THE INTERACTION IN THE THEORY OF BETA DECAY

Thesis

by

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PREFACE

This thesis is an account of research into the theory of beta decay carried out while I was a research student at Glasgow University from October 1948 until September 1951, brought up to date by the inclusion of references to recent work by other authors. The main sources from which this research derives are W.Pauli (Ann. de l'Inst. H. Poincaré, 6, 137 (1936)) and E. Cartan (Leçons sur la Théorie des Spineurs: Hermann et Cie, Paris (1938)) for the discussion of spinors in Chapter 2, and E. Fermi (Zeits. f. Phys. 88, 161 (1934)), E. J. Konopinski and G. E. Uhlenbeck (Phys. Rev. 60, 308 (1941)) and E. Greuling (Phys. Rev. 61, 568 (1942)) for the theory of beta decay discussed in Chapter 3. The transition to configuration space given at the beginning of Chapter 3 derives indirectly from V. Fock (Zeits. f. Phys. 75, 622 (1932)) and R. Becker and G. Leibfried (Phys. Rev. 69, 34 (1946)); I learned this type of approach, however, in the course of discussions with Dr. B. F. X. Touschek.

I am greatly indebted to Professor J. C. Gunn for his guidance of and continued interest in this research, and to Dr. B. F. X. Touschek for innumerable stimulating discussions. I also wish to thank Dr. D. Martin, who introduced me to Cartan's treatment of spinors, Dr. S. C. Curran and his co-workers for various discussions on

experimental aspects of beta decay, and the Department of Scientific and Industrial Research for a maintenance allowance during the period of my research studentship.

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1. INTRODUCTION

This thesis comprises a study of some of the properties of the interaction between heavy and light particles postulated in the Fermi type of theory of beta decay, in the course of which some of the assumptions of the theory are clarified, certain omissions in the literature repaired, and new information about the interaction is obtained by comparing theoretical predictions with experiment. In this introductory chapter, the main experimental features of beta decay are briefly discussed. The enormous historical importance of the neutrino hypothesis and the Fermi theory in guiding experimentation is deliberately ignored, in order to emphasize the weight of the present evidence for the existence of the neutrino. The experimental discussion leads naturally to an account of the main features of the Fermi theory, and a summary of the work described in subsequent chapters.

A beta decaying nucleus emits a beta-particle, identified as an electron, which may have any energy up to a maximum energy W_0 characteristic of the transition. Apart from the continuous energy spectrum of the emitted beta particles, there is no evidence that the product nuclei are not all identical; in particular, there is no continuous gamma ray spectrum of the intensity one would expect from the de-excitation of the final nuclei if they were not all identical. The product nuclei normally have the same mass

as/ and charge numbers of well-known stable nuclei, and there is no reason to doubt that they are identical with these stable nuclei or with excited states of them when, as is frequently the case, gamma rays of discrete energy are emitted in coincidence with the beta particles. The mass difference between the decaying nucleus and the stable product nucleus is equal to the maximum beta particle energy, together with the energies of any gamma rays emitted in coincidence. Energy, therefore, is apparently lost in the beta decay process. These properties are discussed by Ellis, Cockroft, Peierls and Richardson (1937), while the identity of beta particles with electrons has been demonstrated using the exclusion principle by an elegant method due to Goldhaber and Scharff-Goldhaber (1948), and developed by Davies and Grace (1951).

Many experiments have been made on the recoil momentum of the decaying nucleus since the early cloud chamber work of Crane and Halpern (1938), and it is now clear that momentum also is not conserved in the beta decay process. Sherwin (1951) has shown that the missing momentum p and the missing energy $E = W_0 - W$ obey, in the case of the decay of P^{32}, the relation

$$E^2 = p^2 c^2$$

to within 10%. This is just the connection between energy and momentum for a particle of zero rest mass. The relation could not hold between the total energy and momentum of two particles unless there were a strong angular correlation

favouring the emission of both in the same direction. Sherwin's results, however, do not allow any very accurate estimate of the mass associated with the missing energy and momentum, except that it is probably less than the electron mass.

All the measured spins of nuclei are integral or half odd integral according as the mass number of the nucleus is even or odd. This is true in particular for the dozen or so beta decaying nuclei whose spins have been measured, and there is no reason for doubting it for any nucleus, whether stable or unstable. Since the electron can carry away only half odd integral angular momentum, and since the mass number of the nucleus does not change in the decay, it follows that angular momentum also is apparently not conserved.

Most of the observed beta spectra have a shape given by

$$P(W)dW = X F(Z,W) pW(W_o - W)^2 dW \qquad (1.1)$$

where P(W)dW is the probability per unit time of the emission of an electron in the energy range (W, W + dw), p is the electron momentum, F(Z,W) is the ratio of the electron density of an electron in the Conlomb field of a charge Ze, evaluated at a distance of the order of the nuclear radius from the charge, to the electron density at large distances from the charge, and X is a number independent of the beta particle energy. Now the decay process, being a nuclear phenomenon, can depend on the electron density only near

the nucleus, whereas the normalization of the electron wave-function depends essentially on the electron density at large distances from the nucleus. Consequently, the occurrence of the factor F(Z,W) in (1.1) is to be expected. The factor pWdW is proportional to the number of electron states in the energy range (W, W + dW), and its appearance is also to be expected. The remaining energy dependent term, $(W_o-W)^2$, is just the density of states of a zero mass particle with energy Wo-W, and would be expected in (1.1) only if such a particle were emitted in coincidence with the beta particle.

It is natural to suppose that the apparent non-conservation of energy, momentum, and angular momentum, together with the actual spectrum shape, have some common explanation. Granting this, it is difficult to avoid the conclusion that some unobserved particle is emitted along with the beta particle. This particle must be neutral, since it is unobserved, have half odd integral spin, to allow for conservation of angular momentum, and have small or zero mass, because of the spectrum shape and because the maximum beta particle energy is equal to the whole available energy for the transition. Furthermore, no more than one such particle can be emitted, because of Sherwin's results and the spectrum shapes. This particle, normally assumed to have spin $\frac{1}{2}\hbar$ and zero mass, is the neutrino; further evidence for its existence is obtained from the non-conservation of energy and momentum in the decays of

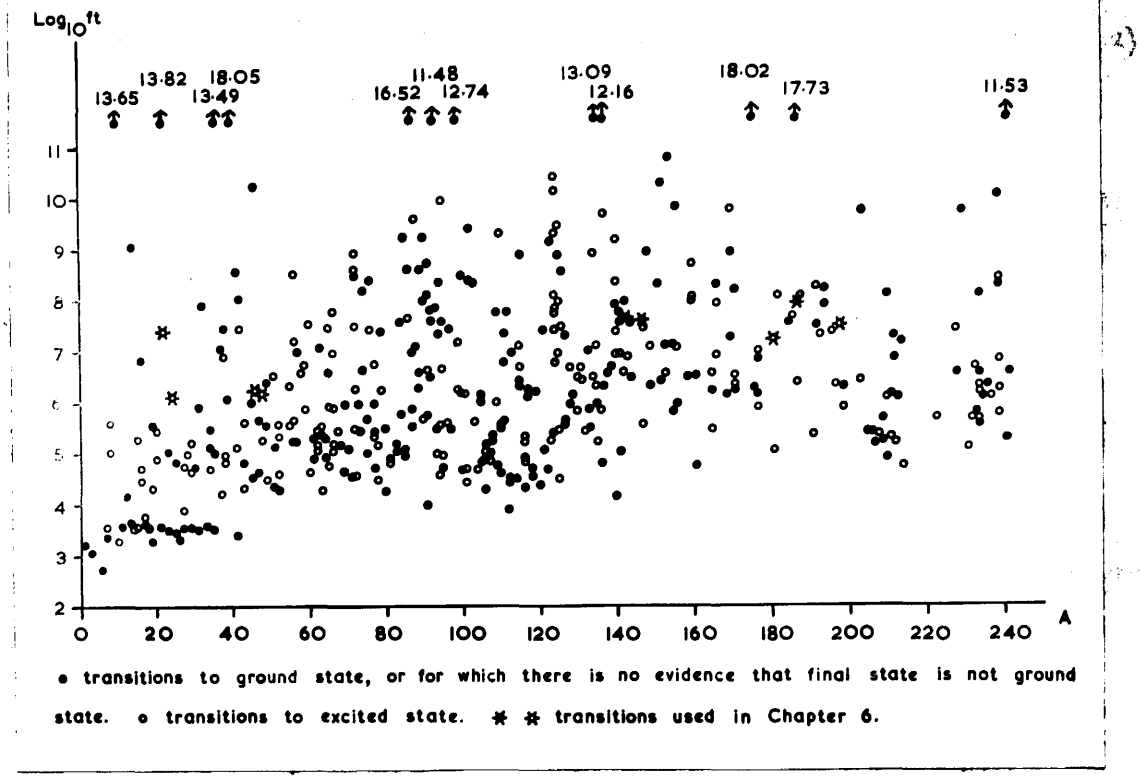


Figure 1. $Log_{10}ft$ plotted against the mass number A. The ft values and the classification of the final states are taken from Feingold (1951). ft values are plotted only for those transitions classified by Feingold as isotope certain or isotope probable.

π – mesons into μ-mesons and of μ-mesons into electrons.

The total probability of the decay of the nucleus per unit time is clearly $X f$, where

$$f = \int_{mc^2}^{W_0} F(Z,W)\, p W (W_0 - W)^2 \, dW, \qquad (1.2)$$

and this must be $\frac{1}{t} \ln 2$, where t is the half-life of the transition. It follows that

$$ft = \frac{1}{X} \ln 2, \qquad (1.3)$$

and since X can depend only on the strength of the interaction causing the transition and on the properties of the initial and final nuclei, it follows that differences in ft for transitions whose beta spectrum is given by (1.1) must be attributed to differences in the properties of the initial and final nuclear states. For this reason, the ft values form a reasonable basis for the empirical classification of transitions.

A fair-sized group of transitions, almost all of light mirror nuclei decaying by K-capture or positron emission, has ft values of about $10^3 - 10^4$. Since for mirror nuclei, the initial and final nuclei can be expected to be very similar in structure, one would expect such transitions to be particularly favoured, and they are therefore said to be superallowed. For larger ft values, the grouping is not so clear cut (see figure 1.), but there appears to be a large group of transitions with ft values in the region of 10^5, while there is a group mainly of medium-weight nuclei with ft values between 2.10^6 and 2.10^7. The first of these groups is interpreted as the group of allowed transitions, while the second is regarded as a

group of first forbidden transitions. There are very few transitions with ft values less than 10^6 for mass numbers between 140 and 200: this is obviously due to some additional orthogonality of the initial and final nuclear states, and supports the view that transitions with ft values greater than 10^6 should be regarded as forbidden. (It is probable that the scatter of ft values in figure 1 could be greatly reduced by considering separately transitions of odd-odd, even-even, odd-even and even-odd nuclei, considering separately transitions to the ground state and to excited states of the product nucleus, and applying a correction for the spin of the final nucleus; such refinements, however, are not required here.).

All transitions whose ft values are less than 10^7, and several transitions with larger ft values, have so far been found to have spectrum shapes given by (1.1). Some transitions with ft values greater than 10^7, however, have spectrum shapes different from that given by (1.1). Since f is essentially a shape correction factor, the ft values for such transitions cannot give more than an approximate indication of the degree of forbiddenness of the transition. Certainly, however, the shapes of the allowed and a sizable group of the first forbidden beta spectra are given by (1.1).

Further experimental evidence will be discussed in Chapters 6 and 7.

The requirements to be satisfied by any theory of

beta decay can now be stated. The theory must postulate some sort of interaction between electrons, neutrinos and nucleons. It must be possible to calculate from this interaction the beta spectrum shape and the lifetime of a transition. Allowed and forbidden transitions must arise naturally from the theory, and the spectrum shapes for both allowed and at least some first forbidden transitions must be given by (1.1). Other beta decay phenomena, not discussed here, such as the shapes of other forbidden spectra and the ratio of K-capture to positron emission for positron emitters must also be satisfactorily described by the theory. The interaction must not be strong enough to give measurable cross-sections for collisions of neutrinos with other particles.

The only suitable theoretical method which has so far been developed is that of quantised field theory with time-dependent perturbation theory. For a field theoretical description of beta decay, there must be an interaction term $\int H' d\tau$ in the Hamiltonian of the fields, where H' is a Lorentz invariant expression, quadrilinear in the electron, neutrino, proton and neutron fields. H' cannot depend on the derivatives of the light particle fields, for if it did, the predicted spectrum shape for allowed transitions would differ from (1.1). This interaction term in the Hamiltonian may be postulated directly, or it may be a consequence of interactions of the heavy and light particles with some intermediate particle such as a meson. The Fermi (1934)

theory of beta decay postulated an interaction H' which is the scalar product of a four-vector formed from the nucleon field and a four-vector formed from the light particle fields; it is now, however, common for any theory whose (directly postulated) interaction is independent of the derivatives of the fields to be called a Fermi type theory, as opposed to the Konopinski and Uhlenbeck (1935) modification, which introduced derivatives of the neutrino field.

There are five linearly independent Lorentz invariant combinations of the fields which are free from derivatives, and the interaction in a Fermi type theory can be an arbitrary linear combination of these five terms. The five "pure" interactions are scalar products of tensor quantities, one formed from the nucleon field, the other from the light particle fields, and are called scalar, polar vector, tensor, axial vector and pseudoscalar (normally denoted by S, V, T, A and P respectively), according to the nature of the constituent tensors. At first sight, the pure interactions appear formally simpler than mixtures; this, however, is spurious, since alternative sets of "pure" interactions can be constructed, for example by forming the tensors from the proton and electron fields and from the neutron and neutrino fields, and these alternative "pure" interactions have been shown by Fierz (1937) and Michel (1950) to be linear combinations of the original set. Symmetry properties

of the interaction with respect to interchanging the fields are likely to be of greater theoretical significance, and consequently Critchfield's suggestion of the interaction S-A-P (Critchfield 1943), which is anti-symmetrical with respect to the interchange of any two fields, is of considerable interest.

Several authors have pointed out that the decay of μ^+-mesons into positrons and the capture of μ^--mesons by protons can both be explained by Fermi type interactions with interaction constants almost identical with that used in the theory of beta decay (see, for example, Tiomno and Wheeler (1949) and Tiomno, Wheeler and Rau (1949)). This has led to the idea of an interaction between spinor fields which is in some sense universal; that is, an interaction between any four spinor fields subject only to some general conditions such as charge conservation.

The importance of the following questions can now be appreciated. What Fermi-type interactions, if any, are capable of explaining all the available experimental evidence on nuclear beta decay? Have any such interactions special symmetry properties? Are any of them "pure" interactions with some particular ordering of the fields? Are any of them capable of being universal interactions; that is, are any of them also capable of explaining the μ-meson decay?
Have any possible universal interactions

special symmetry properties, or are any of them "pure" interactions with some particular ordering of the fields? Answers to these questions, based on comparison of experimental results with theoretical predictions, including work to be described in subsequent chapters, are given in the final chapter. Important for the validity of the theory are the following questions. Can the infinities occurring in the higher order perturbation theory be removed in a consistent way? Is the first order perturbation treatment adequate? Is the assumption of spinor transformation properties for the nucleon fields justified? Renormalisation of the theory has not so far been carried out, and it is therefore impossible at the moment to answer the first question. The perturbation method is essentially an expansion in powers of the interaction constant, and since for the theory of beta decay this is very small, the first order perturbation theory should be adequate provided renormalisation is possible. If, on the other hand, renormalisation proves not to be a universally applicable procedure for the removal of infinities from perturbation theory, some doubt will be thrown on the use of the method in quantum electrodynamics. Because of its success in describing experimental facts, however, the conventional first order perturbation theory in quantum electrodynamics must be a good approximation to the truth,

whatever theory is finally adopted; it is reasonable to suppose the same will be true for other interactions provided, as is the case for the theory of beta decay, the interaction constant is small.

No satisfactory relativistic wave equation for nucleons, taking into account their strong interaction with the meson field, has so far been produced. While it is well-known that of a certain general class of wave equations the only one capable of describing a particle whose spin is $\frac{1}{2}\hbar$ in all Lorentz frames has a spinor wave-function, it is not obvious that other wave equations for spin $\frac{1}{2}\hbar$ particles whose wave-functions are not spinors cannot be constructed. In particular is this not obvious if the particle is assumed to have internal degrees of freedom apart from spin, as may prove necessary if renormalisation procedures fail to remove all the infinities arising in the application of perturbation theory to field theory. By assuming the Lorentz transformations of the field components to be linear, and making reasonable assumptions about the meaning of spin, it is shown at the end of Chapter 2 that any field, irrespective of the field equations, must, if it describes a particle whose spin is $\frac{1}{2}\hbar$ in all Lorentz frames, have spinor transformation properties for all Lorentz transformations not involving reversal of axes, and also for either time reversal or space inversion but not necessarily both. This is consistent

with recent ideas (see, for example, Schwinger 1951) on the nature of the time reversal transformation. The proof of the result is made particularly easy by using the classification of representations of the Lorentz group given by Le Couteur (1950).

Yang and Tiomno (1950) and also Caianiello (1951) have pointed out the possibility of spinors having transformation properties for time reflection and space inversion different from those given by Pauli (1933, 1936) in his well-known treatments. In Chapter 2, therefore, a full discussion of the Lorentz transformation properties of spinors is given which, as a result of making fuller use of the geometrical interpretation of the transformations, is more complete than either of Pauli's treatments. The principles of the method used here are due to Cartan and are described in his book (Cartan 1938), but the approach is considerably altered and most of the derivations of the well-known properties of spinor transformations which are given in this chapter are original. A brief account of Cartan's approach is given in Appendix 1. It is clear from the work of Chapter 2 that the five interaction terms appearing in the Fermi type theory are not necessarily the appropriate ones unless the four fields have identical transformation properties.

In the standard treatments of the theory of beta decay, only the light particle fields and not the nucleon fields are quantised. The interaction is then regarded as a perturbation either in a single particle wave equation, as in Fermi's original formulation, or in the many particle wave equation of a nucleus, as in the generalised formalism of Nordheim and Yost (1937). This situation is clearly unsatisfactory, and therefore the first part of Chapter 3 is devoted to the derivation of Nordheim and Yost's formalism from the formalism of field theory, in which all four fields are quantised. While the methods used are not claimed as original, this particular application of them is thought to be so.

Fermi (1934) calculated the beta spectrum shape for allowed transitions using the polar vector interaction, and such calculations were soon extended to the other pure interactions (see, for example, Bethe and Bacher 1936, Fierz 1937, Hoyle 1937). For the pure interactions, the calculations were extended to forbidden transitions by Konopinski and Uhlenbeck (1941) and Greuling (1942); the last named author calculated the theoretical spectra for the five pure interations up to the fourth forbidden approximation, and inferred gemeral formulae for an arbitrary degree of forbiddenness. The allowed spectra for mixtures of interactions were

were calculated by Fierz (1937) and Hoyle (1937), who neglected the effect of the Coulomb field of the nucleus on the emitted beta particle, and more recently by De Groot and Tolhoek (1950). In order to make the fullest use of experimental data, it is necessary to extend these calculations to forbidden transitions. The shape of the theoretical beta spectrum for an arbitrary mixture of interactions and an arbitrary degree of forbiddenness has therefore been calculated, and the results are given in Appendix 4 in the form of correction factors by which the allowed spectrum shape (1.1) must be multiplied. The method used was essentially that of Konopinski and Uhlenbeck (1941), but the analysis necessary for the general case considered appears to be original. The method of calculation is outlined in Chapter 3, using for the purposes of illustration the polar vector interaction alone; the main analytical result required is proved in Appendix 2, while other analytical results needed for the arbitrary interaction are given in Appendix 3.

The main approximations in these calculations are the rather arbitrary assumptions made about the variation of the electron and neutrino fields within the nucleus, and the neglect of the screening effect of the orbital electrons of the decaying atom. The first of these has been investigated by Rose and Holmes (1951) for pure

interactions, and the necessary correction found to be negligible for allowed and first forbidden transitions, though not for more highly forbidden transitions. The second effect was first investigated by Rose (1936), and later by Longmire and Brown (1949) and Reitz (1950). In Chapter 4, a non-relativistic treatment of the screening effect is used to justify one of the methods used by Rose, which is then extended to include the effect of the residual charge on the product atom. The screening effect for allowed transitions is found to be negligibly small.

Most of the available experimental evidence on spectrum shapes concerns allowed and first forbidden transitions. From a comparison of the theoretically predicted spectrum shapes with experiment for allowed transitions, De Groot and Tolhoek concluded that the interaction is unlikely to contain both scalar and polar vector terms, or both tensor and axial vector terms; more precisely, they estimate that for interactions of the type $aS + bV$, or $aT + bA$, the ratio of the interaction constants a and b must be numerically less than 1/20. The observed first forbidden spectra are of two types, the one type having allowed shape spectra and the other having a shape which is consistent with theory only if the interaction contains either T or A. Detailed comparison of experiment with theory for this second type of spectrum

using a mixture of interactions can only give information about the relative proportions of T and A in the interaction, and since these spectra have not been measured as accurately as many of the allowed spectra, the information obtained will almost certainly be much less accurate than that obtained from the allowed transitions; such a comparison therefore seems at present superfluous. A comparison of the allowed shape first forbidden spectra with theory can, however, give fresh information about the interaction, though to prevent ambiguity in the results transitions must be selected for which selection rules cause any contribution to the theoretical spectrum from the pseudoscalar interaction to vanish. Such a comparison is described in Chapter 6.

Of the twelve nuclear matrix elements occurring in the theoretical description of first forbidden spectra, only six, all matrix elements of vector quantities, are of importance for the work in Chapter 6. In order to reduce the arbitrariness of the theoretical predictions, it is necessary to find relations between these six matrix elements, and this is done in Chapter 5. Using these relations, the only adjustable parameters remaining in the theoretical expressions for the beta spectra are the interaction constants of the four pure interactions which contribute, it being remembered that the transitions are selected so that P does not contribute.

The significance of this work in relation to other evidence on the nature of the beta decay interaction is discussed in Chapter 7, and the questions posed earlier in this chapter are answered as far as seems possible at the present time. In particular, the possibility of a universal quadrilinear interaction, and the significance of its symmetry properties or absence of symmetry properties is discussed.

The work of Chapters 5 and 6, and of course the conclusions derived therefrom in Chapter 7, are entirely original, as is also the direct proof of Greuling's formulae for forbidden beta spectra outlined in Chapter 3, and the extension of Greuling's work to include mixtures of interactions. The work on the screening effect described in Chapter 4 is an extension of one of the treatments given by Rose (1936), while the method used in the non-relativistic treatment using the Hulthén potential is due to Dr. B.F.X. Touschek. While all the other results obtained are already well-known, it is believed that their derivations given here are to some extent original, in that a new approach may be adopted as in the treatment of spinors in Chapter 2, or a proof given of a result previously assumed as in the derivation of Nordheim and Yost's formulation of beta decay theory given in Chapter 3. The proof in Chapter 2 of the spinor nature of spin $\frac{1}{2}\hbar$

fields is thought to be more general than proofs previously given.

The work described in Chapters 5 and 6 and those formulae of Appendix 4 not already given by Greuling (1941) have been published (Pursey 1951).

2. TRANSFORMATION PROPERTIES OF SPIN ½ℏ FIELDS

The best known treatments of the transformation properties of spinors are those given by Pauli (1933,1936). Cartan (1938) has studied the transformations of spinors in space having an arbitrary number of dimensions, using an approach quite different from that of Pauli. In this chapter, the basic principles of Cartan's method are applied to derive all the transformation properties given by Pauli (1933, 1936), sometimes in a more general form, together with the alternative transformations for reflections discussed by Yang and Tiomno (1950) and Caianiello (1951). The method used here differs quite considerably in approach from that of Cartan, although the basic principles are the same, and Cartan's approach is discussed in Appendix 1. At the end of the chapter, the justification for representing spin ½ℏ particles by spinor fields is discussed.

A number of definitions will first be given, followed by a statement of the fundamental theorem used in the method. A Euclidean space is defined as a space in which the square of the length of a vector $\underset{\sim}{x}$ is a positive definite quadratic form $g_{ij}x^i x^j$ (the usual tensor summation convention is used throughout this chapter) known as the fundamental quadratic form; if the length of a vector is given by a quadratic form which is not positive definite, the space is pseudo-Euclidean. The scalar product of two vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$ is $g_{ij}x^i y^j$;

if this is zero, the two vectors are orthogonal. A hyperplane whose normal is a vector $\underset{\sim}{x}$ is the subspace consisting of all vectors orthogonal to $\underset{\sim}{x}$. If a vector $\underset{\sim}{x}$ is self-orthogonal, that is if the square of its length is zero, the vector is called a nul vector: in a pseudo-Euclidean space, or a Euclidean space in which vectors are allowed to have complex components, this does not necessarily imply that all the components of $\underset{\sim}{x}$ are separately zero. The effect on a vector $\underset{\sim}{x}$ of reflection in a hyperplane whose normal $\underset{\sim}{a}$ is not a nul vector is defined by the transformation

$$\left.\begin{array}{c} \underset{\sim}{x} \to \overset{\backprime}{\underset{\sim}{x}} \\ \text{where} \qquad (\underset{\sim}{x} + \overset{\backprime}{\underset{\sim}{x}} \,.\, \underset{\sim}{a}) = 0 \\ \underset{\sim}{x} - \overset{\backprime}{\underset{\sim}{x}} = \lambda \underset{\sim}{a}, \quad \lambda \text{ a number} \end{array}\right\} \qquad (2.1)$$

From these relations it follows at once that $\lambda = 2(\underset{\sim}{x}.\underset{\sim}{a})/a^2$, and

$$\overset{\backprime}{\underset{\sim}{x}} = \underset{\sim}{x} - 2(\underset{\sim}{x}.\underset{\sim}{a})(1/a^2)\underset{\sim}{a} . \qquad (2.2)$$

The concept of a pseudo-Euclidean space is only meaningful if the components of vectors are all real: a pseudo-Euclidean space can always be transformed into a Euclidean space whose vectors may have complex components. In this chapter, pseudo-Euclidean space-time will be treated as a Euclidean space the fourth component of whose vectors are pure imaginary.

A proof of the following fundamental theorem is given by Cartan (1938) t.i., p.13.

Theorem: All linear transformations of co-ordinates leaving invariant the fundamental quadratic form in a Euclidean or pseudo-Euclidean space of n dimensions can be expressed as products of not more than n reflections in hyperplanes whose normals are not nul vectors.

From this theorem, it is obvious that only reflections in hyperplanes need be considered, all other transformations being built up from them.

Pauli (1936) has studied the properties of 4 x 4 matrices satisfying the relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \quad \mu,\nu = 1,2,3,4. \tag{2.3}$$

While in physical applications the γ_μ are normally assumed also to be Hermitian, the complication of the discussion is not increased by removing this restriction, and indeed the meaning of some of the properties of spinor transformations becomes clearer. Some properties of matrices satisfying (2.3) will now be stated: most of these results have been proved by Pauli (1936), and the proofs of the few that have not are trivial.

$$\mathrm{tr}\,\gamma_\mu = 0. \tag{2.4}$$

$$\det\gamma_\mu = 1. \tag{2.5}$$

There exist matrices $\mathcal{A}$ and $\mathcal{B}$ such that

$$\gamma_\mu^+ = \mathcal{A}\gamma_\mu\mathcal{A}^{-1}, \tag{2.6}$$

$$\bar{\gamma}_\mu = \mathcal{B}\gamma_\mu\mathcal{B}^{-1}, \tag{2.7}$$

where $^+$ denotes Hermitian conjugation and $^-$ denotes

transposition. The matrices $\mathcal{A}$ and $\mathcal{B}$ have the properties

$$\mathcal{A}^{+} = \mathcal{A}, \tag{2.8}$$

$$\overline{\mathcal{B}} = -\mathcal{B}, \tag{2.9}$$

$$\mathcal{B}^{+} = \mathcal{A}\mathcal{B}^{-1}\overline{\mathcal{A}}. \tag{2.10}$$

It is convenient to define the matrices $\beta, \mathcal{L}$ as follows:-

$$\beta = \mathcal{A}\gamma_4, \tag{2.11}$$

$$\mathcal{L} = \gamma_5\gamma_4\mathcal{B}^{-1}\overline{\mathcal{A}} = \gamma_5\gamma_4\mathcal{A}^{-1}\mathcal{B}^{+}. \tag{2.12}$$

These two matrices have the properties

$$\left.\begin{aligned} \gamma_k^{+} &= -\beta\gamma_k\beta^{-1}, \quad k=1,2,3, \\ \gamma_4^{+} &= \beta\gamma_4\beta^{-1}, \end{aligned}\right\} \tag{2.13}$$

$$\left.\begin{aligned} \gamma_k &= \mathcal{L}\gamma_k^{*}\mathcal{L}^{-1}, \quad k=1,2,3, \\ \gamma_4 &= -\mathcal{L}\gamma_4^{*}\mathcal{L}^{-1}, \end{aligned}\right\} \tag{2.14}$$

$$\mathcal{L}\mathcal{L}^{*} = 1, \tag{2.15}$$

where * denotes complex conjugation. If the γ_μ are Hermètian, then $\mathcal{A} = 1$ and $\beta = \gamma_4$.

The sixteen matrices $1, \gamma_\mu, \gamma_\mu\gamma_\nu, \gamma_\lambda\gamma_\mu\gamma_\nu, \gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4$, where one product only is taken for each way of selecting the indices from 1,2,3,4, are linearly independent, and since there cannot be more than sixteen linearly independent 4 x 4 matrices, any other 4 x 4 matrix must be expressible as a linear combination of these sixteen products. From this, it is easily shown that, apart from

a numerical factor, the only matrix which anticommutes with all four γ_μ is γ_5.

With any vector $\underset{\sim}{x}$ in space-time, it is possible to associate a 4 x 4 matrix X by the relation

$$X = x_\mu \gamma_\mu \tag{2.16}$$

Throughout this chapter, capital Roman letters will be used to denote the matrices associated in this way with the vectors denoted by the corresponding small letters, and the phrase "the vector X" may be used in the sense of "the vector whose associated matrix is X". From (2.3), (2.5) it is easy to deduce the relations

$$XY + YX = 2(\underset{\sim}{x}.\underset{\sim}{y}) \tag{2.17}$$

$$\det X = x^4 \tag{2.18}$$

(A unit matrix is to be understood on the right hand side of equation (2.17)). From (2.18) it follows that if $\underset{\sim}{x}$ is not a nul vector, X^{-1} exists, while from (2.17) $X^2 = x^2$ and therefore commute with all matrices.

The transformation (2.2) expressed in matrix form becomes

$$\begin{aligned} \acute{X} &= X - [AX + XA]A^{-2}A \\ &= -AXA^{-1} \end{aligned} \tag{2.19}$$

Henceforward, $\underset{\sim}{a}$ will be assumed for simplicity to be a unit vector, by which is meant $a^2 = \pm 1$ according as $\underset{\sim}{a}$ is space-like or time-like.

A column matrix ξ is defined to be a spinor if under a Lorentz transformation its components transform linearly among themselves in the same way as the components of $X\xi$ where X is an arbitrary vector. Only the reflection (2.1) need be considered; suppose under this transformation ξ is transformed to $\xi' = \Gamma\xi$, where Γ is a 4 x 4 matrix. Then

$$\Gamma X \xi = X' \xi' = -AXA^{-1}\Gamma\xi. \tag{2.20}$$

This equation must hold for any spinor ξ , so that

$$\Gamma X = -AXA^{-1}\Gamma,$$

or

$$A^{-1}\Gamma X + XA^{-1}\Gamma = 0. \tag{2.21}$$

This must be true for all vectors X , so that $A^{-1}\Gamma$ must anticommute with all four γ_μ . Hence

$$A^{-1}\Gamma = \alpha\gamma_5,$$

or

$$\Gamma = \alpha A\gamma_5, \tag{2.22}$$

where α is some number as yet undetermined. It is easy to see now that (2.19) may be written

$$X' = \Gamma X \Gamma^{-1} \tag{2.23}$$

The fundamental theorem may now be applied, so that the general Lorentz transformation properties of vectors and spinors are

$$\left.\begin{aligned} X \to X' &= \Lambda X \Lambda^{-1}, \\ \xi \to \xi' &= \Lambda\xi, \end{aligned}\right\} \tag{2.24}$$

where Λ can be written as a product of not more than four

matrices of the type (2.22).

Any hyperplane whose normal is not a nul vector has two unit normals, differing only in sign, and it is impossible to give any unique method of selecting one in preference to the other. Consequently the sign of A, and therefore also of Γ and of any product of Γ's such as Λ, is completely indeterminate, quite independently of the indeterminacy of α. The Λ matrices must therefore form a two valued rather than a single valued representation of the Lorentz group, and in particular the identity must be represented by ± 1. Now two reflections in the same hyperplane must give identity, and from this it is easy to show that

$$\alpha^2 = \pm 1,$$

or

$$\alpha = \pm 1 \text{ or } \pm i. \tag{2.25}$$

Because of the two valued nature of the transformations, it is impossible to distinguish in any absolute sense types of spinors for which α differs only in sign; it is possible, however, to distinguish in an absolute way spinors which transform under reflection with $\alpha = \pm 1$ and spinors which transform with $\alpha = \pm i$. Reflections in hyperplanes will be called time-like or space-like according as the normals to the hyperplanes are time-like or space-like. While α must be the same for all space-like reflections and the same for all time-like reflections, it need not be the same for space-like as for time-like reflections. There can therefore be four types of spinor, which will be called absolute types,

which are classified according to whether $\alpha = \pm 1$ or $\alpha = \pm i$ for space-like and time-like reflections separately. Owing to the two-valued nature of the transformations spinors cannot be further classified according to their transformation properties in any absolute sense. It is possible, however, that two spinors of the same absolute type may always transform under reflections with the same relative sign, which can be either $+$ or $-$, and may be different for space-like and time-like reflections. In this way, spinors of each absolute type can be divided into four relative types, according to whether they transform with the same or opposite sign as a given spinor under space-like and time-like reflections separately. The sixteen spinor types thus introduced are the same as those discussed by Caianiello (1951).

From (2.7), (2.13), (2.14), the commutation relations of γ_5 with γ_μ, and the pure imaginary nature of the fourth component of a vector, the following relations are easily deduced:

$$\gamma_5 \Lambda \gamma_5 = (-1)^N \Lambda, \tag{2.26}$$

$$\mathcal{C} \Lambda^* \mathcal{C}^{-1} = (i\alpha_s)^{2n} (i\alpha_t)^{2m} \Lambda, \tag{2.27}$$

$$\overline{\Lambda} \mathcal{B} \Lambda = (\alpha_s)^{2n} (i\alpha_t)^{2m} \mathcal{B}, \tag{2.28}$$

$$\Lambda^+ \beta \Lambda = (-1)^m \beta, \tag{2.29}$$

where α_s and α_t are the values of α for space-like and time-like reflections respectively, n is the number of space-like reflections, m the number of time-like reflections and $N = n + m$ the total number of reflections in the transformation.

Normally, a Lorentz transformation is defined by

$$x_\mu \to \acute{x}_\mu = a_{\mu\nu} x_\nu, \tag{2.30}$$

where

$$a_{\lambda\nu} a_{\mu\nu} = \delta_{\lambda\mu}. \tag{2.31}$$

Applying (2.30) and using the definition (2.16) of X and the transformation of X given by (2.24), it is found that

$$\Lambda x_\mu \gamma_\mu \Lambda^{-1} = a_{\mu\nu} x_\nu \gamma_\mu,$$

and since this must hold for any vector,

$$\Lambda \gamma_\nu \Lambda^{-1} = a_{\mu\nu} \gamma_\mu,$$

or, on applying (2.31),

$$\Lambda^{-1} \gamma_\mu \Lambda = a_{\mu\nu} \gamma_\nu \tag{2.32}$$

The spinors so far defined are column matrices, and where necessary to avoid ambiguity, they may be called column spinors. A row spinor may be defined to be a row matrix, η say, whose components transform linearly among themselves under a Lorentz transformation in the same way as the components of ηX, where X is any vector. A discussion similar to that given above for column spinors shows that a row spinor transforms according to

$$\eta \to \acute{\eta} = \eta \Lambda^{-1} \epsilon, \tag{2.33}$$

where Λ is the transformation matrix for an arbitrarily chosen column spinor ξ, and ϵ is a number which may be ± 1, or, for reflections, also $\pm i$. Row spinors can be grouped into four absolute types, each of which can be subdivided into four relative types, in a way analogous to the

classification already given for column spinors. A connection can be set up between the types of row spinors and the types of column spinors through the factor ϵ in equation (2.33). If a column spinor ξ and a row spinor η transfonm according to $\xi \to \Lambda \xi$, $\eta \to \epsilon \eta \Lambda^{-1}$, then they can be said to be of the same absolute type if $\epsilon = \pm 1$ for all reflections, and also to be of the same relative type if $\epsilon = +1$ for all reflections. These definitions are consistent with the definitions of absolute and relative types of row and column spinors already given.

A pure number is formed by multiplying any 4 x 4 matrix on the left by a row spinor and on the right by a column spinor; such a number is a bilinear combination of the components of the two spinors, and can clearly have Lorentz transformation properties. Only the sixteen linearly independent 4 x 4 matrices already mentioned need be considered, and the corresponding bilinear combinations of spinor components may be grouped as follows:-

$$\left.\begin{aligned} \Omega^{0} &= \eta\,\xi, \\ \Omega^{1}_{\mu} &= \eta\,\gamma_{\mu}\,\xi, \\ \Omega^{2}_{\mu\nu} &= \eta\,\gamma_{\mu}\gamma_{\nu}\,\xi, \\ \Omega^{3}_{\lambda\mu\nu} &= \eta\,\gamma_{\lambda}\gamma_{\mu}\gamma_{\nu}\,\xi, \\ \Omega^{5} &= \eta\,\gamma_{5}\,\xi. \end{aligned}\right\} \qquad (2.34)$$

From (2.32) and (2.33), it is seen at once that the components of Ω^0, Ω^1, Ω^2, Ω^3, Ω^4 transform like a scalar, polar vector, six vector, axial vector, and pseudoscalar respectively, except that in each transformation the Ω^1s are multiplied by the appropriate value of ϵ. If $\epsilon = -1$ for all reflections, then the Ω^1s transform like the corresponding pseudo-quantities, that is like a pseudoscalar, axial vector, six vector, polar vector and scalar respectively. The Ω^1s may perhaps transform like ordinary quantities under space-like reflections and like pseudo quantities under time-like reflections. If ξ and η are not both of the same absolute type, then ϵ can be $\pm i$ for reflections, and the Ω^1s are not true tensor quantities at all; in fact, it can be seen that the transformations of any Ω would then give a two-valued representation of the Lorentz group.

From a given column spinor, it is possible to construct three other column spinors and four row spinors, all of the same absolute type, but not all of the same relative type. From (2.26), $\gamma_5 \xi$ is a column spinor transforming under all reflections with the opposite sign to ξ; similarly $\eta \gamma_5$ is a row spinor transforming under reflections with the opposite sign to η. $\mathcal{C} \xi^*$, from (2.27), is a column spinor transforming under reflections with the same or

opposite sign to ξ according as $\alpha = \pm i$ or $\alpha = \pm 1$; the third column spinor which can be constructed is $\gamma_5 \mathcal{C} \xi^*$. $\bar{\xi} \mathcal{B}$ is a row spinor which from (2.28) transforms under reflections with the same or opposite sign to ξ according as $\alpha = \pm 1$ or $\pm i$ for space-like reflections and $\alpha = \pm i$ or ± 1 for time-like reflections. $\xi^+ \beta$ is a row spinor which transforms with the opposite sign to ξ under time-like reflections, as follows from (2.29). Column spinors ξ, $\gamma_5 \xi$, $\mathcal{C} \xi^*$, $\gamma_5 \mathcal{C} \xi^*$, and row spinors $\bar{\xi} \mathcal{B}$, $\bar{\xi} \mathcal{B} \gamma_5$, $\xi^+ \beta$, $\xi^+ \beta \gamma_5$ have therefore been formed. In field theoretical applications, $\mathcal{C} \xi^*$ is interpreted as the charge conjugate spinor to ξ. For a neutral particle, $\xi = \mathcal{C} \xi^*$ may be postulated; this equation, which is the basis of the Majorana theory of the neutrino (Majorana 1937, Furry 1938), is self consistent, in that iteration gives $\xi = \xi$. Yang and Tiomno (1950) have pointed out that this equation is Lorentz invariant only if $\alpha = \pm i$ for all reflections; this can be seen from (2.27). If $\alpha = \pm 1$ for all reflections, $\xi = \gamma_5 \mathcal{C} \xi^*$ is Lorentz invariant, but is not self consistent, since on iteration it gives $\xi = -\xi$. The reality properties of the tensor quantities formed as in (2.34) from the row spinor $\xi^+ \beta$ and the column spinor

ξ are independent of the reality properties of ξ and of the representation chosen for the γ_μ. They are therefore suitable for interpretation as physical quantities, and in fact $i\Omega^1$ is normally interpreted as the charge-current density. The $\Omega^0, \Omega^1, \Omega^2, \Omega^3, \Omega^4$ so formed are scalar, polar vector, six vector, axial vector and pseudoscalar quantities under space-like reflections, but are the corresponding pseudo quantities under time reflections; this is the well-known "anomalous" time-reflection property of these quantities.

Five linearly independent scalar quadrilinear combinations of the components of four column spinors ξ_1, ξ_2, ξ_3 and ξ_4, supposed all to be of the same absolute and relative type, can be constructed by taking the scalar products of the five tensors Ω^i formed from $\bar{\xi}_1 \mathcal{B}$ and ξ_2 with the corresponding five constructed from $\bar{\xi}_3 \mathcal{B}$ and ξ_4. Five such scalars can be constructed in this way for each of the 24 possible permutations of the four spinors. Fierz (1937) and Michel (1950), by using rather complicated properties of the γ_μ, have shown that only five of these 120 scalars are linearly independent, and have given a method whereby the scalars formed with one permutation of the spinors can be expressed as linear combinations of the scalars formed with any

other permutation. The same results can be obtained by noticing that $\xi_2 \bar{\xi}_3 \mathcal{B}$ is a 4 x 4 matrix, and can therefore be expressed as a linear combination $\sum_{A=1}^{16} a^A \gamma^A$ of the 16 linearly independent matrices, denoted by γ^A, already given. The coefficients a^A are bilinear combinations of the components of ξ_2 and ξ_3, and can be written $\bar{\xi}_3 \mathcal{B} R^A \xi_2$, where R^A is a 4 x 4 matrix. Since $\bar{\xi}_1 \mathcal{B} \xi_2 \bar{\xi}_3 \mathcal{B} \xi_4$ is a scalar, it is easily seen that $R^A \propto \gamma^A$, and that

$$\bar{\xi}_1 \mathcal{B} \xi_2 \bar{\xi}_3 \mathcal{B} \xi_4 = \sum_{i=0}^{4} c^i \Omega^i(1,4) \Omega^i(3,2), \qquad (2.35)$$

where $\Omega^i(1,4)$ is the Ω^i of equation (2.34) formed from $\bar{\xi}_1 \mathcal{B}$ and ξ_4. Now it is easy to find, from the commutation rules of the γ_μ, an expression for $a^A \gamma^A$ in an expansion of a matrix of the form $\mathcal{S} = \sum_{A=1}^{16} a^A \gamma^A$. In particular, the term involving the unit matrix is

$$\frac{1}{16} \sum_{A=1}^{16} \gamma^A \mathcal{S} \gamma^{A-1} = \frac{1}{16} \sum_{A=1}^{16} \epsilon_A \gamma^A \mathcal{S} \gamma^A, \quad \text{where} \quad \epsilon_A = (\gamma^A)^2 = \pm 1.$$

Applying this to (2.35), one finds

$$c^0 \Omega^0(1,4) \Omega^0(3,2) = \frac{1}{16} \sum_{i=0}^{4} (\pm) \Omega^i(1,2) \Omega^i(3,4)$$

$$= c^0 \sum_{i=0}^{4} c^i \Omega^i(1,2) \Omega^i(3,4). \qquad (2.36)$$

From this, one obtains the coefficients in (2.35), and by picking out other terms on the right hand side, it is possible to express all the scalars formed with the ordering (1432). as linear combinations of these formed with the ordering (1234) The results of interchanging ξ_1, and ξ_2, and of inter-

changing ξ_3 and ξ_4, are easily obtained from the commutation relations of the γ_μ and the properties (2.7), (2.9) of the matrix $\mathcal{B}$, and any permutation of the spinors can be constructed by repeated applications of these three permutations.

All Lorentz transformations can be constructed from spatial rotations leaving invariant ~~two space-like axes, time~~ the time axis and one space-like axis, space-time rotations leaving invariant two space-like axes, time reflection, and space inversion, that is the reversal of all three space-like axes. The corresponding spinor transformations are (from (2.22) and (2.24)):-

Spatial rotations leaving invariant the time axis and the axis

$$\left.\begin{aligned} 1: &\quad \Lambda = \cos\tfrac{\theta}{2} - \gamma_2\gamma_3 \sin\tfrac{\theta}{2}, \\ 2: &\quad \Lambda = \cos\tfrac{\theta}{2} - \gamma_3\gamma_1 \sin\tfrac{\theta}{2}, \\ 3: &\quad \Lambda = \cos\tfrac{\theta}{2} - \gamma_1\gamma_2 \sin\tfrac{\theta}{2}. \end{aligned}\right\} \qquad (2.37)$$

Space-time rotations leaving invariant the space-like axes

$$\left.\begin{aligned} 1 \text{ and } 2: &\quad \Lambda = \cos\tfrac{\theta}{2} - \gamma_4\gamma_3 \sin\tfrac{\theta}{2}, \\ 2 \text{ and } 3: &\quad \Lambda = \cos\tfrac{\theta}{2} - \gamma_4\gamma_1 \sin\tfrac{\theta}{2}, \\ 3 \text{ and } 1: &\quad \Lambda = \cos\tfrac{\theta}{2} - \gamma_4\gamma_2 \sin\tfrac{\theta}{2}. \end{aligned}\right\} \qquad (2.38)$$

Time reflection: $$\Lambda = \pm\alpha_t\gamma_1\gamma_2\gamma_3. \qquad (2.39)$$

Space inversion: $$\Lambda = \pm\alpha_s\gamma_4. \qquad (2.40)$$

The two-valued nature of (2.37) and (2.38) is obvious from the fact that they change sign when θ is increased by 2π.

θ is the angle of the rotation in all cases, and is ofcourse pure imaginary (except for an arbitrary real term $2n\pi$, where n is integral) for the space-time reflections of (2.38). It is convenient to write

$$\left.\begin{aligned} \sigma_x &= i\gamma_2\gamma_3, & \alpha_x &= i\gamma_4\gamma_1, \\ \sigma_y &= i\gamma_3\gamma_1, & \alpha_y &= i\gamma_4\gamma_2, \\ \sigma_z &= i\gamma_1\gamma_2, & \alpha_z &= i\gamma_4\gamma_3. \end{aligned}\right\} \qquad (2.41)$$

If the γ_μ are Hermitian, then so are $\underset{\sim}{\sigma}$ and $\underset{\sim}{\alpha}$, which then have all the properties of the usual $\underset{\sim}{\sigma}$ and $\underset{\sim}{\alpha}$ matrices of Dirac theory. The matrices $\frac{1}{2}i\underset{\sim}{\sigma}$ and $\frac{1}{2}i\underset{\sim}{\alpha}$ are seen from (2.37) and (2.38) to be the generating operators of infinitesimal rotations in the Lorentz group.

Bhabha (1949) has shown on the basis of a number of general postulates that any field which is capable of describing a particle whose spin is $\frac{1}{2}k$ in all Lorentz frames must have spinor transformation properties. Bhabha assumes that the Lorentz transformations of the field are linear and give an irreducible representation of the Lorentz group, that the field components can be uniquely determined at a point on a space-like surface and therefore do not contain any "structural' co-ordinates, that the field equations in the absence of interactions are linear, and that the field equations can be derived from a Lagrangian variational principle. Here, a similar theorem will be proved, but with the last three of

these assumptions replaced by others, which on the surface at least appear more general, based on the ideas of quantised field theory.

The properties of a physical system are assumed to be derivable from the properties of a state vector satisfying a Schrödinger equation whose Hamiltonian is known in terms of a field Ψ. Ψ is regarded as a function of position x_μ in space-time, some further continuous variables λ, and a number of discrete variables s. The discrete variables are supposed to include the numbering of any tensor components, so that the field Ψ can be regarded as having no intrinsic Lorentz transformation properties. Nothing whatever will be assumed about the field equations for Ψ which can be derived from the Hamiltonian together with the commutation relations for Ψ for different values of the variables. It is assumed that a complete set of basic state vectors can be described in terms of a number of single particle states together with occupation numbers for them. The properties of each single particle state are assumed to be given in terms of a wave function ψ which depends on the same set of variables as the field Ψ, which, however, may now be interpreted as the space-time co-ordinates x_μ of the particle together with variables λ, s describing the structure of the particle. Lastly, it is assumed that if for prescribed values of certain of the structural variables each wave-function, for the full range of values of the other structural variables for which it

is defined, forms a basis of a representation of the Lorentz group, the same for all the wave-functions, then for the same prescribed values of the same structural variables and the same range of values of the other structural variables the field Ψ forms a basis of the same representation of the Lorentz group.

The purpose of this last assumption is to overcome the difficulty of defining the angular momentum of the field when the form of the Hamiltonian is completely unknown. As a result of the assumption, only the single particle wave-functions need be considered, and for them the angular momentum can be defined as the linear operator $\underset{\sim}{M} = i\hbar\, \underset{\sim}{m}$, where $\underset{\sim}{m}$ is the generating operator for infinitesimal rotations, that is the effect of an infinitesimal rotation through an angle ω about an axis in the direction $\underset{\sim}{\epsilon}$, where $\underset{\sim}{\epsilon}$ is a unit vector, is given by the operator $(1 + \omega \underset{\sim}{\epsilon}.\underset{\sim}{m})$ (Dirac 1947, p. 142). The generating operator $\underset{\sim}{m}$, and therefore also the angular momentum operator $\underset{\sim}{M}$, can be divided into two commuting parts, the one depending only on position co-ordinates and the other depending only on the structural co-ordinates; the first of these parts must obviously be interpreted as the "orbital" angular momentum, and the second as the "spin" angular momentum.

The effect on a wave-function ψ of an infinitesimal Lorentz transformation

$$x_\mu \to \acute{x}_\mu = x_\mu + \omega_{\mu\nu} x_\nu \qquad (2.42)$$

is given by the operator

$$1 + \omega_{\mu\nu} m_{\mu\nu},$$

where the operators $m_{\mu\nu}$ are the generating operators of the infinitesimal transformations. The generating operators of infinitesimal spatial rotations, which are therefore related to the angular momentum operators, are m_{23}, m_{31} and m_{12}. If the vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are defined by

$$\left.\begin{aligned} p_x &= \tfrac{1}{2} i (m_{23} + m_{14}), \quad p_y = \tfrac{1}{2} i (m_{31} + m_{24}), \quad p_z = \tfrac{1}{2} i (m_{12} + m_{34}) \\ q_x &= \tfrac{1}{2} i (m_{23} - m_{14}), \quad q_y = \tfrac{1}{2} i (m_{31} - m_{24}), \quad q_z = \tfrac{1}{2} i (m_{12} - m_{34}) \end{aligned}\right\} \qquad (2.43)$$

then the components of $\underset{\sim}{p}$ and $\underset{\sim}{q}$ satisfy the usual "angular momentum" commutation relations

$$p_x p_y - p_y p_x = i p_z, \text{ etc.} \qquad (2.44)$$

If now p and q are defined by

$$\left.\begin{aligned} p(p+1) &= p_x^2 + p_y^2 + p_z^2, \\ q(q+1) &= q_x^2 + q_y^2 + q_z^2, \end{aligned}\right\} \qquad (2.45)$$

then p and q commute with each other. The components of $\underset{\sim}{p}$ also commute with the components of $\underset{\sim}{q}$. These commutation rules are a consequence of the multiplication rules for the Lorentz group. These results can easily be shown to hold for the "orbital" and "spin" parts of the generating operators separately. These relations were recently pointed out by Le Couteur (1950).

It is easy to show that all finite irreducible representations of the continuous Lorentz group (obtained by integrating the infinitesimal transformations) are characterised by eigenvalues of p and q which are either integral or half odd integral. Since vectorially the angular momentum operator is $\underset{\sim}{M} = \hbar(\underset{\sim}{P} + \underset{\sim}{Q})$, its magnitude, defined by $j(j+1)\hbar^2 = \underset{\sim}{M}^2$, must have eigenvalues $p+q, p+q-1, \ldots, |p-q|$. Since the representation is irreducible, the angular momentum is not constant under all Lorentz transformations unless either p or q is zero. The operation of space inversion interchanges the roles of $\underset{\sim}{P}$ and $\underset{\sim}{Q}$; consequently, if a representation of the orthochronous Lorentz group (that is, the continuous Lorentz group together with space inversion) contains the representation (p,q) of the continuous Lorentz group, where $p \neq q$, then it must also contain the representation (q,p). The only finite irreducible representation of the orthochronous Lorentz group for which the magnitude of the angular momentum is $\frac{1}{2}\hbar$ in all Lorentz frames is therefore $(\frac{1}{2},0) + (0,\frac{1}{2})$; this representation can be shown to be a spinor representation. By introducing a time reflection transformation, it is possible to construct a finite irreducible representation of the whole Lorentz group, but this cannot be done in a unique manner since owing to the two-valued nature of spinor transformations the space inversion and time reflection operators may either commute or anticommute; for a single valued representation, these operators must commute, while for the spinor representation discussed

earlier in this chapter they anticommute.

Now if, in an irreducible representation of the Lorentz group, the angular momentum operator M_z has only a finite number of eigenvalues, it can be shown that the representation must be finite. For the possible eigenvalues of M_z are all the possible values of $(p_z + \ell_z)\hbar$, and if there are only a finite number of such eigenvalues the number of distinct eigenvalues of p_z and ℓ_z must be finite. Since the representation is irreducible, each eigenvalue of p_z or ℓ_z can occur only a finite number of times. From this the result follows.

The linear operators giving the effect on the wave-function ψ of Lorentz transformations can be divided into two parts, an "orbital" part depending only on the position co-ordinates in space-time and a "spin" part depending only on the structural co-ordinates; the above results can be shown to hold for the orbital and spin parts separately. For the remainder of this chapter, only the spin part will be considered.

A commuting set of operators can now be chosen which is complete in the sense that from operators of the set operators can be constructed which give the effect on the wave-functions of the spin part of any Lorentz transformation whatsoever. The eigenvalues of this set of operators can be regarded as the possible values of structural variables, and

the wave-functions for the full range of these variables must form the basis of a representation of the Lorentz group. This representation can be supposed to be completely reduced, and it must be possible by giving appropriate values to other structural variables to pick out each of its irreducible parts. It is natural to associate each of these irreducible parts with one of the fundamental particles of nature. If one of these fundamental particles has spin $\frac{1}{2}\hbar$ in all states then it follows firstly that its associated representation of the Lorentz group is finite, and secondly that it is a spinor representation at least for the continuous transformations and for space inversion, though it is not necessarily so for time reflection. According to the last of the assumptions on page 36, the field ψ also forms the basis of a representation of the Lorentz group, the irreducible parts of which correspond to the different fundamental particles, while if one particle is always found with spin $\frac{1}{2}\hbar$ the corresponding irreducible representation is a spinor representation at least for continuous transformations and for space inversion.

This discussion differs from the usual treatment in the definition of the spin. Conventionally, an angular momentum tensor is set up in terms of the field, and after quantisation different parts of the angular momentum tensor are interpreted as orbital and spin angular momentum. For this to be possible, an explicit Lagrangian must be assumed.

By defining spin in terms of the single particle states introduced in the second quantised theory, the treatment given above is made independent even of the existence of a Lagrangian. The concept of a single particle wave-function used above is also more general than usual, since such a wave-function is not assumed to describe only one type of fundamental particle. This more general concept could be used to treat, for example, a nucleon with its accompanying meson field as a single particle - such a treatment is not necessarily equivalent to the conventional approach.

3. THE FERMI THEORY OF BETA DECAY.

As was pointed out in the introduction there is some hope that a description of beta decay based on first order time-dependent perturbation theory should give accurate results. For this purpose, a Hamiltonian is assumed of the type $\int H d\tau$ where the integration is over all space, and the Hamiltonian density can be written as

$$H = H_0 + H' \qquad (3.1)$$

with Ho the Hamiltonian density of the unperturbed system and H' the perturbation term. In constructing a suitable perturbation term, one is faced with the choice of regarding protons and neutrons as distinct fundamental particles or as different states of the same particle: the difference in treatment is one of outlook rather than of fundamental physical importance, but in view of the strong internucleon charge exchange forces it is more convenient in this chapter to treat the two particles as different states of the one fundamental particle. For this purpose, an isotopic spin operator τ_3 is introduced in the usual way, neutron and proton states being represented by eigenstates of τ_3 belonging to the eigenvalues $+1$ and -1 respectively.

In setting up the interaction term in the Fermi type of theory, it is implicity assumed that the nucleon field has spinor transformation properties; the work of last chapter shows that this is equivalent to assuming the nucleon spin to be $\frac{1}{2}\hbar$ in all Lorentz frames. It is also assumed that the nucleon, electron and neutrino fields are spinor fields of the same absolute and relative types, or at least that scalar

quantities constructed on this assumption are in fact scalar. Since H' must be Lorentz invariant, and since in the Fermi type theory of beta decay H' cannot involve field derivatives, H' must be a linear combination of the five "pure" interactions, given below in terms of the usual Dirac matrices $\underset{\sim}{\alpha}$, $\underset{\sim}{\sigma}$, β and γ_5:

$$\left.\begin{aligned} S &= (\Psi^+\beta Q \Psi)(\psi^+\beta\varphi) + c.c., \\ V &= (\Psi^+ Q \Psi)(\psi^+\varphi) - (\Psi^+ \underset{\sim}{\alpha} Q \Psi)(\psi^+ \underset{\sim}{\alpha}\varphi) + c.c., \\ T &= (\Psi^+\beta\underset{\sim}{\sigma} Q\Psi)(\psi^+\beta\underset{\sim}{\sigma}\varphi) + (\Psi^+\beta\underset{\sim}{\alpha} Q\Psi)(\psi^+\beta\underset{\sim}{\alpha}\varphi) + c.c., \\ A &= (\Psi^+\underset{\sim}{\sigma} Q\Psi)(\psi^+\underset{\sim}{\sigma}\varphi) - (\Psi^+\gamma_5 Q\Psi)(\psi^+\gamma_5\varphi) + c.c., \\ P &= (\Psi^+\beta\gamma_5 Q\Psi)(\psi^+\beta\gamma_5\varphi) + c.c. \end{aligned}\right\} \qquad (3.2)$$

Here, Ψ, ψ and φ are respectively the nucleon, electron and neutrino fields (the emitted neutral lepton is regarded for convenience as an antineutrino, so that the process may be described equally well as the absorption of a negative energy neutrino), and $Q = \frac{1}{2}(\tau_1 - i\tau_2)$ and its Hermitian conjugate $Q^+ = \frac{1}{2}(\tau_1 + i\tau_2)$ are respectively the operators describing the transition of a neutron into a proton and vice versa.

According to normal time-dependent perturbation theory, the probability per unit time of a transition from an initial state i to a final state f is given by

$$P = \frac{2\pi}{\hbar}\left|(f|\int H' d\underset{\sim}{\tau}|i)\right|^2 \rho(E) \qquad (3.3)$$

where $\rho(E)$ is the density of final states per unit energy range

This formula will be used to obtain the probability per unit time of a nucleus decaying to give an electron in a given energy range. For this purpose, suitable initial and final states must be chosen, and the result of (3.3) summed over all final states having an electron in the required energy range.

In Fermi's original treatment (Fermi, 1934), the interaction was regarded as a perturbation in the Hamiltonian of a single nucleon, only the light particle fields being quantised. The matrix element for the transition was generalised to the many nucleon case by Nordheim and Yost (1937). It is rather unsatisfactory, however, to leave the nucleon field unquantised, and it will be shown that the Nordheim and Yost matrix element can be derived from the interactions (3.2) if all the fields are quantised consistently. The relation between quantised field theory and the many particle wave-functions has already been studied by Fock (1932) and by Becker and Leibfried (1946), and the methods used here are in many ways similar to theirs.

It is convenient to regard the spinor indices as discrete variables which may be denoted by X, and to denote the complete set of field variables $(\underset{\sim}{x}, X)$ by the single symbol ξ. $\delta(\xi-\xi')$ will then be taken to mean $\delta_{XX'}\delta(\underset{\sim}{x}-\underset{\sim}{x}')$.

Th single symbol ξ. $\delta(\xi-\xi')$ will
$\delta_{XX'}\delta(\underset{\sim}{x}-\underset{\sim}{x}')$.

The usual anticommutation rules

$$\left.\begin{aligned} \{\Psi^+(\xi),\Psi(\xi')\} &= \delta(\xi-\xi'), \\ \{\Psi(\xi),\Psi(\xi')\} &= \{\Psi^+(\xi),\Psi^+(\xi')\} = 0 \end{aligned}\right\} \qquad (3.4)$$

are assumed for all three fields, but the nucleon, electron and neutrino fields are assumed to commute with each other. The fields may be expanded in terms of complete sets of orthogonal functions in the following manner:

$$\left.\begin{aligned} \Psi(\xi) &= \sum_\lambda a_\lambda u_\lambda(\xi) + \sum_{\lambda'} \bar{a}^+_{\lambda'} \bar{u}_{\lambda'}(\xi), \qquad \psi(\xi) = \sum_\mu b_\mu v_\mu(\xi) + \sum_{\mu'} \bar{b}^+_{\mu'} \bar{v}_{\mu'}(\xi), \\ \varphi(\xi) &= \sum_\nu c_\nu w_\nu(\xi) + \sum_{\nu'} \bar{c}^+_{\nu'} \bar{w}_{\nu'}(\xi). \end{aligned}\right\} \qquad (3.5)$$

Here for convenience the functions have been classified according as they correspond to positive or negative energy single particle states, the second type of function being denoted by a bar. The orthogonality and completeness relations for the u_λ are

$$\left.\begin{aligned} \int u^*_\lambda(\xi) u_{\lambda'}(\xi)\, d\xi &= \int \bar{u}^*_\lambda(\xi)\, \bar{u}_{\lambda'}(\xi)\, d\xi = \delta_{\lambda\lambda'}, \\ \sum_\lambda u^*_\lambda(\xi) u_\lambda(\xi') + \sum_{\lambda'} \bar{u}^*_{\lambda'}(\xi) \bar{u}_{\lambda'}(\xi') &= \delta(\xi-\xi'), \\ \int u^*_\lambda(\xi)\, \bar{u}_{\lambda'}(\xi)\, d\xi &= \int \bar{u}^*_{\lambda'}(\xi) u_\lambda(\xi)\, d\xi = 0, \end{aligned}\right\} \qquad (3.6)$$

with similar relations for the v_μ, w_ν. From these, together with (3.4), it follows that

$$\left.\begin{aligned} &\{a_\lambda^+, a_{\lambda'}\} = \{\bar{a}_\lambda^+, \bar{a}_{\lambda'}\} = \delta_{\lambda\lambda'}, \\ &\{a_\lambda^+, a_{\lambda'}^+\} = \{a_\lambda, a_{\lambda'}\} = \{\bar{a}_\lambda^+, \bar{a}_{\lambda'}^+\} = \{\bar{a}_\lambda, \bar{a}_{\lambda'}\} = 0, \\ &\{a_\lambda^+, \bar{a}_{\lambda'}\} = \{a_\lambda^+, \bar{a}_{\lambda'}^+\} = \{a_\lambda, \bar{a}_{\lambda'}\} = \{a_\lambda, \bar{a}_{\lambda'}^+\} = 0, \end{aligned}\right\} \quad (3.7)$$

with similar relations for the b_μ, c_ν while any a_λ, b_μ, c_ν commute with each other. The a_λ are interpreted as annihilation operators for nucleons and the $\bar{a}_{\lambda'}^+$ as creation operators for antinucleons, with analogous interpretations for the b's and c's.

The normalised vacuum state vector $|0\rangle$ is defined to be such that

$$\left.\begin{aligned} &a_\lambda|0\rangle = \bar{a}_{\lambda'}|0\rangle = b_\mu|0\rangle = \bar{b}_{\mu'}|0\rangle = c_\nu|0\rangle = \bar{c}_{\nu'}|0\rangle = 0, \\ &\langle 0|0\rangle = 1. \end{aligned}\right\} \quad (3.8)$$

The set of all distinct vectors of the type

$$|\lambda\lambda'\mu\mu'\nu\nu'\rangle = a_{\lambda_1}^+ \cdots a_{\lambda_A}^+ \bar{a}_{\lambda_1'}^+ \cdots \bar{a}_{\lambda_{\bar{A}}'}^+ b_{\mu_1}^+ \cdots b_{\mu_Z}^+ \bar{b}_{\mu_1'}^+ \cdots \bar{b}_{\mu_{\bar{Z}}'}^+ c_{\nu_1}^+ \cdots c_{\nu_n}^+ \bar{c}_{\nu_1'}^+ \cdots \bar{c}_{\nu_{\bar{n}}'}^+ |0\rangle \quad (3.9)$$

forms a complete set of normalised orthogonal basic state vectors, while the most general state vector describing a state with A nucleons, $\bar{A}$ antinucleons, Z electrons,

$\bar{Z}$ positrons, n neutrinos and $\bar{n}$ antineutrinos is

$$|A\rangle = \sum \frac{1}{(A!\bar{A}!z!\bar{z}!n!\bar{n}!)^{1/2}} \alpha(\lambda_1 \ldots \lambda_A, \lambda'_1 \ldots \lambda'_{\bar{A}}, \mu_1 \ldots \mu_z, \mu'_1 \ldots \mu'_{\bar{z}}, \nu_1 \ldots \nu_n, \nu'_1 \ldots \nu'_{\bar{n}}) |\lambda\lambda'\mu\mu'\nu\nu'\rangle \quad (3.10)$$

where the summation is over all values of each of the $\lambda, \lambda', \mu, \mu', \nu$ and ν'. Because of the anticommutation properties of the creation operators, and since the λ, λ', etc., are dummy indices, α must be antisymmetrical with respect to the interchange of any two of the λ, or of the λ', etc. If $|A\rangle$ is normalised, then

$$\sum |\alpha(\lambda\lambda'\mu\mu'\nu\nu')|^2 = 1. \quad (3.11)$$

$|A\rangle$ can be written

$$|A\rangle = \int d\xi_1 \ldots d\xi_A d\xi'_1 \ldots d\xi'_{\bar{A}} d\eta_1 \ldots d\eta_z d\eta'_1 \ldots d\eta'_{\bar{z}} d\zeta_1 \ldots d\zeta_n d\zeta'_1 \ldots d\zeta'_{\bar{n}}$$
$$\alpha(\xi_1 \ldots \xi_A, \xi'_1 \ldots \xi'_{\bar{A}}, \eta_1 \ldots \eta_z, \eta'_1 \ldots \eta'_{\bar{z}}, \zeta_1 \ldots \zeta_n, \zeta'_1 \ldots \zeta'_{\bar{n}}) |\xi, \xi' \eta \eta' \zeta \zeta'\rangle, \quad (3.12)$$

where

$$\left.\begin{aligned} \alpha(\xi\xi'\eta\eta'\zeta\zeta') = \sum \alpha(\lambda\lambda'\mu\mu'\nu\nu') u_{\lambda_1}(\xi_1) \ldots u_{\lambda_A}(\xi_A) \bar{u}^*_{\lambda'_1}(\xi'_1) \ldots \bar{u}^*_{\lambda'_{\bar{A}}}(\xi'_{\bar{A}}) \\ \times v_{\mu_1}(\eta_1) \ldots v_{\mu_z}(\eta_z) \bar{v}^*_{\mu'_1}(\eta'_1) \ldots \bar{v}^*_{\mu'_{\bar{z}}}(\eta'_{\bar{z}}) \omega_{\nu_1}(\zeta_1) \ldots \omega_{\nu_n}(\zeta_n) \omega^*_{\nu'_1}(\zeta'_1) \ldots \omega^*_{\nu'_{\bar{n}}}(\zeta'_{\bar{n}}) \end{aligned}\right\}$$

and

$$\left.\begin{aligned} |\xi\xi'\eta\eta'\zeta\zeta'\rangle = \sum \frac{1}{(A!\bar{A}!z!\bar{z}!n!\bar{n}!)^{\frac{1}{2}}} u^*_{\lambda_1}(\xi_1) \ldots u^*_{\lambda_A}(\xi_A) \bar{u}_{\lambda'_1}(\xi'_1) \ldots \bar{u}_{\lambda'_{\bar{A}}}(\xi'_{\bar{A}}) \\ \times v^*_{\mu_1}(\eta_1) \ldots v^*_{\mu_z}(\eta_z) \bar{v}_{\mu'_1}(\eta'_1) \ldots \bar{v}_{\mu'_{\bar{z}}}(\eta'_{\bar{z}}) \omega^*_{\nu_1}(\zeta_1) \ldots \omega^*_{\nu_n}(\zeta_n) \bar{\omega}_{\nu'_1}(\zeta'_1) \ldots \bar{\omega}_{\nu'_{\bar{n}}}(\zeta'_{\bar{n}}) \\ \times |\lambda\lambda'\mu\mu'\nu\nu'\rangle. \end{aligned}\right\} \quad (3.13)$$

That (3.12) is in fact equivalent to (3.10) may be seen

from the orthogonality relations (3.6). The state vectors $|\xi\xi'\eta\eta'\zeta\zeta'\rangle$ are not orthogonal since the positive energy functions u_λ do not in themselves form a complete set of orthogonal functions. These state vectors, however, must form a complete set of basic state vectors for the system of A nucleons, $\bar{A}$ antinucleons, Z electrons, $\bar{Z}$ positrons, n neutrinos and $\bar{n}$ antineutrinos, since any state vector for the system can be expressed as a linear combination of them. It must be possible, therefore, to form a complete set of normalised orthogonal basic vectors for the system by taking suitable linear combinations of the $|\xi\xi'\eta\eta'\zeta\zeta'\rangle$. A procedure for doing this could be based on the canonical transformations of the Dirac equations discussed by Foldy and Wouthuysen (1950); this, however, is unnecessary for the present purpose, except to note that the transformed vectors form a complete set of basic vectors for the system in the representation in which the position co-ordinates of the particles are diagonal, consequently, the transformed $\alpha(\xi\xi'\eta\eta'\zeta\zeta')$ can be regarded as a many particle wave-function in the Foldy-Wouthuysen representation, and the untransformed $\alpha(\xi\xi'\eta\eta'\zeta\zeta')$ may be regarded as a many particle wave-function in the Dirac representation. From (3.13) and the symmetry

properties of $\alpha(\lambda\lambda'\mu\mu'\nu\nu')$, $\alpha(\xi\xi'\eta\eta'\zeta\zeta')$ is seen to have the correct symmetry properties for this interpretation.

The initial and final state vectors will be linear combinations of vectors of type (3.12) corresponding to definite values of $A-\bar{A}$, $Z-\bar{Z}$, $n-\bar{n}$. If, however, it is assumed that the initial and final states can be described by conventional many particle wave functions, it is necessary to assume definite values for A, $\bar{A}$, Z, $\bar{Z}$, n, $\bar{n}$ separately. the term in the linear combination of state vectors corresponding to these values is then supposed to dominate, and the other terms are regarded as small perturbations which to a first approximation may be ignored. For the emission of electron and antineutrino in beta decay, it will be assumed that $\bar{A}=\bar{Z}=n=0$ for both the initial and final states, and that $\bar{n}=0$ initially. Z will be used to denote the number of electrons in the final state: the number in the initial state will then be Z-1.

The interaction (3.2) can be written as

$$\int d\underset{\sim}{x}\, dX\, dX'\, dx\, dx' \Psi^{+}(\underset{\sim}{x},X)\,\Omega(X,X')\,\Psi(\underset{\sim}{x},X')\,\psi^{+}(\underset{\sim}{x},x)\,\omega(x,x')\,\varphi(\underset{\sim}{x},x')$$

where X, X^{1} denote the discrete variables of the nucleon field, x, x^{1} the discrete variables of the light particle fields, and summation over discrete variables is denoted by an integral sign. The only terms in the expansion of this according to (3.5) which can contribute to the matrix element are those with operator dependence of the type $a^{+}_{\lambda}\, a_{\lambda'}\, b^{+}_{\mu}\, \bar{c}^{+}_{\nu}$, with $\lambda \neq \lambda'$. The matrix element of the interaction between initial and final states of the type (3.10) can now be written down, and is

$$\langle f|\int H' d\tau|i\rangle = A\sqrt{Z}\int d\underset{\sim}{x}\, dX\, dX'\, dx\, dx' \sum \alpha_{f}^{*}(\lambda,\lambda_{2},\ldots\lambda_{A};\mu_{1}\ldots\mu_{z};\nu)$$

$$\times u^{*}_{\lambda}(\underset{\sim}{x},X)\,u_{\lambda_{1}}(\underset{\sim}{x},X')\,\Omega(X,X')\,v^{*}_{\mu_{z}}(\underset{\sim}{x},x)\,\bar{\omega}_{\nu}(\underset{\sim}{x},x')\,\omega(x,x')$$

$$\times \alpha_{i}(\lambda_{1},\lambda_{2}\ldots\lambda_{A};\mu_{1}\ldots\mu_{z-1}), \qquad (3.14)$$

where the summation is over all values of all the λ, μ, ν. Now by using the orthogonality relations (3.6) together with the definition (3.13) of the many particle wave-function, it is easy to show that

$$\alpha(\lambda_1 \ldots \lambda_A; \mu_1 \ldots \mu_z; \nu) = \int d\xi_1 \ldots d\xi_A \, d\eta_1 \ldots d\eta_z \, d\zeta \, \alpha(\xi_1 \ldots \xi_A; \eta_1 \ldots \eta_z; \zeta)$$
$$\times u^*_{\lambda_1}(\xi_1) \ldots u^*_{\lambda_A}(\xi_A) v^*_{\mu_1}(\eta_1) \ldots v^*_{\mu_z}(\eta_z) \bar{w}_\nu(\zeta), \qquad (3.15)$$

while from the definition (3.13) it is clear that

$$\int d\xi \, \alpha(\ldots \xi \ldots) \bar{u}^*_\lambda(\xi) = \int d\eta \, \alpha(\ldots \eta \ldots) \bar{v}^*_\mu(\eta) = \int d\zeta \, \alpha(\ldots \zeta) w_\nu(\zeta) = 0. \qquad (3.16)$$

(3.15) can now be substituted in (3.14), and as a result of (3.16) the summations can be extended over negative energy as well as positive energy states, so that the completeness relation (3.6) can be applied. When this is done, (3.14) reduces to

$$\langle f | \int H' d\tau | i \rangle = \sqrt{Z'} \sum_{i=1}^{A} \int d\xi_1 \ldots d\xi_{i-1} d\xi_{i+1} \ldots d\xi_A \, d\eta_1 \ldots d\eta_{z-1} \, d\underset{\sim}{\xi} \, dX \, dX' \, d\varkappa \, d\varkappa'$$
$$\times \alpha^*_f[\xi_1 \ldots \xi_{i-1}, \xi_i \equiv (\underset{\sim}{\xi}, X), \xi_{i+1} \ldots \xi_A; \eta_1 \ldots \eta_{z-1}, \eta_z \equiv (\underset{\sim}{\xi}, \varkappa); \zeta \equiv (\underset{\sim}{\xi}, \varkappa')]$$
$$\times \Omega(X, X') \omega(\varkappa, \varkappa') \alpha_i[\xi_1 \ldots \xi_{i-1}, \xi'_i \equiv (\underset{\sim}{\xi}, X'), \xi_{i+1} \ldots \xi_A; \eta_1 \ldots \eta_{z-1}]. \qquad (3.17)$$

This is a generalised form of the matrix element given by Nordheim and Yost (1937).

Equation (3.17) is too general in form for convenient application, and several simplifying assumptions must be made. It is assumed in the first place that the wave-functions are separable into factors depending only on the nucleon, electron, and antineutrino co-ordinates respectively. It is also assumed that the final electron

wave-function is an antisymmetrised product of the initial electron wave-function and a single electron wave-function. For convenience, this single electron wave-function can be taken to be one of the orthogonal set ν_μ, and the antineutrino "wave-function" can be taken to be one of the set $\bar{\omega}_\nu^*$; in fact, it is clear from the expansion (3.5) of the neutrino field, and the method of constructing the many particle wave-function, that $\bar{\omega}_\nu^*$ is the complex conjugate of a negative energy neutrino wave-function rather than a positive energy antineutrino wave-function. With these approximations, and reverting once more to the matrix notation for spinor components, (3.17) becomes

$$\langle f|\int H' d\tau|i\rangle = \sum_{i=1}^{A} \int d\underline{x}^1 \ldots d\underline{x}^A \Psi_f^+(\underline{x}^1,\ldots,\underline{x}^A)\,\Omega^i\,[\nu_\mu^+(\underline{x}^i)\,\omega\,\bar{\omega}_\nu(\underline{x}^i)]\,\Psi_i(\underline{x}^1,\ldots,\underline{x}^A) \qquad (3.18)$$

where in the new notation Ψ_i , and Ψ_f are the initial and final nuclear wave-functions, Ω^i and ω are matrix operators, depending on the interaction, operating on the spinor components of the i^{th} nucleon and of the light particle wave-functions respectively. Equation (3.18) is the generalised form of matrix element

for the theory of beta decay proposed by Nordheim and Yost.

The predicted beta spectrum shape for allowed transitions was first calculated by Fermi (1934) for the polar vector interaction (V of(3.2)). Fierz (1937), Hoyle (1937) and De Groot and Tolhoek (1950) have calculated the allowed spectrum shapes for an arbitrary mixture of interactions. Forbidden transitions for the pure interactions were first discussed by Konopinski and Uhlenbeck (1941), while general formulae for the n^{th} forbidden beta spectra for all the pure interactions were inferred by Greuling (1942) from calculations for allowed, first, second, third and fourth forbidden transitions. The direct calculation of the n^{th} forbidden spectrum shape for an arbitrary mixture of interactions is too cumbersome to give in detail, but the method will be illustrated for the pure interaction V.

The following new notations will be used. $\underset{\sim}{A}$ and I will be used to denote the Dirac matrix $\underset{\sim}{\alpha}$ and the unit matrix 1 operating on the nuclear wave-functions, while Greek letters will be used for the Dirac matrices operating on the lepton wave-functions. Operators such as $\underset{\sim}{A}$, $\underset{\sim}{\Delta}$, etc., operating on the nuclear wave-functions, will be denoted by $\underset{\sim}{A}$, $\underset{\sim}{\Delta}$, etc., if they occur in the matrix element $\int \Psi_f^+ \ldots \Psi_i$, and by $\underset{\sim}{A}^*$, $\underset{\sim}{\Delta}^*$, etc., if they

occur in the matrix element $\int \Psi_i^+ \ldots \Psi_f = \left(\int \Psi_f^+ \ldots \Psi_i\right)^*$; it follows that any starred symbol can be commuted with any unstarred symbol, but the ordering of the starred and unstarred symbols among themselves must be preserved unless the corresponding operators commute. Any expression involving both starred and unstarred quantities may be written in the form $X\,Y^*$, and this will be taken to mean $\left(\int \Psi_f^+ X \Psi_i\right)\left(\int \Psi_i^+ Y \Psi_f\right)$?. Units used throughout will be chosen so that $\hbar$, c and m_e are all unity. Any other notations, if not already used in this chapter, will be defined when they are introduced.

The probability per unit time of the emission of an electron with energy in the range $(W,\ W + dW)$ and an antineutrino with energy in the range $(q,\ q+dq)$ where $W + q = W_0$ is the total energy available in the transition, is given by (3.3), where the matrix element is given by (3.18) and its square must be summed over all accessible final electron states with energy W and all accessible negative energy neutrino states with energy -q. A Coulomb wave-function of an electron having definite angular momentum is chosen for ν, while for $\bar{\omega}$ a plane wave is used: $\bar{\omega} = \chi e^{-i\mathbf{q}\cdot\mathbf{r}}$. The density of final states is then

$$\rho(E) = \left(\frac{1}{2\pi}\right)^4 q^2 \frac{2W}{p}\, dW\, d\Omega_\nu , \qquad (3.19)$$

where $d\Omega_\nu$ is an element of solid angle in the direction of $\underset{\sim}{q}$, and $p = \sqrt{W^2 - 1}$. The summation over all final states becomes a summation over all angular momentum states of the electron, summation over spin states of the neutrino, and integration over all directions of the neutrino momentum $\underset{\sim}{q}$.

Denoting the square of the matrix element by $|H'|^2$ and inserting the operators appropriate to the polar vector interaction in (3.18), one obtains

$$|H'|^2 = \left\{ I(\nu^+ \chi) - (\nu^+ \underset{\sim}{A} . \underset{\sim}{\alpha} \chi) \right\} \left\{ I^* (\chi^+ \nu) - (\chi^+ \underset{\sim}{A}^* . \underset{\sim}{\alpha} \nu) \right\} e^{-i \underset{\sim}{q} . \underset{\sim}{R}} \qquad (3.20)$$

where $\underset{\sim}{R} = \underset{\sim}{r} - \underset{\sim}{r}^*$. By introducing the operator $\frac{q - (\underset{\sim}{\alpha} . \underset{\sim}{q})}{2q}$, the summation over spin states of the neutrino can be extended to a summation over all four spinor indices, giving

$$\sum_{\text{spin } \nu} |H'|^2 = \frac{1}{2q} \nu^+(\underset{\sim}{r}) (I - \underset{\sim}{A} . \underset{\sim}{\alpha})(q - \underset{\sim}{\alpha} . \underset{\sim}{q})(I^* - \underset{\sim}{A}^* . \underset{\sim}{\alpha}) \nu(\underset{\sim}{r}^*) e^{-i \underset{\sim}{q} . \underset{\sim}{R}} . \qquad (3.21)$$

(The Hamiltonian for a free particle in the Dirac theory is here taken to be $-\underset{\sim}{\alpha} . \underset{\sim}{p} - \beta m$, in keeping with Dirac (1947), p. 254).

By using the results

$$\left.\begin{aligned}\int d\Omega_q\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{R}} &= 4\pi \sum_{n=0}^{\infty}(-1)^n \frac{(qR)^{2n}}{(2n+1)!}\,,\\ \int d\Omega_q\, \underset{m}{q}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{R}} &= -4\pi i q^2 \underset{m}{R} \sum_{n=0}^{\infty}(-1)^n (qR)^{2n}\frac{2(n+1)}{(2n+3)!}\,,\end{aligned}\right\} \quad (3.22)$$

the integration over the directions of $\underset{m}{q}$ can be performed to give

$$\begin{aligned}\int d\Omega \sum_{\substack{\nu\\ \text{spin}}} |H'|^2 &= 2\pi v^+(\underset{m}{\uparrow})(I - \underset{m}{A}\cdot\underset{m}{\alpha})\sum_{n=0}^{\infty}(-1)^n (qR)^{2n}\left\{\frac{1}{(2n+1)!} + i(\underset{m}{\alpha}\cdot\underset{m}{R})\,q\,\frac{2(n+1)}{(2n+3)!}\right\}\\ &\quad \times (I^* - \underset{m}{A}^*\cdot\underset{m}{\alpha})\, v(\underset{m}{\uparrow}^*)\\ &= 2\pi v^+(\underset{m}{\uparrow})\sum_{n=0}^{\infty}(-1)^n (qR)^{2n}\left\{\frac{1}{(2n+1)!}\left[II^* + \underset{m}{A}\cdot\underset{m}{A}^* + i(\underset{m}{\sigma}\cdot\underset{m}{A}\times\underset{m}{A}^*)\right.\right.\\ &\qquad \left. - (\underset{m}{A}\cdot\underset{m}{\alpha})I^* - I(\underset{m}{A}^*\cdot\underset{m}{\alpha})\right]\\ &\quad + iq\frac{2(n+1)}{(2n+3)!}\left[(II^* - \underset{m}{A}\cdot\underset{m}{A}^*)(\underset{m}{\alpha}\cdot\underset{m}{R}) - I(\underset{m}{A}^*\cdot\underset{m}{R}) - (\underset{m}{A}\cdot\underset{m}{R})I^*\right.\\ &\qquad -iI(\underset{m}{\sigma}\cdot\underset{m}{R}\times\underset{m}{A}^*) - i(\underset{m}{\sigma}\cdot\underset{m}{A}\times\underset{m}{R})I^*\\ &\qquad +(\underset{m}{A}\cdot\underset{m}{R})(\underset{m}{A}^*\cdot\underset{m}{\alpha}) + (\underset{m}{A}\cdot\underset{m}{\alpha})(\underset{m}{A}^*\cdot\underset{m}{R})\\ &\qquad \left.\left. + i\gamma_5[\underset{m}{A}\times\underset{m}{R}\cdot\underset{m}{A}^*)]\right\} v(\underset{m}{\uparrow}^*)\,. \end{aligned} \quad (3.23)$$

The summation over all final states is therefore reduced to the summation of (3.23), or of $v^+(\underset{m}{\uparrow})\,v(\underset{m}{\uparrow}^*)$, $v^+(\underset{m}{\uparrow})\,\underset{m}{\sigma}\,v(\underset{m}{\uparrow}^*)$, $v^+(\underset{m}{\uparrow})\,\underset{m}{\alpha}\,v(\underset{m}{\uparrow}^*)$ and $v^+(\underset{m}{\uparrow})\gamma_5 v(\underset{m}{\uparrow}^*)$, over all the electron angular momentum states.

Because of the large nuclear mass, the nuclear recoil may be neglected, and the centre of mass of the nucleus may be chosen as origin of co-ordinates. Since the integrand in the matrix element contains the nuclear wave-functions, the matrix element cannot depend on the electron and neutrino wave-functions at distances much greater than the nuclear radius ρ from the origin. Now one may expect that the product of the electron and neutrino wave-functions can be expanded in powers of $k\rho$ where k is not much different from $(p+q)$. For most beta transitions, however, $(p+q)\rho$ is less than 1/10, so that in the expansion of the product of the electron and neutrino fields ρ^2 can certainly be neglected compared to 1. This is consistent only because a factor ρ^2 in a matrix element cannot affect the selection rules for a transition: in an expansion of $|H'|^2$ in terms of $\underset{\sim}{\rho}, \underset{\sim}{\rho}^*$ terms proportional to ρ^2 and ρ^{*2} may be neglected, but not terms proportional to $(\underset{\sim}{\rho}.\underset{\sim}{\rho}^*)$ which implies a factor $\underset{\sim}{\rho}$ in the matrix element which does modify the selection rules. Terms proportional to $(\underset{\sim}{A}.\underset{\sim}{\rho})$ and $(\underset{\sim}{A}^*.\underset{\sim}{\rho}^*)$ may also be neglected, since one would expect the magnitude of $\underset{\sim}{A}$ to be of the order of the nucleon velocities in the nucleus, which are thought to be of order 1/10, so that the whole term should be of order 1/100.

The wave-functions of an electron in a Coulomb field have been given in a convenient form by Rose (1937) in terms of mormalised spherical harmonics. For the present purpose, it is more convenient to express the spherical harmonics in Cartesian co-ordinates by means of the relations

$$Y_l^{m\geq 0}(\theta,\varphi)=\left\{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right\}^{\frac{1}{2}}\frac{1}{r^l}\left\{(x+iy)^m z^{l-m}+O(r^2)\right\},\quad Y_l^{-m}=(Y_l^m)^*. \tag{3.24}$$

The wave-functions are of two types, and are given below (for positive m), neglecting terms proportional to r^2.

Type I.

$$\left.\begin{aligned}v_1&=\\v_2&=\\v_3&=\\v_4&=\end{aligned}\right\}\left(\frac{1}{4\pi}\frac{1}{(l+m+\frac{1}{2})!(l-m+\frac{1}{2})!}\right)^{1/2}\frac{(2l)!}{2^l l!}\frac{(x+iy)^{m-\frac{1}{2}}z^{l-m-\frac{1}{2}}}{r^{l+1}}\left\{\begin{aligned}&i(2l+1)z^2 f_l\\&i(2l+1)z(x+iy)f_l\\&(l+m+\tfrac{1}{2})zrg_l\\&-(l-m+\tfrac{1}{2})(x+iy)rg_l\end{aligned}\right. \tag{3.25a}$$

Type II.

$$\left.\begin{aligned}v_1&=\\v_2&=\\v_3&=\\v_4&=\end{aligned}\right\}\left(\frac{1}{4\pi}\frac{1}{(l+m+\frac{1}{2})!(l-m+\frac{1}{2})!}\right)^{1/2}\frac{(2l)!}{2^l l!}\frac{(x+iy)^{m-\frac{1}{2}}z^{l-m-\frac{1}{2}}}{r^{l+1}}\left\{\begin{aligned}&i(l+m+\tfrac{1}{2})zrf_{-l-2}\\&-i(l-m+\tfrac{1}{2})(x+iy)rf_{-l-2}\\&(2l+1)z^2 g_{-l-2}\\&(2l+1)z(x+iy)g_{-l-2}\end{aligned}\right. \tag{3.25b}$$

where

$$r\left\{\begin{matrix}-if_{k-1}\\g_{k-1}\end{matrix}\right\}=\left(\frac{W\mp 1}{W}\right)^{\frac{1}{2}}\frac{(2\rho r)^\gamma e^{\pi\alpha ZW/2\rho}\left|\Gamma(\gamma+i\alpha ZW/\rho)\right|}{2\,\Gamma(2\gamma+1)}\left(k^2+\frac{\alpha^2Z^2}{\rho^2}\right)^{1/2}$$
$$\times\left\{e^{-i\rho r+i\delta}F_1(\gamma+1+i\alpha ZW/\rho,\,2\gamma+1,\,2i\rho r)\mp c.c.\right\} \tag{3.26}$$

$$\left.\begin{aligned} \gamma &= \left(k^2 - \alpha^2 Z^2\right)^{1/2} \\ e^{2i\delta}\left(k^2 + \frac{\alpha^2 Z^2}{\rho^2}\right) &= -(kw - \gamma)(k + \gamma w + i\alpha z\rho)/\rho^2 \end{aligned}\right\} \qquad (3.26)$$

and $\alpha = 1/137$ is the fine structure constant. The wave-functions for $-m$ are obtained from (3.25a) and (3.25b) by the substitutions $v_1(-m) = -v_2^*(m)$, $v_2(-m) = +v_1^*(m)$, $v_3(-m) = -v_4^*(m)$, $v_4(-m) = +v_3^*(m)$. The summation in (3.23) is to be over all half odd integral values of m from $-(l+\frac{1}{2})$ to $+(l+\frac{1}{2})$ and over all positive integral values of l including zero.

To perform this summation, use is made of the following result, which is proved in Appendix 2.

$$\sum_{n=0}^{l} \frac{1}{(l-n)!(l+n)!}(zz^*)^{l-n}\left\{\left[xx^*+yy^*+i(xy^*-yx^*)\right]^n + \left[xx^*+yy^*-i(xy^*-yx^*)\right]^n\right\}$$

$$= \frac{2^l}{(2l)!}(\underset{\sim}{r}\cdot\underset{\sim}{r}^*)^l + \frac{1}{(l!)^2}(zz^*)^l + O(r^2, r^{*2}). \qquad (3.27)$$

It is possible to reduce the sum over m of expressions of the type $v^+(\underset{\sim}{r})v(\underset{\sim}{r}^*)$, $v^+(\underset{\sim}{r})\underset{\sim}{\sigma}v(\underset{\sim}{r}^*)$, etc., to a sum of the type (3.27), together with a term which cancels the term $\frac{1}{(l!)^2}(zz^*)^l$. An example of the procedure, together with the results of the summation for all expressions of this type, is given in Appendix 3. While for the vector interaction, only four such terms have to be summed, for an arbitrary mixture of interactions all

possible terms of this type arise. From Appendix 3, it is seen that

$$\left.\begin{aligned}
&\tfrac{1}{2}\sum_{m,l} v^{+}(\underset{\sim}{r})v(\underset{\sim}{r}^{*}) = \sum_{l=0}^{\infty}\frac{(2l)!}{2^{l}(l!)^{2}}\Big\{(l+1)\,\frac{g_{l}^{2}+f_{l-2}^{2}}{4\pi r^{2l}}(\underset{\sim}{r}\cdot\underset{\sim}{r}^{*})^{l} \\
&\qquad\qquad + (2l+1)\,\frac{g_{l-2}^{2}+f_{l}^{2}}{4\pi r^{2(l+1)}}(\underset{\sim}{r}\cdot\underset{\sim}{r}^{*})^{l+1}\Big\}, \\
&\tfrac{1}{2}\sum_{m,l} v^{+}(\underset{\sim}{r})\underset{\sim}{\sigma}v(\underset{\sim}{r}^{*}) = \sum_{l=0}^{\infty}(-i)[\underset{\sim}{r}\times\underset{\sim}{r}^{*}]\frac{(2l)!}{2^{l}(l!)^{2}}\Big\{-l\,\frac{g_{l}^{2}+f_{l-2}^{2}}{4\pi r^{2l}}(\underset{\sim}{r}\cdot\underset{\sim}{r}^{*})^{l-1} \\
&\qquad\qquad + (2l+1)\,\frac{g_{l-2}^{2}+f_{l}^{2}}{4\pi r^{2(l+1)}}(\underset{\sim}{r}\cdot\underset{\sim}{r}^{*})^{l}\Big\}, \\
&\tfrac{1}{2}\sum_{m,l} v^{+}(\underset{\sim}{r})\underset{\sim}{\alpha}v(\underset{\sim}{r}^{*}) = \sum_{l=0}^{\infty}(-i\underset{\sim}{R})\frac{(2l)!}{2^{l}(l!)^{2}}\,\frac{f_{l}g_{l}-f_{l-2}g_{l-2}}{4\pi r^{2l+1}}(\underset{\sim}{r}\cdot\underset{\sim}{r}^{*})^{l}(2l+1), \\
&\tfrac{1}{2}\sum_{m,l} v^{+}(\underset{\sim}{r})\gamma_{5}v(\underset{\sim}{r}^{*}) = 0.
\end{aligned}\right\} \qquad (3.28)$$

The last result could also be inferred from the impossibility of constructing a pseudoscalar from only two independent vectors.

Inserting (3.28) in (3.23), one obtains

$$\begin{aligned}
\sum_{f}|H'|^{2} = 2\rho^{2}F_{0}\sum_{n=0}^{\infty}\Big\{&II^{*}(\underset{\sim}{r}\cdot\underset{\sim}{r}^{*})^{n}\sum_{\nu=0}^{n}\left[A_{n\nu}q^{2(n-\nu)-2}M_{\nu}-2C_{n\nu}q^{2(n-\nu)-1}N_{\nu}+D_{n\nu}q^{2(n-\nu)}L_{\nu}\right] \\
&+(\underset{\sim}{A}\cdot\underset{\sim}{A}^{*})(\underset{\sim}{r}\cdot\underset{\sim}{r}^{*})^{n-1}\sum_{\nu=0}^{n-1}\left[A_{n-1,\nu}q^{2(n-\nu-1)-2}M_{\nu}+2C_{n-1,\nu}q^{2(n-\nu-1)-1}N_{\nu}+D_{n-1,\nu}q^{2(n-\nu-1)}L_{\nu}\right] \\
&-(\underset{\sim}{A}\cdot\underset{\sim}{r}^{*})(\underset{\sim}{A}^{*}\cdot\underset{\sim}{r})(\underset{\sim}{r}\cdot\underset{\sim}{r}^{*})^{n-2}\sum_{\nu=0}^{n-1}\left[A_{n-1,\nu}q^{2(n-\nu-1)-2}M_{\nu}+2C_{n-1,\nu}q^{2(n-\nu-1)-1}N_{\nu}\right. \\
&\qquad\left.+(D_{n-1,\nu}-E_{n-1,\nu})q^{2(n-\nu-1)}L_{\nu}\right] \\
&+i\left[(\underset{\sim}{A}\cdot\underset{\sim}{r}^{*})I^{*}-(\underset{\sim}{A}^{*}\cdot\underset{\sim}{r})I\right](\underset{\sim}{r}\cdot\underset{\sim}{r}^{*})^{n-1}\sum_{\nu=0}^{n}\left[-A_{n\nu}q^{2(n-\nu)-2}N_{\nu}\right. \\
&\qquad\left.+C_{n\nu}q^{2(n-\nu)-1}L_{\nu}\right]\Big\}
\end{aligned} \qquad (3.29)$$

where

$$\left.\begin{aligned}
&\frac{(1+\gamma)}{4\pi}\rho^2 F_0 = \frac{g_0^2+f_{-2}^2}{4\pi}, \quad \gamma=\sqrt{(1-\alpha^2 Z^2)},\\
&L_\nu = \frac{1}{2\rho^2 F_0}\,\frac{g_\nu^2+f_{-\nu-2}^2}{r^{2\nu}},\\
&M_\nu = \frac{1}{2\rho^2 F_0}\,\frac{g_{-\nu-2}^2+f_\nu^2}{r^{2(\nu+1)}},\\
&N_\nu = \frac{1}{2\rho^2 F_0}\,\frac{f_\nu g_\nu - f_{-\nu-2}g_{-\nu-2}}{r^{2\nu+1}},\\
&A_{n\nu} = \frac{(n-\nu)2^{n-2\nu}(2\nu+1)!}{(2n-2\nu)!\,(\nu!)^2}, \qquad B_{n\nu} = \frac{2^{n-2\nu}(2\nu+1)!}{(2n-2\nu+1)!\,(\nu!)^2},\\
&C_{n\nu} = \frac{(n-\nu)2^{n-2\nu}(2\nu+1)!}{(2n-2\nu+1)!\,(\nu!)^2}, \qquad D_{n\nu} = \frac{2^{n-2\nu}(\nu+1)(2\nu)!}{(2n-2\nu+1)!\,(\nu!)^2}.
\end{aligned}\right\} \quad (3.30)$$

It is reasonable to suppose that the main contribution to the beta decay of nuclei will come from nucleons bound in outer shells, and consequently the main contribution to the matrix elements may be expected to come from distances of the order of the nuclear radius ρ from the centre of mass of the nucleus. In the expressions L_ν, M_ν, N_ν, it is reasonable therefore to replace r by ρ, when they become constants multiplying the matrix elements in (3.29). This is also the justification for writing, for example, $L_\nu = \frac{1}{2\rho^2 F_0}\,\frac{g_\nu^2+f_{-\nu-2}^2}{r^{2\nu}}$,

instead of $\frac{1}{(rr^*)^{\mu}} \frac{g_\nu(\underline{r})g_\nu(\underline{r}^*) + f_{-\nu-2}(\underline{r})f_{-\nu-2}(\underline{r}^*)}{g_0(\underline{r})g_0(\underline{r}^*) + f_{-2}(r)f_{-2}(r^*)}$, in (3.30).

The following n^{th} order tensor constructed from the components of an arbitrary vector $\underline{a}$ and $\underline{r}$ will now be considered.

$$\left.\begin{aligned} Q_{i_1\ldots i_n}(\underline{a}) = \frac{1}{n!}\sum_{m=0}^{\frac{n}{2}(-\frac{1}{2})} \chi_m \Big\{ &(n-2m)r^{2m}\delta_{(i_1i_2}\delta_{i_3i_4}\ldots\delta_{i_{2m-1}i_{2m}}a_{i_{2m+1}}r_{i_{2m+2}}\ldots r_{i_n)} \\ &+2m(\underline{a}\cdot\underline{r})r^{2(m-1)}\delta_{(i_1i_2}\ldots\delta_{i_{2m-1}i_{2m}}r_{i_{2m+1}}\ldots r_{i_n)}\Big\}, \end{aligned}\right\} \quad (3.31)$$

where

$$\chi_m = (-1)^m \frac{1}{2^m m!}\frac{(n-1)(n-2)\ldots(n-2m+1)}{(2n-1)(2n-3)\ldots(2n-2m+1)}, \qquad \chi_0 = \frac{1}{n},$$

the upper limit of the sum is $\frac{n}{2}$ or $\frac{n-1}{2}$ according as n is even or odd, and the brackets round the indices indicate a summation over all permutations of the indices. $Q_{i_1\ldots i_n}(\underline{a})$ is completely symmetrical with respect to interchange of indices, and has been constructed so that all spurs $\sum_{\lambda=1}^{3} Q_{\lambda\lambda i_3\ldots i_n}(\underline{a})$ are zero. $Q_{i_1\ldots i_n}(\underline{a})$ is of the form

$$Q_{i_1\ldots i_n}(\underline{a}) = \frac{1}{n!}a_{(i_1}r_{i_2}\ldots r_{i_n)} + O(r^2, \underline{a}\cdot\underline{r}). \qquad (3.32)$$

The n^{th} order tensor Q will normally be denoted by $Q_n(\underline{a})$.

$Q_n(\underline{a})$ has $2n+1$ linearly independent components which, since they transform linearly among themselves under any rotation, must form a basis of a representation of order $2n+1$ of the three dimensional rotation group; it can be shown that this representation is irreducible.

If $Q_n(\underset{\sim}{a})$ occurs in a matrix element, therefore, the selection rules for the matrix element must be given by the normal conservation laws of angular momentum in quantum theory, an angular momentum of n units being associated with $Q_n(\underset{\sim}{a})$. The parity selection rule will be $\Delta P = (-1)^n$ if $\underset{\sim}{a}$ is a polar vector, $\Delta P = (-1)^{n-1}$ if $\underset{\sim}{a}$ is an axial vector.

The scalar product of $Q_n(\underset{\sim}{a})$ and $Q_n^*(\underset{\sim}{b})$ is easily seen to be given by

$$Q_n(\underset{\sim}{a})Q_n^*(\underset{\sim}{b}) = \frac{1}{n}(\underset{\sim}{a}\cdot\underset{\sim}{b}^*)(\underset{\sim}{r}\cdot\underset{\sim}{r}^*)^{n-1} + \frac{n-1}{n}(\underset{\sim}{a}\cdot\underset{\sim}{r}^*)(\underset{\sim}{r}\cdot\underset{\sim}{b}^*)(\underset{\sim}{r}\cdot\underset{\sim}{r}^*)^{n-2} + \dots \qquad (3.33)$$

As a particular case, by replacing $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in (3.33) by $\underset{\sim}{a}\times\underset{\sim}{r}$ and $\underset{\sim}{b}\times\underset{\sim}{r}$, and simplifying, one obtains

$$Q_n(\underset{\sim}{a}\times\underset{\sim}{r})Q_n^*(\underset{\sim}{b}\times\underset{\sim}{r}) = (\underset{\sim}{a}\cdot\underset{\sim}{b}^*)(\underset{\sim}{r}\cdot\underset{\sim}{r}^*)^{n-1} - (\underset{\sim}{a}\cdot\underset{\sim}{r}^*)(\underset{\sim}{r}^*\cdot\underset{\sim}{b}^*)(\underset{\sim}{r}\cdot\underset{\sim}{r}^*)^{n-2} + \dots \qquad (3.34)$$

From these results it can be shown that, neglecting terms proportional to $(\underset{\sim}{a}\cdot\underset{\sim}{r})$, r^2, $(\underset{\sim}{b}^*\cdot\underset{\sim}{r}^*)$, r^{*2},

$$\left.\begin{aligned} &x(\underset{\sim}{a}\cdot\underset{\sim}{b}^*)(\underset{\sim}{r}\cdot\underset{\sim}{r}^*)^{n-1} + y(\underset{\sim}{a}\cdot\underset{\sim}{r}^*)(\underset{\sim}{r}\cdot\underset{\sim}{b}^*)(\underset{\sim}{r}\cdot\underset{\sim}{r}^*)^{n-2} \\ &\qquad = X\,Q_n(\underset{\sim}{a})Q_n^*(\underset{\sim}{b}) + Y\,Q_{n-1}(\underset{\sim}{a}\times\underset{\sim}{r})Q_n^*(\underset{\sim}{b}\times\underset{\sim}{r}), \end{aligned}\right\} \qquad (3.36)$$

where

$$X = x + y, \qquad Y = \left(\frac{n-1}{n}\right)x - \frac{1}{n}y.$$

By applying (3.36) to (3.29), one obtains Greuling's formula for the theoretical beta spectrum shape for the polar vector interaction.

$$P(w)\,dW = \frac{G^2}{2\pi^3} F_0 p W (W_0 - W)^2 \, dW$$

$$\times \sum_{n=0}^{\infty} \Big\{ \Big| Q_n\Big(\frac{\underline{r}}{u}\Big)\Big|^2 \sum_{\nu=0}^{n} \Big[A_{n\nu} q^{2(n-\nu)-2} M_\nu - 2C_{n\nu} q^{2(n-\nu)-1} N_\nu + D_{n\nu} q^{2(n-\nu)} L_\nu \Big]$$

$$+ \Big| Q_n\Big(\frac{\alpha}{u}\Big)\Big|^2 \sum_{\nu=0}^{n} A_{n\nu} q^{2(n-\nu)-2} L_\nu$$

$$+ i \Big[Q_n\Big(\frac{\alpha}{u}\Big) Q_n^*\Big(\frac{\underline{r}}{u}\Big) - c.c. \Big] \sum_{\nu=0}^{n} \Big[-A_{n\nu} q^{2(n-\nu)-2} N_\nu + C_{n\nu} q^{2(n-\nu)-1} L_\nu \Big]$$

$$+ \Big| Q_{n-1}\Big(\frac{\alpha}{u} \times \frac{\underline{r}}{u}\Big)\Big|^2 \sum_{\nu=0}^{n-1} \Big[A_{n-1,\nu} q^{2(n-\nu-1)-2} M_\nu - 2C_{n-1,\nu} q^{2(n-\nu-1)-1} N_\nu$$

$$+ \Big(D_{n-1,\nu} - \frac{1}{n} B_{n-1,\nu} \Big) q^{2(n-\nu-1)} L_\nu \Big] \Big\}, \qquad (3.37)$$

where G is the interaction constant, and since there is no longer any danger of confusion, Greek letters have been reintroduced to denote the Dirac matrices operating on the nuclear wave-functions.

The terms in this sum corresponding to a definite value of n is interpreted as giving the n^{th} forbidden spectrum shape: this must now be justified. In the first place, it is clear that for a given n, all the matrix elements have the same parity selection rule. Also, in the nuclear matrix elements a factor $\frac{\underline{r}}{u}$ can be expected to contribute an order of magnitude factor ρ, the nuclear radius, with which will be associated a factor of order $\rho + q$ in the spectrum shape, while $\frac{\alpha}{u}$ should be of order v/c where v is a typical nucleon velocity in the nucleus. For most beta transitions, both $(\rho+q)\rho$ and v/c are expected to be of order 1/10, so that

each term in the sum (3.37) should be of order 1/100 times the preceding term. These two reasons - the order of magnitude of the terms and the fact that all the terms for the same n have the same parity selection rule - are the justification for the interpretation of the n^{th} term of (3.37) as giving the n^{th} forbidden spectrum shape. The numerical factors in A_{nr}, B_{nr}, C_{nr}, D_{nr}, L_r, M_r, N_r have been ignored here: it is found that they may affect the relative importance of the different contributions to the n^{th} forbidden spectrum shapes very considerably.

It is very convenient, instead of giving the spectrum shape for an n^{th} forbidden transition directly, to define a correction factor C_n by

$$P_n(W)dW = C_n(W)\,\frac{g^2}{2\pi^3}\,F_0\,pW\,(W_0 - W)^2\,dW\,, \qquad (3.38)$$

where $P_n(W)\,dW$ gives the theoretical n^{th} forbidden spectrum shape. The correction factor C_n for an arbitrary mixture of interactions is very complex, and for this reason it is given in Appendix 4 rather than in the main text.

4. THE SCREENING EFFECT.

Several simplifying assumptions were made in the previous chapter, among which those most likely to affect the result are the assumption of separability for the final electronic wave-function, the use of the Coulomb potential due to a point change in calculating the wave-function of the emitted electron, and the neglect of the screening of this potential due to the orbital electrons. Various processes related to the first of these assumptions have been discussed by Daudel and other workers (see, for example, Daudel and Jean 1949). The second effect has recently been discussed by Rose and Holmes (1951). The screening effect for allowed transitions has been discussed by Rose (1936), Longmire and Brown (1949) and Reitz (1950), the last author performing the calculations on the ENIAC, using the Thomas-Fermi potential. In this chapter, non-relativistic calculations using the Hulthén potential are used to justify a procedure which is used in a relativistic treatment similar to but slightly more general than one of the treatments given by Rose. The Hulthén potential calculation was due to Dr. B.F.X. Touschek.

The effect of the field of the nucleus on an allowed beta spectrum is given by the factor F in equation (1.1), which is essentially the ratio of the electron density at the nucleus of an electron in the nuclear field to the electron density at the nucleus for a free electron:

$$F(Z,W) \sim \frac{|\psi(\rho)|^2}{p^2} \qquad (4.1)$$

where $\psi(\rho)$ is the wave-function of the emitted electron evaluated at the nuclear radius ρ, and p the momentum of the emitted electron. In the non-relativistic case, it is permissible to evaluate ψ at the origin. If $F(Z,W)$ is the F-factor for a screened Coulomb field and $F_c(Z,W)$ is the normal Coulomb factor, then the screening correction can be estimated by calculating $\varphi(Z,W)$, where

$$\varphi(Z,W) = F(Z,W)/F_c(Z,W) - 1 \qquad (4.2)$$

The intention is to calculate φ using a screened Coulomb potential

$$V(r) = \begin{cases} (Z-1)\mu\alpha - Z\alpha/r, & r \leq 1/\mu, \\ -\alpha/r, & r \geq 1/\mu, \end{cases} \qquad (4.3)$$

where Z is the atomic number of the product nucleus, α is the fine structure constant, and μ is the screening constant; $1/\mu$ is of the order of the atomic radius which may be estimated from the Thomas-Fermi atomic model, and in numerical work μ is taken as $Z^{1/3}/121$. The units used are natural units, in which $\hbar = c = m_e = 1$. The potential (4.3) approximates to the Thomas-Fermi potential, and also takes account of the residual charge on the decayed atom. Using this potential, the general solutions of Dirac's equations both for $r \leq \frac{1}{\mu}$ and for $r \geq \frac{1}{\mu}$ can be found from the Coulomb wave-functions given by Rose (1937), but some approximate procedure must be

used to join the solutions at $r = 1/\mu$. In the non-relativistic case, it is possible to solve the Schrödinger equation exactly for the Hulthén potential

$$V(r) = -2\mu Z\alpha \frac{e^{-2\mu r}}{1 - e^{-2\mu r}}, \qquad (4.4)$$

which also approximates to the Thomas-Fermi potential, and the validity of the joining procedure used can be estimated at least in the non-relativistic case, by comparing $\varphi(Z,w)$ calculated from the Hulthén potential with $\varphi(Z,w)$ calculated non-relativistically from the potential

$$V(r) = \begin{cases} Z\mu\alpha - Z\alpha/r, & r \leq 1/\mu, \\ 0, & r \geq 1/\mu, \end{cases} \qquad (4.5)$$

using the joining procedure to be used in the relativistic calculations. This joining procedure consists in equating the asymptotic expansions of the wave-functions for $r \leq 1/\mu$ and $r \geq 1/\mu$ at $r = \frac{1}{\mu}$, and neglecting higher powers of μ than the first. In the non-relativistic case, since the wave equation is second order it is necessary to join both the wave-function and its first derivative by this procedure; in the relativistic case, only the wave-function need be joined, but this must be done for both the large and the small components, so that the number of joining conditions is the same in both cases.

Using the Hulthén potential, the non-relativistic F function is found to be

$$F(Z,W) = \frac{2\pi Z\alpha}{p}\,\frac{1-e^{-2\pi p/\mu}}{\left(1-e^{-\pi(p-P)/\mu}\right)\left(1-e^{-\pi(p+P)/\mu}\right)}\,,$$

provided $\quad p^2 \geq 4\mu\alpha Z\,,$ (4.6)

and where $\quad P^2 = p^2 - 4\mu\alpha Z$

and p is the electron momentum and α the fine structure constant. With the potential (4.5) and the joining procedure, one obtains

$$F(Z,W) = \frac{p'}{p}\,F_N(Z,W')$$

where $\quad W' = W - \mu Z\alpha, \quad p'^2 = 2W', \quad W \geq \mu Z\alpha,$ (4.7)

and $F_N(Z,W)$ is the normal non-relativistic Coulomb factor given by

$$F_N(Z,W) = \frac{2\pi\alpha Z}{p}\,\frac{1}{1-e^{-2\pi\alpha Z/p}}\,. \qquad (4.8)$$

The joining procedure is found to involve an expansion in powers of $\mu/(2p)$, of which powers higher than the first are neglected. The greatest error in using this procedure should therefore occur at low energies, and to estimate the quality of the procedure it should be sufficient to compare the screening effects calculated from the Hulthén potential and the cut-off Coulomb potential at the lowest energy for which the Hulthén potential calculation is valid, that is at $W = 2\mu Z\alpha$. For this energy, the screening corrections φ_H and φ_C calculated from these two potentials according to (4.2) are

$$\left.\begin{aligned} \varphi_H(Z, 2\mu Z\alpha) &= -e^{-a}\,\frac{1-e^{-a}}{1+e^{-a}}, \\ \varphi_c(Z, 2\mu Z\alpha) &= -e^{-a}\,\frac{1-e^{-a(\sqrt{2}-1)}}{1-e^{-a\sqrt{2}-1}}, \end{aligned}\right\} \qquad (4.9)$$

where $a = \pi Z^{1/3}(121/137)^{1/3}$,

and the measure of agreement between these results can be estimated by calculating

$$\chi = \frac{\varphi_c}{\varphi_H} - 1 \approx -e^{-a(\sqrt{2}-1)}. \qquad (4.10)$$

For Z = 8, χ is of the order of $1/10$, and χ decreases as Z is increased. It appears therefore that the two calculations of the screening effect, using the Hulthén potential and using the cut-off Coulomb potential with the joining procedure outlined above, give results of comparable accuracy at least for energies greater than $2\mu Z\alpha$. This quantity increases as $Z^{4/3}$; for Z = 8 it is about 1 KeV.

A remarkable feature of the non-relativistic screening correction appears from (4.9), namely that the effect of screening decreases with increasing Z. Any relativistic contribution to the screening correction can be expected to increase with Z, and consequently relativistic effects may well constitute most of the correction for heavy nuclei.

The relativistic screening correction, using the potential (4.3) which takes account of the residual charge of the atom, may be calculated by a similar procedure to that using the potential (4.5) in the non-relativistic case. The

F function is found to be

$$\left.\begin{aligned} F(Z,W) &= \frac{W'p'}{Wp} F_c(Z,W'), \\ W' &= W - \mu(Z-1)\alpha, \end{aligned}\right\} \qquad (4.11)$$

where

and $F_c(Z,W)$ is the normal relativistic Coulomb factor.(4.11) is clearly the relativistic analogue of (4.7). In order to calculate $\varphi(Z,W)$, the approximation to $F_c(Z,W)$ given by Bethe and Bacher (1936) is used, namely

$$\left.\begin{aligned} F_c(Z,W) &= \frac{4}{[\Gamma(2s+1)]^2} \rho^{2(s-1)} \left[W^2(1+4Z^2\alpha^2)-1\right]^{s-1} F_N(Z,W), \\ \text{where} \quad F_N(Z,W) &= \frac{2\pi Z\alpha W}{p}\left[1-e^{-2\pi Z\alpha W/p}\right]^{-1}, \\ s &= \sqrt{(1-Z^2\alpha^2)}, \end{aligned}\right\} \qquad (4.12)$$

and ρ is the nuclear radius. $F_N(Z,W)$ reduces to the non-relativistic Coulomb factor (4.8) in the non-relativistic limit when $W \approx 1$.

The screening correction $\varphi(Z,W)$ is found to be

$$\left.\begin{aligned} \varphi(Z,W) &= \varphi_N(Z,W) + \varphi_R(Z,W), \\ \text{where} \quad \varphi_N(Z,W) &= \frac{-\exp(-2\pi Z\alpha W'/p')}{1-\exp(-2\pi Z\alpha W'/p')}\left[\exp\left\{2\pi Z\alpha\left(\frac{W'}{p'}-\frac{W}{p}\right)\right\}-1\right], \\ \varphi_R(Z,W) &= -\mu(Z-1)\alpha\frac{d}{dW}\ln\{f(W)\}, \\ f(W) &= W^2\left[W^2(1+4Z^2\alpha^2)-1\right]^{s-1}. \end{aligned}\right\} \qquad (4.13)$$

For low energies, $W \approx 1$ and $W' \approx 1$ and φ_N reduces to the non-relativistic screening correction. φ_R, however, is an effect arising entirely from the use of the relativistic wave

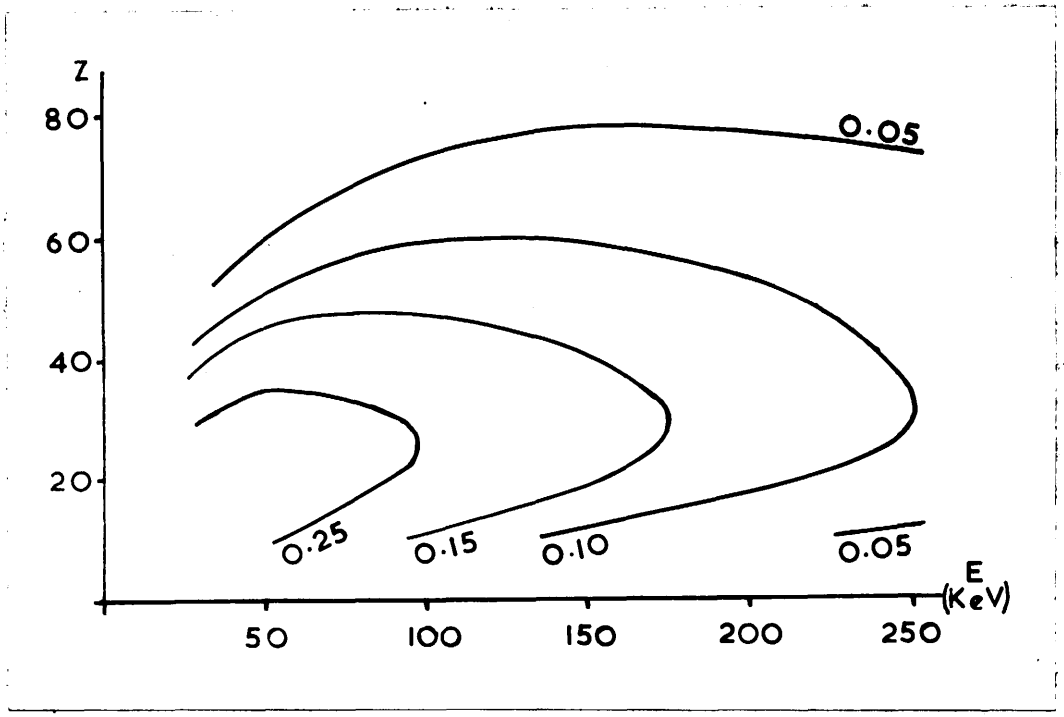


Figure 2. Contour map of $-100\,\varphi_N$ for electrons. $E=W-mc^2$ is the kinetic energy of the emitted beta particle and Z is the atomic number of the product nucleus. The peculiar behaviour of this correction for increasing Z is well illustrated.

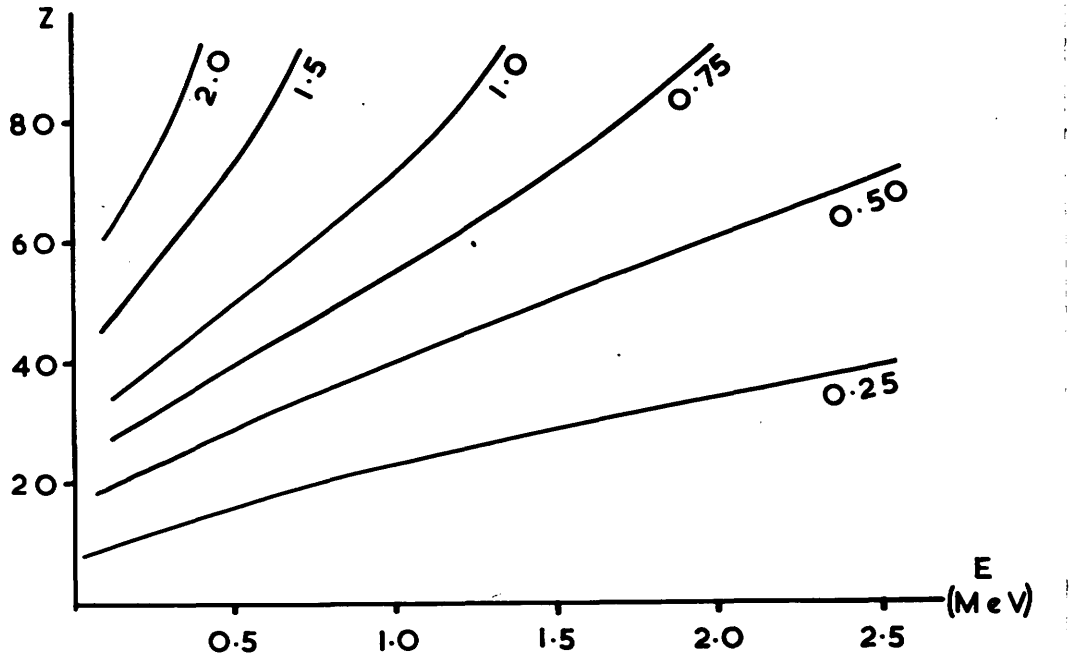


Figure 3. Contour map of $-100\,\varphi_R$ for electrons. The complete screening correction for electrons is the sum of φ_N and φ_R ; for most energies, the relativistic part of the correction predominates. The energy scale in this figure is different from that used in figure 2 for the non-relativistic correction.

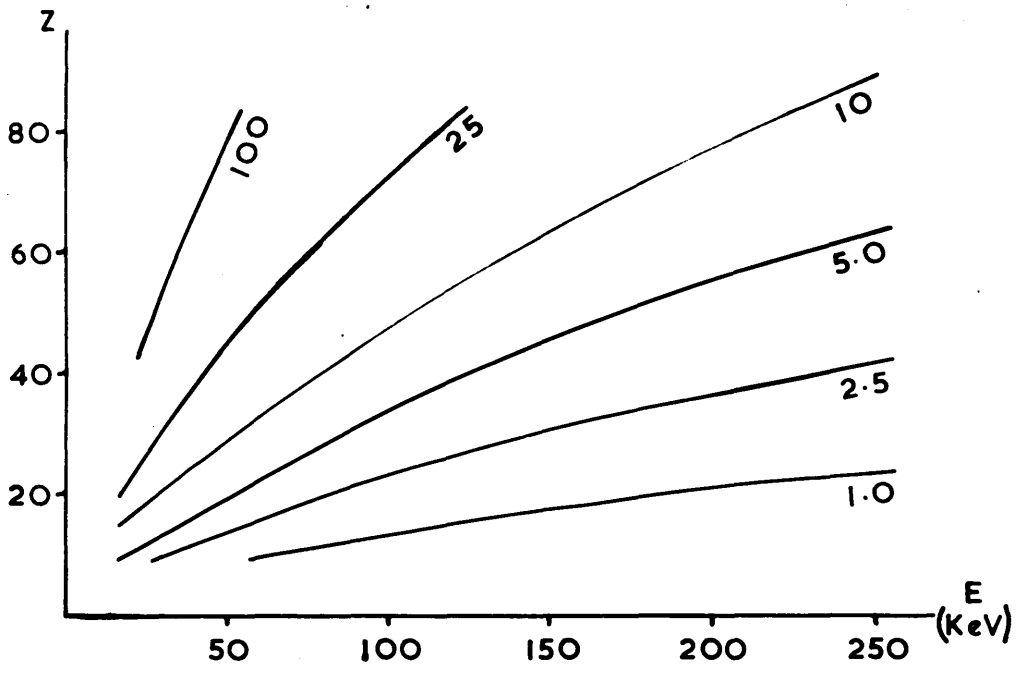


Figure 4. Contour map of $+100\varphi$, the complete screening correction, for positrons. This correction is much larger than the corresponding correction for electron emission, but nevertheless it is doubtful whether the effect could be detected by present experimental techniques.

equations. It is not zero, though it is constant, in the non-relativistic limit $s \approx 1$, $W \approx 1$. It arises as a result of the factor w'/w in (4.11), the explicit occurrence of W and w' in φ_N, and the form of the relativistic Coulomb factor (4.12), none of which by their very nature could appear in a non-relativistic treatment.

The results of numerical calculations are illustrated in the accompanying figures. The strange behaviour of the non-relativistic screening correction is well illustrated by figure 2; this correction, however, is masked by the relativistic correction, ~~and the complete correction~~ which is shown in figure 3. It is clear that the screening correction for electron emission is negligibly small, and may safely be ignored. For positron emission, the sign of Z must be changed in (4.13). In this case, the non-relativistic correction masks the relativistic correction; the screening correction for positron emission is shown in figure 4. For energies less than 50 KeV for the lighter elements or 200 keV for heavier nuclei the screening correction is not negligible. Heavy positron emitting nuclei, however, are very few in number, while as a result of the Coulomb effect the number of low energy positrons emitted is very small and consequently statistics are poor in experimental measurements of the low energy region of positron spectra. For these reasons, the screening correction in positron emission does not seriously

affect the experimental results so far available.

It appears, therefore, that the effect of screening on allowed beta spectra can be ignored when considering the present experimental evidence. While the correction may be larger for forbidden transitions, it is unlikely to be large enough to affect the spectrum shapes to an extent which could be observed experimentally. The results in figures 2, 3, 4 are in qualitative agreement with the numerical calculations of Reitz (1950), using the exact Thomas - Fermi potential.

5. NUCLEAR MATRIX ELEMENTS.

In the next chapter, a comparison of the theoretical predictions with the measured beta spectra of several transitions, all most plausibly of the first forbidden type with allowed shape spectra mentioned in Chapter 1, is used to limit the number of permissible interactions. The correction factor for first forbidden transitions given in Appendix 4 involves twelve distinct nuclear matrix elements, and unless relations can be found between these, or transitions can be chosen to which only a few matrix elements contribute, it will be difficult or impossible to interpret the results of comparing theory with experiment in terms of mixtures of pure interactions. It is the purpose of this chapter to reduce as far as possible the ambiguity in the results due to uncertainties in the ratios of the nuclear matrix elements.

Two of the nuclear matrix elements are second order tensors, $Q_2(C)$ and $Q_2(\beta C)$. They occur in the first forbidden correction factor associated with a shape factor $\frac{1}{3}(p^2+q^2)$, which will be of order of magnitude 1 or 2 for the transitions discussed in the next chapter. The matrix elements themselves should be of order ρ, the nuclear radius (in units of the Compton wave-length of the electron). Now, it will be seen later in this and the next chapter that the other important first forbidden matrix elements are either themselves of order 10ρ or 20ρ, or are associated in the spectrum shape with

factors of order 100. Consequently, the contribution from the matrix elements $Q_2(\underset{\sim}{\Omega})$ and $Q_2(\beta\underset{\sim}{\Omega})$ to the first forbidden correction factor will not normally be more than about 2%, and may be neglected unless the selection rules for the transition cause all the other matrix elements to vanish.

Four of the matrix elements are the pseudoscalars $\langle f|\underset{\sim}{\alpha}.\underset{\sim}{\Omega}|i\rangle$, $\langle f|\beta\underset{\sim}{\alpha}.\underset{\sim}{\Omega}|i\rangle$, $\langle f|\gamma_5|i\rangle$ and $\langle f|\beta\gamma_5|i\rangle$. The last of these is the only first forbidden matrix element occurring with the pseudoscalar interaction, and it is associated with an allowed spectrum shape. It is clear that any allowed shape spectrum in a first forbidden transition to which these matrix elements contribute can be explained merely by supposing P predominates sufficiently strongly in the interaction. Little really significant information about the interaction can be gained from this type of spectrum, therefore, unless transitions are chosen so that the four matrix elements above vanish on account of selection rules. This can be achieved if it is possible to select transitions involving an angular momentum change, and provided this is done these matrix elements need no longer be considered.

The remaining six matrix elements are the vectors $\langle f|\underset{\sim}{\alpha}|i\rangle$, $\langle f|\beta\underset{\sim}{\alpha}|i\rangle$, $\langle f|\underset{\sim}{\Omega}\times\underset{\sim}{\alpha}|i\rangle$, $\langle f|\beta\underset{\sim}{\Omega}\times\underset{\sim}{\alpha}|i\rangle$, $\langle f|\underset{\sim}{\alpha}|i\rangle$ and $\langle f|\beta\underset{\sim}{\alpha}|i\rangle$, and relations will now be derived between them. In the representation of Dirac's equations used (see page 55), the non-relativistic analogues of $\underset{\sim}{\alpha}$ and β are

$-\underset{\sim}{\alpha}/c$ and -1 respectively. Since there is no relativistic theory of the nucleus, it will be permissible to replace β by -1 in the matrix elements above provided it commutes with the other operators in the matrix element.

Consequently

$$\left.\begin{aligned} \langle f|\beta\underset{\sim}{\alpha}|i\rangle &= -\langle f|\underset{\sim}{\alpha}|i\rangle \\ \langle f|\beta\underset{\sim}{\sigma}\times\underset{\sim}{\alpha}|i\rangle &= -\langle f|\underset{\sim}{\sigma}\times\underset{\sim}{\alpha}|i\rangle, \end{aligned}\right\} \qquad (5.1)$$

In recent years, several nuclear shell models have had considerable success in explaining the observed spins of nuclei, and the "magic numbers" of nucleons at which there appears to be some definite discontinuity in nuclear properties. The most generally accepted of these models is the spin-orbit coupling model proposed by Haxel, Jensen and Süss (1949, 1950) and independently by Mayer (1949, 1950). In this model, the nuclear wave-function is regarded as constructed from single particle wave-functions each of which is an eigenfunction of the magnitudes of both the total and the orbital angular momentum; each single particle wave-function is regarded as a solution of the wave-equation for a particle moving in an averaged potential due to the rest of the nucleus. By assuming a suitable form for this averaged potential, one can make plausible a level ordering which can be used to explain the spins of nuclei, the "magic numbers", and several associated phenomena. More relations will be deduced between the nuclear matrix elements by assuming this single particle picture of

nuclei, but it will be unnecessary to assume anything about the particular form of averaged potential producing the single particle states. The results will depend on the single particle interpretation of the transition, and this interpretation must depend on the level scheme assumed: for this purpose, the level scheme proposed by Mayer and by Haxel, Jensen and Süss will be used as a basis, which may be slightly modified to agree with the observed spins of nuclei.

Although the results of the calculations must be regarded as non-relativistic in nature, it is convenient to retain the relativistic notation. Single particle wave-functions for which the magnitudes j and l of the total and orbital angular momenta are constants are eigenfunctions of the operator

$$k = \beta[(\underset{\sim}{\sigma}.\underset{\sim}{m}) + 1], \tag{5.2}$$

where $\underset{\sim}{m}$ is the orbital angular momentum. The eigenvalues of k are

$$k = \begin{cases} -(j+\frac{1}{2}), & j = l+\frac{1}{2}, \\ +(j+\frac{1}{2}), & j = l-\frac{1}{2}. \end{cases} \tag{5.3}$$

k is the well-known angular momentum operator of relativistic theory (see, for example, Dirac 1947, page 266). The following relations, in which $[\,]$ and $\{\}$ denote commutators and anticommutators respectively, are easily proved.

$$\left.\begin{aligned} [k, \underset{\sim}{m}] &= -i\beta\,\underset{\sim}{\sigma}\times\underset{\sim}{m}, \\ [k^2, \underset{\sim}{\alpha}] &= \{k, \beta\underset{\sim}{\alpha}\}, \end{aligned}\right\} \tag{5.4}$$

$$\left.\begin{aligned}[k,[k^2,\underline{\Sigma}]] &= \{k,\underline{\Sigma}\},\\ [k,[k^2,\underline{\alpha}]] &= \{k,\underline{\alpha}\}.\end{aligned}\right\} \qquad (5.5)$$

(5.5) expresses the selection rules $k_f + k_i = 0$ or $k_f - k_i = \pm 1$ for the matrix elements of $\underline{\Sigma}$ and $\underline{\alpha}$; from (5.3) they are seen to be equivalent to $\Delta j = 0, \pm 1$, yes. (5.4) gives the relations

$$\left.\begin{aligned}\langle f|\beta\underline{\alpha}\times\underline{\Sigma}|i\rangle &= -i\epsilon\langle f|\underline{\Sigma}|i\rangle,\\ \langle f|\beta\underline{\alpha}|i\rangle &= -\epsilon\langle f|\underline{\alpha}|i\rangle,\end{aligned}\right\} \qquad (5.6)$$

where $\epsilon = k_i - k_f$ (if $k_i + k_f = 0$, this derivation of the second of equations (5.6) is not valid, but the relation can be shown to hold nevertheless in the non-relativistic case). To ensure that the psudoscalar matrix elements do not contribute, transitions will be selected for comparison with theory only if for them $\Delta j = \pm 1$ seems plausible according to the shell model; for such transitions, $\epsilon = \pm 1$, the sign being obtained from the shell model interpretation of the transition.

The results (5.6), although derived from a single particle model, may well be more general, for one would expect $\langle f|\beta\underline{\alpha}\times\underline{\Sigma}|i\rangle$ to be of the same order as $\langle f|\underline{\Sigma}|i\rangle$ and $\langle f|\beta\underline{\alpha}|i\rangle$ to be of the same order as $\langle f|\underline{\alpha}|i\rangle$, while it is hard to believe that the possibility of ϵ having either sign would occur in only one model. Various attempts can be made to generalise the result by choosing a more general form of angular momentum operator k: however, in the present state

of knowledge of nuclear properties one would still be forced to use the single particle shell model in interpreting transitions, and in this case the single particle interpretation of k in (5.4), and therefore of $\mathbf{\alpha}$ in (5.6), is most convenient in use.

(5.1), together with (5.6), allows the six vector matrix elements to be reduced to two, namely $\langle f|\underset{\sim}{r}|i\rangle$ and $\langle f|\underset{\sim}{\alpha}|i\rangle$. To obtain a relation between these two matrix elements it is necessary to make much more detailed assumptions then hitherto. From (3.18) and (3.2), it is clear that the nuclear matrix element $\langle f|\underset{\sim}{a}|i\rangle$, where $\underset{\sim}{a}$ is any vector operator, is to be interpreted as

$$\langle f|\underset{\sim}{a}|i\rangle = \int d\underset{\sim}{r}^1 \ldots d\underset{\sim}{r}^A \Psi_f^* \sum_{j=1}^{A} \tfrac{1}{2}(\tau_1^j - i\tau_2^j)\,\underset{\sim}{a}^j \Psi_i \,, \qquad (5.7)$$

where Ψ_i and Ψ_f are the initial and final nuclear wavefunctions, τ_1 and τ_2 are the usual isotopic operators, the superscripts refer to the particles of the nucleus, and the integration includes summation over spin and isotopic spin states. The nuclear Hamiltonian is assumed to be

$$\begin{aligned} H = &- \sum_{j=1}^{A} \left\{ (\underset{\sim}{\alpha}^j \cdot \underset{\sim}{p}^j) + \tfrac{1}{2}\beta^j \left[(1+\tau_3^j) M_n + (1-\tau_3^j) M_p \right] \right\} \\ &+ \tfrac{1}{2} \sum_{j=1}^{A} \sum_{k\neq j}^{A} \left\{ V_o(jk) + (\underset{\sim}{\tau}^j \cdot \underset{\sim}{\tau}^k) V_e(jk) - (\underset{\sim}{\sigma}^j + \underset{\sim}{\sigma}^k) \cdot (\underset{\sim}{r}^j - \underset{\sim}{r}^k) \times (\underset{\sim}{p}^j - \underset{\sim}{p}^k) V_s(jk) \right. \\ &\left. + \tfrac{1}{4}(1-\tau_3^j)(1-\tau_3^k) V_c(jk) \right\} , \qquad (5.8) \end{aligned}$$

where M_n and M_p are the neutron and proton masses respectively, and V_o, V_e , V_s , and V_c are respectively the

ordinary, charge exchange, spin-orbit and Coulomb two nucleon potentials. The use of relativistic notation is, as before, for convenience only, and this Hamiltonian is intended only as an approximation valid only in the non-relativistic limit, For the sake of simplicity, and in the absence of any evidence to the contrary, the spin-orbit coupling is supposed not to be of charge exchange type.

By calculating the commutator of $\frac{1}{2}\sum_{j=1}^{A}(\tau_1^j - i\tau_2^j)\hat{r}_\mu^j$ with H, it is found that

$$\langle f|\alpha_\mu|i\rangle = i(W_i - W_f)\langle f|\hat{r}_\mu|i\rangle + i(M_n - M_p)\langle f|\beta\hat{r}_\mu|i\rangle$$
$$-\frac{1}{2}i\sum_{j=1}^{A}\sum_{k\neq j}^{A}\int d\hat{r}_\mu^1 \ldots d\hat{r}_\mu^A \Psi_f^*(\tau_1^j - i\tau_2^j)\Big\{\tau_3^k(\hat{r}_\mu^j - \hat{r}_\mu^k)V_e(jk)$$
$$-i(\sigma_\mu^j + \sigma_\mu^k)\times(\hat{r}_\mu^j - \hat{r}_\mu^k)V_s(jk) - \frac{1}{2}(1-\tau_3^k)V_c(jk)\Big\}\Psi_i . \qquad (5.9)$$

The summed terms in (5.9) will be averaged over the particles labelled k, assuming they form a spin-saturated, spherical core of uniform density and radius ρ equal to the nuclear radius.

For both the exchange force and the spin-orbit coupling contributions in (5.9), the problem is essentially that of averaging $(\hat{r}_\mu^j - \hat{r}_\mu^k)V(r)$, where $r = |\hat{r}_\mu^j - \hat{r}_\mu^k|$. This averaging is performed in Appendix 5, and gives

$$\overline{(\hat{r}_\mu^j - \hat{r}_\mu^k)V(r)} = \hat{r}_\mu^j\left(\frac{3}{16 r_j^3 \rho^3}\right)\int_{|\rho - r_j|}^{\rho + r_j} V(r)\left[(\rho + r_j)^2 - r^2\right]\left[r^2 - (\rho - r_j)^2\right]dr, \qquad (5.10)$$

where $r_j = |\hat{r}_\mu^j|$. Since the transforming single particle state presumably belongs to an outer shell, r_j may be taken approximately equal to ρ. For a Yukawa well of depth v

and range 1/k, this gives

$$(\underset{\sim}{r}^j - \underset{\sim}{r}^k)V(r) = \underset{\sim}{r}^j v\left(\frac{3}{2k^3\rho^3}\right)\left\{(k\rho)^{-1} - 3(k\rho)^{-3} + \exp(-2k\rho)\left[2 + 5(k\rho)^{-1} + 6(k\rho)^{-2} + 3(k\rho)^{-3}\right]\right\}. \quad (5.11)$$

(Hughes and Le Conteur (1950) have also found this result, using, however, a rather different method). For a square well of depth v and range 1/k, (5.10) with $r_3 = \rho$ gives

$$(\underset{\sim}{r}^j - \underset{\sim}{r}^k)V(r) = \underset{\sim}{r}^j v\left(\frac{3}{16k^3\rho^3}\right)\left[(k\rho)^{-1} - \tfrac{1}{6}(k\rho)^{-3}\right]. \quad (5.12)$$

In the charge exchange contribution to $\langle f|\underset{\sim}{\alpha}|i\rangle$, the factor r_3^k gives on averaging a factor $\frac{N-Z}{A}$, where N is the neutron number of the initial nucleus and Z the proton number of the product nucleus, while the summation over k gives a factor A. Ve is assumed to be of the form $V_e(jk) = \left[a_1 + a_2(\underset{\sim}{\sigma}^j \cdot \underset{\sim}{\sigma}^k)\right]V(jk)$, but because of the spin saturation of the core, only the term with the coefficient a_1 will contribute: for the same reason, there will be no contribution from any tensor forces of the usual type. Using the value given by Rosenfeld (1948, Page 233) for a_1, based on the ratio of the singlet and triplet s potentials and the requirement of saturation for heavy nuclei, the effective Ve is found to be $-\frac{1}{10}V(^3s)$, where $V(^3s)$ is the triplet s potential. Taking the nuclear radius ρ to be $r_0 A^{1/3}$, and using both a square well of depth $40mc^2$ and range $2r_0$ and a Yukawa well of depth $100mc^2$ and range r_0 for $V(^3s)$, the exchange force contribution to $\langle f|\underset{\sim}{\alpha}|i\rangle$ is found to be

$$\left.\begin{aligned} &\text{Square Well:} \quad -12i(N-Z)A^{-4/3}\left[1-\tfrac{2}{3}A^{-2/3}\right]\langle f|\underset{\sim}{r}|i\rangle, \\ &\text{Yukawa Well:} \quad -15i(N-Z)A^{-4/3}\left[1-3A^{-2/3}\right]\langle f|\underset{\sim}{r}|i\rangle. \end{aligned}\right\} \qquad (5.13)$$

The two wells give results in fair agreement. This exchange contribution to $\langle f|\underset{\sim}{\alpha}|i\rangle$ is so small in any case that the use of more accurate well parameters is unnecessary.

Hughes and Le Conteur (1950) estimate that for V_s a Yukawa well of range λ_0 and depth $1 MeV$ is consistent with the separation necessary according to the spin-orbit coupling shell model to account for the apparent level order in heavy nuclei. (The factor 2 difference between this well depth and that given by Hughes and Le Conteur is due to a different definition of $\underset{\sim}{\sigma}$; here, the spin angular momentum is $\frac{1}{2}\underset{\sim}{\sigma}$) On averaging, the contribution to $\langle f|\underset{\sim}{\alpha}|i\rangle$ from the spin-orbit coupling is found to be

$$-3A^{-1/3}\left[1-3A^{-2/3}\right]\langle f|\underset{\sim}{\sigma}\times\underset{\sim}{r}|i\rangle = -3i\epsilon A^{-1/3}\left[1-3A^{-2/3}\right]\langle f|\underset{\sim}{r}|i\rangle \qquad (5.14)$$

using (5.6) and (5.1).

The Coulomb contribution to $\langle f|\underset{\sim}{\alpha}|i\rangle$ can be estimated by a similar averaging procedure, and is found to be $i(Z\alpha/\rho)\langle f|\underset{\sim}{r}|i\rangle$.

Collecting these results, one obtains

$$\left.\begin{aligned} &\langle f|\underset{\sim}{\alpha}|i\rangle = i\varkappa\langle f|\underset{\sim}{r}|i\rangle, \\ &\text{where} \\ &\varkappa = (W_i - W_f) - (M_n - M_p) + Z\alpha/\rho - 15A^{-\frac{4}{3}}(N-Z)(1-3A^{-2/3}) \\ &\qquad - 3\epsilon A^{-1/3}(1-3A^{-2/3}). \end{aligned}\right\} \qquad (5.15)$$

For most first forbidden transitions, the first two terms in x

approximately cancel, while the exchange force and spin-orbit coupling terms are small compared to the Coulomb term. For positron emitters, the second, third, and fourth terms in x change sign.

In using the relations derived above, use must be made of the shell model interpretation of the transitions, and it would be very valuable to have independent evidence of the validity of this. Such evidence may be provided by the decay of P^{32}. The spectrum of P^{32} has been measured by many workers (Siegbahn 1946, Langer and Price 1949, Agnew 1950; some uncertainties about the low energy end of the spectrum have been cleared up by Jensen et al. 1952 and by Sheline et al. 1951), and there can be no doubt that the spectrum has the allowed shape. The ft value for the transition is 8×10^{7}, which is considerably greater than one would expect for an allowed or first forbidden transition: it is, however, impossible to fit the beta spectrum with any of the second forbidden correction factors. It will be shown that this apparent contradiction can be resolved using the shell model interpretation of the transition.

The only correction factors that are capable of explaining the shape of the beta spectrum of P^{32} are those for the allowed transitions, for first forbidden transitions with unit spin change according to the tensor and polar vector interactions, and for first forbidden transitions with no

spin change according to the axial vector and pseudoscalar interactions; the spin change in the transition must therefore be 0 or 1. Since no gamma rays are observed, the transition must be to the ground state of S^{32}, and since this has spin zero, the spin of P^{32} must be 0 or 1. As the transition must be either allowed or first forbidden, some mechanism such as an additional selection rule is necessary to explain the large ft value.

According to the shell model, neutron and proton levels are filled in the order $s_{1/2}$, $p_{3/2}$, $p_{1/2}$, $d_{5/2}$, $s_{1/2}$, $d_{3/2}$, ... (this ordering of the $s_{1/2}$ and $d_{3/2}$ levels is necessary to explain the observed spins of nuclei). The shell model interpretation of the decay is therefore that a $d_{3/2}$ neutron is transformed into a $s_{1/2}$ proton. Since this transition does not involve a parity change, it cannot be a first forbidden transition, and must, bearing in mind the results of the previous paragraph, be allowed. Also, the odd neutron and odd proton in P^{32} must be in $d_{3/2}$ and $s_{1/2}$ states respectively, and hence the spin of P^{32} according to the shell model must be either 1 or 2. From the last paragraph, however, the spin of P^{32} is either 0 or 1: consequently if the shell model interpretation is correct, the spin of P^{32} must be 1, and the spin change in the transition must also be 1.

The only allowed matrix elements which permit a unit spin change are $\langle f|\alpha|i\rangle$ and $\langle f|\beta\alpha|i\rangle$, which occur

respectively with the axial vector and tensor interactions: in the next chapter, strong independent evidence will be given for believing the interaction to contain one or other of these interactions, but not both. Now, the transition $d_{3/2} \rightarrow s_{1/2}$ involves a change of two units in the orbital angular momentum, while non-relativistically at least the matrix elements $\langle f|\mathfrak{T}|i\rangle$ and $\langle f|\beta\mathfrak{T}|i\rangle$ cannot change the magnitude of the orbital angular momentum; non-relativistically, therefore, the decay of P^{32} as interpreted by the shell model cannot occur through an allowed matrix element. The fact that the transition does take place, though with a greatly reduced probability, may be taken as evidence that the shell model interpretation is nearly but not completely correct; alternatively, however, it may be that the matrix elements of $\mathfrak{T}$ and $\beta\mathfrak{T}$, although zero non-relativistically, are in fact not strictly zero for the transition. Reasons will now be given for believing this to be so, and for thinking that both the lifetime and the spectrum shape can be explained in this way without supposing any break-down of the shell model interpretation of the transition.

It is easy to find selection rules for $\langle f|\mathfrak{T}|i\rangle$ and $\langle f|\beta\mathfrak{T}|i\rangle$ in a way similar to that used for $\langle f|\mathfrak{t}|i\rangle$; it is found that $k_i = k_f$ or $k_i + k_f = \pm 1$, or in terms of the total angular momentum, $\Delta j = 0, \pm 1$, no. From the

commutation relations of σ_m and k, it is also easy to show that, provided $k_i \neq k_f$,

$$\langle f|\beta\sigma_m|i\rangle = (k_i + k_f)\langle f|\sigma_m|i\rangle. \qquad (5.16)$$

Now β commutes with σ_m, so non-relativistically it is permissible to write

$$\langle f|\beta\sigma_m|i\rangle = -\langle f|\sigma_m|i\rangle. \qquad (5.17)$$

This is inconsistent with (5.16) if $k_i + k_f = +1$; for such transitions, therefore, $\langle f|\sigma_m|i\rangle$ and $\langle f|\beta\sigma_m|i\rangle$ must vanish in the non-relativistic limit. Since, however, the selection rules are satisfied, one would not expect the matrix elements to vanish identically, but would expect them to be of order $(v/c)^2$. The same result may be obtained by noticing that for $k_i + k_f = +1$, the equation (5.16) is self-consistent only if the contribution from the "large" components of the wave-functions vanishes. From (5.3), the selection rule $k_i + k_f = +1$ can be seen to be equivalent to $\Delta j = \pm 1$, $\Delta l = \pm 2$; for the allowed transitions with $k_i + k_f = \mp 1$ or $k_i = k_f$, $\Delta l = 0$.

For the rest of this argument the pure tensor interaction will be assumed; an exactly similar argument, however, holds for the axial vector interaction. Since the matrix element $\langle f|\beta\sigma_m|i\rangle$ for the transition is of order $(v/c)^2$, the contribution from the second forbidden matrix element $\langle f|\beta\alpha_m x_n|i\rangle$, which is expected to be of order $(v/c)\rho$ where ρ is the nuclear radius, cannot be neglected. The correction factor can be calculated by the methods of Chapter 3:

it is found to be essentially similar to $C_1(T)$ given in Appendix 4, but with $Q_1(\beta\alpha_m)$ and $Q_1(\beta\alpha_m x_m^1)$ replaced by $\langle f|\beta\alpha|i\rangle$ and $\langle f|\beta\alpha_m x_m^1|i\rangle$ respectively. If ρ is taken as $(1/274)A^{1/3}$, then the only uncertainties remaining are the value of v/c (or alternatively $x = v/(c\rho)$) and the sign of the cross term in the correction factor, and these can be chosen to fit the observed lifetime of the transition. Using the values of the interaction constant given by Greuling (1942), it is found that the lifetime is well fitted if $x = 12$ $(v/c = 0.14)$ if the cross term is negative, and if $x = 5.5$ $(v/c = 0.064)$ if the cross term is positive: in both these cases, the beta spectrum shape is experimentally indistinguishable from the allowed shape.

Sherwin (1951) has measured the angular correlation between the electron and the neutrino emitted by P^{32}, and found it to be given approximately by $(1 + \cos\theta)$. From the theoretical work of Greuling and Meeks (1951), it is clear that this angular correlation is consistent only with an allowed transition with no spin change on the polar vector interaction and with a first forbidden transition with no spin change on the axial vector interaction; it is not consistent with the explanation of the lifetime and spectrum shape given above, for which one would expect an angular correlation given by $\left(1 + \frac{1}{3}\cos\theta\right)$ for the tensor interaction or $\left(1 - \frac{1}{3}\cos\theta\right)$ for the axial vector interaction. Owing to the extreme difficulty of the experiment, however, it is possible that the

discrepancy between the observed angular correlation and that predicted by the tensor interaction may prove more apparent than real.

If the angular correlation results are accepted, then the problem of the large ft value of the transition remains unsolved, but the spin of P^{32} must be zero, and the interaction must contain either V or A. If the angular correlation results are considered doubtful, then the success of the explanation of the spectrum shape and lifetime given above gives confidence in the use of the shell model interpretation of transitions. The shell model interpretation requires the spin of P^{32} to be one unit, and therefore a direct measurement of the spin of this nucleus would be of great value.

6. COMPARISON WITH EXPERIMENT

In this chapter, allowed shape spectra belonging to empirically first forbidden transitions will be compared with the theoretically predicted spectrum shapes for various mixtures of interactions for first forbidden transitions with unit spin change. Before doing this, however, other evidence will be considered which already greatly restricts the possible interactions.

Because of the arguments favouring the Gamow-Teller selection rules (Gamow and Teller 1936, Konopinski 1943), which are mainly that some transitions for which the angular momentum change is probably one unit nevertheless appear to be allowed, the interaction has long been thought to contain at least a proportion of either T or A. The most convincing evidence for this which is now available is the experimental discovery of several beta spectra having the characteristic shape, predicted only by T and A, for first forbidden transitions with an angular momentum change of two units and parity change. Wu (1950) has given an excellent survey of recent experimental work, including the results just mentioned.

For allowed transitions, cross terms between interactions occur in the theoretical correction factors only for mixtures of S with V and of A with T, and for these mixtures the correction factor is of the form $(1+a/W)$. De Groot and

Tolhoek (1950) have estimated that to agree with the experimental evidence a must be less than $1/10$ and the amount of mixing of S with V or of A with T must be very small.

It was pointed out in the last chapter that transitions involving an angular momentum change would be selected, so that the matrix element $\langle f|\beta\gamma_5|i\rangle$ occurring with the pseudoscalar interaction will not contribute. If P is to contribute at all to the first forbidden transitions with unit angular momentum change, then it must do so through the third forbidden matrix element $Q_2(\beta\gamma_5\vec{\nabla})$, and the coefficient of P in the mixture of interactions must be large enough to overcome the extra degrees of forbiddenness of the matrix element. If this were so, one would expect transitions involving a parity change but no spin change to appear superallowed, occurring through the matrix element $\langle f|\beta\gamma_5|i\rangle$; for none of the observed superallowed transitions, however, does a parity change seem reasonable. Furthermore, if the matrix element $Q_2(\beta\gamma_5\vec{\nabla})$ gave an appreciable contribution to the first forbidden transitions with unit spin change, it should dominate the weaker transition with spin change of two units, and this does not appear to be the case. For these reasons, it is concluded that the possible presence of P in the interaction may be ignored in considering first forbidden transitions

with unit angular momentum change.

Summarising these results, it is seen that the interaction must contain either $\dot{T}$ or A, but not both, and the interaction cannot contain both S and V. For the transitions to be considered, the possible presence of P in the interaction may be ignored. From this, it follows that the only mixtures that need be considered are those of S with T, A with V, V with T and S with A: mixtures involving more than two out of these four interactions are disallowed. The correction factors for these mixtures and the type of transition under consideration may be found from Appendix 4, and may be simplified using the results of the last chapter, to give in low Z approximation:

$$
\left.
\begin{aligned}
T+yS:\ C &= \left\{\xi^2 + (2\xi/3W) + (\eta/18) + \tfrac{1}{9}\right\} \\
&\quad - 2y\epsilon\left\{\xi\left[\left(\tfrac{Z\alpha}{2\rho}\right) + \xi/3\right] + W_0\xi/9 + \left(\tfrac{Z\alpha}{\rho}\right)\tfrac{1}{6W}\right\} \\
&\quad + y^2\left\{\left(\tfrac{Z\alpha}{2\rho}\right)^2 + \left(\tfrac{Z\alpha}{\rho}\right)\xi/3 + \eta/3 - \tfrac{2}{9}(W_0 - W)\left(W - \tfrac{1}{W}\right)\right\}, \\
T+yV:\ C &= (1+y^2)\left\{\xi^2 + (2\xi/3W) + (\eta/18) + \tfrac{1}{9}\right\} + y^2(\eta/6) \\
&\quad - 2y\epsilon\left\{(\xi^2/W) + (2\xi/3) + (W_0/9)(2 - W_0 W)\right\}, \\
A+yS:\ C &= (1+y^2)\left\{\left(\tfrac{Z\alpha}{2\rho}\right)^2 + \left(\tfrac{Z\alpha}{\rho}\right)\xi/3 + \eta/6 - \tfrac{2}{9}(W_0 - W)\left(W - \tfrac{1}{W}\right)\right\} \\
&\quad + y^2(\eta/6) - 2y\epsilon\left\{\left[\left(\tfrac{Z\alpha}{2\rho}\right)^2 - \left(\tfrac{Z\alpha}{\rho}\right)W_0/3\right]/W + \tfrac{Z\alpha}{3\rho}\right\},
\end{aligned}
\right\} \qquad (6.1)
$$

where

$$\xi = x - \frac{Z\alpha}{2\rho} - \frac{W_0}{3},$$

$$\eta = W_0^2 - 1 - 2W_0 W + 2W^2,$$

$$\xi = 2W - W_0 - \frac{1}{W},$$

x is the ratio of $-i\langle f|\alpha|i\rangle$ to $\langle f|1|i\rangle$, and $\epsilon = k_i - k_f = \pm 1$, depending on the transition. A factor $|\langle f|1|i\rangle|^2$ has been omitted from all of these correction factors. The correction factor for the mixture $V + yA$ is almost identical with that for $T + yS$ given above, and is therefore not given. For positron emitters, the sign of Z must be changed throughout, and the sign of the cross term (proportional to y) changed in the mixture $T + yS$. The use of the low Z approximations is unlikely to have a serious effect on the results.

The theoretical spectrum shape is given by

$$P(w)dw = C\,F(Z,w)\,pw(w_0 - w)^2\,dw, \qquad (6.3)$$

where C is the correction factor and is constant for allowed shape spectra. For allowed shape spectra, therefore, a plot of $\sqrt{P/(F_p w)}$ against W will give a straight line cutting the energy axis at the maximum energy of the beta spectrum. If $N(w)\,dw$ is a measured beta spectrum, then the plot of $\sqrt{N/(F_p w)}$ against W is variously called the Fermi plot, F-K plot, and Kurie plot. Since for an allowed shape spectrum the Kurie plot is a straight line, such a plot gives the most convenient method of comparing experimental results with theory. If for a given

transition or a given mixture of interactions C is not a constant, then the theoretical Kurie plot may be obtained from a straight line cutting the energy axis at W_0 by multiplying all the ordinates by $\sqrt{C}$, which will be a function of W. The intercept of the Kurie plot with the $\sqrt{N/(F_p W)}$ axis depends on the life-time of the transition, and as this cannot be predicted theoretically with any certainty, a theoretical Kurie plot may be fitted to experimental results at one point in addition to the end point W_0 of the spectrum.

A search has been made in the literature for first forbidden transitions which have accurately measured allowed shape beta spectra and which are readily interpreted according to the shell model as involving an angular momentum change. Transitions were regarded as sufficiently forbidden and sufficiently accurately investigated only if their ft values were greater than 10^6, and their experimental Kurie plots were straight to within 5% for a fair energy range. The following transitions have been found: Na^{22} (Good, Peaslee and Deutsch 1946), Na^{24} (Siegbahn 1946), Sc^{46} (Peacock and Wilkinson 1948), V^{48} (Peacock and Deutsch 1946), Pr^{143} (Feldman, Lidofsky, Macklin and Wu 1949), Pm^{147} (Agnew 1950; Langer, Motz and Price 1950; Lidofsky; Macklin and Wu 1949), Hf^{181} (Chu and Wiedenbeck 1949), W^{187} (Peacock

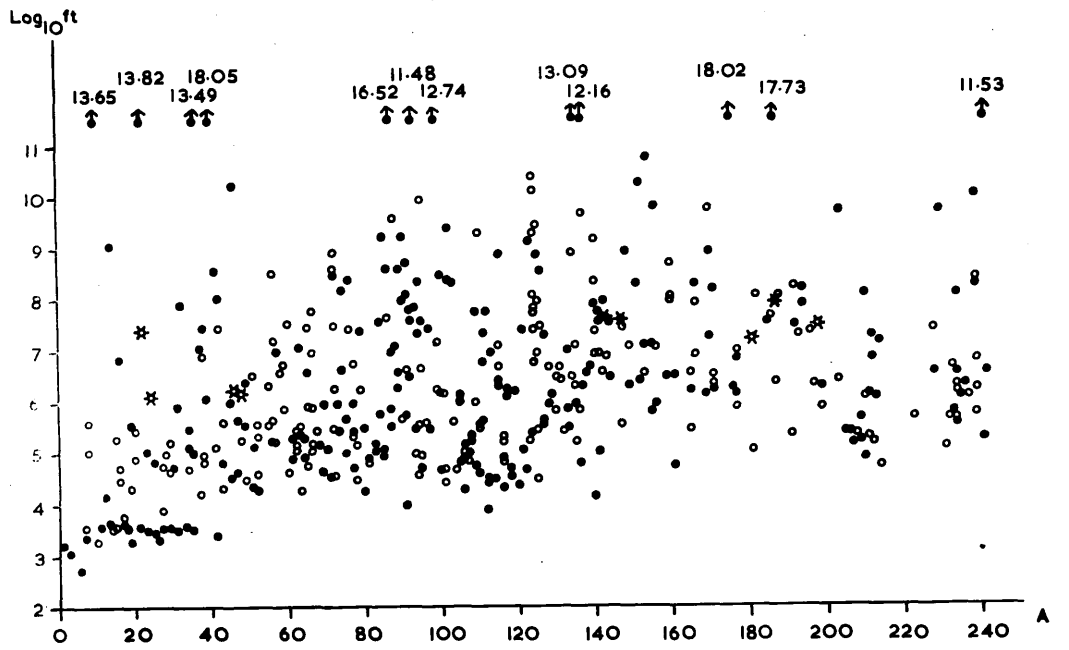


Figure 1 (repeated). $Log_{10}ft$ plotted against the mass number A, showing the transitions used in chapter 6. The ft values and the classification of final states are taken from Feingold (1951). ft values are plotted only for those transitions classified by Feingold as isotope certain or isotope probable.

and Wilkinson 1948), Au^{198} (Langer and Price 1949, Saxon 1948, Steffen, Huber and Humbel 1949). P^{32}, Ga^{66} and Cs^{134} are more probably of the $\Delta j=1$, $\Delta l=2$ type of transition mentioned in the previous chapter, while Mo^{99} and I^{131} could not be readily interpreted on the shell model picture; these five transitions were therefore rejected. The ft values of the above transitions are shown on figure 1, which is repeated here for convenience.

The sign of ϵ is obtained from the shell model interpretation of the transition. The level scheme of Haxel, Jensen and Süss (1950) is used in conjunction with the observed spins of neighbouring odd nuclei to suggest the initial state, and, if the transition is to the ground state of the product nucleus, also the final state, of the transforming nucleon. If the transition is to an excited level of the product nucleus, a reasonable transition involving parity change must be postulated. In this case, for heavy nuclei with a large neutron excess the transforming neutron need not belong to the outermost occupied neutron shell, and several interpretations may be possible.

For Na^{22} and Na^{24}, the shell model interpretation is complicated by an apparent break-down in the usual level scheme. It is possible, however, to explain these transitions if for Na^{22}, $p_{3/2} \rightarrow s_{1/2}$ with $\epsilon = -1$, and for Na^{24}, $d_{5/2} \rightarrow f_{7/2}$ with $\epsilon = +1$, is postulated.

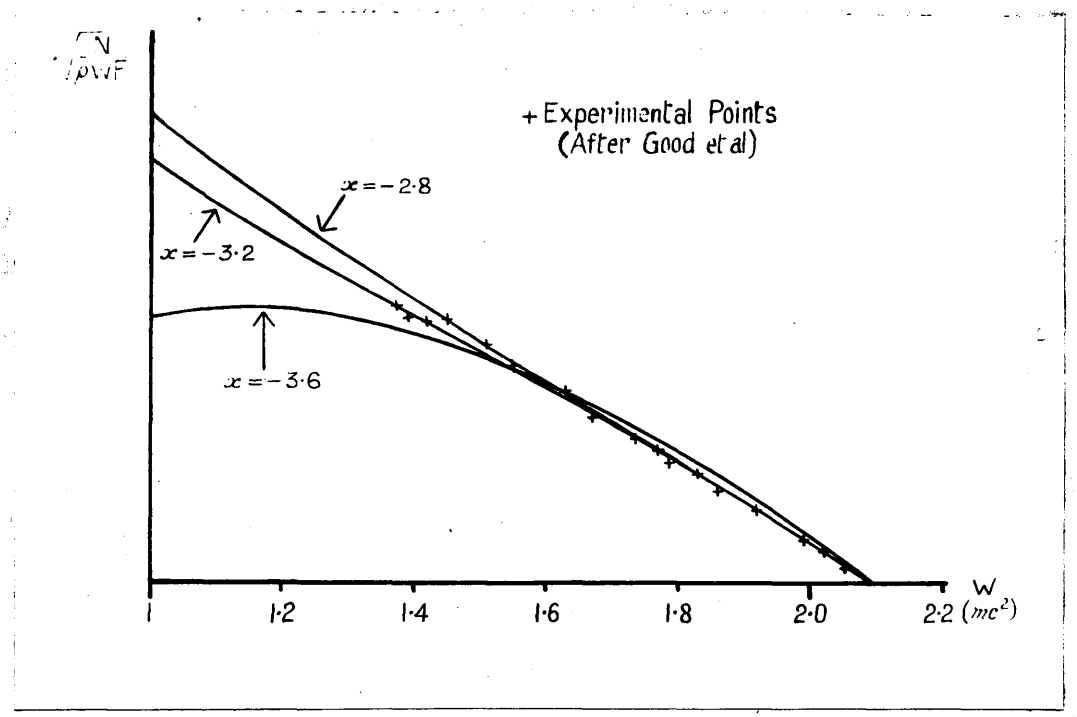


Figure 5. Theoretical Kurie plots for Na^{22} according to the tensor interaction, with different values for the ratio x of $-i\langle f|\alpha|i\rangle$ to $\langle f|\underset{\sim}{\Sigma}|i\rangle$. The estimate of Chapter 5 gives $x = -2.5$.

For both Sc^{46} and V^{48}, the only reasonable transition is $f_{7/2} \rightarrow d_{5/2}$ with $\epsilon = -1$. Pr^{143} and Pm^{147} undergo ground state to ground state transitions which are almost certainly of the type $h_{9/2} \rightarrow g_{7/2}$ with $\epsilon = +1$. For Hf^{181} and W^{187}, the transition may be any of $i_{13/2} \rightarrow h_{11/2}$, $f_{5/2} \rightarrow g_{7/2}$, $p_{1/2} \rightarrow d_{3/2}$, $f_{7/2} \rightarrow d_{5/2}$, all with $\epsilon = -1$, or $h_{9/2} \rightarrow g_{7/2}$ with $\epsilon = +1$, or $f_{7/2} \rightarrow g_{7/2}$, in which case the transition is not of the required type; for these transitions, ϵ was assumed to be -1. For Au^{198}, the transition is most probably $f_{5/2} \rightarrow d_{3/2}$ with $\epsilon = +1$.

The correction factors (6.1) are not very sensitive to the value of $\varkappa$ unless ζ is small. This is the case for Na^{22}, and figure 5 shows theoretical Kurie plots for this transition according to the tensor interaction for different values of $\varkappa$; equation (5.15) of last chapter gives for $\varkappa$ the value -2.5, and this gives an almost straight Kurie plot in fair agreement with experiment. The possibility of error, however, cannot be denied, and in particular if ϵ is $+1$ instead of -1 as assumed, (5.15) gives $\varkappa = -3.9$ and a Kurie plot far from straight. It seems unlikely, therefore, that any reliance can be put on conclusions about mixtures involving T or V based on this transition. It is also found that for the positron emitters Na^{22} and V^{48} the Kurie

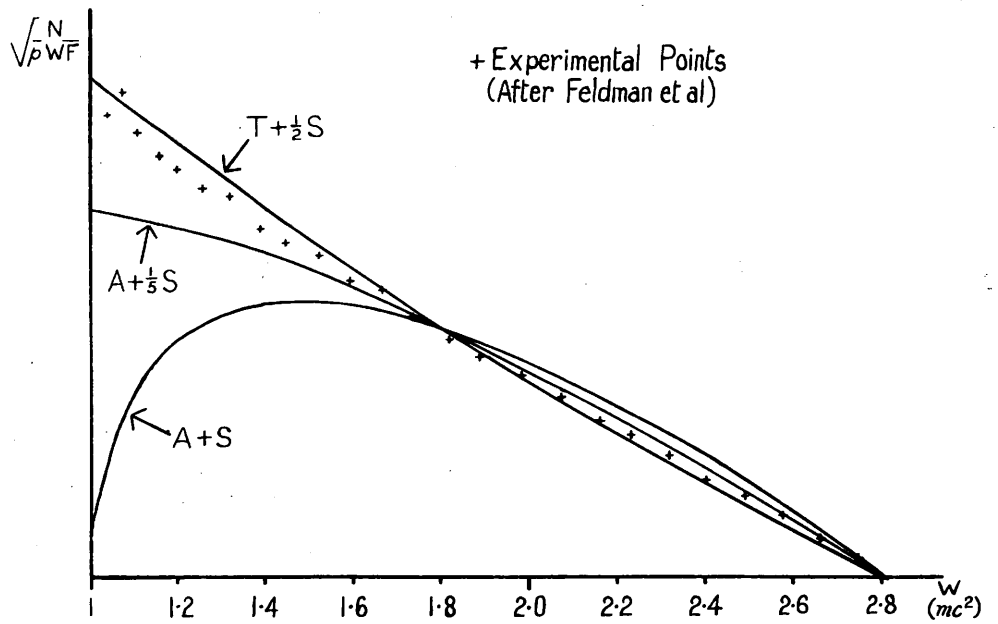


Figure 6. Theoretical Kurie plots for Pr^{143} for various mixtures of interactions.

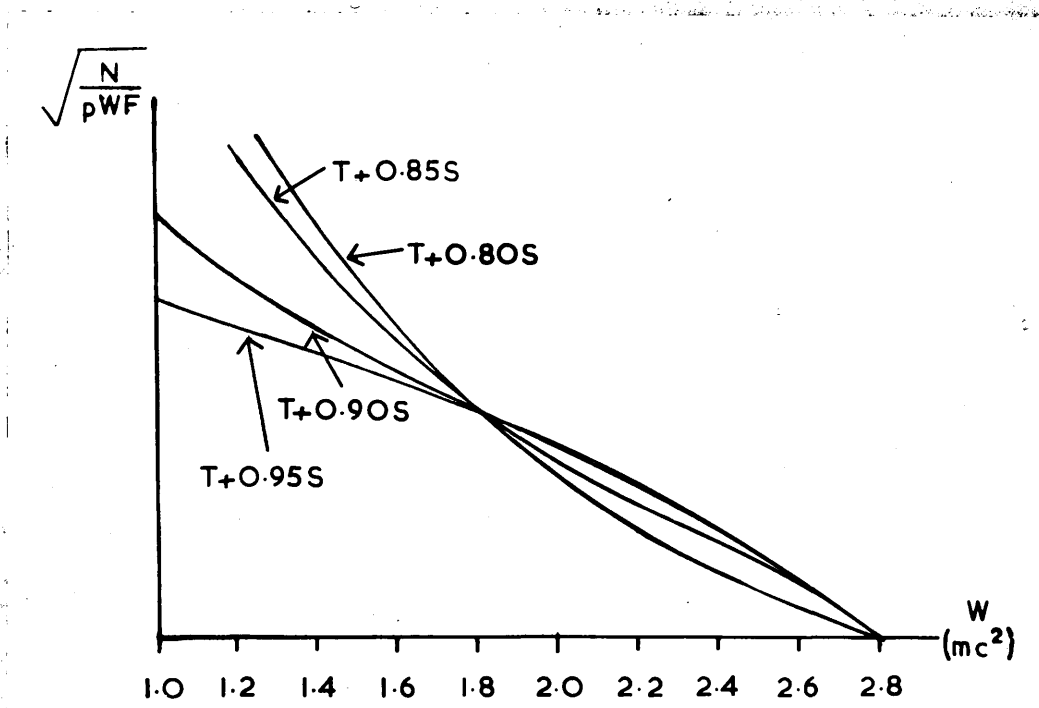


Figure 7. Theoretical Kurie plots for Pr^{143} for various mixtures of T and S, illustrating the transition from the "concave down" type of Kurie plot to the "concave up" type of Kurie plot.

plots for all mixtures of the type $A + y S$ are reasonably straight in the experimentally measured region. Consequently, no reliable information about any mixtures of interactions can be obtained from the available evidence on Na^{22}, and this transition will therefore be rejected. For all the other transitions, the theoretical spectrum shapes are not very sensitive to the value of x, and the estimate of last chapter should give fairly reliable results.

For heavy nuclei with low end points, the spectrum shapes predicted by all the pure interactions are indistinguishable from the allowed shape. For mixtures, however, if the cross term is negative there may be a large cancellation causing the theoretical spectrum to deviate strongly from the allowed shape. This is illustrated for Pr^{143} in figure 6. For all mixtures, if $y \epsilon = +1$ the theoretical Kurie plot is similar to that shown for the mixture $A + S$, while if y increases or decreases the Kurie plot approaches a straight line. The mixtures $T + y S$ and $V + y A$ are remarkable in that the theoretical Kurie plot changes from "concave down" through an S-shape to "concave up" before becoming straight; this change, which is illustrated for Pr^{143} in figure 7, is very rapid. Frequently, the experimental Kurie plot shows an excess of low energy electrons which may be due

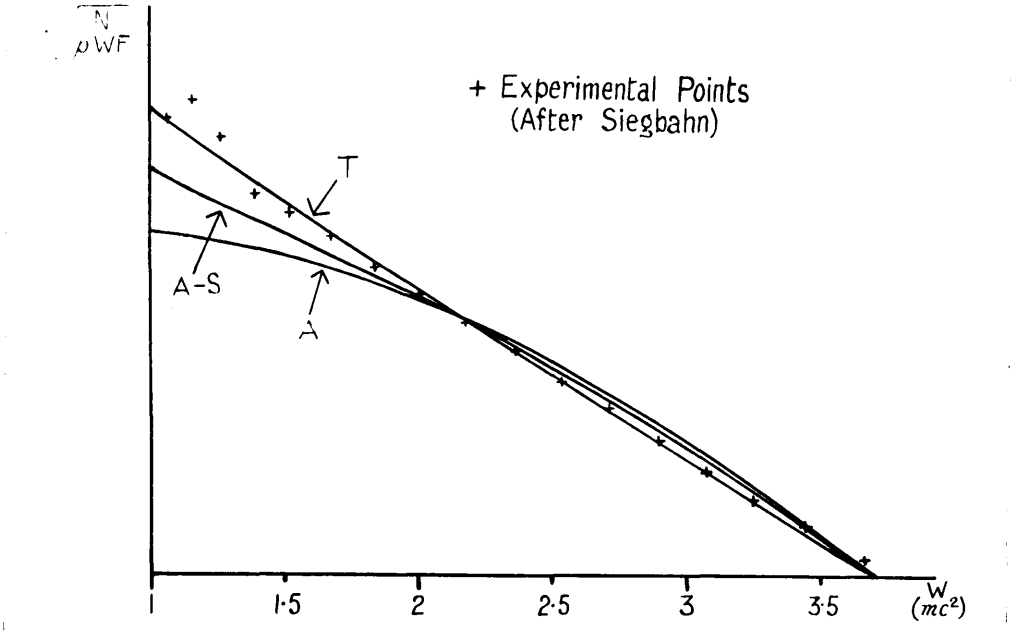


Figure 8. Theoretical Kurie plots for Na^{24} for various interactions. No mixture of the type A + yS is consistent with experiment.

to experimental difficulties such as source thickness effects, or to a second low energy transition, or it may be genuine. Because of this uncertainty, only that part of the Kurie plot which is experimentally straight is compared with theory. Theoretical and experimental Kurie plots are considered consistent with each other if the discrepancy between them is not more than 5% at the low energy end and and 10% at the high energy end of the experimentally straight portion. For Pr^{143}, only that part of the Kurie plot above 1.4 mc^2 is used, and the Kurie plots for the mixtures $A + \frac{1}{5} S$ and $T + \frac{1}{2} S$ both show the maximum allowed discrepancy. Since the sign of the cross term for any mixture depends on ϵ and therefore on the transition, it is clear that by considering different transitions both lower and upper limits can be obtained for the ratio y of the interaction constants in any mixture.

For lighter nuclei such as Na^{24}, the picture is very similar, except that the predicted spectrum shapes for pure A and S are no longer allowed, and indeed for Na^{24} no mixture whatsoever of S with A gives an allowed spectrum shape. Figure 8 gives theoretical Kurie plots for Na^{24} for the interactions A, A-S and T, of which only the last is consistent with experiment. It was pointed out earlier that the interaction must contain either T or A, but not

Mixture	Na^{24}	Sc^{46}	V^{48}	Pr^{143}	Pm^{147}	Hf^{181}	W^{187}	Au^{198}
A + yS	X			0·2	0·2			0·1
		−0·5				−0·33	−0·7	
T + yV	0·2			0·33	0·33			0·33
		−0·5	−0·5			−0·5		
T + yS	0·35		0·25	0·5	0·7			0·5
	−0·7	−0·65				−0·7	−0·65	
S + yT	X		0·7	0·5	0·9			0·6
		−0·75					−0·75	

Figure 9. Upper and lower limits for the mixture parameter y. The first-named interaction is assumed to predominate, so that y is numerically less than 1. The upper and lower rows of figures for each mixture are respectively upper and lower limits for y, while the columns indicate the transitions from which the limits are derived. X denotes that no values of y are consistent with experiment.

both; the spectrum of Na^{24} seems to favour T, but A with a large admixture of V cannot be ruled out. Of the transitions under consideration, Na^{24} is the only one capable of distinguishing between T and A in this way.

The limits to the amount of mixing found from each of the nine transitions considered are given in the table in figure 9. The first named interaction in each mixture is assumed to predominate, so that y is numerically less than 1; since, however, the mixtures $S + yA$ and $V + yT$ have correction factors almost identical to those for $A + yS$ and $T + yV$ respectively, and since the limits are given for both $T + yS$ and $S + yT$, this means no loss of generality. The mixtures $V + yA$ and $A + yV$ are almost identical to the mixtures $T + yS$ and $S + yT$ respectively.

The discussion of the table in figure 9 will be left till next chapter. A glance at the table, however, shows the importance of the evidence from Na^{24}. Since this transition has an ft value close to the limit of 10^6 adopted as a criterion of forbiddenness, since the ft value for the Na^{22} transition is 20 times as large as that for the Na^{24} transition, and since it is difficult to reconcile the parity change necessary if the transition is first forbidden with the shell model level scheme, it is

advisable to give further semi-empirical evidence for the forbidden nature of this transition.

Na^{24} decays by electron emission to an excited state of Mg^{24}, which then emits two gamma rays in cascade. Both gamma rays are almost certainly electric quadrupole (Brady and Deutsch 1948, Rae 1951), and consequently Na^{24} has the same or opposite parity as the ground state of Mg^{24} according as the beta decay is allowed or first forbidden. According to the shell model level scheme, Na^{24} and Mg^{24} have the same parity, so that the beta decay must be allowed, and some additional selection rule must be found to explain the large ft value. On the other hand, the extreme single particle version of the shell model predicts angular momentum $\frac{5}{2}\hbar$ for a group of 11 like nucleons, while the measured spins of odd nuclei with 11 like nucleons are all $\frac{3}{2}\hbar$. The most plausible explanation of this is that more than one nucleon contributes to the angular momentum of such a group. Alternatively, however, the single particle level scheme may be modified for 10 and 11 like nucleons, and this can be done in such a way that the ground states of Na^{24} and Mg^{24} have opposite parity. In order to decide the sign of the cross terms in the correction factors, assuming the Na^{24} transition to be first forbidden, and therefore also to determine the signs of the limits given for the Na^{24} transition in

figure 9, such a modified level scheme had to be adopted.

In addition to the ft value of the transition being greater than 10^6 , further semi-empirical evidence for the forbidden nature of the transition can be obtained by calculating theoretically the ratio of the ft's of Na^{22} and Na^{24}, and also the ft value of Na^{24}, making reasonable assumptions about the nuclear matrix elements. The argument, however, will hold only if the interaction is predominantly T or V. Assuming an interaction either pure T or pure V, and supposing $\langle f|\underset{\sim}{\tau}|i\rangle$ to be of the same order of magnitude both for the Na^{22} transition and the Na^{24} transition, then the ratio of ft values for these transitions depends only on the values of the ratio $\varkappa$ of $-i\langle f|\underset{\sim}{\alpha}|i\rangle$ to $\langle f|\underset{\sim}{\tau}|i\rangle$ for the two transitions. Using the values for these ratios obtained from (5.15) of last chapter, one finds for the theoretical ratio of the ft values for the Na^{22} and Na^{24} transitions the value 25:1, in remarkably good agreement with the observed ratio of 20:1 At the same time, if $\langle f|\underset{\sim}{\tau}|i\rangle$ is assumed to be of the order of the nuclear radius, one obtains a theoretical ft value for the Na^{24} transition. Assuming for the interaction constant $(2\pi^3 \ln 2/G^2) \approx 6.10^3$ and $(2\pi^3 \ln 2/G^2) \approx 3.10^3$ which Greuling (1942) gives as the most appropriate values for the tensor and vector interactions respectively, one finds the theoretical ft

values 2.2×10^6 and 1.1×10^6, both in good agreement with the experimental value 1.3×10^6.

It appears, therefore, that all the empirical and semi-empirical evidence favours the first forbidden interpretation of the Na^{22} and Na^{24} transitions, although this interpretation is inconsistent with the usual shell model level scheme.

7. CONCLUSION.

In the preceding chapters, after considering the foundations of the Fermi type theory of beta decay, the argument was specialised to a discussion of first forbidden transitions with unit angular momentum change. The results of this work will be discussed in this chapter in the course of a wider discussion of the form of the interaction in the theory, based on all the available evidence.

Beta spectrum shapes, lifetimes of transitions, selection rules, and combinations of two or three of these, can all give information about the interaction. Except possibly in the case of the superallowed transitions, one cannot expect to obtain reliable information from a consideration of lifetimes alone, since this involves the absolute magnitudes of nuclear matrix elements. The use of selection rules is also difficult, since for very few transitions have the angular momenta of both the initial and final nuclei been measured directly. Of any single method of approach, therefore, that using spectrum shapes is most likely to give reliable results. At the worst, this method requires a knowledge only of ratios of matrix elements having the same selection rules, and not of their absolute magnitudes.

From the allowed spectrum shapes, De Groot and Tolhoek (1950) have deduced that the ratio of interaction constants in mixtures of S and V, or of A and T, is less than 1:20, while

there is no evidence that any mixing of this sort takes place at all. Since such a small amount of mixing is undetectable with the present accuracy of experiments, and since it is difficult to see what theoretical significance it could have if it did occur, it is consistent and reasonable to suppose that the interaction does not contain both S and V, or both T and A. This is probably the most reliable information available about the interaction.

The discovery of several beta spectra with the shape characteristic of first forbidden transitions with an angular momentum change of two units, possible only with the interactions T and A, is fairly convincing evidence that the interaction contains one or other of these "pure" interactions. This evidence is independent of ratios of matrix elements. It may, however, be possible to explain these transitions as second forbidden, using some particular mixture of interactions. Certainly the ft values for these transitions are rather larger than one would expect for first forbidden transitions, but this has been satisfactorily explained (Davidson 1951, Taketani et al. 1951). A shell model interpretation of the transitions also favours the first forbidden interpretation. W&L (1950) has discussed the experimental evidence on these transitions. Other evidence for an interaction containing either T or A comes from a consideration of the selection rules satisfied by allowed transitions, while the superallowed transitions can be explained using Wigner's

theory of supermultiplets only if one of these two interactions is present. This evidence is discussed by Konopinski (1943). The cumulative evidence makes the conclusion that the interaction contains T or A fairly reliable.

Several authors have recently discussed the lifetimes of superallowed transitions (Moszkowski 1951, Horie and Umezawa 1951, Trigg 1952). For the image transitions, one would expect the initial and final nuclear states to differ only in their dependence on charge coordinates. This belief is supported by the fact that the energy release in these transitions is just the difference in the Coulomb energies of the initial and final nuclei less the neutron-proton mass difference. For these transitions, it is possible to calculate the nuclear matrix elements on the basis of an assumed nuclear model, without making detailed assumptions about the nuclear wave-functions. One would expect

$$\left.\begin{aligned} ft\,|M|^2 &= \text{constant} = C', \text{ say,} \\ \text{where} \quad |M|^2 &= G_1^2 |\langle f|1|i\rangle|^2 + G_2^2 |\langle f|\vec{\sigma}|i\rangle|^2, \end{aligned}\right\} \qquad (7.1)$$

and G_1^2 is the sum of the squares of the interaction constants for the Fermi type interactions S and V, and G_2^2 is the sum of the squares of the interaction constants for the Gamow - Teller type interactions T and A.

Horie and Umezawa have evaluated the nuclear matrix elements for 13 image transitions, and found that $ft|M|^2$ is more nearly constant for a pure Fermi type interaction than for a pure Gamow-Teller type interaction. Trigg writes

$G_1^2 = \lambda G_2^2$, so that (7.1) can be written

$$ft\{\lambda|\langle f|1|i\rangle|^2 + |\langle f|\underline{\alpha}|i\rangle|^2\} = C'' \qquad (7.2)$$

The matrix elements are evaluated according to the extreme single particle model, and an intermediate model in which the deviation from the single particle model is estimated in terms of the position of the magnetic moment of the product nucleus relative to the Schmidt and the Margenau and Wigner limits. This intermediate model has recently been discussed by Davidson (1952). The mean value of C'' is found for 15 image transitions, and λ is varied to make the standard deviation of C'' a minimum. In this way an estimate is obtained for λ, the square of the ratio of the coupling constants for Fermi and Gamow-Teller interactions. The method used, however, is bound to give too large a proportion of Gamow-Teller interaction. The most appropriate treatment along the lines suggested by Trigg's work is to write $G_1^2 = qG^2$, $G_2^2 = \alpha(1-q)G^2$, where $0 \leq q \leq 1$ and

$$\alpha = \overline{ft|\langle f|1|i\rangle|^2} \Big/ \overline{ft|\langle f|\underline{\alpha}|i\rangle|^2},$$

the bar denoting the mean value for the transitions considered. In this case,

$$ft\{q|\langle f|1|i\rangle|^2 + \alpha(1-q)|\langle f|\underline{\alpha}|i\rangle|^2\} = C, \qquad (7.3)$$

and α has been chosen so that the mean value of C is independent of q. q = 0 represents a purely Gamow-Teller interaction and q = 1 denotes a purely Fermi interaction. The ratio λ of the squares of the coupling constants is

$$\lambda = \frac{q}{\alpha(1-q)} . \qquad (7.4)$$

q is now chosen to minimise the standard deviation of C. For the single particle model, one obtains C = 3177 with standard deviation 1050 when $\lambda = 9$. This agrees with the results of Horie and Umezawa, but not with those of Moszkowski, who, however, considered only part of the data. Such a large proportion of Fermi type interaction is inconsistent also with the abnormally low ft value of the He^6 transition. If λ is reduced to 0.61, the standard deviation increases to 2690. For the intermediate model, C = 3177 with standard deviation 600 when $\lambda = 1.44$, while if λ is reduced to 0.61 the standard deviation increases to 860. Thus by the use of the intermediate model, the optimum proportion of Fermi type interaction is greatly reduced, and the standard deviation is nearly halved. If attention is confined to the five transitions for which the magnetic moments of the product nuclei fall exactly on the Schmidt limits, and therefore for which the single particle model is most likely to be valid, one obtains C = 2650 with standard deviation 300 when $\lambda = 0.78$, while if $\lambda = 0.61$ the standard deviation increases to 330. The reduction in the standard deviation, which is the more remarkable since it is associated with a large reduction in the number of transitions considered, is very significant.

Before discussing these results, one must have some

estimate for the probable error in λ. This may be found either by calculating $\bar{q}$ in terms of C from (7.3) and choosing C to minimise the standard deviation of q, or by calculating $\bar{\lambda}$ in terms of C'' from (7.2) and minimising the standard deviation of $\bar{\lambda}$. These methods give different estimates of $\bar{\lambda}$ and its standard deviation, and it is natural to consider that treatment more appropriate which gives the smaller probable errors in q and λ. This is found to be the second treatment, calculating $\bar{\lambda}$ directly. For all fifteen transitions using the intermediate model one finds $\bar{\lambda} = 0.87$ with standard deviation 0.43 when $C'' = 4820$, while for the five transitions already mentioned $\bar{\lambda} = 0.61$ with standard deviation 0.29 when $C'' = 4370$. From the results quoted in the preceding paragraph, it is seen that the use of this last result does not seriously alter the constancy of $ft|M|^2$, provided the intermediate model is used. This value of λ is also in much better agreement with the ft value for He^6 than any larger value. Trigg considers the best compromise value of λ to be 0.5.

Summarising these results, which have been given in considerable detail since Trigg's work has not yet been published, the consideration of the lifetimes of superallowed transitions shows clearly a need for a proportion of S or V in the interaction. It is impossible to reconcile the lifetimes of the image transitions with that of He^6 unless the single particle evaluation of the matrix elements is abandoned - in

this connection, the intermediate model used by Trigg and discussed by Davidson (1952) gives a fair approximation. These results are confirmed when attention is confined to those five transitions for which the single particle model is most reasonable. It seems impossible to assign any sharp upper limit to the proportion of S or V in the interaction, but the most probable value of λ, the ratio of the squares of the coupling constants for Fermi and Gamow-Teller interactions, is 0.6, with standard deviation 0.3. This standard deviation probably gives a fair guide to the minimum probable value of λ.

Smith (1951) has arrived at somewhat similar conclusions to the above from a consideration of the shapes of second forbidden beta spectra. He finds that the best interaction is a mixture of V and T with coupling constants in the ratio of 2:1. Until details are available, however, it is impossible to judge the reliability of this work.

The evidence on the interaction derived in the last chapter from first forbidden transitions will now be discussed against the background of other evidence just given. The work of the last chapter depends on ratios of nuclear matrix elements, which were derived using a single particle model, and also on the use of the shell model for interpreting the transitions. The ratio of $\langle f|\alpha|i\rangle$ to $\langle f|r|i\rangle$ found in Chapter 5 is larger than that used by most authors, but the

Mixture	Na^{24}	Sc^{46}	V^{48}	Pr^{143}	Pm^{147}	Hf^{181}	W^{187}	Au^{198}
A + yS	X	−0·5		0·2	0·2	−0·33	−0·7	0·1
T + yV	0·2	−0·5	−0·5	0·33	0·33	−0·5		0·33
T + yS	0·35 −0·7	−0·65	0·25	0·5	0·7	−0·7	−0·65	0·5
S + yT	X	−0·75	0·7	0·5	0·9		−0·75	0·6

Figure 9 (repeated). Upper and lower limits for the mixture parameter y, derived from first forbidden transitions with allowed shape beta spectra. The first-named interaction is assumed to predominate. The upper and lower rows of figures for each mixture are respectively upper and lower limits for y, while the columns indicate the transitions from which the limits are derived. X denotes that no values of y are consistent with experiment. The limits for the mixtures S+yA and V+yT are the same as for the mixtures A+yS and T+yV respectively, and the limits for the mixtures V+yA and A+yV are the same as for the mixtures T+yS and S+yT respectively.

discussion there given is the most thorough so far published, and in any case a decrease in this ratio would cause a narrowing rather than a widening of the limits of permissible mixing given in figure 9. The only unexpected feature of the other ratios of matrix elements is the dependence of their sign on the transition, but this cannot plausibly be regarded as a property of one particular nuclear model. The shell model is used only to decide the sign of these ratios, but if this should prove to be unjustified, it nevertheless seems unlikely that the sign should be the same for all the eight transitions considered. For these reasons, it is felt that the limits in figure 9 give a fairly reliable guide as to the amount of mixing of interactions possible, without necessarily giving strict limits.

The limits of figure 9 must now be considered in conjunction with the evidence obtained from the superallowed transitions. It must be remembered that y appearing in figure 9 is the ratio of interaction constants, while ρ introduced in the discussion of superallowed transitions is the square of the ratio of interaction constants. No mixture of S and A seems plausible, but if the limits are stretched slightly the interaction T - 0.5 V might be possible. The mixture T - 0.65 S is possible, and if the evidence of the Na^{24} transition is rejected so also is the interaction A-0.7V If also the shell model interpretation of the P^{32} decay is

rejected, then the angular correlation experiments of Sherwin mentioned in Chapter 5 would suggest an interaction containing either A or V. Tentative conclusions, therefore, are that the interaction is one of T - 0.65 S, A - 0.7 V, or less probably T - 0.5 V. None of the evidence excludes the possibility of an admixture of P in the interaction.

It was pointed out in the introduction that symmetry properties with respect to the permutation of fields in the interaction are likely to have theoretical significance. In considering such symmetry properties, it is convenient to suppose that all the fields anticommute with each other - this is easily arranged, for example by the use of dichotomic variables analogous to isotopic spin. The interactions having special symmetry properties and which are consistent with the allowed spectrum shapes are the "pure" interactions, of which S, A and P are symmetrical and V and T antisymmetrical with respect to the interchange either of the light particle fields or of the nucleon fields, the Critchfield interaction S - A - P (Critchfield 1943) which is symmetrical with respect to any permutation of fields, S - T + P and 3S + T + 3P which are respectively symmetrical and antisymmetrical with respect to the interchange of the two neutral or the two charged fields, and S + T + P, A + V, and 3S - T + 3P, the first two of which are symmetrical and the last antisymmetrical with respect to the interchange of the neutron and electron fields, or of the proton and neutrino fields. Of these, the Critchfield inter-

action and the interactions $S \pm T + P$ and $A + V$ are inconsistent with the evidence from first forbidden spectra. The pure interactions are inconsistent with the lifetimes of the superallowed transitions, as also are $3S \pm T + 3P$. Of the interactions considered in the last paragraph, only $T - 0.5\ V$ has symmetry properties, it being antisymmetrical with respect to the interchange of the light particle or the nucleon fields.

The possibility that the beta decay interaction might be in some sense universal was pointed out in Chapter 1, and it is in this context that symmetry properties are significant. It is usual in considering a universal interaction to treat particle and antiparticle fields on an equal footing: this is not unreasonable, since unless two of the particles, such as the electron and neutrino, are regarded as different states of the same particle, the continuity of Fermion lines in a Feynman diagram, a typical feature of quantum electrodynamics, is lost.

A universal interaction may be formed by adding to the field Hamiltonian density interaction terms formed from all possible ordered groups of four spinor fields, and rejecting those terms which violate the requirements of Lorentz invariance and charge conservation. From its mode of construction, this interaction must be symmetrical with respect to the interchange of any two fields. The only such

interaction is the Critchfield interaction S - A - P, which, however, is inconsistent with the first forbidden beta spectrum shapes. It is more reasonable to take charge conservation into account explicitly in setting up the interaction. In this case, interaction terms can be classified according as they contain four, two, or no charged fields, and the symmetry properties of each class may be considered separately. The only class of interaction term for which there is experimental evidence is that involving two charged fields. For such an interaction, it is possible to specify the positions occupied by the positively and negatively charged fields, and the only symmetry that can arise is with respect to the interchange of the two neutral fields. The beta decay interaction does not appear to have such symmetry properties, and it follows that the position in the interaction of the two neutral fields must be uniquely determined by their physical properties. This may be done by associating a definite neutral field with each charged field, for example a neutron with a proton and a neutrino with an electron or a μ-meson. Alternatively the ordering of the neutral fields may be determined by such physical properties as the sign of the magnetic moment relative to the spin. This need not preclude the hypothesis of a Majorana neutrino, since in nuclear beta decay and μ-meson capture the sign of the neutron magnetic moment would serve to determine the positions in the interaction of both neutral fields. This

hypothesis appears to allow the decay of a proton into a positron and two neutrinos, but Yang and Tiomno (1950) have shown that this difficulty may be overcome if the nucleon and light particle fields are of a different absolute transformation type in the sense of Chapter 2. It is interesting that, as a result of the antisymmetry with respect to the interchange of light particle or nucleon fields, if the interaction T - 0.5 V is adopted and the ordering of the fields depends on their charges and the signs of their charges or magnetic moments, then interactions involving four charged or four neutral fields automatically vanish.

From the above discussion, it is clear that in any comparison of the nuclear beta decay interaction with that responsible for μ-meson decay the positions of the positively and negatively charged fields should be the same in both interactions. For nuclear beta decay, the ordering of the fields given in equation (3.2) is most convenient, but Michel (1950) has shown that for the discussion of the μ-meson decay a different ordering, corresponding to the interchange of electron and neutron fields, is preferable. Denoting interactions with Michel's ordering by a bar, it is easily found from the formulae given by Michel that

$$\left.\begin{aligned} S &= \tfrac{1}{4}\bar{S} - \tfrac{1}{4}\bar{V} - \tfrac{1}{4}\bar{T} + \tfrac{1}{4}\bar{A} + \tfrac{1}{4}\bar{P}, \\ V &= -\bar{S} - \tfrac{1}{2}\bar{V} \qquad -\tfrac{1}{2}\bar{A} + \bar{P}, \\ T &= -\tfrac{3}{2}\bar{S} \qquad -\tfrac{1}{2}\bar{T} \qquad -\tfrac{3}{2}\bar{P}, \\ A &= \bar{S} - \tfrac{1}{2}\bar{V} \qquad -\tfrac{1}{2}\bar{A} - \bar{P}, \\ P &= \tfrac{1}{4}\bar{S} + \tfrac{1}{4}\bar{V} - \tfrac{1}{4}\bar{T} - \tfrac{1}{4}\bar{A} + \tfrac{1}{4}\bar{P}. \end{aligned}\right\} \qquad (7.5)$$

For the interactions already considered in nuclear beta decay, one obtains

$$\left.\begin{aligned} T-0.65\,S &= -1.66(\bar{S}+\bar{P}) + 0.16(\bar{V}-\bar{A}) - 0.34\bar{T}, \\ A-0.7\,V &= 1.7(\bar{S}-\bar{P}) - 0.15(\bar{V}+\bar{A}), \\ T-0.5\,V &= -\bar{S} + 0.25(\bar{V}+\bar{A}) - 0.5\bar{T} - 2\bar{P}. \end{aligned}\right\} \qquad (7.6)$$

The shape of the beta spectrum from μ-meson decay for the mixture of interactions $g_1\bar{S} + g_2\bar{V} + g_3\bar{T} + g_4\bar{A} + g_5\bar{P}$ has been calculated by Michel (1950). Lagarrigue and Peyrou (1951) have pointed out that this spectrum shape can be written, to a good approximation, as

$$\left.\begin{aligned} P(w)dw &= C\left[w^3\left(\tfrac{8}{3}\rho - 3\right) + w^2\mu\left(3 - 2\rho\right)\right], \\ \rho &= \frac{3(g_2^2 + g_4^2) + 6g_3^2}{(g_1^2 + g_5^2) + 4(g_2^2 + g_4^2) + 6g_3^2}, \end{aligned}\right\} \qquad (7.7)$$

where

C is a constant and μ is the μ-meson mass in mc^2 units.

Experimental spectra so far published do not show much consistency (Leighton, Anderson and Seriff 1949, Lagarrigue and Peyrou 1951, Sagane, Gardner and Hubbard 1951, Bramsom and Havens 1951). The combined results of the first two references are perhaps the most reliable, and would suggest a value for ρ of order 0.2, although $\rho = 0$ is not completely ruled out. The first and third of the interactions (7.6) are in fair agreement with this value of if the neutrino is a Dirac rather than a Majorana neutral

particle. For the second of the interactions (7.6), and also for the other two if the neutrino is a Majorana particle, ρ is almost zero.

Most of the questions posed in the introductory chapter have now been answered. Three interactions have been found which appear to be consistent with the available experimental evidence on nuclear beta decay. All three are mixed interactions, whatever ordering is chosen for the fields. One has a special symmetry property, and a possible significance of this has been suggested. All three seem capable of being universal interactions. It is remarkable that it has been possible to narrow down the number of possible interactions to such an extent, and still have interactions consistent with all the data. It is quite possible, however, that with refinements in the theory of forbidden transitions and of superallowed transitions, together with more and better experimental measurements, this consistency will be lost. In this connection, it appears from the calculations of Greuling and Meeks (1951) that angular correlation measurements could play an extremely important part in determining the interaction if it were found possible to apply the extremely difficult experimental techniques to transitions whose interpretation is beyond doubt.

APPENDIX 1. CARTAN'S TREATMENT OF SPINORS.

Cartan discusses firstly spinors in a three dimensional Euclidean space with complex coordinates. The equation of the nul cone is then

$$x_1^2 + x_2^2 + x_3^2 = 0. \tag{A1.1}$$

This equation is automatically satisfied if

$$\left.\begin{aligned} x_1 &= \xi_0^2 - \xi_1^2, \\ x_2 &= i(\xi_0^2 + \xi_1^2), \\ x_3 &= -2\xi_0\xi_1. \end{aligned}\right\} \tag{A1.2}$$

These equations may be solved for ξ_0, ξ_1, in terms of x_1, x_2, x_3:-

$$\xi_0 = \pm\sqrt{\frac{x_1 - ix_2}{2}}, \quad \xi_1 = \frac{x_1 + ix_2}{x_3}\xi_0. \tag{A1.3}$$

The number pair (ξ_0, ξ_1) is defined to be a spinor in three dimensional space. From (A1.3), the transformation properties of spinors are seen to be defined in terms of the transformations of the coordinates x_1, x_2, x_3, and it can be shown that ξ_0, ξ_1, transform linearly for all rotations and reflections.

From (A1.1) and (A1.3), it is easy to show that

$$\left.\begin{aligned} \xi_0 x_3 + \xi_1(x_1 - ix_2) &= 0, \\ \xi_0(x_1 + ix_2) - \xi_1 x_3 &= 0, \end{aligned}\right\} \tag{A1.4}$$

which can be written in matrix notation

$$X\xi = 0,$$

where

$$X = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}. \qquad \text{(A1.5)}$$

Clearly

$$X = \sigma_i x_i , \qquad \text{(A1.6)}$$

where the σ_i are the Pauli spin matrices. The matrices X have properties analogous to the matrices associated with 4-vectors in Chapter 2. So far, the vector (x_1, x_2, x_3) has been considered to be a nul vector, but (A1.6) can be used to associate a matrix with any vector.

By a similar argument to that used in Chapter 2, the effect on any vector X of reflection in a plane whose unit normal is A is given by

$$\grave{X} = -AXA^{-1}, \qquad \text{(A1.7)}$$

and this, together with the fact that equations (A1.4) must be invariant, is sufficient to show that

$$\grave{\xi} = \alpha A \xi, \qquad \text{(A1.8)}$$

where α is a pure number. By considering the iteration of the reflection, one finds $\alpha = \pm 1$, and Cartan adopts by convention $\alpha = 1$, giving

$$\grave{\xi} = A\xi \qquad \text{(A1.9)}$$

for the transformed spinor.

In space of more than three dimensions, it is not possible to associate each spinor with a nul vector, as is done in (A1.2) and (A1.3). As a definition of a spinor in four dimensions, Cartan therefore uses the transformation property (A1.9). This definition differs from that used in Chapter 2, since with Cartan's definition $\grave{X}\grave{\xi} = -AX\xi$,

while $\chi\xi = +\Gamma X\xi$ was adopted as the definition of a spinor in Chapter 2. In consequence, in order to achieve the invariance of the Dirac equations, Cartan must associate with each 4-vector $\underset{\sim}{x}$ the matrix

$$\left.\begin{aligned} \chi &= x_\mu E_\mu, \\ \text{where} \qquad E_\mu &= i\gamma_5\gamma_\mu. \end{aligned}\right\} \qquad \text{(A1.10)}$$

If χ, a, denote the matrices associated with the vectors $\underset{\sim}{x}$, $\underset{\sim}{a}$ by Cartan, and X, A are the matrices associated with the same vectors in Chapter 2, then

$$\chi = i\gamma_5 X \quad , \quad a = i\gamma_5 A, \qquad \text{(A1.11)}$$

and a spinor transforms according to

$$\xi' = a\xi = i\gamma_5 A\xi, \qquad \text{(A1.12)}$$

which is the same transformation as is given by (2.22), with $\alpha = i$.

Cartan's definition of spinors is the only one possible in space with an odd number of dimensions. In four dimensional space, however, some simplification is achieved by using only the one set γ_μ of 4 x 4 matrices satisfying the usual commutation rules, instead of the two related sets γ_μ and E_μ.

Watanabe (1951) has given a discussion of spinors covering somewhat similar ground to Chapter 2, but using the Cartan definition.

APPENDIX 2. THE EVALUATION OF

$$\Sigma \equiv \sum_{n=0}^{l} \frac{1}{(l-n)!(l+n)!} (zz^*)^{l-n} \left\{ \left[(xx^*+yy^*)+i(xy^*-yx^*) \right]^n + \left[(xx^*+yy^*)-i(xy^*-yx^*) \right]^n \right\}.$$

Let

$$b = xx^*+yy^*, \qquad c = xy^*-yx^*, \qquad d = zz^*. \tag{A2.1}$$

Then

$$b^2+c^2 = (x^2+y^2)(x^{*2}+y^{*2}) = (r^2-z^2)(r^{*2}-z^{*2}) = d^2 + O(r^2, r^{*2}), \tag{A2.2}$$

and therefore

$$\begin{aligned} X_n &\equiv (b+ic)^n + (b-ic)^n \\ &= 2 \sum_{r=0}^{\frac{n}{2}(-\frac{1}{2})} \binom{n}{2r} (-1)^r b^{n-2r} (d^2-b^2)^r + O(r^2, r^{*2}), \end{aligned} \tag{A2.3}$$

where the upper limit $\frac{n}{2}(-\frac{1}{2})$ is $\frac{n}{2}$ if n is even, $\frac{n-1}{2}$ if n is odd. Σ, therefore, is given by

$$\left.\begin{aligned} \Sigma &= S + O(r^2, r^{*2}), \\ \text{where} \quad S &= \sum_{n=0}^{l} \frac{1}{(l-n)!(l+n)!} d^{l-n} X_n . \end{aligned}\right\} \tag{A2.4}$$

Since the expression $\Sigma - \frac{1}{(l!)^2}(zz^*)^l$ arises after integrating over all directions of the neutrino momentum and summing over the spin states of the neutrino and all the angular momentum states of the electron, it must be a scalar, and must therefore be of form $B(\underset{\sim}{r}.\underset{\sim}{r}^*)^l + O(r^2, r^{*2})$, where B is a constant. B will first be found by evaluating the coefficient of d^l in S given by (A2.4), and this result will then be confirmed by a direct evaluation of S.

From (A2.4) and (A2.3), it is easily seen that

the coefficient of d^l in S is

$$E = \sum_{m=0}^{\frac{1}{2}(-\frac{1}{2})} \frac{2}{(l-2m)!\,(l+2m)!}(-1)^m$$

$$= \sum_{m=0}^{l} \frac{1}{(l-m)!\,(l+m)!}\left[(i)^m + (-i)^m\right]$$

$$= \frac{1}{(2l)!}\left\{\sum_{r=0}^{l}\binom{2l}{r}(i)^{l-r} + \sum_{r=l}^{2l}\binom{2l}{r}(i)^{l-r}\right\}$$

$$= \frac{1}{(2l)!}(-i)^l(2i)^l + \frac{1}{(l!)^2}. \qquad \text{(A2.5)}$$

Hence

$$\mathcal{B} = E - \frac{1}{(l!)^2} = \frac{2^l}{(2l)!}, \qquad \text{(A2.6)}$$

and

$$\mathcal{E} = \frac{2^l}{(2l)!}(\underset{\sim}{t}\cdot\underset{\sim}{t}^*)^l + \frac{1}{(l!)^2}(zz^*)^l + O(t^2, t^{*2}). \qquad \text{(A2.7)}$$

This result will now be confirmed by a direct evaluation of S defined in (A2.4)

The coefficient A_n^m of $b^{n-2m}d^{2m}$ in X_n is

$$A_n^m = 2\sum_{r=m}^{\frac{n}{2}(-\frac{1}{2})}\binom{n}{2r}(-1)^r\binom{r}{m}(-1)^{r-m}$$

$$= (-1)^m \underset{x=1}{\mathcal{D}^m}\sum_{p=0}^{n}\binom{n}{p}x^{p/2}\left[1+(-1)^p\right]\frac{1}{m!},$$

where $\underset{x=1}{\mathcal{D}^m}\ldots \equiv \left[\frac{d^m}{dx^m}\ldots\right]_{x=1}$. Hence

$$A_n^m = \frac{(-1)^m}{m!}\underset{x=1}{\mathcal{D}^m}\left\{(1+x^{\frac{1}{2}})^n + (1-x^{1/2})^n\right\}$$

$$= \frac{(-1)^m}{m!}\underset{x=1}{\mathcal{D}^m}(1+x^{1/2})^n \quad \text{if } n>0, \qquad \text{(A2.8)}$$

since $m \leq r \leq \frac{n}{2} < n$ and therefore $\frac{d^m}{dx^m}(1-x^{1/2})^n$ has at least one factor $(1-x^{1/2})$ and vanishes when $x = 1$.

Obviously $A_0^0 = 2$.

If $\mathop{D}_{x=1}^{m}(1+x^{1/2})^n \equiv D_n^m$, then

$$D_n^{m+1} = \frac{n}{2}\mathop{D}_{x=1}^{m} x^{-\frac{1}{2}}(1+x^{\frac{1}{2}})^{n-1}$$

$$= \frac{n}{2}\sum_{r=0}^{m}\binom{m}{r}(-1)^r\frac{(2r)!}{2^{2r}r!}D_{n-1}^{m-r}, \qquad (A2.9)$$

from which it can be shown by induction that

$$D_n^m = 2^{n-2m}\,n\,\frac{(n-m-1)!}{(n-2m)!} \qquad (A2.10)$$

For, supposing this to be true for $D_{n-1}^0, D_{n-1}^1, \ldots, D_{n-1}^m$,

from (A2.9)

$$D_n^{m+1} = \frac{n}{2}\sum_{r=0}^{m}\binom{m}{r}(-1)^r\frac{(2r)!}{2^{2r}r!}\,\frac{(n-m+r-2)!}{(n-2m+2r-1)!}(n-1)2^{n-2m+2r-1}$$

$$= n(n-1)2^{n-2m-2}\sum_{r=0}^{m}\binom{m}{r}(-1)^r\frac{(2r)!(n-m+r-2)!}{r!(n-2m+2r-1)!}. \qquad (A2.11)$$

Writing $a = n-m-2$, the sum in (A2.11) is

$$\sum_{r=0}^{m}\binom{m}{r}(-1)^r\frac{(2r)!}{(2r-m+a+1)!}\,\frac{(r+a)!}{r!}$$

$$= \sum_{r=0}^{m}\binom{m}{r}(-1)^r\mathop{D}_{x=1}^{m-a-1}x^{2r}\mathop{D}_{y=1}^{a}y^{r+a}, \quad 0 \leq a \leq m-1,$$

$$= \mathop{D}_{x=1}^{m-a-1}\mathop{D}_{y=1}^{a}y^a(1-yx^2)^m$$

$= 0$ if $a = 0, 1, 2, \ldots, m-1$, since $\mathop{D}_{y=1}^{a}y^a(1-yx^2)^m$ has a factor $(1-x^2)^{m-a}$.

Hence the sum has factors

$$(n-m-2)(n-m-3)\ldots(n-2m-1) = \frac{(n-m-2)!}{(n-2m-2)!}$$

But it is easily seen that (n - 1) times this sum is a polynomial of degree m in n, and the coefficient of n^m is 1. Hence

$$(n-1)\times \text{sum in (A2.11)} = \frac{(n-m-2)!}{(n-2m-2)!},$$

and inserting back in (A2.11) one obtains

$$D_n^{m+1} = 2^{n-2m-2}\, n\, \frac{(n-m-2)!}{(n-2m-2)!},$$

in accordance with (A2.10). But (A2.10) is obviously true for m = 0: hence it is true for all m.

Collecting the results (A.210), (A2.8), (A2.4), one obtains

$$S = \sum_{n=0}^{l} \frac{1}{(l-n)!(l+n)!} \sum_{m=0}^{\frac{n}{2}(-\frac{1}{2})} \frac{(-1)^m}{m!}\, n\, \frac{(n-m-1)!}{(n-2m)!}\, 2^{n-2m} b^{n-2m} d^{l-n+2m} \qquad (2.12)$$

The coefficient of $b^s d^{l-s}$ in S is

$$C = \sum_{m=0}^{\frac{l-s}{2}(-\frac{1}{2})} (-1)^m 2^s\, \frac{(s+2m)\,(s+m-1)!}{m!\, s!\, (l-s-2m)!\,(l+s+2m)!}. \qquad (A2.13)$$

The case s = 0 has already been discussed in (A2.5). For $s > 0$, C may be evaluated as follows.

$$C = \frac{2^s}{s!} \sum_{m=0}^{\frac{l-s}{2}(-\frac{1}{2})} \frac{(s+2m)(s+m-1)!}{(l+s+2m)!\,(l-s-2m)!\,m!} (-1)^m$$

$$= \frac{2^{s-1}}{(2l)!\, s!} \left\{ \sum_{r=0}^{l-s} \binom{2l}{l-s-r} (s+r)\left(s+\tfrac{r}{2}-1\right)\left(s+\tfrac{r}{2}-2\right)\cdots\left(\tfrac{r}{2}+1\right)\left[(i)^r + (-i)^r\right] \right.$$

$$= \frac{2^{s-1}}{(2l)!\, s!} \left\{ \sum_{\rho=0}^{l-s} \binom{2l}{\rho} (l-\rho)\left(\frac{l+s-\rho}{2}-1\right)\left(\frac{l+s-\rho}{2}-2\right)\cdots\left(\frac{l-s-\rho}{2}+1\right)(i)^{l-s-\rho} \right.$$

$$\left. \left. + \sum_{\rho=l+s}^{2l} \binom{2l}{\rho} (l-\rho)\left(\frac{l+s-\rho}{2}-1\right)\left(\frac{l+s-\rho}{2}-2\right)\cdots\left(\frac{l-s-\rho}{2}+1\right)(+i)^{l-s-\rho} \right\} \right.$$

$$\therefore C = \frac{2^{s-1}}{(2\ell)!\,s!} \sum_{\rho=0}^{2\ell} \binom{2\ell}{\rho} (\ell-\rho)\left(\frac{\ell+s-\rho}{2}-1\right)\left(\frac{\ell+s-\rho}{2}-2\right)\ldots\left(\frac{\ell-s-\rho}{2}+1\right)(i)^{\ell-s-\rho},$$

since the additional terms either cancel in pairs or vanish in consequence of the factors $(\ell-\rho)$, etc. Hence

$$C = \frac{2^{s-1}}{(2\ell)!\,s!}(-i)^{\ell-s}(-1)^{s} \underset{x=1}{D} x^{-\ell} \underset{y=1}{D}{}^{*\,s-1}\; y^{\frac{-\ell+s}{2}-1}\left(1+ix\,y^{\frac{1}{2}}\right)^{2\ell}$$

$$= \frac{2^{s-1}\ell}{(2\ell)!\,s!}(-i)^{\ell-s}(-1)^{s-1} \underset{y=1}{D}{}^{s-1}\; y^{\frac{-\ell+s}{2}-1}(1+y)\left(1+i\,y^{\frac{1}{2}}\right)^{2\ell-2} \qquad (A2.14)$$

Now

$$\underset{y=1}{D}{}^{m}\left(1+y\right)y^{\frac{-n+m}{2}-1}\left(1+i\,y^{\frac{1}{2}}\right)^{2n}, \quad m \leq n,$$

$$= \underset{y=1}{D}{}^{m}\left(1+y\right)y^{\frac{-n+m}{2}-1}\left(1-y+2i\,y^{1/2}\right)^{n}$$

$$= \underset{y=1}{D}{}^{m} \sum_{r=0}^{n} \binom{n}{r}\left[y^{\frac{m-r}{2}} + y^{\frac{m-r}{2}-1}\right](1-y)^{r}(2i)^{m-r}$$

$$= \sum_{r=0}^{m} \binom{n}{r}(2i)^{m-r}(-1)^{r}\,r! \underset{y=1}{D}{}^{m-r}\left[y^{\frac{m-r}{2}} + y^{\frac{m-r}{2}-1}\right]\binom{m}{r},$$

and $\underset{y=1}{D}{}^{m-r}\left[y^{\frac{m-r}{2}} + y^{\frac{m-r}{2}-1}\right] = 0$ if $m-r$ is even and greater than zero, and

$$\underset{y=1}{D}{}^{m-r}\left[y^{\frac{m-r}{2}} + y^{\frac{m-r}{2}-1}\right] = \left(\frac{m-r}{2}\right)\left(\frac{m-r}{2}-1\right)\ldots\left(-\frac{m-r}{2}+1\right)$$

$$+\left(\frac{m-r}{2}-1\right)\left(\frac{m-r}{2}-2\right)\ldots\left(-\frac{m-r}{2}\right)$$

$$= 0 \quad \text{if} \quad m-r \quad \text{is odd.}$$

Hence

$$\left.\frac{\mathrm{D}^m}{y=1}\, y^{\frac{-n+m}{2}-1}(1+y)(1+iy^{1/2})^{2n} = 2(-1)^m(2i)^{n-m}\frac{n!}{(n-m)!}\right.$$

Inserting this in (A2.14), one finds

$$C = \frac{2^l}{(2l)!}\,\frac{l!}{s!(l-s)!}\,, \qquad \text{(A2.15)}$$

and inserting this result in (A2.12), and using the result (A2.5) for the case s = 0, one obtains

$$S = \frac{2^l}{(2l)!}\sum_{s=0}^{l}\binom{l}{s} b^s d^{l-s} + \frac{1}{(l!)^2}(zz^*)^l$$

$$= \frac{2^l}{(2l)!}(b+d)^l + \frac{1}{(l!)^2}(zz^*)^l$$

$$= \frac{2^l}{(2l)!}(\underset{\sim}{1}\cdot\underset{\sim}{z}^*)^l + \frac{1}{(l!)^2}(zz^*)^l,$$

in agreement with (A2.7).

APPENDIX 3 SUMMATION OVER ELECTRON ANGULAR MOMENTUM STATES

In Chapter 3, it was found necessary to sum expressions of the type $\psi^+(\underset{\sim}{r})\,O\,\psi(\underset{\sim}{r}^*)$, where O is a Dirac matrix and $\psi(\underset{\sim}{r})$ is the electron wave-function, over all angular momentum states of the electron. The summation is performed here with O one component of $\underset{\sim}{\sigma}$, and the results are quoted for all the possible operators O.

With the Dirac representation for $\underset{\sim}{\sigma}$,

$$\psi^+\sigma_x\psi = \psi_1^*\psi_2 + \psi_2^*\psi_1 + \psi_3^*\psi_4 + \psi_4^*\psi_3. \qquad (A3.1)$$

Inserting in this the type I wave-functions given in (3.25) and using the relation between wave-functions for positive and negative m given in Chapter 3, the summation of (A3.1) over all orientations of the angular momentum gives

$$\begin{aligned}\sum_m \psi^+\sigma_x\psi = \frac{1}{4\pi}\left(\frac{(2l)!}{2^l\,l!}\right)^2\Bigg\{ & \frac{(2l+1)^2}{r^{2(l+1)}} f_l^2 \sum_{m=\frac12}^{l+\frac12} \frac{1}{(l+m+\frac12)!\,(l-m+\frac12)!}\,(zz^*)^{l-m+\frac12} \\ & \times\Big[(x-iy)^{m-\frac12}(x^*+iy^*)^{m+\frac12}z + (x-iy)^{m+\frac12}(x^*+iy^*)^{m-\frac12}z^* \\ & -(x+iy)^{m-\frac12}(x^*-iy^*)^{m+\frac12}z - (x+iy)^{m+\frac12}(x^*-iy^*)^{m-\frac12}z^*\Big] \\ & -\frac{1}{r^{2l}}\,g_l^2 \sum_{m=\frac12}^{l+\frac12} \frac{1}{(l+m-\frac12)!\,(l-m-\frac12)!}\,(zz^*)^{l-m-\frac12} \\ & \times\Big[(x-iy)^{m-\frac12}(x^*+iy^*)^{m+\frac12}z + (x-iy)^{m+\frac12}(x^*+iy^*)^{m-\frac12}z^* \\ & -(x+iy)^{m-\frac12}(x^*+iy^*)^{m+\frac12}z - (x+iy)^{m+\frac12}(x^*-iy^*)^{m-\frac12}z^*\Big]\Bigg\}. \qquad (A3.2)\end{aligned}$$

Now let

$$\left.\begin{aligned} D^{+} &= -(x^{*}+iy^{*})\frac{\partial}{\partial z^{*}} + z^{*}\left(\frac{\partial}{\partial x^{*}}+i\frac{\partial}{\partial y^{*}}\right), \\ D^{-} &= (x^{*}-iy^{*})\frac{\partial}{\partial z^{*}} - z^{*}\left(\frac{\partial}{\partial x^{*}}-i\frac{\partial}{\partial y^{*}}\right). \end{aligned}\right\} \qquad (A3.3)$$

Then

$$\left.\begin{aligned} D^{+}(x^{*}+iy^{*})^{\rho}(zz^{*})^{l-\rho} &= -(l-\rho)z(x^{*}+iy^{*})^{\rho+1}(zz^{*})^{l-\rho-1}, \\ D^{+}(x^{*}-iy^{*})^{\rho}(zz^{*})^{l-\rho} &= (l+\rho)z^{*}(x^{*}-iy^{*})^{\rho-1}(zz^{*})^{l-\rho}+O(r^{*2}), \\ D^{-}(x^{*}+iy^{*})^{\rho}(zz^{*})^{l-\rho} &= -(l+\rho)z^{*}(x^{*}+iy^{*})^{\rho-1}(zz^{*})^{l-\rho}+O(r^{*2}), \\ D^{-}(x^{*}-iy^{*})^{\rho}(zz^{*})^{l-\rho} &= (l-\rho)z(x^{*}-iy^{*})^{\rho+1}(zz^{*})^{l-\rho-1}. \end{aligned}\right\} \qquad (A3.4)$$

Applying these relations, one finds for type I wave-functions

$$\begin{aligned} \sum_{m} v^{+}\sigma_{x}v = \frac{1}{4\pi}\left(\frac{(2l)!}{2^{l}\,l!}\right)^{2}\Bigg\{ & \frac{(2l+1)^{2}}{r^{2(l+1)}}\,f_{l}^{2}\,(D^{+}+D^{-}) \\ & \times\Bigg[-\sum_{\rho=0}^{l+1}\frac{1}{(l+\rho+1)!\,(l-\rho+1)!}\,(zz^{*})^{l+1-\rho} \\ & \times\Big\{[xx^{*}+yy^{*}+i(xy^{*}-yx^{*})]^{\rho}+[xx^{*}+yy^{*}-i(xy^{*}-yx^{*})]^{\rho}\Big\} \\ & +\frac{1}{(l+1)!\,(l+1)!}\,(zz^{*})^{l+1}\Bigg] \\ & -\frac{g_{l}^{2}}{r^{2l}}(D^{+}+D^{-})\Bigg[-\sum_{\rho=0}^{l}\frac{1}{(l+\rho)!\,(l-\rho)!}\,(zz^{*})^{l-\rho} \\ & \times\Big\{[xx^{*}+yy^{*}+i(xy^{*}-yx^{*})]^{\rho}+[xx^{*}+yy^{*}-i(xy^{*}-yx^{*})]^{\rho}\Big\} \\ & +\frac{1}{(l!)^{2}}(zz^{*})^{l}\Bigg]\Bigg\}. \end{aligned} \qquad (A3.5)$$

Using the result of Appendix 2, this becomes

$$\sum_m v^+ \sigma_x v = \frac{1}{4\pi}\left(\frac{(2l)!}{2^l l!}\right)^2 \left\{ -\frac{(2l+1)^2}{r^{2(l+1)}} f_l^2 \frac{2^{l+1}}{(2l+2)!}\left(D^+ + D^-\right)\left(\hat{r}\cdot\hat{r}^*\right)^{l+1} \right.$$

$$\left. + \frac{g_l^2}{r^{2l}} \frac{2^l}{(2l)!}\left(D^+ + D^-\right)\left(\hat{r}\cdot\hat{r}^*\right)^l \right\}$$

$$= \frac{1}{4\pi}\left(\frac{(2l)!}{2^l l!}\right)^2 \left\{ \frac{(2l+1)^2}{r^{2(l+1)}} f_l^2 \frac{2^{l+1}}{(2l+2)!}(l+1)(yz^* - zy^*)\left(\hat{r}\cdot\hat{r}^*\right)^l \right.$$

$$\left. - \frac{g_l^2}{r^{2l}} \frac{2^l}{(2l)!} l \left(yz^* - zy^*\right)\left(\hat{r}\cdot\hat{r}^*\right)^{l-1} \right\}(-2i). \qquad (A3.6)$$

A similar result is found for type II wave-functions. Combining the two results, one finds

$$\frac{1}{2}\sum_m v^+ \sigma_x v = (-i)(yz^* - zy^*)\frac{(2l)!}{2^l (l!)^2}$$

$$\times \left\{ (2l+1)\frac{f_l^2 + g_{-l-2}^2}{4\pi r^{2(l+1)}}\left(\hat{r}\cdot\hat{r}^*\right)^l - l\,\frac{g_l^2 + f_{-l-2}^2}{4\pi r^{2l}}\left(\hat{r}\cdot\hat{r}^*\right)^{l-1} \right\}. \qquad (A3.7)$$

For Dirac matrices other than σ_x, the summation can be performed in a similar, but generally easier, manner. The results are

$$\frac{1}{2}\sum_m v^+ v = \frac{(2l)!}{2^l (l!)^2}\left\{ (l+1)\frac{g_l^2 + f_{-l-2}^2}{4\pi r^{2l}}\left(\hat{r}\cdot\hat{r}^*\right)^l + (2l+1)\frac{g_{-l-2}^2 + f_l^2}{4\pi r^{2(l+1)}}\left(\hat{r}\cdot\hat{r}^*\right)^{l+1} \right\},$$

$$\frac{1}{2}\sum_m v^+ \beta v = -\frac{(2l)!}{2^l (l!)^2}\left\{ (l+1)\frac{g_l^2 - f_{-l-2}^2}{4\pi r^{2l}}\left(\hat{r}\cdot\hat{r}^*\right)^l + (2l+1)\frac{g_{-l-2}^2 - f_l^2}{4\pi r^{2(l+1)}}\left(\hat{r}\cdot\hat{r}^*\right)^{l+1} \right\},$$

$$\frac{1}{2}\sum_m v^+ \gamma_5 v = 0,$$

$$\frac{1}{2}\sum_m v^+ \beta\gamma_5 v = 0,$$

$$\frac{1}{2}\sum_m v^+ \underset{m}{\sigma} v = -i\left[\underset{m}{\hat{r}}\times\underset{m}{\hat{r}}^*\right]\frac{(2l)!}{2^l(l!)^2} \times\left\{-l\,\frac{g_l^2+f_{-l-2}^2}{4\pi r^{2l}}\left(\underset{m}{\hat{r}}\cdot\underset{m}{\hat{r}}^*\right)^{l-1} + (2l+1)\,\frac{g_{-l-2}^2+f_l^2}{4\pi r^{2(l+1)}}\left(\underset{m}{\hat{r}}\cdot\underset{m}{\hat{r}}^*\right)^l\right\},$$

$$\frac{1}{2}\sum_m v^+ \beta\underset{m}{\sigma} v = +i\left[\underset{m}{\hat{r}}\times\underset{m}{\hat{r}}^*\right]\frac{(2l)!}{2^l(l!)^2} \times\left\{-l\,\frac{g_l^2-f_{-l-2}^2}{4\pi r^{2l}}\left(\underset{m}{\hat{r}}\cdot\underset{m}{\hat{r}}^*\right)^{l-1} + (2l+1)\,\frac{g_{-l-2}^2+f_l^2}{4\pi r^{2(l+1)}}\left(\underset{m}{\hat{r}}\cdot\underset{m}{\hat{r}}^*\right)^l\right\},$$

$$\frac{1}{2}\sum_m v^+ \underset{m}{\alpha} v = -i\left(\underset{m}{\hat{r}}-\underset{m}{\hat{r}}^*\right)\frac{(2l)!}{2^l(l!)^2}(2l+1)\,\frac{f_l g_l - f_{-l-2}g_{-l-2}}{4\pi r^{2l+1}}\left(\underset{m}{\hat{r}}\cdot\underset{m}{\hat{r}}^*\right)^l,$$

$$\frac{1}{2}\sum_m v^+ \beta\underset{m}{\alpha} v = -i\left(\underset{m}{\hat{r}}+\underset{m}{\hat{r}}^*\right)\frac{(2l)!}{2^l(l!)^2}(2l+1)\,\frac{f_l g_l + f_{-l-2}g_{-l-2}}{4\pi r^{2l+1}}\left(\underset{m}{\hat{r}}\cdot\underset{m}{\hat{r}}^*\right)^l.$$

APPENDIX 4. CORRECTION FACTORS.

The theoretical spectrum shape for an n^{th} forbidden transition is given by

$$P_n(w)dw = C_n(w)\frac{G^2}{2\pi^3}F_0\, pW(W_0-W)^2 dW, \quad (A4.1)$$

where $(1+\gamma)\rho^2 F_0 = g_0^2 + f_{-2}^2$, $\gamma = \sqrt{(1-\alpha^2 Z^2)}$, and $C_n(w)$ is the n^{th} forbidden correction factor. It is convenient to write C_n as a sum of terms in the following way:-

$$C_n(w) = \sum_X \delta_X^2 C_n(X) + \tfrac{1}{2}\sum_X \sum_{Y\neq X} \delta_X \delta_Y C_n(X,Y), \quad (A4.2)$$

where X,Y represent the different interactions S, V, T, A and P, and the interaction constant G_X for the interaction X is $\delta_X G$. Since the matrix element for the most allowed transition with the interaction P has selection rules and magnitude similar to those for first forbidden transitions with the other interactions, this interaction is conveniently regarded as giving no allowed transitions.

The correction factors $C_n(X)$ are

$$C_n(S) = |Q_n(\beta^{\pm})|^2 \sum_{\nu=0}^{n} \left\{ A_{n\nu} q^{2(n-\nu)-2} M_\nu + 2C_{n\nu} q^{2(n-\nu)-1} N_\nu + D_{n\nu} q^{2(n-\nu)} L_\nu \right\},$$

$$C_n(V) = |Q_n(\underline{l}_m)|^2 \sum_{r=0}^{n} \{ A_{nr} q^{2(n-r)-2} M_r - 2C_{nr} q^{2(n-r)-1} N_r + D_{nr} q^{2(n-r)} L_r \}$$

$$+ |Q_n(\underline{\alpha}_m)|^2 \sum_{r=0}^{n} A_{nr} q^{2(n-r)-2} L_r$$

$$+ i[Q_n(\underline{\alpha}_m) Q_n^*(\underline{l}_m) - c.c.] \sum_{r=0}^{n} \{ -A_{nr} q^{2(n-r)-2} N_r + C_{nr} q^{2(n-r)-1} L_r \}$$

$$+ |Q_{n-1}(\underline{\alpha}_m \times \underline{l}_m)|^2 \sum_{r=0}^{n-1} \{ A_{n-1,r} q^{2(n-1-r)-2} M_r + 2C_{n-1,r} q^{2(n-1-r)-1} N_r$$

$$+ \left(D_{n-1,r} - \frac{1}{n} B_{n-1,r} \right) q^{2(n-1-r)} L_r \},$$

$$C_n(T) = |Q_{n+1}(\underline{\beta}_m)|^2 \sum_{r=0}^{n} B_{nr} q^{2(n-r)} L_r.$$

$$+ |Q_n(\underline{\beta}_m \times \underline{l}_m)|^2 \sum_{r=0}^{n} \{ A_{nr} q^{2(n-r)-2} M_r - 2C_{nr} q^{2(n-r)-1} N_r$$

$$+ \left(D_{nr} - \frac{1}{n+1} B_{nr} \right) q^{2(n-r)} L_r \}$$

$$+ |Q_n(\underline{\beta}_m \underline{\alpha}_m)|^2 \sum_{r=0}^{n} A_{nr} q^{2(n-r)-2} L_r$$

$$- [Q_n(\underline{\beta}_m \times \underline{l}_m) Q_n^*(\underline{\beta}_m \underline{\alpha}_m) + c.c.] \sum_{r=0}^{n} \{ -A_{nr} q^{2(n-r)-2} N_r + C_{nr} q^{2(n-r)-1} L_r \},$$

$$C_n(A) = |Q_{n+1}(\underline{\sigma}_m)|^2 \sum_{r=0}^{n} B_{nr} q^{2(n-r)} L_r$$

$$+ |Q_n(\underline{\sigma}_m \times \underline{l}_m)|^2 \sum_{r=0}^{n} \{ A_{nr} q^{2(n-r)-2} M_r + 2C_{nr} q^{2(n-r)-1} N_r$$

$$+ \left(D_{nr} - \frac{1}{n+1} B_{nr} \right) q^{2(n-r)} L_r \},$$

$$C_n(P) = |Q_{n-1}(\underline{\beta}\underline{\gamma}_5 \underline{l}_m)|^2 \sum_{r=0}^{n-1} \{ A_{n-1,r} q^{2(n-1-r)-2} M_r + 2C_{n-1,r} q^{2(n-1-r)-1} N_r$$

$$+ D_{n-1,r} q^{2(n-1-r)} L_r \}.$$

These formula have been given by Greuling (1942). The cross correction factors $C_n(X,Y)$ are

$$C_n(S,V) = -\left[Q_n(\beta\underline{\hat{r}}_m)Q_n^*(\underline{\hat{r}}_m) + c.c.\right]\sum_{\nu=0}^{n}\left\{A_{n\nu}q^{2(n-\nu)-2}Q_\nu + D_{n\nu}q^{2(n-\nu)}P_\nu\right\}$$
$$+i\left[Q_n(\beta\underline{\hat{r}}_m)Q_n^*(\underline{\alpha}_m) - c.c.\right]\sum_{\nu=0}^{n}\left\{A_{n\nu}q^{2(n-\nu)-2}R_\nu + C_{n\nu}q^{2(n-\nu)-1}P_\nu\right\},$$

$$C_n(S,T) = i\left[Q_n(\beta\underline{\hat{r}}_m)Q_n^*(\beta\underline{\alpha}_m) - c.c.\right]\sum_{\nu=0}^{n}\left\{A_{n\nu}q^{2(n-\nu)-2}N_\nu + C_{n\nu}q^{2(n-\nu)-1}L_\nu\right\}$$
$$+i\left[Q_n(\beta\underline{\hat{r}}_m)Q_n^*(\beta\underline{\alpha}\times\underline{\hat{r}}_m) - c.c.\right]\sum_{\nu=0}^{n}\left\{A_{n\nu}q^{2(n-\nu)-2}M_\nu + (D_{n\nu}-B_{n\nu})q^{2(n-\nu)}L_\nu\right\},$$

$$C_n(S,A) = -i\left[Q_n(\beta\underline{\hat{r}}_m)Q_n^*(\underline{\alpha}\times\underline{\hat{r}}_m) - c.c.\right]\sum_{\nu=0}^{n}\left\{A_{n\nu}q^{2(n-\nu)-2}Q_\nu - 2C_{n\nu}q^{2(n-\nu)-1}R_\nu\right.$$
$$\left.+(D_{n\nu}-B_{n\nu})q^{2(n-\nu)}P_\nu\right\},$$

$$C_n(S,P) = 0,$$

$$C_n(V,T) = \left[Q_n(\underline{\alpha}_m)Q_n^*(\beta\underline{\alpha}_m) + c.c.\right]\sum_{\nu=0}^{n}A_{n\nu}q^{2(n-\nu)-2}P_\nu$$
$$-i\left[Q_n(\underline{\hat{r}}_m)Q_n^*(\beta\underline{\alpha}_m) - c.c.\right]\sum_{\nu=0}^{n}\left\{-A_{n\nu}q^{2(n-\nu)-2}R_\nu + C_{n\nu}q^{2(n-\nu)-1}P_\nu\right\}$$
$$-\left[Q_n(\underline{\alpha}_m)Q_n^*(\beta\underline{\alpha}\times\underline{\hat{r}}_m) + c.c.\right]\sum_{\nu=0}^{n}\left\{-A_{n\nu}q^{2(n-\nu)-2}R_\nu + C_{n\nu}q^{2(n-\nu)-1}P_\nu\right\}$$
$$-i\left[Q_n(\underline{\hat{r}}_m)Q_n^*(\beta\underline{\alpha}\times\underline{\hat{r}}_m) - c.c.\right]\sum_{\nu=0}^{n}\left\{A_{n\nu}q^{2(n-\nu)-2}Q_\nu + 2C_{n\nu}q^{2(n-\nu)-1}R_\nu\right.$$
$$\left.+(D_{n\nu}-B_{n\nu})q^{2(n-\nu)}P_\nu\right\},$$

$$C_n(V,A) = \left[Q_n(\underset{\sim}{\alpha}_m)Q_n^*(\underset{\sim}{\sigma}_m \times \underset{\sim}{\hat{q}}_m) + c.c.\right]\sum_{\nu=0}^{n}\left\{A_{n\nu}\, q^{2(n-\nu)-2} N_\nu + C_{n\nu}\, q^{2(n-\nu)-1} L_\nu\right\}$$

$$+ i\left[Q_n(\underset{\sim}{\hat{q}}_m)Q_n^*(\underset{\sim}{\sigma}_m \times \underset{\sim}{\hat{q}}_m) - c.c.\right]\sum_{\nu=0}^{n}\left\{A_{n\nu}\, q^{2(n-\nu)-2} M_\nu + (D_{n\nu} - B_{n\nu})\, q^{2(n-\nu)} L_\nu\right\},$$

$$C_n(V,P) = -i\left[Q_{n-1}(\underset{\sim}{\alpha}_m \times \underset{\sim}{\hat{q}}_m)Q_{n-1}^*(\beta\gamma_5 \underset{\sim}{\hat{q}}_m) - c.c.\right]\sum_{\nu=0}^{n-1}\left\{A_{n-1,\nu}\, q^{2(n-1-\nu)-2} Q_\nu - 2C_{n-1,\nu}\, q^{2(n-1-\nu)-1} R_\nu\right.$$

$$\left. + (D_{n-1,\nu} - B_{n-1,\nu})\, q^{2(n-1-\nu)} P_\nu\right\},$$

$$C_n(T,A) = -\left[Q_{n+1}(\beta \underset{\sim}{\sigma}_m)Q_n^*(\underset{\sim}{\sigma}_m) + c.c.\right]\sum_{\nu=0}^{n} B_{n\nu}\, q^{2(n-\nu)} P_\nu$$

$$- \left[Q_n(\beta \underset{\sim}{\sigma}_m \times \underset{\sim}{\hat{q}}_m)Q_n^*(\underset{\sim}{\sigma}_m \times \underset{\sim}{\hat{q}}_m) + c.c.\right]\sum_{\nu=0}^{n}\left\{A_{n\nu}\, q^{2(n-\nu)-2} Q_\nu + \left(D_{n\nu} - \frac{1}{n+1} B_{n\nu}\right) q^{2(n-\nu)} P_\nu\right\}$$

$$+ \left[Q_n(\beta \underset{\sim}{\alpha}_m)Q_n^*(\underset{\sim}{\sigma}_m \times \underset{\sim}{\hat{q}}_m) + c.c.\right]\sum_{\nu=0}^{n}\left\{A_{n\nu}\, q^{2(n-\nu)-2} R_\nu + C_{n\nu}\, q^{2(n-\nu)-1} P_\nu\right\},$$

$$C_n(T,P) = -i\left[Q_{n+1}(\beta \underset{\sim}{\sigma}_m)Q_{n+1}^*(\beta\gamma_5 \underset{\sim}{\hat{q}}_m) - c.c.\right]\sum_{\nu=0}^{n+1}\left\{A_{n+1,\nu} N_\nu\, q^{2(n+1-\nu)-2} + C_{n+1,\nu}\, q^{2(n+1-\nu)-1} L_\nu\right\},$$

$$C_n(A,P) = i\left[Q_{n+1}(\underset{\sim}{\sigma}_m)Q_{n+1}^*(\beta\gamma_5 \underset{\sim}{\hat{q}}_m) - c.c.\right]\sum_{\nu=0}^{n+1}\left\{A_{n+1,\nu}\, q^{2(n+1-\nu)-2} R_\nu + C_{n+1,\nu}\, q^{2(n+1-\nu)-1} P_\nu\right\}.$$

The last two correction factors represent a mixing of the $(n+2)^{th}$ forbidden P spectrum with the nth forbidden T and A Spectra respectively, and will be important only if the coupling constant for the P interaction is large enough to overcome two degrees of forbiddenmess.

The $Q_n(q)$ are tensor matrix elements, as defined in Chapter 3. The quantities L_ν, M_ν, N_ν, P_ν, Q_ν, R_ν are defined by

$$L_\nu = (2\rho^2 F_0)^{-1}\left[g_\nu^2 + f_{-\nu-2}^2\right]\rho^{-2\nu},$$

$$M_\nu = (2\rho^2 F_0)^{-1}\left[g_{-\nu-2}^2 + f_\nu^2\right]\rho^{-2(\nu+1)},$$

$$N_\nu = (2\rho^2 F_0)^{-1}\left[f_\nu g_\nu - f_{-\nu-2} g_{-\nu-2}\right]\rho^{-2\nu-1},$$

$$P_\nu = (2\rho^2 F_0)^{-1}\left[g_\nu^2 - f_{-\nu-2}^2\right]\rho^{-2\nu},$$

$$Q_\nu = (2\rho^2 F_0)^{-1}\left[g_{-\nu-2}^2 - f_\nu^2\right]\rho^{-2(\nu+1)},$$

$$R_\nu = (2\rho^2 F_0)^{-1}\left[f_\nu g_\nu + f_{-\nu-2} g_{-\nu-2}\right]\rho^{-2\nu-1}.$$

If F_ν is defined to be

$$F_\nu = \left(\frac{(2\nu+2)!}{\nu!}\right)^2 (2\rho\rho)^{2(s_\nu-\nu-1)} e^{\pi(\alpha W/\rho)} \frac{\left|\Gamma\left(s_\nu + \frac{i\alpha W}{\rho}\right)\right|^2}{\Gamma^2(2s_\nu+1)},$$

where

$$s_\nu = \left[(\nu+1)^2 - (\alpha Z)^2\right]^{1/2},$$

then

$$L_\nu = \frac{F_\nu}{F_0}\left(\frac{2^\nu \nu!}{(2\nu+1)!}\rho^\nu\right)^2 \frac{\nu+1+s_\nu}{2(\nu+1)},$$

$$M_\nu = \frac{F_\nu}{F_0}\left(\frac{2^{\nu+1}\nu!}{(2\nu+2)!}\rho^\nu\right)^2\left\{\frac{2\nu+2}{\nu+1+s_\nu}\left(\frac{\alpha Z}{2\rho}\right)^2\right.$$

$$+\left[\frac{s_\nu}{2s_\nu+1}\frac{\rho^2}{w}-\frac{(2\nu+1)(\alpha Z)^2}{(2s_\nu+1)(\nu+1+s_\nu)}w\right]\left(\frac{Z\alpha}{\rho}\right)$$

$$+\frac{(\nu+1)(s_\nu-\nu)s_\nu}{(2s_\nu+1)^2}\left[1-\frac{4s_\nu+3}{s_\nu(s_\nu+1)}(\alpha Z)^2\right]\rho^2$$

$$\left.+\left[1+\frac{\nu(4s_\nu+3)}{(s_\nu+1)(\nu+1+s_\nu)}(\alpha Z)^2\right]\left(\frac{\alpha Z}{2s_\nu+1}\right)^2\right\},$$

$$N_\nu = -\frac{F_\nu}{F_0}\left(\frac{2^\nu\nu!}{(2\nu+1)!}\rho^\nu\right)^2\frac{1}{\nu+1}\left\{\left(\frac{\alpha Z}{2\rho}\right)+\frac{s_\nu}{2s_\nu+1}\frac{\rho^2}{w}-\frac{2(\alpha Z)^2}{2s_\nu+1}w\right\},$$

$$P_\nu = \frac{F_\nu}{F_0}\left(\frac{2^\nu\nu!}{(2\nu+1)!}\rho^\nu\right)^2\frac{s_\nu(\nu+1+s_\nu)}{2(\nu+1)^2}\frac{1}{w},$$

$$Q_\nu = \frac{F_\nu}{F_0}\left(\frac{2^{\nu+1}\nu!}{(2\nu+2)!}\rho^\nu\right)^2\left\{-\frac{2s_\nu}{\nu+1+s_\nu}\left(\frac{\alpha Z}{2\rho}\right)^2\frac{1}{w}+\frac{(\alpha Z)^2}{\nu+1+s_\nu}\left(\frac{\alpha Z}{\rho}\right)\right.$$

$$+\frac{s_\nu(\nu+1)}{(2s_\nu+1)^2}\left[1+\frac{s_\nu(\alpha Z)^2}{(\nu+1)(\nu+1+s_\nu)}\right]\frac{\rho^2}{w}$$

$$\left.+\left[1-\frac{(4s_\nu+3)(Z\alpha)^2}{\nu+1+s_\nu}\right]\left(\frac{\alpha Z}{2s_\nu+1}\right)^2 w\right\},$$

$$R_\nu = -\frac{F_\nu}{F_0}\left(\frac{2^{\nu+1}\nu!}{(2\nu+2)!}\rho^\nu\right)^2\left\{s_\nu\left(\frac{Z\alpha}{2\rho}\right)\frac{1}{w}-(\alpha Z)^2\right\}.$$

For low Z,

$$S_\nu \simeq \nu+1, \qquad F_\nu \simeq F_0, \quad \text{and} \quad (\alpha Z)^2 \quad \left(\text{but not } \left(\frac{\alpha Z}{2\rho}\right)^2\right)$$

may be ignored, to give the simpler expressions

$$L_\nu = \left(\frac{2^\nu \nu!}{(2\nu+1)!}\rho^\nu\right)^2,$$

$$M_\nu = \left(\frac{2^{\nu+1}\nu!}{(2\nu+2)!}\rho^\nu\right)^2\left\{\left(\frac{\alpha Z}{2\rho}\right)^2 + \frac{\nu+1}{2\nu+3}\left(\frac{\alpha Z}{\rho}\right)\frac{\rho^2}{w} + \left(\frac{\nu+1}{2\nu+3}\right)^2\rho^2\right\},$$

$$N_\nu = -\left(\frac{2^\nu \nu!}{(2\nu+1)!}\rho^\nu\right)^2\frac{1}{\nu+1}\left\{\left(\frac{\alpha Z}{2\rho}\right) + \left(\frac{\nu+1}{2\nu+3}\right)\frac{\rho^2}{w}\right\},$$

$$P_\nu = \left(\frac{2^\nu \nu!}{(2\nu+1)!}\rho^\nu\right)^2\frac{1}{w},$$

$$Q_\nu = \left(\frac{2^{\nu+1}\nu!}{(2\nu+2)!}\rho^\nu\right)^2\left\{-\left(\frac{\alpha Z}{2\rho}\right)^2\frac{1}{w} + \left(\frac{\nu+1}{2\nu+3}\right)^2\frac{\rho^2}{w}\right\},$$

$$R_\nu = -\left(\frac{2^\nu \nu!}{(2\nu+1)!}\rho^\nu\right)^2\frac{1}{\nu+1}\left(\frac{\alpha Z}{2\rho}\right)\frac{1}{w}.$$

The numerical coefficients A_{nv}, B_{nv}, C_{nv}, D_{nv}, are

$$A_{nv} = \frac{(n-v)\,2^{n-2v}\,(2v+1)!}{(2n-2v)!\;\;(v!)^2}\,,$$

$$B_{nv} = \frac{2^{n-2v}\,(2v+1)!}{(2n-2v+1)!\;\;(v!)^2}\,,$$

$$C_{nv} = \frac{(n-v)\,2^{n-2v}\,(2v+1)!}{(2n-2v+1)!\;\;(v!)^2}\,,$$

$$D_{nv} = \frac{2^{n-2v}\,(v+1)\,(2v)!}{(2n-2v+1)!\;\;(v!)^2}\,.$$

APPENDIX 5. THE AVERAGING OF $(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)\,V(r)$

In this appendix, $(\underset{\sim}{r}_{m_1} - \underset{\sim}{r}_{m_2})V(r)$, where $r = |\underset{\sim}{r}_{m_1} - \underset{\sim}{r}_{m_2}|$ is averaged with respect to $\underset{\sim}{r}_{m_2}$ over a sphere of uniform density and radius ρ. For this purpose V(r) is expanded in a series of Legendre Polynomials of x, the cosine of the angle between $\underset{\sim}{r}_{m_1}$ and $\underset{\sim}{r}_{m_2}$:-

$$\left.\begin{aligned} V(r) &= \tfrac{1}{2}\sum_{n=0}^{\infty}(2n+1)\,v_n(r_1,r_2)\,P_n(x), \\ \text{where}\qquad v_n(r_1,r_2) &= \int_{-1}^{1} V(r)\,P_n(x)\,dx . \end{aligned}\right\} \quad (A5.1)$$

Now it is easily seen that, if $\underset{\sim}{r}_{m_2}$ is $(r_2\sin\theta\cos\varphi, r_2\sin\theta\sin\varphi, r_2\cos\theta)$,

$$\left.\begin{aligned} \int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\,d\theta\;\underset{\sim}{r}_{m_1}P_n(x) &= 4\pi\,\underset{\sim}{r}_{m_1}\delta_{n0}, \\ \int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\,d\theta\;\underset{\sim}{r}_{m_2}P_n(x) &= \frac{4\pi}{3}\frac{r_2}{r_1}\underset{\sim}{r}_{m_1}\delta_{n1}, \end{aligned}\right\} \quad (A5.2)$$

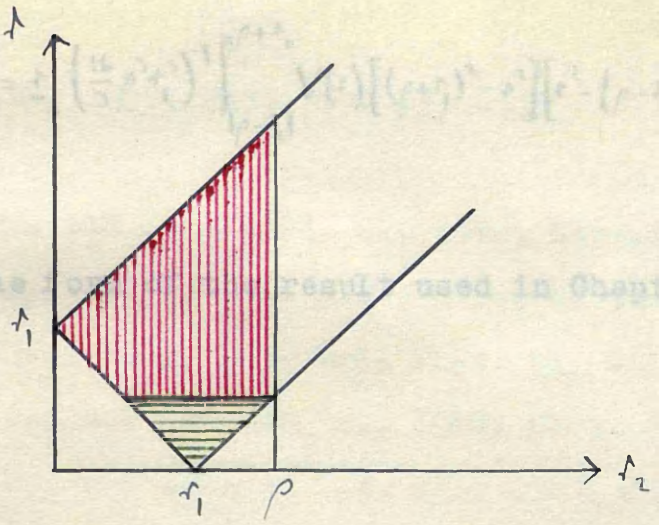
whence

$$\begin{aligned} \overline{(\underset{\sim}{r}_{m_1} - \underset{\sim}{r}_{m_2})V(r)} &= \tfrac{1}{2}\underset{\sim}{r}_{m_1}\left(\tfrac{1}{3}\rho^3\right)^{-1}\int_0^{\rho}\left[v_0(r_1,r_2) - \frac{r_2}{r_1}v_1(r_1,r_2)\right]r_2^2\,dr_2 \\ &= \tfrac{1}{2}\underset{\sim}{r}_{m_1}\left(\tfrac{1}{3}\rho^3\right)^{-1}\int_0^{\rho}r_2^2\,dr_2\int_{-1}^{1}V(r)\left[1-\frac{r_2}{r_1}x\right]dx. \end{aligned} \quad (A5.3)$$

Now since $r^2 = r_1^2 + r_2^2 - 2r_1r_2x$, the variable in the second integral may be changed to r, giving

$$\overline{(\underset{\sim}{r}_{m_1} - \underset{\sim}{r}_{m_2})V(r)} = \underset{\sim}{r}_{m_1}\left(\tfrac{4}{3}\rho^3 r_1^3\right)^{-1}\int_0^{\rho}r_2\,dr_2\int_{|r_1-r_2|}^{r_1+r_2}V(r)\left[r^2+r_1^2-r_2^2\right]r\,dr . \quad (A5.4)$$

The field of integration is shown shaded in the diagram -



If now the order of the integrations in (A5.4) is changed, the integration over r_2 can be performed at once to give

$$\int_a^b \left[r^2 + r_1^2 - r_2^2\right] r_2 \, dr_2 = \tfrac{1}{2}\left(b^2 - a^2\right)\left[\left(r^2 + r_1^2\right) - \tfrac{1}{2}\left(b^2 + a^2\right)\right] . \qquad (A5.5)$$

For the area shaded green in the diagram, the lower and upper limits in (A5.5) are respectively $|r_1 - r|$ and $r_1 + r$; consequently the integral (A5.5) vanishes. For the area shaded red, the lower and upper limits are $|r_1 - r|$ and ρ, so that the integral is

$$\tfrac{1}{4}\left[\rho^2 - (r_1 - r)^2\right]\left[(r_1 + r)^2 - \rho^2\right]$$

and (A5.4) becomes

$$\overline{\left(\frac{1}{r_{m_1}} - \frac{1}{r_{m_2}}\right) V(r)} = r_{m_1}\left(\tfrac{16}{3}\rho^3 r_1^3\right)^{-1} \int_{|\rho - r_1|}^{\rho + r_1} V(r)\left[\rho^2 - (r_1 - r)^2\right]\left[(r_1 + r)^2 - \rho^2\right] r \, dr , \qquad (A5.6)$$

or, what is the same thing,

$$\overline{(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)V(r)} = \underset{\sim}{r}_1 \left(\frac{16}{3}\rho^3 r_1^3\right)^{-1} \int_{|\rho - r_1|}^{\rho + r_1} V(r)\left[(\rho + r_1)^2 - r^2\right]\left[r^2 - (\rho - r_1)^2\right] r \, dr, \qquad (A5.7)$$

which is the form of the result used in Chapter 5.

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