MATHEMATICAL STUDIES OF HEAT CONDUCTION PROBLEMS, USING THE METHOD OF WAVE-TRAINS.

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CONTENTS.

INTRODUCTION.

The wave-train method of finding the solution to heat conduction problems was propounded by Dr. G.Green in a paper in the Philosophical Magazine in April, 1927. 1.* Since then, it has been further developed in a series of later publications which are listed under references 2 to 10 inclusive in the Bibliography. The details of the method, which have appeared in several of the above-mentioned publications will be assumed in the present thesis.

This thesis consists of two parts. In Part 1, the wave-train method of solution is applied to certain convenient experiments for determining conductivity and emissivity. The results afford an accurate theory, from which, in conjunction with certain experimental observations, these coefficients can be calculated. This has been done for J.H.Gray's experimental method of finding the thermal conductivity of metals and, after we have allowed for an error in his working, a close correspondence between his and our value for the conductivity of copper is obtained.

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* The figures 1, 2 ----- refer to the Bibliography at the end.

Part 2 deals with problems of variable heat flow through two media of different materials in contact. It falls naturally into three sections, (a), (b) and (c), the scope of which is briefly outlined below.

(a) In a recent paper in the Philosophical Magazine, Dr. G. Green has summarised the effects throughout two media in contact due to any disturbance whose effect is propagated by wave motion. As illustrations of the application of these general results, he considers three special problems, two of them involving the transmission of elastic vibrations throughout two media in contact and one analagous problem concerning temperature vibrations. He makes no attempt, however, to deal in detail with the two-media problems associated with any one subject.

In the standard textbooks on the theory of heat conduction, detailed solutions of a wide range of one-medium problems are given; but no similar solutions are available for the corresponding problems involving two media in contact. To the best of the writer's knowledge, such results do not appear together in any previously published work. In view of the many applications of these analytical results in different fields of practical science and industry, the task of solving the main two-media problems in heat conduction and the presentation of them in a systematic manner is undertaken in the first section of Part 2. The wave-train method of procedure is adopted in the derivation of these solutions and hence, the general effects summarised by Green can be utilised.

(b) An examination of the solutions to the problems studied in section (a) reveals certain similarities in form between the different results. It is found that the roots of a general equation $\Delta = 0$ [see equation (134) below] determine the normal functions required to express the solution to each problem which involves an instantaneous or continuous heat or temperature source. The method of determining the roots of this equation is investigated in the second section.

(c) Finally, the possibility of applying these exact analytical results to certain practical problems concerning variable heat flow through composite furnace walls is examined.

This latter section is a first attempt to replace the present approximate methods of determining the temperature distribution across insulated furnace walls by an exact theoretical solution. The work done in this section holds out the prospect that the exact method can be adapted so as to give more accurate results without the elaborate graphical methods at present needed for each individual problem. This adaptation involves a certain amount of preliminary work in constructing diagrams using dimensionless groups; but, once constructed, such diagrams will greatly simplify the application of the analytical results to practical work.

The actual construction of these graphs has not yet been undertaken by the writer, who feels that the whole text of Part 2, and particularly of section (c) , requires to be discussed first by the practical scientists who will be using them. Their utility and the best form of their presentation would then be more accurately known.

It is intended, therefore, to submit the work embodied in Part 2 of this thesis to the Philosophical Magazine and to the Journal of the West of Scotland Iron and Steel Institute for publication. The paper to the Philosophical Magazine will be a suitable sequel to that of Dr.G.Green mentioned in Ref. 6 .It will contain less of the detailed working than appears in Part 2 of this thesis. The paper to the other journal will give more prominence to the practical applications dealt with under section (c) and will include some additional numerical and graphical detail of which time did not permit the inclusion in this thesis.

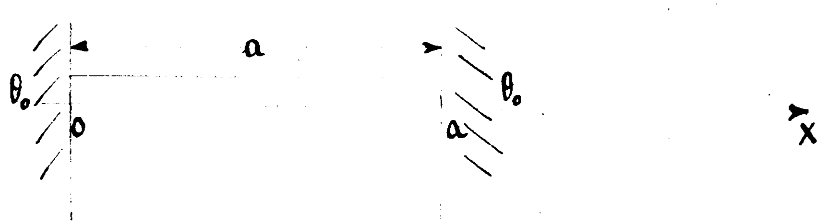
PART 1.

Problem 1.

The details of the process of solution are indicated fairly fully in the first problem, which may be formulated as follows:

To find the temperature distribution throughout a rod, of length a, whose initial temperature is zero, and both ends of which are subsequently kept at constant temperature θ_0, while there are heat losses from the surface of the rod; further, to show how this result could be used to determine the conductivity and emissivity of the material of the rod experimentally. [See Figure 1.]

Figure 1.



For one dimensional flow, the general equation of heat conduction in a uniform medium, where there is loss of heat from the surface is

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} - hv \qquad (1)$$

where v represents temperature, and

$$\kappa = \text{diffusivity} = \frac{K}{s\rho} = \frac{\text{thermal conductivity}}{\text{specific heat} \times \text{density}},$$

$$h = \frac{Hp}{s\rho\omega} = \text{coefficient of surface emissivity},$$

H = emissivity,

p = perimeter,

ω = area of cross-section.

The initial and boundary conditions for the rod are

$$\left.\begin{array}{lll} \text{At } t=0, & v = 0, & 0 \leqq x \leqq a, \\ \text{At any later time,} & v = \theta_0, & x = 0, \\ & v = \theta_0, & x = a. \end{array}\right\} \quad (2)$$

Before proceeding to solve this problem by the wave-train method, we note that there are here two continuous temperature sources, one at each end of the rod. The temperature effects due to these will be considered separately, and the results added together to give the total effect.

The end conditions then are

<u>Case 1.</u> $v = \theta_0, \quad x = 0,$

$v = 0, \quad x = a.$

<u>Case 2.</u> $v = 0, \quad x = 0,$

$v = \theta_0, \quad x = a.$

If we let $v = e^{-ht} u$ in equation (1), it becomes

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}. \quad (3)$$

This is the heat conduction equation for the new function u, and we now proceed to solve it, using the wave-train method and taking Cases 1 and 2 separately.

Case 1.

The new end conditions for u are

$$u = \rho_0, \quad x = 0, \quad \text{where} \quad \rho_0 = e^{ht}\theta_0.$$
$$u = 0, \quad x = a.$$

Consider first a periodic temperature source $\rho_0 e^{ikt}$ situated at $x = 0$. We can regard it as setting up the positively travelling wave-train $\rho_0 e^{ikt - ix\lambda}$ where $i\lambda = \sqrt{\frac{ik}{\kappa}}$. This wave-train will be reflected at the boundary $x = a$ to give a negatively travelling wave-train $A\rho_0 e^{ikt - i(2a-x)\lambda}$, where A is the coefficient of reflection and is determined by the boundary condition of zero temperature at $x = a$.

This wave-train in turn will be reflected at $x = 0$, where the coefficient of reflection is A_0, and so on until the following system of trains is built up

$$\rho_0 e^{ikt - ix\lambda}, \qquad A\rho_0 e^{ikt - i(2a-x)\lambda},$$
$$A_0 A \rho_0 e^{ikt - i(2a+x)\lambda}, \qquad A_0 A^2 \rho_0 e^{ikt - i(4a-x)\lambda},$$
$$A_0^2 A^2 \rho_0 e^{ikt - i(4a+x)\lambda}, \qquad \text{-----}$$

The summation of all these wave-trains gives the temperature effect u' due to a periodic source $\rho_0 e^{ikt}$ at $x = 0$.

$$\therefore u' = \rho_0 e^{ikt} \frac{\left[e^{-ix\lambda} + Ae^{-i(2a-x)\lambda}\right]}{1 - A_0 A e^{-2ia\lambda}}. \qquad (4)$$

The value of A can be determined by considering any wave-train incident on the surface $x=a$ and its reflection. To satisfy the condition that $u'=0$ there, we find $A=-1$. The coefficient A_0 must be such that the temperature effect at $x=0$ due to the initial train and all its reflections is $\rho_0 e^{ikt}$. This means that A_0 must also equal -1.

$$\therefore \; u' = \rho_0 e^{ikt} \frac{\left[e^{-ix\lambda} - e^{-i(2a-x)\lambda}\right]}{1 - e^{-2ia\lambda}}$$

$$= \rho_0 e^{ikt} \frac{\sin (a-x)\lambda}{\sin a\lambda}. \qquad (5)$$

Proceeding now to obtain the temperature effect u due to the instantaneous temperature source ρ_0 at $x=0$, we make use of the fundamental integration[1].

$$u = \frac{1}{\pi}\int_0^\infty u' \, dk$$

$$= \frac{\rho_0}{\pi}\int_0^\infty e^{ikt} \frac{\sin (a-x)\lambda}{\sin a\lambda} \, dk \qquad (6)$$

$$= \frac{\rho_0}{\pi}\int_{0B} e^{-\kappa\lambda^2 t} \frac{\sin (a-x)\lambda}{\sin a\lambda} . 2i\kappa\lambda . d\lambda \qquad (7)$$

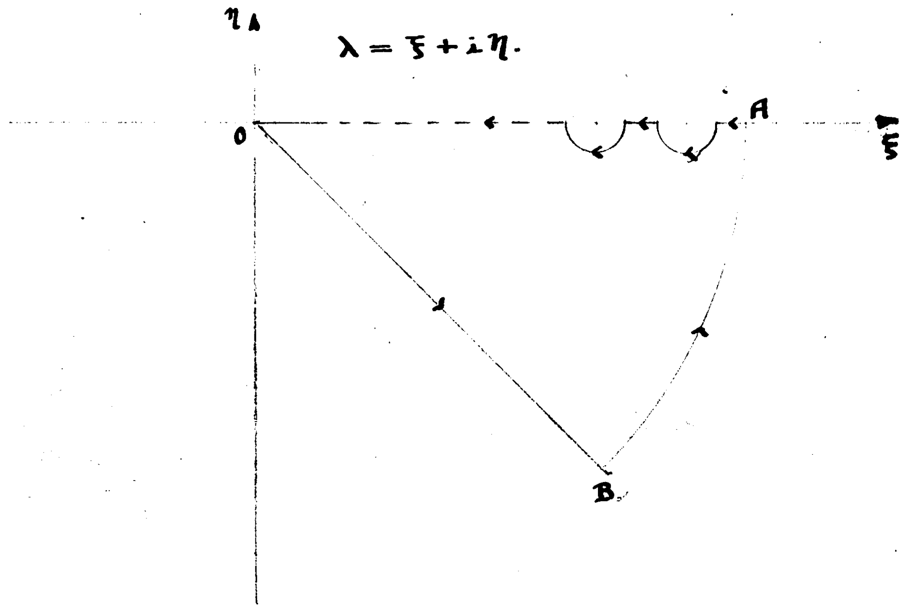
where, for facility in evaluating, we have changed to integration with respect to λ, and $0B$ is the radius vector from the origin to infinity which makes an angle $\theta=-\frac{\pi}{4}$ with the positive ξ-axis in the λ-plane. $[\lambda = \xi + i\eta = Re^{i\theta}]$. This change arises from consideration of the fact that

$$\lambda^2 = -\frac{ik}{\kappa} = \frac{k}{\kappa} e^{-i\pi/2},$$

$$\therefore \; \lambda = \sqrt{\frac{k}{\kappa}} \cdot e^{-i\pi/4}.$$

The evaluation of this integral[10] is most readily performed by integrating round the closed contour in the λ-plane consisting of (1) OB, (2) BA, the arc of the circle $\lambda = Re^{i\theta}$ from $\theta = -\frac{\pi}{4}$ to $\theta = 0$, and (3) AO, the real axis indented at points given by $\sin a\lambda = 0$, i.e. $a\lambda = n\pi$, $n = 0, 1, 2 \cdots$

Figure 2.



If $$f(\lambda) = e^{-\kappa\lambda^2 t}\,\frac{\lambda \sin(a-x)\lambda}{\sin a\lambda},$$

it can be shown that $\int_{BA} f(\lambda)\,d\lambda \to 0$ as $R \to \infty$.

Hence, when $R \to \infty$ and the radii of the indents $\to 0$,

$$\int_{OB} f(\lambda)\,d\lambda = \int_0^\infty f(\xi)\,d\xi + \sum \int_{\text{indents}} f(\lambda)\,d\lambda. \qquad (8)$$

But $$\sum \int_{\text{indents}} f(\lambda)\,d\lambda = \sum_{n=1}^{\infty} (\pi i)\left[-\frac{n\pi}{a}\cdot e^{-\frac{n^2\pi^2}{a^2}\kappa t}\cdot\frac{1}{a}\cdot\sin\frac{n\pi x}{a}\right]. \qquad (9)$$

Therefore, combining results (7), (8) and (9), and equating

u to the real part of the right hand side, we have

$$u = \frac{2\kappa\pi\rho_0}{a^2}\sum_{n=1}^{\infty} n\, e^{-\frac{n^2\pi^2}{a^2}\kappa t} \sin\frac{n\pi x}{a}. \tag{10}$$

Since $v = e^{-ht}u$ and $\rho_0 = \theta_0$ initially, the corresponding solution to equation (1) is

$$v = \frac{2\kappa\pi\theta_0}{a^2}\sum_{n=1}^{\infty} n . e^{-(\kappa.\frac{n^2\pi^2}{a^2}+h)t} . \sin\frac{n\pi x}{a}. \tag{11}$$

where v is now the actual temperature effect due to an <u>instantaneous</u> temperature θ_0 at $x = 0$.

The effect due to a <u>continuous</u> temperature θ_0 at $x = 0$ is obtained by putting $(t-t')$ for t and integrating with respect to t' from 0 to t.

$$\therefore v_1 = \frac{2\kappa\pi\theta_0}{a^2}\sum_{1}^{\infty} n. \sin\frac{n\pi x}{a}\int_0^t e^{-(\kappa\frac{n^2\pi^2}{a^2}+h)(t-t')}dt'.$$

$$= \frac{2\kappa\pi\theta_0}{a^2}\sum_{1}^{\infty} n. \sin\frac{n\pi x}{a}. \frac{a^2}{n^2\pi^2\kappa + a^2h}.\left[1 - e^{-(\kappa\frac{n^2\pi^2}{a^2}+h)t}\right]$$

$$= 2\kappa\pi\theta_0\sum_{1}^{\infty} n. \sin\frac{n\pi x}{a}. \frac{\left[1 - e^{-(\kappa\frac{n^2\pi^2}{a^2}+h)t}\right]}{n^2\pi^2\kappa + a^2h}. \tag{12}$$

<u>Case 2.</u>

In a similar way it would be possible to derive the temperature effect v_2 due to the continuous temperature θ_0 at $x = a$; but it can be seen that this result is obtained by substituting $(a-x)$ for x in (12)

Hence

$$v_2 = -2\kappa\pi\theta_0\sum_{1}^{\infty}(-1)^n n. \sin\frac{n\pi x}{a}. \frac{\left[1 - e^{-(\kappa\frac{n^2\pi^2}{a^2}+h)t}\right]}{n^2\pi^2\kappa + a^2h}. \tag{13}$$

The complete solution to the problem is obtained by the summation of v_1 and v_2. Hence, the temperature at any point x at time t is given by

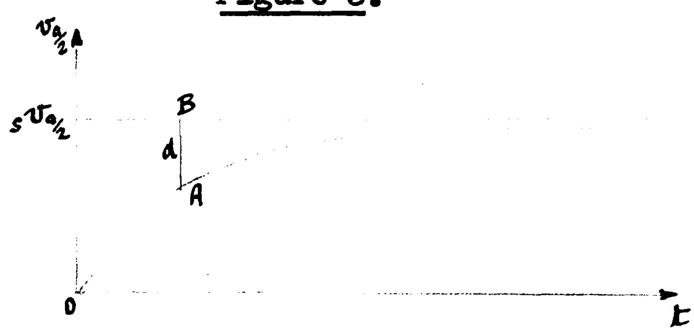
$$v = 2\kappa\pi\theta_0 \sum_1^\infty n \sin\frac{n\pi x}{a} \cdot \frac{[1-(-1)^n][1-e^{-(\kappa\frac{n^2\pi^2}{a^2}+h)t}]}{n^2\pi^2\kappa + a^2h}. \qquad (14)$$

It may be noted that this result agrees with that given in H.S.Carslaw's "The Conduction of Heat", page 70.

Application to an experimental method for finding K and H.

In order to determine K and H experimentally from this result, we may proceed as follows: Take readings of the temperature of the mid-point of the rod until a steady state is approached and plot the results as in Figure 3.

Figure 3.



It is possible to obtain two expressions for the steady temperature to which the mid-point of the rod settles down and their mathematical equivalence provides a verification of result (14). This is now demonstrated.

(a) If we put $t=\infty$ in equation (14), we have

$$_sv = 2\kappa\pi\theta_0 \sum_1^\infty n \sin\frac{n\pi x}{a} \cdot \frac{1-(-1)^n}{n^2\pi^2\kappa + a^2h}. \qquad (15)$$

Hence the steady temperature of the mid-point of the rod is given by

$$ {}_s v_{\frac{a}{2}} = 2\kappa\pi\theta_0 \sum_1^\infty n \sin\frac{n\pi}{2} \cdot \frac{1-(-1)^n}{n^2\pi^2\kappa + a^2 h} $$

$$ = 4\kappa\pi\theta_0 \sum_1^\infty (-1)^{m-1} \frac{2m-1}{(2m-1)^2\pi^2\kappa + a^2 h}. \qquad (16) $$

(b) The steady temperature state can also be determined from first principles; i.e. by solution of the equation

$$ \kappa\frac{\partial^2 v}{\partial x^2} - hv = 0, $$

where $v = \theta_0 \quad , \quad x = 0,$

$v = \theta_0 \quad , \quad x = a.$

Thus $\quad {}_s v = \theta_0 \dfrac{\sinh x\sqrt{\frac{h}{\kappa}} + \sinh(a-x)\sqrt{\frac{h}{\kappa}}}{\sinh a\sqrt{\frac{h}{\kappa}}}$

and $\quad {}_s v_{a/2} = \dfrac{\theta_0}{\cosh \frac{a}{2}\sqrt{\frac{h}{\kappa}}}, \qquad$ putting $x = \frac{a}{2}$. $\qquad (17)$

The values for ${}_s v_{a/2}$ given in equations (16) and (17) are known to be equal from the identity[18.]

$$ \operatorname{sech} x = 4\sum_1^\infty (-1)^{m-1} \frac{(2m-1)\pi}{(2m-1)^2\pi^2 + 4x^2}. $$

Making use of (17) now, we can write for the temperature at the mid-point at any time t

$$ v_{a/2} = \frac{\theta_0}{\cosh \frac{a}{2}\sqrt{\frac{h}{\kappa}}} - 4\kappa\pi\theta_0 \sum_1^\infty (-1)^{m-1} \frac{(2m-1)\, e^{-\left[\kappa\frac{(2m-1)^2\pi^2}{a^2} + h\right]t}}{(2m-1)^2\pi^2\kappa + a^2 h}. \qquad (18) $$

If we insert actual values for κ , h and a , in the summation part of the right side of equation (18) , we find that the exponential part in all terms except the first becomes extremely small after a certain time, which for a copper rod, length 6 cm., is 5 seconds; or for a rod of the same length made of a comparatively poor conductor, iron, is 25 seconds. If we take a reading of the temperature at any time after this, we need only consider the first term in the summation when we apply the theory to the experimental results.

$$\therefore v_{a/2} = {}_s v_{a/2} - \frac{4\kappa\pi\theta_0}{\pi^2\kappa + a^2 h} e^{-\left(\kappa\frac{\pi^2}{a^2} + h\right)t}. \tag{19}$$

Now, from equation (17),

$$\cosh \frac{a}{2}\sqrt{\frac{h}{\kappa}} = \frac{\theta_0}{{}_s v_{a/2}} = r, \text{ say},$$

where r is readily calculated.

Hence $\sqrt{\frac{h}{\kappa}} = \frac{2}{a}\log_e\left(r + \sqrt{r^2 - 1}\right) = y$, say.

Thus $h = \kappa y^2$ where y can be calculated from the observations.

Now, if in Figure 3, $AB = d =$ the difference between the final steady temperature and the rising temperature at any instant after the minimum time [25 seconds for iron, 6 cm. long], then from equation (19)

$$d = {}_s v_{a/2} - v_{a/2} = \frac{4\kappa\pi\theta_0}{\pi^2\kappa + a^2 h} \cdot e^{-\left(\kappa\frac{\pi^2}{a^2} + h\right)t}$$

$$= \frac{4\pi\theta_0}{\pi^2 + a^2 y^2} \cdot e^{-\kappa\left(\frac{\pi^2}{a^2} + y^2\right)t}.$$

$$\therefore \kappa = \frac{a^2 \log_e\left\{\frac{4\pi\theta_0}{d(\pi^2 + a^2y^2)}\right\}}{\left[\pi^2 + a^2y^2\right]t}. \tag{20}$$

From the value of κ obtained by this method $h\,[=\kappa q^2]$ can be calculated. Thermal conductivity, K, and emissivity, H, can then be found since $K = \rho s \kappa$, and $H = \frac{\rho s \omega h}{p}$.

Problem 2.

Application of the wave-train method of solution to J.H.Gray's method of determining the thermal conductivity of metals.

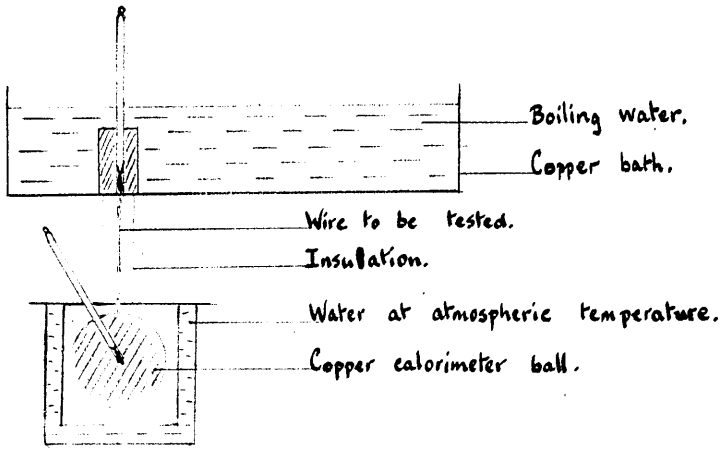
The second problem is related to the experimental method of finding the conductivity of metals used by J.H.Gray.[11] The theory of this experiment has had a partial treatment by J.Robertson in his first paper on the method of wave-trains as applied to the solution of heat conduction problems.[7] The conditions specified by Robertson in his theory, however, are simpler than those actually used by Gray in his experiment.

It is intended here to derive theoretical results which take into account the actual conditions of the experiment. It will then be possible to calculate the conductivity of copper, the metal used by Gray in his first experiment, and by comparing it with Gray's result to estimate the worth of his approximate theory.

The experiment.

In the experiment, a uniform copper wire was kept at the temperature of boiling water at one end, and to the other end a copper calorimeter was soldered. Part of the heat passing from the wire to the ball was used to raise the temperature of the latter; the remaining part was lost by convection and radiation from the surface of the ball to the surrounding enclosure which was maintained at the uniform temperature of the atmosphere. It was assumed that no heat losses occurred from the wire itself, which was insulated. [See Figure 4.]

Figure 4.



In order to allow for the effect of heat lost to the enclosure from the ball during the course of the experiment, Gray performed the experiment in the following way:

He first of all cooled the ball to 6^{o} below atmospheric temperature while he kept the upper end of the wire at the temperature of boiling water. The enclosure box was then brought into position round the ball. Readings of the temperature of the ball were then taken every half-minute from approximately 3° below to 3° above atmospheric temperature. The mean rise in temperature, θ degrees per minute, over this range was used to find the conductivity K of the copper of the wire by substituting in the formula

$$K = \frac{c\theta l}{\pi r^2 (T_1 - T_2)\, 60}$$

where c = thermal capacity of the ball,

r = radius of the wire,

l = lenght of the wire,

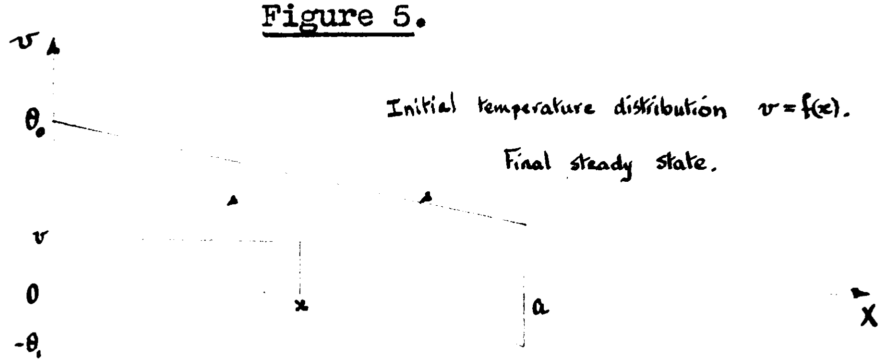
$T_1 - T_2$ = difference in temperature between the hot end of the wire and the atmosphere.

This method depends on the assumption that there is a steady state of heat flow from the wire to the ball; whereas in fact the heat flow varies with time.

Theory.

Let the wire be of length a units, and let the temperature of the hot end, $x = 0$, be θ_0 measured from atmospheric temperature as zero. Suppose the ball end, $x = a$, to be initially at temperature $-\theta_1$.

Figure 5.



Then the heat conduction equation for the wire is

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}\,, \tag{21}$$

and the initial and boundary conditions are

$$v = \theta_0 - \frac{x}{a}(\theta_0 + \theta_1) = f(x)\,, \quad 0 \leqq x \leqq a\,, \quad t = 0, \tag{22}$$

$$v = \theta_0 \qquad , \quad x = 0 \qquad \text{, at any later time.} \tag{23}$$

$$-K\omega \frac{\partial v}{\partial x} = Ms \frac{\partial v}{\partial t} + HSv \qquad , \quad x = a. \tag{24}$$

where ω = cross-sectional area of the wire,

M = mass of the ball,

s = specific heat of copper, the material of the ball,

H = coefficient of surface emissivity of the ball,

S = surface area of the ball.

Condition (24) may be written

$$-K\frac{\partial v}{\partial x} = q\frac{\partial v}{\partial t} + pv \qquad , \quad x = a, \tag{25}$$

where $q = \frac{Ms}{\omega}$, $p = \frac{HS}{\omega}$.

We wish to determine the temperature v at any time after $t=0$ for $0 \leqq x \leqq a$.

Before proceeding any further with the theory, it should be mentioned that no attempt is made here to eliminate one of the possible sources of error foreseen by Gray, namely, that the temperature recorded on the thermometer at the centre of the ball might not read correctly the temperature of the end of the wire, due to the finite thickness of copper intervening. Gray made an allowance for this to which we shall refer later. We are thus assuming that the copper is a good enough conductor of heat to keep this error very small, especially when wires of conductivity less than copper are used for test purposes. The main aim of the following analysis is to eliminate any possible error involved in assuming the heat flow to be steady instead of variable.

The application of the wave-train method to this problem involves its rational subdivision into three parts, (a), (b) and (c) discussed below, the physical interpretations of which are quite obvious. These contribute three temperature effects, v_1, v_2 and v_3 which together make up the final effect, $v = v_1 + v_2 + v_3$.

(a) The effect due to the continuous temperature θ_0 at $x = 0$.

v_1 must satisfy

$$\left.\begin{aligned} v_1 &= \theta_0 , && x = 0, \text{ at all times.} \\ -K\frac{\partial v_1}{\partial x} &= q\frac{\partial v_1}{\partial t} + p v_1 , && x = a. \end{aligned}\right\} \quad (26)$$

(b) The effect due to the initial temperature distribution $v = f(x)$ in the wire.

v_2 must satisfy

$$\left.\begin{aligned} v_2 &= \theta_0 - \frac{x}{a}(\theta_0+\theta_1) , && t = 0. \\ v_2 &= 0 , && x = 0, \text{ at all later times.} \\ -K\frac{\partial v_2}{\partial x} &= q\frac{\partial v_2}{\partial t} + p v_2 , && x = a. \end{aligned}\right\} \quad (27)$$

(c) The effect due to the ball being an instantaneous heat source.

v_3 has to satisfy

$$\left.\begin{aligned} v_3 &= 0 \quad , \quad x = 0. \\ v_3 &= -\theta_1 \text{ , throughout mass } M, \ x = a, \ t = 0. \\ -K\frac{\partial v_3}{\partial x} &= q\frac{\partial v_3}{\partial t} + p v_3 \quad , \quad x = a. \end{aligned}\right\} \tag{28}$$

We discuss these in turn, noticing that v_1 , v_2 and v_3 all satisfy equation (25).

(a) The effect due to the continuous temperature θ_0 at $x = 0$.

First we shall require the temperature effect due to the permanent temperature source θ_0 at $x=0$. This involves solving equation (21) where the end conditions are specified in equations (26) and where initially $v=0$ throughout the wire.

In the usual manner, we find the periodic solution due to a periodic temperature $\theta_0 e^{ikt}$ at $x=0$. The summation of the first wave-train, $\theta_0 e^{ikt-ix\lambda}$ and its successive reflections at $x=0$ and $x=a$ alternately gives

$$v_1' = \theta_0 e^{-\kappa\lambda^2 t} \frac{e^{-ix\lambda} + A e^{-i(2a-x)\lambda}}{1 + A e^{-2ia\lambda}}, \tag{29}$$

where $i\lambda = \sqrt{\frac{ik}{\kappa}}$ and A is the coefficient of reflection at $x=a$ and is determined by condition (25).

The first incident and reflected wave-trains at $x=a$ are $\theta_0 e^{ikt-ix\lambda}$ and $A\theta_0 e^{ikt-i(2a-x)\lambda}$.

In order that equation (25) may be satisfied, we find

$$A = \frac{Ki\lambda - (p - q\kappa\lambda^2)}{Ki\lambda + (p - q\kappa\lambda^2)} = e^{i\pi - 2i\phi} \tag{30}$$

where $$\tan\phi = \frac{K\lambda}{p - q\kappa\lambda^2}. \tag{31}$$

From this, we obtain

$$v_1' = \theta_0 e^{-\kappa\lambda^2 t} \frac{\sin\{(a-x)\lambda + \phi\}}{\sin(a\lambda + \phi)}. \tag{32}$$

Proceeding now to derive the effect due to the <u>instantaneous</u> temperature θ_0 at $x=0$, we have

$$v_1 = \frac{1}{\pi}\int_0^\infty v_1'\,dk.$$

As in problem 1, this integration can be performed most readily by writing it as

$$v_1 = \frac{2i\kappa\theta_0}{\pi}\int_{0B} e^{-\kappa\lambda^2 t} \frac{\sin\{(a-x)\lambda + \phi\}}{\sin(a\lambda + \phi)}\cdot\lambda\,d\lambda, \tag{33}$$

where the path of integration is the infinite radius, $\theta = -\frac{\pi}{4}$, in the λ-plane $[\lambda = Re^{i\theta}]$. We evaluate it by integrating round the contour used in problem 1 [See Figure 3], where the indents in this case are given by $\sin(a\lambda+\phi) = 0$, i.e. $a\lambda + \phi = n\pi$,

i.e. $$\tan a\lambda = -\tan\phi = -\frac{K\lambda}{p - q\kappa\lambda^2}. \tag{34}$$

By reasoning similar to that used in equations (8) and (9) we find

$$v_1 = \frac{2i\kappa\theta_0}{\pi}\sum(\pi i) e^{-\kappa\lambda^2 t} \frac{\sin(n\pi - x\lambda)}{\frac{d}{d\lambda}\{\sin(a\lambda+\phi)\}}$$

$$= -2\kappa\theta_0 \sum\left[-A_n e^{-\kappa\lambda^2 t}\lambda\sin x\lambda\right]$$

$$= 2\kappa\theta_0 \sum A_n e^{-\kappa\lambda^2 t}\lambda\sin x\lambda \tag{35}$$

where $$A_n = \frac{K^2\lambda^2 + (p - q\kappa\lambda^2)^2}{a[K^2\lambda^2 + (p - q\kappa\lambda^2)^2] + K(p + q\kappa\lambda^2)}, \tag{36}$$

and the summation is with respect to all the positive roots of equation (34).

The effect due to the continuous temperature source θ_0 at $x=0$ is obtained from equation (35) by replacing t by $(t-t')$ and integrating with respect to t' from 0 to t. This gives

$$v_1 = 2\theta_0 \sum A_n \frac{\sin x\lambda}{\lambda}\left[1 - e^{-\kappa\lambda^2 t}\right] \tag{37}$$

$$= \theta_0\left(1 - \frac{px}{K+pa}\right) - 2\theta_0 \sum A_n e^{-\kappa\lambda^2 t} \frac{\sin x\lambda}{\lambda}, \tag{38}$$

where $\theta_0\left(1 - \frac{px}{K+pa}\right)$ is the steady temperature effect obtained from first principles and equivalent to the value of the right side of equation (37) when $t=\infty$.

This result (38) is obtained by J.Robertson in Ref. 7, p. 947.

(b) The effect due to the initial temperature distribution $v=f(x)$ in the wire.

In addition to the temperature effect just found, there will be another effect due to the initial temperature distribution $v=f(x)$ throughout the wire [see conditions (27)]. In order to obtain this effect, we consider a periodic heat source, strength Qe^{iht} per unit area situated at a point $x=x_1$ in the wire. This source sends out two initial wave-trains, one in the positive and one in the negative direction. General results for the total temperature effect due to these wave-trains and their reflections at $x=0$ and $x=a$ are given in Refs. 6 and 10. We may use them now.

$$\left.\begin{aligned}\text{Effect at } x<x_1 &= \rho_1 e^{ikt}\frac{[A_0 e^{-ix\lambda}+e^{ix\lambda}][e^{-ix_1\lambda}+Ae^{-i(2a-x_1)\lambda}]}{1-A_0Ae^{-2ia\lambda}}\\ &= \rho_1 e^{ikt}\phi(x,x_1).\\ \text{Effect at } x>x_1 &= \rho_1 e^{ikt}\phi(x_1,x).\end{aligned}\right\} \qquad (39)$$

where $\rho_1 = \dfrac{Q}{2Ki\lambda}$,

and $A_0 =$ coefficient of reflection at $x=0$.

$A =$ coefficient of reflection at $x=a$.

This value of ρ_1 is determined by consideration of the fact that at $x=x_1$ the conditions for a periodic heat source have to be satisfied, namely,

$$v_L = v_R\ , \quad -K\left[\frac{\partial v_R}{\partial x}-\frac{\partial v_L}{\partial x}\right] = Qe^{ikt}.$$

The conditions determining the coefficients A_0, A are (1) that $v=0$ at $x=0$, since the effect due to $v=\theta_0$ at $x=0$ has been accounted for in part (a); and (2) the condition in equation (25) at $x=a$. From these, we find $A_0=-1$, and $A=e^{i\pi-2i\phi}$ as obtained in equations (30) and (31)

We therefore have

$$\left.\begin{aligned}\text{Effect at } x<x_1 &= \rho_1 e^{ikt}\frac{2i\sin x\lambda\,\sin\{(a-x_1)\lambda+\phi\}}{\sin(a\lambda+\phi)}\\ \text{Effect at } x>x_1 &= \rho_1 e^{ikt}\frac{2i\sin x_1\lambda\,\sin\{(a-x)\lambda+\phi\}}{\sin(a\lambda+\phi)}\end{aligned}\right\} \qquad (40)$$

where $\tan\phi = \dfrac{K\lambda}{p-q\chi\lambda^2}$.

Proceeding in the usual manner to obtain the effect due to an instantaneous source Q per unit area situated at $x = x_1$, and inserting the value for β, we have

$$\text{Effect at } x < x_1 = \frac{Q\kappa}{\pi K}\int_{0B} e^{-\kappa\lambda^2 t}\, \frac{2i \sin x\lambda \sin\{(a-x_1)\lambda+\phi\}}{\sin(a\lambda+\phi)}\, d\lambda \tag{41}$$

where $0B$ denotes the usual path of integration. The same contour and points of indentation as are used in part (a) give as the instantaneous solution, for $x < x_1$,

$$v_2 = \frac{2Q\kappa}{K}\sum A_n e^{-\kappa\lambda^2 t} \sin x\lambda \sin x_1\lambda, \tag{42}$$

where A_n is given by equation (36) and the summation is with respect to the positive roots of equation (34).

Integration of the second expression in (40) gives the same value of v_2 for points where $x > x_1$. Hence equation (42) holds throughout the wire and constitutes the effect due to an instantaneous heat source Q situated at $x = x_1$.

Now, the initial temperature distribution $v = f(x)$ may be considered as being due to an infinite number of such instantaneous heat sources. Hence a substitution of $\frac{K}{\kappa} f(x_1)\,dx_1$ for Q in equation (42) and an integration with respect to x_1 from 0 to a will give the total temperature effect due to the initial distribution $v = f(x)$.

We therefore have

$$v_2 = 2\sum A_n e^{-\kappa\lambda^2 t} \sin x\lambda \int_0^a \sin x_1\lambda\; f(x_1)\,dx_1 \tag{43}$$

$$= 2\sum A_n e^{-\kappa\lambda^2 t} \sin x\lambda \int_0^a \sin x_1\lambda\left[\theta_0 - \frac{x_1}{a}(\theta_0+\theta_1)\right]dx_1 \tag{44}$$

$$= 2\sum A_n e^{-\kappa\lambda^2 t}\, \frac{\sin x\lambda}{\lambda}\left[\theta_0 + \theta_1 \cos a\lambda - (\theta_0+\theta_1)\frac{\sin a\lambda}{a\lambda}\right]. \tag{45}$$

(c) The effect due to the ball being an instantaneous heat source.

Thirdly there is the temperature effect throughout the wire due to the ball, mass M, being an instantaneous heat source of temperature $-\theta_1$ at the beginning of the experiment.

Initial and boundary conditions are given in equations (28).

If we consider first that there is a periodic source, strength Qe^{ikt} in the ball, the initial negative wave-train will be of the form $\beta_2 e^{ikt - i(a-x)\lambda}$. The coefficient β_2 must be such that this initial wave-train will satisfy the condition that the ball, mass M, acts as a periodic source Qe^{ikt}. The law of emission in this case is

$$\left.\begin{aligned} K\omega\frac{\partial v}{\partial x} + Ms\frac{\partial v}{\partial t} + HSv &= Qe^{ikt}, \quad x = a. \\ \text{i.e.}\quad K\frac{\partial v}{\partial x} + q\frac{\partial v}{\partial t} + pv &= \frac{Q}{\omega}e^{ikt}, \quad x = a. \end{aligned}\right\} \tag{46}$$

using the same notation as in equation (25).

We therefore obtain β_2 in terms of Q by substituting $v = \beta_2 e^{ikt - i(a-x)\lambda}$ in equation (46). This gives

$$\beta_2 = \frac{Q}{\omega[Ki\lambda + (p - q\kappa\lambda^2)]} \tag{47}$$

Successive wave-trains are built up as usual, the reflection coefficients having the same values as before, and yield as the total effect

$$v_3 = \frac{Q}{\omega[Ki\lambda + (p - qK\lambda^2)]} e^{ikt + i\phi} \cdot \frac{\sin x\lambda}{\sin(a\lambda + \phi)}$$

$$= \frac{Q}{\omega K\lambda} e^{ikt} \cdot \frac{\sin\phi \sin x\lambda}{\sin(a\lambda+\phi)} \qquad (48)$$

where $\tan\phi = \frac{K\lambda}{p - qK\lambda^2}$ as before.

Converting this now to the effect due to the instantaneous heat source Q in the ball, we have*

$$v_3 = \frac{2iK}{\pi}\int_{B} Q e^{-K\lambda^2 t} \frac{\sin\phi \sin x\lambda}{\omega K \sin(a\lambda+\phi)} \cdot d\lambda = \frac{2KQ}{\omega K} \sum A_n e^{-K\lambda^2 t} \sin a\lambda \sin x\lambda . \qquad (49)$$

where A_n is given by equation (36) and the summation is with respect to the positive roots of equation (34).

Initially, the heat content of the mass M is $Ms f(a)$ where $f(a) = -\theta_1$ as obtained from condition (22).

$$\therefore \quad Q = -Ms\theta_1 \qquad (50)$$

Hence $$v_3 = -\frac{2M\theta_1}{\omega\rho} \sum A_n e^{-K\lambda^2 t} \sin a\lambda \sin x\lambda . \qquad (51)$$

The summation of these three temperature effects (38), (45), and (51), dealt with in sections (a), (b) and (c) gives the complete solution to the problem. Thus

$$v = \theta_0\left(1 - \frac{px}{K+pa}\right) - 2\theta_0 \sum A_n e^{-K\lambda^2 t} \frac{\sin x\lambda}{\lambda}$$
$$+ 2\sum A_n e^{-K\lambda^2 t} \frac{\sin x\lambda}{\lambda}\left[\theta_0 + \theta_1 \cos a\lambda - (\theta_0+\theta_1)\frac{\sin a\lambda}{a\lambda}\right] - \frac{2M\theta_1}{\omega\rho}\sum A_n e^{-K\lambda^2 t} \sin a\lambda \sin x\lambda .$$
$$= \theta_0\left(1 - \frac{px}{K+pa}\right) - 2\sum A_n e^{-K\lambda^2 t} \sin x\lambda \left[\frac{\theta_0+\theta_1}{a\lambda^2} \sin a\lambda - \frac{\theta_1}{\lambda}\cos a\lambda + \frac{M\theta_1}{\omega\rho} \sin a\lambda\right]. \qquad (52)$$

* Evaluating the integral by means of the usual contour.

where A_n is given by equation (36) and the summation is in all cases over the roots of equation (34).

An explanation of the fact that parts of v_1 and v_2 neutralise one another is readily afforded when we examine the graphical interpretation of v_1 , v_2 and v_3 .

Mathematical Verification.

It will be observed that apart from the first term on the right side of equation (52) , we have a simple sine summation. It would seem possible, therefore, to obtain the solution to the problem by the ordinary method of assuming that such a sine summation is a possible solution, and thus have a mathematical verification of the above result. [J.Robertson, in Ref. 7, verified that $1-\frac{px}{K+pa}$ could be represented by $2\sum A_n\frac{\sin x\lambda}{\lambda}$, a result of which we made use in equations (37) and (38).]

As stated earlier, the initial and boundary conditions are

$$v=\theta_0-\frac{x}{a}(\theta_0+\theta_1)\,,\quad t=0.$$

$$v=\theta_0\,,\quad x=0.$$

$$-K\frac{\partial v}{\partial x}=q\frac{\partial v}{\partial t}+pv\,,\quad x=a.$$

Let us introduce a new temperature function u , such that
$$u=v-\theta_0\left(1-\frac{px}{K+pa}\right).$$

Then the conditions for u are

$$\left.\begin{aligned} u &= \theta_0 - \frac{x}{a}(\theta_0+\theta_1) - \theta_0\left(1 - \frac{px}{K+pa}\right), \quad t=0. \\ &= g(x). \end{aligned}\right\} \qquad (53)$$

$$u = 0 \qquad\qquad , \quad x=0.$$

$$-K\left(\frac{\partial u}{\partial x} - \frac{\theta_0 p}{K+pa}\right) = q\frac{\partial u}{\partial t} + p\left[u + \theta_0\left(1 - \frac{px}{K+pa}\right)\right], \quad x = a.$$

We can now assume that a solution to our problem exists of the form

$$u = \sum B_n e^{-\kappa\lambda^2 t} \sin x\lambda,$$

where λ denotes λ_n and where the summation is over the roots of equation (34). We evaluate the coefficient B_n by the following method:

$$\begin{aligned} u_{(t=0)} &= \sum B_n \sin\lambda_n x \\ &= g(x). \end{aligned}$$

Now, $$\begin{aligned} \int_0^a g(x)\sin\lambda_m x\,dx &= \sum\int_0^a B_n \sin\lambda_n x \sin\lambda_m x\,dx \\ &= -\frac{qK}{K}\sum B_n \sin\lambda_n a \sin\lambda_m a, \quad m \neq n, \end{aligned}$$

and $$\int_0^a B_m \sin^2\lambda_m x\,dx = \frac{B_m}{2}\left[a - \frac{\sin\lambda_m a\cos\lambda_m a}{\lambda_m}\right].$$

Hence $$\begin{aligned} \int_0^a g(x)\sin\lambda_m x\,dx &= -\frac{qK}{K}\sum{}' B_n \sin\lambda_n a \sin\lambda_m a \\ &\quad + \frac{B_m}{2}\left[a - \frac{\sin\lambda_m a\cos\lambda_m a}{\lambda_m}\right] \\ &= -\frac{qK}{K}\sin\lambda_m a\left[g(a) - B_m\sin\lambda_m a\right] \\ &\quad + \frac{B_m}{2}\left[a - \frac{\sin\lambda_m a\cos\lambda_m a}{\lambda_m}\right]. \end{aligned}$$

Therefore

$$B_m = \frac{\int_0^a g(x)\sin\lambda_m x\,dx + \frac{qK}{K}\sin\lambda_m a\; g(a)}{\frac{qK}{K}\sin^2\lambda_m a + \frac{1}{2}\left[a - \frac{\sin\lambda_m a\cos\lambda_m a}{\lambda_m}\right]}.$$

Substituting for $g(x)$ the expression given in equation (53), we obtain for B_m the value required by equation (52).

Calculation of the conductivity and the emissivity of copper from Gray's experimental results.

The purpose of the following calculation is to obtain a value for the conductivity of copper, using Gray's experimental figures and the theoretical result given in equation (52), the derivation of which is based on the actual experimental conditions. At the same time, our method enables us to find the coefficient of surface emissivity of the copper ball.

In Gray's experiment, measurements of temperature rise were taken at the ball end of the wire, i.e. at $x = a$. Equation (52) gives, for this value of x

$$v_a = \theta_0 \frac{K}{K+pa} - 2\sum A_n e^{-\kappa\lambda^2 t}\sin a\lambda\left[\frac{\theta_0+\theta_1}{a\lambda^2}\sin a\lambda - \frac{\theta_1}{\lambda}\cos a\lambda + \frac{M\theta_1}{\omega\rho}\sin a\lambda\right] \qquad (54)$$

The following data are provided by Gray and in all cases are in c.g.s. units:

Length of wire $= a$ $= 6{\cdot}31.$

Thermal capacity of ball $= Ms$ $= 68{\cdot}9.$

Radius of wire $= r$ $= 0.105.$ Temperature of hot end $= 97.3°$

Radius of ball $= R$ $= 27.$ Temperature of atmosphere $= 9.85°$

From these, we calculate

$$\rho s = .8357.$$

$$q = \frac{Ms}{\omega} = 1994.$$

We shall take atmospheric temperature as our zero. Then, θ_0 becomes $87.45°$ and we make our calculations using the temperature readings of the ball beginning at $-3°$ which corresponds to $-\theta_1$. The graph of Gray's readings, adjusted to this arbitrary zero is given in Figure 6.

It will be appreciated that only if we can effect some simplification of equation (54) will we be able to use it to find values of K and H. Fortunately an examination of the positive roots of equation (34) enables us to neglect all but the first term of the summation in equation (54).

Equation (34) is solved graphically by finding for what values of λ the graphs of $y = \tan a\lambda$ and $y = \frac{K\lambda}{q\kappa\lambda^2 - p}$ intersect.

With the above properties of the ball and wire,

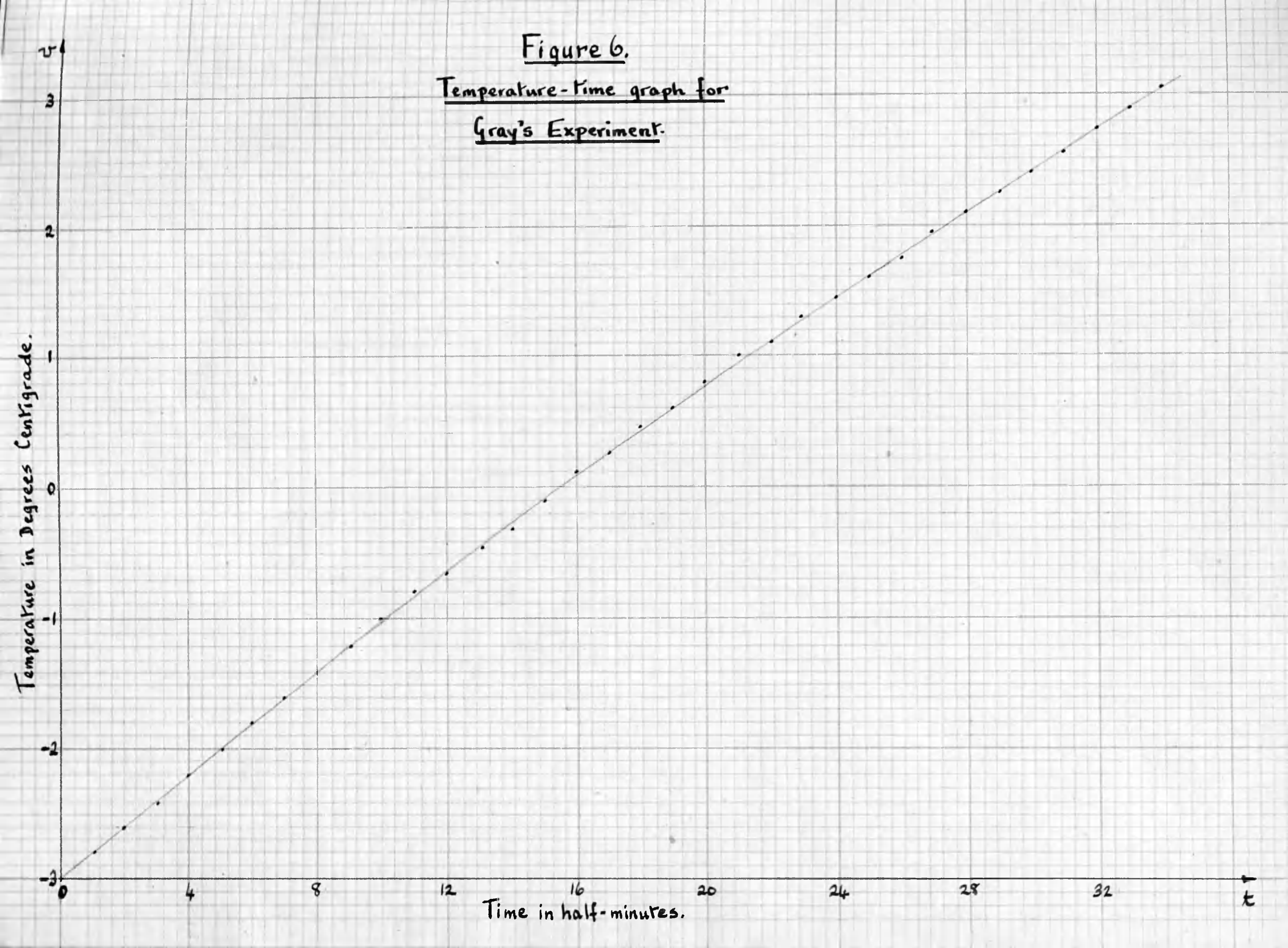
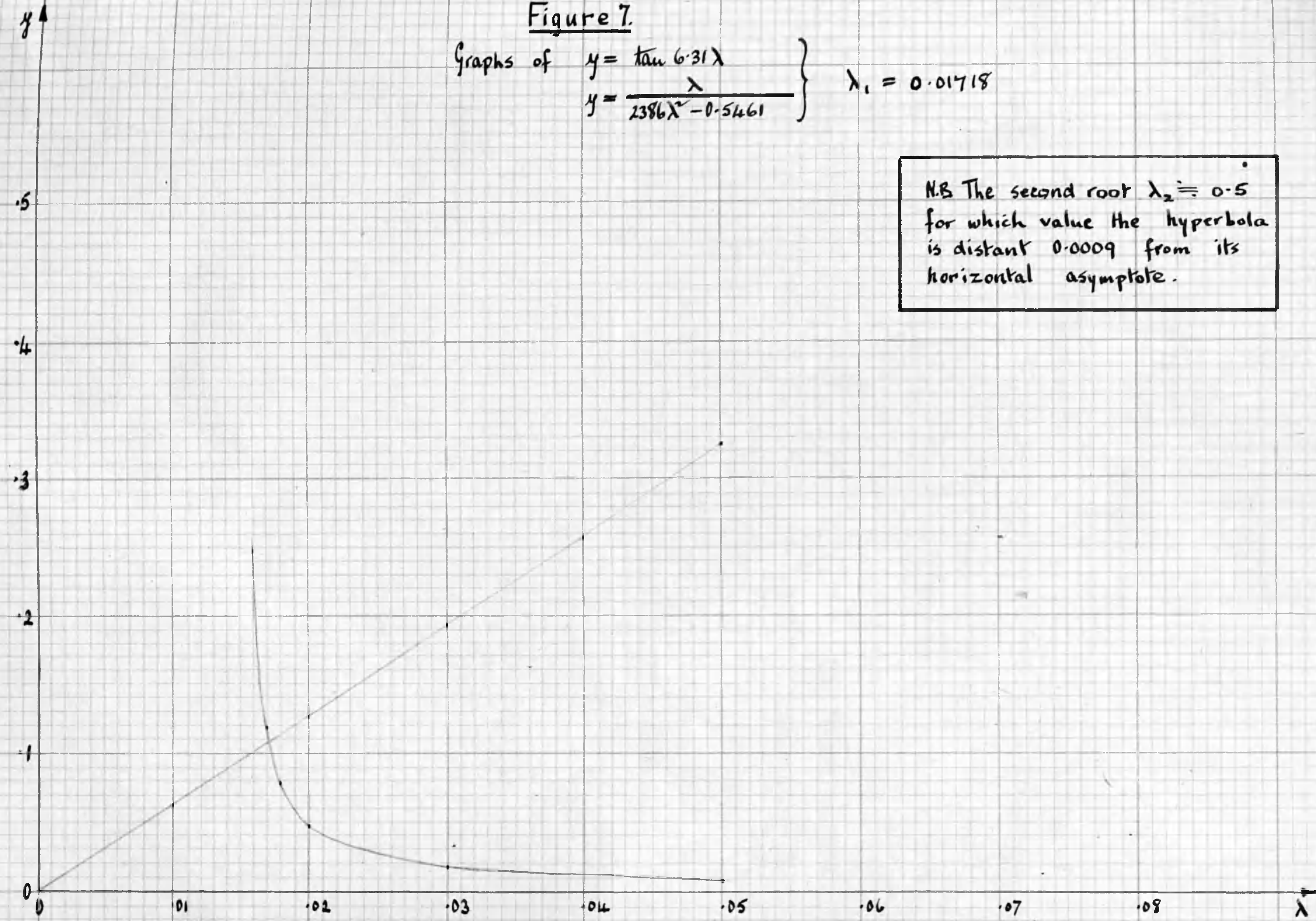


Figure 6.
Temperature-time graph for
Gray's Experiment.
v
3
2
1
0
-1
-2
-3
Temperature in Degrees Centigrade.
0
4
8
12
16
20
24
28
32
t
Time in half-minutes.

Figure 7.

Graphs of $\left.\begin{array}{l} y = \tan 6{\cdot}31\lambda \\ y = \dfrac{\lambda}{2386\lambda^2 - 0{\cdot}5461} \end{array}\right\}$ $\lambda_1 = 0{\cdot}01718$

N.B. The second root $\lambda_2 \doteqdot 0{\cdot}5$ for which value the hyperbola is distant 0·0009 from its horizontal asymptote.



these become $y = \tan 6{\cdot}31\lambda$

$$y = \frac{\lambda}{2386\lambda^2 - 2656\frac{H}{K}}.$$

Even a rough knowledge of the order of magnitude of $\frac{H}{K}$ is sufficient to show that the rectangular hyperbola very soon approaches its horizontal asymptote. The result of this is that the roots of equation (34), after the first, approach nearer and nearer to $\frac{n\pi}{a}$, $n = 1, 2 \cdots\cdots$, the degree of accuracy increasing with n. For these values of λ, it can be seen that $e^{-\kappa\lambda^2 t}\sin a\lambda$ becomes very small and in fact can be neglected; so that in our calculations, we need consider only the first term in the summation in equation (54) which can then be written in the form

$$v_a = C - De^{-\kappa\lambda_1^2 t}.$$

It is now possible for us to measure from the graph three values of v_a at times which are equidistant from one another. We them have

$$v_{t_1} = C - De^{-\kappa\lambda_1^2 t_1}.$$

$$v_{t_2} = C - De^{-\kappa\lambda_1^2 t_2}.$$

$$v_{t_3} = C - De^{-\kappa\lambda_1^2 t_3}.$$

$$\therefore \quad \frac{C - v_{t_1}}{C - v_{t_2}} = e^{-\kappa\lambda_1^2(t_1 - t_2)}$$

$$\frac{C - v_{t_2}}{C - v_{t_3}} = e^{-\kappa\lambda_1^2(t_2 - t_3)}$$

Since $$t_1 - t_2 = t_2 - t_3$$

we have $$(C - v_{t_1})(C - v_{t_3}) = (C - v_{t_2})^2$$

from which C can be found. It gives us the steady temperature to which the ball end of the wire would have settled down. By substitution, D and $\kappa\lambda_1^2$ can also be found.

Using Gray's curve, we calculate

$$C = 19.67. \qquad D = 22.67.$$

$$\kappa\lambda_1^2 = 0.000305.$$

The graph of $v = 19.67 - 22.67\, e^{-0.000305 t}$ fits Gray's curve perfectly.

From equation (54) we know that steady temperature is given by

$$_s v_a = \frac{\theta_0}{1 + \frac{pa}{K}}$$

$$C = \frac{\theta_0}{1 + \frac{HSa}{\rho s \kappa \omega}}$$

$$\therefore \; 19.67 = \frac{87.45}{1 + 20016 \frac{H}{\kappa}}$$

$$\therefore \; \frac{H}{\kappa} = 0.000172.$$

We now require the first root of

$$\tan a\lambda = \frac{K\lambda}{qK\lambda^2 - p} = \frac{\lambda}{\frac{q}{\rho s}\lambda^2 - \frac{HS}{\rho s \kappa \omega}}.$$

i.e. $$\tan 6.31\lambda = \frac{\lambda}{2386\lambda^2 - 0.5461}.$$

We obtain the root graphically and find [see Figure 7].

$$\lambda_1 = 0.01718, \qquad \therefore K = \rho s \kappa = 0.9631.$$

$$\therefore \kappa = 1.033, \qquad H = 0.000178.$$

At first sight, this value of $K = \cdot 8631$ appears to show an unduly large discrepancy with respect to Gray's value of $\cdot 883$, especially in view of the consistency of the results which he quotes for various lengths of the same type of copper wire, and in view of the care with which the experiment was performed. However, an examination of Gray's calculation reveals an error in his working. From his experimental data, he writes

$$K = \frac{68 \cdot 9 \times \cdot 3605 \times 6 \cdot 31}{\pi \times (\cdot 105)^2 \times 87 \cdot 45 \times 60}$$

which, in fact equals $\cdot 8624$ and not $\cdot 883$ as he states. The former value of $\cdot 8624$ differs from ours by less than 1 in 1000.

The correspondence between the two results can partly be attributed to the fact that, for the small range in temperature at the ball end of the wire which is involved in this experiment, the gradient of the temperature-time graph is small and the rate of change of the gradient is very small.

$$v = C - D e^{-\kappa\lambda_1^2 t}$$

$$\frac{dv}{dt} = D \kappa\lambda_1^2 e^{-\kappa\lambda_1^2 t}$$

$$\frac{d^2v}{dt^2} = -D(\kappa\lambda_1^2)^2 e^{-\kappa\lambda_1^2 t}, \qquad \kappa\lambda_1^2 = \cdot 0003.$$

Thus, Gray's mean value, $\frac{\cdot 3605}{60}$, for the gradient, on the assumption of a steady heat flow over this small range of temperature and time, is not far different from [but slightly less than] the actual gradient of the curve at atmospheric temperature, [see Figures 8 and 9].

Figure 8.

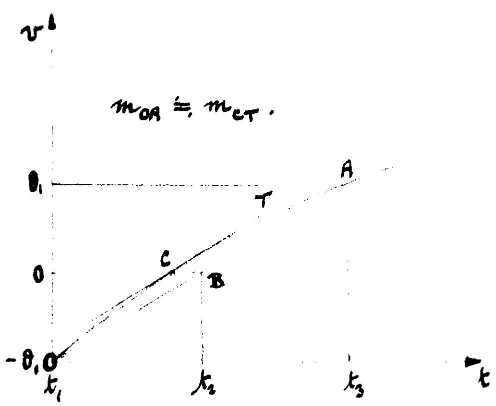
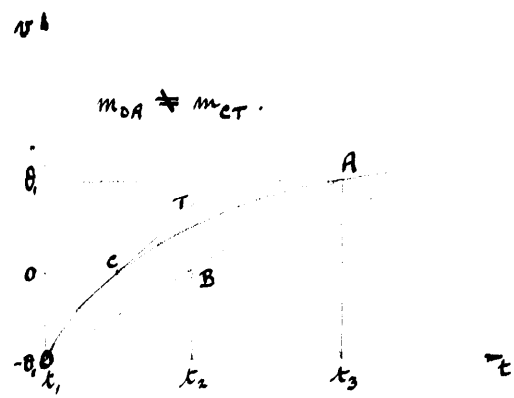


Figure 9.



Our result, which is slightly greater than Gray's thus shows a difference in the direction which we might expect, i.e. an increase.

In order to be certain that our omission of the second and higher terms in the summation part of equation had not introduced an error into our results, the second root of equation (34) was calculated and hence the value of the second term in the summation [using the values of K and H already found as first approximations]. The order of magnitude of this second term was found to be $-\cdot 0035$ after one second. For $t > 1$ second, this term decreases rapidly and higher [redacted] roots of equation (34) yield terms in the summation of alternating signs and even smaller dimensions than ·0035. It is clear, therefore, that, to the degree of accuracy possible in reading the temperature at the ball end of the wire and in calculating C and D [see page 31 above], these terms can be neglected without affecting our result.

One final correction requires to be made to the calculated result for conductivity in order to allow for the thermometer at the centre of the ball not reading the exact temperature at the end of the wire. Gray estimated this correction to be of the order of ·006 of an increase to the uncorrected value, again basing his theory on a steady heat flow into the ball. This correction gives K = ·8684 according to Gray and K = ·8691 according to our result.

The close correspondence between the results constitutes a valuable verification of Gray's method by an exact theoretical solution, even although it has involved the exposure of an unexpected error in his working.

Extension of the method to find the conductivity and emissivity of other metals.

Gray used the same calorimeter ball in a series of similar experiments from which he calculated the conductivities of materials other than copper. The above treatment requires only slight modification in order that a result corresponding to equation (52) may be derived and applied to this new problem, involving a wire made of a different metal to copper.

Consider that the wire is of some other metal, with conductivity K' and emissivity H', and to make the problem perfectly general, consider that there is heat lost from the surface of the wire. The properties of the calorimeter ball remain as before, its emissivity now being known. Its conductivity does not enter into this problem.

The heat conduction equation to be solved is

$$\frac{\partial v}{\partial t} = \kappa' \frac{\partial^2 v}{\partial x^2} - h' v \qquad (55)$$

where $\kappa' = \frac{K'}{\rho' s'}$, $h' = \frac{H' t'}{\rho' s' \omega'}$

and the dashed constants all apply to the material of the wire. The initial and boundary conditions are

$$v = \theta_0 - \frac{x}{a}(\theta_0 + \theta_1) , \quad t = 0.$$

$$v = \theta_0 , \quad x = 0 \text{ at all later times.}$$

$$-K'\frac{\partial v}{\partial x} = q\frac{\partial v}{\partial t} + pv \ , \qquad x = a \ . , \tag{56}$$

where $\quad q = \dfrac{Ms}{w'} \quad , \qquad p = \dfrac{Hs}{w'}.$

By using the substitution $u = e^{h't}v$, we can change equation (55) to

$$\frac{\partial u}{\partial t} = \kappa' \frac{\partial^2 u}{\partial x^2} \tag{57}$$

and equation (56) to

$$-K'\frac{\partial u}{\partial x} = q\frac{\partial u}{\partial t} + (p - h'q) \tag{58}$$

The solution to the problem is now similar to the one already treated, with the difference that the instantaneous solution for u requires multiplication by $e^{-h't}$ to give the solution for v, and that equation (58) differs from equation (25) in that $p - h'q$ has replaced p.

As before, we divide the problem into three parts. Without going into the details of the method again, we find the results to be:

(1) <u>Effect due to the continuous temperature θ_0 at $x = 0$.</u>

$$v_1 = 2\kappa'\theta_0 \sum A'_n \frac{\lambda \sin x\lambda}{\kappa'\lambda^2 + h'}\left[1 - e^{-(\kappa'\lambda^2 + h')t}\right] \tag{59}$$

where $$A'_n = \frac{(K'\lambda)^2 + \{p - q(\kappa'\lambda^2 + h')\}^2}{a[(K'\lambda)^2 + \{p - q(\kappa'\lambda^2 + h')\}^2] + K'[p + q(\kappa'\lambda^2 - h')]} \tag{60}$$

and the summation is over the positive roots of

$$\tan a\lambda = -\frac{K'\lambda}{p - q(\kappa'\lambda^2 + h')} \tag{61}$$

The steady temperature is found from first principles to be

$$ {}_s v_1 = \theta_0 \frac{p \sinh (a-x)\sqrt{\frac{h'}{K'}} + K'\sqrt{\frac{h'}{K'}} \cosh (a-x)\sqrt{\frac{h'}{K'}}}{p \sinh a\sqrt{\frac{h'}{K'}} + K'\sqrt{\frac{h'}{K'}} \cosh a\sqrt{\frac{h'}{K'}}} . \tag{62} $$

Verification that this is equivalent to the steady temperature obtained from equation (59) by putting $t=\infty$ has not been done, but could no doubt be carried out in a manner similar to the verification performed on pages 26-28.

(2) <u>The effect due to the initial temperature distribution $v=f(x)$ in the wire</u> is found by an analysis similar to that used in section (b) and gives

$$ v_2 = 2\sum A'_n e^{-(\kappa'\lambda^2+h')t} \frac{\sin x\lambda}{\lambda}\left[\theta_0 + \theta_1 \cos a\lambda - (\theta_0+\theta_1)\frac{\sin a\lambda}{a\lambda}\right] \tag{63} $$

(3) <u>The effect due to the ball being an instantaneous heat source</u>, temperature $-\theta_1$ is

$$ v_3 = -\frac{2Ms\theta_1}{\rho's'\omega'} \sum A'_n e^{-(\kappa'\lambda^2+h')t} \sin a\lambda \sin x\lambda \tag{64} $$

where in all cases A'_n and the summation are as required by equations (60) and (61).

The total temperature effect is then

$$ \begin{aligned} v &= v_1 + v_2 + v_3 \\ &= \theta_0 \frac{p \sinh (a-x)\sqrt{\frac{h'}{K'}} + K'\sqrt{\frac{h'}{K'}} \cosh (a-x)\sqrt{\frac{h'}{K'}}}{p \sinh a\sqrt{\frac{h'}{K'}} + K'\sqrt{\frac{h'}{K'}} \cosh a\sqrt{\frac{h'}{K'}}} \\ &\quad -2\sum A'_n e^{-(\kappa'\lambda^2+h')t} \sin x\lambda \left[\frac{\theta_0+\theta_1}{a\lambda^2}\sin a\lambda - \frac{\theta_1}{\lambda}\cos a\lambda + \frac{Ms\theta_1}{\rho's'\omega'}\sin a\lambda\right] \end{aligned} \tag{65} $$

This result can now be used in conjunction with experimental readings in order to determine K' and H'.

It should be noted that the steady temperature at $x = a$ is given by

$$ {}_s v_a = \frac{\theta_0}{\frac{h}{K'} \cdot \frac{\sinh a\sqrt{h'/K'}}{\sqrt{h'/K'}} + \cosh a\sqrt{h'/K'}} \doteqdot \frac{\theta_0}{\frac{ha}{K'} + 1} $$

for values of K' and H' likely to occur when dealing with metals.

As we should expect, equation (65) reduces to equation (52) when $h' = 0$, $K' = K$, $\kappa' = \kappa$, etc.

Finally, Gray discusses the desirability of using a smaller calorimeter ball when testing wires of lower conductivity than copper, in order that appreciable temperature rises will be recorded at the ball end of the wire within a reasonably short time.

From a practical stand-point, this reduction in the heat capacity of the ball is essential. Without going into its detailed effect on the roots of equation (61) as compared with those of equation (34), and thus into the effect on the value of v as given by equation (65), we note that a reduction in the size of M in the last term of the R.H.S. of equation (65) does cause an increase in the value of the temperature v after any given length of time.

PART 2.

Two-media problems.

As stated in the Introduction, Part 2 is subdivided into three sections, (a), (b) and (c). In the first of these, a collection of results for some of the main two-media problems in heat conduction is made.

(a) Solution of a variety of problems.

The problems solved concern in all cases a uniform rod, composed of two different materials in contact, medium 1 extending from $x = 0$ to $x = a$, and medium 2 extending from $x = a$ to $x = b$. They involve the determination of the temperatures v_1 in medium 1 and v_2 in medium 2 after time t, where the initial and boundary conditions are stated below.

Statement of problems solved.

1. End $x = 0$ maintained at constant temperature θ_0, end $x = b$ radiating to an atmosphere at zero temperature. Initial temperature zero throughout.

2. End $x = 0$ at temperature θ_0', end $x = b$ impervious to heat. Initial temperature zero throughout.

3. End $x=0$ at temperature θ_0 , end $x=b$ kept at zero temperature. Initial temperature zero throughout.

4. End $x=b$ maintained at constant temperature θ_0 , end $x=0$ radiating to an atmosphere at zero temperature. Initial temperature zero throughout.

5. End $x=b$ at temperature θ_0 , end $x=0$ impervaous to heat. Initial temperature zero throughout.

6. End $x=b$ at temperature θ_0 , end $x=0$ kept at zero temperature. Initial temperature zero throughout.

7. Initial instantaneous temperature distribution $f_1(x)$ in medium 1, and $f_2(x)$ in medium 2. Ends $x=0$ and $x=b$ both radiating to an atmosphere at zero temperature.

8,9,10,11,12,13,14, and 15. Initial instantaneous temperature distribution $f_1(x)$ in medium 1, and $f_2(x)$ in medium 2, but involving each of the eight other possible combinations of end conditions, e.g. radiation to the atmosphere at one end while the other end is impervious to heat or is kept at zero temperature, etc.

16. End $x=0$ at variable temperature $\phi_1(t)$, end $x=b$ radiating to an atmosphere at zero temperature (or impervious to heat or kept at zero temperature.), Initial temperature zero throughout.

17. End $x = b$ at variable temperature $\phi_2(t)$, end $x = 0$ radiating to an atmosphere at zero temperature (or impervious to heat or kept at zero temperature). Initial temperature zero throughout.

18. Problems involving more complex initial and boundary conditions than those stated above.

Since we shall have occasion to refer to Dr. G. Green's results in their general form,[6.] they are for convenience given below in equations (66) to (80), with the appropriate meanings of the notation.

<u>Notation.</u>

For a wave train originating in medium 1, and initially incident on the boundary $x = a$,

$$\left.\begin{array}{lll} A & = \text{reflection coefficient at} & x = a. \\ A' & = \text{transmission coefficient at} & x = a. \\ B & = \text{reflection coefficient at} & x = b. \\ C & = \text{retransmission coefficient at} & x = a. \\ C' & = \text{reflection coefficient at} & x = a. \\ A_0 & = \text{reflection coefficient at} & x = 0. \end{array}\right\} \qquad (66)$$

$$\left.\begin{aligned} S_1 &= \frac{1}{1 - A_0 A e^{-2ia\lambda}} \\ S_2 &= \frac{1}{1 - BC' e^{-2i\mu(b-a)\lambda}} \end{aligned}\right\} \qquad (67)$$

$$E(a\lambda, -A_0 A) = e^{ia\lambda}\left[1 - A_0 A e^{-2ia\lambda}\right] \tag{68}$$

$$S = \frac{1}{1-\alpha}, \qquad \left.\begin{array}{l} \alpha = A_0 A' B C\, S_1 S_2 e^{-2is\lambda} \\ s = a + \mu(b-a) \end{array}\right\} \tag{69}$$

$$D = \frac{1}{S_1 S_2 S}. \tag{70}$$

λ and μ are determined by the differential equations governing the wave motion in the two media [see equation (51)].

Wave-train Summations.

Case 1. Periodic source at point x_1 in medium 1.

Total effect at x in 1 $(x > x_1)$

$$= \frac{\rho_1}{A_0} e^{ikt} E\{x_1\lambda, A_0\}\left[E\{x\lambda, A_0\} S_1 S - e^{ix\lambda}\right] \tag{71}$$

Total effect at x in 1 $(x < x_1)$

= same expression, x and x_1 interchanged. (72)

Total effect at x in 2

$$= \rho_1 e^{ikt - is\lambda} E\{x_1\lambda, A_0\} E\{\mu(b-x)\lambda, B\} \frac{A'}{D}. \tag{73}$$

Case 2. Periodic source at point x_1 in medium 2.

Total effect at x in 2 $(x > x_1)$

$$= \frac{\rho_2}{B} e^{ikt} E\{\mu(b-x)\lambda, B\}\left[E\{\mu(b-x_1)\lambda, B\} S_2 S - e^{i\mu(b-x_1)\lambda}\right] \tag{74}$$

Total effect at x in 2 $(x < x_1)$

= same expression, x and x_1 interchanged. (75)

Total effect at x in 1

$$= \rho_2 e^{ikt - is\lambda} E\{\mu(b-x_1)\lambda, B\} E\{x\lambda, A_0\} \frac{C}{D}. \tag{76}$$

<u>Case 3. Periodic source at boundary $x=0$ in medium 1.</u>

Total effect at x in 1 $= \frac{\rho_0}{A_0} e^{ikt}\left[E\{x\lambda, A_0\} S_1 S - e^{ix\lambda}\right]$. (77)

Total effect at x in 2* $= \rho_0 e^{ikt - is\lambda} E\{\mu(b-x)\lambda, B\} \frac{A'}{D}$. (78)

<u>Case 4. Periodic source at boundary $x=b$ in medium 2.</u>

Total effect at x in 2 $= \frac{\rho}{B} e^{ikt}\left[E\{\mu(b-x)\lambda, B\} S_2 S - e^{i\mu(b-x)\lambda}\right]$. (79)

Total effect at x in 1† $= \rho e^{ikt - is\lambda} E\{x\lambda, A_0\} \frac{C}{D}$. (80)

<u>Procedure.</u>

In the solutions to our problems, we require to use these results and to translate them into terms of the usual notation of heat conduction problems.

Problems 1 to 6 involve finding the temperature effects due to <u>continuous</u> sources at $x=0$ and $x=b$ respectively. We do this by first utilising the <u>periodic</u> source effects given under Cases 3 and 4. An application of Fourier's integral theorem, involving the integration of the periodic solutions with respect to k yields the <u>instantaneous</u> source effect. Finally, the continuous source effect is obtained by integration of the instantaneous

* A_0 in the numerator of the R.H.S. of ref. 6 is wrong and is omitted here.
† B " " " " " " " " " " " " "

result with respect to time.

In the same way, the solutions to problems 7 to 15 are obtained by thinking of the final temperature effect as being due to a series of instantaneous heat sources initially distributed all along the rod, each instantaneous source effect being obtained from the corresponding periodic source effect by the usual integration with respect to k. These require the periodic source effects as given in Cases 1 and 2.

Problems 16 and 17 do not differ in nature from numbers 1 to 6 and are approached by the same method. Those grouped under 18 may involve a subdivision of a particular problem into several parts and an application of Cases 1 and 2 to one part, and of Cases 3 and 4 to another. The final effect will then be a summation of these separate effects.

Coefficients common to all problems.

The following relationships, depending as they do on the properties of the media in contact, and not on the external boundary or initial conditions, are common to all the problems 1 to 18, and are therefore stated at the outset. The heat conduction equations to be satisfied by the solutions are

$$\frac{\partial v_1}{\partial t} = \kappa_1 \frac{\partial^2 v_1}{\partial x^2} \quad , \qquad \frac{\partial v_2}{\partial t} = \kappa_2 \frac{\partial^2 v_2}{\partial x^2} . \qquad (61)$$

in the two media respectively.

A consideration of the wave-trains set up by an initial periodic source proportional to $e^{i\omega t}$ and satisfying equations (81) gives the values of λ and μ to be

$$i\lambda = \sqrt{\frac{i k}{\kappa_1}} \ , \quad \mu = \sqrt{\frac{\kappa_1}{\kappa_2}} \ . \tag{82}$$

The conditions holding at the boundary $x = a$ are

$$v_1 = v_2 \quad , \ K_1\frac{\partial v_1}{\partial x} = K_2\frac{\partial v_2}{\partial x} \ . \tag{83}$$

These determine the coefficients A , A' , C , and C' which remain the same in all the problems.

$$\left.\begin{aligned} A &= \frac{K_1 - \mu K_2}{d} = -C' , \qquad d = K_1 + \mu K_2 \ . \\ A' &= \frac{2K_1}{d} \ , \quad C = \frac{2\mu K_2}{d} \ . \end{aligned}\right\} \tag{84}$$

Problem 1.

End $x = 0$ maintained at constant temperature θ_0 , end $x = b$ radiating to an atmosphere at zero temperature. Initial temperature zero throughout. [Case 3.].

Initial and boundary conditions are

$$v_1 = v_2 = 0 , \qquad t = 0 . \tag{85}$$

$$v_1 = \theta_0 \quad , \ x = 0 , \text{ for all later times.} \tag{86}$$

$$-K_2\frac{\partial v_2}{\partial x} = h_2 v_2 \ , \ \ x = b \tag{87}$$

Equation (87) gives

$$B = \frac{i\mu K_2\lambda - h_2}{i\mu K_2\lambda + h_2} = e^{i(\pi - 2\theta_2)}, \qquad (88)$$

where $\tan\theta_2 = \frac{\mu K_2\lambda}{h_2}$.

All wave-trains which ate reflected at $x=0$ are subject to zero end condition there. Thus

$$A_0 = -1. \qquad (89)$$

From Case 3, we now obtain the temperature effect in medium 1 and medium 2 due to the periodic source $\theta_0 e^{ikt}$ situated at $x=0$. It is convenient to calculate $\frac{d}{S_1}$, $\frac{d}{S_2}$, dD and S_1S which we find to be

$$\frac{d}{S_1} = 2e^{-ia\lambda}\left[K_1\cos a\lambda + i\mu K_2 \sin a\lambda\right]. \qquad (90)$$

$$\frac{d}{S_2} = 2e^{-i(\mu c\lambda+\theta_2)}\left[iK_1\sin(\mu c\lambda+\theta_2) + \mu K_2\cos(\mu c\lambda+\theta_2)\right], \qquad (91)$$

where $b-a=c$.

$$dD = \frac{d}{S_1S_2S} = 4ie^{-i(s\lambda+\theta_2)}\Delta, \qquad (92)$$

where $\Delta = K_1\cos a\lambda\sin(\mu c\lambda+\theta_2) + \mu K_2\sin a\lambda\cos(\mu c\lambda+\theta_2)$. (93)

Also

$$S_1S = \frac{1}{S_2D}$$

$$= \frac{e^{ia\lambda}}{2i\Delta}\left[iK_1\sin(\mu c\lambda+\theta_2) + \mu K_2\cos(\mu c\lambda+\theta_2)\right]. \qquad (94)$$

Hence, substituting in equations (77) and (78), the <u>periodic</u> solutions are

$$v_1 = \frac{\theta_0 e^{ikt}}{\Delta}\left[K_1\cos(a-x)\lambda\sin(\mu c\lambda+\theta_2) + \mu K_2\sin(a-x)\lambda\cos(\mu c\lambda+\theta_2)\right]. \qquad (95)$$

$$v_2 = \frac{\theta_0 e^{ikt}}{\Delta} K_1\sin\{\mu(b-x)\lambda+\theta_2\}. \qquad (96)$$

If in equation (96), we let the right side $= \frac{F(\lambda)}{\Delta}$,
then the instantaneous solutions are

$$\begin{aligned} v_1 &= \frac{1}{\pi}\int_0^\infty \frac{F(\lambda)}{\Delta}\,dk \\ &= \frac{1}{\pi}\int_{0B} \frac{F(\lambda)}{\Delta}\,2i\kappa_1\lambda\,d\lambda \\ &= \frac{2i\kappa_1}{\pi}(\pi i)\sum \frac{\lambda F(\lambda)}{\frac{d\Delta}{d\lambda}} \\ &= -2\kappa_1 \sum \frac{\lambda F(\lambda)}{\frac{d\Delta}{d\lambda}}. \end{aligned}$$

where the summation is over the roots of the equation $\Delta = 0$, and the integration has been performed over the usual contour.

At these roots, $F(\lambda)$ simplifies and gives

$$v_1 = -2\kappa_1 K_1 \sum \frac{\theta_0 e^{-\kappa_1\lambda^2 t}\,\lambda \sin(\mu c\lambda+\theta_2)\sin x\lambda}{\sin a\lambda\,\frac{d\Delta}{d\lambda}} \tag{97}$$

Similarly

$$v_2 = -2\kappa_1 K_1 \sum \frac{\theta_0 e^{-\kappa_1\lambda^2 t}\,\lambda \sin\{\mu(b-x)\lambda+\theta_2\}}{\frac{d\Delta}{d\lambda}} \tag{98}$$

The effect due to a continuous source θ_0 at $x = 0$ is obtained from equations (97) and (98) by integration with respect to time.

$$\begin{aligned} v_1 &= -2\theta_0 K_1 \sum \frac{\sin(\mu c\lambda+\theta_2)\sin x\lambda}{\lambda \sin a\lambda\,\frac{d\Delta}{d\lambda}}\left[1-e^{-\kappa_1\lambda^2 t}\right] \\ &= \theta_0\left[1-\frac{K_2 h_2 x}{\delta}\right] + 2\theta_0 K_1 \sum \frac{\sin(\mu c\lambda+\theta_2)\sin x\lambda\, e^{-\kappa_1\lambda^2 t}}{\lambda\sin a\lambda\,\frac{d\Delta}{d\lambda}} \end{aligned} \tag{99}$$

$$v_2 = -2\theta_0 K_1 \sum \frac{\sin\{\mu(b-x)\lambda + \theta_2\}}{\lambda \frac{d\Delta}{d\lambda}} \left[1 - e^{-\kappa_1\lambda^2 t}\right]$$

$$= \theta_0 K_1 \frac{K_2 + h_2(b-x)}{\delta} + 2\theta_0 K_1 \sum \frac{\sin\{\mu(b-x)\lambda + \theta_2\}}{\lambda \frac{d\Delta}{d\lambda}} \cdot e^{-\kappa_1\lambda^2 t} \qquad (100)$$

where $\delta = K_1 K_2 + h_2[K_1 c + K_2 a]$.

<u>Summary of results</u> for the temperature effect throughout a two-media rod due to temperature θ_0 at $x = 0$, where there is radiation to an atmosphere of zero temperature at end $x = b$. Initial temperature zero throughout.

<u>Periodic.</u>

$$v_1 = \theta_0 \frac{e^{ikt}}{\Delta} \left[K_1 \cos(a-x)\lambda \sin(\mu c\lambda + \theta_2) + \mu K_2 \sin(a-x)\lambda \cos(\mu c\lambda + \theta_2)\right] \qquad (95)$$

$$v_2 = \theta_0 \frac{e^{ikt}}{\Delta} K_1 \sin\{\mu(b-x)\lambda + \theta_2\}. \qquad (96)$$

<u>Instantaneous.</u>

$$v_1 = -2\theta_0 \kappa_1 K_1 \sum \frac{\lambda \sin(\mu c\lambda + \theta_2) \sin x\lambda \; e^{-\kappa_1\lambda^2 t}}{\sin a\lambda \; \frac{d\Delta}{d\lambda}} \qquad (97)$$

$$v_2 = -2\theta_0 \kappa_1 K_1 \sum \frac{\lambda \sin\{\mu(b-x)\lambda + \theta_2\} \; e^{-\kappa_1\lambda^2 t}}{\frac{d\Delta}{d\lambda}} \qquad (98)$$

<u>Continuous.</u>

$$v_1 = \theta_0\left[1 - \frac{K_2 h_2 x}{\delta}\right] + 2\theta_0 K_1 \sum \frac{\sin(\mu c\lambda + \theta_2) \sin x\lambda \; e^{-\kappa_1\lambda^2 t}}{\lambda \sin a\lambda \; \frac{d\Delta}{d\lambda}} \qquad (99)$$

$$v_2 = \theta_0 K_1 \frac{K_2 + h_2(b-x)}{\delta} + 2\theta_0 K_1 \sum \frac{\sin\{\mu(b-x)\lambda + \theta_2\} \; e^{-\kappa_1\lambda^2 t}}{\lambda \frac{d\Delta}{d\lambda}} \qquad (100)$$

where
$$\left.\begin{aligned} \tan\theta_2 &= \frac{\mu K_2 \lambda}{h_2}, \qquad \delta = K_1 K_2 + h_2[K_1 c + K_2 a] \\ \Delta &= K_1 \cos a\lambda \sin(\mu c\lambda + \theta_2) + \mu K_2 \sin a\lambda \cos(\mu c\lambda + \theta_2) \end{aligned}\right\} \qquad (101)$$

and the summations are over the positive roots of $\Delta = 0$.

Problem 2.

End $x = 0$ at temperature θ_0, but end $x = b$ impervious to heat. Initial temperature zero throughout.

Then condition (87) becomes

$$\frac{\partial v_2}{\partial x} = 0 \,, \qquad x = b;$$

thus $\quad B = +1 \,, \quad$ i.e. $\theta_2 = \frac{\pi}{2} \,, \quad h_2 = 0$

With this value of θ_2, results (95) to (101) are then the solutions to problem 2.

Problem 3.

End $x = 0$ at temperature θ_0, but end $x = b$ kept at zero temperature. Initial temperature zero throughout.

Then, condition (87) becomes

$$v_2 = 0 \,, \qquad x = b;$$

Thus $\quad B = -1 \,, \quad$ i.e. $\theta_2 = 0 \,, \quad h_2 = \infty.$

With this value of θ_2, results (95) to (101) are then the solutions to problem 3.

Problem 4.

End $x = b$ maintained at constant temperature θ_0, end $x = 0$ radiating to an atmosphere at zero temperature. Initial temperature zero. [Case 4].

Initial and boundary conditions are

$$v_1 = v_2 = 0, \qquad t = 0 \tag{102}$$

$$v_2 = \theta_0 \quad, x = b \text{ , for all later times.} \tag{103}$$

$$K_1 \frac{\partial v_1}{\partial x} = h_1 v_1 \quad , \quad x = 0. \tag{104}$$

Equations (103) and (104) give

$$B = -1 \tag{105}$$

$$\left.\begin{aligned} A_0 &= \frac{iK_1\lambda - h_1}{iK_1\lambda + h_1} = e^{i(\pi - 2\theta_1)} \\ \text{where} \quad \tan\theta_1 &= \frac{K_1\lambda}{h_1} \end{aligned}\right\} \tag{106}$$

From Case 4, we now obtain the temperature effects in medium 1 and medium 2 due to the periodic source $\theta_0 e^{ikt}$ at $x = b$.

Periodic.

$$v_1 = \frac{\theta_0}{\Delta} e^{ikt} \mu K_2 \sin(x\lambda + \theta_1) \tag{107}$$

$$v_2 = \frac{\theta_0}{\Delta} e^{ikt}\left[K_1 \cos(a\lambda + \theta_1) \sin \mu(x-a)\lambda + \mu K_2 \sin(a\lambda + \theta_1) \cos \mu(x-a)\lambda\right] \tag{108}$$

$$\text{where} \quad \Delta = K_1 \cos(a\lambda + \theta_1)\sin \mu c\lambda + \mu K_2 \sin(a\lambda + \theta_1)\cos \mu c\lambda. \tag{109}$$

Converting these now to the instantaneous solutions by means of the usual contour integration, we have

Instantaneous.

$$v_1 = -2\theta_0 \mu K_2 \kappa_1 \sum \frac{\lambda \sin(x\lambda + \theta_1)}{\frac{d\Delta}{d\lambda}} e^{-\kappa_1\lambda^2 t} \tag{110}$$

$$v_2 = -2\theta_0 \mu K_2 \kappa_1 \sum \frac{\lambda \sin(a\lambda + \theta_1) \sin \mu(b-x)\lambda}{\sin \mu c\lambda \frac{d\Delta}{d\lambda}} e^{-\kappa_1\lambda^2 t} \tag{111}$$

where the summation is over the positive roots of the equation $\Delta = 0$.

Finally, integration with respect to time gives the effects due to the continuous source θ_0 at $x = b$.

Continuous.

$$v_1 = \theta_0 \frac{K_2[K_1 + h_1 x]}{\delta} + 2\theta_0 \mu K_2 \sum \cdot \frac{\sin(x\lambda + \theta_1)\, e^{-\kappa_1 \lambda^2 t}}{\lambda \frac{d\Delta}{d\lambda}} \cdot \qquad (112)$$

$$v_2 = \theta_0 \left[1 - \frac{K_1 h_1 (b-x)}{\delta}\right] + 2\theta_0 \mu K_2 \sum \cdot \frac{\sin(a\lambda + \theta_1)\, \sin \mu(b-x)\lambda \, e^{-\kappa_1 \lambda^2 t}}{\lambda \sin \mu c\lambda \; \frac{d\Delta}{d\lambda}} \cdot \qquad (113)$$

where
$$\left.\begin{aligned} \tan\theta_1 &= \frac{K_1 \lambda}{h_1} \\ \Delta &= K_1 \cos(a\lambda + \theta_1) \sin \mu c\lambda + \mu K_2 \sin(a\lambda + \theta_1) \cos \mu c\lambda . \\ \delta &= K_1 K_2 + h_1 [K_1 c + K_2 a] . \end{aligned}\right\} \qquad (114)$$

and the summations are over the positive roots of $\Delta = 0$.

These results (107) to (114) correspond to results (95) to (10[illegible]) respectively. The one set bears to the other a reciprocal relationship. For the one set, x, λ, K_1, θ_1 in medium 1 correspond to $(b-x)$, $\mu\lambda$, μK_2, θ_2 respectively in medium 2 for the other set.

Problem 5.

End $x = b$ at temperature θ_0, but end $x = 0$ impervious to heat. Initial temperature zero throughout.

Then, condition (104) becomes

$$\frac{\partial v_1}{\partial x} = 0 \quad , \quad x = 0$$

Thus $\quad A_0 = +1 \quad , \quad$ i.e. $\theta_1 = \frac{\pi}{2}$, $h_1 = 0$.

With this value of θ_1, results (107) to (114) are then the solutions to problem 5.

Problem 6.

End $x=b$ at temperature θ_0, but end $x=0$ kept at zero temperature. Initial temperature zero throughout.*

Then, condition (104) becomes

$$v_1 = 0 \quad , \quad x = 0$$

Thus $A_0 = -1$, i.e. $\theta_1 = 0$, $h_1 = \infty$.

With this value of θ_1, results (107) to (114) are then the solutions to problem 6.

Problem 7.

Initial instantaneous temperature distribution $f_1(x)$ in medium 1, and $f_2(x)$ in medium 2. Ends $x=0$ and $x=b$ both radiating to an atmosphere at zero temperature. [Cases 1 and 2.]

Initial and boundary conditions are

$$v_1 = f_1(x) \ , \quad v_2 = f_2(x) \ , \quad t = 0 . \tag{115}$$

$$K_1 \frac{\partial v_1}{\partial x} = h_1 v_1 \ , \quad x = 0 . \tag{116}$$

$$-K_2 \frac{\partial v_2}{\partial x} = h_2 v_2 \ , \quad x = b . \tag{117}$$

* This problem is treated and the same results reached by H.S.Carslaw in "The Conduction of Heat", 1921, p 213, and by G Green in Phil. Mag. Ser.7, vol.xxxv, 1944, p 529.

Periodic Solutions.

A periodic heat source $q_1 e^{ikt}$ at $x = x_1$ in medium 1 gives the effects stated under Case 1, where $\beta_1 = \frac{q_1}{2K_1 i\lambda}$. This value of β_1 is determined by consideration of the fact that at $x = x_1$, the conditions for a periodic heat source $v_L = v_R$, $-K_1\left[\frac{\partial v_R}{\partial x} - \frac{\partial v_L}{\partial x}\right] = q_1 e^{ikt}$ have to be satisfied.

Equations (116) and (117) give

$$A_0 = e^{i(\pi - 2\theta_1)}, \qquad \tan\theta_1 = \frac{K_1\lambda}{h_1} \tag{118}$$

$$B = e^{i(\pi - 2\theta_2)}, \qquad \tan\theta_2 = \frac{\mu K_2\lambda}{h_2} \tag{119}$$

As in problem 1, we find the expressions for $\frac{d}{S_1}$, $\frac{d}{S_2}$, dD and $S_1 S$.

$$\frac{d}{S_1} = 2e^{-i(a\lambda+\theta_1)}\left[K_1\cos(a\lambda+\theta_1) + i\mu K_2\sin(a\lambda+\theta_1)\right]. \tag{120}$$

$$\frac{d}{S_2} = 2e^{-i(\mu c\lambda+\theta_2)}\left[iK_1\sin(\mu c\lambda+\theta_2) + \mu K_2\cos(\mu c\lambda+\theta_2)\right]. \tag{121}$$

$$dD = 4i\,e^{-i(s\lambda+\theta_1+\theta_2)}\Delta,$$

where $\Delta = K_1\cos(a\lambda+\theta_1)\sin(\mu c\lambda+\theta_2) + \mu K_2\sin(a\lambda+\theta_1)\cos(\mu c\lambda+\theta_2)$. (122)

also $$S_1 S = \frac{1}{S_2 D} = \frac{e^{i(a\lambda+\theta_1)}}{2i\Delta}\left[iK_1\sin(\mu c\lambda+\theta_2) + \mu K_2\cos(\mu c\lambda+\theta_2)\right] \tag{123}$$

Then from Case 1, we have

$x > x_1$, $$v_1 = \frac{2i\beta_1}{\Delta}e^{ikt}\sin(x_1\lambda+\theta_1)$$
$$\times\left[K_1\cos(a-x)\lambda\sin(\mu c\lambda+\theta_2) + \mu K_2\sin(a-x)\lambda\cos(\mu c\lambda+\theta_2)\right] \tag{124}$$

$x < x_1$, v_1 = same expression with x and x_1 interchanged. (125)

$$v_2 = \frac{2i\rho_1 K_1}{\Delta} e^{ikt} \sin(x_1\lambda+\theta_1) \sin\{\mu(b-x)\lambda+\theta_2\} \qquad (126)$$

A periodic heat source $q_2 e^{ikt}$ at $x = x_1$ in medium 2 gives the effects stated under Case 2, where $\rho_2 = \frac{q_2}{2\mu K_2 i\lambda}$

Substituting in equations (74), (75) and (76), we have

$$v_1 = \frac{2i\rho_2 \mu K_2}{\Delta} e^{ikt} \sin\{\mu(b-x_1)\lambda+\theta_2\} \sin(x\lambda+\theta_1) \qquad (127)$$

$$x > x_1,\ v_2 = \frac{2i\rho_2}{\Delta} e^{ikt} \sin\{\mu(b-x)\lambda+\theta_2\}$$

$$\times\left[K_1 \cos(a\lambda+\theta_1) \sin\mu(x_1-a)\lambda + \mu K_2 \sin(a\lambda+\theta_1)\cos\mu(x_1-a)\lambda\right] \qquad (128)$$

$x < x_1$, v_2 = same expression with x and x_1 interchanged. (129)

Instantaneous Solutions.

Effects due to the instantaneous heat source q_1 at $x = x_1$ in medium 1 are obtained by integrating the periodic source effects in equations (124), (125) and (126) with respect to k. We find that this integration, performed by means of the usual contour, gives a summation of terms which depend on the roots of the equation $\Delta = 0$ where Δ is defined in equation (122) above.

At these roots the part inside the square bracket in equation (124) becomes modified to

$$\frac{K_1 \sin(\mu c\lambda+\theta_2)\sin(x\lambda+\theta_1)}{\sin(a\lambda+\theta_1)}$$

Hence, instantaneous effects are

$$x > x_1,\ v_1 = \frac{q_1}{\pi}\int_0^\infty \frac{e^{ikt}}{\lambda\ \Delta} \frac{\sin(\mu c\lambda+\theta_2)\sin(x_1\lambda+\theta_1)\sin(x\lambda+\theta_1)}{\sin(a\lambda+\theta_1)}\, dk$$

$$= -2\kappa_1 q_1 \sum \frac{\sin(\mu c\lambda+\theta_2)\sin(x_1\lambda+\theta_1)\sin(x\lambda+\theta_1)\, e^{-\kappa_1\lambda^2 t}}{\sin(a\lambda+\theta_1)\ \frac{d\Delta}{d\lambda}} \qquad (130)$$

Since this is symmetrical in x and x_1, it also constitutes the result for $x < x_1$ in medium 1.

Similarly,

$$v_2 = -2\kappa_1 q_1 \sum \frac{\sin(x_1\lambda + \theta_1)\sin\{\mu(b-x)\lambda + \theta_2\}\, e^{-\kappa_1\lambda^2 t}}{\frac{d\Delta}{d\lambda}} \quad (131)$$

<u>Effects due to the instantaneous heat source q_2 at $x = x_1$ in medium 2</u> are obtained from equations (127), (128), and (129).

$$v_1 = -2\kappa_1 q_2 \sum \frac{\sin\{\mu(b-x_1)\lambda + \theta_2\}\,\sin(x\lambda + \theta_1)\, e^{-\kappa_1\lambda^2 t}}{\frac{d\Delta}{d\lambda}} \quad (132)$$

$$x > x_1, \; v_2 = -2\kappa_1 q_2 \sum \frac{\sin(a\lambda + \theta_1)\sin\{\mu(b-x_1)\lambda + \theta_2\}\sin\{\mu(b-x)\lambda + \theta_2\}\, e^{-\kappa_1\lambda^2 t}}{\sin(\mu c\lambda + \theta_2)\,\frac{d\Delta}{d\lambda}}. \quad (133)$$

where the part in the square bracket in equation (128) has been modified due to the summation again being over the roots of the equation $\Delta = 0$. Equation (133) is symmetrical in x and x_1, and hence gives the result for $x < x_1$ in medium 2.

<u>Summary of results</u> for the temperature effect throughout a two-media rod due to an instantaneous heat source q_1 at $x = x_1$ in medium 1, and due to an instantaneous heat source q_2 at $x = x_1$ in medium 2, where there is radiation to an atmosphere at zero temperature at both ends ofthe rod.

<u>Periodic</u>.

<u>Source at $x = x_1$ in medium 1</u>.

Periodic.

Source at $x = x_1$ in medium 1.

$$x > x_1, \quad v_1 = \frac{2i\rho_1}{\Delta} e^{ikt} \sin(x_1\lambda + \theta_1)$$
$$\times\left[K_1 \cos(a-x)\lambda \sin(\mu c\lambda + \theta_2) + \mu K_2 \sin(a-x)\lambda \cos(\mu c\lambda + \theta_2)\right] \quad (124)$$

$x < x_1$, $v_1 =$ same expression with x and x_1 interchanged. (125)

$$v_2 = \frac{2i\rho_1 K_1}{\Delta} e^{ikt} \sin(x_1\lambda + \theta_1) \sin\{\mu(b-x)\lambda + \theta_2\}. \quad (126)$$

Source at $x = x_1$ in medium 2.

$$v_1 = \frac{2i\rho_2 \mu K_2}{\Delta} e^{ikt} \sin\{\mu(b-x_1)\lambda + \theta_2\} \sin(x\lambda + \theta_1) \quad (127)$$

$$x > x_1, \quad v_2 = \frac{2i\rho_2}{\Delta} e^{ikt} \sin\{\mu(b-x)\lambda + \theta_2\}$$
$$\times\left[K_1 \cos(a\lambda + \theta_1) \sin \mu(x_1 - a)\lambda + \mu K_2 \sin(a\lambda + \theta_1) \cos \mu(x_1 - a)\lambda\right] \quad (128)$$

$x < x_1$, $v_2 =$ same expression, x and x_1 interchanged. (129)

where $\rho_1 = \dfrac{q_1}{2iK_1\lambda}$, $\rho_2 = \dfrac{q_2}{2i\mu K_2\lambda}$

Instantaneous.

Source at $x = x_1$ in medium 1.

$$v_1 = -2\kappa_1 q_1 \sum \frac{\sin(\mu c\lambda + \theta_2) \sin(x_1\lambda + \theta_1) \sin(x\lambda + \theta_1)\, e^{-\kappa_1\lambda^2 t}}{\sin(a\lambda + \theta_1)\, \frac{d\Delta}{d\lambda}} \quad (130)$$

$$v_2 = -2\kappa_1 q_1 \sum \frac{\sin(x_1\lambda + \theta_1) \sin\{\mu(b-x)\lambda + \theta_2\}\, e^{-\kappa_1\lambda^2 t}}{\frac{d\Delta}{d\lambda}} \quad (131)$$

Source at $x = x_1$ in medium 2.

$$v_1 = -2\kappa_1 q_2 \sum \frac{\sin\{\mu(b-x_1)\lambda + \theta_2\} \sin(x\lambda + \theta_1)\, e^{-\kappa_1\lambda^2 t}}{\frac{d\Delta}{d\lambda}} \quad (132)$$

$$v_2 = -2\kappa_1 q_2 \sum \frac{\sin(a\lambda + \theta_1) \sin\{\mu(b-x_1)\lambda + \theta_2\} \sin\{\mu(b-x)\lambda + \theta_2\}\, e^{-\kappa_1\lambda^2 t}}{\sin(\mu c\lambda + \theta_2)\, \frac{d\Delta}{d\lambda}} \quad (133)$$

$$\left.\begin{aligned} &\text{where} \quad \tan\theta_1 = \frac{K_1\lambda}{h_1}, \quad \tan\theta_2 = \frac{\mu K_2\lambda}{h_2}. \\ &\text{and} \quad \Delta = K_1 \cos(a\lambda + \theta_1) \sin(\mu c\lambda + \theta_2) + \mu K_2 \sin(a\lambda + \theta_1) \cos(\mu c\lambda + \theta_2) \end{aligned}\right\} \quad (134)$$

and the summations are over the positive roots of the equation $\Delta = 0$.

The final solution to problem 7, namely the temperature throughout the rod due to an initial distribution $f_1(x)$ in medium 1 and $f_2(x)$ in medium 2 can be obtained from these instantaneous source results by putting $q_1 = \frac{K_1}{\kappa_1} f_1(x_1)\,dx_1$ in equations (130) and (131), and integrating from $x_1 = 0$ to $x_1 = a$, and by putting $q_2 = \frac{K_2}{\kappa_2} f_2(x_1)\,dx_1$ in equations (132) and (133), and integrating from $x_1 = a$ to $x_1 = b$. The sum of the results derived from the equations (130) and (132) will give the temperature at x in medium 1 after time t, while the sum of these derived from equations (131) and (133) will give the temperature at x in medium 2 after time t.

Problems 8,9,10,11,12,13,14 and 15.

Initial temperature distribution $f_1(x)$ in medium 1 and $f_2(x)$ in medium 2, but involving each of the eight other possible combinations of the end conditions; e.g. radiation to an atmosphere at zero temperature at one end, while the other is impervious to heat or kept at zero temperature.

The solutions to problem 7 require to be modified only with regard to the values of θ_1 and θ_2 in order to give the solutions to these eight problems. If the ends $x=0$ and $x=b$ are impervious to heat, then $A_0 = B = 1$, and $\theta_1 = \theta_2 = \frac{\pi}{2}$. If these ends are kept at zero temperature, then $A_0 = B = -1$, and $\theta_1 = \theta_2 = 0$. By choosing the appropriate value for θ_1 and θ_2, we obtain the temperature effect with any combination of these end conditions.

It should be noted that if $K_1 = K_2 = K$, $\theta_1 = \theta_2 = \tan^{-1}\frac{K\lambda}{h}$, $a=b$, $q_1 = q_2 = q$, then results (130), (131), (132), and (133) all reduce to

$$v = -2\kappa q \sum \frac{\sin\theta_1 \sin(x_1\lambda+\theta_1)\sin(x\lambda+\theta_1)\, e^{-\kappa\lambda^2 t}}{\sin(a\lambda+\theta_1)\,\frac{d\Delta}{d\lambda}}$$

which is the correct result for a one medium rod with an instantaneous heat source q situated at $x=x_1$. The summation in this case is over the roots of the equation $\Delta = K\sin(a\lambda+2\theta_1) = 0$.

<u>Problem 16</u>.

End $x=0$ at variable temperature $\phi_1(t)$, end $x=b$ radiating to an atmosphere at zero temperature (or impervious to heat or kept at zero temperature). Initial temperature zero throughout.

The periodic and instantaneous solutions relating to this problem are similar to those given in equations (95) to (98), where in this case $\rho_0 = \theta_0 = \phi_1(t)$. We then have

Instantaneous.

$$v_1 = -2\,\phi_1(t)\,K_1\kappa_1 \sum \frac{\lambda \sin(\mu c\lambda+\theta_2)\sin x\lambda}{\sin a\lambda \;\frac{d\Delta}{d\lambda}}\, e^{-\kappa_1\lambda^2 t}$$

$$= \phi_1(t) \sum Y_1(\lambda)\, e^{-\kappa_1\lambda^2 t} \qquad (135)$$

$$v_2 = -2\,\phi_1(t)\,K_1\kappa_1 \sum \frac{\lambda \sin\{\mu(b-x)\lambda+\theta_2\}}{\frac{d\Delta}{d\lambda}}\, e^{-\kappa_1\lambda^2 t}$$

$$= \phi_1(t) \sum Y_2(\lambda)\, e^{-\kappa_1\lambda^2 t} \qquad (136)$$

where the meanings of $Y_1(\lambda)$ and $Y_2(\lambda)$ are obvious.

Continuous effects are then given by

$$v = \sum Y(\lambda) \int_0^t e^{-\kappa_1\lambda^2(t-t')}\, \phi_1(t-t')\, dt' \qquad (137)$$

for media 1 and 2 respectively, where $\tan\theta_2$ and Δ are dafined in equations (101) and the summations are over the positive rooks of the equation $\Delta = 0$.

Problem 17.

End $x = b$ at variable temperature $\phi_2(t)$, end $x = 0$ radiating to the atmosphere at zero temperature (or impervious to heat or kept at zero temperature). Initial temperature zero throughout.

Just as the solutions to problem 16 beat a simple relationship to results (95) to (98), so can the solutions to problem 17 be derived from results (107) to (111).

The continuous effects are obtained in the same way as result (137) is obtained from (135) and (136) above.

Problem 18.

Problems involving more complex initial and boundary conditions.

Under this heading, it is intended to mention possible extensions of the above results to more complex problems. A two-media rod, for instance, with different constant temperatures maintained at each end and initial temperature zero throughout will require the subdivision of the problem into two parts. Results to problems 3 and 6 then give the two separate effects with the appropriate values for θ_0, the end temperatures at $x=0$ and $x=b$, and their addition gives the total effect.

In the same way, a two-media rod with variable temperatures $\phi_1(t)$ and $\phi_2(t)$ at ends $x=0$ and $x=b$ respectively, and initial temperature zero throughout, requires the addition of the solutions to problems 16 and 17 with $\theta_1 = \theta_2 = 0$.

Similarly, if in addition to having a temperature source at one or both ends, there is an initial temperature distribution throughout the rod, then there will be an effect due to this distribution given by the solution to problem 7 (or one of its modifications 8 to 15). A summation of effects again gives the final result.

Problems involving boundary conditions other than those mentioned here can also be solved by finding the appropriate equations governing the heat transmission at the ends of the rod and hence the value of the coefficients A_0 and B (i.e. θ_1 and θ_2) . For instance, the presence of a large mass at the end of the rod, $x=b$, will affect the value of B (i.e. of θ_2), and will also contribute an initial quantity of heat to the system depending on its mass, specific heat and initial temperature. This problem can then be dealt with by a slight extension to the methods outlined in problems 1 to 17. In every problem, the solution will be found to be dependent on the roots of an equation of the form $\Delta=0$ where θ_1 and θ_2 will have values dependent on the boundary conditions at $x=0$ and $x=b$.

(b) The roots of the general equation $\Delta = 0$.

All the results which have been derived above are dependent on the positive roots of an equation $\Delta = 0$, where Δ is defined for different problems in equations (101), (114) and (134). One cannot fail to notice the similarity of form of these expressions. The equation in its most general form is

$$K_1 \cos(a\lambda+\theta_1)\sin(\mu c\lambda+\theta_2) + \mu K_2 \sin(a\lambda+\theta_1)\cos(\mu c\lambda+\theta_2) = 0$$

where* $\tan\theta_1 = \frac{K_1\lambda}{h_1}$, $\tan\theta_2 = \frac{\mu K_2\lambda}{h_2}$.

If we put $\frac{K_1}{\mu K_2} = \sigma$, $\frac{K_1}{h_1} = \alpha_1$, $\frac{\mu K_2}{h_2} = \alpha_2$, this becomes

$$\cos(a\lambda+\theta_1)\cos(\mu c\lambda+\theta_2)\left[\sigma\tan(\mu c\lambda+\theta_2) + \tan(a\lambda+\theta_1)\right] = 0 \qquad (138)$$

where $\tan\theta_1 = \alpha_1\lambda$, $\tan\theta_2 = \alpha_2\lambda$.

H.S.Carslaw† has made a study of the roots of this equation [written in a slightly different form] for the special case where $\theta_1 = \theta_2 = 0$. He shows that, for this simpler case, there is an infinite number of real, non-repeated roots. The proof used by Carslaw to show that there were no imaginary roots to his equation can be applied to the more general equation (138) in order to establish the same fact with respect to it.

* The values of θ_1 and θ_2 for special problems mentioned under heading 18 will depend on the end conditions.

† Ref. 20.

We show that the roots are infinite in number and non-repeated by recognising that the roots of equation which we require are the positive roots of

$$\sigma \tan(\mu c\lambda+\theta_2) + \tan(a\lambda+\theta_1) = 0 \qquad (139)$$

and the common positive roots, if any, of

$$\cos(a\lambda+\theta_1) = 0 \quad , \quad \cos(\mu c\lambda+\theta_2) = 0 \qquad (140)$$

Roots of equation (139).

General case.

Equation (139) can be written

$$\tan(a\lambda+\theta_1) = -\sigma \tan(\mu c\lambda+\theta_2) \qquad (141)$$

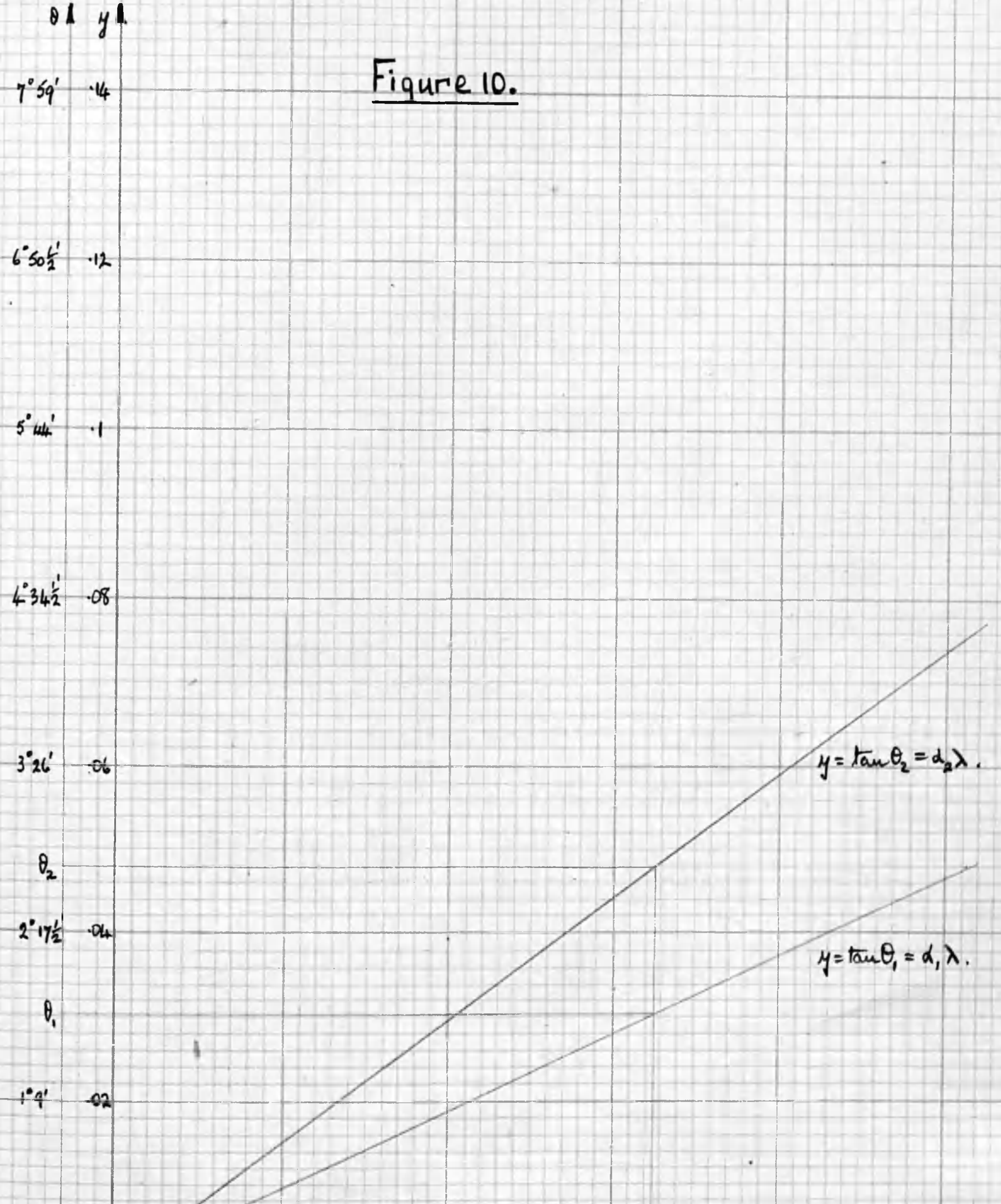
where $\tan\theta_1 = \alpha_1\lambda \quad , \quad \tan\theta_2 = \alpha_2\lambda .$

A graphical method of finding θ_1 and θ_2 for varying values of λ exists only if α_1 and α_2 are very small. λ can then be chosen as the independent variable, while the y-axis is marked off in units to measure $\tan\theta$ with the value of the angle θ marked opposite each measure. If two lines through the origin $y=\alpha_1\lambda$, and $y=\alpha_2\lambda$ are drawn, the values of θ_1 and θ_2 for any value of can be read off [See Figure 10].

For greater accuracy, however, and for all but small values of α_1 and α_2, the most efficient method of finding θ_1 and θ_2 is simply to tabulate $\tan\theta_1$ and $\tan\theta_2$ for varying λ and thus find θ_1 and θ_2.

For every value of λ, the corresponding value of θ_1 is now added to $a\lambda$, and similarly of θ_2 is added to

Figure 10.



θ
y
7°59′
·14
6°50½′
·12
5°44′
·1
4°34½′
·08
3°26′
·06
θ₂
2°17½′
·04
θ₁
1°9′
·02
0
·2
·4
·6
λ
·8
1
λ
y = tan θ₂ = α₂λ.
y = tan θ₁ = α₁λ.

$\mu c \lambda$, and the graphs of

$$y = \tan(a\lambda + \theta_1) \tag{142}$$

$$y = -\sigma \tan(\mu c \lambda + \theta_2) \tag{143}$$

plotted on the same diagram.

The values of λ at their points of intersection constitute the [redacted] roots of equation (139) and can be seen to be infinite in number and non-repeated. [See Fig. 11].

Special Cases.

1. If α_1 and α_2 are very small, this method can be modified by using the first approximation for $\theta_1 = \alpha_1\lambda$ and $\theta_2 = \alpha_2\lambda$ and equations (142) and (143) can be written

$$y = \tan(a + \alpha_1)\lambda \quad , \quad y = -\sigma \tan(\mu c + \alpha_2)\lambda .$$

2. If α_1 and α_2 are very large, then θ_1 and θ_2 may be assumed equal to $\frac{\pi}{2} - \varepsilon_1$, and $\frac{\pi}{2} - \varepsilon_2$ respectively, where $\varepsilon_1 = \tan^{-1}\frac{1}{\alpha_1\lambda}$ and $\varepsilon_2 = \tan^{-1}\frac{1}{\alpha_2\lambda}$ and ε_1 and ε_2 are both small.

Equations (142) and (143) then become

$$y = \tan\left(a\lambda - \frac{1}{\alpha_1\lambda}\right) \quad , \quad y = -\frac{1}{\sigma} \tan\left(\mu c\lambda - \frac{1}{\alpha_2\lambda}\right).$$

3 and 4. α_1 and α_2 equal to zero or infinity.

Finally, if it is known that one or both of the ends of the rod is either impervious to heat or kept at zero temperature, the equations (142) and (143) require to have the appropriate values for θ_1 and θ_2 inserted, namely 0 or $\frac{\pi}{2}$.

The question of when the two curves with equations (142) and (143) settle down to having a regular phase difference depends on the magnitude of α_1 and α_2. If these are small, then over a large range of λ, the values of θ_1 and θ_2 will vary non-linearly, and hence the roots will occur without regularity. If, however, α_1 and α_2 are large, then even for fairly small values of λ, θ_1 and θ_2 will tend to $\frac{\pi}{2}$ and the curves will soon approximate to

$$y = \tan\left(a\lambda + \frac{\pi}{2}\right) \quad , \quad y = -\sigma \tan\left(\mu c\lambda + \frac{\pi}{2}\right).$$

Common roots of equations (140).

General case.

$$\left.\begin{aligned} \cos(a\lambda + \theta_1) &= 0 \\ \cos(\mu c\lambda + \theta_2) &= 0 \end{aligned}\right\} \qquad (140)$$

The common roots are found by drawing on the same diagram the graphs of

$$y = \cos(a\lambda + \theta_1) \quad , \quad y = \cos(\mu c\lambda + \theta_2)$$

where $\tan\theta_1 = \alpha_1\lambda \quad , \quad \tan\theta_2 = \alpha_2\lambda$,

and finding where they cross the λ-axis concurrently.

The condition that the equations $\cos px = 0$ and $\cos qx = 0$ shall have common roots is that

$\frac{p}{q} = \frac{n_1}{n_2}$ = ratio of two odd integers, and the common roots are then at $\frac{n_1\pi}{2p}, \frac{3n_1\pi}{2p}, \frac{5n_1\pi}{2p} \ldots$

Special cases.

1. If α_1 and α_2 are very small, the first approximation for θ_1 and θ_2 may be used in equations (140) and the condition for common roots is that

$$\frac{a+\alpha_1}{\mu c+\alpha_2} = \frac{n_1}{n_2} = \text{ratio of two odd integers.}$$

2. If α_1 and α_2 are very large, we can let $\theta_1 = \frac{\pi}{2} - \epsilon_1$, $\theta_2 = \frac{\pi}{2} - \epsilon_2$ where ϵ_1 and ϵ_2 are small and $\tan\epsilon_1 = \frac{1}{\alpha_1\lambda}$, $\tan\epsilon_2 = \frac{1}{\alpha_2\lambda}$. Equations (140) become

$$\sin\left(a\lambda - \frac{1}{\alpha_1\lambda}\right) = 0 \quad , \quad \sin\left(\mu c\lambda - \frac{1}{\alpha_2\lambda}\right) = 0,$$

and the common roots are found graphically.

3. If α_1 and α_2 are zero, $\theta_1 = \theta_2 = 0$ and equations (140) become

$$\cos a\lambda = 0 \quad , \quad \cos\mu c\lambda = 0$$

These have common roots if

$$\frac{a}{\mu c} = \frac{n_1}{n_2} = \text{ratio of two odd integers.}$$

4. If α_1 and α_2 are infinite, $\theta_1 = \theta_2 = \frac{\pi}{2}$ and equations (140) become

$$\sin a\lambda = 0 \quad , \quad \sin\mu c\lambda = 0$$

These have common roots if

$$\frac{a}{\mu c} = \frac{m_1}{m_2} = \text{rational number.}$$

It will easily be appreciated that only for very few simple cases will equations (140) have common roots which are small and of the same order as the first few roots of equation (139) . Since the results stated in problems 1 to 18 are not of practical value unless they can be simplified by requiring only the first few terms of the summation part, our task in solving the equation $\Delta = 0$ for practical work consists of finding the first few roots of this equation. These are usually given by equation (139) ; and the common roots of equation (140) are usually outwith the range of λ required for the summation. [See section (c) below.].

The writer had hoped, after a study of equation to be able to devise a nomograph which would allow of speedy calculation of the roots for varying values of α_1, α_2, a, μc and σ. There are two main reasons, however, for the fact that a completely graphical method of solution was not found to be possible or practicable.

(1) In the application of this theory to practical work, great accuracy is required in the determination of the first few roots of equation (139) . Such accuracy is impossible where graphical methods alone are being used. Successive approximation methods were found to be hopelessly unweildy due to the complex nature of the derivative of $f(\lambda) = \tan(a\lambda+\theta_1) + \sigma \tan(\mu c\lambda+\theta_2)$.

(2) No straight-forward graphical method was found of incorporating $\theta_1 [= \tan^{-1}\alpha_1\lambda]$ along with $a\lambda$ and $\theta_2 [= \tan^{-1}\alpha_2\lambda]$ along with $\mu c\lambda$ before plotting the curves

necessary to solve equation (139).

(c) Exploration of the possibility of adapting the analytical theory to solve certain practical problems in connection with the insulation of furnace walls.

The complex nature of some of the preceding results (problems 1 to 18) make their application to practical problems seem at first sight rather remote. In the following section, a first attempt is made to investigate how far this application is possible. The heat transfer across insulated furnace walls is chosen as a suitable subject of study and results derived using the exact analytical solution are compared with those obtained using the "Schmidt Graph Method" at present widely used by practising metallurgists and engineers.

Statement of the Problem.

In the firing of open-hearth steel furnaces, one of the causes of inefficiency is the loss of heat due either to its transfer through the furnace or checker walls to the exterior, or to the storage of unutilised heat in the fabric of the walls. The second of these two sources of heat loss assumes the greater importance if the furnace is being operated intermittently. The magnitude of the heat loss due to both causes depends on the material of the walls. In general, the quality of high resistance to heat which is required in a good refractory material is accompanied by high thermal conductivity. This latter feature is an undesirable one on account of the large heat loss which it involves. Low conductivity materials, however desirable they may be in order to overcome the heat loss problem, are uneconomical due to their low resistance to the high temperatures generated in the furnace.

Experience has shown, and theory bears out, that if the outside of the furnace walls (which may be made of high conductivity refractory material) is lined with insulating fire-brick of low conductivity, the heat losses are greatly reduced. The saving in fuel which this involves far outweighs the increased costs of insulation and more expensive refractories.[14, 15]

Insulation, however, introduces a number of other problems which include the variable heat transmission through two media in contact, the quicker wearing of the inside fire-brick due to the forcing up of the internal temperature of the furnace, and the question of the optimum amount of insulation permissible for a given internal refractory material. The second and third of these problems may find their solutions in the future use of basic refractories and are, in any case, outwith the scope of the present paper. It is intended here to investigate the first problem —— that of variable heat transmission through an insulated furnace wall.

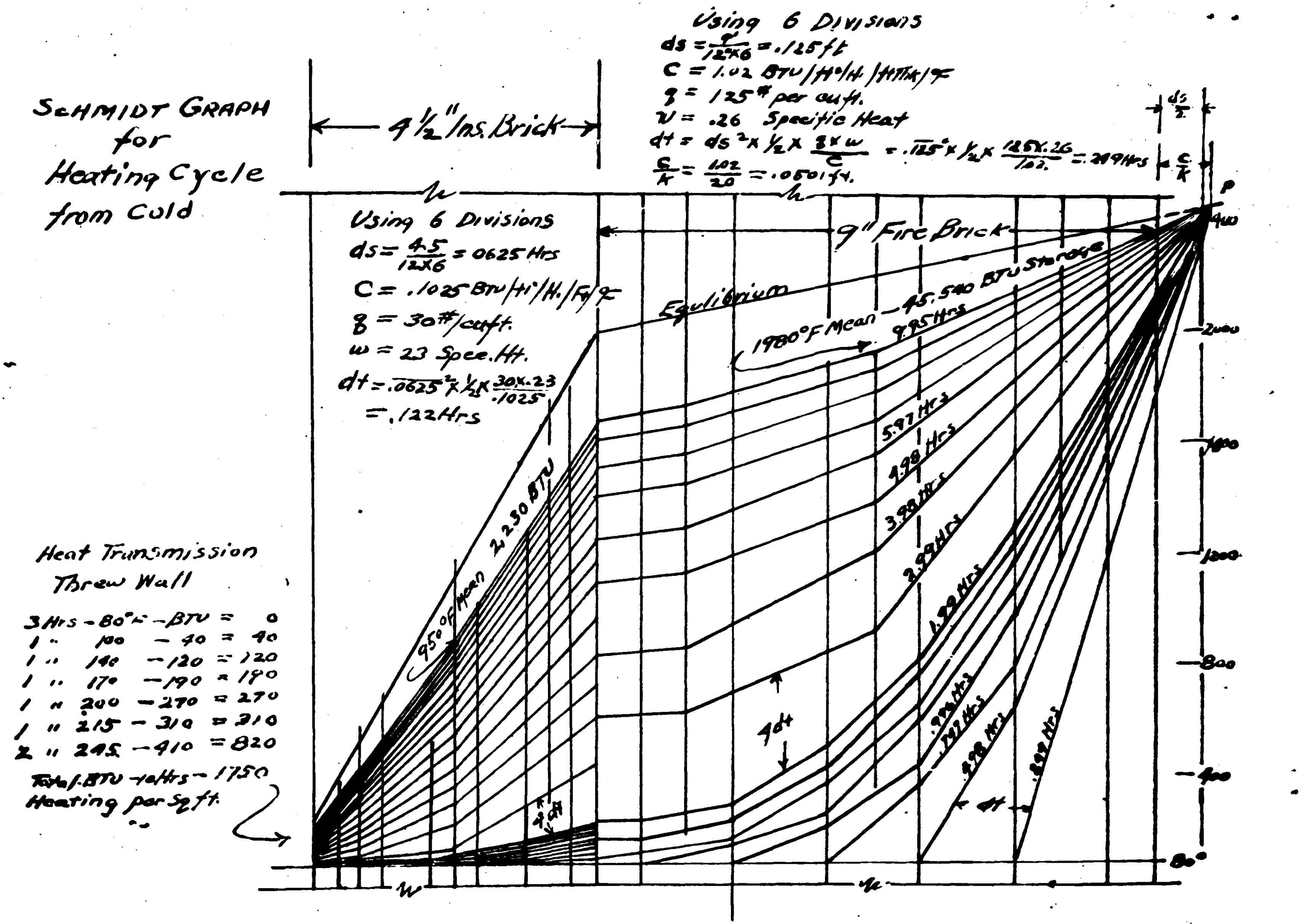
In the standard textbooks[21, 22, 23.] and in various technical articles[14, 15.] on the practical aspects of the subject, the exact theoretical solution to the problem is not given,and instead, several approximate methods are used by which the temperature distribution throughout the furnace walls can be obtained.

Current approximate method of solution by means of Schmidt Graphs.

Most of the approximate theories for the solution of variable linear heat flow problems for a finite medium are based on a method developed by E. Schmidt. Speaking of unsteady heat flow in only one medium, M. Fishenden and O.A. Saunders say, "In many cases where exact mathematical solutions cannot be given, an approximate method due to E. Schmidt may be usefully employed." As expounded by Fishenden and Saunders (see Ref. 22 P.79) this method depends on the solution to the heat conduction equation for a semi-infinite medium. It also involves the use of finite differences instead of the differential equation which governs linear heat flow. The method enables one, by a graphical process, to deduce from an initial temperature distribution what the new distribution will be after a short interval of time. Its successive application gives the temperature after any length of time.

A fuller exposition, taking into account the heat transfer at the surfaces of the plate to or from the surrounding medium, is given by W. Trinks.[24.] This still suffers from the defects associated with choosing finite intervals of time and distance, and unless the width of the wall is divided into more than four sections (laminae) this method will indicate a temperature distribution throughout the wall which rises much too quickly. The method as outlined for one medium can easily be extended to the variable flow of heat through two different media in contact.

In an article in "Blast furnace and steel plant", N.A.Humphrey discusses the advantages of insulation as an economic measure in the running of steel furnaces and uses Schmidt Graphs to study the temperature distributions throughout the two media of the walls for different thicknesses of insulation. One of his diagrams is reproduced in Figure 10A. The work involved in the compilation of these diagrams must have been considerable. Since the problem with which he deals is one to which our theory provides the exact analytical solution, it seems desirable



SCHMIDT GRAPH
for
Heating Cycle
from Cold
4 1/2" Ins. Brick
Using 6 Divisions
ds = 9/(12×6) = .125 ft
C = 1.02 BTU/Ft²/H./Ft Thk/°F
g = 125# per cu.ft.
w = .26 Specific Heat
dt = ds² × 1/2 × (g × w)/C = .125² × 1/2 × (125 × .26)/1.02 = .249 Hrs
C/k = 1.02/20 = .051 ft.
ds/2
C/k
9" Fire Brick
Using 6 Divisions
ds = 4.5/(12×6) = .0625 Hrs
C = .1025 BTU/Ft²/H./Ft/°F
g = 30#/cu.ft.
w = .23 Spec. Ht.
dt = .0625² × 1/2 × (30 × .23)/.1025
= .122 Hrs
Equilibrium
1980°F Mean – 45,540 BTU Storage
9.95 Hrs
5.97 Hrs
4.98 Hrs
3.98 Hrs
2.99 Hrs
1.99 Hrs
.996 Hrs
.747 Hrs
.498 Hrs
.249 Hrs
2,230 BTU
950°F Mean
4 dt
dt
2000
1600
1200
800
400
80°
Heat Transmission
Threw Wall
3 Hrs – 80°F – BTU = 0
1 " 100 – 40 = 40
1 " 140 – 120 = 120
1 " 170 – 190 = 190
1 " 200 – 270 = 270
1 " 215 – 310 = 310
2 " 245 – 410 = 820
Total BTU – 10 Hrs – 1750
Heating per Sq. ft.

Figure 10 A.

to apply this theory and to compare the results with those of Humphrey, while at the same time remembering that practical men want a solution that is speedy as well as accurate.

The problem is as follows: A furnace wall is made of 9 inches of fire-brick and is insulated with $4\frac{1}{2}$ inches of insulating fire-brick. The internal furnace temperature is 2400° F and the external temperature and the initial temperature of the furnace walls is 80° F. Heat is lost by radiation and convection from the outside surface. What is the subsequent temperature distribution throughout the walls and in particular at the outside surface?

In our results (99) to (101) above, we possess the solution to this problem. One condition of their ready application to practice is that only a limited number of terms in the summation should be required. This in fact we find to be the case. It is intended, therefore, to derive numerical results for the temperature distribution using the data supplied by Humphrey and the analytical results (99) to (101) . Comparison with Humphrey's results will then be possible.

It should be mentioned here that no allowance is made in our calculations for the fact that thermal conductivity changes slightly with temperature. This allowance was not made by Humphrey either. A mean value over the range of temperature under consideration is used.

Conversion of the analytical results into terms of dimensionless groups.

Before going into the numerical detail of our problem, some mention should be made of the work which has been done in connection with adapting the analytical solutions for variable heat transfer through one finite medium to suit practical needs. It has been found that the analytical results for the main important shapes of bodies, namely, slab, long cylinder and sphere, can be expressed in terms of four dimensionless ratios and then presented in graphical form.[16, 17, 21, 22] These ratios are: a temperature difference ratio, a "relative time" ratio, a thermal resistance ratio and a position ratio. A useful table and an explanation of these is to be found in Ref. 21, p.31. The simplification which this method of presentation affords for the application of one medium results to practical work is very considerable.

Bearing this work in mind, we now proceed to examine the solution to the furnace wall problem mentioned above, with a view to effecting, if possible, a similar simplification of the analytical results by the introduction of dimensionless groups.

Let us consider the two-media furnace wall as extending from $x=0$ to $x=b$, where $x=0$ is the interior and $x=b$ is the exterior surface. Let the surface of contact between medium 1 and medium 2 be at $x=a$ and let the temperature of the atmosphere be our zero. Then the temperatures in media 1 and 2 respectively are given by equations (99) and (100).

$$v_1 = \theta_0\left[1 - \frac{K_2 h_2 x}{\delta}\right] + 2\theta_0 K_1 \sum \frac{\sin(\mu c\lambda + \theta_2)}{\lambda \sin a\lambda} \frac{\sin x\lambda}{\frac{d\Delta}{d\lambda}} e^{-\kappa_1\lambda^2 t} \qquad (99)$$

$$v_2 = \frac{\theta_0[K_1\{K_2 + h_2(b-x)\}]}{\delta} + 2\theta_0 K_1 \sum \frac{\sin\{\mu(b-x)\lambda + \theta_2\}\, e^{-\kappa_1\lambda^2 t}}{\lambda \frac{d\Delta}{d\lambda}}. \qquad (100)$$

where
$$\left.\begin{aligned} \delta &= K_1K_2 + h_2[K_1c + K_2a], \qquad \tan\theta_2 = \frac{\mu K_2\lambda}{h_2} \\ \Delta &= K_1\cos a\lambda \sin(\mu c\lambda + \theta_2) + \mu K_2 \sin a\lambda \cos(\mu c\lambda + \theta_2), \end{aligned}\right\} \qquad (101)$$

and the summations are over the positive roots of $\Delta = 0$.

We find that

$$\left.\begin{aligned} \frac{d\Delta}{d\lambda} &= \mu K_2 \cos a\lambda \cos(\mu c\lambda + \theta_2)\, G(\lambda) \\ \text{where} \quad G(\lambda) &= a(1 + \tan^2 a\lambda) + \{1 + \tan^2(\mu c\lambda + \theta_2)\}\left\{\sigma\mu c + \frac{K_1}{h_2(1+\tan^2\theta_2)}\right\} \\ \text{and} \quad \sigma &= \frac{K_1}{\mu K_2}. \end{aligned}\right\} \qquad (144)$$

Hence

$$v_1 = \theta_0\left[1 - \frac{K_2 h_2 x}{\delta}\right] - 2\theta_0 \sum \frac{(1 + \tan^2 a\lambda)\sin x\lambda \cdot e^{-\kappa_1\lambda^2 t}}{\lambda\, G(\lambda)} \qquad (145)$$

$$v_2 = \frac{\theta_0 K_1[K_2 + h_2(b-x)]}{\delta} + 2\theta_0\sigma \sum \frac{\sin\{\mu(b-x)\lambda + \theta_2\}\, e^{-\kappa_1\lambda^2 t}}{\lambda \cos a\lambda \cos(\mu c\lambda + \theta_2)\, G(\lambda)}. \qquad (146)$$

If now, in these equations we write $a\lambda = \nu$, ${}_sv_1$ and ${}_sv_2$ for the steady temperatures in media 1 and 2 respectively, we find

$$v_1 = {}_sv_1 - 2\theta_0 \sum \frac{(1 + \tan^2\nu)\sin\frac{x}{a}\nu \quad e^{-\kappa_1\frac{\nu^2}{a^2}t}}{\nu\, G(\nu)} \qquad (147)$$

$$v_2 = {}_sv_2 + 2\theta_0\sigma \sum \frac{\sin\left\{\frac{\mu(b-x)}{a}\nu + \theta_2\right\} e^{-\kappa_1\frac{\nu^2}{a^2}t}}{\nu\cos\nu \cos\left(\frac{\mu c}{a}\nu + \theta_2\right) G(\nu)} \qquad (148)$$

where $\tan\theta_2 = \frac{\mu K_2}{a h_2}\nu$

$$\Delta = \sigma\cos\nu\sin\left(\frac{\mu c}{a}\nu + \theta_2\right) + \sin\nu\cos\left(\frac{\mu c}{a}\nu + \theta_2\right)$$

$$G(\nu) = 1 + \tan^2\nu + \sigma\left\{1 + \tan^2\left(\frac{\mu c}{a}\nu + \theta_2\right)\right\}\left\{\frac{\mu c}{a} + \frac{\mu K_2}{a h_2(1 + \tan^2\theta_2)}\right\}$$

and the summations are over the roots of $\Delta = 0$.

The quantities which appear above in dimensionless group form are:

Two-media comparison ratios. $\sigma = \frac{K_1}{\mu K_2}$

$\gamma = \frac{\mu c}{a}$

Thermal resistance ratio. $m = \frac{\mu K_2}{a h_2}$

"Relative time" ratio. $X = \frac{\kappa_1 t}{a^2}$

Position ratios. $n_1 = \frac{x}{a}$

$n_2 = \frac{\mu(b-x)}{a}$

Temperature difference ratios. $Y_1 = \frac{{}_s v_1 - v_1}{\theta_0}$

$Y_2 = \frac{{}_s v_2 - v_2}{\theta_0}$

Since we wish to calculate absolute temperatures for comparison with Humphrey's results, we shall use this notation with the exception of the temperature difference ratios, Y_1 and Y_2 meantime. Equations (147) and (148) then become

$$v_1 = {}_s v_1 - 2\theta_0 \sum \frac{(1 + \tan^2\nu)\sin n_1\nu .\ e^{-X\nu^2}}{\nu\, G(\nu)} \qquad (149)$$

$$v_2 = {}_s v_2 + 2\theta_0\sigma \sum \frac{\sin(n_2\nu + \theta_2)\ e^{-X\nu^2}}{\nu\cos\nu\cos(\gamma\nu + \theta_2)\, G(\nu)} \qquad (150)$$

where $$\left.\begin{aligned} \tan\theta_2 &= m\nu \\ \Delta &= \sigma\cos\nu\sin(\gamma\nu+\theta_2) + \sin\nu\cos(\gamma\nu+\theta_2) \\ G(\nu) &= (1+\tan^2\nu) + \sigma\{1+\tan^2(\gamma\nu+\theta_2)\}\left\{\gamma + \frac{m}{1+\tan^2\theta_2}\right\} \end{aligned}\right\} \quad (151)$$

and the summation is over the positive roots of $\Delta = 0$.

The equations to determine the roots of $\Delta = 0$ then become

$$\tan\nu = -\sigma\tan(\gamma\nu+\theta_2) \quad (152)$$

and $$\cos\nu = 0 \quad , \quad \cos(\gamma\nu+\theta_2) = 0 . \quad (153)$$

The advantages of this more concise expression of results (99) to (101) are obvious, particularly in connection with the working out of numerical examples.

Calculation of the roots of $\Delta = 0$, and derivation of the temperatures at $x = a$ and $x = b$.

We can now use Humphrey's data to evaluate the various constants in our notation.

Medium 1.	Medium 2.
$a = \cdot 75.$	$c = b - a = \cdot 375.$
$K_1 = 1\cdot 02.$	$K_2 = \cdot 1025.$
$\rho_1 = 125.$	$\rho_2 = 30.$
$s_1 = \cdot 26.$	$s_2 = \cdot 23.$
$\kappa_1 = \frac{K_1}{\rho_1 s_1} = \cdot 0314.$	$\kappa_2 = \frac{K_2}{\rho_2 s_2} = \cdot 0149.$
$\theta_0 = 2320.$	$h_2 = 2\cdot 2.$

$$\mu = \sqrt{\frac{K_1}{K_2}} = 1.453. \qquad \gamma = \frac{\mu c}{a} = .7267.$$

$$\sigma = \frac{K_1}{\mu K_2} = 6.847. \qquad m = \frac{\mu K_2}{a h_2} = .0903.$$

British Units (foot-pound-hour-degrees Fahrenheit) are used throughout in the calculation of the dimensionless ratios, σ , γ , m . The value of the heat transfer coefficient h_2 between the outside surface and the atmosphere is not given explicitly by Humphrey. The value $h_2 = 2.2$ is a mean value over the temperature range of the outside face (see Figure 10A, Heat transmission through wall) and corresponds with the value given by W.Trinks in Ref. 24, p.81, Fig. 70.

Equations (149) to (153) constitute the solution to our problem. Substituting the known numerical quantities, we see that the roots of the equation $\Delta = 0$ are those of

$$\tan \nu = -6.847 \tan(.7267\nu + \theta_2) \tag{154}$$

and the common roots, if any, of

$$\cos \nu = 0 \quad , \quad \cos(.7267\nu + \theta_2) = 0 \tag{155}$$

where $\tan \theta_2 = .0903\,\nu.$

We therefore draw in Figure 11 the graphs of

$$y = \tan \nu , \qquad y = -6.847 \tan(.7267\nu + \theta_2)$$

and find the values of ν at points where they intersect; and in Figure 12, we draw the graphs of

$$y = \cos \nu , \qquad y = \cos(.7267\nu + \theta_2)$$

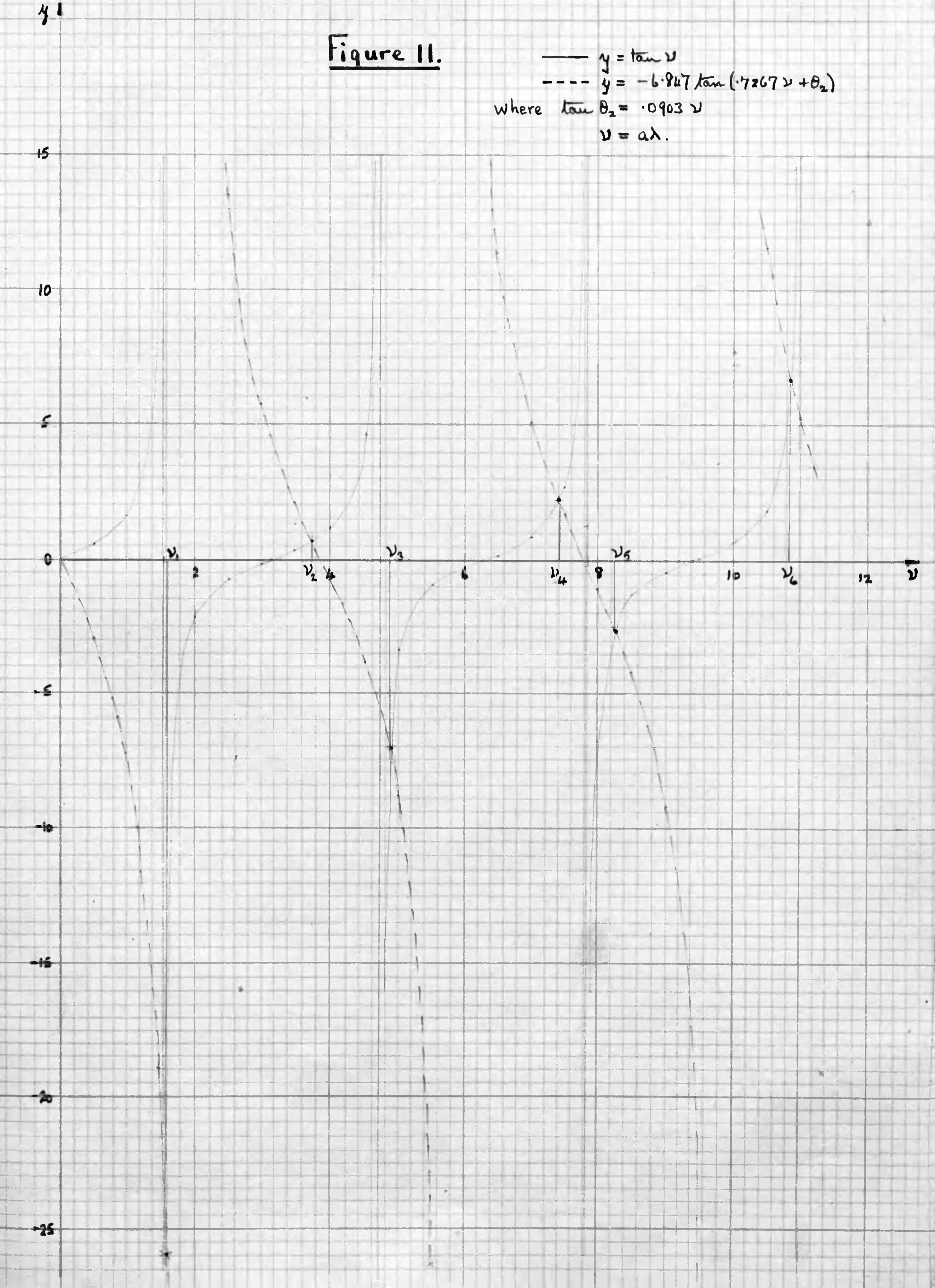


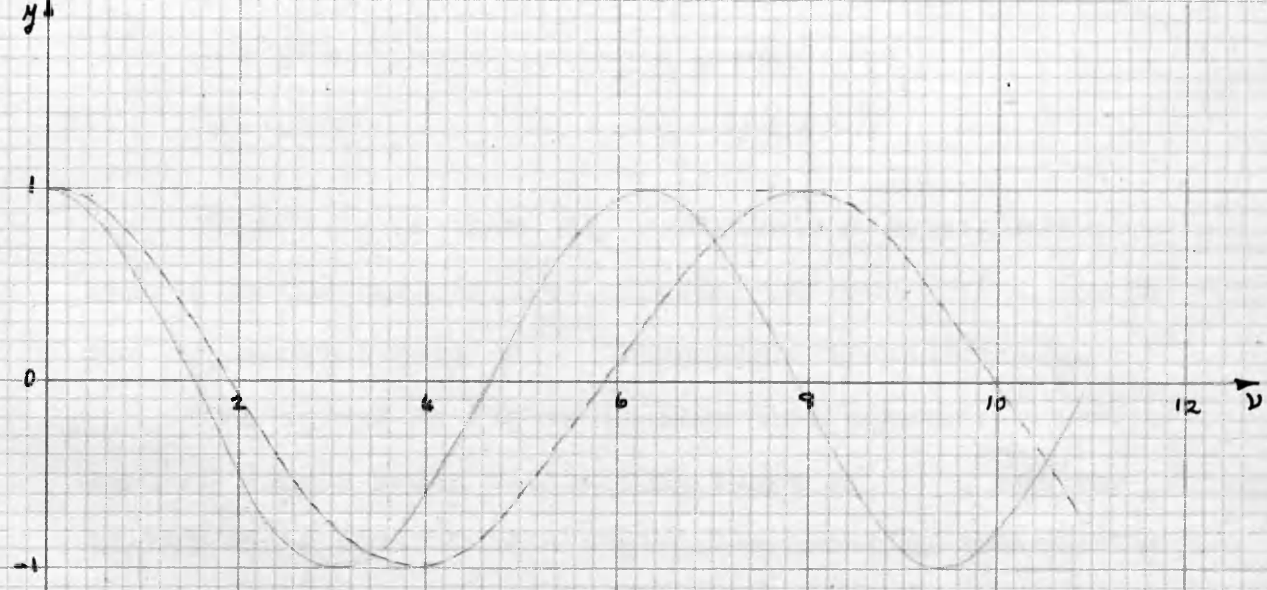
Figure 11.
—— y = tan ν
- - - - y = −6·847 tan (·7267 ν + θ₂)
where tan θ₂ = ·0903 ν
ν = aλ.
y
15
10
5
0
-5
-10
-15
-20
-25
2
4
6
8
10
12
ν
ν₁
ν₂
ν₃
ν₄
ν₅
ν₆

Figure 12.

—— $y = \cos \nu$

- - - - $y = \cos(.7267\nu + \theta_2)$

where $\tan \theta_2 = .0903\nu$.



and find whether they cross the ν-axis concurrently at values of ν in the region of the first few roots of equation (154).

We find that the first six roots of equation (154) are

$\nu_1 = 1.6092.$ $\qquad\qquad$ $\nu_4 = 7.418.$

$\nu_2 = 3.7389.$ $\qquad\qquad$ $\nu_5 = 8.240.$

$\nu_3 = 4.8537.$ $\qquad\qquad$ $\nu_6 = 10.845.$

Higher roots than ν_6 contribute a negligible amount to the temperature effects (149) and (150), due to the exponential term $e^{-X\nu^2}$ becoming extremely small.

The curves in Figure 12 show that equations (155) have no common roots within this range of ν, and any higher roots contribute a negligible quantity towards the temperature effect.

Steady state temperatures are given by the first terms on the R.H.S. of equations (145) and (146); or, expressed in terms of the dimensionless groups, they are

$$ {}_s v_1 = \theta_0 \left[1 - \frac{n_1}{\sigma(m+\gamma)+1} \right] \tag{156}$$

$$ {}_s v_2 = \theta_0 \sigma \frac{m+n_2}{\sigma(m+\gamma)+1} \tag{157}$$

To find the steady temperatures at $x = a$ and $x = b$ we can put $n_1 = 1$ and $n_2 = 0$ in equations (156) and (157) respectively, and find ${}_s v_a = 1968.15$ and ${}_s v_b = 217.50$. The temperatures at $x = a$ and $x = b$ for times after $t = 0$

are obtained by inserting a certain number of values of ν in the expression under the summation sign in equations (149) and (150).

If we let $E = \dfrac{2\theta_0(1+\tan^2\nu)\sin\nu}{\nu\,G(\nu)}\,e^{-\chi\nu^2}$ in equation (149)

and $E = \dfrac{2\theta_0\sigma\sin\theta_2\,e^{-\chi\nu^2}}{\nu\cos\nu\cos(\gamma\nu+\theta_2)\,G(\nu)}$ in equation (150),

we obtain the following temperatures:

Temperature at $x = a$.

After one hour, using the first expression for E, we find

$E_1 = 2212{\cdot}41$ $\quad E_4 = 14{\cdot}12$

$E_2 = -66{\cdot}40$ $\quad E_5 = 6{\cdot}35$

$E_3 = -207{\cdot}74$ $\quad E_6 = -0{\cdot}49$

$$\therefore v_a = 1968{\cdot}15 - 2212{\cdot}41$$
$$66{\cdot}40 - 14{\cdot}12$$
$$207{\cdot}74 - 6{\cdot}35$$
$$0{\cdot}49$$
$$= 2242{\cdot}78 - 2233{\cdot}88$$

$$\therefore v_a = 9{\cdot}90.$$

After two hours,

$E_1 = 1914{\cdot}61$ $\quad E_3 = -55{\cdot}77$

$E_2 = -30{\cdot}43$ $\quad E_4 = 0{\cdot}71$ $\quad \therefore v_a = 139{\cdot}03.$

After four hours,

$E_1 = 1434{\cdot}04$ $\quad E_3 = -4{\cdot}03$

$E_2 = -6{\cdot}40.$ $\quad E_4 =$ negligible. $\quad \therefore v_a = 544{\cdot}54.$

After six hours,

$E_1 = 1074{\cdot}38$ $\quad E_3 = -0{\cdot}29.$

$E_2 = -1{\cdot}34$ $\quad \therefore v_a = 895{\cdot}40.$

After eight hours,

$\epsilon_1 = 804.44.$ $\epsilon_2 = -0.28.$ $\therefore v_a = 1164.00.$

After ten hours,

$\epsilon_1 = 602.76.$ $\epsilon_2 = -0.06.$ $\therefore v_a = 1365.45.$

After twelve hours,

$\epsilon_1 = 451.17.$ $\epsilon_2 =$ negligible. $\therefore v_a = 1516.98.$

It will be noticed that after six hours, only the first two roots, and after eight hours only the first root need be used to give an accuracy of $0.3°F$.

Temperature at $x = b$.

After two hours, using the second expression for ϵ, we find

$\epsilon_1 = -284.69.$ $\epsilon_4 = 1.20.$

$\epsilon_2 = 98.46.$ $\epsilon_5 = -0.25.$

$\epsilon_3 = -30.10.$ $\therefore v_b = 2.12.$

After four hours,

$\epsilon_1 = -213.23.$ $\epsilon_3 = -2.17$

$\epsilon_2 = 20.71.$ $\epsilon_4 =$ negligible. $\therefore v_b = 22.81.$

After six hours,

$\epsilon_1 = -159.75.$ $\epsilon_3 = -0.15.$

$\epsilon_2 = 4.35$ $\therefore v_b = 61.95.$

After eight hours,

$\epsilon_1 = -119.61.$ $\epsilon_2 = 0.92.$ $\therefore v_b = 98.81.$

After ten hours,

$\epsilon_1 = -89.63.$ $\epsilon_2 = 0.19.$ $\therefore v_b = 128.06.$

After twelve hours,

$\epsilon_1 = -67.09.$ $\epsilon_2 =$ negligible. $\therefore v_b = 150.41.$

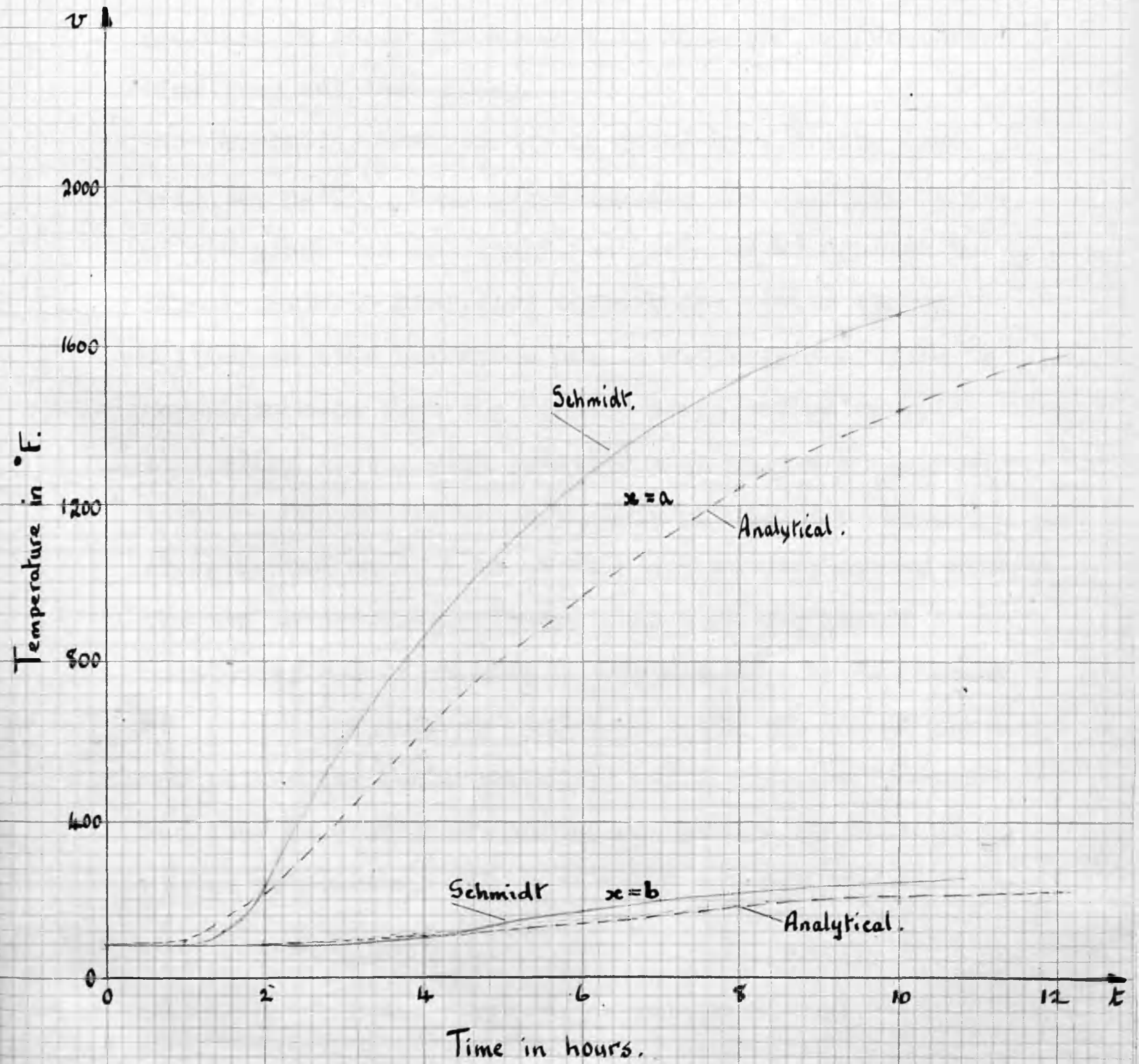
The time required to reach within 10% of the steady state temperature can be calculated, since we know that beyond ten hous only the first term in the summation need be used. We find this time to be, for the end $x = a$, 17.74 hours, and for the end $x = b$, 19.80 hours.

Comparison with Humphrey's results.

The results calculated above are now plotted on a graph in Figure 13 along with the corresponding graphs derived from Humphrey's diagram which was construsted using the Schmidt Graph method. Unfortunately, Humphrey does not publish the exact figures for the temperatures at $x = a$ and $x = b$, but these can be read off his diagram with reasonable accuracy.

The discrepancies between the two sets of results are quite clear. The Schmidt Graph method gives gemperatures which rise more quickly with time than do the

Figure 13.



v
2000
1600
1200
800
400
0
Temperature in °F.
Schmidt.
x = a
Analytical.
Schmidt
x = b
Analytical.
0
2
4
6
8
10
12
t
Time in hours.

temperatures obtained from the analytical results. This is due to the finite thickness of the laminae into which the wall is divided for the purpose of using the graphical method. An increase in the number of laminae and thus a decrease in their thickness will make the temperature-time gradient less steep.

A similar discrepancy is noted by W.Trinks (see Ref. 24, p.405) when investigating the relative merits of the graphical and analytical methods as applied to linear variable heat flow through one finite medium. Neither in this reference nor in Humphrey's article is any claim made that the Schmidt Graph method results correspond better with practice than do the analytical results. The verification of this point would require experimental work to be carried out in conjunction with the two theoretical methods. This work may indeed have been done, but if it has, it has not come to the notice of the writer. Correspondence with the authors of the two references mentioned above would have been possible had they been in this country and not in the United States of America.

Graphical presentation of the results to problems 1 to 18.

Neither the Schmidt Graph method nor the analytical method is quick or easy in its application. It is reasonable to suppose that the analytical results will correspond better with practice than do the Schmidt Graph results provided no other factors than those we have considered are at play. If this is shown to be the case, then it will be desirable to make available the analytical results in a form in which they can be readily applied to practical work. It is intended now to show how their application may be simplified by the construction of graphs using dimensionless groups. Once constructed, such diagrams will obviate the necessity of heavy computational work in connection with each individual problem.

We shall show how the construction of these graphs for Humphrey's problem treated above would greatly simplify future work on problems of the type dealt with under problems 1, 2, and 3 in Section (a), and shall merely indicate how similar modification of results to problems 4 to 18 could also be carried out.

We have already seen how an alternative form of results (99) and (100) is given in equations (149) and (150). At that stage, we purposely omitted using the temperature difference ratios Y_1 and Y_2. The results (149) and (150) can however, be expressed wholly in terms of the dimensionless groups. They then become

$$Y_1 = 2\sum \frac{(1+\tan^2\nu)\,\sin n_1\nu\; e^{-X\nu^2}}{\nu\, G(\nu)} \tag{158}$$

$$Y_2 = -2\sigma \sum \frac{\sin(n_2\nu+\theta_2)\; e^{-X\nu^2}}{\nu \cos\nu \cos(\gamma\nu+\theta_2)\, G(\nu)} \tag{159}$$

where

$$\left.\begin{aligned} \tan\theta_2 &= m\nu \\ G(\nu) &= 1+\tan^2\nu + \sigma\{1+\tan^2(\gamma\nu+\theta_2)\}\{\gamma + \frac{m}{1+\tan^2\theta_2}\} \end{aligned}\right\} \tag{151}$$

and the summations are over the roots of

$$\tan\nu = -\sigma\tan(\gamma\nu+\theta_2) \tag{152}$$

and the common roots, if any, of

$$\cos\nu = 0\,, \qquad \cos(\gamma\nu+\theta_2) = 0\,. \tag{153}$$

The three ratios σ, γ, m are required to find these roots and together with the ratio X are involved in both expressions (158) and (159). Of the other four ratios, n_1 and Y_1 are needed for equation (158) and n_2 and Y_2 for equation (159).

It can be seen then that for given σ, γ and m, the graphs of Y_1 and Y_2 on a logarithmic scale against X to a uniform scale can be plotted for varying n_1 and n_2. This has been done in figure 14 for the data of the previous problem.

A series of diagrams like this and similar to those constructed by Gurney and Lurie[16.] for one medium would enable the metallurgist or engineer to obtain temperature

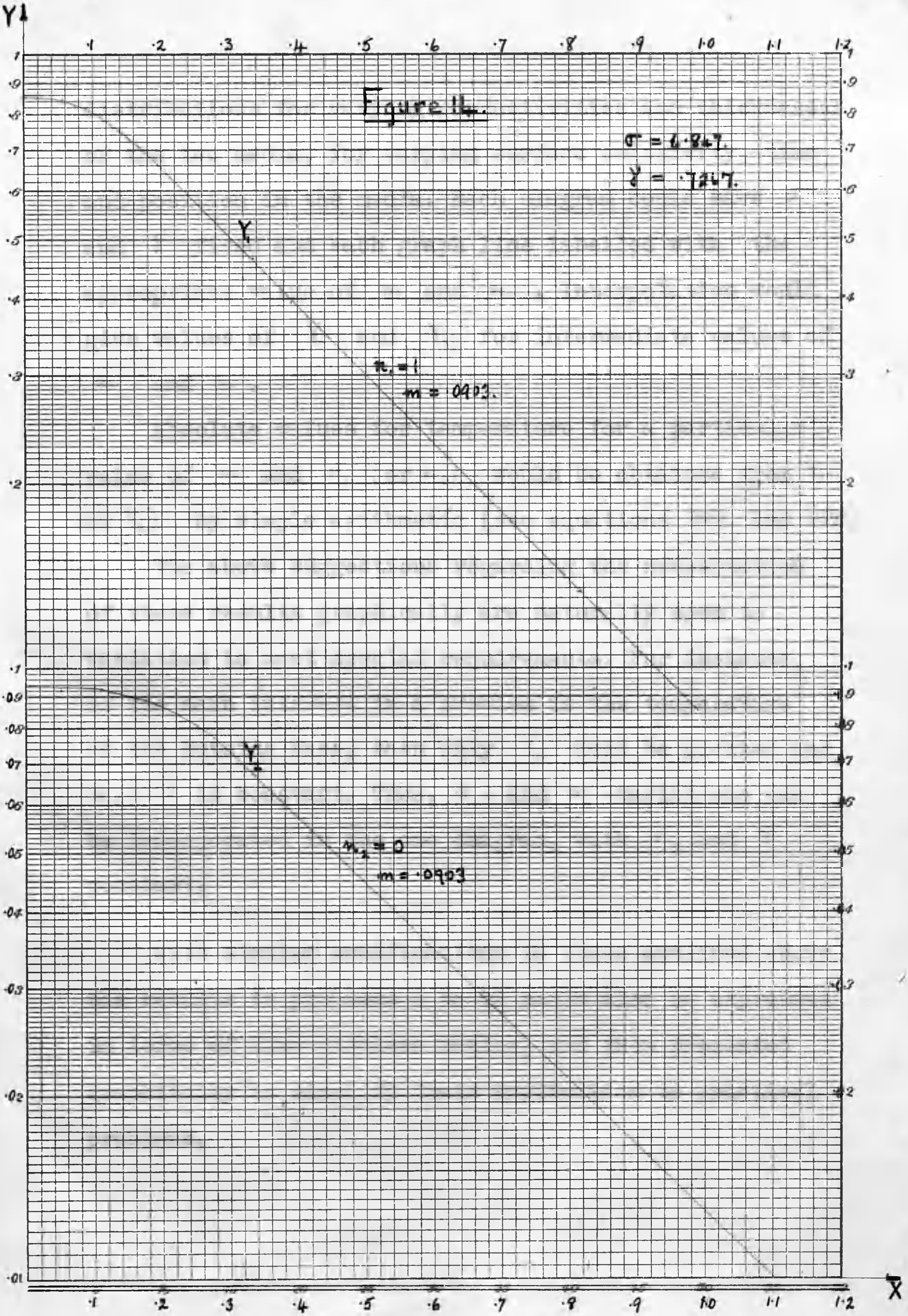

Y
X
σ = 6·847.
γ = ·7247.
Y1
n1 = 1
m = ·0903.
Y2
n2 = 0
m = ·0903
·1 ·2 ·3 ·4 ·5 ·6 ·7 ·8 ·9 1·0 1·1 1·2
1 ·9 ·8 ·7 ·6 ·5 ·4 ·3 ·2 ·1 ·09 ·08 ·07 ·06 ·05 ·04 ·03 ·02 ·01

Figure 14.

distributions for varying conductivities and thicknesses of the two media, for varying surface emissivity, time and position in the media. Each diagram could have σ and γ fixed and each graph line labelled with the appropriate value of m and n. Interpolation would give values of Y_1 and Y_2 for intermediate values of m and n.

Absolute values for temperature for a particular value of m and n_1 (or n_2) would be obtained from Y_1 (or Y_2) by simple arithmetic (see equations (156) and (157))

The above suggestions regarding the presentation of these results graphically are naturally open to variation to meet special requirements. For instance, if the main interest in a problem is the temperature of the outside face, then only Y_2 need be plotted and $n_2 (=0)$ is constant. Thus, σ- and m-variations can be incorporated in the one diagram, with n_2 and γ constant.

With similar modifications to those outlined above the results to problems 4 to 18 could also be expressed in terms of dimensionless ratios, and then presented graphically to simplify their application to practical problems.

Conclusion.

Further work which the writer had planned to do in this field must be postponed meantime due to her early departure from this country to the Continent of Europe.

On the theoretical side, she is well aware of the wide variety of problems in cylindrical and spherical heat flow which still await solution and for which the wave-train method provides a suitable approach, and of the difficulties still to be overcome in modifying these solutions for practical application. It is clear too that this work has its applications in relation not only to problems of furnace wall insulation but also to those involving variable heat transfer through other media in contact, e.g. rubber, glass, steel, etc.

The writer hopes to extend this work at some future date and also hopes that the work embodied in this paper has opened the way up for workers in the same field

1. by providing a compendium of results for the main two media heat conduction problems involving linear flow,

2. by examining in detail the general equation $\Delta = 0$ whose roots determine the normal functions required to express the solutions of the two media problems,

3. by the juxta-position of the approximate and analytical methods of finding the temperature distribution across a two-media wall and showing that the accuracy of the two

theoretical methods requires verification by practice,
4. by indicating how the analytical results may be presented graphically using dimensionless groups in a form suitable for direct use by practical scientists in these fields of work.

Throughout the whole of the work involved in this thesis, the writer has had the guidance and advice of Dr. G. Green and she wishes now to record her great indebtedness to him. Thanks are also due to Dr. T. Carter for advice on the metallurgical side.

Bibliography and References.

1. G.Green: Phil. Mag., Vol.iii. Suppl. April,1927, p.784.
2. " " " " Vol.v, April,1928, p.701.
3. " " " " Vol.ix, February,1930, p.241.
4. " " " " Vol.xii, Suppl. August,1931, p.233.
5. " " " " Vol.xxi, May,1936, p.932.
6. " " " " Vol.xxxv, August,1944, p.519.
7. J.Robertson: Phil. Mag., Vol.xv, May,1933, p.937.
8. " " " " Vol.xviii, July,1934, p.165.
9. " " " " Vol.xviii,Suppl. Nov.,1934,p.1009.
10. " " Thesis: Mathematical theory of heat Conduction. 1935, p.27.
11. J.H.Gray: Phil. Trans. of Royal Soc., Vol.186A,1895,p.165.
12. C.H.Lees: Proc. Phys. Soc.,41, xii, 1928-29, p.204.
13. C.H.Lander: Proc. Inst. of Mech. Eng., Vol.148, No3, 1943. "Review of recent progress in heat transfer."
14. N.A.Humphrey: Blast Furnace and Steel Plant, Dec.,1941. "Insulating Fire Brick in Modern Furnace Construction."
15. J.M.Ferguson: Paper presented to the Scottish Section of the Institute of Fuel on Jan.,26th,1945. "Heat losses and the Insulation of Open-Hearth Furnaces and Blast Furnaces."

16. H.P.Gurney and J.Lurie: Industrial and Engineering Chemistry, 15, 1923, p.1173. "Charts for estimating temperature distributions in heating or cooling solid shapes."

17. E.D.Williamson:and L.H.Adams/ Physical Review, 14, 1919, p.99. "Temperature distribution in solids during heating and cooling."

18. MacRobert and Arthur: Trigonometry, Part 3, p.427.

19. J.Fourier: The Analytical Theory of Heat, C.U.P., 1878.

20. H.S.Carslaw: The Conduction of Heat, Macmillan, 1921.

21. W.H.McAdams: Heat Transmission, McGraw-Hill, 1933.

22. M.Fishenden and O.A.Saunders/ The Calculation of Heat Transmission, H.M.S.O., 1932.

23. Schack, Goldschmidt, Partridge: Industrial Heat Transfer, Wyley, 1933.

24. W.Trinks: Industrial Furnaces, Wyley, 1934.

25. J.H.Chesters: Steelplant Refractories, United Steel Companies Ltd., Sheffield, 1944.

26. Milne-Thomson and Comrie: Standard Four-Figure Mathematical Tables, Macmillan, 1931.