MODEL REDUCTION TECHNIQUES AND

THEIR APPLICATION TO HELICOPTER MODELS

A Thesis Submitted to the

Faculty of Engineering

University of Glasgow

for the Degree of

Master of Science

by

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March 1992

ProQuest Number: 11011467

ProQuest 11011467

To my father's memory

and to my mother

ACKNOWLEDGEMENTS

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NOTATION

A, B, C --- matrices of state space form

a_i --- coefficients of numerator of G(s)

b_i --- coefficients of denominator of G(s)

$D(j\omega)$ --- denominator of G(s)

d1, d2 --- radii concentric circles defined in complex plane

$E(j\omega)$ --- frequency response error

G(s) --- transfer function

$G(j\omega)$ --- transfer function

H --- Hankel matrix

$H(j\omega)$ --- function coinciding with measured data

h_i --- element of eigenvalue Λ

i, j --- integer

$I_{ik}(\omega k)$ --- Imaginaty parts of measured function $H_i(j\omega)$

J_i --- cost function for SISO case

J,E --- cost function for SIMO case

L --- iteration number

m --- number of output

M --- eigenvectors of matrix A

n --- number of points in frequency range considered

N --- form for n transfer function models

$N(j\omega)$ --- numerator of G(s)

n1 --- weighting factor

p --- pole of transfer function

Q_i --- input scaling

Q_o --- output scaling

$R_{ik}(\omega k)$ --- real parts of measured function $H_i(j\omega)$

S --- observability grammian

s --- Laplace operetor

T --- transpose of matrix

T_r --- trance

$\underline{u}$ --- matrix of control

X --- cotrollability grammian

$\underline{x}$ --- matrix of state space

$\underline{y}$ --- output of system

z --- zeros of transfer function

Λ --- eigenvalue of matrix **A**

λ_i --- element of eigenvalue Λ

$\sigma(\cdot)$ --- maximum singular value of $(\cdot)$

σ_i --- element of Hankel singular values

ω_i --- frequency range

ABSTRACT

Techniques of model reduction are important not only because a reduced order model may be needed for control system design or for model validation from measured response data, but also because in many applications accurate information about the plant dynamics may only be available, or required, for a restricted range of frequencies. In the content of aircraft flight mechanics models reduced order descriptions are also of considerable importance for handling qualities studies.

In this document, various model reduction techniques are reviewed. An 'equivalent system approximation' approach has been selected and applied to the reduction of a helicopter flight dynamics model. The adequacy and degree of accuracy of this 'equivalent system approximation' reduced order model was verified through comparison with a high order model using the Puma helicopter as an example. Excellent agreement between the results from the reduced order model and the original high order system model were obtained over selected range of frequency. Another approximate method — extended Levy's complex— curve fitting method using a modified least— squares approach has been extended to the multi— input multi— output and has also applied to the reduction of the helicopter flight dynamics model for a Puma helicopter. Very good agreement between the results from the reduced order models and the original system model were again obtained. Comparison of Levy's method with the 'equivalent system' approach showed that in the latter physical insight can be used in the reduction process whereas Levy's method is purely a curve fitting technique. Both techniques can, however, provide useful reduced— order descriptions for given frequency ranges. The extended Levy approach and the 'equivalent systems' approach have both been implemented using the MATLAB software package.

Chapter 1

The Model Development Process

1.1 Introduction.

The development of dynamic models of complex nonlinear systems is often attempted using a combination of theoretical modelling, based on physical principles, and empirical tuning of model parameters guided by comparisons of model responses with measured response data from the real system. However, such comparisons of system response variables with the corresponding quantities predicted by a theoretical model are not usually sufficient to provide the model developer with real insight concerning the source of any deficiency in the model.

There are two fundamental ways in which a mathematical model may be deficient: it may have an inappropriate structure or it may have inappropriate parameter values. In most cases both types of deficiency are likely to be present in the initial formulation of a model.

Assessment of the adequacy, or otherwise, of a mathematical model must take into account the purpose of the model and the way in which the model is to be used. For most applications empirical model adjustments can lead to unrealistic values of parameters which, although possibly providing an adequate fit between the model and the measured system responses for one particular experiment or rest record, have no physical basis. The larger the number of parameters available for adjustment in this way the more likely it is that some combination can be found which provides a good match of a model response to the corresponding measured quarity. This apparent good fit for one particular case may, however, mask major deficiencies in the model structure and the model may well prove to have no general predictive value.

Modelling from measured response data has now developed into a major activity in a wide range of scientific and engineering disciplines. This is generally termed 'system identification' and can be defined as 'the determination on the basis of inputs and outputs of a system within a specified class of systems to which the system

under test is equivalent'.[1]

Identification and similar inverse problems have a long history and can be traced to the work of Gauss on modelling of planetary motion from astronomical observations. A wide range of identification techniques are now available some of which lead to non—parametric descriptions, such as a frequency response, while others involve parametric models and can be linked directly with the development of models from physical principles.

Although system identification techniques allow complex nonlinear models to be developed and validated with the aid of measured response data, there is also a need for reduced—order linear models for many applications. This is important in control system design where many of the available techniques depend upon the use of a linear description for the plant. In many applications accurate information about the plant dynamics may only be available or required for a restricted range of frequencies and models of reduced order may therefore also be appropriate. Examples of reduced—order models are often encountered as submodels within a larger model of a complex system. Reduced—order models are also of value in real—time simulations of a system where some modelling accuracy, particularly in the high frequency range, has been traded for computational speed.

Since the system identification approach can lead directly to linearised descriptions and involves the use of techniques for the selection of model order there is a close theoretical link between models obtained using system identification methods and reduced—order models obtained by linearisation and subsequent reduction from a more complex nonlinear model.

1.2 Block Diagrams of the Model Development Process.

Figure 1.1 is a block diagram which brings together all the main features of the modelling process. The structure of the diagram emphasises both the iterative nature of mathematical modelling and the relevance of system identification methods within this process. For any given application it must be emphasised that the precise route

followed within the diagram will depend on the nature of the system being considered, the amount of prior information available and the purpose of the model.

Comparison of model behaviour with that of the real system is a central feature of this block diagram. In some cases disagreement between the model and system responses may be a symptom of the need for fundamental structural changes in the model which may require the design of new experiments and tests. If, however, an adequate level of agreement is found between the model and the system for a chosen test case and further tests also show a satisfactory level of model performance then some cautious, assessment of the model beyond the range of available experimental results may be attemped. Since models are never unique it is often useful to consider a number of models at this stage in the process since extrapolation beyond the range of experimental results may highlight situations in which the available models disagree. This may also allow new experiments to be defined which may provide a better basis for discriminating between competing candidate models.

No model can ever be fully validated, but a model which has been tested comprehensively may provide a form of 'working hypothesis' which can be applied when direct experimental investigation of the real system is impossible or inappropriate. Such a model may be used until evidence is obtained from the real system which, in some way, invalidates the model and points to further stages of model refinement.

It is important to note that, even at an early stage of development, a model which is incomplete or inaccurate may still be of considerable practical value in providing a means of designing new tests or experiments to be performed on the real system. The results of these new experiments may well provide new insight concerning the system as well as additional test records for subsequent comparison with model outputs. The process shown in Figure 1.1 can be applied not only to the development of a complex nonlinear model but also in a slightly modified way to the evaluation of a reduced—order linear model. In this case the process becomes
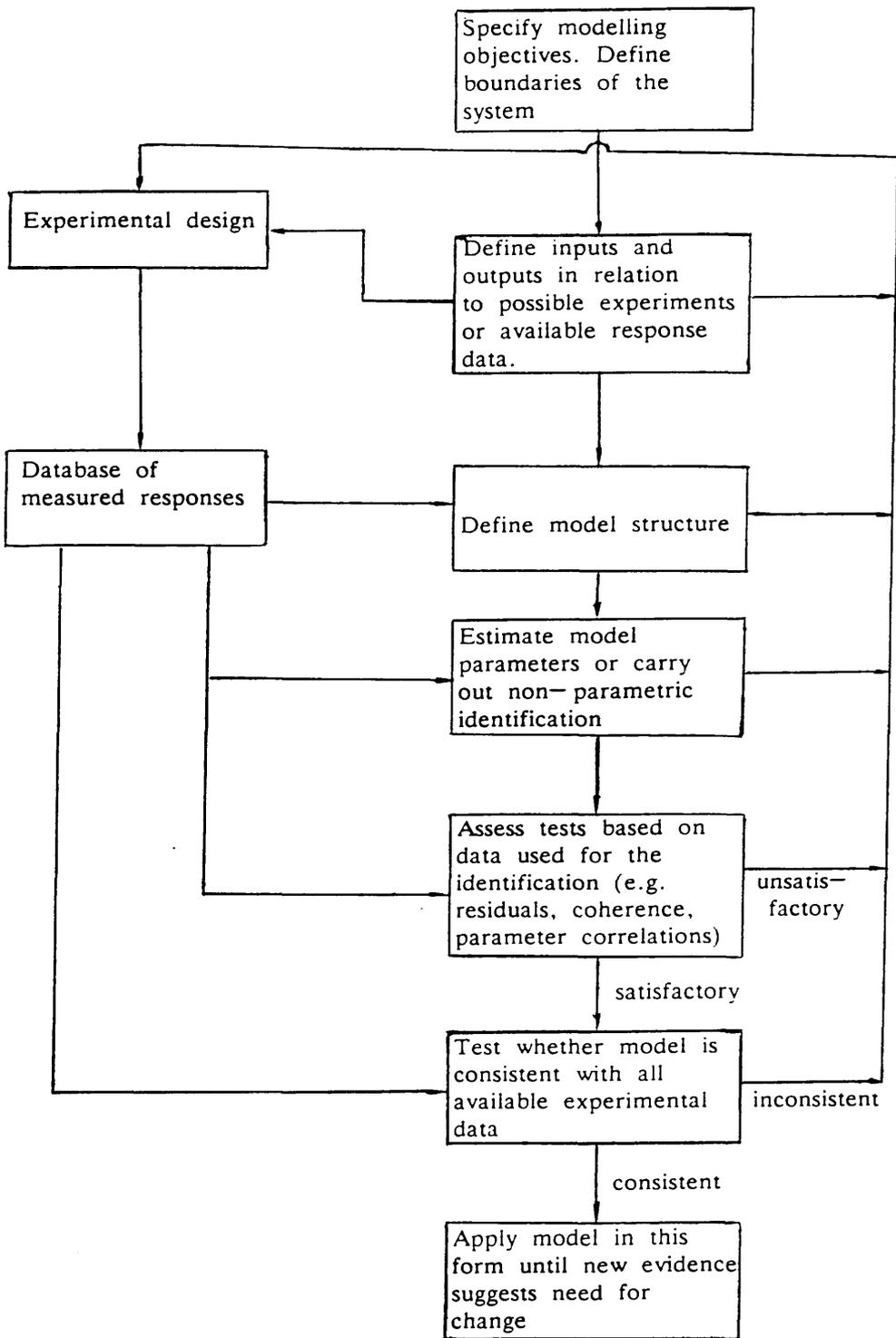
Figure 1.1 The Modelling Process

one of calibration rather than model validation and the objective is to establish the range of validity of the reduced model in terms of both amplitude and frequency. Testing can be carried out with real experimental data and also with simulated response data generated from the high order model. Assessment is likely to be based on residuals both in terms of frequency responses and time histories of output variables. A modified version of the block diagram is shown in Figure 1.2 for the case of reduced—order model development. This process would normally follow the establishment of a more complex nonlinear model using the steps shown in Figure 1.1.

It is important to note that the two block diagrams have many common features. Both involve careful testing of the resulting model and are iterative processes. Both involve databases of experimental or simulated response data and both necessitate careful evaluation of modelling objectives at the outset.

1.3 <u>System Identification Techniques</u>.

. In relation to helicopters, there is a very real need for the use of system identification methods. As outlined above system identification techniques have a role to play in the validation and improvement of existing theoretical flight—mechanics models, as well as for flight testing.

System identification techniques can be separated broadly into two types : time—domain methods and frequency—domain methods. In the literature, identification results from both these methods are reported for helicopter system identification[2,3,4,5,6]. In the Reference [2] it is pointed out that the frequency domain offers some attractive possibilities for helicopter system identification, most notably that it is possible to restrict the frequency range used in the identification, and for which the model is valid, thus enabling models associated with the relatively slow rigid—body dynamics to be obtained which exclude higher—order rotor effects from their description. The measured response and input data is, in this case, effectively transformed into the frequency domain using the Fourier transform; the

Specify modelling objectived define boundaries in terms of amplitude and frequency for reduced order model

High order nonlinear model or linear high order model

Define input and output variables of reduced order model

Simulation of high order nonlinear model

Apply model reduction technique

Database of simulated responses

Assess reduced order model using simulated/ measured data

unsatisfactory

Database of measured responses of real system

Satisfactory for tests considered

Evaluate range of applicability of reduced order model

Apply the reduced order model within the established range

Figure 1.2 Process of reduced order
model development

identification is based on minimising the difference between measured and predicted Fourier transforms rather than measured and predicted time responses. Mathematical details of the problem formulation in terms of the model and cost function minimised are given in the Reference [2]. The models identified are state–space models representing a linearised form of a non–linear helicopter mathematical model for a given flight conditions. The parameters identified include stability and control derivatives. Direct comparisons between estimated values and theoretical values are made. Other identification results using the frequency domain, but based on a spectral–analysis approach are also reported in the literature[7], where fits between measured and predicted transfer functions are obtained. This kind of approach is called non–parametric and does not result in a full set of stability and control parameters.

Leaving aside the question of whether the frequency domain (Fourier domain) or time domain is used in performing the identification, there are three distinct types of optimization methods used for system identification. These are : equation–error, output–error and maximum–likelihood. In fact, the equation–error and output–error methods can be regarded as particular cases of the maximum–likelihood method, and involve simplifying or relaxing the assumptions made about the noise or uncertainty associated with both the measured responses and the model. In the results presented in the literature for helicopter system identification, the equation–error and output–error methods are used[2,8,9]. Because of its simplicity, the equation–error (or regression) approach is often seen as a means of obtaining quickly estimates to be used as initial guesses for the more advanced output–error method. Estimates obtained using the output–error method will be unbiased, even in the presence of measurement noise on the responses used in the identification, whereas the equation–error (or regression) estimates will be biased with respect to the 'true' parameter values. Success with the full maximum likelihood method, which tries to account not only for measurement noise, but also for process noise (i.e.

model uncertainty) has been limited in the context of helicopter system identification because of its complexity.

1.4 Model Reduction Methods.

The modern trend in engineering systems is toward greater complexity, due mainly to the requirements imposed by complex performance specifications and tasks and good accuracy. Complex systems may have multiple inputs and multiple outputs and may be time−varying. Modern control theory, which is a new relatively approach to the analysis and design of complex control systems, has been developed since around 1960. This approach is based on the concept of state. A modern complex system may have many inputs and many outputs, and these may be interrelated in a complicated manner. To analyze such a system, it is essential to reduce the complexity of the mathematical expressions, as well as to resort to computers for most of the tedious computations necessary in the analysis. The state−space approach to system analysis is well suited from this viewpoint. Consider a linear, time−invariant (LTI) system of nth−order; it may been expressed as a set of first order differential equations, which can been futher written as a vector−matrix differential equation. This vector−matrix differential equation is called the state−space representation and has the following form,

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$

And the output equation becomes

$$\underline{y} = C\underline{x}$$

where $x \in R^m$ (R^m is a Cartesian m−space), $u \in R^n$ and A, B and C are constant matrices of appropriate dimensions.

For many multi−input−multi−output (MIMO) systems, the state−space respresentation is in fact the only model convenient to work with. But in many cases the equations the order of the matrix A may be quite large, say 50×50, 100×100 or even 500×500. It is difficult to work with these large matrices and a means of approximating the system matrix by one of lower order is needed.

Therefore, it is very important to reduce the order of such large matrices for control system analysis and design, which can allow, for example, more efficient simulation of dynamic behaviour or may reduce computational demands in on— line applications such as adaptive control. The reduction in the order of matrices or the number of equations is usually called order reduction, or model reduction. This is the main focus of the present work.

A brief survey of the model reduction literature will be presented here. Model reduction research is at present limited mainly to linear systems because order is a measure of complexity only for such systems. But many practical examples, including large space structures, do lead to linear models and linear models are central to most present— day control system analysis and design techniques. The literature pertaining to order reduction of linear systems is vast, as can be seen from the lengthy reference list. Between all the surveys one can obtain an appreciation of the field. However, most of these reviews are not recent, but both Skelton's ideas on cost decomposition[10,11,12] or Moore's "balanced coordinates"[13,14] had important new contributions. Another interesting and quite recent development is Hyland's optimal projection approach to model and controller reduction[15,16,17].

Skelton[11] suggests three categories of model reduction; polynomial approximations, component truncations, and parameter optimization methods. Most polynomial approximation methods are usually based on matching moments of the reduced— order transfer function to those of the original transfer function. Related to moment matching is the fundamental polynomial approximation method using a Pade approximation method. This method investigates basic convergence issues pertinent to all polynomial approximation and has received a good deal of attention[18,19,20,21,22]. The method can be extended directly to multi— input multi— output (MIMO) systems[18]. One problem with all polynomial approximations, shared by the Pade approximations, is that preservation of stability of the original model is not guaranteed in the reduced model[18,19,21].

The parameter optimization approach usually employs a numerical iteration scheme for synthesizing the elements of the reduced model that minimizes an error function of the difference between high- and reduced-order model outputs[23,24,25].

The component truncations methods usually apply to models expressed in state-space form and obtain the reduced-order model by retaining a subset of the original system. Therefore the reduced-model coefficients are more constrained than in the parameter optimization case, where they can be freely synthesized. As such, the reduced models obtained from component truncations may only give "suboptimal" results, but many of these approaches are simple in concept and can be applied to very high-order systems. In addition to discarding rows and columns of the system matrices, various criteria are used to guide these component truncations, and many methods use coordinate transformations to perform the truncation in "more favourable" spaces[37,38,39]. One nice feature of these approaches as compared to the polynomial approximations is that the reduced models produced here always preserve the stability of the high-order models, since their eigenvalues are similar. On other hand, the steady-state response of the reduced model is usually different from that of the high-order model, especially in the simpler approaches. Davison's model simplification technique[26], Marshall's model reduction technique[27] and Moore's "Balanced" approach[13,14] are well known.

In this thesis, the use of a low-order approximations, or equivalent systems, in evaluating aircraft handling qualities is reviewed[28]. This problem is identified as a special case of the more general problem of model reduction in closed-loop systems. In the traditional equivalent system approach, a numerical search algorithm is employed to find the reduced-order model, of 'classical' aircraft form, such that the frequency response of the high-order system (aircraft) is well approximated over a specified frequency range[29]. However questions arise, especially when a good approximation is not obtained with the method. These questions relate to the following: 1) the nonuniqueness of solutions, 2) the interpretation of the matching

cost, 3) the 'good—ness of fit' required, etc..

Because of these general and fundamental difficulties, associated with model reduction, the reduced—order modeling objective of approximating the aircraft's frequency response is being re—examined, in the current work and the question of when and how to match multiple frequency responses is being reviewed. A alternative state—space model—reduction approach developed by Bacon and Schmidt[30] is also being considered. In this method the original transfer function (matrix) G(s) of dynamic order n is reduced via a state—space transformation T. The construction of T involves no numerical search algorithm. In terms of the three classes of problem identified above this method is therefore essentially a form of component truncation method. In addition, the resulting model Gr obtained by this method is unique for the selected dynamic order r and the least effective dynamic order is determined a priori by evaluating a set of frequency—domain matching error bounds. These error bounds, furthermore, are applicable to each i—j element of $[G(s) - Gr(s)]_{s=j\omega}$ over all ω. One important feature of the approach is that it is applicable to multi—input/multi—output systems and is therefore well suited for aircraft applications.

Chapter 2

Helicopter Flight Mechanics Models

2.1 Introduction.

In mathematical terms, the helicopter represents a complex arrangement of in teracting sub— systems that can be viewed from a component (Fig. 2.1) or dynamics prespective. In the former case the subsystems are chosen based on a break down into physical parts, eg. main rotor, fuselage etc., while in the latter case the choice is based on a partitioning into physical or mathematical sub— systems arranged and interconnected according to their frequency and amplitude content. Models of rotorcraft can also be partitioned into three distinct levels[31], differentiated largely by the form of rotor modelling, as summarised in Table 2.1.

Table 2.1 Levels of Helicopter Theoretical Model[31]

	Level 1	Level 2	Level 3

Aerodynamics	Linear 2D
	Dynamic inflow/local
	  momentum theory
	Analytically integrated loads
		Nonlinear (limited 3-D)
		Dynamic inflow/local
		  momentum theory
		Local effects of blade/vortex
		  interaction
		Unsteady 2-D
		Compressibility
		Numerically integrated loads
			Nonlinear (3-D)
			Full wake analysis
			  (free or prescribed)
			Unsteady 2-D
			Compressibility
			Numerically integrated loads

Dynamics	(i) Rigid blades
	6 d.o.f. quasi-steady rotor
	9 d.o.f.-rotor flapping
	12 d.o.f.-flap+lag
	15 d.o.f.-flap+lag+pitch
		(i) Rigid blades
		  Options as in Level 1
		(ii) Limited number of blade elastic
		  modes
			(i) Elastic modes (detailed
			  structural) representation

Application	Parametric trends for flying qualities
	  performance studies
	Within operational flight envelope
	Low bandwidth control
		Parametric trends for flying qualities
		performance studies
		Beyond operational flight envelope
		Medium bandwidth appropiate to
		  high gain active flight control
			Rotor design
			Rotor load prediction across a
			High bandwidth
			Beyond operational flight
			  envelope

Level 1 modelling includes the 'rigid body', six degree of freedom linear or non–linear formulation with quasi–steady rotor dynamics and extends to the inclusion of rotor blade dynamics in multi–blade coordinate forms with analytically integrated blade loadings, actuators etc..   Such models are useful for flying qualities and performance studies within the normal flight envelope where integrated rotor loads are not significantly affected by rotor stall, compressibility effects and the attendant rotor blade dynamic couplings.   Level 3 modelling, at the other extreme, represents the most comprehensive rotor/fuselage modelling necessary to predict, not only integrated, but also vibratory loadings across a wide frequency bandwidth.   This is, in general, unnecessarily complex to be appropriate to flight mechanics work.   Between these extremes Level 2 models provide a means of exploring the design implications for high gain active control systems in a piloted simulation environment.   The need for increased exploitable primary flight control bandwidth, and accurate modelling of flight in conditions where aerodynamic nonlinearities and rotor couplings prevail, require model enhancements beyond Level 1 but do not necessitate the refinements of Level 3 models.   Level 2 model development is still an area of research and Table 2.1 suggests possible elements, under the Level 2 heading.

Although much current research is concerned with the problems of Level 2 and Level 3 models there is a need for Level 1 models which are sufficiently simple to allow computations/simulations in a time scale well suited to:

   a) Real time simulation for control system evaluation in piloted ground–based simulators and for handling qualities studies.

   b) Control system design calculations performed interactively.   Control systems designed initially using a Level 1 model could be tested within a Level 2 simulation or Level 3 using batch mode of processing at a later stage to provide information about rotor loads and the effects of high frequency modes associated with the rotor and fuselage.

2.2 Axis Systems.

In order to analyse a rotorcraft in flight it is necessary to first define a set of axes which will act as a reference frame around which the relevant equations of motion may be developed. Since the rotorcraft is a free body in space, its position and flight path may be defined with respect to a set of earth—fixed axes, which remain fixed relative to the earth. These earth axes assume a flat, non—rotating earth and arbitrary origin, with the x—axis pointing Northward, y—axis Eastward and the z—axis pointing down to the centre of the earth.

However, this axis system is inconvenient for some analysis and therefore a set of axes which remain fixed relative to the airframe can be used. This axis set is called the body—fixed axis and the origin is located at the aircraft's centre of gravity with the x—axis pointing forward, y—axis to starboard and z—axis downwards. It is conventional to define the nomenclature associated with the body—fixed axis system in a standard form and this is summarised in Table 2.2 and Fig. 2.2.

2.3 Equations of Motion.

The helicopter equations of motion of the single rotor helicopter are derived by summing the force and moment contributions of various structural components of the helicopter system. The most important of these components are the fuselage, the main rotor, the tail rotor, engine and transmission system, as illustrated in Fig. 2.1 which is taken from the report [84] by R. Bradley & G. Padfield etc..

The coordinate system used to describe the single rotor helicopter system will be the body fixed axis mentioned in section 2.2. Fig. 2.2 shows the x, y and z axes of this axis system along with the X, Y and Z components of total force and the L, M and N components of the total moment. The derivation of force and moments is given by Padfield[32] along with the nonlinear equations of motion of the fuselage at the centre of gravity.
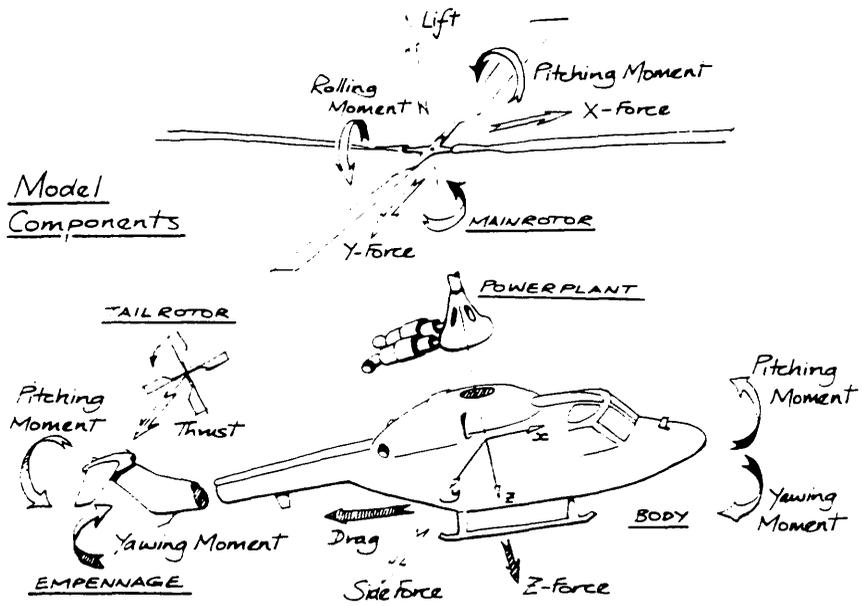
Fig. 2.1 Components of a Helicopter

(taken from report by R. Bradley & G. Padfield etc.[84])
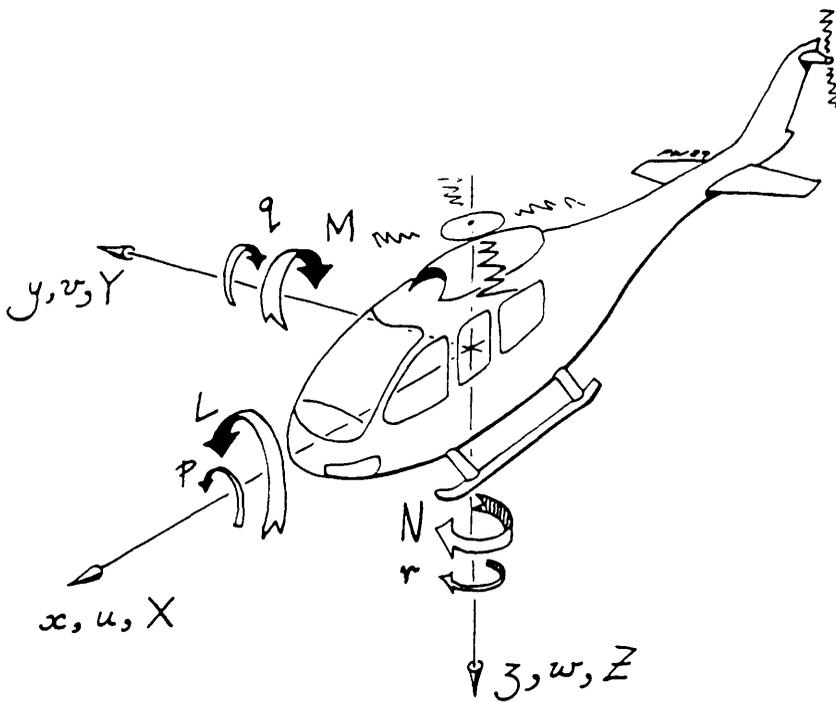
Fig. 2.2 Body Axes System

(taken from report by R. Bradley & G. Padfield etc.[84])

Table 2.2 Body Axes Definitions

AXIS	OX	OY	OZ
NAME	LONGITUDINAL	LATERAL	NORMAL
LINEAR DISPLACEMENT (m)	x	y	z
STEADY STATE VELOCITY (ms⁻¹)	U	V	W
INCREMENTAL VELOCITY (ms⁻¹)	u	v	w
FORCE COMPONENT (N)	X	Y	Z
ROLLING VELOCITY COMPONENT (rads s⁻¹)	ROLL p	PITCH q	YAW r
ANGULAR DISPLACEMENT (rads)	φ	θ	ψ
ROLLING MOMENT (Nm)	L	M	N

Let u, v, w and p, q, r be the vehicle translational and rotational velocity components along and about the X, Y and Z axes respectively; ψ, θ, φ the Euler angle in the transformation from earth to body axes; M the vehicle mass and g the gravitational constant. The equations of transtational motion can then be written in standard form. (see Ref.[32]).

$$
\begin{aligned}
\dot{u} &= -(wg - vr) + X/M - g \sin \theta \\
\dot{v} &= -(ur - wp) + Y/M + g \cos \theta \sin \varphi \\
\dot{w} &= -(vp - uq) + Z/M + g \cos \theta \cos \varphi
\end{aligned}
\qquad (2.1)
$$

Let I_{xx}, I_{yy}, I_{zz} be the moments of inertia about the X, Y and Z axes respectively and I_{xz} the product of inertia about the X and Z axes. Assuming the axes are chosen so that I_{xy} and I_{yz} are negligible, then the equations of rotational

motion can be written in standard form:

$$I_{xx}\dot{p} = (I_{yy} - I_{zz})\, qr + I_{xz}\,(\dot{r} + pq) + L$$

$$I_{yy}\dot{q} = (I_{zz} - I_{xx})\, rp + I_{xz}\,(r^2 - p^2) + M \qquad (2.2)$$

$$I_{zz}\dot{r} = (I_{xx} - I_{yy})\, pq + I_{xz}\,(\dot{p} - qr) + N$$

The Euler angles from the gravitational components of equation (2.1) can be determined from the differential equations relating them to the body angular velocity components. Hence,

$$\dot{\varphi} = p + q\,\sin\varphi\,\tan\theta + r\,\cos\varphi\,\tan\theta$$

$$\dot{\theta} = q\,\cos\varphi - r\,\sin\varphi \qquad (2.3)$$

$$\dot{\psi} = q\,\sin\varphi\,\sec\theta + r\,\cos\varphi\,\sec\theta$$

The importance of the above Euler equations, which are derived from a consideration of Newton's second law of motion, is that they allow the body velocities and accelerations to be defined in terms of the forces and moments acting on the aircraft. In Equ. 2.1, 2.2, 2.3 we only consider the case representing a six degrees of freedom rigid— body (quasi— static model). A more general representing of the helicopter, will also include equations with states representing the behaviour of the rotor, and its coupling with the rigid— body motion of the helicopter. If we just consider the theoretical model of the single main rotor helicopter, it has options for a range of different degrees of freedom in the model: 9 degrees of freedom (with first order modelling of main rotor flap), 12 degrees of freedom (with second order modelling of rotor flap). In these two cases, we obtain 12 equations of motion and 15 equations of motion, respectively.

2.4 External Forces and Moments.

If we neglect the weight of the air, the external forces acting on a helicopter are the gravity and the aerodynamic action. The aerodynamic action can be expressed as

a total force acting on the centre of gravity and a moment around it. For the convenience of using them in our control system model equations, they can be further decomposed into three components which are along three axes of the body fixed coordinate system, separately and three component moments around these axes, namely the pitching moment the yawing moment and the rolling moment as shown in Fig. 2.1. If a rotor is added the rotor is rotating all the time and can change its effective orientation through changes of blade pitch. The resulted aerodynamics will be more complicated, but at any time instant, they can still be expressed by components within the coordinate system used. In this chapter, a theoretical model of the single main rotor helicopter has been derived for flight mechanics studies. The forces and moments from the rotor itself are discussed in detail and the reasons are presented for the choice of a rigid blade and centre spring model to represent flapping with all types of rotor. To solve the differential equations (2.1, 2.2, 2.3, it is necessary to first determine the external forces (X, Y, Z) and moments (L, M, N). The external forces and moments from the defferent elements can be written in component form as:

$$
\left.
\begin{aligned}
X &= X_R + X_T + X_{TP} + X_{FN} + X_F \\
Y &= Y_R + Y_T + Y_{TP} + Y_{FN} + Y_F \\
Z &= Z_R + Z_T + Z_{TP} + Z_{FN} + Z_F
\end{aligned}
\right\} \quad (2.4)
$$

$$
\left.
\begin{aligned}
L &= L_R + L_T + L_{TP} + L_{FN} + L_F \\
M &= M_R + M_T + M_{TP} + M_{FN} + M_F \\
N &= N_R + N_T + N_{TP} + N_{FN} + N_F
\end{aligned}
\right\} \quad (2.5)
$$

where, the suffices, from left to right, refer to main rotor, tail rotor, tailplane, fin and fuselage respectively.

2.5 Levels of Model.

Modelling the ensuing loads in the rotor system and their transmission to the fuselage is commonly described as an aero—servo—elastic problem and can conveniently be formulated at one of the three levels shown in Table 2.1. As already described, the level of complexity relates to the application area. Currently, most organisations work with Level 1 models for the prediction of flying qualities and low bandwidth control up to 1 HZ. At the other extreme of complexity, Level 3 models are generally required for the prediction of rotor loads over a high bandwidth, up to 100 HZ, for vibration analysis and rotor design. Adequate modelling for high bandwidth flight control and the prediction of dynamic characteristics at the flight envelope boundaries are still areas of research. The principal distinguishing features of this, Level 2 model are likely to be non—linear, unsteady aerodynamics integrated along curved elastic model shapes.

2.6 Mathematical Representation of Level 1 Models.

The mathematical representation of Level 1 models can be simplified if we neglect the yaw angle ψ and defining the rigid—body states to be $\underline{x}_F$, and the rotor states to be $\underline{x}_R$, we can then write the following linear constant coefficient equations:

$$\dot{\underline{x}}_F = A_{FF} \, \underline{x}_F + A_{FR} \, \underline{x}_R + B_F \, \underline{u} \tag{2.6}$$

$$\dot{\underline{x}}_R = A_{RF} \, \underline{x}_F + A_{RR} \, \underline{x}_R + B_R \, \underline{u} \tag{2.7}$$

where the matrices A_{FF} and A_{RR} represent the uncoupled systems for the fuselage and rotor. The matrices A_{FR} and A_{RF} represent the coupling between the fuselage and the rotor; B_F and B_R are the fuselage and rotor control dispersion matrices relating the rates of change of state variables to the inputs u.

2.7 Linearisation of a Level 1 Model.

In this section we present a linearization technique that is applicable to many nonlinear systems. The process of linearizing nonlinear systems is important, for by linearizing nonlinear equations, it is possible to apply numerous linear analysis methods that will produce information on the behavior of the underlying nonlinear

systems. The linearization procedure presented here is based on the expansion of the nonlinear function into a Taylor series about the operating point and the retention of only the linear terms. Because we neglect higher—order terms of the Taylor series expansion, these neglected terms must be small so that the variables deviate only slightly from the operating condition.

For flight control systems design it is a common and useful practice to use linearized equations of motion to describe the vehicle's dynamics near a prescribed operating point in the flight envelope. The method of linearizing the equations of motion is through the use of stability and control derivatives[33]. Consider a nonlinear system whose input is u and output is y. Thus the relationship between y and u may be written

$$y = f(u) \tag{2.8}$$

If the normal operating condition corresponds to a point (u_1, y_1), then $y = f(u)$ can be expanded into a Taylor series about this point as follows

$$y = f(u) = f(u_1) + df/du(u-u_1) + 1/2! \ d^2/du^2 \ (u-u_1)^2 + \dots . \tag{2.9}$$

where the derivatives df/du, d^2f/du^2, ... are evaluated at the operating point, $x = u_1$, $y = y_1$. If the variation $u-u_1$ is small, we can neglect the higher—order terms in $u-u_1$. Noting that $y_1 = f(u_1)$, Equ. (2.9) can be written

$$y - y_1 = a \ (u - u_1) \tag{2.10}$$

where $a = df/du|_{u=u_1, y=y_1}$.

Equation (2.10) indicates that $y - y_1$ is proportional to $u - u_1$. It is a linear mathematical model for the nonlinear system given by Equ. (2.8) near the operating point $u = u_1$, $y = y_1$.

The nonlinear force and moment equations are described as a Taylor series expansion about the desired operating point in terms of the degrees of freedom of the aircraft. By truncating each series to first order terms, a linear model of the system is derived. The stability and control derivatives are the coefficients of the linear terms of this Taylor series expansion. The truncated series of linear

representation can be arranged into state space canonical form.

$$\dot{\underline{x}} = A \underline{x} + B \underline{u}$$

$$\underline{y} = C \underline{x}$$

The stability and control derivatives are used to derive the elements of the system matrix, A, and the input distribution matrix, B, respectively. The C matrix is the output matrix for the general system. For Level 1 of models, we consider only a 6 degrees of freedom model which does not include rotor dynamics. The model is therefore an eighth order system.

The state vector, x(t) of the six degrees of freedom model is made up of the eight rigid body states of the fuselage,

$$x(t) = \begin{bmatrix} u \\ w \\ q \\ \theta \\ v \\ p \\ \varphi \\ r \end{bmatrix} \tag{2.11}$$

where u --- longitudinal velocity   (feet/s)

w --- Z - axis velocity   (feet/s)

q --- pitch rate   (rad/s)

θ --- pitch angle   (rad)

v --- lateral velocity   (feet/s)

p --- roll rate   (rad/s)

φ --- roll angle   (rad)

r --- yaw rate   (rad/s)

Following standard practice, the yaw angle is not included as a state variable because the heading on which an aircraft is flying does not effect its stability or control. This can be deduced from the nonlinear equations of motion in which heading or yaw angle, ψ, does not appear in the equations for the other states.

Chapter 3

Review of Model Reduction Methods

3.1. The Necessity and Possibility of Model Reduction.

The mathematical modelling of physical systems or processes often leads to very large models involving a high−order set of equations and reduced order modelling is considered very important for some aspects of control system design and analysis. Usually, it is possible to represent a physical system by a number of simultaneous linear differential equations with constant coefficients,

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

but for many processes (e.g., chemical plants, nuclear reactors), the order of the matrix A may be quite large, say 50×50, 100×100, or even 500×500. In the case of helicopters, the mathematical models obtained have a high order when compared to fixed−wing aircraft. Not only is there appreciable coupling between the longitudinal and lateral rigid−body dynamics − which can often be assumed to be decoupled for some fixed−wing aircraft responses leading to reduced−order models, but there are also the dynamics of the main rotor which can be modelled to varying degrees of complexity and which are also coupled to the rigid−body dynamics.

Helicopter models range in order and complexity from a rigid−body, six−degree−of freedom linear or nonlinear formulation with quasi−steady rotor dynamics (including situations where the longitudinal and lateral body dynamics themselves are considered separately), to those incorporating rotor−blade dynamics in multi−blade coordinate forms with analytically integrated blade loadings, together with a range of additional dynamic elements, e.g. engine/rotorspeed, actuators, etc. Usually, the equations of motion of the single rotor helicopter are derived by summing the force and moment contribution of various structural components of the helicopter system. The most important of these components are the main rotor, the tail rotor, and the fuselage.

The difficulties encountered in attempting to model helicopters arise in two areas.

Firstly, the dynamics of the main rotor are complex, particularly during transient manoeuvres, and in addition, aerodynamic coupling is considerably more pronounced for helicopters than for fixed wing aircraft. G. D. Padfield[32] has derived general equations for a single main rotor helicopter using a body fixed axis system. The derivation of forces and moments is given by Padfield[32] along with the nonlinear equations of motion of the fuselage at the centre of gravity. There are nine equations of motion for the rigid−body dynamics. Following standard practice, the yaw angle is not included as a state variable in this model because the heading on which an aircraft is flying does not affect its stability or control. This can be deduced from the nonlinear equations of motion in which heading or yaw angle, ψ, does not appear in the equations for the other states. If we consider the rotor dynamics we have to include more equations depending on the complexity of the rotor model. For example, if we have a second−order flap model this will, in fact, introduce a further six equations into the overall model. This means there are fourteen equations of motion for the design and analysis of the helicopter plant. It may be judged that such a large model, with the potential for there being significant uncertainties both in the known values of the model parameters and the measurements of the model states, is too impracticable for the design and implementation of a helicopter control system. In this context, there is also a need to take into consideration the size, cost, and capability of the onboard computer system required for control, as well as the availability of appropriate measurement devices to provide the required measured responses. Of course, future technological developments, may in time overcome these problems.

Model reduction or simplification resulting in explicit representation of rotor dynamics in the plant model is the most immediate requirement to make the helicopter model more manageable. The quasi−steady reduction leading to this, is discussed in general terms in section 2.2. It is the large separation in the characteristic frequency ranges between the rigid−body and rotor−state dynamics

which provides the justification for this approach to reduction in the model order for helicopters. The introduction of bearingless rotors in modern designs of helicopter makes this approach more difficult to apply since the frequency separation may then be much smaller.

In mathematical terms, the difficulty of working with large matrices means that some way of approximating the system matrix by one of lower order is needed. In other words, if the system matrix could be reduced by some method, the control system will become simpler and it will be better for design and analysis purposes. When input—output behaviour is considered, then very probably a much simpler model with only a few differential equations would describe the same physical process for many purposes. For many practical purposes including control system design the input—output behaviour is often most important and simple input—output approaches to model reduction are often possible. On the other hand, the practical application of such modern concepts as state estimation, optimal state feedback and even numerical simulation depend upon the availability of intermediate variables, but may also be limited by the capacity of the available digital computer thus requiring a more general approach to reduction which is not limited to input—output considerations.

3.2 The Model Reduction Methods.

3.2.1 Development of the Model Reduction Methods.

A number of techniques have been proposed for the reduction of systems. Much research has been done during the last 23 years concerning the derivation of low—order models from high—order systems, as is evident from the comprehensive bibliography prepared by Genesio and Milanese (1976). Although many different approaches have been published, these may be divided broadly into three main groups. A brief survey of the techniques will be presented here.

3.2.1.1 Dominant Eigenvalues Approach.

In the first group of methods attempts are made to retain the dominant

eigenvalues of the original system and then to obtain the parameters of a low—order model such that its response to given inputs approximates closely that of the original high—order system. This class of method includes those proposed by Davison[26] (1966), Marshall[27] (1966), Mitra[35] (1967) and Aoki[36] (1968). Preservation of dominant eigenvalues (Davison[26,37] 1966, 1968, Davison and Chadha 1972, Chidambara[38] 1969) is an important feature of these techniques although individual methods differ in other respects. For example, Marshall[27] (1966) developed a reduced order model in which the discrepancy in d.c. gain between the model and the original higher order system was eliminated and dominant eigenvalues were retained. For better matching of all phases of the response (initial, intermediate and steady state), a combination of three reduced models has been proposed (A. Kuppurajulu and Elangovan[39] 1970). Mitra[44] (1969) explicity recognized model truncation as a projection, thus anticipating Hyland[40] (1984), and Mitra also suggested retaining the components with the highest entry in a diagonalized controllability matrix, thus preceding Moore's 'balanced' approach[13] (1981). It has been shown (Hickin[41] 1978) that the methods proposed by Davison, Marshall and Mitra may be regarded as special cases of the aggregation method proposed by Aoki[36]. Another approach which preserves the dominant eigenvalues is the method of singular perturbations (Sanuti and Kokotovic[42] 1968), which has certain special properties.

3.2.1.2 The Matching of Impulse— or Step—Responses (or Frequency Response).

The second group of methods involves finding a model of a specified order such that its impulse— or step—response (or, alternatively, its frequency response) provides an optimum match to that of the original system, without restriction on eigenvalue location. Anderson[43] has proposed an optimal projection method using a discretized version of the continuous system (1967). This method used a geometric approach, based on orthogonal projection to obtain a low—order model minimizing the integral square error in the time—domain. Similar optimal projection methods

can be applied to models expressed in transfer function instead of state—space form (e.g. Sinha[45] 1971). Sinha and Pille[46] (1971) have proposed utilizing the matrix pseudoinverse for a least—squares fit. Chen and Shieh[47] (1968) and Liaw, Pan and Chen[48] (1986) showed that if a continued—fraction expansion of a transfer function was truncated, it led to a low—order model the step—response of which matched that of the original system closely. The main attraction of this approach was its computational simplicity, as compared with the methods described in the first two categories. The method usually employs a numerical iteration scheme for synthesizing the elements of the reduced model that minimizes some appropnate function of the difference between full and reduced order outputs. This method can also be extended to multi—input multi—output cases (e.g. C.F. Chen[49] 1974). Other methods for obtaining optimum low—order models have been proposed in the frequency domain (e.g. Langholz and Bistritz[50] 1978, Elliott and Wolovich[51] 1980).

3.2.1.3 <u>The Matching of Some Other Properties of the Responses</u>.

One popular method is based on matching time moments of the reduced order transfer function to those of the original transfer function. (Gibilaro, Lees[52] 1969, Lees[53] 1971, Kropholler[54] 1970, Zakian[55] 1973). The time moment matching method leads to quick convergence of the steady—state response. Convergence of the transient response is improved by also matching the first few Markov parameters (Rossen[56] 1972, Shamash[57] 1975). One other approach which is closely related to moment matching is the fundamental polynomial approximation method using Pade approximants. (Appiah[58] 1979, Shamash[57] 1975, Daly and Colebourn[60] 1979) Although initially these methods were developed for single—input single—output systems only, it has been shown by Hickin and Sinha[61] (1976) that one may also match the time moments and obtain low—order models for multi—input multi—output systems using partial realization with the generalized Markov parameters. An important drawback of the methods using Pade approximation is that the low—order

models obtained may sometimes turn out to be unstable even though the original system is stable. This has led to the development of the Routh approximation method (Hutton and Friedland[62] 1975). In order to ensure simplicity in the procedure and to maintain stability of the model, some mixed methods have also been introduced (Shieh and Wei[63] 1975, 1976, Chen[64] 1980, Therapos and Diamessis[65,66] 1983, 1984).

3.2.2 A Comparison of Selected Model Reduction Methods.

Model reduction research is at present limited to linear systems and it is convenient to classify the model reduction literature according to the main divisions outlined above and to focus on some representative papers. Three classical model reduction methods have been selected for initial review. The first approximate method is that of S. A. Marshall[27] in 1966 and involves models in state-space form. The second approximate method considered in was the stability-equation method and the continued fraction method of T. C. Chen[64] in 1980. The third reduction method considered in this section was proposed by C. M. Liaw[48] in 1986 using dispersion analysis and continued-fraction techniques.

3.2.2.1 Marshall's Method[27].

In 1966, an approximate method in which dominant eigenvalues are presented was proposed by S.A. Marshall. In this method, the high-order system is represented by the vector-differential equation or state-space form

$$\dot{x} = Ax + Bu \tag{3.1}$$

where x is the n - state vector of the system, A and B are respectively n × n and n × r constant coefficient matrices and u is r - input vector. For convenience, we partition equ.(3.1) so that the m variables to be retained in the reduced model are the first m variables of the state vector x, then equ. (3.1) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \hline \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ \hline A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u \tag{3.2}$$

Consider now the transformation

$$x = Uz \qquad (3.3)$$

where U is the modal matrix of A and apply it to equ. (3.1), giving

$$\dot{z}(t) = U^{-1}AUz(t) + U^{-1}Bu(t)$$

i.e. $\qquad \dot{z}(t) = \Lambda z(t) + U^{-1}Bu(t) \qquad (3.4)$

where Λ is the n $\times$ n diagonal matrix whose elements are the eigenvalues of A. Equ. (3.4) in partitioned form becomes

$$\begin{bmatrix} \dot{z}_1 \\ \hline \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ \hline 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ \hline z_2 \end{bmatrix} + \begin{bmatrix} V_1 & V_2 \\ \hline V_3 & V_4 \end{bmatrix} \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u \qquad (3.5)$$

where $V = U^{-1}$

Now, the first m eigenvalues are contained in the submatrix Λ_1 and the remaining n$-$m eigenvalues in Λ_2.

Mathematically, the approximation involved in the development of the reduced$-$order model is equivalent to putting

$$\dot{z}_2 = 0$$

equ.(3.5) then becomes

$$\dot{z}_1 = \Lambda_1 z_1 + (V_1 B_1 + V_2 B_2)\, u$$

and $\qquad 0 = \Lambda_2 z_2 + (V_3 B_1 + V_4 B_2)\, u$

Now from (3.3)

$$z = U^{-1}x = Vx$$

or $\qquad \begin{bmatrix} z_1 \\ \hline z_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \\ \hline V_3 & V_4 \end{bmatrix} \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix}$

giving $\quad z_2 = V_3 x_1 + V_4 x_2 = -\Lambda_2^{-1} (V_3 B_1 + V_4 B_2)\, u$

$$x_2 = -V_4^{-1}V_3 x_1 - V_4^{-1}\Lambda_2^{-1} (V_3 B_1 + V_4 B_2)\, u \qquad (3.6)$$

Substituting equ.(3.6) into (3.2) and using the relationships between the U_i and V_i, one obtains

$$\dot{x}_1 = U_1 \Lambda_1 U_1^{-1} x_1 + [B_1 - A_2 V_4^{-1} \Lambda_2^{-1} (V_3 B_1 + V_4 B_2)] u \qquad (3.7)$$

This set of equations approximates to the original set of n equations and is called the reduced system.   One important aspect is that the steady—state values of the reduced system are identical to the steady—state values of the original system.   This is not so with the techniques presented by Davison[26].

To illustrate the procedures a very simple example for Marshall's method is considered.   The original system is represented by

$$\begin{bmatrix} \dot{x}_1 \\ \hline \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ \hline 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ \hline 2 \end{bmatrix} u \qquad \text{(Example 1)}$$

and it is to be reduced to a system of order 1 by neglecting the dynamic effects associated with the large (in modulus) eigenvalue.   The eigenvalues are -1 and -10, and modal matrix is

$$U = \begin{bmatrix} 1 & 1 \\ \hline 0 & -9 \end{bmatrix}$$

$$U^{-1} = V = \begin{bmatrix} 1 & 1/9 \\ \hline 0 & -1/9 \end{bmatrix}$$

Substituting the required values in to equ.(3.7) yields

$$\dot{x}_1 = -1x_1 + 6/5 \ u \quad \text{and} \quad x_2 = 1/5 \ u$$

The responses of the 2×2 system and the reduced 1×1 system are shown in Fig.(3.1).

3.2.2.2 Stability— Equation Method and Continued— Fraction Method[64].

T. C. Chen, C. Y. Chang and K. W. Han[64] have developed a combined method for model reduction based upon the stability— equation method and the continued— fraction method.   The procedure, as explained below, consists of three steps: 1. to reduce the denominator of a transfer function by the stability— equation method, 2. to obtain partial quotients by the algorithm of the continued— fraction method, and 3. to discard the undesired partial quotients and to reconstruct the

reduced model of which the denominator is obtained from step 1. These three steps, of course, are fully computer-oriented.

Step 1. A transfer function of a high-order system (HOS) can be expressed as:

$$F(s) = \frac{a_{21} + a_{22}s + \dots + a_{2,n}s^{n-1}}{a_{11} + a_{12}s + \dots + a_{2,n+1}s^n} = \frac{F_N(s)}{F_D(s)} \qquad (3.8)$$

where $F_D(s)$ and $F_N(s)$ are the denominator and numerator of $F(s)$, respectively. For a stable system, the stability equations of $F_D(s)$ (i.e., the even part and the odd part of $F_D(s)$) can be factored as (Han and Thaler 1966, Chen and Han 1979)

$$F_{De}(s) = a_{11} \prod_{i=1}^{l_1} (1 + s^2 / z_i^2)$$

$$F_{Do}(s) = a_{12} \, s \prod_{i=1}^{l_2} (1 + s^2 / p_i^2)$$

where l_1 and l_2 are the integer part of $n/2$ and $(n-1)/2$, respectively,

and $z_1^2 < p_1^2 < z_2^2 < p_2^2 < z_3^2 < p_3^2 < \dots\dots$

Since the p_i^2 and z_i^2 are in sequence, discarding the factor with larger magnitudes of p_i^2 or z_i^2 is an approach for reducing the order of stability equations, so that the order of the original system can be reduced. The reduced stability equations with desirable order k are written as

$$F_{De}'(s) = a_{11} \prod_{i=1}^{m_1} (1 + s^2 / z_i^2)$$

$$F_{Do}'(s) = a_{12} \, s \prod_{i=1}^{m_2} (1 + s^2 / p_i^2)$$

where m_1 and m_2 are the integer part of $k/2$ and $(k-1)/2$, respectively. Then the reduced denominator is constructed as

$$F_{Dk}(s) = F_{De}'(s) + F_{Do}'(s) = \sum_{j=0}^{k} a_{1,j+1}' \, s^j$$

Step 2. To find the partial quotients by the algorithm of continued-fraction, the following Routh array of the coefficients of the original function is used.

$$h_1 = a_{11} \, a_{21}^{-1}$$

$$h_2 = a_{21} \, a_{31}^{-1}$$

$$h_3 = a_{31} \, a_{41}^{-1}$$

$$\vdots$$

$$h_i = a_{i1} \, a_{i+1,1}^{-1} \qquad i=1,2,\ldots,2J \text{ for any } J \text{ where } J \leqslant n$$

(3.9)

where a_{ij} is obtained from equ.(3.8) for $i=1,2$ with $j=1,2,3,\ldots$

and $a_{ij} = a_{i-2,j+1} - h_{i-2} \, a_{i-1,j+1}$ for $i=3,4,\ldots$ with $j=1,2,3,\ldots$

Step 3. To construct the reduced model with order k, one can retain the first h_k values and replace the new values of the first row of equ.(3.9) by the coefficients of the reduced denominator obtained from step 1.   The new values of a_{2j}' can be evaluated by the following algorithm

$$a_{i+1,1}' = h_i^{-1} a_{i,1}' \qquad\qquad i=1,2,3,\ldots,k \text{ and } k \leqslant n$$

$$a_{i+1,j+1}' = h_i^{-1}(a_{i,j+1}' - a_{i+2,j}') \quad i=1,2,3,\ldots,k-j \quad j=1,2,\ldots,k-1 \quad (3.10)$$

and then the reduced model is

$$F_k(s) = \frac{a_{21}' + a_{22}' \, s + \ldots + a_{2,k}' \, s^{k-1}}{a_{11}' + a_{12}' \, s + \ldots + a_{1,k+1}' \, s^k}$$

(3.11)

In equ.(3.9) if $a_{i,1}^{-1}$ (i= 2,3,4,...) do not exist, (i.e., $a_{i,1} = 0$,) then $a_{i+1,j}$ and h_i^{-1} in equ.(3.9) and in equ.(3.10) are infinite.   In order to remedy this situation, one can replace the $a_{i,1}$ which are equal to zero by an arbitrary small positive number ε and then proceed to find the remainder of the Routh array. After applying equ.(3.10) and letting ε equal to zero, the new values of $a_{2,j}'$ can be evaluated.

The reduced model is guaranteed to be stable if the original system is stable, but it is noted that, if the zeros of the original system are located in the right half–plane, the stability–equation method proposed by Chen and Han cannot be applied directly for model reduction.

As a example (Example 2), we consider a system with closed-loop transfer function (Han[6 7] 1978) for the stability-equation method and continued-fraction method:

$$F(s) = \frac{s^4 + 35s^3 + 291s^2 + 1093s + 1700}{s^9 + 9s^8 + 66s^7 + 294s^6 + 1029s^5 + 2541s^4 + 4684s^3 + 5856s^2 + 4629s + 1700}$$

The pole-zero configuration is shown in Fig.(3.2). Because all the poles have the same real parts, this is a difficult problem for those techniques based upon the dominant-root principle.

Then, the stability equations of the denomination are

$$F_{De}(s) = 1700 \prod_{i=1}^{4} (1 + s^2/z^2)$$

$$F_{Do}(s) = 4629 \, s \prod_{i=1}^{4} (1 + s^2/p^2)$$

where $z_1^2 = 20.3321$ $p_1^2 = 45.6652$

$z_2^2 = 8.9119$ $p_2^2 = 13.4228$

$z_3^2 = 3.0847$ $p_3^2 = 5.5542$

$z_4^2 = 0.3379$ $p_4^2 = 1.3597$

Discarding the factors with p_1^2, p_2^2, p_3^2 and z_1^2, z_2^2 the reduced denominator is

$$F_{Dk}(s) = \sum_{j=0}^{4} a'_{1,j+1} \, s^j$$

where $a'_{1,j+1} = [1.000 \ 2.0873 \ 3.4219 \ 2.8382 \ 1.0423]$

From equ.(3.9) the first five partial quotients by the algorithm of continued-fraction are

$h_1 = 1$, $h_2 = 0.4808$, $h_3 = -2.2345$, $h_4 = -1.2961$, $h_5 = -1.8895$

This means that the reduced order k = 4. Now we can construct the reduced model with order k= 4. The first h_k values will be retained and be replaced the new values of the first row of equ.(3.9) by the coefficients of the reduced denominator obtained from step 1. The reduced numerator is:

num = [− 0.4123 0.0097 0.6702 1.0423]

Therefore the reduced model is

$$F_4(s) = \frac{-0.4123s^3 + 0.0097s^2 + 0.6702s + 1.0423}{s^4 + 2.0873s^3 + 3.4219s^2 + 2.8382s + 1.0423}$$

The unit− step responses are shown in Fig.(3.3)

3.2.2.3 Dispersion Analysis and Continued− Fraction Method[48].

In 1986, a new method of model reduction based on dispersion analysis and the continued− fraction method was presented by C. M. Liaw, C. T. Pan & Y. C. Chen[48]. From the view point of the energy contribution to the system output, dynamic modes with dominant energy contributions (instead of these with dominant eigenvalues) are preserved by using dispersion analysis. Having determined the denominator of the the reduced model, the parameters of the numerator are calculated by using the continued− fraction method. The reduction procedure is simple, and the reduced model is guaranteed to be stable if the original system is stable. The nth− order transfer function G(s) of the original system is repeated as follows

$$G(s) = \frac{A_2(s)}{A_1(s)} = \frac{a_{21} + a_{22}s + a_{23}s^2 + \ldots + a_{2,n}s^{n-1}}{a_{11} + a_{12}s + a_{13}s^2 + \ldots + a_{1,n+1}s^n}$$

$$= \sum_{i=1}^{n} \frac{h_i}{s - \mu_i} \tag{3.12}$$

The unit-step response can be found as

$$\frac{1}{s} G(s) = \frac{K}{s} + F(s) \tag{3.13}$$

with

$$K = \sum_{i=1}^{n} - \frac{h_i}{\mu_i}, \qquad F(s) = \sum_{i=1}^{n} \frac{h_i/\mu_i}{s - \mu_i} = \sum_{i=1}^{n} \frac{f_i}{s - \mu_i}$$

where k the steady− state value, and F(s) is the transient part of the unit− step response. In order to give a reduced model, putting equal emphasis on its approximation to the transient as well as the steady− state responses, the dispersion analyses of G(s) and F(s) are performed. According to those results, the dynamic

modes with dominant energy contributions either in G(s) or in F(s) will be retained.

Now suppose that the retained dynamic modes are μ_1, μ_2, ..., μ_m, then the reduced model can be written as

$$R(s) = \frac{B_2(s)}{B_1(s)} = \frac{b_{21} + b_{22}s + \ldots + b_{2,m}s^{m-1}}{b_{11} + b_{12}s + \ldots + b_{1,m+1}s^n} \tag{3.14}$$

where b_{11}, b_{12}, ..., $b_{1,m+1}$ can be obtained as outlined below:

The unit impulse response, is obtained from (3.12):

$$G(t) = \sum_{i=1}^{n} h_i \exp(\mu_i t) \tag{3.15}$$

The analogous continuous white noise, denoted by z(t), is defined by the following property

$$E\{z(t)z(t-\tau)\} = \sigma_z^2 \, \delta(\tau)$$

where $\delta(\tau)$ is the Dirac delta function.

In order to give each dynamic mode even weighting, white noise input is assumed and the response of a system with unit impulse response G(t) is

$$y(t) = \int G(t-\tau) \, z(\nu) \, d\nu \tag{3.16}$$

The autocovariance function of output y(t) is defined by

$$\gamma(\tau) = E\{Y(t)Y(t-\tau)\} \tag{3.17}$$

Substituting (3.15) into (3.16)

$$\gamma(\tau) = E\left\{ \int_{-\infty}^{t} G(t-\nu')z(\nu')d\nu' \int G(t+\tau-\nu)z(\nu)d\nu \right\}$$

$$= \int_{-\infty}^{t} \int_{-\infty}^{t+\tau} G(t-\nu')G(t+\tau-\nu)E\{z(\nu')z(\nu)\}d\nu d\nu'$$

$$= \sigma_z^2 \int G(t-\nu')G(t+\tau-\nu') \, d\nu \quad \text{gives} \tag{3.18}$$

Substituting (3.14) into (3.17) and simplifying, one can obtain

$$\gamma(\tau) = \sigma_z^2 \int_{-\infty}^{t} \left[\sum_{i=1}^{n} h_i \exp\{\mu_i(t-\nu')\} \right] \left[\sum_{i=1}^{n} h_i \exp\{\mu_j(t+\tau-\nu')\} \right] d\nu'$$

$$= \sigma_z{}^2 \sum_{j=1}^{n} \exp(\mu_j \tau) \; [\; \sum_{i=1}^{n} \frac{h_i \, h_j}{-(\mu_i + \mu_j)} \;] \qquad (3.19)$$

$$\text{Let} \quad d_j = \sigma_i{}^2 \sum_{i=1}^{n} \frac{h_i \, h_j}{-(\mu_i + \mu_j)} \qquad (3.20)$$

$$\text{then} \quad \gamma(\tau) = \sum_{j=1}^{n} d_j \exp(\mu_j \tau) \qquad (3.21)$$

where d_j is called the coefficient of the autocovariance function. In fact, when the input is white noise , then the energy contribution (corresponding to the dynamic mode μ_j) to the total variance of output $Y(t)$ is exactly equal to d_j.

From equ.(3.21) we can see

$$\gamma(0) = \sum_{j=1}^{n} d_j = d_1 + d_2 + \ldots + d_n$$

$$\frac{\gamma(0)}{\sigma_z{}^2} = \frac{d_1 + d_2 + \ldots + d_n}{\sigma_z{}^2} \quad \text{the total for all modes.}$$

Thus the $\quad \dfrac{d_j}{\sigma_z{}^2} = \sum_{i=1}^{n} \dfrac{h_i \, h_j}{-(\mu_i + \mu_j)} \quad$ for each dynamic mode, (μ_j)

By discarding dynamic modes with small dispersions, the denominator of the reduced model can be found.

$$\text{i.e.,} \quad b_{11} + b_{12} s + \ldots + b_{1,m+1} s^m = \sum_{i=1}^{m} (s - \mu_i{}')$$

where m is the reduced order, and $\mu_i{}'$ are the remaining elements of μ after discarding the dynamic modes. Then, the numerator of the transfer function of the reduced order model can be found by matching a number of time—moments or a combination of Markov parameters and time—moments of the original system.

The transfer function $G(s)$ of the original system as in (3.12) can be expanded into a Cauer—type continued fraction about $s = 0$ of the form:

$$G(s) = G_0 + G_1 s + G_2 s^2 + \ldots$$

where $G_0 = a_{21} / a_{11}$

$$G_k = \frac{1}{a_{11}} [a_{2,k+1} - \sum_{j=1}^{k} a_{1,j+1} G_{k-j}] \quad k>0$$

with $a_{2k} = 0$ for $k > n-1$

and the parameters G_i are proportional to the time-moments of the system. Let the first m coefficients of the continued fraction corresponding to the reduced model $R(s)$ and those of $G(s)$ be identical, i.e.

$$R(s) = \frac{b_{21}+b_{22}s+\ldots+b_{2m}s^{m-1}}{b_{11}+b_{12}s+\ldots+b_{1,m+1}s^m} = G_0+G_1 s+G_2 s^2+\ldots+G_{m-1} s^{m-1}+\ldots$$

Then the parameters b_{2i} are determined as

$$\left.\begin{array}{l} b_{21} = b_{11} G_0 \\[2mm] b_{2m} = b_{1m} G_0 + b_{1,m-1}G_1 + \ldots + b_{11}G_{m-1} \end{array}\right\} \quad (3.22)$$

Now, the numerator of the transfer function of the reduced model can be solved from equation (3.22).

Now we consider a fourth-order transfer function given by

$$G(s) = \frac{13.2s^3+84.8s^2+167.2s+96.8}{s^4+10s^3+35s^2+50s+24} \qquad \text{(Example 3)}$$

Using the residue function of equ.(3.13) the unit-step response is found. The parameters of (3.12) and (3.13) are listed as follows

$$\mu_1 = -1 \quad h_1 = 0.2 \quad f_1 = -0.2$$
$$\mu_2 = -2 \quad h_2 = 2.0 \quad f_2 = -1$$
$$\mu_3 = -3 \quad h_3 = 1.0 \quad f_3 = -0.3333$$
$$\mu_4 = -4 \quad h_4 = 10.0 \quad f_4 = -2.5 \qquad k = 4.0333$$

The dispersion analyses of $G(s)$ and $F(s)$ for output $y(t)$ by the dynamic modes μ_i are performed and listed in Table 3.1. By discarding the dynamic modes corresponding to μ_1 and μ_3, the denominator of the reduced model is obtained and is expressed as

$B_1(s) = [1.0 \ 6.0 \ 8.0]$

If the continued−fraction of G(s) is performed about s= 0, one can find

$G(s) = 4.0333 - 1.43611s + 0.64329s^2 ...$

Then the parameters b_{21} and b_{22} are calculated from Equ.(3.22) as

$b_{21} = 8 \times (-1.43611) + (6 \times 4.0333) = 12.7111$

$b_{22} = 4.0333 \times 8 = 32.26664$

Finally, the reduced model is obtained as

$$R_1(s) = \frac{12.7111s + 32.26664}{s^2 + 6s + 8}$$

The unit−step responses and the frequency response of this model and the original system are compared in Fig. (3.4) and Fig. (3.5).

- for HOS, * for reduced method
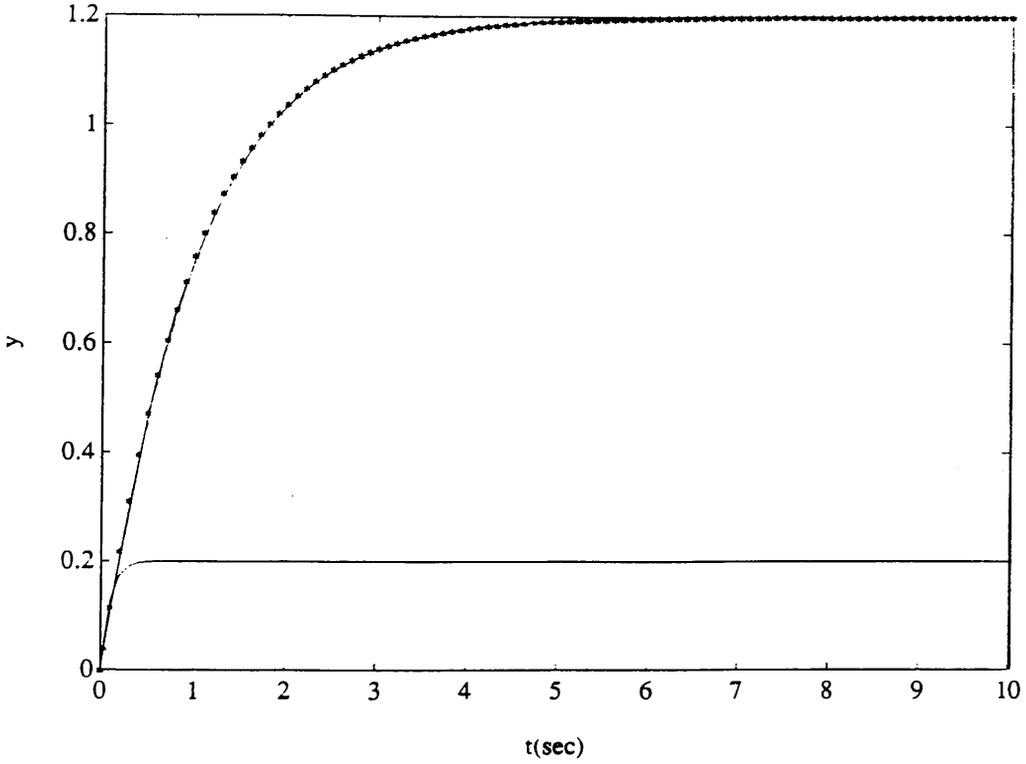
Fig. 3.1 Time Response for Example 1
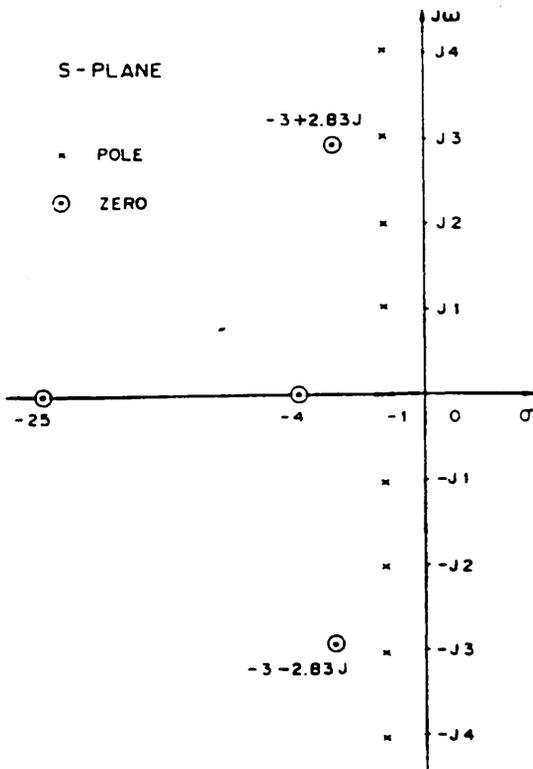
Fig. 3.2 Pole—zero Configuration of Example 2

Table 3.1 Energy dispersion analyses for output y(t)

Dynamic modes	G(s) $\Upsilon(0)/\sigma_Z^2$		F(s) $\Upsilon(0)/\sigma_Z^2$	
μ_1	0.6033	(2.3962%)	0.2033	(7.6976%)
μ_2	4.8667	(19.3300%)	0.7999	(30.2866%)
μ_3	2.0452	(8.1232%)	0.2209	(8.3640%)
μ_4	17.6619	(70.1507%)	1.4170	(53.6519%)
Total	25.1771	(100%)	2.6411	(100%)

- for HOS, * for reduced model
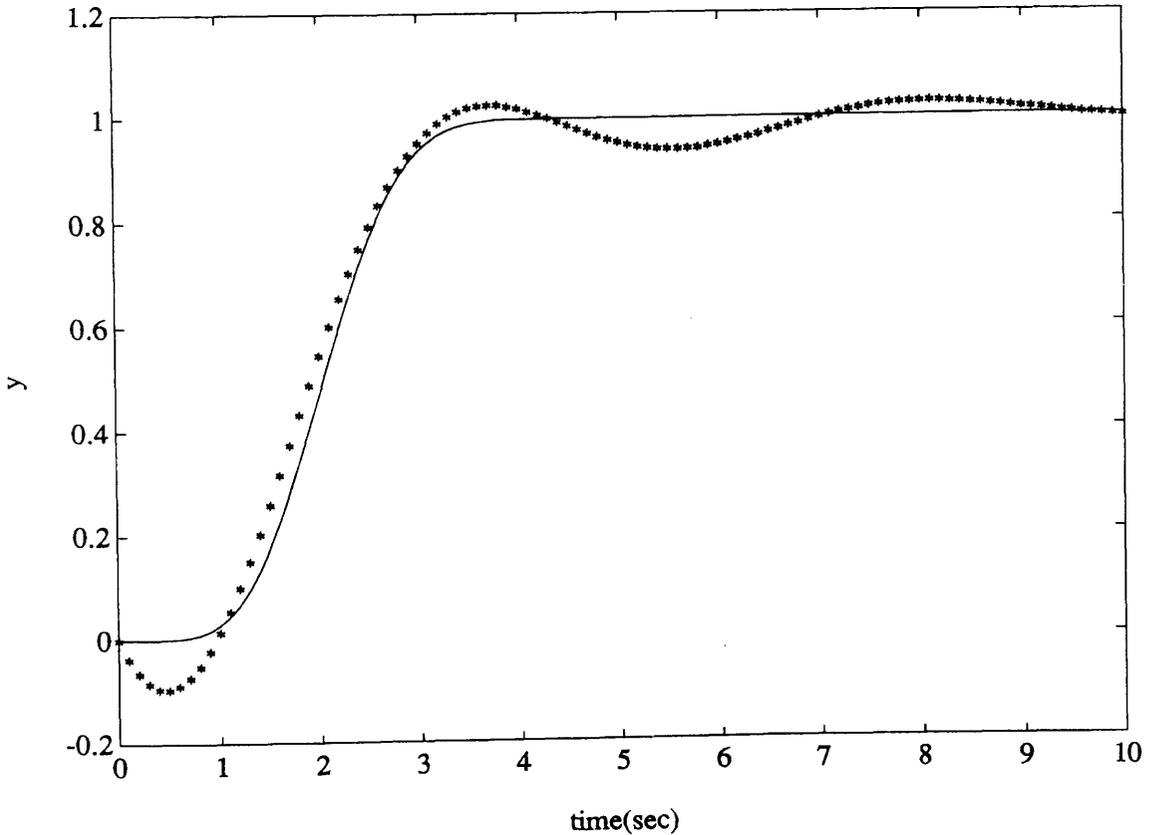
Fig. 3.3 The Unit–Step Response of Example 2
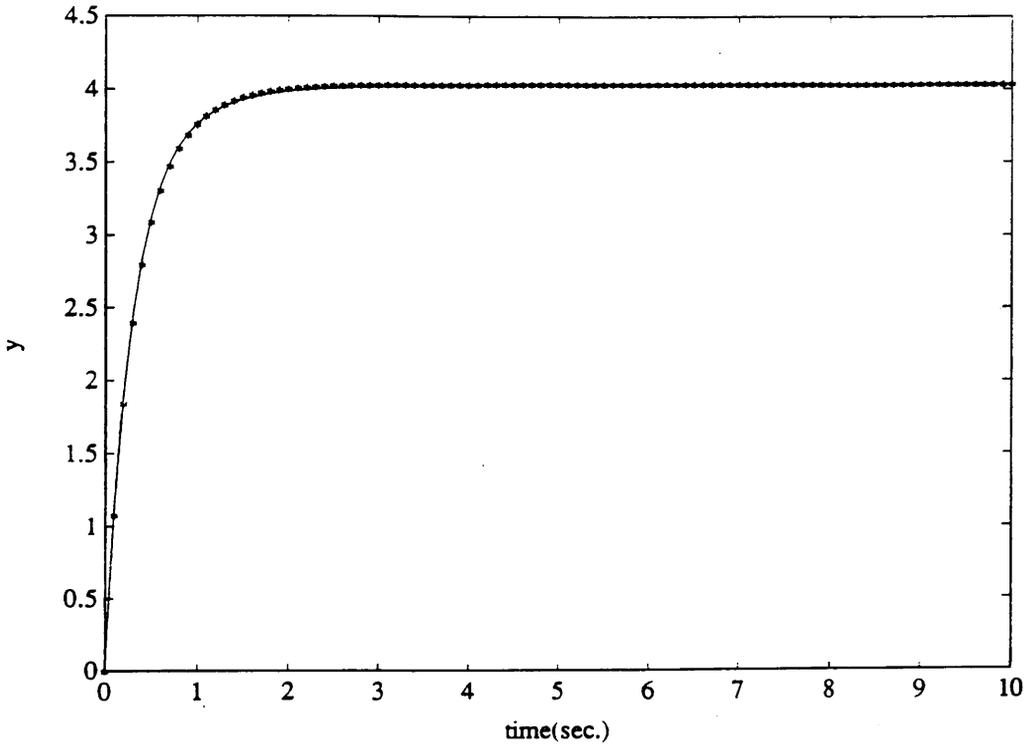
Fig. 3.4 The Unit–Step Response of Example 3

- for HOS, * for reduced model
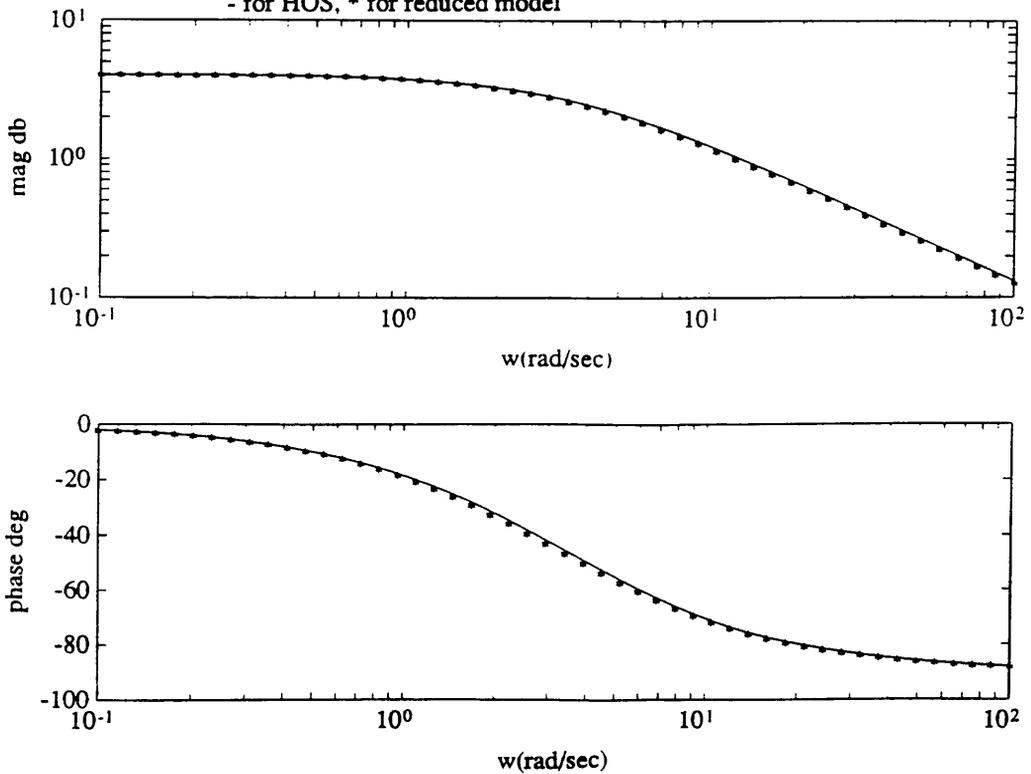
Fig. 3.5 The Frequency Response of Example 3

Chapter 4

Model Reduction Using a Modified Complex

Curve − Fitting Technique in the Frequency Domain

4.1 Introduction.

The mathematical analysis of linear dynamic systems, based on experimental test results, often requires that the frequency response of the system be fitted by an algebraic expression. The form in which this expression is usually desired is that of a ratio of two frequency−dependent polynomials. This transfer function identification technique has been applied to the problem of model reduction. This method is based upon the minimization of the weighted sum of the squares of differences between the frequency response of the known high order system and the approximating reduced order system. It thus provides a means for the evaluation of the optimum set of polynomial coefficients of the approximating transfer function of the reduced order system.

The method was originally developed by Levy[68] for the single−input case and has been extended on the present work to cover multi−output cases. A state−space model representation can be obtained from the reduced transfer−function models using appropriate transformations.

A computer program based on MATLAB has been written to find the coefficients of the numerator and denominator of the transfer function of the low order system equivalent to a given high order description. Computational results are provided for an example which has been used by Bacon and Schmidt[30] to illustrate their approach to model reduction based on equivalent systems. This example relates to a mathematical model of an advanced fighter aircraft.

4.2 Theory.

4.2.1 A Review of Levy's Complex−Curve Fitting Technique.

A complex curve fitting technique for evaluation of the polynomial coefficients was presented by E. C. Levy[68] in 1959. It uses a modified least−squares approach to fit the frequency response curve of a system. The method is based on the minimization of the weighted sum of the squares of the errors between the absolute magnitude of the original frequency response data and the polynomial ratio, taken at various values of frequency.

Levy's method is designed for computer implementation and in its original form was intended to be used for system identification. It uses a modified least−squares approach to fit the frequency response curve of a system to the linear expression

$$G(s) = \frac{b_0 + \sum_{i=1}^{q} b_i s^i}{1 + \sum_{i=1}^{r} a_i s^i}, \qquad r \geqslant q \tag{4.1}$$

Replacing the operator s by $j\omega$ and separating into real and imaginary parts gives

$$G(j\omega) = \frac{\alpha + j\omega\beta}{\sigma + j\omega\tau} = \frac{N(j\omega)}{D(j\omega)} \tag{4.2}$$

It is assumed that a function $H(j\omega)$ exists which coincides exactly with the measured data; $H(j\omega)$ will then also have real and imaginary parts:

$$H(j\omega) = R(\omega) + jI(\omega) \tag{4.3}$$

At any particular value of the frequency, ω_k, the error in fitting becomes

$$\epsilon(\omega_k) = H(j\omega_k) - G(j\omega_k) = H(j\omega_k) - \frac{N(j\omega_k)}{D(j\omega_k)} \tag{4.4}$$

The problem, then, is to minimize this error at each sampling point on the curve. The minimization could be done quite simply by summing the magnitude of the error squared, $\Sigma |\epsilon(\omega_k)|^2$, and setting the partial derivatives with respect to each of the coefficients equal to zero. This corresponds to a least−squares fit, resulting in a set of linear, simultaneous algebraic equations which, in principle, could be solved for the desired coefficients of $G(j\omega)$. However, this approach can result in the optimum approximation of the transfer function having non−minimum phase poles or zeros.

Levy modified Equation (4.4) by multiplying throughout by $D(j\omega_k)$, to give

$$D(j\omega_k)\,\epsilon(\omega_k) \;=\; H(j\omega_k)D(j\omega_k) \;-\; N(j\omega_k) \qquad\qquad (4.5)$$

This modification is valid provided $D(j\omega_k)$ is a non—zero function. This means that the denominator of the overall process transfer function cannot contain any pure integrating elements. At a later date, some authors[69,70,71] have further modified the method in order to obtain better results for high—frequency data. The effect of all these modifications may be embodied in an equation of the form:

$$\epsilon'(\omega_k) \;=\; \frac{\epsilon(\omega_k)\;D(j\omega_k)_L}{|D(j\omega_k)_{L-1}|^{n_1}} \;=\; \frac{H(j\omega_k)\;D(j\omega_k)_L \;-\; N(j\omega_k)_L}{|D(j\omega_k)_{L-1}|^{n_1}} \qquad (4.6)$$

where $\epsilon(\omega_k)$ — error, as defined in equ.(4.4)

   $\epsilon'(\omega_k)$ — modified error, defined by equ.(4.6)

   $H(j\omega_k)$ — function coinciding with measured data

   $D(j\omega_k)$ — denominator of the transfer function of the linear dynamic system in equ.(4.2)

   $N(j\omega_k)$ — numerator of transfer function of the linear dynamic in equ.(4.2)

   L   — Lth iteration

   n_1 — weighting factor

The case where $n_1 = 0$ corresponds to Levy's[68] original, unmodified expression. Sanathanam and Koerner[70] proposed using $n_1 = 1$, whilst 't Mannetje[71] developed the general form given above in equ.(4.6).

Levy's method is usually presented as a means of identifing systems from measured frequency response data. It can however provide a basis for fitting a lower order description to a high order model. This is, in some ways, a simpler application than system identification in that the frequency response cuves of the high order model to which a lower order model is to be fitted are free from measurement noise. This appears to provide a very simple basis for model reduction.

4.2.2 Extension of Levy's Technique from the SISO Case to the SIMO Case.

The contribution of E. C. Levy involved using a least—squares approach to fit the

frequency response curve of a system to a linear transfer function. Sanathanan and Koerner[70], 't Mannetje[71] further modified this approach to give better results by minimization of the "weighted" sum of the squares of the errors in magnitude. All of these approach are only for the single—input and single—output case.

Many practical problems involve multi—input and multi—output descriptions rather than the single—input and single—output type of model to which Levy's method can be applied. To make this approach of any value for model reduction in the case of highly coupled system such as the helicopter it is essential to extend Levy's approach to the SIMO case. This is essentially the same as the single—input single—output approach but involves extending the theory to ensure that the resulting transfer function descriptions all have the same characteristic equation.

The main different between the SISO and SIMO cases lies in the cost function. The cost function for the SISO case can be expressed as :

$$J_i = \sum_{k=1}^{n} | \epsilon(\omega_k) |^2$$

where n is the number of points in the frequency range considered. An appropriate cost function for the multivariable case is

$$J = \sum_{i=1}^{m} J_i = \sum_{i=1}^{m} \sum_{k=1}^{n} | \epsilon(\omega_k) |^2 \tag{4.7}$$

where m is the number of outputs and n is the number of points in the frequency range considered.

We could express the error $\epsilon(\omega_k)$ in terms of real and imaginary parts as $\epsilon(\omega_k) = A(\omega_k) + j B(\omega_k)$. So,

$$| \epsilon(\omega_k) | = \sqrt{A^2 + B^2} \tag{4.8}$$

Now let

$$E = J = \sum_{k=1}^{n} \{ [A_1^2(\omega_k) + B_1^2(\omega_k)] + [A_2^2(\omega_k) + B_2^2(\omega_k)] + ... \} \tag{4.9}$$

where $A_i^2(\omega_k) = (\sigma_k R_{ik} - \omega_k \tau_k I_{ik} - \alpha_{ik})^2 D_{2n_1}(k)$

$B_i^2(\omega_k) = (\omega_k \tau_k R_{ik} + \sigma_k I_{ik} - \omega_k \beta_{ik})^2 D_{2n_1}(k)$

$$\alpha_{ik} = b_{i0} - b_{i2}\omega_k^2 + b_{i4}\omega_k^4 - \ldots$$

$$\beta_{ik} = b_{i1} - b_{i3}\omega_k^2 + b_{i5}\omega_k^4 - \ldots$$

$$\sigma_k = 1 - a_2\omega_k^2 + a_4\omega_k^4 - \ldots$$

$$\tau_k = a_1 - a_3\omega_k^2 + a_5\omega_k^4 - \ldots$$

The partial derivatives of Equation (4.10) with respect to each of the unknown coefficients a_i and b_i must be set to zero, therefore

$$\left. \begin{array}{ll} \dfrac{\partial E}{\partial a_i} = 0 & i = 1, 2, 3, \ldots, r \\[2em] \dfrac{\partial E}{\partial b_{ij}} = 0 & i = 1, 2, 3, \ldots, m; \quad j = 0, 1, 2, \ldots, q \end{array} \right\} \quad (4.10)$$

Each of the equations in (4.9) will contain terms which are functions of the unknown coefficients, and terms which are known. To condense the notation before expanding the above equations, the following relationships are defined :

$$\left. \begin{array}{l} \Gamma_h = \displaystyle\sum_{k=1}^{n} \omega_k^h \, D_{2n_1}(k) \\[1.5em] S_{ih} = \displaystyle\sum_{k=1}^{n} \omega_k^h \, R_{ik}(\omega_k) \, D_{2n_1}(k) \\[1.5em] T_{ih} = \displaystyle\sum_{k=1}^{n} \omega_k^h \, I_{ik}(\omega_k) \, D_{2n_1}(k) \\[1.5em] U_{ih} = \displaystyle\sum_{k=1}^{n} \omega_k^h \, [I_{ik}^2(\omega_k) + R_{ik}^2(\omega_k)] \, D_{2n_1}(k) \end{array} \right\} \quad \begin{array}{l} i = 1, 2, \ldots, m \quad (4.11) \\[1em] h = 0, 1, \ldots. \end{array}$$

where m is the number of outputs.

Here $I_{ik}(\omega_k)$ and $R_{ik}(\omega_k)$ represent the real and imaginary parts of the measured function $H_i(j\omega)$, respectively. Substituting these relationships into equ.(4.10) and separating the coefficients, we obtain a set of equations. Then the problem can be formulated as the solution of a set of linear simutaneous algebraic equations of the form

$$+ \quad [\, P \,]\,[\, X \,] = [\, Y \,] \tag{4.12}$$

where the matrices $[\, P \,]$, $[\, X \,]$ and $[\, Y \,]$ which are for $m = 2$ (two outputs), $r = 5$ and $q = 5$ case (see equ.(4.7) and (4.10)) are written below respectively :

$$[P] =$$

$$
\begin{bmatrix}
\Gamma_0 & 0 & -\Gamma_2 & 0 & \Gamma_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{11} & -S_{12} & -T_{13} & S_{14} & T_{15} \\
0 & -\Gamma_2 & 0 & \Gamma_4 & 0 & -\Gamma_6 & 0 & 0 & 0 & 0 & 0 & 0 & -S_{12} & -T_{13} & S_{14} & T_{15} & -S_{16} \\
-\Gamma_2 & 0 & \Gamma_4 & 0 & -\Gamma_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_{13} & S_{14} & T_{15} & -S_{16} & -T_{17} \\
0 & \Gamma_4 & 0 & -\Gamma_6 & 0 & \Gamma_8 & 0 & 0 & 0 & 0 & 0 & 0 & S_{14} & T_{15} & -S_{16} & -T_{17} & S_{18} \\
\Gamma_4 & 0 & -\Gamma_6 & 0 & \Gamma_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{15} & -S_{16} & -T_{17} & S_{18} & T_{19} \\
0 & -\Gamma_6 & 0 & \Gamma_8 & 0 & -\Gamma_{10} & 0 & 0 & 0 & 0 & 0 & 0 & -S_{16} & -T_{17} & S_{18} & T_{19} & -S_{110} \\
0 & 0 & 0 & 0 & 0 & 0 & \Gamma_0 & 0 & -\Gamma_2 & 0 & \Gamma_4 & 0 & T_{21} & -S_{22} & -T_{23} & S_{24} & T_{25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Gamma_2 & 0 & \Gamma_4 & 0 & -\Gamma_6 & -S_{22} & -T_{23} & S_{24} & T_{25} & -S_{26} \\
0 & 0 & 0 & 0 & 0 & 0 & -\Gamma_2 & 0 & \Gamma_4 & 0 & -\Gamma_6 & 0 & -T_{23} & S_{24} & T_{25} & -S_{26} & -T_{27} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_4 & 0 & -\Gamma_6 & 0 & \Gamma_8 & S_{24} & T_{25} & -S_{26} & -T_{27} & S_{28} \\
0 & 0 & 0 & 0 & 0 & 0 & \Gamma_4 & 0 & -\Gamma_6 & 0 & \Gamma_8 & 0 & T_{25} & -S_{26} & -T_{27} & S_{28} & T_{29} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Gamma_6 & 0 & \Gamma_8 & 0 & -\Gamma_{10} & -S_{26} & -T_{27} & S_{28} & T_{29} & -S_{210} \\
T_{11} & -S_{12} & -T_{13} & S_{14} & T_{15} & -S_{16} & T_{21} & -S_{22} & -T_{23} & S_{24} & T_{25} & -S_{26} & U_{12}+U_{22} & 0 & -U_{14}-U_{24} & 0 & U_{16}+U_{26} \\
S_{12} & -T_{13} & -S_{14} & T_{15} & S_{16} & -T_{17} & S_{22} & -T_{23} & -S_{24} & T_{25} & S_{26} & -T_{27} & 0 & -U_{14}-U_{24} & 0 & U_{16}+U_{26} & 0 \\
-T_{13} & -S_{14} & T_{15} & S_{16} & -T_{17} & -S_{18} & -T_{23} & -S_{24} & T_{25} & S_{26} & -T_{27} & -S_{28} & -U_{14}-U_{24} & 0 & U_{16}+U_{26} & 0 & -U_{18}-U_{28} \\
-S_{14} & T_{15} & S_{16} & -T_{17} & -S_{18} & T_{19} & -S_{24} & T_{25} & S_{26} & -T_{27} & -S_{28} & T_{29} & 0 & U_{16}+U_{26} & 0 & -U_{18}-U_{28} & 0 \\
T_{15} & S_{16} & -T_{17} & -S_{18} & T_{19} & S_{110} & T_{25} & S_{26} & -T_{27} & -S_{28} & T_{29} & S_{210} & U_{16}+U_{26} & 0 & -U_{18}-U_{28} & 0 & U_{110}+U_{210}
\end{bmatrix}
$$

$$[Y] = [S_{10} \ -T_{11} \ -S_{12} \ T_{13} \ S_{14} \ -T_{15} \ S_{20} \ -T_{21} \ -S_{22} \ T_{23} \ S_{24} \ -T_{25}$$

$$0 \ -U_{12}-U_{22} \ 0 \ U_{14}+U_{24} \ 0 \ ]'$$

$$[X] = [b_{10} \ b_{11} \ b_{12} \ b_{13} \ b_{14} \ b_{15} \ b_{20} \ b_{21} \ b_{22} \ b_{23} \ b_{24} \ b_{25} \ a_1 \ a_2$$

$$a_3 \ a_4 \ a_5]'$$

The numerical values of the unknowns a_1, a_2, ...; b_{10}, b_{21}, ... may now be determined once the coefficients (4.11) of equation (4.12) have been evaluated. For any single-input, multi-output case we will obtain the transfer function models:

$$\frac{x_1}{u_1} (s) = \frac{N_1}{\Delta} (s); \quad N_1(s) = k_{11} s^{n-1} + k_{12} s^{n-2} + ... + k_{1n}$$

$$\frac{x_2}{u_1} (s) = \frac{N_2}{\Delta} (s); \quad N_2(s) = k_{21} s^{n-1} + k_{22} s^{n-2} + ... + k_{2n}$$

$$\vdots$$

$$\frac{\dot{x}n}{u_1} (s) = \frac{Nn}{\Delta} (s); \quad Nn(s) = k_{n1} s^{n-1} + k_{n2} s^{n-2} + ... + k_{nn}$$

where $\Delta(s)$ = charcteristic polynominal for all of the transfer functions

$$= s^n + d_1 s^{n-1} + d_1 s^{n-2} + \cdots + d_n$$

A computer program has been implemented using the MATLAB package on the VAX computer for this single-input and multi-output case. This implementation is described in Appendix 1. The size of problem (in terms of the number of poles and zeros of the system) which can be accommodated is limited only by the available computer memory.

4.2.3 The Application of Chen's Nonlinear Inverse Formula.

R. T. N. Chen's nonlinear inverse formula[72] provides an exact and unique state-space model representation from the transfer function models. This inversion is unique because the relationship between the model states and outputs is known. For the presents purpose, let

$$\underline{u} = [u_1] = \text{control input}$$

$$\underline{x} = [x_1, x_2, ..., x_n]^T = \text{state vector}$$

$$\underline{y} = \underline{x} = \text{output vector (simplified case)}$$

and the transfer function models have the form from section 4.2.2

$$N = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \tag{4.13}$$

where K_{ij} is jth coefficient of the numerator of ith transfer function.

$$F_c = \begin{bmatrix} -d_1 & -d_2 & \cdots & -d_n \\ \hline & I_{n-1} & | & 0 \end{bmatrix} \tag{4.14}$$

where d_i is the coefficient of the denominator of the transfer function.

The unique inverse of the transfer function of whole SIMO system is

$$\dot{\underline{x}} = F \underline{x} + G_1 \underline{u}$$

$$\underline{y} = \underline{x}$$

where $F = NF_cN^{-1}$

$$G_1 = [K_{11} \ K_{21} \ \cdots \ K_{n1}]^T \tag{4.15}$$

So, now we have got a low order system which is in state—space form. If we derive a low order description for a high—order system, just using Levy's approach for the SIMO as presented in the section 4.2.2 then R.T.N.Chen's technique as presented above; allows the low—order system based on a set of transfer functions to be translated to state—space form.

4.3 Application of the Extended Levy's Technique & Chen's Method.

A computer programme has been written to find the coefficients a_i and b_{ij} of the denominator and numerator respectively for the transfer function of the reduced order model. The program also obtains the reduced order system in state—space form using Chen's method.

An example which is a used to illustrate the application of the method is taken from Bacon and Schmidt's[30] paper and relates to an advanced fighter aircraft.

Details of the high order system description are given in Table 4.1.

Table 4.1 Example HOS aircraft

Advanced fighter without prefilter

$$\frac{q}{\delta} = \frac{5.26s(s+0.0103)(s+0.773)(s+0.5)(s+1.887)(s+13.986)}{(s+0.418)(s+1.34)(s^2+0.00264s+0.006724)(s^2+3.3916s+7.7284)} \quad \begin{bmatrix} 1 \\ - \\ s \end{bmatrix}$$
$$(s^2+33.0576s+290.3611)$$

$$\frac{n_{zcr}}{\delta} = \frac{1.34s(s+0.00066)(s+49.99)(s+0.5)(s+1.887)(s+13.986)}{\Delta} \quad \begin{bmatrix} g \\ \overline{} \\ rad \end{bmatrix}$$

In order to understand the strengths and limitations of the method a number of cases were investigated involving application of the extended Levy's technique and Chen's method. The frequency response of the reduced model agree with the high order system well over the range of frequencies considered if the weighting factor n1 and the number of points used in calculation are chosen in an appropriate way.

4.3.1 The Effect of the Weighting Factor.

According to Levy and to those who introduced modifications to the method, the introduction of the weighting factor n1 in Equ.(4.6) allows better results to be obtained for high frequency data. However in practical applications the magnitude in the high frequency range may be very small, and it is therefore very difficult to use the resulting error to judge the quality of fit. Fig. 4.1 to Fig. 4.11 and Table 4.2 present the results from a series of tests to investigate the effect of using different values of the weighting factor n1 for the example above for a frequency range of 0.1 rad/sec. to 100 rad/sec.. From these figures and Table 4.2 it can be seen clearly that to obtain a good fit in the high frequency range from 10 rad/sec to 100 rad/sec. it is necessary to use a small value of the weighting factor n1. In some cases negative values of n1 were found to give good results. For this example, when n1 is greater than 0.2, the results in the higher frequency range start to get worse.

When the value of the weighting factor n1 is further increased to over 1.0, the results in all parts of the frequency range get worse.

From Fig. 4.4 and Fig. 4.8, it can be seen that the frequency response of the transfer function for case n1 = 0.2 is much better than the case for n1 = 1. But from Table 4.2 we also find that the error for the case of n1 = 1 is smaller than for the case of n1 = 0.2. That is because for the case of n1 = 1 the effect of the weighting factor n1 in the high frequency range is bigger than in the case of n1 = 0.2. Although the error for the case of n1 = 1 is smaller than for the case of n1 = 0.2 the graphical fit looks worse in the high frequency range. This is due to the fact that the frequency response magnitude information is presented in graphical form on a logarithmic scale (decibels). The error is not however based upon a logarithmic measure. Careful consideration must therefore be given in interpreting the results of the optimisation to the intended use of the reduced order model and the frequency range of importance for that application. In the example being considered, to obtain good results in terms of magnitude (dB) and phase plots as well as satisfying the error criterion it is necessary to use a value of the weighting factor n1 between -1.0 and 0.2.

4.3.2 The Effects of Number of Points Used in the Calculation.

Fig.4.12 $-$ Fig.4.14 and Fig.4.4 show the effects of the number of points taken in the calculation. It is clear from these figures that the number of points taken in the calculation will generally not affect the trend of the solution, although it is also clear that the more points we take, the more smooth the solution is. It is also clear from these results that if the number of points taken exceeds a certain value, such as 200 points for this example, the results start to become worse. There is clearly an optimum number of points and care must be taken in selecting an appropriate number for each application.

Table 4.2 The Error for the Different Value of nl

Weighting Factor	Error		
nl	Iteration 1	Iteration 2	Iteration 3
2	10.2771	1.7973	1.6678
1.5	10.2771	1.8799	2.5220
1.2	10.2771	2.0253	2.0939
1.0	10.2771	2.6121	2.0035
0.8	10.2771	4.6889	2.9673
0.4	10.2771	10.7372	10.6364
0.2	10.2771	10.9150	10.8944
-0.2	10.2771	9.8410	9.8393
-0.5	10.2771	9.9855	9.9855
-1.0	10.2771	11.1129	10.5211
-2.0	10.2771	10.3747	10.3747

4.3.2 The Effects of Number of Points Used in the Calculation.

Fig.4.12 — Fig.4.14 and Fig.4.4 show the effects of the number of points taken in the calculation. It is clear from these figures that the number of points taken in the calculation will generally not affect the trend of the solution, although it is also clear that the more points we take, the more smooth the solution is. It is also clear from these results that if the number of points taken exceeds a certain value, such as 200 points for this example, the results start to become worse. There is clearly an optimum number of points and care must be taken in selecting an appropriate number for each application.

4.3.3 Convergence and Iteration Number.

It has been observed that Levy's method converges very rapidly. Usually, after three to six iterations there are no further significant changes to be found in the solution.

4.4 Extension of Levy's Technique from the SIMO Case to the MIMO Case.

From section 4.2.2, it is very clear that the different between the SISO case and SIMO case lies in cost function. From this concept it is not very difficult to extend the problem of model reduction to the MIMO case. The cost function for the SIMO case can be expressed as equ.(4.7):

$$J_j = \sum_{i=1}^{m} J_i = \sum_{i=1}^{m} \sum_{k=1}^{n} | \epsilon(\omega_k) |^2 \tag{4.7}$$

An appropriate cost function for the multiinput and multioutput case is

$$J = \sum_{j=1}^{l} J_j = \sum_{j=1}^{l} \sum_{i=1}^{m} \sum_{k=1}^{n} | \epsilon(\omega_k) |^2 \tag{4.16}$$

where l is the number of inputs, m is the number of outputs and n is the number of points in the frequency range considered.

We could express the error $\epsilon(\omega_k)$ in term of real and maginary parts as

$$\epsilon(\omega_k) = A(\omega_k) + j \, B(\omega_k) \text{ so,}$$

$$| \epsilon(\omega_k) | = \sqrt{A^2 + B^2}$$

Let $\quad E = J = J_1 + J_2 + \cdots + J_{m \times l}$

$$= \sum_{i=1}^{m \times l} \sum_{k=1}^{n} [A_i^2(\omega_k) + B_i^2(\omega_k)] \tag{4.17}$$

where $A_i^2(\omega_k) = (\sigma_k R_{ik} - \omega_k \tau_k I_{ik} - \alpha_{ik})^2 \, D_{2n_1}(k)$

$$B_i^2(\omega_k) = (\omega_k \tau_k R_{ik} + \sigma_k I_{ik} - \omega_k \beta_{ik})^2 \, D_{2n_1}(k)$$

$$\alpha_{ik} = b_{i0} - b_{i2}\omega_k^2 + b_{i4}\omega_k^4 - \cdots$$

$$\beta_{ik} = b_{i1} - b_{i3}\omega_k^2 + b_{i5}\omega_k^4 - \cdots$$

$$\sigma_k = 1 - a_2\omega_k^2 + a_4\omega_k^4 - \cdots$$

$$\tau_k = a_1 - a_3\omega_k^2 + a_5\omega_k^4 - \cdots$$

The partial derivatives of equ.(4.10) with respect to each of the unknown coefficients a_i and b_i must be set to zero, therefore

$$\left. \begin{array}{ll} \dfrac{\partial E}{\partial a_i} = 0 & i = 1, 2, 3, \ldots, r \\[4mm] \dfrac{\partial E}{\partial b_{ij}} = 0 & i = 1, 2, 3, \ldots, m \times l; \; j = 0, 1, 2, \ldots, q \end{array} \right\} \tag{4.18}$$

To condense the notation before expanding the above equations the relationships in equation (4.11) only can still be used the difference being that

$i = 1, 2, \cdots, m \times l$

where m is the number of outputs

l is the number of inputs.

Here $I_{ik}(\omega_k)$ and $R_{ik}(\omega_k)$ respresent the real and imaginary parts of the measured function $H_i(j\omega)$, respectively. Substituting these relationships into equ.(4.18) and separating the coefficients, we obtain a set of equations. The problem can then be formulated as the solution of a set of linear simultaneous algebraic equations (4.12). The matrices [P], [X] and [Y] which are for l = 2 (two inputs), m = 2 (two outputs), r = 5 and q = 5 (see equ.(4.7) and (4.18)) are written below respectively:

[P]   (see Table 4.3)

$$[Y] = [\ S_{10} \ -T_{11} \ -S_{12} \ T_{13} \ S_{14} \ -T_{15} \ S_{20} \ -T_{21} \ -S_{22} \ T_{23} \ S_{24} \ -T_{25} \ S_{30}$$

$$-T_{31} \ -S_{32} \ T_{33} \ S_{34} \ -T_{35} \ S_{40} \ -T_{41} \ -S_{42} \ T_{43} \ S_{44} \ -T_{45} \ 0$$

$$-(U_{12}+U_{22}+U_{32}+U_{42}) \ 0 \ (U_{14}+U_{24}+U_{34}+U_{44}) \ 0 \ ]'$$

$$[X] = [\ b_{10} \ b_{11} \ b_{12} \ b_{13} \ b_{14} \ b_{15} \ b_{20} \ b_{21} \ b_{22} \ b_{23} \ b_{24} \ b_{25} \ b_{30} \ b_{31} \ b_{32}$$

$$b_{33} \ b_{34} \ b_{35} \ b_{40} \ b_{41} \ b_{42} \ b_{43} \ b_{44} \ b_{45} \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ ]'$$

The matrices [P], [X] and [Y] have the same structure as the matrices [P], [X] and [Y] for the SIMO case. The numerical values of the unknowns a_1, a_2, $\cdots$; b_{10}, b_{21}, $\cdots$ may be determined once the coefficients (4.11) of equation (4.12) have been evaluated. A computer program has been implemented using the MATLAB package on the VAX computer for this Multi- input and Multi- output case. This program has been implemented for an example with two- inputs and two- ouputs for lateral motion of a very large four engined passenger jet aircraft[85]. This example is considered in state- space form :

$\dot{x} = Ax + Bu$

with $x = [\ r \ \beta \ p \ \varphi]'$ as :

where r is the yaw rate

β is the sideslip angle

p is the roll rate

and φ is the roll angle.

The control input vector is

$$u = \begin{bmatrix} \delta_R \\ \delta_A \end{bmatrix}$$

where δ_R is the rudder deflection

δ_A is the aileron deflection.

For this model the A and B matrices involve the following stability and control derivations.

$$A = \begin{bmatrix} N_r' & N_\beta' & N_p' & 0 \\ -1 & Y_\nu' & 0 & g/U_0 \\ L_r' & L_\beta' & L_p & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} N\delta_R' & N\delta_A' \\ Y\delta_R^* & Y\delta_A^* \\ L\delta_R' & L\delta_A' \\ 0 & 0 \end{bmatrix}$$

The order of the HOS of this example is four. Abtompts have been mode to find a ROS of first, second and third order. The behaviour of the ROS for different reduced order depends on the frequency range used. The frequency responses of the third order reduced order system are shown in Fig.4.15 for the frequency range (0.01 — 1.0 rad/sec.). It can be seen that the frequency response of the ROS agrees very well with the HOS in the frequency range used.

4.5 Discussion and Conclusion.

As has been described in above section, the extended Levy's complex— curve fitting technique and Chen's method can provide an effective method for model reduction. The resulting computer program is suitable for handling single— input and multi— output systems and has been applied to an example system relating to an advanced fighter aircraft and to a large passenger transport aircraft. It was found that in order to obtain better results for whole frequency range of interest, the

choice of some parameters must be very carefully made:

1) the weighting factor nl.

   For different inputs and outputs the optimal choice of the nl is fifferent. There is a range of the value nl which is suitable. If the value of nl is outside this range the solution is very sensitive and the error of the frequency response between the high order system and reduced order system can become very large.

2) the number of points used in the calculation for the frequency range of interest.

   In general, the number of points used in the calculation will not affect the trend of the solution. However, if the number of points used exceeds a certain value, the results start to become worse.

3) convergence.

   This technique converges very fast. After only 3 iterations there are no further significant changes in the solution in many cases. Also after three iterations the error of the frequency response and in the time response between the high order system and reduced model show no further obvious changes.
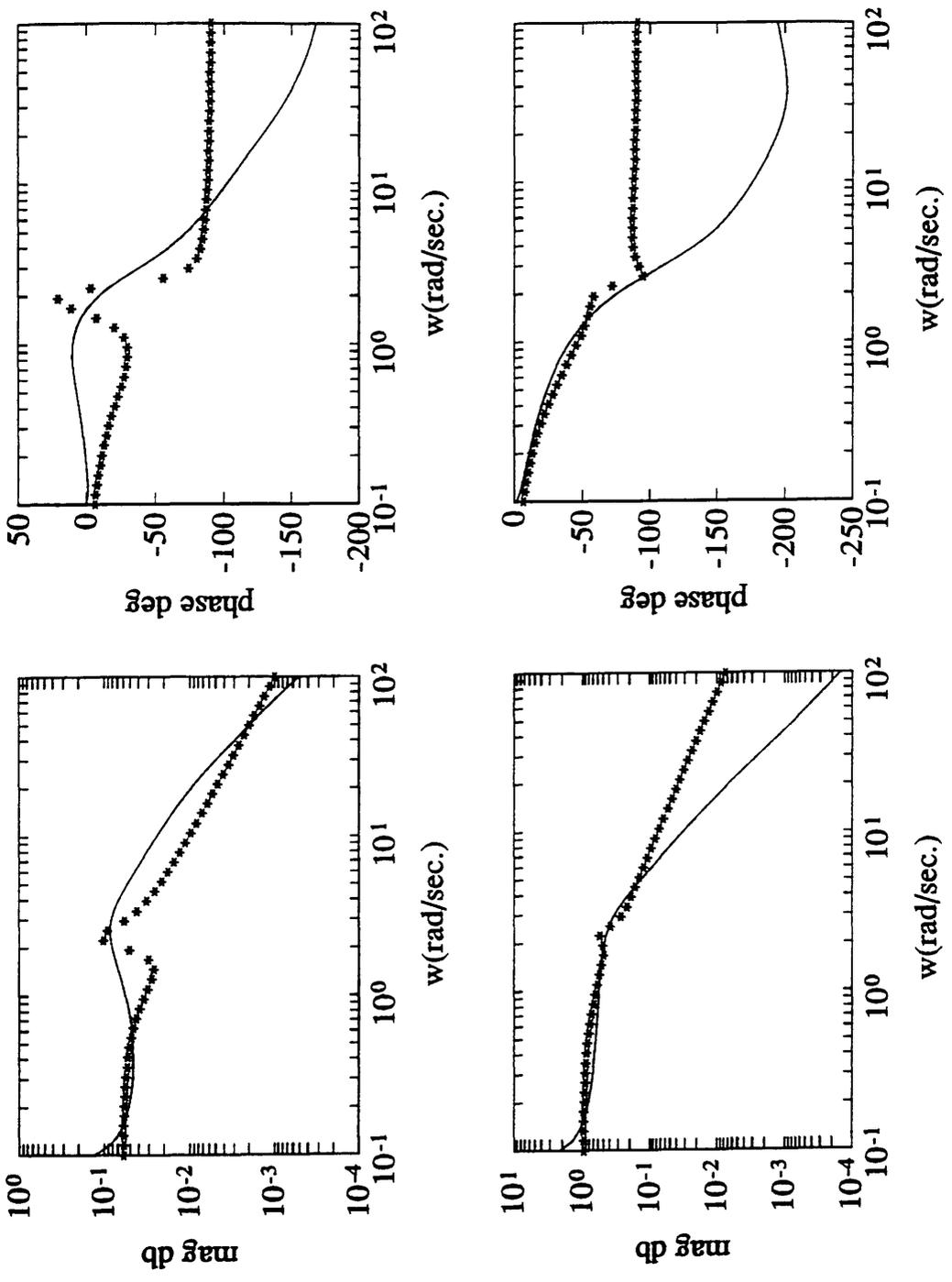
Table 4.3

$[\, P \,] \, =$

The matrix $[P]$ (29 columns):

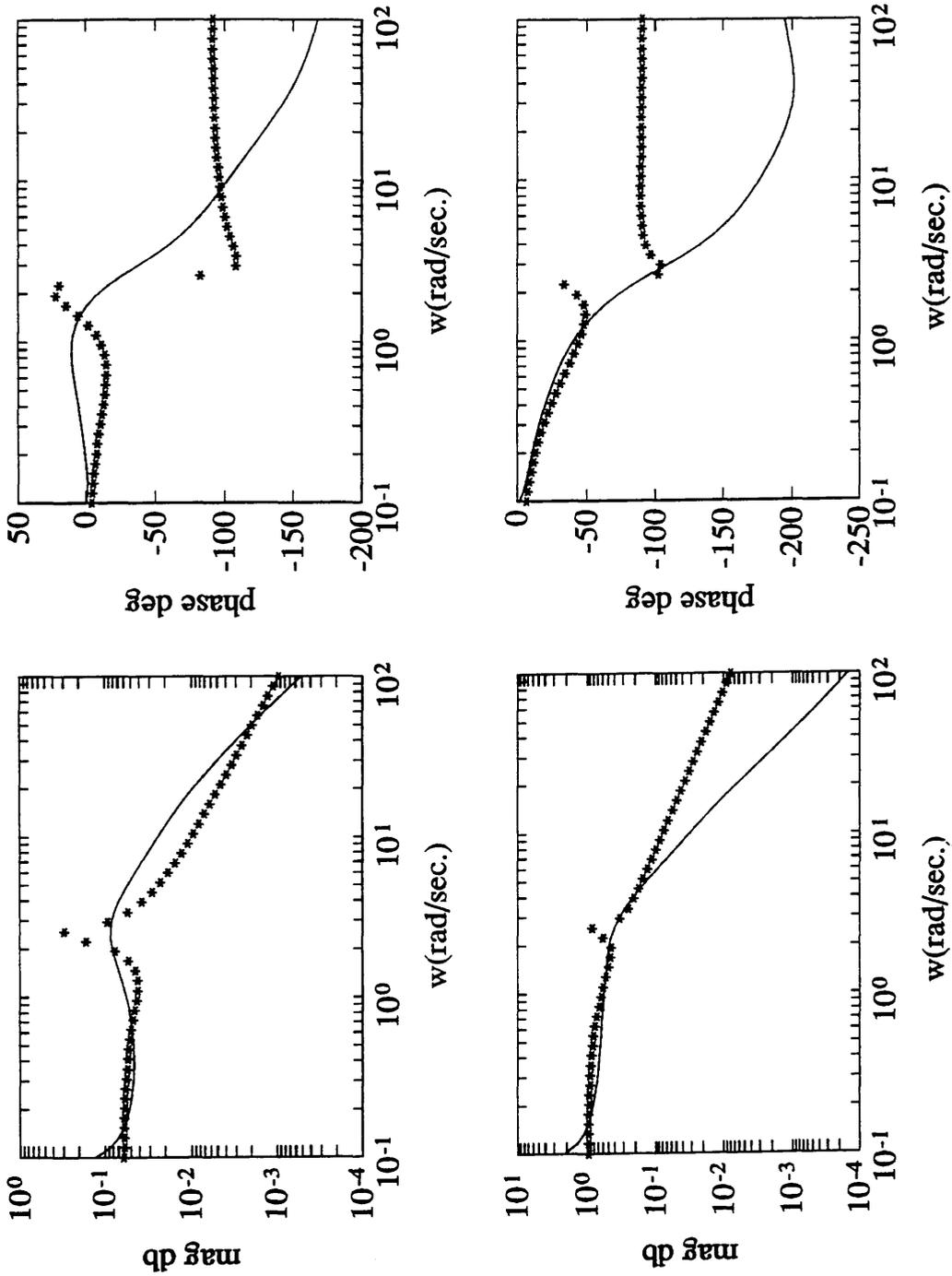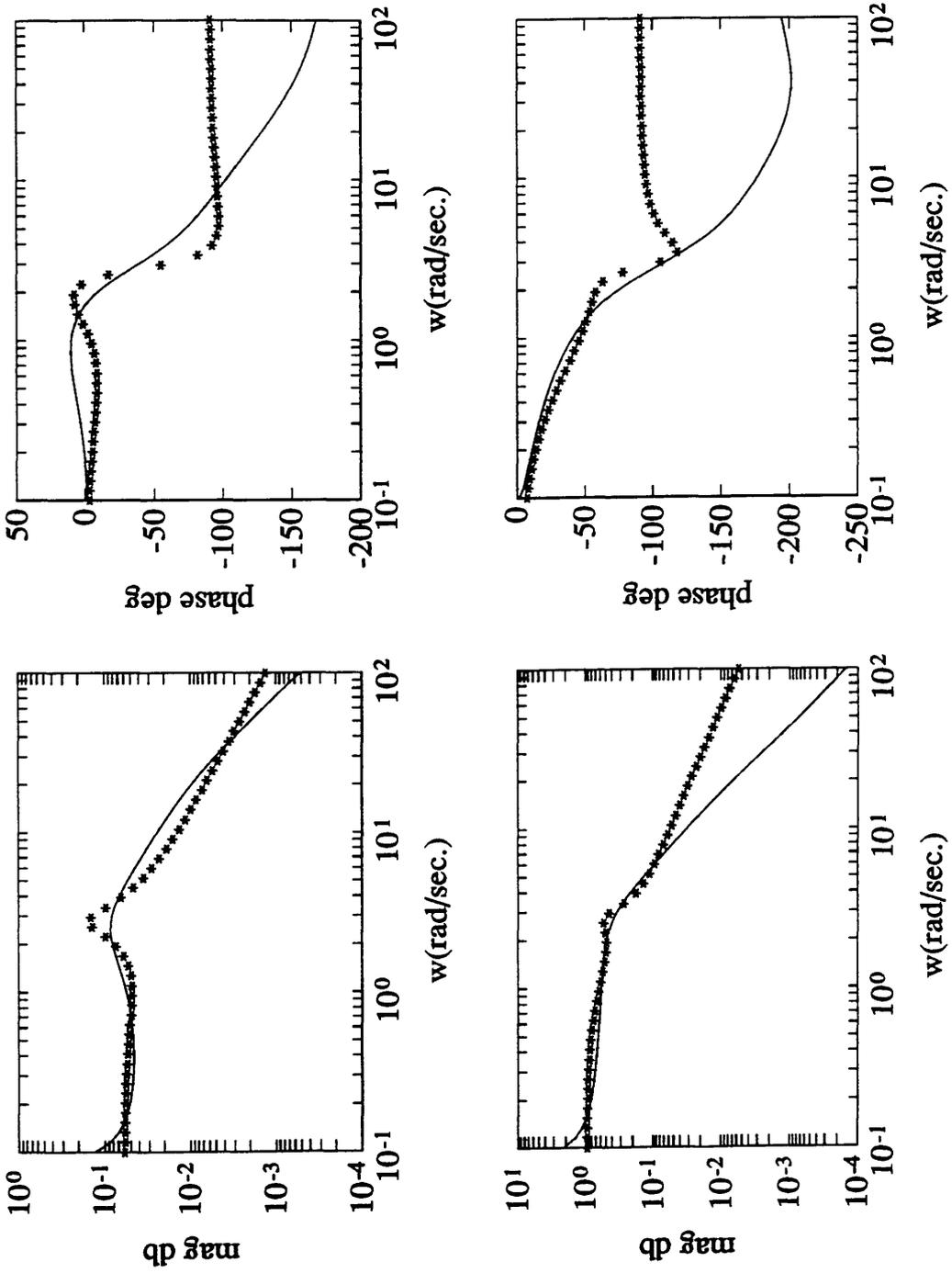|																												
Γ_0	0	$-\Gamma_2$	0	Γ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	T_{11}	S_{12}	$-T_{13}$	$-S_{14}$	T_{15}
0	$-\Gamma_2$	0	Γ_4	0	Γ_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	S_{12}	$-T_{13}$	$-S_{14}$	T_{15}	S_{16}
$-\Gamma_2$	0	Γ_4	0	$-\Gamma_6$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-T_{13}$	$-S_{14}$	T_{15}	S_{16}	$-T_{17}$
0	Γ_4	0	$-\Gamma_6$	0	Γ_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-S_{14}$	T_{15}	S_{16}	$-T_{17}$	$-S_{18}$
Γ_4	0	$-\Gamma_6$	0	Γ_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	T_{15}	S_{16}	$-T_{17}$	$-S_{18}$	T_{19}
0	$-\Gamma_6$	0	Γ_8	0	$-\Gamma_{10}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	S_{16}	$-T_{17}$	$-S_{18}$	T_{19}	S_{110}
0	0	0	0	0	0	Γ_0	0	$-\Gamma_2$	0	Γ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	T_{21}	S_{22}	$-T_{23}$	$-S_{24}$	T_{25}
0	0	0	0	0	0	0	$-\Gamma_2$	0	Γ_4	0	Γ_6	0	0	0	0	0	0	0	0	0	0	0	0	S_{22}	$-T_{23}$	$-S_{24}$	T_{25}	S_{26}
0	0	0	0	0	0	$-\Gamma_2$	0	Γ_4	0	$-\Gamma_6$	0	0	0	0	0	0	0	0	0	0	0	0	0	$-T_{23}$	$-S_{24}$	T_{25}	S_{26}	$-T_{27}$
0	0	0	0	0	0	0	Γ_4	0	$-\Gamma_6$	0	Γ_8	0	0	0	0	0	0	0	0	0	0	0	0	$-S_{24}$	T_{25}	S_{26}	$-T_{27}$	$-S_{28}$
0	0	0	0	0	0	Γ_4	0	$-\Gamma_6$	0	Γ_8	0	0	0	0	0	0	0	0	0	0	0	0	0	T_{25}	S_{26}	$-T_{27}$	$-S_{28}$	T_{29}
0	0	0	0	0	0	0	$-\Gamma_6$	0	Γ_8	0	$-\Gamma_{10}$	0	0	0	0	0	0	0	0	0	0	0	0	S_{26}	$-T_{27}$	$-S_{28}$	T_{29}	S_{210}
0	0	0	0	0	0	0	0	0	0	0	0	Γ_0	0	$-\Gamma_2$	0	Γ_4	0	0	0	0	0	0	0	T_{31}	S_{32}	$-T_{33}$	$-S_{34}$	T_{35}
0	0	0	0	0	0	0	0	0	0	0	0	0	$-\Gamma_2$	0	Γ_4	0	$-\Gamma_6$	0	0	0	0	0	0	S_{32}	$-T_{33}$	$-S_{34}$	T_{35}	S_{36}
0	0	0	0	0	0	0	0	0	0	0	0	$-\Gamma_2$	0	Γ_4	0	$-\Gamma_6$	0	0	0	0	0	0	0	$-T_{33}$	$-S_{34}$	T_{35}	S_{36}	$-T_{37}$
0	0	0	0	0	0	0	0	0	0	0	0	0	Γ_4	0	$-\Gamma_6$	0	Γ_8	0	0	0	0	0	0	$-S_{34}$	T_{35}	S_{36}	$-T_{37}$	$-S_{38}$
0	0	0	0	0	0	0	0	0	0	0	0	Γ_4	0	$-\Gamma_6$	0	Γ_8	0	0	0	0	0	0	0	T_{35}	S_{36}	$-T_{37}$	$-S_{38}$	T_{39}
0	0	0	0	0	0	0	0	0	0	0	0	0	$-\Gamma_6$	0	Γ_8	0	$-\Gamma_{10}$	0	0	0	0	0	0	S_{36}	$-T_{37}$	$-S_{38}$	T_{39}	S_{310}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Γ_0	0	$-\Gamma_2$	0	Γ_4	0	T_{41}	S_{42}	$-T_{43}$	$-S_{44}$	T_{45}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\Gamma_2$	0	Γ_4	0	$-\Gamma_6$	S_{42}	$-T_{43}$	$-S_{44}$	T_{45}	S_{46}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\Gamma_2$	0	Γ_4	0	$-\Gamma_6$	0	$-T_{43}$	$-S_{44}$	T_{45}	S_{46}	$-T_{47}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Γ_4	0	$-\Gamma_6$	0	Γ_8	$-S_{44}$	T_{45}	S_{46}	$-T_{47}$	$-S_{48}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Γ_4	0	$-\Gamma_6$	0	Γ_8	0	T_{45}	S_{46}	$-T_{47}$	$-S_{48}$	T_{49}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\Gamma_6$	0	Γ_8	0	$-\Gamma_{10}$	S_{46}	$-T_{47}$	$-S_{48}$	T_{49}	S_{410}
T_{11}	$-S_{12}$	$-T_{13}$	S_{14}	T_{15}	$-S_{16}$	T_{21}	$-S_{22}$	$-T_{23}$	S_{24}	T_{25}	$-S_{26}$	T_{31}	$-S_{32}$	$-T_{33}$	S_{34}	T_{35}	$-S_{36}$	T_{41}	$-S_{42}$	$-T_{43}$	S_{44}	T_{45}	$-S_{46}$	$(U_{12}+U_{22}+U_{32}+U_{42})$	0	$-(U_{14}+U_{24}+U_{34}+U_{44})$	0	$(U_{16}+U_{26}+U_{36}+U_{46})$
$-S_{12}$	$-T_{13}$	S_{14}	T_{15}	$-S_{16}$	$-T_{17}$	$-S_{22}$	$-T_{23}$	S_{24}	T_{25}	$-S_{26}$	$-T_{27}$	$-S_{32}$	$-T_{33}$	S_{34}	T_{35}	$-S_{36}$	$-T_{37}$	$-S_{42}$	$-T_{43}$	S_{44}	T_{45}	$-S_{46}$	$-T_{47}$	0	$-(U_{14}+U_{24}+U_{34}+U_{44})$	0	$(U_{16}+U_{26}+U_{36}+U_{46})$	0
$-T_{13}$	S_{14}	T_{15}	$-S_{16}$	$-T_{17}$	S_{18}	$-T_{23}$	S_{24}	T_{25}	$-S_{26}$	$-T_{27}$	S_{28}	$-T_{33}$	S_{34}	T_{25}	$-S_{36}$	$-T_{37}$	S_{38}	$-T_{43}$	S_{44}	T_{45}	$-S_{46}$	$-T_{47}$	S_{48}	$-(U_{14}+U_{24}+U_{34}+U_{44})$	0	$(U_{16}+U_{26}+U_{36}+U_{46})$	0	$-(U_{18}+U_{28}+U_{38}+U_{48})$
S_{14}	T_{15}	$-S_{16}$	$-T_{17}$	S_{18}	T_{19}	S_{24}	T_{25}	$-S_{26}$	$-T_{27}$	S_{28}	T_{29}	S_{34}	T_{35}	$-S_{36}$	$-T_{37}$	S_{38}	T_{39}	S_{44}	T_{45}	$-S_{46}$	$-T_{47}$	S_{48}	T_{49}	0	$(U_{16}+U_{26}+U_{36}+U_{46})$	0	$-(U_{18}+U_{28}+U_{38}+U_{48})$	0
T_{15}	$-S_{16}$	$-T_{17}$	S_{18}	T_{19}	$-S_{110}$	T_{25}	$-S_{26}$	$-T_{27}$	S_{28}	T_{29}	$-S_{210}$	T_{35}	$-S_{36}$	$-T_{27}$	S_{38}	T_{39}	$-S_{310}$	T_{45}	$-S_{46}$	$-T_{47}$	S_{48}	T_{49}	$-S_{410}$	$(U_{16}+U_{26}+U_{36}+U_{46})$	0	$-(U_{18}+U_{28}+U_{38}+U_{48})$	0	$(U_{110}+U_{210}+U_{310}+U_{41})$

- HOS, * LEVY METHOD
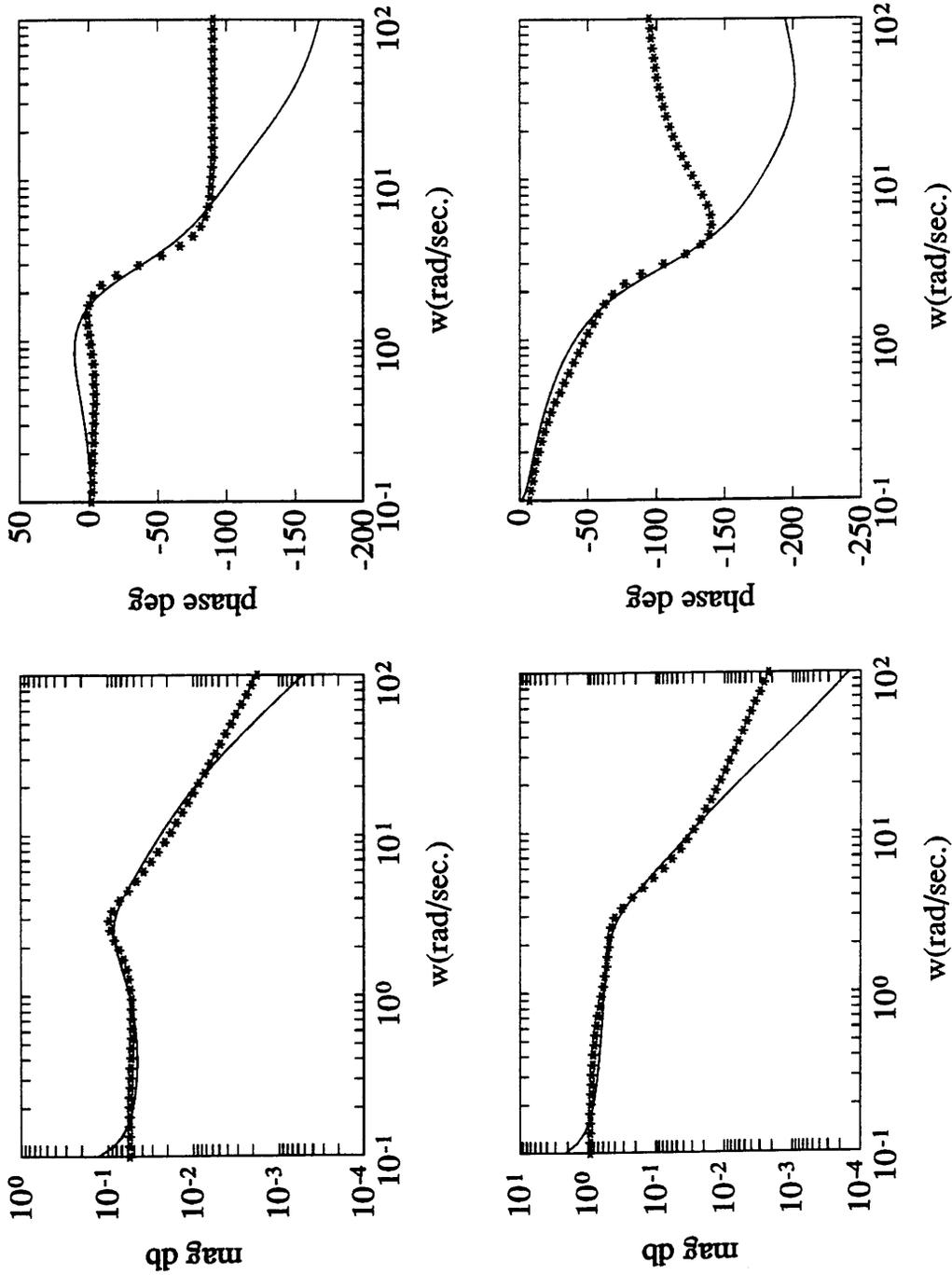
Fig. 4.1 Frequency Response for the Example
$(q/\delta \ \& \ n_{zcr}/\delta, \ n1 = 2.0, \ i = 3)$

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Fig. 4.2 Frequency Response for the Example

$(q/\delta$ & n_{zcr}/δ, $n1 = 1.5$, $i = 3)$
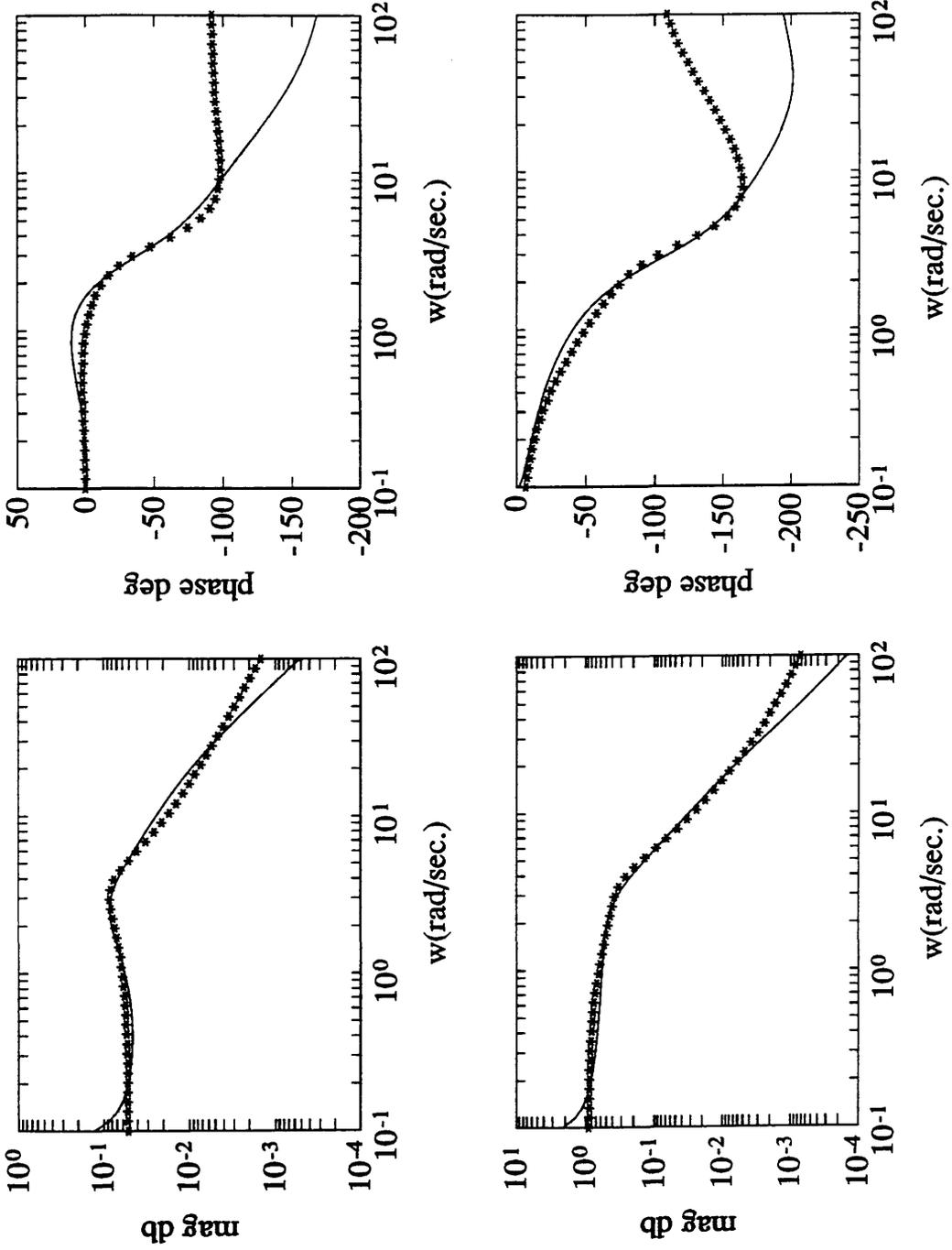
- HOS, * LEVY METHOD

- HOS, * LEVY METHOD

Fig. 4.3 Frequency Response for the Example
$(q/\delta$ & n_{zcr}/δ, n1 = 1.2, i = 3)

Fig. 4.4 Frequency Response for the Example
$(q/\delta \, \& \, n_{zcr}/\delta, \, n1 = 1.0, \, i = 3, \, 50 \text{ points})$

— HOS, * LEVY METHOD

Fig. 4.5 Frequency Response for the Example
(q/δ & n_{zcr}/δ, $n1 = 0.8$, $i = 3$)

- HOS, * LEVY METHOD
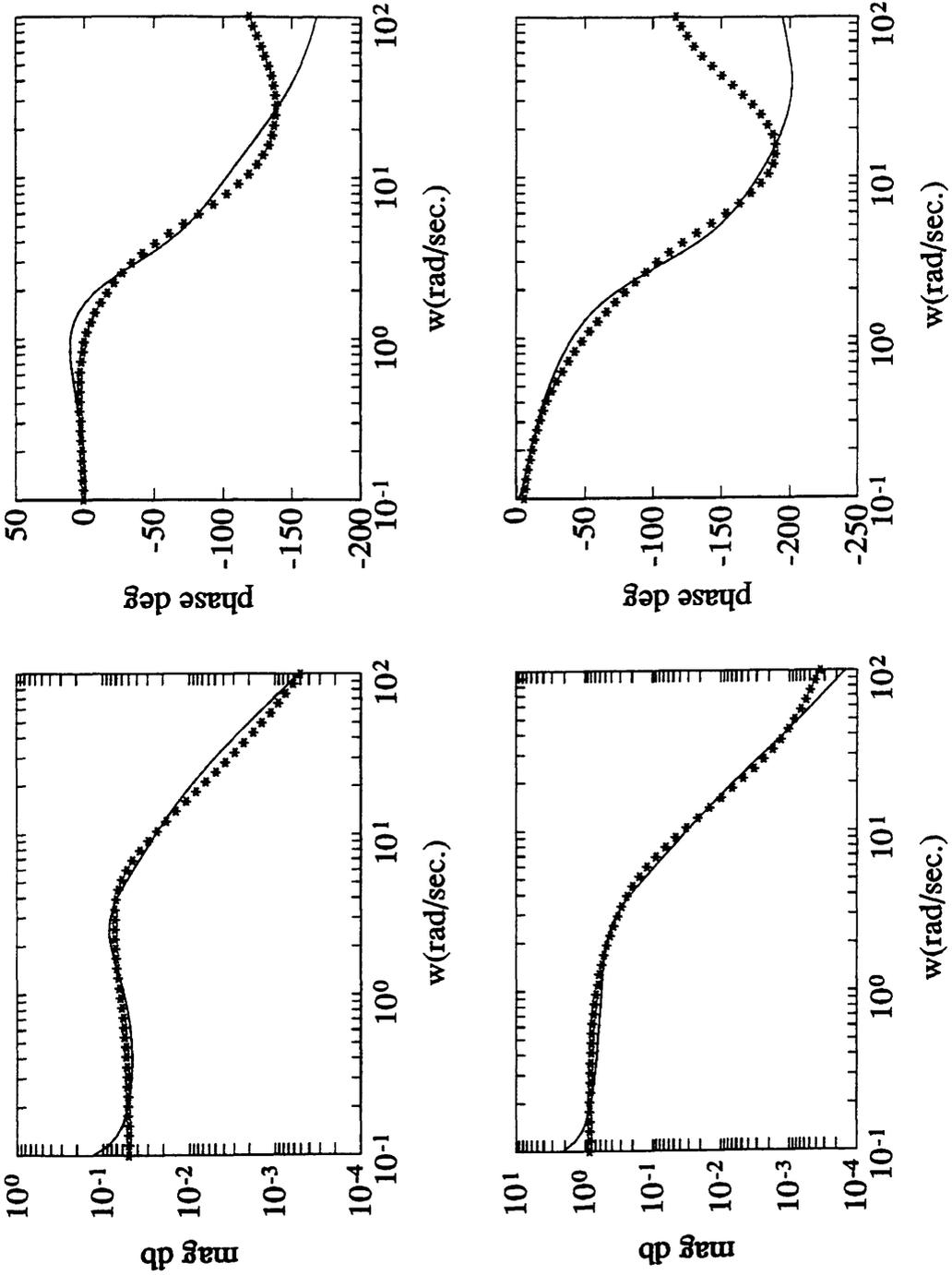
Fig. 4.6 Frequency Response for the Example
$(q/\delta$ & n_{zcr}/δ, $n1 = 0.6$, $i = 3)$

Fig. 4.7 Frequency Response for the Example
$(q/\delta \ \& \ n_{zcr}/\delta, \ n1 = 0.4, \ i = 3)$
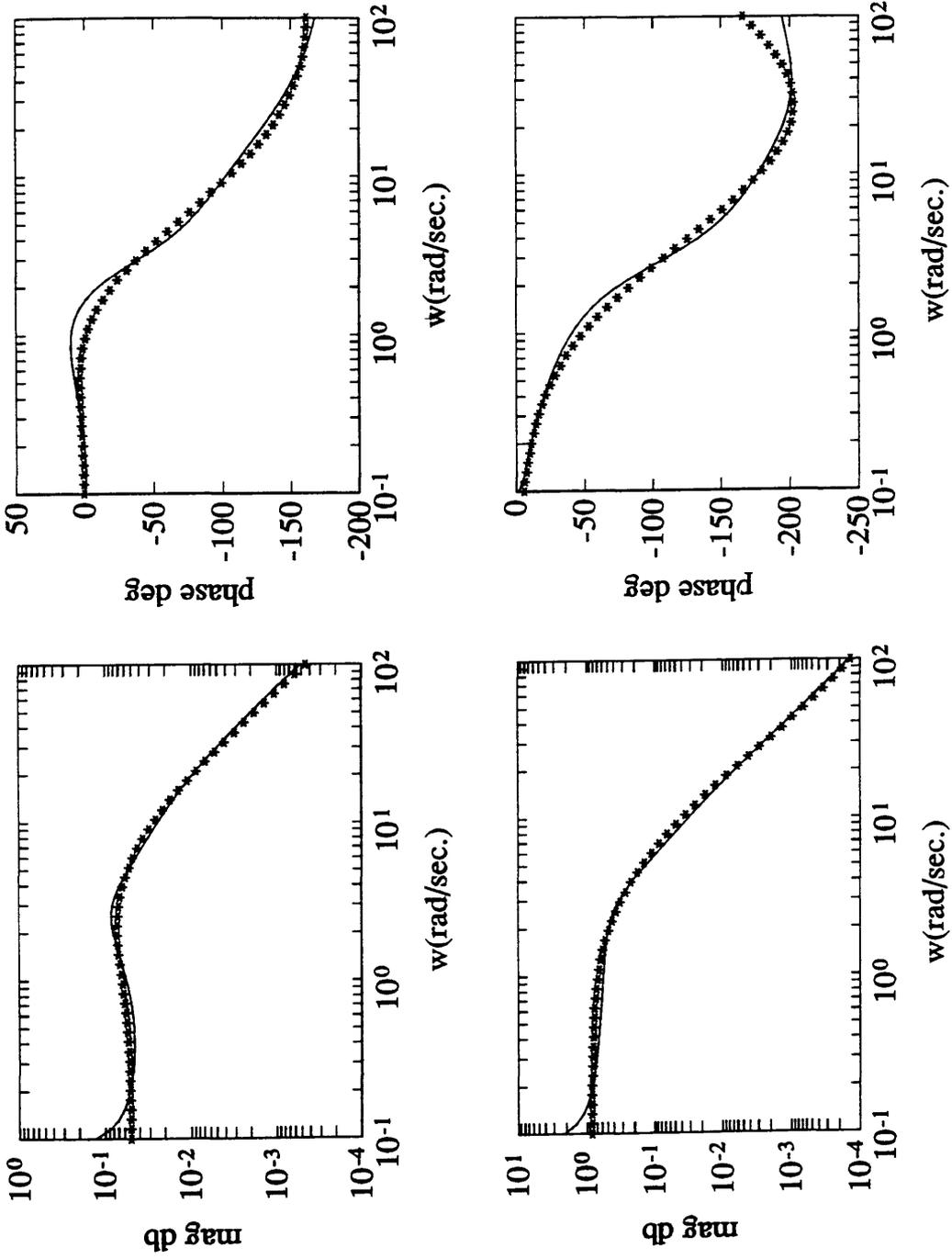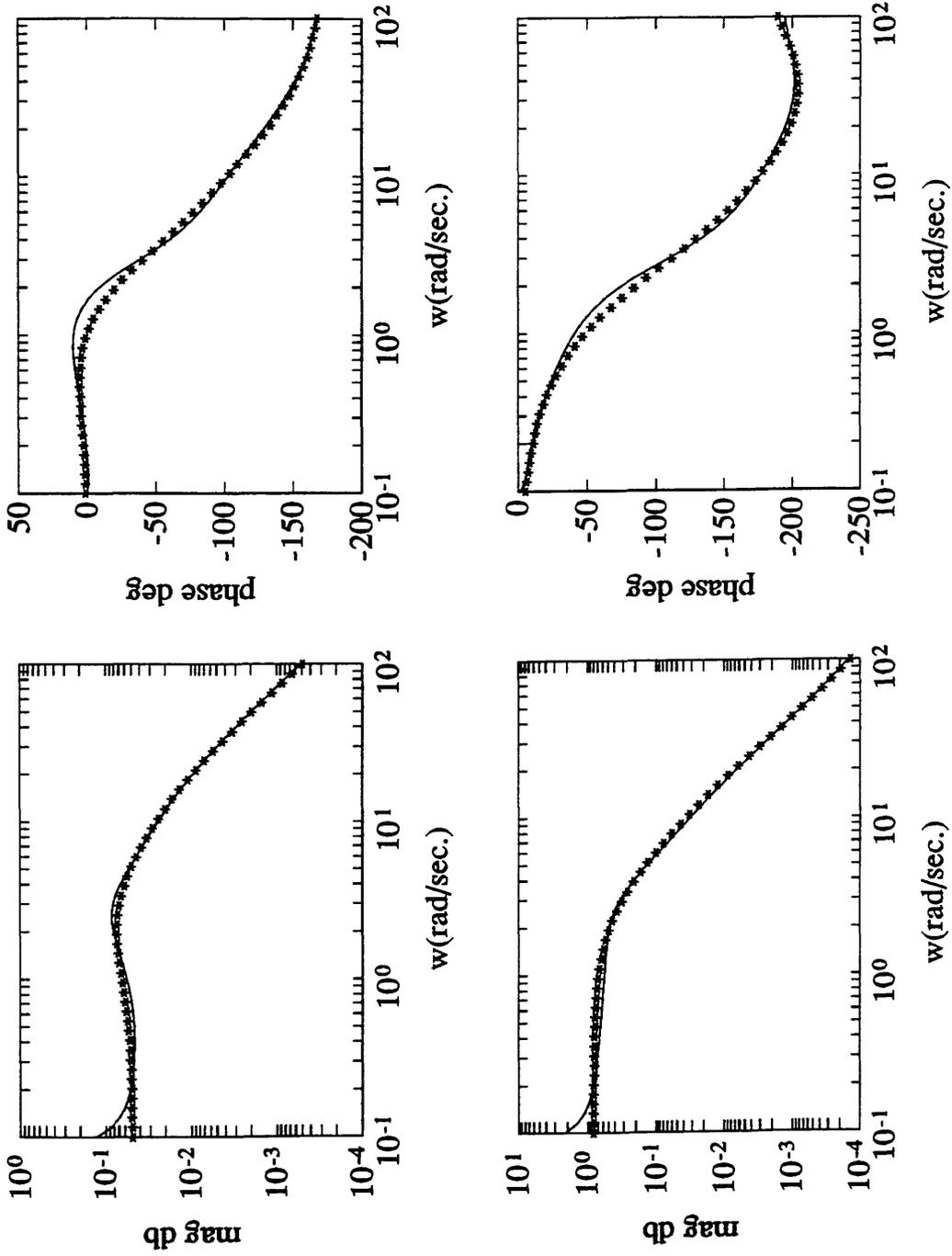
- HOS, * LEVY METHOD

— HOS, * LEVY METHOD

Fig. 4.8 Frequency Response for the Example
(q/δ & n_{zcr}/δ, $n1 = 0.2$, $i = 3$)

Fig. 4.9 Frequency Response for the Example

$(q/\delta$ & n_{zcr}/δ, $n1 = -0.2$, $i = 3)$

- HOS, * LEVY METHOD

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— HOS, * LEVY METHOD

Fig. 4.10 Frequency Response for the Example
(q/δ & n_{zcr}/δ, $n1 = -0.5$, $i = 3$)

— HOS, * LEVY METHOD

Fig. 4.11 Frequency Response for the Example
(q/δ & n_{zcr}/δ, n1 = −1, i = 3)

- 67 -
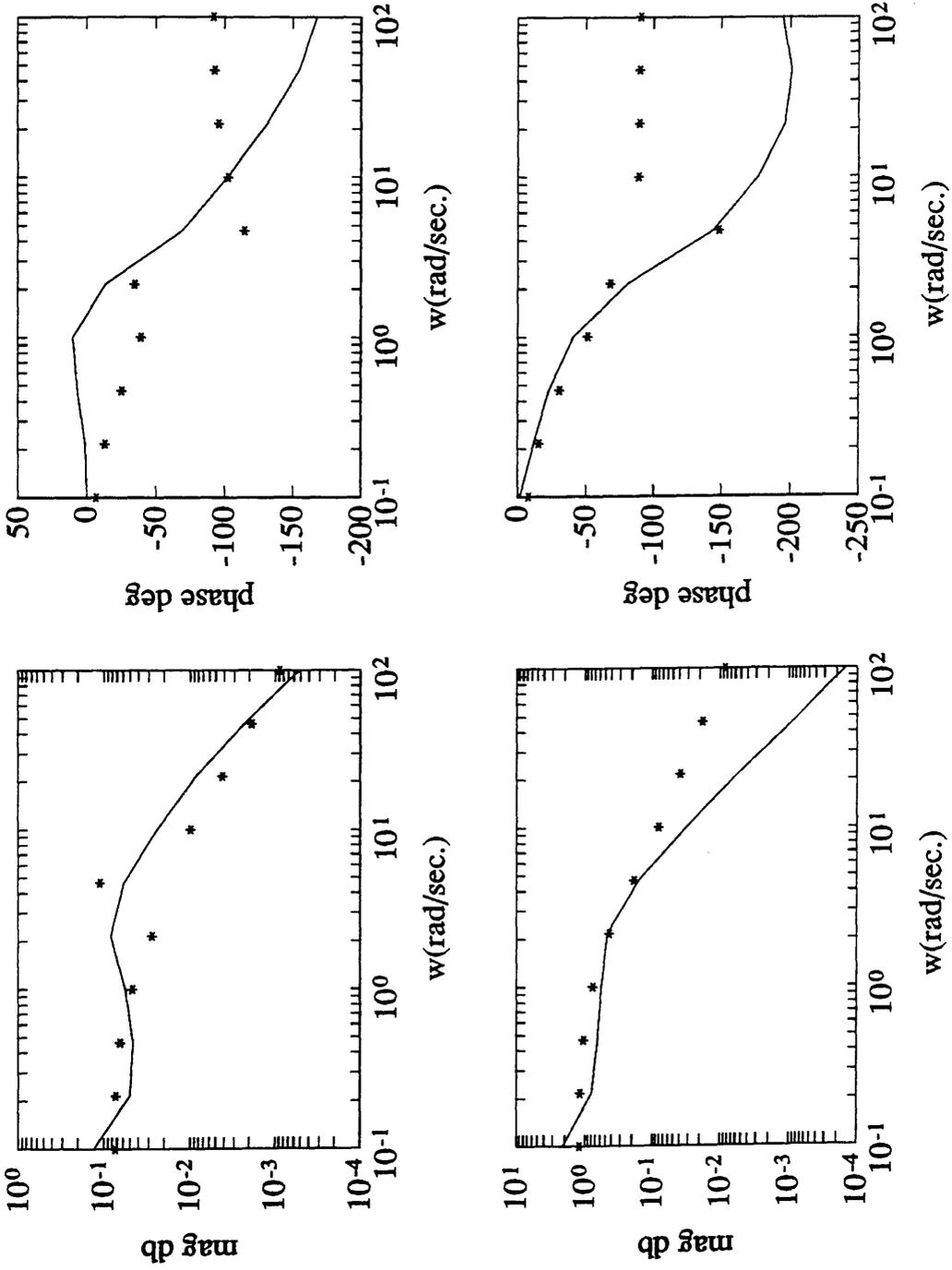
Fig. 4.12 Frequency Response for the Example
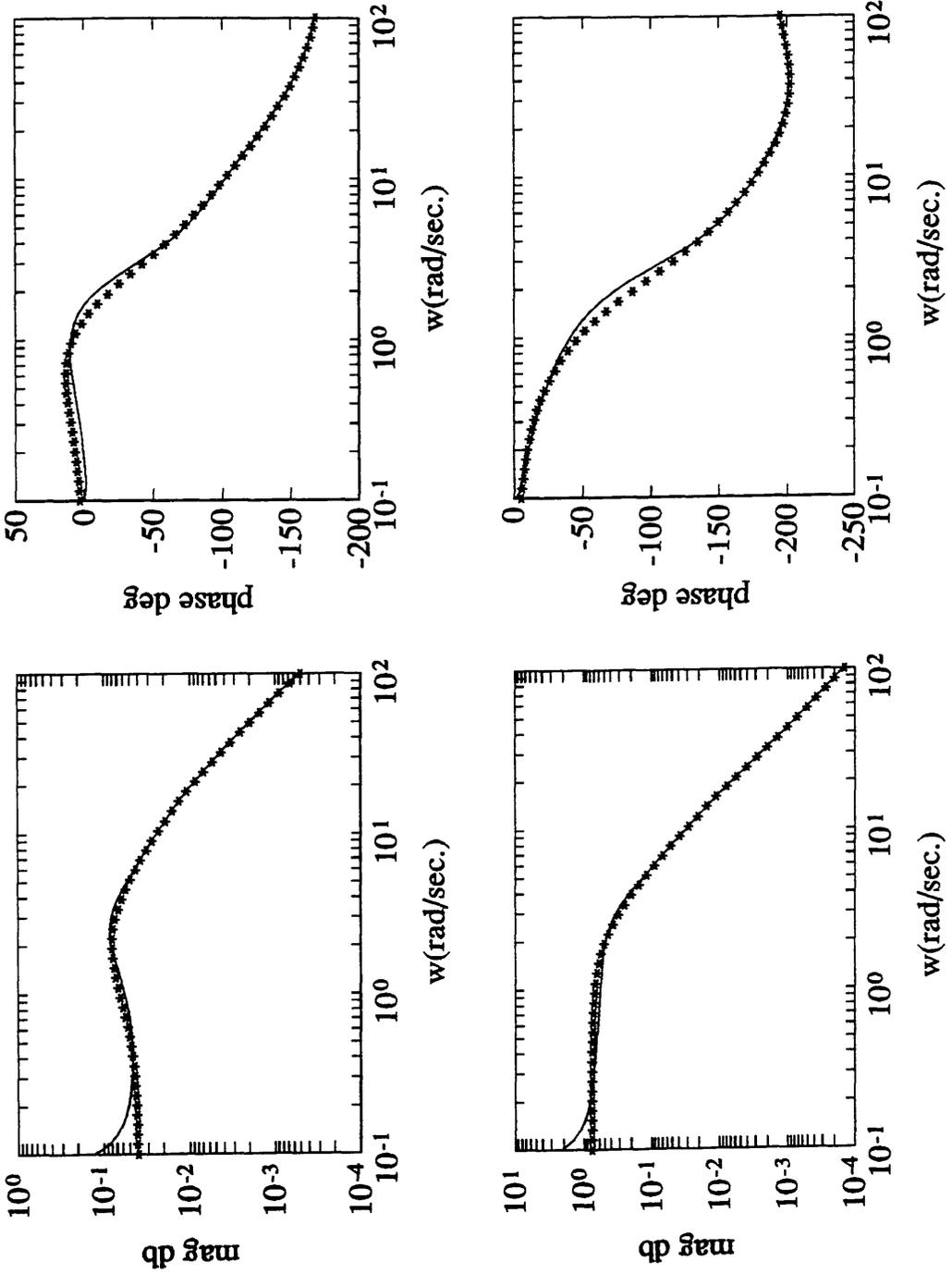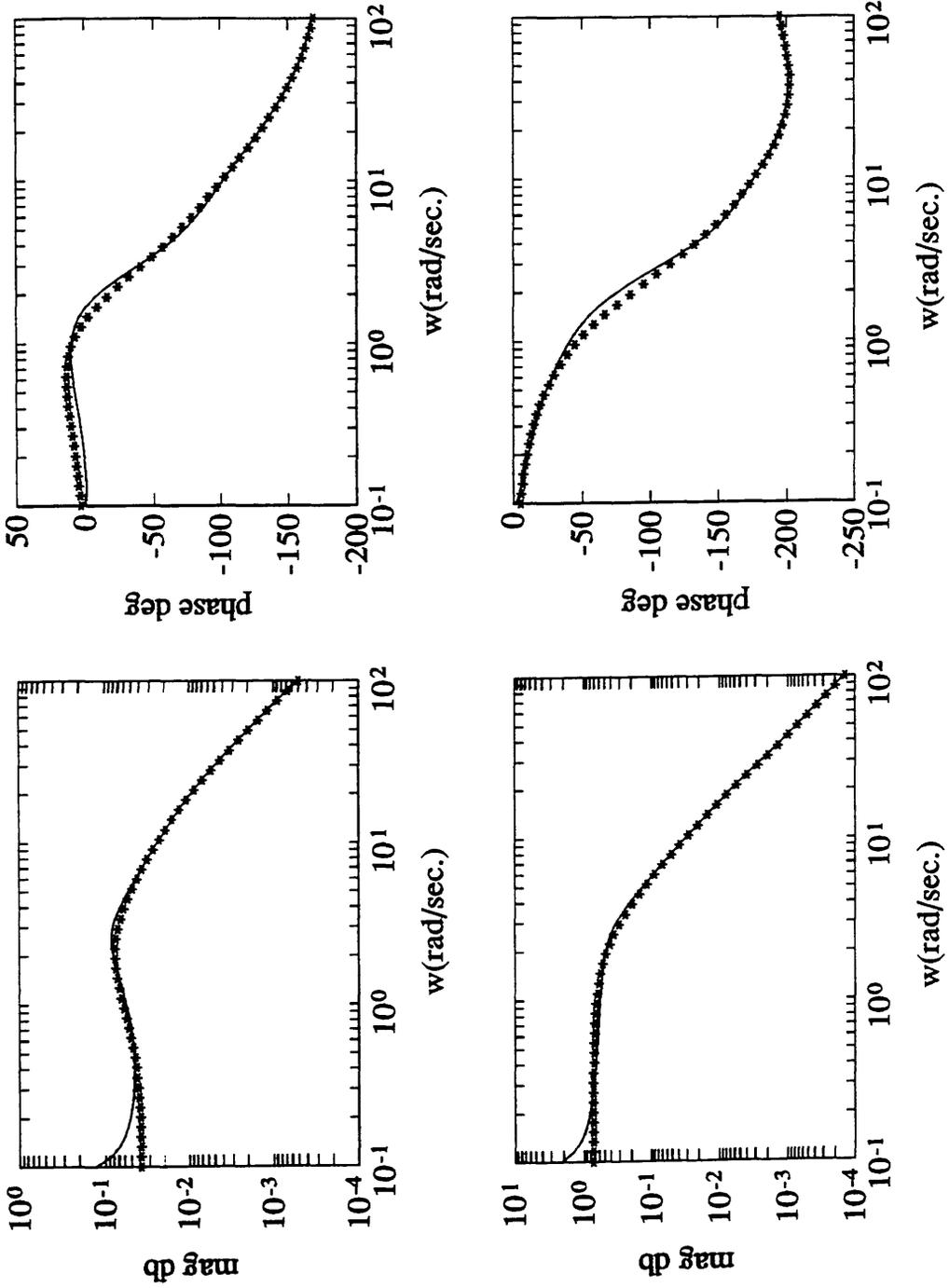(q/δ & n_{zcr}/δ, $n1 = 1.0$, $i = 3$, 10 points)

- HOS, * LEVY METHOD

− HOS, * LEVY METHOD

Fig. 4.13 Frequency Response for the Example
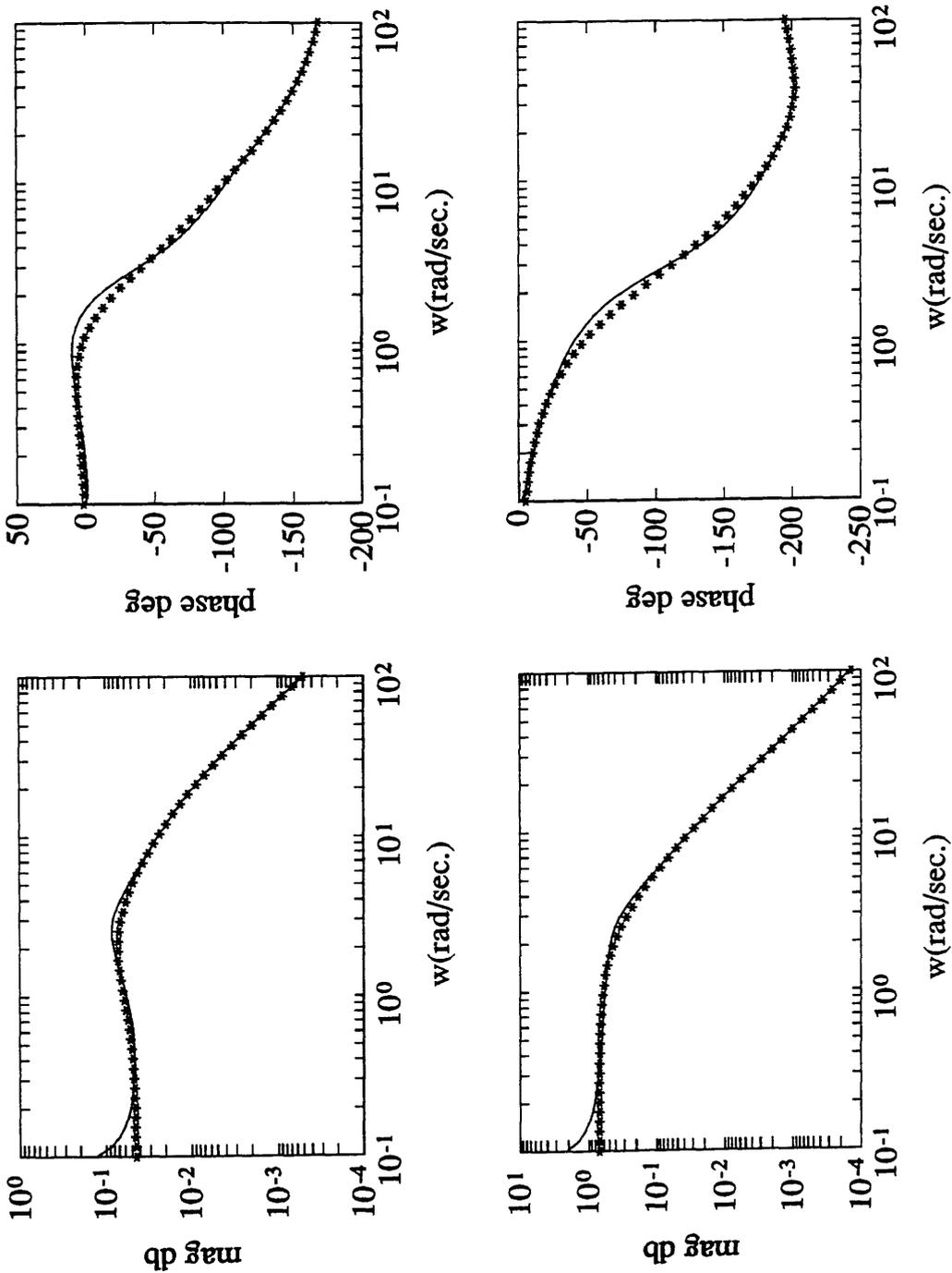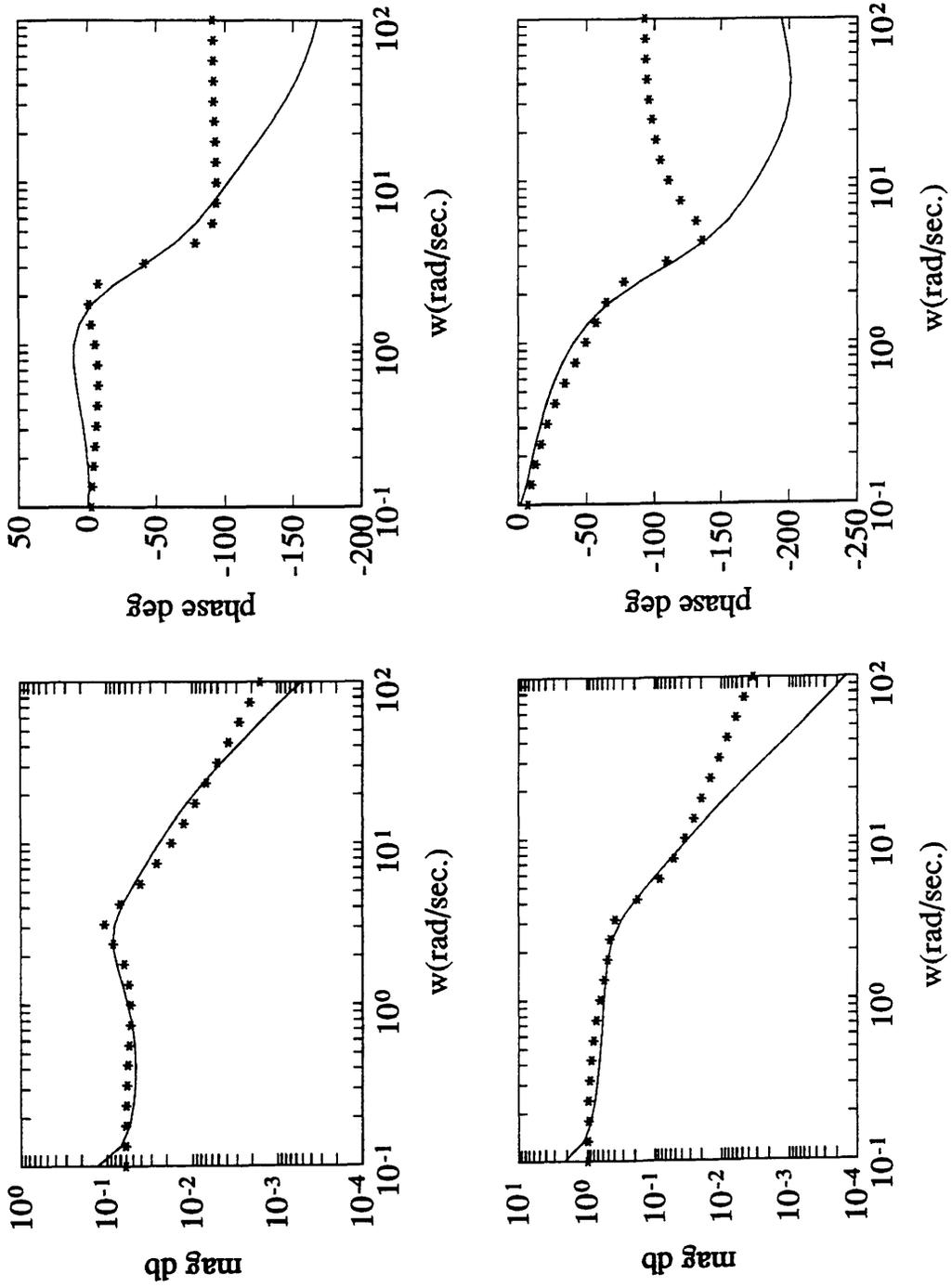(q/δ & n_{zcr}/δ, $n1 = 1.0$, $i = 3$, 25 points)

- HOS, * LEVY METHOD
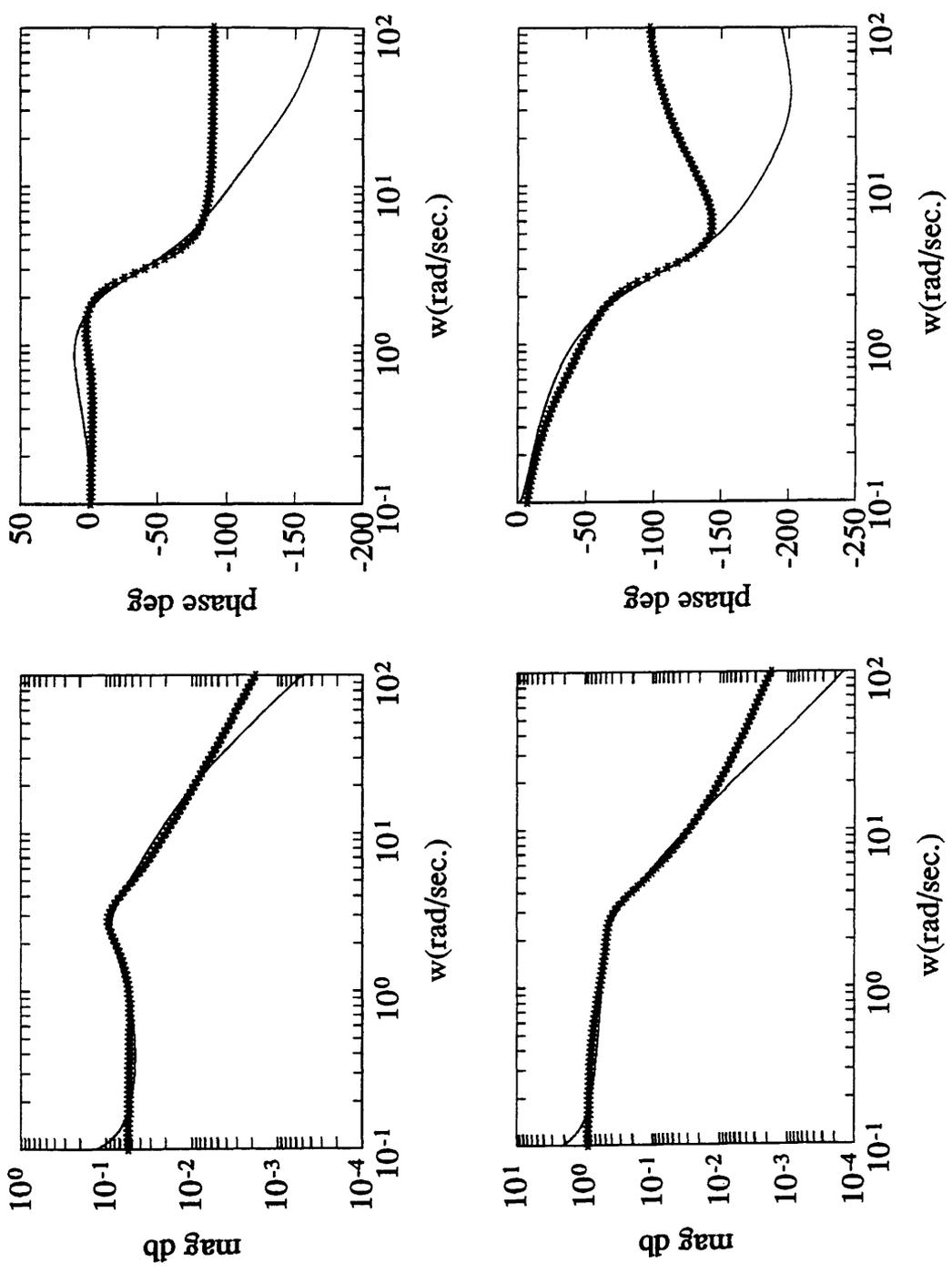
Fig. 4.14 Frequency Response for the Example
(q/δ & n_{zcr}/δ, n1 = 1.0, i = 3, 100 points)
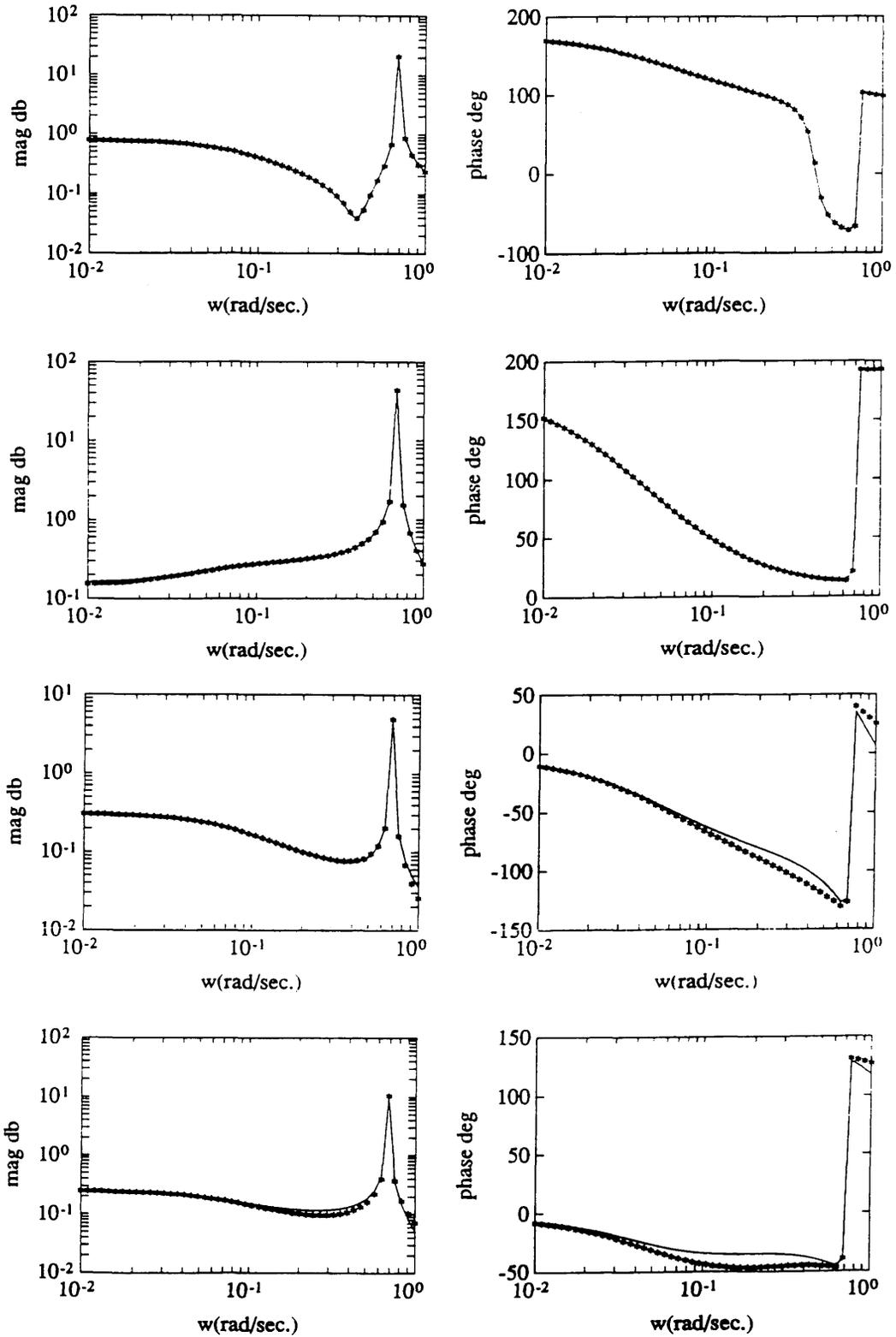
Fig.4.15 Frequency Response of Third Order Model for CHARLIE

$(\gamma/\delta_R, \quad \beta/\delta_R, \quad \gamma/\delta_A \quad \text{and} \quad \beta/\delta_A, \quad n1= 0.5, \quad i= 4)$

Chapter 5

Equivalent Systems Approach to the Reduction

of Multi– Input Multi– Output Systems

5.1 <u>Introduction.</u>

Reduced order modelling can be very important for the purposes of control system design where the designer must attempt to satisfy criteria such as: 1) steady state specifications; and 2) dynamic and transient specifications (e.g. rise time, overshoot, bandwidth etc.) These place emphasise on the low and middle range frequencies. Reduced order models for piloted simulation/handling qualities studies, however, may not be concerned so much with steady state behaviour but may concentrate instead on pilot induced oscillations and other aspects of the overall transient behaviour, with more emphasis often placed on the high and middle frequency ranges.

The development of techniques for model reduction in aircraft applications of this kind has taken place separately from the development of the methods outlined in Chapter 3. Different terminology has therefore arisen and reduced order models developed for use in aircraft handling qualities studies and for applications involving real– time piloted simulation are generally referred to as "equivalent system" models.

The modern trend in aircraft systems is inevitably towards greater complexity, due mainly to requirements imposed by complex performance specifications and tasks and to the need for accuracy. A modern complex system may have many inputs and many outputs, and these may be interrelated in a complicated manner. To analyze such a system, it is essential to reduce the complexity of the mathematical expressions, as well as to resort to computers for most of the tedious computations necessary in the analysis. The state– space approach to system analysis is well suited from this viewpoint. A linear, time– invariant (LTI) system of nth– order, may been expressed as a set of first order differential equations, which can be futher written as

a vector—matrix differential equation. This vector—matrix differential equation is called the state—space representation and has the following form,

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

And the output equation becomes

$$\underline{y} = C\underline{x}$$

where $x \subset R^m$ (R^m is a Cartesian m—space), $u \subset R^n$ and A, B and C are constant matrices of appropriate dimensions.

In this chapter, the use of a low—order approximations, or equivalent systems, in state—space form is considered. Such a formulation is well suited to the treatment of multi—input, multi—output systems and is appropriate for the development of models for aircraft handling qualities assessment. This problem is identified as a special case of the more general problem of model reduction in closed—loop systems.

In the traditional equivalent system approach used in aircraft handling qualities studies[28], a numerical search algorithm is employed to find a reduced—order model, of 'classical' aircraft form, such that the frequency response of the high—order system (the aircraft) is well approximated over a specified frequency range. However, especially when a good approximation is not obtained, there are difficulties in the use of these traditional equivalent systems methods. Non—uniqueness of solutions, the treatment of multi—input multi—output cases and the goodness of fit necessary are factors which present particular problems.

Because of these general and fundamental difficulties, associated with model reduction, the reduced—order modeling objective of approximating the aircraft's frequency response is re—examined in the current work and the question of when and how to match multiple frequency responses is reviewed. An alternative state—space model—reduction approach developed by Bacon and Schmidt[30] is considered both in the content of handling qualities research and for control system design. In this method the original transfer function (matrix) G(s) of dynamic order n is reduced via a state—space transformation T which can be determined without

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the use of any numerical search algorithm. In terms of the three classes of problem identified in Chapter 3 this method is therefore essentially a form of component truncation method. In addition, the resulting model Gr obtained by this method is unique for the selected dynamic order r and the least effective dynamic order is determined a priori by evaluating a set of frequency—domain matching error bounds. These error bounds apply to each i—j element of the difference matrix $[G(s) - Gr(s)]_{s=j\omega}$ over all ω. One important feature of the approach which distinguishes it from the methods outlined in Chapter 3 is that it is applicable to multi—input/multi—output systems and is therefore well suited for aircraft applications.

5.2 The Theory of the Bacon's Method.

The state—space model—reduction approach of Bacon and Schmidt is composed of three stages. The first stage is the modal decomposition. The second stage is the application of an internally balanced approach to model reduction. The third stage is the combination of separate subsystems to form the completed reduced—order approximation.

5.2.1 Modal Decomposition[30]

Consider the state—space model:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \qquad (5.1)$$

$$\underline{y} = C\underline{x} \qquad (5.2)$$

The transfer function G(s) from input $\underline{u}$ to output $\underline{y}$ is given by

$$G(s) = C(sI-A)^{-1}B \qquad (5.3)$$

In the modal decomposition process the given system G(s) is split into three subsystems $G_{lo}(s)$, $G_{mid}(s)$ and $G_{high}(s)$, involving the low—, mid— and high—frequency components of the system separately. This involves four steps:

1) Determination of the eigenvalues and eigenvectors of the matrix A of the state—space equations and thus the diagonal matrix of eigenvalues Λ and the corresponding matrix of eigenvectors M (modal matrix).

2) Determination of radii d1 and d2 which define three frequency regions of the

complex plane.

3) Calculation of the matrices of eigenvectors (modal matrices) M_{lo}, M_{mid}, M_{high} and M_{lo}^{-T}, M_{mid}^{-T}, M_{high}^{-T} where the '$-T$' operation is defined as the transpose of the inverse or the inverse of the transpose which are equivalent operations.

4) Calculation of the three subsystem transfer functions $G_i(s)$ where $i = lo$, mid and hi.

The details of these four steps are described as below.

5.2.1.1 Determination of the Matrices Λ and M Corresponding to the Matrix A of the State—Space Equations.

The matrix A can be modally decomposed to obtain Λ and M, where Λ is the block diagonal, real Jordan form and M is the corresponding (real) modal matrix. Each column of M is, of couse, an eigenvector of A and from standard matrix theory it follows that[71]

$$A = M \Lambda M^{-1} \qquad (5.4)$$

where

Λ --- is a diagonal matrix involving the eigenvalues of the matrix A

M --- is a matrix whose columns are the eigenvectors of the matrix Λ

Conversely it follows that

$$\Lambda = M^{-1}AM \qquad (5.5)$$

5.2.1.2 Determination of Radii d1 and d2 Which Define Three Frequency Regions of the Complex Plane.

Because modal decomposition is used, it is important to examine the implications in terms of the frequency response of the system. The transfer function G(s) describing the input/output behavior of the linear system is:

$$G(s) = \frac{Q(s)}{P(s)} = \frac{\gamma_i \prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

It is these parameters the poles and zeros, that influence the frequency response and the effective order of the system.

The magnitude and phase on the Bode diagram of $G(j\omega)$ are related to the directed line segments in the s–plane as shown in Fig. 5.1. As ω moves along the imaginary axis, these directed segments, rotate and change magnitude. The effect of constraining the frequency ω to lie within a range (ω_1, ω_2) is of particular interest for reduced–order modelling. In Fig. 5.1 two concentric circles of radii ω_1 and ω_2 are shown and these circles separate the s–plane into low–, mid– and high–frequency regions.

The columns of M are now ordered according to the natural frequency, $\omega_{ni} = (\sigma_i^2 + \omega_i^2)^{1/2}$ of the corresponding modes as defined by the positions of the eigenvalues λ_i in the complex plane. Thus, if radii d1 and d2 define concentric circles in the complex plane, and M can be divided as M_{lo}, M_{mid} and M_{high} as shown in Table 5.1, the complex s–plane is divided into three regions by the parameters d1 and d2.
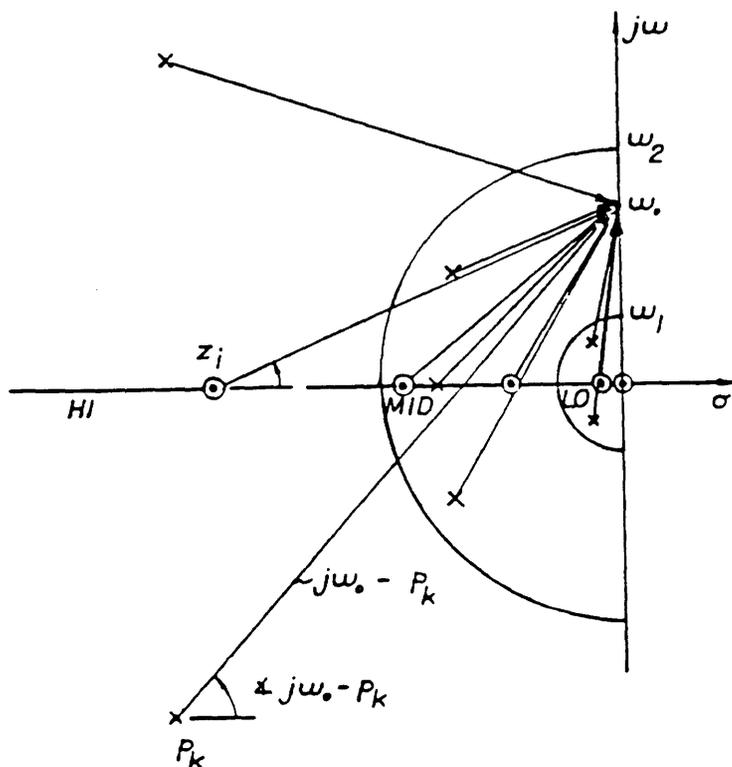

Figure 5.1 Pole/Zero Definition of Frequency Response

Table 5.1

Regions		Column groupings
Low frequency (lo):	$0 \leqslant \omega_{ni} < d1$	M_{lo}
Middle frequency (mid):	$d1 \leqslant \omega_{ni} < d2$	M_{mid}
High frequency (high):	$d2 \leqslant \omega_{ni}$	M_{high}

5.2.1.3 Calculation of M_{lo}, M_{mid}, M_{high} and M_{lo}^{-T}, M_{mid}^{-T}, M_{high}^{-T}.

From Table 5.1, we can see that M is separated into three column groups M_{lo}, M_{mid} and M_{high}. That is

$$M = [\ M_{high}\ \ M_{mid}\ \ M_{lo}\ ] \qquad (5.6)$$

and from the definition of the superscript '$-T$' given in Section 5.2.1 we have

$$M^{-T} = [\ Z_{high}\ \ Z_{mid}\ \ Z_{lo}\ ] \qquad (5.7)$$

where Z_{high} and Z_{lo} are the separate columns of the M^{-T} matrix

5.2.1.4 Calculation of the three subsystem transfer functions $G_i(s)$.

We can find the transfer functions $G_i(s)$ of the three subsystems directly from

$$G_i(s) = C_i(sI - A_i)^{-1}B_i \qquad (5.8)$$

where $i = $ lo, mid and hi. In this equation

$$A_i = Z_i^T A M_i \qquad B_i = Z_i^T B \quad \text{and} \quad C_i = C M_i$$

The HOS (high-order system) transfer function may then be rewritten as

$$G(s) = \gamma \left[\frac{\prod_{i=1}^{mLO}(s-z_i)}{\prod_{i=1}^{nLO}(s-z_i)} \right] \left[\frac{\prod_{i=1}^{mMID}(s-z_i)}{\prod_{i=1}^{nMID}(s-z_i)} \right] \left[\frac{\prod_{i=1}^{mHI}(s-z_i)}{\prod_{i=1}^{nHI}(s-z_i)} \right] \qquad (5.9)$$

or in the form

$$G(s) = G_{lo}(s) + G_{mid}(s) + G_{high}(s) \qquad (5.10)$$

where $G_k(s) = \left[\sum_{i=1}^{n_k} \frac{R_i}{s-p_i} \right]_k \qquad k = $ lo, mid, high.

In equation (5.9) the factor γ represents the gain constant of the orginal high order system expressed in transfer function form as

$$G(s) = \gamma \; \frac{\prod\limits_{i=1}^{m}(s-z_i)}{\prod\limits_{i=1}^{n}(s-z_i)}$$

5.2.2. Application of the Internally Balanced Approach[13,34]

The second stage in the reduction process is to use an internally balanced approach to reduce the order of the G_{high} and G_{mid} which have been obtained by stage 1. After that an effective system will be obtained. For illustration, this stage is separated into several steps as given below:

1) Determination of the controllability grammian X and the observability grammian S and calculation of the eigenvalues and eigenvectors of XS in order to obtain the Hankel singular values which provide a measure of the effective order of a system..

2) Determination of the effective order of each subsystem G_i, i = high, mid.

3) Determination of lower-order system approximations of every subsystem $G_r(s)$.

5.2.2.1 Determination of the Controllability Grammian X and the Observability Grammian S.

Before investigating further the order reduction of a system, it is necessary to consider the concepts of controllability and observability.

A system described by Equ.(5.1) is said to be state controllable at t_0 if it is possible to construct an unconstrained control signal which will transfer an initial state to any final state in a finite time interval $t_0 \leqslant t \leqslant t_1$. If every state is controllable, then the system is said to be completely state controllable. Similarly the system is said to be completely observable if every initial state x(0) can be determined from the observation of y(t) over a finite time interval. The degree of the controllability of a system can be determined by a matrix, called the

Controllability Grammian[79], which is defined by the following integral

$$W_c{}^2 = \int_0^\infty \Phi(t)BB^T \Phi(t)^T dt \qquad (5.11)$$

where $\Phi(t)$ is the state-transion matrix. For a system described by Equ. (5.1), we have $\Phi(t) = e^{At}$ (see Ref. 81).

Similarly, the degree of observability can be found from the Observability Grammian[70], which is defined by

$$W_o{}^2 = \int_0^\infty \Phi(t)^T C^T C \Phi(t) dt \qquad (5.12)$$

In order to calculate $W_c{}^2$ and $W_o{}^2$, it is not necessary to integrate Eqs.(5.11) and (5.12). If $X = W_c{}^2$ and $S = W_o{}^2$, it can been proved[13] that X and S are respectively the solutions of the following equations (5.13) and (5.14)[77], if the eigenvalues of A have negative real parts.

$$XA^T + AX + BB^T = 0 \qquad (5.13)$$

$$SA + A^TS + C^TC = 0 \qquad (5.14)$$

These two equations are known as the Lyapunov equation.

5.2.2.2 Hankel Singular Values.

In order to obtain the effective order of the reduced order model we introduce the concepts of the Hankel matrix and the Hankel singular values. The details of the definition and the calculation of the Hankel matrix and Hankel singular values can be found from APPENDIX 2 (A2.3 & A2.6). From APPENDIX 2 (A2.6 & A2.7) we can obtain the Hankel singular values and obtain a relation between the product XS and H^TH,

$$XS = H^TH$$

This means the matrix XS is a symmetric matrix. Since H^TH is a symmetric matrix, the singular values σ_i of H are equal to the square roots of the eigenvalues

of $H^T H$, namely, equal to the square roots of the eigenvalues of the product of XS[78]. Thus, the singular values S1 of the matrix H are

$$S1 = SVD(H) \quad \text{or} \quad H = U \, [\, S1 \,] \, U^T.$$

where S1 is a n $\times$ n diagonal matrix and where the columns of U are called the left singlar vectors.

$$S1 = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \sigma_n \end{bmatrix} \quad \text{for which } \sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0$$

5.2.2.3 Calculation of the Eigenvalues Λ1 and Eigenvectors M1 of the Product XS.

From the internally balanced approach method we are able to find an optimal rth−order model from the eigenvalues Λ1 and eigenvectors M1 of the product XS by following the procedures given below,

$$XS = (M1)(\Lambda 1)(M1)^{-1}$$

$$\Lambda 1 = \begin{bmatrix} h_1{}^2 & & & & \\ & h_2{}^2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & h_n{}^2 \end{bmatrix}$$

$$h_i = \lambda_i{}^{1/2}(XS)$$

We have seen that

$$h_i = \sigma_i \text{ and } h_1 \geqslant h_2 \geqslant \ldots \geqslant h_r \geqslant h_{r+1} \geqslant \ldots \geqslant h_n \geqslant 0.$$

5.2.2.4 Determination of the Effective Order of the Subsystems G_i(s), (i = high, mid).

Because $h_1 \geqslant h_2 \geqslant \ldots \geqslant h_n \geqslant 0$, if a large separation exists between h_i and h_{i+1} it follows that $h_i / h_{i+1} >> 1$, and hence this i may be taken as the effective order r. The eigenvectors of the product XS, M1, and the inverse of the transpose

of M1, $(M1)^{-T}$, can be expressed as

$$M1 = [(M1)_r, (M1)_{n-r}] \qquad (M1)^{-T} = [U_r, U_{n-r}]$$

and

$$\Lambda1 = \begin{bmatrix} \Sigma_r{}^2 & 0 \\ & \\ & \\ 0 & \Sigma_{n-r}{}^2 \end{bmatrix}$$

where $\Sigma_r = \text{diag}(h_i)$, $i = 1,r$; $\Sigma_{n-r} = \text{diag}(h_i)$, $i = r+1,n$.

This r is effective order of the subsystem $G_i(s)$.

5.2.2.5 <u>Determination of Lower—Order System Approximations of Every Subsystem</u>

<u>$G_r(s)$</u>.

From 5.2.2.4 we have obtained the effective order r_{high}, r_{mid} of the subsystems $G_{high}(s)$, $G_{mid}(s)$. Therefore we can find the approximate lower—order subsystems $G_{ri}(s)$, (i = high, mid).

$$G_r(s) = C_r (sI - A_r)^{-1} B_r$$

where A_r, B_r and C_r are defined by

$$A_r = U_r{}^T A T_r, \qquad B_r = U_r{}^T B, \qquad C_r = C T_r.$$

5.2.3 <u>The Complete Reduced—order Approximation</u>

Now we have found $A_{r,high}$ $C_{r,high}$ and $A_{r,mid}$, $B_{r,mid}$, $C_{r,mid}$. A first order approximation is sought for G_{high}, whereas an approximation of the desired 'classical' order is sought for G_{mid}. The complete reduced—order approximation is the sum of these two subsystem approximations.

The state — space model for a subsystem can be written as

$$\dot{\underline{x}}_i = A_i \underline{x}_i + B_i \underline{u}$$

$$y_i = C_i \underline{x}_i$$

So, for the mid subsystem we have

$$\dot{\underline{x}}_{mid} = A_{mid} \underline{x}_{mid} + B_{mid} \underline{u}$$

$$y_{mid} = C_{mid} \underline{x}_{mid}$$

and for hi subsystem

$$\dot{x}_{high} = A_{high}x_{high} + B_{high}\underline{u}$$

$$y_{high} = C_{high}x_{high}$$

The complete the reduced order system (LOS) of the high order system (HOS) is the sum of the two subsystems, namely

$$\dot{x} = \dot{x}_{mid} + \dot{x}_{high}$$

$$y = y_{mid} + y_{high}$$

The transfer function of the complete reduced-order system (LOS) cab be found from

$$G_{los}(s) = C_{rc} (sI - A_{rc})^{-1}B_{rc}$$

where

$$A_{rc} = \begin{bmatrix} A_{r,high} & 0 \\ & \\ 0 & A_{r,mid} \end{bmatrix} \qquad B_{rc} = \begin{bmatrix} B_{r,high} \\ \\ B_{r,mid} \end{bmatrix} \qquad C_{rc} = [C_{r,high}\ C_{r,mid}]$$

5.2.4 The Measure of Error Bounds.

5.2.4.1 The Scaling Matrices Q_i and Q_o.

The need for scaling arises because the internally balanced approach is more sensitive to the responses with higher magnitudes. Different magnitudes can simply arise from different units in the input or output channels, or different force gradients in manipulators. To obtain a uniform match between the truly dominant response of the system, scaling must be included. Q_i and Q_o are the input scaling matrix and the output scaling matrix and are square diagonal matrices containing the non-negative scaling factors q_{ij} and q_{oj}. By using scaling, one can also bias the resulting low-order system to better approximate a certain response. If the scaling (Q_i and Q_o) has been chosen such that elements of the matrix $G(j\omega)$ are weighted equally (their scaled magnitudes roughly equal), the Bode error bound will be applicable for each element of the frequency response. In Chapter 7 the results

from an example of the advanced fighter[30] with scaling and without scaling are presented to illustrate the effect of the scaling.

5.2.4.2 The Determine of Error Bound.

The measure of how well the reduced order model approximates the high order system is reflected in the frequency response error bound of the model. In multi— input and multi— output closed— loop systems, which include pilot/vehicle systems, a "good" approximation implies that $G_r(j\omega)$ must approximate the $G(j\omega)$ element over the multivariable crossover frequency range.

The frequency response error is defined by

$$E(j\omega) = [\ E_{ij}(j\omega)] = G(j\omega) - G_r(j\omega)$$

each i— j element in $E(j\omega)$ describes the frequency response error associated with the corresponding element in $G(j\omega)$.

A matrix norm, defined by the maximum singular value of the matrix $E(j\omega)$, provides a measure of "smallness" for $E(j\omega)$, and also bounds $|E_{ij}(j\omega)|$. The maximun singular value of E is defined as :

$$\sigma(E) = \lambda^{1/2}(EE^*)$$

$$= \underset{\|v\|=1}{\text{Max}} (v^*EE^*v)^{1/2}$$

where $\lambda(.)$ is the maximun eigenvalue of (.), and $\|v\| = (v^*v)^{1/2}$. The largest value of $\sigma[E(j\omega)]$ over all frequencies $(0 \leqslant \omega < \infty)$ defines the "∞ norm"

$$\|E(j\omega)\|_\infty = \sup_\omega \sigma[E(j\omega)] \tag{5.15}$$

If for some ω $\|E(j\omega)\|_\infty << |G_{ij}(j\omega)|$, then $G_{rij}(j\omega)$ closely approximates $G_{ij}(j\omega)$. The fact that the model reduction procedure to be presented exploits another reduction technique, i.e., Moore's internally balanced approach[13], means that there is another method available for obtaining this norm. Enns[34] has shown that a model obtained from Moore's technique satisfies

$$\|E(j\omega)\|_\infty \leqslant B_\infty$$

where the value of B_∞ may be determined before the reduction is performed. The

procedure to be presented also has such a bound.

Another result involving the internally balanced algorithm was proved by Enns[34], who showed that the frequency response error of the rth order model is bounded for all ω by

$$\sigma\{Q_o[G(j\omega) - G_r(j\omega)]Q_i\} < 2\text{Tr}(\Sigma_{n-r})$$

where $\sigma(\cdot)$ is the maximun singular value of $(\cdot)$, and $\Sigma_{n-r} = \text{diag}(h_j)$, where $j = r+1,n$. The bound is defined by the truncated Hankel singular values of the scaled system $Q_oG(s)Q_i$, which like $G(s)$ are invariant to state transformation.

Chapter 6

Implementation of Bacon and Schmidt's Method in MATLAB

6.1 Introduction to MATLAB Software.

The name MATLAB means "Matrix Laboratory" and it was developed by a group of software professionals under the leadship of Cleve Moler. MATLAB is a very powerful software tool that can be of considerable use for scientific and engineering numerical calculations. It is an easy—to—use interactive system for matrix algebra whose basic data element is a matrix that does not require dimensioning. This allows many numerical problems to be solved in a fraction of the time it would take to write a program in a language like Fortran, Basic, or C. It has a rich collection of functions that are immediately useful to the control engineer or system theorist. Complex arithmetic, eigenvalues, root—finding, matrix inversion etc. are some of the facilities available. More generally, MATLAB's linear algebra, matrix computation, and numerical analysis capabilities provide a reliable foundation for control system engineering as well as many other disciplines. The algorithms used by MATLAB are derived from extensive research and represent the state of the art. This, combined with a two— and three— dimensional graphics capability, provides a very useful environment for the application of linear algebra. Also the MATLAB software is designed to run on various machines including MS—DOS compatible personal computers, Apple Macintosh, Sun Workstations, and VAX computers.

MATLAB has evolved over more than half a decade, with input from many users. In university environments it has become the standard instructional tool used in introductory courses in applied linear algebra, as well as for research and in advanced courses in other areas.

6.2 The Outline of the MATLAB Program for the Bacon and Schmidt's Method.

This chapter is concerned with describing the procedure for implementation of Bacon and Schmidt's method in MATLAB. The program was written in MATLAB and was used on a DEC MicroVAX 3600 computer (VMS2) at the Glasgow

University Computer Centre. The MATLAB program can be described in terms of the folloeing seven steps.

6.2.1 Conversion to State– Space Form from Transfer Function Form.

The given high order system is defined through transfer functions rather than in state– space form. Therefore, we have to convert these transfer functions to state– space form from the original form using an appropriate MATLAB function such as the function TF2SS. The system transfer function is

$$G(s) \ = \ C \ (\ sI \ - \ A \ )^{-1} \ B \tag{6.1}$$

and the application of TF2SS for given a high– order system results in a system matrix A, control matrix B and output matrix C.

6.2.2 Calculation of the Eigenvalues and Eigenvectors of the Matrix A.

In order to decompose a model, it is necessary to obtain the eigenvalues Λ and eigenvectors M of the matrix A. According to the frequency range of interest and the eigenvalues Λ it is possible to establish the radii d1 and d2 which define the concentric circles in the complex plane. These divisions provide the basis on which columns of M are separated into three groups M_{lo}, M_{mid} and M_{high}. Also the columns of M^{-T} can be separated into another three groups Z_{lo}, Z_{mid} and Z_{high} very easily in MATLAB.

6.2.3 Grouping of the Three Subsystem Transfer Functions $G_i(s)$.

Using the formula for transfer function $G_i(s)$, of the form of equation (6.1), we can get the matrices A_i, B_i and C_i of the three subsystems directly from equ. (5.8.1) using MÁTLAB. The subsystems

$G_i(s) \ = \ C_i(sI \ - \ A_i)^{-1} \ B_i$, where i = lo, mid, high

are now

$$A_i \ = \ z_i^T A M_i, \quad B_i \ = \ z_i^T B, \quad C_i \ = \ C M_i. \tag{6.2}$$

6.2.4 Solution of the Controllability Grammian X and the Observability Grammian S.

In order to obtain the effective order of the Low order system, we need firstly to find the effective order for both of the high– and mid– subsystems. From equ.

(5.11, 5.12), we can get the controllability grammian X and the observability grammian S by using MATLAB to calculate

$$X_i = \text{gram} \ (A_i, \ B_i)$$

$$S_i = \text{gram} \ (A_i, \ C_i)$$

where i = high, mid.

The X_i and S_i should satisfy the Lyapunov equations (5.13) and (5.14).

6.2.5 Calculation of the Hankel Singular Values.

By the definition of singular values, a rectangular matrix A can be expressed in the following form:

$$A = UYV^T$$

where U is the matrix of left singular vectors, V is the matrix of right singular vectors, and the Y is the matrix of the singular values of A. We can use the MATLAB function SVD to find the singular values of X and S. As mentioned in APPENDIX 2, the singular values of X and S can be expressed by $\Sigma_c{}^2$ and $\Sigma_o{}^2$, their corresponding left sigular vectors are V_c & V_o and the right singular vectors are $V_c{}^T$ and $V_o{}^T$ separately. Based on the definition of the Hankel matrix we could easily get Hankel matrices H_{high} and H_{mid} of the high– and mid– subsystems. Similarly we could also get the Hankel singular values S_{vh} and S_{vm} directly.

However, it is possible to obtain the Hankel singular values in another way whih is also described here to provide a check on the reliability of the method outlined above. Since we know the relation between the product XS and $H^T H$ already the Hankel singular values S_{vi} could be found from the product of X and S directly as given in APPENDIX 2. Using MATLAB we can find the eigenvalues d_{high} and d_{mid} of the product $(XS)_{high}$ and $(XS)_{mid}$, and using the SQRT function of MATLAB we can obtain the Hankel singular values $S_{vh} = $ SQRT (d_{high}) and $S_{vm} = $ SQRT (d_{mid}):

$$S_{vi} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

where i = high— and mid— . The application of this approach gives the same numerical results as those obtained directly from the Hankel matrices as outlined above.

6.2.6 <u>Determination of the Reduced Order.</u>

Having found the singular values both for the high— and mid— subsystems, according to the theory of Bacon and Schmidt's method we could start to decide the effective order of the high— and mid subsystems. We compare the Hankel singular values σ_i of the high— subsystem first. If there is a large separation between h_i and h_{i+1}, a ratio $h_i/h_{i+1} >> 1$, indicates that the effective order of the high— subsystem is i. We could get the effective order j of the mid— subsystem using same way. Therefore, the reduced order of the system is i+ j.

6.2.7 <u>Determination of the Complete Reduced Order Approximation.</u>

The complete reduced order approximation is the sum of the high— and mid subsystem approximations. The matrices A_r, B_r and C_r of the transfer function of the reduced— order system are then

$$A_r = \begin{bmatrix} A_{rh} & 0 \\ 0 & A_{rm} \end{bmatrix}$$

$$B_r = \begin{bmatrix} B_{rh} \\ B_{rm} \end{bmatrix}$$

$$C_r = \begin{bmatrix} C_{rh} & C_{rm} \end{bmatrix}$$

6.3 <u>Implementation of Bacon and Shmidt's Method in MATLAB Program — an Example of a Fixed Wing Aircraft.</u>

6.3.1 <u>Determination of the Matrices A, B and C of State Space Form.</u>

To illustrate the mathod using an example, we still consider the longitudinal

responses: pitch rate, and normal acceleration to elevator stick force, for a high-order system of an advanced fighter aircraft, presented in table 4.1.

This example is taken from Bacon and Schmidt's paper "Fundamental Approach to Equivalent Systems Analysis"[30]. In Table (4.1) q/δ is the transfer function of a high-order system of an advanced fighter aircraft. i.e. the q is the output variable and the δ is the input variable.

The matrices A, B and C of the state-space form corresponding to the transfer function of Table 4.1 are shown in Appendix 3.

6.3.2 Model Decomposition.

Firstly, we use the 'eig' function of MATLAB to get the eigenvalues Λ and eigenvectors M. The eigenvalues Λ and eigenvectors M for the example are shown in Appendix 3.

According to the frequency range of interest ($\omega_1 = 0.1$, $\omega_2 = 10.0$ rad/s) in this example, the prefilter mode (3.36 rad/s) lies within the range being considered. The radii d1 and d2 define concentric circles in the complex plane. The columns of M are now separated into three groups M_{lo}, M_{mid}, M_{high} using a MATLAB function

$$M_{high} = M \ (:, \ 1:2) \tag{6.3}$$

$$M_{mid} \ = M \ (:, \ 3:6) \tag{6.4}$$

$$M_{lo} \ \ = M \ (:, \ 7:8) \tag{6.5}$$

where in equ. (6.3), (6.4), (6.5) the sign (:) means that all rows of the matrix M are the same as the rows of matrix M_{high}. The notation (1:2) implies that all columns from the 1st to the 2nd of the M matrix are columns of the M_{high} matrix. We are also able to get very easily

$$M^{-T} \ = [\ Z_{high} \ Z_{mid} \ \ Z_{lo} \ ]$$

where the Z_{high}, Z_{mid} and Z_{lo} are shown in Appendix 3.

According to formula for $G_i(s)$ of the form of equation (6.1), we can easily get the matrices A_i, B_i, C_i of the three subsystems directly from equ.(5.8.1) using MATLAB. We have therefore obtained the three description for the high- mid-

and lo- subsystems.

The high- and mid- subsystems for this example are shown in Appendix 3.

6.3.3 To Find the Effective Order of both high- and mid- Subsystems.

The controllability grammiam X and the observability grammian S for the high- and mid- subsystems can be obtained by using MATLAB.

$$X_i = \text{gram} (A_i, B_i)$$

$$S_i = \text{gram} (A_i, C_i) \qquad \text{where} \quad i = \text{high, mid.}$$

The X_i and S_i should satisfy the Lyapunov equations (3.13) and (3.14). They are shown in Appendix 3.

Based on the definition of the Hankel matrix we could easily get Hankel matrices H_{high} and H_{mid} of the high- and mid- subsystems. The Hankel matrices H_{high} and H_{mid} of the high and mid subsystems are shown in Appendix 3. We could also get the Hankel singular values S_{vh} and S_{vm} directly. As outlined in section 6.2.5, since there is a relation between the product XS and $H^T H$, we could also get the Hankel singular values S_{vi} from the product of X and S directly as given in APPENDIX 3.

In the example being considered the Hankel singular values S_{vh} or σ_1 and σ_2 of the high- subsystem are

$$S_{vh} = 0.0062 \qquad \text{or} \qquad \sigma_1 = 0.6005$$

$$0.0007 \qquad \qquad \sigma_2 = 0.0007$$

and the Hankel singular values S_{vm} or $\sigma_3 - \sigma_6$ of the mid-subsystem are

$$S_{vm} = 0.6005 \qquad \text{or} \qquad \sigma_3 = 0.6005$$

$$0.3617 \qquad \qquad \sigma_4 = 0.3617$$

$$0.0588 \qquad \qquad \sigma_5 = 0.0588$$

$$0.0060 \qquad \qquad \sigma_6 = 0.0060$$

6.3.4 Determination of the Complete Reduced- Order Approximation.

From the Internaly Balanced Technique, the low order approximation of the high- subsystem is defined by

$A_{rh} = -23.4804 + 0.0000i$

$B_{rh} = 0.2535 - 3.1606i$

$C_{rh} = -0.0071 - 0.0890i$

The low order approximation of the of mid- subsystem is defined by

$A_{rm} =$

$$\begin{array}{cc} -2.0662 + 0.0000i & -1.3872 - 1.2170i \\ 1.4735 - 1.2926i & -1.1548 - 0.0000i \end{array}$$

$B_{rm} =$

$$\begin{array}{c} 0.0125 - 0.0052i \\ -0.0022 + 0.0045i \end{array}$$

$C_{rm} =$

$$\begin{array}{cc} 16.4064 + 6.7887i & -3.8249 - 7.7523i \end{array}$$

The complete reduced— order approximation is the sum of high— and mid— subsystem approximations.

The matrices A_r, B_r, C_r of the transfer function of the reduced— order system are

$$A_r = \begin{bmatrix} A_{rh} & 0 \\ 0 & A_{rm} \end{bmatrix}$$

$=$

$$\begin{array}{ccc} -23.4804 + 0.0000i & 0 & 0 \\ 0 & -3.2210 - 0.0000i & -6.0030 - 0.0000i \\ 0 & 1.0000 & 0 \end{array}$$

$$B_r = \begin{bmatrix} B_{rh} \\ B_{rm} \end{bmatrix}$$

$=$

$$\begin{array}{c} 1 \\ 1 \\ 0 \end{array}$$

$C_r = [\ C_{rh} \quad C_{rm}\ ]$

=

-0.2832 + 0.0000i   0.2838 - 0.0000i   0.3001 + 0.0000i

Thus we have got a complete reduced–order system, in which the order of the original system is reduced to r = 3 from r = 8.

6.3.5 <u>The Frequency Responses for both the Reduced Order Model and the High Order System</u>.

The frequency response of the reduced order model and the high order system in the frequency range between 0.1 – 100 rad/sec. is shown in Fig. 6.1. From Fig. 6.1 we can see that the matching between the high order system and the reduced order model is excellent except in the low frequency range between 0.1 – 0.3 rad/sec..
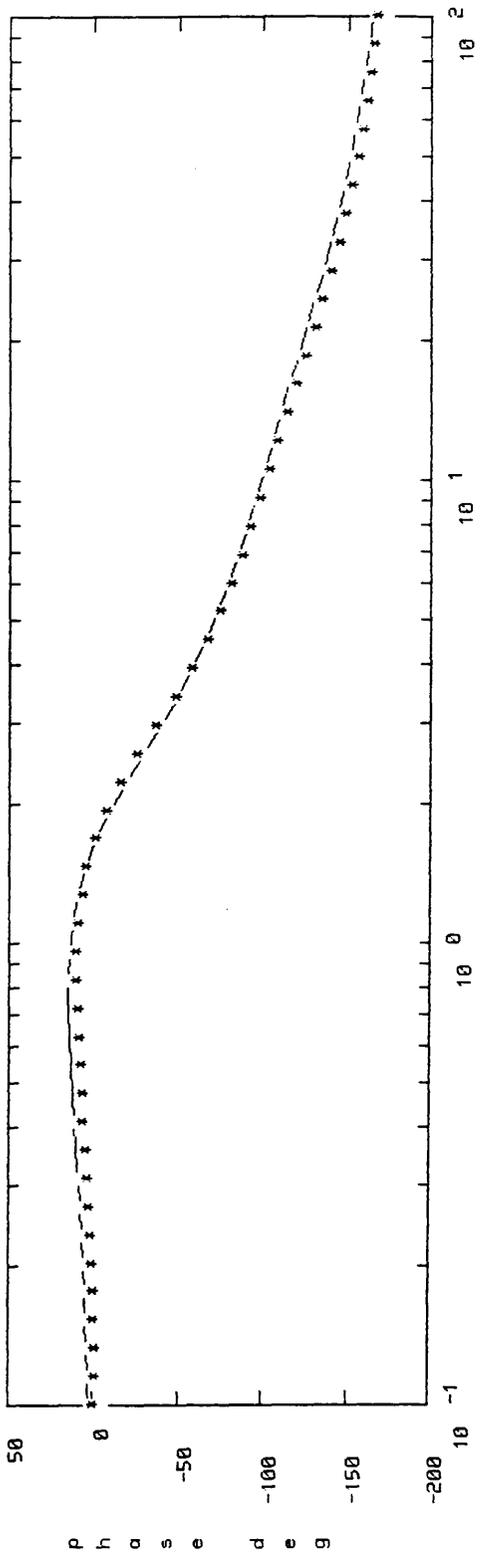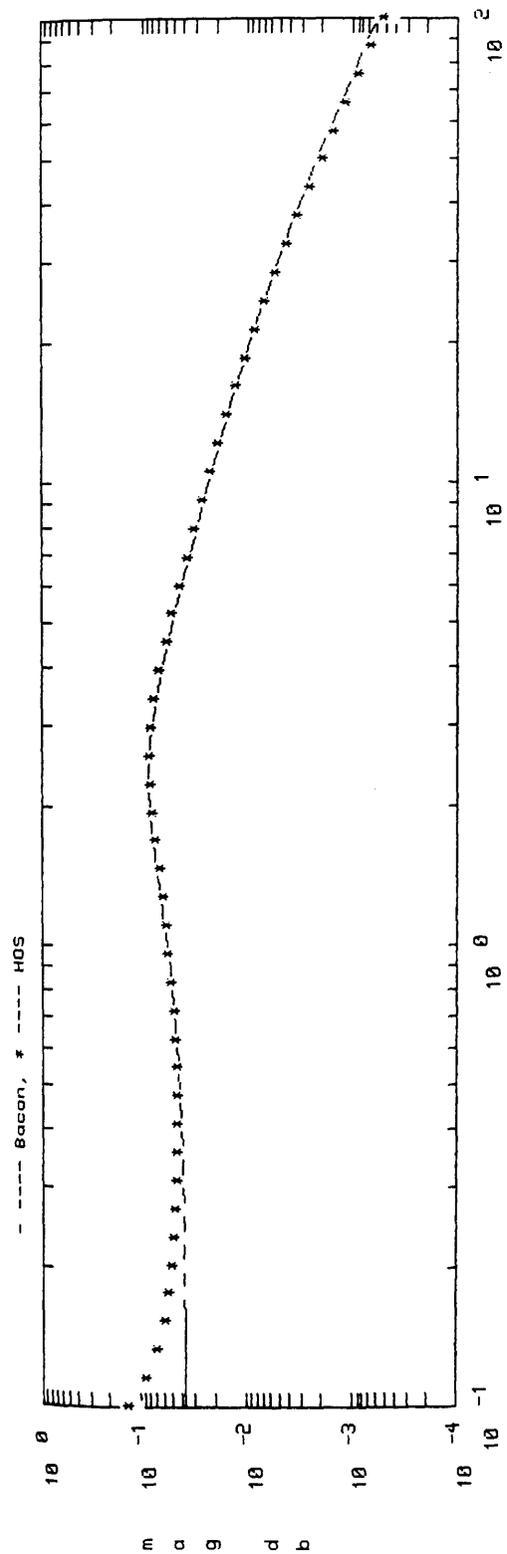
Figure 6.1 The Frequency Response of an Advanced Flighter

Chapter 7

Applications to Helicopter Models

Bacon and Schmidt's approach and the extended Levy's technique have been applied to the Level 1 helicopter model as described in Chapter 2. The particular case considered is a Puma helicopter for which a substantial quantity of flight test data exists. Puma flight test data provided by the Royal Aerospace Establishment has been used at Glasgow University for system identification research.

7.1 Model Reduction for the Puma Helicopter.

For the purposes of flight control system design and handling qualities studies the use of a high order helicopter model is both tedious and costly. It is often desirable to replace a high—order system description by a model of lower order which incorporates the essential characterstics of the vehicle. This is particularly important for real time simulation studies and for the initial stages of design for a flight control system where a number of design methods may be applied. The more complex forms of model can be cumbersome at the preliminaly design stage.

In this chapter we apply Bacon and Schmidt's method and the extended Levy's method respectively to model reduction for a linearised helicopter model in which some eigenvalues may be in the right half plane. Three 14th order state space models, Puma60, Puma80 and Puma100, are considered in the present work. These represent the linearised dynamics of the Puma helicopter for trimmed flight conditions involving 60, 80 and 100 knots level forward flight. In the original model, the matrices A and B in the state—space equations are a 14×14 square matrix and a 14×4 matrix respectively. The C matrix is an identity matrix with dimension 14×14. The A_1, B_1 and C_1 matrices of the Puma60 helicopter control system; the A_2, B_2 and C_2 matrices of the Puma80 helicopter control system and the A_3, B_3 and C_3 matrices of the Puma100 helicopter control system are shown in APPENDIX 3.

The transfer functions are

$$G_{ij}(s) = x_i / u_j \quad (i = 1, \cdots, 14; \; j = 1, \cdots, 4)$$

The computer work was carried on a DEC MicroVAX 3600 at the Glasgow University Computer Centre. The two model reduuction programs are written in MATLAB and have the general form outlined in Chapter 6 . The program listings are shown in APPENDIX 4.

7.2 Inputs and Outputs.

In the present work only the longitudinal responses have been considered and each case is presented in terms of a single–output. For single main rotor helicopters, the control input vector u is made up of the blade pitch angles for the main and tail rotor as follows :

$$
u = \begin{bmatrix}
\theta_{oe} & \text{--- main rotor collective blade pitch angle} \\
\theta_{1c} & \text{--- longitudinal cyclic blade pitch angle} \\
\theta_{1c} & \text{--- lateral cyclic blade pitch angle} \\
\theta_{ot} & \text{--- tail rotor colletive blade pitch angle}
\end{bmatrix}
$$

The main rotor collective and longitudinal cyclic inputs directly affect the magnitude and direction of the main rotor thrust vector, the lateral cyclic input affects the thrust vector to produce lateral motion while the tail rotor thrust is controlled in magnitude by the tail rotor collective input. Lateral cyclic and tail rotor collective inputs thus influence the lateral/directional dynamics of the vehicle. Cross– coupling, which is very strong for certain flight conditions, introduces significant complications and means that the simple separation of dynamics for the longitudinal and lateral axes (as used for most fixed– wing aircraft) is not possible in the case of the helicopter. The complete transfer function matrix is normally needed for flight control system design to allow for these coupling effects.

7.3 Helicopter Model Reduction Applications Using Levy's Method.

As the metioned in section 6.1 the MATLAB software package has been used to obtain 8th order reduced models for Puma60, Puma80 and Puma100 representations.

7.3.1 Frequency Responses.

Some results are shown in Fig.7.1 to Fig.7.17. In some cases excellent agreement has been obtained between the frequency responses of the 14th order system and the

8th order reduced model for most of the frequency range considered. However the choice of weighting factor n1 was found to be of critical importance. From Fig.7.1 to Fig.7.3 we can see that for the case of Puma 80 the frequency response of the reduced model agrees with the high order system very well over the most of range of frequency (0.1 — 10 rad/sec.) if the weighting factor n1 is taken 0.6 for all output x_i, (i = 1 — 8), and input u_3. But from Fig.7.10 — Fig.7.13 we can find that if the weighting factor n1 is taken 1.2 the frequency response of the reduced model agrees with the high order system for all outputs x_i and all inputs u_1 — u_4 over the whole frequency range of interest for Puma100. Also from Fig.7.3 to Fig.7.5 we find that if different values of weighting factor are taken such as 0.2, 0.6 and 1.5 for the same input and output for Puma80 the results can show significant differences. For Puma60 cases for n1 = 0.4, 0.6 and 1.2 again produced results that are very sensitive.(see Fig.7.7 to Fig.7.9) The errors of the frequency response of all the tests tried are shown in Table 7.1.

Fig.7.14 to Fig.7.17 show the effects of the number of points taken in the calculation. From these figures it can be seen that the number of points taken in calculation does not affect the accuracy of fit and only affects the smoothness and resolution of the solution.

7.3.2 Time Responses.

The time responses for different reduced order models have been obtained. From output curves Fig.7.18 to Fig.7.21 we can see from the step response curves that the matching between the high order system and reduced order system is very good in some cases within the period of 0 — 10 seconds for both Puma60 and Puma80. Also Fig.7.22 and Fig.7.23 show the matching within the period of 0 — 10 seconds and 0 — 1 second for both Puma60 and Puma80 helicopter control systems. Cases where the time domain fit is poor (such as Fig.7.18 and 7.21) correspond to frequency domain fits which were also poor (such as Fig.7.5 and Fig.7.8). Cases where the time domain fit is especially good (e.g. Fig.7.19 and Fig.7.20) relate to

reduced order models for which particularly good frequency domain fits were found (e.g. Fig.7.6 and Fig.7.7). Units and variables in the time domain responses are as specified in APPENDIX 3.

7.4 Application of Bacon and Schmidt's Method to Helicopter Model Reduction.

7.4.1 Procedure for Order Reduction.

As the mentioned in Chapter 6 we use MATLAB software to get the eigenvalues d_i and eigenvectors M_i of the original systems matrices A_i (i = 1, 2; 1 for Puma60 and 2 for Puma80). They are shown in APPENDIX 4.

The system is then divided into three groups of subsystems: high, mid and low (see Chapter 5 and Appendix 4) according to the natural frequency ω_{ni}, which is defined by the positions of the eigenvalues λ_i in the complex plane. The radii d1 and d2 define concentric circles in the complex plane. So, the matrices M_{midi}, M_{highi} and $(M_{midi})^{-T}$, $(M_{highi})^{-T}$ can be obtained from the following MATLAB expressions:

$$M_{mid_1} = M_1(:,1:6)$$

$$M_{high_1} = M_1(:,7:14)$$

$$M_{mid_2} = M_2(:,1:6)$$

$$M_{high_2} = M_2(:,7:14)$$

$$(M_{mid_1})^{-T} = inv((M_{mid_1})')$$

$$(M_{high_1})^{-T} = inv((M_{high_1})')$$

$$(M_{mid_2})^{-T} = inv((M_{mid_2})')$$

$$(M_{high_2})^{-T} = inv((M_{high_2})')$$

This process gives the matrices A_i, B_i, C_i of the subsystems $G_i(s)$. After that, the Internally Balanced Technique is applied to reduce the order of the G_{high} and G_{mid} systems. Firstly we solve for the controllability grammian X and observability grammian S, which are the unique solution of the Lyapunov equations(5.13, 5.14). The rank of the product XS equals the number of modes that are both controllable and observable, i.e., the minimal order. It has been shown that this product is

always diagonalizable and its eigenvalues are real and non-negative. Consequently, the rank of the product XS equals n minus the number of zero Hankel singular values. The product XS should be written as

$$XS = T \Lambda T^{-1}$$

where $\Lambda = \begin{bmatrix} \Sigma_r^2 & 0 \\ 0 & \Sigma_{n-r}^2 \end{bmatrix}$

$$T = [T_r, \quad T_{n-r}]$$

$$T^{-T} = [U_r, \quad U_{n-r}]$$

$$\Sigma_r = diag(h_i) \qquad i = 1, r;$$

$$\Sigma_{n-r} = diag(h_i) \qquad i = r+1, n;$$

and $h_1 \geqslant h_2 \geqslant \cdots \geqslant h_r \geqslant h_{r+1} \geqslant \cdots \geqslant h_n \geqslant 0$.

The ratio h_r/h_{r+1} where h_i are ordered such that $h_i \geqslant h_{i+1}$ could be used to infer the effective order. The Hankel singular values of high and mid subsystems for the Puma60 and Puma80 models are shown in Table 7.2.

From the Table 7.2 we can see, for example the ratio $h_3/h_4 = 1.638$ for Puma60 and for Puma80 the $h_1/h_2 = 1.112$ and $h_{13}/h_{14} = 8.000$. The effective orders of the G_{mid} and G_{high} for Puma60 and Puma80 are shown in Table 7.3.

Table 7.3 Effective Orders for high and mid Subsystems

		Puma60	Puma80
G_{high}	HOS	r = 6	r = 6
	LOS	r = 3	r = 1
G_{mid}	HOS	r = 8	r = 8
	LOS	r = 8	r = 7
HOS		r = 14	r = 14
LOS		r = 11	r = 8

7.4.2 Results of the Reduction Model.

For Puma60 the effective order r is taken as 11, and for Puma80 the effective

order r is 8.   The results of the computation are divided into two parts: the results in terms of the frequency response and the results in term of the time response.

7.4.3 Time Response.

For the results in terms of the time response, we can see from the step response curves that the matching between the high order system and the reduced order system is very good for each case within the period of 0 − 100 and 0 − 1 seconds for both Puma60 and Puma 80. (see Fig.7.24 and Fig.7.25)   Fig.7.26 and Fig.7.27 show the matching within the period of 0 − 10 seconds and 0 − 1 second for both Puma60 and Puma80 HELISTAB helicopter control systems.

7.4.4 Frequency Response.

For the results in term of the frequency response, the agreement between the two systems will depend very much on the order taken in the reduced− order system. For Puma60, we found that when the reduced order r is 11, a very good agreement can be obtained in the whole frequency range in some examples.(see Fig.7.28)   For Puma80, when the reduced order r is 8, it can be seen that the matching between the high order system and the reduced order system is excellent in the whole frequency range.(see Fig.7.29)   It may be significant that the 60 knots case for which a poorer fit was obtained involves right half plane eigenvalues both in the original system and in the reduced order model.   Thus, we choose a effective order of the system, r = 11, for Puma60, and r = 8 for Puma80.

7.4.5 Eigenvalues of the High Order System and the Reduced Order System.

The idea of retaining dominant eigenvalues to simplify reduction of a high− order system to a low− order plant was discussed in detail for the first time by Davison[26] in 1966.   This concept has since been further developed and is now widely used in control system design.   For a state− space form of model expressed as Equ. (3.1), the matrix A is of high order.   The reduced model can be expressed as

$$\underline{\dot{x}} = A_r\underline{x} + B_r\underline{u}$$

where the matrix A_r is of lower order but has the same dominant eigenvalues and

eigenvectors.   Comparisions of the eigenvalues of the matrices A and A_r for Puma60 and Puma80 are shown in Table 7.4 and Table 7.5 respectively.   From Table 7.4 and Table 7.5, we can see that the most of eigenvalues of matrix A_r for the reduced order system (11th order) are equal to the eigenvalues of matrix A for the high order system (14th order) for both Puma60 and Puma80.   It means that we obtained a low—order model with the dominant eigenvalue concept which gave a satisfactory dynamic response as well as correct steady—state response.

Table 7.2 Hankel Singular Values for x_1/u_1

			Puma60		Puma80	
		i	h_i	h_i/h_{i+1}	h_i	h_i/h_{i+1}
		1	13.1203	1.086	15.3923	1.112
		2	12.0796	2.402	13.842 2	2.166
	high	3	5.0292	1.638	6.3908	1.697
		4	3.0699	2.158	3.7661	2.004
		5	1.4227	5.599	1.8797	5.771
		6	0.2541		0.3257	
		7	1.4962	1.049	1.6235	1.087
		8	1.4258	12.134	1.4935	13.565
	mid	9	0.1175	1.186	0.1101	1.504
$1.0e+^4\times$		10	0.0991	1.745	0.0732	5.674
		11	0.0568	6.843	0.0129	1.985
		12	0.0083	1.694	0.0065	1.354
		13	0.0049	12.250	0.0048	8.000
		14	0.0004		0.0006	

Table 7.1 The Error for Different Cases

Puma 80

nl	G	L = 1	L = 2	L = 3	L = 4	L = 5
0.6	x1/u3	4.2837×10^8	5.9258×10^7	1.7340×10^6	9.2011×10^5	8.9247×10^5
0.6	x2/u3	1.9675×10^7	6.0596×10^6	1.6886×10^6	1.3986×10^6	1.3502×10^6
0.6	x3/u3	1.4480×10^4	148.2084	243.5019	53.9635	44.2414
0.6	x4/u3	1.5327×10^4	2.6314×10^3	530.6563	301.5158	295.2345
0.6	x7/u3	0.0027	0.0013	0.0011	0.0011	0.0011
0.6	x8/u3	3.2230×10^3	232.1079	35.0603	13.0811	11.2997
1.5	x7/u3	0.0027	0.0011	0.0067	0.0112	0.0112
1.5	x8/u3	3.2230×10^3	18.1810	0.0172	0.0205	0.0209
0.2	x7/u3	0.0027	0.0026	0.0028	0.0027	0.0026
0.2	x8/u3	3.2230×10^3	876.9742	1.1387×10^3	806.5150	694.0652

Puma 60

nl	G	L = 1	L = 2	L = 3	L = 4	L = 5
1.0	x1/u1	1.1801×10^8	1.5012	0.2899	0.2899	0.2899
1.0	x2/u1	1.9741×10^7	1.3127	0.4344	0.4331	0.4331
1.2	x5/u3	1.2974×10^9	3.5235×10^4	68.6969	69.1881	69.1876
1.2	x6/u3	9.6736×10^5	710.2610	29.3468	29.7379	29.7382
1.0	x5/u3	1.2974×10^9	5.8388×10^5	2.3786×10^3	54.1328	54.1633
1.0	x6/u3	9.6736×10^5	1.5794×10^3	150.9394	24.4558	24.4679
0.6	x5/u3	1.2974×10^9	3.3465×10^6	1.3455×10^6	1.2058×10^6	1.2004×10^6
0.6	x6/u3	9.6736×10^5	3.1359×10^4	6.8084×10^3	5.9686×10^3	6.0149×10^3
0.4	x5/u3	1.2974×10^9	3.6298×10^7	1.8560×10^7	1.6329×10^7	1.5974×10^7
0.4	x6/u3	9.6736×10^5	2.3797×10^5	1.9818×10^5	1.8360×10^5	1.8152×10^5

H.O.S	R.O.S.(11TH ORDER)			
	X1/U1	X1/U2	X4/U1	X4/U2
0.0065 + 0.2563i	0.0065 + 0.2563i	0.0065 + 0.2563i	0.0065 + 0.2563i	0.0065 + 0.2563i
0.0065 - 0.2563i	0.0065 - 0.2563i	0.0065 - 0.2563i	0.0065 - 0.2563i	0.0065 - 0.2563i
- 0.1109	- 0.1109	- 0.1109	- 0.1109	- 0.1109
- 0.2072 + 1.0428i	- 0.2072 + 1.0428i	- 0.2072 + 1.0428i	- 0.2072 + 1.0428i	- 0.2072 + 1.0428i
- 0.2072 - 1.0428i	- 0.2072 - 1.0428i	- 0.2072 - 1.0428i	- 0.2072 - 1.0428i	- 0.2072 - 1.0428i
- 1.0522 + 0.7375i	- 1.0522 + 0.7375i	- 1.0522 + 0.7375i	- 1.0522 + 0.7375i	- 1.0522 + 0.7375i
- 1.0522 - 0.7375i	- 1.0522 - 0.7375i	- 1.0522 - 0.7375i	- 1.0522 - 0.7375i	- 1.0522 - 0.7375i
- 1.2288	- 1.2288	- 1.2288	- 1.2288	- 1.2288
-14.9984 + 5.6273i	- 8.8454	-14.4834	- 8.8454	-14.4834
-14.9984 - 5.6273i	-17.6097 + 35.7571i	-14.7613 + 48.6866i	-17.6097 + 35.7571i	-14.7613 + 48.6866i
-15.5691 + 49.1903i	-17.6097 - 35.7571i	-14.7613 - 48.6866i	-17.6097 - 35.7571i	-14.7613 - 48.6866i
-15.5691 - 49.1903i				
-15.8131 + 22.4403i				
-15.8131 - 22.4403i				

Table 7.5 THE EIGENVALUES OF MATRIX [A] FOR ROS (8TH ORDER) AND HOS (14TH ORDER) (PUMA80)

H.O.S	R.O.S.(8TH ORDER)			
	X1/U1	X1/U2	X4/U1	X4/U2
- 0.0092 + 0.2098i	- 0.0092 + 0.2098i	- 0.0092 + 0.2098i	- 0.0092 + 0.2098i	- 0.0092 + 0.2098i
- 0.0092 - 0.2098i	- 0.0092 - 0.2098i	- 0.0092 - 0.2098i	- 0.0092 - 0.2098i	- 0.0092 - 0.2098i
- 0.0001				
- 0.2900 + 0.9357i	- 0.2900 + 0.9357i	- 0.2900 + 0.9357i	- 0.2900 + 0.9357i	- 0.2900 + 0.9357i
- 0.2900 - 0.9357i	- 0.2900 - 0.9357i	- 0.2900 - 0.9357i	- 0.2900 - 0.9357i	- 0.2900 - 0.9357i
- 1.2656 + 0.0000i	- 1.2656 + 0.0000i	- 1.2648 + 0.0000i	- 1.2656 + 0.0000i	- 1.2648 + 0.0000i
- 1.0836 + 1.1670i	- 1.0827 + 1.1664i	- 1.0835 + 1.1669i	- 1.0827 + 1.1664i	- 1.0835 + 1.1669i
- 1.0836 - 1.1670i	- 1.0827 - 1.1664i	- 1.0835 - 1.1669i	- 1.0827 - 1.1664i	- 1.0835 - 1.1669i
- 1.2647	-17.3163 + 0.0000i	-11.0330 + 0.0000i	-17.3163 + 0.0000i	-11.0330 + 0.0000i
-15.1429 + 5.7814i				
-15.1429 - 5.7814i				
-15.4634 + 49.1181i				
-15.4634 - 49.1181i				
-15.7960 + 22.2181i				
-15.7960 - 22.2181i				

Fig. 7.1 Frequency Response for Puma 80
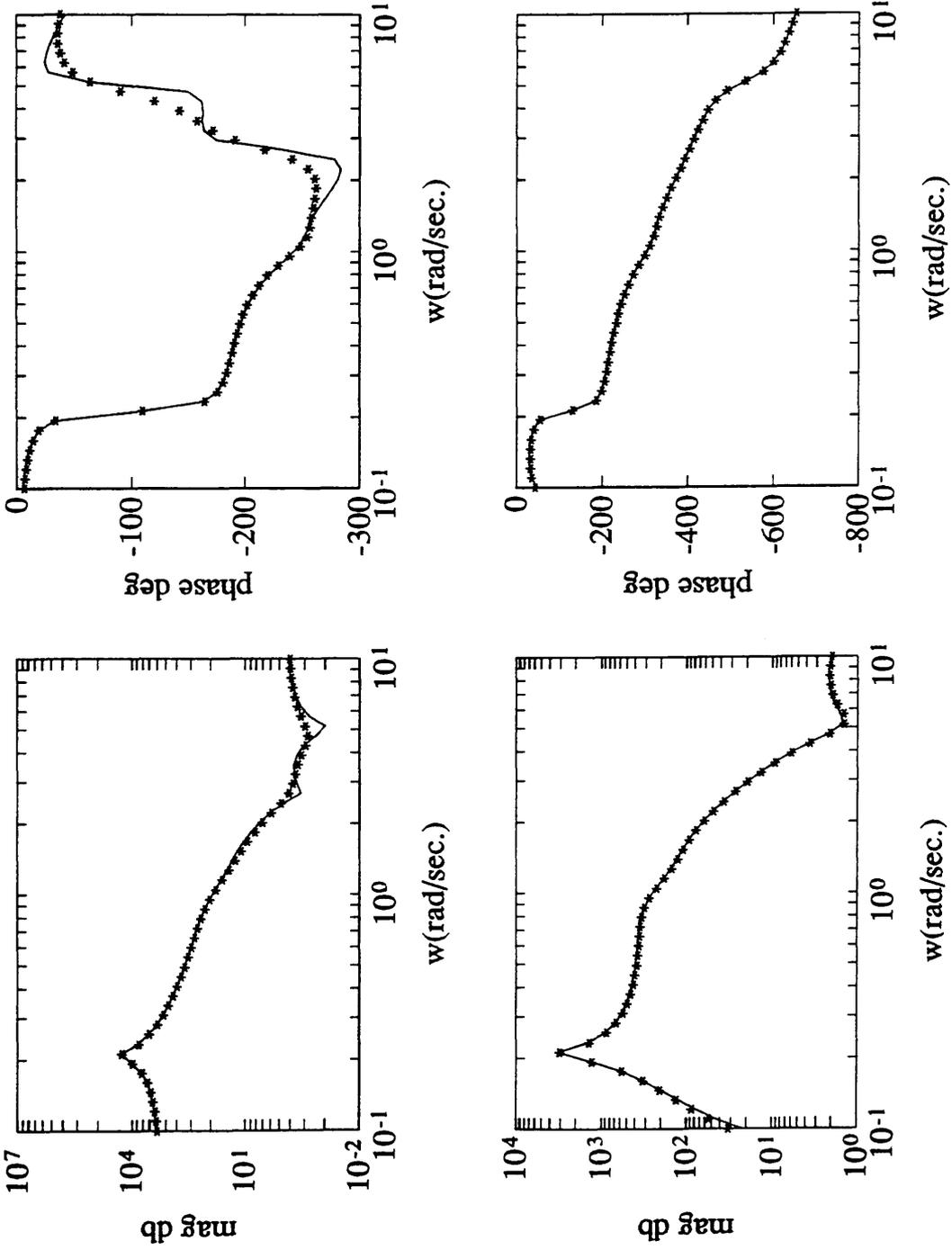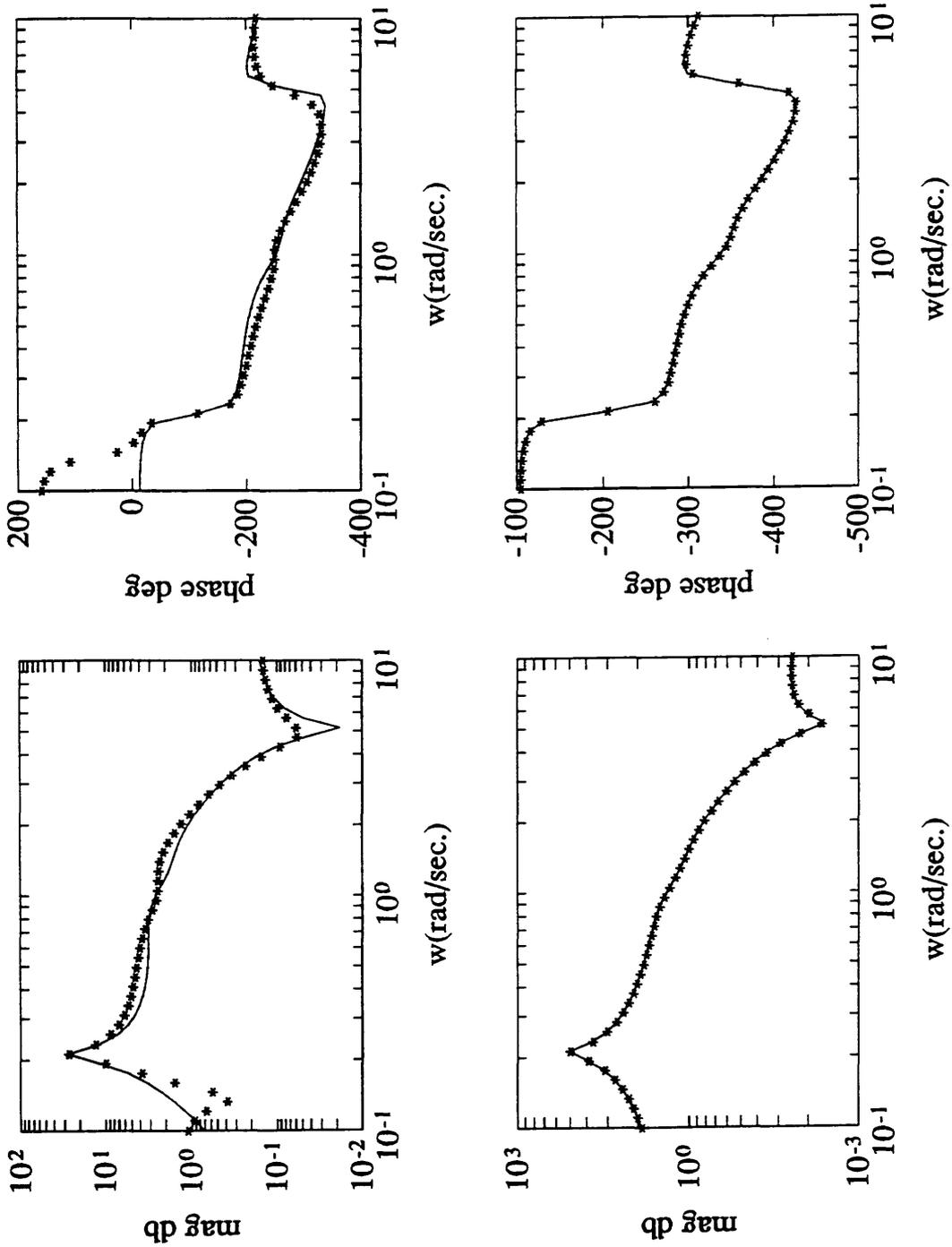(Case x_1/u_3 & x_2/u_3, $n_1 = 0.6$, $i = 5$)

Fig. 7.2 Frequency Response for Puma 80
(Case x_3/u_3 & x_4/u_3 , $n_1 = 0.6$, $i = 5$)

* reduced model, — HOS model

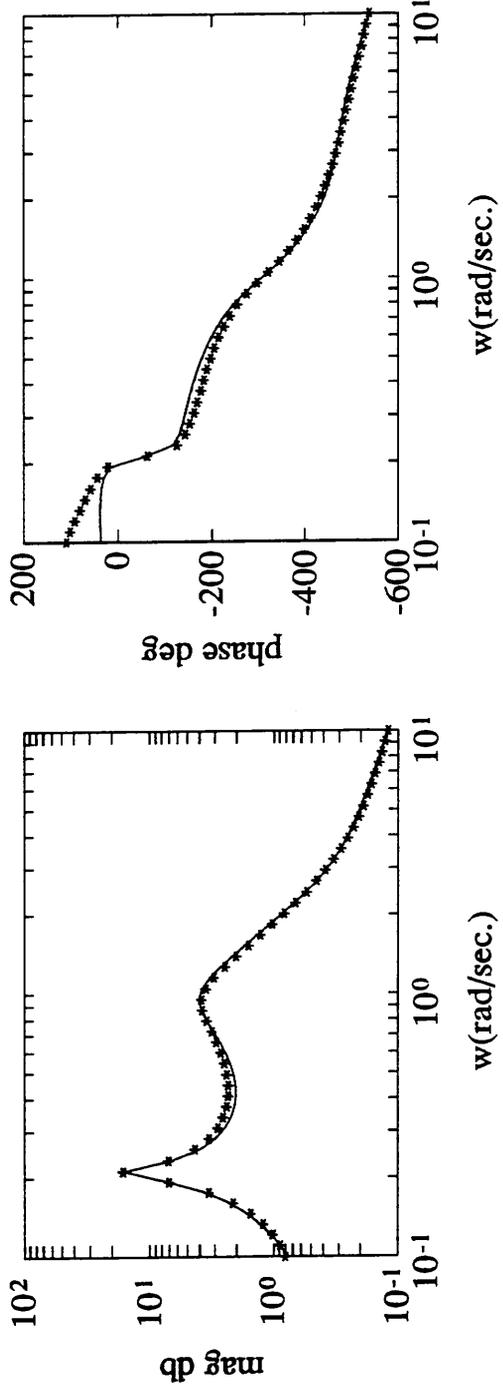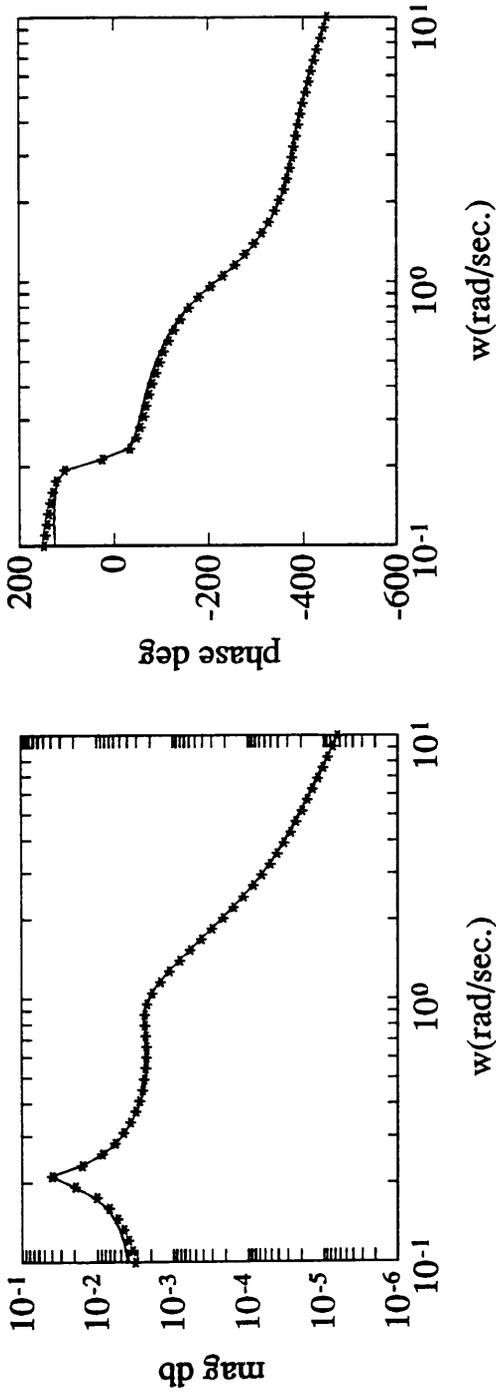* reduced model, − · HOS model

Fig. 7.3 Frequency Response for Puma 80
(Case x_7/u_3 & x_8/u_3 , $n_1 = 0.6$, $i = 5$)

Fig. 7.4 Frequency Response for Puma 80
(Case x_7/u_3 & x_8/u_3 , $n_1 = 1.5$, $i = 5$)

* reduced model,   — HOS model

* Levy s model, — HOS model

Fig. 7.5 Frequency Response for Puma80
(Case x7/u3 & x8/u3, n1 = 0.2, i = 5)

Fig. 7.6 Frequency Response for Puma80
(Case x1/u1 & x2/u1, n1 = 1.0, i = 5)

* Levy s model, - HOS model

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Fig. 7.7 Frequency Response for Puma 60
(Case x_5/u_3 & x_6/u_3 , $n_1 = 1.2$, $i = 5$)
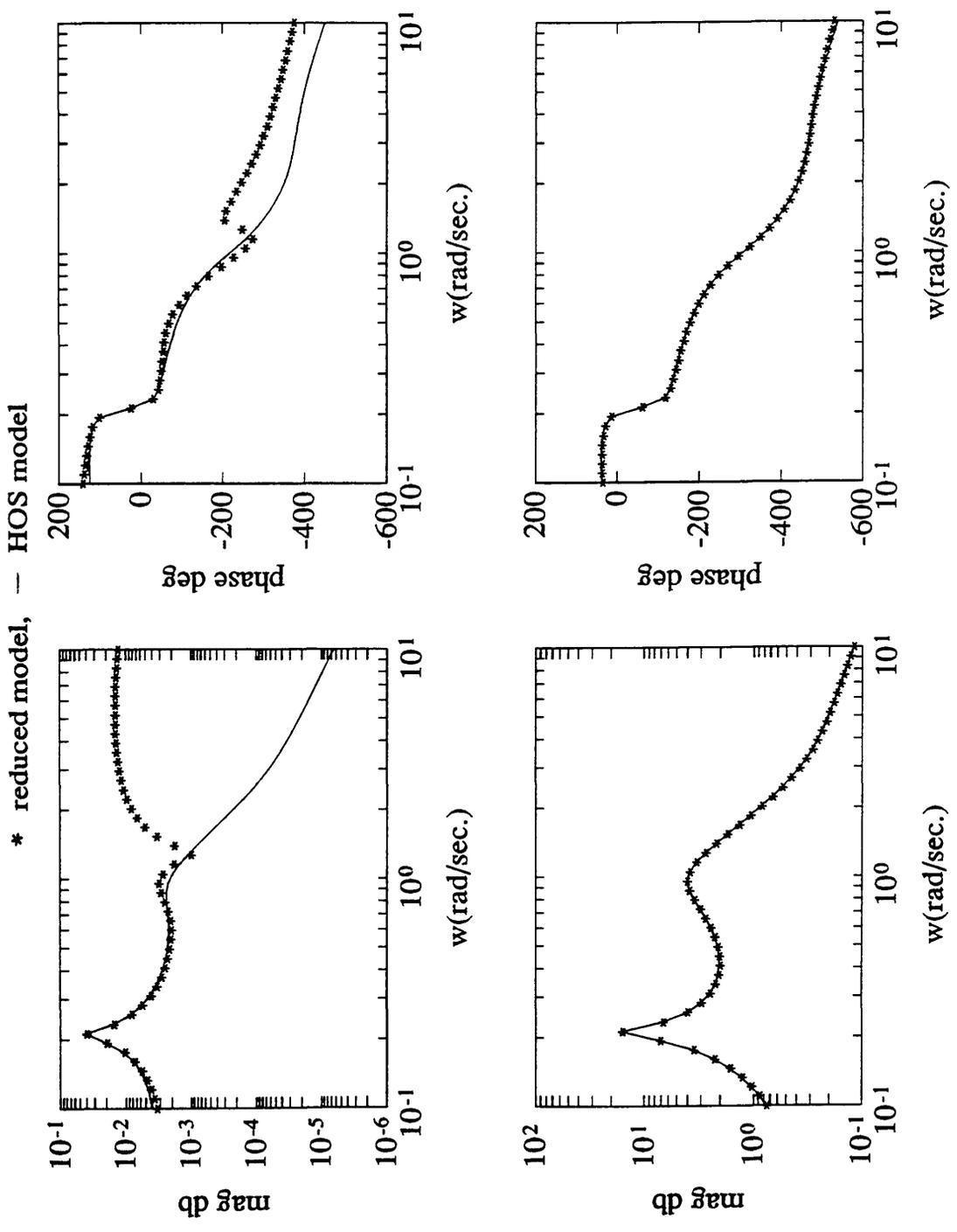
* reduced model, — HOS model

- 110 -

* Levy s model, - HOS model
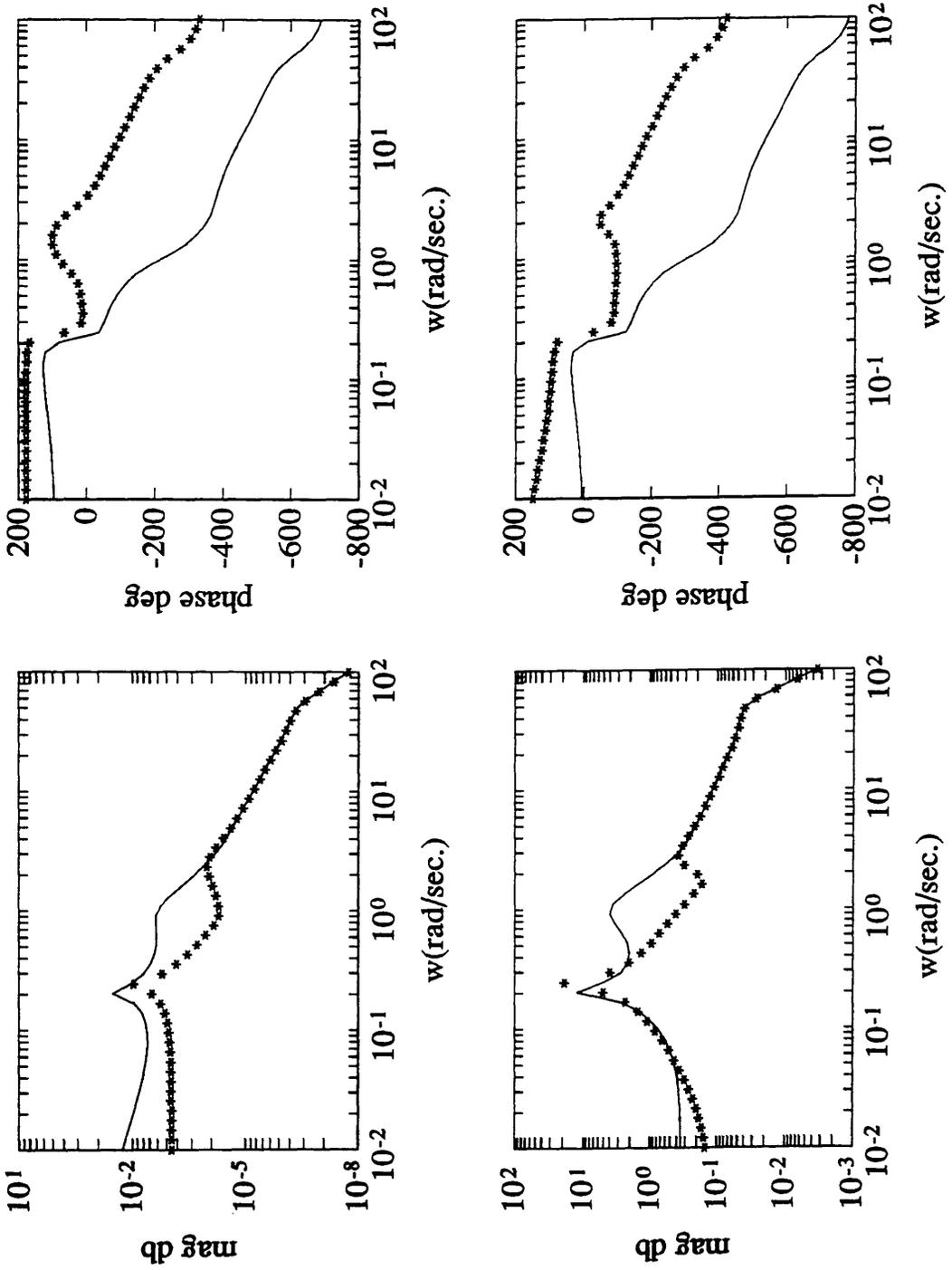
Fig. 7.8 Frequency Response for Puma60
(Case x5/u3 & x6/u3, n1 = 0.6, i = 5)

* Levy s model, - HOS model

Fig. 7.9 Frequency Response for Puma60
(Case x5/u3 & x6/u3, n1 = 0.4, i = 5)

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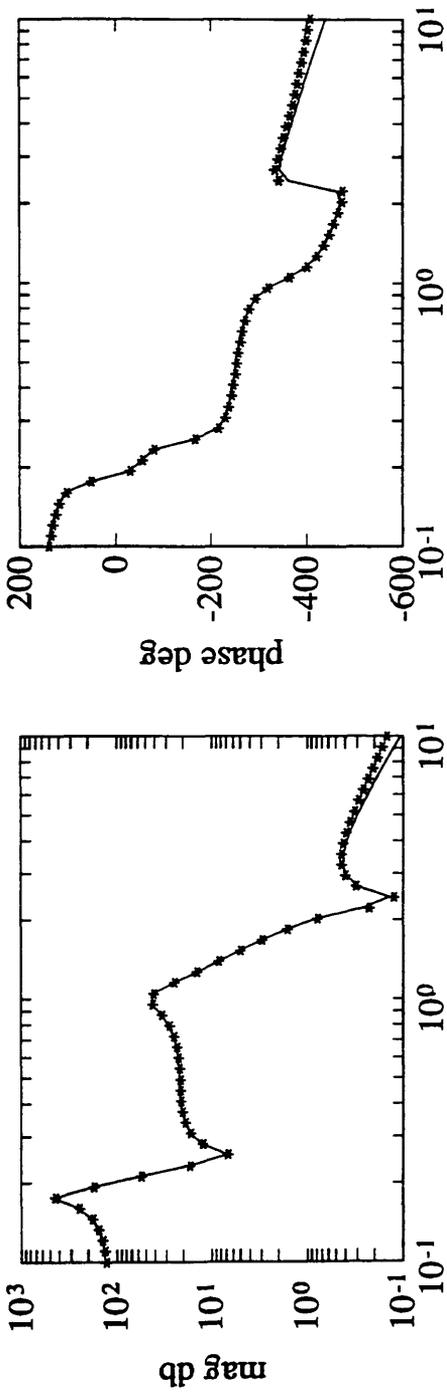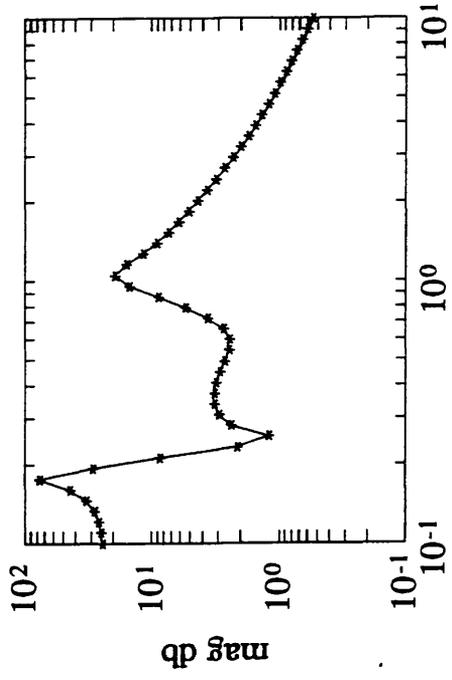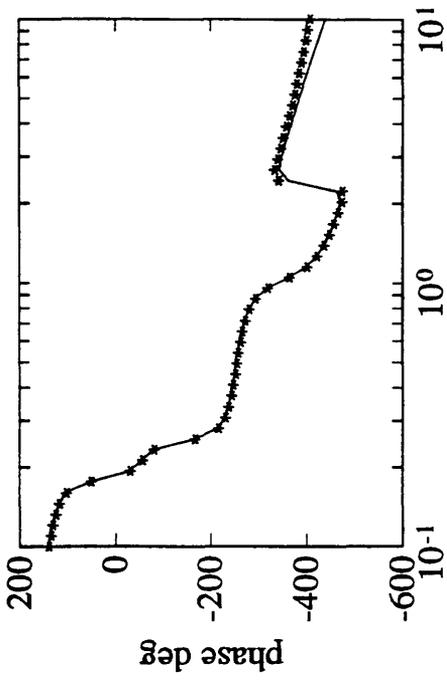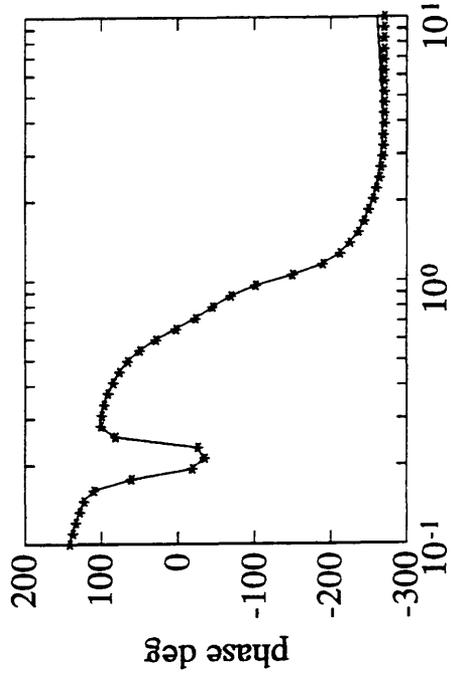
Fig. 7.10 Frequency Response for Puma 100
(Case x_7/u_1 & x_8/u_1 , $n_1 = 1.2$, $i = 5$)

* reduced model, − HOS model

* reduced model, — HOS model
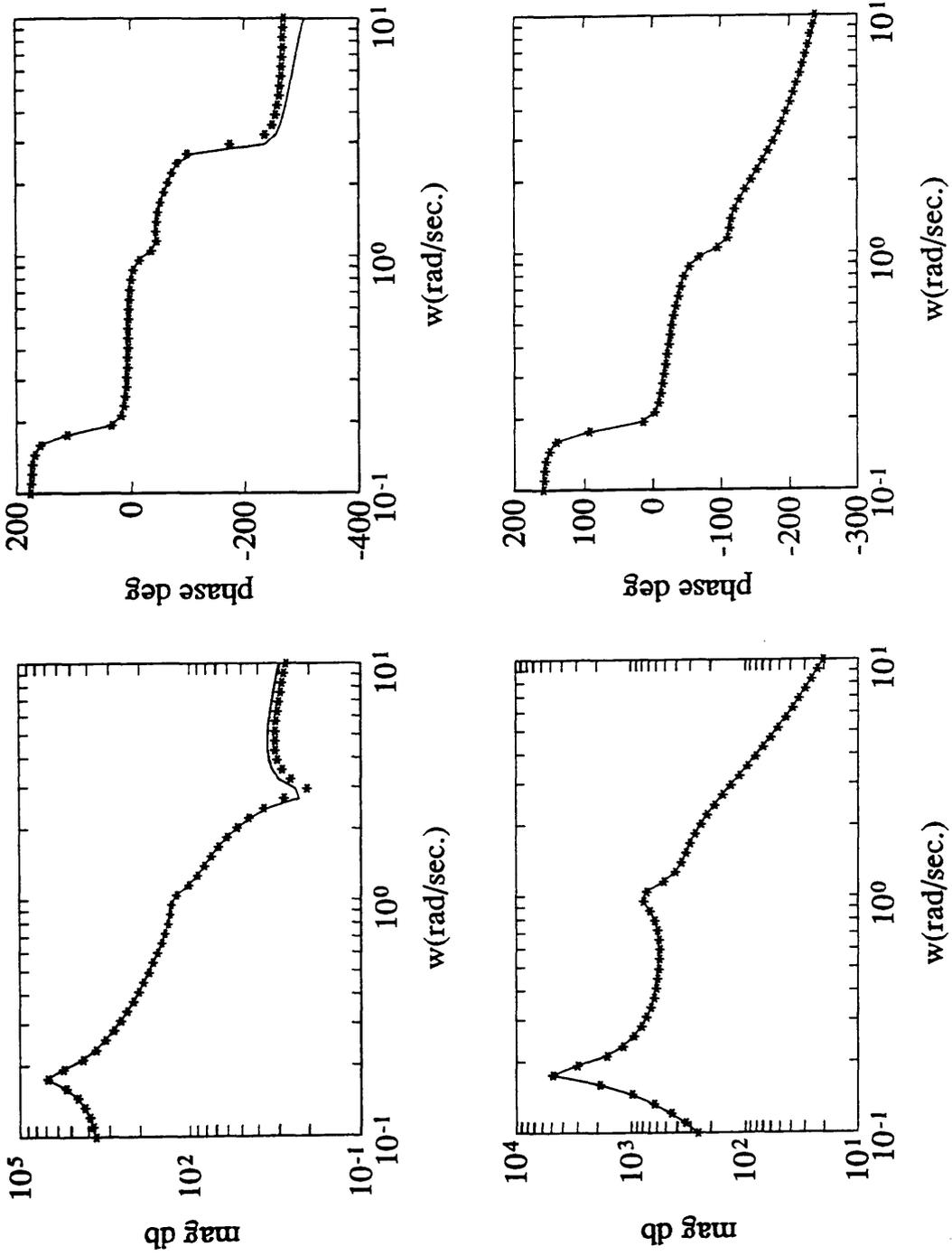
Fig. 7.11 Frequency Response for Puma 100
(Case x_1/u_2 & x_2/u_2 , $n_1 = 1.2$, i = 5)
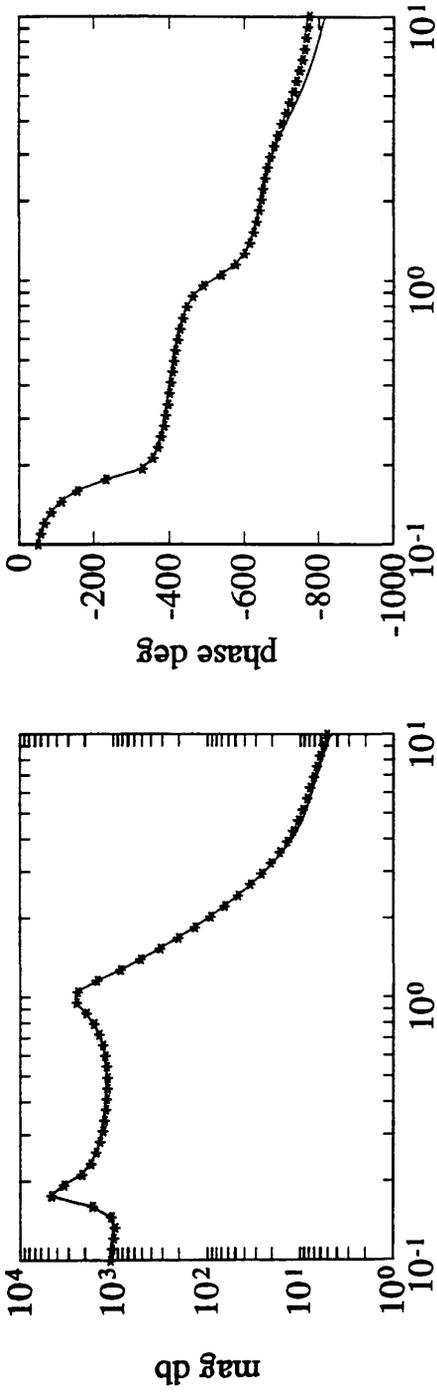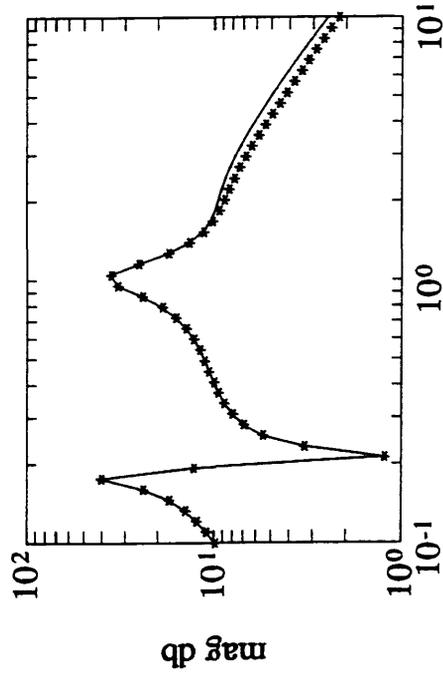
Fig. 7.12 Frequency Response for Puma 100
(Case x_5/u_3 & x_6/u_3 , $n_I = 1.2$, i = 5)

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* reduced model, – HOS model

Fig. 7.13 Frequency Response for Puma 100
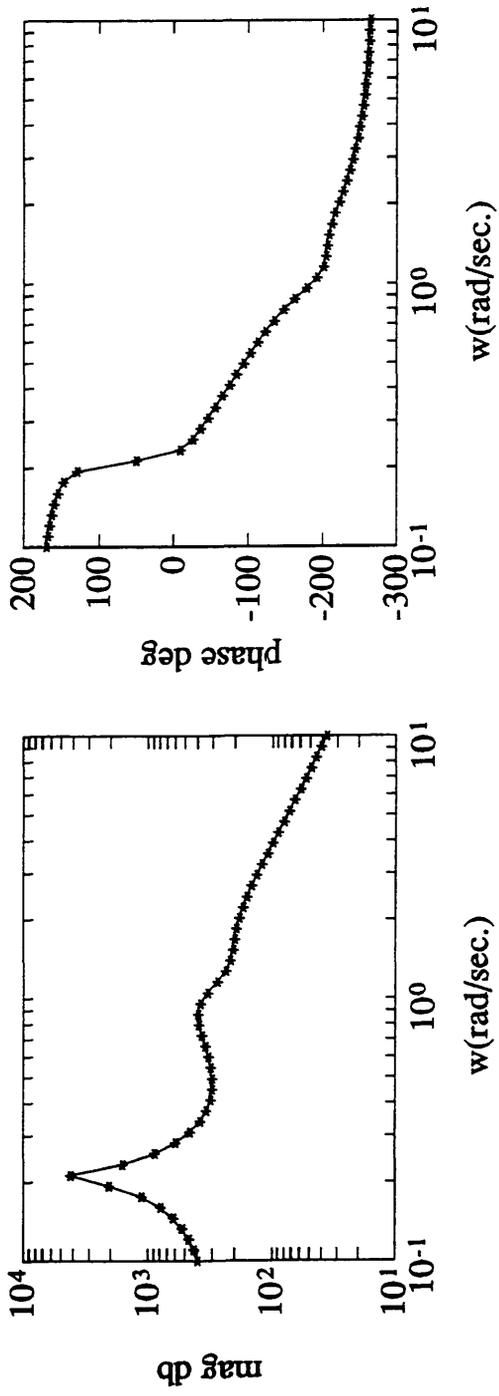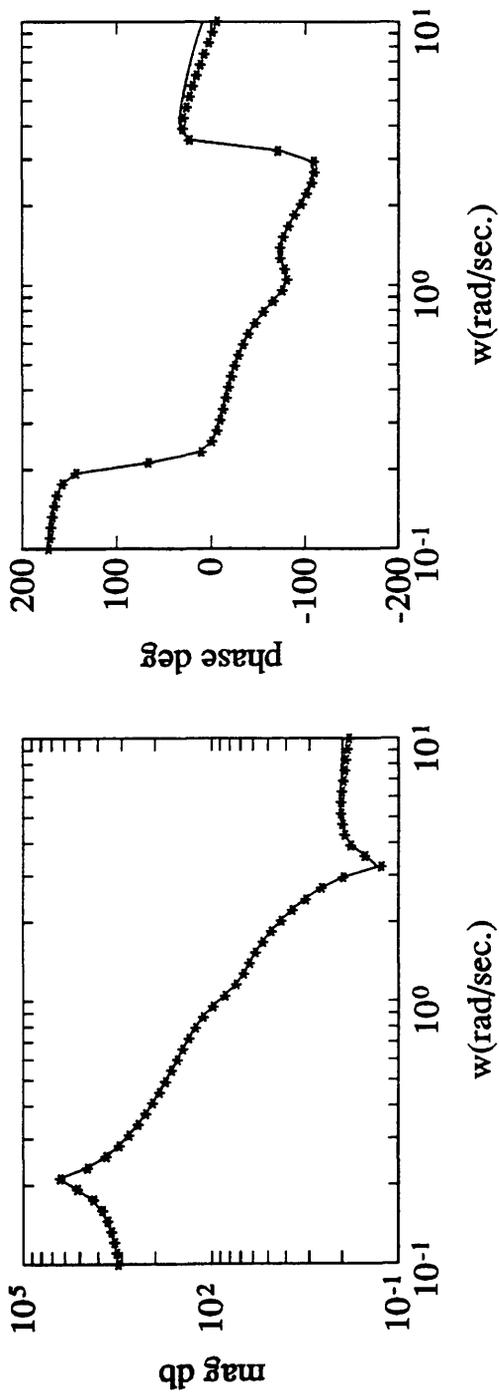(Case x_3/u_4 & x_4/u_4 , $n_1 = 1.2$, $i = 5$)
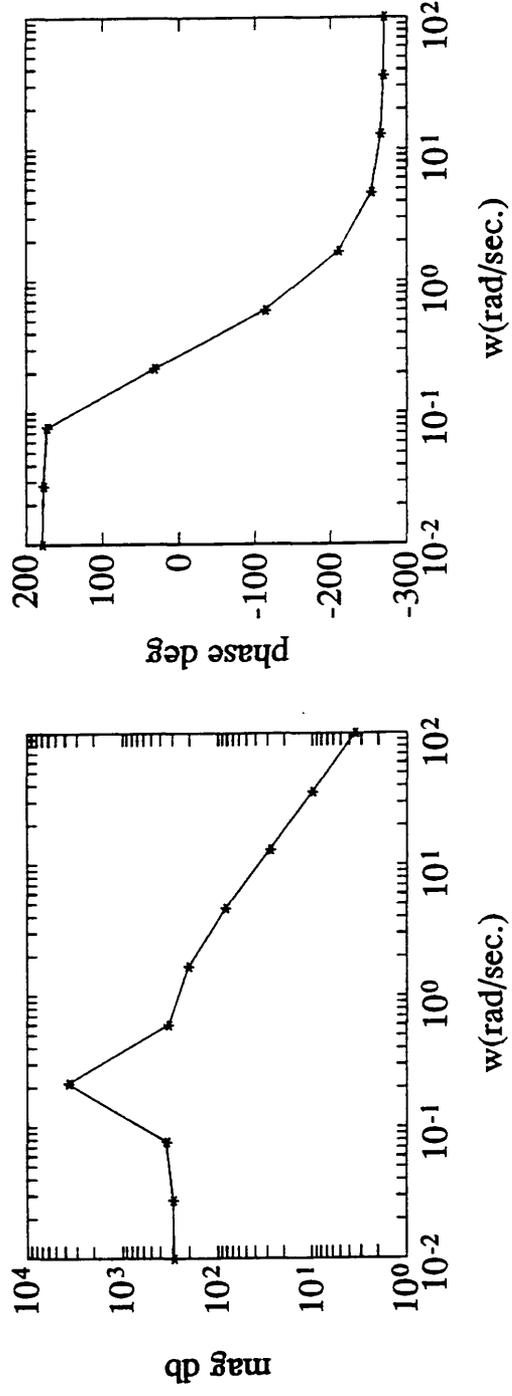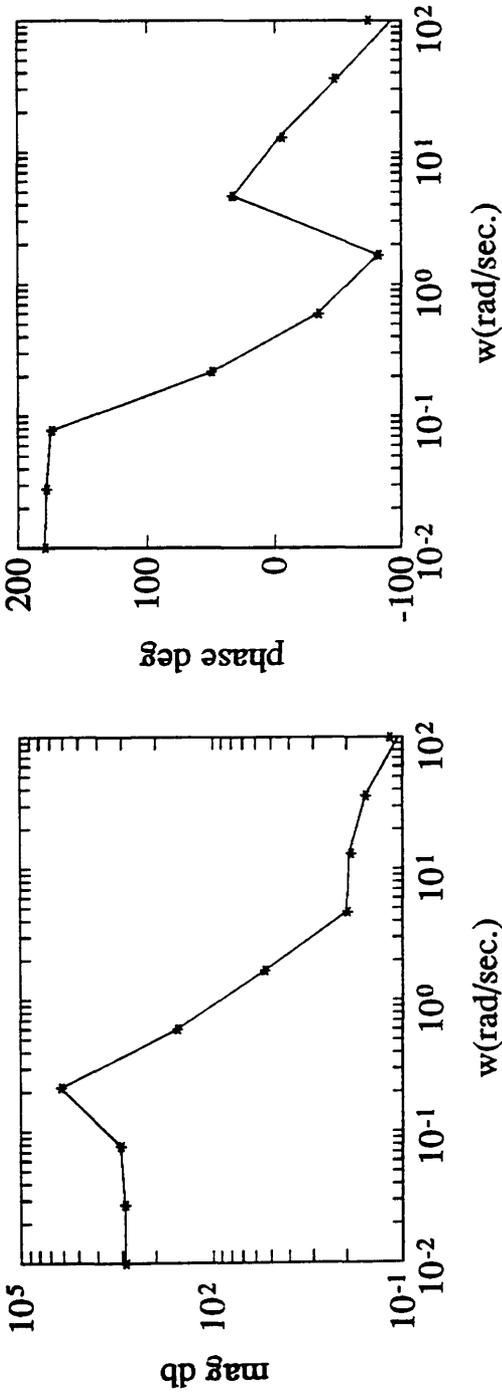
Fig. 7.14 Frequency Response for Puma 80, 10 Points
(Case x_1/u_1 & x_2/u_1 , $n_1 = 1.0$, i = 5)

* reduced model, — HOS model
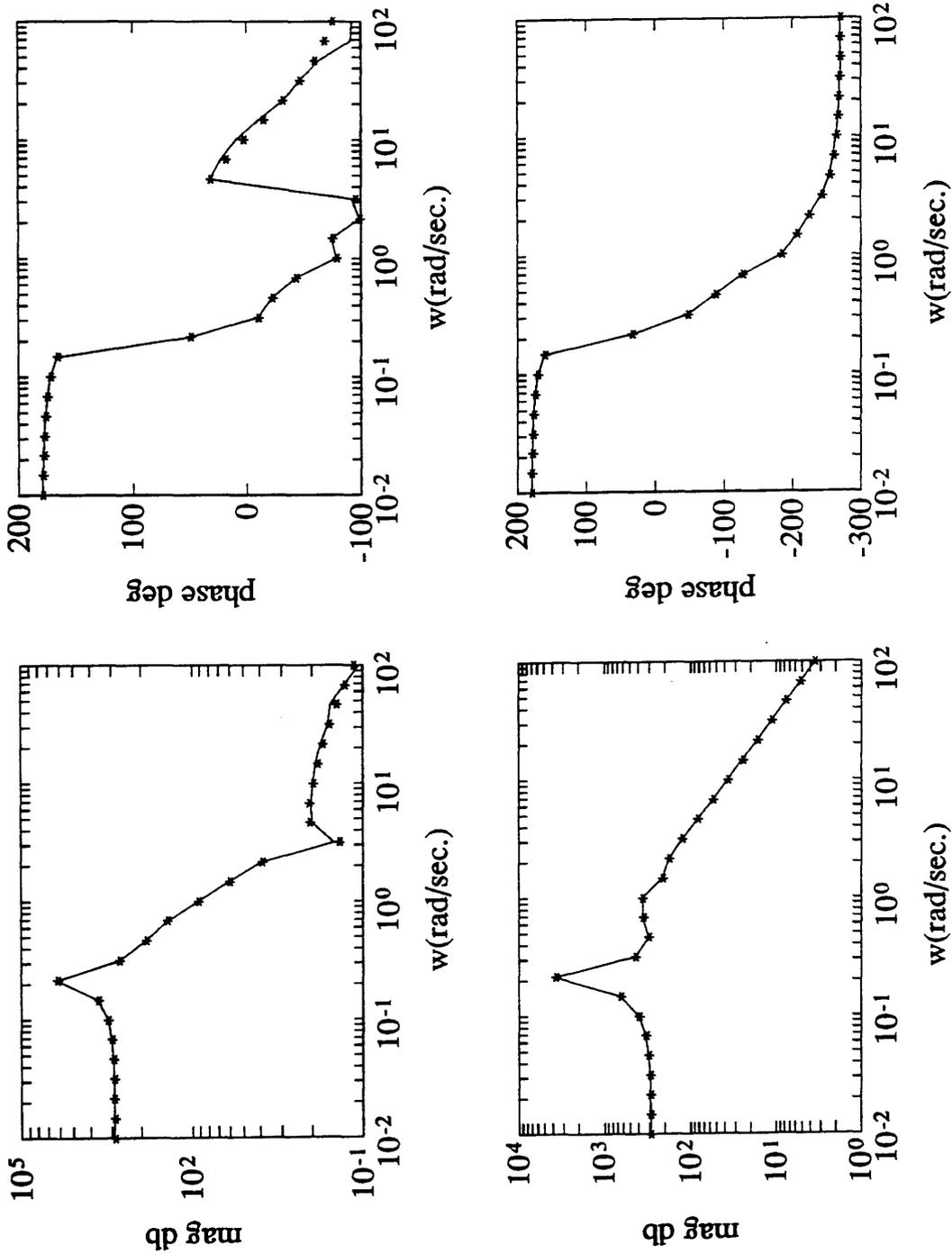
Fig. 7.15 Frequency Response for Puma 80, 25 Points
(Case x_1/u_1 & x_2/u_1 , $n_1 = 1.0$, $i = 5$)

* reduced model, — HOS model
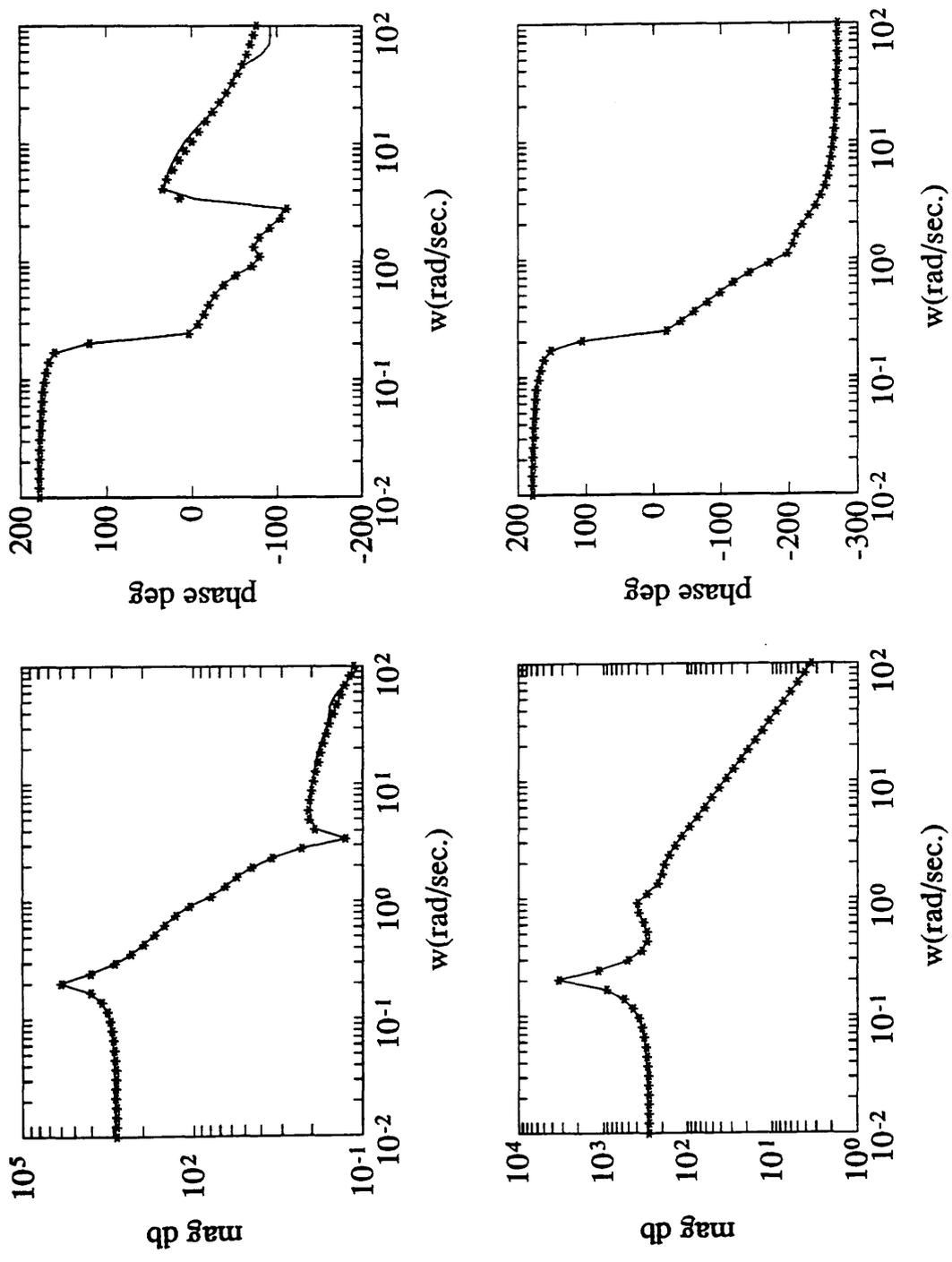
Fig. 7.16 Frequency Response for Puma 80, 50 Points
(Case x_1/u_1 & x_2/u_1 , $n_1 = 1.0$, $i = 5$)
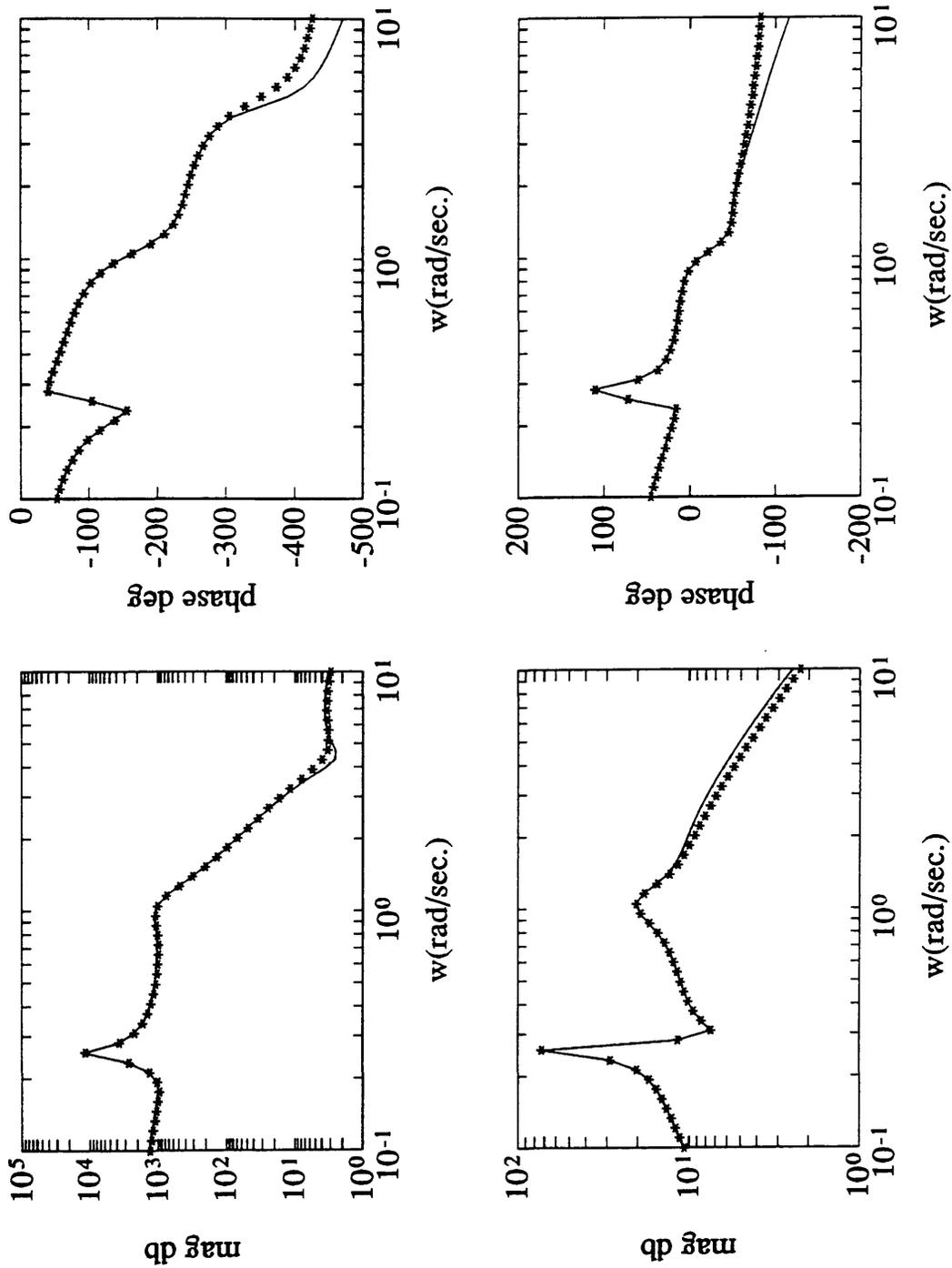
* reduced model, -- HOS model

* reduced model, — HOS model
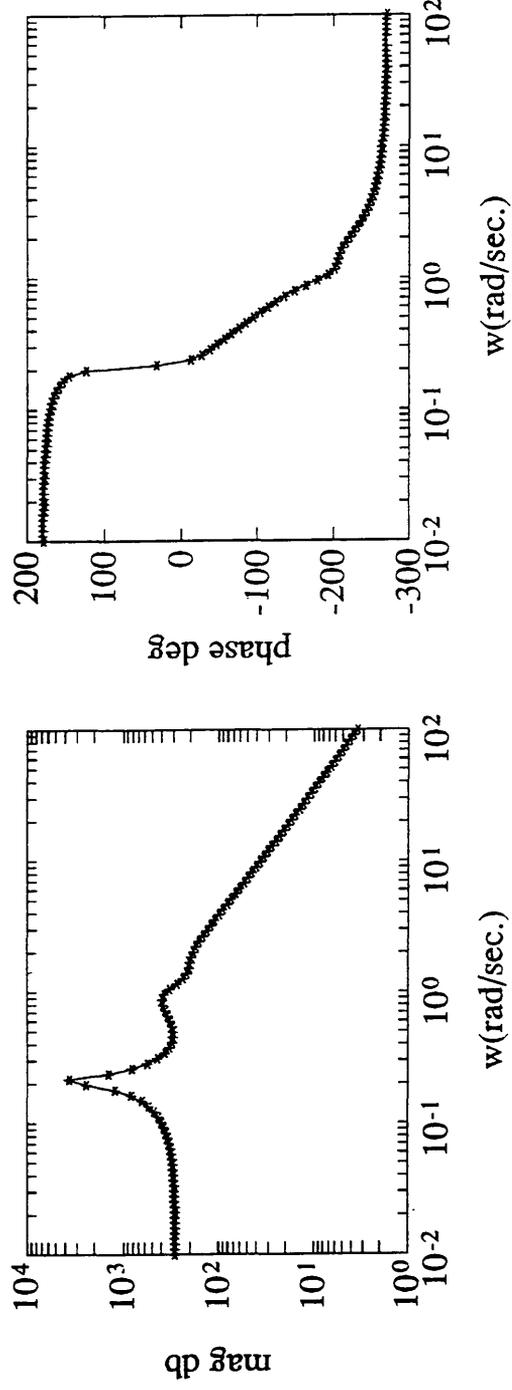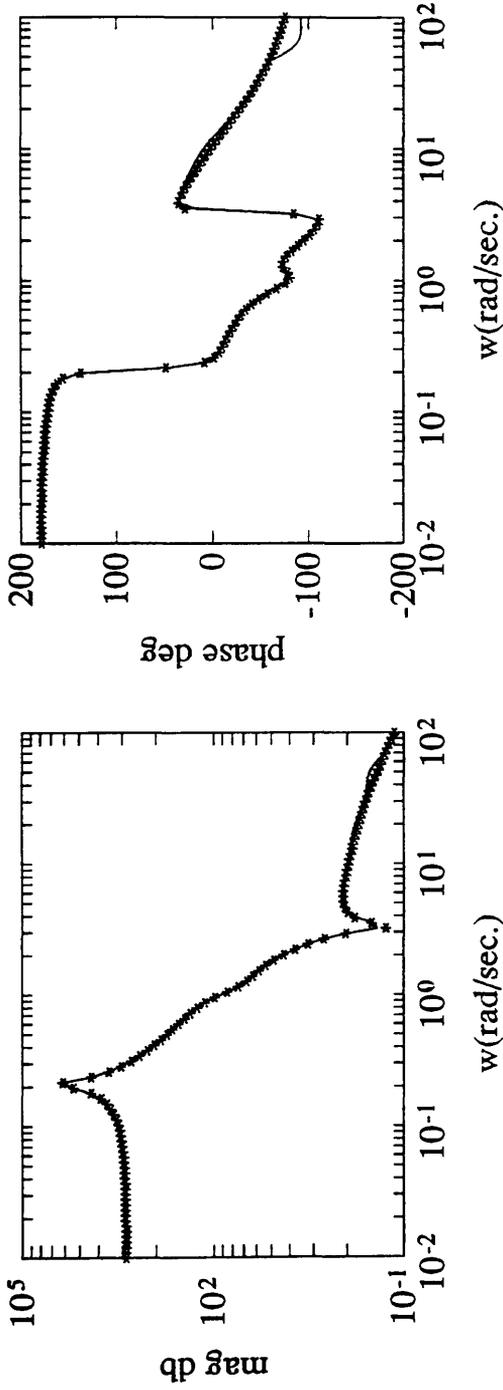
Fig. 7.17 Frequency Response for Puma 80, 100 Points
(Case x_1/u_1 & x_2/u_1 , $n_1 = 1.0$, $i = 5$)

Fig. 7.18   Time Response for Puma80
(Case x7/u3 & x8/u3, n1 = 0.2, i = 5)
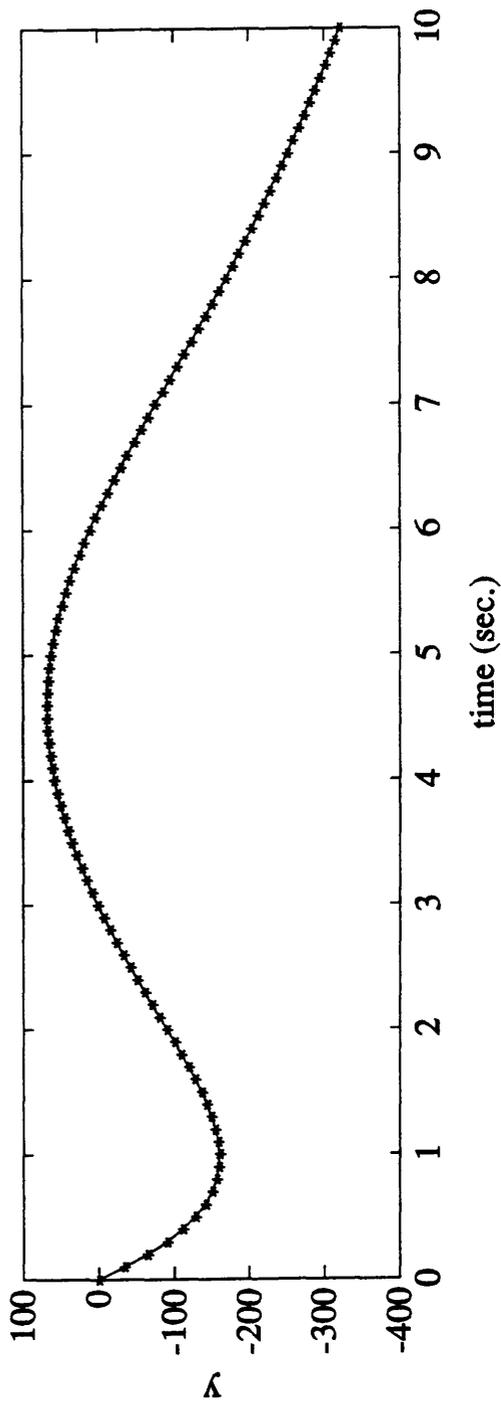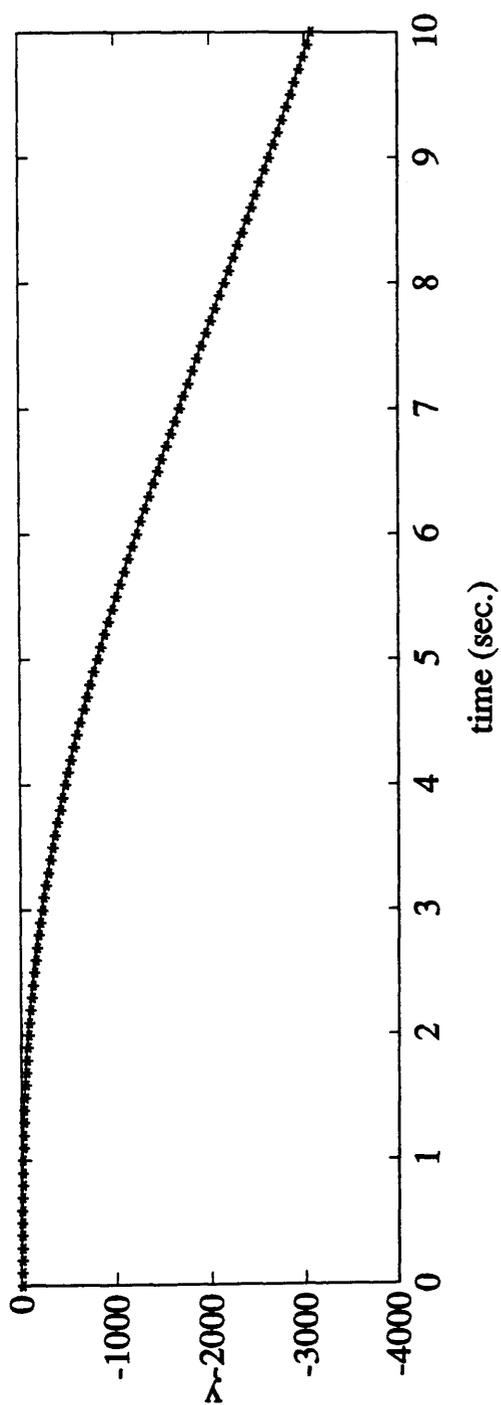
Fig. 7.19 Time Response for Puma80
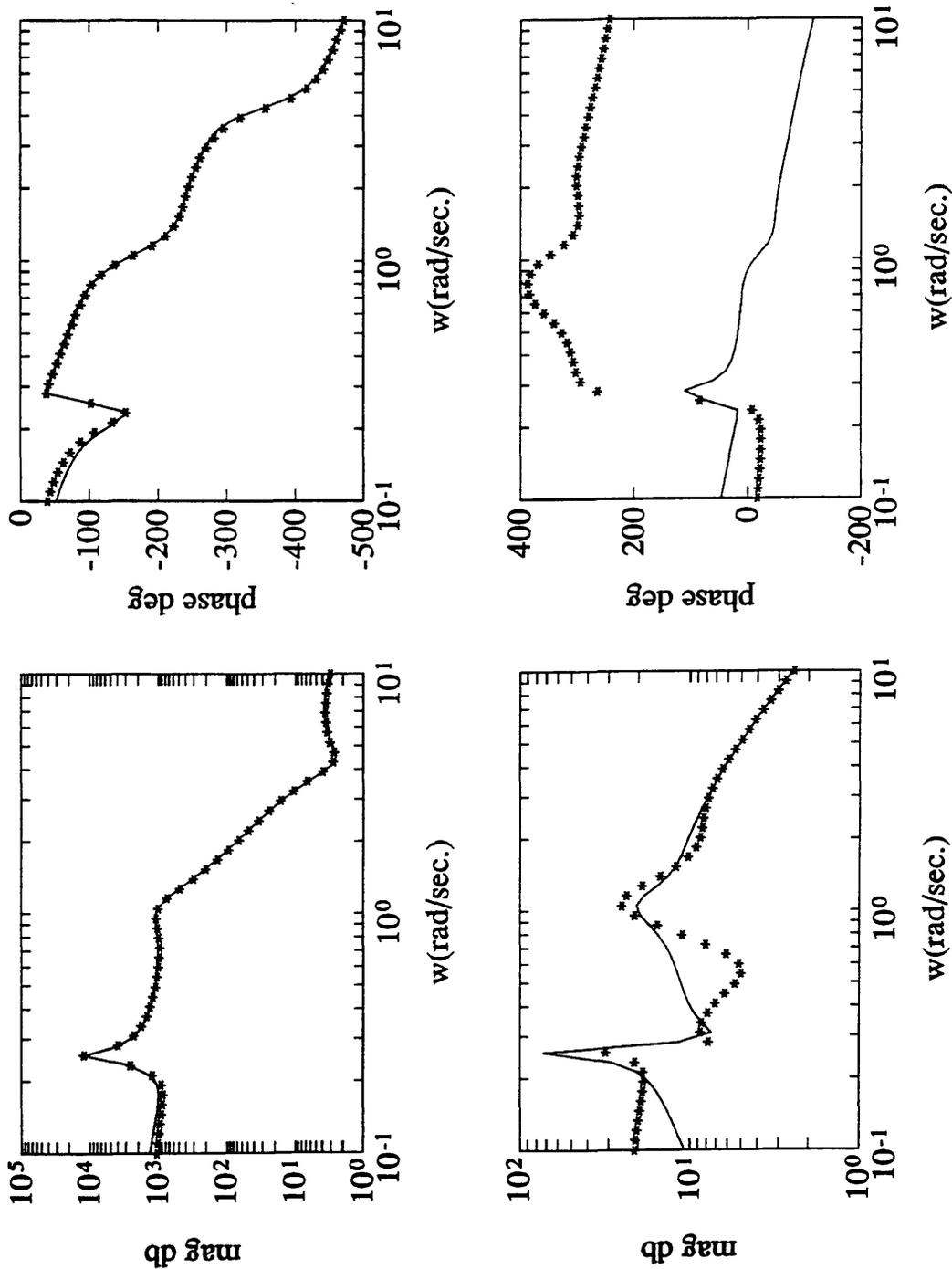(Case x1/u1 & x2/u1, n1 = 1, i = 5)

* Levy's model, - HOS model
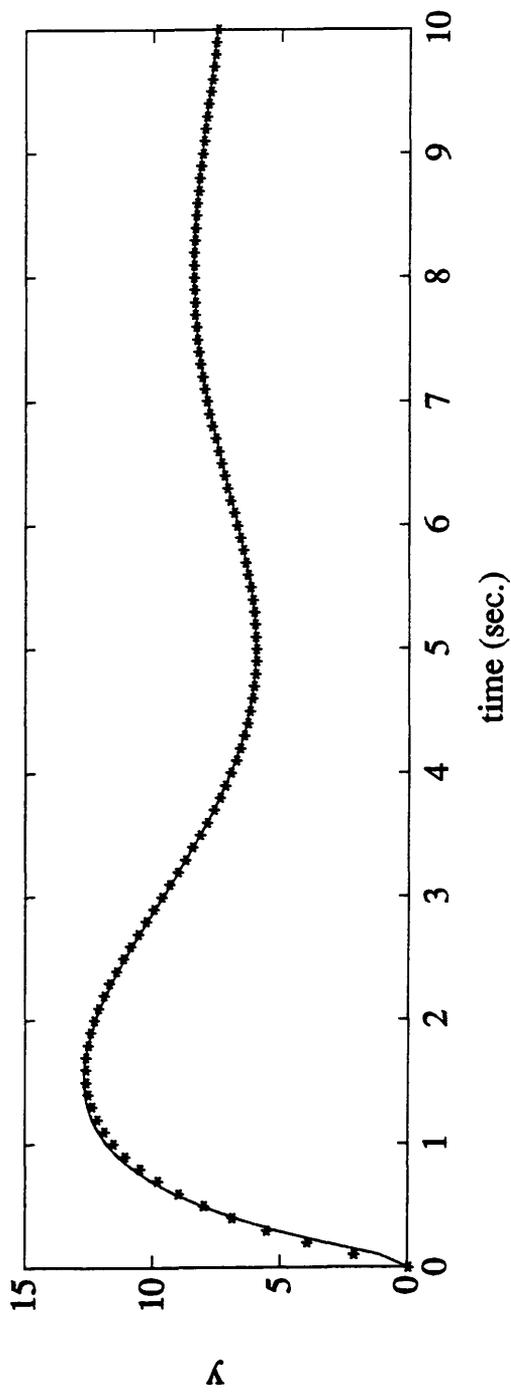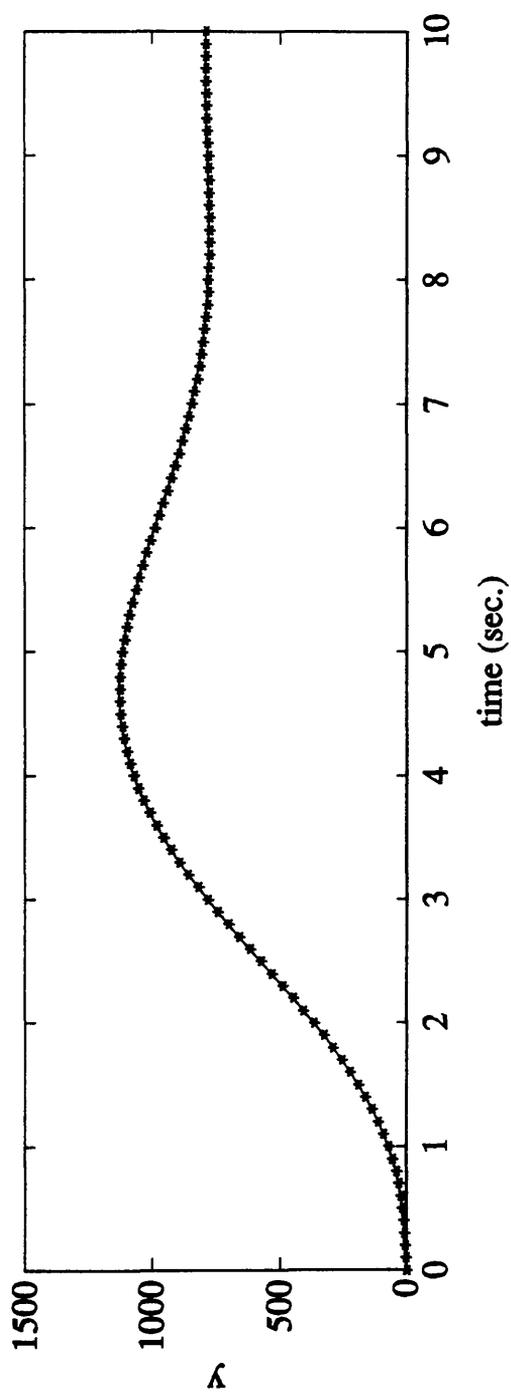
Fig. 7.20 Time Response for Puma60
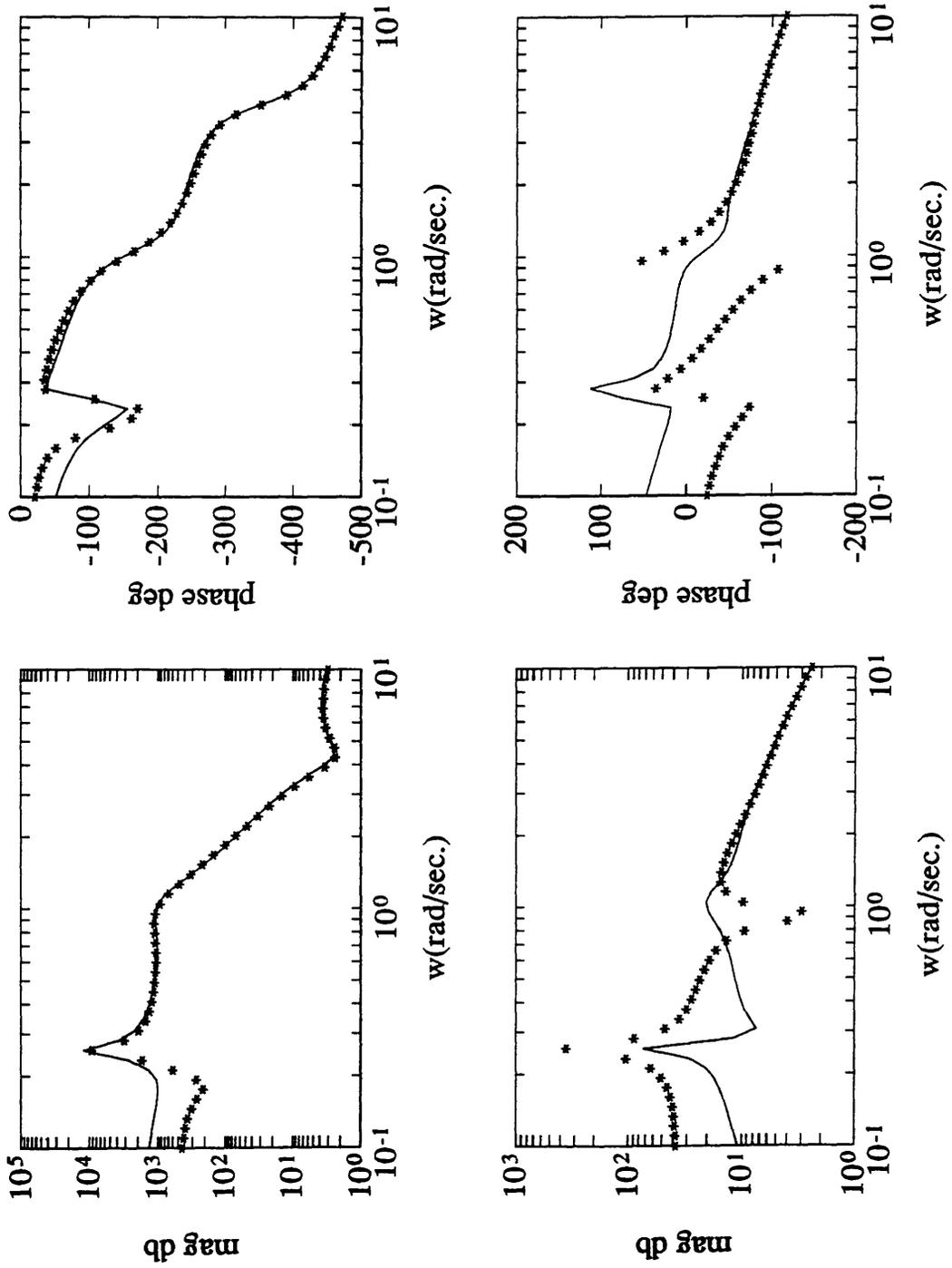(Case x5/u3 & x6/u3, n1 = 1.2, i = 5)

* Levy s model, - HOS model
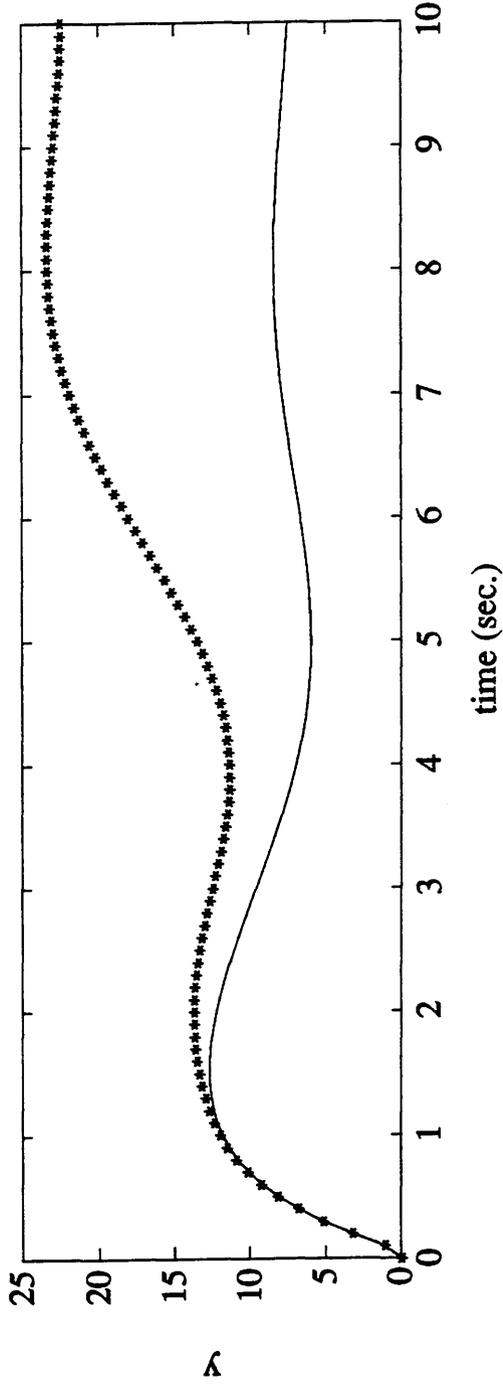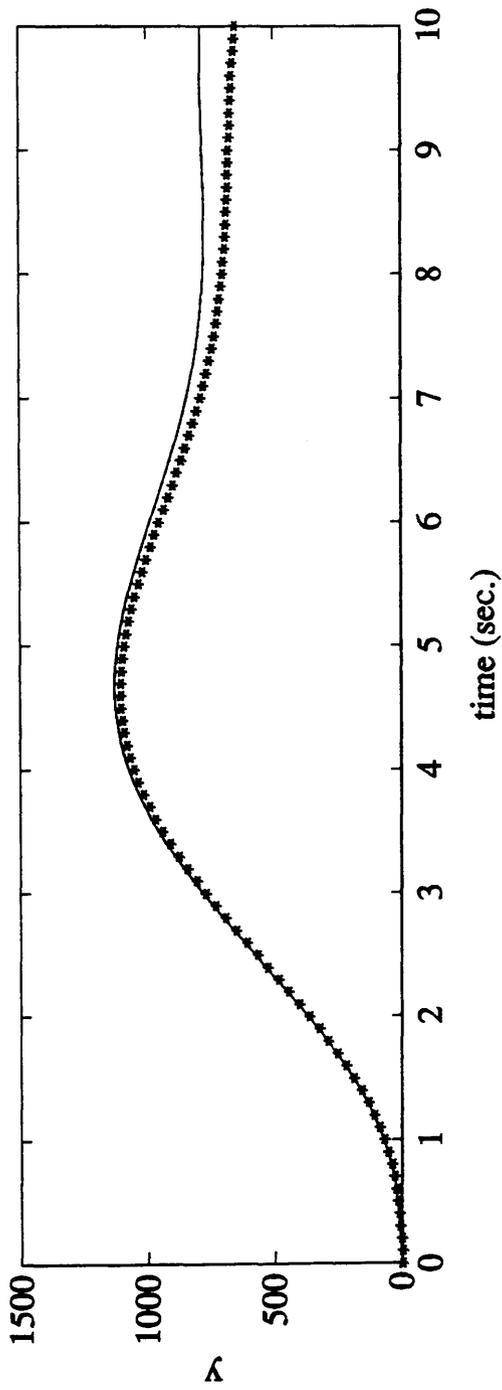
Fig. 7.21 Time Response for Puma60
(Case x5/u3 & x6/u3, n1 = 0.6, i = 5)

Fig. 7.22 Time Response for Puma80 (x1/u1, Levy's Method)
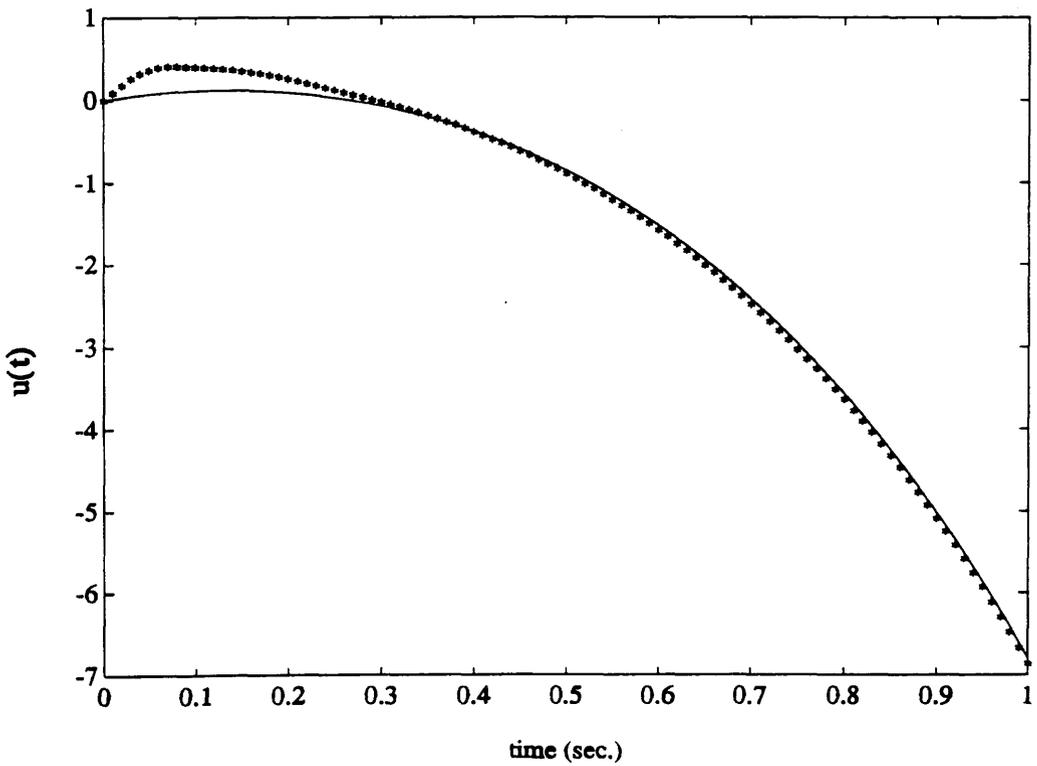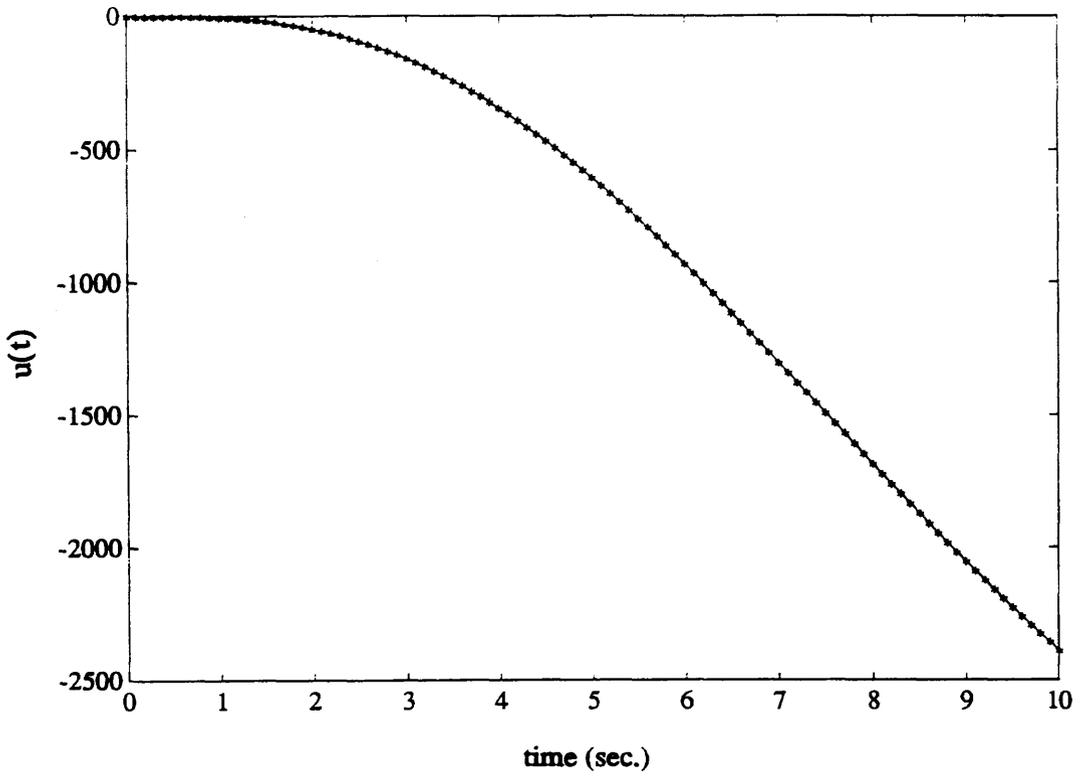
Fig. 7.23   Time Response for Puma60 (x1/u1, Levy's Method)
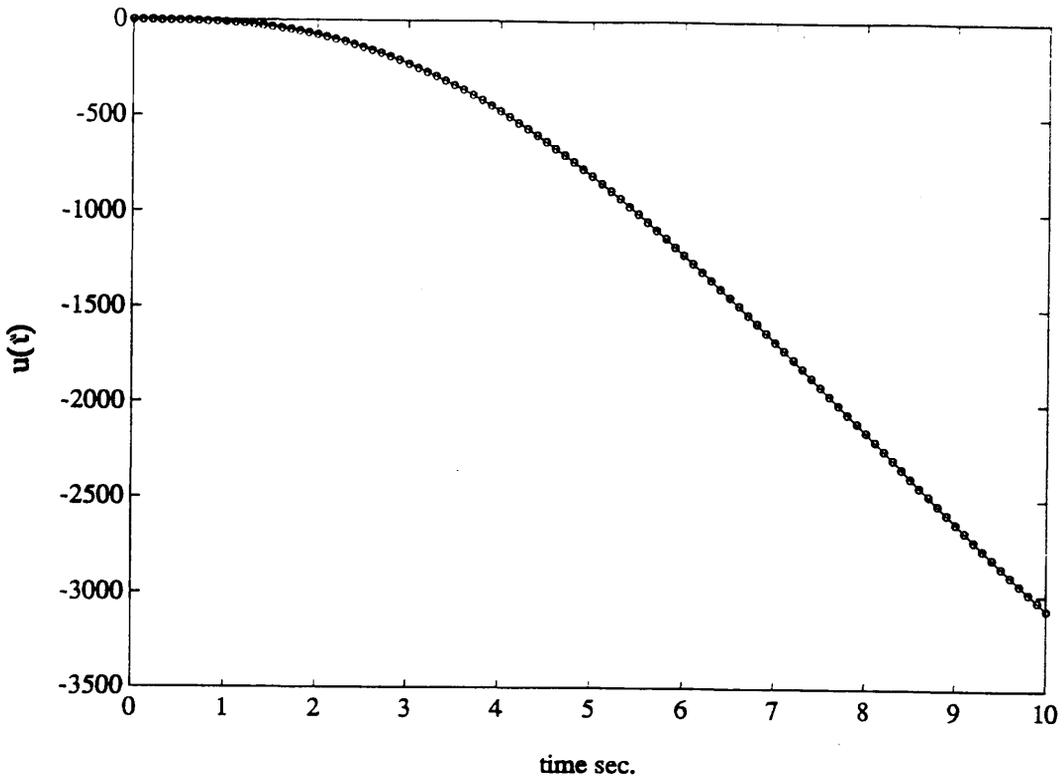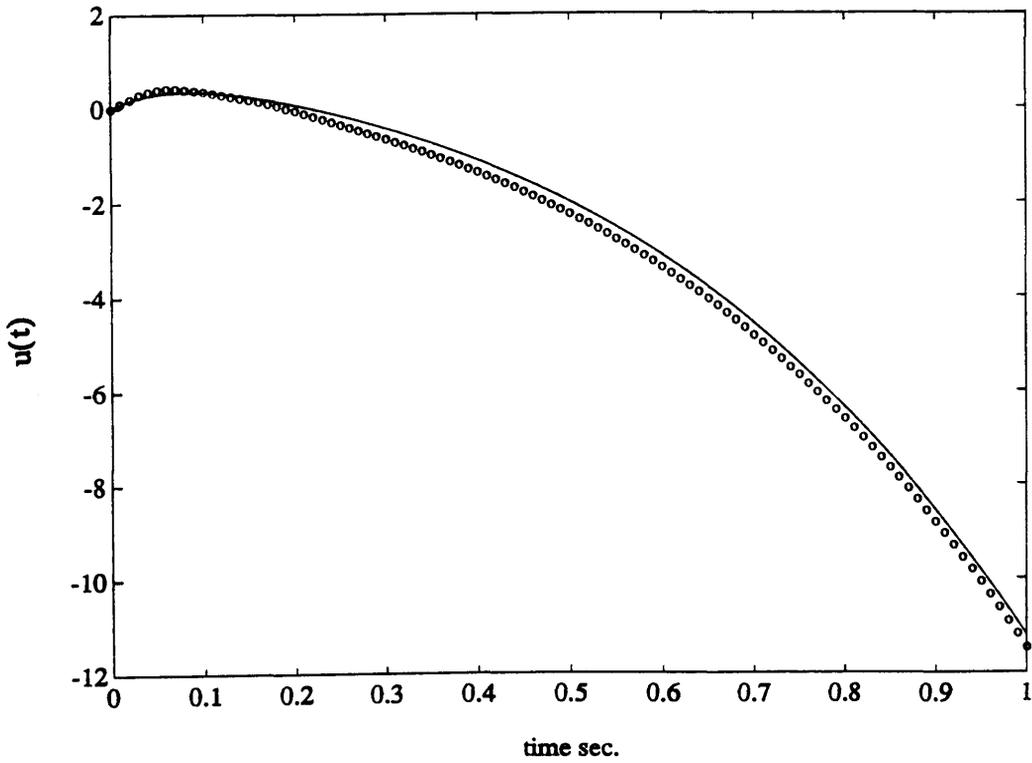
Fig. 7.24 Time Response for Puma80 (x1/u1, Bacon & Schmidt's Method)
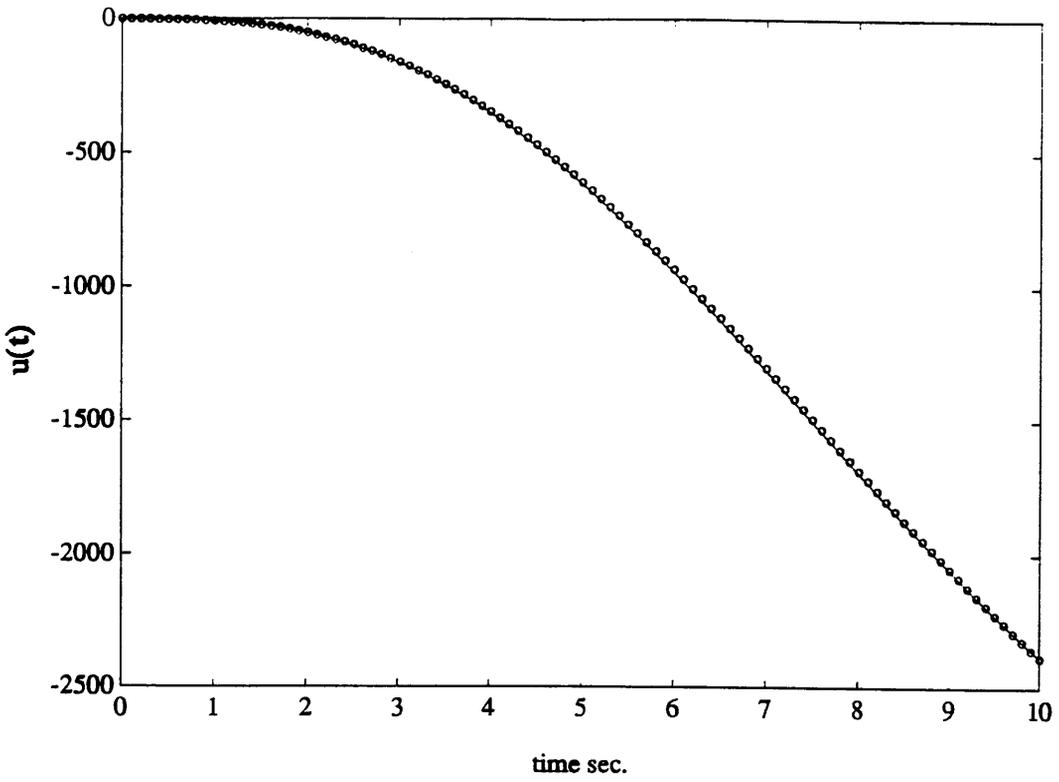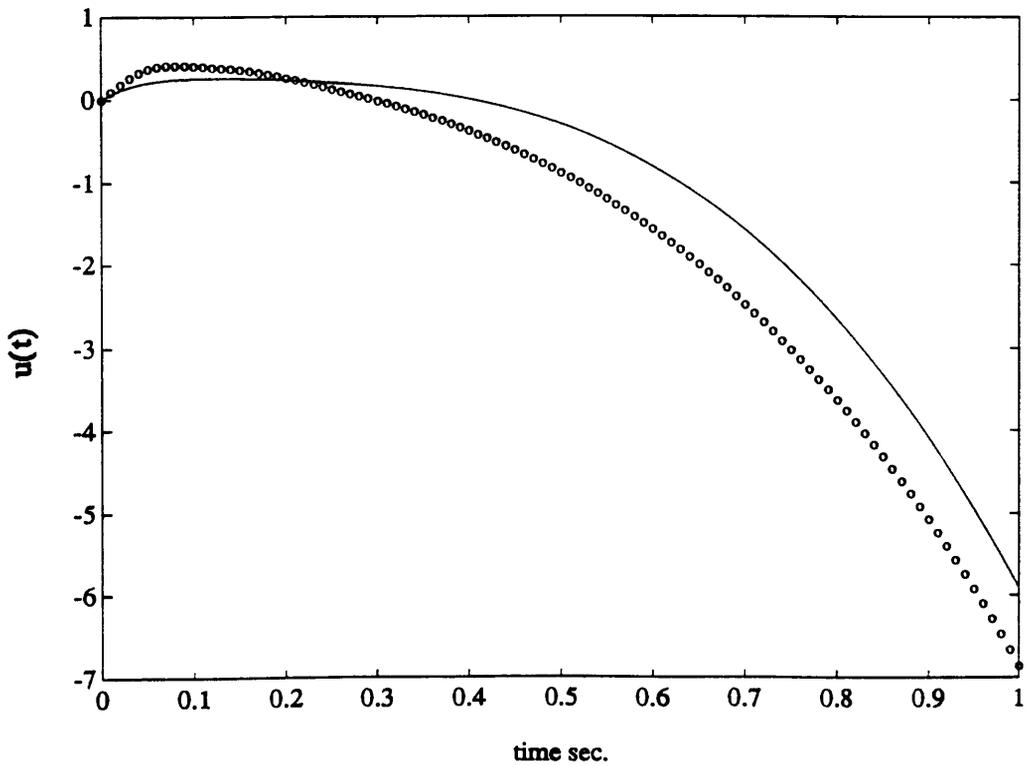
Fig. 7.25   Time Response for Puma60 (x1/u1, Bacon & Schmidt's Method)
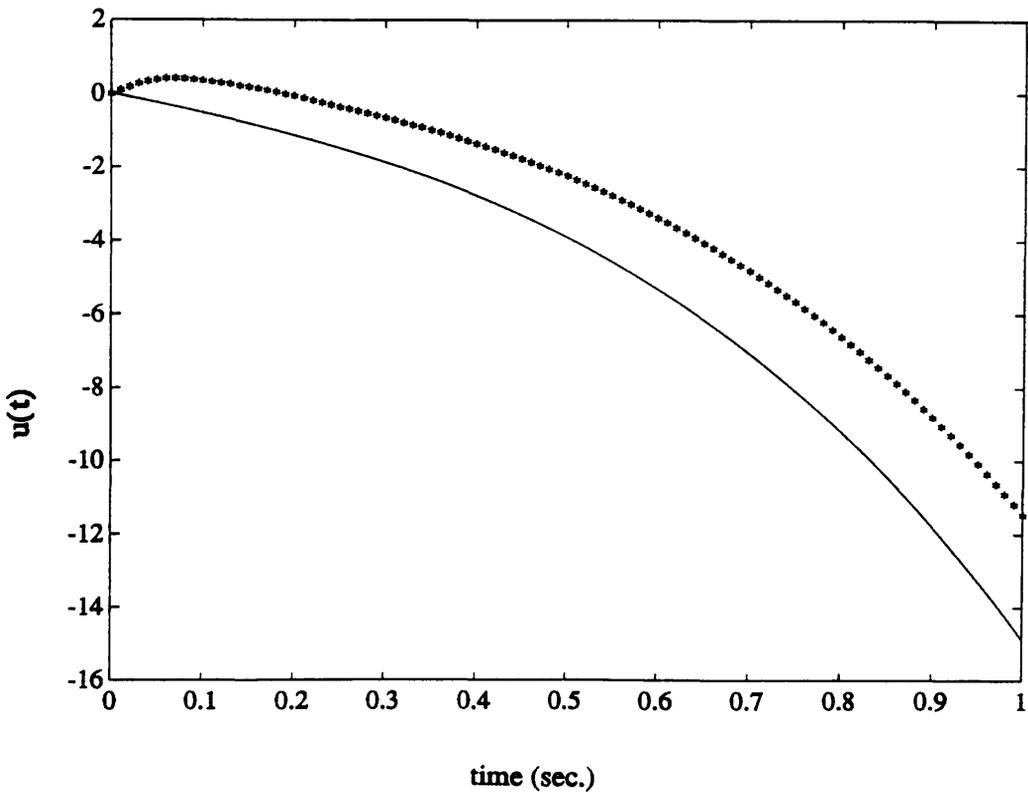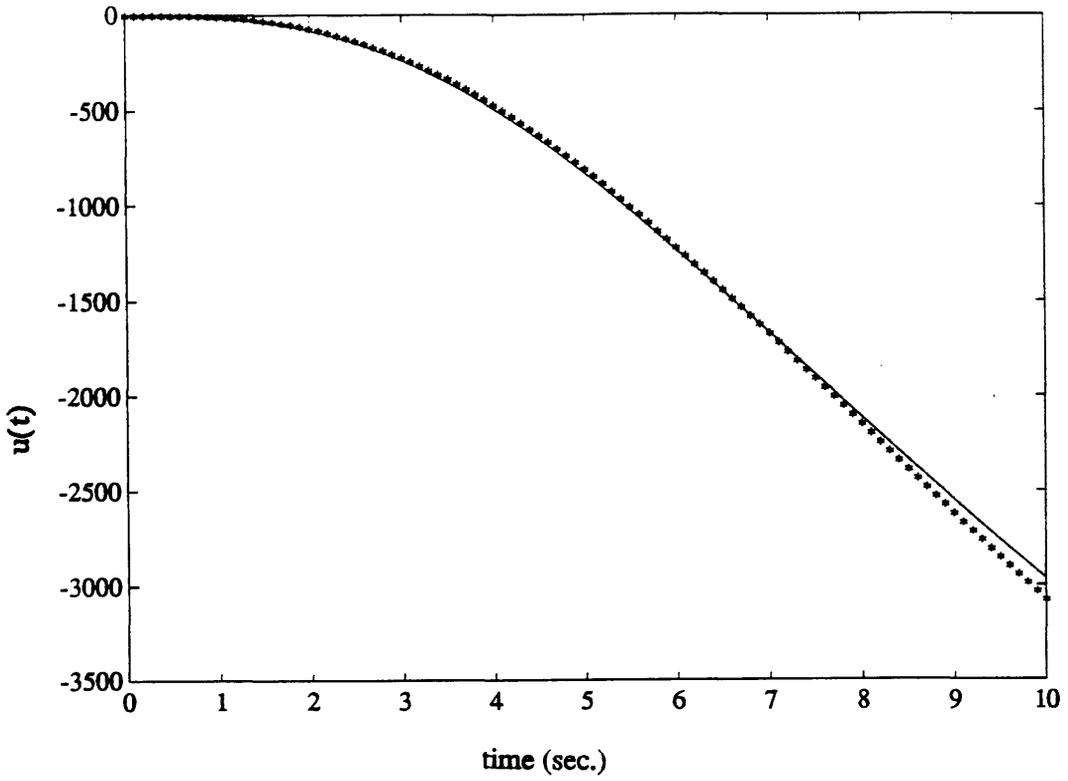
Fig. 7.26   Time Response for Puma80 (x1/u1, HELISTAB Model)
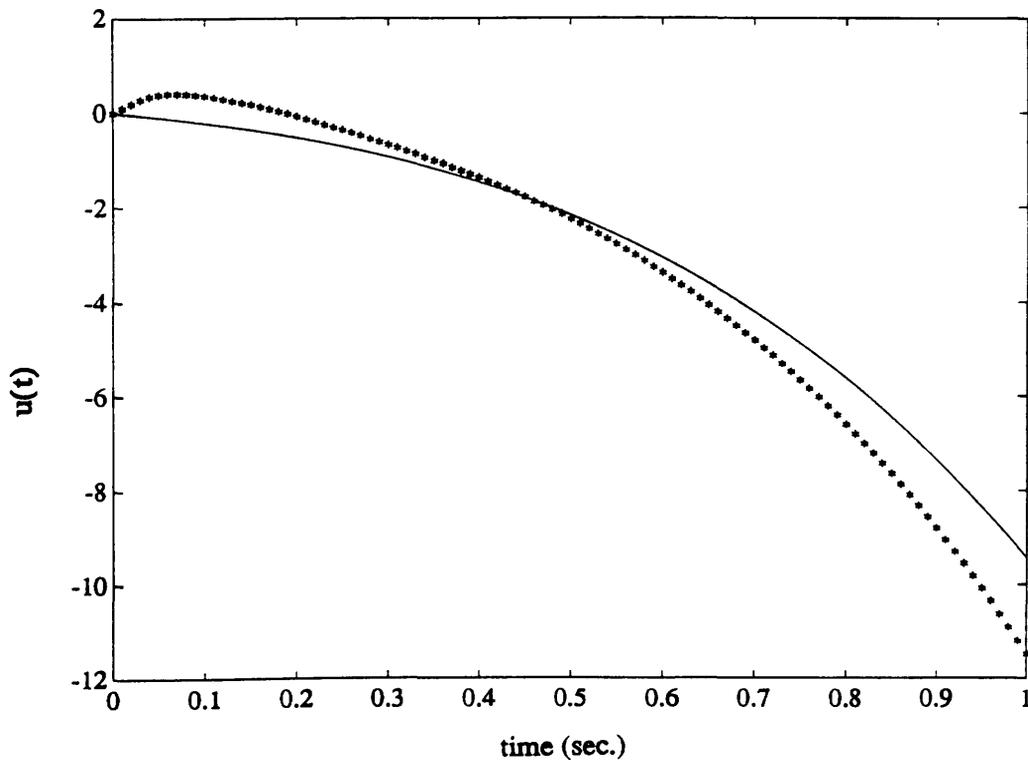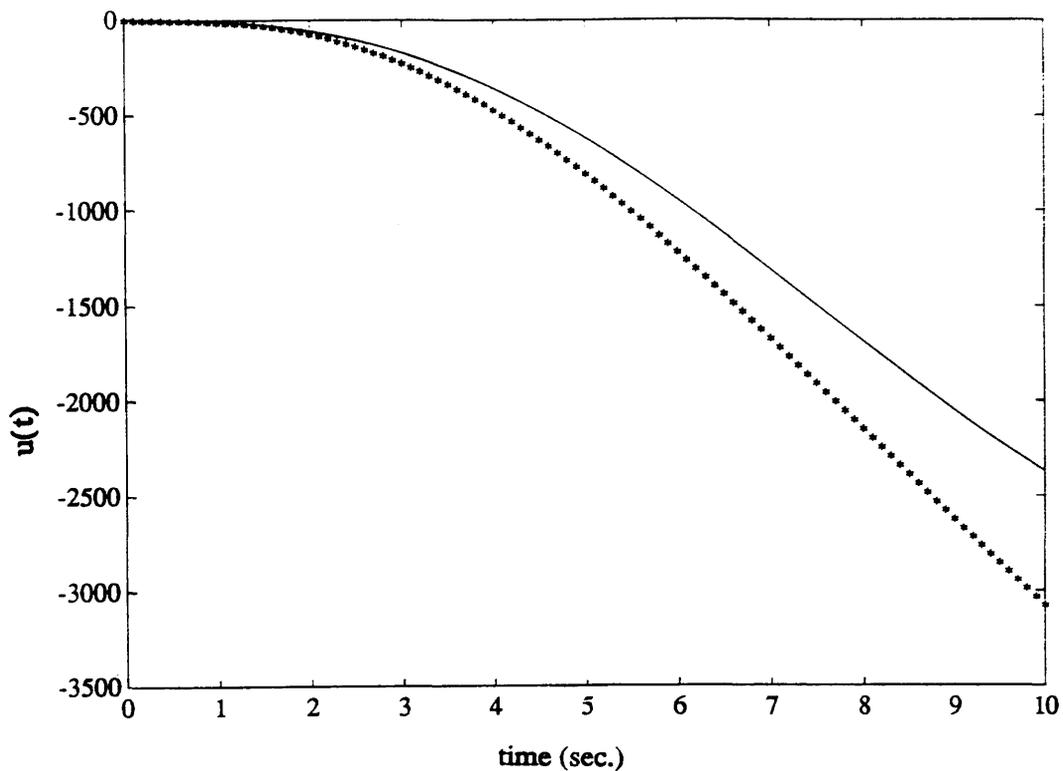
Fig. 7.27   Time Response for Puma60 (x1/u1, HELISTAB Model)
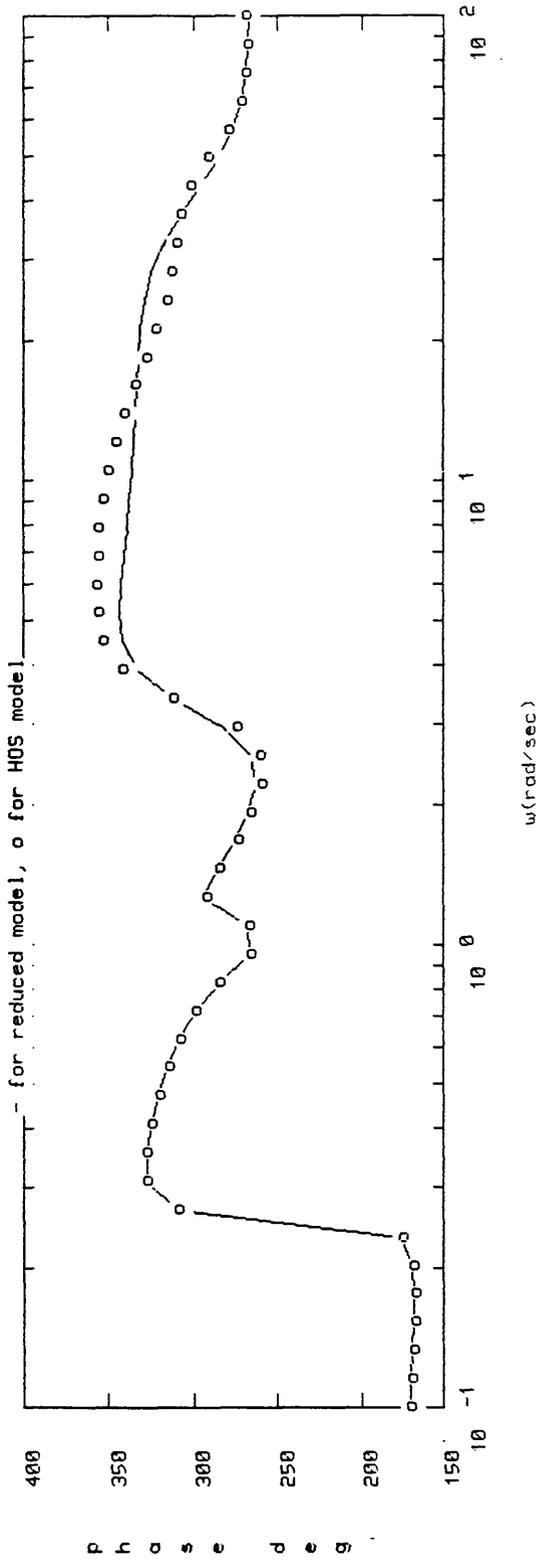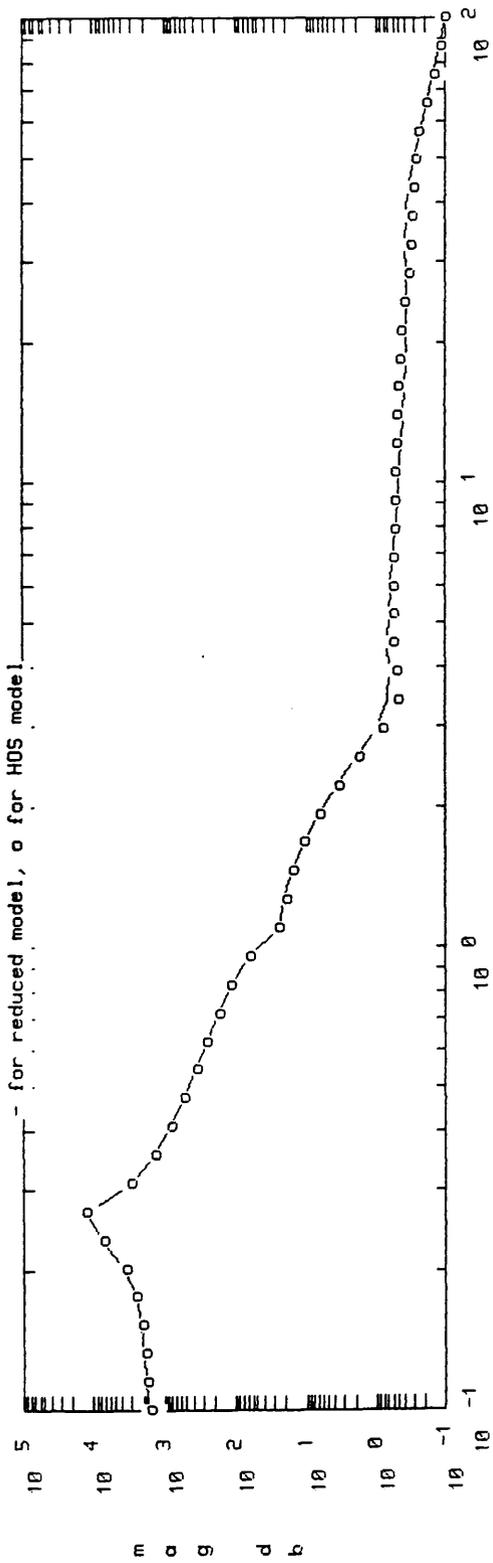
Fig. 7.28 Frequency Response for Puma60
(Bacon & Schmidt's Method, r = 11)
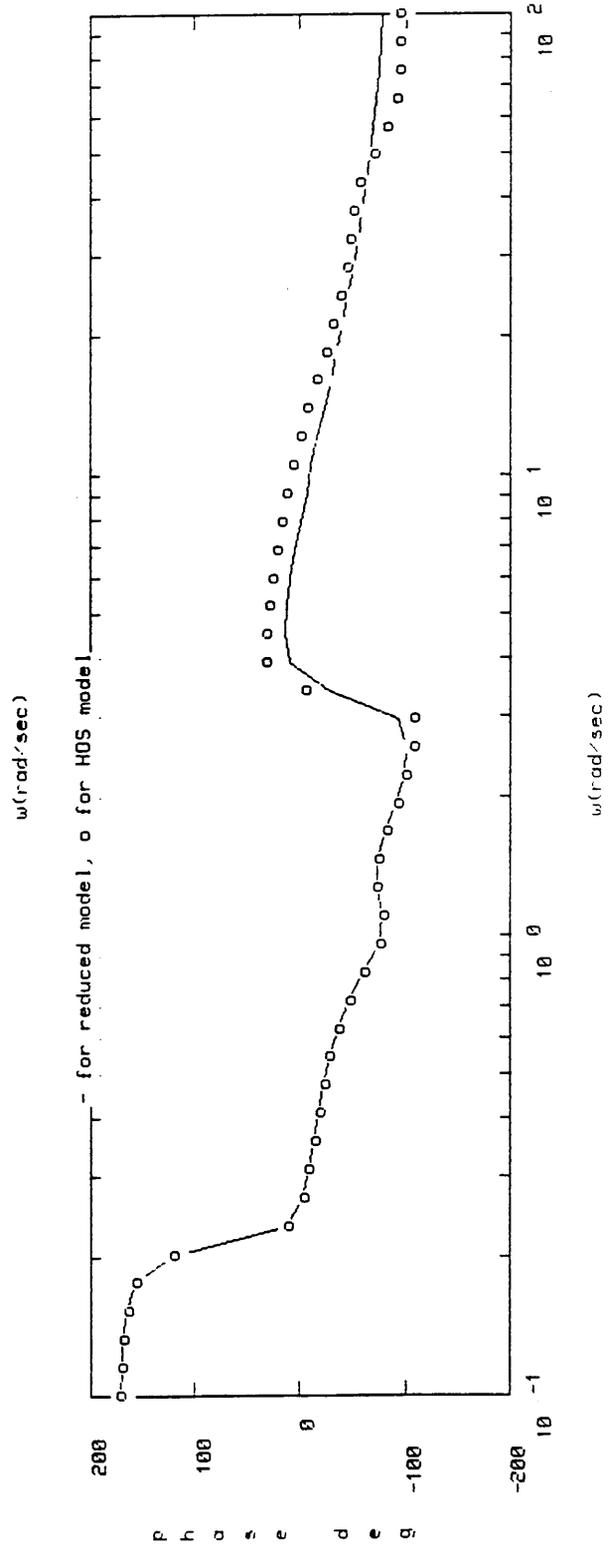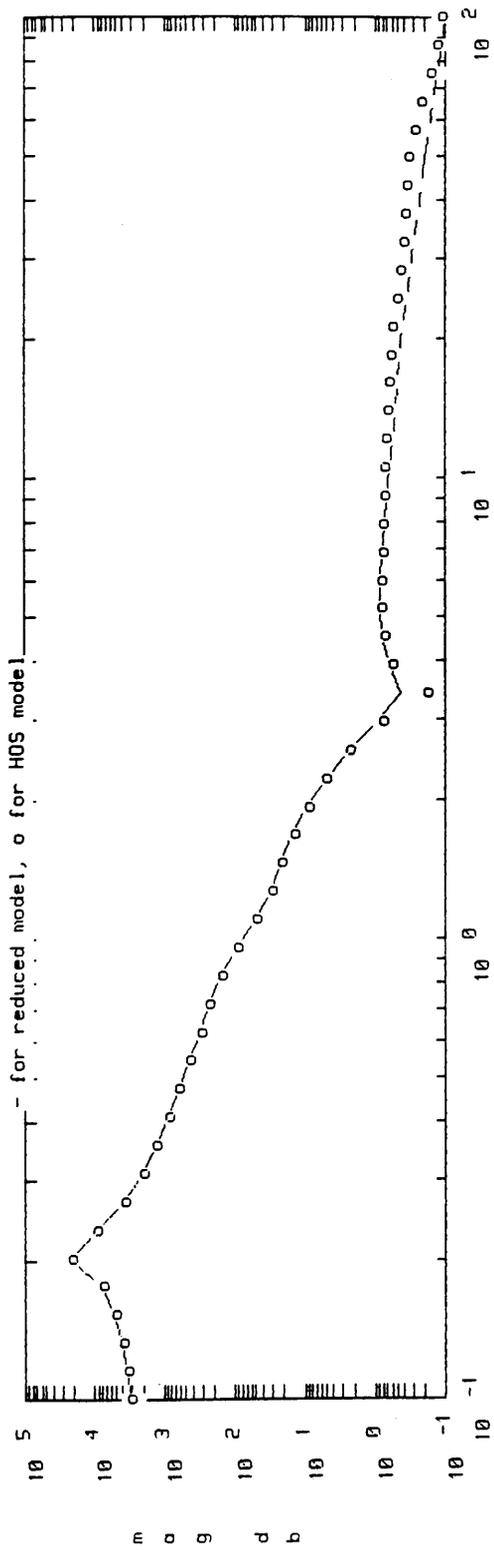
Fig. 7.29 Frequency Response for Puma80
(Bacon & Schmidt's Method, r = 8)

Chapter 8

Discussion and Conclusion

8.1 Comparison of the Results from Different Methods.

In Chapter 3 and Chapter 4 several different methods for model reduction were introduced. They are Marshall's[27] method, Chen's[64] method, Liaw's[48] method and Levy's[68] method which has been further developed by the author to suit the single– input multi– output case Bacon and Schmidt's approach[30] has also been described in Chapter 5. In order to compare these approaches each method has been applied to a model of a Puma helicopter for a condition involving 80 knots forward speed in level flight, as described in Chapter 7. However, R.T.N.Chen's method for obtainning a state space description from a transfer function is not suitable for this application because some zeros of the helicopter flight mechanics models are located in the right half– plane. Since the method cannot be applied to non– minimum phase systems it cannot therefore be applied directly for model reduction of helicopter flight mechanics models. In addition a nonlinear helicopter flight mechanics model HELISTAB[73,74], implemented in terms of the software package developed at the Royal Aerospace Establishment, Bedford can be used to provide theoretical quasi– static parameter values for linearised models of various orders. Only the results from Levy's method, Marshall's method, Liaw's method, Bacon and Schmidt's method and the theoretical model using the HELISTAB package are available for comparison. This includes the comparing of eigenvalues resulting from different methods and a comparison of frequency response and time– domain responses for high and low order models by each method.

8.1.1 Comparison of Eigenvalues of the High– Order System and Reduced Order Systems.

The values of the eigenvalues of the system matrix A are considered to be of vital importance for assessing the quality of approximation of the reduced model. In general, the closer the dominant eigenvalues of the reduced model matrix are to

those of the high order model, the better is the approximation of the model to the real system. The eigenvalues of the high–order system (14th order) and the reduced–order system (8th order) obtained from the five methods are listed in Table 8.1. and Table 8.2, separately. It can readily be seen that Bacon and Schmidt's method and Levy's method produce the eigenvalues which approximate most closely the dominant eigenvalues of the high order system. In general, the eigenvalues from Bacon and Schmidt's method or Levy's method are both in very good agreement with the eigenvalues of the high order system in the low frequency and middle frequency ranges. However in the high frequency range Bacon and Schmidt's method can produce better results than Levy's method.

8.1.2 <u>Comparison of Frequency Responses</u>.

The frequency responses resulting from the different methods of model reduction are shown in Fig.8.1 and Fig.8.2 together with those of the high order system. It can be seen clearly that the frequency response obtained by using Bacon and Schmidt's method gives excellent agreement both in magnitude and phase with the high–order system model over the whole frequency region of interest. From Fig.8.3 and Fig.8.4 it may be seen that the frequency response obtained by using the HELISTAB package for generation of an 8th order linearised model generally agrees well with the high order system model over the frequency region of interest except in the high frequency region. In the high frequency region big differences in phase between the HELISTAB model and original system model can be found. The frequency response produced by Marshall's method and Liaw's method only give agreement in a very small part of the frequency range between 0.01 — 0.25 (rad/sec). Generally Liaw's method gives the poorest results for this application. Again, It is clear from Fig.8.3 and Fig.8.4 that both Bacon and Schmidt's method and Levy's method can produce perfect results in the low and middle frequency range.

One particularly interesting finding which is illustrated clearly in the results of

Figure 8.3 is that by using either Levy's method or the Bacon and Schmidt's approach it is possible to obtain a reduced order model which fits the original 14th order model much more closely than the HELISTAB 8th order model. This is considerable interest for flight control system design where much use is made of linearised 8th order descriptions obtained directly from nonlinear flight mechanics models by considering only the state variables associated with rigid body motion. The results of Fig.8.3 suggest that a better low order description may be obtained if the linearisation is carried out using the full set of state variables in the nonlinear model with appropriate model reduction techniques then being applied as a second and separate stage of the process. The measure of how well the reduced model approximates the original system is reflected in the model's frequency response error bound. Enns[34] has proved that the frequency response error of the rth order model is bounded for all ω by

$$\sigma\{Q_o[G(j\omega) - G_r(j\omega)]Q_i\} < 2T_r(\Sigma_{n-r}) \tag{8.1}$$

where $\sigma(\cdot)$ is the maximum singular value of $(\cdot)$, and $\Sigma_{n-r} = \text{diag}(h_j)$ where $j = r+1, n$. Here, Q_i is an input scaling matrix, $Q_i = \text{diag}(q_{ik})$, $(1 \leqslant k \leqslant m)$; Q_o is an output scaling matrix, $Q_o = \text{diag}(q_{ok})$, $(1 \leqslant k \leqslant p)$, where m is number of inputs u of Equ.(5.1) and p is the number of outputs y of Equ.(5.1). The bound is defined by the truncated Hankel singular values of the scaled system $Q_oG(s)Q_i$, which like G(s) are invariant to state transformation. Also since

$$[G(j\omega)-G_r(j\omega)]=[G_{mid}(j\omega)-G_{mid,r}(j\omega)]+[G_{high}(j\omega)-G_{high,r}(j\omega)]+G_{lo}(j\omega) \tag{8.2}$$

then

$$\sigma[E(j\omega)]\leqslant\sigma[E_{mid}(j\omega)]+\sigma[E_{high}(j\omega)]+\sigma[E_{lo}(j\omega)] \tag{8.3}$$

If the $G_{lo}(s)$ contribution to the frequency response in the region $\omega_1 < \omega < \omega_2$ is negligible, the frequency response error of the reduced−order model described above is bounded by

$$\sup_{\omega_1<\omega<\omega_2} \sigma[Q_oE(j\omega)Q_i] \leqslant 2[\text{Tr}(\Sigma_{mid,n-r}) + \text{Tr}(\Sigma_{high,n-r})] \tag{8.4}$$

If $Q_i = \text{diag}(q_{ij})$ and $Q_o = \text{diag}(q_{oj})$, it can be shown that

$$|E_{ij}(j\omega)| \leqslant B_{\infty ij} \qquad (\omega_1 < \omega < \omega_2) \qquad (8.5)$$

where $B_{\infty ij} = 2[\text{Tr}(\Sigma_{mid,n-r}) + \text{Tr}(\Sigma_{high,n-r})]/(q_{ij}q_{oj})$.

8.2 Further Comparison between Bacon and Schmidt's Method and Levy's Method.

From previous discussion, it has been already seen that among the five model reduction methods considered, only Bacon and Schmidt's method and Levy's method can produce very good results. In this section, we try to further compare these two methods.

Fig. 8.5, Fig. 8.6 and Fig. 8.7 show the flow chart of Bacon and Schmidt's method, Levy's method and Chen's method separately. The Bacon and Schmidt method is based on an Internally Balanced Technique in which we only need to find and to analyse the Hankel Singular Values. The method provides more physical insight than Levy's method so that the resulting reduced order system has a more direct physical connection with the high order system. Levy's method uses the least—square approach which is a pure mathmatical curve fitting technique. This method requires solution of a set of linear equations, and is more complicated to implement and needs more computer CPU time than Bacon and Schmidt's method. However, the application of Bacon and Schmidt's method needs some experience to establish appropriate subsystems whereas the application of Levy's method requires little physical understanding of the system or previous experience.

8.3 The Unique Solution of Bacon and Schmidt's Method.

The resulting reduced system model in Bacon and Schmidt's method is uniquely determined by a small number of parameters: 1)d1, d2; the radii of the concentric circles which define G_{mid} and G_{high}; 2) r_{mid}, r_{high}; the order of subsystems G_{mid}, G_{high}; 3) Q_i, Q_o; the input/output scaling. It is apparent that different choices of d1 and d2 will produce different subsystems and therefore different results will be obtained. On the other hand, if the radii d1, d2 have been decided, then a different choice of order (r_{mid}, r_{high}) of subsystems G_{mid} and G_{high} will produce

different results as well. Also the scaling matrices Q_o and Q_i chosen may influence the resulting model.

8.3.1 The Effect of the Choice of Radii d1 and d2.

Two concentric circles of radii d1 and d2 are chosen according to the location of the poles in the s– plane. These circle will separate the s– plane into a low– , mid– and high– frequency regions and thus can be used to separate the pole/zero constellation into three sets. The choice of different subsystem will give different results.

Fig 8.8 and Fig 8.9 show two different results from two different choices of radii d1 and d2 for the Puma at the 80 knots flight condition. One of them is for radii d1 = 0.001, d2 = 15. The other one is for radii d1 = 0.22 , d2 = 15. Fig. 8.10 and Fig 8.11 also give two results by choosing radii d1 and d2 for an advanced fighter[30]. In this example, it should be noticed that even if we take the same effective order of the subsystems G_{mid} and G_{high} big differences between the two results can be found due to the different choices of d1 and d2.

8.3.2 The Effect of the Choice of the Effective Order of the Subsystems G_{mid} and ~~G_{high}~~.

Using the Puma at 80 knots as an example different effective orders of the subsystems G_{mid} and G_{high} were chosen. One was chosen as $r_{mid} = 7$ and $r_{high} = 1$; another one was chosen as $r_{mid} = 6$ and $r_{high} = 2$. The total effective order of the system is equal to 8 in both cases. From Fig. 8.8 and Fig. 8.12 we can see that very different results are found in these two satuations the difference resulting from the different effective order of subsystems is significant though the effective order of the overall system is the same (r = 8).

Similarly, in the another example – the advanced fighter without prefilter, the results given in Fig.8.10 and Fig.8.13, show that although the final reduced order is same the results still show big differences if the effective order of the subsystems G_{mid} and G_{high} is different.

8.3.3 The Effect of the Scaling Factors.

As mentioned in Chapter 5, to obtain a uniform match between the truly dominant responses of the system, scaling must be included. In order to discuss the importance of the scaling , the example of the advanced fighter without prefilter is again used as an example. In Fig.8.10 and Fig 8.14 the results from using scaling and without scaling are presented respectively. The difference appears only in the high frequency region.

8.3.4 The Equivalent delay in Low Frequency Range.

The use of an equivalent delay parameter is a widely used method in the reduction of complex systems, especially in terms of approximating the dynamics in the high frequency range. The equivalent delay for the Puma helicopter at 80 knots flight condition and a advanced fighter without prefilter are shown in Fig. 8.15 and Fig. 8.16. From Fig. 8.15 and Fig. 8.16 we can see that the reduced order systems with equivalent delay are given. For the Puma at 80 knots the delay time τ is 0.04535 sec., and in the low frequency range between $0.1 - 3$ rad/sec. the reduced order system with delay agrees with the high order system very well. However, in the middle and high frequency ranges the reduced order system with delay does not fit the high order system. For the advanced fighter without prefilter the delay time τ is 0.03216 and as with the Puma, the reduced order system with delay only agrees with the high order system in the low frequency range.

8.4 Conclusion.

The equivalent system approximation technique presented by B.J.Bacon and D.K.Schmidt has for the first time been used in the analysis of a helicopter flight mechanics model. The Puma helicopter for flight conditions of 60 and 80 knots has been used to demonstrate the technique. Reduced order system models have been obtained, which give excellent comparisons with the original system model. Comparing with other methods for obtaining reduced order system, Bacon and Schmidt's method showed superior agreement in terms both of time and frequency

responses. For a given frequency response error each system has an effective order which is determined by a small number of parameters. Therefore, the equivalent system approach technique is easy to apply. In addition, the software package MATLAB provides an easier implementation than other software packages so far considered. The computer CPU time taken using MATLAB is of the order of a few minutes for the SISO case considered using a DEC MicroVAX 3600 (VMS2).

8.5 Future Work.

As described in previous chapters, the equivalent system approach offers a very good representation of a high order system, although the studies involving the Puma helicopter showed that there is still room to further improve the performance in the higher frequency range. Model reduction for use in flight control system design involves requirements that are considerably different from those encountered in other applications such as piloted simulation and wind tunnel validation. Reduced models for use in simulation and wind tunnel validation must be generally accurate over a wide spectrum of frequencies from trim (zero frequency) and phugoid (low frequency) to the dominant transient responses of the longitudinal short− period and roll− subsidence modes (mid/high frequency) . Practical flight control system design requires reduced models that are 1) highly accurate in the crossover frequency range − to exploit the maximum achievable performance from the helicopter − and 2) robust in the crossover range with respect to flight condition and input form and size − to ensure that cloose− loop stability/performance is maintained. Control system design can be made robust to compensate for poor model robustness, but only at the expense of performance. These requirements are especially difficult for advanced high − bandwidth control systems where the crossover range occurs at frequencies near the limit of current model reduction capabilities; the model order must be high enough to capture the important dynamic characteristics. In the frequency domain, this means a sufficient number of states to achieve a "good fit" of the nonparametric response. However, if the model order is excessive model parameters will exhibit

large variability to small changes in flight condition, input form and input size, which will compromise robustness.

The modern trend in aircraft control systems is inevitably towards greater complexity, due mainly to requirements imposed by complex performance specifications and tasks and to the need for accuracy. A modern complex system may have many inputs and many outputs. Therefore Levy's method of model reduction may still have some room to develop. In the present work, a further extension of Levy's method to multi-input multi-output cases has been developed, although more investigation about its application may still be necessary.

It should be possible to use this simpler low order description in applications such as flight control system design and real-time simulation for handling qualities studies. Reduced order models of this kind can also be used in the validation of more complex nonlinear models using system identification methods. Frequency-domain identification of rotorcraft dynamics are of particular importance in this context. Comparisons could be made between the experimental frequency responses and the frequency responses obtained from the reduced order model, either to confirm the validity of the reduced order model or to provide information to further improve the theoretical model from which the reduced model was derived. A spectral analysis approach to the identification and validation of helicopter models could be integrated with the frequency domain methods of model reduction.

H.O.S	R.O.S.(8TH ORDER)			
	BACON	HELISTAB	MARSHALL	LIAW
- 0.0092 + 0.2098i	- 0.0092 + 0.2098i	- 0.0081 + 0.2117i	- 1.0836 + 1.1670i	- 0.0092 + 0.2098i
- 0.0092 - 0.2098i	- 0.0092 - 0.2098i	- 0.0081 - 0.2117i	- 1.0836 - 1.1670i	- 0.0092 - 0.2098i
- 0.0001	- 0.0001	- 0.1299	-15.1429 + 5.7814i	- 0.0001
- 0.2900 + 0.9357i	- 0.2900 + 0.9357i	- 0.1921 + 1.0558i	-15.1429 - 5.7814i	- 0.2900 + 0.9357i
- 0.2900 - 0.9357i	- 0.2900 - 0.9357i	- 0.1921 - 1.0558i	-15.7960 + 22.2181i	- 0.2900 - 0.9357i
- 1.0836 + 1.1670i	- 1.0827 + 1.1664i	- 0.9661 + 1.0097i	-15.7960 - 22.2181i	- 1.2647
- 1.0836 - 1.1670i	- 1.0827 - 1.1664i	- 0.9661 - 1.0097i	-15.4634 + 49.1181i	- 1.0836 + 1.1670i
- 1.2647	- 17.3163	- 1.5133	-15.4634 - 49.1181i	- 1.0836 - 1.1670i
-15.1429 + 5.7814i				
-15.1429 - 5.7814i				
-15.4634 + 49.1181i				
-15.4634 - 49.1181i				
-15.7960 + 22.2181i				
-15.7960 - 22.2181i				

Table 8.2   THE EIGENVALUES OF MATRIX [A]
FOR ROS (8TH ORDER) AND HOS (14TH ORDER)   (PUMA80)

USING BACON & SCHMIDT'S AND LEVY'S METHOD

H.O.S	R.O.S. (8TH ORDER)		
		BACON	LEVY
- 0.0092 + 0.2098i	- 0.0092 + 0.2098i	-0.0092 + 0.2098i	
- 0.0092 - 0.2098i	- 0.0092 - 0.2098i	-0.0092 - 0.2098i	
- 0.0001	- 0.0001	-0.2900 + 0.9354i	
- 0.2900 + 0.9357i	- 0.2900 + 0.9357i	-0.2900 - 0.9354i	
- 0.2900 - 0.9357i	- 0.2900 - 0.9357i	-1.0842 + 1.1606i	
- 1.0836 + 1.1670i	- 1.0827 + 1.1664i	-1.0842 - 1.1606i	
- 1.0836 - 1.1670i	- 1.0827 - 1.1664i	-1.2495	
- 1.2647	- 17.3163	-21.8338	
-15.1429 + 5.7814i			
-15.1429 - 5.7814i			
-15.4634 + 49.1181i			
-15.4634 - 49.1181i			
-15.7960 + 22.2181i			
-15.7960 - 22.2181i			

--- Bacon,o --- Marshall,+ --- Liaw,. --- Helistab,* -- HOS

w(rad/sec)
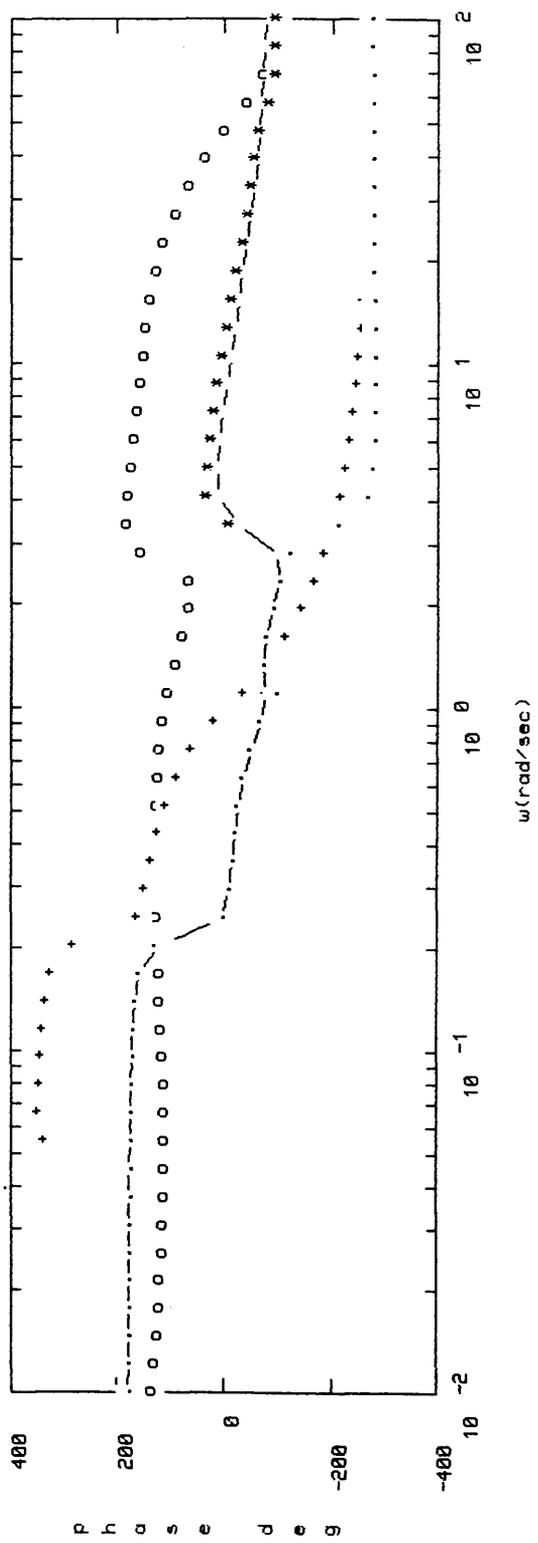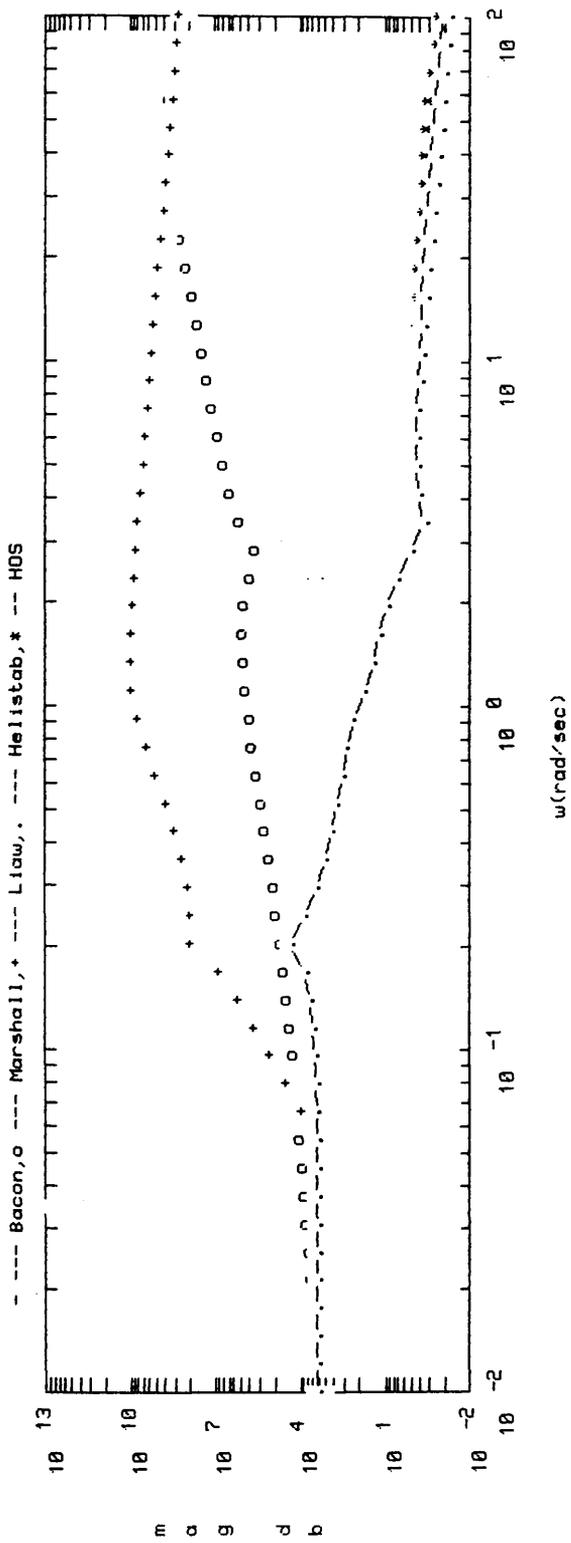
Fig. 8.1 The Frequency Responses from Different

Methods of Model Reduction
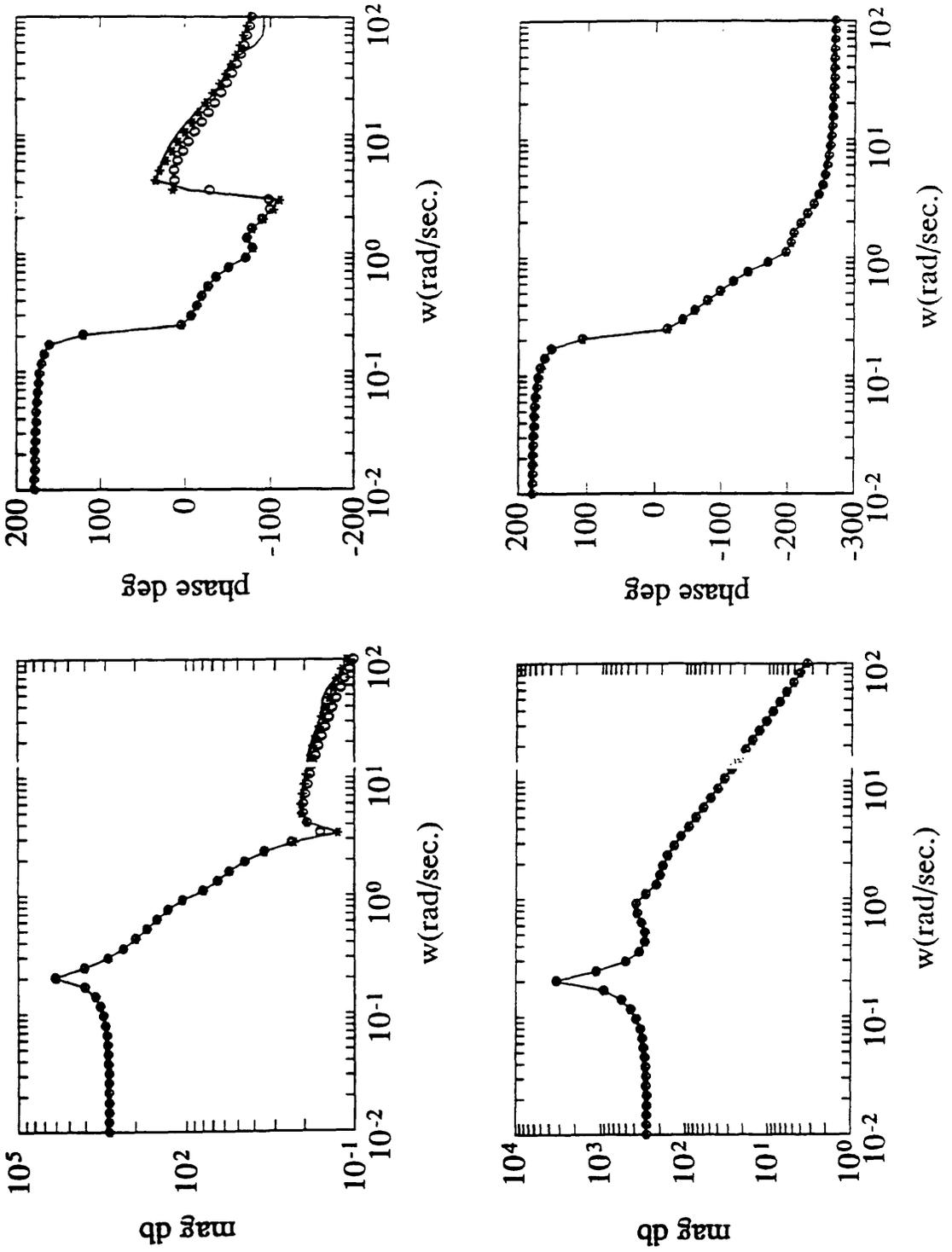
- 143 -

Fig. 8.2 Frequency Responses
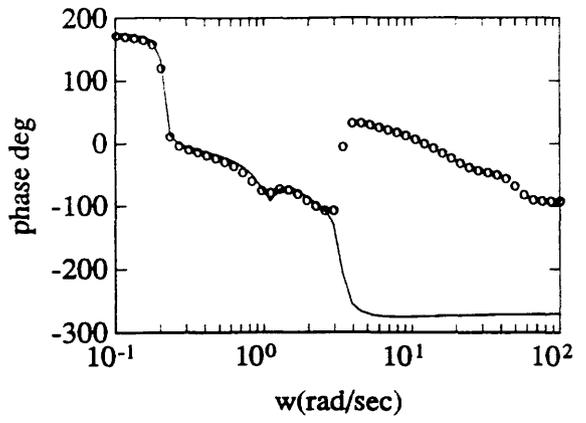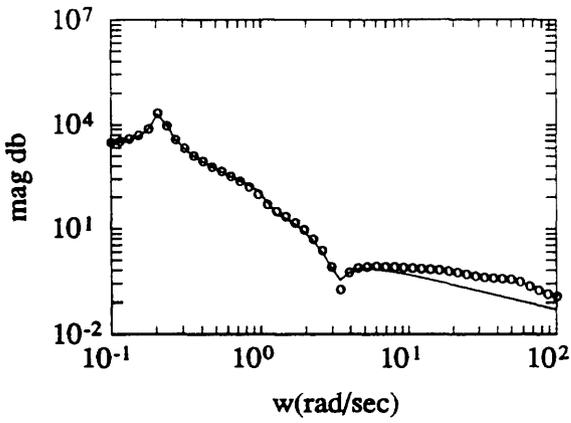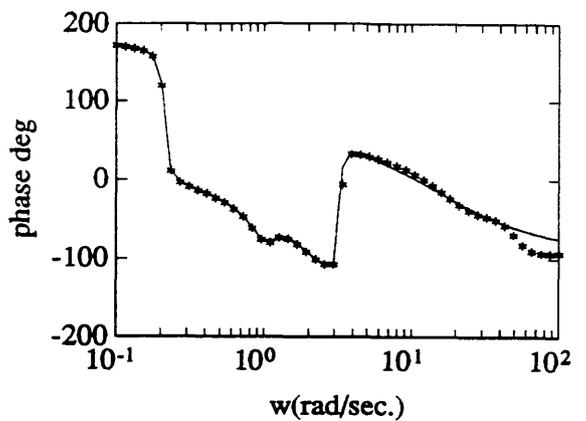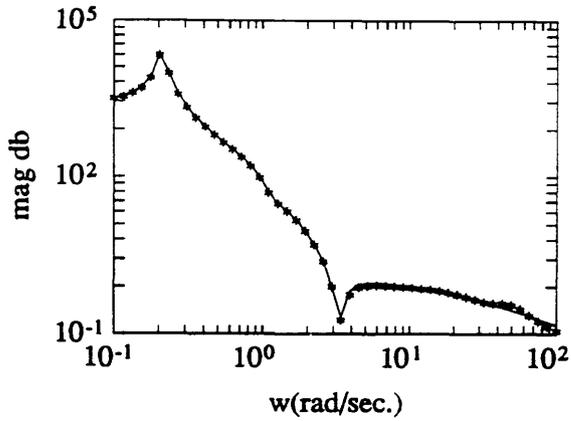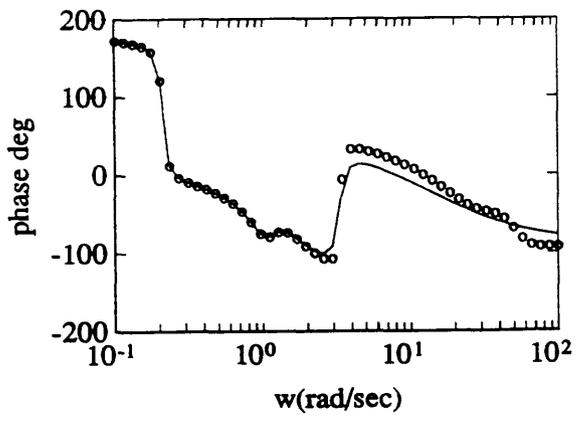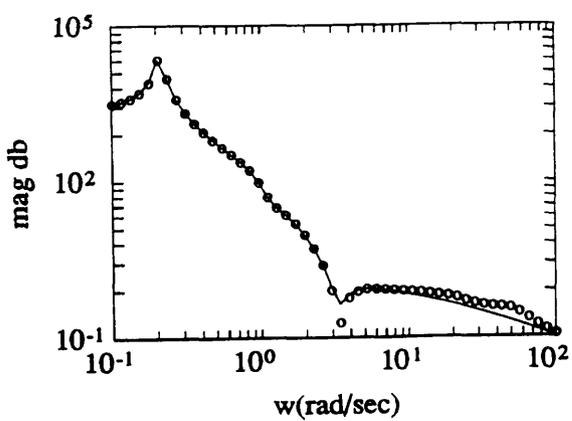
Fig. 8.3 Frequency Responses for Three Models (Puma80)
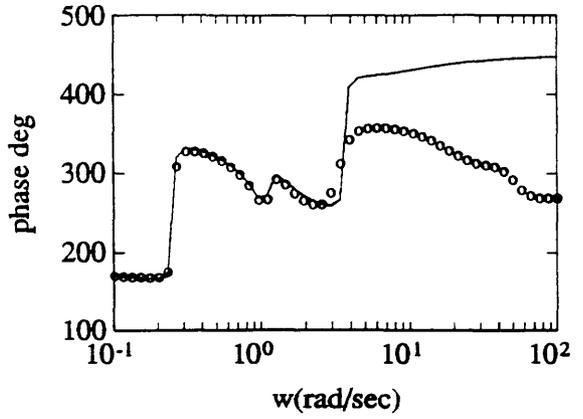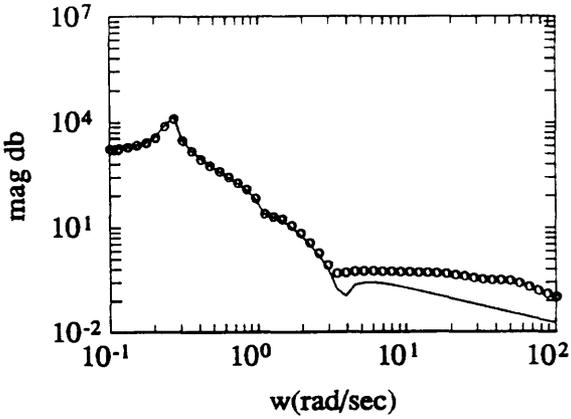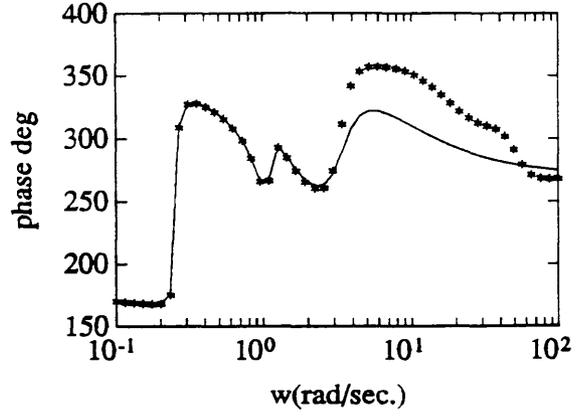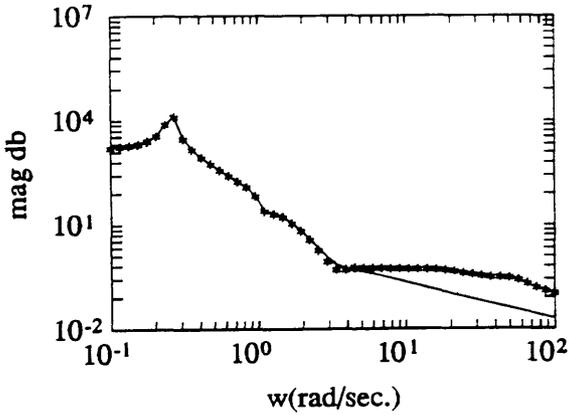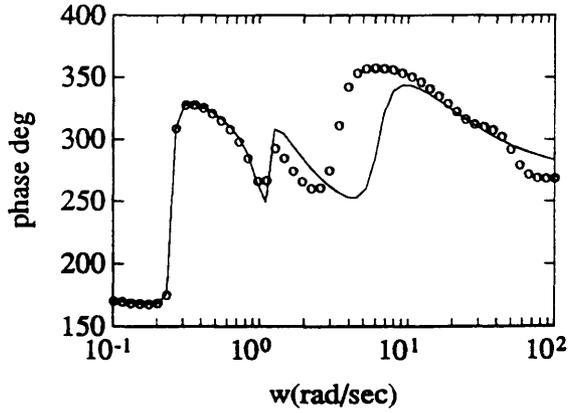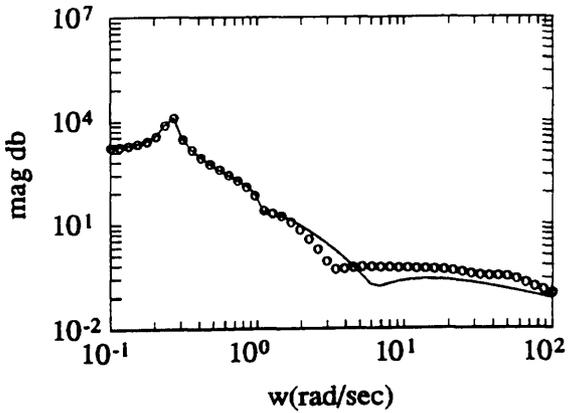
Fig. 8.4 Frequency Responses for Three Models (Puma60)
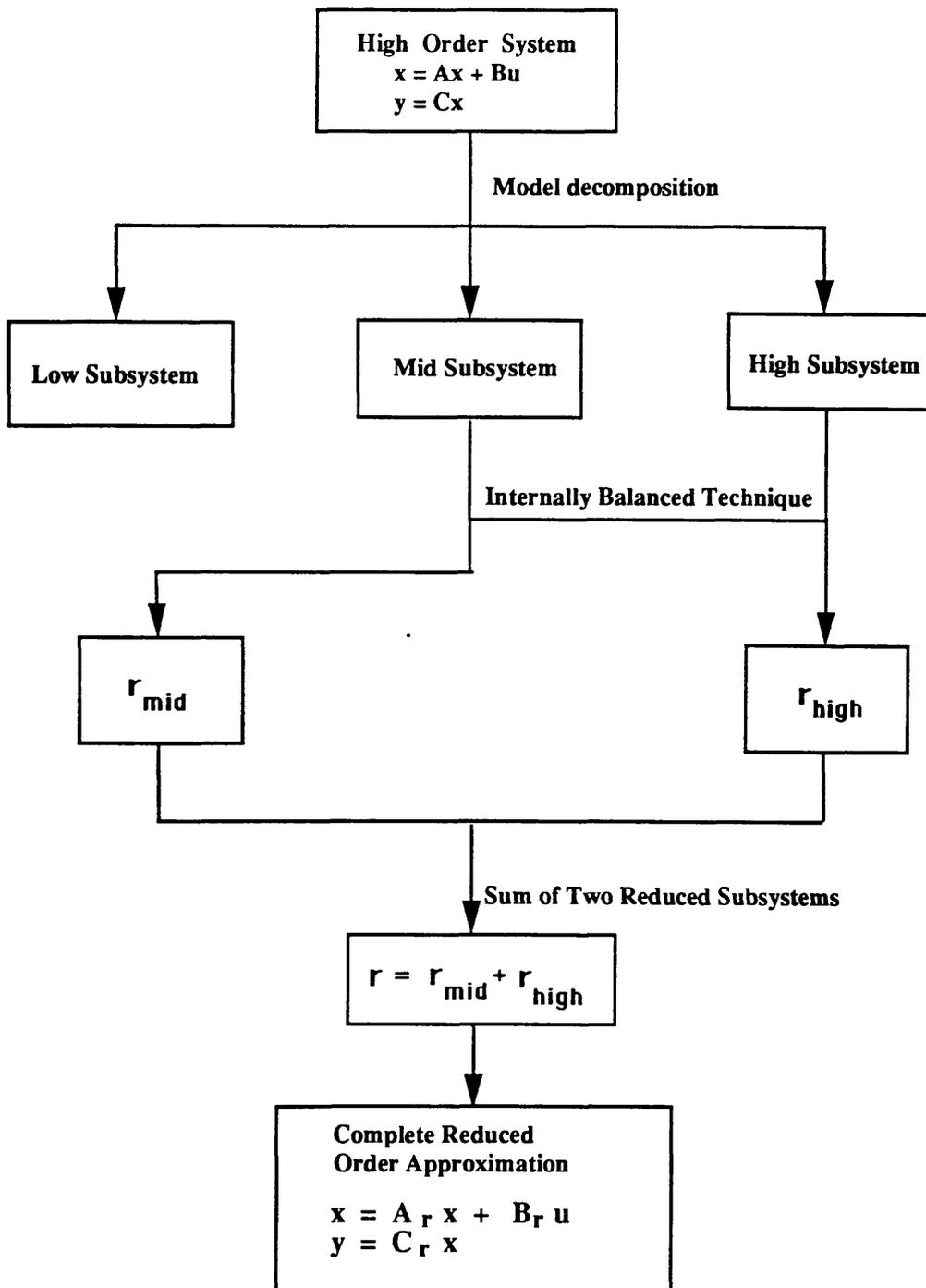
Fig. 8.5 Flow chart for Bacon and Schmidt's method

High-order model

$\dot{x} = Ax + Bu$
$y = Cx$

frequency responce form &
separated into real & imaginary
parts

$H(j\omega) = R(\omega) + jI(\omega)$

assuming the order of the
reduced model, r

$$G(s) = \frac{b_0 + \sum\limits_{i=1}^{q} b_i s^i}{1 + \sum\limits_{i=1}^{r} a_i s^i}$$

using a modified
least-square method

using a modified
least-square method

$$\frac{\partial E}{\partial a_i} = 0 \quad , \quad \frac{\partial E}{\partial b_i} = 0$$

calculating the matrix P and
vector Y of the resulted linear
equations

$Px = Y$

solving the equations

a_i coefficients of Denominator
b_i coefficients of Numerator

obtain the transfer function of
the reduced order model

$$G_k(j\omega) = \frac{b_0 + \sum\limits_{i=1}^{q} b_i (j\omega)^i}{1 + \sum\limits_{i=1}^{r} a_i (j\omega)^i} = \frac{N_k}{\Delta}$$

repeat above
steps

the transfer function of
SIMO

$$G_n(s) = \frac{N_n(s)}{\Delta}$$

Fig. 8.6 Flow chart for Levy's method

Chen's Method

$$\boxed{\begin{array}{c} \text{the transfer function of} \\ \text{SIMO} \\[2mm] G_n\,(s) \;=\; \dfrac{N_n(s)}{\Delta} \end{array}}$$

$$N \;=\; \begin{bmatrix} b_{11} & b_{12}\dots\dots b_{1n} \\ b_{21} & b_{22}\dots\dots b_{2n} \\ & \cdot \\ & \cdot \\ & \cdot \\ b_{n1} & b_{n2}\dots\dots b_{nn} \end{bmatrix}$$

$$F_c \;=\; \begin{bmatrix} -a_1 & -a_2 & \dots & -a_n \\ \hline \multicolumn{2}{c}{I_{n-1}} & \Big| & 0 \end{bmatrix}$$

$$F \;=\; N\,F_c\,N^{-1}$$

$$G_1 \;=\; \begin{bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ \cdot \\ b_{n1} \end{bmatrix}$$

Reduced order model

$$\dot{x} \;=\; Fx + G_1 u$$
$$y \;=\; x$$

Fig. 8.7 Flow Chart for Chen's Method
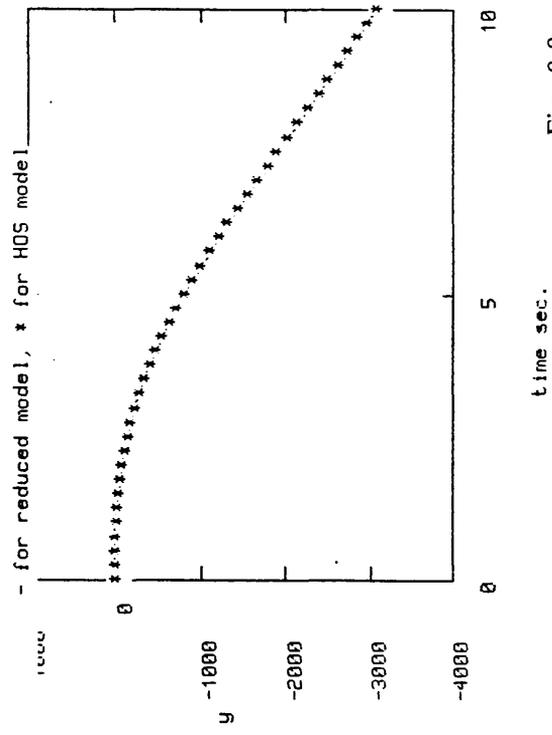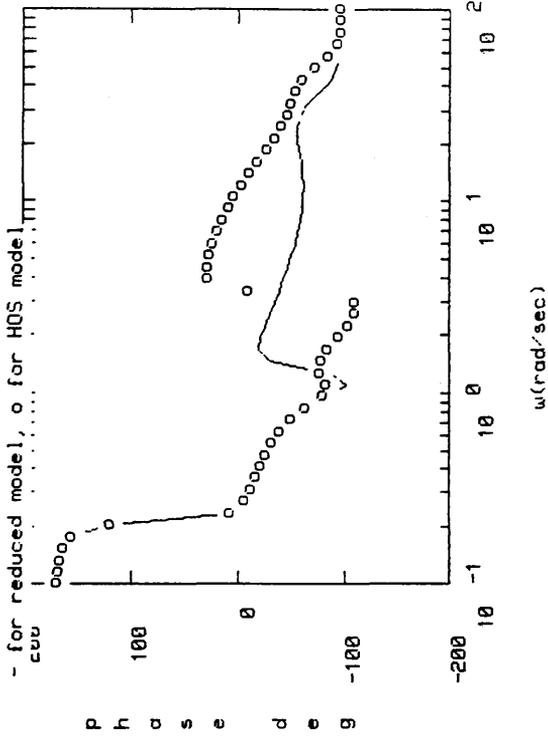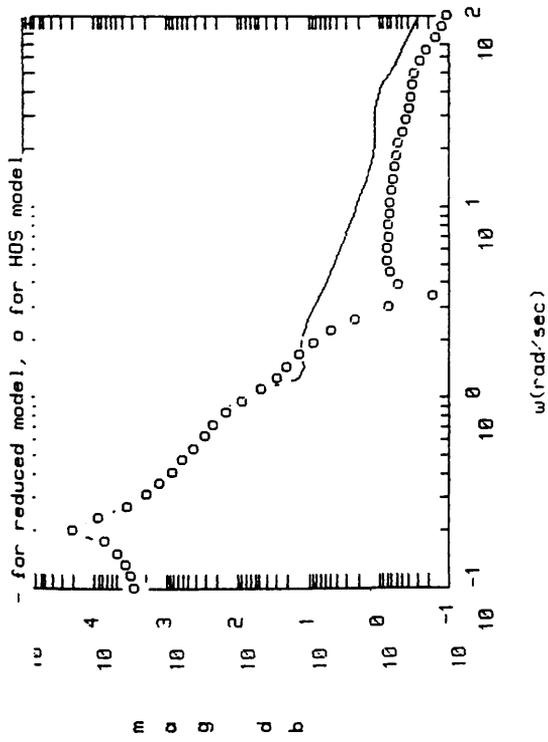
Fig. 8.8   Time & Frequency Response from Choice

Radii d1 = 0.001 & d2 = 15 for Puma80
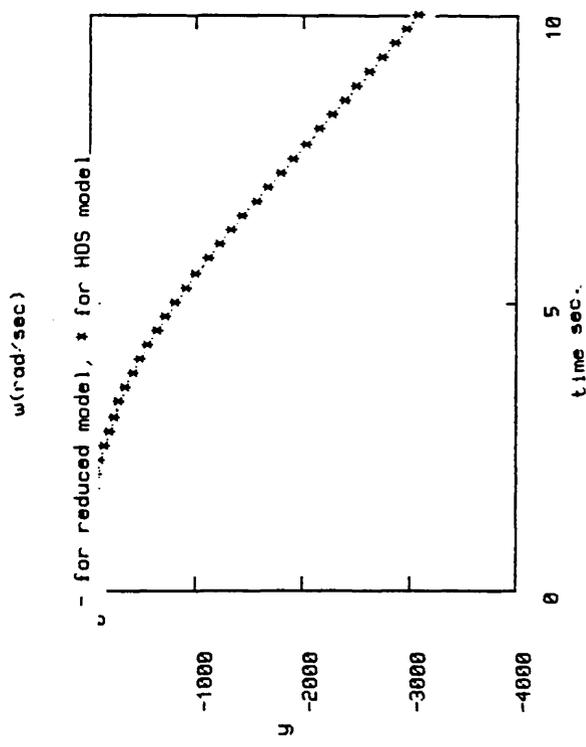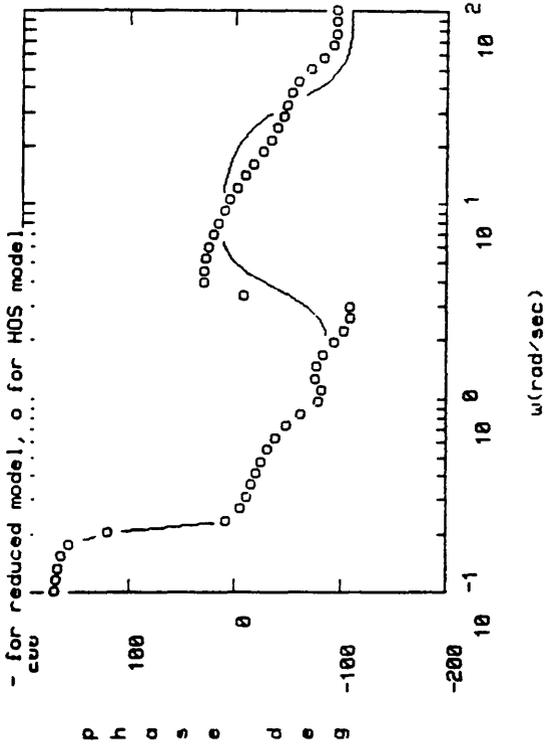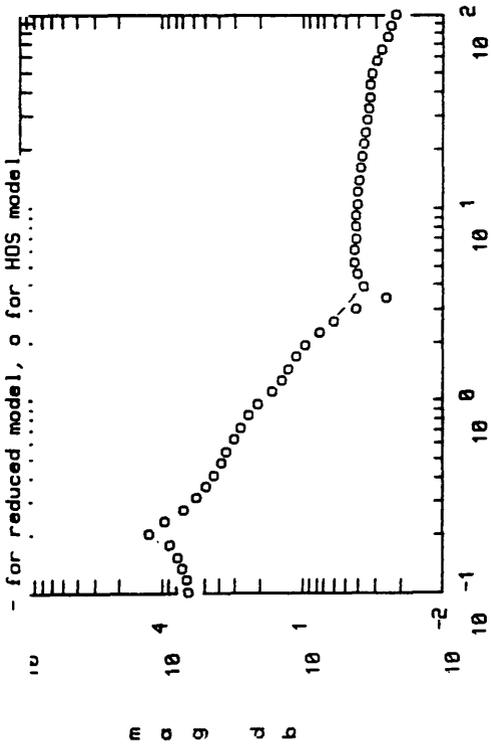
Fig. 8.9   Time & Frequency Response from Choice

Radii d1=0.22 & d2=15 for Puma60

- ---- Bacon, * ---- HOS
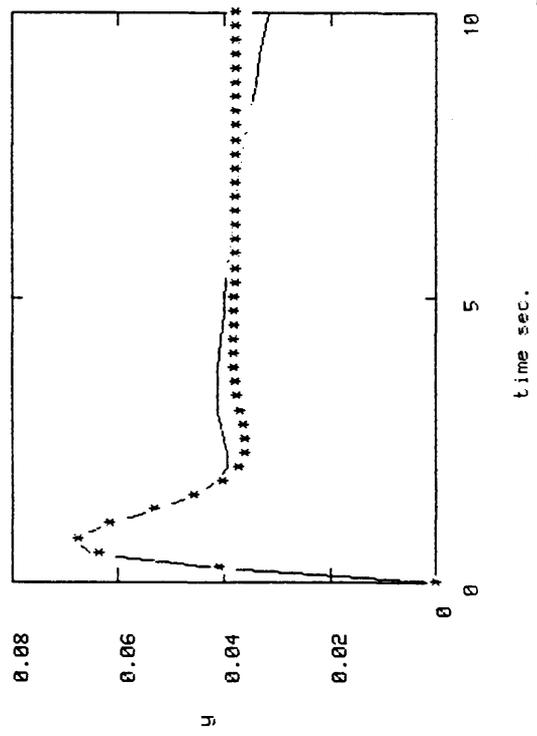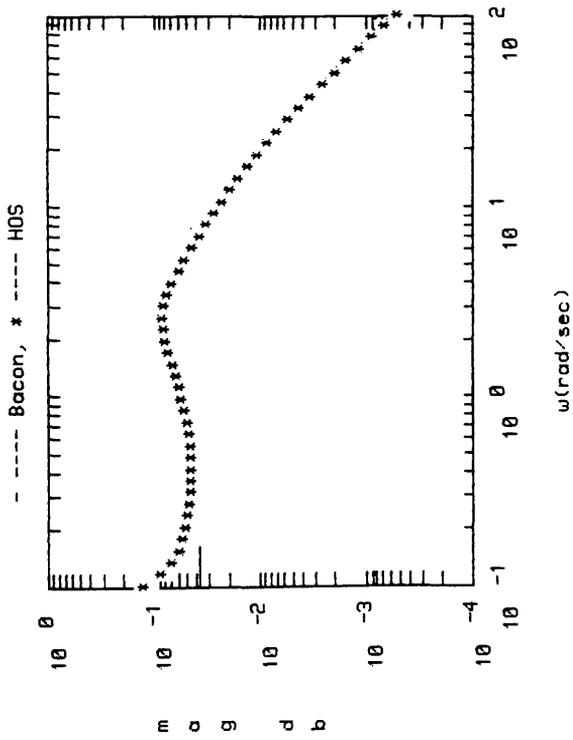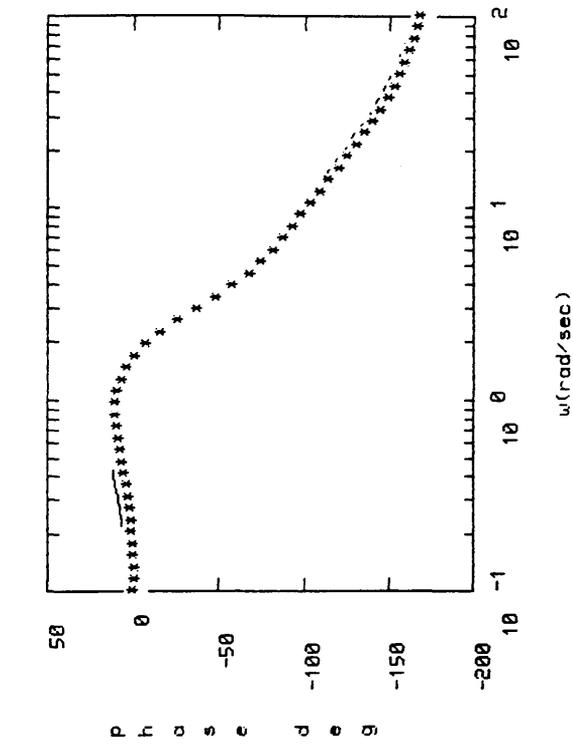
Fig. 8.10   Time & Frequency Response from different
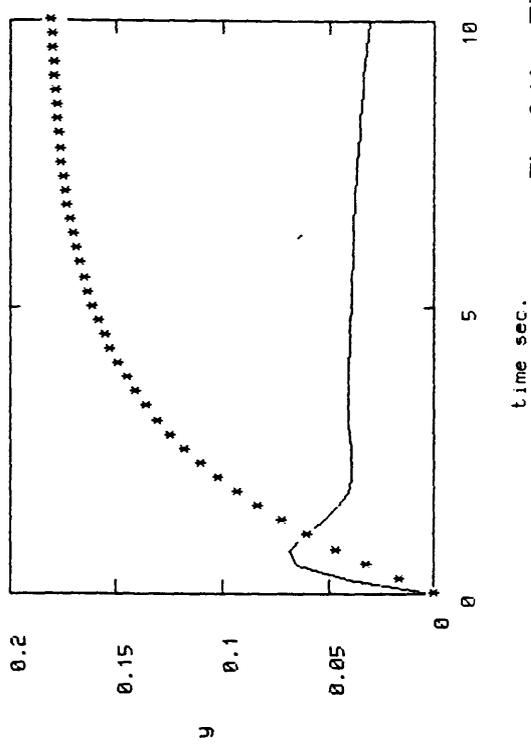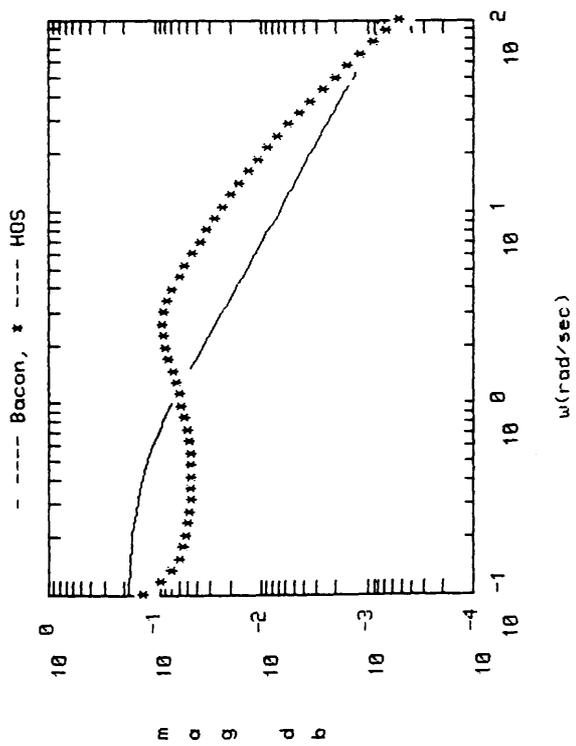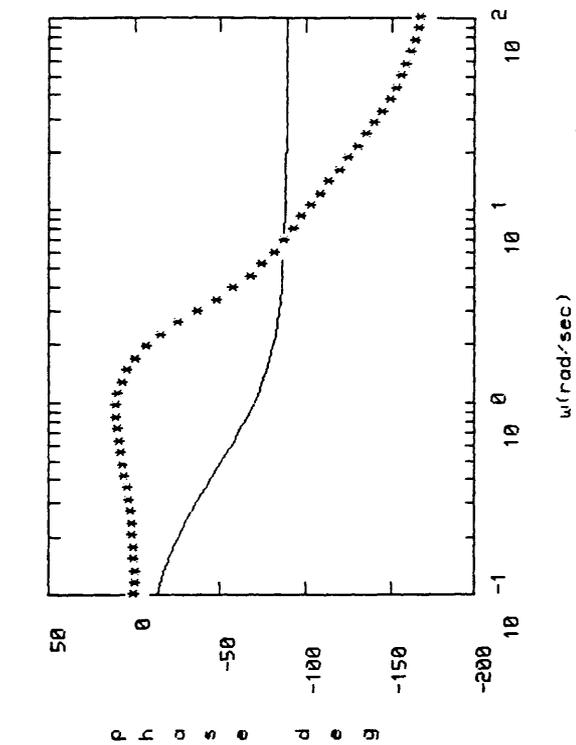
Fig. 8.11 Time & Frequency Response from different
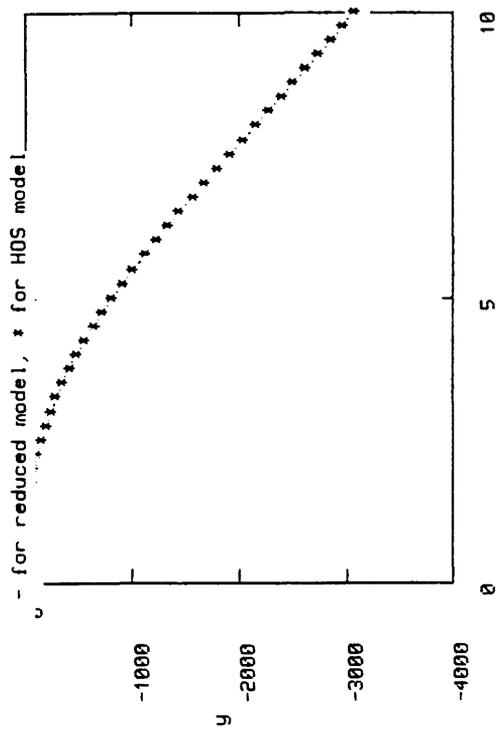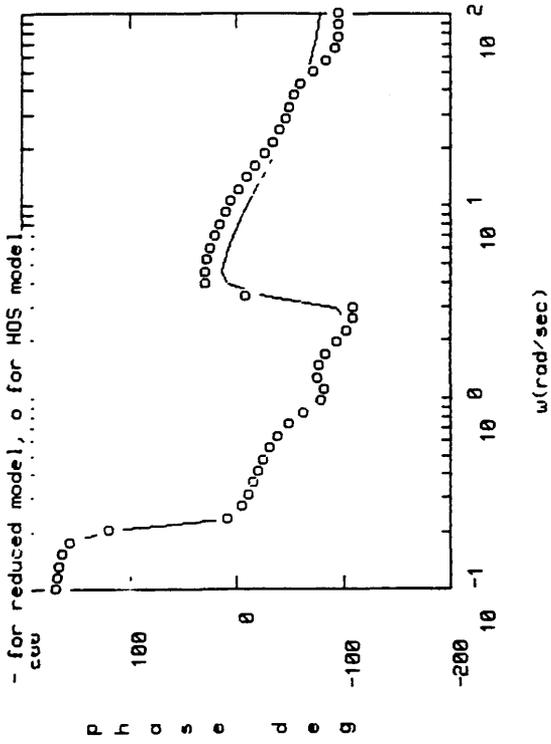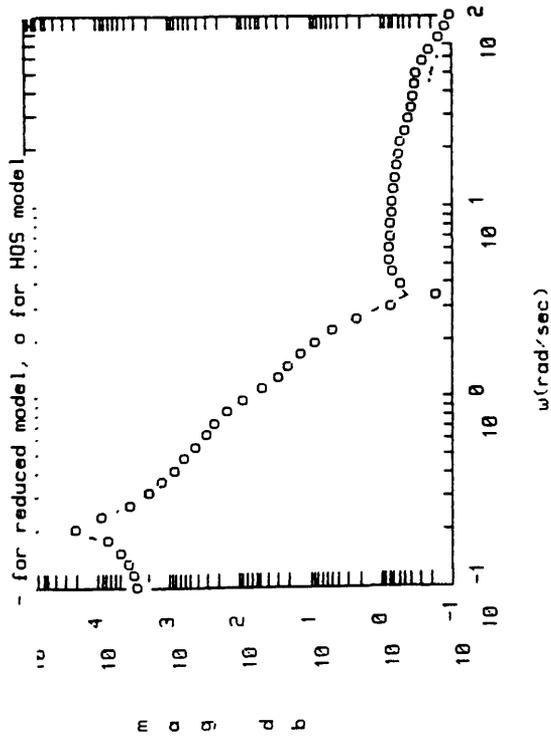
Fig. 8.12 The Effect of Different Effective
Order of the Subsystems for Puma80
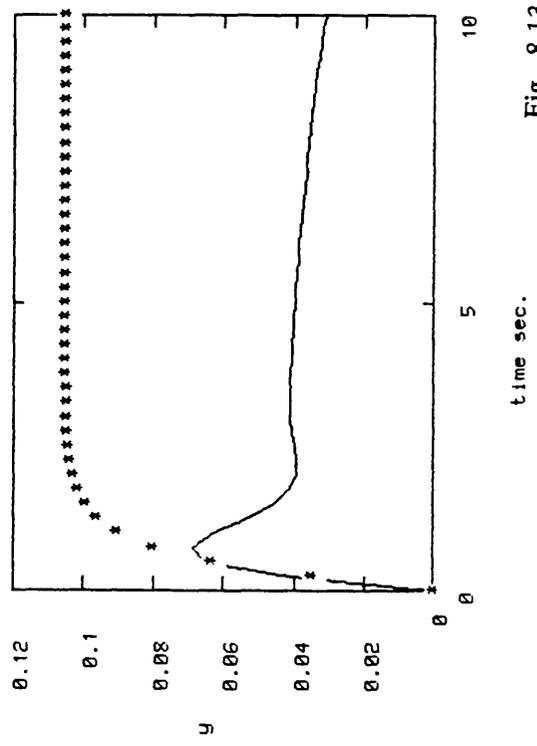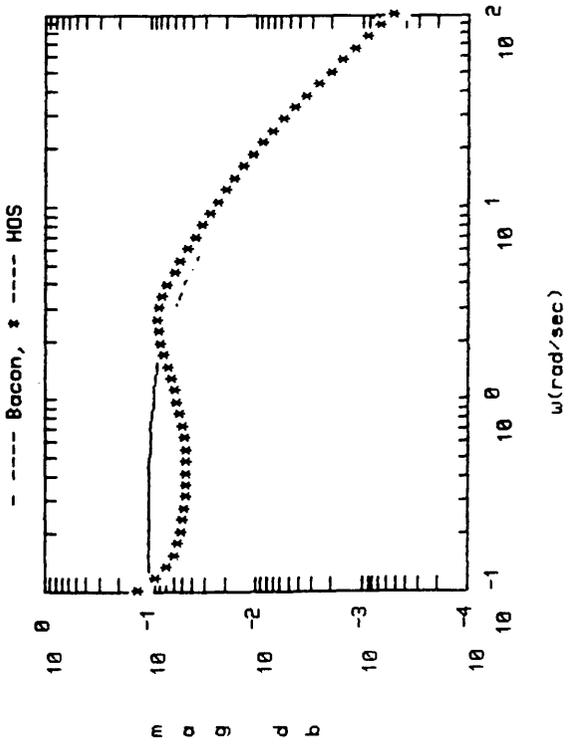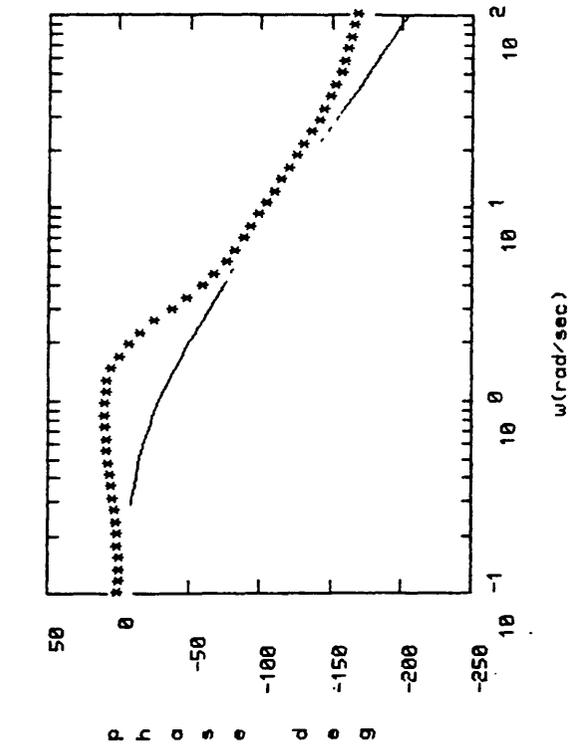
Fig. 8.13 The Effect of Different Effective Order
of the Subsystems for an Advanced Flighter
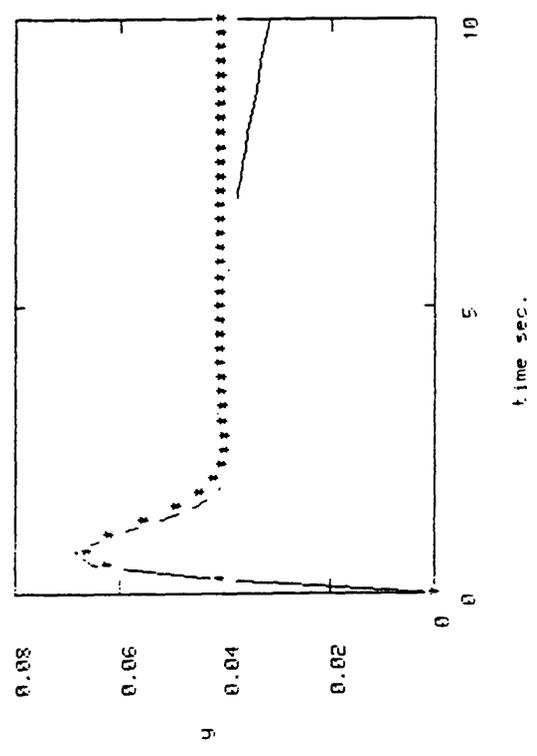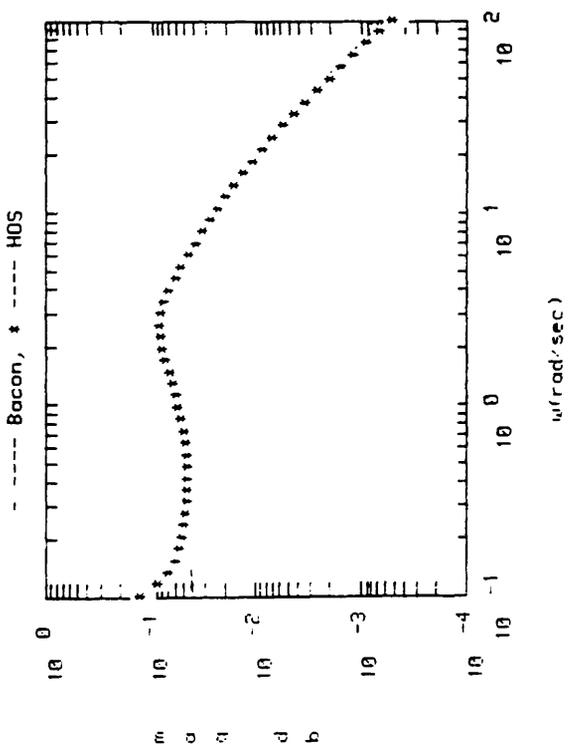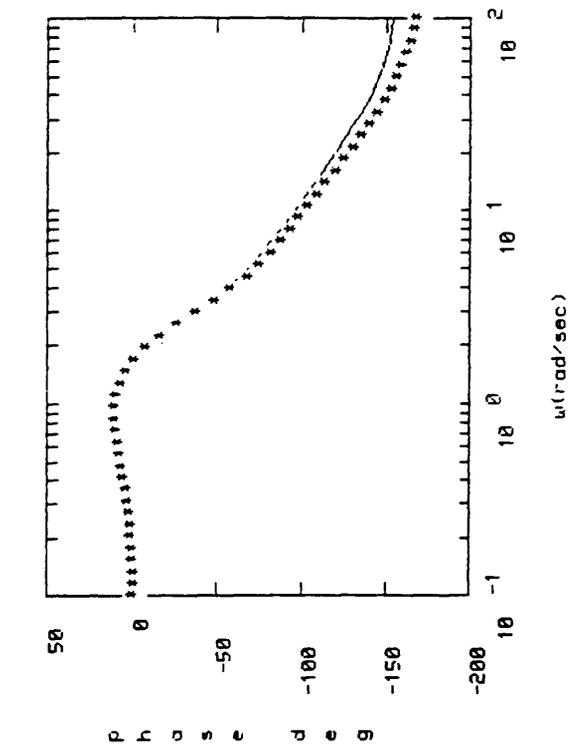
Fig. 8.14 The Resulting without Scaling for an Advanced Flighter
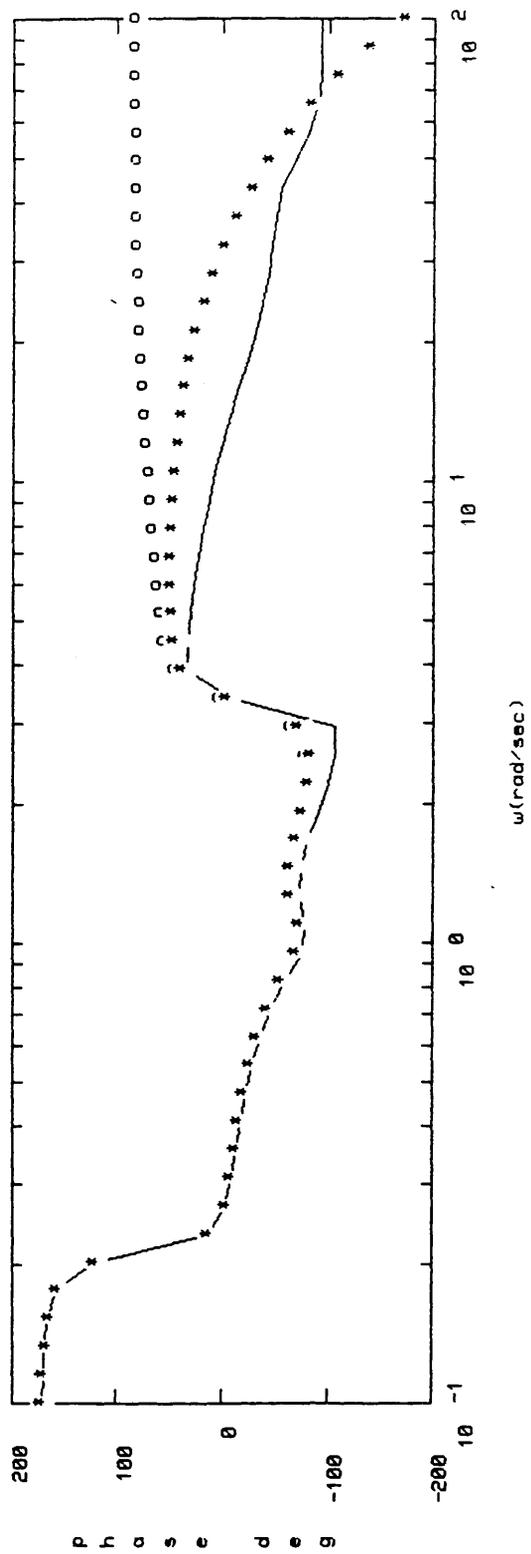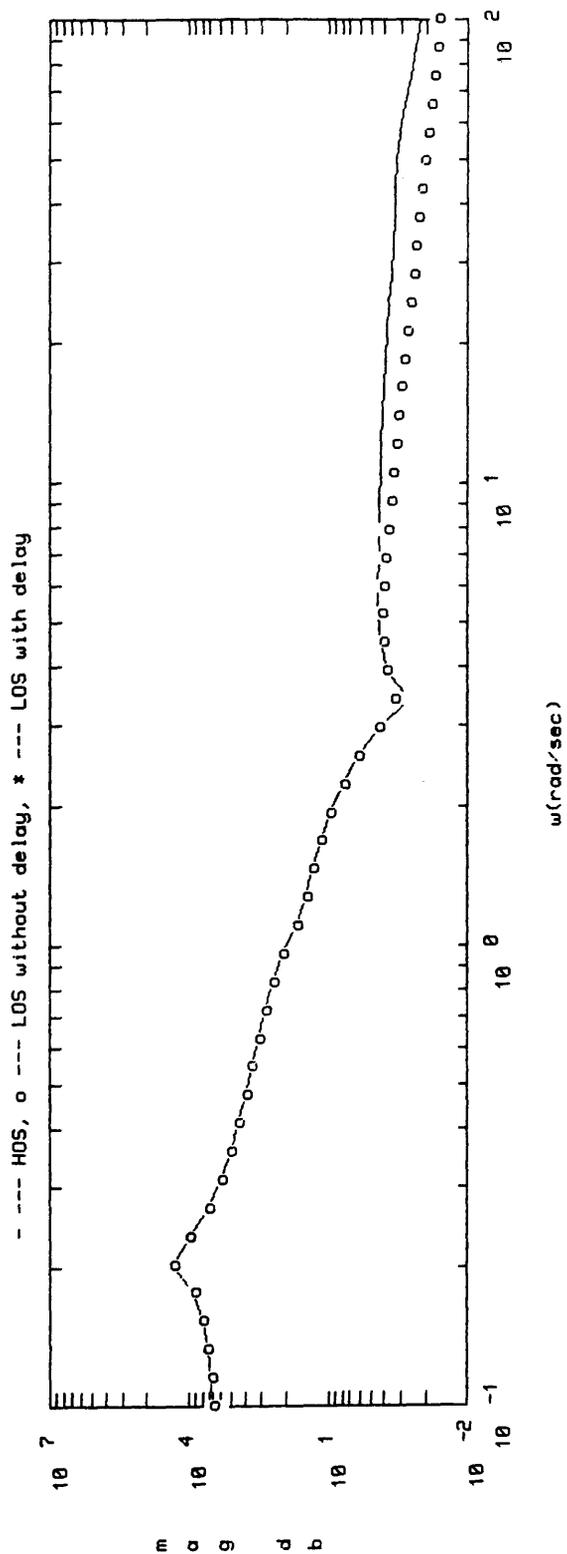
Fig. 8.15   The Equivalent Delay for Puma80
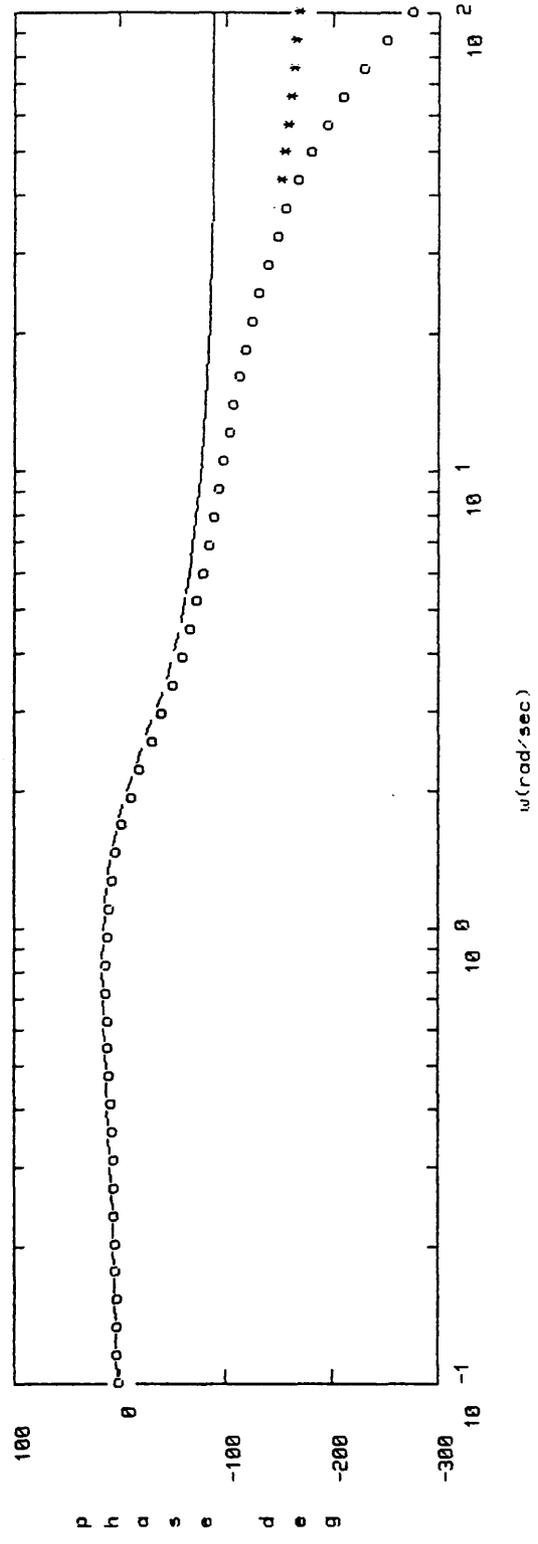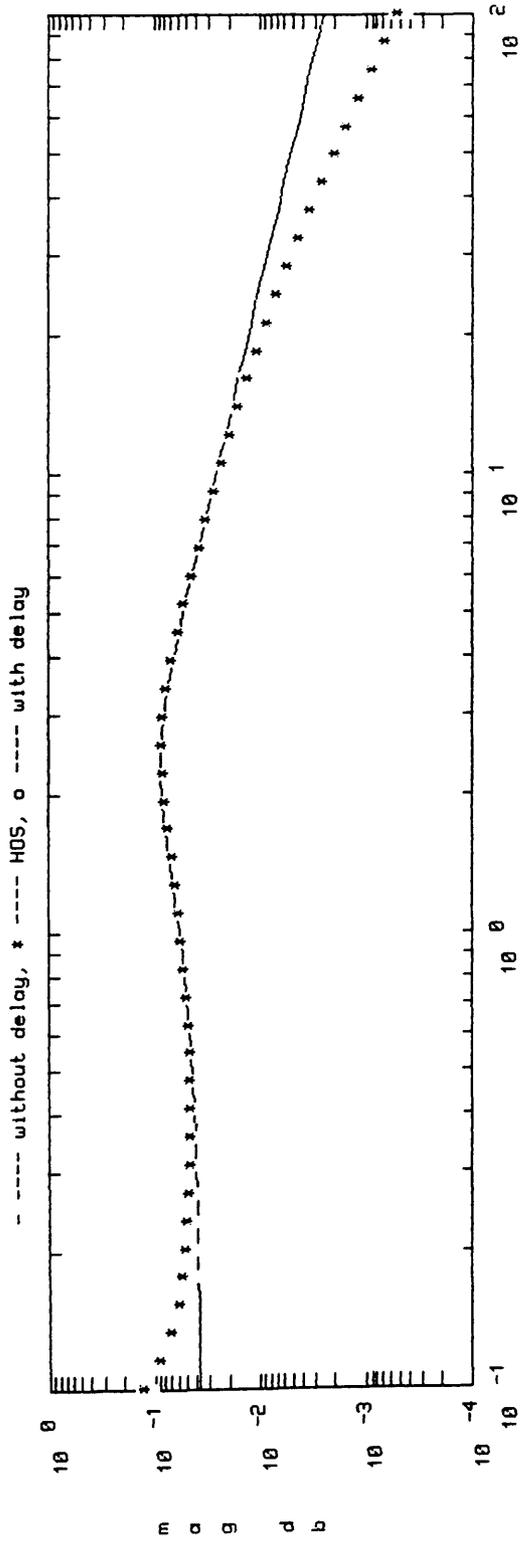
Fig. 8.16 The Equivalent Delay for an Advanced Flighter

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APPENDIX 1

Implementation of the Extended Levy's SIMO Technique in MATLAB.

A1.1 The Outline of the MATLAB Program for Modified Levy's SIMO Technique.

The MATLAB package has been discribed in Chapter 6.

This section is concerned with describing the procedure for implementation of modified Levy's SIMO method in the MATLAB. The program was written using MATLAB for DEC Micro VAX 3600 computer (VMS2) at Glasgow University Computer Centre. The MATLAB program can be described in terms of the following three steps.

A1.1.1 Generation of the Original Data Files.

Because the original model given is of high—order and in state—space form, the frequency response of system can be found using the "Nyquist" function in MATLAB. The numerator and denominator order expected or required for the lower order system must be specified by the user.

A1.1.2 Calculation of the Matrix P and Matrix y.

At first we calculate the values of equations (4.11) for each output u_j, and determine the elements the matrix P and the matrix y. In order to to obtain better results we may increase the number of iteration i and modify the weights $D_{2n_1}(\omega_k)$.

A1.1.3 To Get the Coefficients of the Numerator a_i & Denominator b_{ij} of the Transfer Response.

From section A1.1.1 and A1.1.2 we have obtained the P matrix and y matrix of each output. The numerical value of the unknown coefficients a_i and b_{ij} may thus be obtained from equ.(4.12) once the matrices P, X and Y have been evaluated. Here a and b can be obtained using the "inverse" function of MATLAB.

A1.2 Implementation of Chen's Method in the MATLAB.

From section 4.3 we have obtained the results for the SIMO case in the form of a transfer function only.

A1.2.1 <u>Orgnization of the Matrix N and Matrix Fc</u>.

We can use the MATLAB package to implement Chen's method. According to equ.(4.13) and (4.14) we very easily obtain the matrices N and Fc from the section 4.3.

A1.2.2 <u>Calculation of the Matrix F and G_1</u>.

From Chen's method we know that it is necessary to calculate matrix F and G_1 based on equ.(4.14). Using the inverse function of the MATLAB we can obtain N^{-1} easily. Then from equ.(4.14) we could obtain the F matrix and the G_1 matrix. Thus Chen's method is implemented and we obtain the reduced order system in state space form.

Mathematics Associated with the use of Hankel Singular Values

A2.1 The Definition of the Controllability Grammian and the Observability Grammian[82].

In the $n-$ dimensional linear time$-$invariant state equation

$$\left.\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array}\right\} \qquad \text{(A2-1)}$$

where A, B and C are $n{\times}n$, $n{\times}p$ and $q{\times}n$ constant matrices; time interval of interest is from the present time to infinity; that is $[0,\infty)$. The controllability grammian W_c^2 is defined by

$$W_c^2 = \int_0^\infty e^{\tau A} BB' e^{\tau A'} d\tau \qquad \text{(A2-2)}$$

The observability grammian W_o^2 is defined by

$$W_o^2 = \int_0^\infty e^{\tau A} C' C e^{\tau A'} d\tau \qquad \text{(A2-3)}$$

A2.2 Irreducible Realization[82]

Consider the following scalar proper transfer function:

$$g(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} = \frac{N(s)}{D(s)} \qquad \text{(A2-4)}$$

Let u and y be the input and output of g(s) in (A2$-$4). Then we have

$$D(s)y(s) = N(s)u(s) \qquad \text{(A2-5)}$$

or, in the time domain,

$$D(p)y(t) = N(p)u(t)$$

Consider the nth$-$order differential equation (A2$-$5). Taking the Laplace transform of (A2$-$5) and regrouping the terms associated with the same power of s,

we finally obtain:

$$y(s) = [N(s)/D(s)]u(s) + [1/D(s)]\{y(0)s^{n-1}+[y^{(1)}(0)+a_1y(0)-b_1u(0)]$$

$$s^{n-2}+ \cdots +[y^{(n-1)}(0)+a_1y^{(n-2)}(0)-b_1u^{(n-2)}(0)+a_2y^{(n-3)}(0)$$

$$-b_2u^{(n-3)}(0)+\cdots+a_{n-1}y(0)-b_{n-1}u(0)]\} \qquad (A2-6)$$

The right—hand side of (A2—6) gives the response due to the input u(s), therefore, if all the coefficients associated with s^{n-1}, s^{n-2}, $\cdots$ s^0 in (A2—6) are known, then for any u a unique y can be determined.

The foregoing equations can be arranged in matrix form as the observable canonical form[82]. Similaly, the controllable canonical form[82] can be obtain by same way.

A2.3 Realization of the Hankel Matrix.

Consider the proper rational function (A2—4) again. We expand it into an infinite power series of descending power of s as

$$g(s) = h(0) + h(1)s^{-1} + h(2)s^{-2} + \cdots$$

the parameters, h(i), i = 0, 1, 2, $\cdots$, can be obtained recusively from a_i and b_i as

$$h(0) = b_0$$

$$h(1) = -a_1h(0) +b_1$$

$$\cdots \cdots$$

$$h(n) = -a_1h(n-1) -a_2h(n-2) - \ldots -a_nh(0) +b_n$$

$$h(n+i) = -a_1h(n+i-1) -a_2h(n+i-2) - \ldots -a_nh(i) \qquad i = 1, 2, 3, \cdots$$

We form the $\alpha\times\beta$ matrix:

$$H(\alpha, \beta) = \begin{bmatrix} h(1) & h(2) & h(3) & \ldots & h(\beta) \\ h(2) & h(3) & h(4) & \ldots & h(\beta+1) \\ \cdots \cdots \\ h(\alpha) & h(\alpha+1) & h(\alpha+2) & \ldots & h(\alpha+\beta-1) \end{bmatrix} \qquad (A2-7)$$

It is called a Hankel matrix of order ($\alpha\times\beta$). Usually, we consider α = n+1, β = n.

A2.4 The Application of the Hankel Matrix in Obtaining Irreducible Realizations.

Now we use the Hankel matrix to obtain the irreducible realization. Consider the Hankel Matrix (A2-8) and apply the row-searching algorithm[83] to search the linearly independent rows of $H(n+1,n)$ in (A2-7) in order from top to bottom. We can readily show that the rank of $H(n+1,n)$ is σ. Hence an irreducible realization of $g(s)$ has dimension σ. The row searching algorithm will also yield $\{a_i, i=1,2,...,\}$ such that

$$[a_1 \ a_2 \ \cdots \ a_\sigma \ 1 \ 0 \ \cdots \ 0] \ H(n+1,n) = 0 \tag{A2-8}$$

This equation expresses the primary linearly dependent row as a unique linear combination of its previous rows. The element 1 corresponds to the primary dependent row. We claim that the σ-dimension dynamical equation

$$\dot{x} = Ax + Bu \qquad y = Cx \tag{A2-9}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & 0 & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{\sigma-1} & -a_\sigma \end{bmatrix} \qquad B = \begin{bmatrix} h(1) \\ h(2) \\ \cdots \\ h(\sigma-1) \\ h(\sigma) \end{bmatrix}$$

$$C = [\ 1 \ 0 \ 0 \ \cdots \ 0 \ 0 \ ]$$

is a controllable and observable realization of $g(s)$.

The controllability matrix of (A2-9) is

$$[\ B \ \ AB \ \ \cdots \ \ A^{\sigma-1}B \ ] = H(\sigma,\sigma)$$

the Hankel matrix $H(\sigma,\sigma)$ has rank σ, hence $\{A,B\}$ in (A2-9) is **controllable**. The observability matrix of (A2-9) is

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \cdots \\ CA^{\sigma-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Clearly {A,C} is observable.   Hence (A2–9) is an irreducible realization of g(s).

A2.5 Internally Balanced Realization.

A realization whose controllability and observability matrices have the property

$$V^*V = UU^*$$

is called an internally balanced realization.   Here, V is the nq×n observability matrix,

U is the n×(np) controllability matrix and $U^* = U^{-1}$, $V^* = V^{-1}$.

A2.6 The Calculation of the Hankel Matrix and Hankel Singular Values.

If A is a real m × n matrix, then there exist orthogonal matrices U of size m ×

m and V of size n × n such that

$$A = UYV'$$

where Y is an m × n diagonal matrix.   The matrix Y satisfies

$$Y = diag (y_1, y_2, ..., y_p)$$

for p = min {m,n} and $y_1 \geqslant y_2 \geqslant .... \geqslant y_p \geqslant 0$

The proof of this involves the eigenvalues of the symmetric matrix A'A which can

be shown to be non–negative.   The quantities y_i are the square roots of these

eigenvalues and are called the singular values of A.   The columns of U are called

the left singular vectors and the columns of V are called the right singular vectors.

A quadratic form $\Sigma_{i,k} a_{ik} x_i x_k$ is said to be a Hankel form, if the general

coefficient depends only on the sum of the indices : $a_{ik} = f(i+k)$ (i,k = 1, ... ,

n); likewise, the coefficient matrix of a Hankel form is called a Hankel matrix[76] as

well.   It is written in the form as below:

$$S_h = \begin{bmatrix} s_0 & s_1 & ..... & s_{n-1} \\ s_1 & s_2 & ...... & s_n \\ .... \\ s_{n-1} & s_n & ..... & s_{2n-2} \end{bmatrix}$$

This provides the necessary background to consider the application of principal

component analysis to responses of the model

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$\underline{y} = C\underline{x}$$

In analyzing responses of (A,B,C) over an interval [0,T], it is necessary to be able to extended controllability and observability matrices Q_c and Q_o corresponding to (A,B,C) for time t_s such that

$$Q_c(t_s) = [B \ AB \ \ldots . A^N B]$$

$$Q_o(t_s) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdot \\ \cdot \\ \cdot \\ CA^N \end{bmatrix} \tag{A2-10}$$

By the definition of the Hankel matrix, we have the extended controllability and observability matrices Q_c and Q_o defined by Equ.(A2—10). With T fixed, the corresponding Hankel matrix

$$M_H = Q_o \ Q_c$$

$$= \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^N \end{bmatrix} [\ B \ AB \ \ldots . A^N B]$$

$$= \begin{bmatrix} CB & CAB & CA^2B & \ldots . . & CA^N B \\ CAB & CA^2B & \ldots \ldots \ldots . & CA^{N+1}B \\ CA^2B & CA^3B & \ldots \ldots \ldots \ldots . & \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . & \\ CA^N B. & \ldots \ldots \ldots \ldots . & CA^{2N}B \end{bmatrix} \tag{A2-11}$$

It is clear that the matrix M_H corresponds to the Hankel form and M_H is called a Hankel matrix.

The algorithm (SVD) developed by Golub and Reinsch[75] can be used to compute the singular values of X and S, (i.e. W_c^2 and W_o^2). The singular values of X and S are Σ_c^2 and Σ_o^2 separately and their corresponding left singular vectors are V_c, V_o and right singular vectors are V_c^T, V_o^T[78]

Then it follows that

$$W_c^2 = V_c \Sigma_c^2 V_c^T \tag{A2-12}$$

$$W_o^2 = V_o \Sigma_o^2 V_o^T \tag{A2-13}$$

Because the controllable subspace $e^{At}B$ and the observable subspace $e^{A't}C'$ depend upon the internal coordinate system, we make a coordinate transformation $x(t) = Pz(t)$. The system model (A2-9) becomes

$$\underline{\dot{z}} = A_z\underline{z}(t) + B_z\underline{u} \qquad (A2-14)$$

$$\underline{y_z} = C_z\underline{z}(t)$$

where $A_z = P^{-1}AP$, $\qquad B_z = P^{-1}B$, $\qquad C_z = CP$.

It is important to observe that

$$e^{A_z t}B_z = P^{-1}e^{At}B, \qquad C_z e^{A_z t} = Ce^{At}P.$$

The following notation will be adopted:

$$W_c{}^2(P) = P^{-1}\left(\int_0^\infty e^{tA}BB'e^{tA'}dt\right)P^{-T} = \int_0^\infty e^{A_z t}B_z B_z'e^{A_z' t}dt$$

$$W_o{}^2(P) = P^T\left(\int_0^\infty e^{A't}C'Ce^{At}dt\right)P = \int_0^\infty e^{A_z' t}C_z'C_z e^{A_z t}dt$$

For the case where $P = I$ (original coordinate system) and

from $\quad W_c{}^2 = V_c\Sigma_c{}^2V_c{}^T \qquad$ we have

$$W_c{}^2(P) = P^{-1}W_c{}^2P^{-T}$$

$$= P^{-1}V_c\Sigma_c{}^2V_c{}^TP^{-T}$$

Since the Σ_c matrix is diagonal it follows that

$$W_c{}^2(P) = P^{-1}V_c\Sigma_c\Sigma_c{}^TV_c{}^TP^{-T}$$

Intuitively, by selecting P so that $W_c{}^2(P) = I$

Since $W_c{}^2 = V_c\Sigma_c{}^2V_c{}^{-T}$, it is clear that this can be achieved by

sitting $P = V_c\Sigma_c$. Also we have

$$W_o{}^2(P) = P^TW_o{}^2P$$

$$= (V_c\Sigma_c)^T V_o\Sigma_o{}^2V_o{}^T V_c\Sigma_c$$

$$= \Sigma_c{}^TV_c{}^TV_o\Sigma_o{}^T \Sigma_oV_o{}^TV_c\Sigma_c$$

$$= H^TH. \qquad (A2-15)$$

where

$$H = \Sigma_o V_o^T V_c \Sigma_c \qquad \text{(A2-16)}$$

$$H^T = \Sigma_c^T V_c^T V_o \Sigma_o^T \qquad \text{(A2-17)}$$

The singular values of matrix H will be represented by $\sigma_1^2 \geqslant \sigma_2^2 \geqslant ... \geqslant \sigma_n^2 \geqslant 0$ and will be referred to as second−order models of the system.

From Equ. (A2−10) we can see:

$$M_H = Q_o(t_s) \; Q_c(t_s)$$

$$= U_o^*(t_s)\Sigma_o^{*T}(t_s)V_o^{*T}(t_s)V_c^*(t_s)\Sigma_c^*(t_s)U_c^{*T}(t_s)$$

$$= U_o^*(t_s)\Sigma_o^T V_o^T V_c \Sigma_c U_c^{*T}(t_s)$$

$$= U_o^*(t_s)(\Sigma_o V_o^T V_c \Sigma_c)U_c^{*T}(t_s)$$

Let $\sigma_i^*(t_s)$, $1 \leqslant i \leqslant n$ be the ordered singular values of $M_H(t_s)$. Then for $1 \leqslant i \leqslant n$,

$$\lim_{t_s \to 0} \sigma_i^*(t_s) = \sigma_i^2$$

where σ_i^2 is the singular value of the ith second−order mode of the H matrix.

A2.7 A Proof of $XS = H^T H$

From Equs. (5.11 − 5.14) we have

$$X = W_c^2 = V_c\Sigma_c^2 V_c^T \qquad \text{and}$$

$$S = W_o^2 = V_o\Sigma_o^2 V_o^T$$

Thus $XS = V_c\Sigma_c^2 V_c^T V_o\Sigma_o^2 V_o^T$

We know the fact that the cotrollable subspace $e^{At}B$ and the observable subspace $e^{A't}C'$ depend upon the internal coordinate system. From section A2.6 mationed given a coordinate transformation $z(t) = P \; x(t)$ and selecting $P = V_c\Sigma_c$ we have $\Sigma_c^2(P) = I$. Since $W_c^2 = V_c\Sigma_c^2 V_c^T$ and from Eq.(A2−15) we have

$$W_o^2(P) = \Sigma_c^T V_c^T V_o \Sigma_o^T \Sigma_o V_o^T V_c \Sigma_c$$

$$XS = W_c^2(P) \; W_o^2(P)$$

$$= \Sigma_c^T V_c^T V_o \Sigma_o^T \Sigma_o V_o^T V_c \Sigma_c \qquad \text{(A2-18)}$$

and also from (A2-15) we have

$$H^T H = (\Sigma_c^T V_c^T V_o \Sigma_o^T) \; (\Sigma_o V_o^T V_c \Sigma_c) \qquad \text{(A2-19)}$$

To compare (A2-18) and (A2-19) can easyly see

$XS = H^TH$

APPENDIX 3

Helicopter Puma80 Model

The matrices A, B and C of the state-space form are

A =

1.0e+03 ×

Columns 1 through 8

-0.0382	-0.4750	-1.9833	-4.6626	-4.6652	-1.3004	-0.0345	-0.0085
0.0010	0	0	0	0	0	0	0
0	0.0010	0	0	0	0	0	0
0	0	0.0010	0	0	0	0	0
0	0	0	0.0010	0	0	0	0
0	0	0	0	0.0010	0	0	0
0	0	0	0	0	0.0010	0	0
0	0	0	0	0	0	0.0010	0

B =

1
0
0
0
0
0
0
0

C =

Columns 1 through 8

0	5.2600	90.2421	248.0669	211.5327	55.8064	0.5526	0

The eigenvalues Λ and the eigenvectors for the example system are

M =

Columns 1 through 4

1.0000+0.0000i	1.0000-0.0000i	1.0000	1.0000

-0.0569-0.0143i	-0.0569+0.0143i	-0.2194-0.2850i	-0.2194+0.2850i
0.0030+0.0016i	0.0030-0.0016i	-0.0331+0.1251i	-0.0331-0.1251i
-0.0001-0.0001i	-0.0001+0.0001i	0.0429-0.0180i	0.0429+0.0180i
0.0000+0.0000i	0.0000-0.0000i	-0.0146-0.0083i	-0.0146+0.0083i
0.0000-0.0000i	0.0000+0.0000i	0.0008+0.0060i	0.0008-0.0060i
0.0000+0.0000i	0.0000-0.0000i	0.0015-0.0015i	0.0015+0.0015i
0.0000-0.0000i	0.0000+0.0000i	-0.0008-0.0001i	-0.0008+0.0001i

Columns 5 through 8

1.0000	-0.0022	0.0000 - 0.0000i	0.0000 + 0.0000i
-0.7463	0.0053	0.0000 - 0.0000i	0.0000 + 0.0000i
0.5569	-0.0128	0.0000 + 0.0000i	0.0000 - 0.0000i
-0.4156	0.0305	0.0000 + 0.0000i	0.0000 - 0.0000i
0.3102	-0.0730	0.0000 - 0.0006i	0.0000 + 0.0006i
-0.2315	0.1747	-0.0067 - 0.0002i	-0.0067 + 0.0002i
0.1727	-0.4180	-0.0013 + 0.0820i	-0.0013 - 0.0820i
-0.1289	1.0000	1.0000 - 0.0000i	1.0000 + 0.0000i

$\Lambda =$

$$-16.5288 + 4.1425i$$
$$-16.5288 - 4.1425i$$
$$- 1.6958 + 2.2029i$$
$$- 1.6958 - 2.2029i$$
$$- 1.3400$$
$$- 0.4180$$
$$- 0.0013 + 0.0820i$$
$$- 0.0013 - 0.0820i$$

The columns of the matrix M^{-T} can be splited three groups, Z_{high}, Z_{mid} and Z_{lo}:

$Z_{high} =$

Columns 1 through 2

0.0048 - 0.0270i	0.0048 + 0.0270i
-0.0080 - 0.6056i	-0.0080 + 0.6056i
-0.1000 - 2.7886i	-0.1000 + 2.7886i
-0.3918 - 7.0765i	-0.3918 + 7.0765i

```
-0.4870 - 7.3771i   -0.4870 + 7.3771i
-0.1454 - 2.0857i   -0.1454 + 2.0857i
-0.0036 - 0.0547i   -0.0036 + 0.0547i
-0.0010 - 0.0136i   -0.0010 + 0.0136i
```

Z_{mid} =

Columns 1 through 4

```
0.0002+0.0002i   0.0002-0.0002i   0.0000+0.0000i   0.0000+0.0000i
0.0073+0.0069i   0.0073-0.0069i   0.0014+0.0000i   0.0013-0.0000i
0.0919+0.0678i   0.0919-0.0678i   0.0161-0.0000i   0.0154-0.0000i
0.3651+0.0818i   0.3651-0.0818i   0.0536-0.0000i   0.0600-0.0000i
0.4347-0.0043i   0.4347+0.0043i   0.1049-0.0000i   0.1312-0.0000i
0.1274-0.0111i   0.1274+0.0111i   0.0362-0.0000i   0.1015-0.0000i
0.0032-0.0001i   0.0032+0.0001i   0.0008-0.0000i   0.0011-0.0000i
0.0008-0.0001i   0.0008+0.0001i   0.0002-0.0000i   0.0007-0.0000i
```

Z_{lo} =

Columns 1 through 2

```
 0.0000 + 0.0000i    0.0000 - 0.0000i
-0.0005 + 0.0017i   -0.0005 - 0.0017i
-0.0066 + 0.0217i   -0.0066 - 0.0217i
-0.0263 + 0.0910i   -0.0263 - 0.0910i
-0.0585 + 0.2147i   -0.0585 - 0.2147i
-0.0483 + 0.2173i   -0.0483 - 0.2173i
-0.0005 + 0.0630i   -0.0005 - 0.0630i
 0.0047 + 0.0015i    0.0047 - 0.0015i
```

The high- and mid- subsystems are shown as

A_{high} =

```
-16.5288 + 4.1425i    0.0000 - 0.0000i
  0.0000 + 0.0000i  -16.5288 - 4.1425i
```

B_{high} =

```
0.4794 + 2.7016i
0.4794 - 2.7016i
```

C_{high} =

-0.0611 + 0.0399i   -0.0611 - 0.0399i

A_{mid} =
-1.6958+2.2029i   0.0000-0.0000i   0.0000-0.0000i   0.0000+0.0000i

 0.0000-0.0000i   -1.6958-2.2029i   0.0000+0.0000i   0.0000-0.0000i

 0.0000-0.0000i   0.0000-0.0000i   -1.3400-0.0000i   0.0000+0.0000i

 0.0000-0.0000i   0.0000-0.0000i   0.0000+0.0000i   -0.4180-0.0000i

B_{mid} =
    0.0187 - 0.0201i

    0.0187 + 0.0201i

    0.0038 - 0.0000i

    0.0034 - 0.0000i

C_{mid} =
    3.4745 + 3.9009i   3.4745 - 3.9009i   -3.9803   0.5201

The controllability grammian X and the observability grammian S for
high- and mid- subsystems are shown below:

X_{high} =
     0.2277   -0.2197 + 0.0233i
    -0.2197 - 0.0233i   0.2277 - 0.0000i

S_{high} =
    1.0e-03 ×
    0.2060 - 0.0000i   0.1388 + 0.1300i
    0.1388 - 0.1300i   0.2060 + 0.0000i

X_{mid} =
    1.0e-03 ×
0.2230-0.0000i   0.1016-0.0905i   0.0273-0.0053i   0.0302-0.0005i
0.1016+0.0905i   0.2230+0.0000i   0.0273+0.0053i   0.0302+0.0005i
0.0273+0.0053i   0.0273-0.0053i   0.0054+0.0000i   0.0072-0.0000i
0.0302+0.0005i   0.0302-0.0005i   0.0072+0.0000i   0.0134-0.0000i

S_{mid} =

```
      1.0e+03 ×
1.0195+0.0000i   -0.3779-0.1287i    0.5131+0.8249i    0.1544-0.0441i
-0.3779+0.1287i   1.0195-0.0000i    0.5131-0.8249i    0.1544+0.0441i
0.5131-0.8249i    0.5131+0.8249i    3.4567+0.0000i    0.8263-0.0000i
0.1544+0.0441i    0.1544-0.0441i    0.8263-0.0000i    0.4614+0.0000i
```

The Hankel singular values of the high- and mid- subsystems are

H_{high} =

```
 0.0023 - 0.0050i   -0.0014 - 0.0006i

-0.0022 - 0.0010i    0.0001 - 0.0001i
```

H_{mid} =

```
 0.0234+0.2040i   -0.2989+0.0431i   -0.0483-0.0268i    0.0097+0.0189i

-0.3948+0.3342i    0.1218+0.1358i    0.0028-0.0343i   -0.0001+0.0003i

 0.1781+0.1244i   -0.0341+0.0520i    0.0325-0.0115i   -0.0003-0.0001i

-0.0098-0.0323i    0.0482-0.0162i    0.0109+0.0037i    0.0014+0.0018i
```

APPENDIX 4

EXAMPLES

A EXAMPLE FOR USE PROGRAM - 1
(BACON AND SCHMIDT'S METHOD)


```
**************  FILE AND PARAMETERS TAKEN  ***************
*                                                         *
*           LOAD DATA FILE ------- AB80.MAT               *
*          ORDER OF HOS (n) ------- 14                    *
*                 INPUT (z) ------- 1                     *
*                OUTPUT (y) ------- 1                     *
*       PRINT PARAMETER (PT) ------ 0                     *
*        HIGH SUBSYSTEM (nn) ------ 6                     *
*      MIDDLE SUBSYSTEM (nm) ------ 14                    *
*          ORDER OF HSS (rh) ------ 1                     *
*          ORDER OF MSS (rm) ------ 7                     *
*      CHOICE OF GRAPHIC (tx) ----- 1                     *
*    FREQUENCY RANGE (w1,w1) ------ -1, 2                 *
*                                                         *
************** *** *************** *** ***************
```


(TO ENTER MATLAB PACKAGE)

VMS $ (MATLAB)

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Version 3.34   18-Mar-88

HELP, DEMO, INFO, and TERMINAL are available

(TO CHOOSE AVAILABLE TERMINAL TYPE)

>> (TERMINAL)

------ GRAPHICS TERMINAL TYPES ------

1) TEKTRONIX 4010/4014
2) TEKTRONIX 4100 SERIES
3) VT100 WITH RETROGRAPHICS
4) VT100 WITH SELANAR 100
5) VT100 WITH SELANAR 200
6) VT240
7) VT241 COLOR
8) HUMAN DESIGNED SYSTEMS
9) HP 2647
10) ERGO
11) GRAPHON
12) XTERM
13) MORE SELECTIONS ---

SELECT A MENU NUMBER : (2)

(TO LOAD DATA FILE AND SET THE POSITION OF GRAPHICS)

>> (LOAD AB80)
>> (XX = 0.35)
>> (YY = 0.98)

xx =

    0.2000

yy =

    0.9800

(TYPE THE PROGRAM'S NAME OF WHICH YOU WILL RUN)

>> (GENERAL)

(TO INPUT VALUES OF PARAMETERS)

he size of the A matrix is n=
n =

    14

 Which column of the b matrix will you take ? z=
z =

    1

 which tf do you want to get ? (y= 1 - n) y=
y =

    1

 Do you want to type the A & B matrices ? (1 - yes, 0 - no) pt=
pt =

    0

(DISPLAY THE EIGENVALUES OF MATRIX A OF HOS)

d1 =

```
-15.4634 +49.1181i
-15.4634 -49.1181i
-15.7960 +22.2181i
-15.7960 -22.2181i
-15.1429 + 5.7814i
-15.1429 - 5.7814i
 -1.0836 + 1.1670i
 -1.0836 - 1.1670i
 -1.2647
 -0.2900 + 0.9357i
 -0.2900 - 0.9357i
 -0.0092 + 0.2098i
 -0.0092 - 0.2098i
 -0.0001
```

(TO INPUT VALUES OF PARAMETERS)

 The high subsystem is the columns from first to nh=
nh =

     6

 The mid subsystem is the columns from nh+1 to nm=
nm =

    14

(DISPLAY THE HANKEL SINGULAR VALUES FOR MID- AND HIGH- SUBSYSTEMS)

svh =

    15.3923
    13.8422
     6.3908
     3.7661
     1.8797
     0.3257

svm =

    1.0e+04 *

    1.6235
    1.4935
    0.1101
    0.0732
    0.0129
    0.0068
    0.0048
    0.0006

(TO DECIDE REDUCED ORDER FOR EACH SUBSYSTEM)

How many order do you try to reduce for high subsystem ? rh=
rh =

    1

How many order do you try to reduce for mid subsystem ? rm=
rm =

    7

(DIPLAY THE MATRICES ac, bc, cc AND dc OF THE REDUCED SYSTEM)

ac =

  Columns 1 through 4

-17.3163 + 0.0000i   0   0   0
         0   -0.0070 - 0.0000i -0.2079 - 0.0271i   0.0074 + 0.0024i
         0   0.2077 - 0.0271i -0.0110 + 0.0000i   0.0172 + 0.0032i
         0   0.0165 - 0.0054i -0.0239 + 0.0045i -0.2116 - 0.0000i
         0   0.0040 + 0.0134i -0.0028 - 0.0176i   0.0235 - 0.9082i
         0   0.0057 - 0.0065i -0.0084 + 0.0075i -0.1954 + 0.1171i
         0   0.0017 - 0.0020i -0.0025 + 0.0023i -0.0532 + 0.0319i
         0   -0.0039 + 0.0025i   0.0054 - 0.0026i   0.1028 - 0.0279i

  Columns 5 through 8

         0   0   0   0
 -0.0018 + 0.0059i -0.0016 - 0.0018i -0.0006 - 0.0007i -0.0074 - 0.0048i
 -0.0012 + 0.0072i -0.0030 - 0.0026i -0.0016 - 0.0014i -0.0136 - 0.0066i
 -0.0273 - 1.0529i   0.1990 + 0.1193i   0.0419 + 0.0251i   0.2379 + 0.0645i
 -0.2306 + 0.0000i   0.1332 - 0.2097i -0.0264 + 0.0415i   0.0107 - 0.0359i
  0.1320 + 0.2077i -0.5224 + 0.0000i   0.2201 - 0.0000i -1.2577 + 0.3556i
  0.0468 + 0.0737i -0.3201 - 0.0000i -1.9659 + 0.0000i   1.2884 - 0.3643i
 -0.0445 - 0.1487i   0.7101 + 0.2008i -0.6540 - 0.1849i -1.0807 - 0.0000i

bc =

    1.0e+02 *

    1.2721 + 1.4672i
   -1.0587 - 1.2192i
    1.4534 + 1.2908i
    1.7602 + 1.0555i
   -0.8532 + 1.3427i
    0.9973 + 0.0000i
    0.3005 + 0.0000i
   -0.5123 - 0.1449i

cc =

  Columns 1 through 4

```
   0.0412 - 0.0475i    0.8589 - 0.9891i    1.1166 - 0.9916i   -0.1031 + 0.0618i

   Columns 5 through 8

   0.0631 + 0.0992i   -0.1689 - 0.0000i    0.0019 + 0.0000i    0.3667 - 0.1037i
```

dc =

     0

(DISPLAY THE MATRIX ac IN CONTROLLER CANONICAL FORM)

aba =

   1.0e+02 *

   Columns 1 through 4

```
  -0.2135 + 0.0000i  -0.7811 + 0.0000i  -1.5431 + 0.0000i  -1.7861 + 0.0000i
   0.0100                  0                  0                  0
        0             0.0100                  0                  0
        0                  0             0.0100                  0
        0                  0                  0             0.0100
        0                  0                  0                  0
        0                  0                  0                  0
        0                  0                  0                  0
```

   Columns 5 through 8

```
  -1.3276 + 0.0000i  -0.6312 + 0.0000i  -0.0640 + 0.0000i  -0.0235 + 0.0000i
        0                  0                  0                  0
        0                  0                  0                  0
        0                  0                  0                  0
        0                  0                  0                  0
   0.0100                  0                  0                  0
        0             0.0100                  0                  0
        0                  0             0.0100                  0
```

(TO GRAPHIC AND SET TIME AND FREQUENCY RANGE)

tx =

0 for time response, 1 for frequency response, 2 for both

 which ploting do you want ? tx=
tx =

     1

 The lower limit of frequency range is log(w1)=
w1 =

-1

The upper limit of frequency range is log(w2)=
w2 =

1

(THE GRAPHICS WILL SHOW ON SCREEN)

A EXAMPLE FOR USE PROGRAM - 2
(EXTENDED LEVY'S & T.C. CHEN'S METHOD)


```
***************** DATA FILES AND PARAMETERS TAKEN  *****************
*                                                                 *
*                                                                 *
*                DATA FILES ----- FWRIF1 - FWRIF8                  *
*                                 FLY801 - FLY808                  *
*        ORDER OF DENOMINATOR m ----- 8                           *
*         ORDER OF NUMERATOR nn ----- 7                           *
*       TOTAL NUMBER OF G(s) ml ----- 8                           *
*   POWER OF WEIGHTED FACTOR cn ----- 1.2                         *
* NUMBER OF FREQUENCY POINTS nw ----- 20                          *
*   TOTAL TIMES OF ITERATION nr ----- 3                           *
*          CHOICE OF GRAPHIC gs ----- 12 ( FOR x1/u1 and x2/u1 )  *
*                                                                 *
*                                                                 *
*****************  ****  ****************  ****  *****************
```


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HELP, DEMO, INFO, and TERMINAL are available

(TO CHOOSE AVAILABLE TERMINAL TYPE)

>> (TERMINAL)

----- Graphics Terminal Types -----

1) Tektronix 4010/4014
2) Tektronix 4100 series
3) VT100 with Retrographics
4) VT100 with Selanar 100
5) VT100 with Selanar 200
6) VT-240
7) VT-241 Color
8) Human Designed Systems
9) HP 2647
10) Ergo
11) Graphon
12) xterm
13) More selections - - -

Select a menu number: (2)

(TO LOAD DATA FILES)

>> (LOAD FWRIF1)
>> (LOAD FWRIF2)
>> (LOAD FWRIF3)
>> (LOAD FWRIF4)
>> (LOAD FWRIF5)
>> (LOAD FWRIF6)
>> (LOAD FWRIF7)
>> (LOAD FWRIF8)

(TYPE THE PROGRAM'S NAME OF WHICH YOU WILL RUN)

>> (MLEVY)

(TO CHECK DATA FILES HAVE BEEN LOADED OR NOT)

 Have you loaded the data files? (1 for yes, CTRL Y for no)
x =

     1

(TO INPUT THE VALUES OF PARAMETERS)

 INPUT ORDER OF DENOMINATOR
m =

     8

 INPUT ORDER OF NUMERATOR
nn =

     7

 INPUT NUMBER OF G(s)
m1 =

     8

 PLEASE GIVE THE VALUE OF WEIGHTED FACTOR
cn =

    1.2000

 THE NUMBER OF FREQUENCY POINTS
nw =

    20

 HOW MANY TIMES OF ITERATION DO YOU WANT ?
nr =

3   ʾ

THE NUMBER OF ITERATION ?
x =

        1

(SHOW THE VECTOR OF TRANSFER FUNCTION COEF. AFTER 1ST ITERATION)

THE NUMBER OF ITERATION ?
x =

        2

(SHOW THE TOTAL ERRORS AFTER FIRST ITERATION)

abd2 =

    2.0200e+10

(SHOW THE VECTOR OF TRANSFER FUNCTION COEF. AFTER 2ND ITERATION)

THE NUMBER OF ITERATION ?
x =

        3

(SHOW THE TOTAL ERRORS AFTER 2ND ITERATION)

abd2 =

    1.2627e+10

(SHOW THE VECTOR OF TRANSFER FUNCTION COEF. AFTER 3RD ITERATION)

Do you want to print the transfer function coefficients?
z =

        1

a for the coef. of numerator, b for coef. of denominator
z1 =

        1

(DISPLAY THE COEF. OF NUMERATOR AND DENOMINATOR OF TRANSFER FUNCTION)

a =

```
   1.0e+04 *

 Columns 1 through 7

   -0.2387    -0.0289     0.0000     0.0001     0.1635    -0.0017     0.0000
   -0.3637    -0.0419     0.0001     0.0076     0.2242    -0.0020     0.0000
   -0.3358     0.1570     0.0076     0.0112     1.3779    -0.0242     0.0000
   -0.2509    -0.8012     0.0108     0.0100     3.5724    -0.0372     0.0000
   -0.0469    -0.7671     0.0093     0.0069     2.3636    -0.0262     0.0000
   -0.0152    -0.8405     0.0065     0.0009     0.7092    -0.0050    -0.0001
   -0.0011    -0.3215     0.0008     0.0000     0.0435    -0.0042     0.0000
    0.0005    -0.0227     0.0000     0.0000     0.0008     0.0006     0.0000

 Column 8

   -0.0001
   -0.0014
   -0.0028
   -0.0134
   -0.0289
   -0.0185
   -0.0054
   -0.0004


b =

    1.0000
    2.7356
   26.9108
   56.8247
   76.8654
   66.7577
   34.3388
    9.6205
    0.6209



( DISPLAY THE MATRICES fc , f AND g1 )

fc =

 Columns 1 through 7

  -15.4956    -55.3090  -107.5254  -123.8057    -91.5266    -43.3448     -4.4062
    1.0000           0          0          0           0           0           0
         0      1.0000          0          0           0           0           0
         0           0     1.0000          0           0           0           0
         0           0          0     1.0000           0           0           0
         0           0          0          0      1.0000           0           0
         0           0          0          0           0      1.0000           0
         0           0          0          0           0           0      1.0000

 Column 8

   -1.6107
         0
         0
```

- 189 -

```
                    0
                    0
                    0
                    0
                    0


f =

   Columns 1 through 7

   -0.0285     0.0209     6.7361   -31.9526     0.0493     6.3202    480.3429
   -0.0380    -0.8112   126.3729    -0.4419    -0.0321    -5.8149   -853.1000
    0.0027    -0.0095    -1.4503    -0.0371    -0.0087    -1.1068    -78.1489
    0.0000    -0.0001     0.9554    -0.0024    -0.0003    -0.0425     -4.4773
   -0.0144    -0.0019    10.3897     0.4744    -0.0609     7.2416    967.7565
   -0.0102    -0.0060     7.8388     0.3481     0.0350     5.5235    693.3933
    0.0002    -0.0002    -0.1759    -0.0096    -0.0013    -0.1669    -17.8826
    0.0030     0.0093    -0.4237    -0.0123     0.0060    -0.3151    -26.1371

   Column 8

       3.0912
      -4.8449
      -0.5188
      -0.0504
    -128.2098
       4.7562
      -0.1031
      -0.7831


g1 =

       7.3456
    -364.9653
      -0.6759
      -0.0882
      12.2946
       9.6606
      -0.3436
      -6.0894


( TO GRAPHIC x1/u1 AND x2/u1 )

 WHICH G(s) WOULD YOU LIKE TO GRAPHIC ?
gs =

      12

( THE GRAPHICS WILL SHOW ON THE SCREEN )
```

PROGRAM - 1
(BACON & SCHMIDT'S METHOD)


```
%  **********************************************************************
%                                                                      *
%                                                                      *
%                                                                      *
%        THIS  MATLAB PROGRAM CAN BE USED TO FIND A REDUCED            *
%        ORDER MODEL FROM A HIGH ORDER SYSTEM USING BACON AND          *
%        SCHMIDT'S METHOD.  THE DYNAMICAL SYSTEM IS WRITTEN            *
%        AS THE FOLLOWING STATE-SPACE FORM:                            *
%                                                                      *
%        .                                                             *
%        x = Ax + Bu                                                   *
%                                                                      *
%        AND THE OUTPUT EQUATION BECOMES                               *
%                                                                      *
%        y = Cx                                                        *
%                                                                      *
%                                                                      *
%                                                                      *
%                                                                      *
%        THIS PROGRAM HAS BEEN DEVELOPED ON THE VMS2 OF               *
%        CENTRAL VAX OF GLASGOW UNIVERSITY.                            *
%                                                                      *
%        BY MINGRUI GONG, DEPT. OF ELECTRONICS AND ELECTRICAL         *
%        ENGINEERING OF GLASGOW UNIVERSITY.  AUGUST 1989.             *
%                                                                      *
%                                                                      *
%                                                                      *
%  **********************************************************************
%
%  **********************************************************************
%                                                                      *
%                                                                      *
%              ******** LIST OF THE SYMBOLS ********                    *
%                                                                      *
%           ab ----- NAME OF FILE (.MAT) INVOLVING A,B MATRIX          *
%            a ----- A MATRIX                                          *
%            b ----- B MATRIX                                          *
%            c ----- C MATRIX                                          *
%            d ----- D MATRIX                                          *
%            z ----- NUMBER OF COLUMN OF B MATRIX                      *
%    amid, ahi ----- A MATRIX OF mid, hi SUBSYSTEM                     *
%    bmid, bhi ----- B MATRIX OF mid, hi SUBSYSTEM                     *
%    cmid, chi ----- C MATRIX OF mid, hi SUBSYSTEM                     *
%    dmid, dhi ----- D MATRIX OF mid, hi SUBSYSTEM                     *
%    xmid, xhi ----- CONTROLLABILITY GRAMMIAN FOR mid, hi SUBSYSTEM    *
%    smid, shi ----- OBSERVABILITY GRAMMIAN FOR mid, hi SUBSYSTEM      *
%     hkm, hkh ----- HANKEL MATRIX FOR mid, hi SUBSYSTEM               *
%     svm, svh ----- HANKEL SINGULAR VALUES FOR mid, hi SUBSYSTEM      *
%     hd2, hd1 ----- SQUARE ROOTS OF EIGENVALUES OF XS FOR mid, hi     *
%                    SUBSYSTEM                                         *
%     ar2, ar1 ----- A MATRIX OF REDUCED ORDER MODEL FOR mid, hi       *
%                    SUBSYSTEM                                         *
%     br2, br1 ----- B MATRIX OF REDUCED ORDER MODEL FOR mid, hi       *
%                    SUBSYSTEM                                         *
%     cr2, cr1 ----- C MATRIX OF REDUCED ORDER MODEL FOR mid, hi       *
%                    SUBSYSTEM                                         *
%     dr2, dr1 ----- D MATRIX OF REDUCED ORDER MODEL FOR mid, hi       *
```

```
dhi=0;
amid=(vmid)'*a*mmid;
bmid=(vmid)'*b1;
for i=1:n,
    cmid(i,:)=c(i,:)*mmid;
end
dmid=0;
%
%
% *** DETERMINATION OF THE CONTROLLABILITY GRAMMIAN & OBSERVABILITY GRAMMIAN ***
% ******** CALCULATION OF THE EIGENVALUES AND EIGENVECTORS OF THE PRODUCT XS ***
% ******** CHECK HANKEL SINGULAR VALUES ARE EQUAL TO THE SQUARE ROOTS OF THE
%        EIGENVALUES OF THE PRODUCT XS ********
%
xhi=gram(ahi,bhi);
shi=gram((ahi)',(chi)');
xsh=xhi*shi;
xmid=gram(amid,bmid);
smid=gram((amid)',(cmid)');
xsm=xmid*smid;
[m1,d3]=eig(xsh);
[m2,d2]=eig(xsm);
[vch,sch,uch]=svd(xhi);
[voh,soh,uoh]=svd(shi);
schr=sqrt(sch);
sohr=sqrt(soh);
hkh=sohr*(voh)'*vch*schr;
[vcm,scm,ucm]=svd(xmid);
[vom,som,uom]=svd(smid);
scmr=sqrt(scm);
somr=sqrt(som);
hkm=somr*(vom)'*vcm*scmr;
svh=svd(hkh)
svm=svd(hkm)
hd1=sqrt(d3);
hd2=sqrt(d2);
%
%
% ******** DETERMINATION OF REDUCED ORDER OF THE hi & mid SUBSYSTEMS ********
%
rh=input('order for high subsystem  rh= ')
rm=input('order for mid subsystem   rm= ')
tr1=m1(:,1:rh);
tr2=m2(:,1:rm);
t1=inv((m1)');
t2=inv((m2)');
ur1=t1(:,1:rh);
ur2=t2(:,1:rm);
ar1=(ur1)'*ahi*tr1;
br1=(ur1)'*bhi;
for i=1:n,
    cr1(i,:)=chi(i,:)*tr1;
end
dr1=0;
ar2=(ur2)'*amid*tr2;
br2=(ur2)'*bmid;
for i=1:n,
    cr2(i,:)=cmid(i,:)*tr2;
end
dr2=0;
%
%
% ******** DETERMINATION OF TRANSFER FUNCTIONS OF hi & mid SUBSYSTEMS ********
%
[nh1,dh1]=ss2tf(ar1,br1,cr1(y,:),dr1,1);
[nm1,dm1]=ss2tf(ar2,br2,cr2(y,:),dr2,1);
```

```
%                        SUBSYSTEM                                              *
%           ac ---- A MATRIX OF COMPLETED REDUCED ORDER MODEL                   *
%           bc ---- B MATRIX OF COMPLETED REDUCED ORDER MODEL                   *
%           cc ---- C MATRIX OF COMPLETED REDUCED ORDER MODEL                   *
%           dc ---- D MATRIX OF COMPLETED REDUCED ORDER MODEL                   *
%                                                                               *
%                                                                               *
%   ***********************************************************************
%
%
% ******** INPUT A, B AND C MATRICES ********
% PLEASE LOAD DATA FILE (AB80.mat...) AND INPUT XX=0.35, YY=0.98 FOR GRAPHICS.
%
clc
%load ab80
n=input('The size of the A matrix is n= ')
z=input('Which column of the b matrix will you take ? z= ')
y=input('which tf do you want to get ? (y= 1-n) y= ')
b1=b(:,z);
for i=1:n,
    for j=1:n,
        if i==j,
            c(i,j)=1;
        else
            c(i,j)=0;
        end
    end
end
d=0;
%
%
% ******** TYPE THE A & B MATRICES AND DETERMINATION OF TF OF HOS ********
%
pt=input('Do you want to type the A & B matrices ? (1 - yes) pt= ')
if pt==1,
    a,b,c
else
end
[num1,den1]=ss2tf(a,b1,c(y,:),d,1);
%
%
% ******** DETERMINATION OF RADII D1 AND D2 ********
%
% ******** THE COLUMNS OF M MATRIX ARE SEPARATED THREE COLUMN GROUPS:
%          Mlo, Mmid and Mhi by the D1 AND D2 ********
%
[m,d12]=eig(a);
d12=eig(a)
v=inv((m)');
nh=input('The high subsystem involves the columns from first to nh= ')
nm=input('The mid subsystem involves the columns from nh+1 to nm= ')
mlo=m(:,(nm+1):n);
mmid=m(:,(nh+1):nm);
mhi=m(:,1:nh);
vlo=v(:,(nm+1):n);
vmid=v(:,(nh+1):nm);
vhi=v(:,1:nh);
%
%
% ******** DETERMINATION OF TRANSFER FUNCTIONS OF hi & mid SUBSYSTEM ********
%
ahi=(vhi)'*a*mhi;
bhi=(vhi)'*b1;
for i=1:n,
    chi(i,:)=c(i,:)*mhi;
end
```

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```
%
%
% ******** To COMBINE THE COMPLETED MATRICES OF REDUCED ORDER MODEL ********
%
% ******** THE REDUCED ORDER r = (rh+rm) ********
%
r=rh+rm;
ac=zeros(r);
for i=1:rh,
    for j=1:rh,
        ac(i,j)=ar1(i,j);
    end
end
for i=1:rm,
    for j=1:rm,
        ac(i+rh,j+rh)=ar2(i,j);
    end
end
ac
bc=[br1(:,1)
    br2(:,1)]
cc=[cr1(y,:),cr2(y,:)]
dc=0
[nuc,dec]=ss2tf(ac,bc,cc,dc,1);
[zc,pc,kc]=tf2zp(nuc,dec);
[aba,bba,cba,dba]=tf2ss(nuc,dec);
aba
%
%
% ******** MATCHING WITH TIME RESPONSE AND FREQUENCY RESPONSE ********
%
%
%
tx='0 for time response, 1 for frequency response, 2 for both'
tx=input('which do you want ? tx= ')
%
if tx==0,
    tb=input('The lower limit of time is tb= ')
    delt=input('The increment of time is delt= ')
    te=input('The upper limit of time is te= ')
    t=tb:delt:te;
    y1=step(ac,bc,cc,dc,1,t);
    y2=step(a,b1,c(y,:),d,1,t);
    plot(t,y1,'-',t,y2,'o')
    text(xx,yy,'- for reduced model, o for HOS model','sc')
    xlabel('time sec.')
    ylabel('y')
%
%
%
% ******** TO PLOT THE FREQUENCY RESPONSE ********
%
elseif tx==1,
w1=input('The lower limit of frequency range is log(w1)=  ')
w2=input('The upper limit of frequency range is log(w2)=  ')
w=logspace(w1,w2);
[mag1,phase1]=bode(ac,bc,cc,dc,1,w);
[ma1,pha1]=bode(a,b1,c(y,:),d,1,w);
subplot(211),
  loglog(w,mag1,'-',w,ma1,'o')
ylabel('mag db'),xlabel('w(rad/sec)')
text(xx,yy,'- for reduced model, o for HOS model','sc')
subplot(212),
    semilogx(w,phase1,'-',w,pha1,'o')
    ylabel('phase deg'),xlabel('w(rad/sec)')
else
```

```
%
%
% ******** TO PLOT THE TIME AND FREQUENCY RESPONSE ********
%
tb=input('The lower limit of time is tb= ')
delt=input('The increment of time is delt= ')
te=input('The upper limit of time is te= ')
t=tb:delt:te;
y1=step(ac,bc,cc,dc,1,t);
y2=step(a,b1,c(y,:),d,1,t);
w1=input('The lower limit of frequency range is log(w1)=  ')
w2=input('The upper limit of frequency range is log(w2)=  ')
w=logspace(w1,w2);
[mag1,phase1]=bode(ac,bc,cc,dc,1,w);
[ma1,pha1]=bode(a,b1,c(y,:),d,1,w);
subplot(221),
    loglog(w,mag1,'-',w,ma1,'o')
    ylabel('mag db'),xlabel('w(rad/sec)')
    text(xx,yy,'- for reduced model, o for HOS model','sc')
subplot(222),
    semilogx(w,phase1,'-',w,pha1,'o')
    ylabel('phase deg'),xlabel('w(rad/sec)')
subplot(223),plot(t,y1,'-',t,y2,'o')
    xlabel('time sec.')
    ylabel('y')
end
```

(EXTENDED LEVY'S & T.C. CHEN'S METHODS)

```
%  ********************************************************************
%                                                                    *
%                                                                    *
%                                                                    *
%         THIS MATLAB PROGRAM CAN BE USED TO FIND A REDUCED          *
%         ORDER MODEL FROM A HIGH ORDER SYSTEM USING THE             *
%         EXTENDED LEVY'S METHOD AND T.C.CHEN'S METHOD.              *
%                                                                    *
%         THIS PROGRAM HAS BEEN DEVELOPED ON THE VMS2 OF             *
%         CENTRAL VAX OF GLASGOW UNIVERSITY.                         *
%                                                                    *
%         BY MINGRUI GONG, DEPT. OF ELECTRONICS & ELECTRICAL         *
%         ENGINEERING OF GLASGOW UNIVERSITY.   MAY 1991              *
%                                                                    *
%                                                                    *
%                                                                    *
%  ********************************************************************
%
%  ********************************************************************
%                                                                    *
%                                                                    *
%                                                                    *
%               ******* LIST OF THE SYMBOLS ********                 *
%                                                                    *
%     fwrif1 - fwrif8 ---- DATA FILES                                *
%     fly801 - fly808 ---- DATA FILES                                *
%               m ---- ORDER OF DENOMINATOR                          *
%              nn ---- ORDER OF NUMERATOR                            *
%              ml ---- TOTAL NUMBER OF G(s)                          *
%              cn ---- POWER of WEIGHTING FACTOR                     *
%              nk ---- ITERATION NUMBER                              *
%              nw ---- NUMBER OF FREQUENCY POINTS                    *
%              nr ---- TOTAL TIMES OF ITERATION                      *
%           d2(s) ---- WEIGHTING FACTOR                             *
%            abd2 ---- TOTAL ERRORS                                  *
%          a, ah1 ---- COEF. OF NUMERATOR OF TRANSFER FUNCTION    *
%          b, bh1 ---- COEF. OF DENOMINATOR OF TRANSFER FUNCTION *
%              ab ---- VECTOR OF TRANSFER FUNCTION COEF.             *
%            abh1 ---- RESULTS FILE                                  *
%                                                                    *
%                                                                    *
%                                                                    *
%  ********************************************************************
%
%
% TO CHECK LOAD DATA FILES
x=input('Have you loaded the data files?  (1 for yes, CTRL Y for no)')
%
% SPECIFY THE NUMERATOR AND DENOMINATOR ORDER.
m=input('input order of denominator')
nn=input('input order of numerator')
ml=input('input number of G(s)')
cn=input('please give the value of weighting factor')
n=nn+1;
mn=m+n;
for i=1:(ml*n+m),
   for j=1:(ml*n+m),
     p(i,j)=0;
```

```
          end
       cp(i)=0;
    end
    nw=input('THE NUMBER OF FREQUENCY POINTS')
    for i=1:nw,
       d2(i)=1;
    end
    %
    for jj=1:m1,
       for i=1:nw,
          if jj==1,
             fw1(i,jj)=f1(1,i);
             fr1(i,jj)=f2(1,i);
             fi1(i,jj)=f3(1,i);
          else
            if jj==2,
               fw1(i,jj)=f4(1,i);
               fr1(i,jj)=f5(1,i);
               fi1(i,jj)=f6(1,i);
            else
              if jj==3,
                 fw1(i,jj)=f7(1,i);
                 fr1(i,jj)=f8(1,i);
                 fi1(i,jj)=f9(1,i);
              else
                if jj==4,
                   fw1(i,jj)=f10(1,i);
                   fr1(i,jj)=f11(1,i);
                   fi1(i,jj)=f12(1,i);
                else
                  if jj==5,
                     fw1(i,jj)=f13(1,i);
                     fr1(i,jj)=f14(1,i);
                     fi1(i,jj)=f15(1,i);
                  else
                    if jj==6,
                       fw1(i,jj)=f16(1,i);
                       fr1(i,jj)=f17(1,i);
                       fi1(i,jj)=f18(1,i);
                    else
                      if jj==7,
                         fw1(i,jj)=f19(1,i);
                         fr1(i,jj)=f20(1,i);
                         fi1(i,jj)=f21(1,i);
                      else
                         fw1(i,jj)=f22(1,i);
                         fr1(i,jj)=f23(1,i);
                         fi1(i,jj)=f24(1,i);
                      end
                    end
                  end
                end
              end
            end
          end
       end
    end
    %
    % FOR ITERATION
    nr=input('HOW MANY ITERATIONS DO YOU WANT ?')
    for r=1:nr,
    x=input('THE NUMBER OF ITERATION ?')
    if r~=1,
    abd2=0;
    for jj=1:m1,
    for j=1:nw,
```

```
    w=fw1(j,jj);
    re=fr1(j,jj);
    im=fil(j,jj);
    sr=0;
    si=w;
    dr=1;
    di=0;
    for i=(m1*n+1):(m1*n+m),
       dr=dr+ab(i)*sr;
       di=di+ab(i)*si;
       hs=sr;
       sr=-si*w;
       si=hs*w;
    end
    d2(j)=1/(dr*dr+di*di);
d2(j)=d2(j)^cn;
    sr=0;
    si=w;
    dr1=ab(1);
    di1=0;
    for i=((jj-1)*n+1):(jj*n),
       dr1=dr1+ab(i+1)*sr;
       di1=di1+ab(i+1)*si;
       hs=sr;
       sr=-si*w;
       si=hs*w;
    end
% TO GET BATE & TOU
di=di/w;
di1=di1/w;
aad(j)=((dr*re-w*di*im-dr1)*(dr*re-w*di*im-dr1))*d2(j);
bbd(j)=((w*di*re+dr*im-w*di1)*(w*dr*re+dr*im-w*di1))*d2(j);
abd(j)=aad(j)+bbd(j);
end
abd1=0;
for j=1:nw,
    abd1=abd1+abd(j);
end
abd2=abd2+abd1;
end
abd2
for i=1:(m1*n+m),
   for j=1:(m1*n+m),
     p(i,j)=0;
   end
   cp(i)=0;
end
end
% CALCULATE THE MATRIX p AND CP FOR SISO.
for nk=1:m1,
nk;
   for i=1:mn,
     for j=1:mn,
        p1(i,j)=0;
     end
     cp1(i)=0;
   end
%
% INPUT THE NUMBER OF FREQUENCY POINTS .
%
% INPUT EACH FREQUENCY POINT
% LOAD FWRI.M (F1 - FREQUENCY, F2 - REAL PART, F3 - IMAGINARY PART)
for i=1:nw,
  fw(i)=fw1(i,nk);
  fr(i)=fr1(i,nk);
  fi(i)=fil(i,nk);
```

```
end
%
% SET THE ARRAY SIZES AND CONSTANTS
np=2*nn+1;
n1=n+1;
n2=n+2;
n3=n+3;
mp=mn+nn;
mq=mn+m-1;
zm=mn-1;
%
% THIS IS THE MAIN LOOP SETTING THE SKELETON P MATRIX, CP IS THE VECTOR y
% ASSOCIATED WITH Px, AND ab IS THE VECTOR OF TRANSFER FUNCTION COEFFICIENTS x.
% TO GET P1 AND CP1 FOR SISO.
for ki=1:nw,
    fh=fw(ki)*fw(ki);
    p1(1,1)=p1(1,1)+d2(ki);
    cp1(1)=cp1(1)+fr(ki)*d2(ki);
if n~=1,
  sh=d2(ki);
  for i=3:2:np,
      if i<=n,
        sh=-sh*fh;
        p1(1,i)=p1(1,i)+sh;
      else
        ii=i-n+1;
        sh=sh*fh;
        p1(ii,n)=p1(ii,n)+sh;
      end
  end
end
t1=d2(ki)*fw(ki)*fi(ki);
p1(1,n1)=p1(1,n1)+t1;
if m~=1,
  s2=-d2(ki)*fr(ki);
  for j=n2:2:mp,
    if j<=mn,
      s2=-s2*fh;
      p1(1,j)=p1(1,j)+s2;
      jj=j+1;
      if jj<=mn,
        t1=-t1*fh;
        p1(1,jj)=p1(1,jj)+t1;
      else
        i=jj+1-mn;
        if i<=n,
          t1=t1*fh;
          p1(i,mn)=p1(i,mn)+t1;
        end
      end
    else
      i=j-mn+1;
      s2=s2*fh;
      jj=j+1;
      if i<=n,
        p1(i,mn)=p1(i,mn)+s2;
        i=jj+1-mn;
        if i<=n,
          t1=t1*fh;
          p1(i,mn)=p1(i,mn)+t1;
        end
      end
    end
  end
end
  fx=(fi(ki)*fi(ki)+fr(ki)*fr(ki))*d2(ki)*fh;
```

```
      p1(n1,n1)=p1(n1,n1)+fx;
if m~=1,
     for i=n3:2:mq,
        if i<=mn,
           fx=-fx*fh;
           p1(n1,i)=p1(n1,i)+fx;
        else
           fx=fx*fh;
           j=i-mn+1+n;
           p1(j,mn)=p1(j,mn)+fx;
        end
     end
  end
end
%
% THIS SECTION OF THE PROGRAM FILLS IN THE P MATRIX
if n>=3,
     mh=-1;
     h=1;
     for j=2:n,
        h=h*mh;
        j1=j-1;
        for l=1:nn,
           k=l+1;
           p1(j,l)=p1(j1,k)*h;
        end
     end
end
if n~=1,
     mh=-1;
     h=1;
     for j=2:n,
        h=h*mh;
        j1=j-1;
        for l=n1:zm,
           k=l+1;
           p1(j,l)=p1(j1,k)*h;
        end
     end
end
if m>=3,
     mh=-1;
     h=1;
     for j=n2:mn,
        h=h*mh;
        j1=j-1;
        for l=n1:zm,
           k=l+1;
           p1(j,l)=p1(j1,k)*h;
        end
     end
end
  x1=1;
  x2=-1;
  ii=0;
  for i=1:n,
     ii=ii+1;
     if ii>=3,
        ii=1;
        x1=x1*x2;
     end
     jj=0;
     y1=1;
     for j=n1:mn,
        jj=jj+1;
        if jj>=3,
```

```
                jj=1;
                y1=y1*x2;
            end
            p1(j,i)=p1(i,j)*x1*y1;
        end
    end
%
% VECTOR CP IS FOUND APPROPRIATE ELEMENTS WITHIN THE P MATRIX
if n~=1,
    x1=1;
    for i=1:nn,
            ii=i+1;
            cp1(ii)=p1(i,n1)*x1;
            x1=-x1;
        end
end
if m~=1,
        for i=n2:2:mn,
            j=i-1;
            cp1(i)=p1(j,n1);
        end
end
%
% TO MOVE THE MATRIX FOR SISO P1 & CP1 TO MATRIX P & CP FOR MIMO.
%
for i=1:n,
    for j=1:n,
        p(((nk-1)*n+j),((nk-1)*n+i))=p1(j,i);
    end
  end
%
for i=1:m,
  for j=1:n,
    p(((nk-1)*n+j),(m1*n+i))=p1(j,(n+i));
  end
end
%
for i=1:n,
  for j=1:m,
    p((m1*n+j),((nk-1)*n+i))=p1((n+j),i);
  end
end
%
for i=1:m,
  for j=1:m,
    p((m1*n+j),(m1*n+i))=p((m1*n+j),(m1*n+i))+p1((n+j),(n+i));
  end
end
%
for i=1:n,
  cp((nk-1)*n+i)=cp1(i);
end
for i=1:m,
  cp(m1*n+i)=cp(m1*n+i)+cp1(n+i);
end
end
%
%
% FIND TRANSFER FUNCTION COEFFICIENTS AB
ab=p\(cp)';
ab
end
%
% PRINT OUT THE TRANSFER FUNCTION COEFFICIENTS
z=input('Do you want to print the transfer function coefficients?')
z1=input('a for the coef. of numerator, b for coef. of denominator')
```

```
for i=1:m1,
   a(1:n,i)=ab(((i-1)*n+1):(i*n),1);
end
% EVERY COLUMN OF A IS THE COEF. OF NUMERATOR OF EVERY G(s).
a
b(1,1)=1;
b(2:(m+1),1)=ab((m1*n+1):(m1*n+m),1);
b
%
% THE ORDER OF THE COEF. FROM HIGHER TO LOWER.
for i=1:m1,
    ahl(1:n,i)=a(n:-1:1,i);
end
bhl(1:(m+1),1)=b((m+1):-1:1,1);
% SAVE A AND B TO FILE FOR GRAPHIC.
save abhl ahl bhl
%
% USE CHEN'S METHOD TO GET STATE-SPACE FORM
ahlc=ahl/bhl(1,1);
nc=(ahlc)';
bhlc=bhl/bhl(1,1);
bhlc=-(bhlc)';
bhlc=bhlc(1,2:(m1+1))
fc(1,:)=bhlc;
fc(2:m1,1:(m1-1))=eye((m1-1),(m1-1));
fc(2:m1,m1)=zeros((m1-1),1);
fc
f=nc*fc*inv(nc)
g1=nc(:,1)
%
% gs=12 for x1/u1 and x2/u1, gs=34 for x3/u1 and x4/u1.
% gs=56 for x5/u1 and x6/u1, gs=78 for x7/u1 and x8/u1.
gs=input('WHICH G(s) WOULD YOU LIKE TO PLOT ?')
bhl=(bhl)';
if gs==12,
  load fly801
  load fly802
  ahl1=(ahl(1:n,1))';
  ahl2=(ahl(1:n,2))';
else
  if gs==34,
    load fly803
    load fly804
    ahl1=(ahl(1:n,3))';
    ahl2=(ahl(1:n,4))';
  else
    if gs==56,
      load fly805
      load fly806
      ahl1=(ahl(1:n,5))';
      ahl2=(ahl(1:n,6))';
    else
      load fly807
      load fly808
      ahl1=(ahl(1:n,7))';
      ahl2=(ahl(1:n,8))';
    end
  end
end
% THE LOW ORDER MODEL
[ar1,br1,cr1,dr1]=tf2ss(ahl1,bhl);
[ar2,br2,cr2,dr2]=tf2ss(ahl2,bhl);
eigr1=eig(ar1)
eigr2=eig(ar2)
% HIGH ORDER SYSTEM
at1=ahl;
```

```
bt1=bh1;
ct1=ch1;
dt1=dh1;
at2=ah2;
bt2=bh2;
ct2=ch2;
dt2=dh2;
eigh1=eig(ah1)
eigh2=eig(ah2)
% PLOT FREQUENSE RESPONSES OF THE G(s).
w=logspace(-2,2);
[mag,phase]=bode(at1,bt1,ct1,dt1,1,w);
[mag1,phase1]=bode(ar1,br1,cr1,dr1,1,w);
[mag2,phase2]=bode(at2,bt2,ct2,dt2,1,w);
[mag3,phase3]=bode(ar2,br2,cr2,dr2,1,w);
subplot(221)
loglog(w,mag,'-',w,mag1,'*')
xlabel('w(rad/sec.)'),ylabel('mag db')
subplot(222)
semilogx(w,phase,'-',w,phase1,'*')
xlabel('w(rad/sec.)'), ylabel('phase deg')
text(xx,yy,'- for HOS model, * for ROS model','sc')
subplot(223)
loglog(w,mag2,'-',w,mag3,'*')
xlabel('w(rad/sec.)'),ylabel('mag db')
subplot(224)
semilogx(w,phase2,'-',w,phase3,'*')
xlabel('w(rad/sec.)'),ylabel('phase deg')
```

```
% THIS PROGRAM IS TO GENERATE TWO DATA FILES (FLY80*.MAT AND FWRIF*.MAT)
% FOR PUMA80 OR PUMA60.
%
% THE DATA FILES IS TO FIND 8*8 (1+7) REDUCED MODEL.
% LOAD AB80.M OR AB60.M.
load ab60
z=input('the b matrix is')
w1=input('The lower limit of frequency range is log(w1)= ')
w2=input('The upper limit of frequency range is log(w2)= ')
b=b(:,z);
for i=1:14,
    for j=1:14,
        if i==j,
            c(i,j)=1;
        else
            c(i,j)=0;
        end
    end
end
d=0;
for y=1:8,
  ah1=a;
  bh1=b;
  ch1=c(y,:);
  dh1=d;
  if y==1,
    save fly801 ah1 bh1 ch1 dh1
    f1=logspace(w1,w2);
    [re,im]=nyquist(ah1,bh1,ch1,dh1,1,f1);
    f2=(re)';
    f3=(im)';
    save fwrif1 f1 f2 f3
  else
    if y==3,
      save fly803 ah1 bh1 ch1 dh1
      f7=logspace(w1,w2);
      [re,im]=nyquist(ah1,bh1,ch1,dh1,1,f7);
      f8=(re)';
      f9=(im)';
      save fwrif3 f7 f8 f9
    else
      if y==5,
        save fly805 ah1 bh1 ch1 dh1
        f13=logspace(w1,w2);
        [re,im]=nyquist(ah1,bh1,ch1,dh1,1,f13);
        f14=(re)';
        f15=(im)';
        save fwrif5 f13 f14 f15
      else
        if y==7,
          save fly807 ah1 bh1 ch1 dh1
          f19=logspace(w1,w2);
          [re,im]=nyquist(ah1,bh1,ch1,dh1,1,f19);
          f20=(re)';
          f21=(im)';
          save fwrif7 f19 f20 f21
        else
```

```
            ah2=a;
            bh2=b;
            ch2=c(y,:);
            dh2=d;
            if y==2,
               save fly802 ah2 bh2 ch2 dh2
               f4=logspace(w1,w2);
               [re,im]=nyquist(ah2,bh2,ch2,dh2,1,f4);
               f5=(re)';
               f6=(im)';
               save fwrif2 f4 f5 f6
            else
               if y==4,
                  save fly804 ah2 bh2 ch2 dh2
                  f10=logspace(w1,w2);
                  [re,im]=nyquist(ah2,bh2,ch2,dh2,1,f10);
                  f11=(re)';
                  f12=(im)';
                  save fwrif4 f10 f11 f12
               else
                  if y==6,
                     save fly806 ah2 bh2 ch2 dh2
                     f16=logspace(w1,w2);
                     [re,im]=nyquist(ah2,bh2,ch2,dh2,1,f16);
                     f17=(re)';
                     f18=(im)';
                     save fwrif6 f16 f17 f18
                  else
                     save fly808 ah2 bh2 ch2 dh2
                     f22=logspace(w1,w2);
                     [re,im]=nyquist(ah2,bh2,ch2,dh2,1,f22);
                     f23=(re)';
                     f24=(im)';
                     save fwrif8 f22 f23 f24
                  end
               end
            end
         end
      end
   end
end
```

APPENDIX 6

USER'S GUIDE 1 TO THE MODEL REDUCTION SOFTWARE

(PROGRAM FOR IMPLEMENTATION

OF BACON & SCHMIDT'S METHOD)

A6.1.1 INTRODUCTION

Bacon and Schmidt's method has been coded in MATLAB for a VAX 3000 computer. This Guide gives a brief description of this program. The program can be used to find an equivalent system, with reduced order of a multi—input multi—output high—order system. The high—order system is written as a state—space form, and the low—order approximation, or equivalent system, can be expressed as a state—space form, a pole—zero—gain form, or a transfer function form.

A6.1.2. DETERMINATION OF TRANSFER FUNCTION OF THREE SUBSYSTEMS

The choice of transfer function of the three subsystems, (high—, middle— and low—), very much depends on determination of the radii d1 and d2. The value of radii d1 and d2 are based on the location of poles of the system in the s—place. Physical knowledge of the system and experience are very important for determining d1 and d2. In this program, the value of poles (eigenvalues) are calculated first and displayed on screen; then based on these value we can group them into three subsystem and obtain the range of the high frequency subsystem and the middle frequency subsystem. In Bacon and Schmidt's method the contribution of the low frequency subsystem is considered unimportant for flight control design and handling qualities studies and is ignored.

A6.1.3. THE DETERMINATION OF THE REDUCED ORDER

The order of the reduced system is the sum of the order of the reduced subsystems, r_h and r_m. The order of the reduced subsystems, r_h and r_m, is initially decided from the Hankel singular values by throwing away the parts which correspond

to the smaller values. The final determination should be from the comparison of the frequency response between the HOS and the resulting ROS and from the errors produced by the ROS.

USER'S GUIDE 2 TO THE MODEL REDUCTION SOFTWARE

(PROGRAM FOR IMPLEMENTATION OF THE EXTENDED LEVY'S METHOD)

A6.2.1. INTRODUCTION

This program finds a transfer function for a multi−variable system from frequency response data using a complex curve fitting technique which has been developed recently by M. R. Gong based on Levy's method. The modifications proposed by Sanathanan and Koerner[70] for removing high frequency bias have been included. The method will not be able to deal with poles at the origin of the s−plane. The program was written in MATLAB for use on a DEC MICROVAX 3600 computer. This program is available for single−input and multi−output cases. The original data for the high order system are expressed in state−space form and are converted to required frequency response format by "Nyquist" function of MATLAB.

A6.2.2. THE HIGH ORDER SYSTEM

The high−order system, namely the dynamical system to be represented in reduced order form is given as the frequency response data which must be input in the form of forcing frequency, together with the resulting response in term of a real and imaginary part.

A6.2.3. THE REDUCED ORDER MODEL

The resulted low−order model is first expressed as a transfer function form by using Levy's method, (part one of the program). Then, the reduced order model will further be described as a state−space form by part two of the program using Chen's method.

A6.2.4. GENERATION OF THE DATA FILES

Before runing this program some input data files must be generated. The number of the required input files is equal to the order of the reduced system, m1. These data files are written in "Nyquist" function format which means that every file

includes corresponding frequency part, real value part and imaginary part. The frequency part is at the position (f1, f4, f7, ... f(m1*3−2)) separately, the real part is at (f2, f5, f8, ... f(m1*3−1)) separately and imaginary part at (f3, f6, f9, f(m1*3)).(See APPENDIX 5   Program 3)

A6.2.5. <u>CHOICE OF THE COEFFICIENTS IN THE PROGRAM</u>

In this program, a few coefficients must be given by the user before the program is run.   The right choice of the values very much depends on the particular cases and previous experience of the user.   The choice of the values and the definition of some coefficients may be described as follows:

m   − − −   the order of denominator of transfer function G(s)

nn − − −   the order of numerator of transfer function G(s)

m1 − − −   the number of transfer function G(s) given

nw − − −   the number of the frequency points

nr − − −   the total times of iterations

cn − − −   the power of weighted factor

The numerator nn must be equal to or less than denominator m.   In this program the   maximum m1 can be taken as 8.   In general, more data points may improve the accuracy,   but more data points will increase the cost in terms of computing time.   Fifty points is thought to be suitable for obtaining satisfactory results in most cases.   This program only needs 3 − 6   iterations to be able to achieve convergence.   It is very difficult to say how the value of the constant cn can be determined.   From the examples which we tried, cn in the range 0.6 − 1.5 will give satisfactory results.   However different cases should take different value of cn and sometimes different value of cn are needed in different frequency ranges within one calculation.