T H E S I S

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TWO PROBLEMS IN THE PRODUCTION OF MESONS

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Part I.

The multiple scattering correction to the impulse approximation for the photoproduction of charged mesons at deuterium.

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SUMMARY. Part I deals with the problem of the interaction of the meson with the nucleons in the photoproduction of a charged π meson at deuterium. This is considered as multiple scattering of the meson at alternate nucleons. Earlier work on this problem, and on the multiple scattering correction to the impulse approximation in similar problems, is reviewed. To avoid having to use the meson-nucleon scattering transition operator off the energy shell an approximation, taken from the earlier work, is used. The meaning of this approximation is discussed. In the case of a particular model of meson-nucleon scattering, based on a factorable potential, an estimate is made of the accuracy of this approximation. In obtaining the cross-section the interaction of the nucleons is included when they are in a final state with $\ell = 0$. Results are presented illustrating the behaviour of the meson energy spectrum at a particular angle and photon energy. This has a broad peak around the energy of the meson produced at the same angle from a free nucleon, and a narrow peak near the maximum meson energy, caused by the final state interaction of the nucleons, and important only at forward angles. The multiple scattering correction is -4% to -8% on the free nucleon peak, rising to about -20% on the interaction peak at forward angles. The conclusion is reached that with the present experimental accuracy the multiple scattering correction will not in general affect the interpretation of the experimental results using the impulse approximation.

1. Introduction.

The study of π meson photoproduction and scattering at deuterium is of interest for two reasons. It may be a means of obtaining information about the same processes at free neutrons, which are not directly observable. On the other hand, if we consider that meson theory gives an adequate description of the free nucleon case we can attempt to find how the complex nature of the two nucleon system influences the processes. The work presented here approaches the problem of the photoproduction of charged π mesons from the second point of view. This section contains a summary of the relevant work on free nucleon processes and a review of earlier theoretical work on this particular problem and the related problems of π meson scattering at deuterium, and the elastic photoproduction of neutral π mesons at deuterium and helium. There is also a discussion of the experimental work on the processes,

$$\gamma + D \longrightarrow \pi^+ + n + n$$

$$\gamma + D \longrightarrow \pi^- + p + p$$

Meson scattering and photoproduction at a free nucleon.

We consider this within the framework of theories which treat the nucleon as a static source distribution of a finite size. This is characterised by a source density $\rho(r)$ and the corresponding momentum cut-off function.

$$v(q) = \int e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}} \rho(r)\, d\underset{\sim}{r}$$

The interaction part of the Hamiltonian of the system of a meson and a nucleon is (see Wick(1955)).

$$h = \sum_{q\lambda} \left(a_q^\lambda V_q^\lambda + a_q^{\lambda\dagger} V_q^{\lambda\dagger} \right)$$

with

$$V_q^\lambda = \sqrt{\frac{4\pi}{2\omega(q)}}\, \frac{f}{\mu}\, \tau_\lambda\, i\, \underset{\sim}{\sigma}.\underset{\sim}{q}\; v(q) \qquad (1)$$

In this result a_q^λ destroys, $a_q^{\lambda\dagger}$ creates a meson of momentum q, energy $\omega(q)$, in an isotopic spin state specified by λ. f is the coupling constant, μ the meson mass and $\underset{\sim}{\sigma}$, $\underset{\sim}{\tau}$ the spin and isotopic spin operators of the nucleon.

A simple form of the static nucleon theory is that of Chew (1954). He discusses meson-nucleon scattering, making use of a variational principle of Schwinger. We describe this theory in some detail, in order to introduce various concepts and results which we shall require later. In particular we shall require stationary state scattering theory in both parts of this thesis, while in Section 4 of this part we shall examine a theory similar to that of Chew.

Let H_0 be the Hamiltonian of the free meson field and E the total energy. Then the total Hamiltonian H is $H_0 + h$. In the stationary state scattering theory, as given for example in Lippmann, Schwinger (1950), we make use of eigenfunctions $\Psi_a^{(\pm)}(E)$, $\Psi_a^{(1)}(E)$ of H. They each satisfy the equation

$$(H_0 + h)\Psi_a(E) = E\,\Psi_a(E)$$

but with different boundary conditions, having scattered parts which are respectively outgoing, incoming and standing waves. These boundary conditions are expressed by writing

$$\Psi_a(E) = 2\pi\hbar\, \delta(E - E_a)\, \Phi_a$$

where

$$\Psi_a^{(\pm)} = \Phi_a + \frac{1}{E_a \pm i\varepsilon - H_0} h \Psi_a^{(\pm)} \qquad (3)$$

and

$$\Psi_a^{(1)} = \Phi_a + P \frac{1}{E_a - H_0} h \Psi_a^{(1)}$$

where Φ_a is an eigenstate of H_0 with energy E_a. The denominator $E_a \pm i\varepsilon - H_0$ is defined by the formal result

$$\frac{1}{x \pm i\varepsilon} = \mp \pi i \,\delta(x) + P \frac{1}{x}$$

which is to be understood in the sense that

$$\int dx \frac{f(x)}{x \pm i\varepsilon} = \mp \pi i f(0) + P \int dx \frac{f(x)}{x}$$

the integral on the right being the principal value. We shall denote $E_a + i\varepsilon - H_0$ by a. The transition operator t and the reactance operator K are defined by

$$\begin{aligned} \Psi_a^{(+)} &= \Phi_a + \frac{1}{a} t \,\Phi_a \\ \Psi_a^{(1)} &= \Phi_a + P \frac{1}{E_a - H_0} K \,\Phi_a \end{aligned} \qquad (4)$$

and the matrix elements of these operators between states Φ_a and Φ_b are, from (3),

$$\begin{aligned} t_{ba} &= (\Phi_b , h \Psi_a^{(+)}) \\ &= (\Psi_b^{(-)} , h \Phi_a) \end{aligned} \qquad (5)$$

and $\quad K_{ba} = (\Phi_b , h \Psi_a^{(1)})$

The probability per unit time w_{ba}, for transitions from the state Φ_a to the (different) state Φ_b, is given by

$$w_{ba} = \frac{2\pi}{\hbar} \delta(E_a - E_b) |t_{ba}|^2 \qquad (6)$$

From (3) we can write

$$\Psi_a^{(+)} = \left(1 + \frac{1}{a-h} h\right) \Phi_a$$

so that we have

$$t = h + h \frac{1}{a-h} h \tag{7}$$

which can be written

$$t = \left[V + V\frac{1}{a-V} V\right] + \left[h + V\frac{1}{a-V} h\right] = t_s + t_a \quad \textbf{say,}$$

V being defined as $h \frac{1}{a} h$. The matrix element $t_{\rho a}$ between states with one meson present is effectively $(t_s)_{\rho a}$, because t_a must create or absorb an odd number of mesons. We denote these states by $|\underset{\sim}{q}\rangle$. t_s satisfies the integral equation

$$t_s = V + V \frac{1}{a} t_s \tag{8}$$

in which V acts as a potential. There is a corresponding quantity K_s satisfying a similar equation with $\frac{1}{a}$ replaced by its principal part. The variational principle (Chew (1954a)) states that the solution of (8) on the energy

$$(t_s)_{\rho a} = \frac{(\psi_\rho^{-}, V\Phi_a)(\Phi_\rho, V\psi_a^{(+)})}{(\psi_\rho^{(-)}, V\psi_a^{(+)}) - (\psi_\rho^{(-)} V\frac{1}{a}V\, \psi_a^{(+)})} \tag{9}$$

this being stationary for variations of $\psi_\rho^{(-)}$, $\psi_a^{(+)}$ about the correct solutions $\Psi_\rho^{(-)}$, $\Psi_a^{(+)}$ of (3). Chew

about the correct solutions $\Psi_\rho^{(-)}$, $\Psi_a^{(+)}$ of (3). Chew
uses the simple trial wave functions Φ_a, Φ_ρ so that

$$(t_s)_{\rho a} \doteqdot \frac{(\Phi_\rho, V\Phi_a)^2}{(\Phi_\rho, V\Phi_a) - (\Phi_\rho, V\frac{1}{a}V\, \Phi_a)} \tag{10}$$

The equation (8) can be separated into equations for particular spin and isotopic spin eigenstates. The important one is that for spin $^3/_2$ and isotopic spin $^3/_2$. We have

$$(\underset{\sim}{q}_1|t_s^{33}|\underset{\sim}{q}_2) = (\underset{\sim}{q}_1\cdot\underset{\sim}{q}_2 - \tfrac{1}{3}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{q}_1\,\underset{\sim}{\sigma}\cdot\underset{\sim}{q}_2)\;\frac{2+\underset{\sim}{\tau}\cdot\underset{\sim}{\rho}}{3}\;t_{33}(q_1,q_2)$$

$$(\underset{\sim}{q}_1|K_s^{33}|q_2) = (\underset{\sim}{q}_1\cdot\underset{\sim}{q}_2 - \tfrac{1}{3}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{q}_1\,\underset{\sim}{\sigma}\cdot\underset{\sim}{q}_2)\;\frac{2+\underset{\sim}{\tau}\cdot\underset{\sim}{\rho}}{3}\;K_{33}(q_1,q_2) \qquad (11)$$

the first two factors being projection operators. The phase shift $\delta_{33}(q_E)$ for scattering in this eigenstate is related to t_s^{33} and K_s^{33} by

$$\tan\delta_{33}(q_E) = -\frac{\omega(q_E)\,q_E^3}{2\pi}\;k_{33}(q_E,q_E)$$

$$e^{i\delta_{33}(q_E)}\sin\delta_{33}(q_E) = -\frac{\omega(q_E)\,q_E^3}{2\pi}\;t_{33}(q_E,q_E) \qquad (12)$$

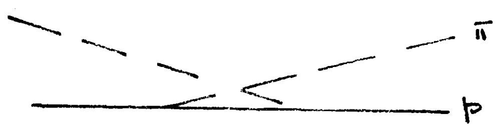
Here q_E is the value of the meson momentum q on the energy shell. The integral equation for $t_{33}(q_1,q_2)$ is

$$t_{33}(q_1,q_2) = V_{33}(q_1,q_2) + \frac{1}{2\pi^2}\int_0^\infty dq\,q^4\,\frac{V_{33}(q_1,q)\,t_{33}(q,q_2)}{\omega(q_E)-\omega(q)+i\epsilon} \qquad (13)$$

and there is a corresponding equation for ,

$$k_{33}(q_1,q_2) = V_{33}(q_1,q_2) + \frac{1}{2\pi^2}P\int_0^\infty dq\,q^4\,\frac{V_{33}(q_1,q)\,k_{33}(q,q_2)}{\omega(q_E)-\omega(q)} \qquad (14)$$

the potential $V_{33}(q_1,q_2)$ corresponds to the graph



and for our form of h it is

$$V_{33}(q_1,q_2) = \frac{8\pi f^2}{3\mu^2}\;\frac{v(q_1)\,v(q_2)}{\{\omega(q_1)\,\omega(q_2)\}^{1/2}}\;\frac{1}{\omega(q_E)-\omega(q_1)-\omega(q_2)} \qquad (15)$$

using the variational principle in (14) we have the following result for $\tan\delta_{33}(q_E)$,

$$\tan\delta_{33}(q_E) = \frac{\frac{2}{3}\frac{f^2}{\mu^2}\frac{q_E^3}{\omega(q_E)}}{1 - \frac{4}{3\pi}\frac{f^2}{\mu^2} P\int_0^{q_{max}} \frac{dq\, q^4\, \omega(q_E)}{\omega^3(q)[\omega(q_E) - \omega(q)]}} \qquad (16)$$

where we have set $v(q) = 1$, $q \leq q_{max}$, $= 0$, $q > q_{max}$. The form of (16) shows that there is a resonance. By choosing suitable values of f and q_{max} Chew was able to obtain the width and position of the resonance in agreement with experiment.

For the purposes of our multiple scattering work we have made use of the results of a more recent development of static nucleon theory, that of Chew and Low (1956a). They find it possible to use an effective range result for $\delta_{33}(q_E)$ and can select the cut-off and coupling constant to obtain agreement with experiment. Their result is

$$q_E^3 \cot\delta_{33}(q_E) = \omega^*(A - B\omega^*) \qquad (17)$$

where ω^* is the sum of the meson energy $\omega(q_E)$ and the nucleon kinetic energy in the centre of mass system, which is added to make some allowance for nucleon recoil, which is, of course, ignored in the static nucleon theory. We have used natural units ($\mu = \hbar = c = 1$). The values of A and B in these units are $A = 8.05$, $B = 3.80$ (Orear (1956)).

Chew and Low (1956b) apply their theory to pion photoproduction. They find that the main contributions to the photoproduction amplitude at a free nucleon T , and also those most likely to remain unchanged in an improved theory, are

$$\frac{-ief}{\{4\omega(q)\nu\}^{1/2}} \frac{(\tau_3\tau_q - \tau_q\tau_3)}{2}\left[\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon} - \frac{2\underset{\sim}{\sigma}\cdot(\underset{\sim}{q}-\underset{\sim}{\nu})\ \underset{\sim}{q}\cdot\underset{\sim}{\varepsilon}}{(\underset{\sim}{q}-\underset{\sim}{\nu})^2 + \mu^2}\right] \qquad (18)$$

and

$$\frac{ef}{\{4\omega(q)\nu\}^{\frac{1}{2}}}\frac{g_p-g_n}{4Mf^2}\quad\frac{2+\underset{\sim}{\tau}\cdot\underset{\sim}{\ell}}{3}\left[2\underset{\sim}{\nu}\cdot\underset{\sim}{q}\times\underset{\sim}{\varepsilon}+i\underset{\sim}{\sigma}\cdot\underset{\sim}{\nu}\times\underset{\sim}{\varepsilon}\times\underset{\sim}{q}\right]\frac{e^{i\delta_{33}(q_E)}\sin\delta_{33}(q_E)}{q_E^3} \tag{19}$$

Here $\underset{\sim}{\nu}$, $\underset{\sim}{\varepsilon}$ are the photon momentum and polarisation and g_p, g_n are the magnetic moments of the proton and neutron in units of the nuclear magneton. $\underset{\sim}{\tau}$, $\underset{\sim}{\ell}$ are the isotopic spin operators for a nucleon and a meson respectively. It is understood that $q = q_E$, and we have again set $\hbar = c = \mu = 1$. The isotopic spin operator in (18) projects out states with a neutral meson, while that in (19) projects out the $t = \frac{1}{2}$ state of the meson and nucleon. f is the renormalised coupling constant.

The expression (18), which is the same in first order perturbation theory, contains an electric dipole term and a meson current term. (19) is a magnetic dipole term giving a final state with spin 3/2 and isotopic spin 3/2, and enhanced by the resonant scattering in that state. We shall use the electric dipole and magnetic dipole terms only. In the notation of (5.12) and (5.14) where we use operators β and δ, containing meson creation operators explicitly, to give the isotopic spin dependence of T, the electric dipole term is

$$\alpha = \beta A = \beta\,\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon}\,E_d \tag{20}$$

where

$$E_d = \frac{-ief}{\{2\omega(q)\nu\}^{1/2}}$$

and the magnetic dipole term is

$$\underset{\sim}{\gamma}\cdot\underset{\sim}{q} = \delta\underset{\sim}{\zeta}\cdot\underset{\sim}{q} = \delta M_{\alpha}(\tfrac{3}{2})\left\{2\underset{\sim}{\nu}\cdot\underset{\sim}{q}\times\underset{\sim}{\varepsilon} + i\underset{\sim}{\sigma}\cdot\underset{\sim}{\nu}\times\underset{\sim}{\varepsilon}\times\underset{\sim}{q}\right\}\frac{1}{\nu q_E} \tag{21}$$

with $$M_{\alpha} = \frac{ef\nu}{\{\omega(q)\nu\}^{\frac{1}{2}}}\,\frac{g_p - g_n}{12Mf^2}\,\frac{e^{i\delta_{33}(q_E)}\sin\delta_{33}(q_E)}{q_E^{\,2}}$$

Instead of starting from meson theory and deducing a form for T one can use a general form (see for example Gell-Mann (1954)) containing parameters which can be adjusted to fit the observed angular distributions of photoproduced pions. For a particular isotopic spin state α the form of T is

$$\begin{aligned} T(\alpha) = {} & \underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon}\,E_d(\alpha) + M_{\alpha}(\tfrac{1}{2},\alpha)\left\{\underset{\sim}{\nu}\cdot\underset{\sim}{q}\times\underset{\sim}{\varepsilon} - i\underset{\sim}{\sigma}\cdot\underset{\sim}{\nu}\times\underset{\sim}{\varepsilon}\times\underset{\sim}{q}\right\}\frac{1}{\nu q_E} \\ & + M_{\alpha}(\tfrac{3}{2},\alpha)\left\{2\underset{\sim}{\nu}\cdot\underset{\sim}{q}\times\underset{\sim}{\varepsilon} + i\underset{\sim}{\sigma}\cdot\underset{\sim}{\nu}\times\underset{\sim}{\varepsilon}\times\underset{\sim}{q}\right\}\frac{1}{\nu q_E} \\ & + E_q(\alpha)\left\{\underset{\sim}{\sigma}\cdot\underset{\sim}{\nu}\;\underset{\sim}{q}\cdot\underset{\sim}{\varepsilon} + \underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon}\;\underset{\sim}{q}\cdot\underset{\sim}{\nu}\right\}\frac{1}{2\nu q_E} \end{aligned} \tag{22}$$

which contains electric dipole and quadrupole terms, and magnetic dipole terms giving states with total spin $\frac{1}{2}$ and $3/2$. To some extent the experiments with deuterium can also be analysed in terms of this form of T. It has recently been pointed out, by Moravscik (1957), that with the accuracy now possible in free nucleon experiments analysis in terms of (22) is inadequate. This is essentially because of the second term in (18), which contains contributions from higher multipole transitions. Our neglect of this term is reasonable because we are mainly concerned with the interaction of the meson and the final state nucleons, and are not attempting to obtain information about T for a free nucleon.

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Multiple scattering.

The simplest approach to the problems of meson photoproduction and scattering in light nuclei is to use the impulse approximation. This was introduced by Chew (1950) in discussing the inelastic scattering of neutrons by deuterons. The validity of the approximation is discussed by Chew and Wick (1952) and by Chew and Goldberger (1952). The transition operator for the process is taken to be the sum of the operators for the corresponding process at each nucleon, as if it were free, and the matrix element is evaluated between appropriate initial and final states. The effect of nuclear binding is ignored except in so far as it determines the wave functions for these states. Also no attempt is made to deal with processes involving the interaction of a meson with more than one nucleon. In the second and third of the papers quoted above the first order correction to the impulse approximation for the scattering of π mesons by deuterium is expressed as two separate terms, one depending on the proton-neutron potential, the other having the form of a double scattering of the meson, first at one nucleon and then at the other, both nucleons taken as free. The corresponding terms are easily written down in the case of photoproduction. Strictly speaking the term "impulse approximation" refers to the neglect of nuclear binding whether or not the meson-nucleus interaction is included in full. (See for example the discussion after equation 21 of Chew and Goldberger). However, it is convenient and convential to use the term in the sense employed here, and we shall continue to do so.

Various papers have appeared which treat the interaction of the meson and the nucleus in terms of multiple scattering at alternate nucleons. We shall consider first the problem, treated by Brueckner (1953a) and by Drell and Verlet (1955) of Λ wave scattering by two heavy point sources. In this the general form of the multiple scattering correction is clearly displayed. Let the initial and final momenta be $\underset{\sim}{q_0}$, $\underset{\sim}{q}$, where $q_0 = q = q_E$, and let the sources be situated at $\underset{\sim}{r_1}$, $\underset{\sim}{r_2}$, with $R = |\underset{\sim}{r_1} - \underset{\sim}{r_2}|$. Then if the phase shift $\delta(q_E)$ refers to scattering at one source we can obtain the amplitude of the scattered wave in the form

$$f(\theta) = \Big\{ e^{i\delta(q_E)} \sin\delta(q_E) \big[e^{i(\underset{\sim}{q_0}-\underset{\sim}{q}).\underset{\sim}{r_1}} + e^{i(\underset{\sim}{q_0}-\underset{\sim}{q_1}).\underset{\sim}{r_2}} \big]$$

$$+ \sin^2\delta\, W \big[e^{i(\underset{\sim}{q_0}.\underset{\sim}{r_1} - \underset{\sim}{q}.\underset{\sim}{r_2})} + e^{i(\underset{\sim}{q_0}.\underset{\sim}{r_2} - \underset{\sim}{q}.\underset{\sim}{r_1})} \big] \Big\} \Big\{ 1 - \sin^2\delta\, W^2 \Big\}^{-1}$$

The impulse approximation is

$$f(\theta) = e^{i\delta(q_E)} \sin\delta(q_E) \big[e^{i(\underset{\sim}{q_0}-\underset{\sim}{q}).\underset{\sim}{r_1}} + e^{i(\underset{\sim}{q_0}-\underset{\sim}{q_1}).\underset{\sim}{r_2}} \big]$$

and the correction consists of terms giving for example the effect of scattering first at $\underset{\sim}{r_1}$ and finally at $\underset{\sim}{r_2}$ after the wave has travelled n times between $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$. The form of W depends on the form of the scattering transition operator at one nucleon. Drell and Verlet work with three different assumptions about scattering at one source. One, also used by Brueckner, is the approximation we shall use below under the name of the "one pole" approximation. This gives $W = e^{iq_E R}/R$. The second assumes that scattering takes place only on the energy shell, and gives $W = i \sin q_E R / R$. We shall discuss these two cases in Section 3 when dealing with our own problem. The third case is that of a potential which is factorable in configuration space. That is, in the equation for scattering at one source,

$$(\nabla^2 + q^2)\,\psi(\underset{\sim}{\rho}) = \lambda \int \mathcal{U}(\underset{\sim}{\rho}, \underset{\sim}{\rho}')\,\psi(\underset{\sim}{\rho}')\,d\underset{\sim}{\rho}'$$

they take $\mathcal{U}(\underset{\sim}{\rho}, \underset{\sim}{\rho}') = u(\underset{\sim}{\rho})\,u(\underset{\sim}{\rho}')$ so that the equation is replaced by the inhomogeneous one,

$$(\nabla^2 + q^2)\,\psi(\underset{\sim}{\rho}) = \lambda\, u(\underset{\sim}{\rho})\,\overline{\Psi}$$

where $\overline{\Psi} = \int u(\underset{\sim}{\rho}')\,\psi(\underset{\sim}{\rho}')\,d\underset{\sim}{\rho}'$

This gives

$$W = \frac{\displaystyle\iint \frac{e^{iq_E|\underset{\sim}{\rho}-\underset{\sim}{\rho}'|}}{|\underset{\sim}{\rho}-\underset{\sim}{\rho}'|}\, u_1(\underset{\sim}{\rho})\, u_2(\underset{\sim}{\rho}')\, d\underset{\sim}{\rho}\, d\underset{\sim}{\rho}'}{\displaystyle\int e^{i\underset{\sim}{q_0}\cdot\underset{\sim}{\rho}}\, u_1(\underset{\sim}{\rho})\, d\underset{\sim}{\rho} \int e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}\, u_2(\underset{\sim}{\rho})\, d\underset{\sim}{\rho}}$$

which reduces to $e^{iq_E R}/R$ where the potentials $u_1(\underset{\sim}{\rho})$, $u_2(\underset{\sim}{\rho})$ do not overlap.

The results given by Drell and Verlet, for the particular case $q_E = 2.2\mu$, $\delta = 45^\circ$, backward scattering and source radius $1/2\mu$ in the third model, are that the ratio of the cross-sections with and without multiple scattering is about $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, taking the cases in the order given above. When double scattering alone is considered the ratio is about $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{3}$ in the three cases. In obtaining these results they use a deuterium wave function for the sources. Their results suggest that in a more realistic problem the form of the scattering amplitude off the energy shell will be important, and that double scattering will give a considerable part of the multiple scattering correction.

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Brueckner (1953a,b) has also studied the scattering of mesons by deuterium, assuming p wave scattering at each nucleon and ignoring spin flip. He finds a considerable reduction from the impulse approximation result. His work has been extended by Rockmore (1957) to the case of a scattering transition operator which is a function of nucleon spin. Where the results of these authors can be compared the correction is smaller in Rockmore's calculation. Rockmore finds that double scattering is important. Using the Born approximation for scattering at a free nucleon he estimates that for the elastic differential cross-section at meson energy 85 MeV. the contributions included and ignored in the one pole approximation are comparable.

An alternative approach to the problem of elastic meson scattering at deuterium is that of Bransden and Moorhouse (1958). They set up the meson-deuteron scattering equation, with the assumption that scattering at individual nucleons is in the $(3/2, 3/2)$ state only, and solve it using the variational principle used by Chew (1954). The equation is

$$\left\{E-\omega(q)+\frac{1}{2M}\nabla_1^2+\frac{1}{2M}\nabla_2^2-V(R)\right\}\Psi(\underset{\sim}{r_1},\underset{\sim}{r_2};i,\underset{\sim}{q})$$
$$=\int\frac{dq'}{(2\pi)^3}H(\underset{\sim}{r_1},\underset{\sim}{r_2};i,\underset{\sim}{q};j,\underset{\sim}{q}')\,\Psi(\underset{\sim}{r_1},\underset{\sim}{r_2};j,\underset{\sim}{q}') \qquad (23)$$

Here i and j are spin indices and $V(R)$ is the deuteron potential.

$$H=K_{33}(\underset{\sim}{r_1};i,\underset{\sim}{q};j,\underset{\sim}{q}')+K_{33}(\underset{\sim}{r_2};i,\underset{\sim}{q};j,\underset{\sim}{q}')$$

K_{33} being essentially our quantity V_{33} of (15). (23) is of

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the same form as (2) and so we have

$$t_{\beta a} \doteq \frac{(\Phi_\beta, H\,\Phi_a)^2}{(\Phi_\beta, H\Phi_a) - (\Phi_\beta, H\frac{1}{a}H\,\Phi_a)}$$

Φ_a, Φ_β being the product of the deuteron wave function and a plane wave meson wave function. The second term in the denominator includes multiple scattering. These authors find that the multiple scattering correction is less than 5% of the impulse approximation cross-section, and they obtain agreement with experiment at meson energies 85 MeV. and 140 MeV. They attribute the disagreement between their results and those of Rockmore to his use of the one pole approximation.

Chappelear (1955) has considered the elastic photoproduction of neutral pions at deuterium. He finds that for photon energy 285 MeV. the cross-section is reduced, at all angles, by 40% to 50%. His results are in agreement with the experiments of Rosengren and Baron (1956). We present in Sections 2 and 3 a modified form of Chappelear's method. Stoodley (1957) has extended the treatment of multiple scattering to the case of photoproduction at a system of more than two nucleons. His result for the matrix element reduces to that of Chappelear for deuterium. Stoodley calculates the correction to the elastic differential cross-section at 90%, for the production of neutral mesons at helium. Like Chappelear, he ignores spin flip scattering, and takes p wave scattering only. He also ignores charge exchange scattering, uses a special simple wave function, and excludes for simplicity certain sequences of multiple scattering. The correction is very large, and the

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experimental results of Bellamy et. al. (1957) lie between the impulse approximation and corrected results. It is of some interest to have the multiple scattering correction for an inelastic process, for comparison with the work on elastic scattering, and elastic π° photoproduction. Watson (1954) gives without details an estimate of 10% for the correction in the case we examine.

As we go to systems with a higher number of nucleons A, multiple scattering theory gives a set of A coupled integral equations. (Watson (1953)). Rather than attempt to solve these equations the method adopted is to transform the multiple scattering problem into that of scattering by a refractive medium. Some work has been done (Butler (1952), Laing and Moorhouse (1957)) on the photoproduction of mesons at complex nuclei, using such an optical model for the meson-nucleus interaction.

Charged meson photoproduction at deuterium.

We now turn to the impulse approximation calculations for the processes

$$\gamma + D \longrightarrow \pi^- + p + p$$

$$\gamma + D \longrightarrow \pi^+ + n + n$$

Probably the most important aspect of these processes is the ratio of π^- to π^+ production near threshold, because of its connection with the wave meson-nucleon scattering and the Panofsky ratio. (See for example Bethe and de Hoffmann (1955), section 33, and Cassels (1957)). However our work is not /

relevant to this, because we confine our attention to energies well above threshold. We expect multiple scattering to be unimportant near threshold because all the scattering phase shifts are small at low meson energies. So we shall not discuss further the papers in which the emphasis lies on the inclusion of the Coulomb interaction in the process

$$\gamma + D \rightarrow \pi^- + p + p$$

and which give results near threshold. (The most recent of these are the papers of Penner (1957) and Baldin (1958)).

There are several papers dealing with higher energies. In these the treatment of the final state is simpler. The Coulomb interaction of the meson with the protons is ignored, while that of the two protons is either ignored or taken into account roughly by using the Coulomb factor $\frac{2\pi e^2 M/k}{e^{2\pi e^2 M/k} - 1}$, which is an approximation for the ratio of the 2 proton wave function to the 2 neutron wave function at $R = 0$. Here M is the nucleon mass and k the relative momentum of the nucleons. Chew and Lewis (1952) use closure in summing over all final states, ignoring the fact that energy conservation restricts the available states, and overestimating the cross-section. Plane wave final states are used by Lax and Feshbach (1952) and by Saito et. al. (1952). This as we shall see can greatly underestimate the cross-section when k is small. Saito et.al. also presents result for a distorted S wave final state, as do Machida and Tamura (1951). We use a plane wave with the partial wave replaced by a distorted wave of the type used by these authors. (Compare Francis (1953) who deals with /

inelastic (π° production). Except at forward angles, as we shall see below, the accuracy with which the final state is described is less important in this energy range than near threshold, because we deal in general with larger values of Hagermann et. al. (1957), in the experimental work mentioned below, state that details of the final state interaction affect the interpretation of their results, and mention work by Tiemann using good wave functions.

<u>Comparison with experiment.</u>

In the papers of Saito et. al. and Machida and Tamura no absolute cross-sections are given. In the other two papers the starting point is the form $T_i = \underset{\sim}{K} \cdot \underset{\sim}{\sigma}_i + L$ for the photo-production transition operator at nucleon i and the aim is to obtain by comparison with experiment the ratio $|\underset{\sim}{K}|^2 / |L|^2$, averaged over $\underset{\sim}{\varepsilon}$. In the experimental work the convenient quantity to measure is the ratio of the cross-sections for positive pions, at a particular angle and energy, from deuterium and hydrogen. Because a ratio is measured the absolute accuracy of the experiments is not important. Taking as a typical case the work of Hagermann et. al. (1957) the cross-sections measured were for pions of around 75 MeV. kinetic energy, the energy spread being 15 MeV., from carbon, ethylene and deuterated ethylene, the last two being corrected for the pions produced from carbon. 350 MeV. bremsstrahlung radiation was used.

In comparing the experimental results with the predictions of the impulse approximation the following difficulty arises.

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We do not have a unique relation between photon energy, meson energy, and meson angle in the deuterium case. Also we have, not a monochromatic photon beam, but a bremsstrahlung spectrum. The impulse approximation calculation leads to an expression for the ratio

$$\left\{ \frac{d^2\sigma(\Theta_q, \omega(q), \nu)}{d\omega(q)\, d\Omega_q} \right\}_D \Bigg/ \left\{ \frac{d^2\sigma(\Theta_q, \nu)}{d\Omega_q} \right\}_H$$

for a particular photon momentum ν and meson angle Θ_q, which determine the value of $\omega(q)$ in the hydrogen case. In the paper of White et. al. (1952) two methods are suggested for comparing this ratio with the experimental results. One is to assume that the energy spectrum of mesons from deuterium is very narrow, and is centred on the line spectrum of mesons from hydrogen. Then only photons of one energy will contribute to the mesons detected at a particular angle and with a particular energy. As these authors point out, and as we shall see in Section 7, the assumption of a narrow energy spectrum is unsound.

The alternative method, which is generally adopted, for example in the papers quoted above and in that of Lebow et. al. (1952), is to integrate $\left\{ \frac{d^2\sigma(\Theta_q, \omega(q), \nu)}{d\omega(q)\, d\Omega_q} \right\}_D$ over the bremsstrahlung spectrum, keeping $\omega(q)$ fixed. This gives the upper term of the ratio which is in fact observed. It is assumed in these papers that $|\underset{\sim}{K}|^2 / |L|^2$ is not strongly dependent on ν. The graph I shows the results of Hagermann et. al. to indicate the accuracy of this kind of

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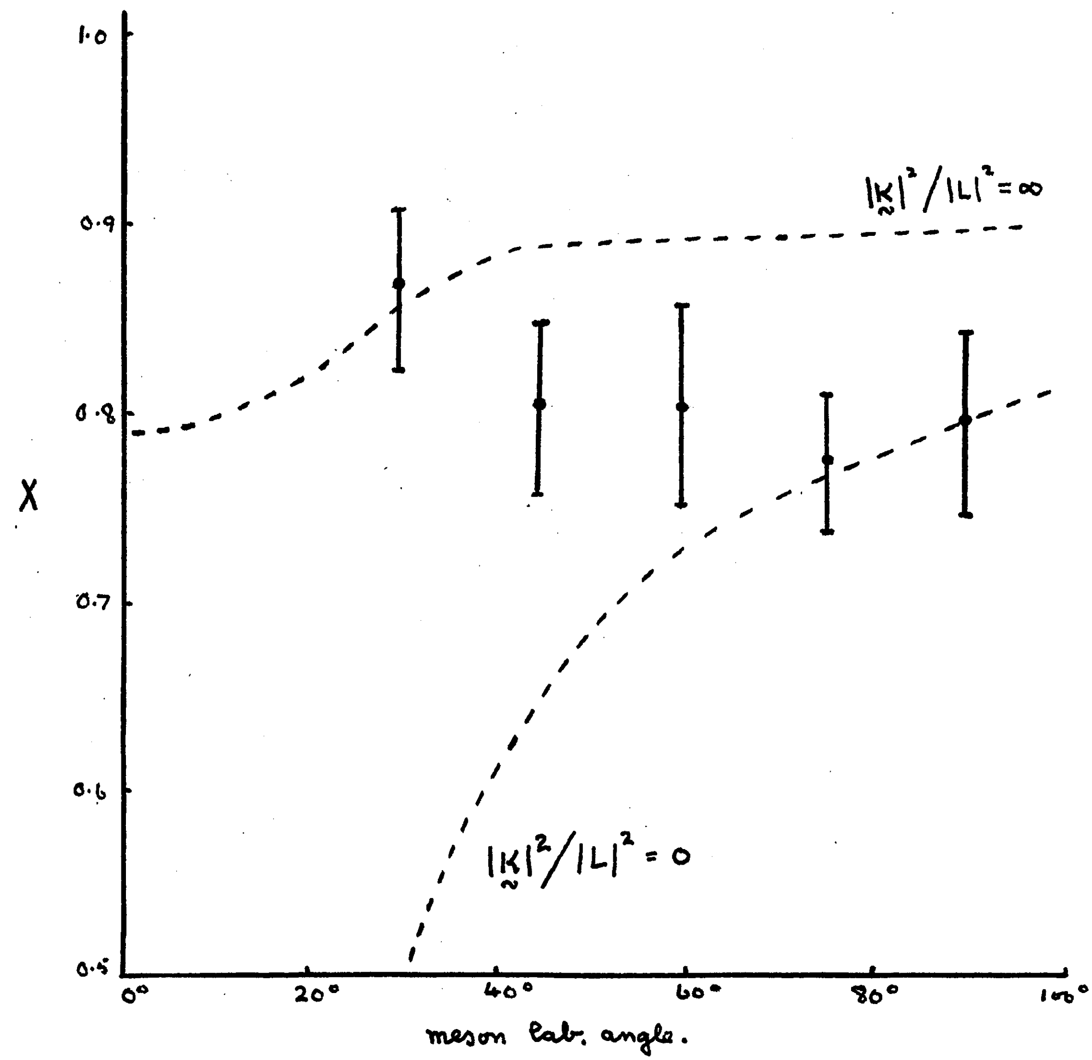


$$X = \frac{d^2\sigma(D)}{d\omega(q)\,d\Omega_q} \div \frac{d^2\sigma(H)}{d\omega(q)\,d\Omega_q}$$

work. The theoretical values they use are calculated from the results of Chew and Lewis. They consider their results consistent with the form of $\underset{\sim}{K}$, $\underset{\sim}{L}$ derived from (22). We can see from (22) that $\underset{\sim}{L}$ must have the form $A\,\underset{\sim}{\nu}.\underset{\sim}{q}\times\underset{\sim}{\varepsilon}$ where A is a scalar, and so $|L|^2 \to 0$ as the angle between $\underset{\sim}{\nu}$ and $\underset{\sim}{q}$ decreases, a result which is consistent with the graph I .

In the case of negative mesons a method which has been adopted (Bandtel et. al. (1958)), involves the measurement of the energy and the direction of one of the two recoiling protons as well as the meson. This has the advantage that the photon energy can be fixed. Also they can distinguish between the cases of low and high energy of relative motion of the nucleons. It is the case of low relative momentum of the nucleons which we shall find most interesting. We discuss in Section 7 the possibility of detecting the effect of multiple scattering in these two types of experiment.

2. Formal multiple scattering theory in a two nucleon system.

The method we use here is derived from the methods used by Chappelear and Stoodley. We shall point out how it differs from these methods and why we do not make use of one or the other of them in its original form. We make use of time-independent scattering theory, as outlined in Section 1, to obtain the transition operator for processes which can occur in a system of two nucleons interacting with a meson field and the radiation field. The Hamiltonian of the system is

$$\mathbb{H} = H_0 + \mathcal{H} \tag{1}$$

where $$\mathcal{H} = h + H = h_1 + h_2 + H_1 + H_2 \tag{2}$$

h_i is the interaction term between nucleon i and the meson field. H_i is the term arising from the interaction of the radiation field with the meson and nucleon currents at nucleon i. H_0 is the sum of the free field Hamiltonians. With $a = E - H_0 + i\varepsilon$ as before the transition operators T_i for production of a meson by a photon incident on nucleon i, t_i for the interaction of the meson field with nucleon i and T for processes involving the whole system, are given by

$$T_i = h_i + H_i + (h_i + H_i)\frac{1}{a - h_i - H_i}(h_i + H_i) \tag{3}$$

$$t_i = h_i + h_i \frac{1}{a - h_i} h_i \tag{4}$$

$$T = \mathcal{H} + \mathcal{H}\frac{1}{a - \mathcal{H}}\mathcal{H} \tag{5}$$

For states with one meson present we use the approximation

$$(\underset{\sim}{q_1}|\frac{1}{a}|\underset{\sim}{q_2}) = (2\pi)^3 \delta(\underset{\sim}{q_1} - \underset{\sim}{q_2})\frac{1}{a(q_2)} \tag{6}$$

where $$a(q) = \omega(q_E) - \omega(q) + i\varepsilon \tag{7}$$

This means that we neglect the nucleon kinetic energy, and that (3) and (4) refer to processes at a free nucleon. As in Section 1 we have effectively

$$t_i = t_{si} = a \frac{1}{a-V_i} V_i \tag{8}$$

with $V_i = h_i \frac{1}{a} h_i$.

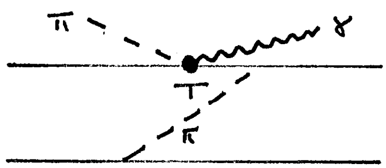
We treat H as a small perturbation, and write (5) to first order in H as

$$T \doteq a \frac{1}{a-h} H \frac{1}{a-h} a \tag{9}$$

In terms of the T_i this is

$$T = a \frac{1}{a-h} \left\{ (a-h_1) \frac{1}{a} T_1 \frac{1}{a} (a-h_1) + (a-h_2) \frac{1}{a} T_2 \frac{1}{a} (a-h_2) \right\} \frac{1}{a-h} a \tag{10}$$

The factors to the right of T_1 and T_2 are set equal to one. Some of the virtual processes represented by these factors are meson exchanges in the initial state, and we expect to take them into account by our deuteron wave function. Other processes ignored are such as



(Here and elsewhere we adopt the convention that the graph reads from right to left, to agree with the order of operators in the relevant formula). The possibility of such processes makes the use of the free nucleon form of T_i incorrect. We have another similar approximation below.

We now have

$$T = \sum_i y_i T_i \tag{11}$$

where $y_i = a\frac{1}{a-h}(a-h_i)\frac{1}{a}$

$$= a\frac{1}{a-V}\left(1-h\frac{1}{a}h_i\frac{1}{a}\right) + a\frac{1}{a-V}\left(h\frac{1}{a}-h_i\frac{1}{a}\right)$$

V being $V_1 + V_2 + h_1\frac{1}{a}h_2 + h_2\frac{1}{a}h_1$. Just as we drop t_{ai} from t_i so we can drop the second term of y_i and use

$$(\underset{\sim}{q}|T|\underset{\sim}{\nu}) = (\underset{\sim}{q}|\sum_i x_i T_i|\underset{\sim}{\nu}) \qquad (11')$$

where

$$x_i = a\frac{1}{a-V}\left(1-h\frac{1}{a}h_i\frac{1}{a}\right)$$

and $|\underset{\sim}{\nu})$, $|\underset{\sim}{q})$ are states of a deuteron and a photon of momentum $\underset{\sim}{\nu}$, and of two nucleons and a meson of momentum $\underset{\sim}{q}$. The part of y_i which we leave out here contributes to deuteron photodisintegration by way of a virtual meson which is scattered and finally absorbed.

We make the further approximation of setting $V = V_1 + V_2$ and $h\frac{1}{a}h_i = V_i$ in x_i . This implies that the only meson present at any stage is that produced by T_i , which suffers a succession of scatterings at the nucleon. We ignore absorption of a meson at one nucleon followed by emission of a meson at the other nucleon. We also ignore the meson exchanges which give the nuclear force. Rockmore (1957) has made an estimate of the effect of the nuclear force in the scattering of mesons at deuterium. Following Chew and Goldberger (1952) he gives a first order correction to the impulse approximation, for the effect of the one meson exchange potential. He finds that at 85 MeV meson energy the correction to the total scattering cross-section is about - 5%. The inclusion of

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binding more completely has not been attempted, but Laing (1953) has made a start by calculating the scattering of a meson at a single nucleon bound in a central potential.

We can now express χ_i in terms of t_{s1} and t_{s2}, since we only have V_1 and V_2 in χ_i. The result is given by Stoodley in the form

$$\chi_i = \frac{1}{-1+Z_1+Z_2} Z_i \tag{12}$$

$$Z_i = 1 - V_i \frac{1}{a} = \left(1+t_{si}\frac{1}{a}\right)^{-1} \tag{13}$$

For A nucleons he obtains the equation

$$\chi_i = \frac{1}{1-A+\sum_j Z_j} Z_i \tag{12'}$$

To obtain the matrix element of T he solves successively (13) and (12'). His method makes it possible to deal with $A > 2$, because (12') is linear in the Z_j, but it is rather clumsy when $A = 2$, compared with our method which is to substitute (13) in (12), giving

$$\chi_i = \left[-1+\left(1+t_{si}\frac{1}{a}\right)^{-1}+\left(1+t_{s\not{i}}\frac{1}{a}\right)^{-1}\right]^{-1}\left(1+t_{si}\frac{1}{a}\right)^{-1} \tag{12''}$$

where we introduce the notation $\not{i} = \left\{\begin{matrix}1\\2\end{matrix}\right.$ when $i = \left\{\begin{matrix}2\\1\end{matrix}\right.$

Therefore
$$\chi_i = \left[\frac{1}{1+t_{si}\frac{1}{a}}\left\{-\left(1+t_{si}\frac{1}{a}\right)\left(1+t_{s\not{i}}\frac{1}{a}\right)+\left(1+t_{si}\frac{1}{a}\right)+\left(1+t_{s\not{i}}\frac{1}{a}\right)\right\}\frac{1}{1+t_{s\not{i}}\frac{1}{a}}\right]^{-1}\left(1+t_{si}\frac{1}{a}\right)^{-1}$$

$$= \left(1+t_{s\not{i}}\frac{1}{a}\right)\left[1-t_{si}\frac{1}{a}t_{s\not{i}}\frac{1}{a}\right]^{-1} \tag{14}$$

We shall work from this equation. Further manipulation /

of (14) gives us

$$T = \sum_i a\left[1 - \frac{1}{a}t_{si}\frac{1}{a}t_{s\not{i}}\right]^{-1}\left[\frac{1}{a}T_i + \frac{1}{a}t_{si}\frac{1}{a}T_{\not{i}}\right] \qquad (14')$$

In Chappelear's paper the form

$$T = \sum_i a\left[1 - \frac{1}{a}t_i\frac{1}{a}t_{\not{i}}\right]^{-1}\left[\frac{1}{a}T_i + \frac{1}{a}t_i\frac{1}{a}T_{\not{i}}\right] \qquad (14'')$$

is derived from (10). He then assumes that at all successive stages in the process the only meson present is that produced by T_1 or T_2, so that the matrix element of t_i required is always that between one meson states $|\underset{\sim}{q})$. Now $(\underset{\sim}{q}_1|t_i|\underset{\sim}{q}_2) = (\underset{\sim}{q}_1|t_{si}|\underset{\sim}{q}_2)$ so that (14') and (14") are identical.

3. Multiple scattering in the one pole approximation.

We use the form $(\underset{\sim}{q_1}|t_{si}|\underset{\sim}{q_2}) = a_i\, b(q_1, q_2)\, \underset{\sim}{q_1}\cdot\underset{\sim}{q_2}\, e^{i(\underset{\sim}{q_2}-\underset{\sim}{q_1})\cdot\underset{\sim}{r_i}}$ (1)

Here a_i is the isotopic spin projection operator which ensures that scattering is only in the $t = 3/2$ state of the meson and nucleon i. We consider only p wave scattering which means we must confine our attention to mesons with sufficient energy for the p wave resonance to dominate the scattering. We use for $b(q_E, q_E)$ the form given by (1.12) using the $(3/2, 3/2)$ phase shift δ_{33} of (1.17) but we ignore in (1) the spin dependence of the scattering.

We first obtain the matrix element of $t_{si}\frac{1}{a}t_{sj}$. From (1) and (2.6), (2.7) this has the form

$$(\underset{\sim}{q_1}|t_{si}\tfrac{1}{a}t_{sj}|\underset{\sim}{q_2}) = a_i a_j e^{i(\underset{\sim}{q_2}\cdot\underset{\sim}{r_j} - \underset{\sim}{q_1}\cdot\underset{\sim}{r_i})} \int \frac{d\underset{\sim}{q}}{(2\pi)^3} \frac{b(q_1,q)\, b(q,q_2)\; \underset{\sim}{q}\cdot\underset{\sim}{q_1}\; \underset{\sim}{q}\cdot\underset{\sim}{q_2}}{\omega(q_E)-\omega(q)+i\varepsilon}\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{R}} \qquad (2)$$

$$= -a_i a_j e^{i(\underset{\sim}{q_2}\cdot\underset{\sim}{r_j} - \underset{\sim}{q_1}\cdot\underset{\sim}{r_i})}\; \underset{\sim}{q_1}\cdot\nabla_R\; \underset{\sim}{q_2}\cdot\nabla_R \left(\frac{1}{2\pi}\right)^2 \frac{1}{iR}\int_0^\infty dq\, \frac{b(q_1,q)\, b(q,q_2)\, q\,(e^{iqR}-e^{-iqR})}{\omega(q_E)-\omega(q)+i\varepsilon}$$

(3)

Here $\underset{\sim}{R} = \underset{\sim}{r_i} - \underset{\sim}{r_j}$, $R = |\underset{\sim}{R}|$. The one pole approximation is that on changing the integral in (3) to the form

$$\int_{-\infty}^{\infty} \frac{dq\, q\, b(q_1,q)\, b(q,q_2)\, e^{iqR}}{\omega(q_E)-\omega(q)+i\varepsilon}$$

and completing the contour in the upper half plane, the only contribution is from the pole at $q = q_E$. Thus we assume that the product $b(q_1,q)\, b(q,q_2)$ is even in q and has no poles for q in the upper half plane. We can compare these

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conditions with the restrictions on the form of $\mathscr{b}$ implicit in the work of Chappelear and Stoodley. In Chappelear's paper the form of t_{si} is

$$(\underset{\sim}{q_1}|t_{si}|\underset{\sim}{q_2}) = \mathscr{b}_i \, \underset{\sim}{q_1}\cdot\underset{\sim}{q_2} \, e^{i(\underset{\sim}{q_2}-\underset{\sim}{q_1})\cdot\underset{\sim}{r_i}}$$

in which $\mathscr{b}_i$ is a function of energy or in our notation

$$\mathscr{b}(q_1, q_2) = \mathscr{b}(q_E) .$$

However for the integral of (3) he has the form

$$\int_{-\infty}^{\infty} \frac{dq \, \mathscr{b}_1 \mathscr{b}_2 \, e^{iqR}}{\omega(q_E) - \omega(q) + i\varepsilon}$$

and states that he ignores poles of $\mathscr{b}_1$ and $\mathscr{b}_2$. So it appears that he is in fact using the same form of $\mathscr{b}$ as we use. In Stoodley's thesis the form of $\mathscr{b}$ is $\mathscr{b}(q_1, q_2) = \mathscr{b}(q_2)$. His method can also be employed using $\mathscr{b}(q_1, q_2)$ but then to solve (2.13) he has to make the one pole approximation in an integral of the form

$$\int_0^{\infty} \frac{dq \, q^4 \, \mathscr{b}(q_1, q) \, \mathscr{b}(q, q_2)}{\omega(q_E) - \omega(q) + i\varepsilon} \qquad (4)$$

This does not contain a factor $\sin qR$, because (2.13) only involves one nucleon, and so the approximation requires more restrictions on the form of $\mathscr{b}$ than in our treatment.

The approximation used here is referred to in various papers, for example those of Chappelear and Rockmore, as corresponding to the neglect of scattering off the energy shell. This is incorrect. If we transform the integral in (3) into

$$2i \int_{\mu}^{\infty} d\omega \, \frac{\omega \sin qR \, \mathscr{b}(q_1, q) \, \mathscr{b}(q, q_2)}{\omega(q_E) - \omega(q) + i\varepsilon} \qquad (3')$$

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and use the result

$$\int \frac{d\omega\, F(q)}{\omega(q_E) - \omega(q) + i\varepsilon} = -\pi i\, F(q_E) + P\int \frac{d\omega\, F(q)}{\omega(q_E) - \omega(q)} \qquad (5)$$

then ignoring scattering off the energy shell means ignoring the principal value integral. This gives

$$2\pi\, \omega(q_E)\, \ell(q_1, q_E)\, \ell(q_E, q_2)\, \sin q_E R$$

while our approximation gives

$$-2\pi i\, \omega(q_E)\, \ell(q_1, q_E)\, \ell(q_E, q_2)\, e^{i q_E R}.$$

The difference between these approximations is recognised by Drell and Verlet (in the work mentioned in Section 1). We have not been able to relate the assumptions of the one pole theory to any physical property of meson-nucleon scattering. We shall see below that in this approximation we only require the energy shell values $\ell(q_E, q_E)$ in our final result.

Continuing from (3) we have

$$(\underset{\sim}{q_1}|\, t_{si} \tfrac{1}{a} t_{sj}\, |\underset{\sim}{q_2}) = a_i a_j\, e^{i(\underset{\sim}{q_2}\cdot\underset{\sim}{r_j} - \underset{\sim}{q_1}\cdot\underset{\sim}{r_i})}\, \ell(q_1, q_E)\, \ell(q_E, q_2) \left\{ f(R)\, \underset{\sim}{q_1}\cdot\underset{\sim}{q_2} + g(R)\, \underset{\sim}{q_1}\cdot\underset{\sim}{R}\; \underset{\sim}{q_2}\cdot\underset{\sim}{R} \right\} \qquad (6)$$

where

$$f(R) = \frac{\omega(q_E)}{2\pi} \frac{1}{R} \frac{d}{dR} \left(\frac{e^{i q_E R}}{R} \right) \qquad (7)$$

and $$g(R) = \frac{1}{R} \frac{d}{dR} f(R).$$

We shall also use the notation $h(R) = f(R) + g(R) R^2$. From (2.14) we have

$$(\underset{\sim}{q_1}|\, \chi_i\, |\underset{\sim}{q_2}) = (2\pi)^3\, \delta(\underset{\sim}{q_1} - \underset{\sim}{q_2}) + a_j \frac{\ell(q_1, q_2)}{a(q_2)}\, \underset{\sim}{q_1}\cdot\underset{\sim}{q_2}\, e^{i(\underset{\sim}{q_2} - \underset{\sim}{q_1})\cdot \underset{\sim}{r_j}} \quad +$$

$$e^{i\underset{\sim}{q_2}\cdot\underset{\sim}{r_{\dot{x}}}}\int\frac{d\underset{\sim}{q}}{(2\pi)^3}(\underset{\sim}{q_1}|x_i|q)\frac{b(q,q_E)b(q_E,q_2)}{a(q_2)} - a_i a_{\dot{x}} e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r_i}}\{f(R)\underset{\sim}{q}\cdot\underset{\sim}{q_2}+g(R)\underset{\sim}{q}\cdot\underset{\sim}{R}\,\underset{\sim}{q_2}\cdot\underset{\sim}{R}\}$$

$$= (2\pi)^3\,\delta(\underset{\sim}{q_1}-\underset{\sim}{q_2}) + a_{\dot{x}}\frac{b(q_1,q_2)}{a(q_2)}\underset{\sim}{q_1}\cdot\underset{\sim}{q_2}\,e^{i(\underset{\sim}{q_2}-\underset{\sim}{q_1})\cdot\underset{\sim}{r_{\dot{x}}}}$$

$$+ e^{i\underset{\sim}{q_2}\cdot\underset{\sim}{r_{\dot{x}}}}\{f(R)\underset{\sim}{q_2}+g(R)\underset{\sim}{q_2}\cdot\underset{\sim}{R}\,\underset{\sim}{R}\}\cdot\underset{\sim}{S_i}(\underset{\sim}{q_1})\,a_i a_{\dot{x}}\frac{b(q_E,q_2)}{a(q_2)} \qquad (8)$$

where $$\underset{\sim}{S_i}(\underset{\sim}{q_1}) = \int\frac{d\underset{\sim}{q}}{(2\pi)^3}(\underset{\sim}{q_1}|x_i|\underset{\sim}{q})\,b(q,q_E)\,\underset{\sim}{q}\,e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r_i}} \qquad (9)$$

We now solve the equation for $\underset{\sim}{S_i}(\underset{\sim}{q_1})$ which follows from (8), (9),

$$\underset{\sim}{S_i}(\underset{\sim}{q_1}) = \int\frac{d\underset{\sim}{q}}{(2\pi)^3}\Big[(2\pi)^3\delta(\underset{\sim}{q_1}-\underset{\sim}{q}) + a_{\dot{x}}\frac{b(q_1,q)}{a(q)}\underset{\sim}{q_1}\cdot\underset{\sim}{q}\,e^{i(\underset{\sim}{q}-\underset{\sim}{q_1})\cdot\underset{\sim}{r_{\dot{x}}}}$$

$$+ e^{i\underset{\sim}{q}\cdot\underset{\sim}{r_{\dot{x}}}}\frac{b(q_E,q)}{a(q)}\{f(R)\underset{\sim}{q}+g(R)\underset{\sim}{q}\cdot\underset{\sim}{R}\,\underset{\sim}{R}\}\cdot\underset{\sim}{S_i}(\underset{\sim}{q_1})\,a_i a_{\dot{x}}\Big]\,b(q,q_E)\,\underset{\sim}{q}\,e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r_i}} \qquad (10)$$

Making the same approximation in the integral,

$$\underset{\sim}{S_i}(\underset{\sim}{q_1}) = b(q_1,q_E)\,\underset{\sim}{q_1}\,e^{-i\underset{\sim}{q_1}\cdot\underset{\sim}{r_i}}$$

$$+ a_{\dot{x}}\,b(q_1,q_E)\,b(q_E,q_E)\{f(R)\underset{\sim}{q_1}+g(R)\underset{\sim}{q_1}\cdot\underset{\sim}{R}\,\underset{\sim}{R}\}\,e^{-i\underset{\sim}{q_1}\cdot\underset{\sim}{r_{\dot{x}}}} \qquad (11)$$

$$+ b^2(q_E,q_E)\{f^2(R)\,\underset{\sim}{S_i}(\underset{\sim}{q_1}) + g(R)[h(R)+f(R)]\,\underset{\sim}{S_i}(\underset{\sim}{q_1})\cdot\underset{\sim}{R}\,\underset{\sim}{R}\}\,a_i a_{\dot{x}}$$

Taking the scalar product with $\underset{\sim}{R}$,

$$\underset{\sim}{S_i}(\underset{\sim}{q_1})\cdot\underset{\sim}{R} = \underset{\sim}{q_1}\cdot\underset{\sim}{R}\;b(q_1,q_E)\;\Big[e^{-i\underset{\sim}{q_1}\cdot\underset{\sim}{r_i}} + a_{\dot{x}}\,e^{-i\underset{\sim}{q_1}\cdot\underset{\sim}{r_{\dot{x}}}}\,b(q_E,q_E)\,h(R)\Big]$$

$$\times\;\Big[1 - b^2(q_E,q_E)\,h^2(R)\,a_i a_{\dot{x}}\Big]^{-1}$$

and hence we obtain

$$\underset{\sim}{S}_i(\underset{\sim}{q}_1) = \Big[\underset{\sim}{q}_1 \, \mathcal{B}(q_1, q_E) \{ e^{-i \underset{\sim}{q}_1 \cdot \underset{\sim}{r}_i} + a_{\dot{x}} f(R, \mathcal{B}(q_E, q_E)) e^{-i \underset{\sim}{q}_1 \cdot \underset{\sim}{r}_{\dot{x}}} \}$$

$$+ \underset{\sim}{q}_1 \cdot \underset{\sim}{R} \underset{\sim}{R} \, \mathcal{B}(q_1, q_E) \mathcal{B}(q_E, q_E) \, g(R) \{ a_{\dot{x}} e^{-i \underset{\sim}{q}_1 \cdot \underset{\sim}{r}_{\dot{x}}} + [f(R) + h(R)] \mathcal{B}(q_E, q_E) \times$$

$$\left(e^{-i \underset{\sim}{q}_1 \cdot \underset{\sim}{r}_i} + a_{\dot{x}} e^{-i \underset{\sim}{q}_1 \cdot \underset{\sim}{r}_{\dot{x}}} \mathcal{B}(q_E, q_E) h(R) \right)\left(1 - \mathcal{B}^2(q_E, q_E) h^2(R) a_i a_{\dot{x}} \right)^{-1} a_i a_{\dot{x}} \} \Big] \qquad (12)$$

$$\times \left[1 - \mathcal{B}^2(q_E, q_E) f^2(R) a_i a_{\dot{x}} \right]^{-1}$$

<u>**The matrix element of T.**</u>

From (2.11') we have

$$(\underset{\sim}{q}_1 | T | \underset{\sim}{\nu}) = \sum_i \int \frac{d\underset{\sim}{q}_2}{(2\pi)^3} (\underset{\sim}{q}_1 | \chi_i | \underset{\sim}{q}_2)(\underset{\sim}{q}_2 | T_i | \underset{\sim}{\nu}) \qquad (13)$$

We take $(\underset{\sim}{q}_2 | T_i | \underset{\sim}{\nu})$ **in the form**

$$\{ \alpha_i(q_2) + \underset{\sim}{\gamma}_i(q_2) \cdot \underset{\sim}{q}_2 \} \, e^{i(\underset{\sim}{\nu} - \underset{\sim}{q}_2) \cdot \underset{\sim}{r}_i} \qquad (14)$$

α_i **and** $\underset{\sim}{\gamma}_i$ **are functions of the meson and photon energies, the photon polarisation and nucleon spin, and also contain isotopic spin operators.** (See Section 5). **With this form of** T_i **we have**

$$(\underset{\sim}{q}_1 | \chi_i T_i | \underset{\sim}{\nu}) = e^{i \underset{\sim}{\nu} \cdot \underset{\sim}{r}_i} \int \frac{d\underset{\sim}{q}_2}{(2\pi)^3} (\underset{\sim}{q}_1 | \chi_i | \underset{\sim}{q}_2) \{ \alpha_i(q_2) + \underset{\sim}{\gamma}_i(q_2) \cdot \underset{\sim}{q}_2 \} e^{-i \underset{\sim}{q}_2 \cdot \underset{\sim}{r}_i}$$

We write $\alpha_i(q_2)$, $\underset{\sim}{\gamma}_i(q_2)$ **in the form** $\bar{\alpha}_i \, \alpha(q_2)$, $\bar{\underset{\sim}{\gamma}}_i \, \gamma(q_2)$ **and define quantities** P_i **and** $\underset{\sim}{Q}_i$ **by**

$$P_i(q_1) = \int \frac{dq_2}{(2\pi)^3} (q_1 | r_i | q_2) e^{-i q_2 \cdot \rho_i} \alpha(q_2)$$

$$Q_i(q_1) = \int \frac{dq_2}{(2\pi)^3} (q_1 | r_i | q_2) e^{-i q_2 \cdot \rho_i} q_2 \gamma(q_2) \qquad (15)$$

so that

$$(q_1 | T | \nu) = \sum_i e^{i \nu \cdot \rho_i} \left[P_i \bar{\alpha}_i + Q_i \cdot \bar{\gamma}_i \right] \qquad (16)$$

We find P_i and Q_i in terms of S_i and use (12) to obtain the final form of $(q_1 | T | \nu)$. From (8) and (15) we have

$$P_i(q_1) = \Big[e^{-i q_1 \cdot \rho_i} + i a_{\bar{i}}\, b(q_E, q_E) f(R)\, q_1 \cdot R\, e^{-i q_1 \cdot \rho_{\bar{i}}} \qquad (17)$$

$$+ i\, b(q_E, q_E) f(R) h(R)\, S_i(q_1) \cdot R \; a_i a_{\bar{i}} \Big] \alpha(q_E)$$

and

$$Q_i(q_1) = S_i(q_1) \frac{\gamma(q_E)}{b(q_E, q_E)} \qquad (18)$$

where we have written $q_1 = q_E$, since q_1 is the final momentum of the meson.

It will be noticed here that the first terms of P_i and Q_i (from (12)) give the impulse approximation when put into (16). We have used the one pole approximation in the integrals (15), with consequent restrictions on the forms of $\alpha(q_2)$ and $\gamma(q_2)$.

Our final result for the matrix element is of the form

$$(q_1 | T | \nu) = (q_1 | T_{I.A.} | \nu) + \sum_{i=1}^{2} \sum_{j=1}^{2} e^{i(\nu \cdot \rho_i - q_1 \cdot \rho_j)} X_{ij} \qquad (19)$$

in which the impulse approximation matrix element, and the corrections for multiple scattering an odd or an even /

number of times, are displayed separately. The full expression is

$$(\boldsymbol{q}_1|T|\boldsymbol{\nu}) = (q_1|T_{IA.}|\nu) + \sum_{i=1}^{2} e^{i(\boldsymbol{\nu}-\boldsymbol{q})\cdot\boldsymbol{r}_i}\, a_i a_{\bar{i}}\, \boldsymbol{q}_1 \cdot$$

$$\left[-\frac{i\,h(R)f(R)b_E^2}{1-a_i a_{\bar{i}} h^2(R) b_E^2}\,\boldsymbol{R}\,\alpha_i + \frac{f^2(R)\,b_E^2}{1-a_i a_{\bar{i}} f^2(R) b_E^2}\,\boldsymbol{\gamma}_i + \frac{g(R)f(R)\,b_E^2}{1-a_i a_{\bar{i}} f^2(R) b_E^2}\,\boldsymbol{\gamma}_i\cdot\boldsymbol{R}\boldsymbol{R} \right.$$

$$\left. + \frac{g(R)h(R)b_E^2}{1-a_i a_{\bar{i}} h^2(R) b_E^2}\,\boldsymbol{\gamma}_i\cdot\boldsymbol{R}\boldsymbol{R} + \frac{f(R)\,b_E}{1-a_i a_{\bar{i}} f^2(R) b_E^2}\, a_i a_{\bar{i}}\, \frac{g(R)[f(R)+h(R)]h(R)b_E^3}{1-a_i a_{\bar{i}} h^2(R) b_E^2} \right]$$

$$+ \sum_{i=1}^{2} e^{i(\boldsymbol{\nu}\cdot\boldsymbol{r}_i - \boldsymbol{q}_1\cdot\boldsymbol{r}_{\bar{i}})}\, a_{\bar{i}}\, \boldsymbol{q}_1 \cdot \left[-\frac{i f(R)\, b_E}{1-a_i a_{\bar{i}} h^2(R) b_E^2}\,\boldsymbol{R}\,\alpha_i + \frac{f(R)\,b_E}{1-a_i a_{\bar{i}} f^2(R)\, b_E^2}\,\boldsymbol{\gamma}_i \right.$$

$$\left. + \frac{g(R)\,b_E}{1-a_i a_{\bar{i}} f^2(R) b_E^2}\,\boldsymbol{\gamma}_i\cdot\boldsymbol{R}\boldsymbol{R} + \frac{1}{1-a_i a_{\bar{i}} f^2(R) b_E^2}\, a_i a_{\bar{i}}\, \frac{g(R)[f(R)+h(R)]h(R)\,b_E^3}{1-a_i a_{\bar{i}}\, h^2(R)\, b_E^2} \right]$$

where we use the abbreviation $b_E = b(q_E, q_E)$. When we use the results of Section 5 and the notation A_i, $\boldsymbol{C}_i$ given there for the parts of α_i, $\boldsymbol{\gamma}_i$ which are independent of isotopic spin, we have for positive (negative) mesons the results

$$-(+)\,(\boldsymbol{q}_1|T|\boldsymbol{\nu}) = e^{i(\boldsymbol{\nu}-\boldsymbol{q}_1)\cdot\boldsymbol{r}_1}\left[-\frac{A_1}{\sqrt{2}} - \frac{\boldsymbol{q}_1\cdot\boldsymbol{C}_1}{2}\right] + e^{i(\boldsymbol{\nu}-\boldsymbol{q}_1)\cdot\boldsymbol{r}_2}\left[\frac{A_2}{\sqrt{2}} + \frac{\boldsymbol{q}_1\cdot\boldsymbol{C}_2}{2}\right]$$

$$+ e^{i(\boldsymbol{\nu}-\boldsymbol{q}_1)\cdot\boldsymbol{r}_1}\,\boldsymbol{q}_1\cdot\left[-\frac{1}{\sqrt{2}}F_1(R)A_1\boldsymbol{R} - \frac{1}{2}F_2(R)\,\boldsymbol{C}_1 - \frac{1}{2}\frac{F_3(R)}{R^2}\,\boldsymbol{C}_1\cdot\boldsymbol{R}\boldsymbol{R}\right]$$

(21)

+

$$+ e^{i(\underset{\sim}{\nu}-\underset{\sim}{q_1})\cdot \underset{\sim}{r_2}} \underset{\sim}{q_1}\cdot\left[-\frac{1}{\sqrt{2}} F_1(R)\, A_2 \underset{\sim}{R} + \frac{1}{2} F_2(R) \underset{\sim}{C_2} + \frac{1}{2} \frac{F_3(R)}{R^2} \underset{\sim}{C_2}\cdot\underset{\sim}{R}\,\underset{\sim}{R}\right]$$

$$+ e^{i(\underset{\sim}{\nu}\cdot\underset{\sim}{r_1} - \underset{\sim}{q_1}\cdot\underset{\sim}{r_2})} \underset{\sim}{q_1}\cdot\left[\frac{1}{\sqrt{2}} F_4(R)\, A_1 \underset{\sim}{R} + \frac{1}{2} F_5(R) \underset{\sim}{C_1} + \frac{1}{2} \frac{F_6(R)}{R^2} \underset{\sim}{C_1}\cdot\underset{\sim}{R}\,\underset{\sim}{R}\right]$$

$$+ e^{i(\underset{\sim}{\nu}\cdot\underset{\sim}{r_2} - \underset{\sim}{q_1}\cdot\underset{\sim}{r_1})} \underset{\sim}{q_1}\cdot\left[\frac{1}{\sqrt{2}} F_4(R)\, A_2 \underset{\sim}{R} - \frac{1}{2} F_5(R) \underset{\sim}{C_2} - \frac{1}{2} \frac{F_6(R)}{R^2} \underset{\sim}{C_2}\cdot\underset{\sim}{R}\,\underset{\sim}{R}\right]$$

where $\underset{\sim}{R}$ now stands for $\underset{\sim}{r_1} - \underset{\sim}{r_2}$, and the functions $F(R)$ are defined by

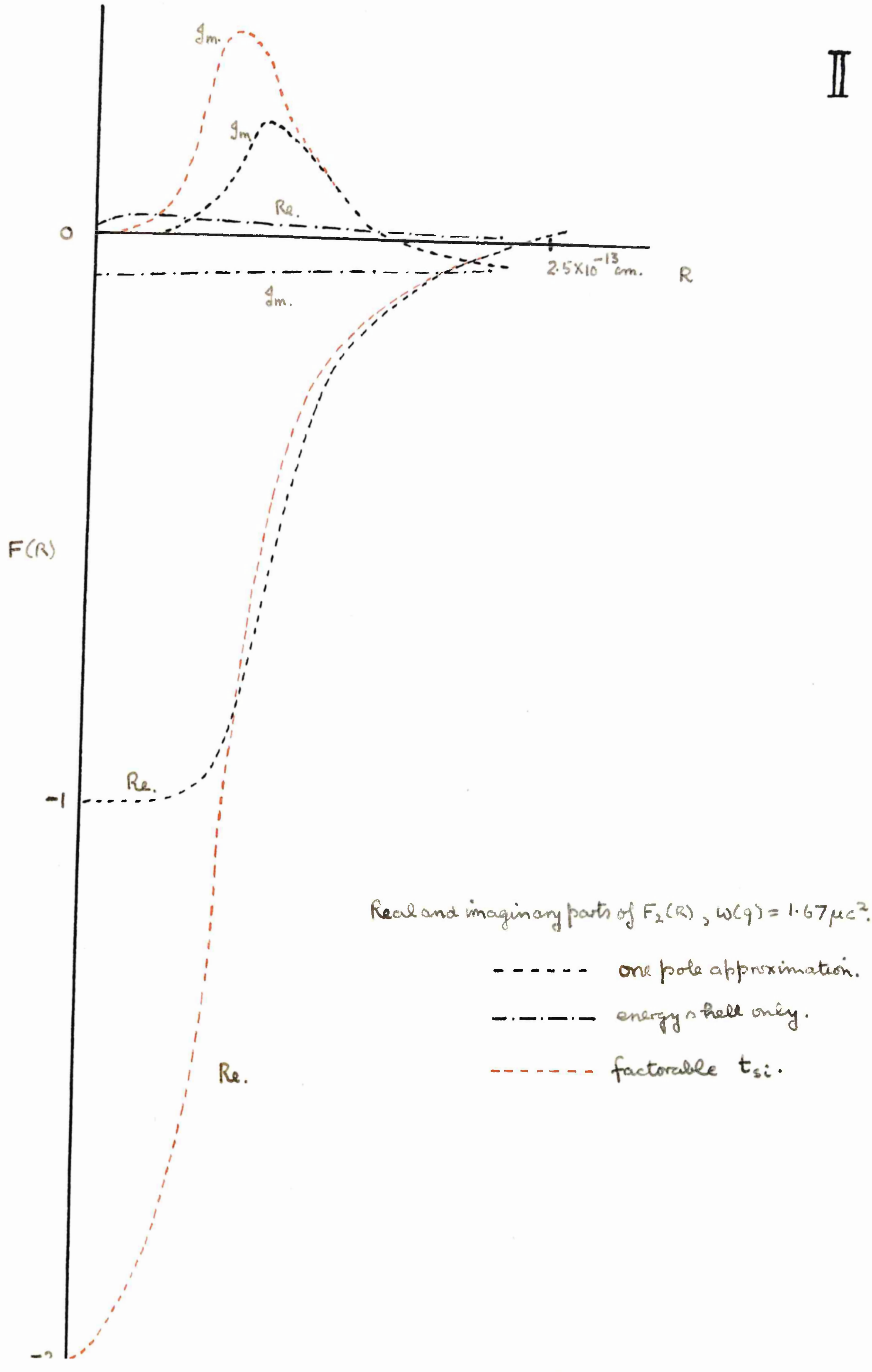
$$F_1(R) = \mathcal{C}_E\, h(R)\, F_4(R) = \frac{i\,h(R)\, f(R)\, \mathcal{C}_E^2}{1 - h^2(R)\,\mathcal{C}_E^2}$$

$$F_2(R) = \mathcal{C}_E\, f(R)\, F_5(R) = \frac{f^2(R)\,\mathcal{C}_E^2}{1 - f^2(R)\,\mathcal{C}_E^2} \qquad (22)$$

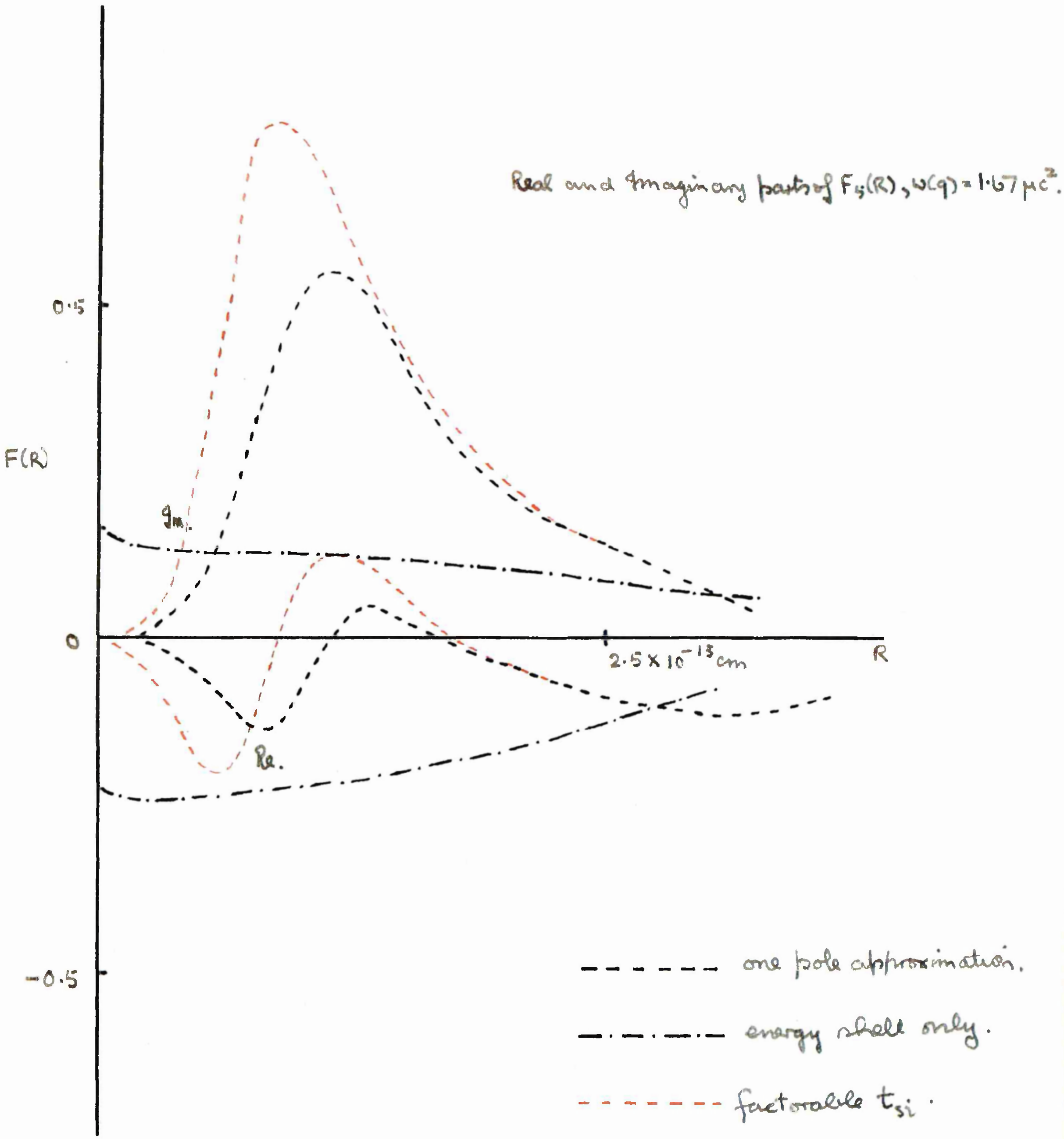
$$F_3(R) = \frac{g(R)\, R^2\,[f(R)+h(R)]\,\mathcal{C}_E^2}{(1-f^2(R)\,\mathcal{C}_E^2)(1-h^2(R)\,\mathcal{C}_E^2)}$$

$$F_6(R) = \frac{g(R)\, R^2\,(1 + h(R)\, f(R)\,\mathcal{C}_E^2)\,\mathcal{C}_E}{(1-f^2(R)\,\mathcal{C}_E^2)(1-h^2(R)\,\mathcal{C}_E^2)}$$

To illustrate the behaviour of these functions we show in graphs II and III the real and imaginary parts of an "even scattering" function $F_2(R)$ and an "odd scattering" function $F_5(R)$. All the odd scattering functions are zero at $R=0$, and some of them are appreciable for larger values of R than any of the even scattering functions. If we evaluate our integrals using the approximation of neglecting scattering off the energy shell we get precisely the same form of result but


II
Im.
Im.
Re.
0
2.5×10^-13 cm.
R
Im.
F(R)
Re.
-1
Re.
-2
Real and imaginary parts of F_2(R), ω(q) = 1·67μc².
one pole approximation.
energy shell only.
factorable t_si.

III



Real and Imaginary parts of F₅(R), ω(q) = 1·67 μc².
0·5
F(R)
Im.
0
2·5 × 10⁻¹³ cm
R
Re.
−0·5
one pole approximation.
energy shell only.
factorable t_{si}.

we have $i \sin q_E R$ in place of $e^{i q_E R}$ in (7). $F_2(R)$ and $F_5(R)$ in this approximation are also shown in graphs II and III. **The third set of curves shown in these graphs is explained in Section 4.**

4. Multiple scattering with a factorable transition operator.

It is of some interest to examine the multiple scattering correction for a particular form of $\mathcal{G}(q_1, q_2)$ which allows us to evaluate the integral (3.3) exactly, rather than in the one pole approximation. In Section 3 this approximation is used so that in our final result we only need to know the values of $\mathcal{G}(q_E, q_E)$, $\alpha(q_E)$ and $\gamma(q_E)$, and so that we can reduce the solution of our problem to the solution of the equation (3.10) for $\underset{\sim}{S}_i$. We can retain these features of our method if we use the form $\mathcal{G}(q_1, q_2) = c(q_1)d(q_2)$, although we must again use the one pole approximation in the integrals (3.15) which involve $\alpha(q_2)$ and $\gamma(q_2)$. We make use of the work of Velibekov and Meshcheryakov (1955) who use a factorable potential

$$V_{33}(q_1, q_2) = \frac{-\lambda\, v(q_1)\, v(q_2)}{\omega(q_1)\,\omega(q_2)} \qquad (1)$$

in (1.14) the equation for meson scattering at a free nucleon, and obtain reasonable values of $\tan\delta_{33}(q_E)$. The fact that these authors obtain the correct behaviour of scattering on the energy shell does not imply that their form of $\mathcal{G}(q_1, q_2)$ is reliable off the energy shell. For this reason, and because we still have to use the one pole approximation in (3.15) we do not use this version of multiple scattering theory in our detailed calculations. We shall take the calculation to the stage of showing that the results of using this form of $\mathcal{G}(q_1, q_2)$ are closer to those of the one pole approximation than to those obtained when we neglect scattering off the energy shell.

With the potential (1) the equation (1.14) has the exact solution

$$k_{33} = \frac{V_{33}(q_1, q_2)}{1 - \lambda I(E)} \tag{2}$$

where

$$I(E) = P\int_0^\infty \frac{dq\, q^4 v^2(q)}{\omega^2(q)[\omega(q_E) - \omega(q)]} \tag{3}$$

Velibekov and Meshcheryakov introduce a new coupling constant

$$\bar{\lambda} = \frac{\lambda}{1 - \lambda I(\mu)} \tag{4}$$

so that

$$k_{33}(q_E, q_E) = -\frac{\bar{\lambda} v^2(q_E)}{\omega^2(q_E)} \left\{ 1 - \bar{\lambda}\left[I(E) - I(\mu) \right] \right\}^{-1} \tag{5}$$

When $v(q)$ is replaced by a cut-off at q_{max} this gives

$$\tan \delta_{33}(q_E) = \frac{\bar{\lambda} q_E^3}{2\pi\, \omega(q_E)} \left\{ 1 + \bar{\lambda}\left[\omega(q_E) - \mu\right] P\int_0^{q_{max}} \frac{dq\, q^4}{\omega^2(q)} \frac{1}{\omega(q_E) - \omega(q)} \frac{1}{\mu - \omega(q)} \right\}^{-1} \tag{6}$$

By suitable choice of $\bar{\lambda}$ and q_{max} these authors are able to fit the experimental phase shifts fairly well. In this theory we have

$$k(q_1, q_2) = -\frac{\bar{\lambda} v(q_1) v(q_2)}{\omega(q_1)\omega(q_2)} \left\{ 1 - \bar{\lambda}\left[I'(E) - I'(\mu) \right] \right\}^{-1} \tag{7}$$

where the integral $I'(E)$ is

$$I'(E) = \int_0^\infty \frac{dq\, q^4 v^2(q)}{\omega^2(q)[\omega(q_E) - \omega(q) + i\epsilon]}$$

With this result for $b(q_1, q_2)$ we have

$$b(q_1,q_2) = \frac{b(q_E,q_E)\,\omega^2(q_E)}{\omega(q_1)\,\omega(q_2)} \tag{8}$$

We shall use this form of $b(q_1,q_2)$ taking $b(q_E,q_E)$ as given by (1.12) and (1.17). Thus the behaviour of $b(q_1,q_2)$ off the energy shell is very simple to deal with. The integral in (3.3) is now

$$b(q_1,q_2)\int_0^{q_{max}} \frac{dq\, q\,(e^{iqR}-e^{-iqR})\, b(q,q)}{\omega(q_E)-\omega(q)+i\varepsilon}$$

$$= b(q_1,q_2)\,b(q_E,q_E)\,\omega^2(q_E)\int_{-q_{max}}^{q_{max}} \frac{dq\, q\, e^{iqR}}{\omega^2(q)\,[\omega(q_E)-\omega(q)+i\varepsilon]} \tag{9}$$

The poles of the integral contribute, on closing the contour round a semicircle in the upper half plane,

$$-2\pi i\,\omega(q_E)\,b(q_1,q_2)\,b(q_E,q_E)\left\{e^{iq_ER}-\tfrac{1}{2}e^{-\mu R}\right\} \tag{10}$$

The integral round the semicircular part of the contour is

$$i q_{max}^2\, b_E\, b(q_1,q_2)\,\omega^2(q_E)\int_0^{\pi} \frac{d\theta\,(\cos 2\theta + i\sin 2\theta)\exp\{iq_{max}R(\cos\theta+i\sin\theta)\}}{[q_{max}^2(\cos 2\theta+i\sin 2\theta)+\mu^2]\{[q_{max}^2(\cos 2\theta+i\sin 2\theta)+\mu^2]^{1/2}-\omega(q_E)\}} \tag{11}$$

Now for consistency we use the large cut-off, $q_{max} = 11\mu$, of Velibekov and Meshcheryakov. It is easily seen that we can neglect (11) in comparison with (10).

We, therefore, have the result that to a good approximation the matrix element of $t_{si}\frac{1}{a}t_{s\not{i}}$ is

$$a_i a_{\not{i}}\, b(q_1,q_2)\, b_E\{\underset{\sim}{q_1}\cdot\underset{\sim}{q_2}\,\hat{f}(R) + \underset{\sim}{q_1}\cdot\underset{\sim}{R}\,\underset{\sim}{q_2}\cdot\underset{\sim}{R}\,\hat{g}(R)\}\, e^{i\underset{\sim}{q_2}\cdot\underset{\sim}{\rho_{\not{i}}} - i\underset{\sim}{q_1}\cdot\underset{\sim}{\rho_i}} \tag{12}$$

where

$$\hat{f}(R) = \frac{\omega(q_E)}{2\pi R}\frac{d}{dR}\left\{\frac{1}{R}\left(e^{iq_E R} - \tfrac{1}{2}e^{-\mu R}\right)\right\}$$

$$\hat{g}(R) = \frac{1}{R}\frac{d}{dR}\hat{f}(R) \tag{13}$$

We shall also use the notation $\hat{h}(R) = \hat{f}(R) + R^2\hat{g}(R)$. For $R \gtrsim 2\mu^{-1}$ $\hat{f}(R)$ and $\hat{g}(R)$ are almost exactly $f(R)$ and $g(R)$. There is a certain resemblance to the third model of Drell and Verlet, discussed in Section 1. They use a potential which is factorable in configuration space, and obtain a result which tends to the result of the one pole approximation as R increases. Using (12) and (13) we can now derive the matrix element of T as in Section 3, obtaining a very similar form. From the result (12) for $(\underset{\sim}{q_1}|t_{si}\frac{1}{a}t_{s\not{i}}|\underset{\sim}{q_2})$ we obtain

$$(\underset{\sim}{q_1}|x_i|\underset{\sim}{q_2}) = (2\pi)^3\delta(\underset{\sim}{q_1}-\underset{\sim}{q_2}) + a_{\not{i}}\frac{b(q_1,q_2)}{a(q_2)}\underset{\sim}{q_1}\cdot\underset{\sim}{q_2}\, e^{i(\underset{\sim}{q_2}-\underset{\sim}{q_1})\cdot\underset{\sim}{\rho_{\not{i}}}}$$

$$+ e^{i\underset{\sim}{q_2}\cdot\underset{\sim}{\rho_{\not{i}}}}\frac{d(q_2)}{a(q_2)}b_E\left\{\hat{f}(R)\underset{\sim}{q_2} + \hat{g}(R)\,\underset{\sim}{q_2}\cdot\underset{\sim}{R}\,\underset{\sim}{R}\right\}\cdot\underset{\sim}{\hat{S}_i}(\underset{\sim}{q_1})\, a_i a_{\not{i}} \tag{14}$$

where

$$\underset{\sim}{\hat{S}_i}(\underset{\sim}{q_1}) = \int\frac{d\underset{\sim}{q}}{(2\pi)^3}(\underset{\sim}{q_1}|x_i|\underset{\sim}{q})\,\underset{\sim}{q}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\rho_i}}\, c(q) \tag{15}$$

In the same way as before we obtain for $\underset{\sim}{\hat{S}_i}$ the result

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$$\hat{S}_i(q_1) = \Big[q_1\, c(q_1) \{ e^{-i q_1 \cdot r_i} + a_{i'}\, b_E\, \hat{f}(R)\, e^{-i q_1 \cdot r_{i'}} \} +$$

$$q_1 \cdot R\, R\, c(q_1)\, b_E\, \hat{g}(R) \{ a_{i'}\, e^{-i q \cdot r_{i'}} + b_E [\hat{f}(R) + \hat{h}(R)] \times$$

$$(e^{-i q_1 \cdot r_i} + a_{i'}\, e^{-i q \cdot r_{i'}}\, b_E\, \hat{h}(R))(1 - \hat{h}^2(R)\, b_E^2\, a_i a_{i'})^{-1} a_i a_{i'} \} \Big] \frac{1}{1 - b_E^2 \hat{f}^2(R) a_i a_{i'}} \qquad (16)$$

We define P_i and Q_i as before and evaluate (3.15) using the one pole approximation, with the result that our expression for $(q_1|T|\nu)$ contains $f(R)$, $g(R)$ as well as $\hat{f}(R)$, $\hat{g}(R)$. We readily obtain the matrix element in the form (3.21) but with the functions $F(R)$ of (3.22) replaced by similarly numbered functions $\hat{F}(R)$, where

$$\hat{F}_1(R) = \hat{h}(R)\, b_E\, \hat{F}_4(R) = \frac{i \hat{h}(R) f(R)\, b_E^2}{1 - \hat{h}^2(R)\, b_E^2}$$

$$\hat{F}_2(R) = \hat{f}(R)\, b_E\, \hat{F}_5(R) = \frac{\hat{f}(R) f(R)\, b_E^2}{1 - \hat{f}^2(R)\, b_E^2}$$

$$\hat{F}_3(R) = R^2 b_E^2 \frac{[\hat{g} h + \hat{f} g + (\hat{f} \hat{g} \hat{h} h - \hat{h}^2 \hat{f} g) b_E^2]}{(1 - \hat{f}^2(R) b_E^2)(1 - \hat{h}^2(R) b_E^2)} \qquad (17)$$

$$\hat{F}_6(R) = \frac{R^2 g(R)\, b_E}{1 - \hat{f}^2(R) b_E^2} + \frac{h(R)\, \hat{g}(R)\, [\hat{f}(R) + \hat{h}(R)]\, b_E^3}{(1 - \hat{f}^2(R) b_E^2)(1 - \hat{h}^2(R) b_E^2)}$$

In the graphs II and III we compare $\hat{F}_2(R)$ and $\hat{F}_5(R)$ with $F_2(R)$ and $F_5(R)$ calculated in the one pole approximation and in the approximation of neglecting scattering off the energy shell. It is clear that the first of these approximations is the better.

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This is to be expected, since deviations from energy conservation can be important for an intermediate state of short duration.

5. Isotopic spin.

We work with representations of the isotopic spin states which are consistent with the use of the form of Clebsch-Gordan coefficients given in Condon and Shortley (1951). We define $Y_{1/2}^{1/2}$ as the one proton state and $Y_{1/2}^{-1/2}$ as the one neutron state, and the operator $\underset{\sim}{\tau}$ by

$$\tau_1 = \tau_+ + \tau_- \quad , \quad i\tau_2 = \tau_+ - \tau_- ,$$

where

$$\left.\begin{aligned} \tau_+ Y_{\frac{1}{2}}^{m} &= \delta_{m,-\frac{1}{2}} Y_{\frac{1}{2}}^{-m} \\ \tau_- Y_{\frac{1}{2}}^{m} &= \delta_{m,\frac{1}{2}} Y_{\frac{1}{2}}^{-m} \end{aligned}\right\} \quad (1)$$

and

$$\tau_3 Y_{\frac{1}{2}}^{m} = 2m\, Y_{\frac{1}{2}}^{m}$$

States of one positive, neutral or negative meson, denoted respectively by Y_1^1, Y_1^0, Y_1^{-1} can be represented in the form

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

When regarded as operating on these column matrices the operator $\underset{\sim}{\ell}$ is

$$\ell_1 = \sqrt{\tfrac{1}{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\ell_2 = \sqrt{\tfrac{1}{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad (2)$$

$$\ell_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

/

The meson creation operators φ act on the vacuum state $|0\rangle$ as follows,

$$\varphi|0\rangle = a Y_1^1$$

$$\varphi^*|0\rangle = b Y_1^{-1} \qquad (3)$$

$$\varphi_3|0\rangle = Y_1^0$$

and we define

$$\varphi_1 = \sqrt{\tfrac{1}{2}}(\varphi+\varphi^*) \,, \quad i\varphi_2 = \sqrt{\tfrac{1}{2}}(\varphi-\varphi^*)$$

We must fix a and b to be consistent with our choice of $\underset{\sim}{\tau}$, $\underset{\sim}{\ell}$ by requiring that the total isotopic spin $\underset{\sim}{t} = \frac{1}{2}\underset{\sim}{\tau} + \underset{\sim}{\ell}$ commutes with the Kemmer (1938) operator $C = \sum_i \varphi_i \tau_i$. We have

$$t^+(C Y_{\frac{1}{2}}^{\frac{1}{2}}) = C(t^+ Y_{\frac{1}{2}}^{\frac{1}{2}}) \equiv 0$$

and $$t^-(C Y_{\frac{1}{2}}^{-\frac{1}{2}}) = C(t^- Y_{\frac{1}{2}}^{-\frac{1}{2}}) \equiv 0$$

Hence $a = -1$ and $b = 1$.

The isotopic spin eigenstates of the system of one nucleon and one meson are, in this representation,

$$X_{3/2}^{\frac{3}{2}} = Y_{\frac{1}{2}}^{\frac{1}{2}} Y_1^1$$

$$X_{3/2}^{\frac{1}{2}} = \sqrt{\tfrac{1}{3}}\, Y_{\frac{1}{2}}^{-\frac{1}{2}} Y_1^1 + \sqrt{\tfrac{2}{3}}\, Y_{\frac{1}{2}}^{\frac{1}{2}} Y_1^0 \qquad (4)$$

$$X_{3/2}^{-\frac{1}{2}} = \sqrt{\tfrac{2}{3}}\, Y_{\frac{1}{2}}^{-\frac{1}{2}} Y_1^0 + \sqrt{\tfrac{1}{3}}\, Y_{\frac{1}{2}}^{\frac{1}{2}} Y_1^{-1}$$

$$X_{3/2}^{-3/2} = Y_{\frac{1}{2}}^{-\frac{1}{2}} Y_1^{-1}$$

$$X_{\frac{1}{2}}^{\frac{1}{2}} = \sqrt{\tfrac{2}{3}}\, Y_{\frac{1}{2}}^{-\frac{1}{2}} Y_1^1 - \sqrt{\tfrac{1}{3}}\, Y_{\frac{1}{2}}^{\frac{1}{2}} Y_1^0$$

$$X_{\frac{1}{2}}^{-\frac{1}{2}} = \sqrt{\tfrac{1}{3}}\, Y_{\frac{1}{2}}^{-\frac{1}{2}} Y_1^0 - \sqrt{\tfrac{2}{3}}\, Y_{\frac{1}{2}}^{\frac{1}{2}} Y_1^{-1}$$

while the eigenstates of a two nucleon system are

$$
\begin{aligned}
X_1^1 &= Y_{\frac{1}{2}}^{\frac{1}{2}}(1)\, Y_{\frac{1}{2}}^{\frac{1}{2}}(2) \\
X_1^0 &= \sqrt{\tfrac{1}{2}} \left\{ Y_{\frac{1}{2}}^{\frac{1}{2}}(1)\, Y_{\frac{1}{2}}^{-\frac{1}{2}}(2) + Y_{\frac{1}{2}}^{-\frac{1}{2}}(1)\, Y_{\frac{1}{2}}^{\frac{1}{2}}(2) \right\} \\
X_1^{-1} &= Y_{\frac{1}{2}}^{-\frac{1}{2}}(1)\, Y_{\frac{1}{2}}^{-\frac{1}{2}}(2) \\
X_0^0 &= \sqrt{\tfrac{1}{2}} \left\{ Y_{\frac{1}{2}}^{\frac{1}{2}}(1)\, Y_{\frac{1}{2}}^{-\frac{1}{2}}(2) - Y_{\frac{1}{2}}^{-\frac{1}{2}}(1)\, Y_{\frac{1}{2}}^{\frac{1}{2}}(2) \right\}
\end{aligned}
\qquad (5)
$$

For the system of two nucleons and one meson we only have to deal with states of unit charge. There are two convenient sets of states. The natural set for describing our final state is

$$
\begin{aligned}
|1\rangle &= X_1^1\, Y_1^{-1}, && \text{that is } \pi^- + 2p \\
|2\rangle &= X_1^0\, Y_1^0, && \text{that is } \pi^0 + \text{triplet } np \text{ state} \\
|3\rangle &= X_1^{-1}\, Y_1^1, && \text{that is } \pi^+ + 2n \\
|4\rangle &= X_0^0\, Y_1^0, && \text{that is } \pi^0 + \text{singlet } np \text{ state.}
\end{aligned}
\qquad (6)
$$

For dealing with multiple scattering in which each scattering is in a $3/2$ state of the isotopic spin of the meson and one nucleon it is more convenient to use eigenstates of t and t_3 . The appropriate states are

$$
\begin{aligned}
|1) &= \sqrt{\tfrac{1}{6}} \left\{ |1\rangle + 2|2\rangle + |3\rangle \right\} \\
|2) &= \sqrt{\tfrac{1}{2}} \left\{ |1\rangle - |3\rangle \right\} \\
|3) &= \sqrt{\tfrac{1}{3}} \left\{ |1\rangle - |2\rangle + |3\rangle \right\} \\
|4) &= |4\rangle
\end{aligned}
\qquad (7)
$$

So for any operator A we have

$$
(\lambda|A|\mu) = \sum_{ij} (\lambda|i\rangle\langle i|A|j\rangle\langle j|\mu) \qquad (8)
$$

where $\langle j|\mu)$ is

$$\begin{bmatrix} \sqrt{\tfrac{1}{6}} & \sqrt{\tfrac{1}{2}} & \sqrt{\tfrac{1}{3}} & 0 \\ \sqrt{\tfrac{2}{3}} & 0 & -\sqrt{\tfrac{1}{3}} & 0 \\ \sqrt{\tfrac{1}{6}} & -\sqrt{\tfrac{1}{2}} & \sqrt{\tfrac{1}{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

The second representation is essentially that used by Chappelear.

Consider the operator $a_i = 2 + \underset{\sim}{\tau}_i \cdot \underset{\sim}{\ell}$ which appears in t_{si}.

$$\langle i | a_1 | j \rangle = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}$$

and (10)

$$\langle i | a_2 | j \rangle = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

while

$$(\lambda | a_1 | \mu) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 2 \end{bmatrix}$$

and (11)

$$(\lambda | a_2 | \mu) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 2 \end{bmatrix}$$

In the form of T_i given by (1.18), (1.19) the operator for wave photoproduction is obtained when $\underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla}\ \underset{\sim}{\tau}\cdot\underset{\sim}{\varphi}$ in the meson-nucleon interaction Hamiltonian for a pseudoscalar meson and a static nucleon, is made gauge invariant. This gives

$$\alpha_i = \beta_i A_i = [\varphi^* \tau_i^+ - \varphi \tau_i^-] \underset{\sim}{\sigma_i} \cdot \underset{\sim}{\varepsilon} E_d \tag{12}$$

In our notation the effect of β_i on the deuteron wave function is given by

$$\begin{aligned} \beta_1 \chi_D^0 &= -|2) \\ \beta_2 \chi_D^0 &= |2) \end{aligned} \tag{13}$$

The p wave part of the photoproduction operator, (1.19), must produce a state with $t = 3/2$. Its isotopic spin part, therefore, is

$$\delta_i = \sqrt{\tfrac{1}{2}}\,[\varphi^* \tau_i^+ - \varphi \tau_i^-] + \varphi_3 \tag{14}$$

Writing $\underset{\sim}{\gamma_i} = \delta_i \underset{\sim}{C_i}$ we have

$$\underset{\sim}{q} \cdot \underset{\sim}{C_i} = \frac{M_d}{\nu q}\left[2\,\underset{\sim}{\nu} \cdot \underset{\sim}{q} \times \underset{\sim}{\varepsilon} + i\,\underset{\sim}{\sigma_i} \cdot \underset{\sim}{\nu} \times \underset{\sim}{\varepsilon} \times \underset{\sim}{q} \right] \tag{15}$$

We have

$$\begin{aligned} \delta_1 \chi_D^0 &= -\sqrt{\tfrac{1}{2}}\,|2) + |4) \\ \delta_2 \chi_D^0 &= \sqrt{\tfrac{1}{2}}\,|2) + |4) \end{aligned} \tag{16}$$

We can now see the advantage of using the states $|\lambda)$ rather than $|i\rangle$. We can take the operator a_i as effectively

$$(\lambda|a_1|\mu) = \begin{bmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$$

$$(\lambda|a_2|\mu) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \tag{11'}$$

Any state reached during the multiple scattering is a linear combination of $|2)$ and $|4)$ and for the amplitude for production of positive (negative) mesons we pick out $-\sqrt{\tfrac{1}{2}}$ $\left(\sqrt{\tfrac{1}{2}}\right)$

/

times the coefficient of $|2\rangle$ in the states obtained from by the various parts of (3.20). This leads to (3.21). Notice **that any difference in the behaviour of positive and negative meson cross-sections must come from the space wave functions, that is it can only come from the effect of the Coulomb interactions in the system of two protons and a negative meson.**

$$\frac{d^3\sigma}{dq\, dk\, d\Omega}\Big\{ \text{Impulse approximation; even part of plane}$$

$$\frac{d^3\sigma}{dq\, dk\, d\Omega}\Big\{ \text{Impulse approximation; } \ell = 0 \text{ partial wave}$$

$$\frac{d^3\sigma}{dq\, dk\, d\Omega}\Big\{ \text{With multiple scattering; distorted } \ell = 0$$

$$\frac{d^3\sigma}{dq\, dk\, d\Omega}\Big\{ \text{Impulse approximation; odd part of plane}$$

$$\frac{d^3\sigma}{dq\, dk\, d\Omega}\Big\{ \text{Impulse approximation; } \ell = 1 \text{ partial wave}$$

$$\frac{d^3\sigma}{dq\, dk\, d\Omega}\Big\{ \text{With multiple scattering; } \ell = 1 \text{ partial wa}$$

6. The cross-section.

The impulse approximation cross-section for a plane wave final state is corrected for the nuclear interaction in the S state, and for multiple scattering when the nucleons are finally in an S or P state. The multiple scattering functions have short ranges so that we can reasonably neglect this correction for $\ell > 1$. The cross-section has the form

$$\frac{d^3\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{k}\,d\underset{\sim}{D}}(\text{even}) + \frac{d^3\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{k}\,d\underset{\sim}{D}}(\text{odd}) =$$

$$\frac{d^3\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{k}\,d\underset{\sim}{D}}\left\{\text{Impulse approximation; even part of plane wave}\right\}$$

$$\frac{d^3\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{k}\,d\underset{\sim}{D}}\left\{\text{Impulse approximation; } \ell = 0 \text{ partial wave}\right\}$$

$$\frac{d^3\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{k}\,d\underset{\sim}{D}}\left\{\text{With multiple scattering; distorted } \ell = 0 \text{ partial wave}\right\} \tag{1}$$

$$\frac{d^3\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{k}\,d\underset{\sim}{D}}\left\{\text{Impulse approximation; odd part of plane wave}\right\}$$

$$\frac{d^3\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{k}\,d\underset{\sim}{D}}\left\{\text{Impulse approximation; } \ell = 1 \text{ partial wave}\right\}$$

$$\frac{d^3\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{k}\,d\underset{\sim}{D}}\left\{\text{With multiple scattering; } \ell = 1 \text{ partial wave}\right\}$$

Each of these cross-sections has the form

$$\frac{d^3\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{k}\,d\underset{\sim}{D}} = 2\pi|\mathcal{Q}|^2\,\delta\left(\omega(q) + \frac{\underset{\sim}{k}^2}{M} + \frac{\underset{\sim}{D}^2}{4M} - \epsilon_D - \omega_\nu\right) \tag{2}$$

$$= 2\pi |Q|^2 \, \delta\left(\frac{k^2}{M} - \frac{k_0^2}{M}\right)$$

Here $\underset{\sim}{q}$, $\underset{\sim}{k}$, $\underset{\sim}{D}$ are the momenta of the meson, the nucleon relative motion and the nucleon centre of mass motion respectively. ϵ_D is the deuteron binding energy, -2.23 MeV., and M is the nucleon mass. $|Q|^2 = |\langle f | T | i \rangle|^2$ for a final state $|f\rangle$ appropriate to the particular cross-section considered. The effect of averaging over $\underset{\sim}{\varepsilon}$ and the weight associated with the final state spin are implicit in $|Q|^2$. T as written in (3.21) already takes account of the isotopic spin parts of $|i\rangle$ and $|f\rangle$ so we have for our initial state

$$|i\rangle = {}^3\chi_m \, u_i(R)$$

where ${}^3\chi_m$ is the triplet spin state and we use the Hulthén deuteron wave function

$$u_i(R) = \left[\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}\right]^{1/2} \frac{e^{-\alpha R} - e^{-\beta R}}{R} = \frac{K}{R}\left(e^{-\alpha R} - e^{-\beta R}\right) \qquad (3)$$

with $\alpha = \sqrt{M|\epsilon_D|}$ and β given by $\rho_1 = \frac{4}{\alpha+\beta} - \frac{1}{\beta}$, ρ_1 being the triplet np effective range, $\rho_1 = 1.704 \times 10^{-13}$ cm.

$$\alpha = .2316 \times 10^{13} \text{ cm}^{-1}$$

$$\beta = 1.434 \times 10^{13} \text{ cm}^{-1}$$

Our final states have the form, for even and odd space parts,

$$(2\pi)^{-3/2} \, u_{fe}(\underset{\sim}{k}.\underset{\sim}{R}) \, e^{i\underset{\sim}{D}.\underset{\sim}{r}} \, {}^3\chi_m$$

$$(2\pi)^{-3/2} \, u_{fo}(\underset{\sim}{k}.\underset{\sim}{R}) \, e^{i\underset{\sim}{D}.\underset{\sim}{r}} \, {}^1\chi_0$$

Here ${}^1\chi_0$ is the singlet spin function, and $\underset{\sim}{r} = \frac{1}{2}(\underset{\sim}{r_1} + \underset{\sim}{r_2})$. The space wave functions we use are, in the order in which

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the cross-sections occur in (1),

$$u_{fe}(\underset{\sim}{k}.\underset{\sim}{R}) = (2\pi)^{-3/2} \cos \underset{\sim}{k}.\underset{\sim}{R}$$

$$u_{fe}(\underset{\sim}{k}.\underset{\sim}{R}) = (2\pi)^{-3/2} \frac{\sin kR}{kR}$$

$$u_{fe}(\underset{\sim}{k}.\underset{\sim}{R}) = (2\pi)^{-3/2} e^{-i\delta_0} \frac{1}{kR}\left[\sin(kR+\delta_0) - e^{-\eta R}\sin\delta_0\right] \qquad (4)$$

$$= (2\pi)^{-3/2} \frac{u(kR)}{kR}$$

$$u_{fo}(\underset{\sim}{k}.\underset{\sim}{R}) = (2\pi)^{-3/2} \sin \underset{\sim}{k}.\underset{\sim}{R}$$

$$u_{fo}(\underset{\sim}{k}.\underset{\sim}{R}) = (2\pi)^{-3/2} j_1(kR)\, 3i \left(\frac{\underset{\sim}{k}.\underset{\sim}{R}}{kR}\right)$$

the last applying to the last two cross-sections in (1). The form given for $u(kR)$ is one frequently used for this purpose. (See for example Saito et. al. (1952)). δ_0, the nucleon - nucleon scattering phase shift for $\ell = 0$, and η are chosen to fit the effective range r_e and the scattering length a which give the non-Coulomb proton-proton scattering in the triplet state, at low energies. The connection between r_e, a, and η is obtained following the method of Bethe (1949). We introduce the following auxiliary functions,

$$u(kR) \longrightarrow w(kR) \quad \text{as} \quad R \longrightarrow \infty$$

$$u(kR) \longrightarrow u_0(R) \quad \text{as} \quad k \longrightarrow 0$$

$$u_0(R) \longrightarrow w_0(R) \quad \text{as} \quad R \longrightarrow \infty$$

/

It is convenient to use the boundary conditions

$$u(0) = u_0(0) = 0$$

$$w(0) = w_0(0) = 1$$

so we work in fact with

$$u(kR) \doteq \frac{\sin(kR+\delta_0)}{\sin\delta_0} - e^{-\eta R}$$

We then have

$$w(kR = \frac{\sin(kR+\delta_0)}{\sin\delta_0}$$

$$u_0(R) = 1-\frac{R}{a} - e^{-\eta R}$$

$$w_0(R) = 1-\frac{R}{a}$$

The effective range is given by

$$r_e = 2\int_0^\infty (w_0^2 - u_0^2)\, dR$$

Using a = -7.7×10^{-13} cm., $r_e = 2.65 \times 10^{-13}$ cm., we obtain $\eta = 1.28 \times 10^{13}$ cm.$^{-1}$

Because we must deal separately with odd and even final states we define Q_o and Q_e where

$$Q_o = \int \frac{d\underset{\sim}{R}\, d\underset{\sim}{r}}{(2\pi)^{3/2}} u_{f_o}^*(\underset{\sim}{k}.\underset{\sim}{R})\, u_i(R)\, e^{-i\underset{\sim}{D}.\underset{\sim}{r}} \langle {}^1\chi_0 | T | {}^3\chi_m \rangle$$

$$Q_e = \int \frac{d\underset{\sim}{R}\, d\underset{\sim}{r}}{(2\pi)^{3/2}} u_{fe}^*(\underset{\sim}{k}.\underset{\sim}{R})\, u_i(R)\, e^{-i\underset{\sim}{D}.\underset{\sim}{r}} \langle {}^3\chi_m | T | {}^3\chi_m \rangle \quad (5)$$

The form of T in(3.2.)can be expressed, recalling the definition of A_i and C_i, as

$$T = e^{i(\underset{\sim}{\nu}-\underset{\sim}{q}).\underset{\sim}{r}}\, T'$$

$$= e^{i(\underset{\sim}{\nu}-\underset{\sim}{q}).\underset{\sim}{r}}\Big[e^{\frac{i}{2}(\underset{\sim}{\nu}-\underset{\sim}{q}).\underset{\sim}{R}}(\underset{\sim}{K}_{11}.\underset{\sim}{\sigma}_1+L_{11}) + e^{\frac{i}{2}(\underset{\sim}{\nu}+\underset{\sim}{q}).\underset{\sim}{R}}(\underset{\sim}{K}_{12}.\underset{\sim}{\sigma}_1+L_{12})$$

$$+ e^{-\frac{i}{2}(\underset{\sim}{\nu}-\underset{\sim}{q}).\underset{\sim}{R}}(\underset{\sim}{K}_{22}.\underset{\sim}{\sigma}_2+L_{22}) + e^{-\frac{i}{2}(\underset{\sim}{\nu}+\underset{\sim}{q}).\underset{\sim}{R}}(\underset{\sim}{K}_{21}.\underset{\sim}{\sigma}_2+L_{21})\Big] \qquad (6)$$

So

$$Q_o = (2\pi)^{3/2}\delta(\underset{\sim}{D}-\underset{\sim}{\nu}+\underset{\sim}{q})\int d\underset{\sim}{R}\, u_{fo}^*(\underset{\sim}{k}.\underset{\sim}{R})\, u_i(R)\, \langle {}^1\chi_o | T' | {}^3\chi_m \rangle$$

$$Q_e = (2\pi)^{3/2}\delta(\underset{\sim}{D}-\underset{\sim}{\nu}+\underset{\sim}{q})\int d\underset{\sim}{R}\, u_{fe}^*(\underset{\sim}{k}.\underset{\sim}{R})\, u_i(R)\, \langle {}^3\chi_m | T' | {}^3\chi_m \rangle \qquad (7)$$

Now write

$$T' = (\underset{\sim}{\sigma}_1+\underset{\sim}{\sigma}_2).\underset{\sim}{T}^+ + (\underset{\sim}{\sigma}_1-\underset{\sim}{\sigma}_2).\underset{\sim}{T}^- + T^o \qquad (8)$$

and define the integrals

$$\underset{\sim}{I}^+ = (2\pi)^{3/2}\int d\underset{\sim}{R}\, u_{fo}^*(\underset{\sim}{k}.\underset{\sim}{R})\, u_i(R)\, \underset{\sim}{T}^+$$

$$\underset{\sim}{I}^- = (2\pi)^{3/2}\int d\underset{\sim}{R}\, u_{fe}^*(\underset{\sim}{k}.\underset{\sim}{R})\, u_i(R)\, \underset{\sim}{T}^- \qquad (9)$$

$$I = (2\pi)^{3/2}\int d\underset{\sim}{R}\, u_{fo}^*(\underset{\sim}{k}.\underset{\sim}{R})\, u_i(R)\, T^o$$

Then

$$|Q_o|^2 = [\delta(\underset{\sim}{D}-\underset{\sim}{\nu}+\underset{\sim}{q})]^2 \left\langle \tfrac{8}{3}|\underset{\sim}{I}^+|^2 + |I|^2 \right\rangle_{Av}$$

$$|Q_e|^2 = [\delta(\underset{\sim}{D}-\underset{\sim}{\nu}+\underset{\sim}{q})]^2\, \tfrac{4}{3}\left\langle |\underset{\sim}{I}^-|^2 \right\rangle_{Av} \qquad (10)$$

the symbol $\langle\ \rangle_{Av}$ indicating the average over $\underset{\sim}{\xi}$. Putting this form into (2) and integrating over $\underset{\sim}{D}$ we get the partial cross-sections for a particular meson momentum $\underset{\sim}{q}$ and any /

compatible $\underset{\sim}{k}$ as

$$\frac{d\sigma}{d\underset{\sim}{q}} = \frac{(2\pi)^{-2}}{\hbar c}\int\frac{d\underset{\sim}{k}}{(2\pi)^3}\,\delta\left(\frac{k^2}{M}-\frac{k_0^2}{M}\right)\frac{4}{3}\left\langle|\underset{\sim}{I}^-|^2\right\rangle_{A\sigma}$$

$$\frac{d\sigma}{d\underset{\sim}{q}} = \frac{(2\pi)^{-2}}{\hbar c}\int\frac{d\underset{\sim}{k}}{(2\pi)^3}\,\delta\left(\frac{k^2}{M}-\frac{k_0^2}{M}\right)\left\langle\frac{8}{3}|\underset{\sim}{I}^+|^2+|I|^2\right\rangle_{A\sigma} \tag{11}$$

for one of the even or odd cross-sections as the case may be. From (11) we have the results,

$$\frac{d\sigma}{d\omega(q)\,d\Omega_q} = \frac{2(2\pi)^{-5}q\,\omega(q)Mk_0}{3\hbar c}\int d\Omega_k\left\langle|\underset{\sim}{I}^-|^2\right\rangle_{A\sigma,\;k=k_0}$$

$$\frac{d\sigma}{d\omega(q)\,d\Omega_q} = \frac{(2\pi)^{-5}q\,\omega(q)Mk_0}{6\hbar c}\int d\Omega_k\left\langle 8|\underset{\sim}{I}^+|^2+3|I|^2\right\rangle_{A\sigma,\;k=k_0} \tag{12}$$

for these cross-sections.

The impulse approximation.

We now give the form of (12) in the impulse approximation case, with the nuclear interaction included in the final state. From the first part of (3.21)

$$\pm(\underset{\sim}{q}|T_{I.A.}|\underset{\sim}{\nu}) = e^{\frac{i}{2}(\underset{\sim}{\nu}-\underset{\sim}{q})\cdot\underset{\sim}{R}}\left\{-\sqrt{\tfrac{1}{2}}\,E_d\,\underset{\sim}{\sigma_1}\cdot\underset{\sim}{\epsilon}-\tfrac{1}{2}M_d(\tfrac{3}{2})\left[2\underset{\sim}{\nu}\cdot\underset{\sim}{q}\times\underset{\sim}{\epsilon}+i\underset{\sim}{\sigma_1}\cdot\underset{\sim}{\nu}\times\underset{\sim}{\epsilon}\times\underset{\sim}{q}\right]\right\}$$

$$+\,e^{-\frac{i}{2}(\underset{\sim}{\nu}-\underset{\sim}{q})\cdot\underset{\sim}{R}}\left\{\sqrt{\tfrac{1}{2}}\,E_d\,\underset{\sim}{\sigma_2}\cdot\underset{\sim}{\epsilon}+\tfrac{1}{2}M_d(\tfrac{3}{2})\left[2\underset{\sim}{\nu}\cdot\underset{\sim}{q}\times\underset{\sim}{\epsilon}+i\underset{\sim}{\sigma_2}\cdot\underset{\sim}{\nu}\times\underset{\sim}{\epsilon}\times\underset{\sim}{q}\right]\right\} \tag{13}$$

It is convenient to use the abbreviations

$$E = E_d\, i\sqrt{2}\,\pi$$

$$M+iN = M_d(\tfrac{3}{2})\,\pi \tag{14}$$

From equations (1.20) and (1.21) we have

$$E = \frac{ef}{(\omega(q)\,\nu)^{1/2}}$$

$$M + iN = E\, m_1 e^{i\delta_{33}(q_E)} \sin\delta_{33}(q_E) \left(\frac{\nu\mu}{q^2}\right) \tag{15}$$

Here we write $m_1 = (g_h - g_n)/12Mf^2$ and include μ which was previously set equal to unity. To average over $\underset{\sim}{\varepsilon}$ we use the following results, in which Θ_q is the angle $\underset{\sim}{q}$ makes with $\underset{\sim}{\nu}$.

$$\langle (\underset{\sim}{\nu} \times \underset{\sim}{\varepsilon} \times \underset{\sim}{q})^2 \rangle_{Av} = \frac{\nu^2 q^2}{2}(1 + \cos^2\Theta_q)$$

$$\langle (\underset{\sim}{\nu} \times \underset{\sim}{\varepsilon} \times \underset{\sim}{q}) \cdot \underset{\sim}{\varepsilon} \rangle_{Av} = \nu q \cos\Theta_q \tag{16}$$

$$\langle (\underset{\sim}{\nu} \times \underset{\sim}{\varepsilon} \cdot \underset{\sim}{q})^2 \rangle_{Av} = \frac{\nu^2 q^2}{2} \sin^2\Theta_q$$

Thus we obtain the results

$$|\underset{\sim}{I}^{\pm}|^2 = K^2 \Big[\tfrac{1}{2}(1+\cos^2\Theta_q)\left(\frac{\nu\mu}{q^2}\right)^2 m_1^2 \sin^2\delta - 2\cos\Theta_q \left(\frac{\nu\mu}{q^2}\right) m_1 \sin\delta\cos\delta + 1 \Big] E^2 (\mathcal{J}^{\pm})^2 \tag{17}$$

$$|I|^2 = K^2 E^2 (\mathcal{J}^{+})^2\, 8 \sin^2\Theta_q \left(\frac{\nu\mu}{q^2}\right)^2 m_1^2 \sin^2\delta$$

The form of $\mathcal{J}^{\pm}$ depends on the particular cross-section we consider. For the even and odd parts of the plane wave we have respectively

$\mathcal{J}^{-} = I_{a1} + I_{a2}$ and $\mathcal{J}^{+} = I_{a1} - I_{a2}$, where

$$I_{a1} = \int_0^\infty dR\,(e^{-\alpha R} - e^{-\beta R})\, R\, j_0 \left(R\,|\underset{\sim}{k} + \tfrac{1}{2}(\underset{\sim}{\nu} - \underset{\sim}{q})|\right) \tag{18}$$

$$I_{a2} = \int_0^\infty dR\,(e^{-\alpha R} - e^{-\beta R})\, R\, j_0 \left(R\,|\underset{\sim}{k} - \tfrac{1}{2}(\underset{\sim}{\nu} - \underset{\sim}{q})|\right)$$

The value of δ^- for the S wave with no interaction is

$$\delta^- = \frac{2}{k} I_s = \frac{2}{k}\int_0^\infty dR\,(e^{-\alpha R} - e^{-\beta R}) \sin kR \; j_0\left(R\,\frac{|\underset{\sim}{\nu} - \underset{\sim}{q}|}{2}\right) \tag{19}$$

while for the distorted wave we have

$$\delta^- = e^{i\delta_0}\frac{2}{k} I_e = e^{i\delta_0}\frac{2}{k}\int_0^\infty dR\,(e^{-\alpha R} - e^{-\beta R})\, u(kR)\; j_0\left(R\,\frac{|\underset{\sim}{\nu} - \underset{\sim}{q}|}{2}\right) \tag{20}$$

In the impulse approximation we do not need to consider the $\ell = 1$ wave separately. If we define $X^{\pm} = \int d\Omega_k (I_{a1} \mp I_{a2})$ we have from (1) and (2)

$$\frac{d\sigma}{d\underset{\sim}{q}} = \frac{(2\pi)^{-2}}{\hbar c}\int dk\, k^2 \delta\left(\frac{k^2}{M} - \frac{k_0^2}{M}\right) \frac{4K^2E^2}{3}\Big\{E^- + 2E^+ + m_1 (C^- + 2C^+) + m_1^2 \left(M^- + 2M^+ + \tfrac{3}{4}M^0\right)\Big\} \tag{11'}$$

where

$$E^- = X^- + \frac{16\pi}{k^2}\left(I_e^2 - I_s^2\right)$$

$$E^+ = X^+$$

$$M^{\mp} = \tfrac{1}{2} E^{\mp} (1 + \cos^2 \Theta_q)\left(\frac{\nu\mu}{q^2}\right)^2 \sin^2\delta$$

$$C^{\mp} = -2E^{\mp} \cos\Theta_q \left(\frac{\nu\mu}{q^2}\right) \sin\delta\cos\delta$$

$$M^0 = 8E^+ \sin^2\Theta_q \left(\frac{\nu\mu}{q^2}\right)^2 \sin^2\delta$$

and from (12),

$$\frac{d\sigma}{d\omega(q)\, d\Omega_q}(\text{even}) = \frac{2K^2e^2f^2Mk_0 q}{3(2\pi)^5\hbar c\nu}\left[E^- + m_1 C^- + m_1^2 M^-\right]_{k=k_0}$$

(12')

$$\frac{d\sigma}{d\omega(q)\, d\Omega_q}(\text{odd}) = \frac{K^2e^2f^2Mk_0 q}{6(2\pi)^5\hbar c\nu}\left[8(E^+ + m_1 C^+ + m_1^2 M^+) + 3m_1^2 M^0\right]_{k=k_0}$$

The integrals $X^{\pm}$, I_{ϵ} and I_{ς} are given in Appendix A, equations 1 to 3.

The multiple scattering correction.

The forms of $E^{\pm}$ and so on required when the correction is included are readily derived from the results of Appendix B. The integrals over R are evaluated numerically. The fact that we only calculate the correction for states with $\ell=0$ or $\ell=1$ simplifies the integration over angles.

Kinematics.

If we specify $\underset{\sim}{\nu}$ and $\underset{\sim}{q}$, the magnitude but not the direction of $\underset{\sim}{k}$ is determined. The two nucleons will have quite different momenta relative to the meson which is scattered at them, while our result for $t_{\varsigma i}$ in Part 1 ((1.12 and (1.17)) is given in the centre of mass system of the meson and nucleon. We treat the nucleons as stationary when dealing with the multiple scattering. We may expect errors caused by this to be partially compensated for when we integrate Ω_k over all angles relative to $\underset{\sim}{q}$. We fix our value of k with the energy δ- function in the laboratory system, for given $\underset{\sim}{\nu}$ and $\underset{\sim}{q}$, and then convert $\underset{\sim}{\nu}$ and $\underset{\sim}{q}$ to the centre of mass system of a photon and a free nucleon and do the calculation in this system. We present results in this system, referred to in Section 7 as the centre of mass system because our choice of values of $\underset{\sim}{q}$ is determined mainly by considerations of convenience in the calculation, and because we are not comparing our results with the data from a particular experiment.

7. Results and discussion.

The impulse approximation.

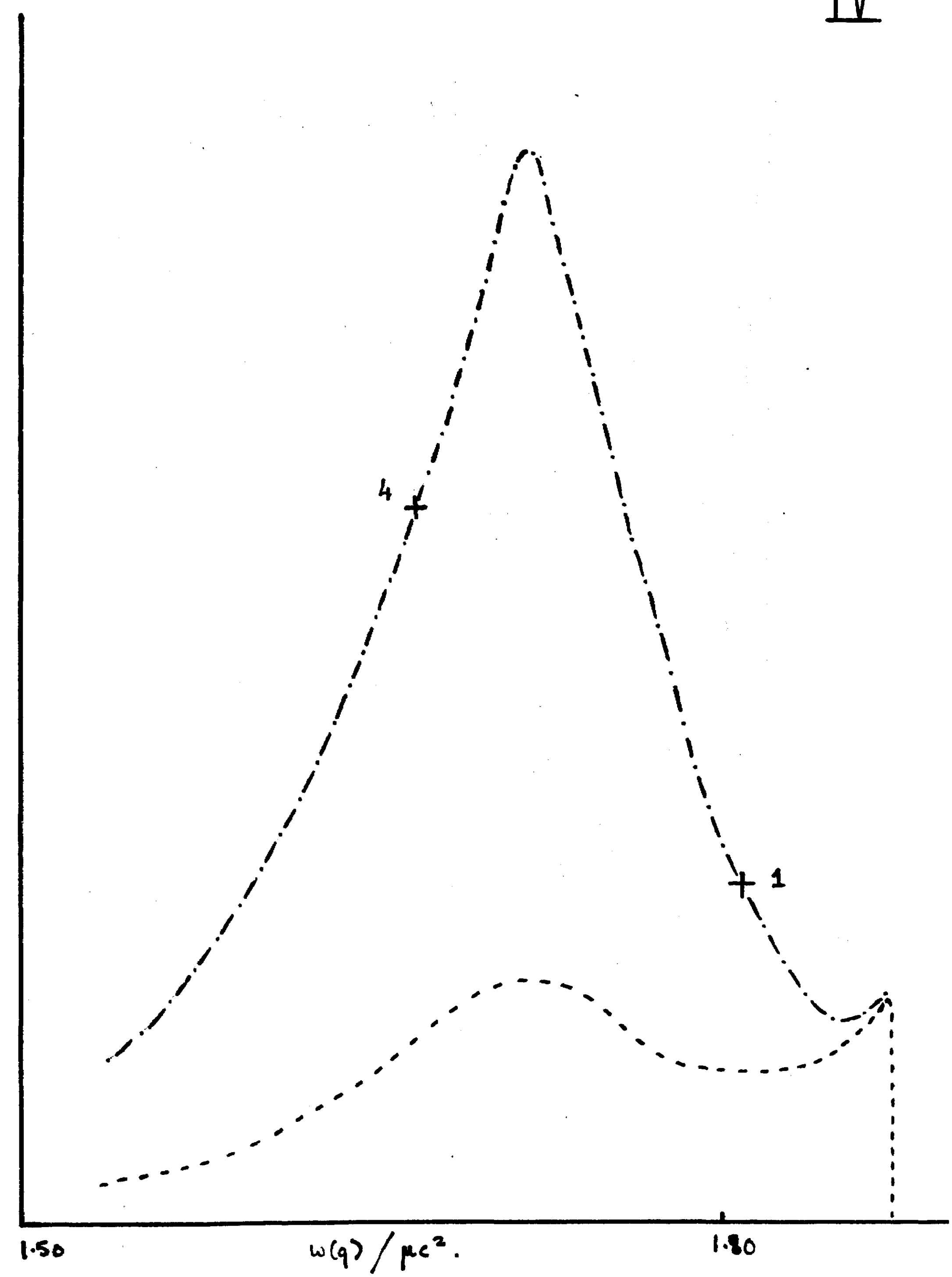
We are only concerned with general features of the cross-section in the impulse approximation. In the graphs $\overline{\text{IV}}$ and $\overline{\text{V}}$ we show the cross-sections at 90° and 30° in the centre of mass system of a photon and a free nucleon, for 300 MeV. photons. The meson energy spectrum has a peak centred on the energy of the meson produced at this angle from a free nucleon, for the same photon energy. As we go to forward angles this peak becomes narrower and its position is nearer the maximum meson energy. (In graph $\overline{\text{V}}$ this peak is only seen on the curve which corresponds to a non-interacting final state).

There is a second peak near the maximum meson energy, caused by the final state nucleon interaction for low values of k, as can be seen from the curves in $\overline{\text{V}}$. At 90° this second peak is unimportant but it dominates the spectrum at forward angles. We refer to the two peaks as the "free nucleon" peak and the "interaction" peak respectively.

The multiple scattering correction.

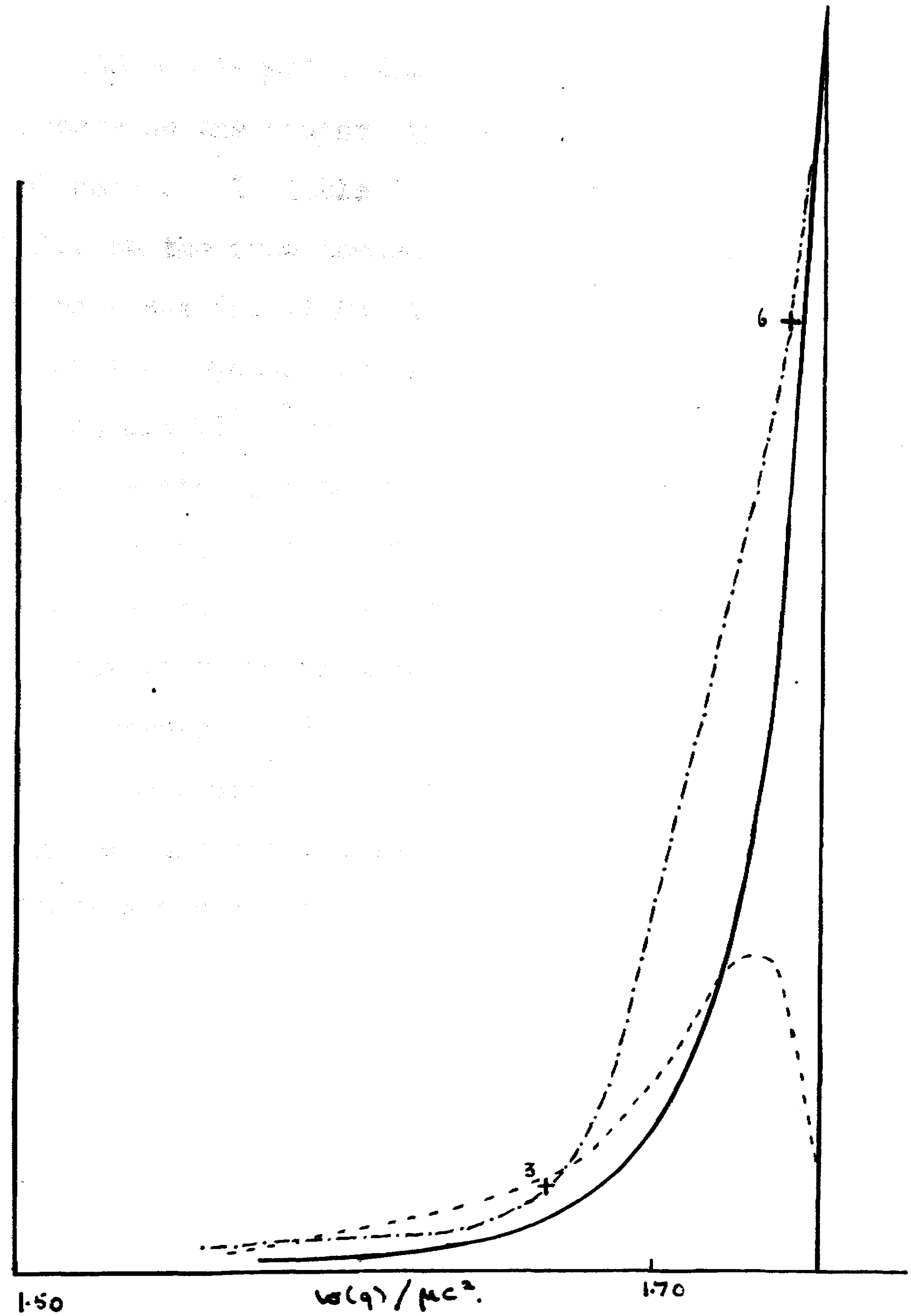
The results here refer to the one pole approximation. The magnitude of the correction is different for the free nucleon and the interaction peaks of the impulse approximation meson energy spectrum. We have calculated the cross-section for the following cases

1. $\omega(q) = 1 \cdot 81\, \mu c^2$, $\Theta_q = 90^\circ$.
2. $\omega(q) = 1 \cdot 81\, \mu c^2$, $\Theta_q = 120^\circ$.
3. $\omega(q) = 1 \cdot 68\, \mu c^2$, $\Theta_q = 30^\circ$.



The I.A. cross-section for $\Theta_q = 90°$. The cross-section is in arbitrary units.

—·—·—·— $\dfrac{d^2\sigma}{d\omega(q)\,d\Omega_q}$ ------ $\dfrac{d^2\sigma}{d\omega(q)\,d\Omega_q}$ (even)



The I.A. cross-section for $\Theta_q = 30°$. Units twice those of IV.

—·—·— Sum of cross-sections for odd and even final states.

——— Cross-section for even final state.

- - - - - - The same, without the nuclear interaction.

4. $\omega(q) = 1{\cdot}68\ \mu c^2$, $\Theta_q = 90°$.

5. $\omega(q) = 1{\cdot}68\ \mu c^2$, $\Theta_q = 150°$.

6. $\omega(q) = 1{\cdot}74\ \mu c^2$, $\Theta_q = 30°$.

The numbers on the graphs IV and V correspond to the 90° and and 30° cases. In table I we show the results of cases 1 to 5, which lie on the free nucleon peak. In the columns giving $E^{\pm}$ and so on (see (6.21) for the notation) the corrected values lie below the impulse approximation values. The units for these quantities are 10^{-26} cm. The last column gives Δ, the percentage correction to the cross-section. It will be seen that the correction can have different signs for terms associated with even and odd space parts of the final state. This, as well as the fact that Δ is small, makes this process less suitable than the process $\gamma + D \rightarrow \pi^{0} + D$ for studying the multiple scattering correction. From table I we see that on the free nucleon peak the correction is about - 4% to - 8%. This will not affect any conclusions drawn from the interpretation of results such as those of graph I in terms of the impulse approximation.

In table II we arrange the cases 1 to 5 in order of increasing value of the parameter $B+k$, where k is the nucleon relative momentum and $B = \frac{1}{2}|\underset{\sim}{\nu} - \underset{\sim}{q}|$. This parameter decreases as we go towards low values of k and forward angles, that is towards the situation in which the interaction peak is important. The corrections Δ (even), Δ (odd) and Δ , to $\frac{d^2\sigma}{d\omega(q)\,d\Omega_q}$ (even), $\frac{d^2\sigma}{d\omega(q)\,d\Omega_q}$ (odd) and $\frac{d^2\sigma}{d\omega(q)\,d\Omega_q}$ respectively, are given in table II. The main feature of this table is the

CASE	E^-	C^-	M^-	E^+	C^+	M^+	M^0	Δ
1	40.70	0	9.14	13.56	0	3.00	47.97	−8.1%
	39.19	−2.00	5.49	11.94	−0.15	3.11	49.38	
2	41.36	12.15	11.60	28.35	8.24	7.96	76.32	−5.7%
	47.11	11.54	8.12	23.81	6.48	7.21	75.33	
3	40.58	−31.09	21.49	28.28	−21.68	14.98	34.23	−3.6%
	35.00	−23.59	10.77	27.14	−19.80	15.81	37.92	
4	81.22	0	24.58	72.97	0	22.08	353.3	−4.8%
	83.31	0.81	18.88	66.68	1.98	20.93	352.7	
5	86.84	66.57	45.99	88.01	67.47	46.61	106.5	−5.6%
	96.54	54.03	37.54	81.41	64.87	42.81	95.4	

CASE	3	1	4	2	5
B+k (10^{13} cm^{-1})	0.89	1.30	1.64	1.74	2.13
Δ (EVEN)	−19.7%	−10.7%	−0.3%	6.2%	−2.4%
Δ (ODD)	5.1%	−5.7%	−6.2%	−12%	−6.9%
Δ	−3.6%	−8.1%	−4.8%	−5.7%	−5.6%

increase in Δ(even) as $\beta+k$ decreases. Now in case 6 (see graph V) $\Delta \doteqdot \Delta$ (even), so we calculate Δ (even) for this case, and find it is - 22%. It should be noted that the 30^o case is an extreme one as far as comparison with experiments detecting the meson is concerned. It corresponds to laboratory angle 24^o, while the furthest forward angle used in the experiments compared by Hagermann et. al. (1957) is 26^o. We conclude that while the correction is in general less than 10% it can rise to 20% in the interaction peak at forward angles. This seems reasonable if we recall that for elastic pion production the correction is large, as described in our account of Chappelear's work in Section 1, while the case of small k is our nearest approach to an elastic process.

The question arises whether the correction to the interaction peak can affect the interpretation of the experimental results. In the work of Hagermann et. al. we notice first that the meson energy resolution is 15 MeV. Assuming that this would be the same at the meson energies we consider, it covers in our 30^o results the range of corrections -4% (case 3) to -22% (case 6). This will have the effect of reducing the cross-section, as will the fact that a bremsstrahlung photon spectrum is used. Mesons coming from the interaction peak for some photon energies, and from well off it for other photon energies, will be detected together. Since the peak is high at forward angles the correction may still approach 20% in an experiment detecting mesons with the energies we consider, at 30^o. The interpretation of such an experiment in terms of the impulse approximation

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would, therefore, have to be corrected, but the effect would not be worth looking for as a way of examining the multiple scattering process. In the other kind of experiment mentioned in Section 1, in which the energy of one photon is measured, it would be possible in principle to examine low k values separately However, in the work of Bandtel et. al. (1958) the accuracy is low, and besides they find it necessary to work with mesons produced at a large angle.

We may remark at this stage that the integrals (see Appendix B) which contain the functions $F_2(R)$ and $F_5(R)$ are increased by about 50% when these functions are replaced by $\hat{F}_2(R)$ and $\hat{F}_5(R)$, which are defined in Section 4. We have not carried out a full calculation with the $\hat{F}(R)$ for reasons given in Section 4, but mention this result because it differs from the results of Drell & Verlet (1955) whose model with a factorable potential gives a smaller correction than the one pole approximation. (See Section 1). Another difference from their results concerns the importance of the double scattering, which gives a major part of their multiple scattering effect. This corresponds in our case to photoproduction followed by one scattering. We cannot evaluate, in our formalism, the effect of this process alone but if it were dominant the contribution to (3.19) from an odd number of scatterings would give the greater part of the correction. We have evaluated Δ (even) in case 3 including only the integrals involving the "odd scattering" functions $F_4(R)$, $F_5(R)$ and $F_6(R)$. (See Appendix B). We find that the even and odd scatterings are of comparable importance.

Appendix A. Integrals used in the impulse approximation cross-section.

We give the values of the integrals $X^{\pm}$, I_s and I_e which appear in Section 6, in the functions defined by (6.21). Writing $\underset{\sim}{B} = \frac{1}{2}(\underset{\sim}{\nu} - \underset{\sim}{q})$ we have

$$X^{\pm} = 4k^2\left\{\frac{1}{(\alpha^2+k^2+B^2)^2-4k^2B^2} + \frac{1}{(\beta^2+k^2+B^2)^2-4k^2B^2}\right\}$$

$$\pm \frac{2k}{B(\alpha^2+\beta^2+2k^2+2B^2)} \log\left\{\frac{\alpha^2+(B+k)^2}{\alpha^2+(B-k)^2} \quad \frac{\beta^2+(B+k)^2}{\beta^2+(B-k)^2}\right\} \tag{1}$$

$$- \frac{2k}{B(\beta^2-\alpha^2)} \log\left\{\frac{\alpha^2+(B+k)^2}{\alpha^2+(B-k)^2} \quad \frac{\beta^2+(B-k)^2}{\beta^2+(B+k)^2}\right\}$$

$$\mp \frac{k}{B(\alpha^2+B^2+k^2)} \log\left\{\frac{\alpha^2+(B+k)^2}{\alpha^2+(B-k)^2}\right\} \mp \frac{k}{B(\beta^2+B^2+k^2)} \log\left\{\frac{\beta^2+(B+k)^2}{\beta^2+(B-k)^2}\right\}$$

$$I_s = \frac{1}{4B} \log\left\{\frac{\alpha^2+(B+k)^2}{\alpha^2+(B-k)^2} \quad \frac{\beta^2+(B-k)^2}{\beta^2+(B+k)^2}\right\} \tag{2}$$

$$I_e = \cos\delta_0\, I_s + \frac{\sin\delta_0}{2B}\left[\tan^{-1}\frac{B+k}{\alpha} + \tan^{-1}\frac{B-k}{\alpha}\right. \tag{3}$$

$$\left. - \tan^{-1}\frac{B+k}{\beta} - \tan^{-1}\frac{B-k}{\beta} + 2\tan^{-1}\frac{B}{\beta+\gamma} - 2\tan^{-1}\frac{B}{\alpha+\gamma}\right]$$

We also give here the integral which appears in the $\ell = 1$ impulse approximation cross-section. This has only to be used when multiple scattering is included. For this cross-section the function J^+ of (6.17) is

$$J^+ = 6\cos(\Theta_k - \Theta_B)\; K\, I_p$$

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where Θ^{*}_{k} , Θ^{*}_{β} are the angles $\underset{\sim}{k}$ and $\underset{\sim}{\beta}$ make with $\underset{\sim}{\nu}'$ and

$$I_p = \int_0^\infty dR\, R\, j_1(kR)\, j_1(\beta R)\, (e^{-\alpha R} - e^{-\beta R})$$

$$= \frac{1}{8k^2\beta^2}\left[(\alpha^2+\beta^2+k^2)\log\frac{\alpha^2+(\beta+k)^2}{\alpha^2+(\beta-k)^2} - (\beta^2+\beta^2+k^2)\log\frac{\beta^2+(\beta+k)^2}{\beta^2+(\beta-k)^2}\right] \quad (4)$$

Appendix B. Formulae required for the matrix element when multiple scattering is included.

We give the results which are to be used in (6.11) to obtain the cross-section including the correction. The general forms of $\underset{\sim}{I}^{\pm}$, I defined by (6.9) are

$$\underset{\sim}{I}^{\pm} = L_1^{\pm}\underset{\sim}{\nu}\times\underset{\sim}{\varepsilon}\times\underset{\sim}{q} + L_2^{\pm}\underset{\sim}{\varepsilon} + \underset{\sim}{\nu}\times\underset{\sim}{\varepsilon}\times\underset{\sim}{L}^{\pm} \tag{1}$$

$$I = 4i\left[L_1^{\pm}\underset{\sim}{\nu}\times\underset{\sim}{\varepsilon}\cdot\underset{\sim}{q} + \underset{\sim}{\nu}\times\underset{\sim}{\varepsilon}\cdot\underset{\sim}{L}^{+}\right]$$

In the impulse approximation $\underset{\sim}{L}^{\pm} = 0$, as can be seen from (6.13). From (1) we obtain the averages over $\underset{\sim}{\varepsilon}$,

$$\langle|\underset{\sim}{I}^{\pm}|^2\rangle_{Av} = \nu^2\left\{\tfrac{1}{2}q_x^2+\tfrac{1}{2}q_y^2+q_z^2\right\}|L_1^{\pm}|^2 + \nu^2\left\{\tfrac{1}{2}|L_x^{\pm}|^2+\tfrac{1}{2}|L_y^{\pm}|^2+|L_z^{\pm}|^2\right\}$$

$$+ |L_2^{\pm}|^2 + \nu L_z^{\pm}(L_2^{\pm})^* + \text{complex conjugate}$$

$$+ q_z\nu L_1^{\pm}(L_2^{\pm})^* + \text{complex conjugate}$$

$$+ \nu^2\left\{\tfrac{1}{2}L_x^{\pm}q_x+\tfrac{1}{2}L_y^{\pm}q_y+L_z^{\pm}q_z\right\}(L_1^{\pm})^* + \text{complex conjugate}$$

$$\langle|I|^2\rangle_{Av} = 8\Big\{|L_1^{+}|^2\nu^2(q_x^2+q_y^2) + \nu^2(|L_x^{+}|^2+|L_y^{+}|^2)$$

$$+ \nu^2(q_xL_x^{+}+q_yL_y^{+})(L_1^{+})^* + \text{complex conjugate}\Big\}$$

If we denote $\tfrac{1}{2}(\underset{\sim}{\nu}+\underset{\sim}{q})$, $\tfrac{1}{2}(\underset{\sim}{\nu}-\underset{\sim}{q})$ by $\underset{\sim}{A}$, $\underset{\sim}{B}$ we have the following results in which we have used the distorted S wave or the P wave final state wave functions as the case may be.

$$L_1^- = \frac{2K}{k}(N-iM)(I_e + I_{fA} - I_{fB})e^{i\delta_0}$$

where I_e is given by (A.3) and

$$I_{fA} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R})\, u(kR)\, F_5(R)\, j_0(RA)$$

$$I_{fB} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R})\, u(kR)\, F_2(R)\, j_0(RB)$$

$$L_2^- = \frac{2KE}{k}\left[iI_e + \cos(\Theta_q - \Theta_B)\, I_{gB} - \cos(\Theta_q - \Theta_A) I_{gA}\right] e^{i\delta_0}$$

where

$$I_{gA} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R}) R\, u(kR)\, F_4(R)\, j_1(RA)$$

$$I_{gB} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R}) R\, u(kR)\, F_1(R)\, j_1(RB)$$

$$\underset{\sim}{L}^- = \frac{2K}{3k}(N-iM)\left[\underset{\sim}{q}(I_{qB} - I_{qA}) + \underset{\sim}{n}(\underset{\sim}{B}) q I_{nB} - \underset{\sim}{n}(\underset{\sim}{A}) q I_{nA}\right] e^{i\delta_0}$$

where

$$\left.\begin{matrix} I_{qA} \\ I_{nA} \end{matrix}\right\} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R})\, u(kR)\, F_6(R) \left\{\begin{matrix} j_0(RA) \\ j_2(RA) \end{matrix}\right.$$

$$\left.\begin{matrix} I_{qB} \\ I_{nB} \end{matrix}\right\} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R})\, u(kR)\, F_3(R) \left\{\begin{matrix} j_0(RB) \\ j_2(RB) \end{matrix}\right.$$

and

$$n_x(\underset{\sim}{A}) = -2\sin\Theta_A \cos(\Theta_q - \Theta_A) + \cos\Theta_A \sin(\Theta_q - \Theta_A)$$

$$n_y(\underset{\sim}{A}) = 0$$

$$n_z(\underset{\sim}{A}) = -2\cos\Theta_A \cos(\Theta_q - \Theta_A) - \sin\Theta_A \sin(\Theta_q - \Theta_A)$$

with a similar form for $\underset{\sim}{n}(\underset{\sim}{B})$.

$$L_1^+ = 6(M+iN)K\left\{\cos(\Theta_k - \Theta_B)(I_p + I_{fB}) - \cos(\Theta_k - \Theta_A)\, I_{fA}\right.$$

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where I_p is given by (A.4) and

$$I_{eA} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R})\, R j_1(kR)\, F_5(R)\, j_1(AR),$$

$$I_{eB} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R})\, R j_1(kR)\, F_2(R)\, j_1(BR).$$

$$L_2^+ = -KE\Big[b\cos(\Theta_k - \Theta_B) I_p + iq\big(c(\underset{\sim}{B}) I_{cB} - c(\underset{\sim}{A}) I_{cA} + d(\underset{\sim}{B}) I_{dB} - d(\underset{\sim}{A}) I_{dA}\big)\Big]$$

where

$$\left.\begin{matrix} I_{cA} \\ I_{dA} \end{matrix}\right\} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R})\, R^2 j_1(kR)\, F_4(R) \left\{\begin{matrix} j_0(AR) \\ j_2(AR) \end{matrix}\right.$$

$$\left.\begin{matrix} I_{cB} \\ I_{dB} \end{matrix}\right\} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R})\, R^2 j_1(kR)\, F_1(R) \left\{\begin{matrix} j_0(BR) \\ j_2(BR) \end{matrix}\right.$$

and

$$c(\underset{\sim}{A}) = \cos(\Theta_q - \Theta_A)\cos(\Theta_k - \Theta_A) + \sin(\Theta_q - \Theta_A)\sin(\Theta_k - \Theta_A)\cos(\Phi_k - \Phi_A)$$

$$d(\underset{\sim}{A}) = -2\cos(\Theta_q - \Theta_A)\cos(\Theta_k - \Theta_A) + \sin(\Theta_q - \Theta_A)\sin(\Theta_k - \Theta_A)\cos(\Phi_k - \Phi_A)$$

with similar forms for $c(\underset{\sim}{B})$ and $d(\underset{\sim}{B})$. Finally

$$\underset{\sim}{L}^+ = \frac{6Kq}{5}(M + iN)\Big\{\underset{\sim}{h}(\underset{\sim}{A}) I_{hA} - \underset{\sim}{h}(\underset{\sim}{B}) I_{hB} + \underset{\sim}{\ell}(\underset{\sim}{A}) I_{\ell A} - \underset{\sim}{\ell}(\underset{\sim}{B}) I_{\ell B}\Big\}$$

where

$$\left.\begin{matrix} I_{hA} \\ I_{\ell A} \end{matrix}\right\} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R})\, R j_1(kR)\, F_6(R) \left\{\begin{matrix} j_1(AR) \\ j_3(AR) \end{matrix}\right.$$

$$\left.\begin{matrix} I_{hB} \\ I_{\ell B} \end{matrix}\right\} = \int_0^\infty dR(e^{-\alpha R} - e^{-\beta R})\, R j_1(kR)\, F_3(R) \left\{\begin{matrix} j_1(BR) \\ j_3(BR) \end{matrix}\right.$$

and

$$h_x(\underset{\sim}{A}) = \cos(\Theta_k - \Theta_A)\{3\sin\Theta_A \cos(\Theta_q - \Theta_A) + \cos\Theta_A \sin(\Theta_q - \Theta_A)\}$$
$$+ \sin(\Theta_k - \Theta_A)\cos(\Phi_k - \Phi_A)\cos(\Theta_q - 2\Theta_A)$$

$$h_y(\underset{\sim}{A}) = \sin(\Theta_k - \Theta_A)\sin(\Phi_k - \Phi_A)\cos(\Theta_q - \Theta_A)$$

$$h_z(\underset{\sim}{A}) = \cos(\Theta_k - \Theta_A)\{3\cos\Theta_A \cos(\Theta_q - \Theta_A) - \sin\Theta_A \sin(\Theta_q - \Theta_A)\}$$
$$+ \sin(\Theta_k - \Theta_A)\cos(\Phi_k - \Phi_A)\sin(\Theta_q - 2\Theta_A)$$

$$\ell_x(\underset{\sim}{A}) = h_x(\underset{\sim}{A}) - 5\cos(\Theta_q - \Theta_A)\sin\Theta_A \cos(\Theta_k - \Theta_A)$$

$$\ell_y(\underset{\sim}{A}) = h_y(\underset{\sim}{A})$$

$$\ell_z(\underset{\sim}{A}) = h_z(\underset{\sim}{A}) - 5\cos(\Theta_q - \Theta_A)\cos\Theta_A \cos(\Theta_k - \Theta_A)$$

with similar forms for $\underset{\sim}{h}(\underset{\sim}{B})$ and $\ell(\underset{\sim}{B})$. Using these results the evaluation of $\int d\Omega_k \langle |\underset{\sim}{I}^{\pm}|^2\rangle_{Av}$ and $\int d\Omega_k \langle |I|^2\rangle_{Av}$ from (2) is straightforward.

References.

A.M. Baldin, Il Nuovo Cimento 8, 569 (1958).

K.C. Bandtel, W.J. Frank and R. Madey. Phys.Rev.109,2117 (1958).

E.H. Bellamy, P. Palit and R.W.P. McWhirter, Proc.Phys.Soc. 70A, 763 (1957.

H.A. Bethe, Phys. Rev. 76,18 (1949).

H.A. Bethe and F. De Hoffmann. Mesons and Fields Vol.2 (1955).

B.H. Bransden and R.G. Moorhouse. Nuclear Physics 6,310 (1958).

K.A. Brueckner, Phys. Rev. 89,834 (1953a).

Phys. Rev. 90,715 (1953b).

K.A. Brueckner and K.M. Watson. Phys. Rev.92,1023 (1953).

S.T. Butler. Phys. Rev. 87,1117 (1952).

J.M. Cassels. Proceedings of the 7th Rochester Conference, Part 2 (1957).

J. Chappelear. Phys. Rev. 99,254 (1955).

G.F. Chew. Phys. Rev. 80,196 (1950).

Phys. Rev. 93,341 (1954a).

Phys. Rev. 94,1755 (1954b).

G.F. Chew and H.W. Lewis. Phys. Rev. 84,779 (1952).

G.F.Chew and F.E. Low. Phys. Rev. 101,1570 (1956a).

Phys. Rev. 101,1579 (1956b).

G.F. Chew and M.L. Goldberger. Phys. Rev. 87,778 (1952).

G.F. Chew and G. - C. Wick. Phys. Rev. 85,636 (1952).

E.U. Condon and G.H. Shortley. Theory of Atomic Spectra,p.71 (1951).

S.D. Drell and L. Verlet. Phys. Rev. 99,849 (1955).

N.C. Francis. Phys. Rev. 89,766 (1953).

M. Gell - Mann and K.M. Watson. Annual Reviews of Nuclear Science 4,219 (1954).

D.C. Hagermann, K.M. Crowe and R.M. Friedman. Phys. Rev.106,818 (1957).

N. Kemmer, Proc. Camb. Phil. Soc. 34,354 (1938).

M. Lax and H. Feshbach. Phys.Rev. 88,509 (1952).

E.W. Laing. Proc. Phys. Soc. 71,932 (1958).

E.W. Laing and R.G. Moorhouse. Proc.Phys.Soc. 70A,629 (1957).

I.L. Lebow, B.T. Feld, D.H. Frisch and L.S. Osborne. Phys.Rev. 85,681 (1952).

B. Lippmann and J. Schwinger. Phys.Rev. 79,469 (1950).

S. Machida and T. Tamura. Prog. Theor. Phys. 6,572 (1951).

J. Moravscik. Phys.Rev. 104,1451 (1957).

J. Orear, Phys.Rev. 100,288 (1956).

S. Penner. Phys.Rev. 105,1113 (1957).

R.M. Rockmore. Phys.Rev. 105,256 (1957).

J.W. Rosengren and N. Baron. Phys.Rev. 101,410 (1956).

Y. Saito, Y. Watanabe and Y. Yamaguchi. Prog. Theor.Phys.7,103 (1952).

K.D.C. Stoodley. Ph.D. Thesis, Glasgow (1957).

V.R. Velibekov and V.A. Meshcheryakov. Doklady Akad.Nauk S.S.S.R. 105,941 (1955).

K.M. Watson. Phys.Rev. 89,575 (1953).

Phys.Rev. 95,228 (1954).

R.S. White, M.J. Jacobsohn and A.G. Schultz. Phys.Rev.88,836 (1952).

G - C. Wick. Rev. Mod. Phys. 27,339 (1955).

Contents.

Part II.

Summary.

The production of K mesons in proton-proton collisions near threshold.

Contents.

SUMMARY. Part II is devoted to the process $p+p \rightarrow p+\Lambda^{\circ}+K^{+}$ near threshold. The relevant earlier work is reviewed. Two models are presented, one of which treats the interaction of the K meson with the two baryons as a small perturbation, while giving a phenomenological treatment of the interaction of the pion field with the baryons. The other uses the formulation of meson theory in terms of physical states. Both are extensions of methods used by other authors for the problem of π meson production. The models give different descriptions of the initial state, and the interpretation of these descriptions is a doubtful point in this work. In each model the final state consists of a free K meson and a proton and Λ° whose interaction is described by a potential. The operator inducing transitions from the initial to the final state is the interaction Hamiltonian for a nucleon and a pseudoscalar $\Lambda^{\circ}+K^{+}$ system, in a static source theory. For the calculations this is generalised to allow for nucleon recoil. The calculation of the cross-section is based on an approximate description of proton-proton scattering in the appropriate energy region, in which the elastic part of the scattering is entirely diffraction scattering associated with the inelastic part. With this approximation the second model has a plane wave initial state. The potential in the initial state in the first model is complex. The cross-sections obtained with the two models differ greatly. A feature of both models is the importance of S wave mesons associated with the nucleon recoil term/

term in the transition operator. Direct comparison with experiment is not possible at present.

[illegible] construction of high energy machines producing [illegible] in large quantities, and in the theory the [illegible] importance of the strangeness quantum number. [illegible] This has made it possible to fit the qualitative [illegible] production and decay of these particles [illegible]. The two main lines of theoretical enquiry have [illegible] hand attempts to find symmetries underlying the [illegible] (see for example d'Espagnat and Prentki [illegible]), [illegible] other the use in this new field of the [illegible] pion physics. Recent work of this nature includes [illegible] and Tamm-Dancoff calculations of the scattering of [illegible] nucleons [illegible] 1957 [illegible]), [illegible] dispersion relations (for example [illegible]) and the study of hyperon-nucleon forces [illegible] nucleon-nucleon forces [illegible]

We shall be concerned with the production of [illegible] proton-proton collisions, which has been studied [illegible] at Berkeley and Brookhaven. (See for example [illegible] (1957) and Lee et. al. [illegible].) [illegible]

1. Introduction.

In the study of heavy mesons and hyperons considerable progress has been made in the last few years. The main factors contributing to this have been, on the experimental side, the construction of high energy machines producing these particles in large quantities, and in the theory the recognition of the importance of the strangeness quantum number. (Gell-Mann(1955)) This made it possible to fit the qualitative features of the production and decay of these particles into a simple scheme. The two main lines of theoretical enquiry have been on the one hand attempts to find symmetries underlying the Gell-Mann scheme (see for example d'Espagnat and Prentki (1958)), and on the other the use in this new field of the techniques developed in pion physics. Recent work of this nature includes weak coupling and Tamm-Dancoff calculations of the scattering of K mesons by nucleons (Ceolin and Taffara 1957 a,b), applications of dispersion relations (for example Matthews and Salam (1958)), and the study of hyperon-nucleon forces by methods developed for nucleon-nucleon forces (Lichtenberg and Ross (1957,1958)).

We shall be concerned with the production of K mesons in proton-proton collisions, which has been studied experimentally at Berkeley and Brookhaven. (See for example Baumel et.al. (1957) and Lea et. al. (1958).) By conservation of charge and strangeness a proton-proton collision can lead to these final states,

$$K^+ + \Lambda^\circ + p \qquad\qquad K^+ + \Sigma^\circ + p$$

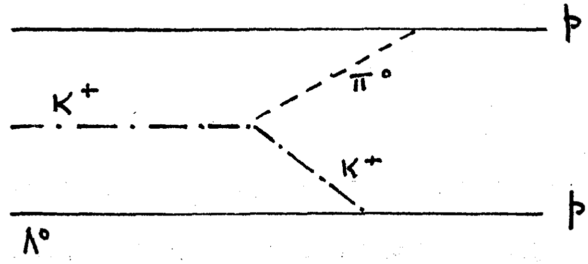
$K^{\circ} + \Sigma^{+} + p$ $\qquad$ $K^{+} + \Sigma^{+} + n$

the threshold in the first case being 1.58 BeV. and in the other three 1.78 BeV. There are no other final states possible containing one K meson and two baryons. We intend to confine our attention to the energy region in which only the first process can occur. This is too near threshold for comparison with experiment to be possible at present. The lowest energy for which a result is available is 1.95 BeV. (Lea et.al.(1958)) and that result is based on one event. However, experience with pions suggests that experiments near threshold will be necessary before we can learn much about the production process.

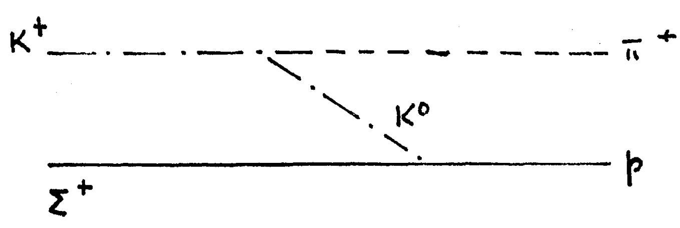
The previously published theoretical work on K meson production in nucleon-nucleon collisions falls into three groups. Several papers have appeared which study, well above threshold, the relative abundance of various K meson and pion production modes, using the statistical methods of Fermi (1950) and Landau (1953). References to this work will be found in the paper of Barashenkov et.al.(1958). Then there are papers by Henley (1957), Costa and Feld (1958), and Feldman and Matthews (1958) which deal with the region near threshold, and discuss features of the process which are not affected by detailed assumptions concerning the mechanism of production. They obtain relations between the cross-sections for different isotopic spin states, examine the behaviour to be expected for different assignments of the parity of the strange particles, and study the effect ofthe final state interaction of the hyperon and the/

the nucleon. They do not attempt to derive absolute values of the cross-sections. We shall discuss below various points treated in these papers.

Thirdly there are papers by Barshay (1956) and Peaslee (1957) giving models which are intended to reproduce the marked forward-backward peaking of the K meson angular distribution in the centre of mass system of the two protons, which was a feature of the early experimental work (Osher(1956)). It should be noted that this marked anisotropy is not apparent in more recent experiments. (See the discussion in Section 5.) In Peaslee's model one nucleon is considered as dissociated into a K meson-hyperon system, the K meson being removed in a "pick-up" process by a pion in the cloud of the other nucleon. It is a rough phenomenological treatment, while Barshay gives a field theoretical (weak coupling) treatment of a similar process. The graph corresponding to this is



Barshay has also studied the process of K meson production when a pion is incident on a nucleon, for example in the process



This will have a forward peak in the K meson angular distribution, which is in contradiction to the observed behaviour (Dalitz(1957), page 187). In addition to the disagreement with experiment there is a theoretical argument against Barshay's approach. The absorption of a pion by a K meson depends on the existence of two types of K mesons with different parity, θ and τ say, so that we can have $\theta \longleftrightarrow \tau + \pi$. Rather than having such a parity doublet the K meson is now considered to have a definite parity. We have looked for a process which does not involve this absorption process, and which might be suitable for calculating the absolute value of the cross-section near threshold. If we examine the work which has been done on pion production in proton-proton collisions we find that one of the most successful methods has been the phenomenological one (Geffen(1955) and Lichtenberg(1955)) in which we take the matrix element $\langle f | U | i \rangle$ of an operator U , which creates one meson, between initial and final states $|i\rangle$, $|f\rangle$ of two nucleons scattered in appropriate potentials. Much of the success of the method has, of course, been due to its lending itself to the inclusion of the scattering of the meson by one of the nucleons. This turns out to be the dominating feature of the process, because of the resonance in π meson nucleon scattering. (See Lichtenberg(1957), Durney(1958) and Mandenstam(1958)). The corresponding scattering of the K meson by the final state proton/

proton should not be so important, since K meson scattering by nucleons corresponds to a weak repulsive interaction(Dalitz(1957) section 4.5). In general repulsive interactions in final states have less effect on the behaviour of cross-sections than attractive interactions of the same strength. (Watson(1952))

If we look for a similar model in our problem we require data on proton-proton elastic and inelastic scattering in this energy region, in order to obtain a potential for our initial state. The fullest treatment available is the analysis by Fowler et.al. (1956) of their experimental results at 0.8 BeV.,1.5 BeV. and 2.75 BeV.. They use a geometrical optical model of an absorbing (and almost black) sphere, of radius 0.93×10^{-13} cm at all energies, and with absorption coefficient K (see Section 3) with the value 4.3×10^{13} cm^{-1} at 0.8 BeV., 3.7×10^{13} cm^{-1} at 1.5 BeV., and 2.7×10^{13} cm^{-1} at 2.75 BeV. They fit the elastic and inelastic total cross-sections, and the differential elastic cross-section, fairly well. In this model the distinction between scattering with and without spin flip is lost. A difficulty in the description of the initial state is the small amount of information available and the possibility of making quite different analysis of the experimental data. (See for example Ito et.al. (1958)). We find in fact that the description of the initial state is the main source of difficulty and ambiguity in this approach to our problem.

2. Two models for the process.

We make the following assumptions about the spin and parity of the strange particles, which are in agreement with their observed behaviour. See for example Dalitz(1957), Walker(1958). The K mesons have spin 0, the hyperons spin $\frac{1}{2}$. Because of the associated production of a K meson together with a hyperon the parity which is defined is that of the system $\Lambda^0 K$ or ΣK relative to a nucleon, which we take to be negative. We adopt the convention that the hyperons have positive parity relative to a nucleon, and refer to the K meson as pseudoscalar. In isotopic spin space the Λ^0 is a scalar, the nucleon, K meson and Ξ are spinors,

$$N = \begin{pmatrix} n \\ p \end{pmatrix} \qquad K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \qquad \Xi = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}$$

and the pion and Σ are vectors

$$\underset{\sim}{\pi} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} \qquad \underset{\sim}{\Sigma} = \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{bmatrix}$$

Here for example $\pi^{\pm} = \pi_1 \mp i\pi_2$, $\pi^0 = \pi_3$.

We use the Hamiltonian $H = H_0 + h_\pi + h_K$, or

$$H_0 + h_\pi(\bar{N}N) + h_\pi(\bar{\Lambda}\Sigma) + h_\pi(\bar{\Sigma}\Lambda) + h_\pi(\bar{\Sigma}\Sigma) + h_\pi(\bar{\Xi}\Xi) + h_K(\bar{N}\Lambda) + h_K(\bar{\Lambda}N) + h_K(\bar{N}\Sigma) + h_K(\bar{\Sigma}N) + h_K(\bar{\Lambda}\Xi) + h_K(\bar{\Xi}\Lambda) + h_K(\bar{\Sigma}\Xi) + h_K(\bar{\Xi}\Sigma) \tag{1}$$

Here for example N destroys and $\bar{N}$ creates a nucleon. The free field Hamiltonian H_0 is the sum of the kinetic energy T of the baryons and the energy of the meson fields,

$$H_0 = T + \sum_p a_p^\dagger a_p \,\omega(p) + \sum_q b_q^\dagger b_q \,\omega(q) \tag{2}$$

Here $a_p^\dagger$ (a_p) creates (destroys) a pion of momentum p, and $b_q^\dagger$ (b_q) a K meson of momentum q. The parts of h_π and h_K which we shall finally require are

$$h_K(\bar{\Lambda}N) + h_K(\bar{N}\Lambda) = \sum_j \sum_q \left(b_q^\dagger \bar{\Lambda}\, U_{jq}^{\circ\dagger} N + b_q \bar{N}\, U_{jq}^{\circ} \Lambda \right)$$

where

$$U_{jq}^{\circ} = \sqrt{\frac{4\pi}{2\omega(q)}}\, \frac{i f_\Lambda^{\circ} u(q)}{\mu_K}\, \underset{\sim}{\sigma}_j \cdot \underset{\sim}{q}\, e^{i \underset{\sim}{q} \cdot \underset{\sim}{r}_j} \tag{3}$$

and

$$h_\pi(\bar{N}N) = \sum_\lambda \sum_p \sum_j \left(a_p^\lambda \bar{N}\, V_{jp\lambda}^{\circ} N + a_p^{\lambda\dagger} \bar{N}\, V_{jp\lambda}^{\circ\dagger} N \right)$$

where

$$V_{jp\lambda}^{\circ} = \sqrt{\frac{4\pi}{2\omega(p)}}\, \frac{i f^{\circ} v(p)}{\mu_\pi}\, \tau_{j\lambda}\, \underset{\sim}{\sigma}_j \cdot \underset{\sim}{p}\, e^{i \underset{\sim}{p} \cdot \underset{\sim}{r}_j}$$

$\underset{\sim}{r}_j$ is the position of nucleon j . $\underset{\sim}{\sigma}_j$, $\underset{\sim}{\tau}_j$ are the spin and isotopic spin operators of nucleon j . f_Λ° , f° are the unrenormalised coupling constants, μ_K and μ_π the meson masses, and $u(q)$, $v(p)$ are momentum cut-off functions. The other terms of the interaction Hamiltonian have similar forms, with

appropriate changes for the different behaviour of the fields in isotopic spin space. In (3) we use static source theory for h_K and h_π . We can expect this to be less useful for h_K than for h_π because of the greater mass of the K meson. We define U_{jq} , $V_{jp\lambda}$ as being the same as U°_{jq}, $V^\circ_{jp\lambda}$ but with the renormalised coupling constants f_Λ , f replacing f_Λ° , f° . A static source treatment of the pion and K meson fields is discussed by Amati and Vitale (1957).

We shall describe two ways of dealing with the problem of the associated production of a K meson and a Λ° , one of which leads to the matrix element $\sum_{j=1}^{2}(\psi_f^{(-)}, U^{\circ\dagger}_{jq}\, \psi_i^{(+)})$ the other to $\sum_{j=1}^{2}(\psi_f^{(-)}, U^{\dagger}_{jq}\, \psi_i^{(+)})$.
The final state $\psi_f^{(-)}$ is dealt with in the same way in the two methods, but the initial states are quite different. The first method treats the interaction term h_K as a small perturbation, while giving a phenomenological treatment of the interaction of the baryons with the pion field. We introduce as in Section 1 of Part I the wave function $\Psi^{(+)}$ which satisfies

$$\Psi^{(+)} = (1 + \tfrac{1}{a}T)\,\Phi = \left\{1 + \frac{1}{a - h_\pi - h_K}(h_\pi + h_K)\right\}\Phi \qquad (4)$$

Φ being an eigenstate of H_0, and $a = E - H_0 + i\varepsilon$, and define the matrix Ω by $\Omega\,\Phi = \Psi^{(+)}$, so that between different eigenstates Φ_a, Φ_b we have

$$\Omega = \frac{1}{a} T \tag{5}$$

We take Φ as a state of two protons, and examine the part of Ω which can lead to states containing one K meson and two baryons, but no pion. We use the notation $D_\pi \Omega$, $D_K \Omega$ and $D_K^+ \Omega$ for the parts of Ω leading from Φ to states having no pions, no K mesons, and one K meson respectively. We thus require $D_\pi D_K^+ \Omega$. From (4)

$$\Omega = 1 + \frac{1}{a - h_\pi - V_K}\left\{ h_K + V_K + h_\pi + h_K \frac{1}{a - h_\pi} h_\pi \right\} \tag{6}$$

where $V_K = h_K \frac{1}{a - h_\pi} h_K$. Therefore

$$D_\pi D_K^+ \Omega = D_\pi D_K^+ \left\{ \frac{1}{a - h_\pi - V_K} h_K \left(1 + \frac{1}{a - h_\pi} h_\pi\right)\right\}$$

$$= D_\pi \left\{ \left(D_K \frac{1}{a - h_\pi - V_K}\right) h_K \left(1 + \frac{1}{a - h_\pi} h_\pi \right)\right\} \tag{7}$$

We now ignore V_K in the denominator to the left of h_K so that we are taking h_K as a small perturbation. When we do this we can write $h_K = h_K(\bar{\Lambda}N) + h_K(\bar{\Sigma}N)$. There are two kinds of process described by the operator

$$\mathcal{D}_\pi \left\{ \frac{1}{a-h_\pi} \left(h_K(\bar{\Lambda}N) + h_K(\bar{\Sigma}N) \right) \left(1 + \frac{1}{a-h_\pi} h_\pi \right) \right\}$$

One kind are described by the part

$$\left(\mathcal{D}_\pi \frac{1}{a-h_\pi} \right) \left(h_K(\bar{\Lambda}N) + h_K(\bar{\Sigma}N) \right) \mathcal{D}_\pi \left(1 + \frac{1}{a-h_\pi} h_\pi \right) \tag{8}$$

This operator can be reduced by the method of Brueckner and Watson (1953) to a form containing the "potentials" $\mathcal{V}_i$ and $\mathcal{V}_f$,

$$\frac{1}{a-\mathcal{V}_f} \left(h_K(\bar{\Lambda}N) + h_K(\bar{\Sigma}N) \right) \left(1 + \frac{1}{a-\mathcal{V}_i} \mathcal{V}_i \right)$$

or (9)

$$\frac{1}{a} \left(1 + \mathcal{V}_f \frac{1}{a-\mathcal{V}_f} \right) \left(h_K(\bar{\Lambda}N) + h_K(\bar{\Sigma}N) \right) \left(1 + \frac{1}{a-\mathcal{V}_i} \mathcal{V}_i \right)$$

So if we consider only (8) we have, from (5),

$$T_{fi} = \left(\Phi_f, \left(1 + \mathcal{V}_f \frac{1}{a-\mathcal{V}_f} \right) \left(h_K(\bar{\Lambda}N) + h_K(\bar{\Sigma}N) \right) \left(1 + \frac{1}{a-\mathcal{V}_i} \mathcal{V}_i \right) \Phi_i \right)$$

$$= \left(\psi_f^{(-)}, \left[h_K(\bar{\Lambda}N) + h_K(\bar{\Sigma}N) \right] \psi_i^{(+)} \right) \tag{10}$$

where $\psi_i^{(+)}$ ($\psi_f^{(-)}$) is scattered by $\mathcal{V}_i$ ($\mathcal{V}_f$) and has outgoing (incoming) scattered part.

We wish to identify v_i and v_j with some kind of phenomeno-logical potential for two protons, and a proton-hyperon system, respectively. The latter will correspond to pion exchanges only, and will allow for such a process as

$$\Lambda^0 + p \longrightarrow \Sigma^0 + p \qquad (11)$$

Lichtenberg and Ross (1957) give such a potential. Feldman and Matthews (1958) emphasise the importance of the coupling of $\Lambda^0 p$ and ΣN states. Lichtenberg and Ross give an effective range and scattering length based on the solution of a pair of coupled equations for $\Lambda^0 p$ scattering, which allow for virtual transitions to a ΣN state, below threshold for the real process (11). By using their result we can allow for the coupling of the $\Lambda^0 p$ and ΣN systems, but not for the production off the energy shell of a Σ, which is scattered and transformed to a Λ^0. However we notice that the potentials $V_{\Lambda\Sigma}$ of Lichtenberg and Ross for the process (11) are much less than those for simple $\Lambda^0 p$ scattering. It is thus consistent with the use of their data to approximate to (10) by the form

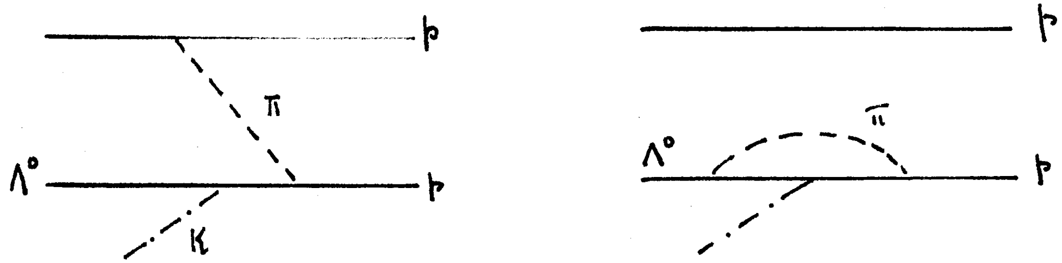
$$(\psi_f^{(-)}, h_K(\bar{\Lambda}N)\, \psi_i^{(+)}) \qquad (10)$$

in which $\psi_f^{(-)}$ is a $\Lambda^0 p$ state.

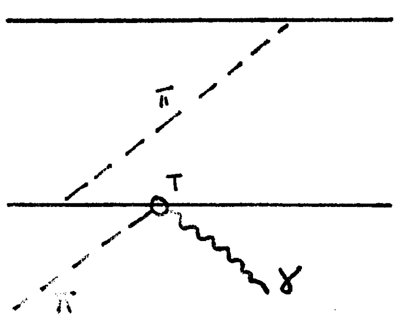
The/

The potential $\mathcal{V}_i$ is more difficult to deal with. We are well above the threshold for the production of real pions, so $\mathcal{V}_i$ can not be identified with an ordinary potential. $\mathcal{V}_i$ as constructed by the method of Brueckner and Watson is Hermitian only when a real pion cannot be produced. We identify $\mathcal{V}_i$ with a complex potential which will reproduce the elastic scattering and inelastic scattering, the latter being almost entirely pion production near the K meson threshold. The inelastic scattering in this model is a result of absorption by the imaginary part of the potential. The protons not absorbed can give rise to K mesons.

The other type of contribution to $\mathcal{D}_\pi \mathcal{D}_K^+ \Omega$ corresponding to such graphs as



can not be dealt with in this method, although we might hope to include the second type by renormalising the coupling constant. The situation here is rather like that of Section 2 in Part 1, in which we ignore such processes as



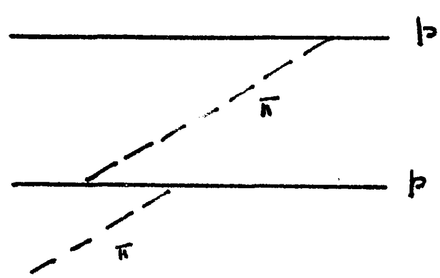
and take the interaction with the radiation field, H_r, as a small perturbation. Taking the first order in h is of course much less likely to be a good approximation than in the case of H_r.

A formally similar treatment of pion production (Aitken et. al. (1954)) leads to

$$D_\pi^+ \Omega = \left(D_\pi \frac{1}{a - V_\pi}\right) h_\pi$$

$$= \frac{1}{a - \mathcal{V}} h_\pi$$

with similar notation to that used above. There is no factor to the right of h_π because only the pion field is considered. $\mathcal{V}$ can be separated into a part giving the nucleon-nucleon potential and a part giving the interaction of the pions with the nucleons. This is treated by considering only the graphs which contribute to resonance scattering, the simplest of which is



Multiple scattering is not considered. As pointed out in Section 1 we have no similar reason for picking out any particular set of graphs. It will be noticed that this treatment of pion production does not lead to the phenomenological method, which has a nucleon-nucleon interaction in the initial state.

Our other approach makes use of the Chew- Low - Wick formulation of meson theory in terms of physical states. (Wick (1955)). The method is that used by Lichtenberg (1955) to introduce the phenomenological theory of pion production, and aims at avoiding the trouble caused in the first method by the processes not included in (7). It is convenient to calculate the matrix element for the reverse process,

$$\Lambda^{\circ} + K^{+} + p \rightarrow p + p$$

Let $\Psi^{(+)}$ be the physical state of a proton and Λ° with energy $E = T + \Delta M$, ΔM being $M_{\Lambda} - M_{p}$, satisfying the condition that its scattered part is outgoing. $\Psi_{q}^{(+)}$ satisfies the same condition and represents the physical state of a proton and a Λ° with the same energy, together with a K meson of momentum q , at infinity. We define Ψ_{s} by

$$\Psi_{q}^{(+)} = \rho_{q}^{\dagger} \Psi^{(+)} + \Psi_{s} \tag{12}$$

together with the boundary conditions on $\Psi^{(+)}$ and $\Psi_{q}^{(+)}$. We have

$$H\left(\rho_{q}^{\dagger}\Psi^{(+)} + \Psi_{s}\right) = \left(E + \omega(q)\right)\left(\rho_{q}^{\dagger}\Psi^{(+)} + \Psi_{s}\right) \tag{13}$$

Now

$$H\, \rho_q^\dagger \Psi^{(+)} = \rho_q^\dagger H \Psi^{(+)} + [H, \rho_q^\dagger]\, \Psi^{(+)}$$

$$= \rho_q^\dagger E \Psi^{(+)} + \rho_q^\dagger \omega(q) \Psi^{(+)} + (U_{1q}^0 + U_{2q}^0)\, \Psi^{(+)} \tag{14}$$

It is seen at this stage that we encounter no complications caused by the presence of the pion field, since $\rho_q^\dagger$ commutes with a_p and $a_p^\dagger$. From (13) and (14) we have $(U_{1q}^0 + U_{2q}^0)\Psi^{(+)} + H\Psi_s = (E+\omega(q))\Psi_s$ or recalling the boundary conditions

$$\Psi_s = \frac{1}{E+\omega(q)-H+i\varepsilon}\,(U_{1q}^0 + U_{2q}^0)\,\Psi^{(+)} \tag{15}$$

Now expand the right hand side of (15) in terms of the complete set of functions $\Psi_n^{(-)}$ with incoming scattered parts.

$$\Psi_s = \sum_n \Psi_n^{(-)} \frac{(\Psi_n^{(-)}, (U_{1q}^0 + U_{2q}^0)\,\Psi^{(+)})}{E-\omega(q)-E_n+i\varepsilon}$$

If in particular $\Psi_m^{(-)}$ is the state of two protons we have the matrix element for the process $K^+ + \Lambda^0 + p \rightarrow p + p$ in the form

$$T = \left(\Psi_m^{(-)}, (U_{1q}^0 + U_{2q}^0)\, \Psi^{(+)} \right) \tag{16}$$

We go from this to the following form for the direct process

$$T = \left(\psi_f^{(-)}, (U_{1q}^{\dagger} + U_{2q}^{\dagger})\, \psi_i^{(+)} \right) \tag{17}$$

in which $\psi_i^{(+)}$, $\psi_f^{(-)}$ represent respectively a bare two proton state and a bare Λ^0 + proton state. Here we assume that the transition from physical two particle states to bare states is made by renormalisation, as for single particle states, together with the use of a phenomenological potential for the initial and final states.

The potential in the final state has to correspond to the effect of the exchange of pions and K mesons between the Λ^0 and the proton. Lichtenberg and Ross (1957) are able to obtain adequate agreement with the data on hyperfragments (nuclear systems with a Λ^0 bound to several nucleons) by using only the pion exchanges. Their later results (Lichtenberg and Ross (1958)) when they include K meson exchanges, are consistent with the assumption that these are less important than pion exchanges. We therefore take the same potential for the final state as in our first model.

Because of the fact that the creation and annihilation operators for the two fields commute, we have for pion production the result

$$T^{\lambda} = (\psi_{2N}^{(-)}, (V_{1p\lambda}^{\dagger} + V_{2p\lambda}^{\dagger})\, \psi_i^{(+)})$$

in which $\psi_i^{(+)}$ is the same state as before. The description of the initial state is thus quite different from that of the first model, in which we cannot have transitions from $\psi_i^{(+)}$ to states containing a pion. We suggest that the way to obtain a suitable potential for the initial state in the second model would be, if this were possible, to separate from the proton-proton elastic scattering the part which is not diffraction scattering corresponding to the inelastic scattering, and look for a potential giving this part. This cannot be done from the data of Fowler et.al. (1956). An exceptional case which can be treated is that in which all the elastic scattering is diffraction scattering and $\psi_i^{(+)}$ can be taken as a plane wave. If we could make a reasonable attempt to find $\psi_i^{(+)}$ in a more general case it is clear that the second model of this section would be preferable to the first.

3. The potentials and wave functions for initial and final states

We adopt the first model of Section 2, and make use of the results of Fowler et.al. as mentioned in the introduction. Calculating the phase shifts for the three lowest values of from the optical model we look for complex wells which will give the same phase shifts. A different well must be found for each value of ℓ . In Section 4 we give our reasons for only using $\ell = 0, 1, 2$. The radius R of the absorbing sphere of Fowler et.al. is taken as the mean radius of the potential, while interpolation of their values of K gives $K = 3.6 \times 10^{13}$ cm at 1.75 BeV., the energy at which we work. The phase shift is given by

$$\delta_\ell = \tfrac{1}{2} i K s_\ell = \tfrac{1}{2} i K \left(k_0^2 R^2 - (\ell + \tfrac{1}{2})^2\right)^{1/2} k_0^{-1} \qquad (1)$$

for momentum k_0 . (See Fernbach et.al. (1949).) We find that for the first three partial waves $\eta_\ell = e^{2i\delta_\ell} = e^{-Ks_\ell}$ is small enough to let us approximate by taking $\eta_\ell = 0$. The error in doing this is less than the effect of taking wells of different shapes which give the same phase shifts. Corresponding to this simplified case of model 1 of Section 2, we have the special case of model 2, already described, in which the initial state is a plane wave. We can expect the assumption that $\eta_\ell = 0$ to be more misleading in model 2 than in model 1.

If data on elastic and inelastic scattering were available at the exact energy required it would be better not to use the absorbing sphere model but to determine the (complex) phase shifts from the data and find wells which will give these phase shifts. The work of Karita (1956) for 1 BeV. suggests that the phase shift analysis would not give a unique result. The use of Schrodinger's equation with a potential $V(r)$ at such high energies is of doubtful value in any case, so refinements in determining the well are probably wasted.

The condition $\eta_\ell = 0$ can be satisfied by a variety of potentials. For $\ell = 0$ we have examined the effect of using different forms of potential. We have also looked at approximate methods which would permit us to use a well with a diffuse boundary for any . We require an analytical solution of the wave equation for each ℓ because we have to find the well by trial and error. The methods are given by Nemirovskii (1956). The form of the potential is taken to be

$$V(r) = -V_0(\rho + i) f(r)$$

where

$$\begin{aligned} f(r) &= 1 & r &\leq r_0 \\ &= f[\alpha(r - r_0)] & r &> r_0 \\ &\rightarrow 0 \text{ as } r \rightarrow \infty \end{aligned}$$

The internal and external solutions are fitted at $r = r_0$.

$$\frac{1}{r}\frac{d\chi_{int}}{dr}\bigg|_{r=r_0} = \frac{1}{r}\frac{d\chi_{ext}}{dr}\bigg|_{r=r_0} \qquad (2)$$

The form of the external solution is

$$\chi_\ell = r^{-1/2}\, u_\ell(r)\, H^{(2)}_{\ell+\frac{1}{2}}(k_0 r)$$

with $u_\ell(r) \to 1$ as $r \to \infty$. Here $H^{(2)}_{\ell+\frac{1}{2}}(k_0 r)$ is Hankel's function of the third kind, with asymptotic form

$(k_0 r)^{-1/2} \exp\{-i(k_0 r - \ell\pi/2)\}$. The Schrodinger equation leads to this equation for $u_\ell(r)$

$$\frac{d^2 u_\ell}{dx^2} + 2\frac{\frac{d}{dx}\left\{x^{1/2} H^{(2)}_{\ell+\frac{1}{2}}(x)\right\}}{x^{1/2} H^{(2)}_{\ell+\frac{1}{2}}(x)}\frac{du_\ell}{dx} + \frac{V_0}{k_0^2} f\left(\frac{x\alpha}{k_0}\right)(\rho + i)u_\ell = 0 \quad (3)$$

in which we write $x = k_0 r$. One of the approximate methods is a quasi-classical one which requires for its validity that $V_0 < k_0^2$. When we use this method however we obtain a value $> k_0^2$. In the second method (3) is solved by a succession of approximations, the first of which consists of setting $u_\ell = 1$ in the third term. The parameter determining the convergence of the process is k_0/α and our value of k_0 is so large that we require a large α, and therefore a well which is almost square. So for general ℓ we have simply used a square well. We have compared the square well result with a rounded well result for $\ell = 0$, an analytic solution of Schrodinger's equation being readily obtained in that case. For the square well r_0 and the mean radius R are the same.

$$V = -V_0(\rho + i) \qquad r \leq R$$

$$= -x_0^2 k_0^2 (\rho + i)$$

$$V = 0 \;, \quad r > R$$

We have the internal solution $j_\ell(rX)$ where $X = X_1 + iX_2$,

$$X_1^2 - X_2^2 = (1 + x_0^2 \rho^2) k_0^2$$

$$2X_1X_2 = x_0^2 k_0^2$$

The boundary condition (2) is equivalent to $f_\ell(\text{int}) = f_\ell(\text{ext})$ where

$$f_\ell(\text{int}) = 1 + \frac{XR\, j_\ell'(XR)}{j_\ell(XR)}$$

and for $\eta_\ell = 0$,

$$f_\ell(\text{ext}) = 1 + \frac{k_0 R\, h_\ell^{(2)\prime}(k_0 R)}{h_\ell^{(2)}(k_0 R)}$$

$j_\ell^{(2)}$ is the spherical Bessel function of the first kind and $h_\ell^{(2)}(k_0 r)$ the spherical Hankel function corresponding to $H^{(2)}_{\ell+\frac{1}{2}}(k_0 r)$. For f_ℓ (int) we have the results (Feshbach et. al. (1954))

$$f_0 = XR \cot(XR)$$

$$f_\ell = \frac{X^2R^2}{\ell - f_{\ell-1}} - \ell$$

Using these results we find the values of x_0 and ρ which satisfy $f_\ell(\text{int}) = f_\ell(\text{ext})$. At incident proton energy 1.75 BeV. these parameters are found to be

$\ell = 0$	$x_0 = \cdot 73$	$\rho = \cdot 66$
$\ell = 1$	$x_0 = \cdot 65$	$\rho = -\cdot 61$
$\ell = 2$	$x_0 = \cdot 74$	$\rho = \cdot 25$

The imaginary parts of the potentials are similar but the real parts are very different. If we compare the values at threshold (1.58 BeV.) which are

$\ell = 0$	$x_0 = \cdot 76$	$\rho = \cdot 85$
$\ell = 1$	$x_0 = \cdot 70$	$\rho = -\cdot 4$
$\ell = 2$	$x_0 = \cdot 75$	$\rho = \cdot 5$

we see that the variation with energy is small for the imaginary part but large for the real part.

For a rounded well we take the form (Scott (1954))

$$V(r) = -\frac{V_0}{2}(\rho+i)\left\{1-\tanh\frac{r-R}{2\ell}\right\}$$

where we use $R = 10^{-13}$ cm., $\ell = k_0^{-1}$, and again write $V_0 = \varkappa_0^2 k_0^2$. The wave number tends to k_0 as $r \to \infty$ and as $r \to 0$ it approaches $X = \varkappa k_0 = k_0\sqrt{\frac{E+V_0}{E}}$ for sufficiently large ℓ. Schrodinger's equation is

$$\frac{d^2\chi_0}{dr^2} + k_0^2\left\{\frac{\varkappa^2+1}{2} - \frac{\varkappa^2-1}{2}\tanh\frac{(r-1)k_0}{2}\right\}\chi_0 = 0$$

where we have taken 10^{-13} cm. as unit length. Let $z = \exp(r-1)k_0$. The equation is equivalent to this equation for $F = z^{-a}\chi$, a being a complex constant,

$$(1+z)z\frac{d^2F}{dz^2} + (1+2a)(1+z)\frac{dF}{dz} + \left\{(a^2+1)+\frac{a^2+\varkappa^2}{z}\right\}F = 0$$

By setting $a^2+\varkappa^2 = 0$, we have a hypergeometric equation. This has two pairs of solutions (Whittaker and Watson (1952))

$$\overrightarrow{\chi}_{int} = z^{i\varkappa}F(i\varkappa+i,\, i\varkappa-i\,;\, 1+2i\varkappa\,;\, -z)$$

$$\overleftarrow{\chi}_{int} = z^{-i\kappa} F(-i\kappa+i, -i\kappa-i; 1-2i\kappa; -z)$$

which tend, as $r \to 0$, to $\exp(\pm i\kappa(r-1))$, and

$$\overrightarrow{\chi}_{ext} = z^{i} F(i\kappa-i, -i\kappa-i; 1-2i; -\tfrac{1}{z})$$

$$\overleftarrow{\chi}_{ext} = z^{-i} F(-i\kappa+i, i\kappa+i; 1+2i; -\tfrac{1}{z})$$

which tend, as $r \to \infty$, to $\exp(\pm ik_0(r-1))$. Using formulae linking these two pairs of solutions we fit $\overleftarrow{\chi}_{ext}$ to $\overrightarrow{\chi}_{int} - \overleftarrow{\chi}_{int}$. Having found κ_0 and ρ in this way we obtain the wave function by numerical integration, using method $\overline{\text{VII}}$ of Fox and Goodwin (1949), of the coupled equations for $Re\,\chi$ and $Im\,\chi$ obtained from (4). The values of κ_0 and ρ are $\kappa_0 = 1{\cdot}235$, $\rho = {\cdot}705$.

Writing $\chi_0(k_0r)$, $\chi_1(k_0r)$, $\chi_2(k_0r)$ for the three wave functions obtained using the square wells, we have the initial state wave function in the form

$$[\chi_0(k_0r) - 5\,\chi_2(k_0r)\,P_2(\underset{\sim}{k_0}.\underset{\sim}{r}/k_0r)]\,X_0^0 + 3i\,\chi_1(k_0r)(\underset{\sim}{k_0}.\underset{\sim}{r}/k_0r)\,X_1^m$$

in which χ_0^0 and χ_1^m are the singlet and triplet spin functions. Now consider the final state, in which we use the results of Lichtenberg and Ross as discussed in Section 2. They have an attractive interaction between the Λ^0 and the proton, stronger in the $^1S_0$ state than in the $^3S_1$ state. We shall see below that we only require the $^3S_1$ state. Their potentials have repulsive cores but they give an equivalent effective range and scattering length. We have therefore ignored the core and used the wave function

$$\frac{u(kr)}{kr} = \frac{e^{-i\delta}}{kr}\left[\sin(kr+\delta) - e^{-\eta r}\sin\delta\right] \qquad (6)$$

where the parameters are obtained in the way described in Part I, Section 6. Here $\underset{\sim}{k}$ is the relative momentum of the Λ^0 and the nucleon.

4. The matrix element and cross section

We confine our attention to S states of the $\Lambda^{\circ} p$ system, and s and p states of the K meson. When we make the spin and parity assignments of Section 2 the possible transitions are

$$^3P_1 \rightarrow {}^3S_1 s$$

$$^3P_0 \rightarrow {}^1S_0 s$$

$$^1S_0 \rightarrow {}^3S_1 p$$

$$^1D_2 \rightarrow {}^3S_1 p$$

For the transition operator $\mathcal{U}$ we use the form (Geffen(1935))

$$\mathcal{U} = \sum_{j=1}^{2} \left\{ \alpha \, \underset{\sim}{\sigma}_j \cdot \underset{\sim}{\nabla}_j \, e^{-i \underset{\sim}{q} \cdot \underset{\sim}{r}_j} + \beta e^{-i \underset{\sim}{q} \cdot \underset{\sim}{r}_j} \, \underset{\sim}{\sigma}_j \cdot \underset{\sim}{\nabla}_j \right\} \qquad (1)$$

Here α and β are complex parameters. This is a generalisation of the form

$$\mathcal{U} = \sum_{j=1}^{2} U_{jq}^{\dagger}$$

obtained from the theory of Section 2, which corresponds to

$$\alpha = \sqrt{\frac{4\pi}{2\omega(q)}}\ \frac{f_\Lambda u(q)}{\mu_K} \quad , \quad \beta = 0 \qquad (2)$$

The term with coefficient β is intended to allow us to take account of the nucleon recoil. The form of β given in the work of Chew et. al. (1952) on pion production is $\beta = \frac{\alpha\,\omega(q)}{M_p}$, M_p being the proton mass. This suggests that we use $\beta = \alpha\,\mu_K / M_p$ or $\beta \doteqdot \alpha/2$. To illustrate the effect of altering β we give results with $\beta = \alpha/2,\ \alpha/4,\ 0$.

With the operator (1) the transition ${}^3P_0 \to {}^1S_0\, s$ cannot occur. So with the final state interaction which we use only the less strongly interacting $\Lambda^0 p$ state is involved. With a scalar meson the ${}^1S_0 \to {}^1S_0\, s$ transition could occur. Writing $\underset{\sim}{R} = \frac{1}{2}(\underset{\sim}{r_1} + \underset{\sim}{r_2})$, $\underset{\sim}{r} = \underset{\sim}{r_1} - \underset{\sim}{r_2}$ and retaining only the first two terms in the expansion of $e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r}}$ in partial waves we have

$$\mathcal{U} = e^{-i\underset{\sim}{q}\cdot\underset{\sim}{R}}\Big[\, i\alpha(\underset{\sim}{\sigma_1} - \underset{\sim}{\sigma_2})\cdot\underset{\sim}{q}\ j_0\!\left(\tfrac{qr}{2}\right) - \beta\, j_0\!\left(\tfrac{qr}{2}\right)(\underset{\sim}{\sigma_1} + \underset{\sim}{\sigma_2})\cdot\underset{\sim}{\nabla} \qquad (3)$$

$$+ 3i\beta\, j_1\!\left(\tfrac{qr}{2}\right)\frac{\underset{\sim}{q}\cdot\underset{\sim}{r}}{qr}\,(\underset{\sim}{\sigma_1} - \underset{\sim}{\sigma_2})\cdot\underset{\sim}{\nabla}\ \Big]$$

Here the second term gives an s meson, the first term contributes to the transition ${}^1S_0 \to {}^3S_1\, p$, and the last to both the transitions giving p mesons.

The initial state is given by (3.5) and the final state by $e^{i\underset{\sim}{K}.\underset{\sim}{R}}\, u(k\rho)/k\rho$, where $\underset{\sim}{K}$ is the momentum of the baryon centre of mass, and $u(k\rho)$ is given by (3.6). We therefore have

$$(f|U|i) = \delta(\underset{\sim}{K}+\underset{\sim}{q})(Q_s + Q_t)$$

where

$$Q_s = \int \frac{d\underset{\sim}{\rho}}{(2\pi)^3}\, \chi_1^m\, \frac{u(k\rho)}{k\rho}\Big[i\alpha(\underset{\sim}{\sigma_1}-\underset{\sim}{\sigma_2}).\underset{\sim}{q}\, j_0(\tfrac{q\rho}{2}) + 3i\beta\frac{\underset{\sim}{q}.\underset{\sim}{\rho}}{q\rho}\, j_1(\tfrac{q\rho}{2})(\underset{\sim}{\sigma_1}-\underset{\sim}{\sigma_2}).\underset{\sim}{\nabla}$$

$$\times\Big[\chi_0(k_0\rho) - 5\chi_2(k_0\rho)P_2(\underset{\sim}{k_0}.\underset{\sim}{\rho}/k_0\rho)\Big]\chi_0^0$$

and

$$Q_t = \int \frac{d\underset{\sim}{\rho}}{(2\pi)^3}\, \chi_1^m\, \frac{u(k\rho)}{(k\rho)}\, j_0(\tfrac{q\rho}{2})\,\beta(\underset{\sim}{\sigma_1}+\underset{\sim}{\sigma_2}).\underset{\sim}{\nabla}\Big[3i\frac{\underset{\sim}{k_0}.\underset{\sim}{\rho}}{k_0\rho}\,\chi_1(k_0\rho)\Big]\chi_1^m$$

Hence when we sum and average over spins and evaluate the angular integrations we have

$$(2\pi)^6\,|(f|U|i)|^2 = [\delta(\underset{\sim}{K}+\underset{\sim}{q})]^2\left(\frac{4\pi}{k}\right)^2\Big\{2|\beta|^2|I_1|^2 \qquad (4)$$

$$+\left|\alpha\underset{\sim}{q}\,I_0 + \frac{\beta\underset{\sim}{q}}{q}I_0' - \frac{\beta I_2}{q}(\underset{\sim}{x}q_x + \underset{\sim}{y}q_y + 2\underset{\sim}{z}q_z)\right|^2\Big\}$$

Here $\underset{\sim}{x}$ $\underset{\sim}{y}$ $\underset{\sim}{z}$ are unit vectors with $\underset{\sim}{z}$ direction that of the incident proton, and the integrals I are defined as

$$
\begin{aligned}
I_0 &= \int_0^\infty dr\, u(kr)\, j_0\!\left(\tfrac{qr}{2}\right) \chi_0(k_0 r) \\
I_0' &= \int_0^\infty dr\, u(kr)\, r\, j_1\!\left(\tfrac{qr}{2}\right) \frac{d}{dr} \chi_0(k_0 r) \\
I_1 &= \int_0^\infty dr\, u(kr)\, j_0\!\left(\tfrac{qr}{2}\right) \left\{ 2\chi_1(k_0 r) + r \frac{d}{dr} \chi_1(k_0 r) \right\} \\
I_2 &= \int_0^\infty dr\, u(kr)\, j_1\!\left(\tfrac{qr}{2}\right) \left\{ 3\chi_2(k_0 r) + r \frac{d}{dr} \chi_2(k_0 r) \right\}
\end{aligned}
\tag{5}
$$

The relative importance of the various terms in (4) can be illustrated by the form of the integrals (5) for a plane wave initial state. Then $\chi_\ell(k_0 r) = 2\, j_\ell(k_0 r)$ and we have

$$
\begin{aligned}
I_0 &= 2\int_0^\infty dr\, u(kr)\, r\, j_0\!\left(\tfrac{qr}{2}\right) j_0(k_0 r) = I_1 / k_0 \\
I_0' &= -2k_0 \int_0^\infty dr\, u(kr)\, r\, j\!\left(\tfrac{qr}{2}\right) j_1(k_0 r) = -I_2
\end{aligned}
\tag{6}
$$

Now k_0 is large ($k_0 = 4{\cdot}59 \times 10^{13}$ cm.$^{-1}$ at $1{\cdot}75$ BeV.) so we may expect the result to be dominated by the $^3S_1$s final state, if β is appreciable.

It is convenient to define the function $S(q)$ by

$$
|(f|U|i)|^2 = \left(\frac{1}{2\pi}\right)^6 |\alpha|^2 \left[\delta(\underset{\sim}{K}+\underset{\sim}{q})\right]^2 S(q) \tag{7}
$$

Then using the form (2) for (we obtain the cross-section

$$\frac{d^2\sigma}{d\underset{\sim}{k}\,d\underset{\sim}{q}} = \frac{(2\pi)^{-4}}{\hbar v}\,\delta(E_i - E_f)\,\frac{f_\Lambda^2}{\mu_K^2}\,\frac{S(q)}{\omega(q)}$$

Here v is the velocity of the incident proton. In performing the integration over $d\underset{\sim}{k}$ we treat the final state non-relativistically and use the Kinetic energies T_q and T_k as variables. Let T_m denote the maximum energy available in the centre of mass system, that is the total energy in the centre of mass system at 1.75 BeV. less the corresponding quantity at threshold. T_m = 61.5 MeV. The energy of the meson and the centre of mass of the baryons is

$$\left\{1 + \frac{\mu_K}{M_p + M_\Lambda}\right\} T_q = 1{\cdot}24\,T_q$$

So we have

$$\frac{d^2\sigma}{dT_q\,d\Omega_q} = \frac{(2\pi)^{-4}}{v\hbar}\,\frac{f_\Lambda^2}{\mu_K^2}\,2(m\mu_K)^{3/2}\int d\Omega_k\,dT_k\,S(q)\,\frac{T_k^{1/2}\,T_q^{1/2}}{\mu_K + T_q}\,\delta(T_m - T_k - 1{\cdot}24\,T_q) \qquad (8)$$

m is the reduced mass, $m = \dfrac{M_p M_\Lambda}{M_p + M_\Lambda}$. When we integrate over T_q we obtain the differential cross-section in the form

$$\frac{d\sigma}{d\Omega_q} = \frac{4(2\pi)^{-3}}{v\hbar}\,\frac{f_\Lambda^2}{\mu_K^2}\,(m\mu_K)^{3/2}\,(P + R\cos^2\Theta)$$

and the total cross-section

$$\sigma = \frac{2}{\pi^2 v\hbar} \frac{f_\Lambda^2}{\mu_K^2} (m\mu_K)^{3/2} (P + \tfrac{1}{3} R)$$

which, using the coupling constant g_Λ instead of f_Λ, is

$$\sigma = \frac{2}{\pi^2 v\hbar} g_\Lambda^2 (P + \tfrac{1}{3} R) \frac{M_p^{3/2} M_\Lambda^{3/2} \mu_K^{3/2}}{(M_p + M_\Lambda)^{3/2}} \tag{9}$$

In view of our crude approximations in the initial state we only look for approximate values of $\frac{d\sigma}{d\Omega_q}$ and σ. So we evaluate the quantities P and R in the following manner (see Watson (1952) and Henley (1957)) which is strictly inconsistent with the form of potential (with a repulsive core) on which our final state data are based. We evaluate $S(q)$ for a particular $q = q_0$. We use the value corresponding to $T_q = 40$ MeV. From (4) $S(q_0)$ has the form

$$S(q_0) = A(q_0, \cos^2\Theta) + B(q_0, \cos^2\Theta) q_0 + C(q_0) q_0^2 \tag{10}$$

Wherever in (4) we have an integral I with a factor $j_1\left(\frac{q_0 r}{2}\right)$, we write it as $q_0\,(I/q_0)$, the factor q_0 representing the behaviour of $j_1\left(\frac{q_0 r}{2}\right)$ as $r \to 0$. Thus (10) is replaced by

$$S(q_0) = a(q_0) + \left[b(q_0) + \cos^2\Theta\, c(q_0)\right] q_0^2 \tag{11}$$

Then defining $\psi_f(0,q)$ to be the value of the $\Lambda^0 p$ wave function at $r = 0$ for the value of k corresponding to q, and $F(T_q)$ to be the ratio $\psi_f(0,q)/\psi_f(0,q_0)$, we assume that the form of the meson energy spectrum is reasonably well represented by

$$S(q) = \left\{a(q_0) + \left[b(q_0) + \cos^2\Theta\, c(q_0)\right] q^2\right\} F(T_q) \tag{12}$$

The integral over T_q is

$$\int_0^{T_m/1\cdot24} dT_q\, T_q^{1/2}\, \frac{(T_m - 1\cdot24\, T_q)^{1/2}}{\mu_K + T_q}\, S(q)$$

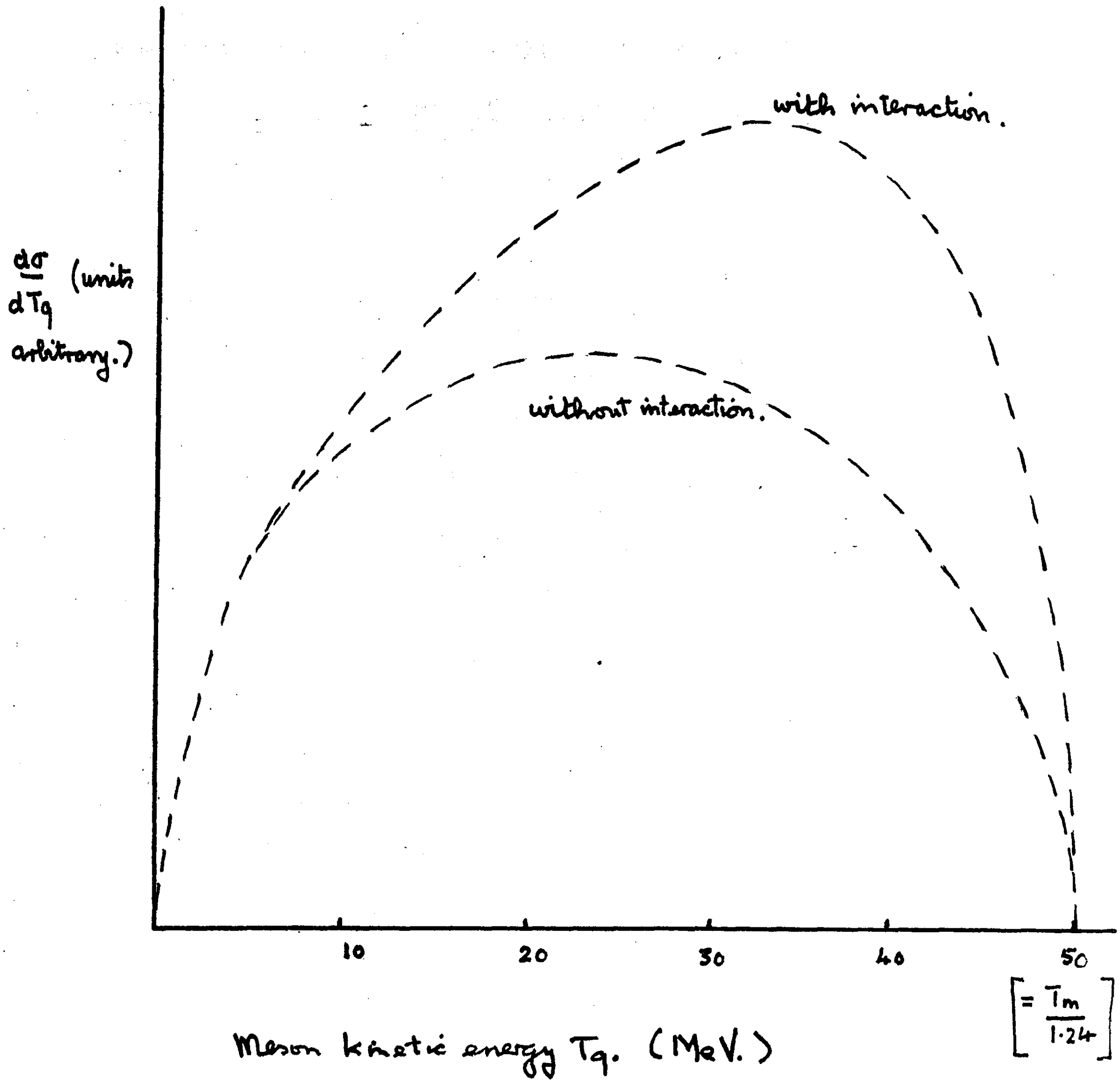
and from (12) we obtain

$$P = a(q_0)\, I + b(q_0)\, J$$

$$R = c(q_0)\, J$$

where

$$I = \int_0^{T_m/1\cdot24} dT_q\, T_q^{1/2}\, \frac{(T_m - 1\cdot24\, T_q)^{1/2}}{\mu_K + T_q}\, F(T_q) \tag{13}$$


with interaction.
without interaction.
$\frac{d\sigma}{dT_q}$ (units arbitrary.)
10
20
30
40
50
$\left[= \frac{T_m}{1.24}\right]$
Meson kinetic energy T_q. (MeV.)

and

$$J = \frac{2}{\mu_K}\int_0^{T_m/1\cdot 24} dT_q \; T_q^{3/2} \frac{(T_m - 1\cdot 24\, T_q)^{1/2}}{\mu_K + T_q} F(T_q)$$

In the graph we show the form of the energy spectrum of s mesons (the integrand in I) with and without the final state interaction. In the paper of Costa and Feld (1953) this problem is treated as if the distribution was much more strongly peaked, in fact as if $T_q \doteqdot T_m / 1\cdot 24$. This is clearly unrealistic for the final ${}^3S_1 s$ state. Finally it should be emphasised that our crude approximations in the final state could readily be improved if it were worth while, by taking wave functions consistent with a repulsive core, using relativistic kinematics, and evaluating $S(q)$ for various q to get the energy spectrum. Also as knowledge of the $\Lambda^0 p$ and $K p$ interactions increases further improvement will be possible. On the other hand because of the very high energies involved the description of the initial state by a potential may be inherently misleading.

5. Results and discussion.

The values of the integrals (4.5) for our square wells, at 1.75 BeV incident proton kinetic energy and $T_q = 40$ MeV., are

$$I_0 = -0.913 + i\,0.756 \quad 10^{-15}\ \text{cm}.$$

$$I_0' = 0.272 + i\,0.453 \quad 10^{-15}\ \text{cm}.$$

$$I_1 = -4.644 + i\,4.192 \quad 10^{-15}\ \text{cm}.$$

$$I_2 = 0.199 + i\,2.061 \quad 10^{-15}\ \text{cm}.$$

These results confirm the predominance of the term involving I_1. For the rounded well we have

$$I_0 = 0.560 + i\,0.615 \quad 10^{-15}\ \text{cm}$$

There is a factor 2 between the values of $|I_0|^2$ in the two cases, which indicates that the results will be sensitive to the choice of the shape of the well. For the case of an incident plane wave, in which (4.6) holds, we have

$$I_0 = 0.312 \quad 10^{-15}\ \text{cm}.$$

$$I_0' = -0.174 \quad 10^{-15}\ \text{cm}.$$

So if we use the approximation of Section 3 for the proton-proton scattering the two models of Section 2 give very different results. From (4.4) and (4.7), remembering that β/α is taken to be real, $\beta = h\alpha$ say, we have

$$S(q) = \left(\frac{4\pi}{k}\right)^2 \Big\{ 2n^2 |I_1|^2 + q^2 |I_0|^2 + n^2 |I_0'|^2$$

$$+ (1+3\cos^2\Theta)\, n^2 |I_2|^2 + 2qn\, \mathrm{Re}(I_0 I_0'^*)$$

$$+ 2(1-3\cos^2\Theta)\, n^2\, \mathrm{Re}(I_0' I_2^*) + 2(1-3\cos^2\Theta)\, qn\, \mathrm{Re}(I_0 I_2^*) \Big\}$$

For the complex well the values of $S(q_0)$ are

$n = 1/2$	$2.278 - 0.123 \cos^2\Theta$
$n = 1/4$	$0.650 - 0.105 \cos^2\Theta \quad (10^{-13}\,\mathrm{cm})^4$
$n = 0$	0.072

When we integrate over T_q in the approximation described in Section 4 we obtain the values of P, R and σ in the table, which also shows the results for model 2. The units of P and R are $(10^{-13}\ \mathrm{cm})^4$ MeV., and σ is in millibarns. The value of σ is obtained using the value $g_\Lambda^2 = 4.2$ of Ceolin and Taffara (1957b). Other estimates of g_Λ^2, for example those of Barshay (1958) and Matthews and Salam (1958) are also of the order of 3 or 4.

As explained in the introduction no direct comparison with experiment is possible at present. The work of Baumel et.al. (1957) at 3 BeV. results in an estimate of 0.2 mb for

Model 1

	P	R	
$n = 0.5$	3.537	−0.133	2.74×10^{-2}
0.25	0.971	−0.113	7.32×10^{-3}
0	0.078	0	6.1×10^{-4}

Model 2 (plane wave initial state).

	P	R	
$n = 0.5$	0.1717	−0.0053	1.33×10^{-3}
0.25	0.0470	−0.0045	3.57×10^{-4}
0	0.0054	0	4.2×10^{-5}

the total cross-section for all K^+ mesons. If we try to estimate the cross-section for our process at 3 BeV., we have two difficulties. We have to decide what is the ratio of our process to the process $p + p \rightarrow K^+ + \Sigma^0 + p$. This involves g_Λ^2 / g_Σ^2, available estimates of which range from 3 (Ceolin and Taffara (1957a)) to 10 (Barshay (1958)). Also we are above threshold for the processes

$$p + p \longrightarrow N + \Lambda^0 + K^+ + \pi$$

$$p + p \longrightarrow N + \Sigma + K^+ + \pi$$

$$p + p \longrightarrow p + p + K^+ + K^-$$

If we take 2mb. as an upper limit for our process at 3 BeV., and take $\sigma \propto T_m$, a rather quicker increase with T_m than is implied by our result (4.13) for S mesons, we get $\sigma \leq$ 0.025 mb. at 1.75 BeV.

One feature of our results is the presence of a strong S meson contribution for appreciable values of β. As mentioned in the introduction early experimental work suggested a very anisotropic cross-section. However Orear (1957) gave an estimate of $\cos^2 \Theta$ for the angular dependence of the available experimental data at that time, while at the Geneva conference (1958) work was reported by Steinberger indicating the presence of an appreciable S meson contribution.

A. Aitken, H. Mahmoud, E. . Henley, M.A. Ruderman and
K.M. Watson. Phys. Rev. 93, 1349 (1954).

D. Amati and B. Vitale. Il Nuovo Cimento 6, 1013 (1957)
Il Nuovo Cimento 6, 1282 (1957).

S. Barshay. Phys.Rev. 104, 853 (1956).
Phys.Rev. 110, 743 (1958).

V.S. Barashenkov, B.M. Barashov, E.G. Bubelov and
V.M. Maximenko. Nuclear Physics 5, 17 (1958).

P. Baumel, G. Harris, J. Orear and S. Taylor.
Phys.Rev. 108, 1322 (1957).

K.A. Brueckner and K.M. Watson. Phys.Rev. 90, 699 (1953).

C. Ceolin and T. Taffara. Il Nuovo Cimento 5, 453 (1957a).
Proceedings of Padua-Venice
Conference (1957b).

G.F. Chew, M.L. Goldberger, I.M. Steinberger and C.N. Yang.
Phys.Rev. 84, 581 (1952).

G. Costa and B. Feld. Phys.Rev. 109, 606 (1958).

R.H. Dalitz. Reports on Progress in Physics 20, 163 (1957).

B. Durney. Proc. Phys. Soc. 71, 654 (1958).

B. d'Espagnat and J. Prentki. Progress in Elementary Particle
and Cosmic Ray Physics 4 (1958).

G. Feldman and P.T. Matthews. Phys.Rev. 109, 546 (1958).

E. Fermi. Prog. Theor. Phys. 5, 570 (1950).

S. Fernbach, R. Serber and T.B.T. Taylor. Phys.Rev. 75, 1352
(1949)

H. Feshbach, C.E. Porter and V.F. Weisskopf. Phys.Rev. 96, 448
(1954)

[illegible] Fowler, [illegible] Shutt, A.[illegible] Thorndike, [illegible].L. Whittemore,
V.T. Cocconi, E. Hart, M.M. Block, E.M. Harth,
E.C. Fowler, J.D. Garrison and T.W. Morris.
Phys.Rev. 103, 1489 (1956).

L. Fox and E.T. Goodwin. Proc. Camb. Phil. Soc. 45, 373 (1949).

D.A. Geffen. Phys.Rev. 99, 1534 (1955).

M. Gell-Mann. Phys.Rev. 92, 833 (1953).

E.M. Henley. Phys.Rev. 106, 1083 (1957).

D. Ito, S. Minami and H. Tanaka. Il Nuovo Cimento 8, 135 (1958).

L.D. Landau, Izv. Akad. Nauk S.S.S.R. 17, 571 (1953).

R. Lea, E.C. Fowler and H.L. Kraybill. Phys.Rev. 110, 748 (1958)

D.B. Lichtenberg. Phys.Rev. 100, 303 (1955).
Phys.Rev. 105, 1034 (1957).

D.B. Lichtenberg and M. Ross. Phys. Rev. 107, 1715 (1957).
Phys.Rev. 109, 2163 (1958).

S. Mandelstam. Proc.Roy. Soc. A244, 491 (1958).

P.T. Matthews and A. Salam. Phys. Rev. 110, 565 (1958).
Phys. Rev. 110, 569 (1958).

P.E. Nemirovskii. Soviet Physics (J.E.T.P.) 3, 484 (1956).

J. Orear. Proceedings of the 7th Rochester Conference,
part 5 (1957).

J. Osher. U.C.R.L. report 3449 (1956). (Quoted in Barshay (1956)

D.C. Peaslee. Phys.Rev. 105, 1034 (1957).

W. Rarita. Phys.Rev. 104, 221 (1956).

J.M.C. Scott. Phil. Mag. 45, 441 (1954).

[illegible].D. [illegible]. Progress in Elementary Particle and Cosmic Ray Physics, 4(1958).

K.M. Watson. Phys.Rev. 88, 1163 (1952).

E. Whittaker and G.N. Watson. Modern Analysis (4th Edition), 289, (1952).

G-C. Wick. Rev.Mod. Phys. 27, 339 (1955).