DETERMINATION OF THE OPTIMUM CONDITIONS OF RESPONSE OF SYSTEMS.

by

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SUMMARY.

In this thesis we consider the transient response of systems satisfying linear differential equations with constant coefficients. Simple mathematical criteria for optimising the response are given in terms of

$$L = \int_0^\infty e^2 d\tau \quad \text{and} \quad L_1 = \int_0^\infty \left(\frac{de}{d\tau} \right)^2 d\tau$$

where e is the error at time τ. Expressions are obtained for both L and L_1 in terms of (i) the roots of the characteristic equation, (ii) the coefficients of the characteristic equation, and (iii) the frequency response spectrum of the system. It is shown how the response due to (i) a step function disturbance, (ii) an initial impulse, (iii) a constant velocity input, and (iv) a constant acceleration input can be simply related to the response in the free motion. The response following an arbitrary disturbance is also considered.

The response of a linear system having one degree of freedom is considered for (i) a zero-displacement-error system, (ii) a zero-velocity-error system, and (iii) a zero-acceleration-error system. By considering the response to a step-function disturbance it is found that systems making L a minimum have a lightly damped oscillatory response. The smaller L_1 is, the "smoother" is the response. Values are obtained for the coefficients of the characteristic equation of any order making L a minimum. An approximate method is given for correcting these coefficients to enable the response to be improved to give equal damping in the least damped modes of oscillation. For the zero-velocity-error and zero-acceleration-error systems the method is extended to allow for the requirement of a zero-displacement-error in the final steady state.

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The method is extended to linear systems with any number of degrees of freedom. The response of a linear first order system with two degrees of freedom is considered in detail, two overall response functions R and R_1 being defined in a similar manner to L and L_1. It is shown that, in the optimum system, there is no coupling; the damping in each mode is the same. A first order system with integral control is also considered; in this case the binomial response is the optimum.

PREFACE.

This thesis is an extension of the work of Mack (reference 6) on the calculation of optimum parameters for a following system. The method of normalizing the equation of motion for a zero-displacement-error system (chapter 2) is based on that given in reference 7. The Laplace transform method of solving linear equations with many degrees of freedom (chapter 3) is given in reference 17.

INTRODUCTION.

When considering the performance of a system (e.g. a dynamical system or servomechanism) we are interested in the accuracy with which the output of the system follows the input. More precise elaboration of this general statement depends upon the particular application, the order of accuracy and the sensitivity of the system varying greatly with different applications. The systems considered may be mechanical, electrical, hydraulic or aerodynamic. In the text the terms used (e.g. forces, equations of motion) are mainly based on mechanical systems, but the analysis is, of course, perfectly general.

We shall in general be concerned with stable systems i.e. systems which when subjected to any disturbance acting for a finite time ultimately return to their initial state. The performance of such a system is intuitively measured by such factors as its overshoot when subjected to a step disturbance, the oscillatory nature and damping in that case (the transient motion) and also by the amplitude of the motion and the resonance peak and frequency when the system is subjected to a steady sinusoidal disturbing force (the frequency response of the motion). Often the designer has freedom to choose the value of many of the parameters of the system (e.g. degree of damping, spring stiffness). The problem of optimisation is the selection of such values of these variables that the response of the system is "the best" or, more often, the most satisfactory for the particular application. The optimum values will depend on the particular input disturbance. Thus at the start we are confronted with the choice of basing our analysis either on the transient behaviour of the system or on its frequency response. The nature of this fundamental choice will depend on which is the most likely type of input the system will have to deal with. Fortunately many systems having a satisfactory transient also have satisfactory frequency response. This is not surprising since the response of a system to any disturbance is governed by the differential equation of motion of the system. In fact the transient and frequency response can be correlated by a Fourier transformation (see chapter I). Empirical relations are often used, based for example on the relationship between the peak overshoot and the resonance peak, or between the resonant frequency and the transient oscillatory frequency, or between the resonant frequency and the speed of response (see reference 1).

We shall endeavour to give simple mathematical criteria for optimising the transient response. Many such criteria have been used in recent years, mainly for systems subjected to input step disturbances. In references 2 and 3

$$I_1 = \int_0^{\infty} e\,d\tau \quad \text{and} \quad I_2 = \int_0^{\infty} \tau e\,d\tau$$

(where e is the error) are minimised. These criteria break down when the response is oscillatory, since an overshoot decreases the value of I_1 and I_2. In references 4 and 5 the criterion for optimisation is that

$$I_3 = \int_0^{\infty} e^2 d\tau$$

should be a minimum. This criterion is widely used and can be readily handled, either analytically or by computing machine. However in general it leads to a slightly underdamped response often with a large undesirable overshoot. To overcome this, in reference 6,

$$I_4 = \int_0^{\infty} \tau^2 e^2 d\tau$$

is considered. The system making I_4 a minimum gives a very satisfactory performance. However the formula for I_4 is often troublesome to evaluate.

In reference 7,

$$I_5 = \int_0^{\infty} \tau\,|e|\,d\tau \quad ,$$

the integral of time-multiplied absolute-value of error (ITAE), is minimised. The ITAE criterion works very well for response of zero-displacement-error systems to a step input, but not so well for zero-velocity-error systems. It may indeed be thought to be over selective in its choice of optimum, distinguishing too sharply between the optimum and systems near the optimum. From the form of the integral it allows sizeable errors for small values of τ; this is seen especially in its choice of optimum response for zero-velocity-error and zero-acceleration-error systems. While I_5 is easy to calculate, it is difficult to handle analytically, and thus it is hard (if not impossible) to extend results of the ITAE criterion to high order differential equations.

In references 8 and 9, optimisation of the transient is sought for by modifying the closed loop frequency spectrum (or attenuation diagram) of the system, avoiding resonance peaks by making the amplitude frequency curve as flat as possible for small frequencies (see/

/ (see also chapter 2). It is noted that the method of reference 8 gives the same response as the Butterworth filters (reference 10) when applied to a zero-displacement-error system. The response, although close to an intuitive optimum, tends to have both an overshoot and a subsequent undershoot (see also chapter 2). For zero-velocity-error systems the analysis of reference 8 is based on the maximum overshoot. To the author it seems inconsistent to choose one method for optimising zero-displacement-error systems and another for zero-velocity-error systems. The methods of reference 8 cannot be extended to high order linear differential equations with any degree of confidence, since they are essentially empirical.

We shall consider the variation of

$$L = \int_0^\infty e^2 d\tau \qquad \text{and} \qquad L_1 = \int_0^\infty \left[\frac{de}{d\tau}\right]^2 d\tau$$

with the parameters of the system. As stated above a system based on minimising L often has an undesirable overshoot. ~~So~~ Often L has a rather flat minimum, which is relatively insensitive to small changes in some of the parameters. We shall indicate how the parameters can be varied from this "mean square" optimum to give a more satisfactory solution. The system having the most satisfactory performance is impossible to define precisely in mathematical terms. For a system with a large overshoot $\frac{de}{d\tau}$ will be large at some time and thus L_1 will be correspondingly large. Our aim will be to investigate the performance of systems for which L is near its minimum value while L_1 is considerably lower than its value at L_{min}. These systems will be of a less oscillatory nature than those given by the mean square optimum. L and L_1 are very simple to handle both analytically or by a computing machine. Being analytical the formulae can immediately be extended to systems based on high order differential equations.

We shall also consider the frequency response characteristics of these solutions. Optimisation techniques based on frequency response characteristics have relied mainly on graphical methods to modify the locus of the transfer function $KG(i\omega)$ especially in the neighbourhood of the point $(-1+0i)$ (see references 1,11,12 and 13). This is mainly due to the numerical labour of calculating transient response curves for systems with high order differential equations. As stated above there is a certain correlation between the transient and the frequency response. This is shown very clearly in reference 14 in which attenuation diagrams are drawn for various positions of the roots of the characteristic equations, using the root-locus method of reference 15. In chapter 2 we shall show how a change in the roots of the characteristic equation affects both the transient and frequency response.

Chapter I.

Optimum Conditions of Response of Linear Systems with Constant Coefficients having One Degree of Freedom.

We consider a system for which the equation of motion is

$$a_n \frac{d^n x}{d\tau^n} + a_{n-1} \frac{d^{n-1} x}{d\tau^{n-1}} + a_{n-2} \frac{d^{n-2} x}{d\tau^{n-2}} + \dots + a_1 \frac{dx}{d\tau} + a_o x = f(\tau) \qquad (1)$$

where a_o , a_1 , a_2 , ... a_n are constants and $f(\tau)$ is an arbitrary known function.

The initial conditions are given by

$$x = x_o \;,\; \dot{x} = \dot{x}_o \;,\; \ddot{x} = \ddot{x}_o \;,\; \text{etc.,}\; \frac{d^{n-1}x}{d\tau^{n-1}} = \left(\frac{d^{n-1}x}{d\tau^{n-1}}\right)_o \text{ at } \tau = 0 .$$

Using the classical methods of reference 16 or the operational methods of reference 17, the solution of (1) for the given initial conditions is

$$x = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{D^{s-1}x_o \; A_{sr}}{\Delta} e^{\lambda_r \tau} + \sum_{r=1}^{n} P_r e^{\lambda_r \tau} \int_0^{\tau} e^{-\lambda_r y} f(y)\,dy \qquad (2)$$

where $\lambda_r (r=1 \text{ to } n)$ are the roots of the equation

$$F(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \dots + a_o = 0 \qquad (3)$$

$$P_r = \frac{1}{a_n \prod\limits_{\substack{s=1 \\ s \neq r}}^{n} (\lambda_r - \lambda_s)} = \frac{1}{F^1(\lambda_r)} \qquad (4)$$

and A_{sr} is the cofactor of the sth row and the rth column in the determinant Δ

$$\Delta = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \dots & \dots & \dots & \dots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix} = \prod_{\substack{r=1 \text{ to } n \\ s=1 \text{ to } n \\ r > s}} (\lambda_r - \lambda_s) \qquad (5)$$

contd.

An alternative expression for $\dfrac{A_{sr}}{\Delta}$ is

$$\frac{A_{sr}}{\Delta} = P_{rs} = \frac{(-1)^{n-s} \sum \text{product of roots (excluding } \lambda_r\text{) taken } (n-s) \text{ at the time}}{\prod\limits_{\substack{s=1 \\ s \neq r}}^{n} (\lambda_r - \lambda_s)} \tag{6}$$

Differentiating (2) we have

$$Dx = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{D^{s-1} x_o \lambda_r A_{sr}}{\Delta} e^{\lambda_r \tau} + \sum_{r=1}^{n} \lambda_r P_r e^{\lambda_r \tau} \int_0^{\tau} e^{-\lambda_r y} f(y)\, dy \tag{7}$$

$$\cdots\cdots\cdots$$

$$D^{n-1}x = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{D^{s-1} x_o \lambda_r^{n-1} A_{sr}}{\Delta} e^{\lambda_r \tau} + \sum_{r=1}^{n} \lambda_r^{n-1} P_r e^{\lambda_r \tau} \int_0^{\tau} e^{-\lambda_r y} f(y)\, dy \tag{8}$$

The Problem of Optimisation.

We shall be concerned primarily with stable systems for which all the roots of the characteristic equation (3) are negative or have negative real parts. Then as $\tau \to \infty$ from (2), (7) and (8),

$$x \sim \sum P_r e^{\lambda_r \tau} \int_0^{\tau} e^{-\lambda_r y} f(y)\, dy$$

$$Dx \sim \sum \lambda_r P_r e^{\lambda_r \tau} \int_0^{\tau} e^{-\lambda_r y} f(y)\, dy$$

$$\cdots\cdots\cdots$$

$$D^{n-1}x \sim \sum \lambda_r^{n-1} P_r e^{\lambda_r \tau} \int_0^{\tau} e^{-\lambda_r y} f(y)\, dy$$

Considering a forcing function f which tends to a finite limit f_1 as τ tends to infinity. Then as $\tau \to \infty$,

$$x \to -f_1 \sum \frac{P_r}{\lambda_r} = \frac{f_1}{a_o} \tag{9}$$

as/

/as can be seen from (1) and Dx , ... $D^{n-1}x \to 0$.

If there were no lag in the system x would equate $\frac{1}{a_o} f(\tau)$ at all times. The error e is given by

$$e = \text{input} - \text{output} = \frac{1}{a_o} f(\tau) - x. \tag{10}$$

As stated above we shall derive formulae for

$$L = \int_0^\infty e^2 d\tau = \int_0^\infty \left[x - \frac{1}{a_o} f(\tau)\right]^2 d\tau \tag{11}$$

and we shall also consider values of

$$L_1 = \int_0^\infty \left[\frac{de}{d\tau}\right]^2 d\tau = \int_0^\infty \left[\frac{dx}{d\tau} - \frac{1}{a_o} f'(\tau)\right]^2 d\tau \tag{12}$$

in the neighbourhood of the minimum value of L.

We shall now consider various forms of the functions $f(\tau)$ and the corresponding formulae for the response coefficients L and L_1.

Free Motion. $f(\tau) = 0$.

Derivation of formulae for response functions in terms of the roots of the characteristic equation.

From (2) and (6) $$x = \sum_{r=1}^{n} \sum_{s=1}^{n} P_{rs} D^{s-1} x_o e^{\lambda_r \tau} = \sum_{r=1}^{n} A_r e^{\lambda_r \tau} \tag{13}$$

If the motion is stable $x \to 0$ as $\tau \to \infty$.

From (13), $$-L = \sum_{r=1}^{n} \sum_{R=1}^{n} \frac{A_r A_R}{\lambda_r + \lambda_R} \tag{14}$$

$$= \sum_{r=1}^{n} \sum_{R=1}^{n} A_r A_r M_{rR}/M$$

where $$M = \prod_{\substack{s=1 \text{ to } n \\ S=1 \text{ to } n \\ S \geq s}} (\lambda_s + \lambda_S)$$

and/

/and $M_{rR} = M/(\lambda_r + \lambda_R)$ (15)

$$\text{Now}\quad M = 2^n \lambda_1 \lambda_2 \ldots \lambda_n (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)\ldots(\lambda_1 + \lambda_n)$$
$$(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)\ldots(\lambda_2 + \lambda_n)$$
$$\ldots \quad (\lambda_{n-1} + \lambda_n) \tag{16}$$

a symmetrical expression in the λ's of degree $\frac{n(n+1)}{2}$ which can therefore be expressed in terms of a_o, $a_1 \ldots a_n$, the coefficients of the powers of λ in (3).

The expression $\sum\sum A_r A_R M_{rR}$ is symmetrical in both the A's and the λ's being of the second degree in the A's and of degree $\frac{(n+2)(n-1)}{2}$ in the λ's.

Now the free motion is determined uniquely when the initial conditions are specified. Thus L is determined by the initial conditions.

We have from (13),

$$\left.\begin{aligned}
\sum A_r &= x_o \\
\sum \lambda_r A_r &= \dot{x}_o = Dx_o \\
\sum \lambda_r^2 A_r &= \ddot{x}_o = D^2 x_o \\
&\ldots\ldots \\
\sum \lambda_r^{n-1} A_r &= \left(\frac{d^{n-1}x}{d\tau^{n-1}}\right)_o = D^{n-1} x_o
\end{aligned}\right\} \tag{17}$$

$$\text{Then}\quad \sum\sum A_r A_R M_{rR} = a_{11}(\sum A_r)^2 + a_{12}(\sum A_r)(\sum \lambda_r A_r)$$
$$+ a_{13}(\sum A_r)(\sum \lambda_r^2 A_r) + \ldots + a_{1n}(\sum A_r)(\sum \lambda_r^{n-1} A_r$$
$$+ \ldots + a_{nn}(\sum \lambda_r^{n-1} A_r)^2 \tag{18}$$

where/

/where α_{pq} is a symmetrical expression in the λ's of degree

$$\frac{(n+2)(n-1)}{2} - (p+q-2) \qquad (p,q = 1 \text{ to } n)$$

and can therefore be expressed in terms of a_0 , a_1, ... a_n . The precise numerical form of the α's can be found by equating coefficients of powers of A_r on both sides of (18). When the degree of $F(\lambda)$ is not too high (say 4 or less) it is a relatively simple matter to determine the α's.

Hence

$$L = -\frac{1}{M}\Big[x_0\,(\alpha_{11}x_0 + \alpha_{12}Dx_0 + \alpha_{13}D^2x_0 + \dots \alpha_{1n}D^{n-1}x_0)$$

$$+ \dots + Dx_0^{n-1}(\alpha_{1n}x_0 + \dots + \alpha_{nn}D^{n-1}x_0)\Big] \qquad (19)$$

We see that L is a second degree expression in the initial conditions From (13),

$$-L_1 = \sum_{r=1}^{n}\sum_{R=1}^{n}\frac{\lambda_r\lambda_R\,A_r\,A_R}{\lambda_r + \lambda_R} \qquad (20)$$

We see that L_1 is obtained from L by replacing A_r, A_R by $\lambda_r A_r$, $\lambda_R A_R$ i.e. by replacing x_0 by Dx_0 , Dx_0 by D^2x_0 , ... etc., in (19).

Thus

$$L_1 = -\frac{1}{M}\Big[Dx_0(\alpha_{11}Dx_0 + \alpha_{12}D^2x_0 + \alpha_{13}D^3x_0 + \dots + \alpha_{1n}D^nx_0\,)$$

$$+ \dots + D^nx_0(\alpha_{1n}Dx_0 + \dots + \alpha_{nn}D^nx_0)\Big] \qquad (21)$$

where from (1), since $f = 0$,

$$a_n\,D^nx_0 + a_{n-1}\,D^{n-1}x_0 + \dots + a_1\,Dx_0 + a_0x_0 = 0\,.$$

We thus see that for given initial conditions the problem of optimisation involves only α_{pq} and M. We note that the particular values of these parameters defining an optimum system will depend on the initial conditions.

<u>Free Motion</u> $f(\tau) = 0$.

<u>Derivation of formulae for response functions in terms of the coefficients of the characteristic equation.</u>

We write the equation of motion in the form

$$a_n D^n x + a_{n-1} D^{n-1} x + a_{n-2} D^{n-2} x + \ldots + a_1 Dx + a_o x = 0 \tag{22}$$

where $$D = \frac{d}{d\tau}$$

Now if $p > q \geqslant 0$,

$$\int_0^\infty D^p x \, D^q x \, d\tau = \Big[D^{p-1} x \, D^q x \Big]_0^\infty - \int_0^\infty D^{p-1} x \, D^{q+1} x \, d\tau$$

$$= - D^{p-1} x_o \, D^q x_o - \int_0^\infty D^{p-1} x \, D^{q+1} x \, d\tau \tag{23}$$

since for the free motion of a stable system $D^m x \to 0$ as $\tau \to \infty$ for $m \geqslant 0$.

We write
$$\left.\begin{aligned} L &= \int_0^\infty x^2 d\tau \\ L_s &= \int_0^\infty \Big[D^s x \Big]^2 d\tau \end{aligned}\right\} \tag{24}$$

Multiplying (22) by x and integrating from 0 to ∞ , using (23), (24),

$$\begin{aligned} & a_o L - a_2 L_1 + a_4 L_2 - a_6 L_3 + \ldots \\ = \; & x_o \left(\tfrac{1}{2} a_1 x_o + a_2 D x_o + a_3 D^2 x_o + a_4 D^3 x_o + \ldots \right) \\ - \; & D x_o \left(\tfrac{1}{2} a_3 D x_o + a_4 D^2 x_o + a_5 D^3 x_o + a_6 D^4 x_o + \ldots \right) \\ + \; & D^2 x_o \left(\tfrac{1}{2} a_5 D^2 x_o + a_6 D^3 x_o + a_7 D^4 x_o + a_8 D^5 x_o + \ldots \right) \\ - \; & D^3 x_o \left(\tfrac{1}{2} a_7 D^3 x_o + a_8 D^4 x_o + a_9 D^5 x_o + a_{10} D^6 x_o + \ldots \right) + \ldots \\ = \; & \beta_1 \end{aligned} \tag{25}$$

Similarly/

Similarly multiplying (22) by Dx and integrating from 0 to ∞,

$$
\begin{aligned}
& a_1L_1 - a_3L_2 + a_5L_3 - \dots \\
&= \tfrac{1}{2}a_o x_o^2 + Dx_o\,(\tfrac{1}{2}a_2 Dx_o + a_3D^2x_o + a_4D^3x_o + \dots) \\
&\qquad - D^2x_o\,(\tfrac{1}{2}a_4D^2x_o + a_5D^3x_o + a_6D^4x_o + \dots) \\
&\qquad + D^3x_o\,(\tfrac{1}{2}a_6D^3x_o + a_7D^4x_o + a_8D^5x_o + \dots) - \dots \\
&= \beta_2
\end{aligned}
\tag{26}
$$

Similarly multiplying (22) by D^2x and integrating from 0 to ∞,

$$
\begin{aligned}
& - a_oL_1 + a_2L_2 - a_4L_3 + a_6L_4 - \dots \\
&= Dx_o(a_ox_o + \tfrac{1}{2}a_1Dx_o) + D^2x_o(\tfrac{1}{2}a_3D^2x_o + a_4D^3x_o + a_5D^4x_o + \dots) \\
&\qquad - D^3x_o(\tfrac{1}{2}a_5D^3x_o + a_6D^4x_o + a_7D^5x_o + \dots) \\
&\qquad + D^4x_o(\tfrac{1}{2}a_7D^4x_o + a_8D^5x_o + a_9D^6x_o + \dots) - \dots \\
&= \beta_3
\end{aligned}
\tag{27}
$$

Similarly multiplying (22) by D^3x and integrating from 0 to ∞,

$$
\begin{aligned}
& - a_1L_2 + a_3L_3 - a_5L_4 + a_7L_5 - \dots \\
&= -\tfrac{1}{2}a_oDx_o^2 + D^2x_o\,(a_ox_o + a_1Dx_o + \frac{a_2}{2}D^2x_o) \\
&\qquad + D^3x_o\,(\tfrac{1}{2}a_4D^3x_o + a_5D^4x_o + a_6D^5x_o + \dots) \\
&\qquad - D^4x_o\,(\tfrac{1}{2}a_6D^4x_o + a_7D^5x_o + a_8D^6x_o + \dots) \\
&\qquad + D^5x_o\,(\tfrac{1}{2}a_8D^5x_o + a_9D^6x_o + a_{10}D^7x_o + \dots) - \dots \\
&= \beta_4
\end{aligned}
\tag{28}
$$

Proceeding in this way (multiplying by D^4x, D^5x, ..., $D^{n-1}x$ and integrating) we obtain n simultaneous linear equations for $L, L_1, \dots L_{n-1}$ in terms of x_o, Dx_o, ... $D^{n-1}x_o$. The equations simplify considerably in the particular case where the initial values of all except one D^mx are zero. In the general case we have

contd.

$$L \begin{vmatrix} a_0 & a_2 & a_4 & a_6 & \cdots \\ 0 & a_1 & a_3 & a_5 & \cdots \\ 0 & a_0 & a_2 & a_4 & \cdots \\ 0 & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} \beta_1 & a_2 & a_4 & a_6 & \cdots \\ -\beta_2 & a_1 & a_3 & a_5 & \cdots \\ \beta_3 & a_0 & a_2 & a_4 & \cdots \\ -\beta_4 & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (29)$$

$$\text{and } L_1 \begin{vmatrix} a_0 & a_2 & a_4 & a_6 & \cdots \\ 0 & a_1 & a_3 & a_5 & \cdots \\ 0 & a_0 & a_2 & a_4 & \cdots \\ 0 & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} \beta_1 & a_0 & a_4 & a_6 & \cdots \\ -\beta_2 & 0 & a_3 & a_5 & \cdots \\ \beta_3 & 0 & a_2 & a_4 & \cdots \\ -\beta_4 & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (30)$$

where the determinants in (29) and (30) are of the n^{th} order.

L and L_1 can be expressed more simply in terms of the n test functions of the characteristic equation (3).

As shown in references 18 and 19, the test functions can be written in determinantal form.

$$T_1 = a_{n-1} \qquad (31)$$

$$T_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} \qquad (32)$$

$$T_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix} \qquad (33)$$

$$T_4 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} \\ a_n & a_{n-2} & a_{n-4} & a_{n-6} \\ 0 & a_{n-1} & a_{n-3} & a_{n-5} \\ 0 & a_n & a_{n-2} & a_{n-4} \end{vmatrix} \qquad (34)$$

$$\cdots\cdots\cdots$$

T_{n-1}/

$$/T_{n-1} = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & & \\ a_n & a_{n-2} & a_{n-4} & \cdots & & \\ 0 & a_{n-1} & a_{n-3} & \cdots & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & a_2 & a_o \\ & \cdots & \cdots & \cdots & a_3 & a_1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & \cdots \\ a_o & a_2 & a_4 & a_6 & \cdots \\ 0 & a_1 & a_3 & a_5 & \cdots \\ 0 & a_o & a_2 & a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (35)$$

$$T_n = a_o \, T_{n-1} \qquad (36)$$

The ~~n~~mth test function T_m is a determinant of order m.

From (29), ~~and~~ (30) and (35),

$$L = \frac{1}{a_o T_{n-1}} \begin{vmatrix} \beta_1 & a_2 & a_4 & a_6 & \cdots \\ -\beta_2 & a_1 & a_3 & a_5 & \cdots \\ \beta_3 & a_o & a_2 & a_4 & \cdots \\ -\beta_4 & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (37)$$

and $$L_1 = \frac{1}{T_{n-1}} \begin{vmatrix} \beta_2 & a_3 & a_5 & a_7 & \cdots \\ -\beta_3 & a_2 & a_4 & a_6 & \cdots \\ \beta_4 & a_1 & a_3 & a_5 & \cdots \\ -\beta_5 & a_o & a_2 & a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (38)$$

As shown in references 20 and 21 for stability (with $a_n > 0$) all the test functions must be positive, or using (36),

contd.

T_1, T_2, ... T_{n-1} and a_o

must be positive. As shown in reference 22 the critical stability criteria are $T_{n-1} > 0$ and $a_o > 0$.

When a_o vanishes, the characteristic equation (3) has a zero root and the system is neutrally stable. When T_{n-1} vanishes, the system has a pair of equal and opposite roots. Thus when (3) has a pair of roots $\pm i\omega$ where ω is real (corresponding to a sinusoidal oscillation) T_{n-1} is zero. In both of these cases the system if displaced from its original position would never return to and remain in that position; ~~and~~ thus L is infinite if either a_o or T_{n-1} vanish as shown by (37).

We can easily obtain expressions for the α's of equation (19) in determinantal form using (25) - (28) and (37). Now as shown in references 20 and 21 the product of the roots of the characteristic equation taken two at the time is

$$(-1)^{\frac{n(n-1)}{2}} \frac{T_{n-1}}{a_n^{n-1}}$$

and the product of the roots taken one at the time is $(-1)^n \frac{a_o}{a_n}$

$$\therefore \text{ from (16), } \left(\frac{a_n}{2}\right)^n M = (-1)^{\frac{n(n+1)}{2}} a_o T_{n-1} \qquad (39)$$

Comparing the coefficients of $x_o^2, x_oDx_o, \ldots Dx_o^2 \ldots$ in (19) and (37) and writing

$$\theta = \frac{2^{n-1}(-1)^{\frac{n^2+n-2}{2}}}{a_n^n} \qquad (40)$$

we find

$$\alpha_{11} = \theta \begin{vmatrix} a_1 & a_2 & a_4 & a_6 & \cdots \\ -a_o & a_1 & a_3 & a_5 & \cdots \\ 0 & a_o & a_2 & a_4 & \cdots \\ 0 & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (41)$$

contd.

$$\alpha_{12} = \alpha_{21} = \theta \begin{vmatrix} a_2 & a_2 & a_4 & a_6 & \cdots \\ 0 & a_1 & a_3 & a_5 & \cdots \\ a_o & a_o & a_2 & a_4 & \cdots \\ 0 & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{42}$$

$$\alpha_{13} = \alpha_{31} = \theta \begin{vmatrix} a_3 & a_2 & a_4 & a_6 & \cdots \\ 0 & a_1 & a_3 & a_5 & \cdots \\ 0 & a_o & a_2 & a_4 & \cdots \\ -a_o & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{43}$$

$$\alpha_{1n} = \alpha_{n1} = \theta a_n T_{n-1} \tag{44}$$

$$\alpha_{22} = \theta \begin{vmatrix} -a_3 & a_2 & a_4 & a_6 & \cdots \\ -a_2 & a_1 & a_3 & a_5 & \cdots \\ a_1 & a_o & a_2 & a_4 & \cdots \\ a_o & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{45}$$

$$\alpha_{23} = \alpha_{32} = \theta \begin{vmatrix} -a_4 & a_2 & a_4 & a_6 & \cdots \\ -a_3 & a_1 & a_3 & a_5 & \cdots \\ 0 & a_o & a_2 & a_4 & \cdots \\ -a_1 & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{46}$$

$$\alpha_{2n} = \alpha_{n2} = \theta a_n \begin{vmatrix} a_2 & a_4 & a_6 & a_8 & \cdots \\ a_o & a_2 & a_4 & a_6 & \cdots \\ 0 & a_1 & a_3 & a_4 & \cdots \\ 0 & a_o & a_2 & a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{47}$$

$$\alpha_{33} = \theta \begin{vmatrix} a_5 & a_2 & a_4 & a_6 & \cdots \\ a_4 & a_1 & a_3 & a_5 & \cdots \\ a_3 & a_o & a_2 & a_4 & \cdots \\ -a_2 & 0 & a_1 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{48}$$

contd.

$$\alpha_{3n} = \alpha_{n3} = \theta a_n \begin{vmatrix} a_2 & a_4 & a_6 & a_8 & \cdots \\ a_1 & a_3 & a_5 & a_7 & \cdots \\ 0 & a_1 & a_3 & a_5 & \cdots \\ 0 & a_0 & a_2 & a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{49}$$

$$\alpha_{nn} = \theta a_n \begin{vmatrix} a_2 & a_4 & a_6 & a_8 & \cdots \\ a_1 & a_3 & a_5 & a_7 & \cdots \\ a_0 & a_2 & a_4 & a_6 & \cdots \\ & a_1 & a_3 & a_5 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{50}$$

The determinants on the right hand sides of equations (41) - (43), (45),(46) and (48) are of order n, while those for α_{1n}, α_{2n}, α_{3n}, α_{nn} on the right hand sides of equations (44), (47) (49) and (50) are of order n-1. In general to write down all the elements of the determinants for a differential equation of the nth order we need to know the equations for β_1, β_2, ... β_n. It is then a simple matter to evaluate the necessary determinants. This method is much quicker than that of using symmetric functions of the roots.

<u>Free Motion</u> $f(\tau) = 0$.

<u>Derivation of integral formulae for response functions in terms of the frequency response spectrum of the system.</u>

As stated in the introduction the transient and the frequency response can be correlated by a Fourier transformation. It is of interest to relate the response functions to the amplitude and phase of the frequency response characteristics of the system.

As shown in reference 17, by Fourier s integral theorem if

$$G(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\, x(\tau)\,d\tau \tag{51}$$

then
$$x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\, G(\omega)\,d\omega \tag{52}$$

provided that $\int_{-\infty}^{\infty} x(\tau)\,d\tau$ is absolutely convergent.

We shall consider only the special case in which $x(\tau) = 0$ for $\tau \leqslant 0$.

From/

From (13), $$G(\omega) = \int_0^\infty e^{-i\omega\tau} \sum A_r e^{\lambda_r \tau} d\tau = \sum \frac{A_r}{i\omega - \lambda_r}$$

provided the system is stable.

Now we can write

$$F(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_o$$

$$= (\lambda - \lambda_r)({}_r b_{n-1}\lambda^{n-1} + {}_r b_{n-2}\lambda^{n-2} + \ldots + {}_r b_o).$$

Equating coefficients and solving for the b s,

$$\left.\begin{aligned} {}_r b_{n-1} &= a_n \\ {}_r b_{n-2} &= a_{n-1} + \lambda_r a_n \\ {}_r b_{n-3} &= a_{n-2} + \lambda_r a_{n-1} + \lambda_r^2 a_n \\ &\quad \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ {}_r b_o &= a_1 + \lambda_r a_2 + \ldots + \lambda_r^{n-2} a_{n-1} + \lambda_r^{n-1} a_n = - a_o/\lambda_r \end{aligned}\right\} \tag{53}$$

$$\therefore \quad G(\omega) = \frac{\sum_{r=1}^{n} \sum_{s=o}^{n-1} A_r \, {}_r b_s (i\omega)^s}{F(i\omega)}$$

i.e. using (17) and (53),

$$G(\omega) = \frac{\sum_{s=o}^{n-1} \left\{ a_{s+1} x_o + a_{s+2} Dx_o + \ldots + a_n D^{n-s-1} x_o \right\} (i\omega)^s}{F(i\omega)}$$

$$= \frac{g_o + g_1(i\omega) + g_2(i\omega)^2 + \ldots + g_{n-1}(i\omega)^{n-1}}{a_o + a_1(i\omega) + a_2(i\omega)^2 + \ldots + a_n(i\omega)^n} \tag{54}$$

where $$g_s = a_{s+1} x_o + a_{s+2} Dx_o + \ldots + a_n D^{n-s-1} x_o \; . \tag{55}$$

$(s = 0 \text{ to } n-1)$

From Parseval's theorem, for stable systems,

contd.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(\omega)|^2 = \int_0^{\infty}|x(\tau)|^2\,d\tau$$

or since $x(\tau)$ is real,

$$\frac{1}{\pi}\int_0^{\infty}|G(\omega)|^2\,d\omega = \int_0^{\infty}x^2(\tau)\,d\tau$$

.·. in the free motion,

$$L = \int_0^{\infty}x^2d\tau = \frac{1}{\pi}\int_0^{\infty}|G(\omega)|^2\,d\omega$$

i.e. $$L = \frac{1}{\pi}\int_0^{\infty}\frac{\left[g_0-g_2\omega^2+g_4\omega^4-\dots\right]^2+\omega^2\left[g_1-g_3\omega^2+\dots\right]^2}{\left[a_0-a_2\omega^2+a_4\omega^4-\dots\right]^2+\omega^2\left[a_1-a_3\omega^2+\dots\right]^2}\,d\omega \qquad (56)$$

where g_s is given by (55), $s = 0$ to $n-1$.

In (56) L is given completely in terms of the coefficients a_r of the characteristic equation and the initial conditions x_0, Dx_0, . . . $D^{n-1}x_0$. It is seen that (56) is an extension of the formula given in reference 6. It is in fact identical with that given in reference 9. For sufficiently large values of ω the integrand tends to $g_{n-1}/(a_n\omega)^2$. Thus the integral converges like $1/\omega$.

The integral for L_1 is obtained from L as above by replacing x_0 by Dx_0, Dx_0 by D^2x_0, ... etc., in the formulae (55) for g_s.

Now the denominator of the integrand in equation (56) is

$$F(i\omega)F(-i\omega) = a_n^2\prod_{r=1}^{n}(\omega^2+\lambda_r^2) \qquad (57)$$

and the integrand can easily be expressed as a sum of terms such as

$$\frac{1}{\omega^2+\lambda_r^2}$$

Then $$\int_0^{\infty}\frac{d\omega}{\omega^2+\lambda_r^2} = \frac{\pi}{2\lambda_r}$$

contd.

Alternatively using the analysis of references 9 and 23 we again arrive at the results given in equations (29) and (30).

Now from equation (52) we see that $|G(\omega)|$ represents the frequency spectrum of the given system. Thus if the ~~given~~ system is given a periodic disturbance $a_o e^{i\omega\tau}$, we find from (1)

$$x = \frac{a_o e^{i\omega\tau}}{a_o + a_1 i\omega - a_2\omega^2 - a_3 i\omega^3 + \ldots} \tag{58}$$

The amplitude (or dynamic magnification) M and phase advance N are given by

$$\frac{1}{M^2} = m = \frac{1}{a_o^2}\left\{\left[a_o - a_2\omega^2 + a_4\omega^4 - \ldots\right]^2 + \omega^2\left[a_1 - a_3\omega^2 + \ldots\right]^2\right\} \tag{59}$$

and
$$N = -\tan^{-1}\frac{a_1\omega - a_3\omega^3 + a_5\omega^5 - \ldots}{a_o - a_2\omega^2 + a_4\omega^4 - \ldots} \tag{60}$$

We see that the expressions on the right hand sides of equations (59) and (60) are precisely those that occur in the denominator of (56). Large amplitudes will occur when the damping is small and the forcing frequency is close to one of the natural frequencies of the system. In frequency response analysis we are often mainly interested in the magnification at frequencies near the natural frequency of the system. As shown in reference 11 both the magnification and phase relations of the response need to be taken into account in the design of the most satisfactory system. From (58) if the dynamic magnification is large over a range of frequencies, the integrand in (56) would be expected to be correspondingly large, and well away from the optimum.

Now in the usual terminology of servomechanisms, the ratio of the input to the output for a periodic response is

$$\frac{e^{i\omega\tau}}{x} = \frac{\theta_i}{\theta_o} = 1 + KG^{-1}(i\omega) = X + iY \tag{61}$$

Now from (58), $\frac{e^{i\omega\tau}}{x} = \frac{1}{a_o}(a_o + a_1 i\omega - a_2\omega^2 - a_3 i\omega^3 + \ldots)$

Thus/

/Thus

$$\left.\begin{aligned} X &= \frac{1}{a_o}\,(a_o - a_2\omega^2 + a_4\omega^4 - \dots\ ) \\ Y &= \frac{1}{a_o}\,(a_1\omega - a_3\omega^3 + a_5\omega^5 - \dots\ ) \end{aligned}\right\} \quad (62)$$

From equations (61) and (62) we see that when the coefficients of the characteristic equation are known we can easily construct the KG^{-1} or the KG locus, enabling us for example to compare the optimum (or most satisfactory)response based on analysis of the transient motion with graphical optimum frequency response systems. These points will be discussed further in chapter 2.

<u>Response of a system with no initial displacement.</u>

We consider a system with initial conditions.

$$x_o = Dx_o = D^2x_o = \quad = D^{n-1}x_o = 0 \qquad (63)$$

the system satisfying the equation of motion (1)

<u>Step function disturbance.</u>

$$f = 0 \text{ for } \tau \leq 0 \ ; \quad f = f_o \text{ for } \tau > 0$$

when f_o is a constant.

From (2) the motion is given by

$$x = f_o \sum_{r=1}^{n} P_r e^{\lambda_r \tau} \int_o^{\tau} e^{-\lambda_r y} dy = f_o \sum_{r=1}^{n} \frac{P_r}{\lambda_r}.(e^{\lambda_r \tau} - 1) \qquad (64)$$

where we are only considering stable systems.

Now $\quad \sum \frac{P_r}{\lambda_r} = \frac{1}{a_o}$

Thus the motion is given by

$$x = \frac{f_o}{a_o} + f_o \sum \frac{P_r}{\lambda_r} e^{\lambda_r \tau} \qquad (65)$$

and as $\tau \to \infty$ for stable systems $x \to \frac{f_o}{a_o}$

From (65)

contd.

$$\left.\begin{aligned} \text{From (65)} \quad Dx &= f_o \sum P_r e^{\lambda_r \tau} \\ &\cdots\cdots \\ D^{n-1}x &= f_o \sum \lambda_r^{n-2} P_r e^{\lambda_r \tau} \end{aligned}\right\} \quad (66)$$

Alternatively writing

$$x^1 = x - \frac{f_o}{a_o}$$

We see from (1) that the response of the system following a step dusturbance is identical in form with that in a free motion with $x_o = -\frac{f_o}{a_o}$ and Dx_o , D^2x_o ,... $D^{n-1}x_o$ zero. The errors in the two responses are the same at any given time. We see that the given system ultimately has no static error since $x' \to 0$. Such a system is sometimes called a "zero-displacement-error system" (see references 1, 7 and 8).

$$\text{From (11)} \quad L = \int_0^\infty \left[x - \frac{1}{a_o} f(\tau)\right]^2 d\tau = \int_0^\infty \left[x^1\right]^2 d\tau \qquad (67)$$

$$\text{and} \qquad Dx^1 = Dx$$

$$D^s x^1 = D^s x$$

Thus the response functions of the given system following a step function disturbance are the same as those of the given system in a free motion with $x_o = -\frac{f_o}{a_o}$ and Dx_o , D^2x_o ,....$D^{n-1}x_o$ zero.

$$\therefore \quad \text{from (19) and (67),} \quad L = -\frac{1}{M}\alpha_{11} \frac{f_o^2}{a_o^2} \qquad (68)$$

α_{11} and M being given in determinantal form by equations (39) - (41).

The corresponding integral formula to (56) is derived directly by the same substitution for x_o , Dx_o, ... $D^{n-1}x_o$.

$$\text{We find} \quad L = \frac{1}{\pi} f_o^2 \int_0^\infty \psi(\omega)\, d\omega \qquad (69)$$

where/

$$\text{where } \psi(\omega) = \frac{\left[a_1-a_3\omega^2+a_5\omega^4\dots\right]^2 + \omega^2\left[a_2-a_4\omega^2+a_6\omega^4\dots\right]^2}{a_o^2\left\{\left[a_o-a_2\omega^2+a_4\omega^4\dots\right]^2 + \omega^2\left[a_1-a_3\omega^2+a_5\omega^4\dots\right]^2\right\}} \qquad (70)$$

This is the same formula as in reference 6 allowing for differences in notation.

Similarly L_1 is found from (21), remembering that $D^n x_o \neq 0$,

$$L_1 = -\frac{1}{M}\alpha_{nn}(D^n x_o)^2$$

Now <u>in the equivalent free motion</u>

$$a_n D^n x_o = -a_o x_o = f_o$$

$$\therefore \quad L_1 = -\frac{1}{M}\alpha_{nn}\frac{f_o^2}{a_n^2} \qquad (71)$$

$$\text{Alternatively} \quad L_1 = \frac{1}{\pi}f_o^2\int_o^\infty \frac{d\omega}{F(i\omega)F(-i\omega)} \qquad (72)$$

From the above analysis it follows that the system having the most satisfactory response to a step function has the most satisfactory response in the free motion.

<u>Response to an initial unit impulse.</u>

For unit impulse $\int_o^\delta f(\tau)d\tau = 1$

where δ is infinitely small and $f(\tau)$ is zero for all other τ.

$$\therefore \quad \text{from (2),} \quad x = \sum_{r=1}^{n} P_r e^{\lambda_r \tau} \qquad (73)$$

Comparing (73) and (65) we see that just as the unit step function is the integral of the unit impulse so the response to a unit step disturbance is the integral of the response to a unit impulse.

Alternatively, integrating (1) over the interval $(0,\delta)$ we see that the motion is the same as that of the free motion with initial conditions $x_o = Dx_o \dots = D^{n-2}x_o = 0$, $D^{n-1}x_o = \frac{1}{a_n}$.

This/

/This also follows from (2) since

$$\frac{A_{nr}}{\Delta} = P_r a_n$$

$\therefore$ the response functions are as for the equivalent free motion. They can be obtained directly from the response functions for a step function disturbance as noted above. We find

$$L_s \text{ (unit impulse)} = L_{s+1} \text{ (unit step disturbance)} \qquad (74)$$

where $L_o \equiv L$

Then from (71), (72), $L = -\dfrac{1}{Ma_n^2}\alpha_{nn}$ (75)

$$L_1 = -\frac{1}{Ma_n^2}\left[\alpha_{n-1,n-1} - 2\alpha_{n,n-1}\frac{a_{n-1}}{a_n} + \alpha_{n,n}\frac{a_{n-1}^2}{a_n^2}\right] \qquad (76)$$

Note: in this case we have to define L by the equation

$$L = \lim_{\delta \to o} \int_\delta^\infty x^2 d\tau$$

<u>Constant velocity input.</u>

$$f = 0 \text{ for } \tau \leqq 0 \,;\; f = f_1 + f_o\tau \text{ for } \tau > 0 \qquad (77)$$

where f_o and f_1 are constants.

When the transient has died away, the steady motion is given by

$$x = a\tau + b \qquad (78)$$

where from (1),

$$\left.\begin{aligned} f_o &= a_o a \\ \text{and} \quad f_1 &= a_1 a + a_o b \end{aligned}\right\} \qquad (79)$$

Writing $x' = x - a\tau - b$

we see from (1) that the response of the given system is identical in form with that in a free motion with $x_o = -b$, $Dx_o = -a$, and $D^2x_o, \dots D^{n-1}x_o$ zero. The errors in the two responses are the same/

/same at any given time. In particular when $\frac{f_o}{a_o} = \frac{f_1}{a_1}$, b = 0 and the equivalent free system has $Dx_o = -\frac{f_o}{a_o}$ and x_o , $D^2x_o, \ldots D^{n-1}x_o$ zero. We see that such a system ultimately has no static error since $x' \to 0$ and there is no position error with constant velocity input (since b = 0). Such a system is sometimes called a "zero-velocity-error system", (see references 1,7 and 8).

From (19) and (67), $$L = -\frac{1}{M}\alpha_{22}\frac{f_o^2}{a_o^2} \qquad (80)$$

<u>Constant acceleration input.</u>

$$f = 0 \text{ for } \tau \leqslant 0 \; ; \; f = f_2 + f_1\tau + \tfrac{1}{2}f_o\tau^2 \text{ for } \tau > 0 \qquad (81)$$

where f_o, f_1 and f_2 are constants.

When the transient has died away, the steady motion is given by

$$x = a\tau^2 + b\tau + c \qquad (82)$$

where from (1), $$\left.\begin{aligned} f_o &= 2a_oa \\ f_1 &= 2a_1a + a_ob \\ f_2 &= 2a_2a + a_1b + a_oc \end{aligned}\right\} \qquad (83)$$

As above we see that the response of the given system is identical in form with that in a free motion with

$$x_o = -c \;,\; Dx_o = -b \;,\; D^2x_o = -\frac{a}{2} \;, \text{ and } D^3x_o \;,\ldots\; D^{n-1}x_o \text{ zero.}$$

In particular when $$\frac{f_o}{a_o} = \frac{f_1}{a_1} = \frac{f_2}{a_2} \;,$$

b = c = 0 and the equivalent free system has $D^2x_o = -\frac{f_o}{a_o}$ and x_o , Dx_o , D^3x_o ,... $D^{n-1}x_o$ zero. Such a system ultimately has no static error and no position error with constant velocity or constant acceleration input. Such a system is sometimes called a "<u>zero-acceleration-error system</u>" (see references 1, 7 and 8).

From (19) and (67), $$L = -\frac{1}{M}\alpha_{33}\frac{f_o^2}{a_o^2} \qquad (84)$$

Response to an arbitrary disturbance.

As is well known the response of a linear system to an arbitrary disturbance can be simply related to its response to unit impulses.

If $$x = \alpha(\tau) = \sum_{r=1}^{n} P_r e^{\lambda_r \tau} \tag{85}$$

is the response of the system initially at rest to a unit impulse and $f(\theta)$ is the arbitrary disturbance input at time $\theta($ $0)$ then the displacement of the system at time τ due to the arbitrary disturbance $f(\theta)$ is

$$x = \int_0^{\tau} \alpha(\tau-\theta) f(\theta)\, d\theta \tag{86}$$

When the input can be expressed as a Fourier integral the response functions can be found by an extension of the method given above (e.g. if the input is a square wave f = const. from $\theta = 0$ to $\theta = T$ and zero outside this range).

We assume that $\int_{-\infty}^{\infty} f(\tau)\, d\tau$ is absolutely convergent.

Let $$f(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[H(\omega) + iK(\omega)\right] e^{i\omega\tau}\, d\omega \tag{87}$$

Then $$H(\omega) + iK(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} f(\tau)\, d\tau \tag{88}$$

Now the response of the system to an input $e^{i\omega\tau}$ is from (1)

$$x(\tau) = \frac{e^{i\omega\tau}}{F(i\omega)}$$

Thus since the system is linear the response to an arbitrary input given by (87) is

$$x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{H(\omega) + iK(\omega)}{F(i\omega)} e^{i\omega\tau}\, d\omega \tag{89}$$

$$\therefore \quad x - \frac{f}{a_0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[H(\omega) + iK(\omega)\right] \frac{a_0 - F(i\omega)}{a_0 F(i\omega)} e^{i\omega\tau}\, d\omega \tag{90}$$

$\therefore$ comparing (52) and (90) and using Parseval's theorem

$$L = \int_0^\infty (x - \frac{f}{a_o})^2 d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[H^2(\omega) + K^2(\omega)\right] ¥(\omega)\omega^2 d\omega \qquad (91)$$

where $¥(\omega) = \dfrac{\left[a_o - F(I\omega)\right]\left[a_o - F(-i\omega)\right]}{a_o^2\, F(i\omega)\, F(-i\omega)\, \omega^2}$

i.e. $$¥(\omega) = \frac{\left[a_1 - a_3\omega^2 + a_5\omega^4 \ldots\right]^2 + \omega^2\left[a_2^2 - a_4\omega^2 + a_6\omega^4 \ldots\right]^2}{a_o^2\left\{\left[a_o - a_2\omega^2 + a_4\omega^4 \ldots\right]^2 + \omega^2\left[a_1 - a_3\omega^2 + a_5\omega^4 \ldots\right]^2\right\}} \qquad (92)$$

For sufficiently large values of ω, $¥(\omega)$ tends to $\dfrac{1}{a_o^2\omega^2}$

For sufficiently small values of ω, $¥(\omega)$ tends to $\dfrac{a_1^2}{a_o^4}$.

This is the same formula as that given in reference 6 allowing for the difference in notation.

Differentiating (90),

$$\frac{dx}{d\tau} - \frac{1}{a_o}\frac{df}{d\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[H(\omega) + iK(\omega)\right]\frac{a_o - F(i\omega)}{a_o F(i\omega)}\, i\omega\, e^{i\omega\tau}\, d\omega$$

provided the latter integral is bounded for the range $(-\infty, +\infty)$.

∴ using Parseval's theorem,

$$L_1 = \int_0^\infty \left[\frac{dx}{d\tau} - \frac{1}{a_o}\frac{df}{d\tau}\right]^2 d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[H^2(\omega) + K^2(\omega)\right]\omega^4 ¥(\omega)\, d\omega \qquad (93)$$

and similarly for L_s.

<u>Response functions for a finite square wave disturbance.</u>

$f(\tau) = f_o$ for $o \leq \tau_o < \tau < \tau_1$; $f(\tau)$ is zero at all other times.

From (88), $$H(\omega) + iK(\omega) = \frac{if_o}{\omega}\left[e^{-i\omega\tau_1} - e^{-i\omega\tau_o}\right]$$

$$= \frac{2f_o}{\omega}\, e^{-i\omega\frac{\tau_o + \tau_1}{2}} \sin\frac{\omega(\tau_1 - \tau_o)}{2} \qquad (94)$$

∴ L /

$$\therefore \quad L = \frac{4f_o^2}{\pi}\int_0^{\infty} \sin^2 \frac{\omega(\tau_1-\tau_o)}{2}\, \mathcal{Y}(\omega)d\omega$$

$$= \frac{f_o^2}{\pi}\int_{-\infty}^{\infty}\left[1 - \cos\omega\,(\tau_1-\tau_o)\right]\mathcal{Y}(\omega)d\omega \qquad (95)$$

These integrals can be evaluated by contour integration (see references 24 and 25).

We consider

$$\int\left[1 - e^{i\omega(\tau_o-\tau_o)}\right]\mathcal{Y}(\omega)d\omega$$

round a contour consisting of a large semicircle radius ρ centre the origin above the real axis together with the real axis. We then let $\rho \to \infty$. The only poles inside the contour are at the points $\omega = -i\lambda_r$ where λ_r is a root of the characteristic equation, the system being stable.

$$\text{Then} \quad L = -2\,f_1^2 \sum_{r=1}^{n} \frac{\left[1-e^{\lambda_r(\tau_1-\tau_o)}\right]\left[a_o-F(-\lambda_r)\right]}{a_o F'(\lambda_r)\,F(-\lambda_r)\,\lambda_r^2} \qquad (96)$$

If the system has rapid damping i.e. $|\lambda_r|$ large or if $(\tau_1-\tau_o)$ is large (but finite) we see that the exponental terms in (96) become negligible and from (95),

$$L = \frac{2f_1^2}{\pi}\int_0^{\infty}\mathcal{Y}(\omega)d\omega \qquad (97)$$

Comparing (69) and (97) we see that L is double the value for a step function disturbance. **This can easily be seen from figure 1.1.**

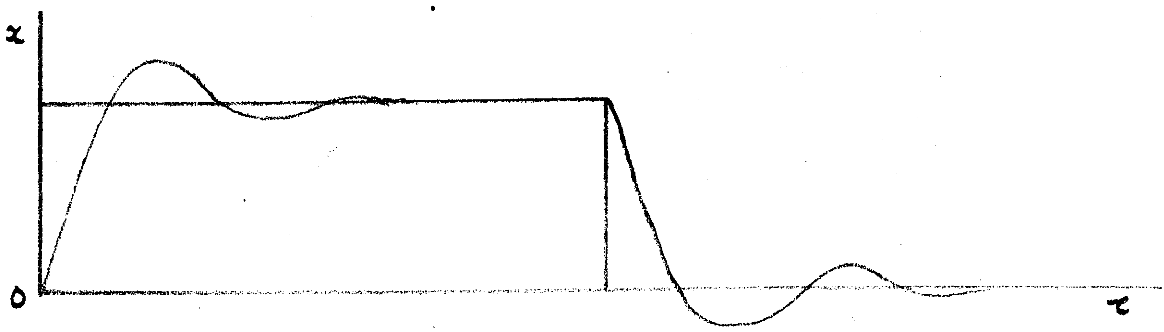


FIGURE 1.1.

RESPONSE OF A HIGHLY DAMPED SYSTEM TO A FINITE SQUARE WAVE DISTURBANCE.

For the finite square wave,

$$L_1 = \int_0^\infty \left(\frac{dx}{d\tau}\right)^2 d\tau$$

Differentiating (89) and remembering the $x(\tau)$ is real we have

$$L_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{H^2(\omega) + K^2(\omega)}{F(i\omega)\ F(-i\omega)}\ \omega^2 d\omega$$

i.e. using (69),

$$L_1 = \frac{4f_1^2}{\pi}\int_0^\infty \sin^2 \frac{\omega(\tau_1-\tau_0)}{2} \quad \frac{d\omega}{F(i\omega)F(-i\omega)} \tag{98}$$

As above we find $L_1 = -2f_1^2 \sum_{r=1}^{n} \dfrac{1-e^{\lambda_r(\tau_1-\tau_0)}}{F'(\lambda_r)F(-\lambda_r)}$ (99)

If the system has rapid damping i.e. $|\lambda_r|$ large or if $(\tau_1-\tau_0)$ is large (but finite),

$$L_1 = \frac{2f_1^2}{\pi}\int_0^\infty \frac{d\omega}{F(i\omega)F(-i\omega)} \tag{100}$$

Comparing (72) and (100) we see that L_1 is double the value for a step function disturbance. As above this can easily be seen from figure 1.1.

Response functions for unstable systems.

The response functions defined above apply only to stable systems. It sometimes happens that the system possesses one slightly unstable mode (e.g. an aircraft with spiral instability). In this case we are primarily interested in the behaviour of the system shortly after the disturbance takes place. The above response functions can be used to give a measure of the response of the system the upper limits of the integrals for L, L_1, etc., being taken to be some convenient finite time.

Chapter 2.

Examples of Optimum Response of Linear Systems with Constant Coefficients having One Degree of Freedom.

In the previous chapter we obtained formulae for the response coefficients L and L_1 in terms of (i) the roots of the characteristic equation, (ii) the coefficients of the characteristic equation and (iii) the frequency response spectrum. We saw too how the response of zero-displacement-error systems, zero-velocity-error systems and zero-acceleration-error systems could be simply related to the response in the free motion. We shall now consider the response of these three systems in greater detail showing how the optimum response of a system can be obtained. We shall show the relation between the transient response, the frequency response and the roots of the characteristic equation for the optimum system.

As in chapter I we consider a system for which the equation of motion is

$$a_n \frac{d^n x}{d\tau^n} + a_{n-1} \frac{d^{n-1} x}{d\tau^{n-1}} + a_{n-2} \frac{d^{n-2} x}{d\tau^{n-2}} + \dots + a_1 \frac{dx}{d\tau} + a_o x = f(\tau) \quad (1)$$

where a_o, a_1, a_2, ... a_n are constants and $a_n \neq 0$.

Response of zero-displacement-error systems to step function disturbance (constant displacement input).

In this case $f = 0$ for $\tau \leq 0$; $f = f_o$ for $\tau > 0$. (2)

where f_o is a constant.

As in reference 7 we shall find it convenient to normalize (1). Let ω_o be the undamped natural frequency of the system.

Then $$a_o = \omega_o^n a_n \quad (3)$$

We define a new time scale by the relation

$$u = \omega_o \tau \quad (4)$$

From (1), (2), (3), (4), the normalized equation of motion is

$$\frac{d^n x}{du^n} + q_{n-1} \frac{d^{n-1} x}{du^{n-1}} + q_{n-2} \frac{d^{n-2} x}{du^{n-2}} + \dots + q_1 \frac{dx}{du} + x = \frac{f_o}{a_o} \; (\tau > 0) \quad (5)$$

where $$q_r = \frac{a_r}{a_n \omega_o^{\,n-r}} \qquad (r=0 \text{ to } n) \quad (6)$$

Now/

/Now the magnitude of the response x is proportional to f_o, but the nature of the response($^o/o$ overshoot, damping time) is independent of f_o. We shall take

$$f_o = a_o \tag{7}$$

Thus we shall determine the response of the system given by (5) to a unit step function disturbance. As shown in chapter (1) the response is identical in form with that in a free motion with $x_o = -1$ and Dx_o , D^2x_o ,... $D^{n-1}x_o$ zero.

We note that equation (4) merely alters the time scale of the damping but does not affect the $^o/o$ overshoot or the general form of the response which are functions of q_r($r=1$ to $n-1$). For a zero-displacement-error system the coefficients of x and $\frac{d^n x}{du^n}$ are unity in the equation of motion. This does not affect the determination of the optimum (since $a_o = 0$ is not an optimum solution in this case).

Second order zero displacement error system.

From Chapter 1, (19), (39) and (41), with $x_o = -1$, $Dx_o = 0$,

$$2La_oa_1 = \begin{vmatrix} a_1 & a_2 \\ -a_o & a_1 \end{vmatrix} = a_1^2 + a_oa_2 \tag{8}$$

or in terms of the normalized coefficient q_1 given by (6),

$$2L\omega_o = \frac{q_1^2+1}{q_1} \tag{9}$$

Similarly $$2L_1a_oa_1 = \begin{vmatrix} a_1 & a_o \\ -a_o & o \end{vmatrix} = a_o^2 \tag{10}$$

i.e. $$\frac{2L_1}{\omega_o} = \frac{1}{q_1} \tag{11}$$

Now for stability with $a_2 > 0$,

$$a_1 > 0 \quad \text{and} \quad a_o > 0$$

i.e. $q_1 > 0$.

We/

/We see that for a system with a given undamped natural frequency ω_o, L and L_1 are both functions of q_1 (which for a mechanical system is proportional to the damping). The minimum value of L occurs for $q_1 = 1$.

$$\text{i.e.} \quad a_1^2 = a_o a_2$$

Then
$$L_{min} = \frac{1}{\omega_o} \tag{12}$$

and for this value of q_1,
$$L_1 = \frac{1}{2}\omega_o \tag{13}$$

This corresponds to an oscillatory motion with overswing of 16 per cent (see Figure 2.1). We shall see that this tendency for the oscillatory motion to be rather lightly damped is a characteristic of systems based on L_{min}. This overswing is lessened by selecting a higher value of q_1 (or a_1) as is seen from the following table.

Table I.

Variation of response functions and overshoot with the damping coefficient for a second order zero displacement error system.

$q_1 = \frac{a_1}{\sqrt{a_o a_2}}$	$L\omega_o$	$\frac{L_1}{\omega_o}$	°/o overshoot.
0.6	1.13	0.83	38
0.8	1.03	0.63	26
1.0	1.00	0.50	16
1.2	1.02	0.42	9
1.4	1.06	0.36	5
1.6	1.11	0.31	2
1.8	1.18	0.28	0
2.0	1.25	0.25	0

We see that the minimum of L is fairly flat; the value of L is not more than 10°/o greater than the minimum for values of q_1 from 0.7/

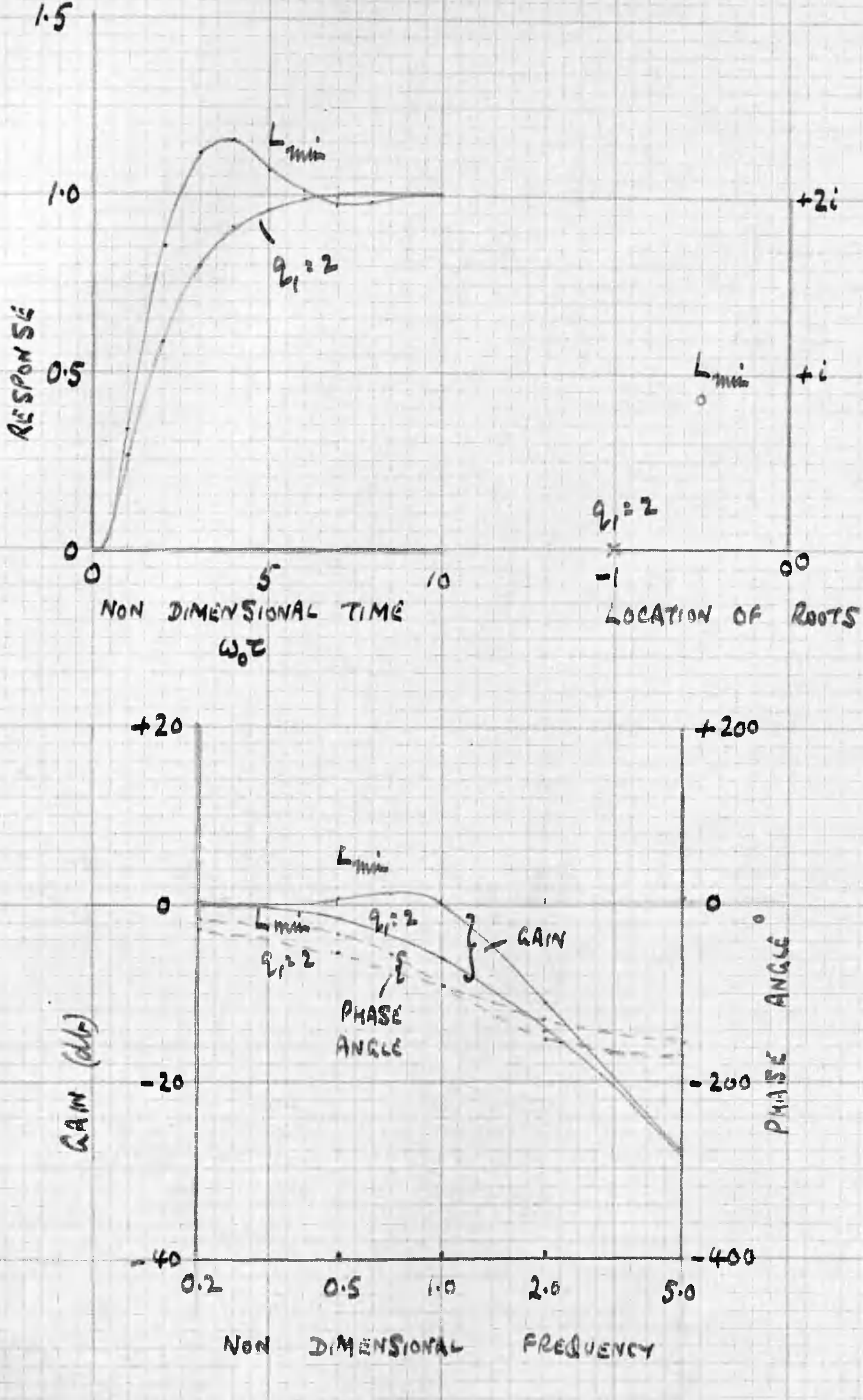


FIGURE 2.1

SECOND ORDER ZERO DISPLACEMENT ERROR SYSTEM

/0.7 to 1.5. This flat minimum has lead other writers to adopt other criteria for optimisation as shown in the introduction. However we see from the above table that there is good correlation between the °/o overshoot and the values of L_1, as would be expected from the definition of L_1. The smaller L_1 is, the "smoother" is the response, the response becoming less oscillatory in nature. Precisely what value of q_1 is chosen for the most satisfactory system is a matter for individual choice, depending upon the acceptable °/o overshoot. From the above table we see that to keep both L and L_1 as small as possible, q_1 should be greater than 1. Now if q_1 is greater than 2 the roots of the characteristic equation will both be real and negative, one root decreasing, the other increasing, as q_1 increases; the motion will then be composed of two subsidences and, in the terminology of servomechanism analysis, would be considered to be overdamped. When $q_1 = 2$, the system is said to be critically damped. We are therefore lead to the criteria

$$1.0 \lesssim q_1 \lesssim 2.0 \tag{14}$$

for satisfactory performance for a second order zero displacement error system. This is in good agreement with current practice with servomechanisms for which $0.8 \lesssim q_1 \lesssim 2.0$ is taken as "a good starting point in adjusting the transient response of a system" (reference 11, p.54).

In the above analysis we have considered the effect on the response functions of varying q_1. This is equivalent to keeping a_0 and a_2 fixed and varying a_1. Considering now the effect of varying a_0 (keeping a_1 and a_2 fixed) we see that the minimum value of L occurs for large (infinite) a_0; then L_1 is large. This corresponds to a system with a large (undesirable) overswing but having a very rapid response. Considering finally the effect of varying a_2 (keeping a_1 and a_0 fixed) we see that the minimum value of L (for stable systems) occurs for $a_2 = 0$; L_1 is independent of a_2. Small values of a_2 correspond to very highly damped systems.

Some general conclusions can be drawn from the above discussion. We see that a satisfactory range of values of the damping coefficient q_1 for a second order zero displacement error system is given by (14). The choice of ω_0 will depend on the desired speed of response of the system; the higher the value of ω_0 the sooner will the system reach its steady state. Changing ω_0 merely/

/merely alters the time scale of the damping but does not affect the °/o overshoot or the general form of the response.

The roots of the characteristic equation for $L_{min}(q_1=1)$ are

$$-0.5 \pm 0.87\,i$$

while those for $q_1=2$ are $-1,-1$. (See Figure 2.1).

The third diagram in Figure 2.1. is an attenuation phase diagram for the two cases $q_1=1$ and $q_1=2$ plotted on a logarithmic scale against the non dimensional frequency ω/ω_o. This is found from chapter 1,(59) and (60). We plot the gain in decibels (see references 1, 11 and 26). If M is the dynamic magnification,

$$\text{the gain} = 20 \log_{10} M \text{ decibels.}$$

We see that the system making L a minimum ($q_1 = 1$) has a maximum magnification M_{max} of 1.15 at a frequency $0.71\omega_o$, whereas for the system with $q_1 = 2$ (sometimes known as a binomial filter) the magnification deceases uniformly with increasing frequency. Thus for the range $1.0 \leq q_1 \leq 2.0$, M_{max} does not exceed 1.15. This is well within the current practice with servomechanisms, "systems for which M_{max} does not exceed about 1.4 probably having a transient performance acceptable for most applications" (reference 11, p.109). As shown in Chapter 1 the KG locus can easily be obtained from the attenuation-phase diagram.

Third order zero displacement error system.

From chapter 1, (19),(39) and (41), with $x_o= -1$, $Dx_o= D^2x_o = 0$,

$$\left.\begin{aligned} 2La_o \begin{vmatrix} a_1 & a_3 \\ a_o & a_2 \end{vmatrix} &= \begin{vmatrix} a_1 & a_2 & 0 \\ -a_o & a_1 & a_3 \\ 0 & a_o & a_2 \end{vmatrix} \\ \text{i.e.}\quad 2La_o\,(a_1a_2 - a_oa_3) &= a_1^2\,a_2 - a_oa_1a_3 + a_oa_2^2 \end{aligned}\right\} \quad (15)$$

or in terms of the normalized coefficients q_1, q_2 given by (6),

$$2L\omega_o = \frac{q_1^2\,q_2 - q_1 + q_2^2}{q_1q_2 - 1} \quad (16)$$

Similarly/

/Similarly $2L_1 a_o \begin{vmatrix} a_1 & a_3 \\ a_o & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_o & 0 \\ -a_o & 0 & a_3 \\ 0 & 0 & a_2 \end{vmatrix}$ (17)

i.e. $$\frac{2L_1}{\omega_o} = \frac{q_2}{q_1 q_2 - 1} \tag{18}$$

Now for stability with $a_3 > 0$,

$a_2 > 0$, $a_1 a_2 - a_o a_3 > 0$ and $a_o > 0$

i.e. $q_2 > 0$ and $q_1 q_2 - 1 > 0$

The minimum value of L is found from

$$\frac{\partial L}{\partial q_1} = \frac{\partial L}{\partial q_2} = 0$$

Then $$q_1 = 2\,, \quad q_2 = 1. \tag{19}$$

i.e. $$a_1^3 = 8\, a_o^2\, a_3$$
$$a_2^3 = a_o a_3^2$$

We see that when the test functions are written in terms of the q s, at L_{min},

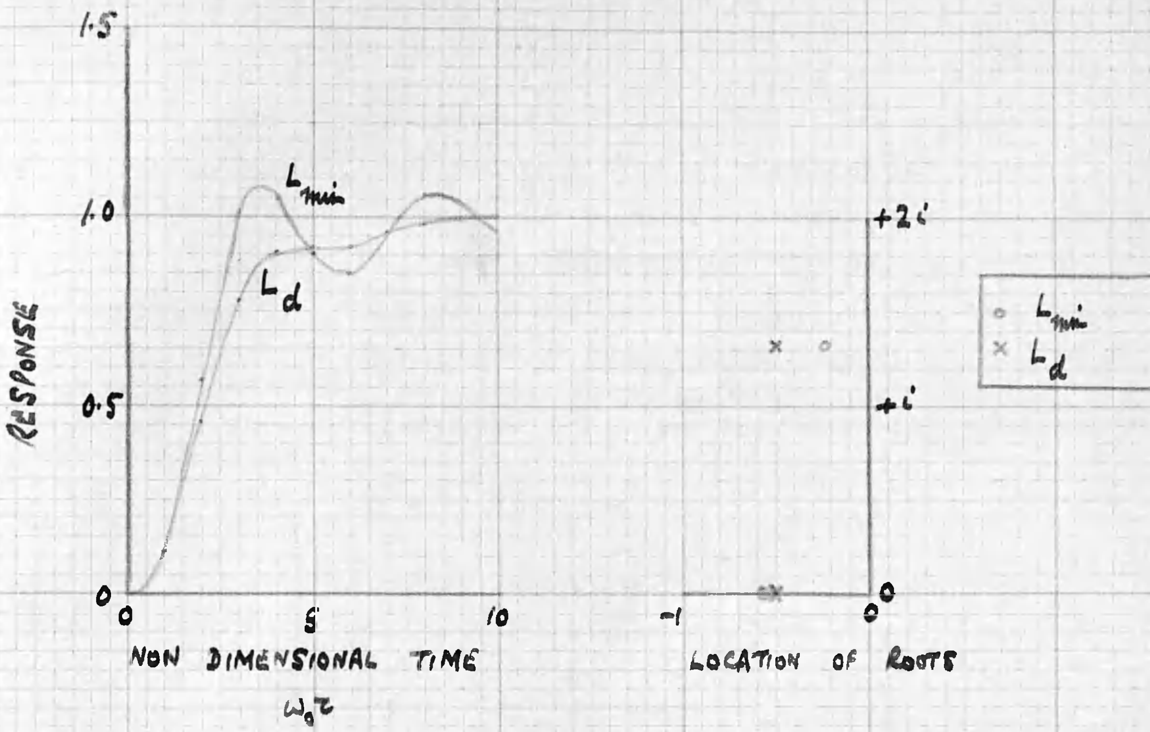
$$\frac{T_1}{\omega_o a_3} = \frac{T_2}{\omega_o^3 a_3^2} = \frac{T_3}{\omega_o^6 a_3^3} = 1 \tag{20}$$

Then $$L_{min} = \frac{3}{2\omega_o} \tag{21}$$

and for these values of q_1 and q_2,

$$L_1 = \tfrac{1}{2}\omega_o \tag{22}$$

This corresponds to a motion composed of a subsidence and a rather lightly damped oscillation, the first overswing for a unit step disturbance being 6 per cent followed by a subsequent underswing of 15 per cent (see figure 2.2). From (16) and (18) we see that L and L_1 are functions of both q_1 and q_2. The variation of the response functions/



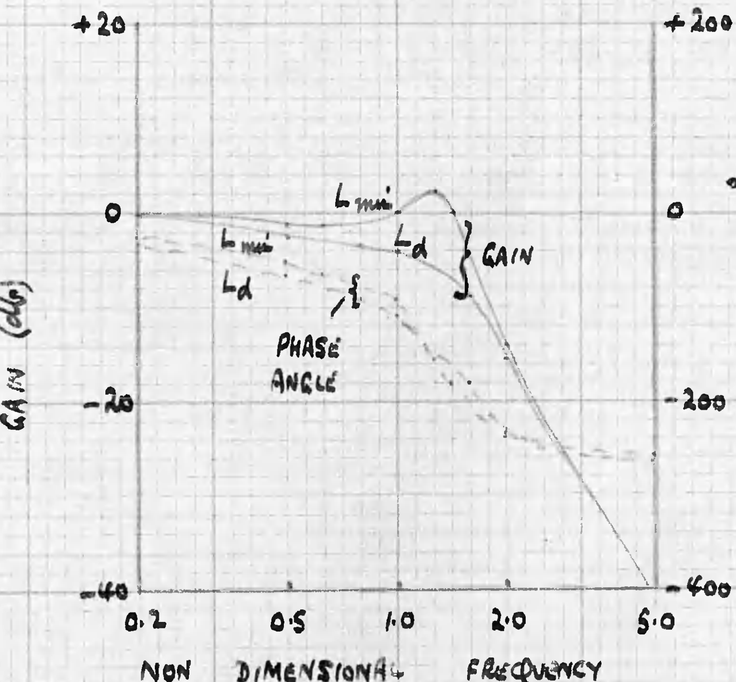


FIGURE 2.2

THIRD ORDER ZERO DISPLACEMENT ERROR SYSTEM

/functions with q_1 and q_2 is shown in Figure 2.3. The curves L = const are oval shaped, while those for L_1 = const. are a family of rectangular hyperbolas. As q_1 and q_2 are varied the roots of the characteristic equation will also vary. Now for q_1 and q_2 in the neighbourhood of L_{min} the characteristic equation has one real negative root and one complex pair of roots with negative real part, representing respectively a subsidence and a damped oscillation, the subsidence being more highly damped than the oscillation. For the characteristic equation

$$\lambda^3 + q_2\lambda^2 + q_1\lambda + 1 = 0$$

to have all its roots with equal negative real part (i.e. equal damping in all modes),

$$q_1 = \frac{2}{9} q_2^2 + \frac{3}{q_2}$$

The curve of equal damping is shown in Figure 2.3. To the left of this curve the subsidence is more highly damped, to the right the oscillatory motion is more highly damped.

To find a system with a more satisfactory transient response than that for L_{min} we shall choose q_1 and q_2 so that L_1 is decreased considerably while L is only increased slightly. We see that these values of q_1 and q_2 lie close to the line

$$q_1 = 1 + q_2$$

which is the normal to the rectangular hyperbola L_1 = const which passes through the point (2,1). L and L_1 are insensitive to small displacements normal to this line.

Consider the system given by

$$q_1 = 2 + \theta$$
$$q_2 = 1 + \theta$$

Then $$q_1 = 1 + q_2 .$$

From Figure 2,3 we see that for $\theta = 0.5$ the ~~roots of the~~ three roots of the characteristic equation ~~have~~ have equal negative real ~~roots~~ parts. For such a system

$$L\omega_0 = L_d\omega_0 = 1.66$$

the/

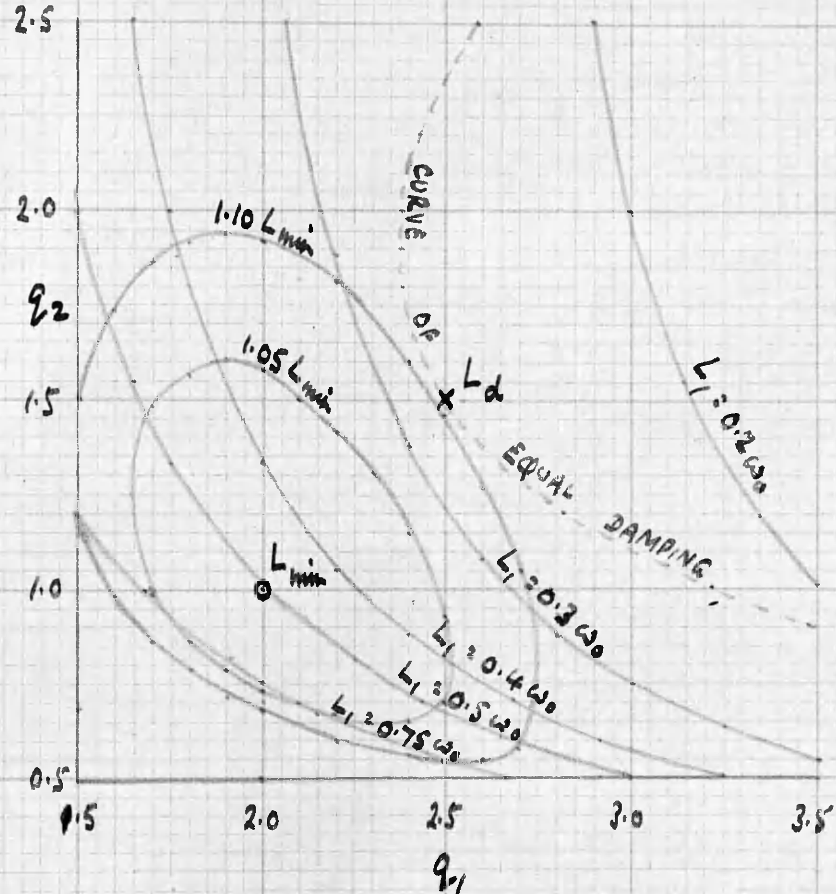


FIGURE 2.3

VARIATION OF RESPONSE FUNCTIONS WITH q_1 AND q_2 FOR THIRD ORDER ZERO DISPLACEMENT ERROR SYSTEM

/the suffix d denoting that the roots are equally damped.

Then $L_1 = 0.273\omega_0$

Thus L_d is 12 per cent greater than L_{min} while L_1 has decreased by 45 per cent. The response to a step function is shown in figure 2,2. We see that there is no overswing for this third order system. As would be expected the response for the system with equal damping (denoted by L_d) is much "smoother" than that for L_{min}, the response increasing practically monotonically to its final value. However the time for the system to reach zero error (momentarily) is much smaller for L_{min}. The corresponding roots of the characteristic equation for the two systems are given in the following table (and in Figure 2,2).

Table 2.

Roots of the characteristic equation for a third order zero displacement error system.

System		
L_{min}	-0.57 ,	$-0.215 \pm 1.31i$
L_d	-0.50 ,	$-0.50 \pm 1.32i$

We see that while the damping of the two systems is different, the frequency of the oscillatory mode is practically unchanged. We note too that the damping of the subsidence is only decreased by 12 per cent while that of the oscillatory mode is more than doubled.

We have arbitrarily considered the case $\theta = 0.5$. The response as shown in Figure 2,2 might well be considered by many to be overdamped. By choosing θ to be rather smaller, say $\theta = 0.3$, the form of the response would be intermediate between that for L_{min} and L_d. As θ is increased the damping of the subsidence increases. By analogy with the second order system $\theta = 0.5$ may be said to correspond to critical damping. Precisely what value of θ is chosen for the most satisfactory system is again a matter for individual choice depending upon the acceptable per cent overshoot. We are therefore led to the criteria

$$\left.\begin{aligned} &0 \leq \theta \leq 0.5 \\ \text{where}\quad &q_1 = 2 + \theta \\ &q_2 = 1 + \theta \end{aligned}\right\} \qquad (23)$$

for/

/for satisfactory performance for a third order zero displacement error system. As for a second order system the higher the value of ω_o the sooner will the system reach its steady state.

The third diagram of Figure 2,2 is an attenuation phase diagram for the two cases L_{min} and L_d. We see that the system making L a minimum has a maximum magnification of 1.29 at a frequency $1.26\omega_o$, whereas for the system L_d the magnification decreases monotonically with increasing frequency. Thus for the range $0 \lesssim \theta \lesssim 0.5$, M_{max} does not exceed 1.29. We note that at the frequency ω_o, the gain is zero for L_{min} while for lower frequencies there is a small negative gain.

<u>Higher order zero displacement error systems.</u>

The analysis given above can easily be extended to higher order systems. From Chapter 1, (19), (39) and (41), with $x_o = -1$, $Dx_o = D^2x_o = \ldots = D^{n-1}x_o = 0$,

$$2La_o \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & \ldots \\ a_o & a_2 & a_4 & a_6 & \ldots \\ 0 & a_1 & a_3 & a_5 & \ldots \\ 0 & a_o & a_2 & a_4 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_4 & a_6 & \ldots \\ -a_o & a_1 & a_3 & a_5 & \ldots \\ 0 & a_o & a_2 & a_4 & \ldots \\ 0 & 0 & a_1 & a_3 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \qquad (24)$$

or in terms of the normalized coefficients $q_1, q_2, \ldots q_{n-1}$ given by (6),

$$2L\omega_o \begin{vmatrix} q_1 & q_3 & q_5 & q_7 & \ldots \\ 1 & q_2 & q_4 & q_6 & \ldots \\ 0 & q_1 & q_3 & q_5 & \ldots \\ 0 & 1 & q_2 & q_4 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} q_1 & q_2 & q_4 & q_6 & \ldots \\ -1 & q_1 & q_3 & q_5 & \ldots \\ 0 & 1 & q_2 & q_4 & \ldots \\ 0 & 0 & q_1 & q_3 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \qquad (25)$$

where the determinants on the left hand side of (24) and (25) are of order n-1 and those on the right hand side are of order n. Also $q_n = 1$.

contd.

Similarly
$$\frac{2L_1}{\omega_o}\begin{vmatrix} q_1 & q_3 & q_5 & q_7 & \cdots \\ 1 & q_2 & q_4 & q_6 & \cdots \\ 0 & q_1 & q_3 & q_5 & \cdots \\ 0 & 1 & q_2 & q_4 & \cdots \\ \cdots & & & & \end{vmatrix} = \begin{vmatrix} q_1 & 1 & q_4 & q_6 & \cdots \\ -1 & 0 & q_3 & q_5 & \cdots \\ 0 & 0 & q_2 & q_4 & \cdots \\ 0 & 0 & q_1 & q_3 & \cdots \\ \cdots & & & & \end{vmatrix} = \begin{vmatrix} q_2 & q_4 & q_6 & \cdots \\ q_1 & q_3 & q_5 & \cdots \\ 1 & q_2 & q_4 & \cdots \\ \cdots & & & \end{vmatrix} \tag{26}$$

the final determinant on the right hand side of (26) being of order n-2. Equation (26) can be written in a simpler form in terms of the test functions of the characteristic equation. We find

$$L_1 = \frac{a_o\, T_{n-2}}{2\, T_{n-1}} \tag{27}$$

Now as stated in Chapter 1, for stability with $a_n > 0$ all the test functions must be positive.

The minimum value of L is found from

$$\frac{\partial L}{\partial q_1} = \frac{\partial L}{\partial q_2} = \cdots = \frac{\partial L}{\partial q_{n-1}} = 0$$

The values of L_{min} and the corresponding values of q_o, q_1, q_2, ... q_n are given in the following table.

contd.

TABLE 3.

Values of L_{min} and the normalized coefficients q_m for zero displacement error systems of order n.

n	$2L_{min}\omega_o$	q_o	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
1	1	1	1							
2	2	1	1	1						
3	3	1	2	1	1					
4	4	1	2	3	1	1				
5	5	1	3	3	4	1	1			
6	6	1	3	6	4	5	1	1		
7	7	1	4	6	10	5	6	1	1	
8	8	1	4	10	10	15	6	7	1	1

From the above table we see that

$$L_{min} = \frac{n}{2\omega_o} \tag{27}$$

It can be seen that the numbers in the diagonals sloping up to the right are the binomial coefficients. We find that the coefficient q_m for an nth order system is equal to ${}_pC_n$, the binomical coefficient, where

$$\left.\begin{aligned} {}_pC_m &= \frac{\lfloor p}{\lfloor m \, \lfloor p-m} \\ \text{and} \quad p &= \frac{m+n}{2} \quad \text{if } m+n \text{ is even} \\ \text{or} \quad & \frac{m+n-1}{2} \quad \text{if } m+n \text{ is odd} \end{aligned}\right\} \tag{28}$$

Thus the above table can easily be extended to any order n.

We find too that, when the test functions are written in terms of the q's with the above coefficients at L_{min},

$$\frac{T_1}{\omega_o a_n} = \frac{T_2}{\omega_o^3 a_n^2} = \frac{T_3}{\omega_o^6 a_n^3} = \dots = \frac{T_{n-1}}{\omega_o^{s\lfloor n-1 \rfloor} a_n^{n-1}} = \frac{T_n}{\omega_o^{s\lfloor n} a_n^n} = 1 \tag{29}$$

where/

/where $s[r] \equiv$ sum of the first r integers $= \frac{r(r+1)}{2}$

From (27) and (29) when L is a minimum,

$$L_1 = \frac{a_o}{2\omega_o^{n-1} a_n} = \frac{\omega_o}{2} \qquad (30)$$

Thus L_{min} is proportional to the order n of the system but the corresponding value of L_1 is independent of the order of the system.

The motion for this system is composed of a number of damped oscillations and possibly a subsidence (if n is odd), as shown in the following table.

TABLE 4.

Roots of the characteristic equation for a zero displamement error L_{min} system of order n.

n	Roots.
2	$-0.5 \pm 0.87i$
3	$-0.57, -0.215 \pm 1.31i$
4	$-0.395 \pm 0.505i, -0.105 \pm 1.57i$
5	$-0.41, -0.235 \pm 0.88i, -0.06 \pm 1.70i$
6	$-0.315 \pm 0.362i, -0.155 \pm 1.5i$, $-0.03 \pm 1.78i$
7	$-0.33, -0.22 \pm 0.665i, -0.09 \pm 1.35i$, $-0.025 \pm 1.83i$
8	$-0.27 \pm 0.283i, -0.15 \pm 0.91i$, $-0.068 \pm 1.50i$, $-0.013 \pm 1.86i$,

For a system of given order the modes of low frequency are more highly damped. As n increases, the damping in all the modes decreases and the corresponding frequencies increase. The mode with the least damping is an oscillatory mode of relatively short period. As would be expected this gives a system with rather a wavy response, due to the terms of high frequency, with a certain amount of hunting(usually of small amplitude compared with the input motion). This is shown in Figures 2.4 and 2.5 for a fourth and fifth order system, the first overswing for a unit step disturbance being 9 per cent for the fourth order system and 10 per cent for the fifth order one. The amplitudes of/

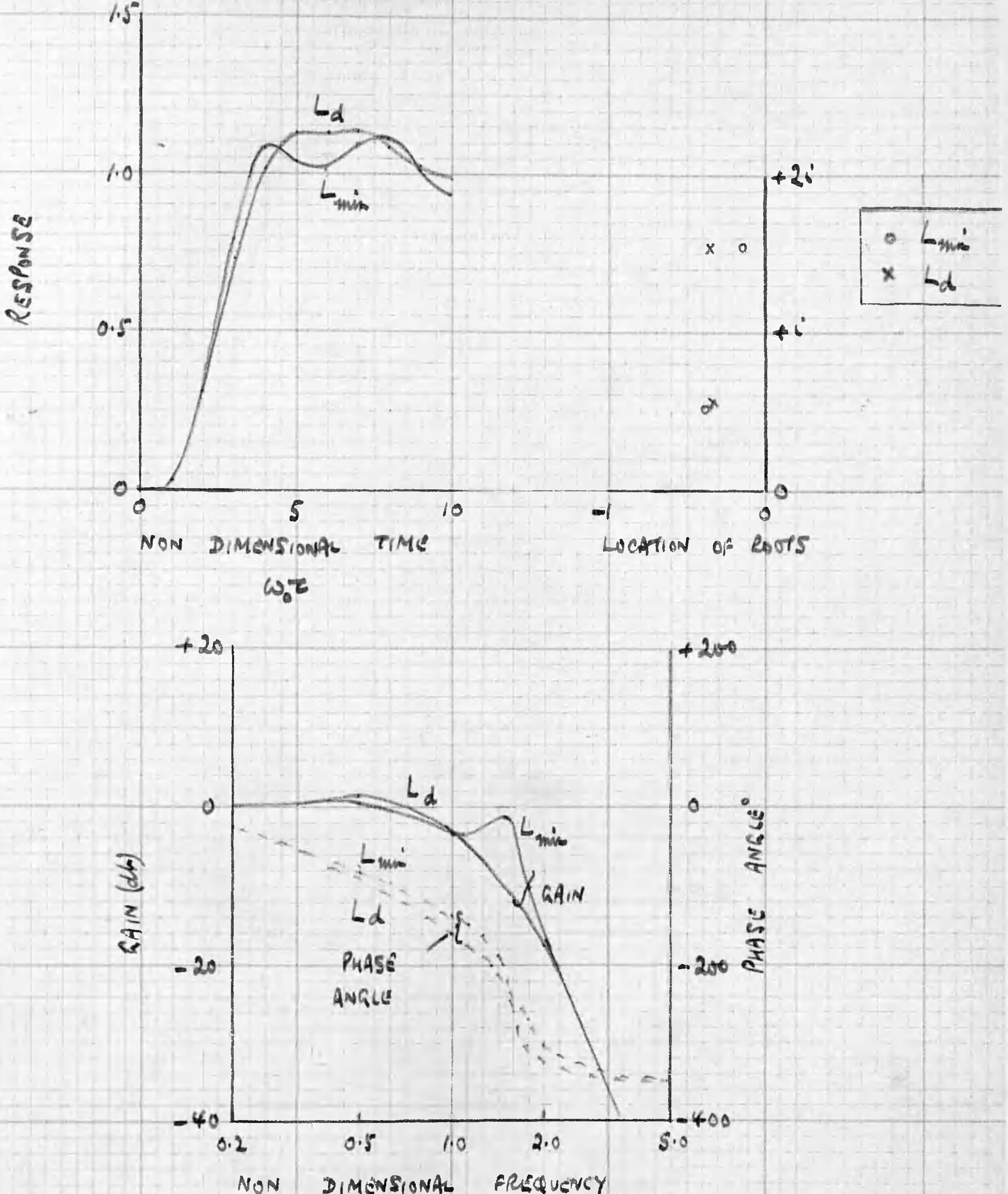
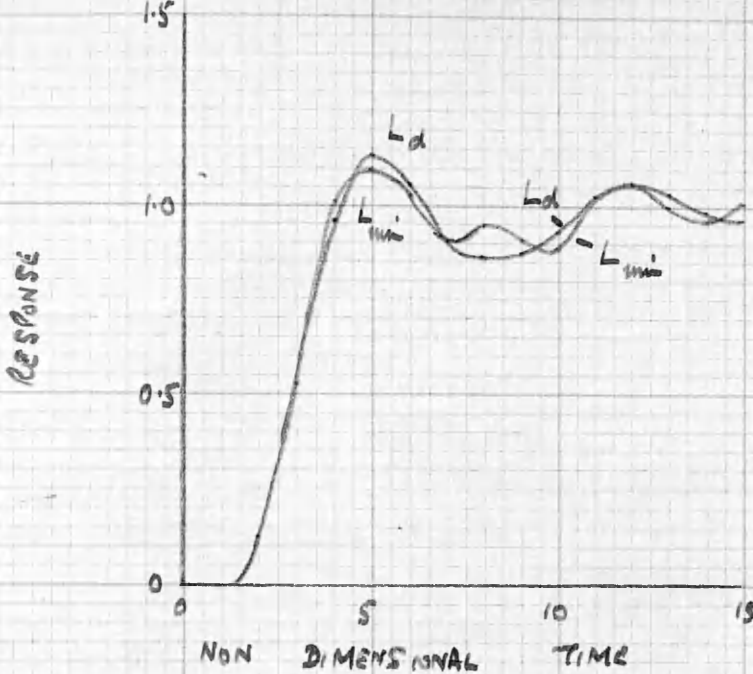


FIGURE 2.4

FOURTH ORDER ZERO DISPLACEMENT ERROR SYSTEM



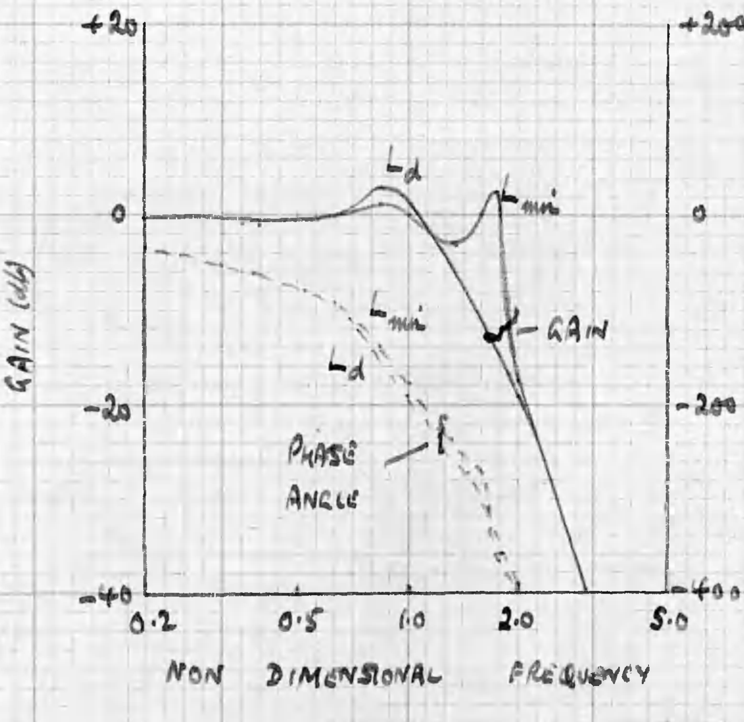


FIGURE 2.5

FIFTH ORDER ZERO DISPLACEMENT ERROR SYSTEM

/of the corresponding motions at $\omega_0\tau=10$ are 6 and 11 per cent respectively.

As with the second and third order systems we shall endeavour to find a system with a more satisfactory response than that for L_{min}, choosing the q's so that L_1 is decreased considerably while L is only increased slightly.

TABLE 5.

Values of L and L_1 for a fourth order zero displacement error system.

q_1	q_2	q_3	$L\omega_0$	L_1/ω_0	~~Minimum damping factor.~~
2	3	1	2.00	0.50	
1	3	1	3.00	1.00	
2	4	1	2.17	0.33	
2	3	2	2.25	0.50	
2	3.38	1.38	2.12	0.39	

From Table 5 we see that for a fourth order system L is least sensitive to small changes in q_2 and q_3. In general we shall proceed as for the third order system, choosing values of the q's which are on the normal to the surface $L_1 = 0.5\omega_0$ at the point corresponding to L_{min}. At this point we find

$$\left.\begin{aligned} \frac{\partial L_1}{\partial q_r} &= 0 \qquad\qquad r = 1 \text{ to } n-3 \\ \frac{\partial L_1}{\partial q_r} &= -\tfrac{1}{2}\omega_0 \qquad r = n-2,\ n-1 \end{aligned}\right\} \quad (31)$$

Thus for a fourth order system points on the normal are given by

$$\left.\begin{aligned} q_1 &= 2 \\ q_2 &= 3 + \theta \\ q_3 &= 1 + \theta \end{aligned}\right\} \quad (32)$$

where, for L_1 to be less than its value at L_{min}, θ must be positive. This is recognised to be an approximate procedure. We should really determine the locus of points at which the surfaces $L = \text{const}$ and $L_1 = \text{const}$ touch one another. This is equivalent to constructing a/

/a graph of dimension (n-1) corresponding to the variables q_1 , q_2 , ... q_{n-1}. However we shall be mainly interested in points close to L_{min} and the present approximate procedure should not be too far from the truth in that region. The exact values of the q's for a fourth order system are given below.

As θ increases the roots of the characteristic equation vary, the damping of the lightly damped oscillation being increased while that of the other modes is either unchanged or slightly decreased. We note that since q_{n-1} increases as θ increases the "total" damping of the system will increase. As above we consider the smallest positive value of θ for which the characteristic equation has at least three (usually four) roots with equal negative real parts. For such a system L is denoted by L_d . The values of the normalized coefficients and the roots of the characteristic equation are given in Tables 6 and 7, obtained by the approximate procedure outlined above.

TABLE 6.

Values of the normalized coefficients q_m for zero displacement error systems of order n with equal damping (L_d system)

(obtained by approximate method)

n	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
3	1	2.5	1.5	1					
4	1	2	3.38	1.38	1				
5	1	3	3	4.17	1.17	1			
6	1	3	6	4	5.07	1.07	1		
7	1	4	6	10	5	6.03	1.03	1	
8	1	4	10	10	15	6	7.014	1.014	1

contd.

TABLE 7.

Roots of the characteristic equation for a zero displacement error L_d system of order n.

n	Roots.
3	$-0.50\ ,\ -0.50 \pm 1.32i$
4	$-0.345 \pm 0.53i\ ,\ -0.345 \pm 1.54i$
5	$-0.41,\ -0.19 \pm 0.90i,\ -0.19 \pm 1.70i$
6	$-0.315 \pm 0.347i\ ,\ -0.105 \pm 1.17i\ ,$ $-0.105 \pm 1.77i$
7	$-0.33\ ,\ -0.22 \pm 0.70i\ ,\ -0.065 \pm 1.36i\ ,$ $-0.065 \pm 1.83i$
8	$-0.27 \pm 0.283i\ ,\ -0.15 \pm 0.91i\ ,$ $-0.0425 \pm 1.50i\ ,\ -0.0425 \pm 1.86i$

We see that, as the order of the system increases, the value of θ for equal damping decreases, and the system approaches that of L_{min}. Here again it must be emphasised that the above tables are based on the approximate procedure outlined above. The exact values of the q's for a fourth order system with equal damping are

$$q_1 = 2.3 \qquad q_2 = 3.8 \qquad q_3 = 1.4$$

giving $\quad L\omega_o = 2.15\ ,\ L_1 = 0.30\omega_o\ ,$

the corresponding values of the roots of the characteristic equation being

$$-0.35 \pm 0.46i\ ,\ -0.35 \pm 1.70i$$

Thus the damping is practically the same as that given by the approximate method while the frequencies differ by between 10 and 15 per cent from those given in Table 7.

Comparing tables 4 and 7 we see that in going from L_{min} to L_d the damping of the least damped oscillation is increased about three times, the frequencies of the various modes being practically unchanged. Thus as shown in Figures 2.4 and 2.5 the response of the system to a unit step disturbance is of a much smoother nature. The peak overshoot has increased slightly to 13 per cent for both systems.

Comparing the response curves of Figures 2.4 and 2.5 with those based/

/based on the ITAE criterion (reference 7) and the Butterworth filters (reference 10), we see that the overshoot is greater and the damping is less than with the ITAE curves. The overshoot is about the same as with the Butterworth filters, the damping is slightly less. The time for the error first to become zero (momentarily) is smaller than for either the ITAE curves or the Butterworth filters (for a given value of ω_o). The advantage of the above determination over that of the ITAE method is that the coefficients given in Table 6 can be readily extended to systems of any order, being based on an analytic formula. The response curves of the systems L_d and L_{min} form a family of curves as the order of the system is increased. Thus the justification of the use of the L criteria must depend on the final form of the response curves, and upon the comparative ease with which the criteria can be calculated especially when the approximate method is used. It is not surprising that the general shape of some of the response curves for L_d is not unlike that with the Butterworth filters. Other investigators have arrived at similar results (see the discussion of reference 7).

The third diagram of Figures 2.4 and 2.5 is an attenuation phase diagram for the two cases L_{min} and L_d. The L_{min} systems have very pronounced resonance peaks for frequencies $1.53\omega_o$ and $1.68\omega_o$ respectively, corresponding to the lightly damped high frequency oscillation shown in Table 4. These peaks are absent in the L_d systems, which have a less pronounced peak at $0.47\omega_o$ and $0.87\omega_o$ respectively, the corresponding values of the maximum magnification being 1.19 and 1.43.

<u>Response of zero-velocity-error systems to step function disturbance (constant velocity input).</u>

As shown in Chapter 1, in this case $f = 0$ for $\tau \leqslant 0$;

$$f = f_1 + f_o\tau \quad \text{for} \quad \tau \geqslant 0 \tag{33}$$

where f_o and f_1 are constants such that

$$\frac{f_o}{a_o} = \frac{f_1}{a_1} \tag{34}$$

As above we shall find it convenient to normalize the equation of motion (1), but in a different manner from that for the zero-displacement-error system.

Let/

Let ω_1 be a frequency defined by

$$a_1 = \omega_1^{n-1} a_n \tag{35}$$

Thus ω_1 would be the undamped natural frequency of a system with $a_0 = 0$.

We define a new time scale by the relation

$$u_1 = \omega_1 \tau \tag{36}$$

From (1), (33) - (36), the normalized equation of motion is

$$\frac{d^n x}{du_1^n} + r_{n-1}\frac{d^{n-1}x}{du_1^{n-1}} + r_{n-2}\frac{d^{n-2}x}{du_1^{n-2}} + \dots + r_2\frac{d^2x}{du_1^2} + \frac{dx}{du_1} + r_0 x$$

$$= \frac{f_1}{a_1\omega_1}(1 + r_0 u_1) \qquad (\tau > 0) \tag{37}$$

where $$r_m = \frac{a_m}{a_n \omega_1^{n-m}} \qquad (m = 0 \text{ to } n) \tag{38}$$

As above we see that the magnitude of the response x is proportional to f_1, but the nature of the response is independent of f_1. We shall take

$$f_1 = a_1 \tag{39}$$

Thus we shall determine the response of the system given by (37) to a constant unit velocity disturbance. ~~of the form~~

$$\cancel{1 + r_0 u_1}$$

As shown in Chapter I the response is identical in form with that in the free motion with $Dx_0 = -1$, and x_0, D^2x_0, D^3x_0, ... $D^{n-1}x_0$ zero.

From (37) we see that for a zero-velocity-error system the coefficients of $\frac{dx}{du_1}$ and $\frac{d^n x}{du_1^n}$ are unity in the equation of motion.

This does not affect the determination of the optimum (since $a_1 = 0$ is not an optimum solution in this case).

For a zero-velocity-error system we are usually interested (in servomechanism theory) with the velocity response at a given time. We shall therefore consider values of

contd.

$$L_1 = \int_0^\infty \left(\frac{dx}{d\tau}\right)^2 d\tau = \int_0^\infty v^2 d\tau \tag{40}$$

and the variation of L_2 where

$$L_2 = \int_0^\infty \left(\frac{d^2x}{d\tau^2}\right)^2 d\tau = \int_0^\infty \left(\frac{dv}{d\tau}\right)^2 d\tau \tag{41}$$

in the neighbourhood of minimum values of L_1 .

where $$v = \frac{dx}{d\tau} \tag{42}$$

<u>Second order zero velocity error system.</u>

From Chapter 1, (38), with $Dx_o = -1\cancel{\omega}_1$, $x_o = 0$,

$$\left.\begin{aligned} 2L_1 a_1 &= a_2 \\ \text{i.e.} \quad L_1 &= \frac{1}{2\omega_1} \end{aligned}\right\} \tag{43}$$

Similarly from Chapter 1, (27),

$$2\, L_2\, a_2 = \frac{a_1^2 + a_o a_2}{a_1} \tag{44}$$

or in terms of the normalized coefficient r_o given by (38),

$$\frac{2L_2}{\omega_1} = 1 + r_o \tag{45}$$

We see that for a system with a given value of ω_1 (i.e. for a second order system, with a given damping factor), L_1 is a constant and L_2 is a function of r_o (which for a mechanical system is proportional to the undamped natural frequency). L_2 is decreased by decreasing r_o . Now for stability $r_o > 0$. Consider the case $r_o = 0$. Equation (37) becomes

$$\frac{d^2x}{du_1^2} + \frac{dx}{du_1} = \frac{1}{\omega_1} \qquad (\tau > 0)$$

or/

/or $$\frac{dv}{du_1} + v = 1$$

This is a first order differential equation showing that v tends monotonically to unity as $u_1 \to \infty$ as shown in Figure 2.6.

Thus the system $r_o = 0$ is admissable and is the optimum system if we are only interested in the response in velocity. For such a system the displacement x tends to $\frac{1}{\omega_1}(u_1 - 1)$ i.e. $\omega_1(x-\tau) \to -1$ giving a displacement lag in the following of a velocity input as shown in Figure 2.6.

In the above analysis we have considered the effect on the response functions of varying r_o. This is equivalent to keeping a_1 and a_2 fixed and varying a_o. Consider now the effect of varying a_1 (keeping a_o and a_2 fixed). This is equivalent to varying ω_1. From (43) we see that the minimum value of L_1(zero) occurs for large (infinite) values of a_1; then L_2 is large. This corresponds to a system with instantaneous response.

In a given system it may be either physically impossible or undesirable for r_o to be zero. We have seen above that when r_o is zero there is a constant position error(at large times) between the output and the input. This is often undesirable; indeed some would say that such a system would be a regulator and not a servomechanism (see reference 11). As shown above, as r_o increases L_1 remains constant but L_2 increases. As with the analysis of zero-displacement error systems we shall choose a value of $r_o(=1/4)$ so that the characteristic equation corresponding to (37) has equal roots. The velocity response and the displacement lag are shown in Fig.2.6 plotted against the non-dimensional time u_1. We see that the velocity response has an overshoot of 14 per cent. The maximum displacement lag $\omega_1(\tau-x)$ is 0.74, the lag becoming less than 0.10 when $\omega_1\tau$ is 9 or more. Systems having smaller (non-zero) values of r_o would have a smaller overshoot in velocity but a higher maximum displacement lag. Considering the velocity response we are therefore led to the criteria

$$0 < r_o \leq 0.25$$

for satisfactory performance for a second order zero velocity error system,/

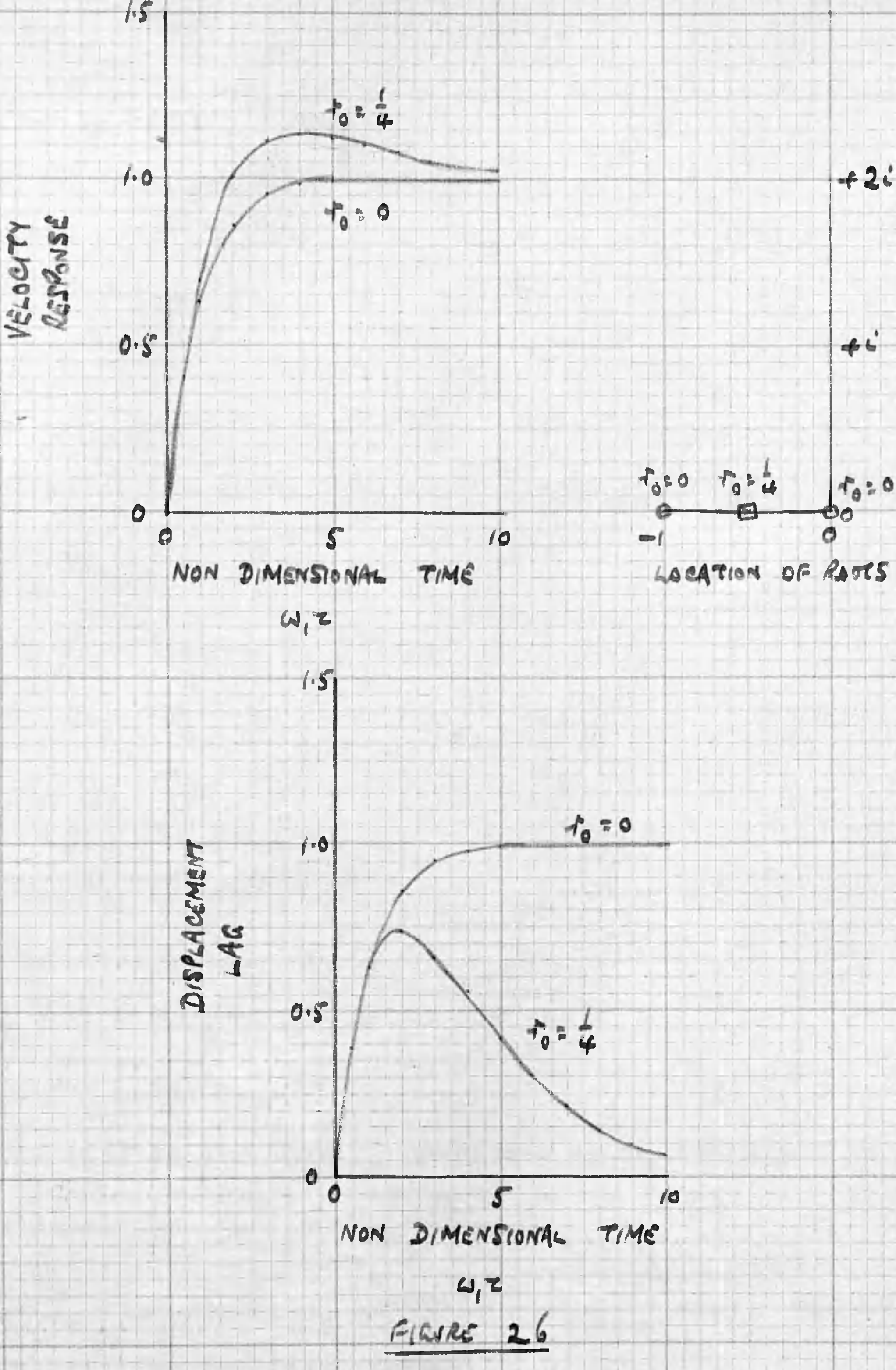


FIGURE 26

SECOND ORDER ZERO VELOCITY ERROR SYSTEM

/system, the precise choice of r_o being governed by the maximum acceptable lag at any time.

Third order zero velocity error system.

From Chapter I, (38), with $Dx_o = -1$, $x_o = D^2x_o = 0$,

$$2L_1 \begin{vmatrix} a_1 & a_3 \\ a_o & a_2 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ -a_1 & a_2 \end{vmatrix}$$

i.e. $$2L_1 \; (a_1a_2 - a_oa_3) = a_2^2 + a_1a_3$$

or in terms of the normalized coefficients r_o , r_2 given by (38),

$$2L_1\omega_1 = \frac{r_2^2 + 1}{r_2 - r_o} \tag{46}$$

Similarly from Chapter I, (27) ,

$$2L_2 \begin{vmatrix} a_1 & a_3 \\ a_o & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & -a_2 \\ a_o & a_1 \end{vmatrix}$$

i.e. $$\frac{2L_2}{\omega_1} = \frac{1 + r_or_2}{r_2 - r_o} \tag{47}$$

Now for stability with $a_3 > 0$,

$$a_2 > 0 \;,\; a_1a_2 - a_oa_3 > 0 \text{ and } a_o > 0$$

i.e. $$r_2 > r_o > 0$$

From (46) we see that L_1 decreases as r_o decreases to zero.

From (47) , $$\frac{2}{\omega_1}\,\frac{\partial L_2}{\partial r_o} = \frac{r_2^2 + 1}{(r_2 - r_o)^2}$$

and thus L_2 also decreases as r_o decreases to zero. Therefore as above we consider the case $r_o = 0$.

Then $$2L_1\omega_1 = \frac{r_2^2 + 1}{r_2} \tag{48}$$

and/

/and
$$\frac{2L_2}{\omega_1} = \frac{1}{r_2} \tag{49}$$

We see that if L_1 , L_2, r_2 and ω_1 are replaced by L, L_1, q_1 and ω_o equations (48) and (49) become identical with (9) and (11). This follows immediately from the equation of motion (37) which with $r_o = 0$ becomes a second order equation in v (for a third order system). The system can therefore be treated in a precisely similar manner to a second order zero displacement system. The velocity response for the L_{1min} and L_{1d} systems is shown in Figure 2.7 together with the displacement lag. In both cases the displacement lag tends to a definite non-zero limit (1 for L_{1min} systems and 2 for L_{1d} systems).

As with the second order zero velocity error system it may be either physically impossible or undesirable for r_o to be zero. Now the characteristic eqyation for the L_{1d} system is

$$\lambda^3 + 2\lambda^2 + \lambda = (\lambda + 1)(\lambda + 1)\lambda = 0$$

We replace the factor λ by $\lambda + 1$ and normalize the resulting equation (with $r_1 = 1$); the characteristic equation is then

$$\lambda^3 + 1.73\lambda^2 + \lambda + 0.19 = (\lambda + 0.58)^3 = 0$$

This corresponds to a system with $r_o = 0.19$ and $r_2 = 1.73$, having three equally damped modes of motion. The velocity response and the displacement lag for this system (denoted by L_{1e}) are shown in Figure 2.7. The velocity response has an overshoot of 25 per cent. The maximum displacement lag is 1.45, the lag becoming less than 0.10 when $\omega_1\tau$ is 12 or more. Values of L_1 and L_2 for the three systems are given in the following table.

TABLE 8.

Values of ~~the~~ L_1 and L_2 for a third order zero velocity error system.

System	r_o	r_2	$L_1\omega_1$	L_2/ω_1
L_{1min}	0	1	1.00	0.50
L_{1d}	0	2	1.25	0.25
L_{1e}	0.19	1.73	1.30	0.43

We/

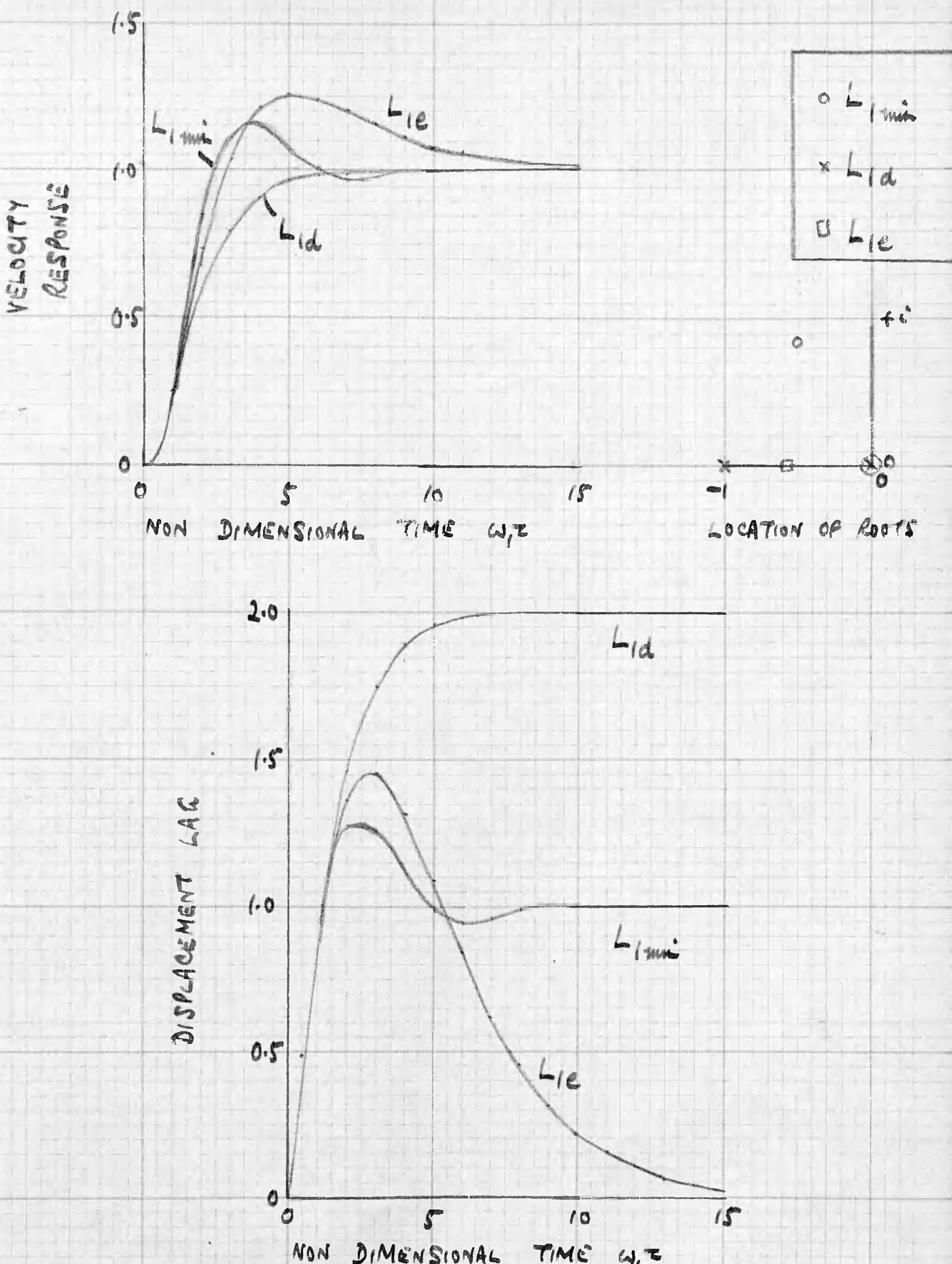


FIGURE 2.7.

THIRD ORDER ZERO VELOCITY ERROR SYSTEM

/We see that in the L_{1e} system the zero displacement error has been obtained mainly at the expense of L_2 ; this is reflected in the increased overshoot. As above, systems having smaller (non-zero) values of r_0 would in general have a smaller overshoot but a higher displacement lag and a longer "tail" to the velocity response, corresponding to a small negative root of the characteristic equation.

Higher order zero velocity error systems.

The above analysis can easily be extended to higher order systems. The formulae for L_1 and L_2 can be derived immediately from chapter I. We find that for stable systems of any order $n(>2)$ both L_1 and L_2 decrease as r_0 tends to zero. The system with $r_0 = 0$ becomes equivalent to a (n-1)th order system in v, and the velocity response for the L_{1min} and L_{1d} systems can be deduced from the preceding analysis (e.g. Tables 3,4,5 and 6 and Figures 2.4 and 2.5). The displacement lag $\omega_1(\tau-x)$ tends to $r_2(\neq 0)$.

To derive the L_{1e} system we replace the zero root of the characteristic equation by a real root having the same damping as the second most lightly damped mode, and normalize the resulting equation (with $r_1 = 1$). The values of the normalized coefficients and the corresponding roots of the characteristic equation are given in Tables 9 and 10.

Table 9.

Values of the normalized coefficients r_m for zero velocity error systems of order n having three modes with equal damping (L_{1e} systems).

n	r_0	r_1	r_2	r_3	r_4	r_5	r_6
3	0.19	1	1.73	1			
4	0.17	1	1.89	1.53	1		
5	0.18	1	2.14	2.97	1.52	1	
6	0.11	1	3.55	2.90	3.66	1.24	1

contd.

Table 10.

Roots of the characteristic equation for a zero velocity error L_{1e} system of order n.

n	Roots.
3	−0.58 , −0.58 , −0.58
4	−0.38 , −0.38 , −0.38 ± 1.01i
5	−0.30 . −0.30 ± 0.47i , −0.30 ± 1.35i
6	−0.37 , −0.17 , −0.17 ± 1.53i , −0.17 ± .82i

As the order of the system increases the damping of the least damped mode decreases. r_o becomes smaller on the whole as n increases; the °/o overshoot in velocity increases as does the maximum displacement lag.

Table 11.

Values of the normalized coefficients q_m for zero velocity error systems of order n having three modes with equal damping (L_{1e} systems).

n	q_0	q_1	q_2	q_3	q_4	q_5	q_6
3	1	3	3	1			
4	1	3.8	4.6	2.4	1		
5	1	4.0	6.0	5.9	2.1	1	
6	1	6.2	10.8	8.7	7.6	1.8	1

In Table 11 the coefficients of the characteristic equation are normalized to make q_o unity, as with the zero displacement error system. This enables a comparison to be made with the results of Whiteley in reference 8. Whiteley's corresponding coefficients are much larger than those shown in Table 11, giving modes with very small damping and hence a velocity response with a long "tail". This is closely connected with the fact that Whiteley's calculations are based on 10 per cent maximum velocity overshoot. No curves of displacement lags are given in reference 8 but these would be correspondingly large. Comparing the above results with the ITAE curves (reference 7) the velocity response of the ITAE system is rather more oscillatory than the L_{1e} curves with a maximum overshoot of about the same order. The L_{1e} curves are not unlike those based on binomial filters and are generally thought to be a reasonable compromise between Whiteley's curves and the ITAE ones.

contd.

Response of zero-acceleration-error systems to step function disturbance (constant acceleration input).

The analysis is carried out in a precisely similar manner to the above. We find it convenient to normalise the equation of motion (1) in such a way that the coefficients of $\frac{d^2x}{du_2^2}$ and $\frac{d^nx}{du_2^n}$ are unity where

$$u_2 = \omega_2\tau \tag{50}$$

and

$$a_2 = \omega_2^{n-2}a_n \tag{51}$$

From (1), (50) - (51), the normalized equation of motion is

$$\frac{d^nx}{du_2^n} + s_{n-1}\frac{d^{n-1}x}{du_2^{n-1}} + s_{n-2}\frac{d^{n-2}x}{du_2^{n-2}} + \ldots + s_3\frac{d^3x}{du_2^3} + \frac{d^2x}{du_2^2} + s_1\frac{dx}{du_2} + s_0x$$

$$= \frac{f_2}{a_2\omega_1^2} \quad (1 + s_1u_2 + s_0u_2^2) \qquad (\tau > 0) \tag{52}$$

where

$$s_m = \frac{a_m}{a_n\omega_2^{n-m}} \qquad (m = 0 \text{ to } n) \tag{53}$$

We take $f_2 = a_2$

The response of the system to a constant unit acceleration disturbance is identical in form with that in the free motion with $D^2x_0 = -1$ and x_0 , Dx_0 , D^3x_0 ... $D^{n-1}x_0$ zero.

We consider values of

$$L_2 = \int_0^\infty \left(\frac{d^2x}{d\tau^2}\right)^2 d\tau = \int_0^\infty a^2 d\tau \tag{54}$$

and the variation of L_3 where

$$L_3 = \int_0^\infty \left(\frac{d^3x}{d\tau^3}\right)^2 d\tau = \int_0^\infty \left(\frac{da}{d\tau}\right)^2 d\tau \tag{55}$$

in the neighbourhood of minimum values of L_2

where/

/where $$a = \frac{d^2x}{d\tau^2} \qquad (56)$$

By a similar analysis to that given above we find that both L_2 and L_3 decrease as δ_0 and δ_1 tend to zero. Considering the case $\delta_0 = \delta_1 = 0$ we find that the relations for L_2 and L_3 become identical with those for L and L_1 for a zero displacement error system of order n-2 when s_m and ω_2 are replaced by q_{m-2} and ω_0.

For systems with $\delta_0 = \delta_1 = 0$, the displacement and velocity lags both tend to non zero limits as $\tau \rightarrow \infty$. This can be remedied as with the zero velocity error systems by replacing the two zero roots of the characteristic equation by two real roots having the same damping as the remaining most lightly damped mode and normalizing the resulting equation (with $s_2 = 1$). The values of the normalized coefficients and the corresponding roots of the characteristic equation are given in Tables 12 and 13.

Table 12.

Values of the normalized coefficients s_m for zero acceleration error systems of order n having four modes with equal damping L_{2f} systems).

n	s_0	s_1	s_2	s_3	s_4	s_5	s_6
3	0.037	0.33	1	1			
4	0.028	0.27	1	1.63	1		
5	0.026	0.27	1	1.72	1.59	1	
6	0.026	0.27	1	2.08	2.67	1.61	1

Note: the third order system in the above table has, of course, only three equally damped modes.

contd.

Table 13.

Roots of the characteristic equation for a zero acceleration error L_{2f} system of order n.

n	Roots
3	−0.33 , −0.33 , − 0.33 .
4	−0.41 , − 0.41 , − 0.41 , − 0.41 .
5	−0.32 , −0.32 , −0.32 , −0.32 ± 0.84i
6	−0.27 , −0.27 , −0.27 ± 0.41i , −0.27 ± ~~1.29i~~ 1.20i

As the order of the system increases (for n > 4) the damping of the least damped mode decreaes, s_0 and s_1 only decrease slightly, and the °/o overshoot in acceleration increases.

Table 14.

Values of the normalized coefficients q_m for zero acceleration error systems of order n having four modes with equal damping (L_{2f} systems).

n	q_0	q_1	q_2	q_3	q_4	q_5	q_6
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5.0	8.9	7.4	3.3	1	
6	1	5.6	11.5	13.1	9.1	3.0	1

Note: the third order system in the above table has, of course, only three equally damped modes.

In the above table the coefficients for both third and fourth order systems are binomial coefficients; this follows immediately from the condition of equal damping. Whiteley's corresponding coefficients (reference 8) are much larger than those in Table 14. As stated above this leads to modes with much smaller damping than those shown in Table 13 (allowing for the different time scale). The L_{2f} curves are not unlike those based on the binomial filters, which are generally thought to be a suitable compromise between overshoot and damping.

Chapter 3.

Optimum Conditions of Response of Linear Systems with Constant Coefficients having many Degrees of Freedom.

In the previous chapters we have investigated the properties of some simple response coefficients L and L_s for linear systems with one degree of freedom. We shall now extend the method to linear systems with a number of degrees of freedom. We shall at first consider only a system of n first order equations. This is quite general since a system of higher order can be reduced to a first order system by substitution.

We consider a system for which the equations of motion are

$$e_{r1}x_1 + e_{r2}x_2 + \dots + e_{rn}x_n = f_r(\tau) \qquad (r = 1 \text{ to } n) \tag{1}$$

where
$$e_{rs} = a_{rs}D + b_{rs}$$
$$D = \frac{d}{d\tau}$$

a_{rs} , b_{rs} are constants and $f_r(\tau)$ $(r = 1$ to $n)$ are arbitrary known functions.

The initial conditions are given by

$$x_r = x_{ro} \quad (r = 1 \text{ to } n) \text{ at } \tau = 0.$$

Using the Laplace transform method (reference 17) the subsidiary equations are

$$\sum_{s=1}^{n} p_{rs}\bar{x}_s = \sum_{s=1}^{n} a_{rs}x_{so} + \bar{f}_r(p) \qquad (r=1 \text{ to } n)$$

where
$$p_{rs} = a_{rs}p + b_{rs}$$

$$\therefore \quad \bar{x}_s = \frac{1}{\Delta}\sum_{r=1}^{n} P_{rs}\left[\sum_{s=1}^{n} a_{rs}x_{so} + \bar{f}_r(p)\right] \tag{2}$$

where
$$\Delta = \begin{vmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \dots & \dots & \dots & \dots \\ p_{n1} & p_{n2} & & p_{nn} \end{vmatrix} \tag{3}$$

and/

/and P_{rs} is the cofactor of $p\,a_{rs}$ in the determinant Δ .

We shall be concerned primarily with stable systems for which all the roots of the characteristic equation (3) are negative or have negative real parts.

Let $$\frac{P_{rs}}{\Delta} = \int_0^\infty e^{-p\tau}\, Q_{rs}(\tau)\, d\tau \tag{4}$$

Then $$x_s = \sum_{r=1}^{n} Q_{rs}\left(\sum_{s=1}^{n} a_{rs}x_{so}\right) + \sum_{r=1}^{n}\int_0^\tau Q_{rs}(\tau-y)f_r(y)dy \qquad (s=1 \text{ to } n) \tag{5}$$

Differentiating(5),

$$\begin{aligned} Dx_s = {} & \sum_{r=1}^{n} Q^1_{rs}\left(\sum_{s=1}^{n} a_{rs}x_{so}\right) + \sum_{r=1}^{n} Q_{rs}(o)\, f_r(\tau) \\ & + \sum_{r=1}^{n}\int_0^\tau Q'_{rs}(\tau-y)\, f_r(y)dy \qquad (s=1 \text{ to } n) \end{aligned} \tag{6}$$

From (3) we see that Δ is of the nth degree in p and we can write

$$\Delta = A(p-\lambda_1)(p-\lambda_2)\ . \ . \ . \ (p-\lambda_n) \tag{7}$$

where $$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{8}$$

We assume that $A \neq 0$.

Now $$\frac{P_{rs}}{\Delta} = \sum_{m=1}^{n} \frac{P_{rs}(\lambda_m)}{A \prod\limits_{\substack{r=1\\ r\neq m}}^{n} (\lambda_m-\lambda_r)(p-\lambda_m)} \tag{9}$$

Hence $$Q_{rs} = \sum_{m=1}^{n} \frac{P_{rs}(\lambda_m)}{A \prod\limits_{\substack{r=1\\ r\neq m}}^{n} (\lambda_m-\lambda_r)}\, e^{\lambda_m\tau} \tag{10}$$

For/

/For large values of p, $$\frac{P_{rs}}{\Delta} \rightarrow \frac{A_{rs}}{A\,p} \qquad (11)$$

where A_{rs} is the cofactor of a_{rs} in (8).

$$\therefore \quad Q_{rs} = \frac{A_{rs}}{A} + C_1\tau + \ldots \text{ higher powers of } \tau \qquad (12)$$

From (12) we see that at $\tau = 0$, $$Q_{rs}(o) = \frac{A_{rs}}{A} \qquad (13)$$

Criteria for Optimisation.

As in Chapter I we shall derive formulae for the response coefficients in terms of the square of the r.m.s. error.

From (1) we see that the values of x_1, x_2 ... x_n which would correspond to a position of equilibrium at time τ are given by

$$b_{r1}\,x_1 + b_{r2}\,x_2 + \ldots + b_{rn}\,x_n = f_r(\tau) \quad (r=1 \text{ to } n)$$

$$\text{i.e.} \quad x_s = \frac{1}{B}\sum_{r=1}^{n} B_{rs}f_r(\tau) = g_s(\tau) \qquad (14)$$

$$\text{where} \quad B = \begin{vmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & & b_{nn} \end{vmatrix} \qquad (15)$$

and B_{rs} is the cofactor of b_{rs} in this determinant.

We assume that $B \neq 0$.

Thus if there were no lag in the system x_s would equal $g_s(\tau)$ at all times. The error e_s is given by

$$e_s = (\text{input} - \text{output})_s = g_s(\tau) - x_s$$

We shall derive formulae for response coefficients L_r and L_{1r} given by

$$L_r = \int_0^{\infty} e_r^2\,d\tau = \int_0^{\infty}\Big[x_r - g_r(\tau)\Big]^2 d\tau \quad (r=1 \text{ to } n) \qquad (16)$$

and/

/and $\bar{L}_{1r} = \int_0^{\infty} \left(\frac{de_r}{d\tau}\right)^2 d\tau = \int_0^{\infty} \left[Dx_r - g^1{}_r(\tau)\right]^2 d\tau \quad (r=1 \text{ to } n)$ (17)

As in the previous section we shall consider values of L_{1r} and L_r in the neighbourhood of a minimum value of a particular L_s. We see that we shall have 2n response coefficients corresponding to the n degrees of freedom. In any physical problem some of the response coefficients may be much more important than others and thus the problem may be considerably simplified.

<u>Free Motion</u> $f_r(\tau) = 0$, $r=1$ to n.

<u>Derivation of formulae for response functions in terms of the roots of the characteristic equation.</u>

From (5) and (10), $x_s = \sum_{r=1}^{n} q_{rs}\left(\sum_{s=1}^{n} a_{rs}\, x_{s0}\right) = \sum_{m=1}^{n} C_{ms} e^{\lambda_m \tau}$ (18)

where $$C_{ms} = \sum_{r=1}^{n} \frac{P_{rs}(\lambda_m)}{A \prod_{\substack{r=1 \\ r \neq m}}^{n} (\lambda_m - \lambda_r)} \left(\sum_{s=1}^{n} a_{rs}\, x_{s0}\right) \qquad (19)$$

If the motion is stable, all the roots of the characteristic equation (7) must be negative or have negative real parts and $x_s \to 0$ ($s = 1$ to n) as $\tau \to \infty$.

The derivation of the formulae for the response functions is precisely analogous to that with one degree of freedom. We find

$$-\bar{L}_s = \sum_{m=1}^{n} \sum_{M=1}^{n} C_{ms}\, C_{Ms}\, M_{mM}/M \qquad (20)$$

where $$M = \prod_{\substack{s=1 \text{ to } n \\ S=1 \text{ to } n \\ S \geqslant s}} (\lambda_s + \lambda_S)$$

and $$M_{mM} = \frac{M}{\lambda_m + \lambda_M} \qquad (21)$$

Now/ $\cancel{M = 2^{n}\,{}_{1}\,{}_{2} \cdots {}_{n}({}_{1}+{}_{2})({}_{1}+{}_{3})\cdots({}_{1}+{}_{n})}$

contd.

/Now
$$M = 2^n \lambda_1 \lambda_2 \ldots \lambda_n (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3) \ldots (\lambda_1 + \lambda_n)$$
$$(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4) \ldots (\lambda_2 + \lambda_n)$$
$$\ldots \qquad (\lambda_{n-1} + \lambda_n) \qquad (22)$$

This is precisely the same as equation (16) of chapter 1.

Now the free motion is determined uniquely when the initial conditions are specified. Thus L_s is determined by the initial conditions.

We have from (18),
$$\left.\begin{aligned} \sum C_{ms} &= x_{so} \\ \sum \lambda_m C_{ms} &= \dot{x}_{so} \\ \sum \lambda_m^2 C_{ms} &= \ddot{x}_{so} \\ &\ldots\ldots \\ \sum \lambda_m^{n-1} C_{ms} &= \left(\frac{d^{n-1} x_s}{d\tau^{n-1}} \right)_o \end{aligned}\right\} \qquad (23)$$

Then
$$L_s = -\frac{1}{M}\Big[x_{so} (a_{11} x_{so} + a_{12} Dx_{so} + \ldots + a_{1n} D^{n-1} x_{so})$$
$$+ \ldots + Dx_{so}^{n-1} (a_{1n} x_{so} + \ldots + a_{nn} D^{n-1} x_{so}) \Big] \qquad (24)$$

This formula is identical with equation (19) of chapter 1 and the a's are identical with those for the linear system of order n with one degree of freedom. We note that L_s is given in terms of x_{so}, Dx_{so}, ... $D^{n-1} x_{so}$ by (24) i.e. in terms of the initial values of x_s and its derivatives only. This follows immediately from (1) since on eliminating x_1, x_2, ... x_{s-1}, x_{s+1}, ... x_n and their first derivati~~es~~ve ~~to~~ the differential equation for x_s is, for the free/

/free motion,

$$A_n \frac{d^n x_s}{d\tau^n} + A_{n-1} \frac{d^{n-1} x_s}{d\tau^{n-1}} + \dots + A_1 \frac{dx_s}{d\tau} + A_o x_s = 0 \qquad (25)$$

where A_r is the coefficient of p^r in the characteristic equation (7).

From (3), (8) and (15) we note that

$$\left.\begin{aligned} A_n &= A , \\ A_o &= B . \end{aligned}\right\} \qquad (26)$$

To find values of Dx_{so} , ... $D^{n-1} x_{so}$ in terms of $x_{1o}, \dots x_{no}$ we use (18) writing

$$U_r = \sum_{s=1}^{n} a_{rs} \, x_{so} \qquad (27)$$

Then in the free motion $x_s = \sum_{r=1}^{n} U_r \, Q_{rs}$

By successive differentiation we find

$$\left.\begin{aligned} Dx_{so} &= \sum Ur \, D \, Q_{rs} \, (o) \\ &\cdots\cdots \\ D^{n-1} x_{so} &= \sum U_r \, D^{n-1} Q_{rs} \, (o) \end{aligned}\right\} \qquad (28)$$

where from (10)

$$\left.\begin{aligned} D \, Q_{rs} \, (o) &= \sum_{m=1}^{n} \frac{\lambda_m P_{rs} (\lambda)_m}{A \prod_{\substack{r=1 \\ r \neq m}}^{n} (\lambda_m - \lambda_r)} \\ &\cdots\cdots \\ D^{n-1} Q_{rs}(o) &= \sum_{m=1}^{n} \frac{\lambda_m^{n-1} P_{rs} (\lambda_m)}{A \prod_{\substack{r=1 \\ r \neq m}}^{n} (\lambda_m - \lambda_r)} \end{aligned}\right\} \qquad (29)$$

Alternatively/

/Alternatively writing

$$Q_{rs} = \frac{A_{rs}}{A} + C_1 + \frac{C_2\tau^2}{2} + \dots \qquad (30)$$

then
$$\left.\begin{aligned} Dx_{so} &= \sum U_r C_1 \\ &\dots\dots \\ D^{n-1}x_{so} &= \sum U_r C_{n-1} \end{aligned}\right\} \qquad (31)$$

where the constants C will of course be different for every P_{rs}.

We see that the initial conditions enter the above equation in the U terms only.

We can immediately derive the formula for L_{1s} where

$$L_{1s} = \int_0^\infty \left[Dx_s\right]^2 d\tau$$

by replacing x_{so} by Dx_{so}, Dx_{so} by D^2x_{so}, ... etc., in (24).

Thus

$$L_{1s} = -\frac{1}{M}\Big[Dx_{so}(a_{11}\,Dx_{so} + a_{12}\,D^2x_{so} + \dots + a_{1n}\,D^nx_{so})$$

$$+ \dots + D^nx_{so}\,(a_{1n}\,Dx_{so} + \dots + a_{nn}\,D^nx_{so})\Big] \qquad (32)$$

where from (28),(29), $D^nx_{so} = U_r\,D^nQ_{rs}\,(o)$

and
$$\left.\begin{aligned} D^nQ_{rs}\,(o) = \sum_{m=1}^{n} \frac{\lambda_m^n P_{rs}(\lambda_m)}{A\prod\limits_{\substack{r=1\\ r\neq m}}^{n}(\lambda_m - \lambda_r)} \end{aligned}\right\} \qquad (33)$$

or from (31), $D^nx_{so} = \sum U_r C_n$

We see that for given initial conditions x_{ro} ($r=1$ to n) we first have to find Dx_{so}, D^2x_{so}, ... D^nx_{so} ($s=1$ to n) for whichever/

/ever variables are important in the physical problem. The response functions L_s and L_{1s} can then be found from (24) and (32) once M and the α's are known. We note that M and the α's depend only on the coefficients of the characteristic equation and the formulae for these parameters can be obtained from those in Chapter I by replacing $a_o, a_1, \ldots a_n$ by $A_o, A_1, \ldots A_n$.

<u>Free Motion</u> $f_r(\tau) = 0 \quad (r=1 \text{ to } n)$

<u>Derivation of formulae for response functions in terms of the coefficients a_{rs}, b_{rs} of the equations of motion.</u>

We define the following integrals

$$\left.\begin{aligned} E_{rs} &= \int_0^\infty x_r\, x_s\, d\tau \\ F_{rs} &= \int_0^\infty Dx_r\, x_s\, d\tau \\ G_{rs} &= \int_0^\infty Dx_r\, Dx_s\, d\tau \end{aligned}\right\} \qquad (34)$$

All these integrals are convergent for the free motion of stable systems. We note that

$$E_{rs} = E_{sr} \qquad (35)$$

$$G_{rs} = G_{sr} \qquad (36)$$

$$F_{rs} = -x_{ro}\, x_{so} - F_{sr} \qquad (37)$$

$$F_{ss} = -\tfrac{1}{2} x_{so}^2 \qquad (38)$$

Also

$$\left.\begin{aligned} L_s &= E_{ss} \\ L_{1s} &= G_{ss} \end{aligned}\right\} (39)$$

Multiplying (1) by x_s and integrating from 0 to ∞, using (34), in the free motion,

$$a_{r1}F_{1s} + b_{r1}E_{1s} + a_{r2}F_{2s} + b_{r2}E_{2s} + \ldots + a_{rn}F_{ns} + b_{rn}E_{ns} = 0 \quad (r,s = 1 \text{ to } n) \qquad (40)$$

Equations/

/Equations (35),(37),(38) and (40) provide $2n^2$ equations for the $2n^2$ variables E_{rs} , F_{rs} . In particular we can solve for L_s . By using the relation (38) before solving for L_s we can reduce the order of the resulting determinants to $2n^2-n$, while by substituting (35),(37) and (38) in (40) before solving, the order of the resulting determinants is reduced to n^2. These last two forms are more suitable for analysis and for calculation respectively.

Similarly multiplying (1) by Dx_s and integrating from 0 to ∞ , using (34), in the free motion,

$$a_{r1}G_{s1} + b_{r1}F_{s1} + a_{r2}G_{s2} + b_{r2}F_{s2} + \dots + a_{rn}G_{sn} + b_{rn}F_{sn} = 0 \qquad (r,s = 1 \text{ to } n) \qquad (41)$$

Equations (36),(37),(38) and (41) provide $2n^2$ equations for the $2n^2$ variables F_{rs} , G_{rs} and can be simplified and solved for L_{1s} as above.

Thus for a first order system with two degrees of freedom

$$\Delta' L_1 = \pm \begin{vmatrix} a_{11}x_{10}^2 & b_{12} & 0 & 0 & a_{12} & 0 \\ a_{21}x_{10}^2 & b_{22} & 0 & 0 & a_{22} & 0 \\ a_{12}x_{20}^2 & 0 & b_{11} & b_{12} & 0 & a_{11} \\ a_{22}x_{20}^2 & 0 & b_{21} & b_{22} & 0 & a_{21} \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 2x_{10}x_{20} & 0 & 0 & 0 & 1 & 1 \end{vmatrix} \qquad (42)$$

contd.

contd.

$$\Delta'_1 L_{11} = \frac{1}{\pi} \begin{vmatrix} b_{11}x_{10}^2 & a_{12} & 0 & 0 & b_{12} & 0 \\ b_{21}x_{10}^2 & a_{22} & 0 & 0 & b_{22} & 0 \\ b_{12}x_{20}^2 & 0 & a_{11} & a_{12} & 0 & b_{11} \\ b_{22}x_{20}^2 & 0 & a_{21} & a_{22} & 0 & b_{21} \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 2x_{10}x_{20} & 0 & 0 & 0 & 1 & 1 \end{vmatrix} \tag{43}$$

where

$$\Delta' = \begin{vmatrix} b_{11} & b_{12} & 0 & 0 & a_{12} & 0 \\ b_{21} & b_{22} & 0 & 0 & a_{22} & 0 \\ 0 & 0 & b_{11} & b_{12} & 0 & a_{11} \\ 0 & 0 & b_{21} & b_{22} & 0 & a_{21} \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{vmatrix} \tag{44}$$

and

$$\Delta'_1 = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 & b_{12} & 0 \\ a_{21} & a_{22} & 0 & 0 & b_{22} & 0 \\ 0 & 0 & a_{11} & a_{12} & 0 & b_{11} \\ 0 & 0 & a_{21} & a_{22} & 0 & b_{21} \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{vmatrix} \tag{45}$$

In general we see that for a first order system with n degrees of freedom,

$$\left.\begin{aligned} \text{if } \Delta' &= f(a_{pq}, b_{rs}) \\ \Delta'_1 &= f(b_{pq}, a_{rs}) \end{aligned}\right\} \tag{46}$$

contd.

Also if

$$\left.\begin{aligned} \Delta L_s &= g(a_{pq}, b_{rs}) \\ \Delta'_1 L_s &= g(b_{pq}, a_{rs}) \end{aligned}\right\} \qquad (47)$$

The determinant Δ' is of degree n^2 in the b's and of degree n^2-n in the a's. From (24) and (32) we see that L_s and L_{1s} are both inversely proportional to M, where from Chapter I, (39),

$$\left(\frac{A_n}{2}\right)^n M = (-1)^{\frac{n(n+1)}{2}} A_0 T_{n-1}$$

i.e.

$$M = (-1)^{\frac{n(n+1)}{2}} 2^n \frac{B}{A^n} T_{n-1} \qquad (48)$$

We find that

$$\left.\begin{aligned} \Delta^1 &= B\,T_{n-1} \\ \Delta'_1 &= A\,T_{n-1} \end{aligned}\right\} \qquad (49)$$

Thus to avoid large values of L_s and L_{1s}, all three of A, B and M should not be small. From the above analysis we see that, when the response functions are expressed in terms of the initial displacements x_{so}, they involve not merely the coefficients of the characteristic equation but also the individual elements a_{rs}, b_{rs} of the determinant Δ. It is thus possible to have two systems which have the same characteristic equation but differ greatly in their response characteristics (see reference 27).

As can be seen from the equations of motion (1), the coupling terms between the motions in x_r and x_s are a_{rs}, a_{sr}, b_{rs} and b_{sr}. Thus the smaller these coupling coefficients the smaller will be the corresponding response coefficients. This is seen from (42), (43) where for a displacement x_{20}, with $x_{10} = 0$, L_1 and L_{11} are both proportional to the determinant

$$c_{12} = \begin{vmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{vmatrix}$$

We note that for the system to be uncoupled it is not necessary that all the coupling terms (e.g. a_{12}, b_{12}, etc) vanish; it is necessary that/

/that all determinants such as c_{12} vanish.

This can also be seen by transforming the equations of motion (1). Multiplying the first equation by B_{1r}, the second by B_{2r}, the third by B_{3r}, etc. and adding, we arrive at n equations of the form

$$c_{r1}\,\dot{x}_1 + c_{r2}\,\dot{x}_2 + \dots + c_{rn}\,\dot{x}_n + Bx_r = 0 \qquad (50)$$

where
$$c_{rs} = \sum_{m=1}^{n} B_{mr}\,a_{ms} \qquad (r,s = 1 \text{ to } n) \qquad (51)$$

The terms c_{rs} $(r \neq s)$ represent the coupling terms. We see that we have reduced the number of parameters in the equations of motion to n^2 terms c_{rs} and B.

We could of course have reduced the equations to the form

$$A\,\dot{x}_r + d_{r1}\,x_1 + d_{r2}\,x_2 + \dots + d_{rn}\,x_n = 0$$

The form (50) is more convenient ~~in~~ for showing the close relation between response following an initial displacement to that following a step disturbance.

The analysis is further simplified by noting that if in two systems the coefficients c_{rs} are related by equations of the form

$$c_{rs} = \mathbf{c}_{rs}\,\theta_s/\theta_r \qquad (r,s = 1 \text{ to } n) \qquad (52)$$

where θ_r, θ_s are <u>any</u> non zero constants (positive or negative), the corresponding amplitudes of response are related by the equations

$$\frac{x_1^1}{\theta_1 x_1} = \frac{x_2^1}{\theta_2 x_2} = \dots = \frac{x_n^1}{\theta_n x_n} \qquad (53)$$

The corresponding formulae for the response functions follow~~s~~ immediately.

The characteristic equation corresponding to (50) is given by/

/by

$$\lambda^n \begin{vmatrix} c_{11}+B\lambda^{-1} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22}+B\lambda^{-1} & c_{23} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & c_{n3} & & c_{nn}+B\lambda^{-1} \end{vmatrix} = 0 \qquad (54)$$

Now (7) and (54) must have the same roots.

Let

$$C = \begin{vmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{vmatrix} \qquad (55)$$

Comparing the coefficients of λ^n and the constant coefficients in (7) and (54) we see that

$$C = AB^{n-1} \qquad (56)$$

Consider the motion resulting from an initial displacement x_{mo} with all x_{so} zero ($s \neq m$). From (50) we see that if every c_{rm} is zero ($r=1$ to n) except c_{rr} there will be no displacement at any time in the rth coordinate following an initial displacement x_{mo}. Then

$$L_r = L_{1r} = 0.$$

If all the coupling terms are zero the equations of motion become

$$c_{rr}\dot{x}_r + B\,x_r = 0 \qquad (r=1 \text{ to } n)$$

with the corresponding modes

$$x_r = x_{ro}e^{-\frac{B}{c_{rr}}\tau}$$

where for stability with $B > 0$, c_{rr} must be positive.
The corresponding response functions are

$$\left.\begin{aligned} L_r &= \frac{c_{rr}}{2B}\,x_{ro}^2 \\ L_{1r} &= \frac{B}{2c_{rr}}\,x_{ro}^2 \end{aligned}\right\} \qquad (57)$$

The ratios c_{rr}/B define the damping of the system, the greater their values the more rapid the damping. The motion in every mode corresponds to a subsidence if the system is stable. For the system to have equal damping in all modes

$$c_{11} = c_{22} = c_{33} = \dots = c_{nn} = C^{\frac{1}{n}} = A^{\frac{1}{n}} B^{\frac{n-1}{n}} \qquad (58)$$

Changing the ratios c_{rr}/B merely changes the time scale of the response.

Another important special case of the general system (50) is that in which one or more of the equations reduce to the form

$$\dot{x}_r = x_s \qquad (r \neq s)$$

i.e.

$$c_{rs} = -B$$

$$c_{rm} = 0 \qquad (m \neq s)$$

It follows immediately that

$$L_{1r} = L_s \qquad (59)$$

<u>Free Motion</u> $f_r(\tau) = 0$ (r=1 to n)

<u>Derivation of formulae for response functioms in terms of the frequency response spectrum of the system.</u>

Using Fourier's integral theorem and Parseval's theorem we can derive integral formulae for L_s and L_{1s} in terms of g_{ms} identical in form with equation (56) of Chapter I.

We have

$$L_s = \frac{1}{\pi}\int_0^\infty \frac{\left[g_{os} - g_{2s}\omega^2 + g_{4s}\omega^4 - \dots\right]^2 + \omega^2\left[g_{1s} - g_{3s}\omega^2 + \dots\right]^2}{\left[A_o - A_2\omega^2 + A_4\omega^4 - \dots\right]^2 + \omega^2\left[A_1 - A_3\omega^2 + \dots\right]^2}\, d\omega \qquad (60)$$

where $\Delta = A_n p^n + A_{n-1}p^{n-1} + \dots + A_o \qquad (61)$

and

$$g_{ms} = A_{m+1} x_{so} + A_{m+2} Dx_{so} + \dots + A_n D^{n-m-1} x_{so}. \qquad (62)$$

(m = 0 to n-1)

The integral for L_{1s} is obtained from L_s as above by replacing x_{so} by Dx_{so} , Dx_{so} by D^2x_{so} , ... etc. in the formula (62) for g_{ms} .

contd.

Response of a system with no initial displacement.

We consider a system with initial conditions

$$x_{ro} = 0 \qquad (r=1 \text{ to } n) \tag{63}$$

the system satisfying the equations of motion (1).

Step function disturbances.

$$f_r = 0 \quad \text{for} \quad \tau \leq 0 \; ; \quad f_r = F_r \quad \text{for} \quad \tau > 0 \quad (r=1 \text{ to } n)$$

where F_r is constant.

From (5) and (10) the motion is given by

$$x_s = \sum_{r=1}^{n} F_r \int_0^{\tau} Q_{rs} (\tau - y)\, dy = \sum_{m=o}^{n} G_{ms}\, e^{\lambda_m \tau} \tag{64}$$

$$\left.\begin{aligned} \text{where} \quad G_{os} &= \sum_{r=1}^{n} \frac{F_r B_{rs}}{B} \\ \text{and} \quad G_{ms} &= \sum_{r=1}^{n} \frac{F_r P_{rs}(\lambda_m)}{A \lambda_m \prod\limits_{\substack{r=1 \\ r \neq m}}^{n} (\lambda_m - \lambda_r)} \qquad (m=1 \text{ to } n) \end{aligned}\right\} \tag{65}$$

$$\text{where} \quad B = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

and B_{rs} is the cofactor of b_{rs} in this determinant. We assume that neither A nor B is zero, and that we are only considering stable systems.

From (14) and (65) we see that

$$G_{os} = g_s \quad \text{(a constant).} \tag{66}$$

As/

/As $\tau \to \infty$ for stable systems $x_s \to g_s$.

From (64), (65),
$$\left.\begin{aligned} Dx_s &= \sum_{m=1}^{n} G_{ms}\lambda_m e^{\lambda_m \tau} \\ &= \sum_{m=1}^{n} \sum_{r=1}^{n} \frac{F_r P_{rs}(\lambda_m)}{A \prod_{\substack{r=1 \\ r\neq m}}^{n} (\lambda_m - \lambda_r)} e^{\lambda_m \tau} \end{aligned}\right\} \qquad (67)$$

Alternatively writing

$$x_r' = x_r - g_r = x_r - G_{or} \qquad (68)$$

We see from (1) that the response of the system following a step disturbance is identical in form with that in a free motion with $x_{ro} = - g_r$ (r=1 to n) since from (65),

$$\sum_{s=1}^{n} b_{rs} g_s = F_r .$$

The errors in the two responses are the same at any given time. The given system ultimately has no static error since $x_r' \to 0$. Such a system corresponds to a zero-displacement-error system with n degrees of freedom.

From (16)
$$L_r = \int_0^\infty \left[x_r - g_r\right]^2 d\tau = \int_0^\infty \left[x_r'\right]^2 d\tau \qquad (69)$$

and $Dx_r' = Dx_r$

Thus the response functions of the given system following step function disturbances are the same as those of the given system in a free motion with

$$x_{so}' = - g_s = - \sum_{r=1}^{n} \frac{F_r B_{rs}}{B} \qquad (s=1 \text{ to } n) \qquad (70)$$

The response functions are then found from (24) and (28) or, more directly, in determinantal form from (40) and (41). We see that Dx_{so}, D^2x_{so} , ... $D^{n-1}x_{so}$ are unchanged by (68). The corresponding integral formula is derived directly by the same substitution for x_{so} . Similarly for L_{1s} .

As above we can transform the equations (1) by multiplying the first equation by B_{1r}, the second by B_{2r}, the third by B_{3r}, etc., and adding. We obtain n equations of the form

$$c_{r1}\dot{x}_1 + c_{r2}\dot{x}_2 + \dots + c_{rn}\dot{x}_n + Bx_r = Bg_r \qquad (71)$$

On using (68), (71) reduces to the form of (50) and can be treated in like manner.

For disturbances such that all the F_r vanish except one F_s, it is often more convenient to keep the equations of motion in their original form (1), or possibly to apply a transformation similar to the above to the remaining n-1 equations. The system could thus be reduced to a set of n equations of the form

$$\left.\begin{array}{l} a_{s1}\dot{x}_1 + b_{s1}x_1 + a_{s2}\dot{x}_2 + b_{s2}x_2 + \dots + a_{sn}\dot{x}_n + b_{sn}x_n = F_s \\ a'_{r1}\dot{x}_1 + b'_{r1}x_1 + a'_{r2}\dot{x}_2 + a'_{r3}\dot{x}_3 + \dots + a'_{rn}\dot{x}_n + B'x_r = 0 \end{array}\right\} \qquad (72)$$

$$(r=1 \text{ to } n,\ r \neq s)$$

In this case we have reduced the number of parameters ~~to~~ in the equations of motion to n^2 terms a'_{rm} (and a_{sm}), n terms b_{sm} and B'.

From (5) we see that if all the F_r vanish except one F_p

$$x_q = F_p \int_0^\tau Q_{pq}(\tau-y)\,dy \qquad (73)$$

Similarly if all the F_r vanish except one F_q

$$x_p = F_q \int_0^\tau Q_{pq}(\tau-y)\,dy \qquad (74)$$

Now if the determinant Δ given by (3) is symmetrical i.e. if both

$$a_{rs} = a_{sr}$$

and $$b_{rs} = b_{sr}$$

then $$Q_{rs} = Q_{sr}$$

and/

/and thus from (73) and (74)

$$\frac{x_q}{F_p} = \frac{x_p}{F_q} \qquad (75)$$

Thus the form of the response in the variables x_p and x_q is identical and ~~thus~~ the response functions are related by the equations

$$\left.\begin{aligned} \frac{L_p}{F_q^2} &= \frac{L_q}{F_p^2} \\ \text{and} \quad \frac{L_{1p}}{F_q^2} &= \frac{L_{1q}}{F_p^2} \end{aligned}\right\} \qquad (76)$$

It must be emphasized that these relations hold only if the determinant Δ is symmetrical. We have proved this for n first order equations, with a step disturbance. However as can easily be seen this is a particular example of a more general theorem which holds for stable systems satisfying linear differential equations of any order with constant coefficients, the systems starting from rest in the equilibrium position and undergoing a general disturbance $f_r(\tau)$ in one "direction", provided that all the integrals

$$\int_0^{\tau} Q_{rs}\ (\tau-y)\ f_r(y)\ dy$$

are convergent.

Response to an initial unit impulse.

We consider a system subject to an initial unit impulse in one degree of freedom, say x_r.

Then
$$\int_0^{\delta} f_s(\tau)d\tau = 1 \qquad (s=r)$$

where δ is infinitely small and $f_r(\tau)$ is zero for all other τ.

$$f_s = 0\ , \quad s \neq r.$$

$\therefore$ from (5), $\qquad x_s = Q_{rs}(\tau) \qquad (77)$

i.e./

/i.e. using (10), $$x_s = \sum_{m=1}^{n} \frac{P_{rs}(\lambda_m)}{A \prod_{\substack{r=1 \\ r \neq m}}^{n} (\lambda_m - \lambda_r)} e^{\lambda_m \tau} \qquad (78)$$

Comparing (78) and (62) we see as in Chapter I that the response to a unit step disturbance is the integral of the response to a unit impulse. Thus L_r (unit impulse in the mth degree of freedom)

$= L_{1r}$ (unit step disturbance in the mth degree of freedom). (79)

From (77) we see that Q_{rs} is identical with the impulsive admittance (or receptance) dealt with in reference 28.

Alternatively we see from (13) that the motion is the same as that of the free motion with initial conditions

$$x'_{s0} = \frac{A_{rs}}{A} \qquad (s=1 \text{ to } n) \qquad (80)$$

Thus the response functions are as for the equivalent free motion.

<u>Extensions of the above theory.</u>

As shown in Chapter I the response of a linear system to an arbitrary disturbance can be simply related to its response to unit impulses. The above theory can be applied to slightly unstable systems if the upper limit of the integrals for L_1 , L_{11} , etc. be taken to be some convenient finite time.

Chapter 4.

Examples of Optimum Conditions of Response of Linear Systems with Constant Coefficients having many Degrees of Freedom.

In the previous chapter we obtained formulae for the response functions L_s and L_{1s} in terms of (1) the roots of the characteristic equation, (11) the coefficients a_{rs} and b_{rs} occurring in the equations of motion and (111) the frequency response spectrum. We saw how the response to step function disturbances and to an initial impulse could be simply related to the response in the free motion. We shall now consider the response of a linear first order system with two degrees of freedom in greater detail, showing how the optimum response of such a system can be obtained.

Free Motion $f_1(\tau) = 0$, $f_2(\tau) = 0$

Response functions for linear first order system with two degrees of freedom.

We consider a system for which the equations of motion (in the free motion) are

$$\left.\begin{aligned} a_{11}\dot{x}_1 + b_{11}x_1 + a_{12}\dot{x}_2 + b_{12}x_2 &= 0 \\ a_{21}\dot{x}_1 + b_{21}x_1 + a_{22}\dot{x}_2 + b_{22}x_2 &= 0 \end{aligned}\right\} \quad (1)$$

where the a's and b's are constants and the dots denote differentiation with respect to τ. The initial conditions are given by

$$x_1 = x_{1o} \quad , \quad x_2 = x_{2o} \quad \text{at} \quad \tau = 0 \, .$$

As shown in Chapter 3, equations (1) can be put in a more convenient form by multiplying the first equation by B_{1r} and the second by B_{2r} and adding. The equations can then be written

$$\left.\begin{aligned} c_{11}\dot{x}_1 + c_{12}\dot{x}_2 + Bx_1 &= 0 \\ c_{21}\dot{x}_1 + c_{22}\dot{x}_2 + Bx_2 &= 0 \end{aligned}\right\} \quad (2)$$

where

$$\begin{aligned} c_{11} &= a_{11}b_{22} - a_{21}b_{12} \\ c_{12} &= a_{12}b_{22} - a_{22}b_{12} \\ c_{21} &= a_{21}b_{11} - a_{11}b_{21} \\ c_{22} &= a_{22}b_{11} - a_{12}b_{21} \end{aligned}$$

and/

/and

$$B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \qquad (3)$$

From (1) and (2) the characteristic equation is $\Delta = 0$ where

$$\Delta = \begin{vmatrix} a_{11}p + b_{11} & a_{12}p + b_{12} \\ a_{21}p + b_{21} & a_{22}p + b_{22} \end{vmatrix} \qquad (4)$$

We write $\Delta = A_2p^2 + A_1p + A_o = A(p-\lambda_1)(p-\lambda_2)$ (5)

where

$$\left.\begin{aligned} A_2 &= A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ A_1 &= a_{11}b_{22} + b_{11}a_{22} - a_{12}b_{21} - b_{12}a_{21} \\ &= c_{11} + c_{22} \\ A_o &= B \end{aligned}\right\} \qquad (6)$$

We note that $C = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = AB$ (7)

We can take A_2 positive without loss of generality. We are only concerned with stable systems for which all the roots of the characteristic equation (5) are negative or have negative real parts.

i.e. with A_2 positive, $A_1 > 0$ and $A_o > 0$.

As in Chapter 2 we shall find it convenient to introduce certain non-dimensional quantities. Let ω_o be the undamped natural frequency of the complete system.

Then $A_o = \omega_o^2 A_2$ (8)

i.e. $B = \omega_o^2 A$

We define a new time scale by the relation

$$u = \omega_o\tau \qquad (9)$$

As in Chapter 2 we note that equation (9) merely alters the time scale of the damping but does not affect the % overshoot or the general form of the response.

Equations/

/Equations (2) become

$$\left.\begin{aligned} q_{11}\frac{dx_1}{du} + q_{12}\frac{dx_2}{du} + x_1 &= 0 \\ q_{21}\frac{dx_1}{du} + q_{22}\frac{dx_2}{du} + x_2 &= 0 \end{aligned}\right\} \quad (10)$$

where

$$\left.\begin{aligned} q_{11} &= \frac{c_{11}\omega_0}{B} \\ q_{12} &= \frac{c_{12}\omega_0}{B} \\ q_{21} &= \frac{c_{21}\omega_0}{B} \\ q_{22} &= \frac{c_{22}\omega_0}{B} \end{aligned}\right\} \quad (11)$$

Then

$$Q = \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} = 1 \quad (12)$$

The characteristic equation is then

$$\left.\begin{aligned} \Delta &= p_1^2 + (q_{11} + q_{22})p_1 + 1 \\ &= (p_1 - \mu_1)(p_1 - \mu_2) = 0 \end{aligned}\right\} \quad (13)$$

Thus for stability $q_{11} + q_{22} > 0$ (14)

We see that the characteristic equation will have real roots (corresponding to subsidences) if

$$q_{11} + q_{22} \geq 2$$

and complex roots (corresponding to a damped oscillation) if

$$0 < q_{11} + q_{22} < 2$$

The solution of (1) for the given initial conditionsis

contd.

$$x_1 = \frac{x_{1o}}{\mu_1 - \mu_2} \left\{ (\mu_1 + q_{11}) e^{\mu_1 u} - (\mu_2 + q_{11}) e^{\mu_2 u} \right\}$$

$$+ \frac{q_{12} x_{2o}}{\mu_1 - \mu_2} \left(e^{\mu_1 u} - e^{\mu_2 u} \right) \tag{15}$$

$$x_2 = \frac{q_{21} x_{1o}}{\mu_1 - \mu_2} \left(e^{\mu_1 u} - e^{\mu_2 u} \right)$$

$$+ \frac{x_{2o}}{\mu_1 - \mu_2} \left\{ (\mu_1 + q_{22}) e^{\mu_1 u} - (\mu_2 + q_{22}) e^{\mu_1 u} \right\} \tag{16}$$

If the roots of (13) are complex, writing

$$\mu_1 = a + ib \quad , \quad \mu_2 = a - ib \tag{17}$$

we find

$$x_1 = \frac{x_{1o}}{b} e^{au} \left\{ (a + q_{11}) \sin bu + b \cos bu \right\}$$

$$+ \frac{q_{12} x_{2o}}{b} e^{au} \sin bu \tag{18}$$

$$x_2 = \frac{q_{21} x_{1o}}{b} e^{au} \sin bu$$

$$+ \frac{x_{2o}}{b} e^{au} \left\{ (a + q_{22}) \sin bu + b \cos bu \right\} \tag{19}$$

If equation (13) has equal roots

$$\mu_1 = \mu_2 = a$$

we find

$$x_1 = x_{1o} e^{au} \left\{ (a + q_{11}) u + 1 \right\} + q_{12} \, x_{2o} \, u e^{au} \tag{20}$$

$$x_2 = q_{21} x_{1o} u e^{au} + x_{2o} \, e^{au} \left\{ (a + q_{22}) u + 1 \right\} \tag{21}$$

We/

/We note that

$$\left.\begin{aligned}\frac{dx_1}{du} &= -q_{22}x_1 + q_{12}x_2\\ \frac{dx_2}{du} &= q_{21}x_1 - q_{11}x_2\end{aligned}\right\}\quad(22)$$

For simplicity we shall examine the response of the system to (i) an initial displacement x_{1o} with $x_{2o} = 0$ and (ii) an initial displacement x_{2o} with $x_{1o} = 0$.

From Chapter 3, (42) - (45), for an initial displacement $x_{1o} = 1$ with $x_{2o} = 0$,

$$\left.\begin{aligned}2L_1\omega_o &= \frac{q_{11}^2+1}{q_{11}+q_{22}}\\ \frac{2L_{11}}{\omega_o} &= \frac{q_{22}^2+1}{q_{11}+q_{22}}\\ 2L_2\omega_o &= \frac{q_{21}^2}{q_{11}+q_{22}}\\ \frac{2L_{12}}{\omega_o} &= \frac{q_{21}^2}{q_{11}+q_{22}}\end{aligned}\right\}\quad(23)$$

Similarly for an initial displacement $x_{2o} = 1$ with $x_{1o} = 0$,

$$\left.\begin{aligned}2L_1\omega_o &= \frac{q_{12}^2}{q_{11}+q_{22}}\\ \frac{2L_{11}}{\omega_o} &= \frac{q_{12}^2}{q_{11}+q_{22}}\\ 2L_2\omega_o &= \frac{q_{22}^2+1}{q_{11}+q_{22}}\\ \frac{2L_{12}}{\omega_o} &= \frac{q_{11}^2+1}{q_{11}+q_{22}}\end{aligned}\right\}\quad(24)$$

We/

/We see that for a system with a given undamped natural frequency ω_0, the response functions L_1, L_{11}, L_2, L_{12} are functions of q_{11}, q_{12}, q_{21} and q_{22} whereas the coefficients of the characteristic equation in the form (13) only depend on $(q_{11}+q_{22})$.

We note that the coupling terms q_{12} and q_{21} only occur in the response functions for x_1 when the initial displacement is in x_2 and vice versa, and then only in the form q_{12}^2, q_{21}^2. If the signs of both q_{12} and q_{21} are changed we see from (15) and (16) that for the initial displacements considered the motion is unaltered apart possibly from the sign of the displacements. More generally as shown in Chapter 3, if in two systems the coupling terms are related by equations of the form

$$q_{12}^1 = q_{12}\,\theta_2/\theta_1$$

$$q_{21}^1 = q_{21}\,\theta_1/\theta_2$$

where θ_1 and θ_2 are any non zero constants (positive or negative), the corresponding amplitudes of response are related by the equations

$$\frac{x_1^1}{\theta_1 x_1} = \frac{x_2^1}{\theta_2 x_2} \qquad (25)$$

Thus the effect of varying q_{12} and q_{21} while keeping their product the same is merely to alter the magnitude of the "coupled" response (provided neither q_{12} nor q_{21} is zero). If in any physical problem the response in x_1, for example, is much more important than that in x_2, q_{12} must be chosen to be small, and vice versa. If the responses in x_1 and x_2 are both equally important then from (23) and (24) considering only the "coupled" responses we should take q_{12} and q_{21} equal in magnitude. When illustrating the nature of the response we shall in general take q_{12} and q_{21} equal in magnitude.

$$\left.\begin{aligned} &\text{Using (12) we take} \quad q_{12} = -q_{21} \quad \text{when} \quad q_{11}q_{22} < 1 \\ &\text{and} \quad q_{12} = q_{21} \quad \text{when} \quad q_{11}q_{22} > 1 \end{aligned}\right\} \qquad (26)$$

Then/

/Then for given values of q_{11} and q_{22},

$$\left.\begin{aligned} & x_1(x_{10} = 0\ ,\ x_{20} = 1) = -\ x_2(x_{10} = 1\ ,\ x_{20} = 0\ ) \\ & \qquad\qquad \text{when } q_{11}q_{22} < 1 \\ \text{and}\quad & x_1(x_{10} = 0\ ,\ x_{20} = 1) = x_2(x_{10} = 1\ ,\ x_{20} = 0\ ) \\ & \qquad\qquad \text{when } q_{11}q_{22} > 1 \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} & L_1(x_{10} = 0\ ,\ x_{20} = 1) = L_2(x_{10} = 1\ ,\ x_{20} = 0\ ) \\ \text{and}\quad & L_{11}(x_{10} = 0\ ,\ x_{20} = 1) = L_{12}(x_{10} = 1\ ,\ x_{20} = 0\ ) \end{aligned}\right\} \tag{28}$$

From (26) we see that the given system is symmetric in the q's if $q_{11}q_{22} > 1$ and asymmetric if $q_{11}q_{22} < 1$. From (27) we see that for a second order system which is either symmetric or asymmetric the response in x_1 due to unit displacement in x_2 is identical in magnitude with the response in x_2 due to unit displacement in x_1, the signs of the two responses being the same for a symmetric system and opposite for an asymmetric system.

From (23),(24) and (26) we see that the 8 response functions corresponding to the two sets of initial conditions are proportional the the 3 parameters X , Y , Z where

$$\left.\begin{aligned} X &= \frac{q_{11}^2 + 1}{q_{11}+q_{22}} \\ Y &= \frac{q_{22}^2 + 1}{q_{11}+q_{22}} \end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} Z &= \frac{q_{12}^2}{q_{11}+q_{22}} = \frac{q_{21}^2}{q_{11}+q_{22}} \\ &= \frac{1 - q_{11}q_{22}}{q_{11}+q_{22}} \end{aligned}\right\} \tag{30}$$

From (29) and (30) we see that the response functions for any system with $q_{11} = a$ where a is some positive quantity will always be less than with $q_{11} = -a$ (where $q_{11}+q_{22}$ is positive). Similarly for/

/for q_{22}. Thus if an unstable system is coupled with a stable system the response can never be the optimum (assuming we are free to change all four of the q_{rs} parameters). We shall therefore only consider positive values of q_{11} and q_{22}.

The variation of X, Y and Z with q_{11} and q_{22} is shown in Figures 4.1 and 4.2. The response curves are shown in Figures 4.3 - 4.6 for values of q_{11} and q_{22} given in the following table.

Table I.

Values of q_{11}, q_{12}, q_{21} and q_{22} for response curves for a first order system with two degrees of freedom. (See Figures 4.3 - 4.6).

Curve	q_{11}	q_{12}	q_{21}	q_{22}	Roots of characteristic equation
a	0.50	1.00	-1.00	0	$-0.25 \pm 0.97i$
b	1.00	1.00	-1.00	0	$-0.50 \pm 0.87i$
c	2.00	1.00	-1.00	0	-1 , -1
~~d~~	0.50	0.87	-0.87	0.50	$-0.50 \pm 0.87i$
e	1.00	0.71	-0.71	0.50	$-0.75 \pm 0.66i$
f	2.00	0	0	0.50	-2 , -0.5
g	2.00	0	-1.00	0.50	-2 , -0.5
h	0.50	0.71	-0.71	1.00	$-0.75 \pm 0.66i$
i	1.00	0	0	1.00	-1 , -1
j	1.00	0	-1.00	1.00	-1 , -1
k	2.00	1.00	1.00	1.00	-2.62, -0.38
l	0.50	0	0	2.00	-2 , -0.5
m	0.50	0	1.00	2.00	-2 , -0.5
n	1.00	1.00	1.00	2.00	-2.62, -0.38
o	2.00	1.73	1.73	2.00	-3.73, -0.27

As stated above, in general we have taken q_{12} and q_{21} to be equal in magnitude, except for curves g, j and m. For these three systems q_{12} is zero; thus an initial displacement x_{20} does not cause any motion in the degree of freedom x_1. However an initial displacement/

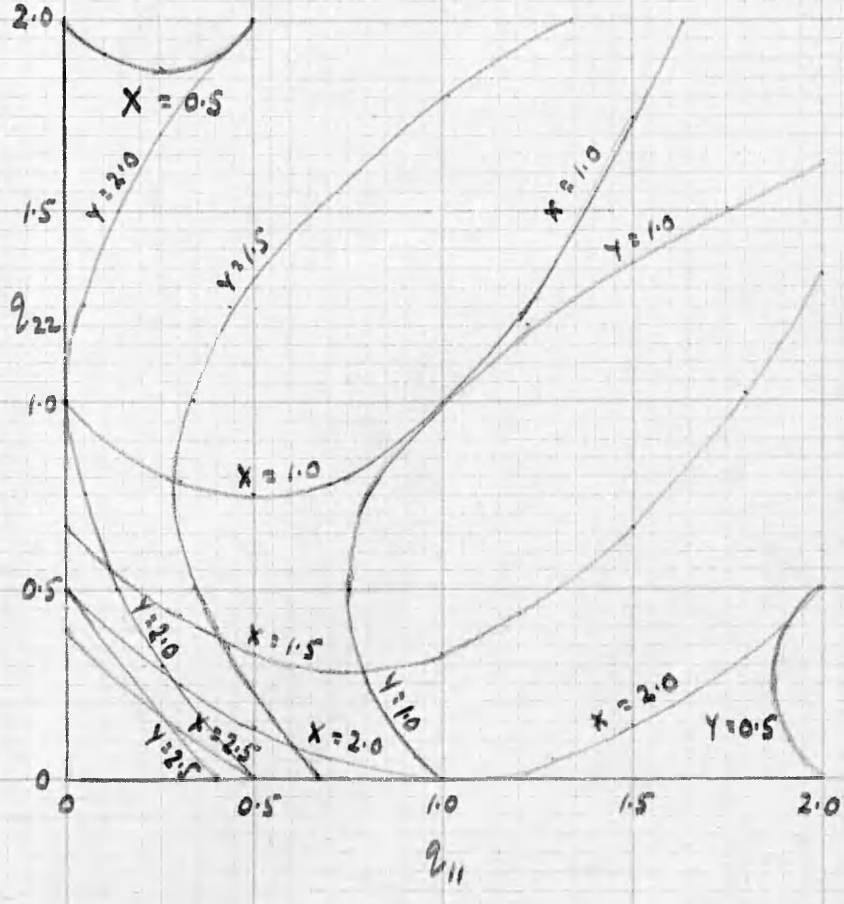


FIGURE 4.1

VARIATION OF RESPONSE FUNCTIONS
WITH q_{11} AND q_{22}

$$X = \frac{2L_1\omega_0}{x_{10}^2} = \frac{2L_{12}}{\omega_0 x_{20}^2}$$

$$Y = \frac{2L_{11}}{\omega_0 x_{10}^2} = \frac{2L_2\omega_0}{x_{20}^2}$$

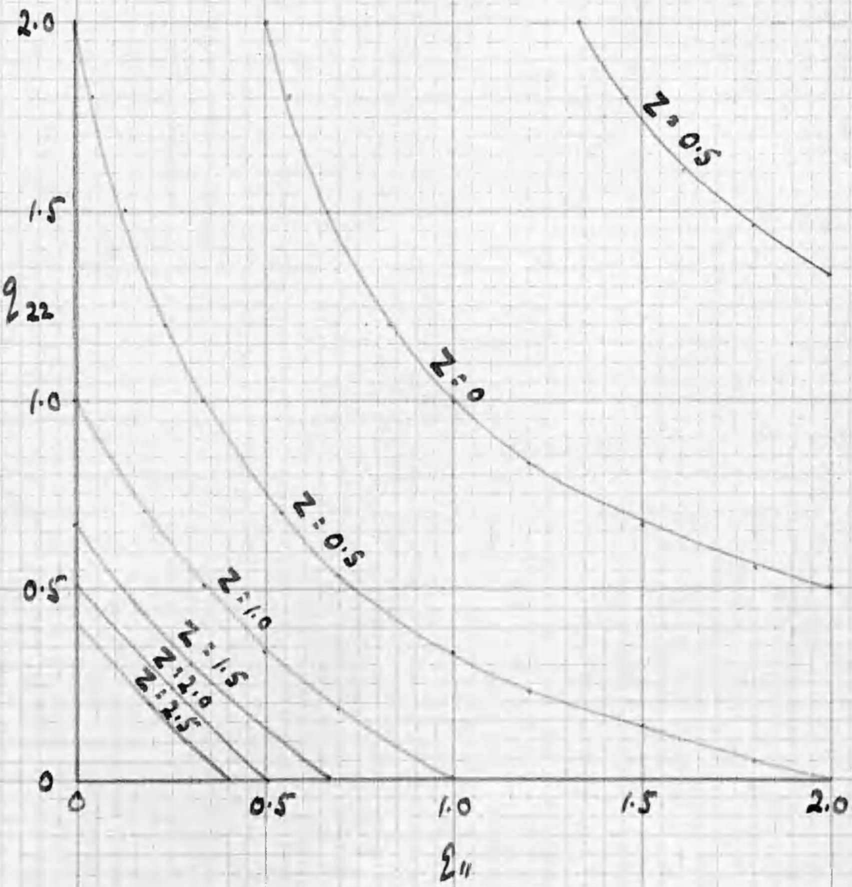
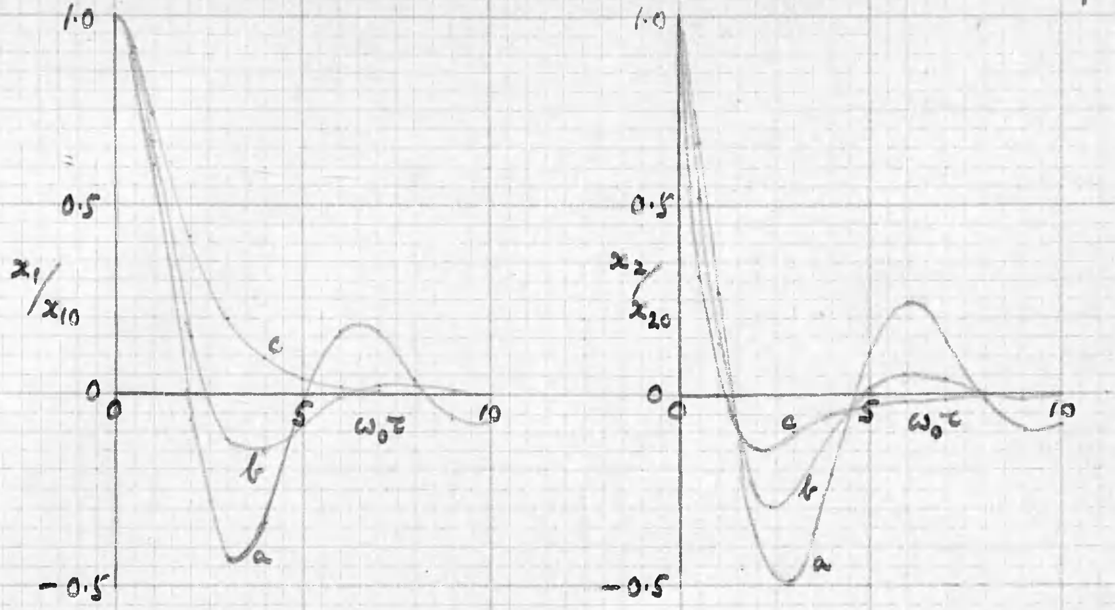
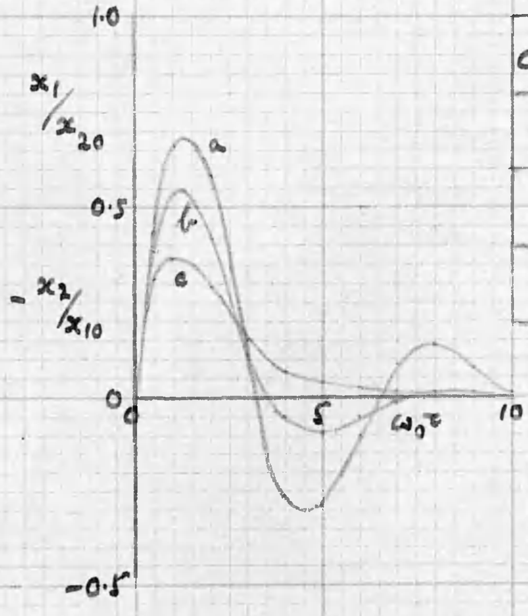


FIGURE 4.2

VARIATION OF RESPONSE FUNCTIONS WITH q_{11} AND q_{22}

$$Z = \frac{2L_1\omega_0}{x_{20}^2} = \frac{2L_{11}}{\omega_0 x_{20}^2} = \frac{2L_2\omega_0}{x_{10}^2} = \frac{2L_{12}}{\omega_0 x_{10}^2}$$



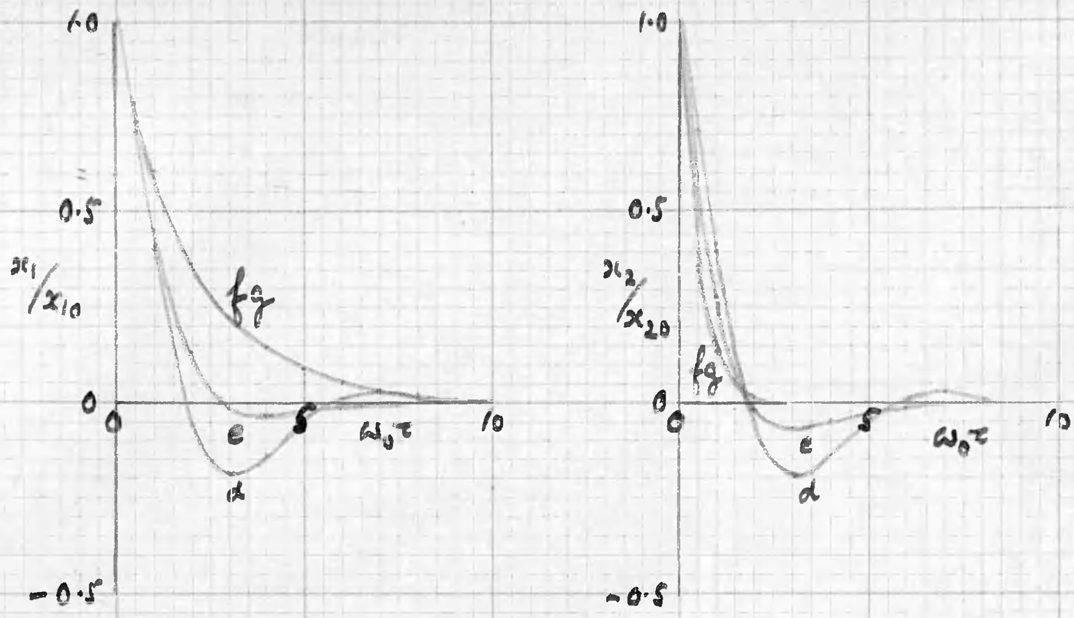


CURVE	q_{11}	q_{12}	q_{21}	q_{22}
a	0.50	1.00	−1.00	0
b	1.00	1.00	−1.00	0
c	2.00	1.00	−1.00	0

FIGURE 4.3

FIRST ORDER SYSTEM WITH TWO DEGREES OF FREEDOM

$q_{22} = 0$



CURVE	q_{11}	q_{12}	q_{21}	q_{22}
d	0.50	0.87	-0.87	0.50
e	1.00	0.71	-0.71	0.50
f	2.00	0	0	0.50
g	2.00	0	-1.00	0.50

NOTE: DOTTED CURVE g APPLIES TO $-x_2/x_{10}$ ONLY

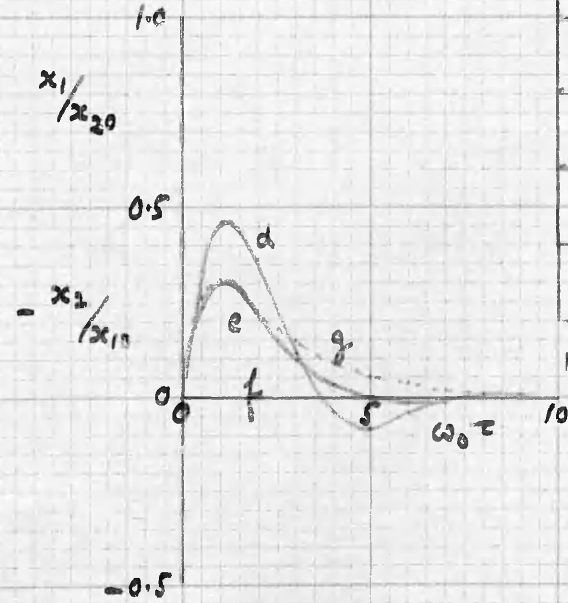
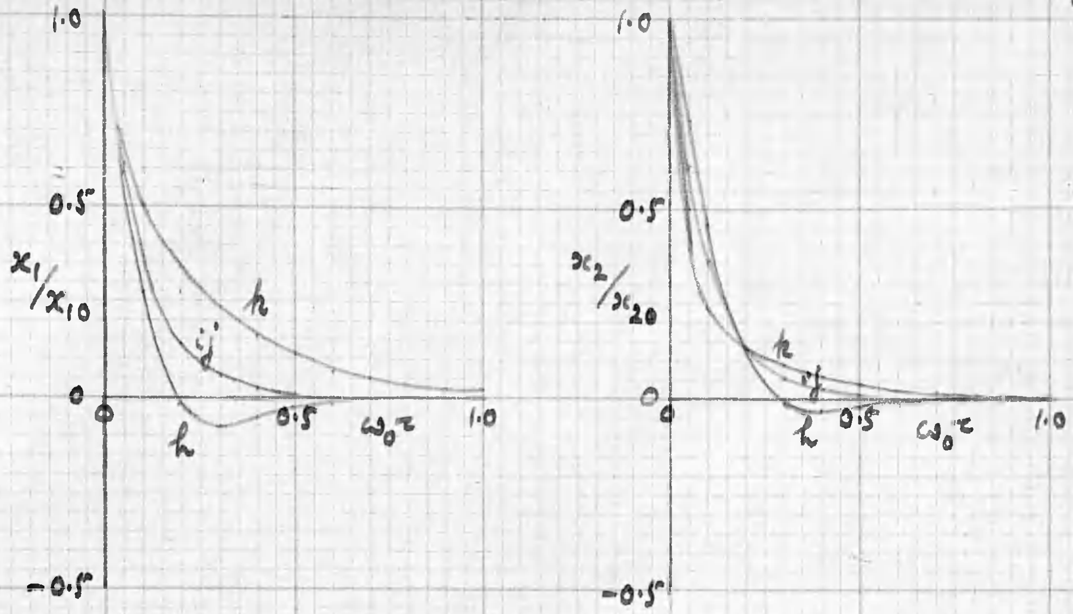


FIGURE 4.4

FIRST ORDER SYSTEM WITH TWO DEGREES OF FREEDOM

$q_{22} = 0.50$



CURVE	q_{11}	q_{12}	q_{21}	q_{22}
h	0.50	0.71	-0.71	1.00
i	1.00	0	0	1.00
j	1.00	0	-1.00	1.00
k	2.00	1.00	1.00	1.00

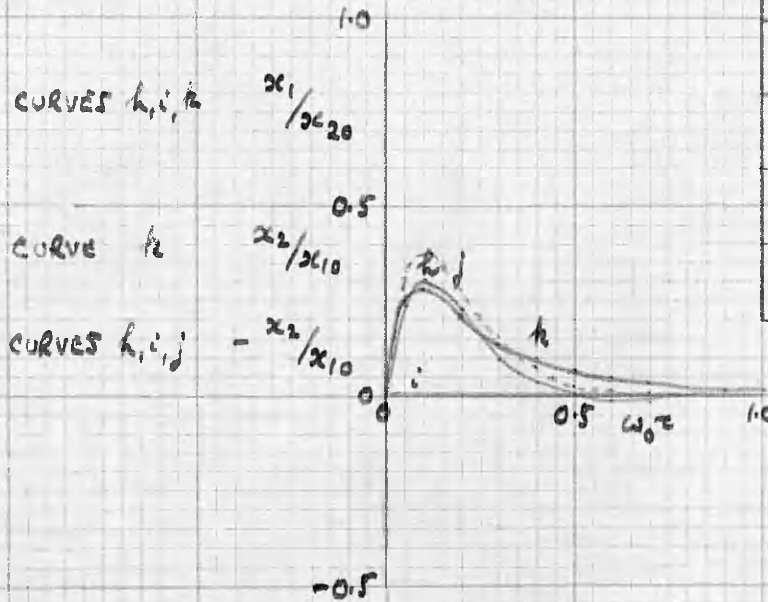
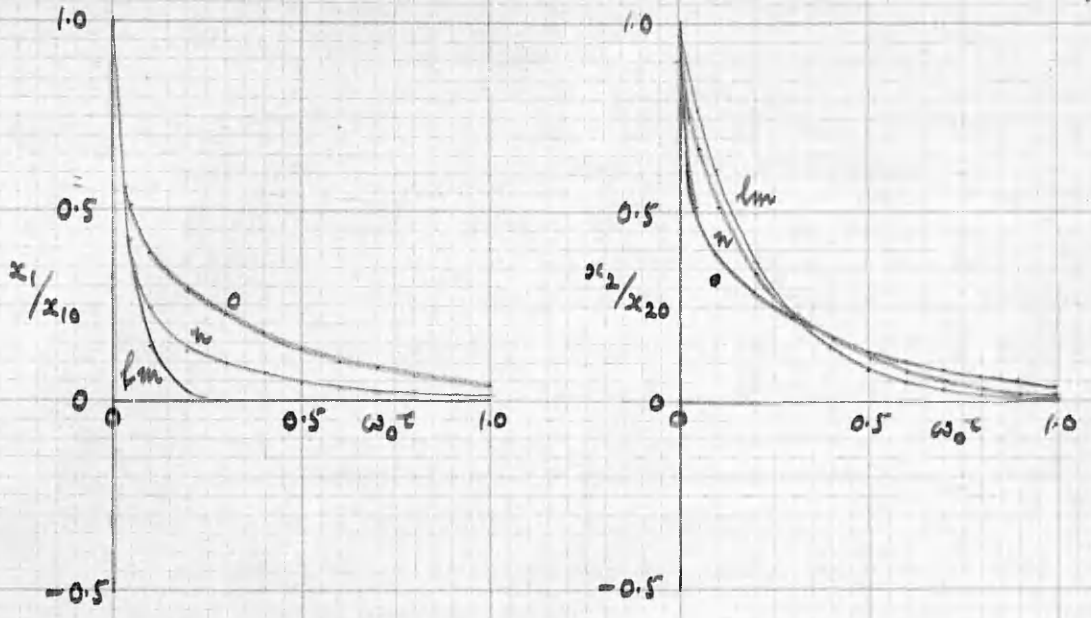


FIGURE 4.5

FIRST ORDER SYSTEM WITH TWO DEGREES OF FREEDOM

$q_{22} = 1.00$



CURVE	q_{11}	q_{12}	q_{21}	q_{22}
l	0.50	0	0	2.00
m	0.50	0	1.00	2.00
n	1.00	1.00	1.00	2.00
o	2.00	1.73	1.73	2.00

NOTE : DOTTED CURVE m APPLIES TO x_2/x_{10} ONLY

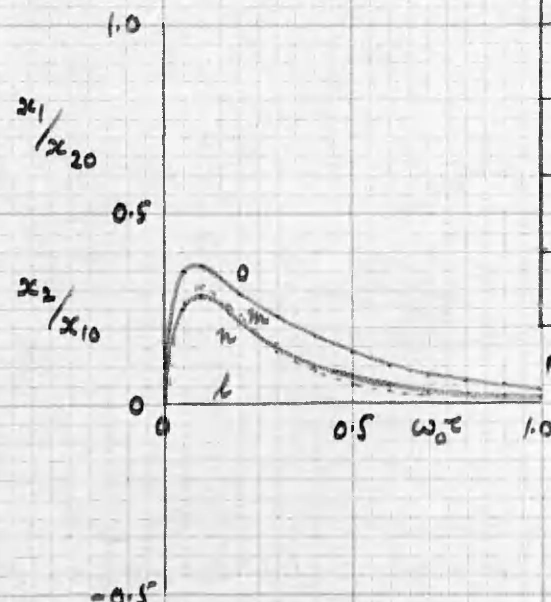


FIGURE 4.6

FIRST ORDER SYSTEM WITH TWO

DEGREES OF FREEDOM

$q_{22} = 2.00$

/displacement x_{10} causes motion in the second degree of freedom x_2. If both q_{12} and q_{21} are zero the motions in the two degrees of freedom are completely independent (uncoupled).

Considering the general system we see from (23) and Figure 4.1 that for a given value of q_{22} there is a certain value of q_{11} making L_1 a minimum for the x_{10} displacement; similarly for a given value of q_{11} there is a certain value of q_{22} making L_2 a minimum for the x_{20} displacement. We see too from (23) and (24) and Figure 4.1 that these values of q_{11} and q_{22} also correspond to minimum values of L_{12} and L_{11} respectively for the two displacements considered.

From (23), for an initial displacement x_{10} with $x_{20} = 0$, the value of q_{11} making L_1 a minimum is given by

$$q_{11} = -q_{22} + \sqrt{q_{22}^2 + 1} \qquad (31)$$

the positive value of the square root being taken.

At this value of q_{11},

$$\frac{L_1}{x_{10}^2} = \frac{q_{11}}{\omega_0} \qquad (32)$$

However as shown in Figures 4.1 and 4.2 it may sometimes be advantageous to select a slightly higher value of q_{11} in order to have smaller values of L_{11}, L_2 and L_{21} to minimise any overswing and coupling. Thus with $q_{22} = 0$, L_{1min} for an x_{10} displacement occurs for $q_{11} = 1$. From Figure 4.3 we see that the overswing in x_1 is 15°/o while that in x_2 is 30°/o, the corresponding maximum displacement in the coupled motion being 55°/o. By choosing a higher value of q_{11} (say 2, curve c) these amplitudes are reduced to 0, 15°/o and 35°/o respectively. For an x_{10} displacement small values of L_1 occur for large q_{22} and small q_{11}; small values of L_{11} occur for large q_{11} and small q_{22}.

From Figures 4.3 - 4.6 we see that if <u>in a certain physical problem</u> we are concerned with optimising the behaviour of the system in the degree of freedom x_1 <u>only</u> (with the given initial disturbance x_{10}) then it is best to choose systems for which

$$1 < q_{11} + q_{22} < 3 \qquad (33)$$

e.g/

/e.g. curves c, e, f, g, h, i, j, l and m.

Systems for which $(q_{11}+q_{22})$ is 1 or less (curves a, b and d) have a large overswing ($15^{o}/o$ or more) while systems for which $(q_{11}+q_{22})$ is 3 or more (curves k, n and o) have rather a low rate of damping.

Consider now a physical system in which we are equally concerned with optimising the behaviour of the system in both the first and second degrees of freedom. We ~~now~~ have to consider the behaviour of the system with both types of initial displacement. As shown above there are now eight response functions. The problem is considerably simplified by grouping together the response functions into two overall response functions.

$$\left.\begin{aligned} R \;=\;& L_1\,(x_{10}=1\,,\ x_{20}=0\,) \\ &+ L_1\,(x_{10}=0\,,\ x_{20}=1\,) \\ &+ L_2\,(x_{10}=1\,,\ x_{20}=0\,) \\ &+ L_2\,(x_{10}=0\,,\ x_{20}=1\,) \\ \text{and}\qquad R_1 \;=\;& L_{11}\,(x_{10}=1\,,\ x_{20}=0\,) \\ &+ L_{11}\,(x_{10}=0\,.\ x_{20}=1\,) \\ &+ L_{12}\,(x_{10}=1\,,\ x_{20}=0\,) \\ &+ L_{12}\,(x_{10}=0\,,\ x_{20}=1\,) \end{aligned}\right\} \qquad (34)$$

From (23), (24), for a linear first order system with two degrees of freedom,

$$2R\omega_o = \frac{2R_1}{\omega_o} = \frac{q_{11}^2+q_{22}^2+q_{12}^2+q_{21}^2+2}{q_{11}+q_{22}} \qquad (35)$$

The coupling terms enter equation (35) in the form $q_{12}^2+q_{21}^2$. As shown above the form of the response is unchanged if q_{12} and q_{21} are varied while their product is kept the same. From (35) we see that for a given value of $q_{12}q_{21}$, R and R_1 will be a minimum when

$$|q_{12}| = |q_{21}| \quad .$$

Then

$$\left.\begin{aligned} 2R\omega_o = \frac{2R_1}{\omega_o} &= \frac{(q_{11}-q_{22})^2+4}{q_{11}+q_{22}} && \text{if } q_{11}q_{22} < 1 \\ &= q_{11}+q_{22} && \text{if } q_{11}q_{22} > 1 \end{aligned}\right\} (36)$$

The/

/The variation of R and R_1 with q_{11} and q_{22} is shown in Figure 4.7. The minimum values of both R and R_1 occurs for $q_{11} = q_{22} = 1$ with $q_{12} = q_{21} = 0$

Then
$$\left.\begin{aligned} R_{min} &= \frac{1}{\omega_o} \\ R_{1\,min} &= \omega_o \end{aligned}\right\} \quad (37)$$

$$\left.\begin{aligned} \frac{L_1}{x_{1o}^2} &= \frac{L_2}{x_{2o}^2} = \frac{1}{2\omega_o} \\ \frac{L_{11}}{x_{1o}^2} &= \frac{L_{12}}{x_{2o}^2} = \frac{\omega_o}{2} \end{aligned}\right\} \quad (38)$$

The response is shown in Figure 4.5 (curve i). The system is uncoupled, the damping in each mode being the same. We note that R_{min} and $R_{1\,min}$ occur for the same system. This system besides being the mathematical optimum is also the "most satisfactory" one, systems with adjacent values of q_{11}, q_{12}, q_{21} and q_{22} having either coupling (curves e, h, j, k and n) or a lower rate of damping in one mode (curves f, k, ℓ and m). Systems having values of R and R_1 up to 25°/o greater than the minimum values will have very satisfactory response characteristics, the coupling being quite small ($Z < 0.23$).

<u>Response functions for linear first order system with two degrees of freedom (integral control).</u>

An important special case of the above analysis is that for which the equations of motion (1) (in the free motion) are

$$\left.\begin{aligned} a_2\dot{x}_1 + a_1x_1 + b_ox_2 &= 0 \\ b_1x_1 - \dot{x}_2 &= 0 \end{aligned}\right\} \quad (39)$$

This could correspond, for example, to a system in x_1 to which integral control has been added by means of x_2. The response is, of course, identical with that of the system

$$a_2\frac{d^2x}{d\tau^2} + a_1\frac{dx}{d\tau} + a_ox = 0$$

dealt with in Chapter 2 where $a_o = b_ob_1$. However if we are equally interested/

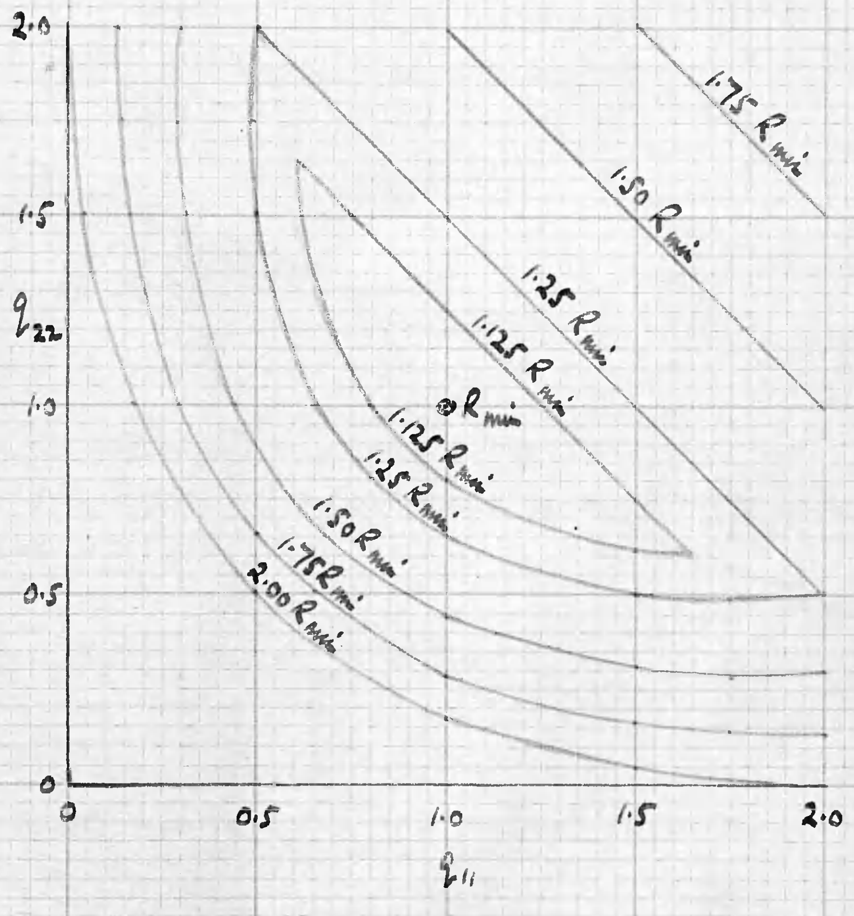


FIGURE 4.7

VARIATION OF OVERALL RESPONSE FUNCTIONS WITH q_{11} AND q_{22}

$$R\omega_0 = \frac{R_1}{\omega_0}$$

/interested in the response to initial displacements in both x_1 and x_2 (separately) the optimum solution will differ from that given in Chapter 2.

As in Chapter 2 we define the normalized coefficient q_1 by the equation

$$q_1 = \frac{a_1}{a_2\omega_o} \tag{40}$$

where ω_o is the undamped natural frequency of the system

i.e. $$a_o = \omega_o^2 a_2 \tag{41}$$

We write $$\left.\begin{aligned} b_o &= \frac{a_o}{\theta\omega_o} \\ b_1 &= \theta\omega_o \end{aligned}\right\} \tag{42}$$

where θ is some numerical constant.

The form of these equations is governed by the dimensions of a_o, b_o, b_1 and ω_o . As stated above the form of the response is unchanged if θ is varied, the product $b_o b_1$ remaining constant.

From (31) and (41) we see that

$$L_{12} = \int_0^\infty \left(\frac{dx_2}{d\tau}\right)^2 d\tau = b_1^2 \int_0^\infty x_1^2 d\tau = \theta^2 \omega_o^2 L_1 \tag{43}$$

From Chapter 1, (25) - (28), with $x_{1o} = 1$, $x_{2o} = 0$ i.e. with $Dx_{2o} = b_1$, with the notation of the present chapter,

contd.

$$\left.\begin{aligned}
L_1 &= \int_0^\infty x_1^2\, d\tau &&= \frac{1}{2q_1\omega_0} \\
L_{11} &= \int_0^\infty (Dx_1)^2 d\tau &&= \frac{q_1^2+1}{2q_1}\,\omega_0 \\
L_2 &= \int_0^\infty x_2^2\, d\tau &&= \frac{\theta^2}{2q_1\,\omega_0} \\
L_{12} &= \int_0^\infty (Dx_2)^2 d\tau &&= \frac{\theta^2\omega_0}{2q_1}
\end{aligned}\right\} \qquad (44)$$

Similarly with $x_{10} = 0$, $x_{20} = 1$,

$$\left.\begin{aligned}
L_1 &= \frac{1}{2\theta^2 q_1\omega_0} \\
L_{11} &= \frac{\omega_0}{2\theta^2 q_1} \\
L_2 &= \frac{q_1^2+1}{2q_1\,\omega_0} \\
L_{12} &= \frac{\omega_0}{2q_1}
\end{aligned}\right\} \qquad (45)$$

From (44), (45), $$2R\omega_0 = \frac{2R_1}{\omega_0} = \frac{q_1^2 + \theta^2 + \frac{1}{\theta^2} + 2}{q_1} \qquad (46)$$

The coupling terms enter equation (46) in the form

$$\theta^2 + \frac{1}{\theta^2}$$

From (46) we see that for a given value of a_0 , R and R_1 will be a minium when

$$\theta^2 = 1$$

Then

$$2R\omega_0 = \frac{2R_1}{\omega_0} = \frac{q_1^2+4}{q_1}$$

The/

/The minimum values of both R and R_1 occurs for $q_1 = 2$ with $\theta = \pm 1$.

$$\left.\begin{aligned} R_{min} &= \frac{2}{\omega_o} \\ R_{1\,min} &= 2\omega_o \end{aligned}\right. \qquad (47)$$

$$\left.\begin{aligned} \frac{L_1}{x_{1o}^2} &= \frac{L_2}{x_{1o}^2} = \frac{L_1}{x_{2o}^2} = \frac{1}{4\omega_o} \\ \frac{L_2}{x_{2o}^2} &= \frac{5}{4\omega_o} \\ \frac{L_{12}}{x_{1o}^2} &= \frac{L_{11}}{x_{2o}^2} = \frac{L_{12}}{x_{2o}^2} = \frac{\omega_o}{4} \\ \frac{L_{11}}{x_{1o}^2} &= \frac{5}{4}\,\omega_o \end{aligned}\right\} \qquad (48)$$

The response is shown in Figures 2.1 and 2.6 (allowing for the different time scale in the latter figure). We see that the binomial response is the optimum.

Linear systems with more than two degrees of freedom.

The analysis given above can be extended to linear systems with more than two degrees of freedom by the methods of Chapter 3. The overall response functions can be deduced by an extension of equations (34) and (35). As shown in Chapter 3 coupling is eliminated by making all the coefficients q_{rs} $(r \neq s)$ zero.

For systems of second and higher orders coupling should first be eliminated. The optimum response in each degree of freedom can then be found by using the methods of Chapter 2.

Bibliography and List of References.

1.	Thaler, G.J. and Brown, R.G.	Servomechanism analysis. McGraw Hill, 1953.
2.	Oldenbourg, R.C. and Sartorius, H.	The dynamics of automatic controls. ASME, 1948.
3.	Nims, P.T.	Some design criteria for automatic controls. AIEE trans. Vol.70(1951) part 1, p.606.
4.	Wiener, N.	Extrapolation, interpolation and smoothing of stationary time series. Wiley, 1950.
5.	Hall, A.C.	The analysis and synthesis of linear servomechanisms. MIT. 1943.
6.	Mack, C.	The calculation of the optimum parameters for a following system. Phil.Mag.7th series. Vol.40(1949),p.922.
7.	Graham, D. and Lathrop, R.C.	The synthesis of optimum transient response: criteria and standard forms. AIEE trans. Vol.72(1953) part 2, p.273.
8.	Whiteley, A.L.	Theory of servo systems, with particular reference to stabilization. Jour.Inst.Electrical Engineers, Vol.93 (1946) part 2, p.353.
9.	Oldenbourg, R.C. and Sartorius, H.	A uniform approach to the optimum adjustment of control loops. ASME trans. Vol.76(1954) No.8, p.1265.
10.	Butterworth, S.	On the theory of filter amplifiers. Wireless Engineer, Vol.7 (1930) p.536.
11.	Brown, G.S. and Campbell, D.P.	Principles of servomechanisms. Wiley, 1948.
12.	Ahrendt, W.R. and Taplin, J.F.	Automatic feedback control. MrGraw Hill, 1951.
13.	Chestnut, H. and Mayer, R.W.	Servomechanisms and regulating system design. Wiley, 1951.

contd.

14. Chu, Y. — Correlation between frequency and transient responses of feedback control systems. AIEE trans. Vol.72(1953) part 2, p.81.

15. Evans, W.R. — Control system synthesis by root locus method. AIEE trans. Vol.69 (1950) part 1, p.66.

16. Ince, E.L. — Ordinary differential equations. Longmans, 1927.

17. Carslaw, H.S. and Jaeger, J.C. — Operational methods in applied mathematics. Oxford, 1941.

18. Hurwitz, A. — Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reelen Theilen besitzt. Math.Ann. Vol.46, 1895. p.273.

19. Weiss, H.K. — Theory of automatic control of airplanes. NACA Tech.Note 700. 1939.

20. Routh, E.J. — On the stability of motion. Macmillan, 1877.

21. Burnside, W.S. and Panton, A.W. — The theory of equations. Vol.1. Longmans, 1904.

22. Frazer, R.A. and Duncan, W.J. — On the criteria for the stability of small motions. Proc.Roy.Soc.London. Vol.A-124,(1929), p.642.

23. James, H.M., Nichols, N.B. and Phillips, R.S. — Theory of servomechanisms. McGraw Hill, 1947.

24. Franklin, P. — Fourier methods. McGraw Hill, 1949.

25. Whittaker, E.T. and Watson, G.N. — Modern analysis. Cambridge, 1935.

26. Macmillan, R.H. — The theory of control in mechanical engineering. Cambridge, 1951.

27. Neumark, S. — The disturbed longitudinal motion of an uncontrolled aircraft and of an aircraft with automatic control. A.R.C. R.and M. 2078, 1943.

28. Duncan, W.J. — Mechanical admittances and their applications to oscillation problems. A.R.C. R.and M.2000.1946.

applications to oscillation problems. A.R.C. R.and M.2000.1946.