Connected Hopf Algebras of Finite Gelfand-Kirillov Dimension

by

Paul Gilmartin

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Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Abstract

Following the seminal work of Zhuang in [67], connected Hopf algebras of finite GK-dimension over algebraically closed fields of characteristic zero have been the subject of several recent papers - see [4], [7] and [62], for example. This thesis is concerned with continuing this line of research and promoting connected Hopf algebras as a natural, intricate and interesting class of algebras.

We begin by discussing the theory of connected Hopf algebras which are either commutative or cocommutative, and then proceed to review the modern theory of arbitrary connected Hopf algebras of finite GK-dimension initiated by Zhuang in [67].

We next focus on the (left) coideal subalgebras of connected Hopf algebras of finite GK-dimension. They are shown to be deformations of commutative polynomial algebras. A number of homological properties follow immediately from this fact. Further properties are described, examples are considered and invariants are constructed.

Let H be a connected Hopf algebra satisfying the equation $\operatorname{GKdim} H = \dim_k P(H) + 1 < \infty$, where $P(H)$ denotes the primitive space of H. Building on the results of [62], we describe a method of constructing such a Hopf algebra, and as a result obtain a host of new examples of such objects. Moreover, we prove that such a Hopf algebra can never be isomorphic to the enveloping algebra of a semisimple Lie algebra, nor can a semisimple Lie algebra appear as its primitive space.

It is asked in [3, Question L] whether connected Hopf algebras of finite GK-dimension are always isomorphic as algebras to enveloping algebras of Lie algebras. We provide a negative answer to this question by constructing a counterexample of GK-dimension 5.

Substantial progress was made in determining the order of the antipode of a finite dimensional pointed Hopf algebra by Taft and Wilson in [54]. Our final main result is to show that the proof of the main result of [54] can be generalised to give an analogous result for arbitrary pointed Hopf algebras.

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Contents

Chapter 1

Introduction

Let k be a field. A Hopf k-algebra is said to be *connected* if its unique simple subcoalgebra is just the base field k. The terminology "connected" dates back to the 1960s, when the notion of a Hopf algebra was first abstracted from the work of Heinz Hopf on the homology of H-spaces: such a space X is connected (as a topological space) if and only if $H_*(X, k)$ is connected as Hopf algebra, [39, p262]. Following this seminal work of Hopf, it was found that connected Hopf algebras appear ubiquitously in several branches of algebra. In algebraic geometry, one obtains from a unipotent group U a natural commutative connected Hopf algebra structure on its coordinate ring $\mathcal{O}(U)$, and in Lie theory one obtains from a Lie algebra $\mathfrak{g}$ a natural cocommutative connected Hopf structure on its enveloping algebra $U(\mathfrak{g})$. This ubiquity in algebra led to the research of connected Hopf algebras as a class of algebras in their own right, with the work done on connected Hopf algebras in the second half of the 20th century mainly focussing on the case where they were also either commutative or cocommutative - see [39], for example. In characteristic zero, a complete classification of such connected Hopf algebras had been achieved before the end of the 1960s. The details of this classification can be found in Chapter 3.

Research into connected Hopf algebras in *positive* characteristic remained active throughout the remainder of the 20th century (see [21], for example), and has done to this day (see [63], for example). However, possibly due to a lack of new examples, after the 1960s classification of connected Hopf algebras which are either commutative or cocommutative, very little progress was made in understanding arbitrary connected Hopf algebras over a field of characteristic zero. That is, until 2012, when G. Zhuang released a paper, [67], in which he dropped the classical assumptions of commutativity or cocommutativity, and instead focused on describing the properties of connected Hopf algebras which have finite GK-dimension. Zhuang's paper included several new beautiful results concerning connected Hopf algebras, and, perhaps most importantly, listed several examples of connected Hopf algebras which were neither commutative nor cocommutative. This work served to reinvigorate the mathematical community's interest in connected Hopf algebras (over fields of characteristic 0), ultimately leading to the publication of several further papers on the matter - see [4], [5], [7] and [62], for example.

It is the ambition of the author that, by detailing both the classical and contemporary theory of connected Hopf algebras in characteristic zero (Chapters 2 - 4), as well as presenting several new results, examples and open questions (Chapters 5 - 8), this thesis will further propagate the recently revitalised notion that connected Hopf algebras in characteristic zero are a class of algebras worth studying in their own right.

1.1 Assumptions and notation

Throughout this thesis, k will denote an algebraically closed field of characteristic 0, unless otherwise stated. All tensor products and vector spaces are assumed to be over k, and all mappings are assumed to be k-linear.

Whenever we speak of an *algebra* we mean a unital associative k-algebra A, and denote by $m : A \otimes A \to A$ the map defining the multiplication in A and by $u : k \to A$ the map defining the unit of A. Given an algebra A and a subspace $V \subseteq A$, the *algebra generated by* V is defined to be the smallest subalgebra of A containing V, denoted by $k\langle V \rangle$.

Given vector spaces V and W, we define the *flip map* as the map $\tau : V \otimes W \to W \otimes V$ such that $\tau(v \otimes w) = w \otimes v$ for all $v \in V, w \in W$.

1.2 Summary

In this section we summarise the layout, aims and results of each chapter in this thesis.

Chapter 2, entilted "Preliminaries", is entirely expository. It includes several of the foundational definitions, objects and methodologies which shall be used repeatedly throughout this thesis.

In Chapter 3 we begin by giving the precise definition of a connected Hopf algebra, as well as detailing some of their basic well known properties. We then go on to discuss "classical" connected Hopf algebras, that is, connected Hopf algebras which are either commutative or cocommutative. We shall see that, in characteristic zero, such Hopf algebras are very well understood: in the commutative setting, an affine connected Hopf algebra is the coordinate ring of an affine unipotent algebraic group (Theorem 3.3.6). In the cocommutative setting, a connected Hopf algebra is the universal enveloping algebra of its Lie algebra of primitive elements (Theorem 3.4.5).

As previously mentioned, after the work done in the early second half of the 20th century, no real advances were made in the understanding of connected Hopf algebras in characteristic zero until 2012, when G. Zhuang released a paper, [67], titled "Properties of pointed and connected Hopf algebras of finite Gelfand-Kirillov dimension". There, Zhuang showed that if H is a connected Hopf algebra of finite Gelfand-Kirillov dimension, then, with respect to its coradical filtration, the associated graded algebra of H is a polynomial algebra in $\mathrm{GKdim}\, H$ variables (see Theorem 4.2.7). A filtered-graded result like this has important homological and ring theoretic consequences for H, as shown in Proposition 4.2.12. To accompany these general results, Zhuang was also able to provide a classification of connected Hopf algebras of Gelfand-Kirillov dimension at most three. Amongst these are the first known examples of connected Hopf algebras over a field of characteristic zero which are neither commutative nor cocommutative (see §4.3.3). In Chapter 4 we give a detailed account of Zhuang's results and how they were proved. This background is necessary as we will be applying and generalising these results and methods in subsequent chapters of this thesis.

Following Zhuang's seminal paper discussed in Chapter 4, connected Hopf algebras of finite Gelfand-Kirillov dimension have been the subject of several recent papers, see for example [5], [7], [62]. None of these, however, has examined their coideal subalgebras, so the primary aim of Chapter 5 is to lay out their basic properties and clarify topics for future research. Our main result in this chapter is to show that if H is a connected Hopf algebra of finite Gelfand-Kirillov dimension and $T \subseteq H$ is a left (or right) coideal subalgebra, then, with respect to its coradical filtration, the associated graded algebra of T is a polynomial algebra in $\mathrm{GKdim}\, T$ variables. This generalises a result of Zhuang's discussed in Chapter 4, Theorem 4.2.7, which considered the case $T = H$. As before, a filtered-graded result like this has important algebraic consequences for T, detailed in Theorem 5.5.5 and §5.5.4. In §5.3 we describe what is known about various particular subclasses of coideal subalgebras T of connected Hopf algebras H of finite GK-dimension. Thus, we discuss the cases where H or T is commutative; where H or simply T is cocommutative; where $\mathrm{GKdim}\, T \leq 2$; and where $\mathrm{GKdim}\, H \leq 3$. We note also that, given that $\mathrm{gr}H$ and $\mathrm{gr}T$ are graded polynomial algebras, their homogeneous generators have specific degrees whose multisets of values constitute invariants $\sigma(T)$ and $\sigma(H)$ of T [resp. H]. We call $\sigma(T)$ the *signature* of T, and write $\sigma(T) = (e_1^{(r_1)}, \ldots, e_s^{(r_s)})$, where e_i and r_i are positive integers with $e_1 < e_2 < \cdots < e_s$, and the term $e_i^{(r_i)}$ indicates that the degree e_i occurs r_i times among the graded polynomial generators of gr$T$. Basic properties of these signature are listed in §5.7. We view one important function of this invariant as being to provide a framework for future work on connected Hopf algebras. We discuss this aspect further in §5.8. The majority of the content of Chapter 5 is based on the work done in [4], to which the author contributed.

A connected Hopf algebra H is said to be *primitively thick* if

$$\operatorname{GKdim} H = \dim_k P(H) + 1 < \infty$$

where $P(H) := \{x \in H : \Delta(x) = 1 \otimes x + x \otimes 1\}$ is the space of primitive elements of H. The study of primitively thick Hopf algebras can be viewed as a natural starting point for an investigation into the general properties of connected Hopf algebras which are noncocommutative (see Proposition 4.2.11). Primitively thick Hopf algebras were first studied in their own right in [62], the main result of which states that, if H is a primitively thick Hopf algebra, there exists a canonical coassociative Lie algebra $P_2(H) \subset H$ such that $H = U_{CLA}(P_2(H))$ (see §6.2.1 for the relevant definitions). The results of [62], however, do not answer the question of describing, in Lie theoretic terms, precisely which Lie algebras can appear as the space $P(H)$ or $P_2(H)$ of H, and so the primary aim of Chapter 6 is to address this matter. In §6.3, using the results of [62], we derive some necessary conditions which a finite dimensional Lie algebra must satisfy in order to be the primitive space of a primitively thick Hopf algebra, and as an immediate consequence of this prove that such Lie algebra can never be semisimple. In §6.4 we define, in purely Lie theoretic terms, a class of Lie algebras, which includes all finite dimensional nilpotent Lie algebras, which we call *admissible* Lie algebras. We then proceed describe a recipe which, given an admissible Lie algebra $\mathfrak{g}$, constructs a primitively thick Hopf structure on the algebra $U(\mathfrak{g})$, and moreover prove that all primitively thick Hopf algebras can be constructed via this recipe.

In [62], Wang, Zhang and Zhuang classify (up to isomorphism) all connected Hopf algebras of GK-dimension at most four, over an algebraically closed field of characteristic 0. Moreover, they are able to show that the Hopf algebras appearing in this classification all have one curious feature in common: each is isomorphic, as an algebra, to the enveloping algebra of a finite dimensional Lie algebra. The same can also be said for any commutative or cocommutative connected Hopf algebra of arbitrary (finite) GK-dimension (see Theorem 3.3.6 and Theorem 3.4.5). It is asked in [3, Question L] whether connected Hopf algebras of finite Gelfand-Kirillov dimension are always isomorphic as algebras to enveloping algebras of Lie algebras. In Chapter 7 we provide a negative answer to this question by constructing a connected Hopf algebra of Gelfand-Kirillov dimension 5 and proving that it is not isomorphic to the universal enveloping algebra of a Lie algebra.

In the final chapter of this thesis, the subject matter differs somewhat from that of the previous chapters, which focused on properties of connected Hopf algebras. In Chapter 8, we examine a much wider class of Hopf algebras known as pointed Hopf algebras, of which connected Hopf algebras form a proper subclass. Motivated by determining the order of the antipode of a connected Hopf algebra, we generalise our aims and explore the matter of determining the order of the antipode of a pointed Hopf algebra over an arbitrary field. Substantial progress was made in determining the order of the antipode of a finite dimensional pointed Hopf algebra H by Taft and Wilson in [55], where the order of the antipode was described in terms of the order of the group of group-likes of H, $G(H)$. The main result of Chapter 8, Theorem 8.1.1, shows that the proof of the main result of [55] can be generalised to give an analogous result for arbitrary pointed Hopf algebras. For the sake of this thesis, an important consequence of this result is Corollary 8.4.4, which states that over a field of characteristic 0, the antipode of a connected Hopf algebra has order either 2 or ∞. Although perhaps well known, this result does not seem to be present in the current literature.

Chapter 2

Preliminaries

2.1 Introduction

This chapter will include some of the foundational definitions, objects and methodologies which shall be used repeatedly throughout this thesis.

Perhaps the most pertinent definition in a thesis about Hopf algebras is the definition of a Hopf algebra itself, which we state in §2.2.1. The precise definition of a Hopf algebra is perhaps an unusual one at first glance, but we see in §2.2.2 that several well known algebras, including enveloping algebras of Lie algebras and group algebras, have canonical Hopf algebra structures.

As will become apparent, a recurring theme in the methodology in the later chapters of this thesis when trying to prove that some mathematical object (a connected Hopf algebra, for example) satisfies a particular property $\mathcal{P}$, is to define a filtration on the object, construct the associated graded space with respect to that filtration, and to show that the associated graded object satisfies property $\mathcal{P}$. Often this is enough to prove that in fact the original object satisfies property $\mathcal{P}$. In §2.3 we begin by recalling the definitions of algebra, coalgebra and Hopf algebra filtrations and gradings. We then define the associated graded algebra (resp. coalgebra, resp. Hopf algebra) of a filtered algebra (resp. coalgebra, resp. Hopf algebra) and state some properties which "pass" from the associated graded algebra of a filtered algebra to the filtered algebra itself.

The *coradical* of a Hopf algebra H is defined as the sum of its simple subcoalgebras. Using the coradical of H and proceeding inductively, we can define a coalgebra filtration of H known as the *coradical filtration.* In Section 2.3.2 we give the precise definition and list some basic properties of the coradical filtration. As we shall see later, the use of the "associated-graded" method outlined in the previous paragraph is particularly fruitful when applied to the coradical filtration of a connected Hopf algebra, the main object of study in this thesis.

Next we discuss some elementary algebraic geometry, mainly focusing on describing the classical correspondence between affine algebraic groups and affine commutative Hopf algebras. A solid foundation in the theory of commutative Hopf algerbras will be imperative in order to understand the noncommutative Hopf algebras which we investigate in the later chapters.

The final sections of this chapter, §2.5 and §2.6, contain some general definitions and results which we call on for various reasons throughout this thesis. In §2.5.1, we discuss the concept of the Gelfand-Kirillov dimension (GK-dimension) of an algebra, which, as the title of this thesis suggests, will play a pivotal role in later chapters. The definitions of AS-regular and GK-Cohen-Macaulay algebras are recalled in §2.5.2. These definitions will be needed later when proving that connected Hopf algebras enjoy these homological properties. The final section of this chapter, §2.6, describes a one to one correspondence between the normal Hopf subalgebras and the normal Hopf ideals for a wide class of Hopf algebras. This correspondence will be utilised most in Chapter 5 when studying the "quantum homogeneous spaces" of connected Hopf algebras.

2.2 Algebras, coalgebras and Hopf algebras

2.2.1 Definitions

In this section we define the fundamental object of study in this thesis - a Hopf algebra. Before we can do this, we must define a *coalgebra*, which is the (categorical) dual of an algebra.

Definition 2.2.1. Let C be a vector space. We say that C is a *coalgebra* if there exist linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ such that

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \qquad \text{(Coassociativity)}$$

and

$$\mu(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id} = \mu(\mathrm{id} \otimes \epsilon)\Delta \qquad \text{(Counit)}$$

where μ denotes the canonical vector space isomorphism given by scalar multiplication.

Dualising the notion of commutativity for an algebra, we obtain the notion of cocommutativity for coalgebras, which we define below.

Definition 2.2.2. Let C be a coalgebra. We say that C is *cocommutative* if $\tau\Delta = \Delta$.

Next, we define bialgebras.

Definition 2.2.3. Let B denote an algebra. We say that B is a *bialgebra* if there exist algebra homomorphisms $\Delta : B \to B \otimes B$ and $\epsilon : B \to k$ such that (B, Δ, ϵ) forms a coalgebra.

Just as we obtained the definition of a coalgebra by dualising categorically the definition of an algebra, we can dualise notions such as algebra morphisms, ideals and modules to obtain coalgebra morphisms, coideals and comodules, and furthermore combine them so that we can talk about bialgebra and morphisms or biideals. The precise definitions of such objects are given below. For further details and properties, see [41, Chapter 1].

Definition 2.2.4. Let C and D be coalgebras, with comultiplications Δ_C and Δ_D, and counits ϵ_C and ϵ_D, respectively.

1. A map $f : C \to D$ is a *coalgebra morphism* if $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\epsilon_C = \epsilon_D \circ f$.
2. A subspace $I \subseteq C$ is a *coideal* if $\Delta_C(I) \subseteq I \otimes C + C \otimes I$ and $\epsilon_C(I) = 0$.

Definition 2.2.5. Let B and B' be bialgebras. A map $f : B \to B'$ is a *bialgebra morphism* if it is both an algebra morphism and a coalgebra morphism. A subspace $I \subseteq B$ is a *biideal* if it is both an ideal and a coideal (or, equivalently, if B/I is a bialgebra).

We need one final definition before we can define Hopf algebras - the convolution product.

Definition 2.2.6. Let C be a coalgebra and A an algebra. Then $\mathrm{Hom}_k(C, A)$ becomes an algebra under the *convolution product*

$$\star : \mathrm{Hom}_k(C, A) \otimes \mathrm{Hom}_k(C, A) \to \mathrm{Hom}_k(C, A)$$

which, for all $f, g \in \mathrm{Hom}_k(C, A)$, $c \in C$, is defined as follows

$$(f \star g)(c) = m(f \otimes g)(\Delta(c)).$$

The unit of this algebra is the map $u\epsilon : C \to A$.

Remark 2.2.7. As a special case of the above, we see that for any coalgebra C, $C^* = \mathrm{Hom}_k(C, k)$ becomes an algebra under the convolution product. Furthermore, we can define a left C^*-module structure on C, known as the *left hit action*, by defining a map

$$\rightharpoonup : C^* \otimes C \to C$$

such that for $f \in C^*$, $c \in C$,

$$f \otimes c \mapsto f \rightharpoonup c = (f \otimes \mathrm{id}) \circ \tau \circ \Delta(c).$$

Similarly, there is also a right C^*-module structure on C by defining a map

$$\leftharpoonup : C \otimes C^* \to C$$

such that, for $f \in C^*, c \in C$,

$$c \otimes f \mapsto c \leftharpoonup f = (f \otimes \mathrm{id}) \circ \Delta(c).$$

For further details, see [41, Example 1.6.5].

Definition 2.2.8. Let H denote a bialgebra. Then H becomes a *Hopf algebra* if there exists a linear map $S : H \to H$ such that S is an inverse to id_H under the convolution operation $\star$ on $\mathrm{Hom}_k(H, H)$. We call such a map S an *antipode* for H.

The antipode of a Hopf algebra is always an anti-algebra and an anti-coalgebra morphism, [41, Proposition 1.5.10]. It is however in general *not* bijective, [59].

Definition 2.2.9. Let H and H' be Hopf algebras, with antipode S_H and $S_{H'}$, respectively. A map $f : H \to H'$ is a *Hopf morphism* if it is an algebra morphism, a coalgebra morphism (i.e. f is a bialgebra morphism) and satisfies the equation

$$f \circ S_H = S_{H'} \circ f.$$

A subspace I of H is a *Hopf ideal* if it is a biideal and $S(I) \subseteq I$ (or, equivalently, if H/I is a Hopf algebra).

Remark 2.2.10. Throughout this thesis, whenever we speak of a Hopf algebra H, we shall assume implicitly that it is equipped with a coproduct denoted Δ, a counit denoted ϵ and an antipode denoted S. We denote by H^+ the kernel of the counit, a Hopf ideal of H, known as the *augmentation ideal* of H. Analogous assumptions are also made for coalgebras and bialgebras throughout.

Notation 2.2.11. We use *Sweedler's sigma notation* (see [41, §1.4.4]) for the coproduct of a Hopf algebra (or coalgebra, or bialgebra): for a Hopf algebra (or coalgebra, or bialgebra) H and $h \in H$, we write

$$\Delta(h) = \sum h_1 \otimes h_2$$

and

$$(\Delta \otimes \mathrm{id})\Delta(h) = \sum h_1 \otimes h_2 \otimes h_3 = (\mathrm{id} \otimes \Delta)\Delta(h).$$

Remark 2.2.12. The following remark appears in [17, §1], and is very useful when trying to construct new Hopf algebras:

When defining a Hopf algebra structure on an algebra A given by generators and relations, and given that the maps in question, namely Δ, ϵ and S are algebra homomorphisms or anti-homomorphisms, it suffices to check the Hopf algebra axioms on a set of algebra generators for A. This is obvious for the counit and coassociativity axioms, which require that various algebra homomorphisms coincide. As for the antipode axiom, it suffices to check it on monomials in the algebra generators, and that follows from the case of a single generator by induction on the length of a monomial.

2.2.2 Examples

Although it may not be immediately clear from the definition, many classically known algebras have an underlying Hopf structure. Listed in this section are three prototypical examples of Hopf algebras.

Our first example of a Hopf algebra is the universal enveloping algebra of a Lie algebra. As we shall see later, this Hopf algebra will play a critical rôle in the study of connected Hopf algebras, the primary objects of study in this thesis.

Example 2.2.13. Let $(\mathfrak{g}, [-,-]_{\mathfrak{g}})$ be a Lie algebra. Recall that the *universal enveloping algebra* of $\mathfrak{g}$ is defined as the algebra

$$U(\mathfrak{g}) = F(\mathfrak{g})/J.$$

where $F(\mathfrak{g})$ is the free algebra generated by $\mathfrak{g}$ and J is the two-sided ideal of $F(\mathfrak{g})$ generated by elements of the form $xy - yx - [x,y]_{\mathfrak{g}}$, for $x, y \in \mathfrak{g}$. The terminology for this algebra derives from the fact that it satisfies the following universal property: for any associative algebra A (equipped with the usual Lie structure coming from the commutator bracket) and any Lie algebra homomorphism $j : \mathfrak{g} \to A$, there exists a unique map $\phi : U(\mathfrak{g}) \to A$ such that the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "j"'] & U(\mathfrak{g}) \arrow[d, dotted, "\phi"] \\ & A \end{tikzcd}$$

commutes.

The algebra $U(\mathfrak{g})$ has a canonical Hopf algebra structure by setting, for each $x \in \mathfrak{g}$, $\Delta(x) = 1 \otimes x + x \otimes 1$, $\epsilon(x) = 0$ and $S(x) = -x$, and extending algebraically or anti-algebraically. With respect to this coproduct, $U(\mathfrak{g})$ is a cocommutative Hopf algebra, since $\tau\Delta(x) = \Delta(x)$ for $x \in \mathfrak{g}$. This Hopf algebra is not, in general, commutative, and is so if and only if $\mathfrak{g}$ is abelian, in which case $U(\mathfrak{g})$ has the structure of a polynomial algebra in $\dim_k \mathfrak{g}$ variables.

Remark 2.2.14. This coalgebra structure on $U(\mathfrak{g})$ is certainly not unique (see Example 2.4.6, for example, where the underlying algebra can be viewed as the universal enveloping algebra of an abelian Lie algebra). To avoid confusion later, when we will work with several distinct coalgebra structures on universal enveloping algebras, we write $U(\mathfrak{g})_c$ to denote the universal enveloping algebra equipped with the cocommutative structure as defined above. As we shall see in Corollary 3.4.6, this is the unique *cocommutative* Hopf structure on $U(\mathfrak{g})$.

Our second example of a Hopf algebra is the group algebra of a group.

Example 2.2.15. Let G be a group and let $A = kG$, the group algebra of G. Then A becomes a Hopf algebra by setting, for each $g \in G$, $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1}$, and extending algebraically or anti-algebraically. With respect to this coproduct, A becomes a cocommutative Hopf algebra, since $\tau\Delta(g) = \Delta(g)$ for $g \in G$. This Hopf algebra is not in general commutative, and is so if and only if G is abelian.

Both Example 2.2.13 and Example 2.2.15 are examples of cocommutative Hopf algebras. To demonstrate that Hopf algebras need not be cocommutative in general, we list below "Sweedler's Hopf algebra", a finite-dimensional noncommutative, noncocommutative Hopf algebra.

Example 2.2.16. ([41, Example 1.5.6]) Consider the algebra with presentation

$$K := k\langle g, x, : g^2 = 1, x^2 = 0, xg = -gx\rangle.$$

Then, when char $k \neq 2$, K becomes a 4-dimensional, noncommutative, noncocommutative Hopf algebra by defining maps $\Delta : K \to K \otimes K$, $\epsilon : K \to k$ and $S : K \to K$ on generators as follows: $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$, $\epsilon(g) = 1$, $\epsilon(x) = 0$, $S(g) = g^{-1}$ and $S(x) = -gx$.

2.3 Filtrations and the associated graded space

We begin by recalling the definitions of vector space, algebra, coalgebra and Hopf filtrations.

Definition 2.3.1. Let V be a vector space. We say that a family of subspaces $\{\mathcal{V}_n\}_{n\geq 0}$ is a *vector space filtration* of V and that V is *filtered as a vector space* if

1. $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$ for all $n \geq 0$.
2. $V = \bigcup_{n\geq 0} \mathcal{V}_n$.

Remark 2.3.2. A filtration $\{\mathcal{V}_n\}$ of a vector space V is said to be *locally finite* if $\dim_k \mathcal{V}_n < \infty$ for all $n \geq 0$. Throughout this thesis, all filtrations are assumed to be locally finite unless otherwise stated.

Definition 2.3.3. Let A be an algebra. We say that a family of subspaces $\{\mathcal{A}_n\}_{n\geq 0}$ is an *algebra filtration* of A and that A is *filtered as an algebra* if $\{\mathcal{A}_n\}$ is a vector space filtration of A and $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for all $i, j \geq 0$.

A fundamental example of an algebra filtration is that of the PBW filtration of an enveloping algebra of a Lie algebra. First, we define new notation.

Notation 2.3.4. For an algebra A and finite dimensional subspace B with basis $\{x_1, \ldots, x_n\}$, with assigned order as shown, define, for each $i \geq 1$, the space

$$(B)^i = \mathrm{span}_k\{x_{j_1} \ldots x_{j_i} : x_{j_t} \in \{x_1, \ldots, x_n\}, \quad 1 \leq t \leq i\}.$$

Example 2.3.5. (The PBW filtration) For any finite-dimensional Lie algebra $\mathfrak{g}$ there exists an algebra filtration on $U(\mathfrak{g})$ known as the PBW filtration. Its construction proceeds as follows.

Let $n \in \mathbb{N}$ and suppose $\mathfrak{g}$ is an n-dimensional Lie algebra, with basis $\{x_1, \ldots, x_n\}$. Let $P_0 = k$, $P_1 = \mathfrak{g} + k1$ and, for $n \geq 1$, define

$$P_n = (P_1)^n.$$

Set $P = \bigcup_{i\geq 0} P_i$. The Poincare-Birkhoff-Witt (PBW) theorem states that $\{P_i\}_{i\geq 0}$ forms an algebra filtration of $U(\mathfrak{g})$ and that $U(\mathfrak{g})$ has a basis of ordered monomials

$$P = \{x_1^{l_1} \dots x_n^{l_n} : l_1, \dots, l_n \geq 0\}.$$

We call the filtration $\{P_i\}_{i\geq 0}$ the *PBW-filtration* of $U(\mathfrak{g})$. As a result of the PBW-theorem, $\{P_i\}$ is independent of our initial choice of basis of $\mathfrak{g}$.

Remark 2.3.6. We remark that the assumption that our Lie algebra $\mathfrak{g}$ is finite-dimensional in the above example is in fact not required. There is in fact a (very similar) version of the PBW theorem for the enveloping algebra of any Lie algebra. However, we shall work only with enveloping algebras of finite-dimensional Lie algebras in this thesis, so we omit the notationally more cumbersome infinite dimensional version.

Definition 2.3.7. Let C be a coalgebra. We say that a family of subspaces $\{\mathcal{C}_n\}_{n\geq 0}$ is a *coalgebra filtration* of C and that C is *filtered as a coalgebra* if $\{\mathcal{C}_n\}$ is a vector space filtration and

$$\Delta(\mathcal{C}_n) \subseteq \sum_{i=0}^{n} \mathcal{C}_i \otimes \mathcal{C}_{n-i}$$

for all $n \geq 0$.

Definition 2.3.8. Let H be a Hopf algebra. We say that a family of subspaces $\{\mathcal{H}_n\}_{n\geq 0}$ is a *Hopf filtration* of H and that H is *filtered as a Hopf algebra* if $\{\mathcal{H}_n\}$ is an algebra and coalgebra filtration such that $S(\mathcal{H}_i) \subseteq \mathcal{H}_i$ for all $i \geq 0$.

Remark 2.3.9. We saw above that for any Lie algebra $\mathfrak{g}$, the PBW filtration on $U(\mathfrak{g})$ forms an algebra filtration. Later, in Proposition 3.4.3, we shall see that, if we equip $U(\mathfrak{g})$ with the usual cocommutative coalgebra structure, the PBW filtration in fact forms a *Hopf* filtration of $U(\mathfrak{g})$.

Definition 2.3.10. A *graded vector space* is a vector space V and a family of subspaces $\{V(n)\}_{n\geq 0}$ such that $V = \bigoplus_{n\geq 0} V(n)$. Such a vector space grading is said to be *locally finite* if $\dim(V(i)) < \infty$ for all $i \geq 0$.

Definition 2.3.11. Let $V = \bigoplus_{n\geq 0} V(i)$ be a graded vector space. Attached to V is a filtration $\{N_i^V\}_{i\geq 0}$ which we shall call the *natural filtration* of V, defined for each $i \geq 0$ by setting

$$N_i^V = \bigoplus_{j=0}^{i} V(j).$$

We then have notions of graded algebra, coalgebra and Hopf algebra.

Definition 2.3.12. 1. Let A be an algebra. We say a family of subspaces $\{A(n)\}_{n\geq 0}$ is an *algebra grading* of A and that A is *graded as an algebra* (or *a graded algebra*) if $\{A(n)\}_{n\geq 0}$ forms a vector space grading of A and $A(i)A(j) \subseteq A(i+j)$ for all $i, j \geq 0$. If, further, $A(0) = k$, we say the graded algebra A is *connected*.

2. Let C be an coalgebra. We say a family of subspaces $\{C(n)\}_{n\geq 0}$ is a *coalgebra grading* of C and that C is *graded as a coalgebra* (or *a graded coalgebra*) if $\{C(n)\}_{n\geq 0}$ forms a vector space grading of C and

$$\Delta(C(n)) \subseteq \sum_{i=0}^{n} C(i) \otimes C(n-i)$$

for all $n \geq 0$.

3. Let H be a Hopf algebra. We say a family of subspaces $\{H(n)\}_{n\geq 0}$ is a *Hopf algebra grading* of H and that C is *graded as a Hopf algebra* (or *a graded Hopf algebra*) if $\{H(n)\}_{n\geq 0}$ is both an algebra and coalgebra grading with the additional property that $S(H(n)) \subseteq H(n)$ for all $n \geq 0$.

Remark 2.3.13. The natural filtration of a graded algebra (resp. coalgebra, resp. Hopf algebra) is an algebra filtration (resp. coalgebra filtration, resp. Hopf filtration). To save space, we leave the straightforward proof of this fact to the reader.

Attached to any filtered space is a graded space, which we call the *associated graded space*. We recall its construction below.

Definition 2.3.14. Let V be a vector space with vector space filtration $\mathcal{V} = \{\mathcal{V}_n\}_{n\geq 0}$. We define the *associated graded space with respect to* $\mathcal{V}$ to be the graded space

$$\operatorname{gr}_{\mathcal{V}} V = \bigoplus_{i=0}^{\infty} \mathcal{V}(i)$$

where $\mathcal{V}_{-1} = 0$ and $\mathcal{V}(i) = \mathcal{V}_i/\mathcal{V}_{i-1}$ for $i \geq 0$. Where no confusion between filtrations can arise, we shorten notation to simply $\operatorname{gr} V$.

The conditions required for a vector space filtration of an algebra (resp. coalgebra, resp. Hopf algebra) to be an algebra (resp. coalgebra, resp. Hopf algebra) filtration, are precisely the conditions required for the associated space to become a graded algebra (resp. graded coalgebra, resp. graded Hopf algebra).

Proposition 2.3.15. *Let H be a vector space with vector space filtration $\mathcal{H} = \{\mathcal{H}_n\}$. If $\mathcal{H}$ is an algebra (resp. coalgebra, resp. Hopf) filtration of H, then $\operatorname{gr}_{\mathcal{H}} H$ inherits a graded algebra (resp. graded coalgebra, resp. graded Hopf algebra) structure from H.*

Proof. To save space, we give only a sketch of the proof of the fact that, for a Hopf algebra H with Hopf filtration $\mathcal{H} = \{\mathcal{H}_i\}_{i\geq 0}$, $\operatorname{gr}_{\mathcal{H}} H = \operatorname{gr} H$ is a graded Hopf algebra - the analogous results about algebras and coalgebras will follow immediately from this proof. As usual, let m, Δ and S denote the multiplication map, comultplication map and antipode of H. For any $n \geq 0$, let $\mathcal{H}(n) = \mathcal{H}_n/\mathcal{H}_{n-1}$.

We begin by showing that $\operatorname{gr} H$ is an algebra. For any $i, j \geq 0$, define a map

$$m_{i,j} : \mathcal{H}(i) \otimes \mathcal{H}(j) \to \mathcal{H}(i + j)$$

such that, for $\bar{x} = x + \mathcal{H}_{i-1}$ and $\bar{y} = y + \mathcal{H}_{j-1}$,

$$m_{i,j}(\bar{x} \otimes \bar{y}) = m(x \otimes y) + \mathcal{H}_{i+j-1}.$$

That this map is well defined is an immediate consequence of the fact that $\mathcal{H}$ is an algebra filtration. We then define a linear map $m_{\operatorname{gr} H} : \operatorname{gr} H \otimes \operatorname{gr} H \to \operatorname{gr} H$ such that $m_{\operatorname{gr} H}|_{H(i)\otimes H(j)} = m_{i,j}$. That this map satisfies the axioms of a graded associative algebra is immediate from the fact that m satisfies the axioms of a filtered associative algebra.

Next we prove that $\operatorname{gr} H$ is a graded coalgebra. First note that, for any $n \geq 0$ and $x \in \mathcal{H}_n$, since $\mathcal{H}$ is a coalgebra filtration, we can write $\Delta(x) = \sum_{i=0}^{n} x_1^{(i)} \otimes x_2^{(n-i)}$, where $x_j^{(r)} \in \mathcal{H}_r$. Now, for any $n \geq 0$ define a map

$$\Delta_n : \mathcal{H}(n) \to \sum_{i=0}^{n} \mathcal{H}(i) \otimes \mathcal{H}(n - i)$$

such that, for $\bar{x} = x + H_{n-1} \in \mathcal{H}(n)$ with $\Delta(x) = \sum_{i=0}^{n} x_1^{(i)} \otimes x_2^{(n-i)}$,

$$\Delta_n(\bar{x}) = \sum_{i=0}^{n} \left(x_1^{(i)} + \mathcal{H}_{i-1}\right) \otimes \left(x_2^{(n-i)} + \mathcal{H}_{n-i}\right).$$

Define also a map

$$\epsilon_n : \mathcal{H}(n) \to k$$

such that, for $\overline{x} = x + \mathcal{H}_{n-1}$,

$$\epsilon_n(\bar{x}) = \epsilon(x).$$

That these maps are well defined is an immediate consequence of the fact that $\mathcal{H}$ is a coalgebra filtration. Now define a linear maps $\Delta_{\operatorname{gr} H} : \operatorname{gr} H \to \operatorname{gr} H \otimes \operatorname{gr} H$ such that

$\Delta_{\operatorname{gr} H}|_{\mathcal{H}(n)} = \Delta_n$ and $\epsilon_{\operatorname{gr} H}$ such that $\epsilon_{\operatorname{gr} H}|_{\mathcal{H}(n)} = \epsilon_n$. That these are graded coassociative coalgebra morphisms on $\operatorname{gr} H$ is an immediate consequence of the fact that Δ and ϵ are filtered coassociative coalgebra morphisms on H. Thus $\operatorname{gr} H$ is a coalgebra.

Finally, to prove $\operatorname{gr} H$ is a Hopf algebra, it suffices to define an antipode on $\operatorname{gr} H$. For any $n \geq 0$, define a map

$$S_n : \mathcal{H}(n) \to \mathcal{H}(n)$$

such that, for $\bar{x} = x + \mathcal{H}_{n-1} \in \mathcal{H}(n)$,

$$S_n(\bar{x}) = S(x) + \mathcal{H}_{n-1}.$$

That this map is well defined is a consequence of the fact that S preserves the filtration $\mathcal{H}$. We then define a linear map $S_{\operatorname{gr} H} : \operatorname{gr} H \to \operatorname{gr} H$ such that $S|_{\mathcal{H}(n)} = S_n$ for all $n \geq 0$. That $S_{\operatorname{gr} H}$ satisfies the axioms of an antipode for $\operatorname{gr} H$ is again an immediate consequence of the fact that S is an antipode for H. $\square$

Example 2.3.16. Fix $n \geq 0$ and let $\mathfrak{g}$ be an n-dimensional Lie algebra and let $H = U(\mathfrak{g})_c$. Then the associated graded algebra (with respect to the PBW filtration) of H is a polynomial algebra in n variables. This classical fact will actually turn out to be a special case of a recent result of Zhuang (Theorem 4.2.7), so we do not prove it here.

In the following result we explicitly list some properties which can be "lifted" from the associated graded structure to the original filtered structure. A proof of this result can be found in [36, §1.6]

Theorem 2.3.17. *Suppose A is an algebra with an algebra filtration $\{A_i\}_{i \geq 0}$. If the associated graded algebra $\operatorname{gr} A = \bigoplus_{i \geq 0} A_i/A_{i-1}$ is noetherian (resp. a domain, resp. prime), then A is noetherian (resp. a domain, resp. prime).*

2.3.1 Graded duals

We recall the notion of a graded dual of a (locally finite) graded vector space.

Definition 2.3.18. Suppose $A = \bigoplus_{i \geq 0} A(i)$ is a locally finite graded vector space. We define the *graded dual* of A as the locally finite graded vector space

$$\mathcal{D}_A = \bigoplus_{i \geq 0} \mathcal{D}_A(i)$$

where $\mathcal{D}_A(i) := A(i)^*$ for $i \geq 0$.

If $K = \bigoplus_{i\geq 0} K(i)$ is a locally finite graded Hopf algebra, the graded dual $\mathcal{D}_K$ inherits the structure of a locally finite graded Hopf algebra by "locally dualising" the Hopf structure of K:

To define a product, coproduct, counit and antipode on the graded dual, it suffices to define them locally for each $i \geq 0$ on the graded components $K(i)^*$ and extend linearly to the whole of $\mathcal{D}_K$. Fix $n, m \geq 0$. Let $f \in K(n)^*, g \in K(m)^*$. For the product, we define maps

$$m_{\mathcal{D}_K} : K(n)^* \otimes K(m)^* \to K(n+m)^*$$

such that

$$m_{\mathcal{D}_K}(f \otimes g) = (f \otimes g)\Delta_K.$$

For the coproduct, define

$$\Delta_{\mathcal{D}_K} : K(n)^* \to \sum_{i=0}^{n} K(i)^* \otimes K(n-i)^*$$

such that

$$\Delta_{\mathcal{D}_K}(f)\left(\sum k^{(i)} \otimes k^{(n-i)}\right) = f\left(m_K\left(\sum k^{(i)} \otimes k^{(n-i)}\right)\right)$$

for $\sum_{i=0}^{n} k^{(i)} \otimes k^{(n-i)} \in \sum_{i=0}^{n} K(i) \otimes K(n-i)$.

For the antipode, define

$$S_{\mathcal{D}_K} : K(n)^* \to K(n)^*$$

such that

$$S_{\mathcal{D}_K}(f) = fS_K.$$

Finally, for the counit, define

$$\epsilon_{\mathcal{D}_K} : K(n)^* \to k$$

such that

$$\epsilon_{\mathcal{D}_K}(f) = fu.$$

where $u : k \to K$ denotes the unit map of K. That the maps $(m_{\mathcal{D}_K}, \Delta_{\mathcal{D}_K}, S_{\mathcal{D}_K}, \epsilon_{\mathcal{D}_K})$ satisfy the axioms of a locally finite graded Hopf algebra is then a consequence of the fact that the corresponding maps define a locally finite graded Hopf structure on K.

2.3.2 The coradical filtration

Let C be a coalgebra. In this section we recall the definition of the *coradical filtration* of C. We omit the proofs of all results presented in this section, all of which can be found in [41, Chapter 5].

Definition 2.3.19. Let C be a coalgebra over a field k. A subcoalgebra $D \subset C$ is said to be a *proper subcoalgebra* if $D \subsetneq C$ and $D \neq 0$. We say that C is *simple* if it has no proper subcoalgebras.

Remark 2.3.20. Coalgebras are "locally finite", that is, given a coalgebra C and a finite set of of elements $\{c_i\} \subset C$, there exists a finite-dimensional subcoalgebra D of C such that $c_i \in D$ for all i, [41, Theorem 5.1.1]. It follows immediately that simple coalgebras are finite dimensional.

Definition 2.3.21. Let C be a coalgebra.

1. The *coradical* C_0 of C is the (direct) sum of all simple subcoalgebras.

2. For $n \geq 1$, define, inductively

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C).$$

Then we say that $\{C_n\}_{n\geq 0}$ is the *coradical filtration* of C.

Definition 2.3.22. For any Hopf algebra H, the *coradical filtration* of H, which we denote by $\{H_n\}$, is defined to be the coradical filtration of the underlying coalgebra of H.

Remark 2.3.23. For a general coalgebra C, the terminology "coradical" for the sum of the simple subcoalgebras derives from the following realisation. As noted in Remark 2.2.7, the linear dual C^* is an algebra under the convolution product and C^* acts on C via the "left hit action". Let $J(C^*)$ denote the Jacobson radical of C^*, that is, the intersection of all maximal right (or left) ideals of C^*. It turns out that it is precisely the sum of the simple subcoalgebras of C which is annihilated by $J(C^*)$ under the left or right hit action on C, i.e., $C_0 = \{c \in C : J(C^*) \rightharpoonup c = 0\} = \{c \in C : c \leftharpoonup J(C^*) = 0\}$. More generally, $C_n = \mathrm{Ann}_C(J^{n+1})$ for all $n \geq 0$. This result, along with its proof, appears as [41, Proposition 5.2.9].

The following appears as [41, Theorem 5.2.2].

Lemma 2.3.24. *Let C be a coalgebra. Then the coradical filtration $\{C_n\}$ forms a coalgebra filtration of C.*

Definition 2.3.25. Let C be a coalgebra (or a Hopf algebra). We define the *associated graded coalgebra* of C, which we denote by $\mathrm{gr}\, C$, to be the associated graded coalgebra with respect to the coradical filtration of C. That is,

$$\operatorname{gr} C = \bigoplus_{i=0}^{\infty} C(i)$$

where $C_{-1} = 0$ and $C(i) = C_i/C_{i-1}$ for all $i \geq 0$.

For a given Hopf algebra H, a natural question is to ask under what circumstances does the coradical filtration form an algebra, or, even stronger, Hopf filtration of H? This latter condition is determined completely by the structure of the coradical H_0, as established by the following lemma. For a proof, see [41, Lemma 5.2.8].

Lemma 2.3.26. *Let H be a Hopf algebra. Then $\{H_n\}$ is a Hopf algebra filtration if and only if H_0 is a Hopf subalgebra of H.*

Note that for a general Hopf algebra H it is certainly *not* true that H_0 is necessarily a Hopf subalgebra. We see an example of such a Hopf algebra later in Example 3.2.4.

When H_0 *is* a Hopf subalgebra of our Hopf algebra H, Lemma 2.3.26 and Proposition 2.3.15 tell us that the associated graded coalgebra $\operatorname{gr} H$ is in fact a graded *Hopf algebra*. Then, more often than not, by virtue of its grading or otherwise, $\operatorname{gr} H$ is a more tractable Hopf algebra than H (as we shall see, this is certainly the case when H is connected), and although they are completely distinct objects, it is often enough to prove a result about $\operatorname{gr} H$ to see that the same result holds for H, à la Lemma 2.3.17.

2.3.3 Coradically graded coalgebras

In this subsection we consider those graded coalgebras C whose natural filtration (see Remark 2.3.11) and coradical filtration coincide.

Definition 2.3.27. Let $C = \bigoplus_{i \geq 0} C(i)$ be a graded coalgebra. We say C is *coradically graded* if

$$C_n = N_n^C = \bigoplus_{i=0}^{n} C(i)$$

for each $n \geq 0$.

Attached to *any* coalgebra C is a coradically graded coalgebra: the associated graded coalgebra (with respect to the coradical filtration) $\operatorname{gr} C$ is coradically graded. This appears as [45, Proposition 4.4.15], so we omit the proof.

Proposition 2.3.28. *Suppose is a C coalgebra. Then $\operatorname{gr} C$ is a coradically graded coalgebra.*

2.4 Algebraic groups and commutative Hopf algebras

Hopf algebras arise naturally in the context of algebraic groups. In fact, in characteristic 0, any (affine) commutative Hopf algebra is the coordinate ring of an (affine) algebraic group, effectively allowing us to translate any statement about affine commutative Hopf algebras into one about affine algebraic groups. The aim of this section is to describe this correspondence and to list some algebraic properties of commutative Hopf algebras.

2.4.1 Affine algebraic groups - definitions and elementary properties

For the reader's convenience, we recall here some definitions from algebraic group theory. The definitions and results of this subsection are adapted from [23], [37] and [64].

We begin by defining the fundamental objects of affine algebraic geometry - algebraic sets and their coordinate rings.

Definition 2.4.1. 1. For any field k and positive integer n, an *algebraic set* of k^n is a subset $X \subseteq k^n$ for which there exists some subset $P \subseteq k[x_1, \ldots, x_n]$ such that

$$X = \{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0 \ \ \forall f \in P\}.$$

For any $f \in k[x_1, \ldots, x_n]$ we denote by $f|_X$ the function determined by evaluating the polynomial f on the set X.

2. For $n \geq 1$ and an algebraic set $X \subseteq k^n$, we define the *coordinate ring* of X, $\mathcal{O}(X)$, as follows:

$$\mathcal{O}(X) = \{f|_X : X \to k : f \in k[x_1, \ldots, x_n]\}.$$

One can identify $\mathcal{O}(X)$ with the factor algebra $k[x_1, \ldots, x_n]/I$, where I is the ideal of $k[x_1, \ldots, x_n]$ generated by those polynomials which vanish on X, and via this identification $\mathcal{O}(X)$ inherits the structure of an affine semiprime commutative algebra.

3. Given algebraic sets $X \subseteq k^n$ and $Y \subseteq k^m$, a *morphism (of algebraic sets)* is a map $\phi : X \to Y$ such that, for each $\underline{x} \in X$,

$$\phi(\underline{x}) = (\psi_1(\underline{x}), \ldots, \psi_m(\underline{x}))$$

for some $\psi_i \in \mathcal{O}(X)$, for $i = 1, \ldots, m$.

4. Let A be an affine semiprime commutative algebra over a field k, so that $A \cong k[x_1, \ldots, x_n]/I$ for some semiprime ideal $I \subset k[x_1, \ldots, x_n]$ and some $n \geq 0$. This ideal I defines an algebraic subset $Z(I)$ of k^n:

$$Z(I) = \{\underline{a} \in k^n : f(\underline{a}) = 0 \quad \forall f \in I\}$$

and this algebraic set can be easily identified with

$$\mathrm{Map}(A) = \{f : A \to k : f \text{ an algebra homomorphism}\}.$$

5. For any $n \geq 1$, we can define a topology on k^n, known as the *Zariski topology*, by setting the closed sets of this topology to be the algebraic subsets of k^n. An algebraic set X is said to be *irreducible* if it is not the union of two proper closed (with respect to the Zariski topology) subsets.

6. For an algebraic set X, the *dimension* of X, denoted $\dim X$, is the maximal length d of chains $V_0 \subset V_1 \subset \ldots \subset V_d$ of distinct nonempty irreducible algebraic subsets.

Next we define affine algebraic groups. For this thesis, an algebraic group will be the same object as an affine algebraic group.

Definition 2.4.2. Let G be an algebraic set with a group structure defined by the maps

$$m : G \times G \to G : (g, h) \mapsto gh$$

and

$$s : G \to G : g \mapsto g^{-1}.$$

We say that G is an *algebraic group* if the maps m and s are morphisms of algebraic sets.

Remark 2.4.3. 1. Defined in the obvious way, there exist notions of morphism of algebraic groups, of algebraic subgroups, of normalisers of algebraic groups and of representations of algebraic groups, for example. We do not define these here explicitly, and refer the reader to [23] or [64] for a detailed account.

2. Whenever we speak of a *closed* subgroup of an algebraic group, we mean closed with respect to the Zariski topology.

Example 2.4.4. 1. Any field k becomes an algebraic group under the usual addition in k, known as the *additive group*, and denoted by $(k, +)$.

2. For any $n \geq 1$ and field k, let $M_n(k)$ denote the algebraic set of $n \times n$ matrices over k. With respect to the usual multiplicative group structure, the general linear group of invertible matrices

$$\mathrm{GL}_n(k) = \{A \in M_n(k) : \det A \neq 0\}$$

is an algebraic group.

3. Any closed subgroup of an algebraic group is an algebraic group. This implies, for example, that for any $n \geq 1$, the subgroup $\mathbb{U}_n$ of strictly upper triangular matrices of $\mathrm{GL}_n(k)$ and the special linear group $\mathrm{SL}_n(k)$ of all matrices with determinant equal to one, are algebraic groups.

2.4.2 Commutative Hopf algebras

For a fixed algebraically closed field k, let AlgSet denote the category of algebraic sets over k and CommAlg the category of semiprime affine commutative k-algebras. For an affine commutative algebra A, a fundamental result of affine algebraic geometry states that, in the notation of Definition 2.4.1 (4), the functors

$$\mathcal{O} : \mathrm{AlgSet} \to \mathrm{CommAlg}$$

$$X \mapsto \mathcal{O}(X)$$

and

$$Z : \mathrm{CommAlg} \to \mathrm{AlgSet}$$

$$A \cong k[x_1, \ldots, x_n]/I \mapsto Z(I)$$

define a contravariant equivalence between the categories AlgSet and CommAlg. Restriction of the functor $\mathcal{O}$ to the subcategory AlgGroup of algebraic groups and of the functor Z to the subcategory CommHopf of affine semiprime commutative Hopf algebras defines a contravariant equivalence of the categories AlgGroup and CommHopf as follows.

Given an algebraic group G, there is a canonical Hopf structure on the coordinate ring $\mathcal{O}(G)$, which arises from the group structure on G: given $f \in \mathcal{O}(G)$, we define the coproduct of f as the map

$$\Delta(f) \in \mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G)$$

such that, for each $x, y \in G$,

$$\Delta(f)(x, y) = f(m(x, y))$$

where $m : G \times G \to G$ is the map defining the group multiplication in G. We define the antipode $S : \mathcal{O}(G) \to \mathcal{O}(G)$ and counit $\epsilon : \mathcal{O}(G) \to k$ by setting, for $f \in \mathcal{O}(G)$ and $g \in G$,

$$S(f)(g) = f(g^{-1}), \quad \epsilon(f) = f(1_G). \tag{2.4.1}$$

That these maps (Δ, S, ϵ) satisfy the axioms of a Hopf k-algebra is a virtue of the fact that the multiplication and inverse maps defining the group structure on G satisfy the axioms of an algebraic group.

Conversely, let H be an affine semiprime commutative Hopf algebra - write $H \cong k[x_1, \ldots, x_n]/I$ for some semiprime ideal I. As noted in Definition 2.4.1 (4), the algebraic set $Z(I)$ can be identified with $\mathrm{Map}(H)$, and so in what follows we view $\mathrm{Map}(H)$ as an algebraic set. There is an induced algebraic group structure on $\mathrm{Map}(H)$ as follows: the product of any two elements $f, g \in \mathrm{Map}(H)$ is the map

$$fg : H \to k$$

such that

$$(fg)(h) = (f \otimes g)(\Delta(h))$$

for each $h \in H$. The inverse of an element $f \in \mathrm{Map}(H)$ is the map

$$f^{-1} : H \to k$$

such that

$$f^{-1}(h) = f(S(h))$$

for all $h \in H$. That these maps satisfy the axioms of an algebraic group follows from the fact that the coproduct and antipode satisfy the axioms of a Hopf algebra.

When working in characteristic 0, it is a result of Cartier, [64, Theorem 11.4], that affine commutative Hopf algebras are always semiprime. Moreover, affine algebraic groups are *smooth* as varieties ([37, Theorem 3.38]). Translated into the language of Hopf algebras, this result says that affine commutative Hopf k-algebras over an algebraically closed field k of characteristic 0 have finite global dimension. We deduce the following fact, which shall be used repeatedly throughout this thesis.

Let k be an algebraically closed field of characteristic 0 and let H be an affine commutative Hopf k-algebra. Then H has finite global dimension and $H = \mathcal{O}(G)$, where $G = \mathrm{Map}(H)$ is an algebraic group.

This result does not hold in positive characteristic. Take for example a prime number p, the cyclic group of order p, which we denote by C_p, and a field k of characteristic p. Then the group algebra kC_p is a commutative Hopf algebra which contains nilpotent elements, and thus cannot be the coordinate ring of an algebraic group, which is always semiprime.

To wrap up the section on commutative Hopf algebras, let's look at an explicit example. First, some notation.

Notation 2.4.5. Let G be a *matrix group*, that is, a closed algebraic subgroup of $\mathrm{GL}_n(k)$, for some $n \geq 1$. For any $1 \leq p, q \leq n$ define a map

$$X_{pq} : G \to k$$

such that, for any $n \times n$-matrix A with entires $(a_{ij})_{1 \leq i,j \leq n}$,

$$X_{pq}(A) = a_{pq}.$$

We call X_{pq} the (p, q)-*coordinate function* on G.

Example 2.4.6. Let $\mathbb{U}_3$ denote the group of 3×3 upper triangular matrices, that is, the group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

for $a, b, c \in k$.

A straightforward calculation shows that, using Notation 2.4.5,

$$\mathcal{O}(\mathbb{U}_3) = k[X_{12}, X_{13}, X_{23}].$$

As discussed above, the algebraic group structure of $\mathbb{U}_3$ gives rise to a commutative Hopf structure on $\mathcal{O}(\mathbb{U}_3)$. The laws for the comultiplication in $\mathcal{O}(\mathbb{U}_3)$ arise from "dualising" the usual matrix multiplication of $\mathbb{U}_3$. On the algebra generators, the coproduct is defined as follows:

$$\Delta(X_{12}) = 1 \otimes X_{12} + X_{12} \otimes 1.$$

$$\Delta(X_{13}) = 1 \otimes X_{13} + X_{13} \otimes 1 + X_{12} \otimes X_{23}.$$

$$\Delta(X_{23}) = 1 \otimes X_{23} + X_{23} \otimes 1.$$

We notice immediately that H is not cocommutative with respect to this coproduct. The antipode and counit are defined by dualising the matrix inverse and identity laws as in (2.4.1). On generators, they take the following values:

$$S(X_{12}) = -X_{12}, \quad S(X_{13}) = -X_{13} + X_{12}X_{23}, \quad S(X_{23}) = -X_{23}.$$

$$\epsilon(X_{12}) = \epsilon(X_{13}) = \epsilon(X_{23}) = 0.$$

2.4.3 Comodules, representations and homogeneous spaces

The purpose of this section is to briefly describe the one to one correspondence between the rational representations of an algebraic group and the comodules of its coordinate ring. For the definition of a comodule over a coalgebra, see [41, Definition 1.6.2], for example.

Definition 2.4.7. Suppose G is an algebraic group and V is a finite dimensional (group) representation of G defined by the group homomorphism $\rho : G \to \mathrm{GL(V)}$. We say V is a *rational* representation of G if $\rho : G \to \mathrm{GL(V)}$ is also a map of algebraic sets.

Remark 2.4.8. Throughout this thesis, whenever we speak of a representation of an algebraic group, we mean a left rational representation.

For a fixed algebraic group G there is a one to one correspondence between finite dimensional rational representations of G and finite dimensional $\mathcal{O}(G)$-comodules as follows:

Let $V = k^n$ (an n-dimensional vector space) and let $(e_i)_{i\in I}$ denote the standard vector basis of V. Let $(r_{ij})_{i,j\in I}$ be a matrix of elements of $\mathcal{O}(G)$ satisfying

$$\Delta(r_{ij}) = \sum_{k\in I} r_{ik} \otimes r_{kj}$$

and

$$\epsilon(r_{ij}) = \delta_{ij}$$

for all $i, j, k \in I$. We can define a right $\mathcal{O}(G)$-comodule structure on V by

$$\rho(e_j) = \sum_{i\in I} e_i \otimes r_{ij}.$$

We can also define a representation of G, that is, a map $r : G \to \mathrm{GL}_n(k)$ which is both a map of algebraic sets and a group homomorphism, by setting

$$r(g) = (r_{ij}(g))_{i,j\in I}$$

for $g \in G$. These constructions yield a one to one correspondence between the representations of G and $\mathcal{O}(G)$ comodules - we refer the reader to [37, Chapter VIII] for a detailed account.

2.5 GK-dimension and homological algebra

This section focuses on describing some elementary properties of the GK-dimension function for algebras, and on listing some basic definitions from homological algebra, namely that of AS-regular and GK Cohen-Macaulay algebras.

2.5.1 GK-dimension

The *Gelfand-Kirillov dimension* (henceforth referred to as *GK-dimension*) of an algebra R, or more generally any R-module M, is a dimension function which assigns to R (or M) a non-negative real number (or ∞) denoted $\operatorname{GKdim} R$ (respectively $\operatorname{GKdim} M$) which, roughly speaking, is a measure of the "rate of growth" of that algebra (respectively module) with respect to any finite generating set. The aim of this section is to define precisely what is meant by the GK-dimension of an algebra (or a module over that algebra) and to highlight some of the key properties of the GK-dimension function. The results in this section have been adapted from [27] and [36, Chapter 8], standard references on the properties of GK-dimension.

Before we can define the GK-dimension of an algebra, we need a few preliminary definitions and lemmas.

Definition 2.5.1. A function $f : \mathbb{N} \to \mathbb{R}^+$ has *polynomially bounded growth* if for some $v \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that for $n > N$, $f(n) \leq n^v$. We then define

$$\gamma(f) = \inf\{v : f(n) \leq n^v, n >> 0\}.$$

If f is not polynomially bounded, define $\gamma(f) = \infty$. We call $\gamma(f)$ the *degree* of f.

Definition 2.5.2. Let R be a k-algebra with filtration $\{R_n\}$ and let M be a right filtered R-module with filtration $\{M_n\}$. We say the filtration on R is *standard* if, in the notation of Notation 2.3.4, $(R_1)^n = R_n$ for each n, and is called *finite dimensional* if $R_0 = k$ and $\dim_k R_n < \infty$ for all n. Similar definitions apply to M, with standard meaning $M_n = M_0 R_n$ for all n.

Definition 2.5.3. Let R be an affine k-algebra. Let S denote a finite set of generators of R and let V denote the vector space spanned by the elements of S. We call such a space V a *generating subspace* of R, and we can define a standard finite dimensional filtration $\{R_n\}$ by

$$R_0^V = V^0 = k \text{ and } R_n^V = \sum_{i=0}^n V^i$$

where by V^i we mean the vector subspace generated by i-fold products of elements from V. It follows that R_n^V is the set of all possible sums of monomials of length at most n, where the monomials are made up of basis elements from V. We define $W_R^V : R \to \mathbb{N}_0$ to be the function such that $W_R^V(R_n) = \dim_k R_n$ for $n \geq 0$.

If M is a finitely generated right R-module, there is a finite dimensional subspace M_0 such that $M_0R = M$ called the *generating subspace* of M. We can then define a finite dimensional standard filtration on M, setting $M_n = M_0R_n^V$ for $n \geq 0$. We define $W_M^{M_0} : R \to \mathbb{N}_0$ to be the function such that $W_M^{M_0}(M_n) = \dim_k M_n$ for $n \geq 0$.

The following result appears as [36, Lemma 8.1.10].

Lemma 2.5.4. *Let R be an affine k-algebra and let M be a right R-module. Let V and V' denote distinct generating spaces of R. Let M_0 and M_0' denote distinct generating subspaces of M. Then, in the notation of Definition 2.5.3, $\gamma(W_R^V) = \gamma(W_R^{V'})$ and $\gamma(W_M^{M_0}) = \gamma(W_M^{M_0'})$.*

We are now in a position to define GK-dimension of an affine algebra and of a finitely generated right module over an affine algebra.

Definition 2.5.5. Let R be an affine k-algebra and M a finitely generated right R-module. Let V and M_0 denote generating spaces of R and M respectively. Define, in the notation of Definition 2.5.3,

$$\operatorname{GKdim} R = \gamma(W_R^V) \text{ and } \operatorname{GKdim} M = \gamma(W_M^{M_0})$$

That this GK-dimension is independent of the choice of initial generating set is Lemma 2.5.4.

We can now extend the definition of GK-dimension to arbitrary algebras and modules.

Definition 2.5.6. Let S be an algebra and let N_S be a right S-module. Let $\mathcal{A}_S$ denote the class of affine subalgebras of S and let $\mathcal{B}_N$ denote the class of all finitely generated

R-submodules of N_S, where R ranges over $\mathcal{A}_S$. Define

$$\operatorname{GKdim} S = \sup\{\operatorname{GKdim} R : R \in \mathcal{A}_S\}$$

and

$$\operatorname{GKdim} N_S = \sup\{\operatorname{GKdim} M : M \in \mathcal{B}_N\}.$$

We now focus on listing some useful properties of the GK-dimension function and examine precisely what it measures when applied to some of the algebras introduced earlier in the chapter. The following results can again be found in [27].

Proposition 2.5.7. *1. If $\mathfrak{g}$ is Lie algebra,* $\operatorname{GKdim} U(\mathfrak{g}) = \dim(\mathfrak{g})$.

2. If X is an algebraic set, $\operatorname{GKdim} \mathcal{O}(X) = \dim(X)$.

Proposition 2.5.8. *1. If B is a subalgebra or homomorphic image of an algebra A,* $\operatorname{GKdim} B \leq \operatorname{GKdim} A$.

2. If A is an algebra with filtration $\{A_i\}$ and M is a filtered A-module with filtration $\{M_i\}$, then

$$\operatorname{GKdim} \operatorname{gr}(M)_{\operatorname{gr} A} \leq \operatorname{GKdim} M_A.$$

If, further, the filtration $\{A_i\}$ is locally finite, $\operatorname{gr} A$ *is finitely generated as an algebra and* $\operatorname{gr}(M)$ *is finitely generated as a* $\operatorname{gr}(A)$*-module,*

$$\operatorname{GKdim} \operatorname{gr}(M)_{\operatorname{gr} A} = \operatorname{GKdim} M_A.$$

Proof. (1) is [27, Lemma 3.1] and (2) is [27, Lemma 6.5, Proposition 6.6].

□

We finish with some remarks and curiosities about the relationship between the GK-dimension function and Hopf algebras.

Remark 2.5.9. 1. For an arbitrary algebra R with finite GK-dimension, $\operatorname{GKdim} R$ will be, in general, a non-integral real number. Examples of such algebras appear in [27]. However, none of these examples are Hopf algebras, and in fact the GK-dimension of any currently known Hopf algebra is either a non-negative integer or ∞. It appears as a question in [67] and [3] whether or not this is true for arbitrary Hopf algebras.

2. It is in general not true that (Hopf) algebras with finite GK-dimension are necessarily affine, nor noetherian. Take for example $H = k(\mathbb{Q}, +)$, the group algebra of the additive group of $\mathbb{Q}$. Since $(\mathbb{Q}, +)$ is locally infinite cyclic, $\operatorname{GKdim} H = 1$. However, H is not noetherian, since $(\mathbb{Q}, +)$ is easily seen not to possess the ACC condition on subgroups. Since $(\mathbb{Q}, +)$ is not a finitely generated group, it is easy to see that H is not affine either.

2.5.2 Homological algebra

We recall the definition of Auslander-regular, AS-Regular and GK Cohen-Macaulay algebras.

Notation 2.5.10. Let A be a ring and M a left A-module. We denote by $\operatorname{inj.dim}({}_AM)$ the injective dimension of M as a left A-module and by $\operatorname{gl.dim}(A)$ the global dimension of the ring A. For the precise definition of the injective dimension of a module and of the global dimension of a ring, see [48, Chapter 8].

Definition 2.5.11. A ring A is said to satisfy the *Auslander condition* if for any noetherian A-module M and $i \in \mathbb{N}_0$,

$$j(N) := \min\{j : \operatorname{Ext}^j_A(N, A) \neq 0\} \geq i$$

for all submodules $N \subset \operatorname{Ext}^i(M, A)$. The ring A is said to be *Auslander-Gorenstein* if it is two-sided noetherian, satisfies the Auslander condition and has finite left and right injective dimension as an A-module. A ring which is Auslander-Gorenstein and has finite global dimension is said to be *Auslander-regular*.

Definition 2.5.12. Let A be a k-algebra which is *augmented*, i.e., there is a distinguished algebra homomorphism $\pi : A \to k$. By virtue of its multiplication, A becomes a left A-module, and by virtue of the homomorphism $\pi : A \to k$, k becomes a left A-module - denote these modules by ${}_AA$ and ${}_Ak$ respectively. Then we say A is *AS-Gorenstein of dimension* n if

1. $\operatorname{inj.dim}({}_AA) = n < \infty$
2. $\operatorname{Ext}^i_A({}_Ak, {}_AA) \cong \delta_{in}k$ as vector spaces

and the analogous conditions hold also for right modules instead of left modules. We say A is *AS-regular* if it is AS-Gorenstein and $\operatorname{gl.dim} A < \infty$.

Definition 2.5.13. Let A be a k-algebra. We say that A is *GK Cohen-Macaulay* if for all finitely generated non-zero A-modules M,

$$j(M) + \operatorname{GKdim} M = \operatorname{GKdim} A$$

where $j(M) = \inf\{n : \operatorname{Ext}^n_A(M, A) \neq \{0\}\}$. The value $j(M)$ is known as the *grade* of M (as an A-module).

Example 2.5.14. For $n \geq 1$, the archetypal example of an AS-regular, GK-Cohen-Macaulay algebra of dimension n is the commutative polynomial algebra in n variables - see [29], for example.

2.6 Normal Hopf ideals and Hopf subalgebras

Let G be a group. As is well known, G acts on itself by conjugation - this is often referred to as the *adjoint action*. A subgroup H of G is said to be *normal* if and only if it is preserved by the adjoint action of G on H. Similarly, for a Lie algebra $\mathfrak{g}$, we can define the *adjoint* action of $\mathfrak{g}$ on itself as the map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by the Lie bracket on $\mathfrak{g}$. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is then said to be a Lie ideal if and only if it is preserved by the adjoint action of $\mathfrak{g}$.

In this section we generalise the concepts of an adjoint action and of normality to an arbitrary Hopf algebra H. We also define the dual notions of an adjoint coaction and of (co)normality of Hopf ideal I of H. Finally, we study a connection between the normal Hopf subalgebras and the normal Hopf ideals of H. All results of this section can be found in [41, §3.4].

Definition 2.6.1. Let H be a Hopf algebra.

1. The *left adjoint action* of H on itself is defined by the map

$$\operatorname{ad}_l := m(m \otimes \operatorname{id})(\operatorname{id} \otimes \operatorname{id} \otimes S)(\operatorname{id} \otimes \tau)(\Delta \otimes \operatorname{id}) : H \otimes H \to H.$$

In Sweedler's notation, $(\operatorname{ad}_l h)(k) := \operatorname{ad}_l(h \otimes k) = \sum h_1 k(Sh_2)$ for $h, k \in H$.

2. The *right adjoint action* of H on itself is defined by the map

$$\operatorname{ad}_r := m(m \otimes \operatorname{id})(S \otimes \operatorname{id} \otimes \operatorname{id})(\operatorname{id} \otimes \tau)(\Delta \otimes \operatorname{id}) : H \otimes H \to H.$$

In Sweedler's notation, $(\operatorname{ad}_r h)(k) := \operatorname{ad}_r(h \otimes k) = \sum (Sh_1) k h_2$ for $h, k \in H$.

3. A Hopf subalgebra K of H is called *normal* if

$$(\operatorname{ad}_l h)(k) \in K \text{ and } (\operatorname{ad}_r h)(k) \in K$$

for all $h \in H$ and $k \in K$.

Example 2.6.2. A simple computation using the standard coproducts shows that, for a group G and $g \in G$, $(\operatorname{ad}_l g)(h) = ghg^{-1}$ for all $h \in kG$, and that for a Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$, $(\operatorname{ad}_l x)(h) = xh - hx$ for all $h \in U(\mathfrak{g})_c$. Thus with this Hopf-theoretic definition we recover the usual classical adjoint actions as described in the opening paragraph of this section.

Example 2.6.3. If H is a commutative Hopf algebra then *any* Hopf subalgebra is normal. Indeed, let K be a Hopf subalgebra, $h \in H$ and $k \in K$. Then

$$\begin{aligned}(\operatorname{ad}_l h)(k) &= \sum h_1 k S(h_2) \\ &= \sum h_1 S(h_2) k \qquad \text{(by commutativity)} \\ &= \epsilon(h) k \in K \qquad \text{(definition of the antipode)}\end{aligned}$$

and similarly $(\operatorname{ad}_r h)(k) \in K$ for all $h \in H$ and $k \in K$.

Dualising Definition 2.6.1, we get the concept of a *normal Hopf ideal.*

Definition 2.6.4. Let H be a Hopf algebra.

1. The *left adjoint coaction* of H on itself is defined by the map

$$\rho_l = (m \otimes \operatorname{id})(\operatorname{id} \otimes \tau)(\operatorname{id} \otimes \operatorname{id} \otimes S)(\Delta \otimes \operatorname{id})\Delta : H \to H \otimes H.$$

This defines a left H-comodule structure on H. In Sweedler's notation,

$$\rho_l(h) = \sum h_1 S(h_3) \otimes h_2$$

for all $h \in H$.

2. The *right adjoint coaction* of H on itself is defined by the map

$$\rho_r = (\operatorname{id} \otimes m)(\tau \otimes \operatorname{id})(S \otimes \operatorname{id} \otimes \operatorname{id})(\Delta \otimes \operatorname{id})\Delta : H \to H \otimes H.$$

This defines a right H-comodule structure on H. In Sweedler's notation,

$$\rho_r(h) = \sum h_2 \otimes S(h_1) h_3$$

for all $h \in H$.

3. A Hopf ideal I is called *normal* if both

$$\rho_l(I) \subseteq H \otimes I \text{ and } \rho_r(I) \subseteq I \otimes H.$$

Normal Hopf ideals arise in the context of algebraic groups. Let G be an algebraic group. In Section 2.4.2 we described the commutative Hopf algebra $H = \mathcal{O}(G)$. In the following example we describe a bijective correspondence between the normal (closed) subgroups of G and the normal Hopf ideals of H.

Example 2.6.5. Suppose $H = \mathcal{O}(G)$ is an affine commutative Hopf algebra, where G is some fixed algebraic group. There is a one-to-one correspondence between the normal Hopf ideals of H and of the normal closed subgroups of G as follows:

Let I be a Hopf ideal of H. Then $\bar{H} = H/I$ is an affine commutative Hopf algebra, so we may write $\bar{H} = \mathcal{O}(D)$ for some unique closed subgroup $D \subset G$. Then I is a normal Hopf ideal if and only if D is a normal subgroup of G, [41, p 36].

2.6.1 A correspondence between normal Hopf subalgebras and normal Hopf ideals

The goal of this section is to describe, for a large class of Hopf algebras, a bijective correspondence between normal Hopf subalgebras and the normal Hopf ideals. As we shall see, this class of Hopf algebras has as a subclass the class of all connected Hopf algebras, the main objects of study in this thesis.

Before we can describe this correspondence, we need a couple of definitions.

Definition 2.6.6. Suppose H is a Hopf algebra and T is a subalgebra of H. If

$$\Delta(T) \subseteq H \otimes T$$

we say that T is a *left coideal subalgebra* of H. If

$$\Delta(T) \subseteq T \otimes H$$

we say T is a *right coideal subalgebra* of H.

Definition 2.6.7. Suppose $\pi : H \to \bar{H}$ is an surjective morphism of Hopf algebras. Define the *right coinvariants* of π,

$$H^{\mathrm{co}\,\pi} := \{h \in H : \sum h_1 \otimes \pi(h_2) = h \otimes \pi(1)\}.$$

Analogously, the *left coinvariants* of π are

$$^{\mathrm{co}\,\pi}H := \{h \in H : \sum \pi(h_1) \otimes h_2 = \pi(1) \otimes h\},$$

We shall sometimes denote $H^{\mathrm{co}\,\pi}$ as $H^{\mathrm{co}\,\bar{H}}$ and $^{\mathrm{co}\,\pi}H$ as $^{\mathrm{co}\,\bar{H}}H$.

Lemma 2.6.8. *Let C be a coalgebra and let $f : M \to N$ be a homomorphism of left (resp. right) C-comodules. Then $\ker f$ is a left (resp. right) C-subcomodule of M.*

Proof. Let ρ_M (resp. ρ_N) denote the left C-comodule structure on M (resp. N). Let $K = \ker f$ and let $m \in K$, so $f(m) = 0$. In the standard Sweedler notation for left-comodules (see [41, §1.6], for example), write $\rho_M(m) = \sum m_{-1} \otimes m_0$, a sum of linearly independent elements in $C \otimes M$. We aim to prove that $m_0 \in K$ for all m_0 appearing in the above expression for ρ_M. Since f is a left C-comodule morphism, by definition

$$\rho_N \circ f = (\mathrm{id} \otimes f) \circ \rho_M. \tag{2.6.1}$$

From (2.6.1) and the fact that $f(m) = 0$, it follows that $\sum m_{-1} \otimes f(m_0) = 0$. By the linear independence of the elements appearing in this sum, it must be that $f(m_0) = 0$ for all m_0 in the expression. It follows $\rho_M(K) \subset C \otimes K$, as required.

□

Lemma 2.6.9. *Suppose $\pi : H \to \overline{H}$ is an surjective morphism of Hopf algebras. Then $H^{\mathrm{co}\,\pi}$ is a left coideal subalgebra of H and $^{\mathrm{co}\,\pi}H$ is a right coideal subalgebra of H.*

Proof. We prove the result for $H^{\mathrm{co}\,\pi}$ - the proof of the corresponding result for $^{\mathrm{co}\,\pi}H$ is similar.

That $H^{\mathrm{co}\,\pi}$ is a subalgebra of H is clear from its definition and the fact that both π and Δ are algebra homomorphisms. Define the map $\rho := (\mathrm{id} \otimes \pi)\Delta : H \to H \otimes \overline{H}$. This map ρ is a homomorphism of left H-comodules, where the left H-comodule structure on H is given by Δ and the left H-comodule structure on $H \otimes \overline{H}$ is given by $\Delta \otimes \mathrm{id}$. Indeed, that ρ is a left H-comodule morphism can be seen at once from the coassociativity of Δ. Define now the map

$$\iota : H \to H \otimes \overline{H}$$

such that

$$h \mapsto h \otimes \overline{1}.$$

A trivial calculation shows that ι is a left H-comodule homomorphism with respect to the aforementioned H-comodule structures on H and $H \otimes \overline{H}$ respectively. Notice now that

$H^{\mathrm{co}\,\pi} = \ker(\rho - \iota)$, hence, by Lemma 2.6.8, $H^{\mathrm{co}\,\pi}$ is a left H-subcomodule of H. Since the left H-comodule structure on H was defined by Δ, $\Delta(H^{\mathrm{co}\,\pi}) \subset H \otimes H^{\mathrm{co}\,\pi}$. This completes the proof.

□

Remark 2.6.10. It is in general *not* true that the left (or right) coinvariants of a surjective Hopf morphism $\pi : H \to \overline{H}$ form a Hopf subalgebra of H. For a counterexample when H is commutative, see Theorem 5.3.3(2).

We are now in a position to describe the claimed correspondence between normal Hopf ideals and normal Hopf subalgebras of an arbitrary Hopf algebra. For any Hopf algebra H, there are, as noted in [52], well-defined maps

$$\phi : \{K : K \text{ a normal Hopf subalgebra of } H\} \to \{I : I \text{ a normal Hopf ideal of } H\}$$

such that

$$K \mapsto K^+H$$

and

$$\psi : \{I : I \text{ a normal Hopf ideal of } H\} \to \{K : K \text{ normal Hopf subalgebra of } H\}$$

such that

$$I \mapsto H^{co(H/I)}.$$

These maps are are known to be mutually inverse bijections when restricted to those normal Hopf subalgebras over which H is (right and left) faithfully flat (equivalently, to those I such that H is an injective right H/I-comodule), [35]. It is in general not true that an arbitrary Hopf algebra is always (left and right) faithfully flat over its Hopf subalgebras - counterexamples were given by Schauenburg in [49]. Faithful flatness over Hopf subalgebras is guaranteed, however, in the setting of the following theorem, which appears as [41, Theorem 3.4.6].

Theorem 2.6.11. *Let H be any Hopf k-algebra. Then the maps ψ and ϕ are mutually inverse bijections if either H is commutative or if H is pointed.*

Proof. When H is commutative, in light of the results of §2.4.2, this is a Hopf algebra theoretic version of the fundamental theorem of algebraic groups: that there is a one-to-one correspondence between closed normal subgroups and quotients of an arbitrary algebraic group - see [64, Theorem 16.3]. For a proof of this result in the language of Hopf algebras, see [57]. The result is proved whenever H is pointed in [35]. $\square$

Remark 2.6.12. We saw in Example 2.6.5 that for an algebraic group G, there is a bijective correspondence between the normal subgroups of G and the normal Hopf ideals of $\mathcal{O}(G)$. In light of Theorem 2.6.11, we can extend this to a correspondence between normal subgroups of G and normal Hopf subalgebras of $\mathcal{O}(G)$ (In fact, since $\mathcal{O}(G)$ is commutative, the specification *normal* for a Hopf subalgebra is superfluous, Example 2.6.3).

Chapter 3

Classical Connected Hopf Algebras

3.1 Introduction

This chapter is our first foray into the world of *connected* Hopf algebras, that is, the world of Hopf algebras which have no non-trivial simple subcoalgebras. As mentioned in the introduction, a complete classification of connected Hopf algebras over a field of characteristic zero which were either commutative or cocommutative had been achieved by the end of the 1960's. The principal aim of this chapter is to describe this classification and to examine some explicit examples of the connected Hopf algebras which occur as part of it.

We begin in §3.2 by giving the precise definition of a connected Hopf algebra, looking at some explicit examples and describing some of their elementary Hopf-theoretic properties. Most important for us is Proposition 3.2.5, stating that the coradical filtration of a connected Hopf algebra H is a Hopf filtration, allowing us to associate to H the graded connected Hopf algebra $\operatorname{gr} H$.

In §2.4.2 we saw that, in characteristic 0, given an affine commutative Hopf algebra H, there exists some algebraic group G such that $H = \mathcal{O}(G)$. In §3.3 we classify those G such that H is connected. Such a classification is a classical result in the theory of algebraic groups, and turns out to be a succinct one: $H = \mathcal{O}(G)$ is connected if and only if G is unipotent.

In §3.4 we turn to affine *cocommutative* connected Hopf algebras. In characteristic 0, there exists a description of connected cocommutative Hopf algebras which is equally as beautiful and succinct as the description of the (affine) commutative ones: any such Hopf algebra H is necessarily of the form $U(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of primitive elements of H. This result is often referred to as the Cartier-Milnor-Moore theorem, and was first proved in the 1960s, [39].

3.2 Preliminary results on connected and pointed coalgebras

In this section we recall what it means for coalgebra or Hopf algebra to be *pointed* or *connected*. We list some classical examples of pointed and connected Hopf algebras and discuss some important elementary properties possessed by such Hopf algebras.

Definition 3.2.1. Let C be a coalgebra.

1. C is *pointed* if every simple subcoalgebra is one-dimensional.

2. C is *connected* if C_0 is one-dimensional, where C_0 denotes the coradical of C (see Definition 2.3.21).

3. A Hopf algebra H is *pointed* (resp. *connected*) if its underlying coalgebra is pointed (resp. connected).

Definition 3.2.2. For a Hopf algebra H, define $G(H) = \{g \in H : \Delta(g) = g \otimes g\}$, the set of *group-like* elements of H. Using the antipode, counit and the fact that Δ is an algebra homomorphism, one can show that $G(H)$ is a group, where $g^{-1} = S(g)$ for any $g \in G(H)$.

Proposition 3.2.3. *Let H be a Hopf algebra.*

1. *Group-like elements of H are linearly independent.*

2. *H is pointed if and only if $H_0 = kG(H)$.*

3. *H is connected if and only if H is pointed and $G(H) = 1_H$.*

Proof. 1. Suppose the opposite. Then there exists some $n \geq 1$ and some elements $h, g_1, \dots, g_n \in G(H)$ such that $\{g_i\}_{i\geq 1}$ are linearly independent and $h = \sum_{i=1}^{n} \lambda_i g_i$ for some $\lambda_i \in k$, $h \neq g_i$ for $1 \leq i \leq n$. Applying the coproduct to this expression yields

$$h \otimes h = \sum_{i=1}^{n} \lambda_i (g_i \otimes g_i).$$

Substituting $h = \sum_{i=1}^{n} \lambda_i g_i$ into the displayed equation, we see that

$$\sum_{i=1}^{n} \lambda_i (h - g_i) \otimes g_i = 0.$$

Since $\{g_i\}_{i\geq 1}$ are linearly independent, it must be that $\lambda_i(h-g_i) = 0$ for all $1 \leq i \leq n$, hence $\lambda_i = 0$ for $1 \leq i \leq n$. This is a contradiction, thus the group-likes must be linearly independent.

2. Necessarily, any one-dimensional subcoalgebra is of the form kg for some $g \in G(H)$. Since H_0 is, by definition, the sum of all simple subcoalgebras of H, it follows that if H pointed then $H_0 = kG(H)$. Conversely, suppose $H_0 = kG(H)$ and let C be a simple subcoalgebra of H. By Remark 2.3.20 C is finite dimensional, hence there exists a finite subset $\{g_1, \ldots, g_n\}$ of G such that $C \subseteq \sum_{i=1}^{n} kg_i := D$. Dualising, we get
$$D^* = \bigoplus_{i=1}^{n} k\rho_i,$$
where $\rho_i(g_j) = \delta_{ij}$ and $\rho_i = \rho_i^2$. Thus D^* is the direct sum of n copies of k. Now D is a completely reducible D^*-module, where, for $i = 1, \ldots, n$, kg_i is a simple D^*-submodule of D and, as D^*-modules, $kg_i \cong kg_j$ if and only if $i = j$. By the basic theory of semisimple modules, the only simple D^*-submodules of D are the $\{kg_i\}$, for $i = 1, \ldots, n$. Thus C must be equal to one of them, and hence must be one-dimensional. The result follows.

3. By definition, any connected Hopf algebra is necessarily pointed. It follows from (2) that a pointed Hopf algebra H is connected if and only if $G(H) = 1_H$.

$\square$

Example 3.2.4. 1. Let G be a group and set $H = kG$, the group algebra of G with the usual cocommutative Hopf structure. By definition of the coproduct, $G \subseteq G(H)$. It follows that $H = kG = kG(H) = H_0$, so H is pointed by Proposition 3.2.3.

2. Let $\mathfrak{g}$ be a Lie algebra and let $H = U(\mathfrak{g})_c$, the universal enveloping algebra of $\mathfrak{g}$ equipped with the usual cocommutative structure. In Corollary 3.4.4 it is proved that H is a connected Hopf algebra.

3. Fix $n \geq 1$ and let $H = \mathcal{O}(\mathbb{U}_n)$, the coordinate ring of the algebraic group of upper triangular $n \times n$-matrices, as introduced in Example 2.4.4. We prove in §3.3.1 that H is connected.

4. Over an algebraically closed field, *any* cocommutative Hopf algebra is pointed. Indeed, suppose k is algebraically closed and H is a cocommutative Hopf k-algebra.

Let C be a simple subcoalgebra of H. Then C^* is a finite dimensional simple commutative algebra over an algebraically closed field, [41, Lemma 5.1.4]. As C^* is a commutative simple algebra it is necessarily a field, and since it is finite dimensional and k is algebraically closed, it must be that $C^* = k$; in particular, $\dim_k C = 1$.

5. (Non-example) Let $H = \mathcal{O}(\mathrm{SL}_2(k))$, which can be realised, using similar notation to that of Example 2.4.6, as the quotient algebra

$$k[X_{11}, X_{12}, X_{21}, X_{22}]/\langle X_{11}X_{22} - X_{12}X_{21} - 1\rangle.$$

As usual, the coalgebra structure on H comes from dualising the usual matrix multiplication in $\mathrm{SL}_2(k)$. A straightforward calculation yields the following equation for any $1 \leq i, j \leq 2$:

$$\Delta(X_{ij}) = X_{i1} \otimes X_{1j} + X_{i2} \otimes X_{2j}.$$

We now show that $C := \mathrm{span}_k\{X_{11}, X_{12}, X_{21}, X_{22}\}$ forms the basis of a 4-dimensional simple subcoalgebra of H, hence H cannot be pointed.

Let $D = C^*$ and let $x_{ij} = X_{ij}^*$ for $1 \leq i, j \leq 2$. As mentioned in Remark 2.2.7, the coalgebra structure of C induces an algebra structure on D defined by the convolution product:

$$x_{ij}x_{pq}(c) := (x_{ij} \otimes x_{pq})\Delta(c)$$

for $c \in C$, $i, j, p, q \in \{1, 2\}$. Using the above formula for the coproduct, it is easy to show that, with respect to this multiplication in D,

$$x_{12}^2 = x_{21}^2 = 0, \quad x_{12}x_{21} = x_{11}, \quad x_{21}x_{12} = x_{22}. \tag{3.2.1}$$

From these relations it is clear to see that D is a 4-dimensional matrix algebra, hence simple. By [41, Lemma 5.1.4 (3)], C is thus a simple coalgebra.

Note that each of the connected Hopf algebras appearing in Example 3.2.4 is either cocommutative or commutative. In §4.3 we give examples of connected Hopf algebras which are both noncommutative and noncocommutative.

In Proposition 2.3.26, it was proved that, for a given Hopf algebra H, $\{H_n\}$ forms a Hopf filtration if and only if H_0 is a Hopf subalgebra. If H is pointed, then its coradical H_0 is the group algebra $kG(H)$, and so a Hopf subalgebra. It follows that the coradical filtration of any pointed Hopf algebra H is necessarily a Hopf filtration, and hence, by

Proposition 2.3.15, $\operatorname{gr} H$ is a Hopf algebra. As we shall see, this fact is invaluable as a tool in the study of connected Hopf algebras, so we state it, and more, as a proposition below.

Proposition 3.2.5. *Suppose is H a pointed (resp. connected) Hopf algebra. Then $\{H_n\}$ forms a Hopf filtration and $\operatorname{gr} H$ is a pointed (resp. connected) Hopf algebra which is graded as a Hopf algebra. Furthermore, $\operatorname{gr} H$ is coradically graded with respect to this Hopf-grading.*

Proof. Let H be a pointed Hopf algebra. Since $H_0 = kG(H)$ is a Hopf subalgebra, $\{H_i\}$ is a Hopf filtration and hence, in the notation of Definition 2.3.25, $\operatorname{gr} H = \bigoplus_{i=0}^{\infty} H(i)$ is graded as a Hopf algebra by Proposition 2.3.15. That $\operatorname{gr} H$ is coradically graded with respect to this Hopf grading is Proposition 2.4.2. In particular, this guarantees that $(\operatorname{gr} H)_0 = H(0) = H_0 = kG(H)$, hence $\operatorname{gr} H$ is pointed, and is connected if (and only if) H is connected. □

3.2.1 Some useful lemmas on connected Hopf algebras

We conclude §3.2 by listing a few useful results pertaining to the structure of connected coalgebras . These can all be found in [41, Chapter 5], but we list them here for the reader's convenience.

Lemma 3.2.6. *Suppose H is a pointed (resp. connected) Hopf algebra. Let K denote a Hopf subalgebra of H and $\overline{H}$ a Hopf quotient of H. Then both K and $\overline{H}$ are pointed (resp. connected).*

Proof. Suppose H is pointed (resp. connected). By [41, Lemma 5.2.12], $K_n = K \cap H_n$ for all $n \geq 0$. In particular, $K_0 = K \cap (kG(H)) = kG(K)$, hence K is pointed (resp. connected).

Let $f : H \to \overline{H}$ denote the canonical surjective Hopf morphism. If H is pointed (resp. connected) [41, Corollary 5.3.5] tells us that $f(H_0) = \overline{H}_0$. Since f is a Hopf morphism it preserves group-like elements, hence $\overline{H}$ is pointed (resp. connected). □

The following result is particularly useful when constructing new examples of connected Hopf algebras.

Lemma 3.2.7. *Suppose B is a connected bialgebra. Then B is a connected Hopf algebra.*

Proof. Suppose B is a bialgebra. To prove that it is a Hopf algebra, it suffices to prove that id_B is invertible with respect to convolution. Now, since B is connected, $B_0 = k$ and

$\mathrm{id}|_{B_0} = \mathrm{id}_k$, which is convolution invertible in $\mathrm{Hom}(B_0, B) = \mathrm{Hom}(k, B)$. The result now follows from [41, Lemma 5.2.10], which states that, for any coalgebra C and algebra A, $f \in \mathrm{Hom}(C, A)$ is convolution invertible if and only if $f|_{C_0}$ is invertible in $\mathrm{Hom}(C_0, A)$.

□

The following result appears as [67, Lemma 6.1]. We offer a different proof to the one given there.

Lemma 3.2.8. *Let H be a connected Hopf algebra and K a sub-bialgebra of H. Then K is a Hopf subalgebra of H.*

Proof. Since H is connected, K is a connected bialgebra by Lemma 3.2.6. By Lemma 3.2.7, K is a connected Hopf algebra. That is, there exists a unique map $S_{(K)} : K \to K$ such that

$$(S_{(K)} \otimes \mathrm{id}_K)\Delta_K = \epsilon_K.$$

Let S denote the antipode of H. If we can show $S_{(K)} = S|_K$, the restriction of the antipode of H to K, then $S(K) \subset K$ and K would be a Hopf subalgebra, which would complete the proof. For this, by uniqueness of the antipode as a convolution inverse of the identity operator, it suffices to prove that

$$(S|_K \otimes \mathrm{id}_K)\Delta_K = \epsilon_K = (\mathrm{id}_K \otimes S|_K)\Delta_K.$$

Now,

$$((S \otimes \mathrm{id})\Delta)|_K = \epsilon|_K = ((\mathrm{id} \otimes S)\Delta)|_K.$$

Since $\Delta(K) \subseteq K \otimes K$,

$$((S \otimes \mathrm{id})\Delta)|_K = (S|_K \otimes \mathrm{id}|_K)\Delta|_K$$

and

$$((\mathrm{id} \otimes S)\Delta)|_K = (\mathrm{id}|_K \otimes S|_K)\Delta|_K.$$

Moreover, since K is a subbialgebra of H, $\Delta_K = \Delta|_K$ and $\epsilon_K = \epsilon|_K$. The result follows.

□

Let H be a connected Hopf algebra. The final result of this section gives us a complete description of H_1 and a useful expression for any element of H under the coproduct.

Definition 3.2.9. Let H be a Hopf algebra. Define $P(H) = \{x \in H : \Delta(x) = 1 \otimes x + x \otimes 1\}$, the space of *primitive elements* of H. An elementary calculation shows that $P(H)$ becomes a Lie algebra under the usual commutator bracket $[x, y] = xy - yx$, for $x, y \in P(H)$.

Lemma 3.2.10. *Let H be a connected Hopf algebra.*

1. $H_1 = k \oplus P(H)$.
2. *For any $n \geq 1$ and $h \in H_n$,*

$$\Delta(h) = h \otimes 1 + 1 \otimes h + y$$

for some $y \in H_{n-1} \otimes H_{n-1}$.

Proof. This is [41, Lemma 5.3.2]. □

3.3 Commutative connected Hopf algebras

The principal aim of this section is to sketch a proof of the classification of affine commutative connected Hopf algebras in characteristic zero. The majority of the results in this section have been adapted from [37, Chapter XV].

3.3.1 Unipotent groups

Recall from §2.4.3 that by a representation of an algebraic group we mean a *rational* representation.

Definition 3.3.1. An algebraic group G is *unipotent* if every nonzero finite dimensional representation of G has a nonzero fixed vector.

In Example 2.4.4 we met the algebraic group $\mathbb{U}_n$ of $n \times n$ strictly upper triangular matrices. That this is a unipotent group is part of the following proposition.

Proposition 3.3.2. *Let G be a unipotent group.*

1. *G is isomorphic to a closed subgroup of $\mathbb{U}_n$ for some $n \geq 1$.*
2. *Closed subgroups and quotients (by normal closed subgroups) of G are unipotent.*
3. *G has a finite chain of subgroups $1 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ with G_i a normal subgroup of G, $\dim G_i = i$ and $G_{i+1}/G_i \cong (k,+)$ for all $0 \leq i \leq n-1$.*
4. *Suppose $\dim G = m$. As an algebraic set, $G \cong (k,+)^m$. If in addition G is abelian, $G \cong (k,+)^m$ as an algebraic group.*
5. *Suppose $K \subset H$ are closed subgroups of G. Then $K = H$ if and only if $\dim K = \dim H$.*

Proof. 1. This is [37, Chapter XV, Theorem 9.4].

2. If G is unipotent and H is a closed subgroup, by (1) H is also isomorphic to a closed subgroup of $\mathbb{U}_n$ for some $n \geq 1$, hence unipotent. To see that quotient groups of G are unipotent, note that a representation of a quotient of G can be regarded as a representation of G, and so has a nonzero fixed vector if it is nontrivial.

3. This is [37, Chapter XV, Corollary 2.13].

4. This is a special case of [23, Theorem 15.5].

5. One direction is clear. For the converse, suppose that $K \subsetneq H$ and $\dim K = \dim H$. By [23, Lemma 17.4 (d)], K is properly contained in its normaliser in H, $N_H(K)$, which is also a closed subgroup of H, [23, Corollary 8.2]. By (2) it follows that $N_H(K)$ and hence $N_H(H)/K$ are unipotent groups. Since $K \subsetneq N_H(K)$, $\dim(N_H(K)/K) \geq 1$ by (4). This contradicts the assumption that $\dim K = \dim H$. The result follows.

□

Remark 3.3.3. By means of the Lie functor, which takes an affine algebraic group to its corresponding Lie algebra, there exists a one-to-one correspondence between affine unipotent algebraic groups and finite dimensional nilpotent Lie algebras, [38, Ch II §4].

Let G be an algebraic group. In §2.4.3 we established a one to one correspondence between (rational) representations V of G and $\mathcal{O}(G)$-comodules. Using this correspondence, we now examine how the property of G being unipotent is reflected in the $\mathcal{O}(G)$-comodule structures on any finite dimensional representation V of G. We shall use the notation set out in §2.4.3 for the following lemma.

Lemma 3.3.4. *Let V be a finite dimensional representation of a unipotent group U. Let $\rho : V \to V \otimes \mathcal{O}(U)$ denote the induced $\mathcal{O}(U)$-comodule structure on V (see* §2.4.3*). Then $v \in V$ is fixed by the action of U if and only if*

$$\rho(v) = v \otimes 1.$$

In particular, since U is unipotent, there exists a one dimensional $\mathcal{O}(U)$-subcomodule of V (defined by a fixed vector).

Proof. Since U is unipotent, V has a nonzero vector fixed by the action of U. Let $v = \sum_{j=1}^{n} \lambda_i e_j$ be such a vector, where $\lambda_j \in k$ and $\{e_j\}$ denote the standard basis vectors of

V. By the discussion in §2.4.3,

$$\rho(v) = \sum_{j=1}^{n} \lambda_j \rho(e_j) = \sum_{j=1}^{n} \lambda_j \sum_{i \in I} e_i \otimes r_{ij}$$

where $r_{ij} \in \mathcal{O}(U)$ define the group action of U on V: $g \cdot v = (r_{ij}(g)_{i,j \in I})(v)$. We assumed v is fixed by G, hence $r_{ij} = \delta_{ij}$, so $\rho(v) = v \otimes 1$, as required. □

Corollary 3.3.5. *Let U be a unipotent group. Then every simple $\mathcal{O}(U)$-comodule is one-dimensional, i.e., $\mathcal{O}(U)$ is a pointed Hopf algebra.*

We are now in a position to state and prove the main result of this section, which completely classifies both those algebraic groups G for which $\mathcal{O}(G)$ is unipotent and those affine commutative algebras which can admit a connected Hopf structure. The latter of these classifications is by far the most difficult to prove, and as mentioned previously is originally due to Lazard in [28]. We do not include his proof here for reasons of space. We follow the proof of [37, Chapter XV, Theorem 2.4] and [64, Theorem 8.3] for the remainder of the following theorem.

Theorem 3.3.6. *Let G be an algebraic group and let $H = \mathcal{O}(G)$. The following are equivalent:*

1. *G is unipotent.*
2. *H is a connected Hopf algebra.*
3. *$H \cong k[x_1, \ldots, x_n]$ as an algebra, where $n = \dim G$.*

Proof. (1) $\implies$ (2): By Proposition 3.3.2, we can assume that G is a closed subgroup of $\mathbb{U}_n$ for some $n \geq 0$. Applying the coordinate ring functor to the inclusion map $G \hookrightarrow \mathbb{U}_n$ gives $H = \mathcal{O}(G)$ as a Hopf quotient of $\mathcal{O}(\mathbb{U}_n)$, so it suffices to prove that $\mathcal{O}(\mathbb{U}_n)$ is connected, Lemma 3.2.6. In the notation of Example 2.4.6, one can show that

$$\mathcal{O}(\mathbb{U}_n) \cong k[X_{ij} : i < j \leq n]$$

and that

$$\Delta(X_{ij}) = X_{ij} \otimes 1 + 1 \otimes X_{ij} + \sum_{i<r<j} X_{ir} \otimes X_{rj}. \tag{3.3.1}$$

Define a filtration of $\mathcal{O}(\mathbb{U}_n)$ as follows: let the degree of a monomial $\prod X_{ij}^{n_{ij}}$ be $\sum n_{ij}(j-i)$ and let

$$C_r = \operatorname{span}_k \left\{ \prod X_{ij}^{n_{ij}} : \sum_{i<j} n_{ij}(j-i) \leq r \right\}.$$

By definition, $\{C_i\}_{i\geq 0}$ is an exhaustive algebra filtration of $\mathcal{O}(\mathbb{U}_n)$, with $C_0 = k$. Moreover, by (3.3.1) if $(j-i) < r$ then $\Delta(X_{ij}) \in \sum_{0\leq k\leq r} C_k \otimes C_{r-k}$. Since Δ is an algebra homomorphism and $\{C_i\}$ is an algebra filtration, we deduce $\Delta(C_r) \in \sum_{0\leq k\leq r} C_k \otimes C_{r-k}$, i.e., that $\{C_i\}$ is a coalgebra filtration. By [41, Lemma 5.3.4], $\mathcal{O}(\mathbb{U}_n)_0 \subseteq C_0 = k$, hence $\mathcal{O}(\mathbb{U}_n)$ is connected.

(2) $\implies$ (1) We use the correspondence between simple rational representations of G and simple subcomodules of $\mathcal{O}(G)$, alongside the fact that the only simple comodules for H are one dimensional. We omit the details, which can be found in [37, Chapter XV, Theorem 2.4, $(c) \implies (a)$].

(1) $\implies$ (3): This follows immediately from Proposition 3.3.2 (4).

(3) $\implies$ (1) This is Lazard's theorem, [28].

□

Remark 3.3.7. As a consequence of Theorem 3.3.6(3) is that connected commutative affine Hopf algebra possess excellent homological properties - they are, for example, AS-regular and GK-Cohen-Macaulay, as noted in §2.5.2. We'll see later in Chapter 4 that these homological properties of connected Hopf algebras survive even when the assumption of commutativity is dropped.

3.4 Cocommutative connected Hopf algberas

The main result of this section is to completely classify, in characteristic zero, any connected cocommutative Hopf algebra H as the universal enveloping algebra of its Lie algebra of primitive elements.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $H = U(\mathfrak{g})_c$, as defined in Example 2.2.13. We saw that this was an example of an affine cocommutative Hopf algebra. The first aim of this section is to prove that in fact H is also a connected Hopf algebra.

Attached to the algebra H is the PBW-filtration $\{P_i\}$, as defined in Example 2.3.5. Also attached to H is the coradical filtration $\{H_i\}$. It turns out that in fact these filtrations on H are identical, and as an immediate consequence we deduce that H is connected. Before we can prove this, we need the following technical lemma, which appears as [41, Lemma 5.5.1].

Lemma 3.4.1. *Let H be a Hopf algebra with subspaces $A_0 \subset A_1$ with A_0 a subalgebra (with 1) such that*

1. *A_1 generates H as an algebra, and $A_0A_1, A_1A_0 \subset A_1$.*

2. *$\Delta(A_0) \subseteq A_0 \otimes A$ and $\Delta(A_1) \subseteq A_1 \otimes A_0 + A_0 \otimes A_1$.*

For $n \geq 1$, define, using the notation of Notation 2.3.4 , $A_n = (A_1)^n$. Then $\{A_n\}$ is a coalgebra filtration of H and $H_0 \subseteq A_0$. If also $A_0 = H_0$, $A_n \subseteq H_n$ for $n \geq 0$.

Remark 3.4.2. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and set $H = U(\mathfrak{g})_c$. Then the filtration $\{P_n\}$ of H satisfies conditions (1) and (2) of Lemma 3.4.1. Indeed, $P_0 = k$ is a unital subalgebra and $P_1 = \mathfrak{g} + k1$ generates H as an algebra. That $P_0P_1, P_1P_0 \subset P_1$ is immediate, as is that $\Delta(P_0) \subset P_0 \otimes H$. That $\Delta(P_1) \subset P_1 \otimes P_0 + P_0 \otimes P_1$ follows from the fact that $\Delta(x) = 1 \otimes x + x \otimes 1$ for all $x \in \mathfrak{g}$.

We are now in a position to prove that, for a finite-dimensional Lie algebra $\mathfrak{g}$, the coradical filtration and PBW filtration of the Hopf algebra $H = U(\mathfrak{g})_c$ are identical. We follow the proof of [41, Proposition 5.5.3], where in fact the result is proved more generally for *any* Lie algebra. We state and prove the finite-dimensional case for two reasons: first, the proof of the infinite-dimensional case requires us to set up a lot more notation, and since we work only with finite-dimensional Lie algebras in this thesis, there is little to gain from this. Second, the proof of this result is enlightening in that it exhibits explicitly the gymnastics of the coradical filtration, a good understanding of which will be imperative in later chapters.

Proposition 3.4.3. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and set $H = U(\mathfrak{g})_c$. Let $\{P_n\}$ denote the usual PBW-filtration of H. Then $H_n = P_n$ for $n \geq 0$.*

Proof. First, by definition, $P_n = (P_1)^n$, hence Lemma 3.4.1 and Remark 3.4.2 imply $H_0 \subseteq P_0$. Clearly, $P_0 = k \subseteq H_0$, thus $P_0 = H_0$. By Lemma 3.4.1, $P_n \subseteq H_n$ for all $n \geq 0$, thus we must prove that $H_n \subseteq P_n$ for $n \geq 0$. We proceed by induction on n.

We already proved the $n = 0$ case. For the inductive hypothesis, fix $n \geq 1$ and suppose $H_{n-1} \subseteq P_{n-1}$. Let $\{x_1, \ldots, x_m\}$ denote a basis of $\mathfrak{g}$ and choose $h \in H_n$. As noted in Example 2.3.5, h can be written as a linear combination of monomials of the form

$$p_{(n_1,\ldots,n_m)} = x_1^{n_1} \ldots x_m^{n_m} \tag{3.4.1}$$

for some $n_1, \ldots, n_m \geq 0$. We define the degree of any such monomial $p_{(n_1,\ldots,n_m)}$ as the non-negative integer $\sum_{i=1}^{m} n_i$. Let $p_{(d_1,\ldots,d_m)}$ be a monomial of maximal degree d in any such expression of h. If we can prove $d \leq n$, then $h \in P_n$, completing the proof. Since $\Delta : H \to H \otimes H$ is an algebra morphism and $\mathfrak{g} \subset P(H)$,

$$\begin{aligned}\Delta(p_{(d_1,\dots,d_m)}) &= \Delta(x_1)^{d_1} \dots \Delta(x_m)^{d_m} \\ &= (1 \otimes x_1 + x_1 \otimes 1)^{d_1} \dots (1 \otimes x_m + x_m \otimes 1)^{d_m} \\ &= \left(\sum_{i=0}^{d_1} \beta_i^{d_1} x_1^i \otimes x_1^{d_1 - i}\right) \dots \left(\sum_{i=0}^{d_m} \beta_i^{d_m} x_m^i \otimes x_m^{d_m - i}\right)\end{aligned} \tag{3.4.2}$$

where, for $1 \leq j \leq m$, $1 \leq i \leq d_j$,

$$\beta_i^{d_j} = \binom{d_j}{i}.$$

As $\operatorname{char} k = 0$, $\beta_i^{d_j} \neq 0$ for all i, j. Noting this and taking the product of the first terms of the first $m-1$ brackets and the second term of the m^{th} bracket in the expression (3.4.2), we see that, up to non-zero scalar, the monomial

$$x_m \otimes x_1^{d_1} \dots x_{m-1}^{d_{m-1}} x_m^{(d_m - 1)}$$

appears in the expression for $\Delta(h)$. By the definition of the coradical filtration and the inductive hypothesis,

$$\begin{aligned}\Delta(h) &\in H \otimes H_{n-1} + H_0 \otimes H \\ &= H \otimes P_{n-1} + k \otimes H.\end{aligned}$$

Since clearly $x_m \otimes x_1^{d_1} \dots x_{m-1}^{d_{m-1}} x_m^{(d_m - 1)} \notin k \otimes H$, it must be that

$$x_m \otimes x_1^{d_1} \dots x_{m-1}^{d_{m-1}} x_m^{(d_m - 1)} \in H \otimes P_{n-1}$$

and hence $d_1 + \ldots + d_{m-1} + d_m - 1 \leq n - 1$, i.e., $d = \sum_{i=1}^m d_i \leq n$, as required. □

Corollary 3.4.4. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and set $H = U(\mathfrak{g})_c$. Then H is a connected Hopf algebra and $P(H) = \mathfrak{g}$.*

Proof. By Proposition 3.4.3, $H_0 = P_0 = k$, hence H is connected. By Proposition 3.2.10, $H_1 = P(H) + k$. Since $P_1 = \mathfrak{g} + k$, the claim follows from Proposition 3.4.3. □

In light of Corollary 3.4.4, there is a correspondence between (finite-dimensional) Lie algebras and connected cocommutative Hopf algebras of finite GK-dimension. It was proved by Milnor and Moore, and by Cartier (independently) in the 1960's that this correspondence is in fact one-to-one. This result is commonly referred to as the Cartier-Milnor-Moore theorem. This result has several different proofs, none of which we give here, as all of them are rather lengthy and not particularly enlightening for the purposes of this thesis. For the interested reader, a proof can be found in [41, §5.6].

Theorem 3.4.5. *(Cartier-Milnor-Moore) Let H be a cocommutative connected Hopf algebra of finite GK-dimension. Then $H = U(\mathfrak{g})_c$, where $\mathfrak{g} = P(H)$ is finite-dimensional.*

Corollary 3.4.6. *For any finite dimensional Lie algebra $\mathfrak{g}$, $U(\mathfrak{g})_c$ is the unique cocommutative Hopf structure on the algebra $U(\mathfrak{g})$.*

Proof. Let H be a cocommutative Hopf algebra which, as an algebra, is isomorphic to $U(\mathfrak{g})$ for some finite dimensional Lie algebra $\mathfrak{g}$. As noted in Example 3.2.4, since k is algebraically closed and H is cocommutative, it must also be pointed. Since $H \cong U(\mathfrak{g})$ as an algebra, H has no non-trivial invertible elements, hence no non-trivial group-like elements. Any pointed Hopf algebra with no non-trivial group-likes is necessarily connected. The result now follows immediately from Theorem 3.4.5.

□

Remark 3.4.7.

1. The Cartier-Milnor-Moore holds more generally for connected cocommutative Hopf algebras of infinite GK-dimension, [41, Theorem 5.6.5].

2. Both Theorem 3.4.5 and Corollary 3.4.6 can be deduced from the more general Cartier-Gabriel-Kostant theorem, which classifies *any* cocommutative Hopf algebra (over an algebraically closed field of characteristic 0) as a smash product of a group algebra and an enveloping algebra of a Lie algebra. Since we are really only concerned with connected Hopf algebras in this thesis, we do not discuss this result here.

Remark 3.4.8. (Positive characteristic) There is no version of the Cartier-Milnor-Moore theorem for connected cocommutative Hopf algebras in positive characteristic - see [41, Example 5.6.8] for a counter-example.

In positive characteristic, *finite dimensional* cocommutative connected Hopf algebras are not even completely understood. As a consequence of Theorem 3.4.5, in characteristic 0, there are no non-trivial finite-dimensional connected cocommutative Hopf algebras. This is certainly *not* the case for cocommutative Hopf algebras in positive characteristic, where, for example, linear duals of the group algebras of finite abelian p-groups (see [41, Theorem 5.7.1]) and restricted enveloping algebras (see [41, Definition 2.3.2]) form a large class of examples. It is an ongoing endeavour to list the finite-dimensional cocommutative connected Hopf algebras over a field of characteristic p. To date this has been accomplished for algebras of dimension less than or equal to p^3, and not all of these low-dimensional algebras can be realised as (restricted) enveloping algebras of Lie algebras - see [21] for details.

Chapter 4

Noncommutative Unipotent Groups

4.1 Introduction

In Chapter 3 we discussed the classification of connected Hopf algebras in characteristic zero which are either commutative or cocommutative. With these foundations laid, we now drop the classical constraints of commutativity or cocommutativity and embark into the realm of arbitrary connected Hopf algebras of finite GK-dimension. Major strides in the understanding such Hopf algebras were made by G. Zhuang in [67], and the chief purpose of this chapter will be to detail precisely what these were. In addition to this, we shall briefly review the main results of [7, §3], which develop a connection between connected Hopf algebras and an interesting class of Hopf algebras known as iterated Hopf Ore extensions.

Let H be a connected Hopf algebra over an algebraically closed field k of characteristic 0, with $\operatorname{GKdim} H < \infty$. In [67, Theorem 6.10] Zhuang proves, among other things, that the associated graded Hopf algebra $\operatorname{gr} H$ is a polynomial algebra in n variables, where $n = \operatorname{GKdim} H$. In §4.2 we reproduce (most of) the steps taken in [67, §6] to prove this statement and finally to give the proof of the statement itself. Zhuang's results allow us to view connected Hopf algebras, in a sense which we shall make precise, as "noncommutative unipotent groups", and in fact we see in §4.2.3 that they share many ring theoretic and homological properties in common with their commutative counterparts.

Zhuang's new results are of little value if there are no concrete examples of connected Hopf algebras which are neither commutative nor cocommutative to accompany them.

In [67], in the course of classifying connected Hopf algebras of GK-dimension at most three, Zhuang constructs two infinite families of such Hopf algebras, all of GK-dimension 3. In fact we shall see that this is the minimal dimension at which such a Hopf algebra can exist. In §4.3 we list each of the low dimensional Hopf algebras appearing in Zhuang's classification and discuss some of their Hopf theoretic properties.

A Hopf Ore extension, as first formally defined by Panov in [44], is a skew polynomial ring $A = R[x; \sigma, \partial]$ with the additional property that A is a Hopf algebra and R is a Hopf subalgebra of A. In [7], Brown, O'Hagan, Zhang and Zhuang investigated the properties of Iterated Hopf Ore extensions, which are those Hopf algebras which can be built by starting with an algebraically closed field k of characteristic 0 and applying Panov's construction iteratively. Several well known classes of Hopf algebras can be obtained from such a construction, for example the (cocommmutative) enveloping algebras of finite dimensional solvable Lie algebras, and the coordinate rings of affine unipotent groups. In §4.4 we review some of the results of [7]. In particular, we'll see that they form a proper subclass of the class of connected Hopf algebras.

4.2 The associated graded Hopf algebra of a connected Hopf algebra

The aim of this section is to reproduce (most of) the steps taken in [67, §6] to prove that, with respect to its coradical filtration, the associated graded algebra of a connected Hopf algebra is a commutative polynomial algebra. We do this for two reasons: first, we shall use several of Zhuang's results repeatedly throughout the rest of our work, so it is convenient to have them at hand. Second, Zhuang's results are key to our understanding of connected Hopf algebras, so it is essential that we understand how and why they work.

4.2.1 Connected coradically graded Hopf algebras

Definition 4.2.1. Suppose $K = \bigoplus_{i \geq 0} K(i)$ is a Hopf algebra which is graded as a Hopf algebra (see Definition 2.3.12). We say that K is a *coradically Hopf graded Hopf algebra* if K is coradically graded as a coalgebra with respect to this Hopf grading. We say K is a *connected coradically Hopf graded Hopf algebra* if in addition $K(0) = k$.

Remark 4.2.2. As an immediate consequence of the definition, connected coradically Hopf graded Hopf algebras are *connected Hopf algebras*, since $K_0 = K(0) = k$.

For any connected Hopf algebra H, Proposition 3.2.5 guarantees that $\mathrm{gr}\, H = \bigoplus_{i\geq 0} H(i)$ is a connected coradically Hopf graded Hopf algebra. The purpose of this section is to investigate the structure of connected coradically Hopf graded Hopf algebras in their own right, and in doing so gain a better understanding of $\mathrm{gr}\, H$.

We begin by proving that affine connected coradically Hopf graded Hopf algebras are commutative. Whilst we follow the proof as in [67, Lemma 6.3], which uses [2, Lemma 5.5], the result is essentially originally due to Sweedler , [53, §11.2] - see Remark 4.2.8.

Lemma 4.2.3. *Let A be a locally finite connected coradically Hopf graded Hopf algebra. Let $\mathcal{D}_A$ denote the graded dual (see §2.3.1). Then $\mathcal{D}_A(1)$ is spanned by primitive elements.*

Proof. By definition, A is locally finite and graded as a Hopf algebra, hence $\mathcal{D}_A$ is also graded as a Hopf algebra. Pick $f \in \mathcal{D}_A(1)$. Since $\{\mathcal{D}_A\}$ is a coalgebra grading,

$$\begin{aligned}\Delta(f) &\in \mathcal{D}_A(0) \otimes \mathcal{D}_A(1) + \mathcal{D}_A(1) \otimes \mathcal{D}_A(0)\\ &= k \otimes \mathcal{D}_A(1) + \mathcal{D}_A(1) \otimes k.\end{aligned}$$

As noted in Remark 4.2.2, A and $\mathcal{D}_A$ are connected as Hopf algebras. The result now follows from Lemma 3.2.10. $\square$

Proposition 4.2.4. *([67, Lemma 6.3, Proposition 6.5]) Let $A = \bigoplus_{i\geq 0} A(i)$ be an affine connected coradically Hopf graded Hopf algebra. Then A is isomorphic to a polynomial algebra in n variables, for some $n \geq 0$.*

Proof. Since A is affine, the grading of A is locally finite. We can therefore define the graded Hopf dual $\mathcal{D}_A = \bigoplus_{i\geq 0} A(i)^*$. Since $\mathcal{D}_A(0) = A(0)^* = k$, it follows from [2, Lemma 5.5] that $\mathcal{D}_A$ is generated in degree one. By Lemma 4.2.3, $\mathcal{D}_A(1)$ is spanned by primitive elements, from which we deduce that $\mathcal{D}_A$ is cocommutative, hence A is an affine connected commutative Hopf algebra. The result now follows from Theorem 3.3.6. $\square$

The following lemma will be key in proving a noncommutative analogue of Proposition 3.3.2 (4).

Lemma 4.2.5. *([67, Lemma 6.8]) Let $A = \bigoplus_{i\geq 0} A(i)$ be a connected coradically Hopf graded Hopf algebra and $B = \bigoplus_{i\geq 0} B(i)$ be a graded subalgebra of A which is a finitely generated Hopf subalgebra of A and which is graded as a Hopf algebra. If $B \neq A$, then* $\mathrm{GKdim}\, A \geq \mathrm{GKdim}\, B + 1$.

Proof. Choose $n \in \mathbb{N}$ minimal such that $B(n) \neq A(n)$. Pick $y \in A(n) \setminus B(n)$. As A is connected coradically Hopf graded, A is connected as a Hopf algebra and $A(n) \subset A_n$. By Lemma 3.2.10,

$$\Delta(y) = 1 \otimes y + y \otimes 1 + w$$

for some $w \in \sum_{i=1}^{n-1} A_i \otimes A_{(n-1-i)}$. Since $\{A(i)\}$ is a coalgebra grading and $A_n = \bigoplus_{i=0}^{n} A(i)$ for $n \geq 0$,

$$\begin{aligned} w &\in \left(\sum_{i=1}^{n-1} A_i \otimes A_{(n-1-i)}\right) \cap \left(\sum_{j=1}^{n-1} A(j) \otimes A(n-1-j)\right) \\ &\subseteq \sum_{j=1}^{n-1} A(j) \otimes A(n-1-j) \\ &= \sum_{j=1}^{n-1} B(j) \otimes B(n-1-j) \end{aligned}$$

where the final equality follows from the minimality of n. If we denote by P the subalgebra of A generated by the set $B \cup \{y\}$, from this we deduce that P is an affine bialgebra in A, hence an affine Hopf subalgebra by Lemma 3.2.8. As A is commutative by Proposition 4.2.4, B is a (normal) Hopf subalgebra of A (see Example 2.6.3), hence B^+P is a normal Hopf ideal of P by Theorem 2.6.11. It follows $H := P/B^+P$ is a connected Hopf algebra by Lemma 3.2.6, and $H \cong \mathcal{O}(G_H)$ for some normal closed subgroup $G_H \subseteq G$ (see Example 2.6.5). We have a short exact sequence of affine connected commutative Hopf algebras

$$0 \longrightarrow B \longrightarrow P \longrightarrow H \longrightarrow 0$$

in the sense that B is a normal Hopf sublalgebra of P and $H = P/B^+P = P/PB^+$. By means of the Map functor described in §2.4.2, this corresponds to a short exact sequence of algebraic groups ([58, Theorem 5.2]),

$$0 \longrightarrow G_H \longrightarrow G_P \longrightarrow G_B \longrightarrow 0$$

where $H = \mathcal{O}(G_H), P = \mathcal{O}(G_P)$ and $B = \mathcal{O}(G_B)$. By [23, 7.4 Proposition B], as algebraic varieties,

$$\dim G_P = \dim G_H + \dim G_B.$$

Then, by Proposition 2.5.7,

$$\text{GKdim}\, P = \text{GKdim}\, H + \text{GKdim}\, B.$$

Now, $\text{GKdim}\, A \geq \text{GKdim}\, P$. Since $B \neq A$, we have $H \neq k$ by Theorem 2.6.11, hence $\text{GKdim}\, H \geq 1$. The result follows. $\square$

For reasons of space, we exclude the straightforward proof of the following result.

Lemma 4.2.6. *[67, Lemma 6.9] Let A be a connected coradically Hopf graded Hopf algebra. Then A has finite GK-dimension if and only if it is affine.*

4.2.2 The main theorem

We can now state and prove the main result of Chapter 4, which gives us a clear and succinct description of the associated graded Hopf algebra of a connected Hopf algebra of finite GK-dimension.

Theorem 4.2.7. *([67, Theorem 6.10]) Let H be a connected Hopf algebra. The following statements are equivalent:*

1. $\operatorname{GKdim} H < \infty$.
2. $\operatorname{GKdim} \operatorname{gr} H < \infty$.
3. $\operatorname{gr} H$ *is affine.*
4. $\operatorname{gr} H$ *is isomorphic as an algebra to the polynomial algebra in ℓ variables for some* $\ell \geq 0$.

If any one of the equivalent conditions are true, $\operatorname{GKdim} H = \operatorname{GKdim} \operatorname{gr} H = \ell \in \mathbb{Z}$.

Proof. (1) $\iff$ (2): Suppose $\operatorname{GKdim} H < \infty$. Then, by Proposition 2.5.8, $\operatorname{GKdim} \operatorname{gr} H \leq \operatorname{GKdim} H < \infty$. For the converse, suppose $\operatorname{GKdim} \operatorname{gr} H < \infty$. Then $\operatorname{gr} H$ is affine by Lemma 4.2.6 and so Proposition 2.5.8 implies $\operatorname{GKdim} H = \operatorname{GKdim} \operatorname{gr} H < \infty$.
(2) $\iff$ (3): Since $\operatorname{gr} H$ is a connected coradically graded Hopf algebra, this follows from Lemma 4.2.6.
(3) $\implies$ (4): Since $\operatorname{gr} H$ is a connected coradically graded Hopf algebra, this follows from Proposition 4.2.4.
(4) $\implies$ (3): Trivial.

If any one of these equivalent conditions hold, $\operatorname{GKdim} H = \operatorname{GKdim} \operatorname{gr} H$ by Proposition 2.5.8.

□

Remark 4.2.8. 1. By a result of Sweedler ([53, §11.2]) with several subsequent rediscoveries, for *any* field k and connected Hopf k-algebra H, $\operatorname{gr} H$ is commutative.

It is not true, however, that over an arbitrary field $\operatorname{gr} H$ is necessarily a polynomial algebra. The following counterexample is adapted from [67, Remark 6.7]: let $k = \mathbb{F}_p$ and $H = k[x]/(x^p)$. Then there exists a unique connected Hopf structure on H (under which x is primitive). Since H is not a domain, neither is $\operatorname{gr} H$, hence $\operatorname{gr} H$ cannot be a polynomial algebra.

2. In the literature, a filtered algebra with a commutative associated graded algebra is often referred to as an *almost commutative* algebra. By the preceding remark connected Hopf algebras thus form a class of almost commutative algebras in this sense.

Corollary 4.2.9. *Let H be a connected Hopf algebra with* $\operatorname{GKdim} H < \infty$. *Then, as a Hopf algebra,* $\operatorname{gr} H \cong \mathcal{O}(U)$ *for some unipotent group U with* $\dim U = \operatorname{GKdim} H$.

Proof. Suppose H is connected and $\operatorname{GKdim} H = n < \infty$. By Proposition 3.2.5, $\operatorname{gr} H$ is a (coradically graded) connected Hopf algebra and Theorem 4.2.7 tells us that $\operatorname{gr} H \cong k[x_1, \dots, x_n]$ as an algebra, where $n = \operatorname{GKdim} H$. The result now follows immediately from Theorem 3.3.6 and Proposition 2.5.7. $\square$

Corollary 4.2.9 gives reasonable justification for labelling connected Hopf algebras (of finite GK-dimension, over an algebraically closed field of characteristic 0) as "noncommutative unipotent groups" or, more precisely, in the sense of [10], for example, *PBW-deformations* of unipotent groups. In line with this philosophy, the next result may be viewed as a noncommutative analogue of the fact that, for a unipotent group U and closed subgroups $M \subseteq N$, $M = N$ if and only if $\dim M = \dim N$.

Lemma 4.2.10. *([67, Lemma 7.4]) Let H be a connected Hopf k-algebra of finite GK-dimension and $K \subseteq H$ a Hopf subalgebra of H. Then $K = H$ if and only if* $\operatorname{GKdim} K = \operatorname{GKdim} H$.

Proof. The left to right implication is obvious. For the converse, suppose $\operatorname{GKdim} H = \operatorname{GKdim} K$. Since $K_i = K \cap H_i$ for all $i \geq 0$, [41, Lemma 5.2.12], $\operatorname{gr} K$ is naturally embedded as a Hopf subalgebra in the Hopf algebra $\operatorname{gr} H$ as follows:

$$\operatorname{gr} K = \bigoplus_{i=0}^{\infty} (K_i + H_{i-1})/H_{i-1} \subseteq \operatorname{gr} H = \bigoplus_{j=0}^{\infty} H_j/H_{j-1}. \tag{4.2.1}$$

Moreover, by Theorem 4.2.7, $\operatorname{GKdim} \operatorname{gr} K = \operatorname{GKdim} K = \operatorname{GKdim} H = \operatorname{GKdim} \operatorname{gr} H$. Since $\operatorname{gr} K$ and $\operatorname{gr} H$ are connected coradically graded Hopf algebras, $\operatorname{gr} H = \operatorname{gr} K$ by Lemma

4.2.5. Examining (4.2.1), we see that this equality implies that $K_i = H_i$ for all $i \geq 0$, hence $H = K$, as required. □

Lemma 4.2.10 is a useful result in practice, and in the context of connected Hopf algebras, tells us that the GK-dimension function closely mirrors the behaviour of the dimension function for finite dimensional vector spaces. One immediate use of this result is to provide us with a another (presumably well known) characterisation of cocommutativity for connected Hopf algebras of finite GK-dimension.

Proposition 4.2.11. *Let H be a connected Hopf algebra of finite GK-dimension. Then H is cocommutative if and only if* $\operatorname{GKdim} H = \dim P(H)$.

Proof. Suppose H is cocommutative. Then $H = U(P(H))$ and $\operatorname{GKdim} H = \dim P(H)$ by Theorem 3.4.5. Conversely, if $\dim P(H) = \operatorname{GKdim} H$, then $K = U(P(H))$ is a Hopf subalgebra of H with $\operatorname{GKdim} K = \dim P(H) = \operatorname{GKdim} H$. The result now follows immediately from Lemma 4.2.10. □

4.2.3 Ring-theoretic consequences

As discussed in the opening paragraph of §2.3 and demonstrated explicitly in Lemma 2.3.17, it is often the case that if we want to show some filtered algebra has a specific property, it is enough to show that its associated graded algebra has this property. Applying this philosophy alongside Theorem 4.2.7 allows us to prove that connected Hopf algebras of finite GK-dimension possess many nice ring and homological theoretic properties.

The following result appears, along with its proof, as [67, Corollary 6.10]. In Theorem 5.5.5 we show that this result holds more generally for any (left) coideal subalgebra of H, with virtually the same proof as the one given in [67, Corollary 6.10]. For this reason, we do not include the proof of Zhuang's result here. For the definitions of the terms used in the following proposition, see §2.5.2. For the definition of the Krull dimension of a noncommutative ring, we refer the reader to [36, Chapter 6].

Proposition 4.2.12. *Let H be a connected Hopf algebra such that* $\operatorname{GKdim} H = n < \infty$. *Then H is*

1. *a noetherian domain of global dimension n and has Krull dimension $\leq n$.*

2. *Auslander-regular.*

3. *GK-Cohen-Macauley.*

4. *AS-regular*

4.3 Connected Hopf algebras of small GK-dimension

In [67, §7], Zhuang gives a complete list of connected Hopf algebras of GK-dimension at most 3. Among these are the first known examples (in characteristic 0) of connected Hopf algebras which are both noncommutative and noncocommutative (the first instance of such a connected Hopf algebra is at GK-dimension 3). In this section we describe explicitly each Hopf algebra in this complete list and examine some of their Hopf-theoretic properties.

Remark 4.3.1. Building on the work of Zhuang, in [62] Wang, Zhang and Zhuang classify all connected Hopf algebras of GK-dimension at most four. We postpone any discussion of the work done in [62] until Chapter 6.

4.3.1 GK-dimension 0 and 1

Let H be a connected Hopf algebra with $\operatorname{GKdim} H < \infty$. One consequence of Theorem 4.2.7 is that $\operatorname{GKdim} H \in \mathbb{Z}^{\geq 0}$. Thus, if we want to classify those H with $\operatorname{GKdim} H \leq 3$, it suffices to look at the cases $\operatorname{GKdim} H = 0, 1, 2$ and 3.

There are two obvious examples of connected Hopf algebras of GK-dimension 0 or 1: at dimension 0 we have the trivial Hopf algebra k and at dimension 1 we have $U(\mathfrak{a}_1)_c$, where $\mathfrak{a}_1$ denotes the 1-dimensional abelian Lie algebra. We now show that these examples are the *only* connected Hopf algebras of GK-dimension 0 or 1. This is contained in [67, Proposition 7.6], but we offer a different proof.

Lemma 4.3.2. *Let T be a left coideal subalgebra of a Hopf algebra H over any field. Let $T_0 = T \cap H_0$ and $T_1 = T \cap H_1$. Suppose $T \nsubseteq H_0$. Then*

$$T_1 \neq T_0.$$

In particular, if H is connected then

$$P(T) := T \cap P(H) \neq 0.$$

Proof. Let $\widehat{T}$ be a finite dimensional left subcomodule of T with $\widehat{T} \nsubseteq H_0$; this exists by [41, Theorem 5.1.1]. Let P denote the subcoalgebra of H generated by $\widehat{T}$; P is finite dimensional by [41, Theorem 5.1.1]. Then the finite dimensional algebra P^* acts on P via the right hit action,

$$\leftharpoondown : P \otimes P^* \to P \; : \; p \otimes f \mapsto p \leftharpoondown f = \sum f(p_1)p_2$$

Since $\widehat{T}$ is a left subcomodule of T, if $t \in \widehat{T}$ then $\Delta(t) = \sum t_1 \otimes t_2 \in P \otimes \widehat{T}$. Thus $\sum f(t_1)t_2 \in \widehat{T}$ for all $t \in \widehat{T}$, so $\widehat{T}$ is a right P^*-submodule of P.

As is well known (see for example Remark 2.3.23 or [41, §5.2]), the terms $P_i = P \cap H_i$ of the coradical filtration of P are precisely the elements annihilated by the powers $J(P^*)^{i+1}$ of the Jacobson radical $J(P^*)$ of P^*. Now $\widehat{T} \leftharpoondown J(P^*) \neq 0$ since $\widehat{T} \not\subset H_0$. Let

$$A := \{t \in \widehat{T} : t \leftharpoondown J(P^*) = 0\},$$

a proper P^*-submodule of $\widehat{T}$. Since $\widehat{T}/A$ is finite dimensional, it contains a simple P^*-submodule, B/A. Then $BJ(P^*)^2 = 0 \neq BJ(P^*)$, hence

$$\mathrm{Ann}_{\widehat{T}}(J(P^*)) \subsetneqq \mathrm{Ann}_{\widehat{T}}(J(P^*)^2).$$

That is, $T_0 \subsetneqq T_1$, as required.

□

While we don't need any assumptions about the ground field in Lemma 4.3.2, we require the usual assumptions that it be algebraically closed of characteristic 0 for the following corollary and everything which proceeds it.

Corollary 4.3.3. *Suppose H is a connected Hopf algebra and T is a left coideal subalgebra of H. If T is finite-dimensional, then T is trivial, i.e., $T = k$.*

Proof. If $T \neq k$ then $P(T) \neq k$, Lemma 4.3.2. Let $x \in P(T) \setminus k$. Then $K = k\langle x \rangle$ is a cocommutative connected subbialgebra of T, hence a cocommutative connected Hopf subalgebra by Lemma 3.2.8. By Theorem 3.4.5, $K = k[x]$, which is infinite dimensional. This completes the proof.

□

Proposition 4.3.4. *Let H be a connected Hopf algebra.*

1. *If* $\mathrm{GKdim}\, H = 0$, *$H = k$, the trivial Hopf algebra.*

2. *If* $\mathrm{GKdim}\, H = 1$, *$H = U(\mathfrak{a})_c$, where $\mathfrak{a}$ is the 1-dimensional abelian Lie algebra.*

Proof.

1. Suppose $\mathrm{GKdim}(H) = 0$. Then H is finite dimensional, so $H = k$ by Corollary 4.3.3.

2. Suppose $\mathrm{GKdim}(H) = 1$. By (the proof of) Corollary 4.3.3, H contains the Hopf subalgebra $U(\mathfrak{a})_c$, where $\mathfrak{a}$ is the 1-dimensional abelian Lie algebra. It is an immediate consequence of Proposition 4.2.10 that in fact $H = U(\mathfrak{a})_c$.

□

4.3.2 GK-dimension 2

It is well known that the only Lie algebras of dimension 2 are the 2-dimensional abelian Lie algebra $\mathfrak{a}_2$ and the unique 2-dimensional non-abelian Lie algebra $\mathfrak{b}$, with basis $\{x, y\}$ and Lie bracket defined by the equation $[x, y] = y$. By the Cartier-Milnor-Moore theorem (Theorem 3.4.5), $U(\mathfrak{a}_2)_c$ and $U(\mathfrak{b})_c$ are the only *cocommutative* connected Hopf algebras of GK-dimension two.

Dually, if we want a list of all *commutative* connected Hopf algebras H of GK-dimension two, we can appeal to Theorem 3.3.6: any such Hopf algebra H is of the form $H = \mathcal{O}(U)$ for 2-dimensional unipotent group U. As noted in Remark 3.3.3, the Lie functor which takes an algebraic group to its corresponding Lie algebra defines a one-to-one correspondence between affine unipotent algebraic groups and finite dimensional nilpotent Lie algebras. Since the only 2-dimensional nilpotent Lie algebra is abelian, [20], the unique 2-dimensional unipotent group is the abelian group $(k, +)^2$. Thus $H = \mathcal{O}((k, +)^2)$ is the only commutative connected Hopf algebra of GK-dimension 2, and is isomorphic as a Hopf algebra to $U(\mathfrak{a}_2)_c$.

We now show that these commutative and cocommutative examples are the only connected Hopf algebras of GK-dimension 2. This follows almost immediately from the following lemma, which originally appeared as [68, Proposition 5.11]. We offer a different (shorter) proof to the one given there (this result, and its proof, reappears later as Theorem 5.7.8 (1)).

Lemma 4.3.5. *Let H be a connected Hopf algebra. If $2 \leq \mathrm{GKdim}\, H < \infty$, then H contains a Hopf subalgebra isomorphic to $U(\mathfrak{g})_c$, where $\mathfrak{g}$ is a 2-dimensional Lie algebra.*

Proof. To prove that H contains a Hopf subalgebra K of the given type, it suffices to show that if $2 \leq \mathrm{GKdim}\, H < \infty$ then $\dim(P(H)) \geq 2$, for then we can appeal to the fact that any finite dimensional Lie algebra of dimension at least 2 contains a Lie subalgebra of dimension 2 (see [68, Lemma 5.10] for a proof of this fact) to find a 2-dimensional Lie subalgebra $\mathfrak{g}$ of $P(H)$ and set $K = U(\mathfrak{g})_c$.

To this end, suppose $\dim(P(H)) \leq 1$, and, as usual, write $\operatorname{gr} H = \bigoplus_{i \geq 0} H(i) = \bigoplus_{i \geq 0} H_i/H_{i-1}$. Since $H(1) \cong P(H)$, Lemma 3.2.10(1), It follows that $\dim(H(1)^*) = \dim(H(1)) = \dim(P(H)) \leq 1$. Proposition 3.2.5 guarantees that $\operatorname{gr} H = \bigoplus_{i \geq 0} H(i)$ is a connected coradically Hopf graded Hopf algebra, so the graded dual $\mathcal{D}_{\operatorname{gr} H} = \bigoplus_{i \geq 0} H(i)^*$ is generated by $H(1)^*$, [2, Lemma 5.5]. Since $\dim(H(1)^*) \leq 1$, we must have

$$1 \geq \operatorname{GKdim} \mathcal{D}_{\operatorname{gr} H} = \operatorname{GKdim} \operatorname{gr} H = \operatorname{GKdim} H$$

where the second equality follows from Theorem 4.2.7.

□

Proposition 4.3.6. *Let H be a connected Hopf algebra with* $\operatorname{GKdim} H = 2$.

1. *If H is commutative, $H = U(\mathfrak{a}_2)_c$, where $\mathfrak{a}_2$ is the* 2*-dimensional abelian Lie algebra.*
2. *If H is noncommutative, $H = U(\mathfrak{b})_c$, where $\mathfrak{b}$ is the unique* 2*-dimensional non-abelian Lie algebra.*

Proof. If H is connected and $\operatorname{GKdim} H = 2$, it contains, by Lemma 4.3.5, a Hopf subalgebra K isomorphic to $U(\mathfrak{g})_c$ where $\mathfrak{g}$ is a 2-dimensional Lie algebra. Since $\operatorname{GKdim} K = 2$, $H = K$ by Lemma 4.2.10. This completes the proof. □

4.3.3 GK-dimension 3

In the concluding subsection of §4.3, we examine connected Hopf algebras of GK-dimension 3. By Theorem 3.4.5, such a Hopf algebra H which is also *cocommutative* is of the form $H = U(\mathfrak{g})_c$ for some 3-dimensional Lie algebra $\mathfrak{g}$. By Theorem 3.3.6, any commutative connected Hopf algebra H of GK-dimension 3 is of the form $H = \mathcal{O}(U)$ for some 3-dimensional unipotent group U. Appealing again to the correspondence between finite dimensional nilpotent Lie algebras an affine unipotent algebraic groups, Remark 3.3.3, and the classification of nilpotent Lie algebras in low dimensions, [20], we see that $(k,+)^3$ and $\mathbb{U}_3$ are the *only* unipotent groups of dimension three. It follows that $\mathcal{O}((k,+)^3)$ and $\mathcal{O}(\mathbb{U}_3)$ are the only *commutative* connected Hopf algebras of GK-dimension 3.

At GK-dimensions 0, 1 and 2, we saw that all connected Hopf algebras arose from the classical cocommutative or commutative constructions. Happily, this does *not* happen at GK-dimension three. In [67, §7], Zhuang introduces two infinite families of GK-dimension

3 connected Hopf algebras, which are (in general) both noncommutative and noncommutative. We list these examples below. That these examples are in fact connected Hopf algebras is proved in [67, Proposition 7.3], so we do not prove it here. In later sections we see very similar proofs when constructing new examples of connected Hopf algebras of GK-dimension 5.

Example 4.3.7. ([67, Example 7.1]) Let $\lambda_1, \lambda_2, \alpha \in k$, where $\alpha = 0$ if $\lambda_1 \neq \lambda_2$ and $\alpha = 0$ or 1 if $\lambda_1 = \lambda_2$. Let A be the algebra generated by elements X, Y, Z satisfying the following relations,

$$[X, Y] = Y, \quad [Z, X] = \lambda_1 X + \alpha Y, \quad [Z, Y] = \lambda_2 Y.$$

Then A is a connected Hopf algebra such that $\operatorname{GKdim} A = 3$, with coproduct, antipode and counit defined on generators as follows

$$\epsilon(X) = 0, \ \Delta(X) = 1 \otimes X + X \otimes 1, \ , S(X) = -X,$$

$$\epsilon(Y) = 0, \ \Delta(Y) = 1 \otimes Y + Y \otimes 1, \ , S(Y) = -Y,$$

$$\epsilon(Z) = 0, \ \Delta(Z) = 1 \otimes Z + X \otimes Y + Z \otimes 1, \ , S(Z) = -Z + XY.$$

We denote this Hopf algebra by $A(\lambda_1, \lambda_2, \alpha)$.

Example 4.3.8. ([67, Example 7.2]) Let $\lambda \in k$ and let B be the algebra generated by elements X, Y, Z satisfying the following relations,

$$[X, Y] = Y, \quad [Z, X] = -Z + \lambda Y, \quad [Z, Y] = \frac{1}{2} Y^2.$$

where $\lambda \in k$. Then B is a connected Hopf algebra such that $\operatorname{GKdim} B = 3$, with coproduct, antipode and counit defined on generators as follows

$$\epsilon(X) = 0, \ \Delta(X) = 1 \otimes X + X \otimes 1, \ , S(X) = -X,$$

$$\epsilon(Y) = 0, \ \Delta(Y) = 1 \otimes Y + Y \otimes 1, \ , S(Y) = -Y,$$

$$\epsilon(Z) = 0, \ \Delta(Z) = 1 \otimes Z + X \otimes Y + Z \otimes 1, \ , S(Z) = -Z + XY.$$

We denote this Hopf algebra by $B(\lambda)$.

We record the following observations about these examples.

Remark 4.3.9. 1. With respect to their coradical filtrations, for any permissible scalars $\lambda_1, \lambda_2, \alpha$ and λ,

$$\mathrm{gr}\, A(\lambda_1, \lambda_2, \alpha) = \mathrm{gr}\, B(\lambda) = \mathcal{O}(\mathbb{U}_3).$$

where $\mathcal{O}(\mathbb{U}_3)$ is the Hopf algebra from Example 2.4.6.

2. The Hopf algebra $A(0, 0, 0)$ is precisely the Hopf algebra $\mathcal{O}(\mathbb{U}_3)$. Thus, the family of connected Hopf algebras $A(\lambda_1, \lambda_2, \alpha)$ has as a subfamily all *commutative* connected Hopf algebras of GK-dimension three which are also noncocommutative.

3. For any $\lambda \in k$, a simple calculation shows that the antipode of $B(\lambda)$ satisfies the following identity: for any $m \geq 1$,

$$S^{2m}(Z) = Z - mY.$$

Since we're working in characteristic 0, it follows $S^{2m}(Z) \neq Z$ for any $m \geq 1$. Thus S (being non-trivial) has either odd or infinite order. However S is an anti-algebra morphism, so can't have odd order, hence

For any $\lambda \in k$, $B(\lambda)$ is a connected Hopf algebra with an antipode of infinite order.

This phenomenon never occurs for the classical cocommutative or commutative connected Hopf algebra, where the antipode must always square to the identity, [41, Corollary 1.5.12]. We'll see in Theorem 5.4.5 that the antipode of any connected Hopf algebra is either of order two or of infinite order.

In [67, Theorem 7.5], it is proved that the families $A(\lambda_1, \lambda_2, \alpha)$ and $B(\lambda)$, along with the usual cocommutative examples, form a complete list of all connected Hopf algebras of GK-dimension three. This result is stated explicitly as a theorem below. The proof of this result is lengthy and not particularly informative, so we do not include it here.

Theorem 4.3.10. *Let H be a connected Hopf algebra of GK-dimension 3. Then H is isomorphic (as a Hopf algebra) to one of the following:*

1. *The cocommutative enveloping algebra $U(\mathfrak{g})_c$ for some three-dimensional Lie algebra $\mathfrak{g}$.*

2. *The Hopf algebras $A(0, 0, 0), A(0, 0, 1), A(1, 1, 1)$ or $A(1, \lambda, 0)$ from Example 4.3.7.*

3. *The Hopf algebras $B(\lambda)$ from Example 4.3.8.*

Moreover, for any $\lambda \in k$, these are mutually non-isomorphic as Hopf algebras, except that $A(1, \lambda, 0) \cong A(1, \mu, 0)$ if and only if $\lambda = \mu$ or $\lambda\mu = 1$, [67, Proposition 7.11].

It is immediate from their definitions that, *as an algebra*, each of the Hopf algebras appearing in parts (1) and (2) of Theorem 4.3.10, along with each appearing in §4.3.1 and §4.3.2, is isomorphic to the enveloping algebra of a finite dimensional Lie algebra. Although this is not obvious from the given presentation, the same can be said for $B(\lambda)$: if we set $W := Z - \frac{1}{2}YX$, a simple calculation shows that, *as an algebra*, $B(\lambda)$ is isomorphic to the enveloping algebra of the solvable Lie algebra $\mathfrak{g}$ with basis X, Y, W and relations

$$[X, Y] = Y, \quad [W, Y] = 0, \quad [X, W] = W - \lambda Y.$$

Proposition 4.3.11. *Let H be a connected Hopf algebra such that $\operatorname{GKdim} H \leq 3$. As an algebra, $H \cong U(\mathfrak{g})$ for some finite dimensional Lie algebra $\mathfrak{g}$.*

Remark 4.3.12. In [62] it is proved that Proposition 4.3.11 extends to any connected Hopf algebra of GK-dimension at most four. It is also true that if H is a commutative or cocommutative connected Hopf algebra of arbitrary (finite) GK-dimension, then H is isomorphic, as an algebra, to the enveloping algebra of a finite dimensional Lie algebra. In the former case this follows from the Cartier-Milnor-Moore theorem (Theorem 3.4.5), and in the latter it follows from Theorem 3.3.6, which states that any connected commutative Hopf algebra of finite GK-dimension is a polynomial algebra. In Chapter 7 we construct an example of a connected Hopf algebra of GK-dimension 5 and give a proof that it is not isomorphic to an enveloping algebra as an algebra.

4.4 Iterated Hopf Ore extensions

In this section we review some of the results of the investigation into the properties of iterated Hopf Ore extensions carried out in [7]. We'll see that such Hopf algebras describe a large class of classical connected Hopf algebras, and also several noncommutative and noncocommutative examples, including the families $A(\lambda_1, \lambda_2, \alpha)$ and $B(\lambda)$ discussed in §4.3.3.

Let's begin by recalling the definition of an Ore-extension as given in [16].

Definition 4.4.1. Let A be a ring, σ a ring automorphism of A, and ∂ a σ-derivation of A (that is, an additive map $\partial : A \to A$ such that $\partial(ab) = \sigma(a)\partial(b) + \partial(a)b$ for all $a, b \in A$). We write $A = R[x; \sigma, \partial]$ and say that A is an *Ore-extension* of R provided

1. A is a ring, containing R as a subring.
2. x is an element of A.
3. A is a free left R-module with basis $\{1, x, x^2, \ldots\}$.
4. $xr = \sigma(r)x + \partial(r)$ for all $r \in R$.

Proceeding inductively, for any $n \geq 1$, an *n-step iterated Ore-extension* of R is one of the form

$$R[x_1; \sigma_1, \partial_1][x_2; \sigma_2, \partial_2] \ldots [x_n; \sigma_n, \partial_n]$$

where, defining $R_m = R[x_1; \sigma_1, \partial_1][x_2; \sigma_2, \partial_2] \ldots [x_m; \sigma_m, \partial_m]$, R_m is an Ore-extension of R_{m-1} for all $1 \leq m \leq n$ (where we set $R_0 = R$).

Definition 4.4.2. An *iterated Hopf Ore extension* (IHOE) is an iterated Ore extension

$$H = k[x_1][x_2; \sigma_2, \partial_2] \ldots [x_n; \sigma_n, \partial_n]$$

where H is a Hopf algebra and $H_{(i)} := k\langle x_1, \ldots, x_i \rangle$ is a Hopf subalgebra of $H_{(i+1)}$ for $1 \leq i \leq n-1$.

The majority of what follows is adapted from [7, Examples 3.1]. We refer the reader to that paper for a more detailed account.

Example 4.4.3.

1. Suppose U is a unipotent group. Then $\mathcal{O}(U)$ is an IHOE. This is the dual of Proposition 3.3.2(3), which tells us that U has a finite normal (even central) series with successive factors isomorphic to $(k, +)$.
2. The enveloping algebra $U(\mathfrak{g})_c$ of a finite dimensional solvable Lie algebra $\mathfrak{g}$ equipped with its usual cocommutative Hopf structure is an IHOE, with $n = \dim_k \mathfrak{g}$ and $\sigma_i = \mathrm{id}$ for all i. This follows from the fact that $\mathfrak{g}$ has a chain of ideals $\mathfrak{g}_i$, $0 \leq i \leq n$, with $\mathfrak{g}_i \subset \mathfrak{g}_{i+1}$ and $\dim_k \mathfrak{g}_i = i$, for all i, [14, 1.3.14].
3. Whilst $U(\mathfrak{sl}_2(k))_c$ *is* an IHOE (see [7, Examples 3.1] for details), if $\mathfrak{g}$ is a semisimple Lie algebra with a simple factor not isomorphic to $\mathfrak{sl}_2(k)$, then $U(\mathfrak{g})_c$ is not an IHOE. To see this, note that the Hopf subalgebras of $U(\mathfrak{g})_c$ are the enveloping algebras of the Lie subalgebras of $\mathfrak{g}$, and there are insufficient of these to form a full flag in $\mathfrak{g}$.

4. For an example of an IHOE which is neither commutative nor cocommutative, consider a member of either of the infinite families $A(\lambda_1, \lambda_2, \alpha)$ or $B(\lambda)$ defined in §4.3.3. That these are in fact IHOEs is perhaps unsurprising upon inspection of their defining relations, and a detailed proof of this claim can be found in [7, §3.4].

Notice that each IHOE appearing in Example 4.4.3 is connected as a Hopf algebra. It is proved in [7, Theorem 3.2 (i)] that this is true in general.

Theorem 4.4.4. *([7, Theorem 3.2 (i)]) Let H be an IHOE. Then H is connected as a Hopf algebra.*

Note that the converse of this statement is in general false - we described a class of a connected Hopf algebras which are not IHOEs in Example 4.4.3(3).

Remark 4.4.5. We noted in Remark 4.3.12 that all connected Hopf algebras of GK-dimension at most 4, as well as all commutative or cocommutative connected Hopf algebras of arbitrary (finite) GK-dimension, are isomorphic, as algebras, to enveloping algebras of finite dimensional Lie algebras. By Theorem 4.4.4, the same can therefore be said for any IHOE which satisfies at least one of these three conditions. In Chapter 7 we construct an IHOE of GK-dimension 5 and give a proof that it is not isomorphic, as an algebra, to the enveloping algebra of a Lie algebra.

Chapter 5

Quantum Homogeneous Spaces

5.1 Introduction

5.1.1 Background

The left and right coideal subalgebras of a Hopf algebra H (defined in §2.6.1) have been an important focus of research since the classical work on the commutative and cocommutative cases in the last century, [13], [42], [57]. Following the seminal papers of Takeuchi [58], Masuoka [35] and Schneider [51], attention has concentrated on the *quantum homogeneous spaces*, that is those coideal subalgebras of H over which H is a faithfully flat module. We continue this line of research for the case where H is a connected Hopf k-algebra of finite GK-dimension n, with k an algebraically closed field of characteristic 0.

As mentioned in earlier chapters, this class of Hopf algebras has been the subject of a series of recent papers, see for example [67], [7], [62]. None of these, however, has examined their coideal subalgebras, so a primary aim here is to lay out their basic properties and clarify topics for future research. Motivating examples from the classical theory are plentiful and offer a rich source of intuition - we saw in Chapter 3 that H as above is *commutative* if and only if it is the coordinate ring $\mathcal{O}(U)$ of a unipotent algebraic k-group U of dimension n, while H is *cocommutative* if and only if it is the enveloping algebra $U(\mathfrak{g})$ of its n-dimensional Lie k-algebra $\mathfrak{g}$ of primitive elements. In the former case the coideal subalgebras of H are the (right and left) homogeneous U-spaces, and are therefore in correspondence with the closed subgroups of U; in the latter case - thanks to cocommutativity - the coideal subalgebras are the Hopf subalgebras, hence are just the enveloping algebras of the Lie subalgebras of $\mathfrak{g}$. For references for these classical facts, see §5.3.

5.1.2 The antipode

As already mentioned, one of the pillars on which our work stands is the paper of Masuoka [35]. The relevant specialisation of the main result of that paper is stated here as Proposition 5.4.2, a central feature being the existence of a bijection between the left coideal subalgebras of H and the quotient right H-module coalgebras of H. Using this bijection and a well-known lemma due to Koppinen [25], we deduce the following, proved in Theorem 5.4.5 and Proposition 5.5.8.

Theorem 5.1.1. *Let H be a connected Hopf algebra of finite GK-dimension n over an algebraically closed field k of characteristic 0 and let T be a left or right coideal subalgebra of H. Then*

1. $S^2(T) = T$
2. *Either* $(S^2)_{|T} = \mathrm{id}$ *or* $|(S^2)_{|T}| = \infty$.

Both possibilities can occur in Theorem 5.1.1 (2): for $T = H$, we saw in Remark 4.3.9 that the smallest example with $|S|$ infinite occurs at GK-dimension 3, namely the infinite family of examples $B(\lambda)$. For $T \neq H$, S^2 can have infinite order already at dimension 2, as we show (for a coideal subalgebra of $B(\lambda)$) in §5.6.2.

5.1.3 The main theorem

In the classical commutative and cocommutative settings, the faithful flatness condition always holds, and both the Hopf algebra H and its coideal subalgebras either are themselves (in the first case), or have associated graded algebras which are (in the second case) commutative polynomial k-algebras. To extend this picture to the non-classical world, we use the *coradical filtration* of a Hopf algebra, as defined in §2.3.2. The starting point is then the result of Zhuang, Theorem 4.2.7, stating that the associated graded algebra $\mathrm{gr}H$ of H with respect to its coradical filtration is a polynomial k-algebra in $n = \mathrm{GKdim}\, H$ commuting variables. Our first main result shows that the whole of the above classical picture extends to the setting of Zhuang's theorem.

Theorem 5.1.2. *Let H be a connected Hopf algebra of finite GK-dimension n over an algebraically closed field k of characteristic 0 and let T be a left or right coideal subalgebra of H. Then*

1. *(*Masuoka*, [35]) H is a free right and left T-module.*

2. *With respect to the coradical filtration of T (defined in* §5.2*), the associated graded algebra* $\operatorname{gr} T$ *is a polynomial k-algebra in m variables.*

3. $\operatorname{GKdim} T = m \leq n$*, and $m = n$ if and only if $T = H$.*

The theorem is given below as Lemma 5.5.1 and Theorem 5.5.5. Just as with parts (1) and (2), so also (3) incorporates familiar classical phenomena: if $W \subset U$ is a strict inclusion of unipotent k-groups, then $\dim W < \dim U$ by Proposition 3.3.2 (5); and a strict inclusion of Lie algebras of course implies a strict inequality of their dimensions. This can be viewed as a noncommutative analogue of Proposition 3.3.2 (4).

As discussed in §4.2.2, $\operatorname{gr} H$ is a connected commutative Hopf algebra, hence the coordinate ring of a unipotent group U, allowing us to view H as "lift" or "PBW-deformation" of a unipotent group. We shall see that Theorem 5.1.2 allows us to come to a similar conclusion about T: it is is a "lift" or "PBW-deformation" of a homogeneous U-space with coordinate ring $\operatorname{gr} T$.

5.1.4 Homological properties

Following [32], we call a left or right coideal subalgebra T of a Hopf algebra H a *quantum homogeneous space* if H is faithfully flat as a left *and* as a right T-module. Thus Theorem 5.1.2 (1) ensures that all coideal subalgebras of H as in the theorem are quantum homogeneous spaces.

As discussed in §2.3, a filtered-graded result such as Theorem 5.1.2 has important homological and ring theoretic consequences. Thus we deduce:

Corollary 5.1.3. *Let H and T be as in Theorem 5.1.2.*

1. *T is a noetherian domain of global dimension m, AS-regular and GK-Cohen-Macaulay.*

2. *T is twisted Calabi-Yau of dimension m.*

For the definitions of the terminology in the above, see §2.5.2. The twisting automorphism in (2) is discussed in the next subsection.

We are also able to determine the Nakayama automorphism of a quantum homogeneous space. For the unexplained terminology in the result below, see §5.5.4.

Theorem 5.1.4. *Suppose T is a right quantum homogeneous space of a connected Hopf algebra of finite GK-dimension. Then the Nakayama automorphism of T is $S^2 \circ \tau^{\ell}_{\chi}$, where*

χ is the character of the right structure of the left integral of T. There is an analogous formula which applies when T is a left quantum homogeneous space

This result depends crucially on earlier work of Krähmer [26] and of Liu and Wu [32], [33].

5.1.5 Classifying quantum homogeneous spaces

In §5.3 we describe what is known about various particular subclasses of quantum homogeneous spaces T of connected Hopf algebras H of finite GK dimension. Thus, we discuss the cases where H or T is commutative; where H or simply T is cocommutative; where $\mathrm{GKdim}\, T \leq 2$; and where $\mathrm{GKdim}\, H \leq 3$. Particularly noteworthy is the fact, recorded in Proposition 5.6.4 in §5.6, that

the Jordan plane $J = k\langle X, Y : [X,Y] = Y^2\rangle$ is a coideal subalgebra of a connected Hopf algebra H with $\mathrm{GKdim}\, H = 3$.

As is well known, and proved in Proposition 5.6.4, J cannot be supplied with a comultiplication with respect to which it is a Hopf algebra.

5.1.6 Invariants of quantum homogeneous spaces

Let H and T be as in Theorem 5.1.2. Given that $\mathrm{gr}H$ and $\mathrm{gr}T$ are graded polynomial algebras, their homogeneous generators have specific degrees whose multisets of values constitute invariants $\sigma(T)$ and $\sigma(H)$ of T [resp. H]. We call $\sigma(T)$ the *signature* of T, and write $\sigma(T) = (e_1^{(r_1)}, \ldots, e_s^{(r_s)})$, where e_i and r_i are positive integers with $e_1 < e_2 < \cdots < e_s$, and the term $e_i^{(r_i)}$ indicates that the degree e_i occurs r_i times among the graded polynomial generators of $\mathrm{gr}T$. A closely related invariant is the *lantern* $\mathcal{L}(T)$ of T, defined in [62, Definition 1.2(d)] for H itself, extended here to a quantum homogeneous space T: namely, $\mathcal{L}(T)$ is the k-vector space of primitive elements of the graded dual $\mathcal{D}_{\mathrm{gr}\, T}$ (as defined in §2.3.1). Basic properties of these signature and lantern are listed in §5.7. We view one important function of these equivalent invariants as being to provide a framework for future work on connected Hopf algebras. We discuss this aspect further in §5.8.

Remark 5.1.5. While (almost) all results in this chapter are stated for left coideal subalgebras, they are equally valid for right coideal subalgebras.

5.2 Preliminaries

Let H be a Hopf algebra and T a left coideal subalgebra of H, that is, a subalgebra of H with the property that $\Delta(T) \subseteq H \otimes T$. Define the *coradical filtration* $\mathcal{T} := \{T_n\}_{n\geq 0}$ of T such that, for each $n \geq 0$,

$$T_n := T \cap H_n$$

where, as usual, $\{H_n\}$ denotes the coradical filtration of H, as defined in §2.3. For each $n \geq 0$, $\Delta(T_n) \subset (H \otimes T) \cap (H_n \otimes H_n) \subseteq H \otimes T_n$, and so the associated graded space

$$\operatorname{gr} T := \bigoplus_{i=0}^{\infty} T(i) \subseteq \operatorname{gr} H; \quad T(i) := ((T \cap H_i) + H_{i-1})/H_{i-1}$$

satisfies the condition

$$\Delta(\operatorname{gr} T) \subset \operatorname{gr} H \otimes \operatorname{gr} T$$

where $\operatorname{gr} H$ denotes the associated graded coalgebra of H with respect to its coradical filtration. In the case where H is connected, $\{H_n\}$ is a Hopf filtration, $\mathcal{T}$ is an algebra filtration and $\operatorname{gr} H$ is a commutative graded connected Hopf algebra (see Proposition 3.2.5, Theorem 4.2.7). Summarising, we get the following.

Lemma 5.2.1. *Let H be a connected Hopf k-algebra of finite GK-dimension, and let T be a left coideal subalgebra of H. Then $\operatorname{gr} T$ is a left coideal subalgebra of the commutative graded connected Hopf algebra $\operatorname{gr} H$.*

5.3 Classical examples

5.3.1 Coideal subalgebras and homogeneous spaces

Suppose G is an affine algebraic group. In §2.4.3 we described a one to one correpsondence between left $\mathcal{O}(G)$-comodules and rational representations of G. Left $\mathcal{O}(G)$-comodules which are also subalgebras of $\mathcal{O}(G)$ are precisely the left coideal subalgebras of $\mathcal{O}(G)$, and under the aforementioned correspondence they give rise to a particular type of rational representation of G known as a *homogeneous space*. The aim of this section is to give a detailed outline of this classical result in the theory of affine algebraic groups.

Definition 5.3.1. Let G be an (affine) algebraic group. We say an algebraic set X is a *homogeneous space* (over G) if it admits a rational transitive G-action.

Let G be an affine unipotent algebraic group and $D \subset G$ a closed subgroup. Applying the coordinate ring functor to the group embedding $D \subset G$, this subgroup defines a surjective Hopf morphism $\mathcal{O}(G) \to \mathcal{O}(D)$, and hence a unique Hopf ideal I such that $\mathcal{O}(D) \cong H/I$ as Hopf algebras. This subgroup D also defines a homogeneous G-space: the quotient algebraic set G/D. Note that this algebraic set is not necessarily an algebraic group (since D is not necessarily normal), but it is equipped with a canonical group-algebraic action from the group G, hence is indeed a homogeneous space over G.

Theorem 5.3.2. *Let G be an affine unipotent algebraic k-group of dimension n, and let D be a closed subgroup of G with $\dim D = m$. Let $H = \mathcal{O}(G)$ and let I be the defining ideal of D, so I is a Hopf ideal of H with $H/I \cong \mathcal{O}(D)$. Let $\pi : H \to H/I$ be the canonical surjective Hopf morphism.*

1. *Let $K = H^{\mathrm{co}\pi}$ and $W = {}^{\mathrm{co}\pi}H$. Then K [resp. W] is a left [resp.right] coideal subalgebra of H, with $W = S(K)$ and $K = S(W)$.*

2. *K is a Hopf subalgebra of $H \Leftrightarrow K = W \Leftrightarrow D$ is normal in $G \Leftrightarrow K^+H$ is a normal ideal of H.*

3. *Every coideal subalgebra of H arises as above. Namely, if K is a left coideal subalgebra of H of dimension $n-m$, then K^+H is a Hopf ideal of H, so $H/K^+H = \mathcal{O}(D)$ for a closed subgroup D of G with $\dim D = m$, and $K = H^{\mathrm{co}\pi}$ where $\pi : H \to H/K^+H$. Moreover, as algebras, $K \cong \mathcal{O}(G/D)$, the coordinate ring of a homogeneous space of G.*

Proof.

1. That $K = H^{\mathrm{co}\pi}$ and $W = {}^{\mathrm{co}\pi}H$ are left (resp. right) coideal subalgebras is Lemma 2.6.9. To see that $W = S(K)$ and $K = S(W)$, use [50, Lemma 3.1.4 (2)(b)].

2. This follows from Theorem 2.6.11 and Remark 2.6.12.

3. This follows from [50, Lemma 3.1.4].

□

5.3.2 Coideal subalgebras in commutative connected Hopf algebras

In Theorem 3.3.6 we saw that, if H is a a commutative connected Hopf algebra of finite GK-dimension, then it is the coordinate ring of some unipotent group U. It follows therefore by the results of §5.3.1 that the study of left coideal subalgebras in affine commutative connected Hopf algebras can be essentially reduced to the study of homogeneous spaces of unipotent groups. We put this philosophy into practice in the following well known theorem, which is the dual version of Proposition 3.3.2. To save space, we give only a sketch of its proof.

Theorem 5.3.3. *Suppose* $H = \mathcal{O}(U)$ *is a connected commutative Hopf algebra with* $\mathrm{GKdim}\, H = n < \infty$ *and* U *an affine unipotent group of dimension* n*. Let* K *denote a left coideal subalgebra of* H *with* $\mathrm{GKdim}\, K = m$*.*

1. *K is polynomial subalgebra of H in m variables, with*

$$H = K[t_1, \ldots, t_{n-m}]$$

   for suitable elements t_i of H, $1 \leq i \leq n-m$.

2. *There is a complete flag*

$$k = K_0 \subset K_1 \subset \cdots \subset K_m = K \subset \cdots \subset K_n = H, \tag{5.3.1}$$

   where the K_j are left coideal subalgebras of H with $\mathrm{GKdim}\, K_j = j$ for all $j = 0, \ldots, n$.

3. *If K is a Hopf subalgebra of H then the K_j in (5.3.1) can be chosen to be Hopf subalgebras.*

4. *If L is a left coideal subalgebra of H with $L \subseteq K$ then*

$$\mathrm{GKdim}\, K = \mathrm{GKdim}\, L \Leftrightarrow K = L.$$

Proof. (Sketch)

(1): By Theorem 5.3.3, K can be realised as the coordinate ring of some homogeneous space of U. The first part now follows from a theorem of Rosenlicht, [47, Theorem 1], which states that any m-dimensional homogeneous space of a unipotent group is, as an algebraic set, isomorphic to $(k,+)^m$. For the second part, note first that $I = K^+H$ is a Hopf ideal by Theorem 5.3.3(4), and $H/I \cong k[z_1, \ldots, z_{n-m}]$ by Theorem 3.3.6. That the extension $K \subset H$ splits as stated now follows from basic commutative algebra, taking $t_1, \ldots, t_m$ to be any lifts of a set of polynomial generators from H/I to H.

(2), (3) and (4): By Theorem 5.3.3(4), there is a closed subgroup D of G, with $\mathrm{dim} D = m$, whose defining ideal is K^+H, and $K = H^{\mathrm{co}\pi}$ where $\pi : H \to H/K^+H$. The theorem follows by dualising parts (3) and (5) of Proposition 3.3.2.

□

5.3.3 Coideal subalgebras in cocommutative connected Hopf algebras

In the case of a cocommutative connected Hopf algebra H in characteristic 0, the left coideal subalgebras of H are precisely the Hopf subalgebras.

Proposition 5.3.4. *Suppose H is a cocommutative connected Hopf algebra and T is a left coideal subalgebra of H. Then T is a Hopf subalgebra and $T = U(\mathfrak{g})_c$, where $\mathfrak{g}$ is the Lie algebra of primitive elements of T.*

Proof. Since H is cocommutative, T is both a left and right coideal subalgebra, hence a subbialgebra of H. Moreover, by Lemma 3.2.6, T is connected (as a bialgebra), hence a connected Hopf subalgbera of H, Lemma 3.2.8. The rest of the result now follows from the Cartier-Milnor-Moore theorem, Theorem 3.4.5. □

5.4 Quantum homogeneous spaces - definitions and basic properties

There is some inconsistency in the literature as to the precise definition of a quantum homogeneous space. For example, Krähmer in [26] defines a left quantum homogeneous space to be a left coideal subalgebra C of a Hopf algebra H such that H is a faithfully flat left C-module. We adopt in this paper the formally more restrictive definition used in [32]:

Definition 5.4.1. Let H be a Hopf algebra with a bijective antipode and let $T \subseteq H$ be a left coideal subalgebra of H. We say T is a *left quantum homogeneous space* of H if H is a faithfully flat left *and* right T-module.

In fact, for the connected Hopf algebras which are the object of study in this paper, the distinction is irrelevant, as shown by the result below.

Proposition 5.4.2. (Masuoka*, [35]) Let H be a connected Hopf algebra and $T \subseteq H$ a subalgebra.*

1. *The antipode S of H is bijective.*

2. *T is a left coideal subalgebra if and only if it is a left quantum homogeneous space. In this case, H is a free left and right T-module.*

3. *The correspondences $T \mapsto \{\pi_T : H \longrightarrow H/T^+H\}$ and $\pi \mapsto H^{\mathrm{co}\,\pi}$ give a bijective correspondence between the left quantum homogeneous spaces of H and the quotient right H-module coalgebras of H.*

4. *There is a parallel version of (3) for right coideal subalgebras of H.*

Proof. 1. This is [35, Proposition 1.2(1)].

2. Suppose T is a left coideal subalgebra of the connected Hopf algebra H. Since $T_0 = T \cap H_0 = k$, $S(T_0) = T_0$, and so, by [35, Theorem 1.3(1)], H is a left and right faithfully flat T-module. The last sentence is a special case of [35, Proposition 1.4].

3. Since $S(T_0) = T_0$ as in (2), this is [35, Theorem 1.3(3)] and the version with right and left swapped.

4. Parallel proof to that of (3).

□

Note that, notwithstanding the above result, it is certainly not true that a left coideal subalgebra T of an arbitrary Hopf algebra H is a quantum homogeneous space. For example, let $H = k\langle x \rangle$, the group algebra of the infinite cyclic group. Then $T := k[x]$ is both a left and right coideal subalgebra of H, but *not* a Hopf subalgebra, since $S(T) \nsubseteq T$; and H is *not* a right or left faithfully flat T-module. Indeed, if we take, for example, the T-module $N = k[x]/\langle x^2 \rangle$,

$$H \otimes_T N = 0.$$

5.4.1 The antipode

Recall the following well-known and easy facts, where k can be any field.

Lemma 5.4.3. *Let H be a Hopf algebra with a bijective antipode.*

1. *The map $C \mapsto S(C)$ gives a bijection between the sets of left and right coideal subalgebras of H. This bijection restricts to a bijection between the left and right quantum homogeneous spaces of H.*

2. *If C is a left or right coideal subalgebra, then $S(C) = C$ if and only if C is a Hopf subalgebra of H.*

Proof. 1. Let C be a left coideal subalgebra of H and $x \in C$. Then

$$\Delta(S(x)) = \tau \circ (S \otimes S) \circ \Delta(x) = \sum S(x_2) \otimes S(x_1) \in S(C) \otimes H.$$

That is, $S(C)$ is a right coideal subalgebra. Applying S^{-1} shows that the correspondence is bijective. Since S is an anti-algebra automorphism, the correspondence preserves any flatness properties.

2. This follows immediately from (1).

□

In the following lemma, we only need to assume that k does not have characteristic 2.

Lemma 5.4.4. *Let H be a connected Hopf k-algebra.*

1. *There exists a k-basis $\mathcal{B}$ of H such that $\mathcal{B} \cap H_n$ spans H_n for all n, and, for all n and all $b \in \mathcal{B} \cap H_n$, there exists $r_b \in H_{n-1}$ such that $S(b) = \pm b + r_b$.*

2. *Let $h \in H_n$. There exists $r \in H_{n-1}$ such that $S^2(h) = h + r$.*

Proof. 1. First, S preserves the coradical filtration, and induces the antipode $\overline{S}$ of $\mathrm{gr}H$. Since $\mathrm{gr}H$ is a commutative Hopf algebra by [53, §11.2], $\overline{S}$ has order 2 by [41, Corollary 1.5.12]. Thus the result follows by constructing $\mathcal{B} \cap H_n$ by induction on n, the new elements at stage n being lifts of a basis of $\overline{S}$-eigenvectors of $H(n)$.

2. This follows immediately from (1).

□

Theorem 5.4.5. *Let H be a connected Hopf k-algebra, with k a field of characteristic 0, and let T be a (left or right) quantum homogeneous space in H.*

1. *$S^2(T) = T$.*

2. *Either $(S^2)_{|T} = \mathrm{id}$ or $|(S^2)_{|T}| = \infty$.*

Proof. 1. Suppose for definiteness that T is a left coideal subalgebra of H. Under the bijective correspondence of Proposition 5.4.2(3), $T \leftrightarrow H/T^+H$, and the right coideal subalgebra $S(T)$ corresponds to $H/H(S(T)^+)$. But

$$H(S(T)^+) = HS(T^+) = S(T^+H) = HT^+,$$

where the final equality is Koppinen's lemma, [25, Lemma 3.1]. Thus, applying S now to the pair $\{S(T), HS(T)^+\}$, it follows that the left coideal subalgebra $S^2(T)$ is paired with $H/S(HT^+) = H/S(T)^+H = H/T^+H$, where this chain of equalities again follows from the other-sided version of Koppinen's lemma. By the bijectivity of the correspondence of Proposition 5.4.2(3), we must have $S^2(T) = T$ as required.

2. Suppose that $(S^2)_{|T} \neq \mathrm{id}$, and choose $h \in T_n$ with n minimal such that $S^2(h) \neq h$. By Lemma 5.4.4(2) there exists $0 \neq r \in H_{n-1}$ such that $S^2(h) = h + r$. By the first part of the theorem, $r \in T$. Hence, by choice of n, $S^2(r) = r$, so that, for all $m \geq 1$,

$$S^{2m}(h) = h + mr.$$

As k has characteristic 0, $S^{2m}(h) \neq h$ for every $m \geq 1$, as required.

□

Note that part (1) of Theorem 5.4.5 holds, with the same proof, for any *pointed* Hopf algebra H and left coideal subalgebra $T \subset H$ with $S(T_0) = T_0$. This is because Proposition 5.4.2 is valid in this wider context. But things go wrong with part (2): consider for instance Sweedler's 4-dimensional pointed Hopf algebra [41, Example 1.5.6],

$$K := k\langle g, x, : g^2 = 1, x^2 = 0, xg = -gx\rangle.$$

with $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$, $\epsilon(g) = 1$, $\epsilon(x) = 0$, $S(g) = g$ and $S(x) = -gx$. Here, $|S| = 4$.

In part (2) of the theorem, both alternatives can occur. Of course $S^2 = \mathrm{Id}$ when H is commutative or cocommutative, [41, Corollary 1.5.12]. On the other hand, as noted in Remark 4.3.9, the connected Hopf algebras $B(\lambda)$, which has GK-dimension 3, has an antipode S with $|S^2| = \infty$. We shall see in Remark 5.6.5 that $B(\lambda)$ in fact has a left coideal subalgebra T with $\mathrm{GKdim}\, T = 2$ and $|S^2|_T| = \infty$.

5.5 Coideal subalgebras of connected Hopf algebras

5.5.1 Associated graded algebras and GK-dimension

The starting point for almost everything which follows is:

Lemma 5.5.1. *Let H be a connected Hopf algebra of finite GK-dimension n and suppose $T \subseteq H$ is a left coideal subalgebra.*

1. *There exists some $m \leq n$ such that* $\operatorname{gr} T$ *forms a (connected) graded polynomial algebra in m variables, with* $\operatorname{gr} H = \operatorname{gr} T[y_{m+1}, \ldots, y_n]$ *for some elements $y_{m+1}, \ldots, y_n$ of* $\operatorname{gr} H$*.*

2. $\operatorname{GKdim} T = \operatorname{GKdim} \operatorname{gr} T = m \in \mathbb{Z}_{\geq 0}$.

Proof. Since $\operatorname{gr} H$ is a commutative connected Hopf algebra with $\operatorname{GKdim} \operatorname{gr} H = n$, (1) is an immediate consequence of Lemma 5.2.1 and Theorem 5.3.3(1). For the final equality of GK-dimensions, use Proposition 2.5.8. □

The first consequence of Lemma 5.5.1 is the quantum analogue of the fact a strict inclusion of subgroups of a unipotent group implies a strict inequality between their dimensions. Using an almost identical proof to that of Lemma 4.2.10, the following proposition follows at once from Lemma 5.5.1 and Theorem 5.3.3(4).

Proposition 5.5.2. *Let H be a connected Hopf k-algebra of finite GK-dimension. Let T and S be (left) coideal subalgebras of H such that $S \subseteq T$. Then $S = T$ if and only if* $\operatorname{GKdim} S = \operatorname{GKdim} T$*.*

5.5.2 Commutative and cocommutative coideals in connected Hopf algebras

Before we move on to discussing some of the ring theoretic consequences of the results of §5.5.1 for general quantum homogeneous spaces, we first show that these results allow us to completely classify the algebraic structure of commutative (or cocommutative) coideal subalgebras in connected Hopf algebras which are *not* necessarily commutative or cocommutative.

Let C be a commutative left quantum homogeneous space in a connected Hopf algebra H of finite GK-dimension n. Then C has finite GK-dimension. By Theorem 5.5.1, there exists some $m \in \mathbb{Z}_{\geq 0}$ such that $m \leq \operatorname{GKdim} \operatorname{gr} H = n$ and $\operatorname{gr} C \cong k[X_1, \ldots, X_m]$. Moreover, $\operatorname{GKdim} C = \operatorname{GKdim} \operatorname{gr} C = m$, with $m = n$ if and only if $C = H$. In particular, C is affine, generated by any choice of lifts of the graded generators X_i to C. Thus C, being commutative, is isomorphic to a factor of the polynomial k-algebra R in m variables. However, proper factors of R have GK-dimension strictly less than m, so $C \cong R$. We have proved:

Proposition 5.5.3. *Let C be a commutative left quantum homogeneous space in a connected Hopf k-algebra H with* $\mathrm{GKdim} H = n < \infty$. *Then C is a polynomial algebra in m variables, where $m \leq n$ and $m = n$ if and only if $C = H$.*

We now turn our attention to describing the cocommutative coideal subalgebras of arbitrary connected Hopf algebras. Note that we do not require the results of §5.5.1 for this, nor do we require any assumptions about finite GK-dimension.

Suppose C is a cocommutative left quantum homogeneous space in a connected Hopf algebra H. Then cocommutativity ensures that C is in fact a connected sub-bialgebra of H, and so a connected Hopf subalgebra, Lemma 3.2.8. Thus one can apply Theorem 3.4.5, characterising cocommutative connected Hopf algebras over a field of characteristic 0; we find that

C is isomorphic as a Hopf algebra to $U(\mathfrak{g})_c$,

where $\mathfrak{g} = P(C)$ is the Lie algebra of primitive elements of C. Conversely, of course, each subalgebra of the Lie algebra $P(H)$ gives rise to a cocommutative coideal subalgebra of H.

5.5.3 Basic properties of quantum homogeneous spaces of connected Hopf algebras

Next, we show that the excellent homological properties enjoyed by connected Hopf algebras of finite GK-dimension extend to their quantum homogeneous spaces. The proof is a standard application of filtered-graded methods, closely following the case $T = H$ dealt with by Zhuang in [67, Corollary 6.10]. The terminology and notation used in the theorem can be found in §2.5.2. For the definition of the Krull dimension of a noncommutative ring, we refer the reader to [36, Chapter 6].

Remark 5.5.4. Note that, for any Hopf algebra H and (left) coideal subalgebra $T \subseteq H$, T is an augmented algebra by virtue of the restriction of the counit $\epsilon|_T : T \to k$. Thus, the counit defines a left (and right) T-module structure on the base field k - we call this the *trivial* T-module.

Theorem 5.5.5. *Let H be a connected Hopf k-algebra of finite GK-dimension n. Let T be a left coideal subalgebra of H with* $\mathrm{GKdim}\, T = m$.

1. *T is a noetherian domain of Krull dimension at most m.*

2. *T is GK-Cohen-Macaulay.*

3. *T is Auslander-regular of global dimension m.*

4. *T is AS-regular of dimension m.*

Proof. (1) This follows from Lemma 5.5.1 and [36, Proposition 1.6.6, Theorem 1.6.9 and Lemma 6.5.6].

(2) Let M be a finitely generated (left or right) T-module. Choose a good filtration of M, in the sense of [22, Definition 5.1]. Then $\mathrm{gr}\, M$ is a finitely generated $\mathrm{gr}\, T$-module by [22, Lemma 5.4]. By Lemma 5.5.1, $\mathrm{gr}\, T$ is a polynomial algebra in m variables, so is GK-Cohen-Macaulay, Example 2.5.14. This implies in particular that,

$$j_{\mathrm{gr}\, T}(\mathrm{gr}\, M) + \mathrm{GKdim}\, \mathrm{gr}\, M = \mathrm{GKdim}\, \mathrm{gr}\, T. \tag{5.5.1}$$

By Lemma 5.5.1(2), $\mathrm{GKdim}\, T = \mathrm{GKdim}\, \mathrm{gr}\, T = m$. Since $\mathrm{gr}\, T$ is affine and $\mathrm{gr}\, M$ is a finitely generated $\mathrm{gr}\, T$-module, $\mathrm{GKdim}\, M = \mathrm{GKdim}\, \mathrm{gr}\, M$ by [27, Proposition 6.6]. Finally, as in the proof of [9, Theorem 3.9], $j_{\mathrm{gr}\, T}(\mathrm{gr}\, M) = j_T(M)$. The result now follows from (5.5.1).

(3) By Lemma 5.5.1 and filtered-graded considerations [36, Corollary 7.6.18], T has (right and left) global dimension at most m. Taking M to be the trivial T-module k in (2) yields $j_T(k) = m$. Hence the global dimension of T is m.

The Hopf algebra H is Auslander-Gorenstein by [67, Corollary 6.11], and so, by [32, Proposition 2.3], the left coideal subalgebra T is too. Being Auslander-Gorenstein of finite global dimension, T is by definition Auslander-regular.

(4) Let k denote the trivial module in what follows. By part (2) and the fact that $\mathrm{gldim} T = m$, it remains only to check that the non-zero space $\mathrm{Ext}^m_T(k, T)$ satisfies

$$\dim_k \mathrm{Ext}^m_T(k, T) = 1. \tag{5.5.2}$$

In the sense of [9, Chapter 2.6], the filtration on the T-module k (coming from the coradical filtration of T) is *good*, [36, 8.6.3]. This yields good filtrations on the T-modules $\mathrm{Ext}^j_T(k, T)$ - let $\mathrm{gr}_*(\mathrm{Ext}^j_T(k, T))$ denote the associated graded $\mathrm{gr}\, T$-modules, for $j = 0, \ldots, m$. By Lemma 5.5.1, $\mathrm{gr}\, T$ is a polynomial algebra in m variables, so is AS-regular of dimension m (Example 2.5.14), so that $\mathrm{Ext}^m_{\mathrm{gr}\, T}(k, \mathrm{gr}\, T) = k$. By [9, Proposition

6.10], $\mathrm{gr}_*(\mathrm{Ext}^j_T(k,T))$ is a sub factor of the $\mathrm{gr}\,T$-module $\mathrm{Ext}^j_{\mathrm{gr}\,T}(k,\mathrm{gr}\,T)$, hence

$$\dim_k \mathrm{Ext}^m_T(k,T) = \dim_k \mathrm{gr}_*(\mathrm{Ext}^j_T(k,T)) \leq \dim_k \mathrm{Ext}^j_{\mathrm{gr}\,T}(k,\mathrm{gr}\,T) = 1.$$

However, by part (2), $j_T(k) = \min\{n : \mathrm{Ext}^n_T(k,T) \neq 0\} = m$, hence

$$\dim_k \mathrm{Ext}^m_T(k,T) \geq 1.$$

The result follows. $\square$

The enveloping algebra $H = U(\mathfrak{g})$ of a finite-dimensional simple complex Lie algebra $\mathfrak{g}$ has Krull dimension $\dim_{\mathbb{C}}(\mathfrak{b})$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$, [30]. Thus the inequality in Theorem 5.5.5(1) is strict in general.

5.5.4 The Calabi-Yau property for quantum homogeneous spaces

Let A be a k-algebra. For a left $A^e = A \otimes A^{\mathrm{op}}$-module (that is, A-bimodule) M and k-algebra endomorphisms ν, σ of A, denote by ${}^{\nu}M^{\sigma}$ the A^e-module whose underlying vector space is M, with A^e-action

$$a \cdot m \cdot b = \nu(a) m \sigma(b).$$

for $a, b \in A, m \in M$. If $\nu = \mathrm{Id}$, write M^{σ} rather than ${}^{\mathrm{Id}}M^{\sigma}$.

Definition 5.5.6. ([8]) An algebra A is *ν-twisted Calabi-Yau of dimension d* for a k-algebra automorphism ν of A and an integer $d \geq 0$ if

1. A is *homologically smooth*, that is, as an A^e-module, A has a finitely generated projective resolution of finite length;

2. $\mathrm{Ext}^i_{A^e}(A, A^e) \cong \delta_{i,d} A^{\nu}$ as A^e-modules, where the A^e-module structure on the Ext group is induced by the right A^e-module structure of A^e.

Then ν is uniquely determined up to an inner automorphism and is called the *Nakayama automorphism* of A. Some authors use the term "ν-skew" rather than "ν-twisted". We omit the adjective "twisted" if ν is inner.

Remark 5.5.7. ([34]) Let A be an AS-Gorenstein k-algebra of dimension d. Taking k to be the *left* A-module annihilated by the augmentation ideal A^+ of A, the one-dimensional space $\mathrm{Ext}^d_A({}_Ak, {}_AA)$ is the *left homological integral* of A, denoted $\int^{\ell}_A$. From its definition, it has an induced A-bimodule structure: the left A-action is induced by the trivial action on k, whereas the right A-module structure on $\int_A^\ell$ is induced from the right A-module structure on A. Thanks to the AS-Gorenstein hypothesis, this right A-module structure induces a character $\chi : A \to k$ such that

$$f \cdot a = \chi(a) f$$

for all $f \in \int_A^l$ and $a \in A$.

Recall that a Hopf algebra A (with bijective antipode) which is noetherian and AS-Gorenstein of dimension n is twisted Calabi-Yau of dimension n, [8, Proposition 4.5]. In view of Proposition 4.2.12, the homological corollary of Theorem 4.2.7, this applies in particular to a connected Hopf k-algebra H of finite GK-dimension n. Moreover, also by [8, Proposition 4.5]), the Nakayama automorphism ν of H is given by

$$\nu = S^2 \circ \tau_\chi^\ell,$$

where S denotes as usual the antipode of H and τ_χ^ℓ denotes the left winding automorphism of the character $\chi : H \to k$ defined by $\int_H^\ell$ as in Remark 5.5.7. That is, $\tau_\chi^\ell(h) = \sum \chi(h_1)h_2$ for $h \in H$.

Naturally, one asks: does this generalise to a right quantum homogeneous space T of a connected Hopf k-algebra H of finite GK-dimension?

By Theorem 5.5.5(4) such a right coideal subalgebra T is AS-Gorenstein, and so has a left homological integral $\int_T^\ell$, with character $\chi : T \to k$. *A priori*, the map

$$\tau_\chi^\ell : T \to H : t \mapsto \sum \chi(t_1)t_2$$

while easily seen to be an algebra homomorphism, might not take values in T. But this *is* in fact so for all such quantum homogeneous spaces T in H, by [33, Lemma 3.9], building on work of [26]. Consequently, we obtain:

Proposition 5.5.8. *Let H be a connected Hopf k-algebra with finite GK-dimension.*

1. *Let T be a right coideal subalgebra of H with* $\mathrm{GKdim}\, T = m$. *Then T is twisted Calabi-Yau of dimension m. Retaining the notation introduced above, so in particular χ is the character of the right structure of the left integral of T, the Nakayama automorphism ν of T is*

$$\nu = S^2 \circ \tau_\chi^\ell.$$

2. *The same conclusions apply to a left coideal subalgebra T of H, with χ as in (1), with Nakayama automorphism ν of T given by*

$$\nu = S^{-2} \circ \tau^r_\chi.$$

Proof. (1) That T is homologically smooth follows from [33, Lemma 3.7]. Given that T is AS-regular by Theorem 5.5.5(4), [33, Theorem 3.6] implies that T is ν-twisted Calabi-Yau, with $\nu = S^2 \circ \tau^\ell_\chi$.

(2) Let T be a left coideal subalgebra of $\mathbf{H} := (H, \mu, \Delta, S, \epsilon)$. By [41, Lemma 1.5.11], $\mathbf{H}' := (H, \mu, \Delta^{op}, S^{-1}, \epsilon)$ is a Hopf algebra, clearly connected, and with the Gel'fand-Kirillov dimensions of H and its subalgebras unchanged, since the algebra structure is the same. However, T is now a *right* coideal subalgebra of $\mathbf{H}'$, so part (1) can be applied to it. Hence, the Nakayama automorphism is as stated in (2), with a *right* winding automorphism appearing now (with respect to Δ) because the coproduct for $\mathbf{H}'$ is Δ^{op}. □

Here is a typical example to illustrate the proposition.

Example 5.5.9. Nakayama automorphism of a coideal subalgebra of $B(\lambda)$.

Consider the connected Hopf algebra $B(\lambda)$ with presentation as in Example 4.3.8. Define a subalgebra $R_\infty := k\langle Y, f\rangle \subseteq B(\lambda)$, where $f := Z - XY$. An easy computation yields

$$[f, Y] = -\frac{1}{2}Y^2$$

and

$$\Delta(f) = 1 \otimes f + f \otimes 1 - Y \otimes X, \quad S(f) = -f - YX.$$

Then R_∞ is a right coideal subalgebra of $B(\lambda)$ with $\mathrm{GKdim} R_\infty = 2$, and R_∞ is isomorphic to the Jordan plane. By Proposition 5.5.8, to compute the Nakayama automorphism of R_∞ we must compute the right R_∞-module structure of $\int^l_{R_\infty} = \mathrm{Ext}^2_{R_\infty}(k, R_\infty)$.

For an automorphism τ of R_∞, call $b \in R_\infty$ τ-*normal* if $\tau(a)b = ba$ for all $a \in R_\infty$. Thus Y is a σ-normal element of R_∞, where $\sigma(Y) = Y$, $\sigma(f) = f + \frac{1}{2}Y$. Set $\overline{R_\infty} = R_\infty / Y R_\infty$. Then $\overline{R_\infty} \cong k[W]$, so that

$$\int^l_{\overline{R_\infty}} = \mathrm{Ext}^1_{\overline{R_\infty}}(k, \overline{R_\infty}) \cong {}^1k^1$$

where ${}^1k^1$ denotes the R_∞-bimodule structure induced by the restriction of the counit map. Hence, by the noncommutative Rees Change of Rings theorem, [8, Lemma 6.6], there is an R_∞-bimodule isomorphism

$$\int_{R_\infty}^{l} \cong \left(\int_{\overline{R_\infty}}^{l}\right)^{\sigma^{-1}} \cong {}^1k^{\sigma^{-1}} \cong {}^1k^1, \tag{5.5.3}$$

where the final isomorphism above holds since twisting by σ^{-1} does not alter the structure of the trivial R_∞-module. Therefore, by Proposition 5.5.8,

$$\nu_{R_\infty} = S^2_{|R_\infty},$$

so that

$$\nu(Y) = Y \text{ and } \nu(f) = f - Y.$$

The above calculation agrees with that carried out for the Jordan plane by other means in, for example [31, §4.2].

Remark 5.5.10. If T is a Hopf subalgebra of $\mathbf{H} = (H, \Delta, S, \epsilon)$, then both parts of the proposition apply to it. The Nakayama automorphism of a skew Calabi-Yau algebra is unique up to an inner automorphism of the algebra, but in this case H and thus T have no non-trivial inner automorphisms, thanks to the fact that the only units of connected Hopf k-algebras of finite GK-dimension are in k^*, since this is true for their associated graded algebras by Zhuang's Theorem 4.2.7. Thus, generalising [8, 4.6] for the particular case of connected algebras, we find that, for a Hopf subalgebra T of the connected Hopf k-algebra H of finite GK-dimension,

$$S^4_{|T} = \tau^{\ell}_{-\chi} \circ \tau^{r}_{\chi}, \tag{5.5.4}$$

where χ is the character of the right structure on $\int_T^\ell$. We find this formula rather curious, given that there is no obvious relationship between $\int_T^\ell$ and $\int_H^r$.

The following corollary of Proposition 5.5.8 ought to have a more direct proof.

Corollary 5.5.11. *Let C be a commutative right or left coideal subalgebra of a connected Hopf k-algebra H of finite GK-dimension. Then $S^2_{|C} = \mathrm{Id}_C$.*

Proof. Apply Proposition 5.5.8 to C. Commutativity of C ensures that both the character χ and the automorphism ν are trivial, meaning $\chi = \epsilon$ and $\nu = \mathrm{Id}_C$. Substituting these values in the formula for ν gives the desired conclusion. □

Note that the corresponding result to the above with "cocommutative" replacing "commutative" is rather trivial, since then C is a cocommutative Hopf subalgebra of H, as was noted in §5.5.2.

5.6 Quantum homogeneous spaces of small GK-dimension

In §4.3.1, §4.3.2 and §4.3.3 we investigated connected Hopf algebras of small GK-dimension, listing results of Zhuang in [67], which classified all such Hopf algebras with GK-dimension at most three. In this section we provide a similar classification for quantum homogeneous spaces of GK-dimension at most one. We'll see that, in dimensions 0 and 1, only classical homogeneous spaces occur and so, in these dimensions, the story for quantum homogeneous spaces is exactly the same as the story for connected Hopf algebras. At GK-dimension two things become more interesting, where we see an example of a quantum homogeneous space which *cannot* admit *any* Hopf structure, in stark contrast to the commutative case. While we are unable to classify all quantum homogeneous spaces of GK-dimension two, we provide a plethora of examples in Proposition 5.5.9, where we give a complete description of the coideal structure of the connected Hopf algebra $B(\lambda)$, as introduced in Example 4.3.8.

5.6.1 GK-dimensions 0 and 1

Here we show that in GK-dimensions 0 and 1 the only quantum homogeneous spaces occuring are in fact Hopf algebras, so the classification is identical to the one described for Hopf algebras in §4.3.1.

Proposition 5.6.1. *Suppose H is a connected Hopf algebra such that $\operatorname{GKdim} H < \infty$. Let C denote a left quantum homogeneous space of H.*

1. *If $\operatorname{GKdim} C = 0$, $C = k$.*

2. *If $\operatorname{GKdim} C = 1$, $C = U(\mathfrak{a})_c$ where $\mathfrak{a}$ is the 1-dimensional abelian Lie algebra.*

Proof. 1. Suppose $\operatorname{GKdim} C = 0$. Then C is finite dimensional, so $C = k$ by Corollary 4.3.3.

2. Suppose $\operatorname{GKdim}(C) = 1$. By Lemma 4.3.2, there exists $0 \neq c \in C \cap P(H)$. Thus C contains the Hopf subalgebra $U(\mathfrak{a})_c$ of H, where $\mathfrak{a}$ denotes the one dimensional abelian Lie algebra kc. It is an immediate consequence of Proposition 5.5.2 that $C = U(\mathfrak{a})_c$.

□

5.6.2 GK-dimension 2

As promised above, we now give a complete description of the coideal structure of the GK-dimension three connected Hopf algebra $B(\lambda)$. By straightforward (but very lengthy) calculations, which we leave to the interested reader, one can verify the following:

Proposition 5.6.2. *Recall the definition of the connected Hopf algebra $B(\lambda)$ (for $\lambda \in k$) from Example 4.3.8. The following is a complete list of the non-trivial proper coideal subalgebras of $B(\lambda)$.*

1. *Set $\mathfrak{g}_\alpha$ to be the 1-dimensional Lie algebra in $B(\lambda)$ with basis $\{X + \alpha Y\}$ ($\alpha \in k$), and $\mathfrak{g}_\infty$ to be the Lie algebra with basis $\{Y\}$. Then $\{U(\mathfrak{g}_\alpha) : \alpha \in k \cup \{\infty\}\}$ is the complete set of Hopf subalgebras (and also left or right coideal subalgebras) in $B(\lambda)$ of GK-dimension 1.*

2. *Let $\delta_\beta, \delta \in \mathrm{Der}_k(U(\mathfrak{g}_\infty))$ be given by*
$$\delta_\beta(Y) = Y + \frac{\beta}{2}Y^2 \text{ and } \delta(Y) = \frac{1}{2}Y^2.$$
Then
$$L_\beta := U(\mathfrak{g}_\infty)[X + \beta Z; \delta_\beta], \text{ for } \beta \in k; \text{ and } L_\infty := U(\mathfrak{g}_\infty)[Z; \delta]$$
are all the left coideal subalgebras of GK-dimension 2 in $B(\lambda)$.

3. *The right coideal subalgebras of GK-dimension dimension 2 in $B(\lambda)$ are*
$$R_\beta := U(\mathfrak{g}_\infty)[X + \beta(Z - XY); \delta_{-\beta}], \text{ for } \beta \in k\}$$
and
$$R_\infty := U(\mathfrak{g}_\infty)[Z - XY; -\delta].$$

4. *$S(L_\beta) = R_\beta$ for $\beta \in k \cup \{\infty\}$, and vice versa. Moreover, $L_0 = R_0$ is the only Hopf algebra in the lists (2) and (3), and is the only algebra which occurs in both lists.*

An obvious question, which we leave open, is the following:

Question 5.6.3. Let $\alpha, \beta \in k$. Under what conditions is $L_\alpha \cong L_\beta$ as an algebra? Under what conditions is $R_\alpha \cong R_\beta$ as an algebra?

Proposition 5.6.4. *There exists a quantum homogeneous space of a connected Hopf algebra of finite GK-dimension whose underlying algebra does not admit any structure as Hopf algebra.*

Proof. We saw in Example 5.5.9 and Proposition 5.6.2 that, up to a change of variable, the subalgebra

$$J = k\langle x, y | [x, y] = y^2 \rangle$$

appears as a right coideal subalgebra of the connected Hopf algebra $B(\lambda)$. One calculates that $\langle [J, J] \rangle = y^2 J$, so that the abelianisation J^{ab} of J is $k[x, y]/\langle y^2 \rangle$. Since, as is easily checked, the abelianisation of a Hopf algebra is always a Hopf algebra ([17, Lemma 3.7]), and commutative Hopf algebras in characteristic 0 are semiprime [64, Theorem 11.4], it must be that the algebra J cannot support any structure as a Hopf algebra. □

Remark 5.6.5. Consider again the right coideal subalgebra R_∞ of $B(\lambda)$ from Example 5.6.2. A simple calculation using the definition of the antipode for $B(\lambda)$ yields, for any $m \geq 1$,

$$S^{2m}(Z - XY) = (Z - XY) - mY.$$

Since we're working over a field of characteristic 0, it follows that $S^{2m}(Z - XY) \neq Z - XY$ for any $m \geq 1$. Thus, R_∞ is an example of a quantum homogeneous space of GK-dimension 2 with $|(S|_{R_\infty})^2| = \infty$. By contrast, we saw in §4.3.2 that a connected Hopf algebra of GK-dimension 2 is either commutative or cocommutative, so has an antipode of order 2, [41, Corollary 1.5.12].

This suggests an obvious project: classify the connected quantum homogeneous spaces of GK-dimension 2. This is listed below as Question 5.8.1.

5.7 Invariants of quantum homogeneous spaces

We now define an invariant of a quantum homogeneous space T of a connected Hopf algebra which we shall call its *signature*, and denote by $\sigma(T)$. We'll examine some of its basic properties and show that it is closely related the *lantern* of a connected Hopf algebra, another invariant first introduced in [62].

5.7.1 Preliminaries on gradings

Let $A = \bigoplus_{i \geq 0} A(i)$ be a locally finite connected graded algebra (see Definition 2.3.12). We write

$$h_A(t) = \sum_{i=0}^{\infty} \dim_k A(i) t^i$$

for the *Hilbert series* of A. We need the following obvious lemma, whose proof is omitted and is a straightforward exercise using the definition.

Lemma 5.7.1. *Let A, B and C be locally finite connected $\mathbb{N}$-graded algebras, with $A \cong B \otimes_k C$ as graded algebras. Then*

$$h_A(t) = h_B(t)h_C(t).$$

The next lemma is key to the definitions in this section.

Lemma 5.7.2. *Let R be a connected graded commutative polynomial algebra, with homogeneous polynomial generators $x_1, \dots, x_n$. Let $\mathfrak{m} = \bigoplus_{i>0} R(i) = \langle x_1, \dots, x_n \rangle$ be the graded maximal ideal of R. Let C and D be graded polynomial subalgebras of R such that $C \subseteq D$ and $R = C[z_1, \dots, z_t] = D[w_1, \dots, w_r]$; that is, R is a polynomial algebra over C and over D.*

1. *Homogeneous elements $y_1, \dots, y_n$ form a set of polynomial generators of R if and only if their images in $\mathfrak{m}/\mathfrak{m}^2$ form a k-basis for this space.*

2. *There exist homogeneous elements $u_1, \dots, u_n$ in $\mathfrak{m}$ such that*

$$C = k[u_1, \dots, u_{n-t}], \qquad D = k[u_1, \dots, u_{n-r}], \qquad R = k[u_1, \dots, u_n].$$

3. *The multiset $\sigma(C)$ of degrees of a homogeneous set of polynomial generators of C equals the multiset of degrees of a homogeneous basis of $\mathfrak{m} \cap C/(\mathfrak{m} \cap C)^2$, and hence is independent of the choice of such a generating set.*

4. *$\sigma(C) \subseteq \sigma(D)$, with equality if and only if $C = D$. Equivalently, the Hilbert polynomial $h_C(t)$, which equals $\prod_{d \in \sigma(C)} \frac{1}{(1-t^d)}$, divides $h_D(t)$.*

Proof. 1. $\Rightarrow$: This is trivial.

$\Leftarrow$: Let $y_1, \dots, y_n$ be homogeneous elements of R whose images *modulo* $\mathfrak{m}^2$ form a k-basis for $\mathfrak{m}/\mathfrak{m}^2$. Define $A = k\langle y_1, \dots, y_n \rangle$. Suppose that $A \subsetneqq R$, and let s be minimal such that

$$A(s) \subsetneqq R(s).$$

Since $R(s) \cap \mathfrak{m}^2$ is spanned by products of pairs of elements in $\{R(i) : i < s\}$,

$$R(s) \cap \mathfrak{m}^2 = A(s) \cap \mathfrak{m}^2 \subseteq A(s). \tag{5.7.1}$$

Choose $x \in R(s) \setminus A(s)$, so $x \notin \mathfrak{m}^2$. By hypothesis, there exist $\lambda_j \in k$, $1 \le j \le n$, such that

$$\widehat{x} := x - \sum_j \lambda_j y_j \in \mathfrak{m}^2.$$

Clearly, for all j in the above expression with $\lambda_j \neq 0$, the degree of y_j is s. But by (5.7.1) this forces $\widehat{x} \in R(s) \cap \mathfrak{m}^2$, so $\widehat{x} \in A(s)$. Hence $x \in A(s)$, contradicting the choice of x. Therefore we must have $A = R$, and finally considering GK-dimension shows that $y_1, \ldots, y_n$ are polynomial generators of A.

2. First, note that since $R = C \otimes_k k[z_1, \ldots, z_t]$, $\mathfrak{m}^2 \cap C = (\mathfrak{m} \cap C)^2$. Choose homogeneous elements $u_1, \ldots, u_{n-t}$ in $\mathfrak{m} \cap C$ whose images *modulo* $\mathfrak{m}^2 \cap C = (\mathfrak{m} \cap C)^2$ form a k-basis for $\mathfrak{m} \cap C / \mathfrak{m}^2 \cap C$. Thus, by part (1), $C = k[u_1, \ldots, u_{n-t}]$. Then $\mathfrak{m} \cap C / \mathfrak{m}^2 \cap C$ embeds in $\mathfrak{m} \cap D / \mathfrak{m}^2 \cap D = \mathfrak{m} \cap D / (\mathfrak{m} \cap D)^2$, so we can extend $\{u_1, \ldots, u_{n-t}\}$ to a set of homogeneous elements $\{u_1, \ldots, u_{n-r}\}$ of D, of positive degree, whose images *modulo* $\mathfrak{m}^2 \cap D$ provide a k-basis for $\mathfrak{m} \cap D / (\mathfrak{m} \cap D)^2$. By (1) again, $D = k[u_1, \ldots, u_{n-r}]$. A further repeat of the argument extends the set to homogeneous polynomial generators $\{u_1, \ldots, u_n\}$ of R.

3. This is clear from (1) applied to C rather than R, since the degrees and dimensions of the homogeneous components of $\mathfrak{m} \cap C / (\mathfrak{m} \cap C)^2$ are fixed.

4. This is immediate from (2) and (3). The equivalent formulation in terms of Hilbert series follows from Lemma 5.7.1.

□

5.7.2 The signature and the lantern

Lemma 5.7.2(3) ensures that the following definitions make sense. That the previous parts of the definition apply to T as in part (4) follows from Lemma 5.5.1.

Definition 5.7.3. 1. Let R be a connected graded polynomial algebra in n variables, $n < \infty$. The *signature* of R, denoted by $\sigma(R)$, is the ordered n-tuple of degrees of the homogeneous generators.

2. Let A be a filtered algebra, with filtration $\mathcal{A} = \{A_i\}_{i \ge 0}$, such that the associated graded algebra $\operatorname{gr} A = \bigoplus_i A_i / A_{i-1}$ is a connected graded polynomial algebra in n variables, $n < \infty$. The $\mathcal{A}-$ *signature* of A, denoted by $\sigma_{\mathcal{A}}(A)$, is the signature of

gr A in the sense of (1). Where no confusion is likely refer simply to the *signature* of A, denoted $\sigma(A)$.

3. With A, $\mathcal{A}$ and gr$A$ as in (2), suppose that gr$A$ has m_i homogeneous generators of degree d_i, $1 \leq i \leq t$, with $1 \leq d_1 < \cdots < d_t$, so that $\sum_{i=1}^t m_i = n$. Then we write

$$\sigma(A) = (d_1^{(m_1)}, \ldots, d_t^{(m_t)}).$$

When $m_i = 1$, the exponent (m_i) is omitted.

4. Let H be a connected Hopf k-algebra of finite GK-dimension, with a left coideal subalgebra T, with coradical filtration $\mathcal{T}$ as defined in §5.2. Then the *signature* of T, denoted $\sigma(T)$, is the $\mathcal{T}$-signature of T as defined in (2).

Dualising the above definition, as follows, gives certain benefits, as we shall see. The definition of the *lantern* of a connected Hopf k-algebra, and the key parts (1), (2), (4) and the corollary of Proposition 5.7.5, are due to Wang, Zhang and Zhuang, [62, Definition 1.2 and Lemma 1.3]. Proofs are given again here for the reader's convenience, in the course of extending their definition.

Definition 5.7.4. Let $R = \bigoplus_{i \geq 0} R(i)$ denote a connected graded polynomial algebra. As noted in §2.3.1, the graded dual $\mathcal{D}_R$ forms a graded cocommutative coalgebra.

1. The *lantern* $\mathfrak{L}(R)$ of R is the space of primitive elements of $\mathcal{D}_R$. That is,

$$\mathfrak{L}(R) := P(\mathcal{D}_R),$$

a graded subcoalgebra of $\mathcal{D}_R$. Note that $\mathfrak{L}(R)$ is non-zero provided $R \neq k$, since then $k = \mathcal{D}_R(0) \subsetneq \mathcal{D}_R$.

2. Let A be a filtered algebra, with filtration $\mathcal{A} = \{A_n\}_{n \geq 0}$, satisfying the hypotheses of Definition 5.7.3(2). Define the $\mathcal{A} -$ *lantern* of A, denoted by $\mathfrak{L}_{\mathcal{A}}(A)$, to be the lantern of the graded polynomial algebra gr A. Where no confusion is likely, shorten notation to $\mathfrak{L}(A)$.

3. Let H be a connected Hopf algebra of finite GK-dimension, with a left coideal subalgebra T with coradical filtration $\mathcal{T}$, as in §5.2. Then the *lantern* $\mathcal{L}(T)$ of T is the $\mathcal{T}$-lantern of T as defined in (2).

Proposition 5.7.5. *Let H be a connected Hopf k-algebra of finite GK-dimension n, and let T be a left coideal subalgebra of H, with* $\operatorname{GKdim} T = m$. *Let* $\sigma(H) = (d_1^{(m_1)}, \ldots, d_t^{(m_t)})$.

1. *The graded dual $\mathcal{D}_{\mathrm{gr}\,H}$ is a Hopf subalgebra of the finite dual $(\mathrm{gr}H)^\circ$; as such, it is isomorphic to the enveloping algebra $U(\mathcal{L}(H))$.*

2. *(a) The lantern $\mathcal{L}(H)$ is a positively graded Lie algebra, with*

$$\mathcal{L}(H) = \bigoplus_{i=1}^{t} \mathcal{L}(H)(d_i), \;\; \dim_k \mathcal{L}(H)(d_i) = m_i, \;\; \dim_k \mathcal{L}(H) = n.$$

   (b) $\dim \mathcal{L}(H) = \mathrm{GKdim}\, H$.

3. *Let W be the unipotent group of dimension n such that* $\mathrm{gr}H \cong \mathcal{O}(W)$*. Then $\mathcal{L}(H)$ is the Lie algebra of W.*

4. *$\mathcal{L}(H)$ is generated in degree 1.*

5. *Let Y_T be the closed subgroup of W defined by the Hopf ideal* $\mathrm{gr}T^+\mathrm{gr}H$ *of* $\mathrm{gr}H$*. Let $\mathfrak{y}_T$ be the Lie subalgebra of $\mathcal{L}(H)$ corresponding to Y_T. The graded dual $\mathcal{D}_{\mathrm{gr}\,T}$ is a graded left $U(\mathcal{L}(H))$-module, isomorphic as such to $U(\mathcal{L}(H))/U(\mathcal{L}(H))\mathfrak{y}_T$.*

6. *The lantern $\mathcal{L}(T)$ is the graded quotient $\mathcal{L}(H)/\mathfrak{y}_T$.*

7. *The following are equivalent.*

   (a) $\sigma(T) = (e_1^{(r_1)}, \ldots, e_s^{(r_s)})$.

   (b) $\dim \mathfrak{L}(T)(e_i) = r_i$ *for* $i \geq 1$.

   (c) $h_{\mathrm{gr}T}(t) = \prod_{i=1}^{s} \frac{1}{(1-t^{e_i})^{r_i}}$.

Proof. 1. Let $\mathfrak{m} = \oplus_{i \geq 1} H(i)$ be the graded maximal ideal of $\mathrm{gr}H$. By construction,

$$\mathcal{D}_H = \{f \in (\mathrm{gr}H)^\circ : f(\mathfrak{m}^j) = 0 \textit{ for } j \gg 0\}.$$

By [41, Proposition 9.2.5], $\mathcal{D}_H$ is a sub-Hopf algebra of $(\mathrm{gr}H)^\circ$; more precisely,

$$\mathcal{D}_H = U(P(\mathcal{D}_H)),$$

the enveloping algebra of the Lie algebra of primitive elements of $\mathcal{D}_H$. Note also that, by Definition 5.7.4(1) and [41, Lemma 9.2.4],

$$\mathcal{L}(H) = P(\mathcal{D}_H) = \{f \in \mathcal{D}_H : f(\mathfrak{m}^2 + k1) = 0\}. \tag{5.7.2}$$

2. Since $\mathcal{D}_H$ is a graded Hopf algebra, its subspace $\mathcal{L}(H)$ of primitive elements is a graded Lie algebra: for $d \geq 1$,

$$\mathcal{L}(H)(d) := \{f \in (H(d))^* : f(\mathfrak{m}^2 \cap H(d)) = 0\}.$$

Thus, for all $d \geq 1$,

$$\dim_k \mathcal{L}(H)(d) = \dim_k (H(d)/\mathfrak{m}^2 \cap H(d)) = \dim_k ((\mathfrak{m}/\mathfrak{m}^2)(d)). \tag{5.7.3}$$

By Lemma 5.7.2(3), this completes the proof of (a). For (b), we have

$$\text{GKdim}\, H = \text{GKdim}\,\text{gr}\, H = \text{GKdim}\,\mathcal{D}_{\text{gr}\, H} = \dim \mathcal{L}(H)$$

where the first equality follows from Theorem 4.2.7.

3. This follows from (5.7.2) and the definition of the Lie algebra of an algebraic group, [23, §9.1].

4. This is [1, Lemma 5.5].

5. Appealing again to [23, §9.1] and noting (5.7.2),

$$\begin{aligned} \mathfrak{y}_T =& \{f \in \mathcal{L}(H) : f((\text{gr}T)^+ \text{gr}H) = 0\} \\ =& \{f \in \mathcal{D}_H : f((\text{gr}T)^+ \text{gr}H + \mathfrak{m}^2 + k1) = 0\}. \end{aligned} \tag{5.7.4}$$

The convolution product makes $\mathcal{D}_T$ into a left module over $\mathcal{D}_H = U(\mathcal{L}(H))$: namely, for $u \in U(\mathcal{L}(H))$, $f \in \mathcal{D}_T$ and $t \in \text{gr}T$,

$$uf(t) = u(t_1)f(t_2).$$

It is clear that, with respect to this action, the restriction map

$$\rho : U(\mathcal{L}(H)) \longrightarrow \mathcal{D}_T$$

is a homomorphism of left $U(\mathcal{L}(H)$-modules. From (5.7.4), $\mathfrak{y}_T \subseteq \ker\rho$, and hence

$$U(\mathcal{L}(H))\mathfrak{y}_T \subseteq \ker\rho. \tag{5.7.5}$$

It remains to check that equality holds in (5.7.5). Take a graded basis $\{u_1, \ldots, u_n\}$ of $\mathcal{L}(H)$, where $\{u_1, \ldots, u_m\}$ is a dual basis to a set $x_1, \ldots x_m$ of graded polynomial generators of gr$T$ and where $\{u_{m+1}, \ldots, u_n\}$ forms a basis of $\mathfrak{y}_T$. Let $u_i \in H(d_i)^*$ for $i = 1, \ldots, n$. Thus $U(\mathcal{L}(H)) = \mathcal{D}_H$ is graded, where, for $j \geq 0$, a basis of

$U(\mathcal{L}(H))(j)$ is given by the ordered monomials $u_1^{r_1} \dots u_n^{r_n}$ for which $\sum_{i=1}^n r_i d_i = j$. Then $U(\mathcal{L}(H)\mathfrak{y}_T(j)$ is spanned by those ordered monomials in the u_i of degree j for which $r_i > 0$ for some $i > m$. Comparing the dimensions of $\mathcal{D}_T(j)$ with $(U(\mathcal{L}(H)/U(\mathcal{L}(H)\mathfrak{y})(j)$ for $j \geq 0$ now yields equality in (5.7.5).

6. It follows from (5.7.3) and (5.7.4) that

$$\begin{aligned}(\mathcal{L}(H)/\mathfrak{y})(d) \cong & \mathcal{L}(H)(d)/\mathfrak{y}(d) \\ \cong & \{f \in (H(d))^* : f((\mathfrak{m}^2 + (\mathrm{gr}T)^+\mathrm{gr}H) \cap H(d)) = 0\} \\ \cong & ((\mathrm{gr}T)^+/(\mathfrak{m}^2 \cap (\mathrm{gr}T)^+)(d))^*.\end{aligned} \tag{5.7.6}$$

   Since the final term above is $\mathcal{L}(T)(d)$, this proves (6).

7. The equivalence of (a) and (c) was noted in Lemma 5.7.2 (4) and its proof. The equivalence of (a) and (b) is (5.7.6) and Lemma 5.7.2 (3).

□

In the literature, a finitely generated positively graded (and hence nilpotent) Lie algebra which is generated in degree 1 is called a *Carnot Lie algebra*; they are important in a number of branches of mathematics, for example in Riemannian geometry. For a brief review with references, one can consult for instance [12].

The *classical* part of the picture described by the theorem, that is the *connected cocommutative* aspect familiar from basic Lie theory, is as follows.

Corollary 5.7.6. *Retain the notation of the above theorem. Then the following are equivalent:*

1. $\mathcal{L}(H)$ *is abelian;*
2. $\mathcal{L}(H) = \mathcal{L}(H)(1)$*;*
3. H *is cocommutative;*
4. $H \cong U(\mathfrak{g})$ *as a Hopf algebra, for some n-dimensional Lie algebra* $\mathfrak{g}$*;*
5. $\mathrm{gr}H$ *is cocommutative;*
6. W *is abelian,* $W \cong (k,+)^n$*;*
7. $\sigma(H) = (1^{(n)})$.

Proof. (1)$\Leftrightarrow$ (2): If $\mathcal{L}(H)$ is abelian, then $\mathcal{L}(H) = \mathcal{L}(H)(1)$ by Proposition 5.7.5(4). Conversely, if $\mathcal{L}(H) = \mathcal{L}(H)(1)$, $\mathcal{L}(H)$ must be abelian, since non-zero commutators have degree greater than 1.

(3)$\Leftrightarrow$(4) This is Theorem 3.4.5.

(5)$\Leftrightarrow$(6): That W is abelian if and only if its coordinate ring gr$H$ is cocommutative is immediate by duality. That, in this case, $W \cong (k,+)^n$ is a consequence of the structure of unipotent abelian algebraic groups in charactersitic 0, Proposition 3.3.2.

(2)$\Leftrightarrow$(7): This is a special case of Proposition 5.7.5 (7).

(7)$\Rightarrow$(4): Suppose $\sigma(H) = (1^{(n)})$. Then gr$H$ is generated by elements of $H(1)$, that is, by primitive elements. Hence gr$H$ is cocommutative.

(3)$\Rightarrow$(5): Trivial.

(5)$\Rightarrow$(7): If gr$H$ is cocommutative, then by Theorem 3.4.5 it isomorphic as a Hopf algebra to $U(P(H))_c$, so generated by the space $H(1)$ of primitive elements. That is, $\sigma(H) = (1^{(n)})$. $\square$

We return to the family of examples $B(\lambda)$ to illustrate aspects of Proposition 5.7.5.

Example 5.7.7. Signature and lantern of $B(\lambda)$. For $\lambda \in k$, recall the family of connected Hopf algebras $B(\lambda)$ defined in Example 4.3.8. Starting from the description of the coideal subalgebras of $B(\lambda)$ in Proposition 5.5.9, one easily calculates the following facts:

1. $\sigma(B(\lambda)) = (1^{(2)}, 2)$.

2. $\mathcal{L}(B(\lambda))$ is the Heisenberg Lie algebra of dimension 3; equivalently, the group W with coordinate ring gr$B(\lambda)$ is the 3-dimensional Heisenberg group, $\mathbb{U}_3$. Here, gr$B(\lambda) = k[X,Y,Z]$ in the obvious "lazy" notation for lifts of elements to the associated graded algebra.

3. The unique two-dimensional Hopf subalgebra of $B(\lambda)$, namely $k\langle X, Y : [X,Y] = Y\rangle$, labelled $L_0 = R_0$ in Proposition 4.3.8, has signature $(1^{(2)})$, with gr$L_0 = k[X,Y]$.

4. The remaining two-dimensional left and right coideal subalgebras L_β, R_β, for $\beta \in \{\infty\} \cup (k \setminus \{0\})$, have signature $(1,2)$. For all such β, gr$L_\beta = k[Y,Z]$ and gr$R_\beta = k[Y, Z - XY]$.

5.7.3 Numerology

The first of the two results in this subsection assembles what we know about the signature of a connected Hopf algebra H. The second theorem gives a parallel account of the known numerical constraints on the signature of a quantum homogeneous space of such an H. For convenience, some results obtained earlier in the thesis are restated here.

Theorem 5.7.8. *Let k be an algebraically closed field of characteristic 0, and let H be a connected Hopf k-algebra of finite GK-dimension n. Let*

$$\sigma(H) = (d_1^{(m_1)}, \ldots, d_u^{(m_t)}), \text{ so } h_{\mathrm{gr}H}(t) = \prod_{i=1}^{t} \frac{1}{(1-t^{d_i})^{m_i}}.$$

1. *(Wang, Zhang, Zhuang, [62, Lemma 1.3(d)]. Also see Lemma 4.3.5) For $n \geq 1$, $d_1 = 1$. If $n > 1$, then $m_1 \geq 2$. That is, $\dim_k P(H) \geq 2$ if $\mathrm{GKdim}\, H \geq 2$.*

2. *(NO GAPS) $\{d_1, \ldots, d_t\} = \{1, \ldots, t\}$.*

3. *For all $i = 1, \ldots, t$,*
$$m_i \leq \frac{1}{i} \sum_{d|i} \mu(d) n^{(i/d)},$$
where $\mu : \mathbb{N} \to \mathbb{N}$ is the Möbius function.

Proof. 1. By Proposition 5.7.5(7), $m_1 = \dim(\mathcal{L}(H)(1))$. Since $\mathcal{L}(H)$ is generated in degree 1 by Proposition 5.7.5(4), $\mathcal{L}(H)$ and hence, by Proposition 5.7.5(2) (b), H would be one-dimensional if $\dim_k \mathcal{L}(H)(1) = 1$.

2. A lemma on $\mathbb{N}$-graded Lie algebras which are generated in degree 1, proved easily by induction, implies that, for all $i \geq 1$,
$$\mathcal{L}(H)(i+1) = [\mathcal{L}(H)(1), \mathcal{L}(H)(i)].$$
Thus (2) follows from this and Proposition 5.7.5(2).

3. Let $1 \leq i \leq t$. By Proposition 5.7.5(7), $m_i = \dim_k \mathcal{L}(H)(i)$, so the bound follows from Proposition 5.7.5(4) and from the well-known Witt formula for the dimension of the ith graded summand of the free Lie algebra on n generators of degree 1, [66, Theorem 3].

$\square$

Theorem 5.7.9. *Let k and H be as in Theorem 5.7.8.Suppose K and L are both right (or both left) coideal subalgebras of H, of GK-dimensions m and ℓ respectively, with $L \subseteq K$. Let $\sigma(K) = (e_1^{(r_1)}, \ldots, e_s^{(r_s)})$ and $\sigma(L) = (f_1^{(q_1)}, \ldots, f_p^{(q_p)})$.*

1. *$m = \sum_i r_i \geq \sum_j q_j = \ell$.*

2. *$\ell = m$ if and only if $L = K$.*

3. *$\sigma(L)$ is a sub-multiset of $\sigma(K)$. That is,*

$$h_{\mathrm{gr}L}(t) | h_{\mathrm{gr}K}(t).$$

   Equality holds (of multisets and of Hilbert polynomials) if and only if $L = K$.

4. *If $K \neq k$ then $e_1 = 1$. Similarly, of course, for L.*

Proof. (1)

1. Immediate from Lemma 5.5.1 and the definition of the signature, Definition 5.7.3(4).

2. This is Proposition 5.5.2.

3. Immediate from the definition and from Lemma 5.7.2(4).

4. This is Lemma 4.3.2.

□

Remarks 5.7.10. (1) One might expect that Theorem 5.7.8(1) applies more generally, namely to quantum homogeneous spaces rather than just to Hopf algebras, especially in the light of Theorem 5.7.9(4). But this is not the case, as is illustrated by $H = B(\lambda)$, see Example 5.7.7(4).

(2) Similarly, the No-Gaps Theorem 5.7.8(3) does *not* extend to quantum homogeneous spaces. This is shown by the example below.

Example 5.7.11. A quantum homogeneous space with signature $(1^{(2)}, 3)$. The example is a qunatum homogeneous space of one of the families listed in the classification of connected Hopf algebras of GK-dimension four, [62, Example 4.5]. Let $a, b, \lambda_1, \lambda_2 \in k$, and let E be the k-algebra generated by X, Y, Z, W, subject to the following relations:

$$[Y, X] = [Z, Y] = 0, \ [Z, X] = X, \ [W, X] = aX,$$
$$[W, Y] = bX, \ [W, Z] = aZ - W + \lambda_1 X + \lambda_2 Y.$$

As an algebra, $E \cong U(\mathfrak{g})$, where $\mathfrak{g}$ is the four dimensional solvable (non-nilpotent) Lie algebra with basis $\{X, Y, Z, W\}$. It is shown in [62] that there is a noncocommutative connected Hopf algebra structure on E. Namely, one defines $X, Y, Z, W \in \ker\epsilon$, and $\Delta : E \to E \otimes E$ is fixed by setting $X, Y \in P(E)$ and

$$\begin{aligned}\Delta(Z) &= 1 \otimes Z + X \otimes Y - Y \otimes X + Z \otimes 1,\\ \Delta(W) &= 1 \otimes W + W \otimes 1 + Z \otimes X - X \otimes Z + X \otimes XY + XY \otimes X.\end{aligned}$$

In [62, Proposition 4.8] it is shown that

$$\sigma(E) = (1^{(2)}, 2, 3);$$

indeed one can see from the definition of Δ that $Z \in E_2 \setminus E_1$ and $W \in E_3 \setminus E_2$. Define $T = k\langle X, Y, W - XZ\rangle$ and confirm easily that

$$T = k[X, Y][(W - XZ); \partial],$$

where $\partial(X) = aX - X^2$ and $\partial(Y) = bX$. Note that $Z \notin T$ and that

$$\begin{aligned}\Delta(W - XZ) = {} & 1 \otimes (W - XZ) + (W - XZ) \otimes 1\\ & + 2(XY \otimes X) + 2(X \otimes Z) + X^2 \otimes Y - Y \otimes X^2\\ & \in T \otimes E.\end{aligned}$$

Thus T is a right coideal subalgebra of E, with

$$\sigma(T) = (1^{(2)}, 3).$$

5.8 Questions and discussion

Some questions concerning connected Hopf algebras are listed in the survey article [3]. We don't repeat those questions here, focussing instead on the possible role of quantum homogeneous spaces in the study of these Hopf algebras. As elsewhere in this paper, k is algebraically closed of characteristic 0.

5.8.1 Small GK-dimension

As mentioned previously, connected Hopf algebras of GK-dimension at most four have been classified up to isomorphism in [62]. An obvious question is then to ask whether or not a similar classification can be achieved for quantum homogeneous spaces of low GK-dimension. As noted in §5.6, if $\mathrm{GKdim}\, T \leq 1$, then T is either k or $k[x]$, with x primitive. Beyond dimension 1, the question is open:

Question 5.8.1. What are the quantum homogeneous spaces T with $\operatorname{GKdim} T = 2$ in connected Hopf k-algebras H of finite GK-dimension?

A good first step towards answering Question 5.8.1 would be to classify all possible signatures of quantum homogeneous spaces T of GK-dimension at most two. In all known examples, such a quantum homogeneous space has signature $(1^{(2)})$ or $(1,2)$ (see Example 5.6.2 for instances of both).

5.8.2 Complete flags of quantum homogeneous spaces

If H is a *commutative* connected Hopf k-algebra of finite GK-dimension n, then, as we noted in Theorem 5.3.3, there is chain of $n+1$ coideal subalgebras (in fact Hopf subalgebras) from k to H. The same is not always true in the *cocommutative* case, where (by an easy argument making use of [7, §3.1, Examples (iii), (iv)]), such a flag exists if and only if $\mathfrak{g}$ has solvable radical $\mathfrak{r}$ with $\mathfrak{g}/\mathfrak{r}$ isomorphic to a finite direct sum of copies of $\mathfrak{sl}_2(k)$. Bearing these cases in mind and aiming to develop a generalisation of the solvable radical in the connected Hopf setting, one might propose the following:

Question 5.8.2. (i) Is there a good structure theory for connected Hopf k-algebras H which possess a complete flag of coideal subalgebras K_i,

$$k = K_0 \subset K_1 \subset \cdots \subset K_n = H,$$

with $K_i^+ H$ a Hopf ideal of H?

(ii) If H is *any* connected Hopf k-algebra of finite GK-dimension, does H have a maximal normal Hopf subalgebra R with the property (i)?

(iii) Given (ii), what can be said about the Hopf algebra H/R^+H?

(iv) How does the class of Hopf algebras in (i) compare with the IHOEs studied in §4.4?

Chapter 6

Primitively Thick Hopf Algebras

6.1 Introduction

For a connected Hopf algebra H (of finite GK-dimension) the following inequality always holds

$$\operatorname{GKdim} H \geq \dim_k P(H) \tag{6.1.1}$$

and this is an equality if and only if H is cocommutative, Proposition 4.2.11. Since the GK-dimension of a connected Hopf algebra H is always an integer value (or infinity) (Theorem 4.2.7), a natural starting point for the study of connected Hopf algebras H which are both noncommutative and noncocommutative is therefore the case where

$$\operatorname{GKdim} H = \dim_k P(H) + 1 < \infty. \tag{6.1.2}$$

If a Hopf algebra H satisfies Equation (6.1.2) and is connected, we say it is *primitively thick*. Primitively thick Hopf algebras were first investigated as a class of Hopf algebras in their own right in [62], although this terminology was not used there.

We begin with §6.2, which starts by listing some examples of primitively thick Hopf algebras (several of which we have already met in earlier chapters). We then proceed to summarise the results of the investigation into the properties of primitively thick Hopf algebras carried out by Wang, Zhang and Zhuang in [62]. Their main result in this direction, which we state as Theorem 6.2.12, is to prove that, for any primitively thick Hopf algebra H, there exists a canonical coassociative Lie algebra $P_2(H) \subset H$ such that $H = U_{CLA}(P_2(H))$ (see §6.2.1 for the relevant definitions).

In §6.3, using the results of [62] and some basic results from Lie theory, we derive some necessary conditions which a finite dimensional Lie algebra must satisfy in order to be the primitive space of a primitively thick Hopf algebra. In fact, our results will hold more generally for any connected Hopf algebra H such that $\dim_k P_2(H)/P(H) = 1$ (recall this equation always holds when H is primitively thick, Theorem 6.2.12 (1)). One consequence of these results is the following, which appears as Proposition 6.3.4.

Proposition 6.1.1. *(Proposition 6.3.4) Let H be a primitively thick Hopf algebra. Then $P(H)$ cannot be semisimple.*

In §6.4 we investigate the matter of listing large classes of examples of algebras which admit a primitively thick Hopf structure. By Theorem 6.2.12, all primitively thick Hopf algebras are enveloping algebras of finite dimensional coassociative Lie algebras. Thus, what we are in fact investigating in this section is which finite dimensional Lie algebras admit a coassociative Lie algebra structure whose enveloping algebra (with respect to the CLA structure) is a primitively thick Hopf algebra. Our main result will be to define, in terms of Lie-theoretic properties, a large class of Lie algebras (which includes all finite dimensional nilpotent Lie algebras) which possess this property and, in Proposition 6.4.2, describe a "recipe" which builds a primitively thick Hopf algebra from such a Lie algebra. Conversely, using results from [62], we show that all primitively thick Hopf algebras arise from this recipe. In the other direction, we use the restrictions imposed by Proposition 6.4.2 to prove in Proposition 6.4.5 that if $\mathfrak{g}$ is a semisimple Lie algebra, $U(\mathfrak{g})$ cannot admit the structure of a primitively thick Hopf algebra.

6.2 Preliminaries

We begin by listing some examples of primitively thick Hopf algebras.

Example 6.2.1.

1. The commutative connected Hopf algebra $H = \mathcal{O}(\mathbb{U}_3)$ from Example 2.4.6 is a primitively thick Hopf algebra of GK-dimension 3. To see why this is true, notice that the Hopf subalgebra generated by the primitive elements of H is isomorphic to a polynomial algebra in two variables, hence has GK-dimension 2.
2. Each member of either of the infinite families $A(\lambda_1, \lambda_2, \alpha)$ or $B(\lambda)$ of (generically) noncommutative, noncocommutative connected Hopf algebras (introduced in §4.3.3) is a primitively thick Hopf algebra of GK-dimension 3.
3. In Wang, Zhang and Zhuang's classification of connected Hopf algebras of GK-dimension 4 (see [62]), several non-isomorphic families of primitively thick Hopf

algebras of GK-dimension 4 are listed.

Remark 6.2.2. 1. We can reformulate the definition of a primitively thick Hopf algebra in terms of its signature (see Definition 5.7.3). Necessarily, if H is a connected Hopf algebra of GK-dimension n, then H is primitively thick if and only if $\sigma(H) = (1^{(n-1)}, m)$ for some $m > 1$. By Theorem 5.7.8 (4), $m = 2$. We deduce:

A connected Hopf algebra H with $\operatorname{GKdim} H = n$ is primitively thick if and only if
$$\sigma(H) = (1^{(n-1)}, 2).$$

2. From Example 6.2.1 (1) and (2), we see that every connected Hopf algebra of GK-dimension at most 3 is either cocommutative or primitively thick - that is, has signature $(1^{(3)})$ or $(1^{(2)}, 2)$. That this must be so is also evident from the numerical constraints in Theorem 5.7.8.

6.2.1 Coassociative Lie algebras

We recall the definition of a coassociative Lie algebra (CLA), as first defined in [61]. As we shall see in §6.2.2, they are intimately linked with primitively thick Hopf algebras.

Definition 6.2.3. [61, Definition 1] A Lie algebra $(L, [,])$ together with a coproduct $\delta : L \to L \otimes L$ is called a *coassociative Lie algebra* (CLA) if

1. (L, δ) is a coassociative coalgebra without counit,
2. δ and $[,]$ satisfy the following compatibility condition in the usual enveloping algebra $U(L)$ of the Lie algebra L:

$$\delta([a, b]) = [\delta(a), \delta(b)] + [\delta(a), 1 \otimes b + b \otimes 1] + [a \otimes 1 + 1 \otimes a, \delta(b)]$$

for all $a, b \in L$. Here $\delta(x) = x_1 \otimes x_2$ is the usual Sweedler notation.

Definition 6.2.4. [61, Definition 1.10] Let (L, δ) be a coassociative Lie algebra. We say L is *conilpotent* if $\delta^n(x) = 0$ for some $n > 0$.

Definition 6.2.5. [61, Definition 1.8] Let L be a CLA. The enveloping algebra of L, denoted by $U_{CLA}(L)$, is defined to be the bialgebra whose algebra structure is identical to that of the usual enveloping algebra of the Lie algebra L and whose coproduct and counit is defined as follows

$$\Delta(a) = 1 \otimes a + a \otimes 1 + \delta(a), \quad \epsilon(a) = 0,$$

for all $a \in L$.

Remark 6.2.6. Given an arbitrary CLA L, it will not always be the case that $U_{CLA}(L)$ is a Hopf algebra - see [61, Theorem 0.1] for details.

We do not discuss any general properties of CLAs (or their enveloping algebras) here (for a detailed account, see [61]). Instead, we proceed to describe their connection with primitively thick Hopf algebras.

6.2.2 Primitively thick Hopf algebras and CLAs

In this section we describe the connection between coassociative Lie algebras and primitively thick Hopf algebras.

Definition 6.2.7. ([62, Definition 2.4]) Let H be a Hopf algebra. Define a map

$$\delta_H : H \to H \otimes H$$

such that

$$\delta_H(h) = \Delta(h) - (1 \otimes h + h \otimes 1)$$

for all $h \in H$. Define now the space

$$P_2(H) = \{h \in H : \delta_H(h) \in P(H)^{\otimes 2},\ \tau\delta_H(h) = -\delta_H(h)\}.$$

Lemma 6.2.8. *[62, Lemma 2.5] Let H be a Hopf algebra.*

1. $\ker \delta_H = P(H) \subseteq P_2(H)$.

2. $(P_2(H), \delta_H)$ *is an* anti-cocommutative *($\tau\delta_H(x) = -\delta_H(x)$ for $x \in P_2(H)$) coassociative coalgebra (without counit).*

3. $P_2(H) \subseteq H_2$.

4. *For any $x, y \in H$,*

$$\delta_H([x, y]) = [\delta_H(x), 1 \otimes y + y \otimes 1] + [1 \otimes x + x \otimes 1, \delta_H(y)] + [\delta_H(x), \delta_H(y)].$$

Proof. Parts (1) and (2) are clear. For (3), if $x \in P_2(H)$,

$$\delta_H(x) \in P(H)^{\otimes 2} \subset H_1 \otimes H_1$$

so,

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \delta_H(x) \in H_0 \otimes H + H \otimes H_1$$

hence $P_2(H) \subseteq H_2$ by the definition of the coradical filtration. Part (4) is an elementary calculation using the definition, proved in the proof of [62, Lemma 2.5 (c)].

$\square$

In [62, Lemma 2.5] an extensive list of basic properties of $P_2(H)$ is given, which we reproduce here for the reader's convenience (but exclude the proofs, which are straightforward). We may drop any assumptions about the ground field in the following lemma.

Lemma 6.2.9. *([62, Lemma 2.5]) Let H be a connected Hopf algebra.*

1. *$(P_2(H), \delta_H)$ is the largest anti-cocommutative subcoalgebra of (H, δ_H).*
2. *$P_2(H)$ is a Lie subalgebra of H if and only if $[\delta_H(x), \delta_H(y)] = 0$ for all $x, y \in P_2(H)$.*
3. *Suppose $P(H)$ is abelian. Then $(P_2(H), [,])$ is a Lie subalgebra of H and $[P_2(H), P_2(H)] \subseteq P(H)$.*
4. *If $P_2(H)/P(H)$ is 1-dimensional, $P_2(H)$ is a Lie subalgebra of H.*
5. *$P_2(H)$ is a Lie-module over $P(H)$ (with respect to the adjoint action).*

Next, we focus on properties of $P_2(H)$ when H is a connected Hopf algebra. We resume our usual assumptions about the ground field for the rest of the section.

Notation 6.2.10. For any Hopf algebra H, let

$$p(H) = \dim_k P(H),\ p_2(H) = \dim_k P_2(H) \text{ and } p_2(H)' = \dim_k P_2(H)/P(H).$$

Lemma 6.2.11. *([62, Lemma 2.6]) Let H be a Hopf algebra. In parts $(3, 4, 5)$, assume that H is a connected Hopf algebra, with associated graded algebra $\operatorname{gr} H$ given by the coradical filtration.*

1. *$p_2(H) \leq \operatorname{GKdim} H$.*
2. *Let $\mathfrak{g}$ be a Lie algebra. Then $P_2(U(\mathfrak{g})_c) = P(U(\mathfrak{g})_c) = \mathfrak{g}$.*
3. *Let U be the Hopf subalgebra of H generated by $P(H)$. If $U \neq H$, then $P(H) \neq P_2(H)$ and $\operatorname{GKdim} U < \operatorname{GKdim} H$.*
4. *$P(H) \cong \operatorname{gr} H(1) = P(\operatorname{gr} H)$ as vector spaces.*
5. *$P_2(H) \cong P_2(\operatorname{gr} H)$ as vector spaces and $P_2(\operatorname{gr} H) \oplus \operatorname{gr} H(1)^2 = \operatorname{gr} H(1) \oplus \operatorname{gr} H(2)$.*

We now sketch a proof of the main result of this section. For a full proof, see [62].

Theorem 6.2.12. *[62, Theorem 2.7] Suppose H is a primitively thick Hopf algebra.*

1. *$(P_2(H), [,], \delta_H)$ is a CLA with $\dim_k P_2(H) = \dim_k P(H) + 1$.*
2. *$H = U_{CLA}(P_2(H))$.*

Proof. (Sketch) Suppose H is a primitively thick Hopf algebra. By Lemma 6.2.11 (1) and (3), $p_2(H) = p(H)+1$. Then, by Lemma 6.2.9 (4), $(P_2(H), [,])$ forms a Lie subalgebra of H. An elementary calculation (carried out explicitly in the proof of [62, Theorem 2.7]) proves that $(P_2(H), [,], \delta_H)$ is in fact an anti-cocommutative CLA. Since an anti-cocommutative CLA is conilpotent ([61, Lemma 2.8(b)]), it must be that $U_{CLA}(P_2(H))$ is a connected Hopf algebra sitting inside H, [61, Theorem 0.1]. Moreover,

$$\operatorname{GKdim} U_{CLA}(P_2(H)) = p_2(H) = \operatorname{GKdim} H$$

and hence $H = U_{CLA}(P_2(H))$ by Lemma 4.2.10, as required.

□

Remark 6.2.13. For an arbitrary connected Hopf algebra H, $P_2(H)$ is *not* necessarily a Lie algebra. An example of a Hopf algebra with this property is constructed in Chapter 7.

6.3 The primitive space

In this section we investigate the Lie-theoretic properties of $P(H)$ for a primitively thick Hopf algebra H.

Notation 6.3.1.

1. For any Lie algebra L and L-modules V and W, $V \otimes W$ becomes an L-module via the action

$$x \cdot (\sum v \otimes w) = \sum (v \otimes (x \cdot w)) + ((x \cdot v) \otimes w)$$

for $x \in L, v \in V, w \in W$.

2. For any Lie algebra $(L, [,])$, the vector space $L \otimes L$ becomes an L-module via the adjoint action:

$$x \cdot (\sum y \otimes z) = \sum y \otimes [x, z] + [x, y] \otimes z$$

for $x, y, z \in L$. Whenever we speak of a Lie algebra L and an L-submodule of L or $L \otimes L$, we mean with respect to the adjoint action.

3. For any vector space V, let $\wedge^2 V$ denote the subspace of $V^{\otimes 2}$ spanned by skew-symmetric elements. If V is a Lie algebra, $\wedge^2 V$ becomes a V-submodule of $V \otimes V$ via the adjoint action.

Lemma 6.3.2. *Suppose H is a connected Hopf algebra. Then the map $\delta_H : H \to H \otimes H$ is a $P(H)$-module map. Moreover, it defines a $P(H)$-submodule $\delta_H(P_2(H))$ of $\wedge^2 P(H)$ such that*

$$P_2(H)/P(H) \cong \delta_H(P_2(H))$$

as $P(H)$-modules.

Proof. First, recall that by Lemma 6.2.9, $P_2(H)$ is a $P(H)$-module under the adjoint action and if we restrict the map δ_H to this submodule it takes values in $\wedge^2 P(H)$. By Lemma 6.2.8 (4), if $p \in P(H), z \in H$, then, in the notation of Notation 6.3.1,

$$\delta_H([p,z]) = [1 \otimes p + p \otimes 1, \delta_H(z)] = p \cdot \delta_H(z)$$

hence δ_H is indeed a $P(H)$-module map. The required isomorphism now follows from the first isomorphism theorem and the fact that $\ker \delta_H = P(H)$. □

We'll now show that for any connected Hopf algebra H with $p_2(H)' = 1$, $P(H)$ cannot be a semisimple Lie algebra. First, some notation.

Notation 6.3.3. For any Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-module M,

$$M^{\mathfrak{g}} := \{m \in M : x \cdot m = 0 \ \forall x \in \mathfrak{g}\}.$$

Proposition 6.3.4. *Suppose H is a connected Hopf algebra with* $\operatorname{GKdim} H < \infty$ *and $p_2(H)' = 1$. Then $P(H)$ cannot be a semisimple Lie algebra.*

Proof. Suppose H is a connected Hopf algebra with $p_2(H)' = 1$ and $P(H)$ semsimple. By Lemma 6.3.2, $\delta_H(P_2(H))$ is a one-dimensional submodule of $\wedge^2 P(H)$. Since one dimensional modules of a semisimple Lie algebra $\mathfrak{g}$ are necessarily trivial, as $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, it must be that $\delta_H(P_2(H))$ is a one dimensional submodule of $(\wedge^2 P(H))^{P(H)}$. However, the semisimplicity of $P(H)$ implies

$$(\wedge^2 P(H))^{P(H)} \cong H^2(P(H)) = 0$$

where the isomorphism follows from [65, Corollary 7.7.3] and the equality is Whitehead's Second Lemma, [65, Corollary 7.8.12]. This is a contradiction, hence it must be that $P(H)$ cannot be semisimple. □

In the current literature, the only known examples of connected Hopf algebras with semisimple primitive spaces are the cocommutative enveloping algebras of semisimple Lie algebras. This observation, coupled with Proposition 6.3.4, prompts the following question.

Question 6.3.5. Suppose H is a connected Hopf algebra with $\operatorname{GKdim} H < \infty$. If $P(H)$ is a semisimple Lie algebra, is H necessarily cocommutative?

6.4 A recipe for constructing primitively thick Hopf algebras

In this section we investigate the matter of listing large classes of examples of algebras which admit a primitively thick Hopf structure. Our method will be to examine the Lie theoretic properties of $P_2(H)$ for a primitively thick Hopf algebra H.

Definition 6.4.1. Let $n \geq 3$. Suppose $\mathfrak{g}$ is an n-dimensional Lie algebra. We say $\mathfrak{g}$ is *admissible* if the following conditions are satisfied.

1. There exists a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\dim(\mathfrak{g}/\mathfrak{h}) = 1$.

2. If $\chi : \mathfrak{h} \to k$ denotes the character defining the 1-dimensional $\mathfrak{h}$-module $\mathfrak{g}/\mathfrak{h}$, there exists a 1-dimensional $\mathfrak{h}$-submodule of $\wedge^2\mathfrak{h}$ which is also defined by the character χ.

The (proof of the) following proposition describes a recipe for building a primitively thick Hopf algebra from any admissible Lie algebra.

Proposition 6.4.2. *Let $\mathfrak{g}$ be an admissible Lie algebra. Then there exists a primitively thick Hopf structure on $U(\mathfrak{g})$. Conversely, every primitively thick Hopf algebra arises from an admissible Lie algebra.*

Proof. Let $\mathfrak{g}$ an admissible Lie algebra. Let $\mathfrak{h}$ denote a Lie subalgebra of codimension 1 and let $\chi : \mathfrak{h} \to k$ denote the character defining the $\mathfrak{h}$-module $\mathfrak{g}/\mathfrak{h}$. We begin by proving that there exists a non-trivial map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ so that $(\mathfrak{g}, \delta)$ forms a CLA. By Definition 6.2.3, such a linear map δ must satisfy the following conditions:

1. δ is coassociative.

2. $\delta([a, b]) = [\delta(a), \delta(b)] + [\delta(a), 1 \otimes b + b \otimes 1] + [a \otimes 1 + 1 \otimes a, \delta(b)]$ for all $a, b \in \mathfrak{g}$.

Let $\{x_1, \ldots, x_{n-1}\}$ be a basis of $\mathfrak{h}$. Extend this to a basis $\{x_1, \ldots, x_{n-1}, z\}$ of $\mathfrak{g}$. Let $S = kw$ denote a 1-dimensional simple $\mathfrak{h}$-submodule of $\wedge^2\mathfrak{h}$ which is determined by the character $\chi : \mathfrak{h} \to k$. By definition we have that

$$x \cdot w = [1 \otimes x + x \otimes 1, w] = \chi(x)w \tag{6.4.1}$$

for all $x \in \mathfrak{h}$. Define now a map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\delta(x_i) = 0$$

for $i = 1, \ldots, n-1$ and

$$\delta(z) = w.$$

To prove $(\mathfrak{g}, \delta)$ is a CLA, it remains to verify conditions (1) and (2) above. Condition (1), the coassociativity of δ, is straightforward. Since $\{x_1, \ldots, x_{n-1}, z\}$ is a basis of $\mathfrak{g}$, it remains to check condition (2) for each pair (x_i, x_j) and (x_i, z), for $i, j \in \{1, \ldots, n-1\}$. Since $\delta(x_i) = 0$ for each i, this is trivial for each pair (x_i, x_j). To check (2) for each pair (x_i, z), it suffices to check that

$$\delta([x_i, z]) = [1 \otimes x_i + x_i \otimes 1, \delta(z)] = x_i \cdot \delta(z).$$

In other words, that δ is a $\mathfrak{g}$-module homomorphism, which is clear from its definition. This verifies condition (2), hence $(\mathfrak{g}, \delta)$ forms a CLA.

Consider now the usual enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Define a map

$$\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}).$$

such that

$$x \mapsto 1 \otimes x + x \otimes 1 + \delta(x).$$

By [61, Lemma 1.3], since δ is coassociative, so too is Δ, and since δ satisfies condition (2) above, Δ becomes an algebra homomorphism. In this way $H := (U(\mathfrak{g}), \Delta)$ becomes a Hopf algebra (where the algebra, unit and counit maps are defined in the same way as in the cocommutative case). With respect to this comultiplication, it is easy to see that $P(H) = \mathfrak{h}$ and hence that

$$\operatorname{GKdim} H = \dim \mathfrak{g} = \dim \mathfrak{h} + 1 = \dim(P(H)) + 1.$$

In other words, $U(\mathfrak{g})$ admits a primitively thick Hopf structure, as required.

For the converse, let H be a primitively thick Hopf algebra. By Theorem 6.2.12, $H = U(P_2(H))$, where $P_2(H)$ is a finite dimensional CLA with $P(H)$ a subalgebra of codimension 1. We aim to prove that $P_2(H)$ is admissible. Let $\chi : P(H) \to k$ be the character defining the 1-dimensional $P(H)$-module $P_2(H)/P(H)$. Choosing $0 \neq z \in P_2(H) \setminus P(H)$, we have by Lemma 6.3.2 that $0 \neq w := \delta_H(z) \in \wedge^2 P(H)$. It remains to prove that this 1-dimensional submodule is also defined by the character $\chi : P(H) \to k$, i.e., that

$$[1 \otimes x + x \otimes 1, w] = \chi(x)w \tag{6.4.2}$$

for $x \in P(H)$. We have that $(P_2(H), \delta_H)$ is a CLA, so that δ_H satisfies equation (2) above. Moreover, $P_2(H)/P(H) = k\bar{z}$ is a one-dimensional $P(H)$-module defined by χ, hence, for any $x \in P(H)$ there exists some $h \in P(H)$ such that $[x, z] = \chi(x)z + h$. Thus,

$$\begin{aligned}
&[1 \otimes x + x \otimes 1, \delta_H(z)] \\
&= \delta_H([x, z]) - [\delta_H(x), \delta_H(z)] - [\delta_H(x), 1 \otimes z + z \otimes 1] \\
&= \delta_H([x, z]) = \delta_H(\chi(x)z + h) \\
&= \chi(x)\delta_H(z)
\end{aligned}$$

as required. □

Corollary 6.4.3. *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra (of dimension greater than 2). Then $U(\mathfrak{g})$ admits the structure of a primitively thick Hopf algebra.*

Proof. Let $\mathfrak{g}$ be an n-dimensional nilpotent Lie algebra, where $n \geq 3$. By Proposition 6.4.2, it suffices to prove that $\mathfrak{g}$ is admissible. Since $\mathfrak{g}$ is nilpotent, $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$. Choose any subspace A of codimension 1 in $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$. Thus A is an ideal of $\mathfrak{g}$ of codimension 1. Since A is an ideal, the A-module action on $\mathfrak{g}/A$ is trivial, so it remains to find a one dimensional trivial A-submodule of $\wedge^2 A$.

By Lie's theorem there exists a 1-dimensional A-submodule of $\wedge^2 A$. In other words, there exists some $w = w_1 \otimes w_2 \in \wedge^2 A$ and some character $\chi : A \to k$ such that

$$[x, w_1] \otimes w_2 + w_1 \otimes [x, w_2] = \chi(x)w$$

for all $x \in A$. Since A is nilpotent, there exists some $m \in \mathbb{N}$ such that $\mathrm{ad}^m(x)(w_i) = 0$. Thus $\chi^m(x) = 0$ and hence $\chi(x) = 0$ for each $x \in A$. This proves kw forms a 1-dimensional trivial submodule of $\wedge^2 A$, as required.

□

In [61, Theorem 3.9] it is proved that the only CLA structure on $\mathfrak{sl}_2(k)$ is the trivial one. This implies in particular that there can be no primitively thick structure on $U(\mathfrak{sl}_2(k))$. We prove that this latter statement holds for the enveloping algebra of *any* finite dimensional semisimple Lie algebra. First, a preparatory lemma. The following result is originally due to Golubitsky, [19], in the language of Lie groups. A more accessible algebraic version of this lemma, which we state below, appears as [24, Lemma 1.2.1].

Lemma 6.4.4. *(Golubitsky) Suppose $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h}$ is a maximal Lie subalgebra of $\mathfrak{g}$ which contains no non-trivial ideals of $\mathfrak{g}$. Then either $\mathfrak{g} \cong \mathfrak{sl}_2(k)$ or there exists a simple Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ such that $\mathfrak{g} \cong \mathfrak{g}_0 \oplus \mathfrak{g}_0$ and $\mathfrak{h} = \{(x, x) : x \in \mathfrak{g}_0\} \cong \mathfrak{g}_0$.*

Proposition 6.4.5. *Suppose $\mathfrak{g}$ is a semisimple Lie algebra. Then $U(\mathfrak{g})$ cannot admit the structure of a primitively thick Hopf algebra.*

Proof. Let $\mathfrak{g} = \bigoplus_{i=1}^{t} \mathfrak{g}_i$, where $\mathfrak{g}_i$ is a simple ideal for $1 \leq i \leq t$. Let $H = U(\mathfrak{g})$ and suppose for a contradiction that H admits a primitively thick Hopf structure. It follows then that $\mathfrak{g}$ contains a Lie subalgebra $\mathfrak{h} = P(H)$ with $\dim_k \mathfrak{g}/\mathfrak{h} = 1$. Let $\mathfrak{s} = \bigoplus_{i=1}^{s} \mathfrak{g}_i$ be the largest ideal of $\mathfrak{g}$ inside $\mathfrak{h}$. Define $\bar{\mathfrak{h}} = \mathfrak{h}/\mathfrak{s}$ and $\mathfrak{g} = \mathfrak{g}/\mathfrak{s}$. Then $\bar{\mathfrak{h}}$ is a maximal Lie subalgebra of $\bar{\mathfrak{g}}$. By Lemma 6.4.4, it must therefore be the case that either $\bar{\mathfrak{g}} = \mathfrak{sl}_2(k)$ or that $\bar{\mathfrak{g}} \cong \bar{\mathfrak{h}} \oplus \bar{\mathfrak{h}}$. Clearly the fact that $\dim_k \bar{\mathfrak{g}}/\bar{\mathfrak{h}} = 1$ prohibits the possibility that $\bar{\mathfrak{g}} \cong \bar{\mathfrak{h}} \oplus \bar{\mathfrak{h}}$, thus we have

$$\mathfrak{g} = \bigoplus_{i=1}^{t-1} \mathfrak{g}_i \oplus \mathfrak{sl}_2(k)$$

and

$$\mathfrak{h} = \bigoplus_{i=1}^{t-1} \mathfrak{g}_i \oplus \mathfrak{b}$$

where $\mathfrak{b}$ denotes a 2-dimensional borel subalgebra of $\mathfrak{sl}_2(k)$. Now $\mathfrak{g}/\mathfrak{h}$ is a one-dimensional $\mathfrak{h}$-module on which $\mathfrak{s}$ acts trivially (since it is an ideal of $\mathfrak{g}$ inside $\mathfrak{h}$), and $\mathfrak{b}$ acts with weight 2 or -2, depending on the choice of $\mathfrak{b}$. Without loss of generality, assume $\mathfrak{b}$ acts with weight 2 on $\mathfrak{g}/\mathfrak{g}$. Then, considering $\wedge^2\mathfrak{h}$ as an $\mathfrak{h}$-module via the adjoint action, since $[\mathfrak{b}, \mathfrak{s}] = 0$, $\mathfrak{b}$ is acting with weight -2. Thus $\mathfrak{g}/\mathfrak{h}$ has no submodule isomorphic to $\wedge^2\mathfrak{h}$, contradicting Proposition 6.4.2, hence there is no primitively thick structure on H.

□

Proposition 6.4.5 prompts the following question, which we leave open.

Question 6.4.6. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Can $U(\mathfrak{g})$ admit any connected Hopf algebra structure, other than the usual cocommutative one?

Chapter 7

A New Example

7.1 Introduction

In [62], Wang, Zhang and Zhuang classify (up to isomorphism) all connected Hopf algebras of GK-dimension at most four, over an algebraically closed field of characteristic 0. Moreover, they are able to show that the Hopf algebras appearing in this classification all have one curious feature in common: each is isomorphic, as an algebra, to the enveloping algebra of a finite dimensional Lie algebra (we saw this explicitly in §4.3.3 for connected Hopf algebras of GK-dimension at most three). The same can also be said for any commutative or cocommutative connected Hopf algebra of arbitrary (finite) GK-dimension. In the latter case this follows from the Cartier-Milnor-Moore theorem (Theorem 3.4.5), and in the former it follows from Theorem 3.3.6, which states that any connected commutative Hopf algebra of finite GK-dimension is a polynomial algebra (hence isomorphic, as an algebra, to an enveloping algebra of an abelian algebra). It is asked in [3, Question L] whether connected Hopf algebras of finite GK-dimension are always isomorphic as algebras to enveloping algebras of Lie algebras. The main result of this section, which appears as Theorem 7.2.6, gives a negative answer to this question.

Theorem 7.1.1. *There exists a connected Hopf algebra L with* $\operatorname{GKdim} L = 5$ *which is not isomorphic, as an algebra, to the enveloping algebra of a Lie algebra. Furthermore, this Hopf algebra is minimal (with respect to GK-dimension) with this property.*

The Hopf algebra L appearing in Theorem 7.1.1 has further attractive properties.

Theorem 7.1.2. *Retain the notation of Theorem 7.1.1.*

1. *L admits a connected algebra (in fact, Hopf) grading.*

2. *L is an IHOE.*

3. *L is a (Hopf) quotient of a Hopf algebra which is isomorphic, as an algebra, to an enveloping algebra of a Lie algebra.*

Hopf algebras which admit a connected algebra grading are investigated in [5], and the results of this section appear as [5, §5]. Among other things, it was demonstrated in that paper that such Hopf algebras share many algebraic properties in common with enveloping algebras of finite dimensional nilpotent Lie algebras, and in fact the Hopf algebra L appearing in Theorem 7.1.1 is the first, and so far only, known example of a Hopf algebra of finite GK-dimension which is connected graded as an algebra, but which isn't isomorphic to an enveloping algebra of a Lie algebra. It is currently unknown, and asked as a question in [5], whether or not any Hopf algebra which admits a connected algebra grading is necessarily connected as a Hopf algebra.

7.2 The construction

The main result of this section is Theorem 7.2.6, which answers a number of open questions by giving an example of a connected Hopf algebra L with $\operatorname{GKdim} L = 5$, which is also connected graded as a (Hopf) algebra, but which is not isomorphic to the enveloping algebra of a Lie algebra. The construction proceeds by way of another connected Hopf algebra, denoted by J, which also has interesting features.

7.2.1 Algebra J

First, take two copies of the Heisenberg Lie algebra of dimension 3. Thus, let

$$\mathfrak{h}_1 = ka \oplus kb \oplus kc$$

be the Lie algebra with $[a, b] = c$ and $c \in Z(\mathfrak{h}_1)$. Let

$$\mathfrak{h}_2 = kz \oplus kw \oplus kd$$

where $[z, w] = d$ and $d \in Z(\mathfrak{h}_2)$. Set $J := U(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. By the Poincaré-Birkhoff-Witt theorem J has basis $\mathcal{B}$ consisting of the ordered monomials in

$$\{a, b, c, z, w, d\},$$

which we call the *PBW-generators* of J. Since J is an enveloping algebra it comes equipped with a natural cocommutative coproduct. We show next that there also exists a non-cocommutative Hopf structure on the algebra J.

7.2.2 Algebra J: defining a new coproduct

We now define a non-cocommutative bialgebra structure (J, Δ, ϵ).

Lemma 7.2.1. *Retain the above notation. Then there exist algebra homomorphisms* $\Delta : J \to J \otimes J$ *and* $\epsilon : J \to k$ *such that*

$$\Delta(a) = 1 \otimes a + a \otimes 1; \quad \Delta(b) = 1 \otimes b + b \otimes 1; \quad \Delta(c) = 1 \otimes c + c \otimes 1; \tag{E4.1.1}$$

$$\Delta(z) = 1 \otimes z + a \otimes c - c \otimes a + z \otimes 1; \tag{E4.1.2}$$

$$\Delta(w) = 1 \otimes w + b \otimes c - c \otimes b + w \otimes 1; \tag{E4.1.3}$$

$$\Delta(d) = 1 \otimes d + c \otimes c^2 + c^2 \otimes c + d \otimes 1; \tag{E4.1.4}$$

and

$$\epsilon(a) = \epsilon(b) = \epsilon(c) = \epsilon(z) = \epsilon(w) = \epsilon(d) = 0. \tag{E4.1.5}$$

With these definitions (J, Δ, ϵ) *is a bialgebra.*

The proof of the lemma is a straightforward but long computation. The interested reader can read it in the Appendix, §7.3.

Proposition 7.2.2. *Retain the above notation. Then* J *is a noncommutative, noncocommutative connected Hopf algebra with* GKdim $J = 6$.

Proof. From Section 7.2.1, J has a PBW-basis $\mathcal{B}$. Define the degree of a PBW-monomial $p = a^{n_1} b^{n_2} c^{n_3} z^{n_4} w^{n_5} d^{n_6} \in \mathcal{B}$ to be

$$N(p) := \sum_{i=1}^{3} n_i + 2(n_4 + n_5) + 3n_6.$$

Define $F_0 = k$, and for each $n \geq 1$ let F_n be the vector space spanned by all monomials in $\mathcal{B}$ of degree at most n. In particular, $z, w \in F_2, d \in F_3$, and

$$\Delta(z) = 1 \otimes z + z \otimes 1 + (a \otimes c - c \otimes a) \in F_0 \otimes F_2 + F_1 \otimes F_1 + F_0 \otimes F_2; \tag{E4.2.2}$$

$$\Delta(w) = 1 \otimes w + w \otimes 1 + (b \otimes c - c \otimes b) \in F_0 \otimes F_2 + F_1 \otimes F_1 + F_0 \otimes F_2; \tag{E4.2.3}$$

$$\Delta(d) = 1 \otimes d + d \otimes 1 + c^2 \otimes c + c \otimes c^2 \in \sum_{i=0}^{3} F_i \otimes F_{3-i}. \tag{E4.2.4}$$

Claim: $\{F_n\}$ is an algebra and a coalgebra filtration of J.

That $\{F_n\}$ is an exhaustive vector space filtration of J is immediate. That it is then an algebra filtration is clear from the defining relations of J. Thus it suffices to prove $\{F_n\}$ is a coalgebra filtration. To this end, let $p = a^{n_1}b^{n_2}c^{n_3}z^{n_4}w^{n_5}d^{n_6} \in \mathcal{B}$ be a PBW monomial with $N(p) = n$. We claim

$$\Delta(p) \in \sum_{i=0}^{n} F_i \otimes F_{n-i}. \tag{E4.2.5}$$

Indeed, noting (E4.2.2), (E4.2.3) and (E4.2.4), we have

$$\begin{aligned}
\Delta(p) &= \Delta(a)^{n_1}\Delta(b)^{n_2}\Delta(c)^{n_3}\Delta(z)^{n_4}\Delta(w)^{n_5}\Delta(d)^{n_6} \\
&\subseteq \left(\sum_{j_1=0}^{1} F_{j_1} \otimes F_{1-j_1}\right)^{n_1} \cdots \left(\sum_{j_6=0}^{1} F_1 \otimes F_{1-j_3}\right)^{n_3} \\
&\quad \left(\sum_{j_4=0}^{2} F_{j_4} \otimes F_{2-j_4}\right)^{n_4} \left(\sum_{j_5=0}^{2} F_{j_5} \otimes F_{2-j_5}\right)^{n_5} \left(\sum_{j_6=0}^{3} F_{j_6} \otimes F_{3-j_6}\right)^{n_6} \\
&\subseteq \left(\sum_{j_1=0}^{n_1} F_{j_1} \otimes F_{n_1-j_1}\right) \cdots \left(\sum_{j_3=0}^{n_3} F_{j_3} \otimes F_{n_3-j_3}\right) \\
&\quad \left(\sum_{j_4=0}^{2n_4} F_{j_4} \otimes F_{2n_4-j_4}\right) \left(\sum_{j_5=0}^{2n_5} F_{j_5} \otimes F_{2n_5-j_5}\right) \left(\sum_{j_6=0}^{3n_6} F_{j_6} \otimes F_{3n_6-j_6}\right) \\
&\subseteq \sum_{i=0}^{n} F_i \otimes F_{n-i}
\end{aligned}$$

where the last two inclusions follow from the fact that $\{F_n\}$ is an algebra filtration. This proves (E4.2.5). By [41, Lemma 5.3.4], it follows that $J_0 \subset F_0 = k$, and hence J is a connected bialgebra. By Lemma 3.2.7, J is thus a connected Hopf algebra. Moreover, since J is, as an algebra, the enveloping algebra of a Lie algebra of dimension 6, GKdim $J =$ GKdim $U(\mathfrak{g}) = 6$ by Example 2.5.7. □

7.2.3 Definition of L

Let J be the Hopf algebra defined in Section 7.2.2. Using the defining relations of J and the definition of its coproduct, a straightforward calculation shows that $d - \frac{1}{3}c^3$ is a nonzero primitive element central element of J. Let I be the principal ideal $(d - \frac{1}{3}c^3)J$ of J, and define $L := J/I$. Recall that an ideal S in a ring R is called *completely prime* if R/S is a domain.

Proposition 7.2.3. *Retain the above notation. Then L is a connected Hopf algebra with* GKdim $L = 5$*, and I is a completely prime Hopf ideal of J.*

Proof. From the above comments and since $\epsilon(I) = 0$, I is a Hopf ideal of J and hence L is a Hopf algebra, and is connected because it is a factor of J, which is connected by Proposition 7.2.2. By Proposition 4.2.12, L, being connected, is a domain. Hence I is a completely prime ideal.

It remains to show that $\operatorname{GKdim} L = 5$. It is easy to check that $I' = \langle d, c\rangle$ is a ideal of J and that the factor ring $L' := J/(d, c)$ is isomorphic as an algebra to the enveloping algebra of the Lie algebra $\mathfrak{g}/(kd + kc)$. By Example 2.5.7, $\operatorname{GKdim} L' = 6 - 2 = 4$. We have natural algebra surjections of domains $J \to L \to L'$, so that

$$6 = \operatorname{GKdim} J \geq \operatorname{GKdim} L \geq \operatorname{GKdim} L' = 4. \tag{E4.3.1}$$

Moreover, since L is a proper factor of the domain J, the first inequality above is strict [27, Proposition 3.15]. Similarly, L' is a proper factor of L, since a short exercise with the PBW monomials $\mathcal{B}$ shows that $c \notin \langle d - \frac{1}{3}c^3\rangle$. Thus the second inequality in (E4.3.1) is also strict. As $\operatorname{GKdim} L \in \mathbb{Z}$ by Theorem 4.2.7, it follows that $\operatorname{GKdim} L = 5$. $\square$

Of course, the fact that I is completely prime can easily be proved by a direct ring theoretic argument, or, alternatively, one could use the fact that $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a nilpotent Lie algebra, and apply [36, Theorem 14.2.11].

Since $L' = L/cL$ and L' is an enveloping algebra of a Lie algebra with basis the images of a, b, z, w, it is easy to deduce that L also has a PBW-basis, namely the ordered monomials in $\mathcal{B}_L := \{a, b, c, z, w\}$. Here and below, we are abusing notation by omitting "bars" above these elements.

For the reader's convenience, we give an explicit presentation for the Hopf algebra L in the remark below. To calculate the effect of the antipode on its algebra generators, we use the fact that $m_L(S_L \otimes \mathrm{id})\Delta = \epsilon$ (where $m_L : L \otimes L \to L$ denotes the multiplication map).

Remark 7.2.4. Retain the above notation. Then L is the connected Hopf algebra with algebra generators a, b, c, z, w subject to the relations

$$[a, c] = [a, z] = [a, w] = [b, c] = [b, z] = [b, w] = [c, z] = [c, w] = 0$$

and

$$[a, b] = c, \quad [z, w] = \frac{1}{3}c^3$$

with coproduct $\Delta : L \to L \otimes L$, counit $\epsilon : L \to k$ and antipode $S : L \to L$ defined on generators as follows:

$$\Delta(a) = 1 \otimes a + a \otimes 1, \quad \Delta(b) = 1 \otimes b + b \otimes 1, \quad \Delta(c) = 1 \otimes c + c \otimes 1,$$

$$\Delta(z) = 1 \otimes z + a \otimes c - c \otimes a + z \otimes 1,$$

$$\Delta(w) = 1 \otimes w + b \otimes c - c \otimes b + w \otimes 1,$$

and

$$S(x) = -x; \quad \epsilon(x) = 0$$

for $x \in \{a, b, c, z, w\}$.

7.2.4 Properties of L

For the definition and basic properties of the *signature* of a connected Hopf algebra of finite GK-dimension, see Definition 5.7.3; the signature records the degrees of the homogeneous generators of the commutative polynomial algebra $\operatorname{gr} H$, the associated graded algebra of H with respect to its coradical filtration. An *iterated Hopf Ore extension (IHOE)* is a Hopf algebra constructed as a finite ascending sequence of Hopf subalgebras, each an Ore extension of the preceding one - see §4.4. Recall a Hopf algebra is said to be *involutory* if its antipode squares to the identity.

Lemma 7.2.5. *Retain the above notation.*

(1) L is an involutory IHOE.

(2) $\operatorname{GKdim} L = 5$.

(3) $\sigma(L) = (1^{(3)}, 2^{(2)})$.

(4) The centre $Z(L)$ of L is $k[c]$.

(5) L is connected graded as a Hopf algebra.

Proof. 1. Adjoin the generators of L in the order c, a, b, z, w to produce an iterated Ore extension

$$L(0) = k, \quad L(1) = L(0)[c], \quad L(2) = L(1)[a],$$
$$L(3) = L(2)[b; -c\frac{\partial}{\partial a}], \quad L(4) = L(3)[z], \quad L(5) = L(4)[w; -\frac{1}{3}c^3\frac{\partial}{\partial z}].$$

It is clear from the definition of the coproduct in §7.2.2 that each step in this construction gives a Hopf subalgebra of L; that is, it yields an iterated Hopf Ore extension. That L is involutory is immediate from the calculation of the antipode as given in Remark 7.2.4.

2. This follows from (1) and [7, Theorem 2.6].

3. Since a, b, c are primitive, the signature $\sigma(L)$ contains $(1^{(3)})$. By the definition of z and w, these two elements are linearly independent in $P_2(L)/P(L)$ (see the definitions given in §6.2.2). Thus, by [62, Lemma 2.6(e)], z and w are linearly independent in $P_2(\operatorname{gr} L)/P(\operatorname{gr} L)$. Hence the signature $\sigma(L)$ contains $(2^{(2)})$. Since $\operatorname{GKdim} L = 5$, the assertion follows.

4. Clearly, $k[c] \subseteq Z(L)$. For the reverse inclusion, from (1), we can write L as a free (left, say) $k[c]$-module, with $k[c]$-basis

$$\{a^{n_1} b^{n_2} z^{n_3} w^{n_4} : (n_1, n_2, n_3, n_4) \in \mathbb{Z}^{(4)}_{\geq 0}\}.$$

Define a lexiographic ordering on $\mathbb{Z}^{(4)}_{\geq 0}$, so that, for $\underline{i}, \underline{i}' \in \mathbb{Z}^{(4)}_{\geq 0}$, $\underline{i} > \underline{i}'$ if $i_1 > i'_1$. If $i_1 = i'_1$, $\underline{i} > \underline{i}'$ if $i_2 > i'_2$, and so on. Let $Z = Z(L)$ and suppose for a contradiction that $h \in Z \setminus k[c]$. Write h in terms of the $k[c]$-basis above, so that,

$$h = \sum_{\underline{i}} \alpha_{\underline{i}} a^{i_1} b^{i_2} z^{i_3} w^{i_4} \tag{7.2.1}$$

for some $\alpha_{\underline{i}} \in k[c]$, $\underline{i} = (i_1, i_2, i_3, i_4) \in \mathbb{Z}^{(4)}_{\geq 0}$, where $\underline{0} = (0,0,0,0)$ and $\alpha_{\underline{0}} = 0$. With respect to our ordering on $\mathbb{Z}^{(4)}_{\geq 0}$, let $\alpha_{\underline{i}'}$ be the least non-zero term in (7.2.1), where $\underline{i}' = (i'_1, i'_2, i'_3, i'_4) \neq \underline{0}$. By definition,

$$[a, -] = c\frac{\partial}{\partial b}, \quad [b, -] = -c\frac{\partial}{\partial a}, \quad [z, -] = \frac{1}{3}c^3\frac{\partial}{\partial w}, \quad [w, -] = -\frac{1}{3}c^3\frac{\partial}{\partial z}.$$

Let i'_j be the first non-zero entry in $\underline{i}'$. We consider four separate cases. Suppose $i'_j = i'_1$. Then

$$0 = [b, h] = -\sum_{\underline{i}} i_1 \alpha_{\underline{i}} a^{i_1 - 1} b^{i_2} z^{i_3} w^{i_4}.$$

If $i_j = i'_2$,

$$0 = [a, h] = \sum_{\underline{i}} \alpha_{\underline{i}} i_2 a^{i_1} b^{i_2 - 1} z^{i_3} w^{i_4}.$$

If $i_j = i'_3$,

$$0 = [w, h] = -\frac{1}{3}\sum_{\underline{i}} i_3 c^3 \alpha_{\underline{i}} a^{i_1} b^{i_2} z^{i_3 - 1} w^{i_4}.$$

If $i_j = i'_4$,

$$0 = [z, h] = \frac{1}{3} \sum_{\underline{i}} i_4 c^3 \alpha_{\underline{i}} a^{i_1} b^{i_2} z^{i_3} w^{i_4 - 1}.$$

Since L is a free $k[c]$-module on the stated basis, if we examine the $\underline{i'}$ in each of these four equations, linear independence yields that $\alpha_{\underline{i'}} = 0$ in each case. This is a contradiction, completing the proof.

5. Assign degrees to the generators as follows:

$$\deg a = \deg b = 1;$$
$$\deg z = \deg w = 3;$$
$$\deg c = 2.$$

Then it is easy to check from the relations that L is then connected graded as an algebra. Checking the definition of Δ on the generators, one finds that L is a graded Hopf algebra with these assigned degrees.

□

We can now prove the main result of this section. For convenience of reference, we restate in it parts of Lemma 7.2.5.

Theorem 7.2.6. *Let L be the algebra defined in 7.2.3.*

(1) L is an involutory connected Hopf algebra.

(2) L is connected graded as a Hopf algebra.

(3) As an algebra, L is not isomorphic to $U(\mathfrak{n})$ for any finite dimensional Lie algebra $\mathfrak{n}$.

(4) L is an IHOE.

(5) $\operatorname{GKdim} L = 5$.

(6) The signature $\sigma(L)$ of L is $(1^{(3)}, 2^{(2)})$.

Proof. It remains only to prove (3). Let $\mathfrak{n}$ be a five-dimensional Lie algebra, and write $U := U(\mathfrak{n})$. Suppose that, as algebras,

$$L \cong U \tag{E4.5.1}$$

via the map $\theta : L \to U$. Easy calculations show that the commutator ideal $\langle [L, L] \rangle$ of L is the principal ideal cL, and hence that $L/\langle [L, L] \rangle$ is a commutative polynomial algebra in 4 variables. By (E4.5.1), since $U/\langle[U,U]\rangle \cong U/([\mathfrak{n},\mathfrak{n}]U) \cong U(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])$ (the first isomorphism following from the fact that the objects on both the left and right of the isomorphism are the largest abelian factors of U),

$$\dim_k(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]) = 4. \tag{E4.5.2}$$

So $\dim_k([\mathfrak{n},\mathfrak{n}]) = 1$. Write $[\mathfrak{n},\mathfrak{n}] = kx$ for some $x \in \mathfrak{n}$. Then the isomorphism (E4.5.1) takes $\langle[L,L]\rangle = cL$ to $\langle[U,U]\rangle = xU$. Since L and U are domains with only scalars as units, we may assume without loss of generality that $\theta(c) = x$. In particular, $[\mathfrak{n},\mathfrak{n}]$ is a 1-dimensional space contained within $Z(\mathfrak{n})$. More precisely, from Lemma 7.2.5(4), $Z(\mathfrak{n}) = kx$. That is,

$$\mathfrak{n} \textit{ is nilpotent of class 2, with } Z(\mathfrak{n}) = kx. \tag{E4.5.3}$$

By the classification of 5-dimensional nilpotent Lie algebras (see [20], for example) (or use linear algebra), there is up to isomorphism precisely one Lie algebra of dimension 5 satisfying (E4.5.3). Namely, $\mathfrak{n}$ has a basis $\{x, x_1, x_2, x_3, x_4\}$ such that

$$[x_1, x_2] \quad = \quad x \quad = \quad [x_3, x_4],$$

and all other brackets are 0.

Let L^+ denote the augmentation ideal of L with respect to its given Hopf algebra structure. Since the winding automorphisms of U (and similarly those of L) transitively permute the character ideals of U (and, respectively, of L), we can follow the map θ of (E4.5.1) by an appropriate winding automorphism of U, and thus assume without loss of generality that

$$\theta(L^+) = \mathfrak{n}U = \langle x, x_1, x_2, x_3, x_4\rangle =: I. \tag{E4.5.4}$$

Define now

$$B := L^+/(L^+)^3 \textit{ and } S := I/I^3. \tag{E4.5.5}$$

By (E4.5.1) and (E4.5.4), θ induces an isomorphism (of algebras without identity) of B with S. We shall show however that

$$\dim_k(Z(B)) = 13, \textit{ and } \dim_k(Z(S)) = 11. \tag{E4.5.6}$$

This is manifestly a contradiction, so (E4.5.1) must be false, and the proof is complete. It remains therefore to prove (E4.5.6). This is done in Proposition 7.2.7 below. □

Proposition 7.2.7. *Retain the notation introduced in Theorem 7.2.6.*

(1) $\dim_k(B) = \dim_k(S) = 15$.

(2) $\dim_k(Z(B)) = 13$.

(3) $\dim_k(Z(S)) = 11$.

Proof. (1) To prove that

$$\dim_k(B) = 15, \tag{E4.6.1}$$

we show that

$$B \textit{ has basis (the images of) } \mathcal{B}_B, \tag{E4.6.2}$$

where

$$\mathcal{B}_B := \{a, b, c, z, w, ab, az, aw, a^2, b^2, bz, bw, z^2, zw, w^2\}.$$

Since (abusing notation regarding images), $L/cL \cong k[a, b, z, w] =: T$, it follows that

$$\begin{aligned} T^+/(T^+)^3 &= L^+/((L^+)^3 + cL) \\ &= L^+/((L^+)^3 + ck) \\ &= B/\overline{cL}, \end{aligned}$$

where $\overline{cL}$ is the image of cL in B, and where the second equality holds since c multiplied by any element of L^+ is in $(L^+)^3$. Now $T^+/(T^+)^3$ is clearly a vector space of dimension 14 with basis the image of $\mathcal{B}_B \setminus \{c\}$.

Moreover,

$$\overline{cL} = ((L^+)^3 + cL)/(L^+)^3 = ((L^+)^3 + ck)/(L^+)^3,$$

so that $\dim_k(cB) \leq 1$. Thus, to prove (E4.6.1) it remains to show that

$$c \notin (L^+)^3. \tag{E4.6.3}$$

By definition, $(L^+)^3$ is the k-span of the words of length at least 3 in $\{a, b, c, z, w\}$. By a routine straightening argument using the defining relations for L, using for definiteness the ordering $\{a, b, c, z, w\}$ of the PBW-generators, we calculate that

$$(L^+)^3 = \mathcal{S} + \mathcal{C}, \tag{E4.6.4}$$

where

$$\mathcal{S} := k - \textit{span of ordered words in } a, b, z, w \textit{ of length at least } 3,$$

and

$$\mathcal{C} := c \times \{k - \textit{span of non-empty ordered words in } a, b, c, z, w\}.$$

By the linear independence part of the PBW-theorem for L, which holds because L is an IHOE, Lemma 7.2.5(1), (E4.6.3) now follows from (E4.6.4).

The argument for S is similar to the one for B. In particular, one shows that S has basis

$$\mathcal{B}_S := \{x, x_1, x_2, x_3, x_4\} \cup \{\textit{ordered words of length 2 in } x_i : 1 \leq i \leq 4\}.$$

(2) By inspection,

$$k - \textit{span}\{\mathcal{B}_B \setminus \{a, b\}\} \subset Z(B).$$

Let $f = \lambda a + \mu b + d \in B$, where $d \in k - \textit{span}\{\mathcal{B} \setminus \{a, b\}\}$. Then, if $f \in Z(B)$, $0 = [a, f] = \mu c$. Hence, $\mu = 0$. Similarly, $\lambda = 0$, and (2) is proved.

(3) By inspection,

$$\mathcal{B}_S \setminus \{x_1, x_2, x_3, x_4\} \subset Z(S).$$

Let $f = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 + d$, where $d \in \mathcal{B}_S \setminus \{x_1, x_2, x_3, x_4\}$. If $f \in Z(B)$, $0 = [x_2, f] = \lambda_1 x + [x_2, d] = \lambda_1 x$, hence $\lambda_1 = 0$. Similarly, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, and (3) is proved.

$\square$

Remarks 7.2.8. Let L be the connected Hopf algebra in Theorem 7.2.6.

(1) L provides a negative answer to Question L of [3], which asked if every connected Hopf algebra of finite GK-dimension is isomorphic, as an algebra, to the universal enveloping algebra of a Lie algebra.

(2) L is also an example of connected Hopf algebra with signature $(1^{(3)}, 2^{(2)})$ which is not isomorphic as a Hopf algebra to the enveloping algebra of any coassociative Lie algebra (see §6.2.2 for the relevant definition).

(3) L is not connected coradically graded as a Hopf algebra, nor is it generated in degree one as an algebra. Indeed, if L satisfied either one of these conditions, it would be commutative. In the former case this follows from Proposition 4.2.4, and in the latter it follows from [5, Theorem 0.3(4)].

Corollary 7.2.9. *Let L be the connected Hopf algebra in Theorem 7.2.6. Then the following hold.*

(1) There is a connected $\mathbb{N}$-filtration of L, namely, the coradical filtration c, such that the associated graded ring $\operatorname{gr}_c H \cong k[x_1, \cdots, x_5]$ where $\deg x_i \geq 1$ (but these degrees are not all equal to 1).

(2) There is no connected $\mathbb{N}$-filtration such that $\operatorname{gr} L \cong k[x_1, \cdots, x_5]$ with $\deg x_i = 1$ for all i.

(3) There is a negative $\mathbb{N}$-filtration $\mathcal{F}$ of L such that the associated graded ring $\operatorname{gr}_{\mathcal{F}} L \cong U(\mathfrak{g})$ for a certain 5-dimensional graded Lie algebra $\mathfrak{g}$. Nevertheless, L itself is not isomorphic to $U(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$.

Proof. 1. Since L is a connected Hopf algebra, this follows from Theorem 4.2.7.

2. This is a variant of a well known result of Duflo, [27, Theorem 7.2]. The precise result is stated in [5, Corollary 0.6] and proved in [5, Lemma 5.9].

3. This follows from [17, Proposition 3.4(a)].

□

7.3 Appendix: Proof of Lemma 7.2.1

Now we give a complete proof of Lemma 7.2.1.

Lemma 7.3.1. *Let J be the algebra defined as in Section 7.2.1. Let $\Delta : J \to J \otimes J$ and $\epsilon : J \to k$ be the maps such that*

$$\Delta(a) = 1 \otimes a + a \otimes 1; \quad \Delta(b) = 1 \otimes b + b \otimes 1; \quad \Delta(c) = 1 \otimes c + c \otimes 1; \tag{7.3.1}$$

$$\Delta(z) = 1 \otimes z + a \otimes c - c \otimes a + z \otimes 1; \tag{7.3.2}$$

$$\Delta(w) = 1 \otimes w + b \otimes c - c \otimes b + w \otimes 1; \tag{7.3.3}$$

$$\Delta(d) = 1 \otimes d + c \otimes c^2 + c^2 \otimes c + d \otimes 1; \tag{7.3.4}$$

and

$$\epsilon(a) = \epsilon(b) = \epsilon(c) = \epsilon(z) = \epsilon(w) = \epsilon(d) = 0. \tag{7.3.5}$$

Then these maps satisfy the following properties:

(1) Δ defines an algebra homomorphism.

(2) ϵ defines an algebra homomorphism.

(3) Δ *is coassociative.*

(4) $(1 \otimes \epsilon) \circ \Delta = 1 = (\epsilon \otimes 1) \circ \Delta$*, where* 1 *denotes the identity map on* J*.*

As a result, (J, Δ, ϵ) *is a bialgebra.*

Proof. (1) Define a free algebra F on generators

$$\{a, b, c, z, w, d\}$$

and a map $\Delta' : F \to F \otimes F$ defined on these generators as in equations (7.3.1) – (7.3.4). (Here we are abusing notation by using the same letters for generators of F and their images in J.) Since F is a free algebra on these generators, we can extend Δ multiplicatively so that it defines an algebra homomorphism $F \to F \otimes F$. Setting I to be the ideal generated by the relations defining the Lie algebra $\mathfrak{g}$, the algebra J is the factor algebra F/I. Therefore, to check that there exists an algebra homomorphism $\Delta : J \to J \otimes J$ defined on generators as in equations (7.3.1) – (7.3.4), it suffices to prove that the algebra homomorphism $\Delta' : F \to F \otimes F$ defined above satisfies

$$\Delta'(I) \subseteq I \otimes F + F \otimes I. \tag{7.3.6}$$

Since $\Delta' : F \to F \otimes F$ is an algebra homomorphism, it suffices to check this on the generators of I, that is, on the relations defining $\mathfrak{g}$. We list these relations below

$$\begin{aligned}
&[a, b] - c, \quad [a, c], \quad [b, c], \\
&[a, z], \quad [b, z], \quad [c, z], \quad [a, w], \quad [b, w], \quad [c, w], \\
&[z, w] - d, \\
&[a, d], \quad [b, d], \quad [c, d], \quad [z, d], \quad [w, d].
\end{aligned}$$

This is already immediately clear for the ideal generators which involve only PBW-generators set to be primitive under Δ', so it suffices to check only those involving z, w or d,. For each $t \in F$, define $\delta'(t) = \Delta'(t) - t \otimes 1 - 1 \otimes t$. It is trivial that, for $t \in I$, $\Delta'(t) \in I \otimes F + F \otimes I$ if and only if $\delta'(t) \in I \otimes F + F \otimes I$. It is easy to check that

$$\delta'([t_1, t_2]) = [t_1 \otimes 1 + 1 \otimes t_1, \delta'(t_2)] + [\delta'(t_1), t_2 \otimes 1 + 1 + \otimes t_2] + [\delta'(t_1), \delta'(t_2)].$$

Next we check that $\delta'(t) \in I \otimes F + F \otimes I$ for all t in the third and fourth row of ideal generators.

1. Generator: $[a, z]$.

$$\begin{aligned}\delta'([a,z]) =&[1\otimes a + a\otimes 1, \delta'(z)] = [1\otimes a + a\otimes 1, a\otimes c - c\otimes a]\\ =&a\otimes [a,c] + [a,c]\otimes a \quad \in \quad I\otimes F + F\otimes I.\end{aligned}$$

2. Generator: $[b, z]$.

$$\begin{aligned}\delta'([b,z]) =&[1\otimes b + b\otimes 1, \delta'(z)] = [1\otimes b + b\otimes 1, (a\otimes c - c\otimes a)]\\ =&a\otimes [b,c] - c\otimes [b,a] + [b,a]\otimes c - [b,c]\otimes a\\ =&a\otimes [b,c] - [b,c]\otimes a - c\otimes ([b,a]+c) + ([b,a]+c)\otimes c\\ &\in I\otimes F + F\otimes I.\end{aligned}$$

3. Generator: $[c, z]$.

$$\begin{aligned}\delta'([c,z]) =&[1\otimes c + c\otimes 1, (a\otimes c - c\otimes a)]\\ =&-c\otimes [c,a] + [c,a]\otimes c \quad \in \quad I\otimes F + F\otimes I.\end{aligned}$$

4. Generator: $[a, w]$

$$\begin{aligned}\delta'([a,w]) =&[1\otimes a + a\otimes 1, (b\otimes c - c\otimes b)]\\ =&b\otimes [a,c] - c\otimes [a,b] + [a,b]\otimes c - [a,c]\otimes b\\ =&b\otimes [a,c] - c\otimes ([a,b]-c) + ([a,b]-c)\otimes c - [a,c]\otimes b\\ &\in I\otimes F + F\otimes I.\end{aligned}$$

5. Generator: $[b, w]$

$$\begin{aligned}\delta'([b,w]) =&[1\otimes b + b\otimes 1, (b\otimes c - c\otimes b)]\\ &\in I\otimes F + F\otimes I.\end{aligned}$$

6. Generator: $[c, w]$

$$\begin{aligned}\delta'([c,w]) =&[1\otimes c + c\otimes 1, (b\otimes c - c\otimes b)]\\ &\in I\otimes F + F\otimes I.\end{aligned}$$

7. Generator: $[z, w] - d$.

$$\begin{aligned}
\delta'([z,w]-d) =&([1\otimes z + z\otimes 1, (b\otimes c - c\otimes b)] + [(a\otimes c - c\otimes a), 1\otimes w + w\otimes 1]\\
&+ [(a\otimes c - c\otimes a), (b\otimes c - c\otimes b)] - \delta'(d)\\
=&(b\otimes [z,c] - c\otimes [z,b] + [z,b]\otimes c - [z,c]\otimes b)\\
&+ (a\otimes [c,w] + [a,w]\otimes c - c\otimes [a,w] - [c,w]\otimes a)\\
&- ac\otimes cb + ca\otimes bc - cb\otimes ac + bc\otimes ac\\
&+ ([a,b]-c)\otimes c^2 + c^2\otimes ([a,b]-c))\\
=&(b\otimes [z,c] - c\otimes [z,b] + [z,b]\otimes c - [z,c]\otimes b)\\
&+ (a\otimes [c,w] + [a,w]\otimes c - c\otimes [a,w] - [c,w]\otimes a)\\
&+ [c,a]\otimes cb - ca\otimes [c,b] + [b,c]\otimes ac - bc\otimes [a,c]\\
&+ ([a,b]-c)\otimes c^2 + c^2\otimes ([a,b]-c))\\
&\in I\otimes F + F\otimes I.
\end{aligned}$$

8. Generator: $[a, d]$.

$$\begin{aligned}
\delta'([a,d]) =&c\otimes [a,c^2] + c^2\otimes [a,c] + [a,c]\otimes c^2 + [a,c^2]\otimes c\\
&\in I\otimes F + F\otimes I.
\end{aligned}$$

9. Generator: $[b, d]$.

$$\begin{aligned}
\delta'([b,d]) =&c\otimes [b,c^2] + c^2\otimes [b,c] + [b,c]\otimes c^2 + [b,c^2]\otimes c\\
&\in I\otimes F + F\otimes I.
\end{aligned}$$

10. Generator: $[c, d]$.

$$\begin{aligned}
\delta'([c,d]) =&c\otimes [c,c^2] + c^2\otimes [c,c] + [c,c]\otimes c^2 + [c,c^2]\otimes c\\
=&0 \quad \in \quad I\otimes F + F\otimes I.
\end{aligned}$$

11. Generator: $[z, d]$

$$\begin{aligned}\delta'([z,d]) =& [1\otimes z + z\otimes 1, c\otimes c^2 + c^2\otimes c] + [b\otimes c - c\otimes b, 1\otimes d + d\otimes 1]\\ &+ [b\otimes c - c\otimes b, c\otimes c^2 + c^2\otimes c]\\ =& [z,c]\otimes c^2 + [z,c^2]\otimes c + c\otimes [z,c^2] + c^2\otimes [z,c]\\ &+ b\otimes [c,d] + [b,d]\otimes c - c\otimes [b,d] - [c,d]\otimes b\\ &+ [b,c]\otimes c^3 + [b,c^2]\otimes c^2 - c^2\otimes [b,c^2] - c^3\otimes [b,c]\\ &\in I\otimes F + F\otimes I.\end{aligned}$$

12. Generator: $[w, d]$

$$\begin{aligned}\delta'([w,d]) =& [1\otimes w + w\otimes 1, c\otimes c^2 + c^2\otimes c] + [a\otimes c - c\otimes a, 1\otimes d + d\otimes 1]\\ &[a\otimes c - c\otimes a, c\otimes c^2 + c^2\otimes c]\\ =& [w,c]\otimes c^2 + [w,c^2]\otimes c + c\otimes [w,c^2] + c^2\otimes [w,c]\\ &+ [a,d]\otimes c - c\otimes [a,d] + a\otimes [c,d] - [c,d]\otimes a\\ &+ [a,c]\otimes c^3 + [a,c^2]\otimes c^2 - c^2\otimes [a,c^2] - c^3\otimes [a,c]\\ &\in I\otimes F + F\otimes I.\end{aligned}$$

Thus there exists an algebra homomorphism $\Delta : J \to J\otimes J$ defined on the generators $\{a, b, c, z, w, d\}$ as in equations (7.3.1) – (7.3.4).

(2) Retain the notation of part (1). Define an algebra homomorphism $\epsilon' : F \to k$ which takes the value 0 on all generators $\{a, b, c, z, w, d\}$ of F. To check that there exists an algebra homomorphism $\epsilon : J \to k$ defined on generators as in (7.3.5), it suffices to check that $\epsilon'(I) = 0$. This is clear.

(3) As noted in [17, §1], to check that the algebra homomorphism $\Delta : J \to J\otimes J$ is coassociative, it suffices to check this on the algebra generators

$$\{a, b, c, z, w, d\}$$

of J. This is already clear for each of the generators a, b, c, so we are left to check for the generators z, w and d. Let $\delta(t) = \Delta(t) - t\otimes 1 - 1\otimes t$ for any $t \in J$. It is routine to check that Δ is coassociative if and only if δ is. So it suffices to show that δ is coassociative on z, w and d. We calculate

$$\begin{aligned}(\delta\otimes 1)\circ\delta(z) &= (\delta\otimes 1)(a\otimes c - c\otimes a)\\ &= \delta(a)\otimes c - \delta(c)\otimes a = 0,\end{aligned}$$

on the other hand,

$$\begin{aligned}(1 \otimes \delta) \circ \delta(z) &= (1 \otimes \delta)(a \otimes c - c \otimes a) \\ &= a \otimes \delta(c) - c \otimes \delta(a) = 0.\end{aligned}$$

So $(\delta \otimes 1) \circ \delta(z) = (\delta \otimes 1) \circ \delta(z)$. Similarly, $(\delta \otimes 1) \circ \delta(w) = 0 = (1 \otimes \delta) \circ \delta(w)$.

Note that $\delta(c^2) = 2c \otimes c$. Then

$$\begin{aligned}(\delta \otimes 1) \circ \delta(d) =&(\delta \otimes 1)(c \otimes c^2 + c^2 \otimes c) \\ =&\delta(c) \otimes c^2 + \delta(c^2) \otimes c \\ =&2c \otimes c \otimes c.\end{aligned}$$

By symmetry, $(1 \otimes \delta) \circ \delta(d) = 2c \otimes c \otimes c$. Therefore $(\delta \otimes 1) \circ \delta(d) = (1 \otimes \delta) \circ \delta(d)$. Therefore δ (and then Δ) is coassociative.

(4) Again, as noted in [17, §1], it suffices to check that

$$(\epsilon \otimes 1) \circ \Delta(g) = g = (1 \otimes \epsilon) \otimes \Delta(g)$$

for each algebra generator g of J. Noting that $\epsilon(1) = 1_k$ and that $\epsilon(g) = 0$ for all $g \in \{a, b, c, z, w, d\}$, this is clear. □

Chapter 8

The Antipode of a Pointed Hopf Algebra

8.1 Introduction

In this chapter, we are interested in determining the order of the antipode of a pointed Hopf algebra over an arbitrary field k. For a pointed Hopf k-algebra H, we introduce an invariant, which we denote by m_H, which is, in a sense which we shall make precise, a measure of the extent to which a group like element $x \in G(H)$ commutes with any $h \in H$ such that $\Delta(h) = h \otimes x + 1 \otimes h$. Whilst in general m_H can take values in $\mathbb{Z}^{\geq 0} \cup \{\infty\}$, the condition that m_H is finite is valid in a variety of natural settings, for example whenever $G(H)$ is finite or central in H (see Proposition 8.2.8). We record our main results below (where part (3) appears as Proposition 8.4.1 and part (4) appears as Corollary 8.5.2). These results connect the order of the antipode of a pointed Hopf algebra H to the value of m_H in both the cases of zero and positive characteristic.

Theorem 8.1.1. *Let k be a field. Let H be a pointed Hopf k-algebra and let $\{H_n\}_{n\geq 0}$ denote the coradical filtration of H.*

1. *If $m_H = \infty$, $|S| = \infty$.*
2. *(Taft, Wilson) If $m_H < \infty$, $(S^{2m_H} - \mathrm{id})(H_n) \subseteq H_{n-1}$ for $n \geq 1$.*
3. *Suppose $\operatorname{char} k = 0$ and that $H \neq H_0$. Then either $|S| = \infty$ or $|S| = 2m_H$.*
4. *Suppose $\operatorname{char} k = p > 0$. If $m_H < \infty$ and $H = k\langle H_n\rangle$ for some $n \geq 0$, then $|S|$ divides $2m_H p^l$, where $l \in \mathbb{N}$ is such that $p^l \geq n \geq p^{l-1}$.*

Theorem 8.1.1 (2), from which everything else quickly follows, has exactly the same proof as part (1) of the following result of Taft and Wilson from 1974, and so we credit it to them. Parts (3) and (4) of Theorem 8.1.1 should be compared to the analogous results for H finite-dimensional as presented in part (2) of the result below. It will be clear from the definition that m_H divides the exponent of $G(H)$ whenever the exponent is finite.

Theorem 8.1.2. *(Taft, Wilson, [55]) Let k be a field and let H be a pointed Hopf k-algebra. Assume that $G(H)$ has finite exponent e.*

1. *([55, Proposition 3, Proposition 4]) For $n \geq 1$, $(S^{2e} - \mathrm{id})(H_n) \subseteq H_{n-1}$.*
2. *([55, Corollary 6]) Assume that H is finite-dimensional and that $H = H_n$ for some $n \geq 0$. If $\operatorname{char} k = 0$ and S has finite order, then $|S|$ divides $2e$. If $\operatorname{char} k = p > 0$, $S^{2ep^m} = \mathrm{Id}$, where $p^m \geq n > p^{m-1}$.*

Remark 8.1.3. It was subsequently proved by Radford, [46], that the order of the antipode of a finite dimensional Hopf algebra is always finite, allowing us to drop the assumption that S has finite order in the finite dimensional setting of Theorem 8.1.2(2).

The converse of Theorem 8.1.1 (1) is in general not true, even in the connected case, where $m_H = 1$ always. In §4.3.3 we gave an example of a connected Hopf algebra over a field of characteristic 0 with an antipode of infinite order. However, as an almost immediate consequence of of Theorem 8.1.1(2), in the case where H is known to be coradically Hopf graded (see Definition 4.2.1), the condition that $m_H < \infty$ is equivalent to the condition that S has finite order. That is, we deduce the following.

Proposition 8.1.4. *Let k be a field and let H be a pointed coradically Hopf graded Hopf k-algebra.*

1. *$|S| = \infty$ if and only if $m_H = \infty$.*
2. *If $m_H < \infty$, $|S| = 2m_H$.*

Since the associated graded Hopf algebra of a pointed Hopf algebra is always a pointed coradically Hopf graded Hopf algebra by Proposition 3.2.5, the following corollary is immediate.

Corollary 8.1.5. *Let k be a field and let H be a pointed Hopf algebra with $m_H < \infty$. Then $m_{\operatorname{gr} H} < \infty$ and $|S_{\operatorname{gr} H}| = 2m_{\operatorname{gr} H}$.*

Remark 8.1.6. 1. In positive characteristic, there exist examples of pointed Hopf algebras where the antipode has order strictly less than the bound obtained in Theorem 8.1.1 (4). Take, for example, a field k such that $\operatorname{char} k = p > 0$ and let H be a pointed coradically Hopf graded Hopf k-algebra with $m_H < \infty$ and $H \neq H_0$ (see Example 8.2.9 for an explicit example of such a Hopf algebra). By Proposition 8.1.4, $|S| = 2m_H$, which is strictly less than the bound obtained in Theorem 8.1.1.

   On the other hand, there exist examples of pointed Hopf algebras over fields of positive characteristic where the bound obtained in Theorem 8.1.1 (4) is actually attained. In Example 8.5.4 we give an example, originally due to Taft and Wilson, [54], of a finite dimensional connected Hopf algebra R over a field of characteristic $p \geq 3$ with $R = R_2$, $m_R = 1$ and an antipode of order $2p$.

2. Note that for an arbitrary pointed Hopf algebra H, m_H will be in general strictly less than the exponent of $G(H)$ - in Example 8.2.9 we see that, if q is a primitive n^{th} root of unity, the pointed coradically Hopf graded Hopf algebra $H = U_q(\mathfrak{b}^+)$ has the property that $m_H = n$ and that $G(H)$ has infinite exponent.

8.2 Preliminaries

Throughout this chapter, k will denote an arbitrary field (unless otherwise stated), H will denote a pointed Hopf algebra, $\{H_n\}$ will denote the coradical filtration of H and $G := G(H)$ the group of group like elements of H (see §2.3.2 and Definition 3.2.1 for the relevant definitions). We denote the coradical degree of an element $h \in H$ by $d(h)$.

Remark 8.2.1. Recall that $\{H_n\}$ is always a Hopf filtration (see Definition 2.3.8) whenever H is pointed, Proposition 3.2.5.

Definition 8.2.2. Suppose H is a pointed Hopf algebra. For any $x, y \in G$, define the space of (x, y)-*skew-primitive elements* of H,

$$P_{x,y}(H) := \{h \in H : \Delta(h) = h \otimes x + y \otimes h\}.$$

Notation 8.2.3. As noted in the opening remarks of [41, §5.4], if H is a pointed Hopf algebra and $x, y \in G$, then $P_{x,y}(H) \cap H_0 = k(x - y)$. For each such pair $x, y \in G$, let $P_{x,y}(H)'$ denote a subspace such that

$$P_{x,y}(H) = k(x - y) \oplus P_{x,y}(H)'.$$

The following result, which appears as stated below as [41, Theorem 5.4.1], but is originally due to Taft and Wilson, [55], is the crux of the proof of our main theorem, Theorem 8.1.1.

Theorem 8.2.4. *Let H be a pointed Hopf algebra. Then*

$$H_1 = kG \oplus (\oplus_{x,y \in G} P_{x,y}(H)').$$

Lemma 8.2.5. *Let H be a pointed Hopf algebra, let $x \in G$ and let $\langle x \rangle$ denote the subgroup of G generated by x.*

1. *$P_{x,1}(H)$ is an $\langle x \rangle$-invariant subspace of H, where x acts by conjugation.*

2. *$P_{x,1}(H) \subseteq \ker \epsilon$.*

Proof. 1. Let $h \in P_{x,1}(H)$. Since Δ is an algebra homomorphism,

$$\Delta(xhx^{-1}) = (x \otimes x)(h \otimes x + 1 \otimes h)(x^{-1} \otimes x^{-1}) = xhx^{-1} \otimes x + 1 \otimes xhx^{-1},$$

hence $xhx^{-1} \in P_{x,1}(H)$, as required.

2. Let $x \in G$ and $h \in P_{x,1}(H)$. By the counit axiom of the coproduct, $h\epsilon(x) + \epsilon(h) = h$. Since x is group-like, $\epsilon(x) = 1$. The result follows.

□

In light of Lemma 8.2.5 (1), we make the following definition.

Definition 8.2.6. Let H be a pointed Hopf algebra. For any $x \in G$, define

$$a_x = |\langle x \rangle : C_{\langle x \rangle}(P_{x,1}(H))|$$

where $C_{\langle x \rangle}(P_{x,1}(H))$ denotes the centraliser of $P_{x,1}(H)$ in $\langle x \rangle$ under the conjugation action by $\langle x \rangle$.

Definition 8.2.7. Let H be a pointed Hopf k-algebra.

1. If $H = k$, $m_H := 0$.

2. If $H \neq k$, $m_H := \operatorname{lcm}\{a_x : x \in G\}$.

Proposition 8.2.8. *Let H be a pointed Hopf algebra.*

1. *Suppose $G(H)$ is central in H. Then $m_H = 1$. In particular, $m_H = 1$ if H is connected (that is, if $G(H)$ is the trivial group).*

2. *Suppose $G(H)$ is finite. Then m_H is finite and divides the exponent of the group $G(H)$.*

Proof. This is immediate from the way we defined m_H. □

For an arbitrary pointed Hopf algebra H, m_H can take values in $\mathbb{Z}^{\geq 0} \cup \{\infty\}$ and will be in general strictly less than the exponent of the group $G(H)$, as shown in the following example.

Example 8.2.9. Let k be a field and let $0, 1 \neq q \in k$. Set $H = U_q(\mathfrak{b}^+)$, the quantised enveloping algebra of the positive two dimensional Borel. This is defined as the algebra generated by the letters E, K and K^{-1}, subject to the relations $KK^{-1} = 1 = K^{-1}K$ and

$$KE = qEK.$$

Then, as proved in [6, I.3.4], for example, H becomes a pointed Hopf algebra, with coproduct $\Delta : H \to H \otimes H$ and antipode $S : H \to H$ defined on generators as follows

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(K) = K \otimes K$$

$$S(K) = K^{-1}, \quad S(E) = -K^{-1}E.$$

Set $E' := EK^{-1} \in P_{K^{-1},1}(H)$. An elementary calculation shows that $S^{2n}(E') = q^{-n}E'$ and $K^nE' = q^nE'K^n$ for all $n \geq 1$. Since K, K^{-1} and E' form a set of generators of H, it is clear that the value of m_H depends only on the action of K on E'. Thus

1. If q is an n^{th} primitive root of unity for some $1 \leq n < \infty$, $G(H)$ has infinite exponent, $m_H = n$ and $|S| = 2n$.

2. If q is *not* a root of unity, then $G(H)$ has infinite exponent, $m_H = \infty$ and $|S| = \infty$.

8.3 Preliminary computations

The main results of this section, Proposition 8.3.3 and Proposition 8.3.7, along with their proofs, are almost identical to [55, Proposition 3, Theorem 5], the only difference being that, for a pointed Hopf algebra H, we state our results in terms of m_H, instead of the exponent of $G(H)$, and do not restrict ourselves to stating the result for H being finite-dimensional only.

The results of this section are valid over any field.

Lemma 8.3.1. *Let H be a pointed Hopf algebra. Let $x \in G(H)$. Then, for $h \in P_{x,1}(H)$, $S(h) = -hx^{-1}$.*

Proof. Let $x \in G$ and choose $h \in P_{x,1}(H)$, so that $\Delta(h) = h \otimes x + 1 \otimes h$. By Lemma 8.2.5(2), $\epsilon(h) = 0$. By the counit axiom of the antipode,

$$S(h)x + h = 0.$$

That is, $S(h) = -hx^{-1}$. □

The following lemma is valid over any field.

Lemma 8.3.2. *Let H be a pointed Hopf algebra. Let $x \in G(H)$ and let $h \in P_{x,1}(H)$.*

1. *For any $m \geq 1$, $S^{2m}(h) = x^m h x^{-m}$.*
2. *If in addition $m_H < \infty$, $S^{2m_H}(h) = h$.*

Proof. Fix $h \in P_{x,1}(H)$. By Lemma 8.3.1, $S(h) = -hx^{-1}$. This gives

$$S^2(h) = -S(x^{-1})S(h) = xhx^{-1}. \tag{8.3.1}$$

Proceeding inductively, for any $m \geq 1$,

$$S^{2m}(h) = x^m h x^{-m}.$$

It is then clear from the definition of m_H that if $m_H < \infty$, $S^{2m_H}(h) = h$.

□

Proposition 8.3.3. *Let H be a pointed Hopf algebra with $m_H < \infty$. Then*

$$(S^{2m_H} - \mathrm{Id})(H_1) = 0.$$

Proof. Let $h \in H_1$. If $h \in H_0$ then $S^2(h) = h$ since $H_0 = kG(H)$ and $S^2|_{kG(H)} = \mathrm{id}$. Thus by Theorem 8.2.4 and linearity, we can without loss of generality assume that $h \in P_{x,y}(H)$ for some $x, y \in G$. Using the fact that Δ is an algebra homomorphism, an elementary calculation then shows that $hy^{-1} \in P_{xy^{-1},1}(H)$. Then, using Lemma 8.3.2(2) and the fact that S^{2m_H} is an algebra morphism,

$$hy^{-1} = S^{2m_H}(hy^{-1}) = S^{2m_H}(h)S^{2m_H}(y^{-1}) = S^{2m_H}(h)y^{-1},$$

giving $S^{2m_H}(h) = h$, as required.

□

Next we shall prove a sufficient condition for the antipode of a pointed Hopf algebra to have infinite order. Before we do this, we need the following well-known lemma.

Lemma 8.3.4. *Let H be a Hopf algebra with $|S| < \infty$. Then either $S = \mathrm{id}$ or $|S| = 2k$ for some $k \in \mathbb{N}$.*

Proof. Suppose $S \neq id$. If H is commutative, $|S| = 2$ by [41, Corollary 1.5.12], so without loss of generality assume that H is noncommutative. Choose $x, y \in H$ such that $xy \neq yx$. Let $|S| = m < \infty$. Suppose m is odd - write $m = 2q + 1$ for some $q \geq 1$. Since S is an anti-algebra morphism, so too is S^{2q+1}, hence

$$xy = S^{2q+1}(xy) = S^{2q+1}(y)S^{2q+1}(x) = yx.$$

This contradicts the assumption that x and y do not commute, thus S must have even order.

□

Proposition 8.3.5. *Let H be a pointed Hopf algebra with $m_H = \infty$. Then $|S| = \infty$.*

Proof. Suppose $|S| < \infty$. By Lemma 8.3.4, we can write $|S| = 2t$ for some $t \in \mathbb{N}$. Then by Lemma 8.3.2(1), for any $x \in G$ and $h \in P_{x,1}(H)$,

$$h = S^{2t}(h) = x^t h x^{-t}.$$

If $m_H = \infty$, there exists some $y \in G$ and $f \in P_{y,1}(H)$ such that $y^t f y^{-t} \neq f$. Thus it must be that $m_H < \infty$. The result follows. □

Following the arguments of [55], the following result allows us to extend Proposition 8.3.3 to higher terms in the coradical filtration.

Proposition 8.3.6. *Let H be a pointed Hopf algebra, let $i \geq 1$ and let $\psi : H \to H$ be a coalgebra homomorphism. Suppose $(\psi - \mathrm{id})(H_j) \subseteq H_{j-1}$ for all $0 \leq j \leq i$. Then $(\psi - \mathrm{id})(H_{i+1}) \subseteq H_i$.*

Proof. This is [55, Proposition 4]. □

Proposition 8.3.7. *Let H be a pointed Hopf algebra with $m_H < \infty$. Then, for any $n \geq 1$,*

1. $(S^{2m_H} - \mathrm{id})(H_n) \subseteq H_{n-1}$.

2. $(S^{2m_H} - \mathrm{id})^n(H_n) = 0$.

Proof. This is immediate from Proposition 8.3.3, Proposition 8.3.6 and the fact that S^{2m_H} is a coalgebra morphism. $\square$

Recall from Definition 4.2.1 the definition of a coradically Hopf graded Hopf algebra.

Proposition 8.3.8. *Let H be a pointed coradically Hopf graded Hopf algebra. Then*

1. $m_H = \infty$ *if and only if* $|S| = \infty$.
2. *If* $m_H < \infty$, $S = \mathrm{id}$ *or* $|S| = 2m_H$.

Proof. If $m_H = \infty$, $|S| = \infty$ by Proposition 8.3.5. For the converse, let $H = \bigoplus_{i=0}^{\infty} H(i)$ be a pointed coradically Hopf graded Hopf algebra, so that, for $n \geq 1$, $H_n = \bigoplus_{i=0}^{n} H(i)$, and let $m_H < \infty$. Suppose $S \neq \mathrm{id}$. Fix $n \geq 1$. By Proposition 8.3.7, it follows that

$$(S^{2m_H} - \mathrm{id})(H(n)) \subseteq \bigoplus_{j=0}^{n-1} H(j-1).$$

However, since $\{H(n)\}_n$ is a Hopf grading, $(S^{2m_H} - \mathrm{id})(H(n)) \subseteq H(n)$. It must therefore be that $(S^{2m_H} - \mathrm{id})(H(n)) = 0$, hence $S^{2m_H} = \mathrm{id}$. To complete the proof, it suffices to prove that for any $q < m_H$, there exists $h \in H$ such that $S^{2q}(h) \neq h$, since Lemma 8.3.4 guarantees that the order of the antipode is always either 1 or divisible by 2. Suppose for a contradiction that there exists some $q < m_H$ such that $|S| = 2q$. By Lemma 8.3.2(1), for any $x \in G$, $h \in P_{x,1}(H)$,

$$h = S^{2q}(h) = x^q h x^{-q}.$$

However, since $q < m_H$, by definition there exists some $y \in G$ and $f \in P_{y,1}(H)$ such that

$$f \neq y^q f y^{-q},$$

a contradiction. This completes the proof.

$\square$

Since the associated graded Hopf algebra of a pointed Hopf algebra is always a pointed coradically Hopf graded Hopf algebra, Proposition 3.2.5, the following corollary is immediate.

Corollary 8.3.9. *Let k be a field and let H be a pointed Hopf algebra with $m_H < \infty$. Then $m_{\mathrm{gr}\,H} < \infty$ and $|S_{\mathrm{gr}\,H}| = 2m_{\mathrm{gr}\,H}$.*

8.4 The antipode in characteristic zero

We now consider what happens when we work over a field of characteristic 0.

Proposition 8.4.1. *Let H be a pointed Hopf k-algebra. Suppose* $\operatorname{char} k = 0$*. If $m_H = \infty$ then $|S| = \infty$. If $m_H < \infty$ then either $|S|$ divides $2m_H$, or there exists $h \in H$ such that the orbit of S on h is infinite. In particular, either $|S|$ divides $2m_H$ or $|S| = \infty$.*

Proof. We can, without loss of generality, assume that $m_H < \infty$, since otherwise Proposition 8.3.5 guarantees that $|S| = \infty$. Suppose $S^{2m_H} \neq \mathrm{id}$, and choose n minimal such that $h \in H_n$ and $S^{2m_H}(h) \neq h$. We shall prove that $S^{2l}(h) \neq h$ for any $l \geq 1$. By Proposition 8.3.7(1), $S^{2m_H}(h) = h + r$ for some $r \in H_{n-1}$. By the choice of h, we can assume $r \neq 0$.

Claim 8.4.2. Retain the above notation. For $t \geq 1$, $S^{2m_H t}(h) = h + tr$.

Proof. of Claim 8.4.2: We proceed by induction on $t \geq 1$. The $t = 1$ case was Proposition 8.3.7(1). Fix $t \geq 1$. Then, by Proposition 8.3.7(1),

$$S^{2(t+1)m_H}(h) = S^{2tm_H} S^{2m_H}(h) = S^{2tm_H}(h) + S^{2tm_H}(r). \tag{8.4.1}$$

By the minimality of n, $S^{2tm_H}(r) = r$. By the inductive hypothesis, equation (8.4.1) becomes

$$S^{2(t+1)m_H}(h) = (h + tr) + r = h + (t+1)r$$

proving the claim by induction.

□

So, $S^{2m_H t}(h) = h + tr$ for all $t \geq 1$. Since $\operatorname{char} k = 0$, this implies $S^{2m_H t}(h) \neq h$ for all $t \geq 1$. Thus $|S^{2m_H}| = \infty$, and so $|S| = \infty$. In particular, if i and j are distinct integers, then $S^i(h) \neq S^j(h)$, since otherwise $S^{2m_H(i-j)}(h) = h$, contradicting the above. This completes the proof.

□

Theorem 8.4.3. *Let H be a pointed Hopf k-algebra, where* $\operatorname{char} k = 0$*.*

1. *If $H = H_0$, either $S = \mathrm{id}$ or $|S| = 2$.*

2. *If $H \neq H_0$, $|S| = \infty, |S| = 2m_H$ or $S = \mathrm{id}$.*

Proof. 1. This is immediate, since for any $x \in H_0$, either $S(x) = x$ or $S(x) = x^{-1}$.

2. Suppose $S \neq \mathrm{id}$. By Proposition 8.4.1 and Lemma 8.3.4, it suffices to show that if $|S| < \infty$, then, for any $q < m_H$, there exists $h \in H$ such that $S^{2q}(h) \neq h$.

   Suppose for a contradiction that there exists some $q < m_H$ such that $|S| = 2q$. By Lemma 8.3.2 (1), for any $x \in G$, $h \in P_{x,1}(H)$,

$$h = S^{2m_H}(h) = x^q h x^{-q}.$$

   However, since $q < m_H$, by definition there exists some $y \in G$ and $f \in P_{y,1}(H)$ such that

$$f \neq y^q f y^{-q},$$

   a contradiction. This completes the proof.

□

Corollary 8.4.4. *Let H be a pointed Hopf k-algebra. Assume* $\mathrm{char}\, k = 0$ *. Then*

1. *If H_0 is central in H, then either $|S| = \infty, |S| = 2$ or $S = \mathrm{id}$.*

2. *If H is a connected Hopf algebra (i.e. $H_0 = k$), then either $|S| = \infty, |S| = 2$ or $S = \mathrm{id}$.*

Proof. Clearly if H_0 is central in H then $m_H = 1$, so (1) follows from Proposition 8.4.1. Part (2) is immediate from (1).

□

Remark 8.4.5. Note that in Proposition 8.4.1 both alternatives can occur, even in the case where H is connected. In §4.3.3 we saw an example of a connected Hopf algebra of GK-dimension 3 with an antipode of infinite order.

8.5 The antipode in positive characteristic

In [55, Corollary 6], a bound is obtained for the order of the antipode of a finite dimensional Hopf algebra H over a field of positive characteristic, in terms of the exponent of $G(H)$. It turns out that the proof of that result is equally valid if we drop the assumption that H is finite dimensional, and also that, in light of the results of §8.3, the result can be restated in terms of m_H, rather than the exponent of $G(H)$.

Proposition 8.5.1. *Let H be a pointed Hopf k-algebra with $m_H < \infty$. Let $n \geq 1$ and $h \in H_n$. Choose $l \in \mathbb{N}$ such that $p^l \geq n \geq p^{l-1}$. If* $\mathrm{char}\, k = p > 0$*, $S^{2m_H p^l}(h) = h$.*

Proof. Let $n \geq 1$ and $h \in H_n$. Choose $l \in \mathbb{N}$ such that $p^l \geq n \geq p^{l-1}$. By Proposition Lemma 8.3.7(2),

$$0 = (S^{2m_H} - \mathrm{id})^{p^l}(h) = S^{2m_H p^l}(h) - h.$$

where the final equality follows from the binomial theorem in characteristic $p > 0$, which works here since S and id commute in $\mathrm{End}_k(H)$. The result follows. □

The following corollary is immediate.

Corollary 8.5.2. *Let H be a pointed Hopf k-algebra with $m_H < \infty$. Assume* $\mathrm{char}\, k = p > 0$.

1. *Suppose that $H = k\langle H_n \rangle$ for some $0 \leq n < \infty$. Choose $l \in \mathbb{N}$ such that $p^l \geq n \geq p^{l-1}$. Then $|S|$ divides $2m_H p^l$.*

2. *If H is affine (that is, finitely generated as an algebra), then $|S|$ divides $2m_H p^n$ for some $0 \leq n < \infty$. In particular, $|S| < \infty$.*

Corollary 8.5.3. *Let H be an affine pointed Hopf k-algebra, where* $\mathrm{char}\, k = p > 0$.

1. *Suppose $G(H)$ is central in H. Then there exists some $n \geq 0$ such that $|S|$ divides $2p^n$.*

2. *Suppose H is connected. Then there exists some $n \geq 0$ such that $|S|$ divides $2p^n$.*

Proof. Part (1) follows immediately from the Corollary 8.5.2 and fact that if H_0 is central in H then $m_H = 1$. Part (2) is a special case of (1). □

As shown by the following example, originally due to Taft and Wilson, [54], the bound on the order of the antipode obtained in Corollary 8.5.2 is in not attained, even in the finite-dimensional connected case.

Example 8.5.4. Let k be a field with $\mathrm{char}\, k = p \geq 3$. Let R be the algebra with generators X, Y and Z subject to the following relations.

$$[X, Y] = X \quad [Y, Z] = -Z, \quad [X, Z] = \frac{1}{2}X^2,$$

$$X^p = 0, \quad Y^p = Y, \quad Z^p = 0.$$

It is proved in [54] that R is a connected Hopf algebra of vector space dimension p^3 with coproduct, counit and antipode defined on generators as follows:

$$\epsilon(X) = 0, \quad \Delta(X) = 1 \otimes X + X \otimes 1,$$

$$\epsilon(Y) = 0, \quad \Delta(Y) = 1 \otimes Y + Y \otimes 1,$$

$$\epsilon(Z) = 0, \quad \Delta(Z) = 1 \otimes Z + X \otimes Y + Z \otimes 1,$$

$$S(X) = -X, \quad S(Y) = -Y, \quad S(Z) = -Z + XY.$$

Since R is connected, $m_R = 1$. Morover, notice that $X, Y \in R_1$ and $Z \in R_2$, so R can be generated in at least coradical degree 2. If R was generated in coradical degree one, it would be cocommutative, since R being connected implies $R_1 = k \oplus P(R)$, [41, Lemma 5.3.2]. Since $p \geq 3$, the bound on the order of the antipode of R as determined by Corollary 8.5.2 is therefore $2p$. A simple calculation yields the identity

$$S^{2t}(Z) = Z - tX$$

for any $t \geq 1$. In particular, $S^{2p}(Z) = Z$. Since S is an anti-algebra morphism, it follows that $|S| = 2p$.

Example 8.5.5. For an example of a pointed Hopf k-algebra H over a field of positive characteristic which has an antipode of infinite order, see Example 8.2.9: when q is not a root of unity, the Hopf algebra $H = U_q(\mathfrak{b}^+)$ has an antipode of infinite order over *any* field.

We know of no example of a connected Hopf algebra H in positive characteristic with an antipode of infinite order. This prompts the following question.

Question 8.5.6. Suppose H is a connected Hopf k-algebra, where $\operatorname{char} k = p > 0$. Does the antipode of H always have finite order?

By Corollary 8.5.3 (2), an example which gives a negative answer to the above question would be necessarily non-affine. More generally, we could ask the following.

Question 8.5.7. Suppose H is a pointed Hopf k-algebra, where $\operatorname{char} k = p > 0$ and $m_H < \infty$. Does the antipode of H always have finite order?

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