# Bifurcation of Thick-walled Electroelastic Cylindrical and Spherical Shells at Finite Deformation

by

**Andrey Melnikov**

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**Abstract** In this dissertation we consider some boundary value and stability problems for electro-active soft rubberlike materials which withstand finite deformations elastically.

In the beginning we consider in detail the problem of finite deformation of a pressurized electroelastic circular cylindrical tube with closed ends with compliant electrodes at its curved boundaries. Expressions for the dependence of the pressure and reduced axial load on the deformation and a potential difference between the electrodes, or uniform surface charge distributions, are obtained in respect of a general isotropic electroelastic energy function. To illustrate the behaviour of the tube specific forms of energy functions accounting for different mechanical properties coupled with a deformation independent quadratic dependence on the electric field are used for numerical purposes, for a given potential difference and separately for a given charge distribution. Numerical dependences of the non-dimensional pressure and reduced axial load on the deformation are obtained for the considered energy functions. Results are then given for the thin-walled approximation as a limiting case of a thick-walled cylindrical tube without restriction on the energy function. The theory provides a general basis for the detailed analysis of the electroelastic response of tubular dielectric elastomer actuators, which is illustrated for a fixed axial load in the absence of internal pressure and fixed internal pressure in the absence of an applied axial load.

Using the theory of small incremental electroelastic deformations superimposed on an electroelastic finitely deformed body, we then look for solutions of underlying configurations which are different from perfect cylindrical shape of the tube. First, we consider prismatic bifurcations. We obtain the solutions which show that for neo-Hookean electroelastic material prismatic modes of bifurcation become possible under inflation. This result is different from the pure mechanical case considered previously in Haughton & Ogden (1979), because in Haughton & Ogden (1979) prismatic bifurcation modes were found only for an externally pressurised tube. Second, we consider axisymmetric bifurcations, and we obtain results for neo-Hookean and Mooney-Rivlin electroelastic energy functions. Our solutions show that in the presence of an electric field the electroelastic tube become more unstable: axisymmetric bifurcations become possible at lower values of circumferential stretches as compared with the values of circumferential stretches found for analogous problems solved for electromechanically indifferent materials, or equivalently, when electric field is not present.

Within similar lines we consider the bifurcation of a thick-walled electroelastic spherical shell with compliant electrodes at its curved boundaries under internal and external pressure. The solutions obtained for neo-Hookean electroelastic energy function show that in some cases axisymmetric modes of bifurcation become possible under inflation in the presence of electric field.

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**Statement** This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

No part of this thesis has previously been submitted by me for a degree at this or any other university.

The contents of Chapters 3, 4, 5 represent the original work (except for some cases where it is stated by appropriate references) of the author with his supervisor Professor R. W. Ogden.

The material of Chapter 3 in revised form has been published in Zeitschrift für angewandte Mathematik und Physik (Melnikov & Ogden, 2016).

The contents of Chapters 4 and 5 will be submitted to the appropriated journals.

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# Chapter 1

# Introduction

In this thesis we consider some boundary-value and stability problems for electro-active elastomeric materials which recently received a strong interest in the literature due to their specific properties suitable for many applications in engineering science, for example, production of actuators, sensors and other devices.

Before proceeding any further in discussion of problems for electroelastic materials we will discuss here briefly analogous problems for pure elastic materials. In the series of papers Haughton & Ogden (1978, 1979) have given an extensive bifurcation analysis for thin- and thick-walled cylindrical and spherical shells of elastic material under internal and external pressure. A more recent analysis which involves some new aspects of the solutions of these problems can be found in Zhu et al. (2008), deBotton et al. (2013).

It is a well known fact that an inflated tube made of a rubber material may develop a bulge at some point of deformation caused by internal pressure[1]. Likewise a spherical shell may become aspherical under internal pressure at some point of deformation (Alexander, 1971). The studies of spherical shells were motivated by applications in meteorology which employ high altitude weather balloons. It is interesting to note that these configurations which deviate from perfect cylindrical or spherical configurations of shells may arise under symmetrical load.

In order to model these cases when deviations from perfect cylindrical and spherical configurations are possible we use the theory of incremental deformations superimposed on an underlying finite deformation. The solutions obtained using this theory may contain cases when cylindrical or spherical shape is still preserved, for example, rigid-body trans-

[1]Experimental data for internal pressure as a function of a volume ratio up to the critical pressure where bulging may occur can be found in Charrier & Li (1977) and Skala (1970).

lation and other cases. We are not interested in these solutions and we discard them from our consideration. It is worth noting that the theory of incremental deformations used here can essentially model the onset of small deformations deviating from perfect original underlying configuration. In order to model full finite bifurcation configurations full nonlinear equations are needed (as opposed to linearized versions of these equations for small incremental deformations). Some progress in this regard can be found in Haughton (1980).

Successes in the technological production of electro-active polymers instigated a development of theories which account for electromechanical coupling. The theories which account for electromechanical coupling in continuum may be traced to the middle of the last century in the seminal work of Toupin (1956), who was concerned with the theory governing elastic dielectric materials. Books dealing with the theory include Eringen & Maugin (1990); Hutter & van de Ven (1978); Maugin (1988); Nelson (1979). The approach to the theory in the form described by Dorfmann & Ogden (2005), however, has led to further developments and has proved to be amenable to the solution of boundary-value problems, as exemplified in Dorfmann & Ogden (2006) and the recent monograph by Dorfmann & Ogden (2014c) and references therein.

In this PhD dissertation we analyze the response of an electroelastic tube to the combination of a radial electric field, an internal pressure and an axial load using the nonlinear theory of electroelasticity developed in Dorfmann & Ogden (2005). Then we superimpose incremental small deformations and electric displacements on the deformed underlying configuration of a cylinder and initial electric field. This allows us to consider the problem of stability of a cylinder made of electroactive material at the presence of electric field under internal and external pressure. The analysis of the nonlinear response of electroelastic spherical shell was done in Dorfmann & Ogden (2014b). We use some results and notation from this work and perform stability analysis for electroelastic shell using the theory of small electroelastic deformations. For stability analysis we use simple strain energy functions which can be expressed in terms of invariants: Neo-Hookean and Mooney-Rivlin electroelastic models.

This thesis is structured in the following manner. In Chapter 2 we give the most important ingredients of a general theory of Electroelasticity within the lines proposed by Dorfmann & Ogden (2005). The electroelastic constitutive laws are based on the so-called total energy density function which allows us to write constitutive laws in a simple form, and thus constitutive laws can be regarded as direct generalizations of pure mechanical counterparts. Incremental electroelastic constitutive laws are formulated in this chapter, and we also give the explicit formulae for electroelastic moduli tensors. This chapter ends with some important connections between the electroelastic moduli tensors used further in the text of the thesis.

In Chapter 3 we consider in great detail the problem of inflation and extension of electroelastic tubes with closed ends in the presence of electric field. The electric field is generated by compliant electrodes attached to the inner and outer surfaces of the tube. This construction with compliant electrodes can essentially be considered as an actuator where the actuating force can be generated by both inflation and electric field. General expressions are obtained for the internal pressure in a tube with closed ends and the axial load on its ends. Next, by considering a simple specific form of energy function, we obtain explicit expressions for the pressure and axial load in terms of the deformation and the electrostatic potential (or charge) applied to the compliant electrodes.

From the formulas for a thick-walled tube we provide numerical results which illustrate the dependence of the pressure and (reduced) axial load on the tube radius (via the azimuthal stretch on its inner boundary) and length (via the axial stretch). This is done for different values of the applied potential or charge for three different forms of the elastic part of the energy function for two different wall thicknesses (one relatively thin and one thicker) and we compare results with the results for the purely elastic case. It was found that there is very little difference qualitatively between the results for different tube thicknesses. Thus, it is appropriate to specialize to the thin-walled tube approximation, and we obtain explicit expressions for the pressure and (reduced) axial load in respect of a general electroelastic constitutive law.

In Chapter 4 we present a bifucation/stability analysis for the electroelastic tube with flexible electrodes in the presence of electric field under internal and external pressure. Without taking into account electromechanical coupling this problem was discussed by Haughton & Ogden (1979) and we return to this problem with the view to include the affect of electric field on the stability of electroelastic tube. First, we considered prismatic bifurcations. In this case we are looking for configurations of the tube with cross-sections deviating from a perfect circle, but remaining in the same shape along the axis of the tube, i.e. if $z$ is the axis of the tube, then the shape of cross-sections does not depend on $z$ axis along the tube. For the neo-Hookean electroelastic model we received quite a striking result: prismatic modes of bifurcations become possible under inflation in the presence of an electric field. Second, we considered axisymmetric bifurcations. These are the configurations of the tube with perfect circular cross-sections, the radius of which depends on the axis of the tube $z$. We compared our results with those obtained for pure elastic materials: the presence of electric field makes the electroelastic tube more unstable for both used energy functions neo-Hookean and Mooney-Rivlin, i.e. we could see from the obtained bifurcation curves that unstable configurations become possible at lower circumferential stretches as opposed to the cases with pure elastic materials without electromechanical interactions. A more general case of asymmetric bifurcations can be considered along the same lines, although in this case the number of equations in the system of ordinary differential equations increases and they become even more cumbersome.

In Chapter 5 we consider bifurcation/stability analysis of an electroelastic shell with flexible electrodes at the boundaries at the presence of electric field under internal and external pressure. For the purely mechanical case in Haughton & Ogden (1978) dependence on a spherical coordinate $\phi$ was omitted, because inclusion of it does not have influence on bifurcation criteria for pure mechanical case. We adopt this approach here, and therefore, we consider only axisymmetric bifurcations. The results show that a neo-Hookean electroelastic shell may develop axisymmetric modes of bifurcation under inflation in the presence of electric field. This result is different from pure elastic case where it was shown that axisymmetric bifurcation can be possible only under external pressure for neo-Hookean material.

Unlike Haughton & Ogden (1978, 1979) we used a more general formulation of governing equations and boundary conditions, because, first, they were expressed in terms of functions $\phi$ and $\psi$ and then we specified these functions appropriately.

In all cases the electroelastic term in the energy functions was expressed as $\varepsilon^{-1}I_5/2$. It can be shown that this leads to a linear constitutive law for $E_r$ and $D_r$: $E_r = \varepsilon^{-1}D_r$ with electric permittivity being independent of deformation. At least for some materials this can be viewed as a limitation and we discuss it briefly at the end of Chapter 3.

In Appendix A we give the derivations of some relations, used in Chapter 3. In Appendix B we give MATLAB code for our numerical calculations. We also discuss and explain briefly some important aspects of this code. The code employs a numerical scheme used in Haughton & Ogden (1979). We note that for the type of the problems considered here another numerical scheme known as matrix compound method can be successfully used. A good reference for this method can be found in Haughton (1997), for example.

# Chapter 2

# General Theory of Nonlinear Electroelasticity

## 2.1 The equations of nonlinear electroelasticity

In continuum mechanics we work with physical quantities either in reference or current configurations. We consider a deformable electrosensitive body which occupies the reference configuration $\mathcal{B}_\mathrm{r}$ with the boundary $\partial\mathcal{B}_\mathrm{r}$ in the absence of mechanical loads and electric fields. Application of an electric field and mechanical loads induces deformation which results in a new configuration $\mathcal{B}$ with the boundary $\partial\mathcal{B}$, normally called the current configuration. We label a material point in the reference configuration $\mathcal{B}_\mathrm{r}$ by a position vector $\mathbf{X}$, and this point in the current configuration $\mathcal{B}$ by a position vector $\mathbf{x}$. Deformation is described by the vector field $\boldsymbol{\chi}$, which relates the position of a particle in the reference configuration to the position of the same particle in the current configuration: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. The deformation gradient tensor, denoted $\mathbf{F}$, is defined by

$$\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}, \tag{2.1}$$

where Grad is the operator defined with respect to $\mathbf{X}$.

Along with the deformation gradient we use the right and left Cauchy–Green deformation tensors, defined by

$$\mathbf{c} = \mathbf{F}^\mathrm{T}\mathbf{F}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^\mathrm{T}. \tag{2.2}$$

The quantity, defined by

$$J = \det\mathbf{F} \tag{2.3}$$

accounts for volumetric changes.

We note that for incompressible materials

$$\det \mathbf{F} = 1. \tag{2.4}$$

### 2.1.1 Governing equations and boundary conditions

Here we give specializations of Maxwell's equations for the electric field variables provided that we do not have magnetic fields, free currents, free volumetric electric charges (free surface charges can be present on the boundaries $\partial\mathcal{B}$), and we have no time dependence. In this case we have

$$\operatorname{curl} \mathbf{E} = 0, \quad \operatorname{div} \mathbf{D} = 0, \tag{2.5}$$

where $\mathbf{E}$ denotes electric field vector, and $\mathbf{D}$ is electric displacement vector. Operators curl and div are defined with respect to $\mathbf{x}$. We assume that outside the body we have vacuum. We will use a star to denote the respective quantities outside the body. In this case we have a standard relation between the electric field and the electric displacement

$$\mathbf{D}^\star = \epsilon_0 \mathbf{E}^\star, \tag{2.6}$$

where $\epsilon_0$ is the vacuum permittivity. In vacuum we have

$$\operatorname{curl} \mathbf{E}^\star = 0, \quad \operatorname{div} \mathbf{D}^\star = 0. \tag{2.7}$$

Fields $\mathbf{E}^\star$ and $\mathbf{D}^\star$ have to satisfy the boundary conditions

$$\mathbf{n} \times (\mathbf{E}^\star - \mathbf{E}) = 0, \quad \mathbf{n} \cdot (\mathbf{D}^\star - \mathbf{D}) = \sigma_f \quad \text{on} \quad \partial\mathcal{B}, \tag{2.8}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{B}$, $\sigma_f$ is the free surface charge on $\partial\mathcal{B}$ per unit area. Derivations of (2.8) can be found in Dorfmann & Ogden (2014c).

In Dorfmann & Ogden (2005) it was shown that electromechanical equilibrium equation can be conveniently written as

$$\operatorname{div} \boldsymbol{\tau} = 0, \tag{2.9}$$

where $\boldsymbol{\tau}$ is the total Cauchy stress tensor. We note that the total Cauchy stress tensor depends on the deformation and electric field via a constitutive law, which will be discussed in Section 2.1.3. We assume that there are no mechanical body forces, whereas electrical body forces are incorporated in (2.9) implicitly. Tensor $\boldsymbol{\tau}$ is symmetric, provided that the mechanical angular moments are balanced or not present at all.

The boundary condition for the total Chauchy stress is

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \mathbf{t}_{\mathrm{e}}^{\star} \quad \text{on } \partial\mathcal{B}_{\mathrm{t}}, \tag{2.10}$$

where $\partial\mathcal{B}_t$ is the part of the boundary where the mechanical traction $\mathbf{t}_a$ is prescribed along with $\mathbf{t}_{\mathrm{e}}^{\star} = \boldsymbol{\tau}_{\mathrm{e}}^{\star}\mathbf{n}$ which is the load due to the Maxwell stress $\boldsymbol{\tau}_{\mathrm{e}}^{\star}$, calculated from the fields outside the body $\mathcal{B}$. The Maxwell stress is defined by

$$\boldsymbol{\tau}_{\mathrm{e}}^{\star} = \varepsilon_0 \mathbf{E}^{\star} \otimes \mathbf{E}^{\star} - \frac{1}{2}\varepsilon_0 (\mathbf{E}^{\star} \cdot \mathbf{E}^{\star})\mathbf{I}, \tag{2.11}$$

where $\mathbf{I}$ is the identity tensor.

### 2.1.2 Lagrangian forms of the electric fields

Lagrangian forms of electric fields are given by the following relations

$$\mathbf{E}_{\mathrm{L}} = \mathbf{F}^{\mathrm{T}}\mathbf{E}, \quad \mathbf{D}_{\mathrm{L}} = J\mathbf{F}^{-1}\mathbf{D}, \tag{2.12}$$

where we recall that $J = \det \mathbf{F}$. Justification of these relations can be found in Dorfmann & Ogden (2005). The counterparts of equations (2.5) in the reference configuration are

$$\operatorname{Curl} \mathbf{E}_{\mathrm{L}} = 0, \quad \operatorname{Div} \mathbf{D}_{\mathrm{L}} = 0. \tag{2.13}$$

Here operators Curl and Div are defined with respect to $\mathbf{X}$.

In order to obtain a Lagrangian form of the equilibrium equation (2.9) we introduce the total nominal stress tensor $\mathbf{T}$ defined by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}. \tag{2.14}$$

We note that expression (2.14) is a generalization of the nominal stress tensor in nonlinear elasticity. Using identity $\operatorname{Div} \mathbf{A} = J \operatorname{div}(J^{-1}\mathbf{F}\mathbf{A})$ we obtain a Lagrangean form of the electromechanical equilibrium equation (2.9)

$$\operatorname{Div} \mathbf{T} = 0. \tag{2.15}$$

The boundary condition associated with (2.15) can be obtained with the help of relation

$$\boldsymbol{\tau}\mathbf{n}\mathrm{d}s = \mathbf{T}^{\mathrm{T}}\mathbf{N}\mathrm{d}S, \tag{2.16}$$

connecting infinitesimal areas $\mathrm{d}s$ and $\mathrm{d}S$ in the current and reference configurations, $\mathbf{n}$ and $\mathbf{N}$ being respective normals to these areas. Relation (2.16) has been obtained from Nanson's formula $\mathbf{n}\mathrm{d}s = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\mathrm{d}S$ and (2.14).

Therefore, (2.10) transforms to

$$\mathbf{T}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_{\mathrm{A}} + \mathbf{t}_{\mathrm{E}}^{\star} \quad \text{on} \quad \partial\mathcal{B}_{\mathrm{rt}}, \tag{2.17}$$

where $\partial\mathcal{B}_{\mathrm{rt}}$ is the part $\partial\mathcal{B}_{\mathrm{r}}$ on which electromechanical tractions are defined, $\mathbf{t}_{\mathrm{A}}$ and $\mathbf{t}_{\mathrm{E}}^{\star} = \mathbf{T}_{\mathrm{E}}^{\star\,\mathrm{T}}\mathbf{N}$ (with $\mathbf{T}_{\mathrm{E}}^{\star} = J\mathbf{F}^{-1}\boldsymbol{\tau}_{\mathrm{e}}^{\star}$) being the mechanical traction and the Maxwell traction per unit reference area, respectively.

Using Nanson's formula and relations (2.12) boundary conditions (2.8) in Lagrangian form can be written as

$$(\mathbf{F}^{\mathrm{T}}\mathbf{E}^{*} - \mathbf{E}_{\mathrm{L}}) \times \mathbf{N} = 0, \quad (J\mathbf{F}^{-1}\mathbf{D}^{*} - \mathbf{D}_{\mathrm{L}}) \cdot \mathbf{N} = \sigma_{\mathrm{F}} \quad \text{on} \quad \partial\mathcal{B}_{r}, \tag{2.18}$$

where $\mathbf{N}$ is the unit outward normal to $\partial\mathcal{B}_{r}$, $\sigma_{\mathrm{F}}$ is free surface charge density per unit area of $\partial\mathcal{B}_{r}$.

### 2.1.3 Constitutive equations

In the problems discussed in this thesis it is convenient to choose $\mathbf{D}_{\mathrm{L}}$ as an independent variable. For mechanically unconstrained and incompressible materials the total stress tensor and the electric field in Lagrangian form are, respectively,

$$\mathbf{T} = \frac{\partial\Omega^{*}}{\partial\mathbf{F}}, \quad \mathbf{E}_{\mathrm{L}} = \frac{\partial\Omega^{*}}{\partial\mathbf{D}_{\mathrm{L}}}, \tag{2.19}$$

$$\mathbf{T} = \frac{\partial\Omega^{*}}{\partial\mathbf{F}} - p\mathbf{F}^{-1}, \quad \mathbf{E}_{\mathrm{L}} = \frac{\partial\Omega^{*}}{\partial\mathbf{D}_{\mathrm{L}}}, \tag{2.20}$$

where $\Omega^{*}$ is a total energy density function (Dorfmann & Ogden, 2005), which depends on $\mathbf{F}$ and $\mathbf{D}_{\mathrm{L}}$ through the invariants of the right Cauchy-Green deformation tensor $\mathbf{c}$

$$I_1 = \mathrm{tr}\,\mathbf{c}, \quad I_2 = \frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,(\mathbf{c}^2)], \tag{2.21}$$

$$I_4 = \mathbf{D}_{\mathrm{L}} \cdot \mathbf{D}_{\mathrm{L}}, \quad I_5 = \mathbf{D}_{\mathrm{L}} \cdot (\mathbf{c}\mathbf{D}_{\mathrm{L}}), \quad I_6 = \mathbf{D}_{\mathrm{L}} \cdot (\mathbf{c}^2\mathbf{D}_{\mathrm{L}}), \tag{2.22}$$

and $p$ is a Lagrange multiplier associated with the incompressibility constraint (2.4).

For incompressible deformations $I_3 = \det\mathbf{c} = J^2 = 1$, therefore, we did not include it in (2.21). In general $I_3$ must be used to account for volumetric changes for compressible materials. Following the same convention used by Dorfmann and Ogden in their papers on Electroelasticity, we retained the asterisk in $\Omega^{*}$, which signifies that the total energy function was defined in terms of $\mathbf{D}_{\mathrm{L}}$.

From (2.14) the total stress $\boldsymbol{\tau}$ can be defined as a (partial) push forward of a total nominal stress $\mathbf{T}$: $\boldsymbol{\tau} = \mathbf{F}\mathbf{T}$, and push forward of $\mathbf{E}_{\mathrm{L}}$ according to (2.12): $\mathbf{E} = \mathbf{F}^{-\mathrm{T}}\mathbf{E}_{\mathrm{L}}$.

Therefore, pushing forward the quantities (2.20) and using the fact that $\Omega^*$ is defined through the invariants (2.21) and (2.22) we can get the following expressions:

$$\boldsymbol{\tau} = 2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega_5^*\mathbf{D}\otimes\mathbf{D} + 2\Omega_6^*(\mathbf{D}\otimes\mathbf{bD} + \mathbf{bD}\otimes\mathbf{D}), \tag{2.23}$$

$$\mathbf{E} = 2(\Omega_4^*\mathbf{b}^{-1}\mathbf{D} + \Omega_5^*\mathbf{D} + \Omega_6^*\mathbf{bD}), \tag{2.24}$$

where $\Omega_i^*$ is a partial derivative $\partial\Omega^*/\partial I_i$ for $i = 1, 2, 4, 5, 6$, and the deformation tensor $\mathbf{b}$ was defined earlier by $(2.2)_2$.

## 2.2 Incremental Formulation

In this section we give the equations governing incremental deformations and electric displacements superimposed on a deformed configuration and an initial electric field. A more detailed discussion of this theory and relevant equations can be found in the book by Dorfmann & Ogden (2014c).

### 2.2.1 Incremental equations and boundary conditions

We denote the increment for a certain variable by a superimposed dot. For example, $\dot{\mathbf{x}}$ is the increment in the displacement, $\dot{\mathbf{F}} = \mathrm{Grad}\,\dot{\mathbf{x}}$ being a corresponding increment in the deformation gradient. Increments of $\dot{\mathbf{E}}_\mathrm{L}$, $\dot{\mathbf{D}}_\mathrm{L}$, $\dot{\mathbf{T}}$ must satisfy the incremental governing equations

$$\mathrm{Curl}\,\dot{\mathbf{E}}_\mathrm{L} = \mathbf{0}, \quad \mathrm{Div}\,\dot{\mathbf{D}}_\mathrm{L} = 0, \quad \mathrm{Div}\,\dot{\mathbf{T}} = \mathbf{0}. \tag{2.25}$$

Outside the material increments in electric displacement field and electric field are connected by $\dot{\mathbf{D}}^\star = \varepsilon_0\dot{\mathbf{E}}^\star$ and must satisfy the equations

$$\mathrm{curl}\,\dot{\mathbf{E}}^\star = 0, \quad \mathrm{div}\,\dot{\mathbf{D}}^\star = 0. \tag{2.26}$$

Incrementing electric and traction boundary conditions (2.18) and (2.17) with $J = 1$ we have

$$(\dot{\mathbf{F}}^\mathrm{T}\mathbf{E}^\star + \mathbf{F}^\mathrm{T}\dot{\mathbf{E}}^\star - \dot{\mathbf{E}}_\mathrm{L}) \times \mathbf{N} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}_r, \tag{2.27}$$

$$(\mathbf{F}^{-1}\dot{\mathbf{D}}^\star - \mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{D}^\star - \dot{\mathbf{D}}_\mathrm{L}) \cdot \mathbf{N} = \dot{\sigma}_\mathrm{F} \quad \text{on} \quad \partial\mathcal{B}_r, \tag{2.28}$$

$$\dot{\mathbf{T}}^\mathrm{T}\mathbf{N} = \dot{\mathbf{t}}_\mathrm{A} + \dot{\boldsymbol{\tau}}_\mathbf{e}^\star\mathbf{F}^{-\mathrm{T}}\mathbf{N} - \boldsymbol{\tau}_\mathbf{e}^\star\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{F}}^\mathrm{T}\mathbf{F}^{-\mathrm{T}}\mathbf{N} \quad \text{on} \quad \partial\mathcal{B}_r, \tag{2.29}$$

where $\dot{\boldsymbol{\tau}}_\mathbf{e}^\star$ is the incremental Maxwell stress which can be calculated from (2.11) and expressed as

$$\dot{\boldsymbol{\tau}}_\mathbf{e}^\star = \varepsilon_0[\dot{\mathbf{E}}^\star\otimes\mathbf{E}^\star + \mathbf{E}^\star\otimes\dot{\mathbf{E}}^\star - (\mathbf{E}^\star\cdot\dot{\mathbf{E}}^\star)\mathbf{I}]. \tag{2.30}$$

We will work with the push-forward versions of increments in $\dot{\mathbf{E}}_{\rm L}$, $\dot{\mathbf{D}}_{\rm L}$ and $\dot{\mathbf{T}}$ defined by

$$\dot{\mathbf{E}}_{\rm L0} = \mathbf{F}^{-\rm T}\dot{\mathbf{E}}_{\rm L}, \quad \dot{\mathbf{D}}_{\rm L0} = \mathbf{F}\dot{\mathbf{D}}_{\rm L}, \quad \dot{\mathbf{T}}_0 = \mathbf{F}\dot{\mathbf{T}}. \tag{2.31}$$

The above mentioned quantities with a zero subscript can be also referred to as the updated quantities with respect to the current configuration $\mathcal{B}$. A more detailed discussion of this concept can be found in Ogden (1997) for pure mechanical problems, and here we use a similar approach of updating variables with respect to the current configuration for an electromechanical problem.

Therefore, using the previous relations it can be shown that governing equations (2.25) are updated to

$$\operatorname{curl}\dot{\mathbf{E}}_{\rm L0} = \mathbf{0}, \quad \operatorname{div}\dot{\mathbf{D}}_{\rm L0} = 0, \quad \operatorname{div}\dot{\mathbf{T}}_0 = \mathbf{0}, \tag{2.32}$$

and corresponding boundary conditions are updated to

$$(\dot{\mathbf{E}}^\star + \mathbf{L}^{\rm T}\mathbf{E}^\star - \dot{\mathbf{E}}_{\rm L0}) \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}, \tag{2.33}$$

$$(\dot{\mathbf{D}}^\star - \mathbf{L}\mathbf{D}^\star - \dot{\mathbf{D}}_{\rm L0}) \cdot \mathbf{n} = \dot{\sigma}_{\rm F0} \quad \text{on} \quad \partial\mathcal{B}, \tag{2.34}$$

$$\dot{\mathbf{T}}_0^{\rm T}\mathbf{n} = \dot{\mathbf{t}}_{\rm A0} + \dot{\boldsymbol{\tau}}_{\mathbf{e}}^\star\mathbf{n} - \boldsymbol{\tau}_{\mathbf{e}}^\star\mathbf{L}^{\rm T}\mathbf{n} \quad \text{on} \quad \partial\mathcal{B}, \tag{2.35}$$

where $\mathbf{L} = \operatorname{grad}\mathbf{u}$, $\mathbf{u}$ being the increment in the displacement vector: $\mathbf{u} = \dot{\mathbf{x}}$. When increment $\dot{\mathbf{x}}$ is treated as a function of $\mathbf{x}$ we can obtain $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. Incrementing (2.3) we have $\dot{J} = J\operatorname{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1})$. Since for incompressible materials $\dot{J} = 0$, a linearized form of the incompressible condition (2.4) follows

$$\operatorname{tr}\mathbf{L} = \operatorname{div}\mathbf{u} = 0. \tag{2.36}$$

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the unit basis vectors in the orthogonal curvilinear coordinate system. Therefore, expression $(2.32)_3$ gives three scalar equations

$$\dot{T}_{0ji,j} + \dot{T}_{0ji}\mathbf{e}_k \cdot \mathbf{e}_{j,k} + \dot{T}_{0kj}\mathbf{e}_i \cdot \mathbf{e}_{j,k} = 0 \quad (i = 1, 2, 3), \tag{2.37}$$

where summation over repeated indices $j$ and $k$ from 1 to 3 is implied.

### 2.2.2 Incremental constitutive equations

The increments in the deformation gradient $\dot{\mathbf{F}}$ and Lagrangian electric displacement $\dot{\mathbf{D}}_{\rm L}$ will induce the increments in stress $\dot{\mathbf{T}}$ and Lagrangian electric field $\dot{\mathbf{E}}_{\rm L}$. Incrementing the constitutive laws (2.19) we obtain linearized incremental constitutive equations for unconstrained materials

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}^*\dot{\mathbf{F}} + \mathbb{A}^*\dot{\mathbf{D}}_{\mathrm{L}}, \quad \dot{\mathbf{E}}_{\mathrm{L}} = \mathbb{A}^{*\mathrm{T}}\dot{\mathbf{F}} + \mathsf{A}^*\dot{\mathbf{D}}_{\mathrm{L}}, \tag{2.38}$$

where $\boldsymbol{\mathcal{A}}^*$, $\mathbb{A}^*$, $\mathsf{A}^*$ denote electroelastic moduli associated with the total energy $\Omega^* = \Omega^*(\mathbf{F}, \mathbf{D}_{\mathrm{L}})$. These quantities are the fourth-, third- and second-order tensors, respectively. In component form they can be represented by

$$\mathcal{A}^*_{\alpha i\beta j} = \frac{\partial^2\Omega^*}{\partial F_{i\alpha}\partial F_{j\beta}}, \quad \mathbb{A}^*_{\alpha i|\beta} = \frac{\partial^2\Omega^*}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}}, \quad \mathsf{A}^*_{\alpha\beta} = \frac{\partial^2\Omega^*}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}}. \tag{2.39}$$

Mixed derivatives in (2.39) allows us to see the following symmetries

$$\mathcal{A}^*_{\alpha i\beta j} = \mathcal{A}^*_{\beta j\alpha i}, \quad \mathsf{A}^*_{\alpha\beta} = \mathsf{A}^*_{\beta\alpha}. \tag{2.40}$$

The vertical bar in the component form of $\mathbb{A}^*$ separates the first 2 indices with the third index, because the first 2 indices are associated with a second-order tensor, whereas the third one is associated with a vector. The tensor $\mathbb{A}^*$ maps a vector into a second-order tensor, whereas the transpose of it does the opposite: maps a second-order tensor into a vector. In the component form we can write $\mathbb{A}^*_{\alpha i|\beta} = \mathbb{A}^{*\mathrm{T}}_{\beta|\alpha i}$.

In component form equations (2.38) can be written as

$$\dot{T}_{\alpha i} = \mathcal{A}^*_{\alpha i\beta j}\dot{F}_{j\beta} + \mathbb{A}^*_{\alpha i|\beta}\dot{D}_{\mathrm{L}\beta}, \quad \dot{E}_{\mathrm{L}\alpha} = \mathbb{A}^*_{\beta i|\alpha}\dot{F}_{i\beta} + \mathsf{A}^*_{\alpha\beta}\dot{D}_{\mathrm{L}\beta}. \tag{2.41}$$

Taking into account that $\Omega^*$ in (2.39) depends on $\mathbf{F}$ and $\mathbf{D}_{\mathrm{L}}$ through invariants $I_i$, $i \in \{1, .., 6\}$ we can expand electroelastic moduli tensors and write

$$\begin{aligned}
\mathcal{A}^*_{\alpha i\beta j} &= \sum_{m=1,m\neq4}^{6}\sum_{n=1,n\neq4}^{6}\Omega^*_{mn}\frac{\partial I_m}{\partial F_{i\alpha}}\frac{\partial I_n}{\partial F_{j\beta}} + \sum_{n=1,n\neq4}^{6}\Omega^*_n\frac{\partial^2 I_n}{\partial F_{i\alpha}\partial F_{j\beta}},\\
\mathbb{A}^*_{\alpha i|\beta} &= \sum_{m=4}^{6}\sum_{n=1,n\neq4}^{6}\Omega^*_{mn}\frac{\partial I_m}{\partial D_{\mathrm{L}\beta}}\frac{\partial I_n}{\partial F_{i\alpha}} + \sum_{n=5}^{6}\Omega^*_n\frac{\partial^2 I_n}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}},\\
\mathsf{A}^*_{\alpha\beta} &= \sum_{m=4}^{6}\sum_{n=4}^{6}\Omega^*_{mn}\frac{\partial I_m}{\partial D_{\mathrm{L}\alpha}}\frac{\partial I_n}{\partial D_{\mathrm{L}\beta}} + \sum_{n=4}^{6}\Omega^*_n\frac{\partial^2 I_n}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}},
\end{aligned} \tag{2.42}$$

where $\Omega^*_n = \partial\Omega^*/\partial I_n$, $\Omega^*_{mn} = \partial^2\Omega^*/\partial I_m\partial I_n$, $m, n \in \{1, \ldots, 6\}$. In (2.42) we need to calculate derivatives of invariants $I_i$, $i \in \{1, .., 6\}$ with respect to $\mathbf{F}$ and $\mathbf{D}_{\mathrm{L}}$. In component form the non-zero first derivatives have the following expressions

$$\begin{aligned}
&\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 2(c_{\gamma\gamma}F_{i\alpha} - c_{\alpha\gamma}F_{i\gamma}), \quad \frac{\partial I_3}{\partial F_{i\alpha}} = 2I_3F^{-1}_{\alpha i},\\
&\frac{\partial I_5}{\partial F_{i\alpha}} = 2D_{\mathrm{L}\alpha}(F_{i\gamma}D_{\mathrm{L}\gamma}), \quad \frac{\partial I_6}{\partial F_{i\alpha}} = 2(c_{\alpha\beta}D_{\mathrm{L}\beta}F_{i\gamma}D_{\mathrm{L}\gamma} + D_{\mathrm{L}\alpha}F_{i\gamma}c_{\gamma\beta}D_{\mathrm{L}\beta}),\\
&\frac{\partial I_4}{\partial D_{\mathrm{L}\alpha}} = 2D_{\mathrm{L}\alpha}, \quad \frac{\partial I_5}{\partial D_{\mathrm{L}\alpha}} = 2c_{\alpha\beta}D_{\mathrm{L}\beta}, \quad \frac{\partial I_6}{\partial D_{\mathrm{L}\alpha}} = 2c^2_{\alpha\beta}D_{\mathrm{L}\beta}.
\end{aligned} \tag{2.43}$$

The non-zero second derivatives with respect to $\mathbf{F}$ are

$$
\begin{aligned}
\frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} &= 2\delta_{ij}\delta_{\alpha\beta},\\
\frac{\partial^2 I_2}{\partial F_{i\alpha}\partial F_{j\beta}} &= 2(2F_{i\alpha}F_{j\beta} - F_{i\beta}F_{j\alpha} + c_{\gamma\gamma}\delta_{ij}\delta_{\alpha\beta} - b_{ij}\delta_{\alpha\beta} - c_{\alpha\beta}\delta_{ij}),\\
\frac{\partial^2 I_3}{\partial F_{i\alpha}\partial F_{j\beta}} &= 2I_3(2F^{-1}_{\alpha i}F^{-1}_{\beta j} - F^{-1}_{\alpha j}F^{-1}_{\beta i}), \quad \frac{\partial^2 I_5}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}D_{\mathrm{L}\alpha}D_{\mathrm{L}\beta},\\
\frac{\partial^2 I_6}{\partial F_{i\alpha}\partial F_{j\beta}} &= 2[\delta_{ij}(c_{\alpha\gamma}D_{\mathrm{L}\gamma}D_{\mathrm{L}\beta} + c_{\beta\gamma}D_{\mathrm{L}\gamma}D_{\mathrm{L}\alpha}) + \delta_{\alpha\beta}F_{i\gamma}D_{\mathrm{L}\gamma}F_{j\delta}D_{\mathrm{L}\delta}\\
&\quad + F_{i\gamma}D_{\mathrm{L}\gamma}F_{j\alpha}D_{\mathrm{L}\beta} + F_{j\gamma}D_{\mathrm{L}\gamma}F_{i\beta}D_{\mathrm{L}\alpha} + b_{ij}D_{\mathrm{L}\alpha}D_{\mathrm{L}\beta}]. \qquad (2.44)
\end{aligned}
$$

The second derivatives of $I_4, I_5, I_6$ with respect to $\mathbf{D}_\mathrm{L}$ are

$$
\frac{\partial^2 I_4}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}} = 2\delta_{\alpha\beta}, \quad \frac{\partial^2 I_5}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}} = 2c_{\alpha\beta}, \quad \frac{\partial^2 I_6}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}} = 2c^2_{\alpha\beta}. \qquad (2.45)
$$

The mixed derivatives of $I_1, I_2, I_3$ and $I_4$ with respect to $\mathbf{F}$ and $\mathbf{D}_\mathrm{L}$ are equal to zero. For $I_5, I_6$ we calculate

$$
\begin{aligned}
\frac{\partial^2 I_5}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}} &= 2\delta_{\alpha\beta}F_{i\gamma}D_{\mathrm{L}\gamma} + 2D_{\mathrm{L}\alpha}F_{i\beta},\\
\frac{\partial^2 I_6}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}} &= 2F_{i\beta}c_{\alpha\gamma}D_{\mathrm{L}\gamma} + 2F_{i\gamma}D_{\mathrm{L}\gamma}c_{\alpha\beta} + 2F_{i\gamma}c_{\gamma\beta}D_{\mathrm{L}\alpha} + 2\delta_{\alpha\beta}F_{i\gamma}c_{\gamma\delta}D_{\mathrm{L}\delta}.
\end{aligned}
$$

We introduce the notation $\bar{\mathbf{b}} = I_1\mathbf{b} - \mathbf{b}^2$, $\mathbf{D}^{(1)} = \mathbf{b}\mathbf{D}$ and $\mathbf{D}^{(-1)} = \mathbf{b}^{-1}\mathbf{D}$. We obtain the following expressions for the components of electroelastic moduli tensors

$$
\begin{aligned}
J\mathcal{A}^*_{0piqj} = {} & 4\{\Omega^*_{11}b_{ip}b_{jq} + \Omega^*_{12}(b_{ip}\bar{b}_{jq} + b_{jq}\bar{b}_{ip}) + \Omega^*_{22}\bar{b}_{ip}\bar{b}_{jq} + I_3^2\Omega^*_{33}\delta_{ip}\delta_{jq} \\
& + I_3\Omega^*_{13}(\delta_{ip}b_{jq} + \delta_{jq}b_{ip}) + I_3\Omega^*_{23}(\delta_{ip}\bar{b}_{jq} + \delta_{jq}\bar{b}_{ip}) + I_3^2\Omega^*_{55}D_iD_jD_pD_q \\
& + I_3\Omega^*_{15}(b_{ip}D_jD_q + b_{jq}D_iD_p) + I_3\Omega^*_{25}(D_iD_p\bar{b}_{jq} + D_jD_q\bar{b}_{ip}) \\
& + I_3^2\Omega^*_{35}(D_iD_p\delta_{jq} + D_jD_q\delta_{ip}) \\
& + I_3^2\Omega^*_{66}(D_i^{(1)}D_p + D_p^{(1)}D_i)(D_j^{(1)}D_q + D_q^{(1)}D_j) \\
& + I_3\Omega^*_{16}[b_{ip}(D_j^{(1)}D_q + D_q^{(1)}D_j) + b_{jq}(D_i^{(1)}D_p + D_p^{(1)}D_i)] \\
& + I_3\Omega^*_{26}[\bar{b}_{ip}(D_j^{(1)}D_q + D_q^{(1)}D_j) + \bar{b}_{jq}(D_i^{(1)}D_p + D_p^{(1)}D_i)] \\
& + I_3^2\Omega^*_{36}[\delta_{ip}(D_j^{(1)}D_q + D_q^{(1)}D_j) + \delta_{jq}(D_i^{(1)}D_p + D_p^{(1)}D_i)] \\
& + I_3^2\Omega^*_{56}(D_p^{(1)}D_iD_qD_j + D_i^{(1)}D_pD_qD_j + D_q^{(1)}D_pD_iD_j + D_j^{(1)}D_pD_iD_q)\} \\
& + 2\{\Omega^*_1\delta_{ij}b_{pq} + \Omega^*_2[2b_{ip}b_{jq} - b_{iq}b_{jp} + \delta_{ij}\bar{b}_{pq} - b_{ij}b_{pq}] \\
& + I_3\Omega^*_3(2\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) + I_3\Omega^*_5\delta_{ij}D_pD_q \\
& + I_3\Omega^*_6[\delta_{ij}(D_p^{(1)}D_q + D_q^{(1)}D_p) \\
& + b_{pq}D_iD_j + b_{jp}D_iD_q + b_{iq}D_jD_p + b_{ij}D_pD_q]\},
\end{aligned}
\tag{2.46}
$$

$$
\begin{aligned}
J^{-1}\mathbb{A}^*_{0pi|q} = {} & 4[\Omega^*_{14}b_{ip}D_q^{(-1)} + \Omega^*_{24}\bar{b}_{ip}D_q^{(-1)} + I_3\Omega^*_{34}\delta_{ip}D_q^{(-1)} \\
& + I_3\Omega^*_{45}D_iD_pD_q^{(-1)} + I_3\Omega^*_{46}(D_i^{(1)}D_p + D_p^{(1)}D_i)D_q^{(-1)} \\
& + \Omega^*_{15}b_{ip}D_q + \Omega^*_{25}\bar{b}_{ip}D_q + I_3\Omega^*_{35}\delta_{ip}D_q + I_3\Omega^*_{55}D_iD_pD_q \\
& + I_3\Omega^*_{56}(D_p^{(1)}D_iD_q + D_i^{(1)}D_pD_q + D_q^{(1)}D_iD_p) + \Omega^*_{16}b_{ip}D_q^{(1)} \\
& + \Omega^*_{26}\bar{b}_{ip}D_q^{(1)} + I_3\Omega^*_{36}\delta_{ip}D_q^{(1)} + I_3\Omega^*_{66}(D_i^{(1)}D_p + D_p^{(1)}D_i)D_q^{(1)}] \\
& + 2[\Omega^*_5(\delta_{pq}D_i + \delta_{iq}D_p) + \Omega^*_6(\delta_{iq}D_p^{(1)} + \delta_{pq}D_i^{(1)} + b_{pq}D_i + b_{iq}D_p)],
\end{aligned}
\tag{2.47}
$$

$$
\begin{aligned}
J^{-1}\mathsf{A}^*_{0ij} = {} & 4I_3[\Omega^*_{44}D_i^{(-1)}D_j^{(-1)} + \Omega^*_{55}D_iD_j + \Omega^*_{66}D_i^{(1)}D_j^{(1)} \\
& + \Omega^*_{45}(D_i^{(-1)}D_j + D_j^{(-1)}D_i) + \Omega^*_{46}(D_i^{(-1)}D_j^{(1)} + D_j^{(-1)}D_i^{(1)}) \\
& + \Omega^*_{56}(D_i^{(1)}D_j + D_j^{(1)}D_i)] + 2(\Omega^*_4b_{ij}^{(-1)} + \Omega^*_5\delta_{ij} + \Omega^*_6b_{ij}),
\end{aligned}
\tag{2.48}
$$

for an unconstrained material. In order to obtain the expressions for an incompressible material we set $J = I_3 = 1$ in the above formulas and omit the terms involving derivatives of $\Omega^*$ with respect to $I_3$.

We can evaluate the previous expressions with respect to the principal axes. Here we give the expressions of tensors $\boldsymbol{\mathcal{A}}^*$, $\mathbb{A}^*$, $\mathbf{A}^*$ referred to principal axes of the left Cauchy-Green tensor $\mathbf{b}$, i.e. in terms of principal stretches $\lambda_1$, $\lambda_2$, $\lambda_3$ and components ($D_1$, $D_2$, $D_3$) of the electric displacement vector $\mathbf{D}$. In these expressions we note that the indices are such that $i \neq j \neq k \neq i$.

$$
\begin{aligned}
J\mathcal{A}^*_{0iiii} = {} & 2\lambda_i^2[\Omega_1^* + (\lambda_j^2+\lambda_k^2)\Omega_2^* + \lambda_j^2\lambda_k^2\Omega_3^* + \lambda_j^2\lambda_k^2 D_i^2(\Omega_5^* + 6\lambda_i^2\Omega_6^*)] \\
& + 4\lambda_i^4\{\Omega_{11}^* + 2(\lambda_j^2+\lambda_k^2)\Omega_{12}^* + (\lambda_j^2+\lambda_k^2)^2\Omega_{22}^* \\
& + \lambda_j^2\lambda_k^2[2\Omega_{13}^* + 2(\lambda_j^2+\lambda_k^2)\Omega_{23}^* + \lambda_j^2\lambda_k^2\Omega_{33}^*] + 2\lambda_j^2\lambda_k^2 D_i^2[\Omega_{15}^* + 2\lambda_i^2\Omega_{16}^* \\
& + (\lambda_j^2+\lambda_k^2)\Omega_{25}^* + 2\lambda_i^2(\lambda_j^2+\lambda_k^2)\Omega_{26}^* + \lambda_j^2\lambda_k^2\Omega_{35}^* + 2I_3\Omega_{36}^*] \\
& + \lambda_j^4\lambda_k^4 D_i^4(\Omega_{55}^* + 4\lambda_i^2\Omega_{56}^* + 4\lambda_i^4\Omega_{66}^*)\},
\end{aligned}
$$

$$
\begin{aligned}
J\mathcal{A}^*_{0iiij} = {} & 4D_iD_jI_3\lambda_i^2\{\Omega_6^* + \Omega_{15}^* + (\lambda_j^2+\lambda_k^2)\Omega_{25}^* + \lambda_j^2\lambda_k^2\Omega_{35}^* \\
& + (\lambda_i^2+\lambda_j^2)[\Omega_{16}^* + (\lambda_j^2+\lambda_k^2)\Omega_{26}^* + \lambda_j^2\lambda_k^2\Omega_{36}^*] \\
& + \lambda_j^2\lambda_k^2 D_i^2[\Omega_{55}^* + (3\lambda_i^2+\lambda_j^2)\Omega_{56}^* + 2\lambda_i^2(\lambda_i^2+\lambda_j^2)\Omega_{66}^*]\},
\end{aligned}
$$

$$
\begin{aligned}
J\mathcal{A}^*_{0iiji} = {} & 2D_iD_jI_3\{\Omega_5^* + (\lambda_j^2+3\lambda_i^2)\Omega_6^* \\
& + 2\lambda_i^2[\Omega_{15}^* + (\lambda_j^2+\lambda_k^2)\Omega_{25}^* + \lambda_j^2\lambda_k^2\Omega_{35}^*] \\
& + 2\lambda_i^2(\lambda_i^2+\lambda_j^2)[\Omega_{16}^* + (\lambda_j^2+\lambda_k^2)\Omega_{26}^* + \lambda_j^2\lambda_k^2\Omega_{36}^*] \\
& + 2I_3D_i^2[\Omega_{55}^* + (3\lambda_i^2+\lambda_j^2)\Omega_{56}^* + 2\lambda_i^2(\lambda_i^2+\lambda_j^2)\Omega_{66}^*]\},
\end{aligned}
$$

$$
\begin{aligned}
J\mathcal{A}^*_{0iijj} = {} & 4\lambda_i^2\lambda_j^2\{\Omega_2^* + \lambda_k^2\Omega_3^* + \Omega_{11}^* + (I_1+\lambda_k^2)\Omega_{12}^* + (I_2+\lambda_k^4)\Omega_{22}^* \\
& + \lambda_k^2[(\lambda_i^2+\lambda_j^2)\Omega_{13}^* + (I_2+\lambda_i^2\lambda_j^2)\Omega_{23}^* + I_3\Omega_{33}^*] \\
& + \lambda_k^2(\lambda_j^2D_i^2 + \lambda_i^2D_j^2)(\Omega_{15}^* + \lambda_k^2\Omega_{25}^*) \\
& + 2I_3(\lambda_i^2D_i^2 + \lambda_j^2D_j^2)(\Omega_{26}^* + \lambda_k^2\Omega_{36}^*) \\
& + I_3(D_i^2+D_j^2)(2\Omega_{16}^* + \Omega_{25}^* + 2\lambda_k^2\Omega_{26}^* + \lambda_k^2\Omega_{35}^*) \\
& + I_3\lambda_k^2D_i^2D_j^2[\Omega_{55}^* + 2(\lambda_i^2+\lambda_j^2)\Omega_{56}^* + 4\lambda_i^2\lambda_j^2\Omega_{66}^*]\},
\end{aligned}
$$

$$
\begin{aligned}
J\mathcal{A}^*_{0ijij} = {} & 2\lambda_i^2\{\Omega_1^* + \lambda_k^2\Omega_2^* + D_i^2\lambda_j^2\lambda_k^2\Omega_5^* + \lambda_j^2\lambda_k^2(2D_i^2\lambda_i^2 + D_i^2\lambda_j^2 + D_j^2\lambda_i^2)\Omega_6^* \\
& + 2D_i^2D_j^2I_3\lambda_j^2\lambda_k^2[\Omega_{55}^* + 2(\lambda_i^2+\lambda_j^2)\Omega_{56}^* + (\lambda_i^2+\lambda_j^2)^2\Omega_{66}^*]\},
\end{aligned}
$$

$$
\begin{aligned}
J\mathcal{A}^*_{0ijji} = {} & 2\lambda_i^2\lambda_j^2\{-\Omega_2^* - \lambda_k^2\Omega_3^* + \lambda_k^2(\lambda_j^2 D_i^2 + \lambda_i^2 D_j^2)\Omega_6^* \\
& + 2D_i^2 D_j^2 I_3 \lambda_k^2[\Omega_{55}^* + 2(\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + (\lambda_i^2 + \lambda_j^2)^2\Omega_{66}^*]\},
\end{aligned}
$$

$$
\begin{aligned}
J\mathcal{A}^*_{0iijk} = {} & 4D_j D_k I_3 \lambda_i^2\{\Omega_{15}^* + (\lambda_j^2 + \lambda_k^2)(\Omega_{25}^* + \Omega_{16}^*) + (\lambda_j^2 + \lambda_k^2)^2\Omega_{26}^* \\
& + \lambda_j^2\lambda_k^2\Omega_{35}^* + \lambda_j^2\lambda_k^2(\lambda_j^2 + \lambda_k^2)\Omega_{36}^* + D_i^2\lambda_j^2\lambda_k^2[\Omega_{55}^* + (I_1 + \lambda_i^2)\Omega_{56}^* \\
& + 2\lambda_i^2(\lambda_j^2 + \lambda_k^2)\Omega_{66}^*]\},
\end{aligned}
$$

$$
\begin{aligned}
J\mathcal{A}^*_{0ijki} = {} & J\mathcal{A}_{0ijik} = 2D_j D_k I_3\{\lambda_i^2\Omega_6^* + 2D_i^2 I_3[\Omega_{55}^* + (I_1 + \lambda_i^2)\Omega_{56}^* \\
& + (I_2 + \lambda_i^4)\Omega_{66}^*]\},
\end{aligned}
$$

$$
\begin{aligned}
J\mathcal{A}^*_{0jiki} = {} & 2D_j D_k I_3\{\Omega_5^* + I_1\Omega_6^* + 2D_i^2 I_3[\Omega_{55}^* + (I_1 + \lambda_i^2)\Omega_{56}^* \\
& + (I_2 + \lambda_i^4)\Omega_{66}^*]\},
\end{aligned}
$$

$$
\begin{aligned}
J^{-1}\mathbb{A}^*_{0ii|i} = {} & 4D_i\{\Omega_5^* + 2\lambda_i^2\Omega_6^* + \Omega_{14}^* + \lambda_i^2\Omega_{15}^* + \lambda_i^4\Omega_{16}^* \\
& + (\lambda_j^2 + \lambda_k^2)(\Omega_{24}^* + \lambda_i^2\Omega_{25}^* + \lambda_i^4\Omega_{26}^*) + \lambda_j^2\lambda_k^2(\Omega_{34}^* + \lambda_i^2\Omega_{35}^* + \lambda_i^4\Omega_{36}^*) \\
& + D_i^2\lambda_j^2\lambda_k^2[\Omega_{45}^* + \lambda_i^2\Omega_{55}^* + \lambda_i^4\Omega_{56}^* + 2\lambda_i^2(\Omega_{46}^* + \lambda_i^2\Omega_{56}^* + \lambda_i^4\Omega_{66}^*)]\},
\end{aligned}
$$

$$
\begin{aligned}
J^{-1}\mathbb{A}^*_{0ii|j} = {} & 4D_j\lambda_i^2\lambda_j^{-2}\{\Omega_{14}^* + \lambda_j^2\Omega_{15}^* + \lambda_j^4\Omega_{16}^* + (\lambda_j^2 + \lambda_k^2)(\Omega_{24}^* + \lambda_j^2\Omega_{25}^* \\
& + \lambda_j^4\Omega_{26}^*) + \lambda_j^2\lambda_k^2(\Omega_{34}^* + \lambda_j^2\Omega_{35}^* + \lambda_j^4\Omega_{36}^*) + D_i^2\lambda_j^2\lambda_k^2[\Omega_{45}^* + \lambda_j^2\Omega_{55}^* \\
& + \lambda_j^4\Omega_{56}^* + 2\lambda_i^2(\Omega_{46}^* + \lambda_j^2\Omega_{56}^* + \lambda_j^4\Omega_{66}^*)]\},
\end{aligned}
$$

$$
\begin{aligned}
J^{-1}\mathbb{A}^*_{0ij|i} = {} & 2D_j\{\Omega_5^* + (\lambda_i^2 + \lambda_j^2)\Omega_6^* + 2D_i^2\lambda_j^2\lambda_k^2[\Omega_{45}^* + \lambda_i^2\Omega_{55}^* + \lambda_i^4\Omega_{56}^* \\
& + (\lambda_i^2 + \lambda_j^2)(\Omega_{46}^* + \lambda_i^2\Omega_{56}^* + \lambda_i^4\Omega_{66}^*)]\},
\end{aligned}
$$

$$
\begin{aligned}
J^{-1}\mathbb{A}^*_{0ij|k} = {} & 4D_i D_j D_k\lambda_i^2\lambda_j^2[\Omega_{45}^* + \lambda_k^2\Omega_{55}^* + \lambda_k^4\Omega_{56}^* \\
& + (\lambda_i^2 + \lambda_j^2)(\Omega_{46}^* + \lambda_k^2\Omega_{56}^* + \lambda_k^4\Omega_{66}^*)],
\end{aligned}
$$

$$
\begin{aligned}
J^{-1}\mathsf{A}^*_{0ii} = {} & 2\lambda_i^{-2}\{\Omega_4^* + \lambda_i^2\Omega_5^* + \lambda_i^4\Omega_6^* + 2D_i^2\lambda_j^2\lambda_k^2[\Omega_{44}^* + \lambda_i^2\Omega_{45}^* + \lambda_i^4\Omega_{46}^* \\
& + \lambda_i^2(\Omega_{45}^* + \lambda_i^2\Omega_{55}^* + \lambda_i^4\Omega_{56}^*) + \lambda_i^4(\Omega_{46}^* + \lambda_i^2\Omega_{56}^* + \lambda_i^4\Omega_{66}^*]\},
\end{aligned}
$$

$$
\begin{aligned}
J^{-1}\mathsf{A}^*_{0ij} = {} & 4D_i D_j\lambda_k^2[\Omega_{44}^* + \lambda_i^2\Omega_{45}^* + \lambda_i^4\Omega_{46}^* + \lambda_j^2(\Omega_{45}^* + \lambda_i^2\Omega_{55}^* + \lambda_i^4\Omega_{56}^*) \\
& + \lambda_j^4(\Omega_{46}^* + \lambda_i^2\Omega_{56}^* + \lambda_i^4\Omega_{66}^*)].
\end{aligned}
$$

For an incompressible material these relations remain valid if we omit all terms with derivatives of $\Omega^*$ with respect to $I_3$ and set $J = I_3 = 1$.

Incrementing a constitutive law (2.20) for incompressible materials we can obtain

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}^*\dot{\mathbf{F}} + \mathbb{A}^*\dot{\mathbf{D}}_{\mathrm{L}} + p^*\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1} - \dot{p}^*\mathbf{F}^{-1}, \tag{2.49}$$

while equation $(2.38)_2$ is not affected by the incompressibility constraint. Electroelastic moduli are defined by the expressions (2.39) with $J = 1$ in this case. The updated versions of (2.38) and (2.49) are

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{D}}_{\mathrm{L}0}, \quad \dot{\mathbf{E}}_{\mathrm{L}0} = \mathbb{A}_0^{*\mathrm{T}}\mathbf{L} + \mathsf{\mathbf{A}}_0^*\dot{\mathbf{D}}_{\mathrm{L}0}, \tag{2.50}$$

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{D}}_{\mathrm{L}0} + p^*\mathbf{L} - \dot{p}^*\mathbf{I}, \quad \dot{\mathbf{E}}_{\mathrm{L}0} = \mathbb{A}_0^{*\mathrm{T}}\mathbf{L} + \mathsf{\mathbf{A}}_0^*\dot{\mathbf{D}}_{\mathrm{L}0}, \tag{2.51}$$

obtained from the connections (2.31).

Calculations show that electroelastic moduli tensors (2.39) are updated according to the following relations in the component form

$$\mathcal{A}^*_{0jilk} = J^{-1}F_{j\alpha}F_{l\beta}\mathcal{A}^*_{0\alpha i\beta k}, \tag{2.52}$$

$$\mathbb{A}^*_{0ji|k} = F_{j\alpha}F^{-1}_{\beta k}\mathbb{A}^*_{0\alpha i|\beta}, \tag{2.53}$$

$$\mathsf{A}^*_{0ij} = JF^{-1}_{\alpha i}F^{-1}_{\beta j}\mathsf{A}^*_{0\alpha\beta}, \tag{2.54}$$

where $J = 1$ for incompressible materials.

The symmetries (2.40) for updated versions of electroelastic moduli tensors remain valid and an additional symmetry for tensor $\mathbb{A}_0^*$ can be obtained

$$\mathbb{A}^*_{0ij|k} = \mathbb{A}^*_{0ji|k}. \tag{2.55}$$

For unconstrained and incompressible materials we also mention here the following useful connections

$$\mathcal{A}^*_{0jisk} - \mathcal{A}^*_{0ijsk} = \tau_{js}\delta_{ik} - \tau_{is}\delta_{jk}, \tag{2.56}$$

$$\mathcal{A}^*_{0jisk} - \mathcal{A}^*_{0ijsk} = (\tau_{js} + p\delta_{js})\delta_{ik} - (\tau_{is} + p\delta_{is})\delta_{jk}. \tag{2.57}$$

# Chapter 3

# Finite Deformations of Electroelastic Tube

## 3.1 Introduction

Recent successes in the technological production of new dielectric elastomeric materials instigated a rapid development of devices which employ the properties of such materials. For instance, actuators, sensors and even artificial muscles can be manufactured from dielectric elastomers. We note that the theories which account for the nonlinear electromechanical interaction can be traced to the middle of the last century (Toupin, 1956), but Dorfmann & Ogden (2014c) indicated that the present theories cannot be used easily for applications and for solutions of boundary-value problems. In this chapter we consider a cylindrical configuration which is one the possible geometries for actuators (Pelrine et al., 1998). Using the theory of Dorfmann & Ogden (2005), we analyzed the nonlinear response of a pressurized thick-walled tube in the presense of radial electric field, which is generated by two compliant electrodes attached to the lateral internal and external surfaces of the tube. Previously, a similar problem was considered by Dorfmann & Ogden (2006) without electrodes.

The boundary-value problem considered in this chapter can be used as a model for an actuator, the actuating force of which can be generated by inflation and electric field. This type of actuator can be deemed as multipurpose and versatile, because it has a potential to be used for more applications where advantages of both actuation mechanisms are required. For example, actuation by inflation can be used for handling fragile objects where we need a soft touch (Reynolds et al., 2003). On the other hand, actuation by an electric field can be advantageous for some applications where we need quick and precise deformation by actuation (Goulbourne, 2009). A mathematical model for cylindrical, fibre-reinforced pneumatic actuators was considered in Goulbourne (2009). In this model a purely elastic strain energy potential was used, which does not account for the interaction between deformation and electric properties of dielectric material. The effect of the electric field was modelled by Maxwell stress. The augmented Cauchy stress was calculated as a sum of Maxwell stress and mechanical stress derived from a purely elastic strain energy potential. Also purely elastic potentials were used in Zhu et al. (2010). The previous models were based on nonlinear elasticity which allows us to model large deformations of dielectric elastomers. For small range deformations a model based on linear elasticity was proposed by Carpi & Rossi (2004).

Prototype actuators were initially proposed by Pelrine et al. (1998) as a proof of concept for actuating dielectric elastomers by an electric field. We can mention briefly some applications of actuators with cylindrical geometry. Cylindrical fiber actuators can be used in building blocks mimicking the structure of real biological muscles (Arora et al., 2007). Also they have a potential to be used in textiles to produce active, smart structures (Arora et al., 2007). Tubular actuators were reported to be used in refreshable Braille displays (Chakraborti et al., 2012). Since dielectric elastomer actuators are advantageous in many aspects (Pelrine et al., 2001), we can expect further developments and expansion of the areas of application of cylindrical and other types actuators in the future. Technological aspects of production of tubular elastomer actuators are discussed in Cameron et al. (2008). Cameron et al. (2008) proposed to use a commonly used procedure of coextrusion, which allows us to produce elastomer tubes filled with conductive core. The authors indicated that this method combined with inexpensive commercially available materials makes an actuator of this type easily available to mass production.

This chapter is organized in the following order. In Section 3.2 we give general expressions for pressure and reduced axial load. In Section 3.3 we consider a simple energy function and give specialized expressions for pressure and reduced axial load. In Sections 3.4 – 3.5 we derive expressions for pressure and reduced axial load for a thin-walled cylindrical shell. In Section 3.6 we obtained numerical dependences of nondimensional pressure and reduced axial load on deformation for specific strain energy functions which account for the pure mechanical properties of material. A short discussion of activation is contained in Section 3.7, based on the thin-walled formulas from Section 3.4, by considering either zero internal pressure and activation at fixed axial load or zero reduced axial load at fixed internal pressure. Specific results are illustrated in respect of the neo-Hookean elastic model. Finally, some short concluding remarks are provided in Sect. 3.8. We can mention that a similar analysis for an electroelastic spherical shell was done in Dorfmann & Ogden (2014b).

## 3.2 Application to a thick-walled electroelastic circular tube

### 3.2.1 Extension and inflation of a tube

The geometry of a circular tube and its extension and inflation can be conveniently described by cylindrical polar coordinates $R$, $\Theta$, $Z$. In the reference configuration the tube is described by

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L, \tag{3.1}$$

where $A$ and $B$ the internal and external radii, $L$ is the length of a tube.

Assuming that the circular symmetry is maintained in the current configuration we have the counterpart of (3.1)

$$a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq l, \tag{3.2}$$

where $r$, $\theta$, $z$ are cylindrical polar coordinates, and $a$, $b$ and $l$ are the radii and the length in the current (deformed) configuration.

Since we have incompressible deformation and the tube is extended according to the relation $l = \lambda_z L$, the resulting deformation is

$$r^2 = a^2 + \lambda_z^{-1}(R^2 - A^2), \quad \theta = \Theta, \quad z = \lambda_z Z. \tag{3.3}$$

We will define $\lambda = r/R$ as the azimuthal stretch and $\lambda_z = z/Z$ as the axial stretch. By the incompressibility condition (2.4), the stretch in the radial direction can be expressed as $\lambda_r = \lambda^{-1}\lambda_z^{-1}$. From (3.3) we can calculate

$$\lambda_a^2\lambda_z - 1 = \frac{R^2}{A^2}(\lambda^2\lambda_z - 1) = \frac{B^2}{A^2}(\lambda_b^2\lambda_z - 1), \tag{3.4}$$

where

$$\lambda_a = \frac{a}{A}, \quad \lambda_b = \frac{b}{B}, \quad b = f(B). \tag{3.5}$$

When the tube is inflated the following inequalities hold

$$\lambda_a^2\lambda_z \geq 1, \quad \lambda_a \geq \lambda \geq \lambda_b. \tag{3.6}$$

Note, that with respect to the chosen cylindrical polar coordinates the matrix of the deformation gradient is diagonal: $\mathrm{F} = \mathrm{diag}[\lambda_r, \lambda, \lambda_z]$. The invariants $I_1$ and $I_2$ can be specialized for this deformation gradient:

$$I_1 = \lambda^{-2}\lambda_z^{-2} + \lambda^2 + \lambda_z^2, \quad I_2 = \lambda^2\lambda_z^2 + \lambda^{-2} + \lambda_z^{-2}. \tag{3.7}$$

### 3.2.2 Boundary conditions

In this problem we consider an electroelastic tube, the lateral boundaries of which have flexible electrodes. The charges on both electrodes are equal and have the opposite sign. Therefore, by Gauss's Theorem and because of the given geometry, we do not have a field outside the material. We will denote a total charge at $r = a$ by $Q(a)$, and at $r = b$ by $Q(b)$. Therefore, we have

$$Q(a) + Q(b) = 0. \tag{3.8}$$

The free surface charge densities per unit area on the inner and outer boundaries in the current deformed configuration will be

$$\sigma_{\mathrm{f}a} = \frac{Q(a)}{2\pi a l}, \quad \sigma_{\mathrm{f}b} = \frac{Q(b)}{2\pi b l}, \tag{3.9}$$

where $l$ is the length of the cylinder in the deformed configuration. Therefore, we can rewrite (3.8) as

$$a\sigma_{\mathrm{f}a} + b\sigma_{\mathrm{f}b} = 0. \tag{3.10}$$

Referred to the undeformed configuration we have the following analogues of the expressions (3.9)

$$\sigma_{\mathrm{F}A} = \frac{Q(a)}{2\pi AL}, \quad \sigma_{\mathrm{F}B} = \frac{Q(b)}{2\pi BL}, \tag{3.11}$$

where $L$, $A$, $B$ are the length, the inner and the outer radii of the cylinder in the undeformed configuration. In the undeformed configuration we have the following connection between free surface charge densities

$$A\sigma_{\mathrm{F}A} + B\sigma_{\mathrm{F}B} = 0. \tag{3.12}$$

For the considered cylindrical geometry the radial electic displacement $D_r$ ($D_\theta = 0$, $D_z = 0$) will depend only on $r$ and expression $(2.5)_2$ will be equivalent to

$$\frac{1}{r}\frac{\mathrm{d}(rD_r)}{\mathrm{d}r} = 0. \tag{3.13}$$

Therefore, $rD_r$ is a constant, which can be expressed at the boundaries $r = a$ and $r = b$ as $aD_r(a)$ and $bD_r(b)$, respectively. And we have

$$rD_r = aD_r(a) = bD_r(b) = \text{const.} \tag{3.14}$$

Using the boundary condition $(2.8)_2$, where $\mathbf{D}^\star = 0$, we can relate radial electric field components at the boundaries to free surface charge densities per unit area in the deformed configuration

$$D_r(a) = \sigma_{\text{f}a}, \quad D_r(b) = -\sigma_{\text{f}b}. \tag{3.15}$$

Therefore, using (3.9) solutions (3.14) can be expressed as

$$rD_r = \frac{Q(a)}{2\pi l} = -\frac{Q(b)}{2\pi l}. \tag{3.16}$$

We note that for a finite length tube boundary condition $(2.8)_1$ applied to the ends of the tube and the boundary condition $(2.8)_2$ applied to the lateral cylindrical surface are not compatible. Boundary condition $(2.8)_2$ implies for this problem that we have a jump in $E_r$ through the lateral cylindrical surface at $r = a$ and $r = b$, since we do not have electric field outside, and inside at the boundaries $E_r$ can be found from (3.23), whereas condition $(2.8)_1$ applied to the ends of the tube implies that tangential component $E_r$ is continuous. We assume that we deal with a long enough tube so that the edge effects can be neglected. We refer to the work of Bustamante et al. (2007), where the edge effects are discussed for a magnetoelastic problem in more detail.

### 3.2.3 Electric field components

In this problem it is natural to choose the electric displacement as an independent variable. We can control the electric field by prescribing a certain charge on the boundaries, and the charge on the boundaries is related to the electric displacement field through the boundary condition $(2.8)_2$. We will consider a radial field ($D_\theta = 0$, $D_z = 0$). Since the constitutive law

$$\mathbf{E}_{\text{L}} = \frac{\partial \Omega^*}{\partial \mathbf{D}_{\text{L}}} \tag{3.17}$$

is expressed in terms of Lagrangian variable $\mathbf{D}_{\text{L}}$ we will switch to this variable using relation

$$\mathbf{D}_{\text{L}} = \mathbf{F}^{-1}\mathbf{D}. \tag{3.18}$$

Since electric displacement vector is aligned along the radial direction of strain, we have

$$[D_L] = \begin{bmatrix} \lambda\lambda_z D_r \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} D_{LR} \\ 0 \\ 0 \end{bmatrix}. \tag{3.19}$$

Using (2.22), we calculate the invariants

$$I_4 = \lambda^2\lambda_z^2 D_r^2 = D_{LR}^2, \tag{3.20}$$

$$I_5 = \lambda^{-2}\lambda_z^{-2} D_{LR}^2 = \lambda^{-2}\lambda_z^{-2} I_4 = D_r^2, \tag{3.21}$$

$$I_6 = \lambda^{-4}\lambda_z^{-4} D_{LR}^2 = \lambda^{-4}\lambda_z^{-4} I_4 = \lambda^{-2}\lambda_z^{-2} D_r^2. \tag{3.22}$$

The components of electric field can be found using equation (2.24).

Since the deformation gradient is diagonal and $D_\theta = D_z = 0$ we have $E_\theta = E_z = 0$ and the third component will be

$$E_r = 2(\Omega_4^*\lambda^2\lambda_z^2 D_r + \Omega_5^* D_r + \Omega_6^*\lambda^{-2}\lambda_z^{-2} D_r). \tag{3.23}$$

For cylindrical symmetry (assuming no dependence on either $\theta$ or $z$) $\text{curl}\,\mathbf{E} = 0$ will be equivalent to $rE_\theta = \text{const}$ and $E_z = \text{const}$, which are satisfied automatically. At this point we do not need to impose any condition on the function $\Omega^*$. For some types of deformations we do need such a condition. We can refer to Dorfmann & Ogden (2006) for an example of such a condition, where azimuthal shear deformation is considered.

### 3.2.4 Stress components

Stress components can be calculated with the help of (2.23)

$$\tau_{rr} = 2\Omega_1^*\lambda^{-2}\lambda_z^{-2} + 2\Omega_2^*(\lambda_z^{-2} + \lambda^{-2}) - p + 2\Omega_5^* D_r^2 + 4\Omega_6^*\lambda^{-2}\lambda_z^{-2} D_r^2, \tag{3.24}$$

$$\tau_{\theta\theta} = 2\Omega_1^*\lambda^2 + 2\Omega_2^*\left[\lambda_z^{-2} + \lambda_z^2\lambda^2\right] - p, \tag{3.25}$$

$$\tau_{zz} = 2\Omega_1^*\lambda_z^2 + 2\Omega_2^*\left[\lambda^{-2} + \lambda^2\lambda_z^2\right] - p. \tag{3.26}$$

Since the invariants are the functions of two independent stretches and $I_4$, we can define a reduced energy function in the form

$$\omega^*(\lambda, \lambda_z, I_4) = \Omega^*(I_1, I_2, I_4, I_5, I_6). \tag{3.27}$$

Therefore, we can calculate

$$\frac{\partial \omega^*}{\partial \lambda} = \Omega_1^*(-2\lambda_z^{-2}\lambda^{-3} + 2\lambda) + \Omega_2^*(2\lambda\lambda_z^2 - 2\lambda^{-3}) + \Omega_5^*(-2\lambda^{-3}\lambda_z^{-2}I_4) + \Omega_6^*(-4\lambda^{-5}\lambda_z^{-4}I_4),$$

$$\frac{\partial \omega^*}{\partial \lambda_z} = \Omega_1^*(-2\lambda^{-2}\lambda_z^{-3} + 2\lambda_z) + \Omega_2^*(2\lambda^2\lambda_z - 2\lambda_z^{-3}) + \Omega_5^*(-2\lambda^{-2}\lambda_z^{-3}I_4) + \Omega_6^*(-4\lambda^{-4}\lambda_z^{-5}I_4). \tag{3.28}$$

From (3.24), (3.25) and (3.26) we have

$$\tau_{\theta\theta} - \tau_{rr} = \Omega_1^*(2\lambda^2 - 2\lambda^{-2}\lambda_z^{-2}) + \Omega_2^*(2\lambda_z^2\lambda^2 - 2\lambda^{-2}) - 2\Omega_5^*D_r^2 - 4\Omega_6^*\lambda^{-2}\lambda_z^{-2}D_r^2, \tag{3.29}$$

$$\tau_{zz} - \tau_{rr} = \Omega_1^*(2\lambda_z^2 - 2\lambda^{-2}\lambda_z^{-2}) + \Omega_2^*(2\lambda^2\lambda_z^2 - 2\lambda_z^{-2}) - 2\Omega_5^*D_r^2 - 4\Omega_6^*\lambda^{-2}\lambda_z^{-2}D_r^2. \tag{3.30}$$

Therefore, we have the following connections featuring the stress differences:

$$\tau_{\theta\theta} - \tau_{rr} = \lambda\frac{\partial \omega^*}{\partial \lambda}, \tag{3.31}$$

$$\tau_{zz} - \tau_{rr} = \lambda_z\frac{\partial \omega^*}{\partial \lambda_z}. \tag{3.32}$$

Also

$$\frac{\partial \omega^*}{\partial I_4} = \Omega_4^* + \Omega_5^*\lambda^{-2}\lambda_z^{-2} + \Omega_6^*\lambda^{-4}\lambda_z^{-4}. \tag{3.33}$$

Therefore, expression (3.23) can be rewritten as

$$E_r = 2\lambda^2\lambda_z^2\frac{\partial \omega^*}{\partial I_4}D_r. \tag{3.34}$$

According to Gauss's theorem we have no field outside the tube, therefore by (2.11) the Maxwell stress is zero. Thus, we have only mechanical load due to a pressure $P$ inside the tube applied to the inner surface at $r = a$ and no loads at $r = b$, and hence

$$\tau_{rr} = -P \quad \text{on} \quad r = a, \quad \tau_{rr} = 0 \quad \text{on} \quad r = b. \tag{3.35}$$

In this problem the equilibrium equation div$\boldsymbol{\tau} = 0$ reduces to

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \tau_{\theta\theta} - \tau_{rr} = \lambda\omega^*_\lambda. \tag{3.36}$$

In the previous expression we have used (3.31). Integrating (3.36) and using the boundary conditions (3.35) we have

$$\int_{-P}^{0} \mathrm{d}\tau_{rr} = \int_a^b \lambda\omega^*_\lambda \frac{\mathrm{d}r}{r}. \tag{3.37}$$

Therefore,

$$P = \int_a^b \lambda\omega^*_\lambda \frac{\mathrm{d}r}{r}. \tag{3.38}$$

In some cases it is convenient to change the variable of integration from $r$ to $\lambda$. To this end, we rearrange and differentiate $(3.3)_1$ with respect to $r$, taking into account that $\lambda$ depends on $r$. We have

$$r\frac{\mathrm{d}\lambda}{\mathrm{d}r} = -\lambda(\lambda^2\lambda_z - 1). \tag{3.39}$$

The details of the calculation which lead to (3.39) can be found in Appendix A of this thesis. Therefore, expression (3.38) can be rewritten as

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1}\omega^*_\lambda \mathrm{d}\lambda. \tag{3.40}$$

From (3.4) we see that $\lambda_b$ depends on $\lambda_a$. Therefore, assuming that $\lambda_z$ is known, the previous relation gives $P$ as a function of $\lambda_a$ and invariant $I_4 = Q^2(a)/4\pi^2L^2R^2$, which is known for a given charge $Q(a) = -Q(b)$.

Similarly, since $b = \sqrt{a^2 + \lambda_z^{-1}(B^2 - A^2)}$ we see that (3.38) provides a relationship between pressure and the inner radius $a$ and invariant $I_4$.

The total axial load $N$ can be calculated from

$$N = 2\pi \int_a^b \tau_{zz} r \mathrm{d}r. \tag{3.41}$$

Using (3.32), (3.31) and the equilibrium equation (3.36), the axial stress $\tau_{zz}$ can be expressed as

$$\tau_{zz} = \frac{1}{2}\Big[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\tau_{rr})\Big] - \frac{\lambda\omega^*_\lambda}{2} + \lambda_z\omega^*_{\lambda_z}. \tag{3.42}$$

Therefore, the total axial load can be rewritten as

$$\begin{aligned} N &= 2\pi \int_a^b \Big[\frac{1}{2}\Big[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\tau_{rr})\Big] - \frac{\lambda\omega^*_\lambda}{2} + \lambda_z\omega^*_{\lambda_z}\Big] r\mathrm{d}r \\ &= \pi \int_a^b \mathrm{d}(r^2\tau_{rr}) + \pi \int_a^b (2\lambda_z\omega^*_{\lambda_z} - \lambda\omega^*_\lambda) r\mathrm{d}r. \end{aligned} \tag{3.43}$$

Using the limits of integration for the given problem, finally, we have

$$N = \pi \int_a^b (2\lambda_z \omega^*_{\lambda_z} - \lambda\omega^*_\lambda) r\mathrm{d}r + \pi a^2 P. \tag{3.44}$$

We assume that the cylinder has closed ends. The quantity $F = N - P\pi a^2$ can be interpreted as a reduced axial load, because the action of pressure on the ends of the cylinder is removed from the total load. Using the previous result (3.39) and (3.4), we can change the variable of integration from $\lambda$ to $r$

$$F = \pi A^2(\lambda_a^2\lambda_z - 1)\int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-2}(2\lambda_z\omega^*_{\lambda_z} - \lambda\omega^*_\lambda)\lambda \mathrm{d}\lambda. \tag{3.45}$$

## 3.3 Illustrative example

We will consider the simple energy function

$$\Omega^* = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{2}\varepsilon^{-1} I_5, \tag{3.46}$$

where the constant $\mu$ is the shear modulus of the neo-Hookean material in the absence of an electric field and the constant $\varepsilon$ is the electric permittivity of the electroelastic material. Using $(2.21)_1$ and $(2.22)_2$ with $\mathrm{F} = \mathrm{diag}[\lambda_r, \lambda, \lambda_z]$ we can write potential (3.46) in the reduced form

$$\omega^* = \frac{1}{2}\mu(\lambda^{-2}\lambda_z^{-2} + \lambda^2 + \lambda_z^2 - 3) + \frac{1}{2}\varepsilon^{-1}\lambda^{-2}\lambda_z^{-2} I_4. \tag{3.47}$$

To find the pressure inside the cylinder we need to calculate $\lambda\omega^*_\lambda$

$$\lambda\omega^*_\lambda = \mu(-\lambda^{-2}\lambda_z^{-2} + \lambda^2) - \varepsilon^{-1}\lambda^{-2}\lambda_z^{-2} I_4. \tag{3.48}$$

Integral (3.38) for neo-Hookean material can be calculated explicitly and the result is

$$P = \mu\big[\lambda_z^{-1}\ln\frac{\lambda_a}{\lambda_b} + \lambda_z^{-2}\frac{\lambda_a^2 - \lambda_b^2}{2\lambda_b^2\lambda_a^2}\big] - \varepsilon^{-1} q \lambda_z^{-2}\frac{b^2 - a^2}{2a^2b^2}, \tag{3.49}$$

where $q$ is defined as $q = \sigma_{\mathrm{F}A}^2 A^2$ and related to the charge $Q(a)$ via

$$q = \big(\frac{Q(a)}{2\pi L}\big)^2. \tag{3.50}$$

Expression (3.49) gives $P$ in terms of the charge $q$ and $\lambda_a$. Again, recall that $\lambda_b$, $a$ and $b$ can be expressed in terms of $\lambda_a$. The reduced axial load can also be evaluated explicitly for neo-Hookean material

$$F = \pi A^2\mu[(\lambda_z - \lambda_z^{-2})(\eta^2 - 1) - \lambda_z^{-2}(\lambda_a^2\lambda_z - 1)\log(\lambda_a/\lambda_b)] - \frac{\pi q}{\varepsilon\lambda_z^2}\log(b/a). \tag{3.51}$$

In (3.47) instead of a neo-Hookean material we can write more generally

$$\omega^* = \omega(\lambda, \lambda_z) + \frac{1}{2}\varepsilon^{-1}\lambda^{-2}\lambda_z^{-2}I_4. \tag{3.52}$$

Therefore, we can rewrite (3.49) as

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1}\omega_\lambda \mathrm{d}\lambda - \varepsilon^{-1}q\lambda_z^{-2}\frac{b^2 - a^2}{2a^2b^2}, \tag{3.53}$$

and the reduced axial load will have the following representation

$$F = \pi A^2(\lambda_a^2\lambda_z - 1)\int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-2}(2\lambda_z\omega_{\lambda_z} - \lambda\omega_\lambda)\lambda \mathrm{d}\lambda - \frac{\pi q}{\varepsilon\lambda_z^2}\log\frac{b}{a}. \tag{3.54}$$

## 3.4 Application to a thin-walled cylindrical shell

For a thin-walled cylindrical shell we can approximate expression (3.40) using the mean value theorem

$$P \simeq (\lambda_a - \lambda_b)(\lambda^2\lambda_z - 1)^{-1}\omega^*_\lambda(\lambda, \lambda_z, I_4). \tag{3.55}$$

From the expression (3.4) to the first order in $\delta = (B - A)/A$ we can obtain the following approximation

$$\lambda_a \simeq \lambda_b + \delta\lambda^{-1}\lambda_z^{-1}(\lambda^2\lambda_z - 1), \tag{3.56}$$

where $\lambda$ can be taken as either $\lambda_a$ or $\lambda_b$ to the first order approximation in $\delta$. Therefore, expression (3.55) can be rewritten as

$$P \simeq \delta\lambda^{-1}\lambda_z^{-1}\omega^*_\lambda(\lambda, \lambda_z, I_4). \tag{3.57}$$

Approximation of $I_4$ gives the following result

$$I_4 \simeq \sigma_{\mathrm{F}A}^2 = \frac{q}{A^2} = \tilde{q}, \tag{3.58}$$

where we defined the notation $\tilde{q} = \sigma_{\mathrm{F}A}^2$. Therefore, for a fixed $\lambda_z$, $P$ will be a function of stretch $\lambda$ and the charge $\sigma_{\mathrm{F}A}$. For the reduced potential with a general elastic term we can rewrite (3.57) as

$$P \simeq \delta\lambda^{-1}\lambda_z^{-1}[\omega_\lambda(\lambda, \lambda_z) - \varepsilon^{-1}\lambda^{-3}\lambda_z^{-2}\tilde{q}]. \tag{3.59}$$

We can see from this relation that the influence of the charge $\tilde{q}$ on the pressure $P$ becomes less and less with increasing azimuthal stretch $\lambda$ ($\lambda_z$ is fixed).

Using the mean value theorem and expression (3.56) we can approximate the reduced axial load

$$F \simeq \delta\pi A^2(2\omega^*_{\lambda_z} - \lambda\lambda_z^{-1}\omega^*_\lambda). \tag{3.60}$$

In a similar way for a reduced energy potential with a general elastic term we can write

$$F \simeq \delta\pi A^2(2\omega_{\lambda_z} - \lambda\lambda_z^{-1}\omega_\lambda - \varepsilon^{-1}\lambda^{-2}\lambda_z^{-3}\tilde{q}). \tag{3.61}$$

We can interpret this result in the following way. With increasing circumferential stretch $\lambda$ the influence of the electric field expressed in terms of charge $\tilde{q}$ becomes less and less.

## 3.5 Charge and potential

Since $\operatorname{curl}\mathbf{E} = 0$, there exists a scalar field $\phi$ (electrostatic potential) such that $\mathbf{E} = -\operatorname{grad}\phi$. For cylindrically symmetric problem $\phi$ depends only on $r$, therefore, $E_r = -\mathrm{d}\phi/\mathrm{d}r$. Previously, we found that $E_r = 2\lambda^2\lambda_z^2\omega^*_{I_4}D_r$. Therefore, we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = -2\lambda^2\lambda_z^2\omega^*_{I_4}D_r. \tag{3.62}$$

Integration of the previous expression and use of (3.16) will give us an expression for potential difference between the surfaces, and we have

$$\phi(b) - \phi(a) = -\frac{Q(a)l}{\pi L^2}\int_a^b \lambda^2\omega^*_{I_4}\frac{\mathrm{d}r}{r}. \tag{3.63}$$

For the simple model (3.46) and fixing $\lambda_z$ we have

$$\phi(b) - \phi(a) = -\frac{Q(a)}{2\pi l}\varepsilon^{-1}\log\frac{b}{a}. \tag{3.64}$$

The obtained expression provides a relationship between potential difference at the boundaries, the charge $Q(a) = -Q(b)$, the inner radius $a$ and the length of the cylinder $l$.

We can rewrite the previous expression (3.64)

$$\frac{\phi(b) - \phi(a)}{B - A} = \frac{\sigma_{\mathrm{F}A}}{\lambda_z(\eta - 1)}\varepsilon^{-1}\log\frac{\lambda_a}{\lambda_b\eta}, \tag{3.65}$$

where we defined $\eta = B/A$. Expression (3.65) provides the relationship between potential and the charge $\sigma_{\mathrm{F}A}$, azimuthal stretch $\lambda_a$, $\eta$ and dielectric permittivity of material $\varepsilon$. We will define reference electric field as

$$E_0 = \frac{\phi(b) - \phi(a)}{B - A}, \tag{3.66}$$

and we can approximate (3.64) for the membrane

$$E_0 = \frac{Q(a)}{2\pi\lambda_z^2\lambda^2 LA\varepsilon}. \tag{3.67}$$

Therefore,

$$\tilde{q} = E_0^2 \varepsilon^2 \lambda^4 \lambda_z^4, \tag{3.68}$$

and we can rewrite (3.59) as

$$P \simeq \delta \lambda^{-1} \lambda_z^{-1} [\omega_\lambda(\lambda, \lambda_z) - \varepsilon \lambda \lambda_z^2 E_0^2]. \tag{3.69}$$

We can observe from this relation that for the considered case when the electric field is defined by (3.66) through the potential difference, the second term in (3.69) is not affected by the azimuthal stretch stretch $\lambda$. Therefore, in this case the effect of electric field is uncoupled from mechanical stretch $\lambda$, provided that $\lambda_z$ remains fixed.

For the reduced axial load we have the following result

$$F \simeq \delta \pi A^2 (2\omega_{\lambda_z} - \lambda \lambda_z^{-1} \omega_\lambda - \varepsilon \lambda^2 \lambda_z E_0^2). \tag{3.70}$$

Therefore, we can conclude that if electric field is expressed in terms of potential difference, the reduced axial load will be affected more and more significantly with increasing circumferential stretch $\lambda$.

## 3.6 Numerical results

Here we give explicit relations, based on which the figures were produced. In figures 3.1(a)–3.3(a) we used the following expression for a non-dimensional pressure $P^* = P/\mu$

$$P^* = \frac{1}{\mu} \int_{\lambda_b}^{\lambda_a} (\lambda^2 \lambda_z - 1)^{-1} \omega_\lambda \mathrm{d}\lambda - \frac{\tilde{q}(\eta^2 - 1)}{2\varepsilon \mu \lambda_z^3 \lambda_a^2 [\lambda_a^2 + \lambda_z^{-1}(\eta^2 - 1)]}, \tag{3.71}$$

obtained from (3.53), using (3.58), $(3.5)_1$, $(3.5)_3$, $(3.3)_1$ and the definition $\eta = B/A$.

In figures 3.1(b)–3.3(b) we used

$$P^* = \frac{1}{\mu} \int_{\lambda_b}^{\lambda_a} (\lambda^2 \lambda_z - 1)^{-1} \omega_\lambda \mathrm{d}\lambda - \frac{E_0^2 \varepsilon (\eta^2 - 1)(\eta - 1)^2}{2\mu \lambda_z \lambda_a^2 (\lambda_a^2 + \lambda_z^{-1}(\eta^2 - 1)) \left[\log \frac{\lambda_a}{\lambda_b \eta}\right]^2}, \tag{3.72}$$

obtained from (3.71), (3.58), (3.65) and (3.66).

In figures 3.4(a)–3.6(a) we used the following expression for a non-dimensional reduced axial load $F^* = (N - \pi a^2 P)/\mu \pi A^2$

$$F^* = \frac{1}{\mu} (\lambda_a^2 \lambda_z - 1) \int_{\lambda_b}^{\lambda_a} (\lambda^2 \lambda_z - 1)^{-2} (2\lambda_z \omega_{\lambda_z} - \lambda \omega_\lambda) \lambda \mathrm{d}\lambda + \frac{\tilde{q}}{\varepsilon \mu \lambda_z^2} \log \frac{\lambda_a}{\lambda_b \eta}, \tag{3.73}$$

obtained from (3.54) and the definition for $\eta$.

In figures 3.4(b)–3.6(b) we used

$$F^* = \frac{1}{\mu}(\lambda_a^2\lambda_z - 1)\int_{\lambda_b}^{\lambda_a}(\lambda^2\lambda_z - 1)^{-2}(2\lambda_z\omega_{\lambda_z} - \lambda\omega_\lambda)\lambda \mathrm{d}\lambda + \frac{E_0^2\varepsilon(\eta-1)^2}{\mu\log\frac{\lambda_a}{\lambda_b\eta}}, \tag{3.74}$$

obtained from (3.73), (3.58), (3.65) and (3.66). In the expressions for $P^*$ and $F^*$ $\lambda_b$ can be expressed in terms of $\lambda_a$ through the relation (3.4).

In this section we show numerical results for different elastic models, which are accounted for by the term $\omega(\lambda, \lambda_z)$ in (3.52). We used Mathematica (Wolfram Research, 2013) for this purpose. First, we will consider a neo-Hookean model. In Fig. 3.1 the dimensionless ratio $P/\mu$ is plotted for different charges meassured by dimensionless quantity $\tilde{q}/\mu\varepsilon$, and different potential differencies, measured by dimensionless quantity $\varepsilon E_0^2/\mu$. We

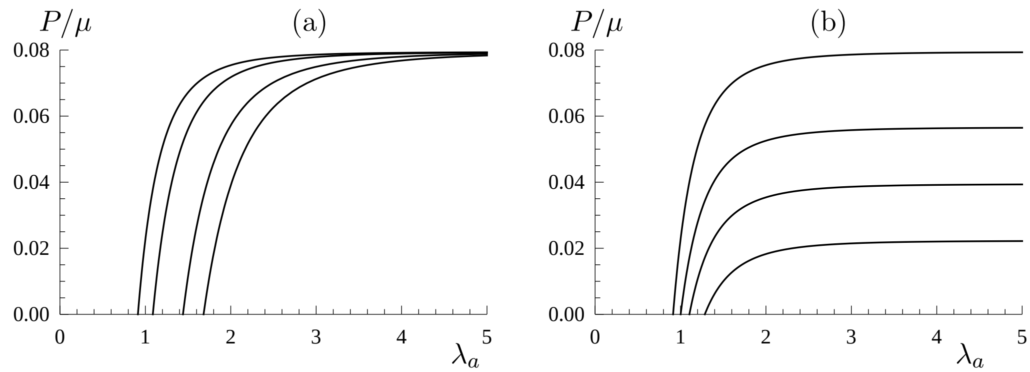


Figure 3.1: Plots of $P/\mu$ versus $\lambda_a$ for the neo-Hookean electroelastic material based on Eq. (3.71) and (3.72) with $\eta = 1.1$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $P/\mu$ decreases as the magnitude of the field measure increases.

see that the results for a thick walled tube ($B/A = 1.1$) are in conformity with formulas, obtained for a thin-walled cylindrical shell. For a constant charge influence of the field becomes less and less with increasing azimuthal stretch $\lambda_a$. For a thin-walled shell this feature can be observed from a factor $\lambda^{-4}$ in (3.59). If we prescribe different constant potential differences, we see that the effect of electric field measured by dimensionless potential differences $\varepsilon E_0^2/\mu$ is now uncoupled from the mechanical deformation. We can observe the same situation in the case of a thin-walled cylindrical shell. Observe that in the second term of (3.69) the azimuthal stretch $\lambda$ becomes unity.

Next, we will consider the Ogden model, (Ogden, 1972). The strain energy potential is expressed in terms of principal stretches

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{n=1}^{N} \frac{\mu_n}{\alpha_n}(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3), \tag{3.75}$$

where $N$ is a positive integer, $\mu_n$ and $\alpha_n$ are material constants, and shear modulus $\mu$ satisfies

$$2\mu = \sum_{n=1}^{N} \mu_n \alpha_n. \tag{3.76}$$

We will consider a three term version of this model: $N = 3$. It was found that this model gives a good approximation for vulcanized natural rubber with the following values of material constants: $\alpha_1 = 1.3$, $\alpha_2 = 5.0$, $\alpha_3 = -2.0$. To plot the dimensionless pressure $P/\mu$ we will use material constants $\mu_n$ divided by shear modulus $\mu$: $\mu_n^* = \mu_n/\mu$, with the following values: $\mu_1^* = 1.491$, $\mu_2^* = 0.0028$, $\mu_3^* = -0.0237$. For the considered deformation of a cylinder the potential (3.75) specifies to

$$w(\lambda, \lambda_z) = \sum_{n=1}^{3} \frac{\mu_n}{\alpha_n}(\lambda^{\alpha_n} + \lambda_z^{\alpha_n} + \lambda_z^{-\alpha_n}\lambda^{-\alpha_n} - 3). \tag{3.77}$$

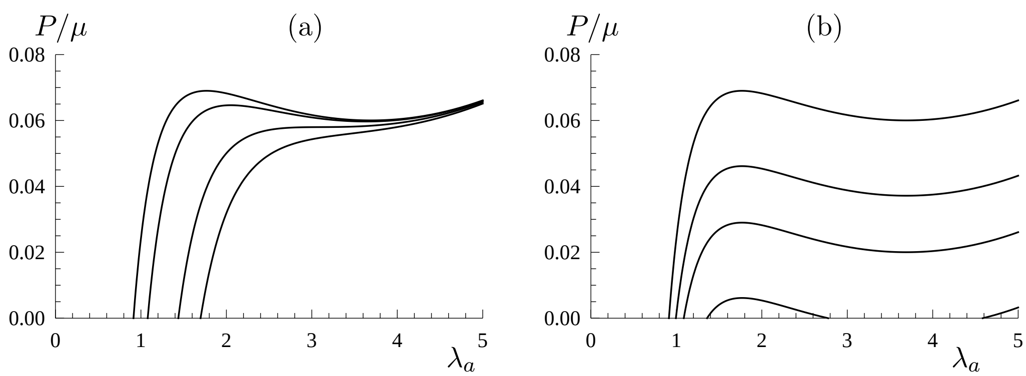


Figure 3.2: Plots of $P/\mu$ versus $\lambda_a$ for the Ogden electroelastic material based on Eq. (3.71) and (3.72) with $\eta = 1.1$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.55$. In each of (a) and (b) the value of $P/\mu$ decreases as the magnitude of the field measure increases.

We can observe in Fig. 3.2 that most plots with low charge have maxima and minima, which is in conformity with pure elastic case. Again, we have the same trend. When electric field is expressed as a constant charge, with higher circumferential stretch, the influence of the field becomes less and less.

Finally, we will consider the Gent model (Gent, 1996). This is an isotropic model. Its distinctive feature is that it has an asymptote, which reflects the fact that polymeric chains

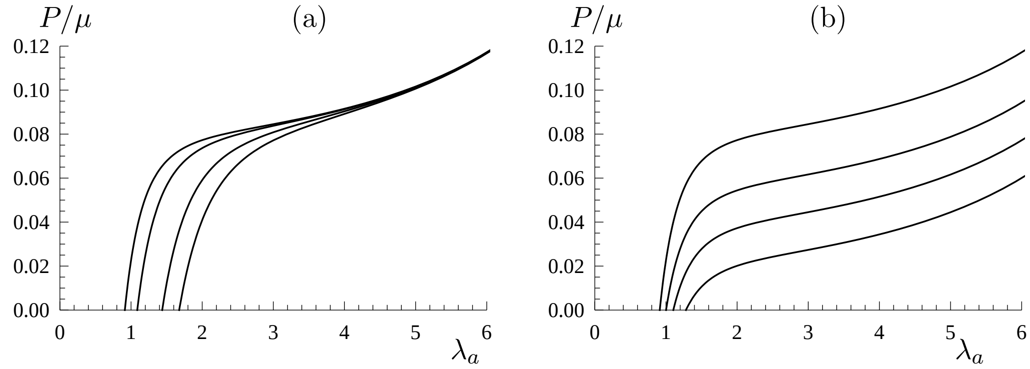


Figure 3.3: Plots of $P/\mu$ versus $\lambda_a$ for the Gent electroelastic material based on Eq. (3.71) and (3.72) with $\eta = 1.1$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $P/\mu$ decreases as the magnitude of the field measure increases.

in rubber cannot be extended beyond a certain threshold. The elastic potential has the following representation for this model

$$W(I_1) = -\frac{\mu G}{2} \log \left[ 1 - \frac{(I_1 - 3)}{G} \right], \tag{3.78}$$

where $G$ is a material constant. We took $G = 97.2$ for our calculations. This value was used by Gent for the unfilled rubber vulcanizate. The results of numerical calculations are shown at Fig. 3.3. We observed an asymptote at $\lambda \simeq 10$, which is not shown here.

In general we can note that electric field predeforms a cylinder by increasing its circumferential stretch, therefore we can observe that in order to obtain a certain circumferential stretch a lower pressure is required with respect to pure elastic case.

We assume that the cylinder has closed ends. In order to keep $\lambda_z$ fixed, we need to apply an external axial load. For the sign convention we accept that positive load tries to extend the cylinder, and negative load tries to compress it. In this part of the thesis we will consider how non-dimensional reduced axial load is affected by an electric field. We define non-dimensional reduced axial load as $F^* = (N - \pi a^2 P)/\mu\pi A^2$. We will consider the previous models in the same order. The same trend can be observed in all figures. With increasing electric field a lower axial load is required to keep $\lambda_z$ fixed. Therefore, we can conclude that according to this model the electric field tries to stretch the cylinder in the axial direction, therefore a lower axial load is required for stronger electric field. As an example, Fig. 3.4(a) can be interpreted in the following way. Initially, a positive extensional load is required to have a prestretch $\lambda_z = 1.2$. Then due to the inflating pressure, which extends the tube in the axial direction, we observe that reduced axial load is decreasing with increasing circumferential stretch $\lambda_a$. Next, if we apply electric field, it will give an initial circumferential prestretch $\lambda_a$, and then as it was before, the axial load will be decreasing due to inflation. We note that we used the same parameters (electric field, axial stretch $\lambda_z$, ratio $\eta = B/A$), and there is a direct correspondence between the figures which depict nondimensional pressure and reduced axial load for each model. Essentially similar trends can be observed for models in Fig. 3.5–3.6.

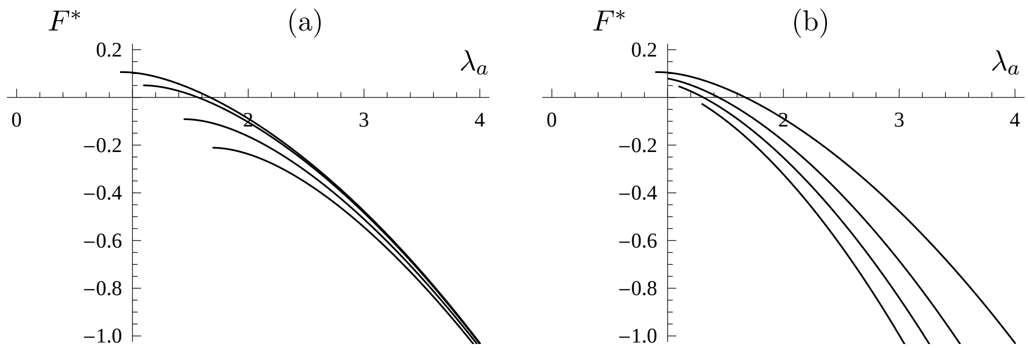


Figure 3.4: Plots of nondimensional reduced axial load $F^*$ versus $\lambda_a$ for the neo-Hookean electroelastic material based on Eq. (3.73) and (3.74) with $\eta = 1.1$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $F^*$ decreases as the magnitude of the field measure increases.

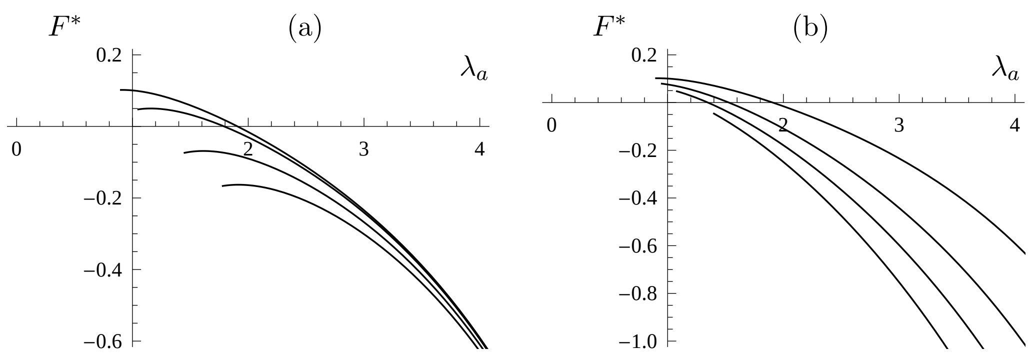


Figure 3.5: Plots of nondimensional reduced axial load $F^*$ versus $\lambda_a$ for the Ogden elastic material based on Eq. (3.73) and (3.74) with $\eta = 1.1$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $F^*$ decreases as the magnitude of the field measure increases.

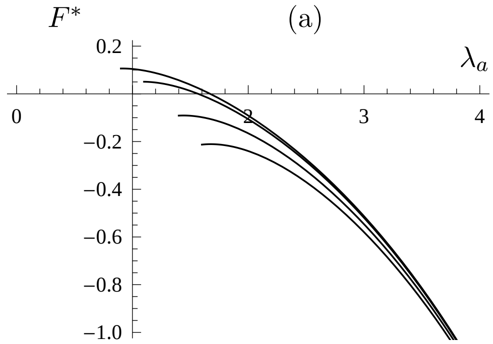


Figure 3.6: Plots of nondimensional reduced axial load $F^*$ versus $\lambda_a$ for the Gent elastic material based on Eq. (3.73) and (3.74) with $\eta = 1.1$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $F^*$ decreases as the magnitude of the field measure increases.

Now we investigate the behaviour of pressure and reduced axial load if we increase the wall thickness $\eta$. In Fig. 3.7–3.9 for both cases of electric field expressed in terms of charge and potential difference we can observe that for a thicker-walled tube higher levels of pressure are required to achieve the same level of radial deformation, which is understandable intuitively.

In Fig. 3.10–3.12 reduced axial load as a function of radial deformation for thicker-walled tubes is plotted. We observe in these plots that for thicker-walled tubes higher levels of axial load (either extensional or compressive) are required to keep $\lambda_z$ unchanged for the same radial deformation in comparison with thinner-walled tubes. This behaviour is understandable intuitively.

## 3.7 A note on activation

We now express the formulas for $P$ and $F$ from Section 3.4 in the dimensionless forms

$$P^* = \lambda_a^{-1}\lambda_z^{-1}\bar{\omega}_\lambda(\lambda_a, \lambda_z) - \lambda_a^{-4}\lambda_z^{-3}q^*, \quad F^* = 2\bar{\omega}_{\lambda_z}(\lambda_a, \lambda_z) - \lambda_a\lambda_z^{-1}\bar{\omega}_\lambda(\lambda_a, \lambda_z) - \lambda_a^{-2}\lambda_z^{-3}q^*, \tag{3.79}$$

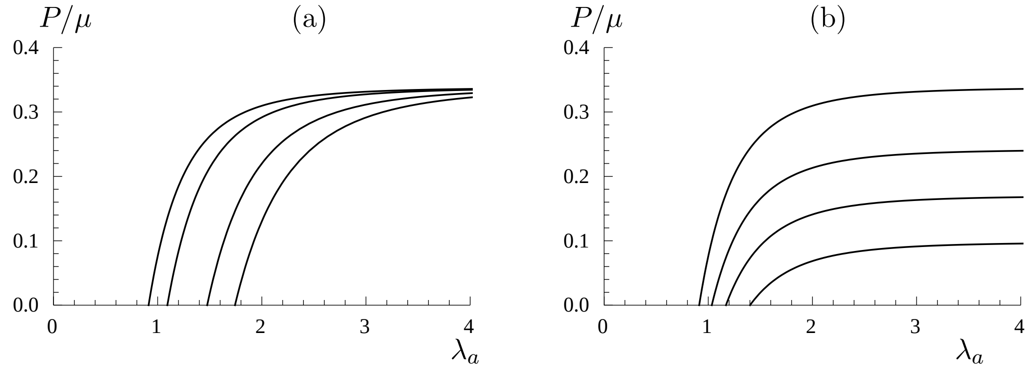


Figure 3.7: Plots of $P/\mu$ versus $\lambda_a$ for the neo-Hookean electroelastic material based on Eq. (3.71) and (3.72) with $\eta = 1.5$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $P/\mu$ decreases as the magnitude of the field measure increases.

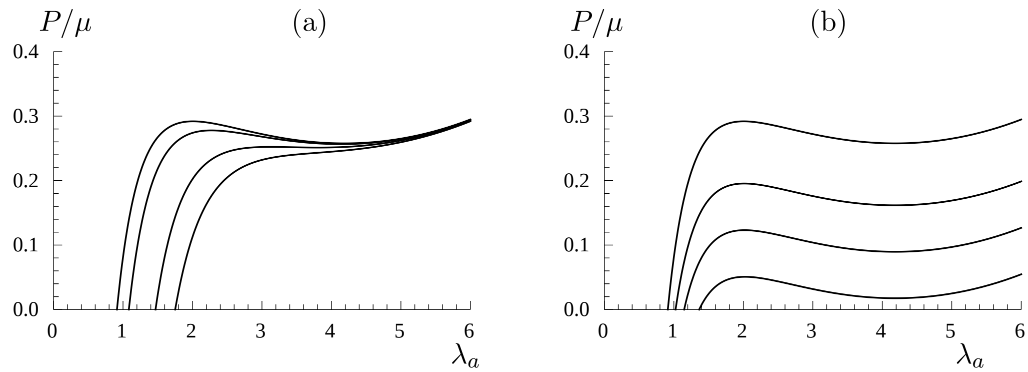


Figure 3.8: Plots of $P/\mu$ versus $\lambda_a$ for the Ogden electroelastic material based on Eq. (3.71) and (3.72) with $\eta = 1.5$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10, 15$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $P/\mu$ decreases as the magnitude of the field measure increases.

from (3.59) and (3.61), and

$$P^* = \lambda_a^{-1}\lambda_z^{-1}\bar{\omega}_\lambda(\lambda_a, \lambda_z) - \lambda_z e^*, \quad F^* = 2\bar{\omega}_{\lambda_z}(\lambda_a, \lambda_z) - \lambda_a\lambda_z^{-1}\bar{\omega}_\lambda(\lambda_a, \lambda_z) - \lambda_a^2\lambda_z e^*, \tag{3.80}$$

from (3.69) and (3.70), where $q^* = \tilde{q}/(\mu\varepsilon)$ and $e^* = \varepsilon E_0^2/\mu$, $\bar{\omega} = \omega/\mu$, and $P^* = P/(\delta\mu)$ and $F^* = F/(\delta\mu\pi A^2)$, the latter two non-dimensionalizations being different from those used in Section 3.6.

From either of (3.79) or (3.80) it follows that

$$F^* - \lambda_a^2 P^* = 2\bar{\omega}_{\lambda_z}(\lambda_a, \lambda_z) - 2\lambda_a\lambda_z^{-1}\bar{\omega}_\lambda(\lambda_a, \lambda_z). \tag{3.81}$$

If there is no internal pressure ($P^* = 0$) then for a given (fixed) axial load $F^*$ this determines a connection between $\lambda_a$ and $\lambda_z$ (in general implicit), and, for an applied voltage (in terms

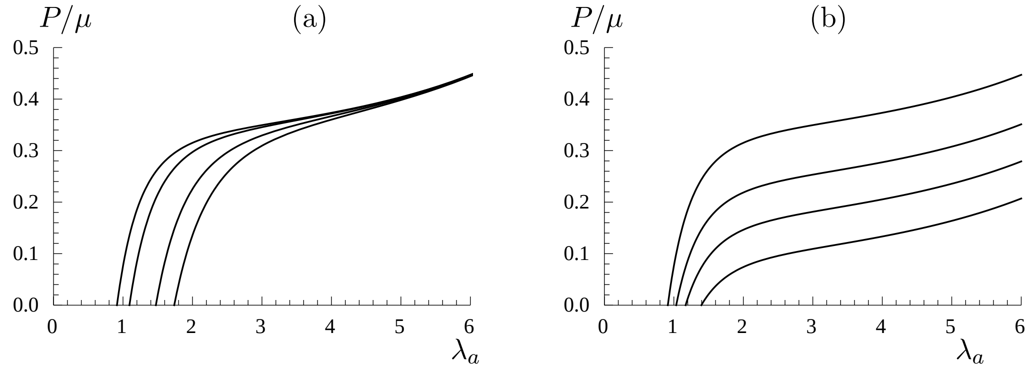


Figure 3.9: Plots of $P/\mu$ versus $\lambda_a$ for the Gent electroelastic material based on Eq. (3.71) and (3.72) with $\eta = 1.5$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $P/\mu$ decreases as the magnitude of the field measure increases.

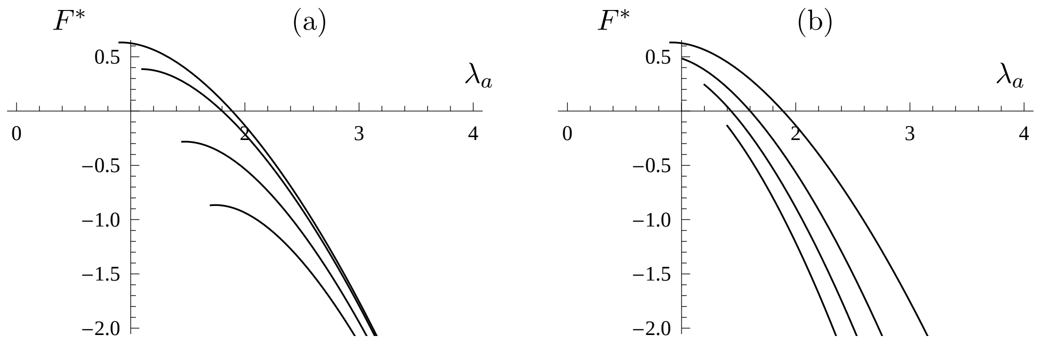


Figure 3.10: Plots of nondimensional reduced axial load $F^*$ versus $\lambda_a$ for the neo-Hookean electroelastic material based on Eq. (3.73) and (3.74) with $\eta = 1.5$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $F^*$ decreases as the magnitude of the field measure increases.

of $e^*$) for example, Eq. $(3.80)_1$ provides a connection between $\lambda_z$ and $e^*$, i.e. it determines the change in $\lambda_z$ due to activation from its initial value at $e^* = 0$. Similarly, if $F^* = 0$ and $P^*$ is fixed activation with $e^*$ causes a change in $\lambda_z$.

For simplicity these general principles are now illustrated in respect of the neo-Hookean elasticity model (3.47), for which

$$F^* - \lambda_a^2 P^* = 2(\lambda_z - \lambda_a^2 \lambda_z^{-1}). \tag{3.82}$$

For $P^* = 0$ we then have

$$\lambda_a^2 \lambda_z = (1 - \lambda_z^2 e^*)^{-1/2}, \tag{3.83}$$

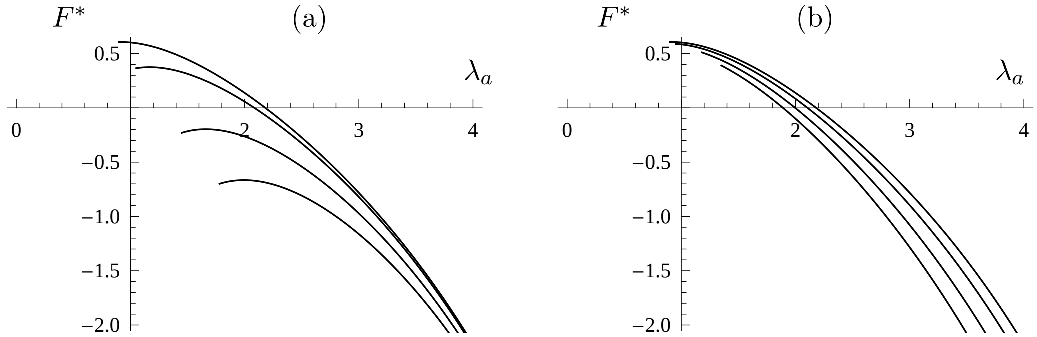


Figure 3.11: Plots of nondimensional reduced axial load $F^*$ versus $\lambda_a$ for the Ogden elastic material based on Eq. (3.73) and (3.74) with $\eta = 1.5$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $F^*$ decreases as the magnitude of the field measure increases.

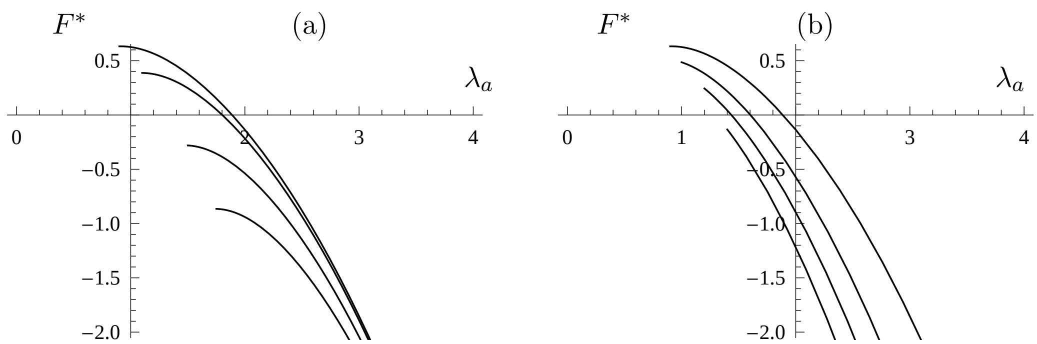


Figure 3.12: Plots of nondimensional reduced axial load $F^*$ versus $\lambda_a$ for the Gent elastic material based on Eq. (3.73) and (3.74) with $\eta = 1.5$ and $\lambda_z = 1.2$: (a) for fixed charge with $\tilde{q}/\mu\varepsilon = 0, 1, 5, 10$; (b) for fixed potential difference with $\varepsilon E_0^2/\mu = 0.0, 0.2, 0.35, 0.5$. In each of (a) and (b) the value of $F^*$ decreases as the magnitude of the field measure increases.

which requires that $\lambda_z^2 e^* < 1$. Note, in particular, that in the limit $\lambda_z^2 e^* \to 1$, $\lambda_a \to \infty$ and the wall thickness decreases to zero! Equation (3.82) requires that $\lambda_z > \lambda_a$ for $F^* > 0$. From (3.82) it also follows that

$$F^* = 2\lambda_z - 2\lambda_z^{-2}(1 - \lambda_z^2 e^*)^{-1/2}. \tag{3.84}$$

For several fixed positive values of $F^*$ the interdependence of $e^*$ and $\lambda_z$ is illustrated in Fig. 3.13(a). In terms of different variables similar plots were provided in Zhu et al. (2010) for different values of the initial axial stretch (equivalently, different values of $F^*$) and for a thick-walled tube with $B/A = 2$. In Zhu et al. (2010) the maxima on the curves were interpreted as corresponding to loss of electromechanical stability.

For contrast we now consider activation at fixed pressure and zero axial load, so that, from (3.82),

$$P^* = 2\lambda_z^{-1} - 2\lambda_z\lambda_a^{-2}, \tag{3.85}$$

which requires $\lambda_a > \lambda_z$ for $P^* > 0$, while $F^* = 0$ yields the quadratic

$$(\lambda_z^2 e^* + 1)\lambda_a^4 - 2\lambda_z^2\lambda_a^2 + \lambda_z^{-2} = 0 \tag{3.86}$$

for $\lambda_a^2$, the only solution of which consistent with $\lambda_a > \lambda_z$ being

$$\lambda_a^2 = \frac{\lambda_z^2 + \sqrt{\lambda_z^4 - \lambda_z^{-2} - e^*}}{\lambda_z^2 e^* + 1}, \tag{3.87}$$

which requires $\lambda_z^4 - \lambda_z^{-2} > e^*$. Hence

$$P^* = 2\lambda_z^{-1} - \frac{2\lambda_z(\lambda_z^2 e^* + 1)}{\lambda_z^2 + \sqrt{\lambda_z^4 - \lambda_z^{-2} - e^*}}, \tag{3.88}$$

and this equation is the basis for the plots in Fig. 3.13(b) in which the interdependence of $e^*$ and $\lambda_z$ is illustrated for several fixed values of $P^*$.

As for the case with $P^* = 0$ and fixed $F^*$ there is a maximum actuation voltage for each considered value of $P^*$ and again the maxima are associated with loss of electromechanical stability. However, for the considered neo-Hookean model in the absence of a voltage the radius can increase indefinitely as the pressure approaches a finite asymptote, and this behaviour is a reflection of the limited applicability of the neo-Hookean model, which is only realistic for stretches up to about 2. This should be borne in mind when assessing the results of activation. For models such as those in Ogden (1972) and Arruda & Boyce (1993) that are valid for a wider range of deformations than for the neo-Hookean model there is no theoretical limit to the allowable voltage, which can increase indefinitely with the axial stretch, possibly with an intermediate maximum followed by a minimum, as is the case for a particular Arruda–Boyce model considered in Zhu et al. (2010).

Next, based on the equations in (3.79), we consider activation with specified charge rather than a potential, in which case, with $P^* = 0$ we obtain

$$\lambda_a^2\lambda_z = \sqrt{1 + q^*} \tag{3.89}$$

and

$$F^* = 2\lambda_z - 2\lambda_z^{-2}\sqrt{1 + q^*}. \tag{3.90}$$

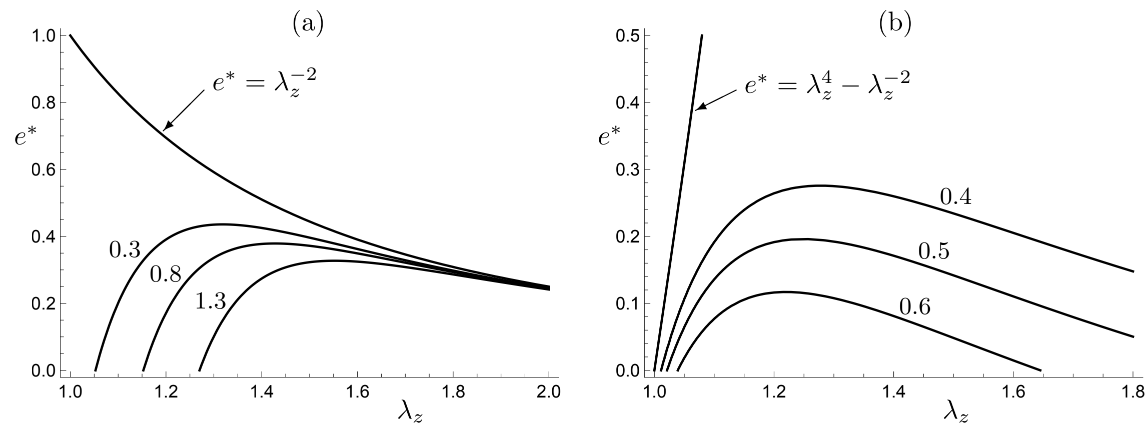


Figure 3.13: (a) For $P^* = 0$: plots of the activation potential (as measured by $e^*$) versus the resulting axial stretch $\lambda_z$ for the indicated fixed values $0.3, 0.8, 1.3$ of the dimensionless axial load $F^*$ (corresponding to initial stretches, for $e^* = 0$, of approximately $1.053, 1.153, 1.27$, respectively), together with the limiting curve defined by $\lambda_z^2 e^* = 1$. (b) For $F^* = 0$: plots of the activation potential (as measured by $e^*$) versus the resulting axial stretch $\lambda_z$ for the indicated fixed values $0.4, 0.5, 0.6$ of the dimensionless pressure $P^*$ (corresponding to initial stretches of approximately $1.01, 1.02, 1.04$, respectively), together with the limiting curve defined by $e^* = \lambda_z^4 - \lambda_z^{-2}$.

For $F^* = 0$, on the other hand, we have

$$\lambda_a^2 \lambda_z = \lambda_z^3 + \sqrt{\lambda_z^6 - 1 - q^*} \tag{3.91}$$

and

$$P^* = 2\lambda_z^{-1} - \frac{2\lambda_z^2}{\lambda_z^3 + \sqrt{\lambda_z^6 - 1 - q^*}}. \tag{3.92}$$

Results for $P^* = 0$ and $F^* = 0$, respectively, are illustrated in Fig. 3.14(a), (b) with $q^*$ plotted against $\lambda_z$ analogously to those in Fig. 3.13(a), (b) for $e^*$ against $\lambda_z$. In Fig. 3.14(a) the plots are for $F^* = 0, 0.3, 1, 2$ and in Fig. 3.14(b) for $P^* = 0.52, 0.56, 6$. In Fig. 3.14(a), in contrast to Fig. 3.13(a), there is no maximum and the stretch $\lambda_z$ increases monotonically with the applied charge, whereas in Fig. 3.14(b) there is a maximum for any pressure below the maximum attainable ($P^* \simeq 0.75$) with $q^* = 0$ for the neo-Hookean material and this has a similar 'instability' interpretation as for a fixed $F^*$ at $P^* = 0$.

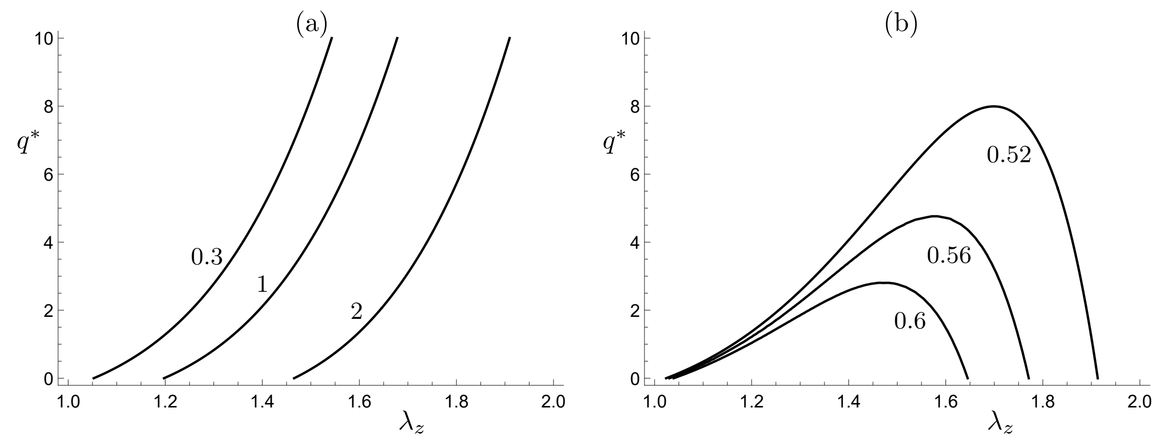


Figure 3.14: (a) For $P^* = 0$: plots of the activation charge (as measured by $q^*$) versus the resulting axial stretch $\lambda_z$ for the indicated fixed values $0.3, 1, 2$ of the dimensionless axial load $F^*$ (corresponding to initial stretches, for $q^* = 0$, of approximately $1.05, 1.2, 1.47$, respectively). (b) For $F^* = 0$: plots of the activation charge (as measured by $q^*$) versus the resulting axial stretch $\lambda_z$ for the indicated fixed values $0.52, 0.56, 0.6$ of the dimensionless pressure $P^*$ (corresponding to initial stretches of approximately $1.02, 1.03, 1.04$, respectively).

## 3.8 Concluding remarks

In this chapter the general formulation of nonlinear isotropic electroelasticity in the form developed by Dorfmann & Ogden (2005) has been applied to the prototype problem of a circular cylindrical tube of dielectric elastomer with compliant electrodes on its major surfaces. Without specialization of the constitutive law general expressions have been obtained for the internal pressure in the tube and axial load on its ends when subject to a radial electric field generated by a potential difference between the electrodes while the circular cylindrical geometry is maintained. The general results are then applied to a material model for which the electrostatic part of the constitutive law is linear with a deformation independent permittivity, and the electroelastic response of the tube has been illustrated for three different models of the elastic contribution to the constitutive law from rubber elasticity.

It is, of course, a simplifying assumption that the permittivity of the material is independent of the deformation, an assumption that runs counter to experimental evidence, at least for some dielectric elastomers. For example, in an extensive series of experiments on the acrylic elastomer VBH 4910 Wissler & Mazza (2007) showed that the permittivity decreases with stretching, and this should be taken into account in the modelling in situations where the deformations are relatively large. Such an influence is easily accommodated within the general constitutive framework presented in Section 2.1.3 and its specialization to the considered geometry in Section 3.2.1. However, in general this leads to a more complicated analysis and numerical solution will for the most part be required. Specific models which do include deformation dependent permittivity have been examined in a variety of boundary-value problems by Dorfmann & Ogden (2005, 2006, 2010a,b, 2014a) and Dorfmann & Ogden (2014c), while the influence of deformation dependent permittivity on stability considerations has been addressed in Zhao & Suo (2008), Liu et al. (2010) and Jimenez & McMeeking (2013). The problem of stability of an electroelastic tube under internal and external pressure is considered in the next chapter of this thesis.

To incorporate a fibre structure within the constitutive law is feasible but requires a more involved theory with a much larger set of invariants than those considered here in general, as exemplified in the case of a transversely isotropic electroelastic material by Bustamante (2009).

# Chapter 4

# Bifurcation of Electroelastic Circular Cylinders

## 4.1 Introduction

In the previous chapter we studied in detail the problem of inflation and extension of a cylindrical circular electroelastic tube with closed ends with compliant electrodes at its curved boundaries. The obtained solution for this problem preserves the perfect cylindrical shape of the tube, although we know that inflation of a tube may lead sometimes to its bulging, for example, as it was discussed briefly in Introduction of this thesis. In order to capture these additional solutions (now for the more general case which accounts for electromechanical effects) we use the theory of small incremental deformations superimposed on a finitely deformed electroelastic body. The solutions represent curves which show for which values of circumferential stretch and axial stretch with fixed wall thicknesses and fixed electric parameters the configuration of the tube may become unstable and the tube may adopt a configuration which differs from perfect cylindrical shape. We start this chapter formulating stress components. In this chapter we use a slightly different formulation for stress components and we repeat this Section here with some appropriate changes. Note that in order to have equations consistent with Haughton & Ogden (1979) in this chapter we use a different order for the cylindrical polar coordinates and corresponding principal stretches. Therefore, with respect to the cylindrical polar coordinates $\theta$, $z$, $r$ and their respective counterparts in the reference configuration $\Theta$, $Z$, $R$ we have the following sequence of stretches

$$\lambda_1 = \lambda_\theta = \lambda, \quad \lambda_2 = \lambda_z, \quad \lambda_3 = \lambda_r. \tag{4.1}$$

Thus, the deformation gradient is diagonal $\mathrm{F} = \mathrm{diag}[\lambda, \lambda_z, \lambda_r]$. Also, changing this sequence affects expression (3.19), which for the present case changes to

$$[D_L] = \begin{bmatrix} 0 \\ 0 \\ \lambda\lambda_z D_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ D_{LR} \end{bmatrix}. \tag{4.2}$$

## 4.2 Stress components

Let us now consider $\Omega^*$ as a function of principal stretches $\lambda_1$, $\lambda_2$, $\lambda_3$ and electromechanical invariants $I_4$, $I_5$, $I_6$. Recognizing the fact that in the present problem the only electrical variable is $I_4$ we can consider a function $\hat{\Omega}^*$ such that

$$\hat{\Omega}^*(\lambda_1, \lambda_2, \lambda_3, I_4) = \Omega^*(\lambda_1, \lambda_2, \lambda_3, I_4, I_5, I_6). \tag{4.3}$$

This allows us to obtain simple expressions for principal components of the Cauchy stress tensor $\tau_{ii}$ ($i$=1, 2, 3)[1]

$$\tau_{ii} = \tau_i - p^* \quad (i = 1, 2, 3), \tag{4.4}$$

where

$$\tau_i = \lambda_i \frac{\partial \hat{\Omega}^*}{\partial \lambda_i} \quad (i = 1, 2, 3). \tag{4.5}$$

From the incompressibility condition (2.4) we can conclude that we have only two independent principal stretches. Therefore, we can express $\lambda_3$ in terms of $\lambda_1$ and $\lambda_2$ and we introduce a new function $w^*$, such that

$$\omega^*(\lambda_1, \lambda_2, I_4) = \hat{\Omega}^*(\lambda_1, \lambda_2, \lambda_3, I_4). \tag{4.6}$$

This allows us to write

$$\tau_{11} - \tau_{33} = \lambda\omega^*_\lambda, \quad \tau_{22} - \tau_{33} = \lambda_z\omega^*_{\lambda_z}, \tag{4.7}$$

where $\omega^*_\lambda$, $\omega^*_{\lambda_z}$ denote derivatives $\partial\omega^*/\partial\lambda$, $\partial\omega^*/\partial\lambda_z$.

Expression (3.23) can now be rewritten as

$$E_r = 2\lambda^2\lambda_z^2 \frac{\partial \omega^*}{\partial I_4} D_r. \tag{4.8}$$

According to Gauss's theorem, we have no field outside the tube, therefore by (2.11) the Maxwell stress is zero. Thus, we have only mechanical load due to a pressure $P$ inside the tube applied to the inner surface at $r = a$ and no loads at $r = b$

$$\tau_{rr} = -P \quad \text{on} \quad r = a, \quad \tau_{rr} = 0 \quad \text{on} \quad r = b. \tag{4.9}$$

[1] no summation for the subscript $i$.

In this problem the equilibrium equation div$\boldsymbol{\tau} = 0$ reduces to

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \tau_{\theta\theta} - \tau_{rr} = \lambda\omega^*_\lambda. \tag{4.10}$$

In the previous expression we have used (3.31). Integrating (3.36) and using the boundary conditions (3.35) we have

$$\int_{-P}^{0} \mathrm{d}\tau_{rr} = \int_a^b \lambda\omega^*_\lambda \frac{\mathrm{d}r}{r}. \tag{4.11}$$

Therefore,

$$P = \int_a^b \lambda\omega^*_\lambda \frac{\mathrm{d}r}{r}. \tag{4.12}$$

In some cases it is convenient to change the variable of integration from $r$ to $\lambda$. To this end, we rearrange and differentiate $(3.3)_1$ with respect to $r$, taking into account that $\lambda$ depends on $r$. We have

$$r\frac{\mathrm{d}\lambda}{\mathrm{d}r} = -\lambda(\lambda^2\lambda_z - 1). \tag{4.13}$$

Therefore, expression (3.38) can rewritten as

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1}\omega^*_\lambda \mathrm{d}\lambda. \tag{4.14}$$

From (3.4) we see that $\lambda_b$ depends on $\lambda_a$. Therefore, assuming that $\lambda_z$ is known, the previous relation gives $P$ as a function of $\lambda_a$ and invariant $I_4 = Q^2(a)/4\pi^2L^2A^2$, which is known for a given charge $Q(a) = -Q(b)$.

Similarly, since $b = \sqrt{a^2 + \lambda_z^{-1}(B^2 - A^2)}$ we see that (3.38) provides a relationship between pressure and the inner radius $a$ and invariant $I_4$.

## 4.3 Bifurcation analysis

In the present setting we use cylindrical polar coordinates $\theta$, $z$, $r$ with the corresponding unit basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Derivatives in (2.37) denoted by subscripts with commas $(\cdot)_{,k}$ can now be specified as $\partial(\cdot)/r\partial\theta$, $\partial/\partial z$, $\partial/\partial r$ for $k = 1, 2, 3$, respectively. For the cylindrical polar coordinates in (2.37) the only non-zero scalar products $\mathbf{e}_i \cdot \mathbf{e}_{j,k}$ are

$$\mathbf{e}_1 \cdot \mathbf{e}_{3,1} = -\mathbf{e}_3 \cdot \mathbf{e}_{1,1} = \frac{1}{r}. \tag{4.15}$$

The increment in the position vector $\mathbf{x}$ of a point in the current configuration is

$$\dot{\mathbf{x}} = v\mathbf{e}_1 + w\mathbf{e}_2 + u\mathbf{e}_3. \tag{4.16}$$

The components of $\mathbf{L}$ on the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ can be calculated as

$$\begin{bmatrix} (u+v_\theta)/r & v_z & v_r \\ w_\theta/r & w_z & w_r \\ (u_\theta - v)/r & u_z & u_r \end{bmatrix}, \tag{4.17}$$

where the subscripts $\theta$, $z$, $r$ are corresponding partial derivatives.

For an incompressible material we can write

$$\mathrm{tr}\,\mathbf{L} = (u+v_\theta)/r + w_z + u_r = 0. \tag{4.18}$$

### 4.3.1 Prismatic bifurcations

For prismatic bifurcations we assume that $u$, $v$ and $w$ are independent of $z$. Furthermore, we assume that $w = 0$, the justification of which will be mentioned later in this section. We will specialize here previous expressions.

The gradient of the deformation displacement vector $\dot{\mathbf{x}}$ will specialize to

$$\begin{bmatrix} (u+v_\theta)/r & 0 & v_r \\ 0 & 0 & 0 \\ (u_\theta - v)/r & 0 & u_r \end{bmatrix}, \tag{4.19}$$

Therefore, incompressibility condition reduces to

$$u + v_\theta + r u_r = 0. \tag{4.20}$$

Equation (4.20) is satisfied if we define function $\phi(\theta, r)$ such that

$$u = \frac{\phi_{,\theta}}{r}, \quad v = -\phi_{,r}. \tag{4.21}$$

For $i = 1, 3$ expression (2.37) gives respectively

$$\dot{T}_{011,1} + \dot{T}_{031,3} + \frac{1}{r}(\dot{T}_{031} + \dot{T}_{013}) = 0, \tag{4.22}$$

$$\dot{T}_{013,1} + \dot{T}_{033,3} + \frac{1}{r}(\dot{T}_{033} - \dot{T}_{011}) = 0. \tag{4.23}$$

In what follows we will consider the case when the electric field is generated by the electrodes attached to the boundaries of the hollow tube. Therefore, according to Gauss's theorem there is no field outside the material. For the considered underlying deformation, we have $F_{ij} = 0$ for $i \neq j$, and for radial electric displacement field $D_{L1} = D_{L2} = 0$ the required non-zero values of electroelastic moduli tensors $\boldsymbol{\mathcal{A}}_0^*$, $\mathbb{A}_0^*$, $\mathbf{A}_0^*$ can be obtained from the general expressions given in Section 2.2.2. Therefore, we can write using $(2.51)_1$

$$\dot{T}_{011} = \mathcal{A}^*_{01111}L_{11} + \mathcal{A}^*_{01133}L_{33} + pL_{11} - \dot{p} + \mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03}, \tag{4.24}$$

$$\dot{T}_{013} = \mathcal{A}^*_{01313}L_{31} + \mathcal{A}^*_{01331}L_{13} + pL_{13} + \mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01}, \tag{4.25}$$

$$\dot{T}_{031} = \mathcal{A}^*_{03131}L_{13} + \mathcal{A}^*_{03113}L_{31} + pL_{31}, \tag{4.26}$$

$$\dot{T}_{033} = \mathcal{A}^*_{03311}L_{11} + \mathcal{A}^*_{03333}L_{33} + pL_{33} - \dot{p} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03}. \tag{4.27}$$

Since there is no dependence on $z$, respective derivatives with respect to the variable $z$ are zero in (4.22) and (4.23).

Substituting these expressions into (4.22) and (4.23) and using incompressibility condition (4.20) more than once we find that (4.22) and (4.23) give respectively

$$\begin{aligned}\dot{p}_\theta =& [r(\mathcal{A}^{*'}_{03113} + p^{*'}) + \mathcal{A}^*_{01313}](u_\theta - v)/r + (r\mathcal{A}^{*'}_{03131} + \mathcal{A}^*_{03131})v_r + \\ &+ \mathcal{A}^*_{03131}rv_{rr} + (\mathcal{A}^*_{01331} + \mathcal{A}^*_{01133} - \mathcal{A}^*_{01111})u_{r\theta} + \mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03,\theta} + \mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01},\end{aligned} \tag{4.28}$$

$$\begin{aligned}\dot{p}_r =& [r(\mathcal{A}^{*'}_{03333} + p^{*'} - \mathcal{A}^{*'}_{01133}) + \mathcal{A}^*_{03333} + \mathcal{A}^*_{01111} - 2\mathcal{A}^*_{01133}]u_r/r + \\ &+ (\mathcal{A}^*_{03333} - \mathcal{A}^*_{01133})u_{rr} + \mathcal{A}^*_{01313}(u_{\theta\theta} - v_\theta)/r^2 + \mathcal{A}^*_{01331}v_{r\theta}/r + \\ &+ \mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01,\theta}/r + \mathbb{A}^{*'}_{033|3}\dot{D}_{\mathrm{L}03} + \mathbb{A}^*_{033|3}(\dot{D}_{\mathrm{L}03,r} + \dot{D}_{\mathrm{L}03}/r) - \mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03}/r,\end{aligned} \tag{4.29}$$

where prime denotes differentiation with respect to $r$.

In the beginning of this section we assumed that $w = 0$. Without this assumption calculations shows that we can still obtain expressions (4.28) and (4.29). For $i = 2$ from (2.37) we get

$$\dot{T}_{012,1} + \dot{T}_{032,3} + \frac{1}{r}\dot{T}_{032} = 0, \tag{4.30}$$

where derivative with respect to $z$ again was omitted for prismatic case. The expressions for the other terms are

$$\dot{T}_{012} = \mathcal{A}^*_{01212}L_{21}, \quad \dot{T}_{032} = \mathcal{A}^*_{03232}L_{23}. \tag{4.31}$$

Therefore, (4.30) gives uncoupled equation for $w$

$$\mathcal{A}^*_{01212}w_{\theta\theta}/r + \mathcal{A}^{*'}_{03232}rw_r + \mathcal{A}^*_{03232}(rw_{rr} + w_r) = 0, \tag{4.32}$$

which can be solved for $w$. This solution does not affect the shape of a cross-section of a cylinder, and therefore, we may set it equal to zero, which was done in the beginning of this section.

For the present case the governing equation $(2.32)_1$ reduces to

$$\frac{\partial(r\dot{E}_{\mathrm{L}0\theta})}{\partial r} - \frac{\partial \dot{E}_{\mathrm{L}0r}}{\partial\theta} = 0. \tag{4.33}$$

From $(2.51)_2$ we calculate

$$\dot{E}_{\mathrm{L}0\theta} = \dot{E}_{\mathrm{L}01} = \mathbb{A}^*_{013|1}L_{31} + \mathsf{A}^*_{011}\dot{D}_{\mathrm{L}01}, \tag{4.34}$$

$$\dot{E}_{\mathrm{L}0r} = \dot{E}_{\mathrm{L}03} = \mathbb{A}^*_{011|3}L_{11} + \mathbb{A}^*_{033|3}L_{33} + \mathsf{A}^*_{033}\dot{D}_{\mathrm{L}03}. \tag{4.35}$$

Therefore, equation (4.33) gives

$$\begin{aligned}&\mathbb{A}^*_{013|1}\frac{u_\theta - v}{r} + \mathsf{A}^*_{011}\dot{D}_{\mathrm{L}01} + \mathbb{A}^{*\prime}_{013|1}(u_\theta - v) + \mathbb{A}^*_{013|1}\frac{(u_{\theta r} - v_r)r - u_\theta + v}{r}\\ &+ \mathsf{A}^{*\prime}_{011}r\dot{D}_{\mathrm{L}01} + \mathsf{A}^*_{011}r\dot{D}_{\mathrm{L}01,r} - \mathbb{A}^*_{011|3}\frac{u_\theta + v_{\theta\theta}}{r} - \mathbb{A}^*_{033|3}u_{r\theta} - \mathsf{A}^*_{033}\dot{D}_{\mathrm{L}03,\theta} = 0.\end{aligned} \tag{4.36}$$

The governing equation $(2.32)_2$ reduces to

$$\frac{\partial(r\dot{D}_{\mathrm{L}0r})}{\partial r} + \frac{\partial \dot{D}_{\mathrm{L}0\theta}}{\partial\theta} = 0. \tag{4.37}$$

The previous equation will be satisfied if we define a function $\psi(\theta, r)$ such that

$$\dot{D}_{\mathrm{L}0r} = \frac{\psi_{,\theta}}{r}, \quad \dot{D}_{\mathrm{L}0\theta} = -\psi_{,r}. \tag{4.38}$$

Eliminating $\dot{p}_r$ and $\dot{p}_\theta$ in (5.49) and (5.52) by cross-differentiation and using expressions (4.21) and (4.38), after some rearrangements, we can get two coupled equations for $\phi$ and

$\psi$:

$$
\begin{aligned}
&r^4\mathcal{A}^*_{03131}\phi_{,rrrr} + \mathcal{A}^*_{01313}\phi_{,\theta\theta\theta\theta} \\
&-(2\mathcal{A}^*_{01331} + 2\mathcal{A}^*_{01133} - \mathcal{A}^*_{01111} - \mathcal{A}^*_{03333})r^2\phi_{,\theta\theta rr} \\
&+(2r^4\mathcal{A}^{*'}_{03131} + 2r^3\mathcal{A}^*_{03131})\phi_{,rrr} \\
&-\big[(\mathcal{A}^*_{03333} + \mathcal{A}^*_{01111} - 2\mathcal{A}^*_{01331} - 2\mathcal{A}^*_{01133})r \\
&\quad + (2\mathcal{A}^{*'}_{01331} + 2\mathcal{A}^{*'}_{01133} - \mathcal{A}^{*'}_{01111} - \mathcal{A}^{*'}_{03333})r^2\big]\phi_{,\theta\theta r} \\
&-\big[2\mathcal{A}^*_{01331} - \mathcal{A}^*_{01111} - \mathcal{A}^*_{03333} + 2\mathcal{A}^*_{01133} - 2\mathcal{A}^*_{01313} \\
&\quad + (\mathcal{A}^{*'}_{01313} - 2\mathcal{A}^{*'}_{01331} - 2\mathcal{A}^{*'}_{01133} + \mathcal{A}^{*'}_{01111} + \mathcal{A}^{*'}_{03333})r \\
&\quad + (\mathcal{A}^{*''}_{01331} + p^{*''})r^2\big]\phi_{,\theta\theta} \\
&-\big[\mathcal{A}^{*'}_{01331} + p^{*'} + \mathcal{A}^*_{01313}/r - 2\mathcal{A}^{*'}_{03131} - r\mathcal{A}^{*''}_{03131}\big]r^3\phi_{,rr} \\
&-\big[\mathcal{A}^{*''}_{01331} + p^{*''} + \mathcal{A}^{*'}_{01313}/r - \mathcal{A}^*_{01313}/r^2\big]r^3\phi_{,r} \\
&-(\mathbb{A}^*_{011|3} + \mathbb{A}^*_{013|1} - \mathbb{A}^*_{033|3})r^2\psi_{,\theta\theta r} - (\mathbb{A}^{*'}_{011|3} - \mathbb{A}^{*'}_{033|3})r^2\psi_{,\theta\theta} \\
&+\mathbb{A}^*_{013|1}r^3\psi_{,rr} + \mathbb{A}^{*'}_{013|1}r^3\psi_{,r} = 0,
\end{aligned}
\tag{4.39}
$$

$$
\begin{aligned}
&(\mathbb{A}^*_{013|1} + \mathbb{A}^*_{011|3} - \mathbb{A}^*_{033|3})r\phi_{,\theta\theta r} + (\mathbb{A}^*_{033|3} - \mathbb{A}^*_{013|1} - \mathbb{A}^*_{011|3} + \mathbb{A}^{*'}_{013|1}r)\phi_{,\theta\theta} \\
&+\mathbb{A}^*_{013|1}r^2\phi_{,rr} + \mathbb{A}^{*'}_{013|1}r^2\phi_{,r} - r^3\mathsf{A}^*_{011}\psi_{,rr} - \mathsf{A}^*_{033}r\psi_{,\theta\theta} - (\mathsf{A}^*_{011} + r\mathsf{A}^{*'}_{011})r^2\psi_{,r} = 0.
\end{aligned}
\tag{4.40}
$$

Equation (4.40) was obtained from (4.36).

Now we will specialize the boundary condition (2.35). Since for the present case when electric field is generated by electrodes there is no field outside the material. We have

$$
\dot{\mathbf{T}}_0^{\mathrm{T}}\mathbf{n} = \dot{\mathbf{t}}_{\mathrm{A}0} = \begin{cases} P\mathbf{L}^{\mathrm{T}}\mathbf{n} - \dot{P}\mathbf{n} & \text{on } r = a, \\ 0 & \text{on } r = b. \end{cases} \tag{4.41}
$$

Calculations show that

$$
rv_r + u_\theta - v = 0 \quad \text{on} \quad r = a, b. \tag{4.42}
$$

$$
(\mathcal{A}^*_{03333} - \mathcal{A}^*_{03311} + \tau_3)u_r - \dot{p} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03} = \begin{cases} -\dot{P} & \text{on } r = a, \\ 0 & \text{on } r = b. \end{cases} \tag{4.43}
$$

The third component of (4.41) is satisfied automatically.

Boundary conditions (4.42) and (4.43) can be written in terms of functions $\phi$ and $\psi$:

$$
r^2\phi_{,rr} - r\phi_{,r} - \phi_{,\theta\theta} = 0 \quad \text{on} \quad r = a, b, \tag{4.44}
$$

$$
\begin{aligned}
& r(\mathcal{A}^*_{03333} - 2\mathcal{A}^*_{01133} - 2\mathcal{A}^*_{01331} + \mathcal{A}^*_{03131} + \mathcal{A}^*_{01111})\phi_{,\theta\theta r} + r^3\mathcal{A}^*_{03131}\phi_{,rrr} \\
& - (r\mathcal{A}^{*'}_{03131} + \mathcal{A}^*_{03333} - 2\mathcal{A}^*_{01133} - 2\mathcal{A}^*_{01331} + 2\mathcal{A}^*_{03131} + \mathcal{A}^*_{01111})\phi_{,\theta\theta} \\
& + r^2(r\mathcal{A}^{*'}_{03131} + \mathcal{A}^*_{03131})\phi_{,rr} - r(r\mathcal{A}^{*'}_{03131} + \mathcal{A}^*_{03131})\phi_{,r} \\
& + r(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{011|3})\psi_{,\theta\theta} + r^2\mathbb{A}^*_{013|1}\psi_{,r} = 0 \quad \text{on} \quad r = a, b.
\end{aligned} \tag{4.45}
$$

In order to obtain (4.45) we differentiated (4.43) with respect to $\theta$, set $\dot{P} = 0$ and used (4.28).

In order to have equations consistent with Haughton & Ogden (1979) we write

$$
\phi = r f_n(r)\sin n\theta \quad \text{and} \quad \psi = g_n(r)\sin n\theta. \tag{4.46}
$$

The governing equations (4.39) and (4.40) are now can be expressed in terms of functions $f_n(r)$ and $g_n(r)$ and their derivatives

$$
\begin{aligned}
& r\{\mathcal{A}^*_{03131} r^3 f'''_n + (r\mathcal{A}^{*'}_{03131} + 3\mathcal{A}^*_{03131})r^2 f''_n \\
& + [r\mathcal{A}^{*'}_{03131} - \mathcal{A}^*_{03131} + n^2(2\mathcal{A}^*_{01331} + 2\mathcal{A}^*_{01133} - \mathcal{A}^*_{01111} - \mathcal{A}^*_{03333})]r f'_n\}' \\
& + (n^2-1)[r^2\mathcal{A}^{*''}_{03131} + r\mathcal{A}^{*'}_{03131} + (n^2-1)\mathcal{A}^*_{03131} + n^2(\tau_1 - \tau_3)]f_n \\
& + \mathbb{A}^*_{013|1} r^2 g''_n + \mathbb{A}^{*'}_{013|1} r^2 g'_n + (\mathbb{A}^*_{011|3} + \mathbb{A}^*_{013|1} - \mathbb{A}^*_{033|3}) r n^2 g'_n \\
& + (\mathbb{A}^{*'}_{011|3} - \mathbb{A}^{*'}_{033|3}) r n^2 g_n = 0,
\end{aligned} \tag{4.47}
$$

$$
\begin{aligned}
& r^2\mathbb{A}^*_{013|1} f''_n + [r^2\mathbb{A}^{*'}_{013|1} + 2r\mathbb{A}^*_{013|1} - r n^2(\mathbb{A}^*_{013|1} + \mathbb{A}^*_{011|3} - \mathbb{A}^*_{033|3})]f'_n \\
& + r(\mathbb{A}^{*'}_{013|1} - n^2\mathbb{A}^{*'}_{013|1})f_n - r^2\mathsf{A}^*_{011} g''_n - (\mathsf{A}^*_{011} + r\mathsf{A}^{*'}_{011}) r g'_n + \mathsf{A}^*_{033} n^2 g_n = 0.
\end{aligned} \tag{4.48}
$$

In the governing equation (4.47) we have used the connection

$$
p^{*''} = \mathcal{A}^{*''}_{03131} - \mathcal{A}^{*''}_{01331} - (\mathcal{A}^*_{03131} - \mathcal{A}^*_{01313})/r^2 + (\mathcal{A}^{*'}_{03131} - \mathcal{A}^{*'}_{01313})/r, \tag{4.49}
$$

which can be obtained from (3.36), (4.4) and (2.57).

Boundary conditions (4.44) and (4.45) can be rewritten as

$$
r^2 f''_n + r f'_n + (n^2-1)f_n = 0 \quad \text{on} \quad r = a, b, \tag{4.50}
$$

$$
\begin{aligned}
& \mathcal{A}^*_{03131} r^3 f'''_n + (r\mathcal{A}^{*'}_{03131} + 4\mathcal{A}^*_{03131})r^2 f''_n \\
& + [r\mathcal{A}^{*'}_{03131} - (n^2-1)\mathcal{A}^*_{03131} + n^2(2\mathcal{A}^*_{01331} + 2\mathcal{A}^*_{01133} - \mathcal{A}^*_{01111} - \mathcal{A}^*_{03333})]r f'_n \\
& + (n^2-1)(r\mathcal{A}^{*'}_{03131} + \mathcal{A}^*_{03131})f_n - n^2(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{011|3})g_n \\
& + \mathbb{A}^*_{013|1} r g'_n = 0 \quad \text{on} \quad r = a, b.
\end{aligned} \tag{4.51}
$$

Electrical boundary conditions (2.33) reduce to

$$\dot{E}_{\mathrm{L}01} = \dot{E}_{\mathrm{L}0\theta} = 0 \quad \text{on} \quad r = a, b. \tag{4.52}$$

Boundary conditions (4.52) can be written in terms of functions $\phi$ and $\psi$ and $f_n$ and $g_n$, respectively, as

$$\mathbb{A}^*_{013|1}(\phi_{,\theta\theta} + r\phi_{,r})/r^2 - \mathsf{A}^*_{011}\psi_{,r} = 0 \quad \text{on} \quad r = a, b, \tag{4.53}$$

$$\mathbb{A}^*_{013|1}[rf'_n + (1-n^2)f_n] - \mathsf{A}^*_{011}rg'_n = 0 \quad \text{on} \quad r = a, b. \tag{4.54}$$

Boundary conditions (2.34) reduce to

$$\dot{D}_{\mathrm{L}0r} = \begin{cases} -\dot{\sigma}_{\mathrm{F}0b} & \text{on } r = b, \\ \dot{\sigma}_{\mathrm{F}0a} & \text{on } r = a, \end{cases} \tag{4.55}$$

where $\dot{\sigma}_{\mathrm{F}0} = \dot{\sigma}_{\mathrm{F}}\mathrm{d}S/\mathrm{d}s$ is the increment of the free surface charge $\sigma_{\mathrm{F}}$ per unit area of $\partial\mathcal{B}$, and $\mathrm{d}S/\mathrm{d}s$ is the ratio of area elements in $\partial\mathcal{B}_r$ and $\partial\mathcal{B}$. For the considered problem free surface charges at the boundaries per unit area are different by the absolute value (and sign, of course). Therefore, in general increments will be also different at the boundaries. Thus, we can write

$$\dot{\sigma}_{\mathrm{F}0a} = \dot{\sigma}_{\mathrm{F}}|_{r=a}\frac{\mathrm{d}S}{\mathrm{d}s} = \dot{\sigma}_{\mathrm{F}}|_{r=a}\frac{A}{a}\lambda_z^{-1} = \dot{\sigma}_{\mathrm{F}}|_{r=a}\lambda_a^{-1}\lambda_z^{-1} \tag{4.56}$$

at the inner boundary, and

$$\dot{\sigma}_{\mathrm{F}0b} = \dot{\sigma}_{\mathrm{F}}|_{r=b}\frac{\mathrm{d}S}{\mathrm{d}s} = \dot{\sigma}_{\mathrm{F}}|_{r=b}\frac{B}{b}\lambda_z^{-1} = \dot{\sigma}_{\mathrm{F}}|_{r=a}\lambda_b^{-1}\lambda_z^{-1} \tag{4.57}$$

at the outer boundary.

In the present and following chapter we assume that boundary condition (2.34) is satisfied implicitly, so to speak, and we do not use it directly in our calculations. Since $\dot{D}_{\mathrm{L}0r}$ was defined as $\dot{D}_{\mathrm{L}0r} = \psi_{,\theta}/r$ the solution will lead to function $\psi(\theta, r)$ defined at the boundaries $r = a, b$. Therefore, boundary condition (4.55) will be adjusted according to the solution for function $\psi$. The same approach for incremental electric boundary conditions was used in Dorfmann & Ogden (2014a).

In what follows we give non-dimensional equations. Relations (4.21) and $(4.46)_1$ suggest that non-dimensional function is defined as

$$\hat{f}(\hat{r}) = \frac{f_n(r)}{A}. \tag{4.58}$$

Also, relations (4.38) and $(4.46)_2$ imply that

$$\hat{g}_n(\hat{r}) = \frac{g_n(r)}{D_r(a)A}. \tag{4.59}$$

The other non-dimensional quantities are defined as they are for axisymmetric bifurcations. Axisymmetric bifurcations will be discussed in the next section of this chapter. We also rearrange the governing equation (4.47) to make it more suitable for MATLAB (2014) and we introduce new variables

$$\begin{aligned} &\hat{y}_1(\hat{r}) = \hat{f}_n(\hat{r}), \quad \hat{y}_2(\hat{r}) = \hat{f}_n'(\hat{r}), \quad \hat{y}_3(\hat{r}) = \hat{f}_n''(\hat{r}), \\ &\hat{y}_4(\hat{r}) = \hat{f}_n'''(\hat{r}), \quad \hat{y}_5(\hat{r}) = \hat{g}_n(\hat{r}), \quad \hat{y}_6(\hat{r}) = \hat{g}_n'(\hat{r}), \end{aligned} \tag{4.60}$$

so that the governing equations (4.47) and (4.48) can be rewritten as a non-dimensional system of 6 ordinary differential equations. The result of this manipulation is as follows

$$\begin{aligned}
&\hat{y}_1' = \hat{y}_2, \\
&\hat{y}_2' = \hat{y}_3, \\
&\hat{y}_3' = \hat{y}_4, \\
&\hat{\mathcal{A}}^*_{03131}\hat{r}^4\hat{y}_4' + (6\hat{r}^3\hat{\mathcal{A}}^*_{03131} + 2\hat{r}^4\hat{\mathcal{A}}^{*\prime}_{03131})\hat{y}_4 + \{7\hat{r}^3\hat{\mathcal{A}}^{*\prime}_{03131} + \hat{r}^4\hat{\mathcal{A}}^{*\prime\prime}_{03131} + 5\hat{r}^2\hat{\mathcal{A}}^*_{03131} + n^2\hat{r}^2\hat{Q}(\hat{r})\}\hat{y}_3 + \\
&(\hat{r}^3\hat{\mathcal{A}}^{*\prime\prime}_{03131} + n^2\hat{r}^2\hat{Q}'(\hat{r}) + \hat{r}^2\hat{\mathcal{A}}^{*\prime}_{03131} - \hat{r}\hat{\mathcal{A}}^*_{03131} + n^2\hat{r}\hat{Q}(\hat{r}))\hat{y}_2 + (n^2-1)(\hat{r}^2\hat{\mathcal{A}}^{*\prime\prime}_{03131} + \hat{r}\hat{\mathcal{A}}^{*\prime}_{03131} + \\
&(n^2-1)\hat{\mathcal{A}}^*_{03131} + n^2(\hat{\tau}_1 - \hat{\tau}_3))\hat{y}_1 + \hat{\mathbb{A}}^*_{013|1}\hat{r}^2\hat{\sigma}^2_{fa}\hat{y}_6' + \{\hat{\mathbb{A}}^{*\prime}_{013|1}\hat{r}^2 + (\hat{\mathbb{A}}^*_{011|3} + \hat{\mathbb{A}}^*_{013|1} - \hat{\mathbb{A}}^*_{033|3})\hat{r}n^2\}\hat{\sigma}^2_{fa}\hat{y}_6 + \\
&(\hat{\mathbb{A}}^{*\prime}_{011|3} - \hat{\mathbb{A}}^{*\prime}_{033|3})\hat{r}n^2\hat{\sigma}^2_{fa}\hat{y}_5 = 0, \\
&\hat{y}_5' = \hat{y}_6, \\
&\hat{r}^2\hat{\mathbb{A}}^*_{013|1}\hat{y}_3 + [\hat{r}^2\hat{\mathbb{A}}^{*\prime}_{013|1} + 2\hat{r}\hat{\mathbb{A}}^*_{013|1} - \hat{r}n^2(\hat{\mathbb{A}}^*_{013|1} + \hat{\mathbb{A}}^*_{011|3} - \hat{\mathbb{A}}^*_{033|3})]\hat{y}_2 \\
&+ \hat{r}(\hat{\mathbb{A}}^{*\prime}_{013|1} - n^2\hat{\mathbb{A}}^{*\prime}_{013|1})\hat{y}_1 - \hat{r}^2\hat{\mathsf{A}}^*_{011}\hat{y}_6' - (\hat{\mathsf{A}}^*_{011} + \hat{r}\hat{\mathsf{A}}^{*\prime}_{011})\hat{r}\hat{y}_6 \\
&+ \hat{\mathsf{A}}^*_{033}n^2\hat{y}_5 = 0,
\end{aligned} \tag{4.61}$$

where for brevity we defined function

$$\hat{Q}(\hat{r}) = 2\hat{\mathcal{A}}^*_{01331} + 2\hat{\mathcal{A}}^*_{01133} - \hat{\mathcal{A}}^*_{01111} - \hat{\mathcal{A}}^*_{03333}, \tag{4.62}$$

and non-dimensional electric parameter was defined as

$$\hat{\sigma}^2_{fa} = \frac{D_r^2(a)}{\varepsilon\mu}. \tag{4.63}$$

Also, we non-dimensionalize the boundary conditions (4.50), (4.51) and (4.54) and express them in terms of new variables

$$\hat{r}^2\hat{y}_3 + \hat{r}\hat{y}_2 + (n^2-1)\hat{y}_1 = 0 \quad \text{on} \quad \hat{r} = \hat{a}, \hat{b}, \tag{4.64}$$

$$\hat{\mathcal{A}}^*_{03131}\hat{r}^3\hat{y}_4 + (\hat{r}\hat{\mathcal{A}}^{*\prime}_{03131} + 4\hat{\mathcal{A}}^*_{03131})\hat{r}^2\hat{y}_3 + \tag{4.65}$$
$$[\hat{r}\hat{\mathcal{A}}^{*\prime}_{03131} - (n^2-1)\hat{\mathcal{A}}^*_{03131} + n^2(2\hat{\mathcal{A}}^*_{01331} + 2\hat{\mathcal{A}}^*_{01133} - \hat{\mathcal{A}}^*_{01111} - \hat{\mathcal{A}}^*_{03333})]\hat{r}\hat{y}_2 +$$
$$+ (n^2-1)(\hat{r}\hat{\mathcal{A}}^{*\prime}_{03131} + \hat{\mathcal{A}}^*_{03131})\hat{y}_1 - n^2(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{011|3})\hat{\sigma}^2_{fa}\hat{y}_5 +$$
$$+ \hat{\mathbb{A}}^*_{013|1}\hat{r}\hat{\sigma}^2_{fa}\hat{y}_6 = 0 \quad \text{on} \quad \hat{r} = \hat{a}, \hat{b},$$

$$\hat{\mathbb{A}}^*_{013|1}[\hat{r}\hat{y}_2 + (1-n^2)\hat{y}_1] - \hat{\mathsf{A}}^*_{011}\hat{r}\hat{y}_6 = 0 \quad \text{on} \quad \hat{r} = \hat{a}, \hat{b}. \tag{4.66}$$

We use the neo-Hookean electroelastic material (3.46) and we calculate electroelastic moduli for this material:

$$\begin{aligned}
&\mathcal{A}^*_{03131} = \mathcal{A}^*_{03333} = 2\lambda_3^2\Omega_1^* + 2D_3^2\Omega_5^*, \\
&\mathcal{A}^*_{01313} = \mathcal{A}^*_{01111} = 2\lambda_1^2\Omega_1^*, \\
&\mathcal{A}^*_{01331} = \mathcal{A}^*_{01133} = 0, \\
&2\mathbb{A}^*_{013|1} = \mathbb{A}^*_{033|3} = 4D_3\Omega_5^*, \quad \mathbb{A}^*_{011|3} = 0, \\
&\mathsf{A}^*_{011} = \mathsf{A}^*_{033} = 2\Omega_5^*, \quad \tau_1 - \tau_3 = \mathcal{A}^*_{01313} - \mathcal{A}^*_{03131}.
\end{aligned} \tag{4.67}$$

In non-dimensional form these moduli can be expressed as functions of $\hat{r}$ as below

$$\begin{aligned}
&\hat{\mathcal{A}}^*_{03131} = \hat{\mathcal{A}}^*_{03333} = \frac{\lambda_z^{-1}(\hat{r}^2A^2 - a^2) + \lambda_z^{-2}A^2 + \hat{\sigma}^2_{fa}a^2}{\hat{r}^2A^2}, \\
&\hat{\mathcal{A}}^*_{01313} = \hat{\mathcal{A}}^*_{01111} = \frac{\hat{r}^2A^2}{\lambda_z(\hat{r}^2A^2 - a^2) + A^2}, \\
&\hat{\mathcal{A}}^*_{01331} = \hat{\mathcal{A}}^*_{01133} = 0, \\
&2\hat{\mathbb{A}}^*_{013|1} = \hat{\mathbb{A}}^*_{033|3} = \frac{2a}{\hat{r}A}, \quad \hat{\mathbb{A}}^*_{011|3} = 0, \\
&\hat{\mathsf{A}}^*_{011} = \hat{\mathsf{A}}^*_{033} = 1, \quad \hat{\tau}_1 - \hat{\tau}_3 = \hat{\mathcal{A}}^*_{01313} - \hat{\mathcal{A}}^*_{03131}.
\end{aligned} \tag{4.68}$$

The results of our calculations are given in Table 4.1.

We used the electroelastic neo-Hookean model (3.46) and we set $\lambda_z = 1$ for all cases. We were changing electroelastic parameter $\hat{\sigma}_{fa}$ and we calculated the values of $\lambda_a$ (and hence $\lambda_b$) at which prismatic bifurcations become possible for mode number $n = 2$. The numerical scheme for this calculation is described in Section 4.3.2. First, we note that the results for neo-Hookean electroelastic material with $\hat{\sigma}_{fa} = 0$ are almost identical to those reported in Haughton & Ogden (1979) for Three-term energy function. Haughton & Ogden (1979) also confirmed that under external pressure the values of $\lambda_b$ (or equivalently $\lambda_a$) remain almost the same for many strain energy functions for pure elastic materials, i.e. essentially they do not depend on a particular form of energy function.

Table 4.1: Bifurcation values $\lambda_a$, $\lambda_b$ and non-dimensional pressure $P/\mu$ for neo-Hookean electroelastic material for different values of parameter $\hat{\sigma}_{fa}$ and different values $A/B$; $\lambda_z = 1$, mode number $n = 2$ in all cases. Also, for comparison bifurcation values of $\lambda_b$ from Haughton & Ogden (1979) for three-term strain-energy function are given here.

| | | neo-Hookean electroelastic material (4.117) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\sigma}_{fa} = 0$ | | | $\hat{\sigma}_{fa} = 0.25$ | | | $\hat{\sigma}_{fa} = 0.5$ | | | $\hat{\sigma}_{fa} = 0.75$ | | |
| $A/B$ | $\lambda_b$ [a] | $\lambda_a$ | $\lambda_b$ | $P/\mu$ | $\lambda_a$ | $\lambda_b$ | $P/\mu$ | $\lambda_a$ | $\lambda_b$ | $P/\mu$ | $\lambda_a$ | $\lambda_b$ | $P/\mu$ |
| 0.95 | 0.999 | 0.9993 | 0.999 | -0.000 | 1.0651 | 1.058 | 0.008 | 1.3013 | 1.275 | 0.025 | 1.7789 | 1.718 | 0.036 |
| 0.9 | 0.997 | 0.9969 | 0.997 | -0.001 | 1.0628 | 1.051 | 0.155 | 1.3032 | 1.251 | 0.050 | 1.8014 | 1.678 | 0.074 |
| 0.85 | 0.994 | 0.9923 | 0.994 | -0.004 | 1.0581 | 1.042 | 0.020 | 1.3029 | 1.226 | 0.075 | 1.8248 | 1.638 | 0.115 |
| 0.8 | 0.990 | 0.9848 | 0.990 | -0.011 | 1.0503 | 1.032 | 0.022 | 1.2992 | 1.200 | 0.099 | 1.8488 | 1.596 | 0.157 |
| 0.7 | 0.979 | 0.9576 | 0.979 | -0.046 | 1.0211 | 1.010 | 0.005 | 1.2761 | 1.143 | 0.136 | 1.8963 | 1.507 | 0.247 |
| 0.6 | 0.968 | 0.9066 | 0.967 | -0.139 | 0.9647 | 0.987 | -0.068 | 1.2148 | 1.082 | 0.135 | 1.9339 | 1.409 | 0.343 |

[a] Data for three-term model from Haughton & Ogden (1979)

Second, an important difference for the present electroelastic material is that unlike the case for pure elastic materials the present example shows that due to electric field electrically sensitive elastic material may bifurcate into a prismatic configuration under internal pressure ($P > 0$). In Haughton & Ogden (1979) it was reported that prismatic modes are possible for neo-Hookean and Three-term pure elastic materials under external pressure only ($P < 0$), this can also be observed here in Table 4.1 for the case $\hat{\sigma}_{fa} = 0$: all non-dimensional pressures at which prismatic bifurcations are possible are negative. The values $P/\mu$ in Table 4.1 were calculated using formula (3.53) and connection $q = \hat{\sigma}_{fa}^2 \varepsilon \mu a^2 \lambda_z^2$.

### 4.3.2 Axisymmetric bifurcations

For axisymmetric bifurcations we assume that $u$, $v$ and $w$ are independent of $\theta$. Furthermore, we assume that $v = 0$.

The gradient of the deformation displacement vector $\dot{\mathbf{x}}$ will specialize to

$$\begin{bmatrix} u/r & 0 & 0 \\ 0 & w_z & w_r \\ 0 & u_z & u_r \end{bmatrix}. \tag{4.69}$$

Therefore, incompressibility condition will reduce to

$$u/r + w_z + u_r = 0. \tag{4.70}$$

Equation (4.70) is satisfied if we define function $\phi(z, r)$ such that

$$u = \frac{\phi_{,z}}{r}, \quad w = -\frac{\phi_{,r}}{r}. \tag{4.71}$$

For the axisymmetric motions expression (2.37) gives for $i = 3, 2$, respectively

$$\dot{T}_{023,2} + \dot{T}_{033,3} + \frac{1}{r}(\dot{T}_{033} - \dot{T}_{011}) = 0, \tag{4.72}$$

$$\dot{T}_{022,2} + \dot{T}_{032,3} + \frac{1}{r}\dot{T}_{032} = 0. \tag{4.73}$$

In (4.72) and (4.73) derivatives with respect to $\theta$ were omitted. Calculating from $(2.51)_1$ we have

$$\dot{T}_{023} = \mathcal{A}^*_{02323} L_{32} + \mathcal{A}^*_{02332} L_{23} + pL_{23} + \mathbb{A}^*_{023|2} \dot{D}_{\mathrm{L}02} \tag{4.74}$$

$$\dot{T}_{032} = \mathcal{A}^*_{03232} L_{23} + \mathcal{A}^*_{03223} L_{32} + pL_{32}. \tag{4.75}$$

$$\dot{T}_{011} = \mathcal{A}^*_{01111} L_{11} + \mathcal{A}^*_{01122} L_{22} + \mathcal{A}^*_{01133} L_{33} + pL_{11} - \dot{p} + \mathbb{A}^*_{011|3} \dot{D}_{\mathrm{L}03}, \tag{4.76}$$

$$\dot{T}_{022} = \mathcal{A}^*_{02211}L_{11} + \mathcal{A}^*_{02222}L_{22} + \mathcal{A}^*_{02233}L_{33} + pL_{22} - \dot{p} + \mathbb{A}^*_{022|3}\dot{D}_{\mathrm{L}03}, \tag{4.77}$$

$$\dot{T}_{033} = \mathcal{A}^*_{03311}L_{11} + \mathcal{A}^*_{03322}L_{22} + \mathcal{A}^*_{03333}L_{33} + pL_{33} - \dot{p} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03}. \tag{4.78}$$

Substitution of the previous expressions in (4.72) and (4.73) and use of (4.70) give respectively

$$\begin{aligned}\dot{p}_r =&(r\mathcal{A}^{*'}_{01133} - \mathcal{A}^*_{01111})u/r^2 + (r\mathcal{A}^{*'}_{03333} + rp^{*'} + \mathcal{A}^*_{03333})u_r/r \\ &+ \mathcal{A}^*_{03333}u_{rr} + \mathcal{A}^*_{02323}u_{zz} + (r\mathcal{A}^{*'}_{02233} + \mathcal{A}^*_{02233} - \mathcal{A}^*_{01122})w_z/r \\ &+ (\mathcal{A}^*_{02233} + \mathcal{A}^*_{03223})w_{rz} + \mathbb{A}^*_{023|2}\dot{D}_{\mathrm{L}02,z} \\ &+ \mathbb{A}^{*'}_{033|3}\dot{D}_{\mathrm{L}03} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03,r} + (\mathbb{A}^*_{033|3} - \mathbb{A}^*_{011|3})\dot{D}_{\mathrm{L}03}/r,\end{aligned} \tag{4.79}$$

$$\begin{aligned}\dot{p}_z =&\mathcal{A}^*_{03232}w_{rr} + (r\mathcal{A}^{*'}_{03232} + \mathcal{A}^*_{03232})w_r/r + \mathcal{A}^*_{02222}w_{zz} + (\mathcal{A}^*_{02233} + \mathcal{A}^*_{03223})u_{rz} \\ &+ (r\mathcal{A}^{*'}_{03223} + rp^{*'} + \mathcal{A}^*_{03223} + \mathcal{A}^*_{01122})u_z/r + \mathbb{A}^*_{022|3}\dot{D}_{\mathrm{L}03,z}.\end{aligned} \tag{4.80}$$

For $i = 1$ we have

$$\dot{T}_{021,2} + \dot{T}_{031,3} + \frac{1}{r}(\dot{T}_{031} + \dot{T}_{013}) = 0, \tag{4.81}$$

where again the derivative with respect to $\theta$ was omitted. The other terms will be

$$\dot{T}_{021} = \mathcal{A}^*_{02121}L_{12}, \tag{4.82}$$

$$\dot{T}_{031} = \mathcal{A}^*_{03131}L_{13} + \mathcal{A}^*_{03113}L_{31} + pL_{31}, \tag{4.83}$$

$$\dot{T}_{013} = \mathcal{A}^*_{01313}L_{31} + \mathcal{A}^*_{01331}L_{13} + pL_{13} + \mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01}. \tag{4.84}$$

Note that we assume that $\dot{D}_{L0\theta} = \dot{D}_{L01} = 0$. Substitution of the previous expressions into (4.81) and use of (4.70) give

$$(r\mathcal{A}^{*'}_{03131} + \mathcal{A}^*_{03131})(rv_r - v)/r^2 + \mathcal{A}^*_{02121}v_{zz} + \mathcal{A}^*_{03131}v_{rr} = 0, \tag{4.85}$$

which is satisfied, since we assumed that $v = 0$. Non-zero solutions of (4.85) are of little interest, and we set $v = 0$. In (4.85) we have used connection

$$p^{*'} = \mathcal{A}^{*'}_{03131} - \mathcal{A}^{*'}_{01331} + (\mathcal{A}^*_{03131} - \mathcal{A}^*_{01313})/r, \tag{4.86}$$

which can be obtained from the equilibrium equation (3.36), relation (4.4) and connections for electroelastic moduli (2.57).

The governing equation $(2.32)_1$ reduces to

$$\frac{\partial \dot{E}_{\mathrm{L}0r}}{\partial z} - \frac{\partial \dot{E}_{\mathrm{L}0z}}{\partial r} = 0. \tag{4.87}$$

Using $(2.51)_2$ we calculate

$$\dot{E}_{\mathrm{L}0z} = \dot{E}_{\mathrm{L}02} = \mathbb{A}^*_{023|2} u_z + \mathsf{A}^*_{022} \dot{D}_{\mathrm{L}02}, \tag{4.88}$$

$$\dot{E}_{\mathrm{L}0r} = \dot{E}_{\mathrm{L}03} = \mathbb{A}^*_{011|3} u/r + \mathbb{A}^*_{022|3} w_z + \mathbb{A}^*_{033|3} u_r + \mathsf{A}^*_{033} \dot{D}_{\mathrm{L}03}. \tag{4.89}$$

Substituting (4.88) and (4.89) into (4.87) we have

$$\begin{aligned} &\mathbb{A}^*_{011|3} u_z/r + \mathbb{A}^*_{022|3} w_{zz} + \mathbb{A}^*_{033|3} u_{rz} + \mathsf{A}^*_{033} \dot{D}_{\mathrm{L}03,z} \\ &\quad - \mathbb{A}^{*'}_{023|2} u_z - \mathbb{A}^*_{023|2} u_{zr} - \mathsf{A}^{*'}_{022} \dot{D}_{\mathrm{L}02} - \mathsf{A}^*_{022} \dot{D}_{\mathrm{L}02,r} = 0. \end{aligned} \tag{4.90}$$

The governing equation $(2.32)_2$ reduces to

$$\frac{\partial (r\dot{D}_{\mathrm{L}0r})}{\partial r} + \frac{\partial (r\dot{D}_{\mathrm{L}0z})}{\partial z} = 0. \tag{4.91}$$

Equation (4.91) will be satisfied if we define a function $\psi(z, r)$ such that

$$\dot{D}_{\mathrm{L}0r} = \frac{\psi_{,z}}{r}, \quad \dot{D}_{\mathrm{L}0z} = -\frac{\psi_{,r}}{r}. \tag{4.92}$$

Again cross-differentiation of (4.79) and (4.80) and some rearrangement give

$$\begin{aligned} &\mathcal{A}^*_{03232} r^3 \phi_{,rrrr} + \mathcal{A}^*_{02323} r^3 \phi_{,zzzz} - (2\mathcal{A}^*_{02233} + 2\mathcal{A}^*_{03223} - \mathcal{A}^*_{02222} - \mathcal{A}^*_{03333}) r^3 \phi_{,rrzz} \\ &- (2r^2 \mathcal{A}^*_{03232} - 2r^3 \mathcal{A}^{*'}_{03232}) \phi_{,rrr} - \big[ (\mathcal{A}^*_{02222} - 2\mathcal{A}^*_{02233} - 2\mathcal{A}^*_{03223} + \mathcal{A}^*_{03333}) r^2 \\ &+ (2\mathcal{A}^{*'}_{02233} + 2\mathcal{A}^{*'}_{03223} - \mathcal{A}^{*'}_{02222} - \mathcal{A}^{*'}_{03333}) r^3 \big] \phi_{,rzz} \\ &+ (3\mathcal{A}^*_{03232} r - 3\mathcal{A}^{*'}_{03232} r^2 + \mathcal{A}^{*''}_{03232} r^3) \phi_{,rr} \\ &- \big[ (2\mathcal{A}^*_{02233} - 2\mathcal{A}^*_{01122} - r\mathcal{A}^{*'}_{01133} + \mathcal{A}^*_{01111} - \mathcal{A}^*_{03333}) r \\ &\quad + (\mathcal{A}^{*'}_{01122} - \mathcal{A}^{*'}_{02233} - \mathcal{A}^{*'}_{03223} + \mathcal{A}^{*'}_{03333}) r^2 + (\mathcal{A}^{*''}_{03223} + p^{*''}) r^3 \big] \phi_{,zz} \\ &- (3\mathcal{A}^*_{03232} - 3\mathcal{A}^{*'}_{03232} r + \mathcal{A}^{*''}_{03232} r^2) \phi_{,r} - (\mathbb{A}^*_{022|3} + \mathbb{A}^*_{023|2} - \mathbb{A}^*_{033|3}) r^3 \psi_{,zzr} \\ &- \big[ (\mathbb{A}^*_{011|3} - \mathbb{A}^*_{022|3}) r^2 + (\mathbb{A}^{*'}_{022|3} - \mathbb{A}^{*'}_{033|3}) r^3 \big] \psi_{,zz} = 0. \end{aligned} \tag{4.93}$$

From (4.90) we have

$$\begin{aligned} &r(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{022|3} - \mathbb{A}^*_{023|2}) \phi_{,rzz} \\ &+ (\mathbb{A}^*_{011|3} - \mathbb{A}^*_{033|3} + \mathbb{A}^*_{023|2} - r\mathbb{A}^{*'}_{023|2}) \phi_{,zz} \\ &+ \mathsf{A}^*_{033} r \psi_{,zz} + \mathsf{A}^*_{022} r \psi_{,rr} + (r\mathsf{A}^{*'}_{022} - \mathsf{A}^*_{022}) \psi_{,r} = 0. \end{aligned} \tag{4.94}$$

Specialization of the boundary condition (2.35) leads to

$$w_r + u_z = 0 \quad \text{on} \quad r = a, b, \tag{4.95}$$

$$(\mathcal{A}^*_{03333} - \mathcal{A}^*_{02233} + \tau_3) u_r + (\mathcal{A}^*_{03311} - \mathcal{A}^*_{03322}) u/r - \dot{p} + \mathbb{A}^*_{033|3} \dot{D}_{\mathrm{L}03} = 0 \quad \text{on} \quad r = a, b. \tag{4.96}$$

In (4.95) and (4.96) we set $\dot{P} = 0$. The third component of (2.35) is satisfied automatically.

Boundary conditions (4.95) and (4.96) can be expressed in terms of functions $\phi$ and $\psi$:

$$r\phi_{,zz} - r\phi_{,rr} + \phi_{,r} = 0 \quad \text{on} \quad r = a, b, \tag{4.97}$$

$$\begin{aligned} &r^2(\mathcal{A}^*_{02222} + \mathcal{A}^*_{03333} - 2\mathcal{A}^*_{02233} + \tau_3 - \mathcal{A}^*_{03223})\phi_{,zzr} + r^2\mathcal{A}^*_{03232}\phi_{,rrr} \\ &+ r(\mathcal{A}^*_{01133} - r\mathcal{A}^{*'}_{03232} + r\tau_{33}^{'} - \mathcal{A}^*_{01122} - \mathcal{A}^*_{03333} - \tau_3 + \mathcal{A}^*_{02233})\phi_{,zz} \\ &+ r(r\mathcal{A}^{*'}_{03232} - \mathcal{A}^*_{03232})\phi_{,rr} + (\mathcal{A}^*_{03232} - r\mathcal{A}^{*'}_{03232})\phi_{,r} \\ &+ r^2(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{022|3})\psi_{,zz} = 0 \quad \text{on} \quad r = a, b. \end{aligned} \tag{4.98}$$

In order to have a consistency with Haughton & Ogden (1979) we write

$$\phi(z, r) = rf(r)\cos\alpha z \quad \text{and} \quad \psi(z, r) = g(r)\cos\alpha z. \tag{4.99}$$

The governing equations (4.93) and (4.94) can be expressed respectively

$$\begin{aligned} &r^4[\mathcal{A}^*_{03232}f''' + (r\mathcal{A}^{*'}_{03232} + 2\mathcal{A}^*_{03232})f''/r + (r\mathcal{A}^{*'}_{03232} - \mathcal{A}^*_{03232})f'/r^2 \\ &- (r\mathcal{A}^{*'}_{03232} - \mathcal{A}^*_{03232})f/r^3]' + \alpha^2r^2[(2\mathcal{A}^*_{02233} + 2\mathcal{A}^*_{03223} - \mathcal{A}^*_{03333} - \mathcal{A}^*_{02222})r^2f'' \\ &+ (2r\mathcal{A}^{*'}_{03223} + 2r\mathcal{A}^{*'}_{02233} - r\mathcal{A}^{*'}_{03333} - r\mathcal{A}^{*'}_{02222} - \mathcal{A}^*_{03333} - \mathcal{A}^*_{02222} + 2\mathcal{A}^*_{02233} + 2\mathcal{A}^*_{03223})rf' \\ &+ (r^2\mathcal{A}^{*''}_{03223} + r^2p^{*''} + r\mathcal{A}^{*'}_{03223} + r\mathcal{A}^{*'}_{01122} - r\mathcal{A}^{*'}_{01133} - r\mathcal{A}^{*'}_{02222} \\ &+ r\mathcal{A}^{*'}_{02233} + \mathcal{A}^*_{01111} + \mathcal{A}^*_{02222} - 2\mathcal{A}^*_{01122} - 2\mathcal{A}^*_{03223})f] + \alpha^4r^4\mathcal{A}^*_{02323}f \\ &+ \alpha^2r^3(\mathbb{A}^*_{022|3} + \mathbb{A}^*_{023|2} - \mathbb{A}^*_{033|3})g' + \alpha^2[(\mathbb{A}^*_{011|3} - \mathbb{A}^*_{022|3})r^2 + (\mathbb{A}^{*'}_{022|3} - \mathbb{A}^{*'}_{033|3})r^3]g = 0, \end{aligned} \tag{4.100}$$

$$\begin{aligned} &\alpha^2r^2(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{022|3} - \mathbb{A}^*_{023|2})f' + \alpha^2(\mathbb{A}^*_{011|3}r - \mathbb{A}^*_{022|3}r - \mathbb{A}^{*'}_{023|2}r^2)f \\ &- \mathsf{A}^*_{022}rg'' - (\mathsf{A}^{*'}_{022}r - \mathsf{A}^*_{022})g' + \alpha^2r\mathsf{A}^*_{033}g = 0. \end{aligned} \tag{4.101}$$

The boundary conditions (4.97) and (4.98) can be rewritten as

$$r^2f'' + rf' + (\alpha^2r^2 - 1)f = 0 \quad \text{on} \quad r = a, b, \tag{4.102}$$

$$\begin{aligned} &\mathcal{A}^*_{03232}r^3f''' + (r\mathcal{A}^{*'}_{03232} + 2\mathcal{A}^*_{03232})r^2f'' + (r\mathcal{A}^{*'}_{03232} - \mathcal{A}^*_{03232})rf' \\ &- (r\mathcal{A}^{*'}_{03232} - \mathcal{A}^*_{03232})f - \alpha^2r^2[(\mathcal{A}^*_{03333} + \mathcal{A}^*_{02222} - 2\mathcal{A}^*_{02233} - \mathcal{A}^*_{03223} + \tau_3)rf' \\ &- (r\mathcal{A}^{*'}_{03232} - r\tau_{33}^{'} + \mathcal{A}^*_{03232} - \tau_3 + \mathcal{A}^*_{01122} - \mathcal{A}^*_{02222} + \mathcal{A}^*_{02233} - \mathcal{A}^*_{01133})f] \\ &- \alpha^2r^2(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{022|3})g = 0 \quad \text{on} \quad r = a, b. \end{aligned} \tag{4.103}$$

The electrical boundary conditions (2.33) reduce to

$$\dot{E}_{\mathrm{L}02} = \dot{E}_{\mathrm{L}0z} = 0 \quad \text{on} \quad r = a, b. \tag{4.104}$$

Boundary conditions (4.104) can be written in terms of functions $\phi$ and $\psi$ and $f$ and $g$, respectively, as

$$\mathbb{A}^{*}_{023|2}\phi_{,zz} - \mathsf{A}^{*}_{022}\psi_{,r} = 0 \quad \text{on} \quad r = a, b, \tag{4.105}$$

$$\mathbb{A}^{*}_{023|2}\alpha^2 r f + \mathsf{A}^{*}_{022} g' = 0 \quad \text{on} \quad r = a, b. \tag{4.106}$$

Boundary conditions (2.34) reduce to

$$\dot{D}_{\mathrm{L}0r} = \begin{cases} -\dot{\sigma}_{\mathrm{F}0b} & \text{on } r = b, \\ \dot{\sigma}_{\mathrm{F}0a} & \text{on } r = a. \end{cases} \tag{4.107}$$

Here again we do not require boundary condition (4.107) to be satisfied explicitly. We assume that this boundary condition is adjusted according the solution obtained with the boundary condition (4.104), satisfied explicitly.

We introduce new variables

$$y_1 = f(r), \quad y_2 = f'(r), \quad y_3 = f''(r), \quad y_4 = f'''(r), \quad y_5 = g(r), \quad y_6 = g'(r). \tag{4.108}$$

Thus, we can rewrite the governing equations (4.100) and (4.101) as a system of six ordinary differential equations

$$\begin{aligned}
&y_1' = y_2, \quad y_2' = y_3, \quad y_3' = y_4, \quad y_5' = y_6, \qquad (4.109)\\
&r^4\mathcal{A}^{*}_{03232} y_4' + (2r^4\mathcal{A}^{*'}_{03232} + 2r^3\mathcal{A}^{*}_{03232}) y_4 \\
&+ [r^3(3\mathcal{A}^{*'}_{03232} + r\mathcal{A}^{*''}_{03232}) - 3r^2\mathcal{A}^{*}_{03232} + \alpha^2 r^4(2\mathcal{A}^{*}_{02233} + 2\mathcal{A}^{*}_{03223} - \mathcal{A}^{*}_{03333} - \mathcal{A}^{*}_{02222})] y_3 \\
&+ [r^3\mathcal{A}^{*''}_{03232} - 3r^2\mathcal{A}^{*'}_{03232} + 3r\mathcal{A}^{*}_{03232} + \alpha^2 r^3(2r\mathcal{A}^{*'}_{03223} + 2r\mathcal{A}^{*'}_{02233} - r\mathcal{A}^{*'}_{03333} \\
&- r\mathcal{A}^{*'}_{02222} - \mathcal{A}^{*}_{03333} - \mathcal{A}^{*}_{02222} + 2\mathcal{A}^{*}_{02233} + 2\mathcal{A}^{*}_{03223})] y_2 \\
&+ [3(r\mathcal{A}^{*'}_{03232} - \mathcal{A}^{*}_{03232}) - r^2\mathcal{A}^{*''}_{03232} + \alpha^2 r^2(r^2\mathcal{A}^{*''}_{03223} + r^2 p^{*''} \\
&+ r\mathcal{A}^{*'}_{03223} + r\mathcal{A}^{*'}_{01122} - r\mathcal{A}^{*'}_{01133} - r\mathcal{A}^{*'}_{02222} + r\mathcal{A}^{*'}_{02233} + \mathcal{A}^{*}_{01111} + \mathcal{A}^{*}_{02222} \\
&- 2\mathcal{A}^{*}_{01122} - 2\mathcal{A}^{*}_{03223}) + \alpha^4 r^4\mathcal{A}^{*}_{02323}] y_1 \\
&+ \alpha^2 r^3(\mathbb{A}^{*}_{022|3} + \mathbb{A}^{*}_{023|2} - \mathbb{A}^{*}_{033|3}) y_6 + \alpha^2[(\mathbb{A}^{*}_{011|3} - \mathbb{A}^{*}_{022|3}) r^2 + (\mathbb{A}^{*'}_{022|3} - \mathbb{A}^{*'}_{033|3}) r^3] y_5 = 0, \\
&\alpha^2 r^2(\mathbb{A}^{*}_{033|3} - \mathbb{A}^{*}_{022|3} - \mathbb{A}^{*}_{023|2}) y_2 + \alpha^2(\mathbb{A}^{*}_{011|3} r - \mathbb{A}^{*}_{022|3} r - \mathbb{A}^{*'}_{023|2} r^2) y_1 \\
&- \mathsf{A}^{*}_{022} r y_6' - (\mathsf{A}^{*'}_{022} r - \mathsf{A}^{*}_{022}) y_6 + \alpha^2 r \mathsf{A}^{*}_{033} y_5 = 0.
\end{aligned}$$

Boundary conditions in terms of new variables take the form

$$
\begin{aligned}
& r^2 y_3 + r y_2 + (\alpha^2 r^2 - 1) y_1 = 0, \\
& \mathcal{A}^*_{03232} r^3 y_4 + (r\mathcal{A}^{*'}_{03232} + 2\mathcal{A}^*_{03232}) r^2 y_3 + (r\mathcal{A}^{*'}_{03232} - \mathcal{A}^*_{03232}) r y_2 \\
& - (r\mathcal{A}^{*'}_{03232} - \mathcal{A}^*_{03232}) y_1 - \alpha^2 r^2 [(\mathcal{A}^*_{03333} + \mathcal{A}^*_{02222} - 2\mathcal{A}^*_{02233} - \mathcal{A}^*_{03223} + \tau_3) r y_2 \\
& - (r\mathcal{A}^{*'}_{03232} - r\tau'_{33} + \mathcal{A}^*_{03232} - \tau_3 + \mathcal{A}^*_{01122} - \mathcal{A}^*_{02222} + \mathcal{A}^*_{02233} - \mathcal{A}^*_{01133}) y_1] \\
& - \alpha^2 r^2 (\mathbb{A}^*_{033|3} - \mathbb{A}^*_{022|3}) y_5 = 0, \\
& \mathbb{A}^*_{023|2} \alpha^2 r y_1 + \mathsf{A}^*_{022} y_6 = 0 \quad \text{on} \quad r = a, b.
\end{aligned}
\tag{4.110}
$$

In order to proceed further we use incremental boundary condition

$$
u = 0 \quad \text{on} \quad z = 0, l,
\tag{4.111}
$$

thus radial displacements at the ends of the cylinder are not allowed. Therefore, from the previous relation, $(4.71)_1$ and $(4.99)_1$ we obtain the condition for $\alpha$

$$
\alpha = \frac{\pi n}{l} = \frac{\pi n}{\lambda_z L},
\tag{4.112}
$$

where $n = 1, 2, 3, ...$ is the axisymmetric mode number. We see from 4.112 that $\alpha$ may be changed either by mode number $n$ or the length of the cylinder $L$. We fix $n = 1$ and we perform our analysis for different lengths of the cylinder.

We define initial values for the system (4.109) in the form

$$
y_i(a) = \delta_{ik} \quad (i = 1, ..., 6),
\tag{4.113}
$$

where $\delta_{ik}$ is the Kronecker delta. Each $k$ ($k = 1, ..., 6$) in (4.131) corresponds to the solution $\mathbf{y}^k$ of the system (4.109). The general solution of (4.109) can be written in the form

$$
\mathbf{y} = \sum_{k=1}^{6} c_k \mathbf{y}^k,
\tag{4.114}
$$

where $c_k$ are constants.

Now we require the solution (4.114) to satisfy boundary conditions (4.110). We are interested in the solutions (4.114), where at least one constant $c_k$ is non-zero. Substitution of (4.114) into (4.110) leads to the vanishing of $6 \times 6$ determinant of coefficients of $c_k$. Thus, vanishing $6 \times 6$ determinant of coefficients of $c_k$ is a bifurcation criterion for this problem.

Nondimensional equations and boundary conditions (with no energy function applied):

$$
\begin{aligned}
&\hat{y}_1' = \hat{y}_2, \quad \hat{y}_2' = \hat{y}_3, \quad \hat{y}_3' = \hat{y}_4, \quad \hat{y}_5' = \hat{y}_6, \qquad (4.115)\\
&\hat{r}^4\hat{\mathcal{A}}^*_{03232}\hat{y}_4' + (2\hat{r}^4\hat{\mathcal{A}}^{*'}_{03232} + 2\hat{r}^3\hat{\mathcal{A}}^*_{03232})\hat{y}_4\\
&+ [\hat{r}^3(3\hat{\mathcal{A}}^{*'}_{03232} + \hat{r}\hat{\mathcal{A}}^{*''}_{03232}) - 3\hat{r}^2\mathcal{A}^*_{03232} + \hat{\alpha}^2\hat{r}^4(2\hat{\mathcal{A}}^*_{02233} + 2\hat{\mathcal{A}}^*_{03223} - \hat{\mathcal{A}}^*_{03333} - \hat{\mathcal{A}}^*_{02222})]\hat{y}_3\\
&+ [\hat{r}^3\hat{\mathcal{A}}^{*''}_{03232} - 3\hat{r}^2\hat{\mathcal{A}}^{*'}_{03232} + 3\hat{r}\hat{\mathcal{A}}^*_{03232} + \hat{\alpha}^2\hat{r}^3(2\hat{r}\hat{\mathcal{A}}^{*'}_{03223} + 2\hat{r}\hat{\mathcal{A}}^{*'}_{02233} - \hat{r}\hat{\mathcal{A}}^{*'}_{03333}\\
&- \hat{r}\hat{\mathcal{A}}^{*'}_{02222} - \hat{\mathcal{A}}^*_{03333} - \hat{\mathcal{A}}^*_{02222} + 2\hat{\mathcal{A}}^*_{02233} + 2\hat{\mathcal{A}}^*_{03223})]\hat{y}_2\\
&+ [3(\hat{r}\hat{\mathcal{A}}^{*'}_{03232} - \hat{\mathcal{A}}^*_{03232}) - \hat{r}^2\hat{\mathcal{A}}^{*''}_{03232} + \hat{\alpha}^2\hat{r}^2(\hat{r}^2\hat{\mathcal{A}}^{*''}_{03223} + \hat{r}^2\hat{p}^{*''}\\
&+ \hat{r}\hat{\mathcal{A}}^{*'}_{03223} + \hat{r}\hat{\mathcal{A}}^{*'}_{01122} - \hat{r}\hat{\mathcal{A}}^{*'}_{01133} - \hat{r}\hat{\mathcal{A}}^{*'}_{02222} + \hat{r}\hat{\mathcal{A}}^{*'}_{02233} + \hat{\mathcal{A}}^*_{01111} + \hat{\mathcal{A}}^*_{02222}\\
&- 2\hat{\mathcal{A}}^*_{01122} - 2\hat{\mathcal{A}}^*_{03223}) + \hat{\alpha}^4\hat{r}^4\hat{\mathcal{A}}^*_{02323}]\hat{y}_1\\
&+ \hat{\alpha}^2\hat{r}^3(\hat{\mathbb{A}}^*_{022|3} + \hat{\mathbb{A}}^*_{023|2} - \hat{\mathbb{A}}^*_{033|3})\hat{\sigma}^2_{fa}\hat{y}_6 + \hat{\alpha}^2[(\hat{\mathbb{A}}^*_{011|3} - \hat{\mathbb{A}}^*_{022|3})\hat{r}^2 + (\hat{\mathbb{A}}^{*'}_{022|3} - \hat{\mathbb{A}}^{*'}_{033|3})\hat{r}^3]\hat{\sigma}^2_{fa}\hat{y}_5 = 0,\\
&\hat{\alpha}^2\hat{r}^2(\hat{\mathbb{A}}^*_{033|3} - \hat{\mathbb{A}}^*_{022|3} - \hat{\mathbb{A}}^*_{023|2})\hat{y}_2 + \hat{\alpha}^2(\hat{\mathbb{A}}^*_{011|3}\hat{r} - \hat{\mathbb{A}}^*_{022|3}\hat{r} - \hat{\mathbb{A}}^{*'}_{023|2}\hat{r}^2)\hat{y}_1\\
&- \hat{\mathsf{A}}^*_{022}\hat{r}\hat{y}_6' - (\hat{\mathsf{A}}^{*'}_{022}\hat{r} - \hat{\mathsf{A}}^*_{022})\hat{y}_6 + \hat{\alpha}^2\hat{r}\hat{\mathsf{A}}^*_{033}\hat{y}_5 = 0.
\end{aligned}
$$

Nondimensional boundary conditions (with no energy function applied):

$$
\begin{aligned}
&\hat{r}^2\hat{y}_3 + \hat{r}\hat{y}_2 + (\hat{\alpha}^2\hat{r}^2 - 1)\hat{y}_1 = 0, \qquad (4.116)\\
&\hat{\mathcal{A}}^*_{03232}\hat{r}^3\hat{y}_4 + (\hat{r}\hat{\mathcal{A}}^{*'}_{03232} + 2\hat{\mathcal{A}}^*_{03232})\hat{r}^2\hat{y}_3 + (\hat{r}\hat{\mathcal{A}}^{*'}_{03232} - \hat{\mathcal{A}}^*_{03232})\hat{r}\hat{y}_2\\
&- (\hat{r}\hat{\mathcal{A}}^{*'}_{03232} - \hat{\mathcal{A}}^*_{03232})\hat{y}_1 - \hat{\alpha}^2\hat{r}^2[(\hat{\mathcal{A}}^*_{03333} + \hat{\mathcal{A}}^*_{02222} - 2\hat{\mathcal{A}}^*_{02233} - \hat{\mathcal{A}}^*_{03223} + \hat{\tau}_3)\hat{r}\hat{y}_2\\
&- (\hat{r}\hat{\mathcal{A}}^{*'}_{03232} - \hat{r}\hat{\tau}_{33}' + \hat{\mathcal{A}}^*_{03232} - \hat{\tau}_3 + \hat{\mathcal{A}}^*_{01122} - \hat{\mathcal{A}}^*_{02222} + \hat{\mathcal{A}}^*_{02233} - \hat{\mathcal{A}}^*_{01133})\hat{y}_1]\\
&- \hat{\alpha}^2\hat{r}^2(\hat{\mathbb{A}}^*_{033|3} - \hat{\mathbb{A}}^*_{022|3})\hat{\sigma}^2_{fa}\hat{y}_5 = 0,\\
&\hat{\mathbb{A}}^*_{023|2}\hat{\alpha}^2\hat{r}\hat{y}_1 + \hat{\mathsf{A}}^*_{022}\hat{y}_6 = 0 \quad \text{on} \quad r = \hat{a}, \hat{b}.
\end{aligned}
$$

We consider neo-Hookean electroelastic material

$$\Omega^* = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{2}\varepsilon^{-1}I_5, \tag{4.117}$$

where the constant $\mu$ is the shear modulus of the neo-Hookean material in the absence of an electric field and the constant $\varepsilon$ is the electric permittivity of the electroelastic material.

With the energy function specified above, electroelastic moduli take the following values

$$
\begin{aligned}
&\mathcal{A}^*_{03131} = \mathcal{A}^*_{03232} = \mathcal{A}^*_{03333} = \lambda_3^2(\mu + D_3^2\lambda_1^2\lambda_2^2\varepsilon^{-1}), \qquad (4.118)\\
&\mathcal{A}^*_{01122} = \mathcal{A}^*_{01133} = \mathcal{A}^*_{03223} = \mathcal{A}^*_{02233} = \mathcal{A}^*_{01331} = 0,\\
&\mathcal{A}^*_{02222} = \mathcal{A}^*_{02323} = \lambda_2^2\mu,\\
&\mathcal{A}^*_{01111} = \mathcal{A}_{01313} = \lambda_1^2\mu\\
&\mathbb{A}^*_{033|3} = 2\mathbb{A}^*_{023|2} = 2D_3\varepsilon^{-1}, \quad \mathbb{A}^*_{022|3} = \mathbb{A}^*_{011|3} = 0, \quad \mathsf{A}^*_{022} = \mathsf{A}^*_{033} = \varepsilon^{-1}.
\end{aligned}
$$

We introduce new non-dimensional variables

$$\hat{r} = \frac{r}{A}, \quad \hat{\alpha} = \alpha A, \quad \hat{f}(\hat{r}) = \frac{f(r)}{A^2}, \quad \hat{g}(\hat{r}) = \frac{g(r)}{D_r(a)A^2}, \tag{4.119}$$
$$\hat{\boldsymbol{\tau}}(\hat{r}) = \frac{\boldsymbol{\tau}}{\mu}, \quad \hat{p}(\hat{r}) = \frac{p(r)}{\mu}, \quad \hat{\boldsymbol{\mathcal{A}}}^*_0(\hat{r}) = \frac{\boldsymbol{\mathcal{A}}^*_0}{\mu}, \quad \hat{\mathbb{A}}^*_0(\hat{r}) = \mathbb{A}^*_0(r)\frac{\varepsilon}{D_r(a)},$$
$$\hat{\mathsf{A}}^*_0(\hat{r}) = \mathsf{A}^*_0\varepsilon.$$

For the specified energy function the governing equation (4.100) or an equivalent (4.109) can be rewritten in a non-dimensional form

$$\begin{aligned}
&\hat{r}^4\hat{\mathcal{A}}^*_{03232}\hat{f}'''' + (2\hat{r}^4\hat{\mathcal{A}}^{*\prime}_{03232} + 2\hat{r}^3\hat{\mathcal{A}}^*_{03232})\hat{f}''' \\
&+ [\hat{r}^3(3\hat{\mathcal{A}}^{*\prime}_{03232} + \hat{r}\hat{\mathcal{A}}^{*\prime\prime}_{03232}) - 3\hat{r}^2\hat{\mathcal{A}}^*_{03232} - \hat{\alpha}^2\hat{r}^4(\hat{\mathcal{A}}^*_{03333} + \hat{\mathcal{A}}^*_{02222})]\hat{f}'' \\
&+ [\hat{r}^3\hat{\mathcal{A}}^{*\prime\prime}_{03232} - 3\hat{r}^2\hat{\mathcal{A}}^{*\prime}_{03232} + 3\hat{r}\hat{\mathcal{A}}^*_{03232} + \hat{\alpha}^2\hat{r}^3(-\hat{r}\hat{\mathcal{A}}^{*\prime}_{03333} - \hat{\mathcal{A}}^*_{03333} - \hat{\mathcal{A}}^*_{02222})]\hat{f}' \\
&+ [3(\hat{r}\hat{\mathcal{A}}^{*\prime}_{03232} - \hat{\mathcal{A}}^*_{03232}) - \hat{r}^2\hat{\mathcal{A}}^{*\prime\prime}_{03232} + \hat{\alpha}^2\hat{r}^2(\hat{r}^2\hat{p}'' + \hat{\mathcal{A}}^*_{01111} + \hat{\mathcal{A}}^*_{02222}) + \hat{\alpha}^4\hat{r}^4\hat{\mathcal{A}}^*_{02323}]\hat{f} \\
&+ \hat{\alpha}^2\hat{r}^3(\hat{\mathbb{A}}^*_{023|2} - \hat{\mathbb{A}}^*_{033|3})\hat{g}'\frac{D_r^2(a)}{\varepsilon\mu} - \hat{\alpha}^2\hat{r}^3\hat{\mathbb{A}}^{*\prime}_{033|3}\hat{g}\frac{D_r^2(a)}{\varepsilon\mu} = 0.
\end{aligned} \tag{4.120}$$

We introduce a non-dimensional quantity

$$\hat{\sigma}_{fa} = \frac{\sigma_{fa}}{\sqrt{\varepsilon\mu}}. \tag{4.121}$$

Using $(3.15)_1$ we obtain

$$\frac{D_r^2(a)}{\varepsilon\mu} = \hat{\sigma}^2_{fa}. \tag{4.122}$$

Therefore, we use $\hat{\sigma}_{fa}$ as a non-dimensional electrical parameter which accounts for free surface charge per unit area on the inner boundary of a tube in the deformed configuration.

Governing equation (4.101) in a non-dimensional form can be rewritten as

$$\begin{aligned}
&\hat{\alpha}^2\hat{r}^2(\hat{\mathbb{A}}^*_{033|3} - \hat{\mathbb{A}}^*_{023|2})\hat{\sigma}^2_{fa}\hat{f}' \\
&- \hat{\alpha}^2\hat{r}^2\hat{\mathbb{A}}^{*\prime}_{023|2}\hat{\sigma}^2_{fa}\hat{f} - \hat{\mathsf{A}}^*_{022}\hat{r}\hat{\sigma}^2_{fa}\hat{g}'' + \hat{\mathsf{A}}^*_{022}\hat{\sigma}^2_{fa}\hat{g}' + \hat{\alpha}^2\hat{r}\hat{\mathsf{A}}^*_{033}\hat{\sigma}^2_{fa}\hat{g} = 0.
\end{aligned} \tag{4.123}$$

Also boundary conditions (4.102), (4.103), and (4.106) specialise in a non-dimensional form to

$$\hat{r}^2\hat{f}'' + \hat{r}\hat{f}' + (\hat{\alpha}^2\hat{r}^2 - 1)\hat{f} = 0 \quad \text{on} \quad \hat{r} = \hat{a}, \hat{b}, \tag{4.124}$$

$$\begin{aligned}
&\hat{\mathcal{A}}^*_{03232}\hat{r}^3\hat{f}''' + (\hat{r}\hat{\mathcal{A}}^{*\prime}_{03232} + 2\hat{\mathcal{A}}^*_{03232})\hat{r}^2\hat{f}'' + (\hat{r}\hat{\mathcal{A}}^{*\prime}_{03232} - \hat{\mathcal{A}}^*_{03232})\hat{r}\hat{f}' \\
&- (\hat{r}\hat{\mathcal{A}}^{*\prime}_{03232} - \hat{\mathcal{A}}^*_{03232})\hat{f} - \hat{\alpha}^2\hat{r}^2[(\hat{\mathcal{A}}^*_{03333} + \hat{\mathcal{A}}^*_{02222} + \hat{\tau}_3)\hat{r}\hat{f}' \\
&- (\hat{r}\hat{\mathcal{A}}^{*\prime}_{03232} - \hat{r}\hat{\tau}'_{33} + \hat{\mathcal{A}}^*_{03232} - \hat{\tau}_3 - \hat{\mathcal{A}}^*_{02222})\hat{f}] \\
&- \hat{\alpha}^2\hat{r}^2\hat{\mathbb{A}}^*_{033|3}\hat{\sigma}^2_{fa}\hat{g} = 0 \quad \text{on} \quad \hat{r} = \hat{a}, \hat{b},
\end{aligned} \tag{4.125}$$

$$\hat{\alpha}^2\hat{r}\hat{\mathbb{A}}^*_{023|2}\hat{f} + \hat{\mathsf{A}}^*_{022}\hat{g}' = 0 \quad \text{on} \quad \hat{r} = \hat{a}, \hat{b}. \tag{4.126}$$

Using definitions (4.119) electroelastic moduli and other quantities can be expressed in a non-dimensional form for particular choice of energy function (4.117)

$$\begin{aligned}
&\hat{\mathcal{A}}^*_{03131}(\hat{r}) = \hat{\mathcal{A}}^*_{03232}(\hat{r}) = \hat{\mathcal{A}}^*_{03333}(\hat{r}) = \frac{\lambda_z^{-1}(\hat{r}^2A^2 - a^2) + \lambda_z^{-2}A^2 + \hat{\sigma}^2_{fa}a^2}{\hat{r}^2A^2}, \\
&\hat{\mathcal{A}}^*_{01122}(\hat{r}) = \hat{\mathcal{A}}^*_{01133}(\hat{r}) = \hat{\mathcal{A}}^*_{03223}(\hat{r}) = \hat{\mathcal{A}}^*_{02233}(\hat{r}) = \hat{\mathcal{A}}^*_{01331}(\hat{r}) = 0, \\
&\hat{\mathcal{A}}^*_{02222}(\hat{r}) = \hat{\mathcal{A}}^*_{02323}(\hat{r}) = \lambda_z^2, \\
&\hat{\mathcal{A}}^*_{01111}(\hat{r}) = \hat{\mathcal{A}}^*_{01313}(\hat{r}) = \lambda_\theta^2 = \frac{\hat{r}^2A^2}{\lambda_z(\hat{r}^2A^2 - a^2) + A^2}, \\
&\hat{\mathbb{A}}^*_{033|3}(\hat{r}) = 2\hat{\mathbb{A}}^*_{023|2}(\hat{r}) = \frac{2a}{\hat{r}A}, \quad \hat{\mathbb{A}}^*_{022|3}(\hat{r}) = \hat{\mathbb{A}}^*_{011|3}(\hat{r}) = 0, \quad \hat{\mathsf{A}}^*_{022}(\hat{r}) = \hat{\mathsf{A}}^*_{033}(\hat{r}) = 1, \\
&\hat{\tau}_3(\hat{r}) = \hat{\mathcal{A}}^*_{03131}(\hat{r}), \quad \hat{\tau}'_{33}(\hat{r}) = (\hat{\mathcal{A}}^*_{01313}(\hat{r}) - \hat{\mathcal{A}}^*_{03131}(\hat{r}))/\hat{r}, \quad \hat{\alpha} = \frac{\pi A}{\lambda_z L}, \\
&\hat{p}''(\hat{r}) = \hat{\mathcal{A}}^{*''}_{03131}(\hat{r}) - \frac{1}{\hat{r}^2}(\hat{\mathcal{A}}^*_{03131}(\hat{r}) - \hat{\mathcal{A}}^*_{01313}(\hat{r})) + \frac{1}{\hat{r}}(\hat{\mathcal{A}}^{*'}_{03131}(\hat{r}) - \hat{\mathcal{A}}^{*'}_{01313}(\hat{r})).
\end{aligned} \tag{4.127}$$

We introduce new variables

$$\begin{aligned}
&\hat{y}_1(\hat{r}) = \hat{f}(\hat{r}), \quad \hat{y}_2(\hat{r}) = \hat{f}'(\hat{r}), \quad \hat{y}_3(\hat{r}) = \hat{f}''(\hat{r}), \\
&\hat{y}_4(\hat{r}) = \hat{f}'''(\hat{r}), \quad \hat{y}_5(\hat{r}) = \hat{g}(\hat{r}), \quad \hat{y}_6(\hat{r}) = \hat{g}'(\hat{r}).
\end{aligned} \tag{4.128}$$

We can rewrite the governing equations (4.120) and (4.123) in terms of new variables as a system of six ordinary differential equations

$$\begin{aligned}
&\hat{y}'_1 = \hat{y}_2, \quad \hat{y}'_2 = \hat{y}_3, \quad \hat{y}'_3 = \hat{y}_4, \quad \hat{y}'_5 = \hat{y}_6, \\
&\hat{r}^4\hat{\mathcal{A}}^*_{03232}\hat{y}'_4 + (2\hat{r}^4\hat{\mathcal{A}}^{*'}_{03232} + 2\hat{r}^3\hat{\mathcal{A}}^*_{03232})\hat{y}_4 \\
&+ [\hat{r}^3(3\hat{\mathcal{A}}^{*'}_{03232} + \hat{r}\hat{\mathcal{A}}^{*''}_{03232}) - 3\hat{r}^2\hat{\mathcal{A}}^*_{03232} - \hat{\alpha}^2\hat{r}^4(\hat{\mathcal{A}}^*_{03333} + \hat{\mathcal{A}}^*_{02222})]\hat{y}_3 \\
&+ [\hat{r}^3\hat{\mathcal{A}}^{*''}_{03232} - 3\hat{r}^2\hat{\mathcal{A}}^{*'}_{03232} + 3\hat{r}\hat{\mathcal{A}}^*_{03232} + \hat{\alpha}^2\hat{r}^3(-\hat{r}\hat{\mathcal{A}}^{*'}_{03333} - \hat{\mathcal{A}}^*_{03333} - \hat{\mathcal{A}}^*_{02222})]\hat{y}_2 \\
&+ [3(\hat{r}\hat{\mathcal{A}}^{*'}_{03232} - \hat{\mathcal{A}}^*_{03232}) - \hat{r}^2\hat{\mathcal{A}}^{*''}_{03232} + \hat{\alpha}^2\hat{r}^2(\hat{r}^2\hat{p}'' + \hat{\mathcal{A}}^*_{01111} + \hat{\mathcal{A}}^*_{02222}) + \hat{\alpha}^4\hat{r}^4\hat{\mathcal{A}}^*_{02323}]\hat{y}_1 \\
&+ \hat{\alpha}^2\hat{r}^3(\hat{\mathbb{A}}^*_{023|2} - \hat{\mathbb{A}}^*_{033|3})\hat{\sigma}^2_{fa}\hat{y}_6 - \hat{\alpha}^2\hat{r}^3\hat{\mathbb{A}}^{*'}_{033|3}\hat{\sigma}^2_{fa}\hat{y}_5 = 0. \\
&\hat{\alpha}^2\hat{r}^2(\hat{\mathbb{A}}^*_{033|3} - \hat{\mathbb{A}}^*_{023|2})\hat{y}_2 \\
&- \hat{\alpha}^2\hat{r}^2\hat{\mathbb{A}}^{*'}_{023|2}\hat{y}_1 - \hat{\mathsf{A}}^*_{022}\hat{r}\hat{y}'_6 + \hat{\mathsf{A}}^*_{022}\hat{y}_6 + \hat{\alpha}^2\hat{r}\hat{\mathsf{A}}^*_{033}\hat{y}_5 = 0.
\end{aligned} \tag{4.129}$$

Corresponding boundary conditions in terms of new variables are

$$\begin{aligned}
&\hat{r}^2\hat{y}_3 + \hat{r}\hat{y}_2 + (\hat{\alpha}^2\hat{r}^2 - 1)\hat{y}_1 = 0, \qquad (4.130)\\
&\hat{\mathcal{A}}^*_{03232}\hat{r}^3\hat{y}_4 + (\hat{r}\hat{\mathcal{A}}^{*'}_{03232} + 2\hat{\mathcal{A}}^*_{03232})\hat{r}^2\hat{y}_3 + (\hat{r}\hat{\mathcal{A}}^{*'}_{03232} - \hat{\mathcal{A}}^*_{03232})\hat{r}\hat{y}_2\\
&- (\hat{r}\hat{\mathcal{A}}^{*'}_{03232} - \hat{\mathcal{A}}^*_{03232})\hat{y}_1 - \hat{\alpha}^2\hat{r}^2[(\hat{\mathcal{A}}^*_{03333} + \hat{\mathcal{A}}^*_{02222} + \hat{\tau}_3)\hat{r}\hat{y}_2\\
&- (\hat{r}\hat{\mathcal{A}}^{*'}_{03232} - \hat{r}\hat{\tau}^{'}_{33} + \hat{\mathcal{A}}^*_{03232} - \hat{\tau}_3 - \hat{\mathcal{A}}^*_{02222})\hat{y}_1]\\
&- \hat{\alpha}^2\hat{r}^2\hat{\mathbb{A}}^*_{033|3}\hat{\sigma}^2_{fa}\hat{y}_5 = 0,\\
&\hat{\alpha}^2\hat{r}\hat{\mathbb{A}}^*_{023|2}\hat{y}_1 + \hat{\mathsf{A}}^*_{022}\hat{y}_6 = 0 \quad \text{on} \quad \hat{r} = \hat{a}, \hat{b}.
\end{aligned}$$

We define initial values for the system (4.129) in the form

$$\hat{y}_i(\hat{a}) = \delta_{ik} \quad (i = 1, ..., 6), \tag{4.131}$$

where $\delta_{ik}$ is the Kronecker delta. Each $k$ ($k = 1, ..., 6$) in (4.131) corresponds to the solution $\mathbf{y}^k$ of the system (4.129). The general solution of (4.129) can be written in the form

$$\hat{\mathbf{y}} = \sum_{k=1}^{6} c_k \hat{\mathbf{y}}^k, \tag{4.132}$$

where $c_k$ are constants.

Now we require the solution (4.132) to satisfy boundary conditions (4.130). We are interested in the solutions (4.132), where at least one constant $c_k$ is non-zero. Substitution of (4.132) into (4.130) leads to the vanishing of $6 \times 6$ determinant of coefficients of $c_k$. Thus, vanishing $6 \times 6$ determinant of coefficients of $c_k$ is a bifurcation criterion for this problem.

Now we consider augmented Mooney-Rivlin model

$$\Omega^* = \frac{1}{2}\mu_1(I_1 - 3) - \frac{1}{2}\mu_2(I_2 - 3) + \frac{1}{2}\varepsilon^{-1}I_5, \tag{4.133}$$

where $\mu_1 \geq 0$ and $\mu_2 \leq 0$ are material constants satisfying $\mu_1 - \mu_2 = \mu$. In what follows we use $\mu_1 = 0.8\mu$ and $\mu_2 = -0.2\mu$.

We calculate (dimensional) electroelastic moduli:

$$\begin{aligned}
&\mathcal{A}^*_{03131} = 2\lambda_3^2(\Omega_1 + \lambda_2^2\Omega_2 + D_3^2\lambda_1^2\lambda_2^2\Omega_5) = 0.8\mu\lambda_3^2 + 0.2\mu\lambda_1^{-2} + D_3^2\varepsilon^{-1}, \qquad (4.134)\\
&\mathcal{A}^*_{03232} = 2\lambda_3^2(\Omega_1 + \lambda_1^2\Omega_2 + D_3^2\lambda_1^2\lambda_2^2\Omega_5) = 0.8\mu\lambda_3^2 + 0.2\mu\lambda_2^{-2} + D_3^2\varepsilon^{-1},\\
&\mathcal{A}^*_{03333} = 2\lambda_3^2(\Omega_1 + (\lambda_1^2+\lambda_2^2)\Omega_2 + D_3^2\lambda_1^2\lambda_2^2\Omega_5), \quad \mathcal{A}^*_{01313} = 2\lambda_1^2(\Omega_1 + \lambda_2^2\Omega_2),\\
&\mathcal{A}^*_{01122} = 4\lambda_1^2\lambda_2^2\Omega_2 = 4\lambda_3^{-2}\Omega_2 = 0.4\lambda_3^{-2}\mu, \quad \mathcal{A}^*_{01133} = 4\lambda_1^2\lambda_3^2\Omega_2 = 4\lambda_2^{-2}\Omega_2 = 0.4\lambda_2^{-2}\mu,\\
&\mathcal{A}^*_{03223} = -2\lambda_3^2\lambda_2^2\Omega_2 = -0.2\mu\lambda_1^{-2}, \quad \mathcal{A}^*_{02233} = 4\lambda_2^2\lambda_3^2\Omega_2 = 4\lambda_1^{-2}\Omega_2 = 0.4\mu\lambda_1^{-2},\\
&\mathcal{A}^*_{01331} = -2\lambda_1^2\lambda_3^2\Omega_2 = -0.2\mu\lambda_2^{-2}, \quad \mathcal{A}^*_{02222} = 2\lambda_2^2(\Omega_1 + (\lambda_1^2+\lambda_3^2)\Omega_2),\\
&\mathcal{A}^*_{02323} = 2\lambda_2^2(\Omega_1 + \lambda_1^2\Omega_2), \quad \mathcal{A}^*_{01111} = 2\lambda_1^2(\Omega_1 + (\lambda_2^2+\lambda_3^2)\Omega_2),\\
&\mathbb{A}^*_{033|3} = 4D_3\Omega_5 = 2D_3\varepsilon^{-1}, \quad \mathbb{A}^*_{023|2} = 2D_3\Omega_5 = D_3\varepsilon^{-1},\\
&\mathbb{A}^*_{022|3} = 0, \quad \mathbb{A}^*_{011|3} = 0, \quad \mathsf{A}^*_{022} = 2\Omega_5 = \varepsilon^{-1}, \quad \mathsf{A}^*_{033} = 2\Omega_5 = \varepsilon^{-1}.
\end{aligned}$$

Now we rewrite these moduli as nondimensional quantities:

$$\begin{aligned}
&\hat{\mathcal{A}}^*_{03131}(\hat{r}) = \frac{0.8[\lambda_z^{-1}(\hat{r}^2A^2 - a^2) + \lambda_z^{-2}A^2] + 0.2[\lambda_z(\hat{r}^2A^2 - a^2) + A^2] + \hat{\sigma}_{fa}^2a^2}{\hat{r}^2A^2}, \qquad (4.135)\\
&\hat{\mathcal{A}}^*_{03232}(\hat{r}) = \frac{0.8[\lambda_z^{-1}(\hat{r}^2A^2 - a^2) + \lambda_z^{-2}A^2 + \hat{\sigma}_{fa}^2a^2]}{\hat{r}^2A^2} + 0.2\lambda_z^{-2},\\
&\hat{\mathcal{A}}^*_{03333}(\hat{r}) = \frac{0.8[\lambda_z^{-1}(\hat{r}^2A^2 - a^2) + \lambda_z^{-2}A^2] + 0.2[\lambda_z(\hat{r}^2A^2 - a^2) + A^2] + \hat{\sigma}_{fa}^2a^2}{\hat{r}^2A^2} + 0.2\lambda_z^{-2},\\
&\hat{\mathcal{A}}^*_{01122}(\hat{r}) = 0.4\frac{\hat{r}^2A^2}{\lambda_z^{-1}(\hat{r}^2A^2 - a^2) + \lambda_z^{-2}A^2},\\
&\hat{\mathcal{A}}^*_{01133}(\hat{r}) = 0.4\lambda_z^{-2}, \quad \hat{\mathcal{A}}^*_{01331}(\hat{r}) = -0.2\lambda_z^{-2},\\
&\hat{\mathcal{A}}^*_{03223}(\hat{r}) = \frac{-0.2[\lambda_z(\hat{r}^2A^2 - a^2) + A^2]}{\hat{r}^2A^2}, \quad \hat{\mathcal{A}}^*_{02233}(\hat{r}) = \frac{0.4[\lambda_z(\hat{r}^2A^2 - a^2) + A^2]}{\hat{r}^2A^2},\\
&\hat{\mathcal{A}}^*_{02222}(\hat{r}) = 0.8\lambda_z^2 + 0.2\Big[\frac{\hat{r}^2A^2}{\lambda_z^{-1}(\hat{r}^2A^2 - a^2) + \lambda_z^{-2}A^2} + \frac{\lambda_z(\hat{r}^2A^2 - a^2) + A^2}{\hat{r}^2A^2}\Big],\\
&\hat{\mathcal{A}}^*_{02323}(\hat{r}) = 0.8\lambda_z^2 + \frac{0.2\hat{r}^2A^2}{\lambda_z^{-1}(\hat{r}^2A^2 - a^2) + \lambda_z^{-2}A^2},\\
&\hat{\mathcal{A}}^*_{01111}(\hat{r}) = \frac{0.8\hat{r}^2A^2}{\lambda_z(\hat{r}^2A^2 - a^2) + A^2} + 0.2\Big[\frac{\hat{r}^2A^2}{\lambda_z^{-1}(\hat{r}^2A^2 - a^2) + \lambda_z^{-2}A^2} + \lambda_z^{-2}\Big],\\
&\hat{\mathcal{A}}^*_{01313}(\hat{r}) = \frac{0.8\lambda_z^{-2}\hat{r}^2A^2 + 0.2\hat{r}^2A^2}{\lambda_z^{-1}(\hat{r}^2A^2 - a^2) + \lambda_z^{-2}A^2}, \quad \hat{\mathbb{A}}^*_{033|3}(\hat{r}) = \frac{2a}{\hat{r}A}, \quad \hat{\mathbb{A}}^*_{023|2}(\hat{r}) = \frac{a}{\hat{r}A},\\
&\hat{\mathbb{A}}^*_{022|3}(\hat{r}) = 0, \quad \hat{\mathbb{A}}^*_{011|3}(\hat{r}) = 0, \quad \hat{\mathsf{A}}^*_{022}(\hat{r}) = 1, \quad \hat{\mathsf{A}}^*_{033}(\hat{r}) = 1.
\end{aligned}$$

We also need the following derivatives of moduli:

$$\hat{\mathcal{A}}^{*\prime}_{03131}(\hat{r}) = \frac{-1.6A^2 + 1.6A^2\lambda_z + (-0.4A^2 - 2A^2\hat{\sigma}^2_{fa})\lambda_z^2 + 0.4a^2\lambda_z^2}{A^2\hat{r}^3\lambda_z^2}, \tag{4.136}$$

$$\hat{\mathcal{A}}^{*\prime\prime}_{03131}(\hat{r}) = \frac{4.8A^2 - 4.8a^2\lambda_z + (1.2A^2 + 6a^2\hat{\sigma}^2_{fa})\lambda_z^2 - 1.2a^2\lambda_z^2}{A^2\hat{r}^4\lambda_z^2},$$

$$\hat{\mathcal{A}}^{*\prime}_{03232}(\hat{r}) = \frac{-1.6A^2 + 1.6a^2\lambda_z - 2a^2\hat{\sigma}^2_{fa}\lambda_z^2}{A^2\hat{r}^3\lambda_z^2}, \quad \hat{\mathcal{A}}^{*\prime\prime}_{03232}(\hat{r}) = \frac{4.8A^2 + a^2\lambda_z(-4.8 + 6\hat{\sigma}^2_{fa}\lambda_z)}{A^2\hat{r}^4\lambda_z^2},$$

$$\hat{\mathcal{A}}^{*\prime}_{01313}(\hat{r}) = \frac{A^2\hat{r}(1.6A^2 - 1.6a^2\lambda_z + 0.4A^2\lambda_z^2 - 0.4a^2\lambda_z^3)}{[A^2 + (-a^2 + A^2\hat{r}^2)\lambda_z]^2},$$

$$\hat{\mathcal{A}}^{*\prime}_{03223}(\hat{r}) = \frac{0.4A^2 - 0.4a^2\lambda_z}{A^2\hat{r}^3},$$

$$\hat{\mathcal{A}}^{*\prime\prime}_{03223}(\hat{r}) = \frac{-1.2A^2 + 1.2a^2\lambda_z}{A^2\hat{r}^4},$$

$$\hat{\mathcal{A}}^{*\prime}_{02233}(\hat{r}) = \frac{-0.8A^2 + 0.8a^2\lambda_z}{A^2\hat{r}^3},$$

$$\hat{\mathcal{A}}^{*\prime}_{03333}(\hat{r}) = \frac{-1.6A^2 + 1.6a^2\lambda_z + (-0.4A^2 - 2a^2\hat{\sigma}^2_{fa})\lambda_z^2 + 0.4a^2\lambda_z^2}{A^2\hat{r}^3\lambda_z^2},$$

$$\hat{\mathcal{A}}^{*\prime}_{01122}(\hat{r}) = \frac{A^2\hat{r}\lambda_z^2(0.8A^2 - 0.8a^2\lambda_z)}{[A^2 + (-a^2 + A^2\hat{r}^2)\lambda_z]^2},$$

$$\hat{\mathcal{A}}^{*\prime}_{02222}(\hat{r}) = \frac{(-0.4A^6 + \lambda_z(1.2a^2A^4 - 0.8A^6\hat{r}^2 + a^2\lambda_z(-1.2a^2A^2 + 1.6A^4\hat{r}^2 + (0.4a^4 - 0.8a^2A^2\hat{r}^2)\lambda_z)))}{\hat{r}^3(A^3 + (-a^2A + A^3\hat{r}^2)\lambda_z)^2},$$

$$\hat{\mathbb{A}}^{*\prime}_{033|3}(\hat{r}) = \frac{-2a}{\hat{r}^2A}, \quad \hat{\mathbb{A}}^{*\prime}_{023|2}(\hat{r}) = \frac{-a}{\hat{r}^2A}.$$

Also we use the following connections in the governing system of ODEs and boundary conditions

$$\hat{\tau}_3 = \hat{\mathcal{A}}^*_{03131}(\hat{r}) - \hat{\mathcal{A}}^*_{01331}(\hat{r}), \tag{4.137}$$

$$\hat{\tau}'_{33}(\hat{r}) = (\hat{\mathcal{A}}^*_{01313}(\hat{r}) - \hat{\mathcal{A}}^*_{03131}(\hat{r}))/\hat{r}, \tag{4.138}$$

$$\hat{p}''(\hat{r}) = \hat{\mathcal{A}}^{*\prime\prime}_{03131}(\hat{r}) - \hat{\mathcal{A}}^{*\prime\prime}_{01331}(\hat{r}) - \frac{1}{\hat{r}^2}(\hat{\mathcal{A}}^*_{03131}(\hat{r}) - \hat{\mathcal{A}}^*_{01313}(\hat{r})) + \frac{1}{\hat{r}}(\hat{\mathcal{A}}^{*\prime}_{03131}(\hat{r}) - \hat{\mathcal{A}}^{*\prime}_{01313}(\hat{r})). \tag{4.139}$$

In Fig. (4.1 – 4.5) we show pairs of $\lambda_z$ and $\lambda_a$ such that bifurcation criterion is satisfied. We were able to reproduce exactly the results obtained by Haughton & Ogden (1979) for neo-Hookean pure elastic material. We note though that Haughton & Ogden (1979) associated their results with the wrong values of ratios $L/B$. Our calculations show that in order to obtain correct values $L/B$ in Fig. 3 in Haughton & Ogden (1979) we need to divide them by 2. This was also confirmed by Zhu et al. (2008).

As far as Mooney-Rivlin material is concerned, Haughton & Ogden (1979) did not report in their paper the values of material parameters $\mu_1$ and $\mu_2$ they used for their calculations. In our calculations we used $\mu_1 = 0.8\mu$ and $\mu_2 = -0.2\mu$. Because of our assumption for the values of material parameters we obviously could not reproduce the results of Haughton & Ogden (1979) in the exact form, nonetheless qualitatively our results for Mooney-Rivlin material is in accordance with the results reported in their paper in Fig. 3.

From our results we can conclude that the presence of an electric field make an electroelastic tube more unstable. We see that in the presence of an electric field an electroelastic tube can bifurcate into unstable axisymmetric configuration at lower values of circumferential stretch $\lambda_a$. In each figure reported here the uppermost curve corresponds to pure elastic case (or, equivalently, to the case when there is no potential between electrodes and thus no electric field). Increasing of an electric field results in placing the curves one under another with the downmost curve corresponding to the highest value of an electric field.

With or without presence of an electric field we can also note that decreasing $L/B$, i.e. making a tube shorter leads to steeper bifurcation curves, thus limiting the range of values $\lambda_z$ where bifurcation is possible. This can be seen clearly here for Mooney-Rivlin electroelastic material.

In Fig. (4.6–4.10) we showed the results for a thicker cylindrical shell with $A/B = 0.5$. We can observe that for a thicker shell a higher circumferential stretch is required to make the tube unstable.

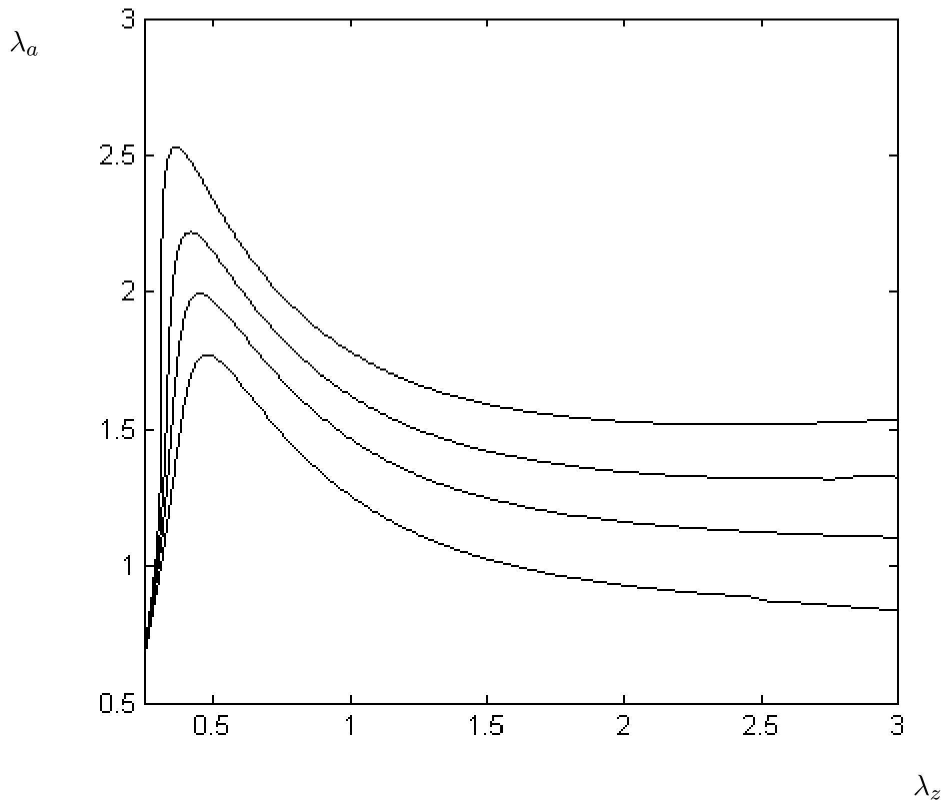


Figure 4.1: Plot of axisymmetric bifurcation curves for neo-Hookean electroelastic material with $L/B$=10, $A/B = 0.85$, $\hat{\sigma}_{fa} = 0, 0.75, 1.1, 1.5$.

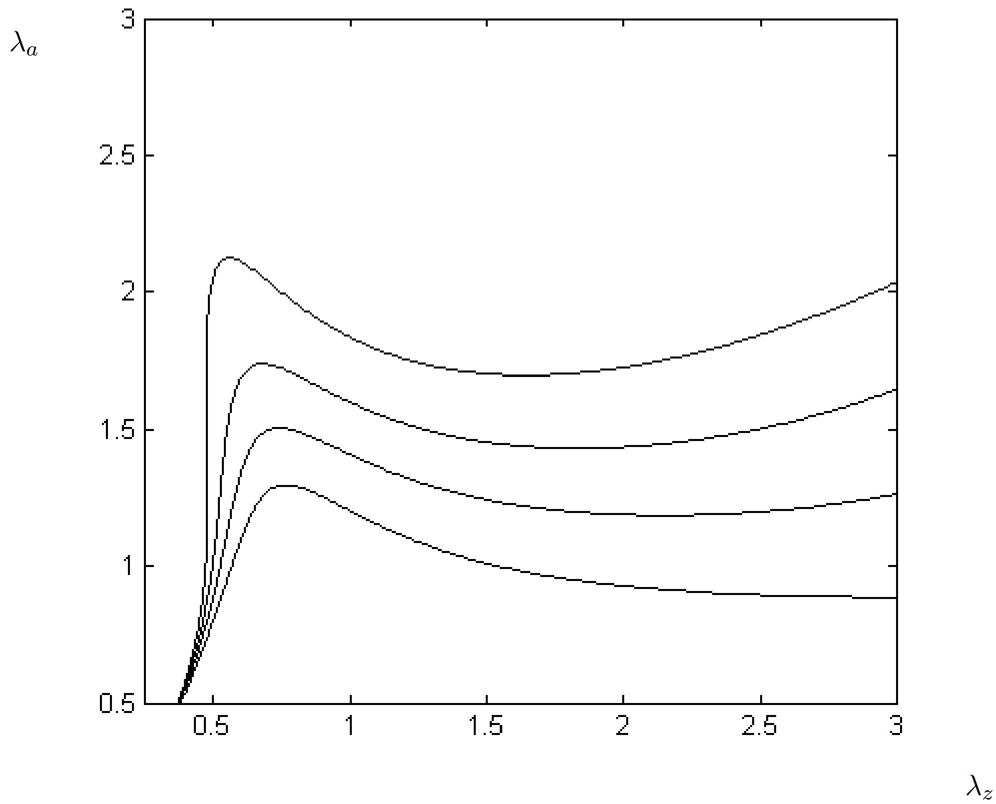


Figure 4.2: Plot of axisymmetric bifurcation curves for neo-Hookean electroelastic material with $L/B$=5, $A/B = 0.85$, $\hat{\sigma}_{fa} = 0, 0.75, 1.1, 1.5$.

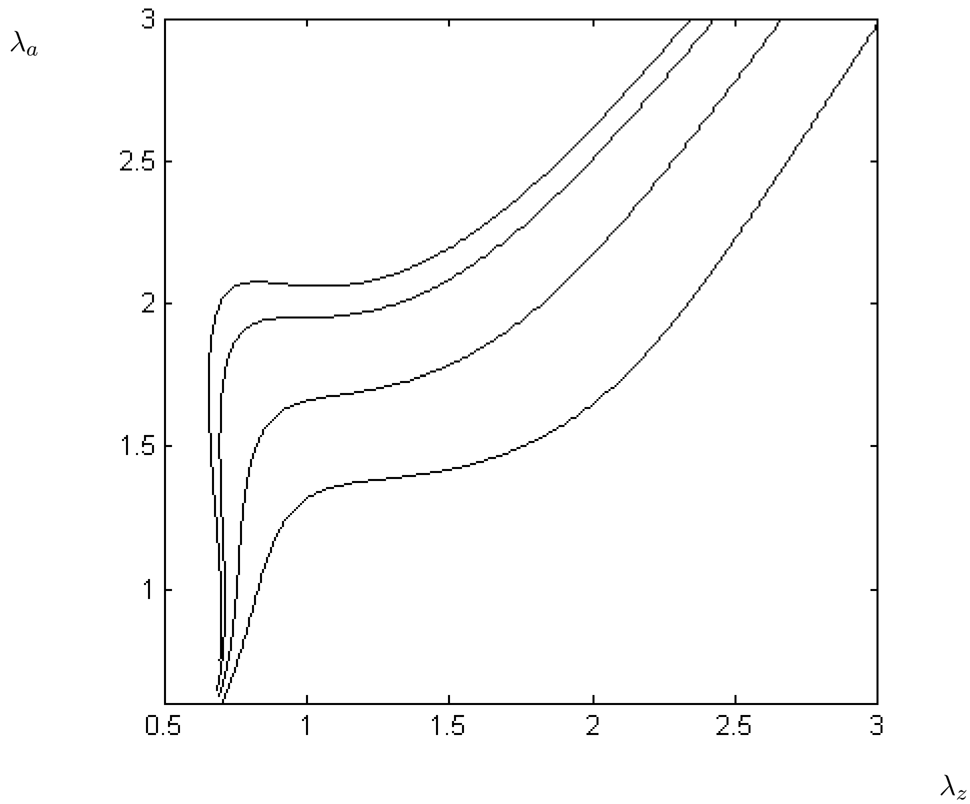


Figure 4.3: Plot of axisymmetric bifurcation curves for neo-Hookean electroelastic material with $L/B$=2.5, $A/B = 0.85$, $\hat{\sigma}_{fa} = 0, 0.3, 0.6, 0.9$.

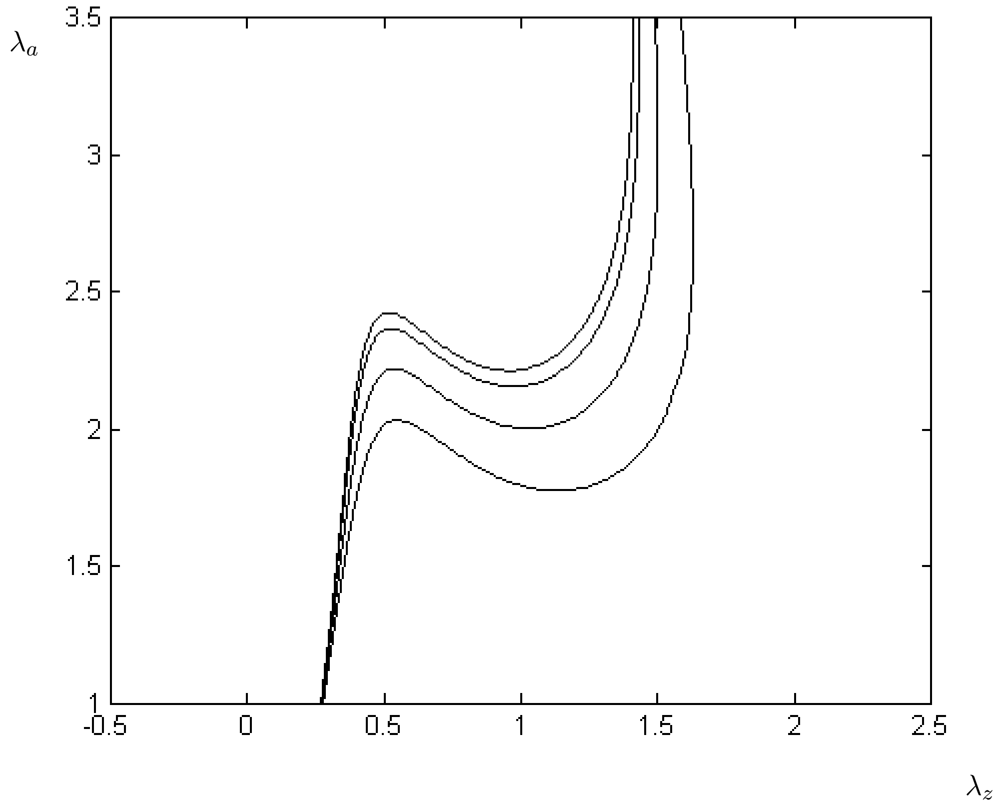


Figure 4.4: Plot of axisymmetric bifurcation curves for Mooney-Rivlin electroelastic material with $L/B$=10, $A/B = 0.85$, $\hat{\sigma}_{fa} = 0, 0.3, 0.6, 0.9$.

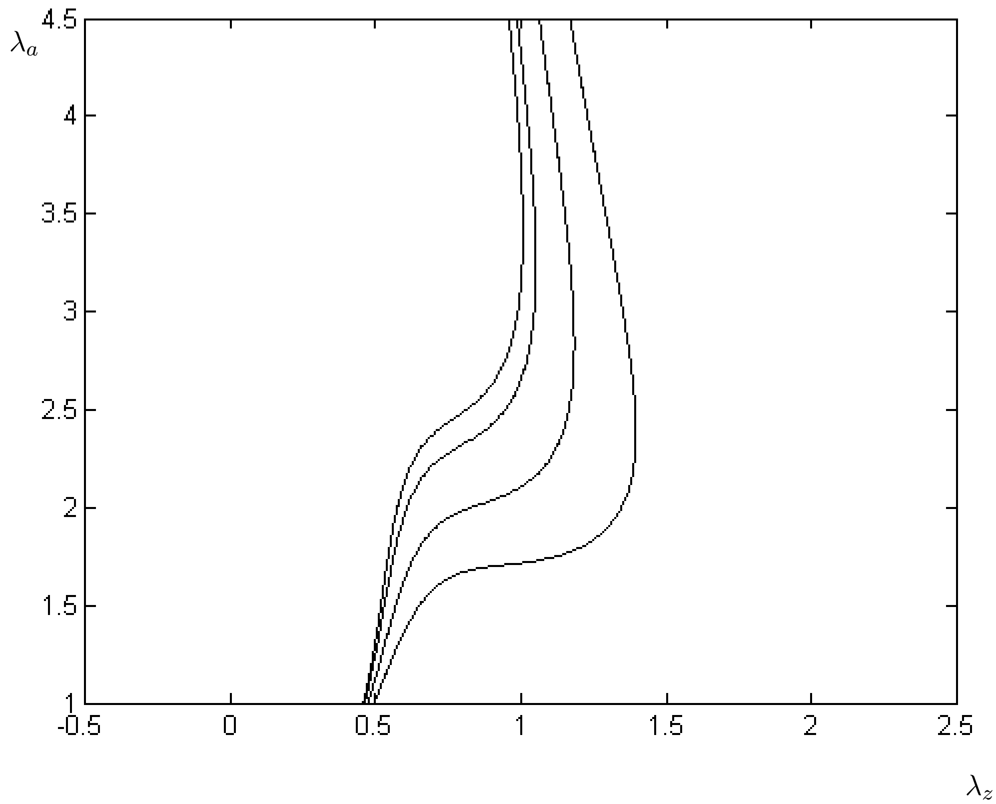


Figure 4.5: Plot of axisymmetric bifurcation curves for Mooney-Rivlin electroelastic material with $L/B$=5, $A/B = 0.85$, $\hat{\sigma}_{fa} = 0, 0.3, 0.6, 0.9$.

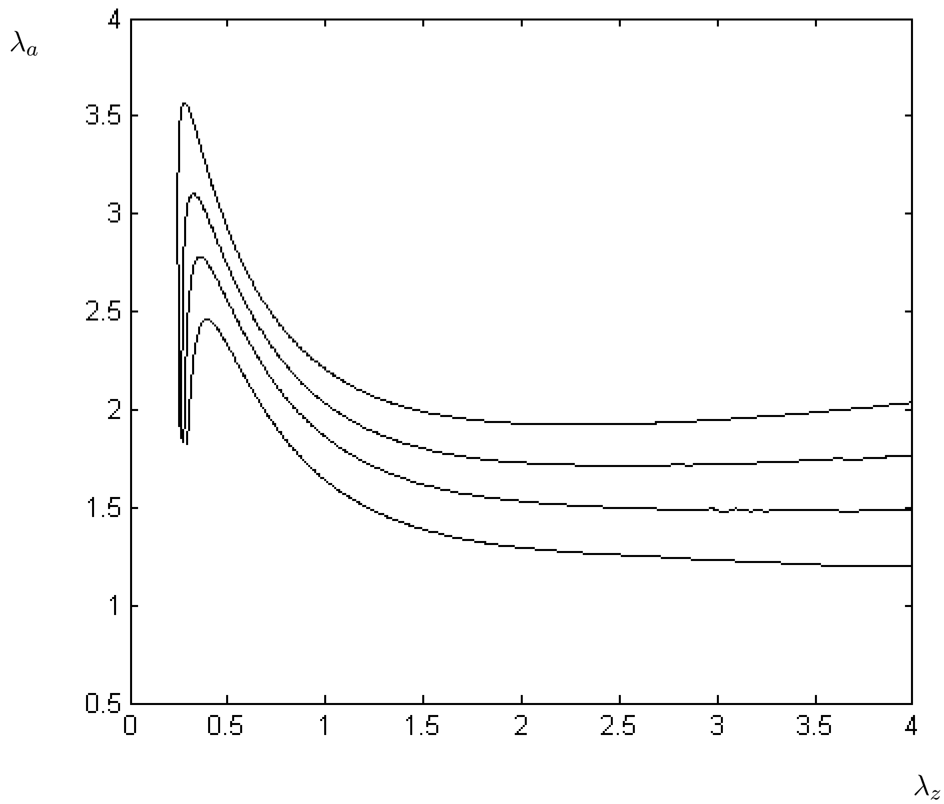


Figure 4.6: Plot of axisymmetric bifurcation curves for neo-Hookean electroelastic material with $L/B$=10, $A/B = 0.5$, $\hat{\sigma}_{fa} = 0, 0.75, 1.1, 1.5$.

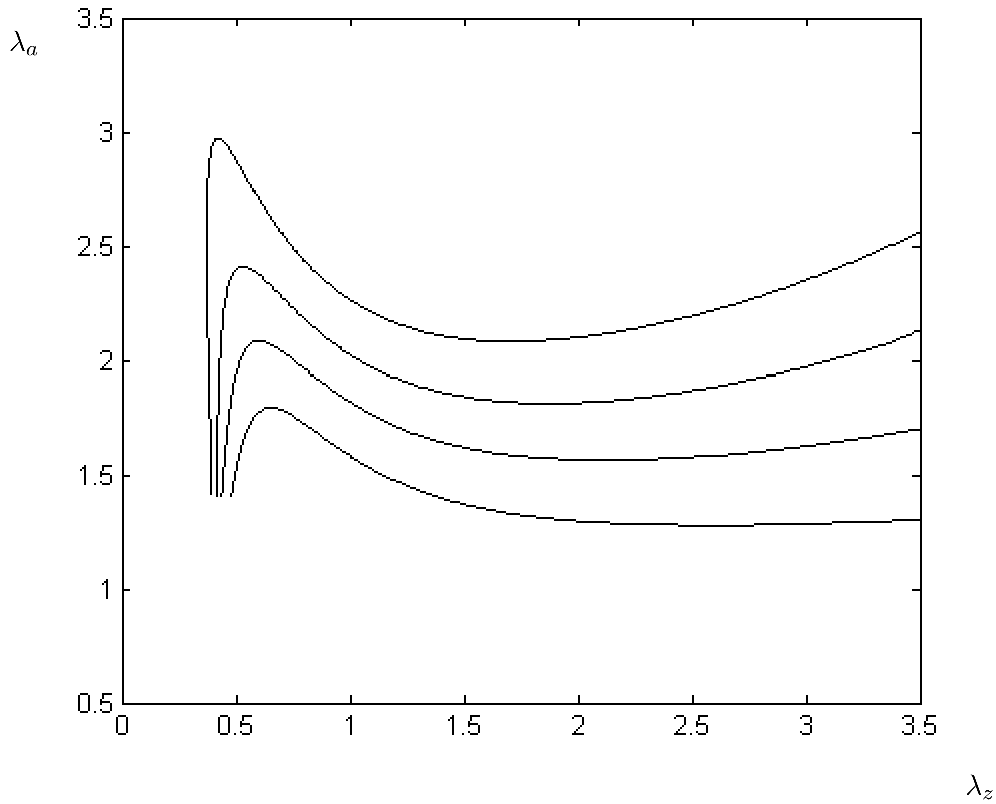


Figure 4.7: Plot of axisymmetric bifurcation curves for neo-Hookean electroelastic material with $L/B$=5, $A/B = 0.5$, $\hat{\sigma}_{fa} = 0, 0.75, 1.1, 1.5$.

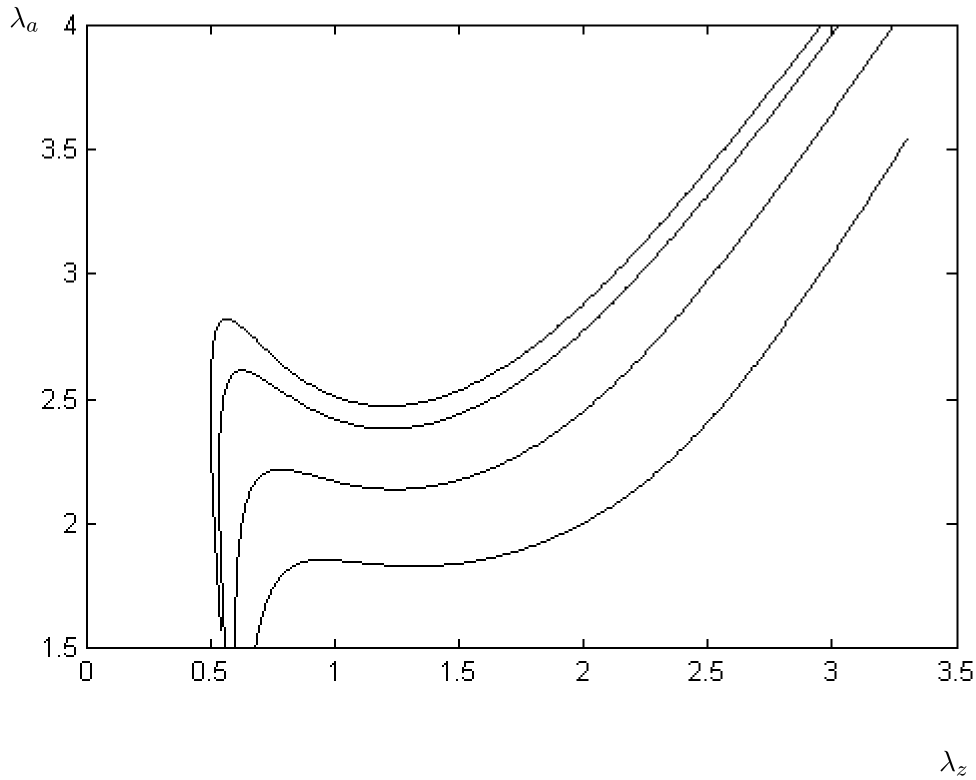


Figure 4.8: Plot of axisymmetric bifurcation curves for neo-Hookean electroelastic material with $L/B$=2.5, $A/B = 0.5$, $\hat{\sigma}_{fa} = 0, 0.3, 0.6, 0.9$.

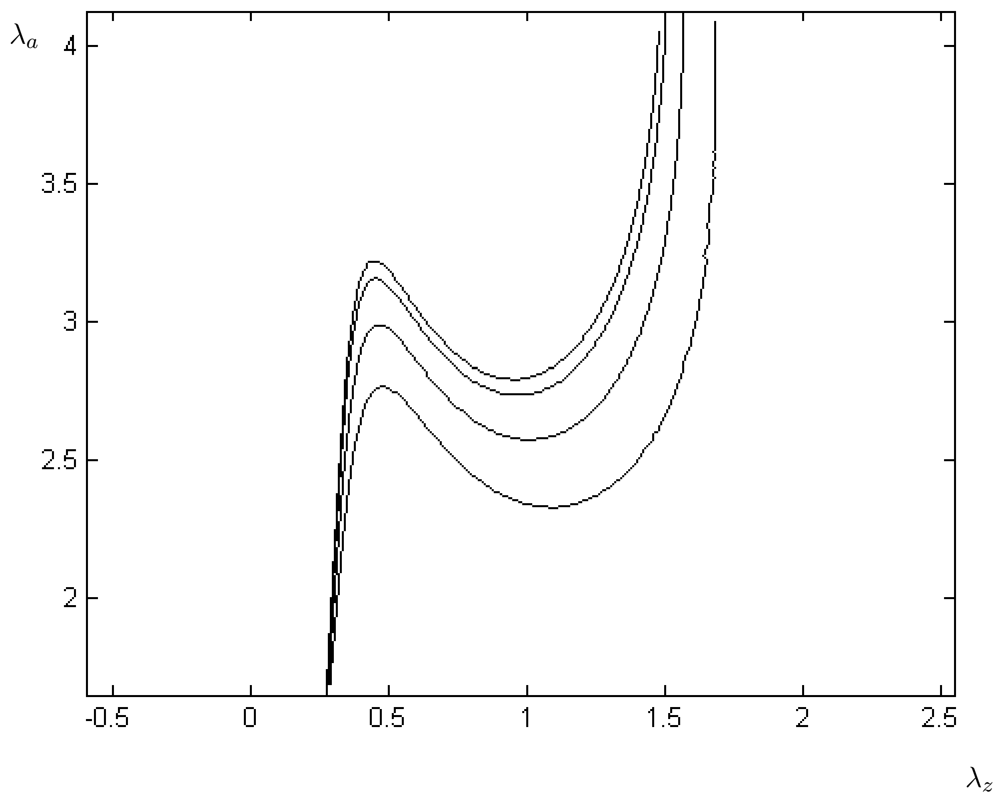


Figure 4.9: Plot of axisymmetric bifurcation curves for Mooney-Rivlin electroelastic material with $L/B$=10, $A/B = 0.5$, $\hat{\sigma}_{fa} = 0, 0.3, 0.6, 0.9$.

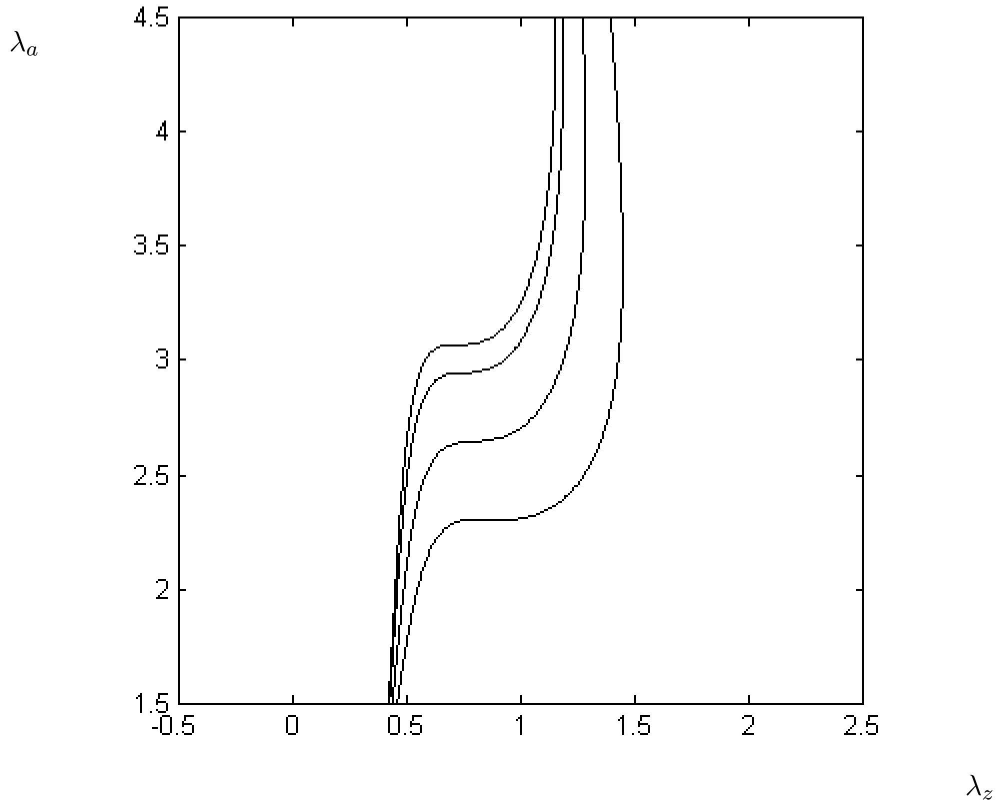


Figure 4.10: Plot of axisymmetric bifurcation curves for Mooney-Rivlin electroelastic material with $L/B$=5, $A/B = 0.5$, $\hat{\sigma}_{fa} = 0, 0.3, 0.6, 0.9$.

# Chapter 5

# Bifurcation of Electroelastic Spherical Shells

## 5.1 Introduction

In this chapter we give bifurcation analysis of an electorelastic thick-walled spherical shell with compliant electrodes at its boundaries under inflation and compression. We start with considering the underlying configuration: a finitely deformed electroelastic spherical shell. The problem of the inflation of an electroelastic spherical shell was considered in Dorfmann & Ogden (2014b). We use some results and notation from this work and then we develop a bifurcation analysis within the similar lines as for a thick-walled electroelastic cylinder in the previous Chapter 4. For the pure mechanical case in Haughton & Ogden (1978) it was found that inclusion of $\phi-$dependence does not affect the bifurcation criteria. Here, we adopt this approach and we consider only axisymmetric bifurcations. We complete this chapter with an analysis performed for the neo-Hookean energy function.

## 5.2 The underlying configuration

### 5.2.1 Spherically symmetric inflation of a spherical shell

The geometry of a spherical shell can be conveniently described by spherical polar coordinates $R$, $\Theta$, $\Phi$. In the reference configuration the shell is described by

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq \Phi \leq 2\pi, \tag{5.1}$$

where $A$ and $B$ are the internal and external radii.

Assuming that the spherical symmetry is maintained in the current configuration we have the counterpart of (5.1)

$$a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq 2\pi, \tag{5.2}$$

where $r$, $\theta$, $\phi$ are spherical polar coordinates and $a$, $b$ are the radii in the current (deformed) configuration.

Since we have an incompressible deformation the shell is expanded (preserving spherical symmetry) according to the relations

$$r = (R^3 + a^3 - A^3)^{1/3}, \quad \theta = \Theta, \quad \phi = \Phi. \tag{5.3}$$

The resulting deformation gradient with respect to the spherical polar coordinate axes is diagonal. The associated principal stretches $\lambda_1$ and $\lambda_2$ corresponding to the $\theta$ and $\phi$ directions are equal and we can write

$$\lambda_\theta = \lambda_\phi = \lambda = r/R > 1. \tag{5.4}$$

The principal stretch corresponding to the (third) radial direction is

$$\lambda_r = \frac{dr}{dR} = \lambda^{-2}. \tag{5.5}$$

We define the circumferential stretches at the inner and outer boundaries as $\lambda_a = a/A$ and $\lambda_b = b/B$. Using (5.3) we write

$$\lambda = \frac{r}{R} = \big(1 + \frac{a^3 - A^3}{R^3}\big)^{1/3}, \tag{5.6}$$

therefore the following relation follows

$$\lambda^3 - 1 = \frac{A^3}{R^3}(\lambda_a^3 - 1). \tag{5.7}$$

Evaluating the previous relation at $R = B$ we obtain the connection between the stretches at the inner and outer boundaries

$$(\lambda_a^3 - 1) = \left(\frac{B}{A}\right)^3 (\lambda_b^3 - 1). \tag{5.8}$$

Since $B/A > 1$ we can conclude from the previous relations that

$$\lambda_a \geq \lambda \geq \lambda_b \geq 1. \tag{5.9}$$

The invariants $I_1$ and $I_2$ can be specialized for this deformation gradient:

$$I_1 = 2\lambda^2 + \lambda^{-4}, \quad I_2 = \lambda^4 + 2\lambda^{-2}. \tag{5.10}$$

### 5.2.2 Electrical boundary conditions

In this problem we consider an electroelastic spherical shell the lateral boundaries of which have flexible electrodes. The charges on both electrodes are equal and have the opposite signs. Therefore, by Gauss's Theorem and because of the given geometry, we do not have an electric field outside the material. We will denote a total charge at $r = a$ by $Q(a)$, and at $r = b$ by $Q(b)$, respectively. Therefore, we have

$$Q(a) + Q(b) = 0. \tag{5.11}$$

The free surface charge densities per unit area on the inner and outer boundaries in the current deformed configuration will be

$$\sigma_{\mathrm{f}a} = \frac{Q(a)}{4\pi a^2}, \quad \sigma_{\mathrm{f}b} = \frac{Q(b)}{4\pi b^2}. \tag{5.12}$$

Therefore, we can rewrite (5.11) as

$$a^2\sigma_{\mathrm{f}a} + b^2\sigma_{\mathrm{f}b} = 0. \tag{5.13}$$

Referred to undeformed configuration we have the following analogues of the expressions (5.12)

$$\sigma_{\mathrm{F}A} = \frac{Q(a)}{4\pi A^2}, \quad \sigma_{\mathrm{F}B} = \frac{Q(b)}{4\pi B^2}, \tag{5.14}$$

where $A$, $B$ are the inner and the outer radii of the spherical shell in the undeformed configuration. In the undeformed configuration we have the following connection between free surface charge densities

$$A^2\sigma_{\mathrm{F}A} + B^2\sigma_{\mathrm{F}B} = 0. \tag{5.15}$$

For the considered spherical geometry the radial electric displacement $D_r$ ($D_\theta = 0$, $D_\phi = 0$) will depend only on $r$ and expression $(2.5)_2$ will be equivalent to

$$\frac{1}{r^2}\frac{\mathrm{d}(r^2 D_r)}{\mathrm{d}r} = 0. \tag{5.16}$$

Therefore, $r^2 D_r$ is a constant, which can be expressed at the boundaries $r = a$ and $r = b$ as $a^2 D_r(a)$ and $b^2 D_r(b)$, respectively. And we have

$$r^2 D_r = a^2 D_r(a) = b^2 D_r(b) = const. \tag{5.17}$$

Using the boundary condition $(2.8)_2$, where $\mathbf{D}^\star = 0$, we can relate radial electric field components at the boundaries to free surface charge densities per unit area in the deformed configuration

$$D_r(a) = \sigma_{\mathrm{f}a}, \quad D_r(b) = -\sigma_{\mathrm{f}b}. \tag{5.18}$$

Therefore, using (5.12) solutions (5.17) can be expressed as

$$r^2 D_r = \frac{Q(a)}{4\pi} = -\frac{Q(b)}{4\pi}. \tag{5.19}$$

### 5.2.3 Electric field components

In this problem it is natural to choose the electric displacement as an independent variable. We can control the electric field by prescribing a certain charge on the boundaries, and the charge on the boundaries is related to the electric displacement field through the boundary condition $(2.8)_2$. We will consider a radial field ($D_\theta = 0$, $D_\phi = 0$). Since constitutive law

$$\mathbf{E}_\mathrm{L} = \frac{\partial \Omega^*}{\partial \mathbf{D}_\mathrm{L}} \tag{5.20}$$

is expressed in terms of Lagrangian variable $\mathbf{D}_\mathrm{L}$ we will switch to this variable using relation

$$\mathbf{D}_\mathrm{L} = \mathbf{F}^{-1}\mathbf{D}. \tag{5.21}$$

Since the electric displacement vector is aligned along the radial direction of strain, we have

$$[D_L] = \begin{bmatrix} 0 \\ 0 \\ \lambda^2 D_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ D_{LR} \end{bmatrix}. \tag{5.22}$$

Using (2.22), we calculate the invariants

$$I_4 = \lambda^4 D_r^2 = D_{LR}^2, \tag{5.23}$$

$$I_5 = D_r^2 = \lambda^{-4} I_4, \tag{5.24}$$

$$I_6 = \lambda^{-4} D_r^2 = \lambda^{-8} I_4. \tag{5.25}$$

The components of electric field can be found using equation (2.24).

Since the deformation gradient is diagonal and $D_\theta = D_\phi = 0$, we have $E_\theta = E_\phi = 0$ and the remaining radial component will be

$$E_r = 2(\Omega_4^* \lambda^4 D_r + \Omega_5^* D_r + \Omega_6^* \lambda^{-4} D_r). \tag{5.26}$$

For the spherical symmetry (assuming that there is no dependence on either $\theta$ or $\phi$) $\mathrm{curl}\,\mathbf{E} = \mathbf{0}$ will be equivalent to $rE_\theta = \mathrm{const}$ and $rE_\phi = \mathrm{const}$, which are satisfied for this problem. Here we do not need to impose any condition on the function $\Omega^*$. Unlike the present problem for some types of deformations we do need such a condition. We can refer to Dorfmann & Ogden (2006) for an example of such a condition, where azimuthal shear deformation is considered.

### 5.2.4 Stress components

Let us now consider $\Omega^*$ as a function of principal stretches $\lambda_1$, $\lambda_2$, $\lambda_3$ and electromechanical invariants $I_4$, $I_5$, $I_6$. Recognizing the fact that in the present problem the only electrical variable is $I_4$ we can consider a function $\hat{\Omega}^*$ such that

$$\hat{\Omega}^*(\lambda_1, \lambda_2, \lambda_3, I_4) = \Omega^*(\lambda_1, \lambda_2, \lambda_3, I_4, I_5, I_6). \tag{5.27}$$

This allows us to obtain simple expressions for the principal components of the Cauchy stress tensor $\tau_{ii}$ ($i$=1, 2, 3)[1]

$$\tau_{ii} = \tau_i - p^* \quad (i = 1, 2, 3), \tag{5.28}$$

where

$$\tau_i = \lambda_i \frac{\partial \hat{\Omega}^*}{\partial \lambda_i} \quad (i = 1, 2, 3). \tag{5.29}$$

From (5.4) and (5.5) we can conclude that the principal stretches are functions of a sole variable $\lambda$. Therefore, we can introduce a new function $w^*$ such that

$$\omega^*(\lambda, I_4) = \hat{\Omega}^*(\lambda_1, \lambda_2, \lambda_3, I_4). \tag{5.30}$$

This allows us to write

$$\tau_{11} - \tau_{33} = \frac{1}{2}\lambda\omega^*_\lambda, \tag{5.31}$$

where $\omega^*_\lambda$ denote derivatives $\partial\omega^*/\partial\lambda$.

Expression (5.26) can now be rewritten as

$$E_r = 2\lambda^4 \frac{\partial \omega^*}{\partial I_4} D_r. \tag{5.32}$$

According to Gauss's theorem we have no field outside the tube, therefore by (2.11) the Maxwell stress is zero. Thus, we have only mechanical load due to a pressure $P$ inside the shell applied to the inner surface at $r = a$ and no loads at $r = b$

$$\tau_{rr} = -P \quad \text{on} \quad r = a, \quad \tau_{rr} = 0 \quad \text{on} \quad r = b. \tag{5.33}$$

In this problem the equilibrium equation $\operatorname{div} \boldsymbol{\tau} = 0$ reduces to

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = 2(\tau_{\theta\theta} - \tau_{rr}) = \lambda\omega^*_\lambda. \tag{5.34}$$

[1]no summation for the subscript $i$ is implied here.

In the previous expression we have used (5.31). Integrating (5.34) and using the boundary conditions (5.33) we have

$$\int_{-P}^{0} \mathrm{d}\tau_{rr} = \int_{a}^{b} \lambda\omega_{\lambda}^{*}\frac{\mathrm{d}r}{r}. \tag{5.35}$$

Therefore,

$$P = \int_{a}^{b} \lambda\omega_{\lambda}^{*}\frac{\mathrm{d}r}{r}. \tag{5.36}$$

In some cases it is convenient to change the variable of integration from $r$ to $\lambda$. To this end, we rearrange and differentiate $(5.3)_1$ with respect to $r$, taking into account that $\lambda$ depends on $r$. We have

$$r\frac{\mathrm{d}\lambda}{\mathrm{d}r} = -\lambda(\lambda^3 - 1). \tag{5.37}$$

Therefore, expression (5.36) can rewritten as

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^3 - 1)^{-1}\omega_{\lambda}^{*}\mathrm{d}\lambda. \tag{5.38}$$

From (5.8) we see that $\lambda_b$ depends on $\lambda_a$. Therefore, the previous relation gives $P$ as a function of $\lambda_a$ and invariant $I_4 = Q^2(a)/16\pi^2A^4$, which is known for a given charge $Q(a) = -Q(b)$.

Similarly, since $b = (B^3 + a^3 - A^3)^{1/3}$ we see that (5.36) provides a relationship between pressure and the inner radius $a$ and invariant $I_4$.

## 5.3 Bifurcation analysis

In the present setting we use spherical polar coordinates $\theta$, $\phi$, $r$ with the corresponding unit basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Derivatives in (2.37) denoted by subscripts with commas $(\cdot)_{,k}$ can now be specified as $\partial(\cdot)/r\partial\theta$, $\partial(\cdot)/r\sin\theta\partial\phi$, $\partial(\cdot)/\partial r$ for $k = 1, 2, 3$, respectively. For spherical polar coordinates in (2.37) the only non-zero scalar products $\mathbf{e}_i \cdot \mathbf{e}_{j,k}$ are

$$\begin{aligned} -\mathbf{e}_3 \cdot \mathbf{e}_{1,1} = -\mathbf{e}_3 \cdot \mathbf{e}_{2,2} = \mathbf{e}_1 \cdot \mathbf{e}_{3,1} = \mathbf{e}_2 \cdot \mathbf{e}_{3,2} = r^{-1}, \\ \mathbf{e}_1 \cdot \mathbf{e}_{2,2} = -\mathbf{e}_2 \cdot \mathbf{e}_{1,2} = -r^{-1}\cot\theta. \end{aligned} \tag{5.39}$$

### 5.3.1 Axisymmetric bifurcations

The increment in the position vector $\mathbf{x}$ of a point in the current configuration is

$$\dot{\mathbf{x}} = v\mathbf{e}_1 + w\mathbf{e}_2 + u\mathbf{e}_3. \tag{5.40}$$

We will consider axisymmetric bifurcations, hence $u$, $v$ and $w$ are independent of $\phi$, and we also accept that $w = 0$. Therefore, the components of $\mathbf{L}$ on the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ can be calculated as

$$\begin{bmatrix} (u+v_\theta)/r & 0 & v_r \\ 0 & (u+v\cot\theta)/r & 0 \\ (u_\theta - v)/r & 0 & u_r \end{bmatrix}, \tag{5.41}$$

where subscripts $\theta$, $r$ are corresponding partial derivatives.

For an incompressible material we can write

$$\mathrm{tr}\,\mathbf{L} = 2u + v_\theta + v\cot\theta + r u_r = 0. \tag{5.42}$$

The incompressibility condition (5.42) is satisfied if we define $u$ and $v$ in terms of function $\phi(\theta, r)$ such that

$$u = \frac{1}{r^2 \sin\theta}\frac{\partial\phi}{\partial\theta}, \quad v = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial r}. \tag{5.43}$$

For $i = 1$ the expression (2.37) gives

$$\dot{T}_{011,1} + \dot{T}_{021,2} + \dot{T}_{031,3} + 2r^{-1}\dot{T}_{031} + r^{-1}\dot{T}_{013} + r^{-1}\cot\theta(\dot{T}_{011} - \dot{T}_{022}) = 0. \tag{5.44}$$

In what follows we will consider the case when electric field is generated by the electrodes attached to the boundaries of a spherical shell. Therefore, according to Gauss's theorem there is no field outside the material. For the considered underlying deformation we have $F_{ij} = 0$ for $i \neq j$, and for radial electric displacement field ($D_{\mathrm{L}1} = D_{\mathrm{L}2} = 0$) required non-zero values of electroelastic moduli tensors $\boldsymbol{\mathcal{A}}_0^*$, $\mathbb{A}_0^*$, $\mathbf{A}_0^*$ can be obtained from the general expressions given in Section 2.2.2. Therefore, we can write using $(2.51)_1$

$$\dot{T}_{011} = \mathcal{A}^*_{01111}L_{11} + \mathcal{A}^*_{01122}L_{22} + \mathcal{A}^*_{01133}L_{33} + pL_{11} - \dot{p} + \mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03}, \tag{5.45}$$

$$\dot{T}_{022} = \mathcal{A}^*_{02211}L_{11} + \mathcal{A}^*_{01111}L_{22} + \mathcal{A}^*_{01133}L_{33} + pL_{22} - \dot{p} + \mathbb{A}^*_{022|3}\dot{D}_{\mathrm{L}03}, \tag{5.46}$$

$$\dot{T}_{013} = \mathcal{A}^*_{01313}L_{31} + \mathcal{A}^*_{01331}L_{13} + pL_{13} + \mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01}, \tag{5.47}$$

$$\dot{T}_{031} = \mathcal{A}^*_{03131}L_{13} + \mathcal{A}^*_{03113}L_{31} + pL_{31}. \tag{5.48}$$

Since there is no dependence on $\phi$ the derivative of $\dot{T}_{021}$ with respect to variable $\phi$ is zero in (5.44). Because the underlying deformation is radially symmetric in (5.46) we have used $\mathcal{A}^*_{01111} = \mathcal{A}^*_{02222}$ and $\mathcal{A}^*_{02233} = \mathcal{A}^*_{01133}$.

Substituting these expressions into (5.44) and using incompressibility condition (5.42) we find that (5.44) gives

$$\begin{aligned} r\dot{p}_\theta =&(u_\theta - v)[r(\mathcal{A}^{*'}_{01331} + p') + \mathcal{A}^*_{01331} + \mathcal{A}^*_{01313} + \mathcal{A}^*_{01122} - \mathcal{A}^*_{01111}] \\ &+ ru_{r\theta}[\mathcal{A}^*_{01331} + \mathcal{A}^*_{01133} - \mathcal{A}^*_{01111}] + rv_r(r\mathcal{A}^{*'}_{03131} + 2\mathcal{A}^*_{03131}) \\ &+ r^2 v_{rr}\mathcal{A}^*_{03131} + \mathbb{A}^*_{011|3} r\dot{D}_{\mathrm{L}03,\theta} + \mathbb{A}^*_{013|1} r\dot{D}_{\mathrm{L}01} + r\cot\theta\dot{D}_{\mathrm{L}03}(\mathbb{A}^*_{011|3} - \mathbb{A}^*_{022|3}), \end{aligned} \tag{5.49}$$

where prime denotes differentiation with respect to $r$.

For $i = 3$ in (2.37) we have

$$\dot{T}_{013,1} + \dot{T}_{023,2} + \dot{T}_{033,3} + 2\dot{T}_{033}r^{-1} + \dot{T}_{013}r^{-1}\cot\theta - (\dot{T}_{011} + \dot{T}_{022})r^{-1} = 0. \tag{5.50}$$

From $(2.51)_1$ we can calculate

$$\dot{T}_{033} = \mathcal{A}^*_{03311}L_{11} + \mathcal{A}^*_{02233}L_{22} + \mathcal{A}^*_{03333}L_{33} + pL_{33} - \dot{p} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03}. \tag{5.51}$$

Note that derivative $\dot{T}_{023,2} = 0$. Substituting expressions (5.45)-(5.47) and (5.51) in (5.50) and using (5.42) we have

$$\begin{aligned} r^2\dot{p}_r =&ru_r\{r(\mathcal{A}^{*'}_{03333} - \mathcal{A}^{*'}_{01133} + p') - 3\mathcal{A}^*_{01331} + \mathcal{A}^*_{01313} + \mathcal{A}^*_{01122} \\ &- 4\mathcal{A}^*_{01133} + 2\mathcal{A}^*_{03333} + \mathcal{A}^*_{01111}\} + r^2 u_{rr}(\mathcal{A}^*_{03333} - \mathcal{A}^*_{01331} - \mathcal{A}^*_{01133}) \\ &+ \mathcal{A}^*_{01313}(u_{\theta\theta} + u_\theta\cot\theta + 2u) + \mathbb{A}^*_{013|1}(r\dot{D}_{\mathrm{L}01,\theta} + r\cot\theta\dot{D}_{\mathrm{L}01}) \\ &+ \mathbb{A}^{*'}_{033|3}r^2\dot{D}_{\mathrm{L}03} + \mathbb{A}^*_{033|3}(r^2\dot{D}_{\mathrm{L}03,r} + 2r\dot{D}_{\mathrm{L}03}) \\ &- r\dot{D}_{\mathrm{L}03}(\mathbb{A}^*_{011|3} + \mathbb{A}^*_{022|3}). \end{aligned} \tag{5.52}$$

For the present case the governing equation $(2.32)_1$ reduces to

$$\dot{E}_{\mathrm{L}0\theta} + r\frac{\partial \dot{E}_{\mathrm{L}0\theta}}{\partial r} - \frac{\partial \dot{E}_{\mathrm{L}0r}}{\partial \theta} = 0. \tag{5.53}$$

From $(2.51)_2$ we calculate

$$\dot{E}_{\mathrm{L}0\theta} = \dot{E}_{\mathrm{L}01} = \mathbb{A}^*_{013|1}L_{31} + \mathsf{A}^*_{011}\dot{D}_{\mathrm{L}01}, \tag{5.54}$$

$$\dot{E}_{\mathrm{L}0r} = \dot{E}_{\mathrm{L}03} = \mathbb{A}^*_{011|3}L_{11} + \mathbb{A}^*_{022|3}L_{22} + \mathbb{A}^*_{033|3}L_{33} + \mathsf{A}^*_{033}\dot{D}_{\mathrm{L}03}. \tag{5.55}$$

Note that due to radial symmetry $\mathbb{A}^*_{011|3} = \mathbb{A}^*_{022|3}$. Therefore, equation (5.53) gives

$$\begin{aligned} &u_{r\theta}(\mathbb{A}^*_{013|1} + \mathbb{A}^*_{011|3} - \mathbb{A}^*_{033|3}) - \mathbb{A}^*_{013|1}v_r + \mathbb{A}^{*'}_{013|1}(u_\theta - v) \\ &+ \mathsf{A}^*_{011}(\dot{D}_{\mathrm{L}01} + r\dot{D}_{\mathrm{L}01,r}) + \mathsf{A}^{*'}_{011}r\dot{D}_{\mathrm{L}01} - \mathsf{A}^*_{033}\dot{D}_{\mathrm{L}03,\theta} = 0. \end{aligned} \tag{5.56}$$

The governing equation $(2.32)_2$ in spherical coordinates reduces to

$$\frac{1}{r^2}\frac{\partial(r^2\dot{D}_{L0r})}{\partial r}+\frac{1}{r}\frac{\partial\dot{D}_{L0\theta}}{\partial\theta}+\frac{\cot\theta}{r}\dot{D}_{L0\theta}=0, \tag{5.57}$$

which after some rearrangements can be written as

$$\frac{\partial(r^2\sin\theta\dot{D}_{L0r})}{\partial r}+\frac{\partial(r\sin\theta\dot{D}_{L0\theta})}{\partial\theta}=0. \tag{5.58}$$

Therefore, the governing equation $(2.32)_2$ is satisfied if we introduce function $\psi(\theta,r)$ such that

$$\dot{D}_{L0r}=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\quad \dot{D}_{L0\theta}=-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{5.59}$$

Differentiating $\dot{p}_\theta$ from (5.49) with respect to $r$ and $\dot{p}_r$ from (5.52) with respect to $\theta$ and using (5.43), (5.59) we obtain the governing equation in terms of functions $\phi$ and $\psi$

$$\begin{aligned}
&\mathcal{A}^*_{03131}\phi_{,rrrr}+\mathcal{A}^*_{01313}\phi_{,\theta\theta\theta\theta}+\frac{1}{r^2}(\mathcal{A}^*_{03333}-2\mathcal{A}^*_{01331}-2\mathcal{A}^*_{01133}+\mathcal{A}^*_{01111})\phi_{,\theta\theta rr} \qquad (5.60)\\
&+2\mathcal{A}^{*'}_{03131}\phi_{,rrr}-\frac{2}{r^4}\cot\theta\mathcal{A}^*_{01313}\phi_{,\theta\theta\theta}+\frac{1}{r^2}\cot\theta(2\mathcal{A}^*_{01331}+2\mathcal{A}^*_{01133}-\mathcal{A}^*_{01111}-\mathcal{A}^*_{03333})\phi_{,\theta rr}\\
&\frac{1}{r^3}(r\mathcal{A}^{*'}_{03333}-2r\mathcal{A}^{*'}_{01133}-2r\mathcal{A}^{*'}_{01331}+r\mathcal{A}^{*'}_{01111}+4\mathcal{A}^*_{01331}+4\mathcal{A}^*_{01133}-2\mathcal{A}^*_{03333}-2\mathcal{A}^*_{01111})\phi_{,\theta\theta r}\\
&\{\frac{1}{r^4}(3\cot^2\theta+4)\mathcal{A}^*_{01313}+\frac{1}{r^4}(3r\mathcal{A}^{*'}_{01331}-3\mathcal{A}^*_{01331}+\mathcal{A}^*_{01313}+\mathcal{A}^*_{01122}+\mathcal{A}^*_{01111}\\
&-2r(\mathcal{A}^{*'}_{03333}-\mathcal{A}^{*'}_{01133})-4\mathcal{A}^*_{01133}+2\mathcal{A}^*_{03333})+\frac{1}{r^3}(2\mathcal{A}^{*'}_{01133}-r(\mathcal{A}^{*''}_{01331}+p^{*''})\\
&-\mathcal{A}^{*'}_{01313}-\mathcal{A}^{*'}_{01122}-\mathcal{A}^{*'}_{01111})\}\phi_{,\theta\theta}+\{\frac{1}{r^3}\cot\theta(2r\mathcal{A}^{*'}_{01331}-r\mathcal{A}^{*'}_{03333}+2r\mathcal{A}^{*'}_{01133}\\
&-4\mathcal{A}^*_{01331}+2\mathcal{A}^*_{01111}-4\mathcal{A}^*_{01133}+2\mathcal{A}^*_{03333})-\frac{1}{r^2}\cot\theta\mathcal{A}^{*'}_{01111}\}\phi_{,\theta r}\\
&+\{\mathcal{A}^{*''}_{03131}-\frac{1}{r}\mathcal{A}^{*'}_{03131}-\frac{1}{r^2}[r(\mathcal{A}^{*'}_{01331}+p^{*'})+\mathcal{A}^*_{01331}+\mathcal{A}^*_{01313}+\mathcal{A}^*_{01122}-\mathcal{A}^*_{01111}]\}\phi_{,rr}\\
&+\{\frac{1}{r^4}\mathcal{A}^*_{01313}(-3\cot^3\theta-5\cot\theta)+\frac{1}{r^4}[2r(\mathcal{A}^{*'}_{03333}-\mathcal{A}^{*'}_{01133})+3\mathcal{A}^*_{01331}-\mathcal{A}^*_{01313}\\
&-\mathcal{A}^*_{01122}+4\mathcal{A}^*_{01133}-2\mathcal{A}^*_{03333}-\mathcal{A}^*_{01111}-3r\mathcal{A}^{*'}_{01331}]+\frac{1}{r^3}\cot\theta[r(\mathcal{A}^{*''}_{01331}+p^{*''})\\
&+\mathcal{A}^{*'}_{01313}+\mathcal{A}^{*'}_{01122}+\mathcal{A}^{*'}_{01111}-2\mathcal{A}^{*'}_{01133}]\}\phi_{,\theta}+\{\frac{2}{r^3}(rp^{*'}+\mathcal{A}^*_{01331}+\mathcal{A}^*_{01313}\\
&+\mathcal{A}^*_{01122}-\mathcal{A}^*_{01111})-\frac{1}{r^2}[p^{*'}+r(\mathcal{A}^{*''}_{01331}+p^{*''})+\mathcal{A}^{*'}_{01313}+\mathcal{A}^{*'}_{01122}-\mathcal{A}^{*'}_{01111}\\
&-\mathcal{A}^{*'}_{03131}+r\mathcal{A}^{*''}_{03131}]\}\phi_{,r}+\frac{1}{r^2}(\mathbb{A}^*_{033|3}-\mathbb{A}^*_{013|1}-\mathbb{A}^*_{011|3})\psi_{,r\theta\theta}\\
&+\frac{1}{r^2}\cot\theta(\mathbb{A}^*_{013|1}-\mathbb{A}^*_{033|3}+\mathbb{A}^*_{011|3})\psi_{,r\theta}+\frac{1}{r}\mathbb{A}^*_{013|1}\psi_{,rr}\\
&+\{\frac{1}{r^2}\mathbb{A}^{*'}_{033|3}-\frac{1}{r^3}(\mathbb{A}^*_{011|3}+\mathbb{A}^*_{022|3})-\frac{1}{r^2}\mathbb{A}^{*'}_{011|3}+\frac{2}{r^3}\mathbb{A}^*_{011|3}\}\psi_{,\theta\theta}\\
&+\{\frac{1}{r}\mathbb{A}^{*'}_{013|1}-\frac{1}{r^2}\mathbb{A}^*_{013|1}\}\psi_{,r}+\cot\theta\{\frac{1}{r^3}(\mathbb{A}^*_{011|3}+\mathbb{A}^*_{022|3})-\frac{1}{r^2}\mathbb{A}^{*'}_{033|3}\\
&+\frac{1}{r^2}\mathbb{A}^{*'}_{011|3}-\frac{2}{r^3}\mathbb{A}^*_{011|3}\}\psi_{,\theta}=0.
\end{aligned}$$

From (5.56) we obtain the second governing equation in terms of derivatives of functions

$\phi$ and $\psi$

$$
\begin{aligned}
&\frac{1}{r^2}(\mathbb{A}^*_{013|1}+\mathbb{A}^*_{011|3}-\mathbb{A}^*_{033|3})\phi_{,\theta\theta r}-\frac{1}{r^2}\cot\theta(\mathbb{A}^*_{013|1}+\mathbb{A}^*_{011|3}-\mathbb{A}^*_{033|3})\phi_{,\theta r} \\
&+\frac{1}{r}\mathbb{A}^*_{013|1}\phi_{,rr}+\{\frac{1}{r^2}\mathbb{A}^{*'}_{013|1}-\frac{2}{r^3}(\mathbb{A}^*_{013|1}+\mathbb{A}^*_{011|3}-\mathbb{A}^*_{033|3})\}\phi_{,\theta\theta} \\
&+(\frac{1}{r}\mathbb{A}^{*'}_{013|1}-\frac{1}{r^2}\mathbb{A}^*_{013|1})\phi_{,r}+\{\frac{2}{r^3}\cot\theta(\mathbb{A}^*_{013|1}+\mathbb{A}^*_{011|3}-\mathbb{A}^*_{033|3})-\frac{1}{r^2}\cot\theta\mathbb{A}^{*'}_{013|1}\}\phi_{,\theta} \\
&-\mathsf{A}^*_{011}\psi_{,rr}-\frac{1}{r^2}\mathsf{A}^*_{033}\psi_{,\theta\theta}-\mathsf{A}^{*'}_{011}\psi_{,r}+\frac{1}{r^2}\cot\theta\mathsf{A}^*_{033}\psi_{,\theta}=0.
\end{aligned}
\tag{5.61}
$$

Now we will specialize the boundary condition (2.35). Since for the present case when electric field is generated by electrodes there is no field outside the material. We have

$$
\dot{\mathbf{T}}_0^{\mathrm{T}}\mathbf{n}=\dot{\mathbf{t}}_{\mathrm{A}0}=\begin{cases}P\mathbf{L}^{\mathrm{T}}\mathbf{n}-\dot{P}\mathbf{n} & \text{on } r=a,\\ 0 & \text{on } r=b.\end{cases}
\tag{5.62}
$$

Calculations show that

$$
rv_r+u_\theta-v=0 \quad \text{on} \quad r=a,b. \tag{5.63}
$$

$$
(\mathcal{A}^*_{03333}-\mathcal{A}^*_{03311}+\tau_3)u_r-\dot{p}+\mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03}=\begin{cases}-\dot{P} & \text{on } r=a,\\ 0 & \text{on } r=b.\end{cases}
\tag{5.64}
$$

The remaining component of (5.62) is satisfied automatically.

Boundary condition (5.63) in terms of function $\phi$ and its derivatives can be written as

$$
r^2\phi_{,rr}-\phi_{,\theta\theta}+\cot\theta\phi_{,\theta}-2r\phi_{,r}=0 \quad \text{on} \quad r=a,b. \tag{5.65}
$$

In (5.64) we accept that $\dot{P}=0$ and differentiate (5.64) with respect to $\theta$ and use (5.49). Thus we obtain boundary condition

$$
\begin{aligned}
&\mathcal{A}^*_{03131}\phi_{,rrr}+\{\frac{1}{r^2}(\mathcal{A}^*_{03333}-2\mathcal{A}^*_{01133}+\tau_3-\mathcal{A}^*_{01331}+\mathcal{A}^*_{01111})\}\phi_{,\theta\theta r} \\
&-\frac{1}{r^3}[r(\mathcal{A}^{*'}_{01331}+p^{*'})-\mathcal{A}^*_{01331}+\mathcal{A}^*_{01313}+\mathcal{A}^*_{01122}+\mathcal{A}^*_{01111}+2\mathcal{A}^*_{03333} \\
&-4\mathcal{A}^*_{01133}+2\tau_3]\phi_{,\theta\theta}+\frac{1}{r^2}\cot\theta(\mathcal{A}^*_{01331}+2\mathcal{A}^*_{01133}-\mathcal{A}^*_{01111}-\mathcal{A}^*_{03333}-\tau_3)\phi_{,\theta r} \\
&+\mathcal{A}^{*'}_{03131}\phi_{,rr}+\frac{1}{r^3}\cot\theta[r(\mathcal{A}^{*'}_{01331}+p^{*'})-\mathcal{A}^*_{01331}+\mathcal{A}^*_{01313}+\mathcal{A}^*_{01122}+\mathcal{A}^*_{01111} \\
&+2\mathcal{A}^*_{03333}-4\mathcal{A}^*_{01133}+2\tau_3]\phi_{,\theta}-\frac{1}{r^2}[r(\mathcal{A}^{*'}_{01331}+p^{*'})+\mathcal{A}^*_{01331}+\mathcal{A}^*_{01313} \\
&+\mathcal{A}^*_{01122}-\mathcal{A}^*_{01111}+r\mathcal{A}^{*'}_{03131}]\phi_{,r}+\frac{1}{r^2}(\mathbb{A}^*_{033|3}-\mathbb{A}^*_{011|3})\psi_{,\theta\theta}+\frac{1}{r}\mathbb{A}^*_{013|1}\psi_{,r} \\
&+\frac{1}{r^2}\cot\theta(\mathbb{A}^*_{022|3}-\mathbb{A}^*_{033|3})\psi_{,\theta}=0 \quad \text{on} \quad r=a,b.
\end{aligned}
\tag{5.66}
$$

The electrical boundary condition (2.33) will reduce to

$$
\dot{E}_{\mathrm{L}01}=\dot{E}_{\mathrm{L}0\theta}=0 \quad \text{on} \quad r=a,b, \tag{5.67}
$$

which can be rewritten as

$$\mathbb{A}^*_{013|1}(\phi_{,\theta\theta} - \cot\theta\phi_{,\theta} + r\phi_{,r}) - r^2\mathsf{A}^*_{011}\psi_{,r} = 0 \quad \text{on} \quad r = a, b. \tag{5.68}$$

The boundary condition (2.34) will reduce to

$$\dot{D}_{\mathrm{L}0r} = \begin{cases} -\dot{\sigma}_{\mathrm{F}0b} & \text{on } r = b, \\ \dot{\sigma}_{\mathrm{F}0a} & \text{on } r = a, \end{cases} \tag{5.69}$$

where $\dot{\sigma}_{\mathrm{F}0} = \dot{\sigma}_{\mathrm{F}}\mathrm{d}A/\mathrm{d}a$ a is the increment of the free surface charge $\sigma_{\mathrm{F}}$ per unit area of $\partial\mathcal{B}$, and $\mathrm{d}A/\mathrm{d}a$ is the ratio of area elements on $\partial\mathcal{B}_r$ and $\partial\mathcal{B}$. For the considered problem free surface charges at the boundaries per unit area are different by the absolute value (and sign, of course). Therefore, in general the increments will be also different at the boundaries. Thus, we can write

$$\dot{\sigma}_{\mathrm{F}0a} = \dot{\sigma}_{\mathrm{F}}|_{r=a}\frac{\mathrm{d}A}{\mathrm{d}a} = \dot{\sigma}_{\mathrm{F}}|_{r=a}\frac{A^2}{a^2} \tag{5.70}$$

at the inner boundary, and

$$\dot{\sigma}_{\mathrm{F}0b} = \dot{\sigma}_{\mathrm{F}}|_{r=b}\frac{\mathrm{d}A}{\mathrm{d}a} = \dot{\sigma}_{\mathrm{F}}|_{r=b}\frac{B^2}{b^2} \tag{5.71}$$

at the outer boundary.

Here again we require only the boundary condition (5.67) to be satisfied explicitly. Boundary condition (5.69) will be adjusted according to the solution of the problem with the boundary condition (5.67) satisfied explicitly.

In order to have equations consistent with Haughton & Ogden (1978) we write

$$\phi = -\frac{1}{m}r^2 f_n(r)\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_n(\cos\theta), \quad \psi = -\frac{1}{m}g_n(r)\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_n(\cos\theta), \tag{5.72}$$

where $P_n(\cos\theta)$ is the Legendre polynomial of degree $n$ and $m = n(n+1)$.

Using the identity

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}P_n(\cos\theta) + \cot\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_n(\cos\theta) + n(n+1)P_n(\cos\theta) = 0 \tag{5.73}$$

and (5.72) we can rewrite governing equations (5.60) and (5.61) as

$$\begin{aligned} &r^4\mathcal{A}^*_{03131}f_n'''' + (8r^3\mathcal{A}^*_{03131} + 2r^4\mathcal{A}^{*'}_{03131})f_n''' + \{10r^3\mathcal{A}^{*'}_{03131} + r^4\mathcal{A}^{*''}_{03131} \\ &+ 12r^2\mathcal{A}^*_{03131} + r^3\tau_{33}' + r^2(mG - F)\}f_n'' + \{3r^2(2\mathcal{A}^{*'}_{03131} + \tau_{33}') \\ &+ r^3(2\mathcal{A}^{*''}_{03131} + \tau_{33}'') + 2r(mG - F) + r^2(mG' - F')\}f_n' \\ &+ (m-2)(r^2\mathcal{A}^{*''}_{03131} - r^2\tau_{33}'' + rF'(r) - F(r) + m\mathcal{A}^*_{01313})f_n + \mathbb{A}^*_{013|1}rg_n'' \\ &+ \{m(\mathbb{A}^*_{013|1} - \mathbb{A}^*_{033|3} + \mathbb{A}^*_{011|3}) + (r\mathbb{A}^{*'}_{013|1} - \mathbb{A}^*_{013|1})\}g_n' - m(\mathbb{A}^{*'}_{033|3} - \mathbb{A}^{*'}_{011|3})g_n = 0, \end{aligned} \tag{5.74}$$

$$\mathbb{A}^*_{013|1} r f_n'' + \{3\mathbb{A}^*_{013|1} + r\mathbb{A}^{*'}_{013|1} - m(\mathbb{A}^*_{013|1} + \mathbb{A}^*_{011|3} - \mathbb{A}^*_{033|3})\} f_n' \tag{5.75}$$
$$+ (2\mathbb{A}^{*'}_{013|1} - m\mathbb{A}^{*'}_{013|1}) f_n - \mathsf{A}^*_{011} g_n'' - \mathsf{A}^{*'}_{011} g_n' + m\mathsf{A}^*_{033} g_n / r^2 = 0.$$

As in Haughton & Ogden (1978) in (5.74) for brevity we denoted

$$F(r) = \mathcal{A}^*_{01331} + \mathcal{A}^*_{01313} + \mathcal{A}^*_{01122} - \mathcal{A}^*_{01111}, \tag{5.76}$$

$$G(r) = 2\mathcal{A}^*_{01331} + 2\mathcal{A}^*_{01133} - \mathcal{A}^*_{03333} - \mathcal{A}^*_{01111}. \tag{5.77}$$

Also we have used expressions for the first and second derivatives of Lagrange multiplier $p^*$ with respect to $r$

$$p^{*'} = \mathcal{A}^{*'}_{03131} - \mathcal{A}^{*'}_{01331} + 2(\mathcal{A}^*_{03131} - \mathcal{A}^*_{01313})/r, \tag{5.78}$$

$$p^{*''} = \mathcal{A}^{*''}_{03131} - \mathcal{A}^{*''}_{01331} - 2(\mathcal{A}^*_{03131} - \mathcal{A}^*_{01313})/r^2 + 2(\mathcal{A}^{*'}_{03131} - \mathcal{A}^{*'}_{01313})/r \tag{5.79}$$

and the following connections

$$\tau_3 = \mathcal{A}^*_{03131} - \mathcal{A}^*_{01331}, \tag{5.80}$$

$$\tau_{33}' = \tau_3' - p^{*'} = \frac{2}{r}(\mathcal{A}^*_{01313} - \mathcal{A}^*_{03131}), \tag{5.81}$$

$$r^2 \tau_{33}'' = 2(r\mathcal{A}^{*'}_{01313} - r\mathcal{A}^{*'}_{03131} - \mathcal{A}^*_{01313} + \mathcal{A}^*_{03131}). \tag{5.82}$$

Boundary conditions (5.65), (5.66) and (5.68) take the form, respectively

$$r^2 f_n'' + 2r f_n' + (m-2) f_n = 0 \quad \text{on} \quad r = a, b, \tag{5.83}$$

$$r^3 \mathcal{A}^*_{03131} f_n''' + r^2(r\mathcal{A}^{*'}_{03131} + 6\mathcal{A}^*_{03131}) f_n'' + r[2r\mathcal{A}^{*'}_{03131} + r\tau_{33}' - F \tag{5.84}$$
$$+ m(G - \mathcal{A}^*_{03131}) + 6\mathcal{A}^*_{03131}] f_n' + (m-2)(r\mathcal{A}^{*'}_{03131} - r\tau_{33}' + F) f_n$$
$$+ \mathbb{A}^*_{013|1} g_n' - \frac{m}{r}(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{011|3}) g_n = 0 \quad \text{on} \quad r = a, b,$$

$$\mathbb{A}^*_{013|1}\{(m-2) f_n - r f_n'\} + \mathsf{A}^*_{011} g_n' = 0 \quad \text{on} \quad r = a, b. \tag{5.85}$$

In what follows we give non-dimensional equations. Expressions (5.43) and $(5.72)_1$, (5.59) and $(5.72)_2$ suggest the following definition for non-dimensional functions $\hat{f}_n(\hat{r})$ and $\hat{g}_n(\hat{r})$

$$\hat{f}_n(\hat{r}) = \frac{f(r)}{A}, \quad \hat{g}_n(\hat{r}) = \frac{g_n(r)}{D_r(a)A^2}. \tag{5.86}$$

The other quantities are defined in the same way as they were defined for the cylindrical shell. We also introduce new variables to transform the system of the governing equations (5.74) and (5.75) into the system of 6 ODEs

$$\hat{y}_1(\hat{r}) = \hat{f}_n(\hat{r}), \quad \hat{y}_2(\hat{r}) = \hat{f}_n'(\hat{r}), \quad \hat{y}_3(\hat{r}) = \hat{f}_n''(\hat{r}), \tag{5.87}$$
$$\hat{y}_4(\hat{r}) = \hat{f}_n'''(\hat{r}), \quad \hat{y}_5(\hat{r}) = \hat{g}_n(\hat{r}), \quad \hat{y}_6(\hat{r}) = \hat{g}_n'(\hat{r}).$$

For our numerical calculations MATLAB ODE solver requires representation of the governing equations as a system of 6 first-order ODEs.

Now we can rewrite the governing equations and the boundary conditions in non-dimensional form. The governing equations transform into the following system of non-dimensional equations

$$
\begin{aligned}
&\hat{y}_1' = \hat{y}_2, \\
&\hat{y}_2' = \hat{y}_3, \\
&\hat{y}_3' = \hat{y}_4, \\
&\hat{r}^4\hat{\mathcal{A}}^*_{03131}\hat{y}_4' + (8\hat{r}^3\hat{\mathcal{A}}^*_{03131} + 2\hat{r}^4\hat{\mathcal{A}}^{*'}_{03131})\hat{y}_4 + \{10\hat{r}^3\hat{\mathcal{A}}^{*'}_{03131} + \hat{r}^4\hat{\mathcal{A}}^{*''}_{03131} \\
&+ 12\hat{r}^2\hat{\mathcal{A}}^*_{03131} + \hat{r}^3\hat{\tau}'_{33} + \hat{r}^2(m\hat{G} - \hat{F})\}\hat{y}_3 + \{3\hat{r}^2(2\hat{\mathcal{A}}^{*'}_{03131} + \hat{\tau}'_{33}) \\
&+ \hat{r}^3(2\hat{\mathcal{A}}^{*''}_{03131} + \hat{\tau}''_{33}) + 2\hat{r}(m\hat{G} - \hat{F}) + \hat{r}^2(m\hat{G}' - \hat{F}')\}\hat{y}_2 \\
&+ (m-2)(\hat{r}^2\hat{\mathcal{A}}^{*''}_{03131} - \hat{r}^2\hat{\tau}''_{33} + \hat{r}\hat{F}' - \hat{F} + m\hat{\mathcal{A}}^*_{01313})\hat{y}_1 + \hat{\mathbb{A}}^*_{013|1}\hat{r}\hat{\sigma}^2_{fa}\hat{y}_6' \\
&+ \{m(\hat{\mathbb{A}}^*_{013|1} - \hat{\mathbb{A}}^*_{033|3} + \hat{\mathbb{A}}^*_{011|3}) + (\hat{r}\hat{\mathbb{A}}^{*'}_{013|1} - \hat{\mathbb{A}}^*_{013|1})\}\hat{\sigma}^2_{fa}\hat{y}_6 - m(\hat{\mathbb{A}}^{*'}_{033|3} - \hat{\mathbb{A}}^{*'}_{011|3})\hat{\sigma}^2_{fa}y_5 = 0, \\
&\hat{y}_5' = \hat{y}_6, \\
&\hat{\mathbb{A}}^*_{013|1}\hat{r}\hat{y}_3 + \{3\hat{\mathbb{A}}^*_{013|1} + \hat{r}\hat{\mathbb{A}}^{*'}_{013|1} - m(\hat{\mathbb{A}}^*_{013|1} + \hat{\mathbb{A}}^*_{011|3} - \hat{\mathbb{A}}^*_{033|3})\}\hat{y}_2 \\
&+ (2\hat{\mathbb{A}}^{*'}_{013|1} - m\hat{\mathbb{A}}^{*'}_{013|1})y_1 - \hat{\mathsf{A}}^*_{011}\hat{y}_6' - \hat{\mathsf{A}}^{*'}_{011}\hat{y}_6 + m\hat{\mathsf{A}}^*_{033}\hat{y}_5/\hat{r}^2 = 0.
\end{aligned}
\tag{5.88}
$$

The boundary conditions in terms of new variables in non-dimensional form are

$$\hat{r}^2\hat{y}_3 + 2\hat{r}\hat{y}_2 + (m-2)\hat{y}_1 = 0 \quad \text{on} \quad r = \hat{a}, \hat{b}, \tag{5.89}$$

$$
\begin{aligned}
&\hat{r}^3\hat{\mathcal{A}}^*_{03131}\hat{y}_4 + \hat{r}^2(\hat{r}\hat{\mathcal{A}}^{*'}_{03131} + 6\hat{\mathcal{A}}^*_{03131})\hat{y}_3 + \hat{r}[2\hat{r}\hat{\mathcal{A}}^{*'}_{03131} + \hat{r}\hat{\tau}'_{33} - \hat{F} \\
&+ m(\hat{G} - \hat{\mathcal{A}}^*_{03131}) + 6\hat{\mathcal{A}}^*_{03131}]\hat{y}_2 + (m-2)(\hat{r}\hat{\mathcal{A}}^{*'}_{03131} - \hat{r}\hat{\tau}'_{33} + \hat{F})\hat{y}_1 \\
&+ \hat{\mathbb{A}}^*_{013|1}\hat{\sigma}^2_{fa}\hat{y}_6 - \frac{m}{\hat{r}}(\hat{\mathbb{A}}^*_{033|3} - \hat{\mathbb{A}}^*_{011|3})\hat{\sigma}^2_{fa}\hat{y}_5 = 0 \quad \text{on} \quad r = \hat{a}, \hat{b},
\end{aligned}
\tag{5.90}
$$

$$\hat{\mathbb{A}}^*_{013|1}\{(m-2)\hat{y}_1 - \hat{r}\hat{y}_2\} + \hat{\mathsf{A}}^*_{011}\hat{y}_6 = 0 \quad \text{on} \quad r = \hat{a}, \hat{b}. \tag{5.91}$$

In the above mentioned non-dimensional expressions we used non-dimensional parameter

$$\hat{\sigma}^2_{fa} = \frac{D^2_r(a)}{\varepsilon\mu}. \tag{5.92}$$

We calculate required electroelastic moduli for neo-Hookean electroelastic material

(4.117):

$$\begin{aligned}
&\mathcal{A}^*_{03131} = \mathcal{A}^*_{03333} = 2\lambda_3^2\Omega_1^* + 2D_3^2\Omega_5^*, \\
&\mathcal{A}^*_{01313} = \mathcal{A}^*_{01111} = 2\lambda_1^2\Omega_1^*, \\
&\mathcal{A}^*_{01331} = \mathcal{A}^*_{01122} = \mathcal{A}^*_{01133} = 0, \\
&2\mathbb{A}^*_{013|1} = \mathbb{A}^*_{033|3} = 4D_3\Omega_5^*, \quad \mathbb{A}^*_{011|3} = 0, \\
&\mathsf{A}^*_{011} = \mathsf{A}^*_{033} = 2\Omega_5^*, \quad F(r) = 0.
\end{aligned} \tag{5.93}$$

In non-dimensional form the expressions for moduli can be rewritten as

$$\begin{aligned}
&\hat{\mathcal{A}}^*_{03131}(\hat{r}) = \mathcal{A}^*_{03333}(\hat{r}) = \frac{(\hat{r}^3A^3 - a^3 + A^3)^{4/3}}{\hat{r}^4A^4} + \frac{\hat{\sigma}_{fa}^2 a^4}{\hat{r}^4A^4}, \\
&\hat{\mathcal{A}}^*_{01313}(\hat{r}) = \hat{\mathcal{A}}^*_{01111}(\hat{r}) = \frac{\hat{r}^2A^2}{(\hat{r}^3A^3 - a^3 + A^3)^{2/3}}, \\
&\hat{\mathcal{A}}^*_{01331}(\hat{r}) = \hat{\mathcal{A}}^*_{01122}(\hat{r}) = \hat{\mathcal{A}}^*_{01133}(\hat{r}) = 0, \\
&2\hat{\mathbb{A}}^*_{013|1}(\hat{r}) = \hat{\mathbb{A}}^*_{033|3}(\hat{r}) = \frac{2a^2}{\hat{r}^2A^2}, \quad \hat{\mathbb{A}}^*_{011|3}(\hat{r}) = 0, \\
&\hat{\mathsf{A}}^*_{011}(\hat{r}) = \hat{\mathsf{A}}^*_{033}(\hat{r}) = 1, \quad \hat{F}(\hat{r}) = 0, \\
&\hat{G}(\hat{r}) = \frac{-(\hat{r}^3A^3 - a^3 + A^3)^{4/3} - \hat{\sigma}_{fa}^2 a^4}{\hat{r}^4A^4} - \frac{\hat{r}^2A^2}{(\hat{r}^3A^3 - a^3 + A^3)^{2/3}}.
\end{aligned} \tag{5.94}$$

For our calcualtions we used numerical scheme described in Chapter 4, the results of our calculations are given in Table 5.1.

Haughton & Ogden (1978) found that for neo-Hookean material axisymmetric bifurcations are possible at external pressure ($P < 0$) only, no bifurcation solutions were found for internally pressurized spherical shells. From the results in Table 5.1 we see that the significant difference for the present problem of bifucation analysis of electrically sensitive material is that some modes become possible for internally pressurized spherical shell ($P > 0$). We note that here we used a numerical scheme described in Chapter 4. For the purely mechanical case our calculations are very close to those reported in Haughton & Ogden (1978); for convenience we reproduce the results from Haughton & Ogden (1978) here in Table 5.2. We note that Haughton & Ogden (1978) used a different numerical scheme in their work for bifurcation analysis of spherical shells. For thin shells Haughton & Ogden (1978) reported that their method becomes increasingly sensitive.

The purely mechanical case of axisymmetric bifurcations of inflated and compressed spherical shells was considered in a more recent work of deBotton et al. (2013). They used the same theory as in Haughton & Ogden (1978) with different strain energy functions.

Table 5.1: Bifurcation values $\lambda_a$, $\lambda_b$ and non-dimensional pressure $P/\mu$ for neo-Hookean (4.117) electroelastic spherical shell for different values of parameter $\hat{\sigma}_{fa}$ and different values $A/B$

| $A/B$ | mode $n$ | $\hat{\sigma}_{fa} = 0$ | | | $\hat{\sigma}_{fa} = 0.25$ | | | $\hat{\sigma}_{fa} = 0.5$ | | | $\hat{\sigma}_{fa} = 0.75$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\lambda_a$ | $\lambda_b$ | $P/\mu$ | $\lambda_a$ | $\lambda_b$ | $P/\mu$ | $\lambda_a$ | $\lambda_b$ | $P/\mu$ | $\lambda_a$ | $\lambda_b$ | $P/\mu$ |
| 0.95 | 7 | 0.9817 | 0.9844 | -0.0112 | 1.0017 | 1.0015 | -0.0048 | 1.0712 | 1.0616 | 0.01128 | | | |
| 0.9 | 5 | 0.9600 | 0.9712 | -0.0505 | 0.9783 | 0.9843 | -0.0367 | 1.0430 | 1.0317 | 0.000489 | 1.1973 | 1.1503 | 0.0457 |
| 0.85 | 4 | 0.9345 | 0.9608 | -0.1276 | 0.9508 | 0.9704 | -0.1065 | 1.0090 | 1.0055 | -0.045378 | 1.1551 | 1.1004 | 0.0433 |
| 0.8 | 3 | 0.9063 | 0.9543 | -0.2493 | 0.9200 | 0.9607 | -0.2235 | 0.9683 | 0.9840 | -0.14549 | 1.0913 | 1.0487 | -0.0109 |
| 0.7 | 2 | 0.8356 | 0.9499 | -0.6995 | 0.8440 | 0.9522 | -0.8440 | 0.8724 | 0.9600 | -0.5882 | 0.9367 | 0.9792 | -0.4230 |
| 0.6 | 2 | 0.7744 | 0.9598 | -1.3002 | 0.7803 | 0.9607 | -1.2724 | 0.8000 | 0.9636 | -1.18229 | 0.8413 | 0.9700 | -1.0092 |

Table 5.2: Bifurcation values $\lambda_b$ for neo-Hookean spherical shell with different values $A/B$ and mode numbers $n$ from Haughton & Ogden (1978)

| $A/B$ | $n$ | $\lambda_b$ |
|---|---|---|
| 0.95 | 7 | 0.985 |
| 0.90 | 5 | 0.971 |
| 0.85 | 4 | 0.961 |
| 0.80 | 3 | 0.955 |
| 0.70 | 2 | 0.950 |
| 0.60 | 2 | 0.960 |

They also reported that additional solutions were found, which were not reported in literature before, for one-term Ogden material. Unlike Haughton & Ogden (1978), they used a different numerical scheme known as matrix compound method. More details about this numerical technique can be found in their work and in references mentioned therein.

# Appendix A

# Some Details

Here we provide some details of some non-trivial calculations.

## A.1 Formula which allows to change the variable of integration from $r$ to $\lambda$

In Section 3.2.4 we used formula (3.39), which allows us to change the variable of integration. The details for this calculation are as follows.

Since we have incompressible deformation, the volume of a cylinder remains unchanged:

$$\pi(B^2 - A^2)L = \pi(b^2 - a^2)l. \tag{A.1}$$

Therefore, we have

$$\begin{aligned} r^2 - a^2 &= \lambda_z^{-1}(R^2 - A^2) \\ &= \lambda_z^{-1}(\lambda^{-2}r^2 - A^2), \end{aligned} \tag{A.2}$$

where we introduced notation $\lambda_z = l/L$.

We rearrange the previous expression as follows:

$$r^2(1 - \lambda^{-2}\lambda_z^{-1}) = a^2 - \lambda_z^{-1}A^2. \tag{A.3}$$

Differentiating (A.3) with respect to $r$, and taking into account that the vertical stretch $\lambda_z$ does not depend on $r$, whereas circumferetial stretch $\lambda$ does depend on $r$ we have:

$$2r(1 - \lambda^2\lambda_z^{-1}) + r^2(2\lambda^{-3}\lambda_z^{-1})\frac{\mathrm{d}\lambda}{\mathrm{d}r} = 0. \tag{A.4}$$

We have

$$r\frac{\mathrm{d}\lambda}{\mathrm{d}r} = -(\lambda^2\lambda_z - 1)\lambda, \tag{A.5}$$

or

$$\frac{\mathrm{d}r}{r} = -\frac{\mathrm{d}\lambda}{\lambda(\lambda^2\lambda_z - 1)}. \tag{A.6}$$

Thus, we obtained the required relation. Using a similar reasoning a corresponding relation for a spherical shell used in Chapter 5 can be obtained.

## A.2 Expression for axial stress

Using (3.32), (3.31) and the equilibrium equation (3.36), axial stress $\tau_{zz}$ can be expressed as

$$\begin{aligned}
\tau_{zz} &= \tau_{rr} + \lambda_z\omega^*_{\lambda_z} \qquad (A.7)\\
&= \tau_{rr} - \tau_{\theta\theta} + \tau_{\theta\theta} + \lambda_z\omega^*_{\lambda_z}\\
&= -\lambda\omega^*_{\lambda} + \tau_{\theta\theta} + \lambda_z\omega^*_{\lambda_z}\\
&= \tau_{\theta\theta} - \tau_{rr} + \tau_{rr} - \lambda\omega^*_{\lambda} + \lambda_z\omega^*_{\lambda_z}\\
&= \frac{\tau_{\theta\theta} - \tau_{rr}}{2} + \frac{\tau_{\theta\theta} - \tau_{rr}}{2} + \tau_{rr} - \lambda\omega^*_{\lambda} + \lambda_z\omega^*_{\lambda_z}\\
&= \frac{1}{2}r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \tau_{rr} + \frac{1}{2}\lambda\omega^*_{\lambda} - \lambda\omega^*_{\lambda} + \lambda_z\omega^*_{\lambda_z}\\
&= \frac{1}{2}\left(r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + 2\tau_{rr}\right) - \frac{\lambda\omega^*_{\lambda}}{2} + \lambda_z\omega^*_{\lambda_z}\\
&= \frac{1}{2}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\tau_{rr})\right] - \frac{\lambda\omega^*_{\lambda}}{2} + \lambda_z\omega^*_{\lambda_z}.
\end{aligned}$$

# Appendix B

# Listing of computer programs

Here we give the MATLAB code we used for our numerical calculations in Chapter 4. The following code will reproduce the results shown in Fig. 4.1. The other results can be obtained with some minor modifications of this code. The code consists of 3 MATLAB files. Discussion of this code can be found after MATLAB files, given here below.

File calculate_branch.m has the following contents:

```
function branch=calculate_branch(par,start,delta,Nb,dir)
% Now we initialize a matrix in which we will store the calculated current radius a.
% We take Nb steps along the branch, each step along the branch is size delta.
branch = zeros(2, Nb);

initial_guess=start(2); %Here the intial guess for radious a.

options = optimset('Display','iter','TolFun',1e-16);

%Here we find a point on the branch
[a, cond_val] = fsolve(@(a)determinant(lz,a,par), initial_guess,options);

branch(1,1)=lz;
branch(2,1)=a;

```

```
%plot(branch(1,2), branch(2,2),'*');

options = optimset('Display','iter','TolFun',1e-16);

for i = 3:Nb

    [x,fval]=fsolve(@(x) branch_fun(x,[branch(1,i-1);branch(2,i-1)],delta)
       ,[2*branch(1,i-1)-branch(1,i-2);2*branch(2,i-1)-branch(2,i-2)],
       options);

    branch(1, i) = x(1);
    branch(2, i) = x(2);

    plot(branch(1,1:i), branch(2,1:i),'-k'); hold on;
    drawnow;
end

function out=branch_fun(x,x0,d)
out=[determinant(x(1),x(2),par);
    (x(1)-x0(1))^2+(x(2)-x0(2))^2-d^2;];
end

end
```

File determinant.m has the following lines:

```
function [answer] = determinant(lz,a,par)

% Specifying the initial geometry

a0=par.a0;
b0=par.b0;
L=par.L;
s=par.s;
```

```

% sigma denoted here as s is an electrical parameter

alpha = pi*a0/(lz*L);

% Specifying the deformation

b = sqrt(a^2 + (b0^2 - a0^2)/lz);

% Specifying the radii in non-dimensional form
ra = a/a0;
rb = b/a0;

% Solving the IVP
Tspan = [ra,rb];
%%%%
RelTol=1e-7; %relative tolerance for ODE solver
%%%%
options = odeset('Mass', @MASS, 'RelTol',RelTol,'MStateDependence','none');

[~,Y1] = ode15s(@ode, Tspan, [1 0 0 0 0 0],options);
[~,Y2] = ode15s(@ode, Tspan, [0 1 0 0 0 0],options);
[~,Y3] = ode15s(@ode, Tspan, [0 0 1 0 0 0],options);
[~,Y4] = ode15s(@ode, Tspan, [0 0 0 1 0 0],options);
[~,Y5] = ode15s(@ode, Tspan, [0 0 0 0 1 0],options);
[~,Y6] = ode15s(@ode, Tspan, [0 0 0 0 0 1],options);

 function dydx = ode(r,y)
        dydx = [y(2);
            y(3);
            y(4);
            -(2*r^4*der_a03232(r)+2*r^3*a03232(r))*y(4)-(r^3*(3*der_a03232(r
               )+r*der2_a03232(r))-3*r^2*a03232(r)-alpha^2*r^4*(a03333(r)+
```

```
                a02222(r)))*y(3)-(r^3*der2_a03232(r)-3*r^2*der_a03232(r)+3*r
                *a03232(r)+alpha^2*r^3*(-r*der_a03333(r)-a03333(r)-a02222(r)
                ))*y(2)-(3*(r*der_a03232(r)-a03232(r))-r^2*der2_a03232(r)+
                alpha^2*r^2*(r^2*der2_p(r)+a01111(r)+a02222(r))+alpha^4*r^4*
                a02323(r))*y(1)-s^2*alpha^2*r^3*(a0232(r)-a0333(r))*y(6)+s
                ^2*alpha^2*r^3*der_a0333(r)*y(5);
            y(6);
            -alpha^2*r^2*(a0333(r)-a0232(r))*y(2)+alpha^2*r^2*der_a0232(r)*y
                (1)-a022(r)*y(6)-alpha^2*r*a033(r)*y(5)];
        % defining the ODE
    end

  function ret = MASS(r,y)
        ret = [1 0 0 0 0 0;
               0 1 0 0 0 0;
               0 0 1 0 0 0;
               0 0 0 r^4*a03232(r) 0 0;
               0 0 0 0 1 0;
               0 0 0 0 0 -a022(r)*r];
        % Mass matrix of the ODE

    end
 %
m11 = bc1(ra,Y1(1,:));
m12 = bc1(ra,Y2(1,:));
m13 = bc1(ra,Y3(1,:));
m14 = bc1(ra,Y4(1,:));
m15 = bc1(ra,Y5(1,:));
m16 = bc1(ra,Y6(1,:));
%
m21 = bc3(ra,Y1(1,:));
m22 = bc3(ra,Y2(1,:));
m23 = bc3(ra,Y3(1,:));
```

```
m24 = bc3(ra,Y4(1,:));
m25 = bc3(ra,Y5(1,:));
m26 = bc3(ra,Y6(1,:));
%
m51 = bc2(ra,Y1(1,:));
m52 = bc2(ra,Y2(1,:));
m53 = bc2(ra,Y3(1,:));
m54 = bc2(ra,Y4(1,:));
m55 = bc2(ra,Y5(1,:));
m56 = bc2(ra,Y6(1,:));
%
%
m31 = bc1(rb,Y1(end,:));
m32 = bc1(rb,Y2(end,:));
m33 = bc1(rb,Y3(end,:));
m34 = bc1(rb,Y4(end,:));
m35 = bc1(rb,Y5(end,:));
m36 = bc1(rb,Y6(end,:));
%
%
m41 = bc3(rb,Y1(end,:));
m42 = bc3(rb,Y2(end,:));
m43 = bc3(rb,Y3(end,:));
m44 = bc3(rb,Y4(end,:));
m45 = bc3(rb,Y5(end,:));
m46 = bc3(rb,Y6(end,:));
%
m61 = bc2(rb,Y1(end,:));
m62 = bc2(rb,Y2(end,:));
m63 = bc2(rb,Y3(end,:));
m64 = bc2(rb,Y4(end,:));
m65 = bc2(rb,Y5(end,:));
m66 = bc2(rb,Y6(end,:));
```

```
matr = [m11 m12 m13 m14 m15 m16; m21 m22 m23 m24 m25 m26; m31 m32 m33 m34 m35 m36; m41 m42 m43 m44 m45 m46; m51 m52 m53 m54 m55 m56; m61 m62 m63 m64 m65 m66];

answer = 1/(cond(matr));

% Boundary conditions

    function ret = bc1(r,y)
        ret = r^2*y(3)+r*y(2)+(alpha^2*r^2-1)*y(1);
    end

    function ret = bc2(r,y)
        % The electrical boundary condition
        ret = alpha^2*r*a0232(r)*y(1)+a022(r)*y(6);
    end

    function ret = bc3(r,y)
        ret = a03232(r)*r^3*y(4)+r^2*(r*der_a03232(r)+2*a03232(r))*y(3)+((r*der_a03232(r)-a03232(r))*r-alpha^2*r^3*(a03333(r)+a02222(r)+tau_3(r)))*y(2)+(a03232(r)-r*der_a03232(r)+alpha^2*r^2*(r*der_a03232(r)-r*der_tau33(r)+a03232(r)-tau_3(r)-a02222(r)))*y(1)-alpha^2*r^2*a0333(r)*s^2*y(5);
    end

 % Moduli

    function ret = a03131(r)
        ret = (lz^(-1)*(r^2*a0^2-a^2)+lz^(-2)*a0^2+s^2*a^2)/(r^2*a0^2);
    end
    function ret = a03232(r)
        ret = (lz^(-1)*(r^2*a0^2-a^2)+lz^(-2)*a0^2+s^2*a^2)/(r^2*a0^2);
```

```
    end
    function ret = a03333(r)
        ret = (lz^(-1)*(r^2*a0^2-a^2)+lz^(-2)*a0^2+s^2*a^2)/(r^2*a0^2);
    end
    function ret = a02222(r)
        ret = lz^2;
    end
    function ret = a02323(r)
        ret = lz^2;
    end
    function ret = a01111(r)
        ret = r^2*a0^2/(lz*(r^2*a0^2-a^2)+a0^2);
    end
    function ret = a0333(r)
        ret = 2*a/(r*a0);
    end
    function ret = der_a0333(r) % first derivative of a0333(r) with respect
       to r
        ret = (-2*a)/(r^2*a0);
    end
    function ret = a0232(r)
        ret = a/(r*a0);
    end
    function ret = der_a0232(r) % first derivative of a0232(r) with respect
       to r
        ret = -a/(r^2*a0);
    end
    function ret = a022(r)
        ret = 1;
    end
    function ret = a033(r)
        ret = 1;
    end
```

```
function ret = tau_3(r)
    ret = a03131(r);
end
function ret = der_a03131(r) %first derivative of a03131(r) with respect to r
    ret = 2*(lz*a^2-a0^2-s^2*a^2*lz^2)/(r^3*lz^2*a0^2);
end
function ret = der_a03333(r) % first derivative of a03333(r) with respect to r
    ret = 2*(lz*a^2-a0^2-s^2*a^2*lz^2)/(r^3*lz^2*a0^2);
end
function ret = der2_a03131(r) %second derivative of a03131(r) with respect to r
    ret = -6*(lz*a^2-a0^2-s^2*a^2*lz^2)/(r^4*a0^2*lz^2);
end
function ret = der_a01313(r)
    ret = 2*a0^2*r*(a0^2-a^2*lz)/(a0^2+(a0^2*r^2-a^2)*lz)^2;
end
function ret = a01313(r)
    ret = r^2*a0^2/(lz*(r^2*a0^2-a^2)+a0^2);
end
function ret = der2_p(r) % second derivative of Lagrange multiplier p(r) with respect to r. It can be expressed interms of electroelastic moduli.
    ret = der2_a03131(r)-(a03131(r)-a01313(r))/r^2+(der_a03131(r)-der_a01313(r))/r;
end
function ret = der_tau33(r) % derivative of stress component tau_33(r); it can be expressed in terms of electroelastic moduli.
    ret = (a01313(r)-a03131(r))/r;
end
function ret = der_a03232(r) % first derivative of a03232(r) with respect to r
```

```
        ret = 2*(lz*a^2-a0^2-s^2*a^2*lz^2)/(r^3*lz^2*a0^2);
    end
    function ret = der2_a03232(r) % second derivative of a03232(r) with
        respect to r
        ret = -6*(lz*a^2-a0^2-s^2*a^2*lz^2)/(r^4*a0^2*lz^2);
    end

end
```

File experiment experiment1.m has the contents:

```
% Experiment 1

par1.a0=1;
par1.b0=1/.85;
par1.L=10/0.85;
par1.s=0;

branch1=calculate_branch(par1,[0.25,0.71],0.01,450,1);
axis equal;

% Experiment 2

par2.a0=1;
par2.b0=1/.85;
par2.L=10/0.85;
par2.s=0.75;

branch2=calculate_branch(par2,[0.25,0.71],0.01,450,1);
axis equal;

% Experiment 3

par3.a0=1;
```

```
par3.b0=1/.85;
par3.L=10/0.85;
par3.s=1.1;

branch3=calculate_branch(par3,[0.25,0.71],0.01,450,1);
axis equal;

% Experiment 4

par4.a0=1;
par4.b0=1/.85;
par4.L=10/0.85;
par4.s=1.5;

branch4=calculate_branch(par4,[0.25,0.71],0.01,450,1);
axis equal;
```

The file experiment1.m should be run in order to start the calculation.

Some comments about this code follow here. According to the numerical scheme discussed in Chapter 4 we need to find pairs of $\lambda_z$ and $\lambda_a$[1] such that the matrix comprised of coefficients $c_k$ mentioned after relation (4.132) becomes singular, i.e. the determinant of this matrix becomes zero. In file determinant.m in line 103 instead of a direct calculation of determinant of this matrix we used another test for singularity which involves a condition number of a matrix. The condition number is a measure how close a matrix to being singular. A very large condition number suggests that matrix is almost singular. The inverse of a condition number for the singular matrix is zero. We found that the test for singularity which involves a condition number works better here, because in some cases the code which involved a calculation of a determinant as a test of singularity could not reproduce some parts of the branches correctly for purely elastic material which were obtained by Haughton & Ogden (1979).

Lines $20-34$ in file calculate_branch.m the program calculates pairs $\lambda_z$ and $a$ such

[1]This code actually calculates pairs $\lambda_z$ and internal deformed radius $a$, because $\lambda_a = a/A$ and we set here in the code $A = 1$

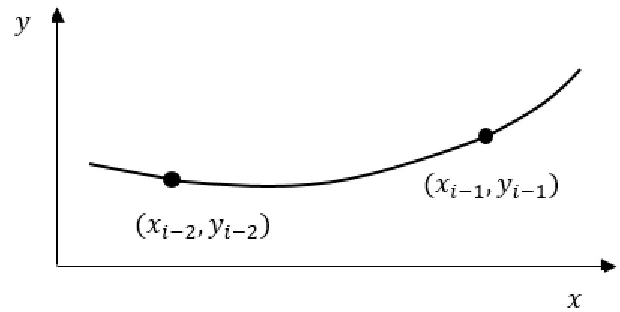


Figure B.1: Solution points which constitute a bifurcation curve.

that the following system of equations is satisfied

$$\begin{cases} f(\lambda_z, a) = 0, \\ (\lambda_z - \lambda_z^0)^2 + (a - a^0)^2 = \Delta^2, \end{cases} \tag{B.1}$$

where function $f$ is the inverse of the condition number of the matrix which depends on $\lambda_z$ and $a$, $\lambda_z^0$ and $a^0$ are the known values from the previous step of calculation, $\Delta^2$ is the length of the step between the previous and the current points on the bifurcation curve.

The second equation of (B.1) represents the equation of a circle with the centre at $(\lambda_z^0, a^0)$ and radius $\Delta$. Standard mathematical solution of this system can normally give 2 solutions on the curve, but MATLAB solver fsolve looks for the one nearest solution near the initial guess, which is updated in the code for each step. This ensures a correct progressing along the curve. Formulation (B.1) is advantageous, because it allows to find bifurcation curves even when they start turning back, for example, when there are 2 or more values of $a$ for each $\lambda_z$ in some region of $(\lambda_z, a)$- plane.

The solver fsolve in the loop of the file calculation_branch.m has the initial guess which is updated at each step. It can be calculated in this way. Let us assume that we know two points with coordinates $(x_{i-2}, y_{i-2})$ and $(x_{i-1}, y_{i-1})$ from calculations for the previous steps. Now we want to find the coordinates of the initial guess point where the solver fsolve has to start looking. Tangent unit vector to the curve between the two points $(x_{i-2}, y_{i-2})$ and $(x_{i-1}, y_{i-1})$ can be calculated approximately as

$$\mathbf{t}_{i-1} = \frac{(x_{i-1} - x_{i-2}, y_{i-1} - y_{i-2})}{\sqrt{(x_{i-1} - x_{i-2})^2 + (y_{i-1} - y_{i-2})^2}} = \frac{(x_{i-1} - x_{i-2}, y_{i-1} - y_{i-2})}{\Delta}. \tag{B.2}$$

[2] In the code it is denoted as delta and d

Therefore, the coordinates for the initial guess can be found as

$$\begin{aligned}(x_{i-1}, y_{i-1}) + \mathbf{\Delta t}_{i-1} &= (x_{i-1}, y_{i-1}) + (x_{i-1} - x_{i-2}, y_{i-1} - y_{i-2}) \\ &= (2x_{i-1} - x_{i-2}, 2y_{i-1} - y_{i-2}).\end{aligned} \tag{B.3}$$

This result is used in the line 22 of calculate_branch.m file.

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