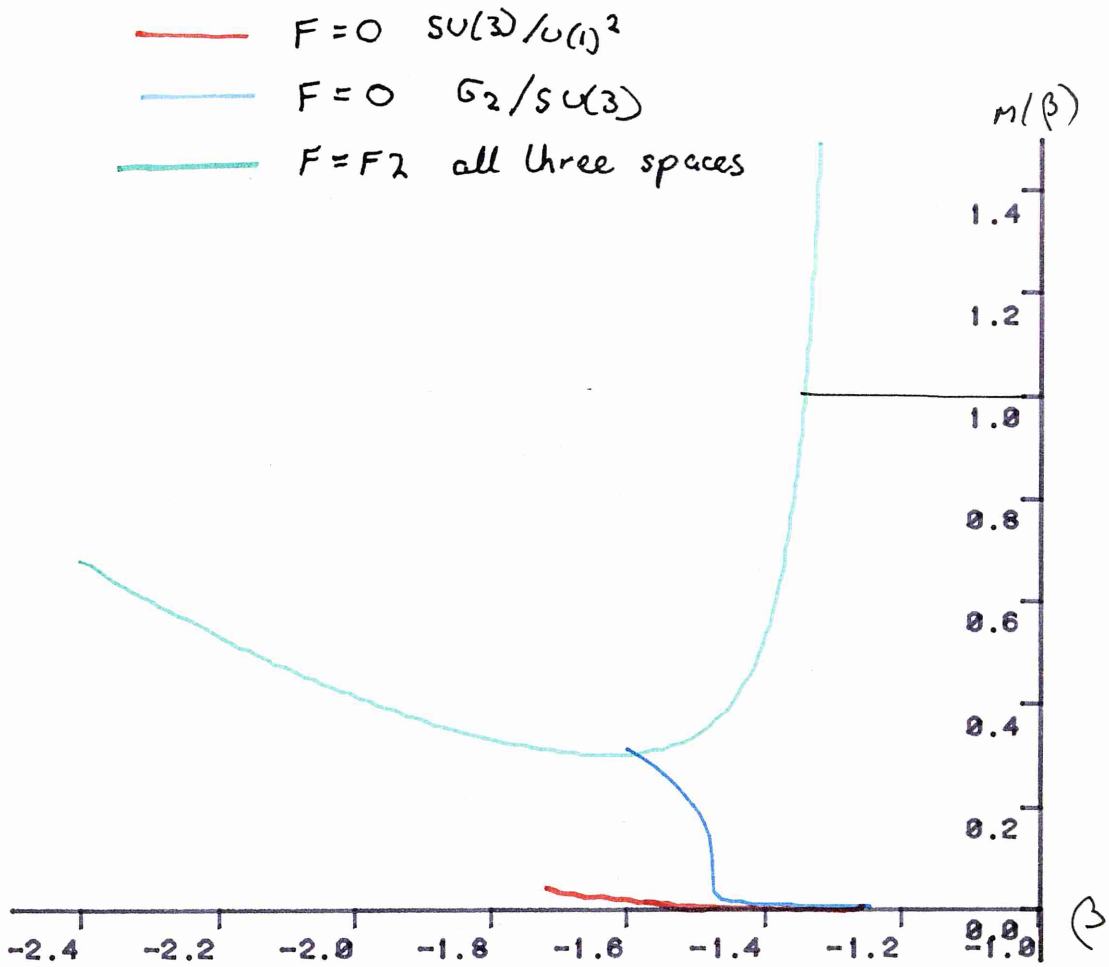
At this solution both R_3 and r are negative.

Diagram 5.16 $G(\beta)$ For $SU(3)/U(1)xU(1)$ with a multiple (8-times) imbedding of $U(1)xU(1)$



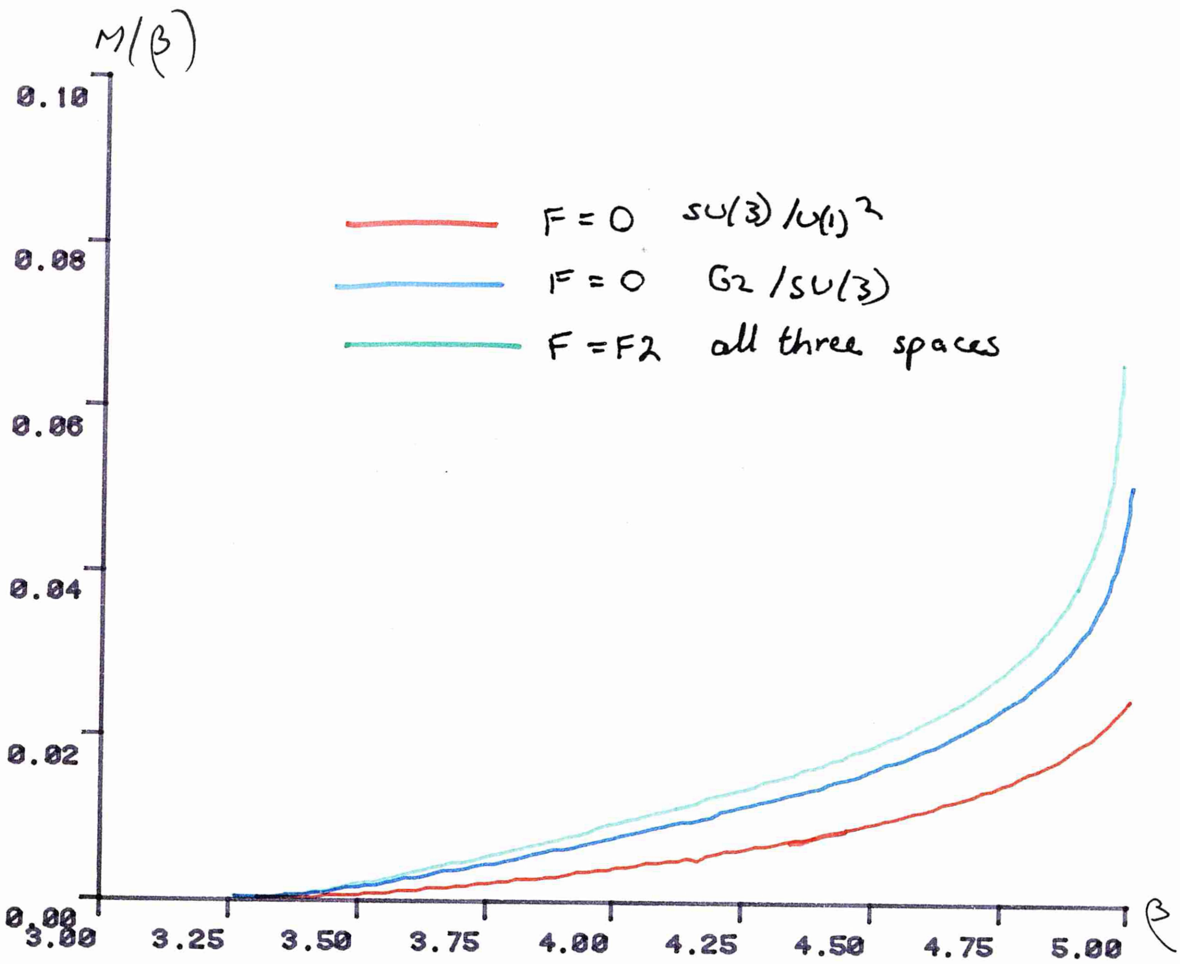
In Chapter 6 it is of interest whether a multiple imbedding of the U(1)xU(1) field will still have solutions. In particular we wish to know if Minkowski solutions exist for a eight-fold imbedding. As can be seen we have four roots to the equation $G(\beta)=0$ r1-r4. THe positivity conditions on λ^2 and h^2 are satisfied at r_1 and r_4 So we have two valid solutions.

Diagram 5.17 $M(\beta)$ For the three cases on the -ve branch



As we can see we have a single point where $M(\beta)=1$. This is for the case of all three coset spaces have field F2.

Diagram 5.18 $M(\beta)$ for the three cases on the +ve branch



As we can see there are no solutions where $M(\beta)=1$ (At the end of the range where it looks as if the curves will turn up to reach one in actual fact h_0^2 becomes negative before ths occurs)

Chapter 6 Physics from Non-Symmetric Coset Spaces

We shall now examine the consequences of our solutions from chapter 5. We shall exclude some possibilities first.

If $F=0$ then we will have ,upon compactification to four dimensions, non-chiral fermions. Also the 4-D gauge group will be SO(32) or $E_8 x E_8$. In actual fact E_8 is a reasonable unification group in its own right [40] as is E_7 ($SU(5) \approx E_4$ and $SO(10) \approx E_5$ so all the E_4-E_8 are possible unification groups !). However we need chiral representations at the unification scale so an unbroken E_8 will be no good. It is possible that some other symmetry breaking mechanism will occur between the compactification scale and the unification scale. However we have no positive suggestions to make for a realistic scenario as to how this takes place. So this is not a realistic picture so we shall not consider $F=0$ further.

For the case F3 (ie for coset space S/R we imbed F as S) we have a single solution for the case $RxHS^3$ with $G_2/SU(3)$. This single solution is not terribly attractive since we do not have M_4 as a solution. When we consider the Yang-Mills field we find $Tr(F \wedge F \wedge F$)is zero and $Tr(F \wedge F$)is a total derivative so F is topologically trivial and we will not have chiral fermions in 4-D. The Yang-Mills symmetries will however be broken. eg imbedding G_2 within E_8 yields , amongst others, SU(2)xSp(6), SU(3)xSU(3) and F_4

depending on how the imbedding is done. The non-chiral nature of the fermions is a major problem and we will not consider this case further.

So we shall concentrate on the case where F is a F2 field ie for coset space S/R we imbed F as R. We still have three different theories to consider $SO(32)/Z_2$, SO(32) and $E_8 x E_8$. We shall carry out calculations with $E_8 x E_8$ first returning to the others briefly later. The consequences of a solution depend very much upon the particular imbedding of the gauge field within the overall gauge group. We shall not attempt to classify completely the imbeddings as these are very numerous! (especially when we have U(1)s to consider) but shall be selective looking only at physically hopeful imbeddings. We are most interested in imbeddings which will lead to one of the possible unification groups . The best candidates for a unification group are E_6 ,SO(10) and SU(5). We would like to obtain three or four families of chiral fermions belonging to the appropriate representation of the unification groups. (The fermions originally lie in the adjoint of $E_8 x E_8$ or SO(32)). These are

E_6	27
SO(10)	16
SU(5)	$10 + \bar{5}$

When we imbed our fields within $E_8 x E_8$ we must imbed any simple group entirely within one E_8 or the other. With non-simple groups like SU(2)xU(1) we can put the U(1) in one E_8 and the SU(2) within the other. However it proves that keeping the Yang-Mills group within a single E_8 gives better results. When we imbed our Yang-Mills fields within one of the E_8s we will obtain at low energy some smaller group which we take as the physical 'visible' fields. The other E_8 will be unbroken at the compactification scale but presumably not at low energies. These fields will only interact with the 'visible' fields gravitationally and will be unobservable otherwise. These fields have been termed 'dark matter' and their existence is not incompatable with cosmological evidence. So we will concentrate on the 4-D fields obtained from breaking a single E_8. We shall look at our three coset spaces in turn first $G_2/SU(3)$.

$G_2/SU(3)$

Our Yang-Mills field is a SU(3) field. Imbedding SU(3) into E_8 breaks the E_8 symmetry down to E_6 ($SU(3)xE_6$ is a maximal subgroup of E_8). Under this imbedding the $\underline{248}$ of $\underline{E_8}$ (this is the adjoint representation). breaks as follows

$$\underline{248} \longrightarrow (\underline{1},\underline{78}) \oplus (\underline{8},\underline{1}) \oplus (\underline{3},\underline{27}) \oplus (\underline{\bar{3}},\underline{\overline{27}}) \qquad (6.1)$$

We are primarily interested in the 27s. The imbalance between massless 27s and $\overline{27}$s in four dimensions is given by the imbalance between massless 3s and $\bar{3}$s of SU(3) in the background field of the internal space. This is given by the index theorem for a six dimensional manifold

$$n_+ - n_- = \frac{1}{6}\int(3c_3 -3c_2c_1 +c_1^3) - \frac{1}{24}\int p_1c_1 \qquad (6.2)$$

Where c_i is the ith Chern class ie [35]

$$c_1 = \frac{i}{2\pi}\mathrm{Tr}(F)$$

$$c_2 = \frac{1}{8\pi^2}\left(\mathrm{Tr}(F\wedge F) - \mathrm{Tr}(F)\wedge\mathrm{Tr}(F)\right)$$

$$c_3 = \frac{-i}{24\pi^3}\Big(-2\mathrm{Tr}(F\wedge F\wedge F) + 3\mathrm{Tr}(F\wedge F)\wedge\mathrm{Tr}(F) - \mathrm{Tr}(F)\wedge\mathrm{Tr}(F)\wedge\mathrm{Tr}(F)\Big) \quad (6.3)$$

and p_1 is the first Pontrjagin class

$$p_1 = \frac{1}{4\pi^2}\mathrm{Tr}(R\wedge R) \quad (6.4)$$

Using these we can rewrite (6.3)

$$n_+ - n_- = -\frac{1}{24\pi^3} i\int \mathrm{Tr}(F\wedge F\wedge F) \quad -\frac{1}{24}\int p_1 . c_1 \quad (6.5)$$

The trace is in the 3 of SU(3) for this case.

Now $G_2/SU(3) \simeq S^6$ [42] so the first Pontrjagin class p_1, which is zero for S^6, must be zero for $G_2/SU(3)$. c_1 and c_2 are zero for a SU(3) bundle also so we have

$$n_+ - n_- = 1/2 \int c_3 \quad (6.6)$$

When $\beta=0$ F=R and this chern class is identical to the Euler class. The Euler characteristic is 2 for a six sphere so we obtain

$$n_+ - n_- = 1 \quad (6.7)$$

For different β this does not alter since it is a topological invarient. That

$$n_+ - n_- = 1/2.(\text{Euler characteristic})$$

for a SU(3) bundle was first given in [29]. This conclusion differs from that given in [43]. One excess chiral fermion in the 3 of SU(3) on the internal space leads to one massless chiral fermion on 4-d space time in the 27 of E_6. Thus we have a single 27 of E_6 in 4-D. Of course we would like to have three or four and it may be possible to gain a horizontal symmetry from the G group of isometries associated with $G_2/SU(3)$. However if such a mechanism were to work one would expect a minimum of seven families (seven being the lowest dimensional representation of G_2 ,excluding the singlet) which is incompatable with cosmology [ ]. Thus if this idea were to work we must find some way of of breaking the G_2 down to some smaller group. At present we have no suggestions to make as to how this may be done.

Sp(4)/SU(2)xU(1)

The fermion spectra for this manifold has been considered in [43], where the topology is discussed. Considering Sp(4) as a S^3 bundle over S^7 (it has the same cohomology as $S^3 x S^7$) the coset space is formed by allowing SU(2) to act on S^7 and U(1) to act on S^3 so as to induce the fibrations $S^7 \to S^4$ and $S^3 \to S^2$, resulting in a S^2 bundle over S^4 (since $\pi_3(S^3) = Z$ these bundles are classified by the integers). Hence Sp(4)/SU(2)xU(1) is an S^2 bundle over S^4. Imbedding a non-zero background SU(2)xU(1) field into one E_8 produces the following decompositions

$$
\begin{array}{lll}
E_8 & \longrightarrow E_7 \, xSU(2) & \longrightarrow E_6 \, x \, U(1) \, xSU(2) \\
(\underline{248}) & (\underline{133},\underline{1})\oplus(\underline{1},\underline{3})\oplus(\underline{56},\underline{2}) & (\underline{78},\underline{1})_0 \oplus(\underline{27},\underline{1})_{-2} \oplus(\overline{\underline{27}},\underline{1})_2 \\
 & & \oplus(\underline{1},\underline{1})_0 \oplus(\underline{1},\underline{3})_0 \oplus(\underline{27},\underline{2})_1 \\
 & & \oplus(\overline{\underline{27}},\underline{2})_{-1} \oplus(\underline{1},\underline{2})_3 \oplus(\underline{1},\underline{2})_{-3}
\end{array}
\tag{6.7}
$$

Where the subscripts denote the U(1) quantum numbers of the representations. The $(\underline{27},\underline{1})_2$,the $(\underline{27},\underline{2})_1$ and their conjugates would be interpreted as fermion families in four dimensions. We now examine the index theorem to discover the excess of 27s over $\overline{27}$s. Since the first Pontrjagin class of S^4 vanishes the index theorem for a fermion with U(1) charge p in a background SU(2)xU(1) bundle over Sp(4)/SU(2)xU(1) is

$$
\begin{aligned}
n_+ - n_- &= \int ch\{SU(2)xU(1)\} \\
&= \int ch\{SU(2)\} x ch\{U(1)\} \\
&= \int (-c_2\{SU(2)\} + \tfrac{1}{2} c_1\{SU(2)\}^2) \wedge c_1\{U(1)\} \\
&= - \int c_2\{SU(2)\} \wedge c_1\{U(1)\} \\
&= -mnp
\end{aligned}
\qquad (6.9)
$$

(p =-2,1 depending on which 27 we are considering.) Where m=monopole charge on S^2 and n=instanton number on S^4. This formula disagrees with that in [43] by a factor of 1/2. It is argued in Ref [43] that mn must be a multiple of two hence we always obtain an even munber of families in 4-D eg if mn=2 we obtain 4 more massless 27_2s than $\overline{27}_2$s and 2 more massless 27_1s than $\overline{27}_{-1}$s.

In [43] other schemes are discussed in particular one which breaks E_8 to SU(5)xSU(3)xSU(2)xU(1) giving three families of ($\overline{5}$ +10) of SU(5).

SU(3)/U(1)xU(1)

The fermion spectra on this manifold has been considered in [43] and [44] but here we shall consider an alternate scheme.

SU(3)/U(1)xU(1) can be constructed as a CP^1 bundle over CP^2. This structure is obtained by considering SU(3) as a S^3 bundle over S^5. (There is one and only one non-trivial S^3 bundle over S^5, since $\pi_4(S^3) = Z_2$ [45] this is SU(3)). By allowing one U(1) to act on S^5 and the other on S^3, so as to induce Hopf fibrations $S^3 \to S^2 \simeq CP^1$ and $S^5 \to CP^2$, we reduce the S^3 bundle over S^5 (SU(3)) to a S^2 bundle over CP^2 (SU(3)/U(1)xU(1)). Imbeding a U(1)xU(1) field within E_8 gives two of the gauge bosons of the cartan subalgebra of E_8 a mass [34], due to expectation values of the Chern-Simons terms in the field strength of the antisymmetric tensor. There are various ways of imbedding U(1)xU(1) into E_8. We shall discuss one of these which gives E_6 as a residual gauge group in 4-D. E_8 contains SU(2)xE_7 as a maximal subgroup. We imbed one of the U(1)s into the SU(2). This gives us E_7 which has U(1)xE_6 as a maximal subgroup. Identifying the remaining U(1) with the U(1) subgroup of E_7 leaves us with E_6. Under this breaking the $\underline{248}$ of E_8 decomposes as follows

$$E_8 \to E_7 \times SU(2)$$

$$\underline{248} \to (\underline{133},\underline{1}) + (\underline{1},\underline{3}) + (\underline{56},\underline{1}) + (\overline{\underline{56}},-\underline{1})$$

$$\to E_7 \times U(1)$$

$$(\underline{133})_0 + (1)_0 + (1)_{-2} + (\underline{1})_2 + (\underline{56})_1 + (\overline{\underline{56}})_{-1}$$

$$\to E_6 \times U(1) \times U(1)$$

$$\underline{78}_{00} + 27_{0,-2} + \overline{27}_{0,2} + 1_{00} + 1_{00} + 1_{-2,0} + 1_{2,0} + 1_{1,3}$$

$$+ 1_{1-3} + 27_{11} + \overline{27}_{1-1} + 1_{-1,3} + 1_{-1,-3} + \overline{27}_{-1,-1} + 27_{-1\,1} \qquad (6.9)$$

The index theorem for fermions of charge (1,1) yields

$$n_+ - n_- = \frac{1}{6}\int c_1{}^3 - \frac{1}{24}\int c_1 p_1 \qquad (6.10)$$

Since $c_2 = c_3 = 0$ for U(1) fibres.

Let x be the Kahler two form (volume form) on S^2, $x^2=0$, and y be the Kahler two form on CP^2 (y =volume form) $y^3=0$ then

$$p_1 = p_1(CP^2) = 3y^2 \quad \text{see ref [35]} \qquad (6.11)$$

$$c_1 = mx + ny \qquad (6.12)$$

where m = the monopole number of the U(1) in S^2 and n = monopole number of the other U(1) field in CP^2, then (6.10) reduces to ($c_1^3 = 3mn^2xy^2$, $\int xy = 1$)

$$n_+ - n_- = 1/2mn^2 - 1/8m \qquad (6.13)$$

For fermions of charge (p,q) this is modified to

$$n_+ - n_- = 1/2(pm)(qn)^2 - 1/8(pm) \qquad (6.14)$$

When the U(1) which breaks $E_8 \rightarrow E_7$ lives on S^2 and the U(1) which breaks $E_7 \rightarrow E_6$ lives on CP^2. Alternatively

$$n_+ - n_- = 1/2(qm)(pn)^2 - 1/8(qm) \qquad (6.15)$$

When the fermions change roles.

For a fermion of charge (1,1) on SU(3)/U(1)xU(1) (6.13) shows that m must be a multiple of 8. This reflects the fact that SU(3)/U(1)xU(1) does not admit a spinor structure coupled to a U(1) field unless m is a multiple of 8. As an example we take the simplest case non-trivial case n=1,m=8 From the decomposition (6.9) there are three different 27s to consider (p,q)= (0,-2) ,(1,1) and (-1,1) When the U(1) field on S^2 is used to break $E_8 \rightarrow E_7$ (6.15) gives

$$n_+ - n_- = p(4q^2 - 1) \qquad (6.16)$$

Hence the number of massless $\underline{27}_{1,1}$ exceeds that of the massless $\overline{27}_{-1-1}$ s by 3, the number of massless $\overline{27}_{1-1}$ s exceeds the number of $\overline{27}_{-1,1}$ s by 3 and there is no imbalence between the 27_{0-2} s and $\overline{27}_{0,2}$ s If the difference in U(1) numbers shows up as a physical difference in four dimensions then it is possible that the $\overline{27}_{1-1}$ s behave differently from the $27_{1\,1}$ s and so we will obtain a chiral theory.

When the U(1) field on CP^2 is used for the first step $E_8 \to E_7$ (6.15) gives

$$n_+ - n_- = q(4p^2-1) \qquad (6.17)$$

Hence the number of massless 27_{0-2}s exceeds the $\overline{27}_{02}$s by 2, the number of maassless 27_{11} s exceeds the $\overline{27}_{-1-1}$s by 3 and the number of massless 27_{-11}s exceeds the $\overline{27}_{1-1}$s by 3. thus we have a total of 8 massless 27s in 4-d, though again the different U(1)xU(1) quantum numbers may give different physics in four dimensions.

All this looks very interesting for phenomonology unfortunately our ansatz for solving the dynamics has used m=n on SU(3)/U(1)xU(1) and m=n=8 leads to an unacceptably large number of chiral fermion families. However should it prove possible to relax this, the above scheme is an interesting alternative to previous proposals. Since we have really got $E_8 x E_8$ we can contrive a situation which will give this. If we take a U(1)xU(1)xU(1) field and imbed U(1)xU(1) within one E_8 as a m=8,n=1 field and we imbed the other U(1) within the remaining E_8 as a n=7 field then this we appear in the Einstein equations in the same way as a m=n=8 field but the fields arising from the E_8 with two U(1)s imbedded would be as descibed for a m=8,n=1 case. It may be possible that in this case we will find at low energies that the 'dark matter' interacts in not quite so dark a manner !. Other schemes have been explored in references [43] and [44]. In particular [44] discusses a scheme with $E_8 \to SO(10)$ and three massless 16s of SO(10) in four dimensions.

For the case when we have SO(32) or SO(32)/Z_2 gauge group we do not find any appealing schemes. Since SO(32) has rank 16 and our background F fields have gauge groups with rank 2 a single imbedding will leave us with a rank 14 gauge group. This is much larger than any of the popular candidates for a unification group. To obtain E_6 (rank 6), SO(10) (rank 5) or SU(5) (rank 4) we would have to have a multiple imbedding and imbed F 5-6 times. It is possible to do so Eg For a U(1)xU(1) field since SO(32) SO(22)xSO(10) then imbedding 11 U(1)s within the SO(22) will leave us with SO(10) since SO(10) $\supset$ SU(5)xU(1) imbedding a further U(1) could leave us with SU(5). So we can obtain SU(5) by imbedding (U(1)xU(1)) as our gauge group. When we do this we have the problem of why only 6 times why not 7 or 8 ? so these imbeddings are not very natural.

So in conclusion we can ,when we take the $E_8 x E_8$ theory, find compactification schemes ,for all three coset spaces, which result in Yang-Mills groups of suitable Unification groups upon compactification. In all three cases we find the fermions lie in chiral representations however not always with the appropriate number of representations. In particular we find only one 27 of E when we compactify on G_2/SU(3) (we can obtain more but probably only $\geq$ 7).

As for our group manifolds our

(RxS^3/HS^3)x(internal manifold)

solutions are interesting from a cosmological viewpoint. In particular ,for the case where we have $F \neq 0$ (case F2), we find solutions extending, in a smooth set, from the case where space-time and the internal manifold are both curved (on the planck scale) to the case where space-time is flat but the coset space is still highly curved. This is interesting because it might explain why the internal dimensions have such a large curvature relative to the present measured curvature of 4-D space-time.

As for our group manifold case (chapter 4) we shall not discuss whether we have residual supersymmetry when we compactify.

<u>Chapter 7 Symmetric Coset Spaces</u>

We now consider Symmetric Coset Spaces (S.C.S). Symmetric coset spaces have the defining property that the structure constants $C^{a}{}_{bc}$ are all zero (see p70-71 for notation). The only non-zero structure constants being $C^{\bar{a}}{}_{bc}$ & $C^{\bar{a}}{}_{\bar{b}\bar{c}}$. If $C^{a}{}_{bc}=0$ then our ansatz for the torsion

$$T^{a} \sim C^{a}{}_{bc}\, E^{bc} \tag{7.1}$$

is zero as is our ansatz for the three form H. If the torsion is zero then we lose a great deal of the motivation for considering coset spaces. However for completeness we shall investigate whether solutions exist for our ansatz.

Our ansatz for the curvature reduces to

$$R^{a}{}_{b} = \frac{\lambda^{2}}{2} C^{a}{}_{b\bar{c}}\, C^{\bar{c}}{}_{de}\, E^{de} \tag{7.2}$$

We have two choices for the Yang-Mills field either

$$F = 0 \tag{7.3}$$

or

$$F^{\bar{a}} = -\frac{\lambda^{2}}{2} C^{\bar{a}}{}_{bc}\, E^{bc} \tag{7.4}$$

The only free parameter in R and F is λ^{2}.

For S.C.S. the combination of structure constants

$$C^{a}{}_{\bar{c}[b}\, C^{\bar{c}}{}_{de]} \quad (\ [\quad]\ \text{denotes antisymmetrisation}\ ) \tag{7.5}$$

is zero hence both tr($F\wedge F$) and tr($R\wedge R$) are individually zero so the Bianchi identity for H

$$dH = \mathrm{tr}(\ R\wedge R\ ) - \mathrm{tr}(\ F\wedge F\ ) \tag{7.6}$$

is satisfied without leaving any constraints.

If space-time is Minkowski then we will have two independent equations of motion (two Einstein say) and one parameter.

If space-time is AdS or dS then we will have three equations (two Einstein and the scalar) and two parameters λ^2 and R_4.

If space-time is $R\times S^3/HS^3$ then we will have three equations (three Einstein) and three parameters h_0, R_3 & λ^2.

So unless for Minkowski ,AdS and dS the equations are degenerate we will not find solutions however we may find solutions for the case of space-time being $R\times S^3/HS^3$.

We shall use the notation of chapter 3 but now we have functions of λ^2 alone.

If we take space time to be Minkowski we have (as always)

$$g(H,H) = -2\ r(\lambda) \tag{7.7}$$

as H=0 we have that

$$r(\lambda)=2\lambda^2= 0 \tag{7.8}$$

so $\lambda^2=0$ is the only solution. This is just 10-D Minkowski.

If we take 4-D to be AntideSitter or deSitter the we have the two Einstein equations

$$\frac{1}{2}\{\ r(\lambda)+\frac{1}{2}R_3\ \}+\frac{1}{4}\{\ z(\lambda)+R_4.r(\lambda)\ \}+\frac{1}{2}F(\lambda)=0 \tag{7.9}$$

$$\frac{1}{2}\{\ \frac{2}{3}r(\lambda)+\ R_3\ \}+\frac{1}{4}\{\frac{1}{3}z(\lambda)+R_4.r(\lambda)\}\frac{2}{3}\}+\frac{1}{6}F(\lambda)=0 \tag{7.10}$$

These (plus the scalar equation) imply

$$0 = g(H,H) = -2\ (\ r(\lambda)\ +\ R_4\ ) \tag{7.11}$$

$$\text{so}\qquad R_4=-r(\lambda) \tag{7.12}$$

substituting back into the Einstein and scalar equations we obtain

$$\frac{1}{4}r(\lambda)-\frac{1}{4}r^2(\lambda)+\ \frac{1}{4}\{\ z(\lambda)+\ 2F(\lambda)\ \}=0 \tag{7.13}$$

$$-\frac{1}{6}r(\lambda)-\frac{1}{3}r^2(\lambda)+\frac{1}{12}\{\ z(\lambda)+\ 2F(\lambda)\ \}=0 \tag{7.14}$$

(7.13)-3(7.14) gives

$$\frac{3}{4}r(\lambda)+\frac{3}{4}r^2(\lambda) = 0 \qquad (7.15)$$

so $\quad r(\lambda) = 0$ or $r(\lambda) = -1$

$r(\lambda) = -1$ is impossible and $r(\lambda) = 0$ is again the trivial solution so we find no non-trivial solutions in this case.

We turn now to our last (but best !) possibility $R\times S^3/HS^3$. For $R\times S^3/HS^3$. We have $g(H_1,H_1) = 0$ and our three Einstein equations (no scalar see p40) are

$$\frac{1}{2}\{ r(\lambda)+ R_3 \}+\frac{1}{4}\{ z(\lambda)+2r(\lambda).R_3 \} - \frac{1}{4}g(H_0H_0) + \frac{1}{2}F(\lambda) = 0 \qquad (7.16)$$

$$\frac{1}{2}\{ r(\lambda)+\frac{1}{3}R_3 \}+\frac{1}{4}\{ z(\lambda)+\frac{2}{3}r(\lambda).R_3 \} + \frac{1}{4}g(H_0,H_0) + \frac{1}{2}F(\lambda) = 0 \qquad (7.17)$$

$$\frac{1}{2}\{ \frac{2}{3}r(\lambda)+ R_3 \}+\frac{1}{4}\{\frac{1}{3}z(\lambda)+\frac{4}{3}r(\lambda).R_3 \} - \frac{1}{4}g(H_0,H_0) + \frac{1}{6}F(\lambda) = 0 \qquad (7.18)$$

(7.16) we can take as defining $g(H_0H_0)$ so we can eliminate $g(H_0,H_0)$leaving two independent equations which are

$$\frac{1}{2}\{ 2r(\lambda)+\frac{4}{3}R_3 \}+ \frac{1}{4}\{ 2z(\lambda)+\frac{8}{3}r(\lambda).R_3 +4F(\lambda) \} = 0 \qquad (7.19)$$

$$\frac{1}{2}\{ \frac{1}{3}r(\lambda) \}+ \{ \frac{2}{3}z(\lambda)+\frac{2}{3}r(\lambda).R_3 + \frac{4}{3}F(\lambda) \} = 0 \qquad (7.20)$$

(7.19)-4(7.20) gives

$$\frac{1}{2}\{ \frac{4}{3}R_3+ \frac{2}{3}r(\lambda) \} + \frac{1}{4}\{- \frac{2}{3}z(\lambda)- \frac{4}{3}F(\lambda) \} = 0 \qquad (7.21)$$

so

$$\frac{4}{3}R_3 = \frac{1}{3}z(\lambda)- \frac{2}{3}r(\lambda) + \frac{2}{3}F(\lambda) \qquad (7.22)$$

substituting back we find we have one equation

$$\frac{1}{6}r(\lambda) + \frac{1}{6}\{ z(\lambda)+2F(\lambda) \} - \frac{1}{12}r^2(\lambda) + \frac{1}{24}\{ z(\lambda)+2F(\lambda) \}.r(\lambda) = 0 \qquad (7.23)$$

now $z(\lambda)\sim\lambda^4$,$F(\lambda)\sim\lambda^4$ and $r(\lambda) \sim \lambda^2$

so

$$z(\lambda)+2F(\lambda) = z_0 r^2(\lambda) \qquad (7.24)$$

hence upon substituting (7.24) into (7.23)

$$\tfrac{1}{6} r(\lambda).\{ 1+z_0 r(\lambda)+\tfrac{1}{4} z_0 r^2(\lambda)-\tfrac{1}{2} r(\lambda) \} = 0 \qquad (7.25)$$

so $r(\lambda)=0$ or $2r(\lambda)= (2/z_0 -4) \pm\sqrt{((4-2/z_0)^2-16/z_0)}$

We require real positive solutions for $r(\lambda)$.

This will be true iff

$$2/z_0 -4 \geqslant 0 \text{ and } (4 -2/z_0)^2 -16/z_0 \geqslant 0 \qquad (7.26)$$

The first implies we need $0\leqslant z_0 \leqslant \frac{1}{2}$.

The second implies $(z_0^2 - \frac{3}{2}z_0 + \frac{1}{4}) \geqslant 0$

or $(z -a_+).(z -a_-) \geqslant 0$

where $a_\pm = \{ \frac{3}{4} \pm\sqrt{(\frac{3}{4})^2-\frac{1}{4}} \}$

ie $a_+ = 1.31$ and $a_- = 0.19$

putting the two conditions together we need

$0\leqslant z_0 \leqslant a_- = 0.19$

Now $r(\lambda) = 6\lambda^2 n_1$, $z(\lambda) = \frac{1}{6}\lambda^4 q$ where $q = 36(2n_1^2+n_1 n_2)$

and $F(\lambda) = -\lambda^4\chi^2$ or 0

hence $z_0 = \frac{1}{3}(1 + 3/\dim R) - \frac{2}{9}\chi^2$

(if F=0 we neglect the last term), we need $z_0 < 0.19$. If F=0 then $z_0 > 0.33$ so will not find solutions. If $F\neq 0$ then we must consider the value of χ^2. We find we must have

$$\chi^2 \geqslant \tfrac{9}{2}((\tfrac{1}{3}-a_-) +1/\dim R) \qquad (7.27)$$

$$\text{and } \chi^2 \leqslant \tfrac{9}{2}(1/3 +1/\dim R) \qquad (7.28)$$

these are a fairly restrictive for χ^2.

What are the six-dimensional symmetric coset spaces ?

We find the following [46]

1. $SO(7)/SO(6)$ ($\simeq S^6$)
2. $SU(2)xSU(2)xSU(2)/\{ U(1)xU(1)xU(1) \}$ ($S^2xS^2xS^2$)
3. $Sp(4)xSU(2)/\{ SO(4)xU(1)xU(1) \}$ (S^2xS^4)

4. $SU(4)/\{SU(3)xU(1)\}$ ($\approx CP^3$)

5. $Sp(4)/\{SU(2)xU(1)\}$

6. $SU(3)xSU(2)/\{U(2)xU(1)\}$ (7.29)

6. is a complicated case. If the SU(2) in the U(2) is factored out with the SU(2) in the top then we have the case of $SU(3)/\{U(1)xU(1)\}$ - a nonsymmetric coset space. If the SU(2) in the bottom is factored out of the SU(3) then we have a symmetric coset space.

As discussed previously $Sp(4)/\{SU(2)xU(1)\}$ has both a symmetric and a non- symmetric imbeding. Of interest immediately is $SU(2)xSU(2)xSU(2)/\{U(1)xU(1)xU(1)\}$ because since R is just U(1)s we can have the fields/normalisations as large as we want (see Appendix 3) this means we will be able to satisfy (7.27).

For several of our coset spaces ,S/R, we have R of the form

(simple group) x U(1)

This will lead to difficulties in the equations of motion analogous to those encoumtered for $Sp(4)/SU(2)xU(1)$ in Chapter 5. We can deal with these in the same way (see Appendix 2). This means we must take the normalisations appropriate for the simple group. We then have no possibilities of multiple monopole charges for the U(1)s when this is done. We find the normalisations and give them in Appendix 3. As we can see other than $\{SU(2)\}^3/\{U(1)^3\}$ none of these have χ^2 large enough to admit any solutions.

For $\{SU(2)^2\}/\{U(1)^2\}$ we find (7.27) and (7.28) reduce to

$$\chi^2 \geqslant 2.1 \quad \text{and} \quad \chi^2 \leqslant 3.0$$

The only possibility is $\chi^2 = 3$ (χ^2 must be an integer). If we substitute back we find

$$r(\lambda) = 2 \ , \ R_3 = -1 \ , \ g(H,H) = -1/2$$

As g(H,H) is negative we do not in fact have a valid solution !

So in conclusion , for the ansatz tried , we do not find any solutions for symmetric coset spaces for any of the possible space-times

In [29] the particular case of SO(7)/SO(6) is considered and a non-trivial torsion is given.

Chapter 8 Additional Terms For The Lagrangian

When in chapter 2 we modified the Chapline-Manton lagrangian to take account of the extra terms which we would expect from superstring theories we were not performing a consistent truncation of the infinite number of terms which we would have. The Chapline-Manton lagrangian contains terms of order ($k/M_S)^2$ (recall from Chapter 2 that we regarded our terms of an expansion of terms which had 'sizes' of different powers of (k/M_S) and we regarded this parameter as small) We included the Zwiebach term which is of order ($k/M_S)^4$ (every derivative gives us another factor of k) and the three form was modified

$$H = H_0 + \Omega_{LOR} \qquad (8.1)$$

H_0 is order (k/M_S) and Ω_{LOR} is ($k/M_S)^3$ so our modified

$$H \wedge {}^*H = H_0 \wedge {}^*H_0 + 2H_0 \wedge {}^*\Omega_{LOR} + \Omega_{LOR} \wedge {}^*\Omega_{LOR} \qquad (8.2)$$

Will include both $(k/M_S)^4$ and $(k/M_S)^6$ terms.

If we include these terms then we should also expect any other terms of order $(k/M_S)^4$ to also be needed (not mention the $(k/M_S)^6$). These would be terms of the form H^4, dHH^2, RH^2, F^4, RF^2, dHF^2, H^2F^2 plus lots involving $d\rho$!.

In this chapter we shall attempt to produce some of these terms. We shall only look at when F=0 (ie we forget those terms involving F). In principle these terms should be derivable from string theory but this procedure is difficult and well out of the scope of this work. Instead we shall make some assumptions for which there is no rigourous justification but for which there is a little evidence to suppose might hold.

We start off be looking at the Chapline-Manton lagrangian with F set to zero ie

$$\frac{1}{2} R_{AB} \wedge {*E}^{AB} + \frac{1}{4} e^{2\nu} H \wedge {*H} - d\nu \wedge {*d\nu} \qquad (8.3)$$

If we take the connection to have zero torsion then this can be written as

$$\frac{1}{2} R'_{AB} \wedge {*E}^{AB} \qquad (8.4)$$

Where $R'_{AB} = d\omega'_{AB} + \omega_{AC} \wedge \omega^{C}{}_{B}$

and ω'_{AB} is defined by

$$\omega'_{ab} = \omega^{0}_{AB} \pm \sqrt{3}\, e^{\nu} H_{ABC}\, E^{C} \pm \frac{1}{6}\{ i_a d\nu\, E_b - i_b d\nu\, E_a \} \qquad (8.5)$$

($H = H_{ABC} E^{ABC}$ and ω^{0}_{AB} is the initial connection)

So we can write all our R/H/$d\nu$ terms as just a curvature term. We know the curvature squared term is the Zwiebach form

$$\frac{1}{4} e^{\nu} R_{AB} \wedge R_{CD} \wedge {*E}^{ABCD} \qquad (8.6)$$

If we assume the same trick occurs as above ie assume

$$\frac{1}{4} e^{\nu} R'_{AB} \wedge R'_{CD} \wedge {*E}^{ABCD} \qquad (8.7)$$

includes all the H/R/$d\nu$ terms to order $(k/M_S)^4$ then we have a means by which we can explicitely calculate these terms. Although this seem a rather unjustifyable assumption work done in ref [33] which is calculating terms directly from string scattering amplitudes does suggest that this does work.

When we evaluate (8.7) we find we have the following additions to the lagrangian

$$-(1/4).e^{3\nu} R_{AB} \wedge H \wedge {*(E^{AB} \wedge H)} \qquad (8.8)$$

$$+(1/288).e^{5\nu}\{ 2 i_A H \wedge i_B H \wedge {*(i^A H \wedge i^B H)} \qquad (8.9)$$

$$+ i_A H \wedge i^A H \wedge {*(i_B H \wedge i^B H)} \qquad (8.10)$$

$$\pm (1/12\sqrt{3}) e^{4\nu} dH \wedge {*(i^A H \wedge i_A H)} \qquad (8.11)$$

$$-(7/12) e^{\nu} R_{AB} \wedge d\nu \wedge {*(E^{AB} \wedge d\nu)} \qquad (8.12)$$

$$-(7/12)e^{\nu}(d\nu\wedge *d\nu)\wedge *(d\nu\wedge *d\nu) \tag{8.13}$$

$$+(19/36)e^{3\nu}i_A d\nu\, i^A d\nu\, H\wedge *H \tag{8.14}$$

$$+(184/48)e^{3\nu}H\wedge d\nu\wedge *(H\wedge d\nu) \tag{8.15}$$

$$\pm(7/2)i_A d\nu\, i^A H\wedge d*(e^{\nu}H).\,e^{2\nu} \tag{8.16}$$

$$\pm(3/2)D\{i_A d\nu\}i^A H\wedge *H.\,e^{3\nu} \tag{8.17}$$

$$\pm(15/4)d*d\nu\wedge *(H\wedge *H)e^{3\nu} \tag{8.18}$$

We would like to see the effect of these terms on our ansatzes of Chpt 3 and 5.

For both our ansatzes $\nu =$ const so any terms involing $d\nu$ to a power greater than one will give no contributions. so we can neglect (8.12)-(8.15). The terms involving a single $d\nu$ will contribute only to the scalar equation. We shall look at the effect of adding (8.8)-(8.18) to the lagrangian on our equations of motion in turn.

First we shall look at the Equation arising from varying B ($H = dB + \Omega$)

This is possibly the most important equation since it is satisfied trivially for the two ansatzes and hence any constraint arising ,when we have our additional terms, would lead us into difficulties. When we vary B we find we have a generalised (2.20)

$$d*H + T_1 + T_2 + T_3 + T_4 = 0 \tag{8.19}$$

where

$$T_1 = \tfrac{1}{4}d\{\, R_{AB}\wedge *(E^{AB}\wedge H)\,\} \tag{8.20}$$

$$T_2 = -\tfrac{1}{36}d\{\, i_A\{\, i_B H\wedge *(i^A H\wedge i^B H)\,\}\,\} \tag{8.21}$$

$$T_3 = -\tfrac{1}{72}d\{\, i_A\{\, i^A H\wedge *(i^B H\wedge i_B H)\,\}\,\} \tag{8.22}$$

$$T_4 = \mp\tfrac{1}{6\sqrt{3}}d\{\, i_A\{\, i^A H\wedge *dH\,\}\,\} \tag{8.23}$$

We shall look first at the case of SU(2)xSU(2) (the other manifolds are not significantly different from this).

T_4 is zero since $dH = 0$ for this case. Since $R_{AB} \sim E_{AB}$ then $R_{AB} \wedge *(E^{AB} H) \sim *H$ so T_1 will reduce to $d*H$ which is zero. For the term T_3, $i_b H \wedge i^b H$ will be zero since if b is for the first SU(2) say then $i_b H$ will be a two form of the first SU(2) and hence $i_b H i^b H$ will be a four form on the first SU(2) and hence must be zero. So T_3 is zero also . If we look at our remaining term T_2 then unless a and b are for different SU(2)s we will have zero for the same reasoning as for T so we will find

$$
\begin{aligned}
T_2 \sim\ & d\,(i_B H \wedge i_A *(i^B H \wedge i^A H)) \qquad (\text{take } A \in 1^{st} SU(2) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad B \in 2^{nd} SU(2)) \\
& d\,(i_B H \wedge * (H_1 \wedge i_B H)) \\
& d\,(i_B H_2 \; *(H_1 \wedge H_2)) \\
& d\, *H_1
\end{aligned}
$$

This is zero. So we find the generalisation of $d*H=0$ is still satisfied for the SU(2)xSU(2) ansatz.

The case of $SU(2)xU(1)^3$ being the internal manifolds follows in close parallel.

We now take our coset space ansatz (chpt 5).

Terms T_2 and T_3 must reduce to the form

$$
\begin{aligned}
d\{ \; & aH_{ABC}\, H_{DEF}\, H^{DEF} *E^{ABC} \\
& +bH_{ABC}\, H_{CDG}\, H_{FDG} *E^{ABC} \\
& +cH_{ADE}\, H_{BDF}\, H_{CEF} *E^{ABC} \}
\end{aligned}
\tag{8.24}
$$

(where a,b,c are some numbers) since these are the only possible tensors available. For our ansatz we had $H_{ABC} \sim C_{ABC}$. For our Coset spaces where we had our structure constants normalised so that $C_{ABC}\, C^{ABC}$ was some number and $C_{ABC}\, C^{DBC} = \text{const} \times \delta^{D}_{A}$ so the first and second terms in (8.24) must just reduce to

$$d(\, C_{ABC}\, {*E}^{ABC}\,) \qquad (8.25)$$

Which is zero. The third term is proves upon evaluation to be zero for our coset spaces (this was mentioned in page 71)

So terms of the form (8.24) ie T_2 and T_3 will be zero

Since $dH \sim C_{ABC}\, C^{C}{}_{DE}\, E^{ABDE}$ then T_4 must also reduce to the form (8.24) so T_4 will also be zero

For T_1 we note that

$$R_{AB} = R^{0}{}_{AB} + \beta R^{1}{}_{AB} + \beta^{2} R^{2}{}_{AB} \qquad (8.25)$$

Now $R^{1}{}_{AB}$ and $R^{2}{}_{AB}$ both only involve $C^{A}{}_{BC}$ and not $C^{\bar{A}}{}_{BC}$ so upon substitution of $R^{1}{}_{AB}$ and $R^{2}{}_{AB}$ into T_1 we will obtain the form (8.24) again so giving zero. So we only need consider R^{0}_{AB} Now

$$R^{0}_{AB}\, *(\, E^{AB} \wedge H\,) = \sim\ R^{0}{}_{AB}\, i^{A} i^{B} *H$$

$$\sim\ i^{A} i^{B}(\, R^{0}_{AB} \wedge *H\,) + i^{A}(\, R^{0}_{A} \wedge *H\,) + \overset{0}{R} *H$$

R= a constant and $R_A = \text{const} \times E_A$ so

$$= \quad i^{A} i^{B}(\, R^{0}_{AB} \wedge *H\,) + \text{const} \times *H$$

explicit evaluation of $R^{0}_{AB} \wedge *H$ reveals it to be zero. So T_1 reduces to $d*H$ and is hence zero. So all four of the terms T_i will give zero and so this equation is also satified by our coset space ansatz of chpter 5 (also for chpter 7)

Before looking at the scaler/Einstein equations as a whole we shall look at the effect of terms (8.16)-(8.18) ie those terms which only effect the scalar equation.

Term (8.18) will upon variation yield a contribution to the scalar equation of the form

$$d{*}d(* (H \wedge {*H})) \qquad (8.26)$$

For both group and coset spaces $H \wedge {*H}$ is a constant $x{*}1$ so in both case this will reduce to $d{*}d(1)$ which is simply zero.

Term (8.16) will give

$$d\{ i_A(i^A H \wedge d{*H}) \} \qquad (8.27)$$

In both cases again this reduces to $d{*H}$ so this will also vanish.

Term (8.17) gives

$$d\{ i^A(D(i_A H \wedge {*H})) \} \qquad (8.28)$$

in both cases $i_A H \wedge {*H}$ (const)$x{*}E_A$ so this will be of the form

$$d\{ i^A(D{*}E_A) \} \qquad (8.29)$$

now $D{*}E^A \sim T_B \wedge {*}E^{AB}$ (T^B is the torsion) so $i^A(D{*}E_A)$ will reduce to $i_A T_B \, {*}E^{AB}$ In both our cases $T^A \sim C^A{}_{BC} E^{BC}$ (where $C^A{}_{BC}$ are different objects in the two situations) so this will be $C_{AB}{}^C E_C \, {*}E^{AB}$. This is zero since $E_C \wedge {*}E^{AB} = (\delta^B_C {*}E^A - \delta^A_C {*}E^B)$ and $C_{AB}{}^B = C_{AB}{}^A = 0$. So this term gives no contribution either so for our ansatzes the terms (8.16)-(8.18) have no effect on the equations of motion.

When we consider the Einstein equations we will only have terms (8.8)-(8.11) to consider. These are the H^4 and RH^2 terms. These will in fact give non-zero alterations to the Einstien equations. It is important to check whether the energy-momentum tensor is still of the form (constant depending on λ, β etc)x$*E^E$ for E an internal space index. Otherwise we will find we have more constraints arising from the Einstein equations than we can satisfy.

For our group manifolds it follows very easily that this is the case.

For the Coset space ansatz (and actually for the group one also) we must have the energy momentum form for the H^4 to be

$$= \tau^E{}_F *E^F \tag{8.30}$$

Where $\tau^E{}_F$ is formed from four $C^a{}_{bc}$s. The only four such tensors are

$$C^E{}_{bc}\, C^{bc}{}_F\, C^{deg}\, C_{deg} \tag{8.31}$$

$$C^E{}_{bc}\, C^{bc}{}_d\, C^{deg}\, C_{egF} \tag{8.32}$$

$$C^E{}_{bc}\, C_{bdg}\, C_{hdg}\, C_{hcF} \tag{8.33}$$

$$C^E{}_{bc}\, C_{bdg}\, C_{cdh}\, C_{ghF} \tag{8.34}$$

(8.31),(8.32) and (8.33) immediately reduce to (const)x$*E^E$ so these will be fine. Explicit evaluation of (8.34) reveals it also to be (const)x $\delta^E{}_F$ so the energy-momentum form $\tau^E{}_F$ will not cause any problems in the Einstein equations.

The energy-momentum form for the RH^4 terms is

$$-\tfrac{1}{4}e^{3\nu}\left\{ \iota^E\left[R_{AB}\wedge H\wedge *(E^{AB}\wedge H)\right] - \iota^E(R_{AB}\wedge H\wedge *(E^{AB}\wedge H) - E_{AB}\wedge \iota^E H\wedge *(R_{AB}\wedge H)\right\} \tag{8.34}$$

For the RH terms recall that

$$R_{ab} = \overset{0}{R}_{ab} + \overset{1}{R}_{ab} + \overset{2}{R}_{ab}$$

Where R and R only involve the $C^a{}_{bc}$. So for the $\overset{1}{R}H^2$ and $\overset{2}{R}H^2$ the energy-momentum form will reduce to something like (8.30) so these terms will also not give problems with the Einstein equations. For the $\overset{0}{R}H^2$ term we find when we evaluate it we obtain zero or (const)$x*E^{\varepsilon}$ so this is also fine.

So with the addition of our extra terms we still find the Einstein equations reduce to two or three algebraic constraints. Generally these will involve higher powers of λ^2 and h^2 than before. Eg we might find we have to solve a sixth order polynomial for λ^2 rather than a third order , which we had before. In general this cannot be done analytically. This does not mean solutions don't exist only that we can't express then in terms of standard functions. (Since the equations are non-linear it may be however that the new set of solutions will not in fact have any solutions)

So the addition of the extra terms to the lagrangian does not alter the way the equations reduce to constraints but is merely (?) to make the system much more complicated.

Conclusions

We have examined the low energy field theory limit of Superstring theories and attempted to find alternate compactification schemes to the standard Calabi-Yau spaces. We have taken a lagrangian which describes the low energy limit and we have found solutions to the equations of motion for two forms of the internal, compactified, manifold. Namely Group spaces and Non-symmetric Coset spaces (with the fermion feilds set to zero).

For the solutions which were Group spaces we have analysed the consequences of these solutions and it is very difficult to regard them as serious physical possibilities. In particular we have not obtained solutions where the 4-D fermions lie within Chiral representations. For the Coset spaces however the physical implications of such spaces can be realistic. For the three types of Non-Symmetric Coset Space we have found solutions which give realistic gauge groups in 4-D and chiral fermions. For $G_2/SU(3)$ it is difficult to obtain three or four families of Chiral fermion but for the other two cases ($Sp(4)/SU(2)xU(1)$ & $SU(3)/U(1)xU(1)$) a realistic number of chiral families could be found.

In both cases solutions were only found in the presense of non-zero torsion (in fact for Symmetric Coset spaces where the torsion was zero we found no solutions at all). So for the type of spaces we were considering the presense of torsion is crucial to finding solutions to the equations of motion.

The question of whether we were using an appropriate lagrangian was considered in Chapter 8 and the possible consequences of adding additional terms considered.

An important question which has not been answered within this work is whether thier will be residual supersymmetry ,left in four dimensions, after compactification. It is desirable to have some left over otherwise the Hierarchy problem is not solved.

Appendix 1 Details of structure constants etc for the three Coset spaces

In this appendix we shall produce the structure constants, in an appropriate form, for the three groups we use in chapter 5 ie G_2, Sp(4)& SU(3). We shall also give information on the imbeddings which yield the non-symetric coset spaces which are used in chapter 5. We also give the explicit form of the curvature two forms on these coset spaces and the explicit form of the Yang-Mills fields when they are the F2 case (this is this most interesting form of the Yang-Mills fields).

If we have a simple Lie algebra L with rank rank(L) we can choose a basis called the Cartan basis. Within this basis there is an abelian subalgebra called the the cartan subalgebra which has generators H_i i=1..rank(L)

$$[H_i , H_j] = 0 \tag{A1.1}$$

The remaining generators E have a rank(L) dimensional label $\underline{\alpha}$ (which we call a root) and obey with the H_i

$$[H_i , E_{\underline{\alpha}}] = \alpha_i E \tag{A1.2}$$

If $\underline{\alpha}$ is a root then so is $-\underline{\alpha}$ and we have

$$[E_{-\underline{\alpha}} , E_{\underline{\alpha}}] = \alpha^i H_i \tag{A1.3}$$

The α^i are related to the α_i by a metric g^{ij} which we can take as δ^{ij} to give $\alpha^i = \alpha_i$. If $\underline{\alpha} + \underline{\beta}$ is not a root then $E_{\underline{\alpha}}$ and $E_{\underline{\beta}}$ will have zero commutator ,If however $\underline{\alpha} + \underline{\beta}$ is another root then $E_{\underline{\alpha}}$ and $E_{\underline{\beta}}$ will obey

$$[E_{\underline{\alpha}} , E_{\underline{\beta}}] = N_{\alpha\beta} \, E_{\underline{\alpha}+\underline{\beta}} \tag{A1.4}$$

Where the $N_{\alpha\beta}$ are defined by

$$N^2_{\alpha,\beta} = n(m+1)\alpha_i \alpha^i \tag{A1.5}$$

Where n is the largest integer such that $\underline{\alpha}+n\underline{\beta}$ is a root and m is the largest integer such that $\underline{\alpha}-m\underline{\beta}$ is a root.

A graph upon which all the $\underline{\alpha}$ are plotted is called the "root diagram" of L and is rank(L) dimensional. Not any choice of roots $\underline{\alpha}$ will correspond to a Lie algebra there are various consistency conditions which must be satisfied (arising from the Jacobi identities). These conditions are very limiting. From the root diagram one can read off the $\underline{\alpha}$ and calculate the $N_{\alpha,\beta}$ so the diagram contains all the commutation relations. The cartan basis is not a convenient basis for some purposes in particular the the structure constants are not cyclic. to obtain a basis where the structure constants are cyclic we change the basis to

$$E^{+}_{\alpha} = \frac{i}{\sqrt{2}}(E_{\underline{\alpha}} + E_{-\underline{\alpha}})$$
$$E^{-}_{\alpha} = \frac{1}{\sqrt{2}}(E_{\underline{\alpha}} - E_{-\underline{\alpha}})$$
$$H'_{i} = iH_{i} \tag{A1.6}$$

We can further normalise these generators to obtain a basis where

$$C^{a}{}_{bc}\, C^{dbc} = \delta^{ad} \tag{A1.7}$$

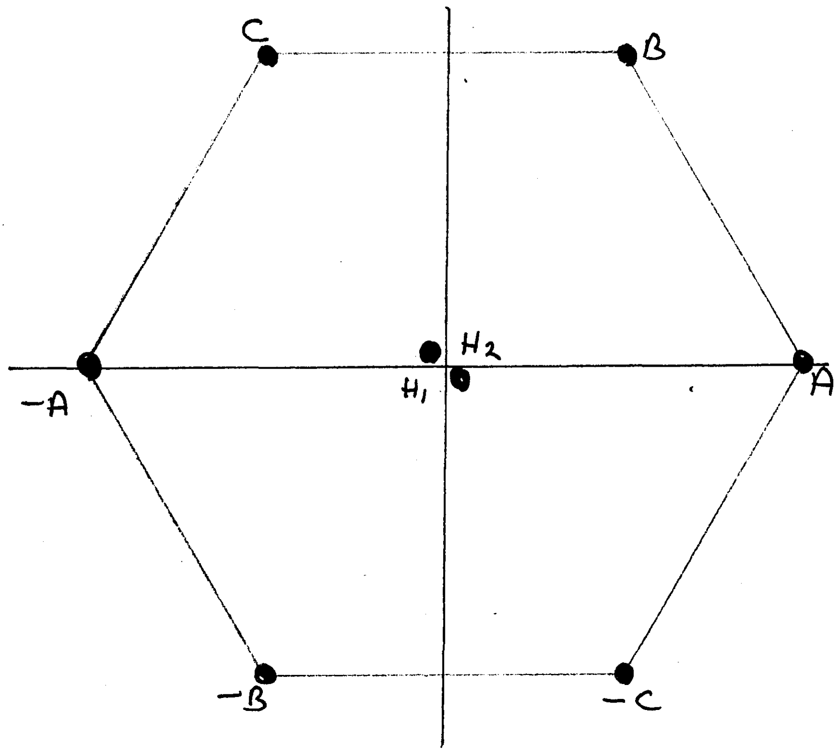
This is the type of basis which we require for our work on coset spaces. We present here the root diagrams for SU(3) , Sp(4) and G_2 ,which are all two-dimensional , and we give the structure constants which are cyclic and normalised to (A1.7) We also indicate the subalgebra which is used to form the non-symmetric coset space.

In all three cases the roots indicated in red are those which will form the subalgebra

We shall be conserned whether the additional normalisations (5.11)-((5.14) are satisfied or not and shall give details of the n_i where appropriate.

For Sp(4) the roots in green are those which will form (together with the two Hs) the symmetric embeding of SU(2)xU(1) within Sp(4)

SU(3)



We form a cyclic basis via

$$T_1 = \frac{i}{\sqrt{2}}(E_{\underline{A}} + E_{-\underline{A}}) ,\ T_2 = \frac{i}{\sqrt{2}}(E_{\underline{B}} + E_{-\underline{B}}),\ T_3 = \frac{i}{\sqrt{2}}(E_{\underline{C}} + E_{-\underline{C}})$$

$$T_4 = \frac{1}{\sqrt{2}}(E_{\underline{A}} - E_{-\underline{A}}) ,\ T_5 = \frac{1}{\sqrt{2}}(E_{\underline{B}} - E_{-\underline{B}}),\ T_6 = \frac{1}{\sqrt{2}}(E_{\underline{C}} - E_{-\underline{C}})$$

$$T_7 = iH_1 \quad ,\ T_8 = iH_2 \tag{A1.8}$$

We find the structure constants are cyclic the non-zero ones being

$$C_{126} = C_{135} = C_{234} = C_{456} = -\frac{1}{\sqrt{12}}$$

$$-C_{257} = C_{367} = +\frac{1}{\sqrt{12}},\ C_{147} = -\frac{1}{\sqrt{3}}$$

$$C_{258} = C_{368} = -\frac{1}{2} \tag{A1.9}$$

We find (5.11)-(5.14) are all obeyed when we imbed U(1)xU(1) within SU(3) as indicated with

$$n_1 = \frac{1}{3} \quad , n_2 = \frac{1}{3}$$

$$n_3 = 1 \quad , n_4 = 0 \tag{A1.10}$$

We give here the curvatures for the coset space SU(3)/U(1)xU(1)

$$
\begin{aligned}
R_{12} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{12} + (\beta + \tfrac{1}{2}\beta^2)E_{45} \} \\
R_{13} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{13} - (\beta + \tfrac{1}{2}\beta^2)E_{46} \} \\
R_{14} &= \frac{\lambda^2}{24}\{ 8E_{14} + (4-\beta^2)(E_{25} - E_{36}) \} \\
R_{15} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{15} + (\beta + \tfrac{1}{2}\beta^2)E_{24} \} \\
R_{16} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{16} - (\beta + \tfrac{1}{2}\beta^2)E_{34} \} \\
R_{23} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{23} + (\beta + \tfrac{1}{2}\beta^2)E_{56} \} \\
R_{24} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{24} + (\beta + \tfrac{1}{2}\beta^2)E_{15} \} \\
R_{25} &= \frac{\lambda^2}{24}\{ 8E_{25} + (4-\beta^2)(E_{14} - E_{36}) \} \\
R_{26} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{26} + (\beta + \tfrac{1}{2}\beta^2)E_{35} \} \\
R_{34} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{34} - (\beta + \tfrac{1}{2}\beta^2)E_{16} \} \\
R_{35} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{35} + (\beta + \tfrac{1}{2}\beta^2)E_{26} \} \\
R_{36} &= \frac{\lambda^2}{24}\{ 8E_{36} + (4-\beta^2)(E_{25} - E_{14}) \} \\
R_{45} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{45} + (\beta + \tfrac{1}{2}\beta^2)E_{12} \} \\
R_{46} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{46} - (\beta + \tfrac{1}{2}\beta^2)E_{13} \} \\
R_{56} &= \frac{\lambda^2}{24}\{ (\beta - \tfrac{1}{2}\beta^2)E_{56} + (\beta + \tfrac{1}{2}\beta^2)E_{23} \}
\end{aligned}
\qquad (A1.11)
$$

And we also give the Yang-Mills ansatz F2 explicitly

$$
\begin{aligned}
F^1 &= \frac{\lambda^2}{12}(+2E_{14} + E_{57} \quad E_{36}) \\
F^2 &= \sqrt{3}.\frac{\lambda^2}{12}(+ E_{36} + E_{25})
\end{aligned}
\qquad (A1.12)
$$

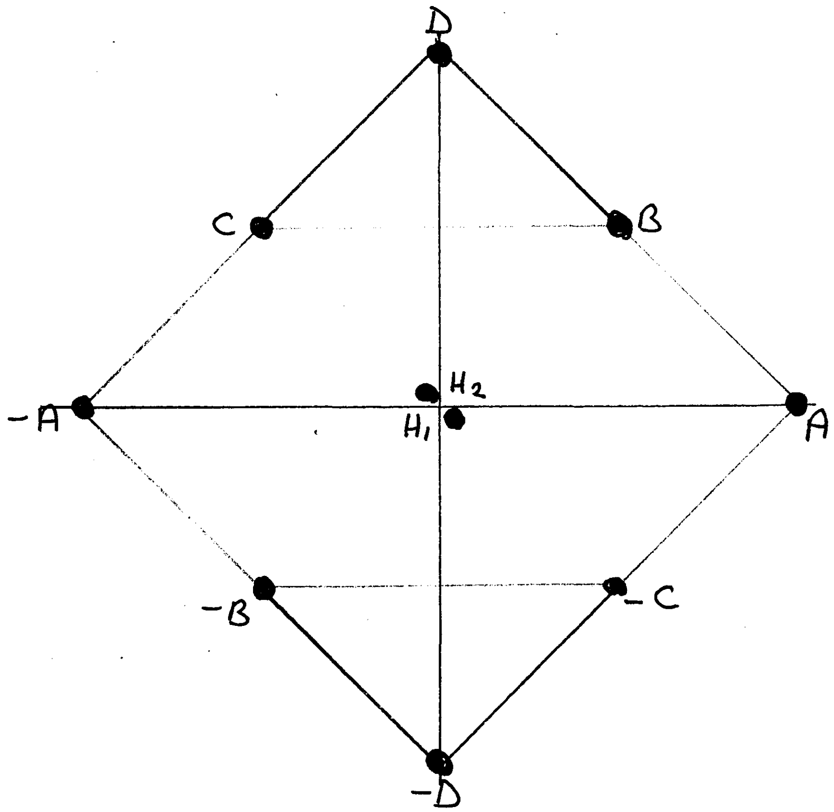
For SU(3)/{U(1)xU(1)} and in fact for the other two cases also we find the Ricci one-forms to be

$$R^a = \frac{\lambda^2}{6}(2+\beta-\tfrac{1}{2}\beta^2)E^a \qquad (A1.13)$$

and the curvature scalar

$$R = \lambda^2(2+\beta-\tfrac{1}{2}\beta^2) \qquad (A1.14)$$

Sp(4)



We find a cyclic basis via

$$T_1 = \frac{i}{\sqrt{2}}(E_{\underline{A}} + E_{-\underline{A}}), \quad T_2 = \frac{i}{\sqrt{2}}(E_{\underline{B}} + E_{-\underline{B}}), \quad T_3 = \frac{i}{\sqrt{2}}(E_{\underline{C}} + E_{-\underline{C}})$$
$$T_4 = \frac{1}{\sqrt{2}}(E_{\underline{A}} - E_{-\underline{A}}), \quad T_5 = \frac{1}{\sqrt{2}}(E_{\underline{B}} - E_{-\underline{B}}), \quad T_6 = \frac{1}{\sqrt{2}}(E_{\underline{C}} - E_{-\underline{C}})$$
$$T_7 = \frac{i}{\sqrt{2}}(E_{D} + E_{-D}), \quad T_8 = \frac{1}{\sqrt{2}}(E_{D} - E_{-D}), \quad T_9 = iH_1$$
$$T_{10} = iH_2 \qquad \text{(A1.15)}$$

We find the structure constants are cyclic and we normalise then to

$$-C_{126} = C_{135} = C_{324} = C_{456} = -1/\sqrt{12}$$
$$C_{238} = C_{259} = c_{267} = C_{357} = C_{568} = -1/\sqrt{12}$$
$$C_{25,10} = C_{369} = C_{36,10} = +1/\sqrt{12}$$
$$C_{14,10} = C_{789} = -2/\sqrt{12} \qquad \text{(A1.16)}$$

We find of (5.11)-(5.14) only (5.11) and (5.12) are satisfied with

$$n_1 = \frac{1}{3}, \quad n_2 = \frac{1}{3} \qquad \text{(A1.17)}$$

For (5.13) and (5.14) we find

$$C^{\bar{a}}{}_{bc}\, C^{bc\bar{d}} = n_3(\bar{a})\delta^{\bar{a}\bar{d}}, \quad C^{\bar{a}}{}_{\bar{b}\bar{c}}\, C^{\bar{b}\bar{c}\bar{d}} = n_4(\bar{a}) \tag{A1.18}$$

where $n_3(7)=n_3(8)=n_3(9)=\frac{1}{3}$, $n_3(10)= 1$

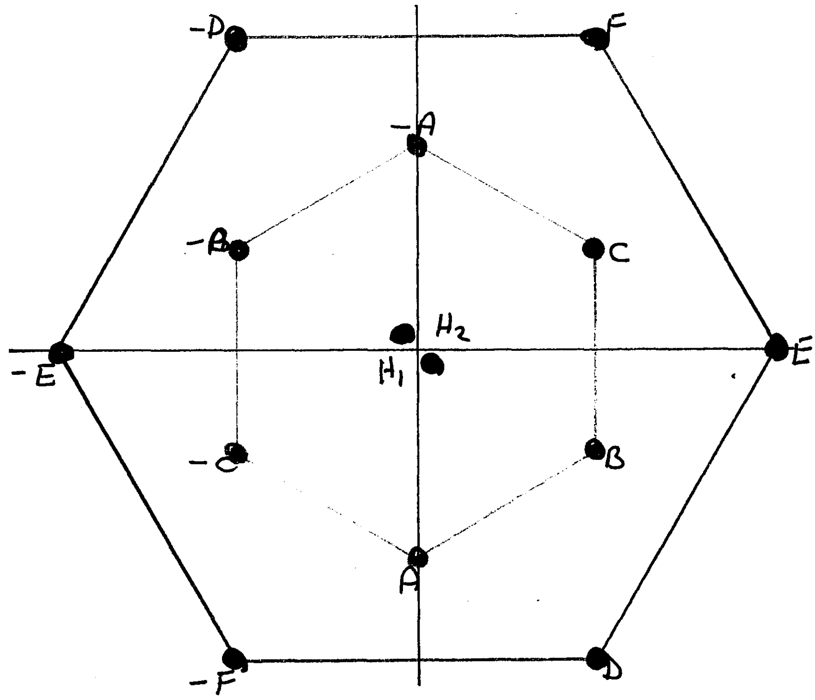
and $n_4(7)=n_4(8)=n_4(9)=\frac{2}{3}$, $n_4(10)= 0$ (A1.19)

We give here the curvatures for the coset space $Sp(4)/SU(2)xU(1)$ for the antisymmetric imbeding

$$
\begin{aligned}
R_{12} &= \frac{\lambda^2}{24}\{ (\beta-\tfrac{1}{2}\beta^2)E_{12} -(\beta+\tfrac{1}{2}\beta^2)E_{45} \}\\
R_{13} &= \frac{\lambda^2}{24}\{ (\beta-\tfrac{1}{2}\beta^2)E_{13} -(\beta+\tfrac{1}{2}\beta^2)E_{46} \}\\
R_{14} &= \frac{\lambda^2}{24}\{ 8E_{14} -(4-\beta^2)(E_{25}+E_{36}) \}\\
R_{15} &= \frac{\lambda^2}{24}\{ (\beta-\tfrac{1}{2}\beta^2)E_{15} -(\beta+\tfrac{1}{2}\beta^2)E_{24} \}\\
R_{16} &= \frac{\lambda^2}{24}\{ (\beta-\tfrac{1}{2}\beta^2)E_{16} -(\beta+\tfrac{1}{2}\beta^2)E_{34} \}\\
R_{23} &= \frac{\lambda^2}{24}\{ (2+\beta-\tfrac{1}{2}\beta^2)E_{23} +(2-\beta-\tfrac{1}{2}\beta^2)E_{56} \}\\
R_{24} &= \frac{\lambda^2}{24}\{ (\beta-\tfrac{1}{2}\beta^2)E_{24} -(\beta+\tfrac{1}{2}\beta^2)E_{15} \}\\
R_{25} &= \frac{\lambda^2}{24}\{ 4E_{25} +(-4+\tfrac{1}{2}\beta^2)E_{14} +\tfrac{1}{2}\beta^2E_{36} \}\\
R_{26} &= \frac{\lambda^2}{24}\{ (2+\beta-\tfrac{1}{2}\beta^2)E_{26} +(2-\beta-\tfrac{1}{2}\beta^2)E_{35} \}\\
R_{34} &= \frac{\lambda^2}{24}\{ (\beta-\tfrac{1}{2}\beta^2)E_{34} -(\beta+\tfrac{1}{2}\beta^2)E_{16} \}\\
R_{35} &= \frac{\lambda^2}{24}\{ (2+\beta-\tfrac{1}{2}\beta^2)E_{35} +(2-\beta-\tfrac{1}{2}\beta^2)E_{26} \}\\
R_{36} &= \frac{\lambda^2}{24}\{ 4E_{36} +(-4+\tfrac{1}{2}\beta^2)E_{14} +\tfrac{1}{2}\beta^2E_{25} \}\\
R_{45} &= \frac{\lambda^2}{24}\{ (\beta-\tfrac{1}{2}\beta^2)E_{45} -(\beta+\tfrac{1}{2}\beta^2)E_{12} \}\\
R_{46} &= \frac{\lambda^2}{24}\{ (\beta-\tfrac{1}{2}\beta^2)E_{46} -(\beta+\tfrac{1}{2}\beta^2)E_{13} \}\\
R_{56} &= \frac{\lambda^2}{24}\{ (2+\beta-\tfrac{1}{2}\beta^2)E_{56} +(2-\beta-\tfrac{1}{2}\beta^2)E_{23} \}
\end{aligned} \tag{A1.20}
$$

The $SU(2)xU(1)$ Yang-Mills ansatz F2 is

$$
\begin{aligned}
F^1 &= \frac{\lambda^2}{12}\{ E_{26} + E_{35} \}\\
F^2 &= \frac{\lambda^2}{12}\{ E_{23} + E_{56} \}\\
F^3 &= \frac{\lambda^2}{12}\{ E_{25} - E_{36} \}\\
F^4 &= \frac{\lambda^2}{12}\{ 2E_{14} - E_{25} - E_{36} \}
\end{aligned} \tag{A1.21}
$$



We find a cyclic basis via

$$
\begin{aligned}
T_1 &= \tfrac{i}{\sqrt{2}}(E_A + E_{-A}) , & T_2 &= \tfrac{i}{\sqrt{2}}(E_B + E_{-B}) , & T_3 &= \tfrac{i}{\sqrt{2}}(E_C + E_{-C}) \\
T_4 &= \tfrac{1}{\sqrt{2}}(E_A - E_{-A}) , & T_5 &= \tfrac{1}{\sqrt{2}}(E_B - E_{-B}) , & T_6 &= \tfrac{1}{\sqrt{2}}(E_C - E_{-C}) \\
T_7 &= \tfrac{i}{\sqrt{2}}(E_D + E_{-D}) , & T_8 &= \tfrac{i}{\sqrt{2}}(E_E + E_{-E}) , & T_9 &= \tfrac{i}{\sqrt{2}}(E_F + E_{-F}) \\
T_{10} &= \tfrac{1}{\sqrt{2}}(E_D - E_{-D}) , & T_{11} &= \tfrac{1}{\sqrt{2}}(E_E - E_{-E}) , & T_{12} &= \tfrac{1}{\sqrt{2}}(E_F - E_{-F}) \\
T_{13} &= iH_1 , & T_{14} &= iH_2
\end{aligned}
\tag{A1.22}
$$

We find the structure constants are cyclic and we normalise them leaving the following

$$
\begin{aligned}
&C_{234} = C_{135} = -C_{456} = -C_{126} = +\frac{1}{\sqrt{12}} \\
&C_{159} = C_{167} = C_{23,11} = C_{268} = C_{358} = C_{45,12} = -\frac{1}{4} \\
&C_{12,12} = C_{13,10} = C_{249} = -C_{46,10} = C_{347} = -C_{56,11} = +\frac{1}{4} \\
&C_{25,14} = C_{36,14} = +\frac{1}{2\sqrt{12}} , \quad C_{14,14} = +\frac{1}{\sqrt{12}} , \quad C_{25,13} = -C_{36,13} = -\frac{1}{4} \\
&C_{78,14} = C_{9,12,14} = +\frac{\sqrt{3}}{4} , \quad C_{8,11,13} = +\frac{1}{2} , \quad C_{9,12,13} = -C_{7,10,13} = +\frac{1}{4} \\
&C_{78,12} = C_{79,11} = -C_{89,10} = -C_{10,11,12} = +\frac{1}{4}
\end{aligned}
\tag{A1.23}
$$

We find (5.11)-(5.14) are all satisfied with

$$n_1 = \frac{1}{3}\ , \ n_2 = \frac{1}{3}$$
$$n_3 = \frac{1}{4}\ , \ n_4 = \frac{3}{4} \qquad (A1.24)$$

We should note that our structure constants are different from those of Lust [26]. Analysis of those used by Lust shows that the Bianchi identity

$$[T_a,[T_b,T_c]] + [T_b,[T_c, T_a]] + [T_c,[T_a, T_b]] = 0$$

is not satisfied when we choose a= 2 ,b= 9 and c= 11 (this is Lust's numbering). In [26] it is claimed that for G2/SU(3) the dH equation is satisfied without constraint because

$$dH \sim C^a{}_{bc} C_{ade} E^{bcde}$$

and for those structure constants used this is zero. However we find using our structure constants that dH $\neq$ 0 so we do obtain a constriant from the dH equation

We give the curvature two forms for $G_2/SU(3)$

$$R_{12} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{12} +(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{45} \}$$
$$R_{13} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{13} +(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{46} \}$$
$$R_{14} = \frac{\lambda^2}{24}\{ 2E_{14} +(1-\beta^2)(E_{25} -E_{36}) \}$$
$$R_{15} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{15} +(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{24} \}$$
$$R_{16} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{16} +(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{34} \}$$
$$R_{23} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{23} -(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{56} \}$$
$$R_{24} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{24} +(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{15} \}$$
$$R_{25} = \frac{\lambda^2}{24}\{ 2E_{25} +(1-\beta^2)(E_{14} -E_{36}) \}$$
$$R_{26} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{26} -(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{35} \}$$
$$R_{34} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{34} +(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{16} \}$$
$$R_{35} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{35} -(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{26} \}$$
$$R_{36} = \frac{\lambda^2}{24}\{ 2E_{36} +(1-\beta^2)(E_{14} -E_{25}) \}$$
$$R_{45} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{45} +(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{12} \}$$
$$R_{46} = \frac{\lambda^2}{24}\{ (\tfrac{3}{2}+\beta-\tfrac{1}{2}\beta^2)E_{46} +(-\tfrac{3}{2}+\beta+\tfrac{1}{2}\beta^2)E_{13} \}$$

$$R_{56} = \frac{\lambda^2}{24} \left\{ \left(\frac{3}{2} + \beta - \frac{1}{2}\beta^2\right) E_{56} - \left(-\frac{3}{2} + \beta + \frac{1}{2}\beta^2\right) E_{23} \right\}$$

(A1.11)

If we take the case $\beta = 1$ (ie take our space to be torsion free) we find that

$$R_{AB} = \frac{\lambda^2}{12} E_{AB} \qquad \text{(A1.25)}$$

This is the same curvature as the six-sphere ($G_2/SU(3)$ is isomorphic to the six-sphere).

And the SU(3) Yang-Mills ansatz F2 for the coset space is

$$F^1 = \frac{\sqrt{3}}{24} \lambda^2 \left(E_{16} - E_{34} \right)$$

$$F^2 = \frac{\sqrt{3}}{24} \lambda^2 \left(E_{26} + E_{35} \right)$$

$$F^3 = \frac{\sqrt{3}}{24} \lambda^2 \left(E_{15} - E_{24} \right)$$

$$F^4 = \frac{\sqrt{3}}{24} \lambda^2 \left(-E_{13} + E_{46} \right)$$

$$F^5 = \frac{\sqrt{3}}{24} \lambda^2 \left(E_{23} + E_{56} \right)$$

$$F^6 = \frac{\sqrt{3}}{24} \lambda^2 \left(-E_{12} + E_{45} \right)$$

$$F^7 = \frac{\sqrt{3}}{24} \lambda^2 \left(-E_{25} + E_{36} \right)$$

$$F^8 = -\frac{\lambda^2}{24} \left(2E_{14} + E_{25} + E_{36} \right) \qquad \text{(A1.26)}$$

Appendix 2 The case of $Sp(4)/\{SU(2)xU(1)\}$

We shall now take a look at the case of $Sp(4)/\{SU(2)xU(1)\}$ Which is different from the other two coset spaces because of two facts

1. $tr(R \wedge R)$ was not proportional to dH
2. $R_{AB} \wedge R_{CD} \wedge i^{E} *E^{ABCD}$ is not simply proportional to $*E^{E}$

Why is there a problem ? if we take 1. first

now

$$R^{a}{}_{b} = R^{a\,0}_{b} + \beta R'^{a}{}_{b} + \beta^{2} R^{2a}{}_{b} \qquad \text{(see 5.20)} \qquad \text{(A2.1)}$$

$R'^{a}{}_{b}$ and $R^{2a}{}_{b}$ involve only the S-R structue constants $C^{a}{}_{bc}$ and so since these are essentially the same for all three coset spaces (see Appendix 1) the difficulty cannot arise from these terms. the problem must must be with R^{0} which involves $C^{\bar{a}}{}_{bc}$ which are different .So we need only look at $R^{0} \wedge R^{0}$ terms (mixed terms $R^{0}R^{1}$ and $R^{0}R^{2}$ do not contribute)

$$\begin{aligned} tr(R \wedge R) &= \frac{\lambda^4}{4} C_{ab\bar{c}} C^{\bar{c}}{}_{de} C^{ab}{}_{\bar{f}} C^{\bar{f}}{}_{gh} E^{degh} \\ &= \frac{\lambda^4}{4} C_{ab\bar{c}} C^{ab}{}_{\bar{f}} C^{\bar{c}}{}_{de} C^{\bar{f}}{}_{gh} E^{degh} \end{aligned} \qquad \text{(A2.2)}$$

if $C_{ab\bar{c}} C^{ab}{}_{\bar{f}} = n_3 \delta_{\bar{c}\bar{f}}$ ie (5.13) is satisfied then we simply find $tr(R \wedge R)$ proportional to dH. However for $Sp(4)/\{SU(2)xU(1)\}$ we find this is not obeyed $C_{ab\bar{c}} C^{ab}{}_{\bar{f}}$ is still proportional to $\delta_{\bar{c}\bar{f}}$ but the factor is different for the two case - $\bar{c}$ a SU(2) index and $\bar{c}$ a U(1) index

$$C_{ab\bar{c}} C^{ab}{}_{\bar{f}} = n_3(\hat{c}) \delta_{\bar{c}\bar{f}} \text{ for } \bar{c} \text{ a SU(2) index} \qquad \text{(A2.3)}$$

$$C_{ab\bar{c}} C^{ab}{}_{\bar{f}} = n_3(\overset{o}{c}) \delta_{\bar{c}\bar{f}} \text{ for } \bar{c} \text{ a U(1) index} \qquad \text{(A2.4)}$$

splitting our $\bar{c}$ indices up into $\tilde{c}$ for SU(2) and c^{o} for U(1) we can write

$$\begin{aligned} tr(R \wedge R) &= \frac{\lambda^4}{4}\{ n_3(\hat{c}) C_{\hat{e}de} C^{\hat{e}}{}_{gh} E^{degh} + n_3(\overset{o}{c}) C_{\overset{o}{e}de} C^{\overset{o}{e}}{}_{gh} E^{degh} \} \\ &= \frac{\lambda^4}{4} n_3(\hat{c}) C_{\hat{e}de} C^{\hat{e}}{}_{gh} E^{degh} \quad + \end{aligned}$$

$$+ \ (n_3(\overset{\circ}{c})-n_3(\tilde{c}))C_{\ell de}\, C^{c^{\circ}}{}_{gh} E^{degh} \} \qquad (A2.5)$$

The first term is just proportional to dH as usual however the second is not. How can we solve this problem ? If we take the case of the Yang-Mills fields being case F2 then

$$\begin{aligned} tr(F\wedge F) &= -\chi^2_{su(2)} F^{\tilde{c}} \wedge F^{\tilde{c}} - \chi^2_{u(1)} F^{c^\circ} \wedge F^{c^\circ} \\ &= -\chi^2_{su(2)} F^{\bar{c}} \wedge F^{\bar{c}} - (\chi^2_{u(1)} - \chi^2_{su(2)}) F^{c^\circ} \wedge F^{c^\circ} \qquad (A2.6) \end{aligned}$$

the first term is just proportional to dH the second is

$$-(\chi^2_{u(1)} - \chi^2_{su(2)}) \frac{\lambda^4}{4} C_{\ell de}\, C^{c^\circ}{}_{gh} E^{degh} \qquad (A2.7)$$

This can cancel the problem in tr(R $\wedge$ R) provided

$$\{ n_3(\tilde{c})-n_3(\overset{\circ}{c}) \} = \{ \chi^2_{su(2)} - \chi^2_{u(1)} \} \qquad (A2.8)$$

Taking the values of n_3 from appendix 1 and the values of χ^2 from appendix 3 we do indeed find that the difficulties in the curvature terms are cancelled by the Yang-Mills fields.

Returning to problem 2. we find that again provided (A2.8) is satisfied we have a cancelation of our awkward terms (this is not unexpected $R_{AB} \wedge R_{CD} \wedge *E^{ABCD}$ is closely related to tr (R $\wedge$ *R) and if the U(1) part of tr(R $\wedge$ R) is cancelled by that of tr(F $\wedge$ F) then we might expect a similar cancellation with tr(R $\wedge$ *R) and tr(F $\wedge$ *F)

If we wish to deal with a generalisation of case F1 we have no SU(2) field and we find (A2.8) reduces to

$$\{ n_3(\overset{\circ}{c}) - n_3(\tilde{c}) \} = \chi^2_{u(1)}$$

This means $\chi^2_{u(1)} = 2/3$. Now we can (effectivly) have for our U(1) field fields which give an integral multiple of the fundamental value which is 1 but we cannot have fractions so this will not work ie having just a U(1) field will not work.

Can we have a generalisation of F3 namely a Sp(4)xU(1) field ? If we do this we find the anologue of (A2.8) is

$$\{ n_3(\overset{\circ}{c}) - n_3(\hat{c}) \} = \chi^2_{U(1)} + \chi^2_{Sp(4)}$$

This again will require $\chi^2_{U(1)}$ being a fraction and so will not work (The problem with the $R_{AB} \wedge R_{CD} \wedge i_E \cdot E^{ABCDE}$ term will in fact be remedied by using a Sp(4)xU(1) field).

So we can only take case F2 for the coset space Sp(4)/SU(2)xU(1) the other two cases leading to extra constraints. We find for our ansatz that we can just treat this case along with the others but we must use normalisations appropriate for the SU(2) part alone. In particular we find we should in the following equation use the value of 1/3 for χ^2 which is the SU(2) value.

$$K_1 = 12 - 12\chi^2 \quad , \; K_2 = 4/3 - 4\chi^2$$

(See p53 for the definition of K_i)

giving $K_1 = 8$ and $K_2 = 0$

We can now treat this coset space on the some footing as the other two coset spaces. This is done within the text.

Appendix 3 Normalisation of the Yang-Mills generators

In Chapter 5 it is importortant to obtain the correct normalisation of the generators of the Yang-Mills fields. We have two ansatzes for a coset space S/R . The Yang-Mills potential can be imbedded as a R-field (F2)or as a S-field (F3) Our total gauge group can be $E_8 x E_8$ SO(32)/Z_2 or SO(32).

Assuming our structure constants are normalised so that

$$C^{\hat{a}}{}_{\hat{b}\hat{c}}\, C^{\hat{d}\hat{b}\hat{c}} = +\,\delta^{\hat{a}\hat{d}}$$
$$C^{\bar{a}}{}_{\bar{b}\bar{c}}\, C^{\bar{d}\bar{b}\bar{c}} = +n_4\, \delta^{\bar{a}\bar{d}} \qquad (A3.1)$$

Hence in the adjoint representation

$$Tr_S(Q_{\hat{a}} Q_{\hat{d}})_{adj} = -\,\delta^{\hat{a}\hat{d}}$$
$$Tr_R(Q_{\bar{a}} Q_{\bar{d}})_{adj} = -n_4\, \delta_{\bar{a}\bar{d}} \qquad (A3.2)$$

The ratio $n_4/1$ is independent of the normalisation used for the structure constants and is , by definition, the ratio of the 'second index of the representation's ie

$$I_2\{ adj(R) \}/I_2\{ adj(S) \} \qquad (A3.3)$$

We will be imbedding S and R into E_8 and we will need the fact that [47]

$$I_2\{ fund(E_8) \} = I_2(adj(E_8) = 60 \qquad (A3.4)$$

In the lagrangian we have written tr( ) for E_8 we mean by this

$\frac{1}{30}\, Tr_{E_8}(\quad)_{fund}$ (this was mentioned on P29)

So to calculate χ^2 we must evaluate ,for the case of a R-field,

$$\frac{1}{30} Tr_{E_8}\{ Q_{\bar{a}} Q_{\bar{b}}\} \qquad (A3.5)$$

This is

$$= \frac{1}{30} tr_R \{ Q_{\bar{a}} Q_{\bar{b}})_{fund} \;(I_2(fund(E_8)/I_2(fund(R))$$
$$= \frac{1}{30} tr_R \{ Q_{\bar{a}} Q_{\bar{b}})_{adj} \cdot \frac{I_2(fund(R)) \,.\, I_2(fund(E_8))}{I_2(adj(R)) \,.\, I_2(fund(R))}$$

Using (A3.2) we can deduce

$$=+\delta_{\bar{a}\bar{d}}\{ \quad -2n_4 \quad /I_2 \quad (\quad adj(R) \quad) \quad \}.$$

(A3.6)

For an S-field we will obtain

$$= \{ \quad -2/I_2(\; adj(S) \;) \quad \} \, \delta_{\bar{a}\bar{d}} \tag{A3.7}$$

For $G_2/SU(3)$ [47]

$n_4 = \frac{3}{4}$, $I_2(\; adj(SU(3) \;) = 6$ & $I_2(\; adj(G \;) \;) = 8$

so we obtain

$$\begin{aligned} \frac{1}{30} Tr_{E_8}\{ \; Q_{\bar{a}}Q_{\bar{b}}\}_{fund} &= -\frac{1}{4} \\ \frac{1}{30} Tr_{E_8}\{ \; Q_{\hat{a}}Q_{\hat{b}}\}_{fund} &= -\frac{1}{4} \end{aligned} \tag{A3.8}$$

For $SU(3)/\{U(1)xU(1)\}$

The U(1) generators have a normalisation within E_8 given by

$$\frac{1}{30} Tr_{E_8}(\; Q_7Q_7)_{fund} = \quad Tr_{E_8}(\; Q_8Q_8 \;)_{fund} = -1 \tag{A3.9}$$

Since we have U(1) solutions we can take $\chi^2 = -m$ where m is the monopole number for the U(1) field over S^2 and CP^2 repsectively.

As $I_2(\; adj(SU(3) \;) = 6$ We will have

$$\frac{1}{30} Tr_{E_8}(\; Q_{\hat{a}}Q_{\hat{b}})_{fund} = -\frac{1}{3} \, \delta_{\hat{a}\hat{b}} \tag{A3.10}$$

For $Sp(4)/\{SU(2)xU(1)\}$

We will obtain different normalisations for the SU(2) and U(1). We have (for the SU(2))

$n_4 = \frac{2}{3}$, $I_2(\; adj(SU(2) \;) = 4$ and $I_2(\; adj(Sp(4)) \;) = 6$

So we obtain ($\tilde{a}$ a SU(2) index)

$$\begin{aligned} \frac{1}{30} Tr_{E_8}(\; Q_{\tilde{a}}Q_{\tilde{b}})_{fund} &= -\frac{1}{3} \, \delta_{\tilde{a}\tilde{b}} \\ \frac{1}{30} Tr_{E_8}(\; Q_{\hat{a}}Q_{\hat{b}})_{fund} &= -\frac{1}{3} \, \delta_{\hat{a}\hat{b}} \end{aligned} \tag{A3.11}$$

For the U(1) we will have

$$\frac{1}{30}\, Tr_{E_8}(\ Q_{10} Q_{10})_{fund} = \quad -1 \qquad (A3.12)$$

For the U(1) we again have $\chi^2 = n$ where n is the monopole charge.

We also have imbeddings within SO(32) to consider. In the lagrangian we do not have factors of 1/30 appearing so we wish to evaluate

$$Tr_{SO(32)}(\ Q_{\bar{a}} Q_{\bar{b}})_{fund}$$

$$= tr_R(\ Q_{\bar{a}} Q_{\bar{b}})_{fund} \ \frac{I_2(\ fund(\ R\ )\ )\ .\ I_2(\ fund(SO(32))}{I_2(\ adj(\ R\ )\ )\ .\ I_2(\ fund(\ R\ )\ )}$$

as in (A3.5) Since $I_2($ fund(SO(32)) = 2 we have

$$= -(\ 2n_4/I\ (\ adj(R)\ )\ )\,\delta_{\bar{a}\bar{b}} \qquad (A3.12)$$

exactly as for the $E_8 x E_8$ case. So the factors we obtain are identical in both cases so in chapter 5 they can be considered together

In chapter 7 we dealt with symmetric coset spaces S/R and we again imbed the Yang-Mills potential as a R field and agian it is importrant to know the normalisations. We calculate the appropriate normalisations as above. We have U(1)s appearing in R for several of the cases so we must be careful to define exactly what we are normalising. In the following list we give the normalisations appropriate to that required in Chapter 7 ie if R is of the form

(simple group) x $U(1)^p$

Then we are interested in the simple group for normalisation purposes. (As for Sp(4)/SU(2)xU(1) which was dealt with in detail in Appendix 2).

Coset Space	χ^2:
SO(7)/SO(6)	-1/5
SU(3)xSU(2)/U(2)xU(1)	-1/5
SU(4)/SU(3)xU(1)	-1/4
Sp(4)/SU(2)xU(1)	-1/3
Sp(4)xSU(2)/SO(4)xU(1)xU(1)	-1/5
SU(2)xSU(2)xSU(2)/U(1)xU(1)xU(1)	-(integer)

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