COLLISIONS BETWEEN ELECTRONS AND NUCLEI.

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PREFACE.

This thesis represents a review of the work on electron excitation carried out by the author in collaboration with Dr. I. N. Sneddon during the years 1947 to 1949. It resulted from a lecture on isomeric states, which Mr. Angus delivered early in 1947 at the Physics Colloquium at this university. In this lecture the experiments of Wiedenbeck[1] on the excitation of nuclei by electrons were discussed in some detail.

Most of the work was done independently from the work of other investigators (a paper on the disintegration of Be by Mamasachlisov[2] was found when a letter on electron excitation[3] had already been published by the author), but where in the meantime investigations of a similar type have come to our notice they were found to be in fair agreement with our results.

The basic methodological idea of the present thesis in which the transition of an electron from one state to another is understood as equivalent to a certain electromagnetic field which interacts with the nucleus stands somewhere between the Weizsäcker Williams[4] method

and the expansion method used in the quantum theory of radiation.[5] The results obtained are identical with those of the latter method.

This interpretation of the basic phenomenon which for some time appeared to be new has been previously used by other authors and is reviewed in relation to the disintegration of deuterium in Rosenfeld's 'Nuclear Forces'.[6] Out of the whole complex of problems related to the interaction between electrons and nuclei that of the electron disintegration of deuterium has received maximum attention, owing partly to the fact that most of the properties of this nucleus (wavefunction) are well known and owing to the small energy expenditure in this process. Since the treatment of disintegration phenomena (as disintegration phenomena) is in general very similar to that reviewed in Rosenfeld's book this subject has not been included in the present thesis. Here disintegration is treated as a special case of nuclear excitation a concept closely related to the idea of the compound nucleus:[7] the compound nucleus is formed under electron impact; if it decays under emission of a heavy particle we have a nuclear disintegration, but the mode of decay has little influence on

the formation of the compound nucleus.[8] This concept seems to be better adapted to the treatment of heavy nuclei.

The main purpose of this investigation was to gain some insight into the most general properties of electrons as nuclear projectiles, mainly because it was felt that the machines shortly expected to operate in this department would make such knowledge useful and necessary. This purpose defines the frame of the present thesis from which all considerations related to particular nuclei have been excluded.

Paragraphs 1-3 contain a general development of the theory. The nuclear transition is described in terms of the matrix element of the current corresponding to this transition; the justification of this procedure in the case of magnetic transitions is given in appendix 2.

Paragraphs 4-7 contain the application of this theory to simple multipole transitions.

The following sections 8-10 discuss 'sum-rules' for the excitation of nuclei, similar to those known

in the theory of atomic spectra.[9] These considerations establish the electron on the bottom of the scale of nuclear projectiles: their total cross sections are very small and the average energy transfer in a collision is only of the order of 1Mev for heavy nuclei.

Paragraph 11 gives a brief account of the present status of the work on the behaviour of electronic excitation curves near the threshold. The agreement reached with the experimental data given by Wiedenbeck is poor and the work on this topic is still going on.

Paragraphs 12 and 13 establish a reasonable agreement with the experimental estimates of the order of magnitude of the cross section for electron excitation given by Collins and Waldmann.[10]

Paragraphs 14 and 15 deal with the production of pseudoscalar mesons in collisions between electrons and nuclei, a question suggested by the development of electron accelerators in the 300 Mev region.

Appendix 1 gives the formalism for the determination of nuclear matrix elements. This formalism is applied to magnetic transitions in appendix 2.

The work on all the sections of this thesis has been greatly stimulated by numerous discussions in this department. I wish to thank in particular Professor P.I.Dee for his kindness, his criticisms and the encouragement given at all stages of the work, Professor J.C. Gunn to whom I am indebted for a valuable suggestion on the subject of meson production and Dr. I.N. Sneddon for many discussions and the hours of joint work.

CONTENTS.

The Interaction between Electrons and Nuclei.

Since the discovery of the neutron it is generally assumed that electrons do not form a part of the atomic nucleus. To explain the various phenomena related to β-decay one has to assume a peculiar and essentially quantised interaction between electrons and nuclear matter in consequence of which electrons and (11) neutrinos can be created and annihilated in the nucleus. Since the lifetimes of β-emitters are very long in comparison with the unit nuclear time R/v (R= nuclear radius, v= some average velocity of a nucleon in an atomic nucleus) the forces arising from this interaction will be very small. Mathematically this smallness of interaction manifests itself in the form of a dimensionless constant $\gamma \simeq 10^{-24}$ which in the theory of β-decay plays a rôle similar to that played by the Sommerfeld constant $\alpha = 1/137$ in the quantum theory of radiation. In addition to the specific interaction arrising from the possibility of β-decay we have to assume electromagnetic forces between the electrons and the charges

in the nucleus.

The introduction of the meson [12] has brought some additional complication into the pattern of forces acting between electrons and nuclear matter. If a charged meson theory is assumed there will be an electromagnetic interaction between the electrons and the mesons. But since the mesons can be thought of as bound to the nucleons with an energy approximately equal to their rest energy their presence will only become ~~for~~ apparent for very fast electrons that is when the de Broglie wavelength of the electron becomes small enough to resolve the meson cloud in the neighbourhood of a nucleon. In this case the exchange of charge (and magnetic moment) between the nucleons and the mesons will give rise to an electromagnetic field which in its turn will interact with the electrons. This interaction will give rise to the production of mesons in collisions between electrons and nuclei, the production of quanta and electron pairs in collisions between mesons and nuclei and other processes of this type.

The mesons are generally considered to be radio-

active - an assumption which has recently been put on a solid quantitative basis [19]. Whatever the detailed nature of this radioactivity - which today is believed to lead over an intermediate longlived μ-meson λ - one branch of it at least should in the end lead to the simultaneous production of an electron (carrying the charge of the meson) and a neutrino. In this way and in agreement with the original proposal made by Yukawa the mesons can be made responsible for the whole complex of β-decay phenomena, such that e.g. the β-activity of the neutron could be described in terms of the (virtual) processes

$$n \rightarrow p + \pi^- \rightarrow \ldots \rightarrow p + e + \nu$$

where the ... indicate processes involving the μ-meson. As a consequence of the radio activity of the mesons there results a specific β-type interaction between the mesons and the electrons similar to that previously assumed for nucleons. The force resulting from this interaction will be describable in terms of a dimensionless constant γ', the magnitude of which can be estimated in the following way. Since it is known

that nucleons interact strongly with one another (the force acting between two closely spaced nucleons is certainly very much greater than the electrostatic force) the creation of a π -meson in a virtual process is a very frequent ~~moment~~ event. It follows that a nucleon will spend a considerable portion of its time in the resolved state, i.e. in the form: nucleon + one or several mesons. It can therefore be concluded that the constant γ' will be of approximately the same order (though possibly by a factor 10 larger) as the constant γ appearing in the original Fermi theory of β-decay.

The smallness of this constant justifies the neglect of any specific β -interaction, whenever the times involved in the process under consideration are small. This is particularily true for all scattering processes, ~~since~~ because there the natural unit of time is the time required by the scattered particle to cross the nucleus. This can be easily verified by estimating the order of magnitude of the cross section for a typical β -process, e.g. the production of a neutrino in a collision between an electron and a proton: $e + p \rightarrow n + \nu$. For

moderate energies (a few Mev above the threshold) this cross section will be of the order $\gamma\lambda_0^2$ where λ_0 is the Compton wavelength of the electron. With $\gamma \simeq 10^{-24}$ this gives 10^{-46} cm^2. The absorption length corresponding to this cross section would be larger than the diameter of the sun. Cross sections of this order may be of considerable cosmological importance but seem hardly detectable under laboratory conditions. The probability of the above process increases rapidly with energy up to a point where in the centre of mass system the wavelenghts of the light particles are comparable with nuclear dimensions. Beyond this point the applicability of theory becomes doubtful and it seems probable that radiation-damping will check the increase to infinity of the cross section. But even if the increase in cross section beyond the critical energy region were real it would only be of importance for the fastest particles in cosmic radiation ($>10^{18}$ ev).

Fermi's original theory of β-decay [11] shows a peculiar divergence phenomenon [14] associated with a 'resonant' continuum of intermediate states and it might be argued, that this may give rise to cross sections in

excess of the value of 10^{-46} cm^2. This argument would hold in particular for processes such as the eleastic scattering of an electron or a neutrino by a nucleon. This divergence difficulty is however a characteristic of Fermi's 'quadrilinear' formulation of the theory of β-decay. It does not appear if mesons are assumed as carriers of β-activity.

We are therefore left with a purely electromagnetic interaction between the electrons and the nucleus, particularly in an energy range of up to approximately 500 Mev to which the calculations in this paper are restricted. The limitation to energies of this order allows an approximately non-relativistic treatment of the nucleons. Recent calculations by Heitler and coworkers (15) show that radiation damping - which will be neglected in the following - is of minor importance in this energy region.

2)

The Electromagnetic Field Produced in an Electron - Transition.

In any scattering experiment with electrons as primary particles the electron undergoes a change of state. This change of state is accompanied by an electro-magnetic field which in its turn may interact with the scatterer.

We assume that the initial state of the electron can be described completely in terms of a Dirac wave function corresponding to a plane wave with momentum $\underline{p}$ and energy E ($E^2 = 1 + p^2$) at an infinite distance from the scatterer. In the following radiation units will be used: m, c, $\hbar = 1$ (m = mass of the electron, c = velocity of light, $\hbar = h/2\pi$, h = Planck's constant). In this system of units the unit of length is the Compton- wavelength of the electron

$$\lambda_0 = 3.85\ 10^{-11}\ \text{cm}$$

and the elementary charge is given by

$$e^2 = \alpha = 1/137.$$

The system itself suggests the use of Born's [13] approximation in which the effect of the charge is treated as a small perturbation. The final state of the electron

will be indicated by primed quantities: E' describing the energy of the final state. To the transition $0 \to f$ from the initial state to the final state we may then ascribe a current density $(f|s_\nu(\underline{r})|0)\, e^{-i\kappa t}$, where $\kappa = E-E'$. For inelastic collisions we will have $\kappa > 0$, for elastic collisions $\kappa = 0$. The 4 components of s satisfy the equation of continuity

$$\partial_\nu s_\nu = 0 \qquad (1)$$

where $\partial_\nu = \frac{\partial}{\partial x_\nu}$ and $x_1, x_2, x_3, x_4 = x, y, z, it$. In the following we will have s denote the kinematical current, i.e. the electric current divided by e. The electric forces may now be derived from a 4-potential A_ν, the first 3 components of which form the vector potential. The 4-potential will be understood to be the retarded solution of the equation

$$\partial_\nu^2 A_\mu = -4\pi s_\mu$$

which with the assumed time dependence of s gives

$$A_\mu(\underline{r},t) = \int \underline{dr}'\, s_\mu(\underline{r}') \frac{e^{i\kappa(t-|\underline{r}-\underline{r}'|)}}{|\underline{r}-\underline{r}'|} \qquad (2)$$

From (2) one obtains a very simple form for the 4-potentials if the electron waves of the initial and final

states can be taken to be plane waves. In this case

$$s_\nu = \alpha_\nu e^{i(\underline{k}\underline{r})} \qquad (3)$$

with $\underline{k} = \underline{p} - \underline{p}'$. α_ν represents the expectation value for the transition $0 \to f$ of the matrices $\alpha_\nu = (\underline{\alpha}, i)$. Inserting from equation (3) into equation (2) one obtains after an elementary integration and by enforcing convergence by a suitable 'screening-factor'

$$A_\nu(\underline{r},t) = \frac{4\pi e\, \alpha_\nu}{k^2 - \kappa^2} e^{i((\underline{k}\underline{r}) - \kappa t)} \qquad (4)$$

The 4 components A_ν are related by the Lorentz-equation

$$\partial_\nu A_\nu = 0 \qquad (5)$$

which follows from the equation of continuity (1). The exponent in equation (4) shows that the field produced by a change of state of the electrons has the form of a wave with a vector of propagation $\underline{k}$. In the case of an inelastic collision it is interesting to note that $(\underline{k}\underline{p})$ is always positive, so that the momentum carried by the wave can never be directed opposite to the momentum of the incident electron. The phase velocity of the wave is always smaller than the velocity of light, which it only approaches when $\underline{p}$ and $\underline{p}'$ are nearly parallel and both $p, p' \gg \max(1, \kappa)$. In this case

the transverse part of the field is nearly equal to that of a suitable spectrum of light quanta. The Weizsäcker-Williams method is based on this fact and therefore restricted to the above inequality. This is the reason why this method should not be applied to large energy transfers.

The spacepart of the exponent in equation (4) defines the momentum transferred from the electron to the scatterer, the time part the energy. Because of the conservation of energy the energy of the scatterer must change from E_o to $E_o + \kappa$. A part of the energy of the electronic transition will turn up as nuclear recoil, another as a change of the intrinsic energy of the scatterer. The recoil-energy is usually negligible being at most of the order $k^2/4000$ (for a proton with mass $M \simeq 2000$). However for electrons with energies of approximately 120 Mev the recoil energy of a single proton will be of the order of 8 Mev that is roughly equal to its binding energy within the nucleus.

It follows from equation (4) that - the energy

κ being given - the most frequent transitions will lead to a final state near to that corresponding to a minimum transfer of momentum. This effect will be most pronounced for extremely high electron energies and small energy transfers. It is due to the denominator $D = k^2 - \kappa^2 = p^2 + p'^2 - 2pp'\cos\theta - \kappa^2$ (θ being the angle between $\underline{p}$ and $\underline{p}'$). Using extreme relativistic approximation we have

$$p = E - \frac{1}{2E} + O(E^{-2}) \qquad (6)$$

so that

$$D = 2pp'(1-\cos\theta) + \frac{\kappa^2}{pp'} \qquad (7)$$

We see therefore that the 4-potential corresponding to 'forward-transitions' is of the order of $4p^2p'^2/\kappa^2$ times stronger than for inelastic processes leading to backward scattering. This factor is approximately 1400 for electrons of 20 Mev energy and an energy transfer of 5 Mev. For the width of the maximum we obtain from equation (7): $\theta_{1/2} = \kappa/pp'$. It is defined in such a way that the amplitude of the 4 potential averaged over the solid angle $\theta^2_{1/2}\pi$ is roughly the same as the average

over the rest of the solid angle.

The perturbation formalism will in general lead to terms proportional to D^0 , D^{-1} and D^{-2} . For the term proportional to D^0 the large scattering angles will give the major contribution to the angular integration, for D^{-1} large and small scattering angles will be of equal importance, whereas for terms proportional to D^{-2} the backward component can be neglected. One would therefore expect that - since all cross sections calculated in this paper are proportional to the square of the amplitude - the behaviour of D^{-2} would introduce a considerable simplification of the calculations. That this will not be the case is due to the fact that the angles occur in the factors α_ν as well as in the denominator (this makes terms proportional to D^0 and D^{-1} appear in the differential cross sections) and that the equation of continuity for the 4-current corresponding to the change of state of the nuclear scatterer causes a considerable cancellation in the forward direction. The important terms are those proportional to 1/D. They are in the end responsible for the slow logarithmic increase

with energy of the cross sections of most processes involving electrons as primary particles. This logarithmic increase can be deduced from the fact that because of $d\cos\theta = -dk^2/2pp'$, the angular integration over the directions of the electron in the final state gives $\int d\cos\theta/D = \log(p+p')/(p-p')$ and this is indeed the asymptotic behaviour for large energies of all the cross sections derived in this paper.

We have yet to specify the quantities a_ν occurring in equation (3). They can be expressed in terms of the Dirac matrices [6]

$$\underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix}$$

with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a_ν then becomes

$$a_k = (a'^* \alpha_k a) \qquad a_4 = i\,(a'^* a) \tag{8}$$

$(k = 1,2,3)$, where $a = a(\underline{p}, s)$ are the Dirac spin amplitudes for a plane wave. They are defined by the equation

$$((\underline{\alpha}\underline{p}) + \beta - E_s)\, a(\underline{p}, s) = 0 \tag{9}$$

and the additional requirementsthat for $s = 1,2$ $\quad E_s > 0$

for $s = 3,4$ $E_s < 0$, for $s = 1,3$ $a^*(\sigma p)a = p$ and for $s = 2,4$ $a^*(\sigma p)a = -p$. The $a(p,s)$ represent a complete set of eigensolutions to equation (9).

3)

The Electromagnetic Interaction between Electrons and Nuclei.

In this paragraph we will discuss the form of the interaction matrix between the field corresponding to the transition of the electrons and the current corresponding to the transition carried out by the nucleus. Neglecting electrostatic forces acting between the nucleus as a whole and the electrons we may represent the electronic states by plane waves and use the method of approximation which has been developed by Born [17] and applied by Møller [18] and Bethe [19] to the calculation of the energy-losses due to ionisation.

The limitations of this procedure will be discussed in paragraph II . In this place it may suffice to state that the method should be applicable to

light nuclei ($2\pi Z < 137$) and for electron energies $\gg mc^2$.

The phyical picture underlying the method used in this paper is a nucleus situated at the origin and exposed to the electromagnetic field corresponding to the transition of the electron from its initial state to its final state. The electromagnetic field is defined in equation (4). We assume that under the influence of this field the nucleus will make transitions from a state 0 with energy 0 to a state f with energy κ . The final state f may belong to the continuous as well as to the discrete spectrum of the nucleus, though this distinction in nuclear theory is not nearly as rigid as in the field of atomic spectra. The theory can be carried quite far without specifying the nature of the final state. The nuclear transition will be characterised by a current 4-vector $j_\nu e^{i\kappa t}$. The determination of j_ν - assuming the nuclear eigenfunctions of the initial and final state to be known - will be carried out in appendix 1. The 4 components of the current vector are related by the equation of continuity

$$\partial_\nu j_\nu \equiv (\nabla \underline{j}) + \kappa j_4 = 0 \qquad (10)$$

The interaction energy corresponding to the forces acting between the field and the nucleus is then given by

$$H' = e\int \underline{dr}\; A_\nu(\underline{r},t)\, j_\nu(\underline{r})\, e^{i\kappa t} \qquad (11)$$

which after inserting from equation (4) and with the abbreviation

$$J_\nu = \int \underline{dr}\; j_\nu(\underline{r})\, e^{i(\underline{k}\,\underline{r})} \qquad (12)$$

gives

$$H' = \frac{4\pi a}{k^2 - \kappa^2}\, a_\nu J_\nu \qquad (13)$$

By means of the equations of continuity and the Lorentz-condition we can eliminate the components with $\nu = 4$ in equation (13). For a_ν we obtain by using equations (1) and (4)

$$i(\underline{a}\,\underline{k}) - \kappa a_4 = 0 \qquad a_4 = \frac{i}{\kappa}(\underline{a}\,\underline{k}) \qquad (14)$$

Inserting for j_4 from equation (10) into equation (12):

$$J_4 = \int e^{i(\underline{k}\,\underline{r})}\, j_4(\underline{r})\,\underline{dr} = -\frac{1}{\kappa}\int e^{i(\underline{k}\,\underline{r})}\,(\nabla \underline{j})\,\underline{dr}$$

we obtain by using Gauss's theorem

$$J_4 = \frac{i}{\kappa}(\underline{k}\,\underline{J}) \qquad (15)$$

Inserting this into the expression (13) for the per-

turbation energy we finally obtain

$$H' = \frac{4\pi a}{k^2 - \kappa^2} \left(\underline{a} \cdot \underline{J} - \underline{k} \frac{(\underline{J}\,\underline{k})}{\kappa^2} \right) \qquad (16)$$

From this equation we may now determine the cross section (in units of λ_0^2) by means of the well known relation

$$d\Phi = \pi \frac{\epsilon}{p} \sum_{ss'} |H'|^2 \varrho_f \, d\omega \qquad (17)$$

Here $d\Phi$ denotes the differential cross section corresponding to an inelastic scattering process in which the electron is scattered into an element of solid angle $d\omega$, ϱ_f is the density of final states of the electron:

$$\varrho_f = (2\pi)^{-3} p'\epsilon' \qquad (18)$$

and $\sum_{ss'}$ represents a summation over the spin-eigenvalues of the initial and final state of the electron. The cross section (17) therefore is the differential cross-section averaged over the directions of spin of the incident electron and summed over the directions of the final electron. Inserting from equations (16) and (18) into equation (17) one obtains

$$d\Phi = 2a^2 \frac{\epsilon\epsilon' p'}{p} (k^2 - \kappa^2)^{-2} \, p \qquad (19)$$

where P is defined by the relation

$$P = |\int \underline{dr}\, s_\nu j| = \underline{\mathfrak{J}}^* S \underline{\mathfrak{J}} \qquad (20)$$

S is the symmetric tensor

$$S = \sum_{s s'} ((I - \underline{k}\underline{k}/\kappa^2)\, \underline{a}\, \underline{a}\, (I - \underline{k}\underline{k}/\kappa^2)) \qquad (21)$$

In this equation$ all the vectorproducts are dyadic.

Equation (19) is identical with a formula given by Bethe[19] on the basis of the Möller[18] theory of retarded electromagnetic interaction. The term P is of the form

$$P = |\int j_\nu s_\nu \underline{dr}|^2 .$$

and appears in this form in Bethe's theory of ionisation processes.

Equation (19) contains all information about the process of inelastic electromagnetic scattering in the range of validity of Born's approximation and the further calculations will proceed in three steps:

1) The determination of the tensor S - defined in equation (21). Its value when inserted into (20) and (19) will give the differential cross section for a

given direction of $\underline{p}'$ averaged over the spin of the electron. The cross section so obtained will still depend upon the orientation of the nuclear spin in the initial and final states.

2) An averaging process over the spin-orientations of the nucleus in the initial state and a summation over the - physically equivalent - spin-orientations of the final state. When these summations are carried out the cross section will depend upon the direction of $\underline{p}'$ only.

3) Finally the integrations over all the possible values of the vector $\underline{p}'$ will be carried out. We then obtain the total cross section for the production of a given final state of the nucleus in a collision between an electron and the nucleus.

If the energy of the electron is sufficiently large a further summation may be carried out over the possible final states of the nucleus. These states will be allowed to have different energies and simple relations are obtained when a certain average excitation energy is small compared with the energy of the electron. The result of this summation will then re-

present the total cross section for an inelastic collision between the electron and the nucleus. The average energy loss can be determined in a similar manner.

4)

Determination of the Tensor S.

Because of equation (21) the determination of the tensor S may be reduced to the determination of the tensor $\sum_{ss'} \underline{a}\,\underline{a}$. According to equation (8) we may write for this quantity:

$$\sum_{ss'} \underline{a}\,\underline{a} = \sum_{ss'} (a^* \underline{\alpha}\, a')(a'^* \underline{\alpha}\, a) \qquad (22)$$

The spin amplitudes $a_\nu(\underline{p}, s)$ can be interpreted as unitary matrices with the spinor indices ν . The Dirac-matrices are matrices in these spinor indices. For a given value of $\underline{p}$ the $a_\nu(\underline{p}s)$ form a complete system such that

$$\sum_s a_\nu(\underline{p}\,s)\, a^*_\mu(\underline{p}\,s) = \delta_{\nu\mu} \qquad (23)$$

This fact may be used for the evaluation of sums of the type (22). In a manner similar to that used by Heitler[20] we exclude the negative energy values by intro-

ducing an operator

$$K = \frac{1}{2E}((\underline{a}\underline{p}) + \beta + E) \qquad (24)$$

which, when applied to a spin amplitude describing a state of positive energy, acts as a unit operator, but gives 0 when applied to a negative energy amplitude. Since the amplitudes occurring in equation (22) represent states of positive energy we may write

$$a = Ka \qquad a' = K'a' \qquad (25)$$

and after inserting into equation (22) sum over all values of the spin variables s and s'. Then by applying the identity (23) we obtain

$$\sum_{ss'} \underline{a}\,\underline{a} = \frac{1}{4} sp(\underline{a} K' \underline{a} K) \qquad (26)$$

where sp denotes the trace with respect to the spinor indices: $sp(S_{\mu\nu}) = \sum_{\nu} S_{\nu\nu}$. This trace can be readily evaluated by considering that the trace of all odd products of the matrices α and β vanishes and that

$$sp(\beta^2) = sp(\alpha_i^2) = 4 \qquad i = 1, 2, 3$$

By using the obvious relation

$$\underline{a}(\underline{a}\underline{p}) + (\underline{a}\underline{p})\underline{a} = 2\underline{p}$$

we obtain in this way:

$$EE' \sum_{ss'} \underline{a}\,\underline{a} = I(EE' - (\underline{p}\,\underline{p}') - 1) + \underline{p}\,\underline{p}' + \underline{p}'\,\underline{p} \qquad (27)$$

where I is the unit tensor and the products $\underline{pp}'$, $\underline{p}'\underline{p}$ and $\underline{a}\,\underline{a}$ are dyadic products. From this the tensor S may be obtained by means of equation (21). This gives

$$\begin{aligned} EE'S = {} & I(EE' - (\underline{p}\,\underline{p}') - 1) + \underline{p}\,\underline{p}' + \underline{p}'\,\underline{p} \\ & - \frac{1}{k}\{E'(\underline{k}\,\underline{p} + \underline{p}\,\underline{k}) + E(\underline{k}\,\underline{p}' + \underline{p}'\,\underline{k}) \\ & - \frac{1}{k^2}\,\underline{k}\,\underline{k}\{EE' + (pp') + 1\} \end{aligned} \qquad (28)$$

This rather complicated expression will be simplified considerably in the special cases treated in this paper. The calculation up to this state represents a small step forward in the direction indicated by the methods used by Weizsäcker and Williams. It allows to determine the field equivalent to an electron transition (rather than that equivalent to an electron state as in the Weizsäcker Williams method) and in the form (28) frees us of the troublesome complications brought about by the electron spin - the electron being described by only two characteristic parameters: mass and charge. The quantity inextricably connected with the electronic spin -

the magnetic moment of the electron has been eliminated in the spin summations.

It will be noted that S is a symmetric tensor. Its components are effectively of the order $1/p^2$ (which can be seen by considering the case $\underline{p}' \parallel \underline{p}$. This is due to the fact that the forward component of the expression (4) is purely longitudinal but that on the other hand the transverse part of the field will cause the majority of nuclear transitions - a result to be expected from the Weizsäcker Williams method.

5) (21)

The Multipole Approximation.

According to equations (19) and (20) the cross section for the excitation of a certain nuclear level is proportional to the matrix element $\underline{J}$ defined by equation (12). This matrix element contains the field parameters as well as the nuclear eigenfunctions. However, if the fieldpart can be considered to vary only little within nuclear dimensions an expansion of the exponential $e^{i\underline{k}\underline{r}}$ will become possible, so that (12)

can be written in the form

$$\underline{J} = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \underline{k}^n \int \underline{r}^n \underline{j}(\underline{r}) \underline{dr} \qquad (29)$$

where $\underline{r}^n$ as well as $\underline{k}^n$ stand for tensors of order n so that equation (29) may be interpreted to represent a scalar product betwwen the tensor $\underline{k}^n$ (of order n) and the tensor $\underline{r}^n \underline{j}$ (of order n+1). The result of such a multiplication is a vector - or a'tensor'of order 1. The first term in the expansion (29) represents the dipole approximation, the second an electric quadripole if $\int \underline{drrj}$ is a symmetric tensor and a magnetic dipole if the integral is antisymmetric in the tensorindices. Generally the nth term will represent an electric 2^n -pole transition together with a magnetic 2^{n-1}-pole transition.

The expansion (29) will give convergent results for all those values of $\underline{k}$ for which $kR < \pi$, say. The convergency condition in our case is not uniform since to a given electron energy E there may belong a large variety of values k. A sufficient - but far from necessary-condition is that pR should be less than π . That

this condition cannot be necessary follows from the fact that it may just be the small values of k which give a maximum contribution to the cross section. Table 1. below gives the critical values of energy. They are chosen in such a way that for energies below this ~~this~~ critical value the multipole approximation should be good. For the nuclear radius R we have assumed $R = 1.56\ 10^{-13}\ A^{1/3}$ cm, giving approximately $.96\ 10^{-12}$ cm

Table 1.

A	5	10	15	20	50	100	200
E_{crit} (Mv)	114	92	81	73	53	42	34

for uranium. E_{crit} is determined from the equation $E_{crit}\ R = \frac{1}{2}\pi$.

In the following two paragraphs we shall determine the energy dependence of the cross section for electric dipole transitions and for electric quadripoles and magnetic dipoles. An upper limit for the validity of the calculation is given by the values of table 1, a lower limit is defined by the application of Born's method.

In the electric dipole case it follows from equation (29) that

$$\underline{J} = \int \underline{j}\, d\underline{r} \tag{30}$$

should be a good approximation. For this expression we may write because of the equation of continuity (10)

$$\underline{J} = i\kappa \int \rho\, \underline{r}\, d\underline{r} \qquad \rho = -i j_4 \tag{31}$$

In the case of an electric quadripole or a magnetic dipole we may write

$$\underline{J} = i\kappa T \underline{k} \tag{32}$$

where because of equation (29) T is defined by

$$i\kappa T = i \int \underline{j}\, \underline{r}\, d\underline{r} \tag{33}$$

the product between $\underline{r}$ and $\underline{j}$ being dyadic. In the case of electric quadripole transitions T is a symmetric tensor and we may restrict ourselves to the case in which T in a suitable system of coordinates reduces to the form

$$T = \tau \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{34}$$

where τ is the electric quadripole moment of the tran-

sition. For electri magnetic dipole transitions on the other hand the tensor T is antisymmetric and we shall treat the case

$$T = \mu \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \quad (35)$$

where $\mu\chi_\kappa$ is the magnetic dipole moment of the transition.

It should be observed that the choice of the tensor T can not be made arbitrarily but must be such as not to be in contradiction to the conditions

$$\int \varrho \, dr = 0 \qquad \int \varrho \underline{r} \, dr = 0$$

the first of which expresses the orthogonality of the wave functions in the initial and final state and the second of which expresses the impossibility of electric dipole transitions. If the second condition would not be satisfied the tensor T would only represent a correction to the dipole term and this correction will be small - of the order ER - if the multipole approximation is at all justified. It may be easily verified that the choice (34) and (35) satisfies these conditions.

Electric Dipole Transitions.

Since the expression (30) for the matrix element $\underline{J}$ is independent of $\underline{k}$ in the dipole approximation the summation over the directions of $\underline{j}$ (i.e. the summation over the orientations of the nucleus in the final state and the average over the orientations of the nucleus in the initial state) becomes identical with the summation over the orientations of $\underline{J}$, so that by equation (20)

$$\overline{P} = \frac{1}{3}(S)|\underline{J}|^2 = (S)\kappa^2|\varkappa|^2 \qquad (36)$$

where the bar denotes the result of the averaging and the summation process and

$$\varkappa = \int \varrho \, \underline{r} \, d\tau$$

is the 'geometrical' dipole moment with λ_0 as a unit. (S) denotes the trace of S with respect to the tensor-indices. In the derivation of (36) use is made of the theorem $\overline{\underline{e}\underline{e}} = \frac{1}{3}I$ where the upper bar denotes that we take an average over all directions of the unit vector $\underline{e}$ and I denotes the unit tensor. We therefore have to evaluate (S). This can be done by means of equation (28) and by replacing the dyadic products occuring in this

expression by scalar products. We then have because of (I) = 3 :

$$EE'(S) = 3(EE' - (\underline{p}\,\underline{p}') - 1) + 2(\underline{P}\,\underline{P}') - \frac{2}{\kappa}\{E'(\underline{k}\,\underline{p}) + E(\underline{k}\,\underline{p}')\} - \frac{k^2}{\kappa^2}(EE' + (\underline{p}\,\underline{p}') + 1) \quad (37)$$

The occurrence of $(k^2 - \kappa^2)^2$ in the denominator of the expression (19) for the cross section suggests an expansion of (S) in powers of $k - \kappa^2$. This can easily be accomplished by replacing the scalar product $(\underline{p}\,\underline{p}')$ whereever it occurs in equation (37) by $\frac{1}{2}(p^2 + p'^2 - k^2)$. An elementary calculation then gives

$$EE'(S) = -2 + \frac{1}{\kappa^2}(E^2 + E'^2)(k^2 - \kappa^2) - \frac{1}{2\kappa^2}(k^2 - \kappa^2)^2 \quad (38)$$

This expression does not contain the azimuthal angle of the electron in the final state. The integration over the angles of $\underline{p}'$ therefore reduces to an integration over the angle θ . Now

$$d\cos\theta = -dk^2/2pp'$$

and k^2 varies from $(p-p')^2$ to $(p+p')^2$, so that the integration over $d\omega$ can be replaced by an integration over k^2 . This integration can be carried out and gives immediately

$$\phi = 4\pi a^2 |x|^2 \frac{p'}{p}\left\{\frac{E^2 + E'^2}{pp'} \log \frac{EE' + pp' - 1}{\kappa} - 2\right\} \quad (39)$$

for the total cross section for a process leading to a specified nuclear level with energy κ . In the limiting case $p' \to 0$, i.e. near the threshold of the reaction, we have $E = 1 + \kappa$ and $E' = 1$ so that equation (39) reduces to

$$\text{N.R.} \qquad \Phi = 4\pi\alpha^2 |x|^2 \frac{p'}{p} \left[\frac{1}{\kappa} + \kappa\right] \qquad (40)$$

for nonrelativistic energies of the electron in the final state. It should however be noted that the expression (39) does not take account of the electrostatic field of the nucleus and will therefore become unreliable for heavy nuclei. The threshold behaviour in this case will be estimated in paragraph 11 .

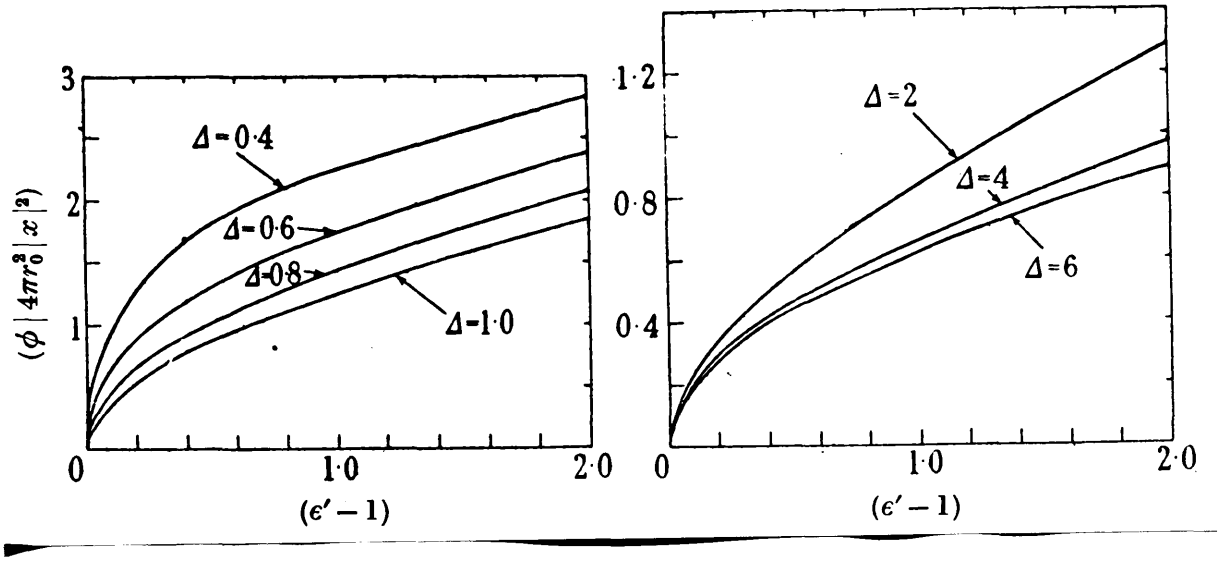
In the limiting case $p \gg 1$ and $p' \lesssim p$ we have

$$\text{E.R.} \qquad \Phi = 4\pi\alpha^2 |x|^2 \left(\log \frac{2E^2}{\kappa} - 1\right) \qquad (41)$$

The close resemblance of this relation to a formula given by Bohr [12] (and extended by Bethe [19] to cover the quantum theoretical aspects of the process) for the ionisation losses of charged fast particles should be observed.

The validity of equation (41) is restricted to

Figure 1.



One level excitation curves for dipole transitions. $\epsilon'-1$ is the kinetic energy of the scattered electron, Δ the excitation energy in units of mc^2. The figure does not take account of the electrostatic interaction between the nucleus and the electron.

energies less than the critical energy listed in table 1. Beyond this limit the higher order components of the transition will no longer be negligible. If further electrostatic forces are taken into account even equation (41) might have to be corrected by a factor of the order $2\pi Z/137$. This factor should be nearly constant since the deformation of the electronic eigenfunctions by the electrostatic field of the nucleus will be mainly defined by the velocity of the electron which is constant in the energy-range in which equation (41) is applicable.

The function $\Phi / 4\pi\alpha^2 |\varkappa|^2$ was calculated for various values of the excitation energy $\varkappa$ and the energy E' of the scattered electron by means of equation (39). The results of this calculation are shown in figure 1.

7)
Electric Quadripole and Magnetic Dipole Transitions.

If we insert from equation (32) into equation (20) we obtain

$$P = \pm (\underline{k} T S T \underline{k}) \qquad (42)$$

The + sign refers to electric quadripole transitions the - sign to magnetic dipoles. In the quadripole case T is symmetric and may be expressed in terms of the transition density ϱ . Multiplying the equation of contimity (10) by the tensor $\underline{r}\,\underline{r}$ we obtain by integrating over the whole space:

$$\tfrac{1}{2}\int(\mathrm{div}\,\underline{j})\,\underline{r}\,\underline{r}\,d\underline{r} = -\tfrac{i}{2}\kappa\int\varrho\,\underline{r}\,\underline{r}\,d\underline{r}$$

from which we obtain - because of the symmetry of T:

$$\tfrac{1}{2}\int(\mathrm{div}\,\underline{j})\,\underline{r}\,\underline{r}\,d\underline{r} = -\int\underline{j}\,\underline{r}\,d\underline{r} = -i\kappa T$$

$\frac{1}{2}\int\varrho\,\underline{r}\,\underline{r}\,d\underline{r}$ is the conventional expression for the electric quadripole moment.

With the tensor T defined in equation (34) we may write

$$T = \underline{a}\,\underline{a} - \underline{b}\,\underline{b} \qquad (43)$$

where - in the suitably chosen system of coordinates

$$\underline{a} = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) \qquad \underline{b} = \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0\right)$$

so that $(\underline{ab}) = 0$. $\underline{a}$ and $\underline{b}$ are unit vectors. This gives for the product $T\underline{k}$

$$T\underline{k} = \underline{a}(\underline{ak}) - \underline{b}(\underline{bk}) = \underline{k}\times\underline{c}$$

where $\underline{c}$ is a unit vector. Inserting this into equation

(42) we obtain therefore

$$P = \tau^2 ((\underline{c}\times\underline{k})\, S\, (\underline{c}\times\underline{k}))$$

To determine the average of this expression over all the orientations of the nucleus, we choose the direction of $\underline{k}$ as the x-axis of a new coordinate system so that $\underline{k} = (k,0,0)$. Then

$$(\underline{c}\times\underline{k}) = (0, c_z, -c_y)$$

so that

$$P = \tau^2 k^2 (S_{yy} c_z^2 + S_{zz} c_y^2 - 2 S_{zy} c_z c_y)$$

and therefore

$$\overline{P} = \frac{\tau^2}{3} k^2 (S_{yy} + S_{zz}) = \frac{\tau^2}{3} (k^2 (S) - (\underline{k} S \underline{k})) \qquad (44)$$

By using equation (28) we thus obtain

$$\kappa^2 E E' (\underline{k} S \underline{k}) = k^4 (E E' + (\underline{p}\underline{p}') + 1) + 2\kappa^2 k^2 + \{E' (\underline{k}\underline{p}) + E (\underline{k}\underline{p}')\} + \kappa^2 k^2 (E E' - (\underline{p}\underline{p}') - 1) + 2\kappa^2 (\underline{p}\,\underline{k})(\underline{p}'\underline{k}) \qquad (45)$$

and after a few reductions similar to those carried out in the dipole case:

$$(\underline{k} S \underline{k}) = \frac{1}{2 E E' \kappa^2} \{4 E E' (k^2 - \kappa^2)^2 - (k^2 - \kappa^2)^3\} \qquad (46)$$

The expression (S) has been determined in equation (37) of the previous paragraph. Substituting from this equa-

tion and from equation (46) into equation (44) we have

$$\overline{P} = \frac{\tau^2}{6EE'}\left\{(k^2-\kappa^2)^2 + 2(p^2+p'^2)(k^2-\kappa^2) - 4\kappa^2\right\} \qquad (47)$$

Inserting from here into equation (19) and integrating over the directions of the scattered electron we obtain for the total cross section in the case of quadripole transitions

$$\phi = \frac{4\pi}{3}\alpha^2\tau^2 F(E',\kappa) \qquad (48)$$

where the function $F(E',\kappa)$ is defined by the equation

$$F(E',\kappa) = \kappa^2\,\frac{p^2+p'^2}{p^2}\log\frac{EE'+pp'-1}{\kappa} \qquad (49)$$

In the limiting case $p' \to 0$ we have

$$\text{N.R.} \qquad \phi = \frac{4\pi}{3}\alpha^2\tau^2(2\kappa+\kappa^2)^{1/2}p' \qquad (50)$$

and for $p, p' \to \infty$

$$\text{E.R.} \qquad \phi = \frac{8\pi}{3}\alpha^2\tau^2\kappa^2\log\left(\frac{2p}{\kappa}\right) \qquad (51)$$

Values of the function $F(E',\kappa)$ were calculated for various values of the excitation energy κ by means of equation (49). The variation of this function with E' and κ is shown graphically in figure 2.

In the case of a magnetic dipole transition the tensor T occurring in equation (42) may be taken to have the form (35). Introducing unit vectors $\underline{e}_x$ and $\underline{e}_y$ perpendicular to one another we may write

$$T = \mu (\underline{e}_x \underline{e}_y - \underline{e}_y \underline{e}_x) \tag{52}$$

from which it follows that

$$T\underline{k} = \mu (\underline{e}_x k_y - \underline{e}_y k_x) \tag{53}$$

Inserting this into equation (42) we obtain after some elementary transformations

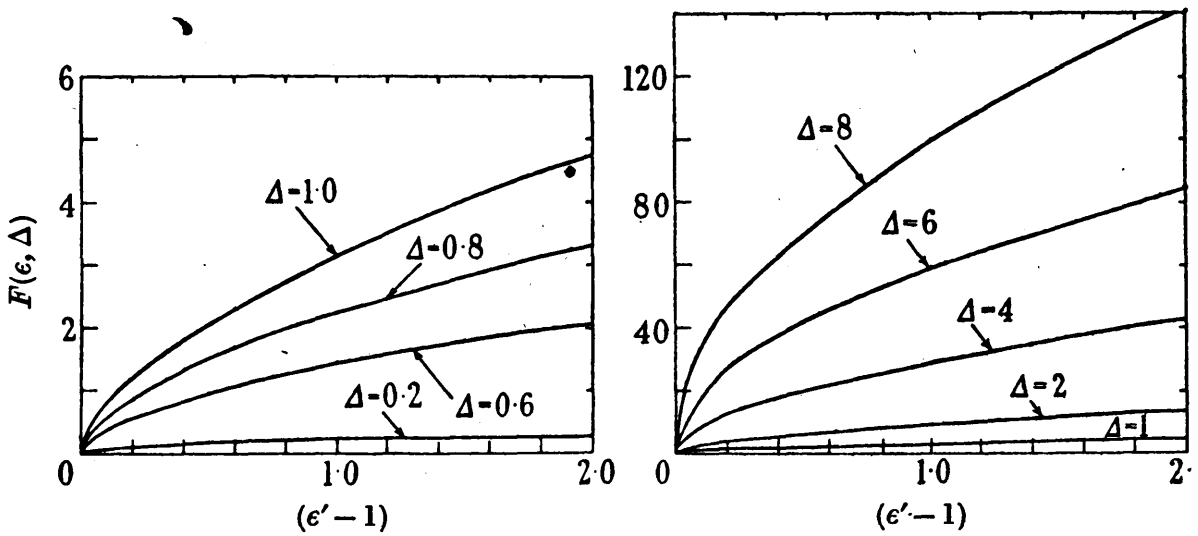
$$\overline{P} = -\frac{1}{3} \mu^2 (k \times S \times k) \tag{53}$$

It follows immediately that all those terms in the dyadic representation of S which contain a factor $\underline{k}$ cannot contribute to equation (53). $\underline{k} \times I \times \underline{k}$ can be easily determined by writing it down in coordinates. We have $\underline{k} \times I \times \underline{k} = -2k^2$. We therefore obtain by inserting from equation (28) into equation (53)

$$\overline{P} = \frac{\mu^2}{6EE'} \left\{ (k^2 - \kappa^2) + 2(p^2 + p'^2)(k^2 - \kappa^2) - 4\kappa^2 \right\} \tag{54}$$

Comparing this formula with equation (47) we see that

Figure 2.



One level excitation curves for electric quadripole and magnetic dipole transitions. ϵ' -1 is the kinetic energy of the scattered electron, Δ the excitation energy in units of mc . The figure does not take account of the electrostatic interaction between the electron and the nucleus and should therefore only be valid for very light nuclei.

the integrations over the directions of the scattered electron are the same as in the case of electric quadripole transitions. The cross section for the magnetic dipole case is therefore obtained from the cross section for the electric quadripole case - given in equation (49) by replacing τ - the electric quadripole moment - by μ - the magnetic dipole moment.

It should be noted that the result for the magnetic dipole is different from that obtained for the electric dipole. The difference - which is rather small - is due to the fact that the electron carries an electric charge - and not a magnetic charge. It is due mainly to the longitudinal component of the transition field.

8)
Sum Rules for very High Energies.

In the previous paragraphs we have been considering the excitation functions corresponding to the excitation of one particular nuclear level. If the energy of the electron is very large compared with the

average spacing of energy levels the excitation of one particular level in the primary process will lose its interest in favour of problems as that of the average energy expenditure in a collision between an electron and a nucleus or the total cross section for the excitation of a nucleus into any one of its excited states. Some of the levels reached in transitions of this kind will be deexcited by the emission of γ-rays, so that in general a particular level will be occupied directly (i.e. in the process of collision and from the groundstate) as well as indirectly over an intermediate state of higher energy.

In the following paragraphs we shall derive some expressions for the total cross section for nuclear excitation and for the average nuclear energy loss of electrons. The calculations will be based on the completeness of the set of nuclear eigenfunctions and represent an analogue to what is known as sum rules in the theory of atomic spectra. It will be assumed that the energy of the electron is large and for definiteness larger than the energy values given in table 2.

The mathematical procedure will be the following. The total cross-section for the excitation of a level f with excitation energy κ_f will have the form.

$$\Phi_f = \Phi(E, E-\kappa_f, f) \tag{55}$$

The total cross section for the excitation of any nuclear level - and these levels include the 'continuous' levels of the nucleus - is obtained from equation (55) by summing over all possible nuclear states f. If for a given state f the cross section Φ_f varies only slowly with energy and if the levels with an energy $\gg \bar{\kappa}$ do not contribute appreciably to the total cross section then a good approximation should be obtained by using the mean value theorem:

$$\sum \Phi_f \cong \sum_f \Phi(E, E-\bar{\kappa}, f) \tag{56}$$

The summation over the final states f now only concerns the nuclear part of the matrix element of the transition - i.e. the vector integral $\underline{J}$ defined in equation (12). With this simplification the summation over the final states of the nucleus can be car-

ried out by using the fact that the nuclear wavefunctions to states with energy less than a certain maximum energy form a nearly complete set - the completeness of which increases with increasing maximum energy. If $\sum_f \Phi_f$ is known the average energy loss may be defined by

$$\overline{\kappa_f} = \sum_f \kappa_f \Phi_f / \sum_f \Phi_f \qquad (57)$$

The sum in the enumerator may be evaluated in the following way. We replace the nuclear Eigenfunction u_f of the final state by Hu_f/κ_f where H is the nuclear Hamiltonian normalised in such a way that the grundstate has energy 0. With this substitution use can be made of the completeness relations. The values of $\overline{\kappa_f}$ thus obtained can then be inserted into the expression for the total cross section.

The method for the calculation of these average values becomes very cumbersome in the multipole region the lower limit of which is given in table 1. It will however be seen from a comparison of tables 1 and 2, that there is a considerable range of energy in which

$E \gg \bar{\kappa}$, so that the sum rules can be applied and also $E \ll E_{crit}$ so that the dipole approximation is justified. For very high energies and in particular for very heavy nuclei the applicability of the dipole approximation is no longer obvious. We shall derive sum rules ~~for~~ for electric quadripole transitions and magnetic dipole transitions in pragraph 10 and it will be seen that their contribution to the average cross section is very small. A reason for this is that the multipole expansion is equivalent to an expansion in v/c of the nucleons. The contributions of successive multipole terms therefore decrease by factors T/M where T is an average kinetic energy of a nucleon in the nucleus. Since this ratio is hardly ever greater than 1/10 it can therefore be expected - and this expectation is verified for the lowest multipole orders - that higher multipole transitions - though of extreme importance for the excitation of particular levels - only represent a small portion of the total nuclear energy loss of very fast electrons.

9)

Sum Rules for the Dipole Approximation.

In this approximation it follows from equation

(41) together with equation (56) that - provided that a $\bar{\kappa}$ can be chosen in such a way that $E \gg \bar{\kappa}$ -

$$\bar{\Phi} = 8\pi\alpha^2 \left(\log \frac{2\epsilon^2}{\bar{\kappa}} - 1 \right) \sum_f |x_{f0}|^2 \qquad (58)$$

The matrix element x_{f0} can be evaluated by means of the method given in the appendix. We may write

$$\sum_f |x_{f0}|^2 = \sum_f \sum_{ij} \int d\xi_1 \ldots d\xi_A \int d\xi_1' \ldots d\xi_A' \, u_0^*(\xi_1 \ldots \xi_A) q_i x_i$$
$$u_f(\xi_1 \ldots \xi_A) u_f^*(\xi_1' \ldots \xi_A') x_j' q_j u_0(\xi_1' \ldots \xi_A')$$

Here ξ_i represents all the coordinates of the ith nucleon (including the space variables $\underline{r}_i = (x_i, y_i, z_i)$, the spinvariables s_i and the variables corresponding to the isotopic spin t_i). The indices i and j run from 1 to A. q_i is an operator acting on the isotopic spin of the ith particle. It has the eigenvalue 1 for protons and 0 for neutrons. The completeness relation for the Schrödinger functions may be assumed to be of the form

$$\sum_f u_f(\xi_1 \ldots \xi_A) u_f^*(\xi_1' \ldots \xi_A') = \delta(\xi - \xi') \qquad (58)$$

where $\delta(\xi - \xi')$ is the antisymmetric δ -function defined in the appendix. (δ is a Kronnecker symbol with regard to the discrete variables s and t). Using equation

(59) we obtain ~~from equation (58)~~

$$\sum_f |x|^2 = \sum_{i,j} \int d\xi_1 \ldots d\xi_A \, u_o^* \, q_i x_i y_j q_j \, u_o$$

Here the terms with $i \neq j$ do not contribute to the summation. (If u_o represents an S state this is trivial and in the case of spatial asymmetry of the ground state the terms $i \neq j$ vanish after averaging over all the possible orientations of the nucleus).

It follows from the antisymmetry of the u's that all the terms with $i = j$ are equal. Now, since the expectation value of q_i^2 is Z/A and there are A terms with $i = j$ we have

$$\sum_f |x|^2 = Z\overline{x^2} = \frac{1}{3} Z \overline{R^2}$$

where $\overline{R^2}$ is the average ~~radius~~ square radius of the charge distribution in the nucleus. Assuming a constant charge distribution all over the nucleus (of radius R) we may write

$$\sum_f |x|^2 = \frac{1}{5} Z R^2 \qquad (60)$$

Inserting this result into equation (58) we obtain for the total cross section for all dipole transitions

$$\phi = \frac{8\pi a^2}{5} R^2 Z \left(\log \frac{2\epsilon^2}{\kappa} - 1 \right) \qquad (61)$$

It will be noticed that with $R = R_0 A^{1/3}$ this cross section increases roughly as $A^{5/3}$ thus favouring strongly the excitation of very heavy nuclei. The increase of the cross section with A however is not strong enough to make the cross section absurdly small for the highest values of the atomic weight. Even for the heaviest nuclei the cross section remains well below 1/10 of the geometrical cross section of the nucleus. Equation (61) is incomplete as long as we can not propose a suitable value for $\bar{\kappa}$. It will however be seen that the result does not depend critically on this value.

The determination of $\bar{\kappa}$ can be carried out in the manner outlined in the previous paragraph. Combining equation (57) with the result (41) for the dipole case we have - provided that $E \gg \bar{\kappa}$ -

$$\overline{\kappa\Phi} = \bar{\kappa}\bar{\Phi} \simeq 8\pi a^2 \left\{ \log \frac{2E^2}{\bar{\kappa}} - 1 \right\} \sum_f \kappa_f |x_{f0}|^2 \qquad (62)$$

The summation on the right hand side can now be carried out in the following manner: if H is the Hamiltonian of the nuclear system normalised in such a way that $Hu_0 = 0$, then for every state $f \neq 0$

$$\kappa_f u_f = Hu_f$$

and therefore

$$\sum_f \kappa_f |x_{f0}|^2 = \sum_{ij} \int d\xi_1 \ldots d\xi_A \, u_0^* q_i x_i H x_j q_j u_0 \qquad (63)$$

Here again only the diagonal terms i = j need be considered since the non diagonal terms cancel out in the average over the orientations. Since the expectation value of q^2 is Z/A we may write

$$\sum_f \kappa_f |x_{f0}|^2 = Z \int d\xi_1 \ldots d\xi_A \, u_0^* x_1 H x_1 u_0 \qquad (63')$$

This integral may now be evaluated by using the commutation properties of the nuclear Hamiltonian H. In a nonrelativistic theory H can be assumed to be of the form $H = \sum_i \frac{1}{2M} P_i^2 + F$ where F is a function of the space coordinates of the nucleons and an operator with respect to the spin and isotopic spin coordinates. P_i is the momentum of the ith nucleon and M is an average nucleon mass. The function F will commute with x. Now, since u_0 is an eigen function to $Hu_0 = 0$ we have

$$Hx_1 u_0 = [Hx_1] u_0$$

where the square brackets denote the commutator Hx-xH. The only non commuting term in the Hamiltonian is $P_1^2/2M$

Now, because of $[Px] = -i$ we have

$$[P_l^2 x] = -2iP_l$$

Inserting this into equation (63') we obtain after a partial integration (remembering that P_l may be represented by the operator $-i\partial/\partial x_l$)

$$\bar{\kappa}\bar{\phi} = 4\pi a^2 \frac{Z}{M}\left\{\log\frac{2E^2}{\bar{\kappa}} - 1\right\} \qquad (64)$$

Comparing this with equation (51) we find

$$\bar{\kappa} = \frac{5}{2MR^2} \qquad (65)$$

The expression $1/MR^2$ is the natural nuclear energy unit ($\hbar^2/MR$ in dimensional units) which for a given value of R gives a rough indication of the spacings of the lowest energy levels. The result of this analysis can therefore be said to be, that the average dipole excitation energy is 2.5 nuclear energy units for all nuclei. We see that this average excitation energy is very small - even for the lightest nuclei, so that the application of the completeness relations is justified for moderately high energies. The result (65) will represent an upper limit for energies only little in excess of the mean excitation energy $\bar{\kappa}$.

In table 2. we have listed the values of $\bar{\kappa}$ for different atomic weights A. The 2nd line gives the values in units of mc^2 the third in Mev. R has been assumed to be of the form $R = R_0 A^{1/3}$ with $R_0 = 1.56\ 10^{-13}$ cm.

Table 2.

A	5	10	15	20	50	100	200
	27.4	17.2	13.1	10.9	5.9	3.7	2.3
(Mev)	14.0	8.8	6.7	5.6	3.0	1.9	1.2

Table 3 gives the values for the total cross section for different values of A and E. The cross sections are calculated from equation (61), in which $\bar{\kappa}$ has been replaced by the values given in table 2. The unit of cross section was taken to be 10^{-28} cm^2. The range of the energy values E has been chosen in such a way that the dipole approximation as well as the application of sum rules is justifyable. It is seen that near the upper limit $E \sim E_{crit}$ the cross section varies only slowly with energy.

It is seen from table 3 that the total electron

Table 3.

Total cross sections for electron excitation; unit = 10^{-18} cm^2.

E \ A	5	10	15	20	50	100	200
10					47	179	602
20		4.34	8.7	15.9	76	261	845
30	1.52	5.7	11.0	19.7	92	309	986
40	1.79	6.4	12.5	22.4	101	342	1080
50	2.00	7.07	13.7	24.5	111	368	1160
60	2.19	7.65	14.8	26.3	119	392	1230
70	2.33	8.11	15.7	27.8	125	410	
80	2.46	8.54	16.4	29.1	130	426	
90	2.58	8.89	17.1	30.2	135		
100	2.68	9.20	17.7	31.3	139		
120	2.84	9.54	18.6	32.8			
140	3.00	10.2	19.5	34.5			
160	3.12	10.6	20.4				
180	3.25	11.0					
200	3.34						

cross sections are fairly large, particularly for heavy elements. On the other hand it has to be considered that

according to equation (64) the total energy loss /proton is practically independent of the atomic number (there is a slight increase with increasing atomic number due to the variation $\bar{\kappa}$ in the logarithmic term.) For a heavy nucleus the average energy loss (of the order of 1 Mev) is so small that most of the levels excitable by electrons will lead to β-active or γ-active states rather than to particle emission.

10)

Estimate of the Error in Extrapolating the Dipole Sum Rules into the Multipole Region.

In this paragraph we want to give reasons which make us believe that the sum rules given in the previous paragraph hold even for energies for which the dipole approximation should no longer be reliable. We want to show that

(i) the contribution to the total cross section of electric quadripole and magnetic dipole transitions is small in comparison with the electric dipole contribution and

(ii) that with a simplified model in which all

energy transfers $< E_{crit}$ are treated as dipole transitions and all energy transfers with $\kappa > E$ as due to individual collisions between electrons and protons, the cross-section obtained in the dipole approximation only suffers a small correction and the total ~~eress~~ energy loss remains unaltered.

To prove (i) we have to go back to equation (51) which is valid if $E \gg \bar{\kappa}$ - which we shall assume in the following. We the have to determine $\sum_f \kappa_f^2 \tau^2$ to obtain the total cross section. By a consideration similar to that carried out in the previous paragraph we obtain

$$\sum_f \kappa_f^2 \tau^2 = Z \int d\xi_1 \dots d\xi_A \, u_0^* \, x_1 y_1 H x_1 y_1 u_0 \qquad (66)$$

Here again use can be made of $Hu_0 = 0$ and the products $H_{(1)} xy$ can be replaced by commutators. In this way one obtains

$$\sum \kappa^2 \tau^2 = \frac{1}{M^2} \int d\xi_1 \dots d\xi_A \, u_0^* \, (y_1 P_{x_1} + x_1 P_{y_1})^2 u_0$$

and this integral is of the order $P^2 R^2 / M^2$. Comparing this result with that obtained in the electric dipole case, namely equation (62), we see that the quadripole contribution is of the order P^2/M^2 times smaller than the dipole contribution. It should be noted that for

a rigorous proof of this statement we should have to determine the average energy loss $\overline{\kappa_q}$ for the quadripole case. But since this energy loss will only show up under the logarithm of equation (51) an accurate determination of $\overline{\kappa_q}$ does not seem necessary.

This argument can be generalised to all electric transitions in such a way that the contribution of successive multipole orders has a ratio of the order T/M. It therefore appears that the majority of the transitions caused by electron impact are dipole transitions.

The contribution of magnetic dipole transitions can be estimated by using the expressions for the magnetic dipole moment given in appendix 2. For the orbital part of the dipole moment the considerations are quite similar to the electric quadripole case and lead to the same result. For the part due to the magnetic moments of protons and neutrons we obtained by applying the completeness relation (59) to equation (5) of appendix 2:

$$\sum \kappa_f^2 \mu^2 = \frac{A}{4M^2} \int d\xi_1 \dots d\xi_A \, u_0^* (\mu_p q_1 + \mu_N (1-q_1))^2 u_0 \qquad (67)$$

where μ_p and μ_N are the magnetic moments of proton and neutron respectively. ($\mu_p = 2.79$, $\mu_N = -1.91$ nuclear magnetons.) The factor $1/4M^2$ is due to the nuclear magneton which in our units is $1/2M$. Putting $A \sim 2Z$ we obtain from equation (67)

$$\sum_f \kappa_f^2 \mu^2 = \frac{Z}{4M^2}\{\mu_p^2 + \mu_N^2\} = \frac{10.5\,Z}{4M^2} \qquad (68)$$

Comparing this with the result obtained in the electric dipole case we obtain

$$\bar{\Phi}^{(e)}/\bar{\Phi}^{(m)} \simeq 12.8\, A^{2/3} \qquad (69)$$

where $\bar{\Phi}^{(e)}$ denotes the total cross section for the electric dipole and $\bar{\Phi}^{(m)}$ that for the magnetic dipole. It is seen that the dipole correction is of the order of less than 10% for the lightest nuclei. It is relatively less (a fraction of 1%)for very heavy nuclei.

For the comparison of electric and magnetic transition we have assumed that the average excitation energy of magnetic dipole transitions is of the same order as that for electric dipole transitions. Though this seems very plausible it is not quite easy to prove. The average energy loss comes out to depend strongly

on the assumptions made about the nuclear forces e.g. their spin- and isotopic spin-dependence. To make sure that the higher order transitions do not affect the average energy loss it seems useful to investigate the model proposed under alternative (ii) of this paragraph.

The physical justification for the application of model (ii) is that with increasing energy the electron becomes more and more 'shortsighted', so that in the end the fact that the electron nucleon is bound in a nucleus can no longer influence the behaviour of the cross section. In quantum theory the critical quantity is the energy transferred by the electron rather than the energy of the electron, so that for large energy transfers we should be entitled to treat the nucleons as free particles.

If $\overline{\phi}_1$ denotes the cross section (in dipole approximation) corresponding to all those processes which lead to a transfer of energy less than E_{crit} and as long as $E\,\overline{\kappa}$ is small in comparison with E_{crit} (which is always the case as can be seen from a comparison of tables 1. and 3.) we have

$$\overline{\Phi_1} = \frac{8\pi a^2}{5} R^2 \log \frac{2EE_{crit}}{\kappa} \qquad (70)$$

if only the leading term is taken into account: $\log 2EE_{crit}/\kappa \gg 1$.

In the case of a free proton the cross section for the production of a proton with momentum $\underline{k}$ can be derived from equations (19),(20) and (21) by putting

$$\underline{J} = \underline{k}/2M \qquad (71)$$

(For a free proton the magnetic moment does not contribute in non relativistic approximation owing to the lack of spin orbit coupling.) Inserting this into equations (19) and (20) we obtain for the differential cross section:

$$d\Phi_2 = \frac{a^2}{2M^2} \frac{EE'p'}{p} (k^2 - \kappa^2)^{-2} (\underline{k} S \underline{k}) \qquad (72)$$

The expression $(\underline{k}S\underline{k})$ has been derived in equation (46), which when inserted into (72) gives

$$d\Phi_2 \simeq a^2 \frac{E^2}{M^2 \kappa^2} \qquad (73)$$

where it has been assumed that $E^2 \gg k^2$. Now the energy loss in the case of a free proton is simply $k^2/2M$. Inserting this value for κ and integrating over all

angles leading to an energy transfer $> E_{crit}$ we have

$$\overline{\Phi_2} = Z \frac{\pi a^2}{E_{crit}^2} \qquad (74)$$

where again only the leading term has been written down. Comparing this with the cross section $\overline{\Phi_1}$ (equation (70)) we see that the free particle contribution is comparatively small. The cross section (70) is larger by a factor of the order $\log \frac{E^2}{\kappa}$ which is approximately 10 for very high energies. The replacement of E by E_{crit} in the logarithm of equation (70) also gives a correction of approximately 10% - compared with the uncorrected dipole result. The two corrections have opposite sign, so that by using this picture of the process we are again led to assume that the dipole approximation is very good even beyond its natural range of validity.

The situation is even better with regard to the total energy loss. Here

$$\overline{\kappa \Phi_1} = 4\pi a^2 \frac{Z}{M} \log \frac{2E E_{crit}}{\kappa} \qquad (75)$$

and for the free proton

$$\overline{\kappa \Phi_2} = 4\pi a^2 \frac{Z}{M} \log \frac{E}{E_{crit}} \qquad (76)$$

and these two expression add up to exactly the leading

term of equation (64) obtained for the dipole approximation.

11)

The Threshold Problem.

The results obtained in the previous sections are only valid if the electron in its initial as well as in its final state can be represented by a plane Dirac wave. This will never be the case since the electrostatic field of the nucleus may cause a considerable deformation of the electronic eigenfunctions. However, for very large energies this deformation will be comparatively small and will not depend very strongly on the energy, so that the sum rules should remain unaltered provided that the nuclear charge is not too high. The Coulomb-deformation of the eigenfunctions may be mitigated a little by the cut-off of the electric field strength in the neighbourhood of the nuclear radius. The influence of the electrostatic charge of the nucleus will always tend to increase the cross-section for processes initiated by particles of negative charge, since the attraction between unlike charges makes the

~~makes the~~ electronic eigenfunctions large in the neighbourhood of the nucleus. The magnitude of the effect however can not be immediately estimated from the Coulomb-correction of the electronic eigenfunctions at the origin: the electric field produced by an electronic transition will depend on the values of the eigenfunction at any distance from the origin owing to the large range of the electric interaction.

The mathematical programme for the determination of the electrostatic correction can be seen clearly from equation (2) into which the transition density or the transition current for an electron in a Coulomb-field would have to be inserted. The following calculations then proceed in a manner identical to the analysis of the previous paragraphs.

We shall here give an estimate of the electrostatic effect based on the Sommerfeld-Maue[24] approximation and restrict ourselves to the dipole approximation. The shortcomings of the various approximations made will be discussed in paragraph 13 . An accurate determination of the electrostatic correction - though possible - involves quite considerable analytical dif-

ficulties comparable and very similar to those in the relativistic theory of the internal conversion coefficient.[23] Work on this problem is still in progress and seems necessary in view of the lack of agreement with experimental data - i.e. Wiedenbecks experiments.

Since the Coulomb force will be of particular importance for very slow electrons and since the first 'accessible' excited levels are usually about 1 Mev above the ground state a maximum correction will have to be expected near the threshold for the excitation of a given level. There the electron in the final state is very slow, but it is to be expected that the Coulomb correction for the initial state will not be very large. We shall therefore neglect the electrostatic correction for the initial state of the electron. The Coulomb correction for the final state on the other hand is described in terms of a zero order Sommerfeld Maue function

$$\psi' = N a' e^{i(\underline{p}'\underline{r})} L_n(\varrho) \qquad (77)$$

where $L_n(\varrho)$ is the Laguerre transcendental function, n is defined by

$$n = -i\alpha Z/\beta \tag{78}$$

β is the velocity of the scattered electron and $i\varrho$ is the 'transverse' part of $(\underline{p}'\underline{r})$ i.e.

$$\varrho = -i\,((\underline{p}'\underline{r}) - p'r) \tag{79}$$

N is a normalisation factor which has to be chosen in such a way that the outgoing part of the wave (76) represents a spherical wave of unit amplitude. For this

$$|N|^2 = \frac{2\pi|n|}{1 - e^{-2\pi|n|}} \tag{80}$$

With these abbreviations we obtain in dipole approximation, i.e. in that approximation in which $\underline{A}(\underline{r})$ is replaced by $\underline{A}(0)$ - the value of $\underline{A}$ at the origin:

$$A^*_\nu(0) = eN^* a^*_\nu J \tag{81}$$

where J is the integral

$$J = \int \frac{dr}{r} e^{-i(\kappa r + (\underline{k}\underline{r}))} L_n(\varrho) \tag{82}$$

For the evaluation of the integral (81) we may use the integral representation of the Laguerrefunction, namely

$$2\pi i\, L_n(\varrho) = \oint (1+y)^n y^{-(1+n)} e^{-\varrho y}\, dy \tag{83}$$

in which the path of integration goes round the points 0 and -1 in the positive sense. Inserting this into equation (82) the integration over $\underline{r}$ can be carried out and one obtains

$$J = \frac{2}{i}\oint dy\,(1-y)^n y^{-n-1}\left\{k^2-\kappa^2-2y(p'k+(\underline{p}'\underline{k}))\right\}^{-1} \qquad (84)$$

Putting as an abbreviation

$$\omega = 2(p'k+(\underline{p}'\underline{k}))/(k^2-\kappa^2) > -1 \qquad (85)$$

we may write instead of (84)

$$J = \frac{2}{i(k^2-\kappa^2)}\oint dy\,(1+y)^n y^{-n-1}(1-\omega y)^{-1} \qquad (84')$$

We note that in the limit $\underline{p}' = 0$ $\omega \to 0$. Further, with $Z = 0$ that is by taking a plane wave approximation for the final state, we obtain

$$J = \frac{2}{i(k^2-\kappa^2)}\oint \frac{dy}{y} = \frac{4\pi}{k^2-\kappa^2}$$

and this is identical with the equation of paragraph .

The integral (84') may be evaluated by noting that

$$\oint u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}du = (1-e^{-2\pi i\alpha})\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F(\alpha,\beta,\gamma;x)$$

where $F(\alpha,\beta,\gamma;x)$ is the hypergeometric function. The integral (84') represents a special case of this integral with $\alpha = -n$, $\beta = \gamma = 1$ and therefore

$$J = \frac{2(-)^{-n-1}}{i(k^2-\kappa^2)}(1-e^{2\pi i n})F(-n,1,1,-\omega)\Gamma(-n)\Gamma(1+n) \quad (86)$$

Now by using the relation $F(-n,1,1;-\omega) = (1+\omega)^n$ (25) and a well known property of the Γ - function we obtain:

$$J = \frac{4\pi}{k^2-\kappa^2}(1+\omega)^n \quad (87)$$

Since n is purely imaginary we have $|(1+\omega)^n| = 1$ and therefore by disregarding a phasefactor which does not affect the final result

$$J = \frac{4\pi N a_v}{k^2-\kappa^2} \quad (88)$$

This expression replaces equation (4) in the Sommerfeld Maue approximation for the outgoing electron. It is therefore seen that as far as this approximation goes the influence of the electrostatic interaction will just multiply the results obtained earlier by the factor $|N|^2$ defined in equation (80). The following table 4. gives the values of the factor $|N|^2$ for electron energies ranging from 1 to 1.15 electron units and for a medium heavy nucleus (Z = 48). The factor $|N|^2$ is familiar from the theory of β decay and numerical determinations have been carried out by many authors. (26)

Table 4.

E'	1.005	1.010	1.015	1.020	1.03	11.04	11.10	1.15
$\|N\|^2$	22.1	15.6	12.9	11.2	9.2	8.02	5.3	4.5

$$\lim_{p' \to \infty} |N|^2 = 2.2$$

The result obtained in this section is surprising when one considers the argument given above, namely that the amplitude of the electromagnetic potential should not only depend on the values of the wavefunction near the nucleus. One excuse may be perhaps be found in the fact that the wavelength of the equivalent quanta being very short, the argument about the long range of electro magnetic simply implies that the volume in which the electronic eigenfunction has to be known accurate is large compared with the cube of the wave length of the quantum. This can still be small compared with the de Broglie wavelength of the outgoing electron. The principal objection however can be raised against the application of the Sommerfeld-Maue method, which in the case of a Coulomb type field does not converge properly near the origin: the wellknown factor $(pR)^{2s-2}$ which should accompany the Coulombfactor $|N|^2$ can not be represented-

in terms of an expansion into powers of e in a uniformly convergent manner. This together with the fact that owing to the long range character of electro-magnetic forces it is very hard to decide to which part of the electronic wavefunction one should attribute maximum weight, makes the Sommerfeld Maue method not very reliable for our purposes. A decision on the threshold problem (i.e. a decision as to whether such a problem exists at all) can only be expected from the application of the full Dirac eigenfunctions.

12)
Estimate of the Nuclear Quadripole Moment.

In the foregoing sections the only nuclear property which entered the calculations was the radiative moment of the transition in question. Transition moments of this kind can be measured directly by observing resonance widths and these measurements turn out to be in rough agreement with the theoretical expectations derived from various nuclear models. One way of forming a theoretical opinion on the magnitude of nuclear transition moments is offered by the liquid drop

model. There certainly are many objections against the application of this model arrising from the unsatisfactory way in which it deals with internal degrees of freedom. On the other hand it seems the only model capable of making systematic predictions on this subject.

According to the liquid drop model (27) there should be no electric dipole moment and only quadripole and higher moments should occur. This arrises from the fact that in this model the centre of gravity of the protons coincides with the centre of gravity of the nucleus as a whole. That there is no elee magnetic dipole moment is due to the lack of the rotational components of the displacement current and the logical neglect of exchange currents which represent one of the internal degrees of freedom necessarily neglected in a semiclassical theory of the nucleus.

In the liquid drop model the frequency ω of the lth harmonic of the surface wave is given by

$$\omega^2 = \frac{G}{\rho S R^3}(l+2)l(l-1) \qquad (89)$$

where G is the total surface energy, namely $9.6A^{2/3}$ (Mev),

ϱ is the density of nuclear matter and S the surface of the nucleus. Only the mode $l = 2$ gives a contribution to the electric quadripole moment. The energy of the first excited level is $\hbar\omega$ so that by taking the nuclear radius to be $1.56\ 10^{-13}\ A^{1/3}$ cm we obtain

$$\kappa = 39\, A^{-1/2} \tag{90}$$

This would give a first excitation level at 4.9 Mev in oxygen and at 1.3 Mev for a heavy nucleus like uranium. The quadripole moment of a transition corresponds to

$$\tau = \frac{Z R_0\, b\, A^{1/2}}{5\, \lambda_0^2} \tag{91}$$

where b - the amplitude of the second Harmonic - is related to κ by

$$mc^2\, \kappa = 2\varrho b^2 / 15 R^2 \tag{92}$$

Combining these equations we obtain for the quadripole moment in electronic units:

$$\tau = 1.2\, Z\, A^{11/12}\, 10^{-5} \tag{93}$$

This value may now be inserted into equation (48), to obtain the order of magnitude of the cross section.

Discussion.

As a general result of the foregoing sections we conclude that any reaction which can be caused by γ-rays can also by caused by electrons. This result is nearly trivial in view of the contents of the Weizsäcker Williams theory of Bremsstrahlung according to which - in a suitable frame of reference - an electron is equivalent to a continuous spectrum of quanta with an energy distribution $n(k) \propto dk/137k$. However the method applied here goes a little further than the Weizsäcker Williams method: formally it is capable of extrapolation to the threshold in contrast to the Weizsäcker Williams method which may become very unreliable in this region - the results depending on the cut off parameter. Formally our method is identical with Born's approximation in radiation theory, though the classical treatment of the electromagnetic field represents some simplification of the mathematical procedure, as we shall show in particular in paragraph 14 for the case of meson production.

From an experimental point of view there is one

important difference between electrons and γ-quanta: in the energy region above a few Mev quanta can only be produced in the form of a continuous spectrum, where as the new machines are capable of delivering a 'monochromatic' beam of electrons. Can any use be made of this property of electrons? The answer seems to be in the affirmative, with the qualification that even with a monochromatic beam of electrons the electrons are equivalent to a continuous spectrum of γ-rays. In Borns approximation there is very little difference between this equivalent spectrum and the continuous spectrum produced in the process of Bremsstrahlung. However, since the Brems-spectrum used is usually a thicktarget spectrum, the use of electrons will at least have the advantage of allowing one to work with something closely resembling a thin target spectrum under geometrical conditions which resemble the thick-target situation.

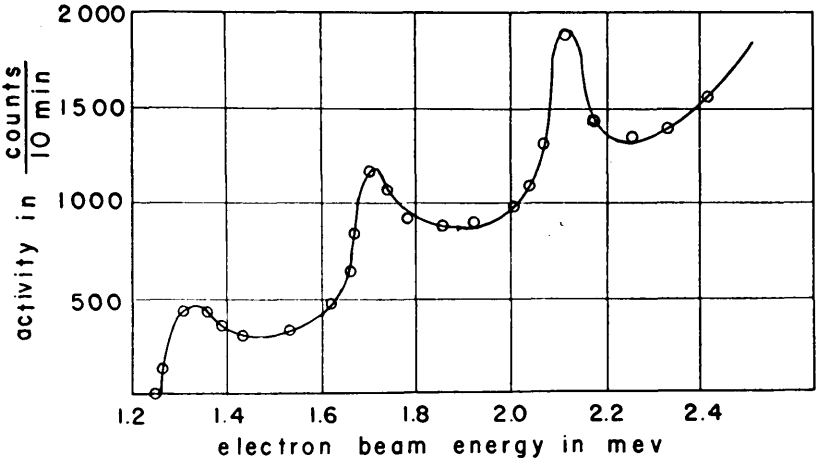
This advantage has indeed been successfully used by Wiedenbeck, who, in the 4 Mev region, finds excitation curves closely resembling the 'resonance curves'

observed in a Franck-Hertz experiment. This resemblance however is deceptive: in the Franck Hertz experiment the typical resonance behaviour is not due to a resonance at all but is caused by a peculiar property of the circuit together with the $U^{3/2}$-law of thermionic valves.(28) A typical Wiedenbeck curve is shown in figure 3. For the first level the excitation curve derived from this measurement is shown in figure 4. One can see that the threshold is considerably better marked than in the case of γ rays represented in figure 5. The threshold behaviour in Wiedenbecks experiments is even more favourable than one would expect from the rough theoretical model given in paragraph II .The prediction derived from this model is indicated by the dotted line in figure 4 and it is seen that the agreement of the two curves is very poor. It is hard to decide which of the assumptions leading to the theoretical estimate should be responsible for this discrepancy. The only approximately relativistic method may be one of the reasons, since it results in the neglect of the divergent behaviour of the s and p eigenfunctions of Dirac's theory. A correct consideration ~~of~~

of this property of the Dirac wave functions might be expected to add a factor $(p'R)^{2s-2}$ where $s = (1-\alpha^2 Z^2)^{1/2}$ to the Coulomb factor $|N|^2$ defined in equation (80). It can however be seen that this correction would vary too slowly to give a noticeable decrease of the excitation function near the threshold.

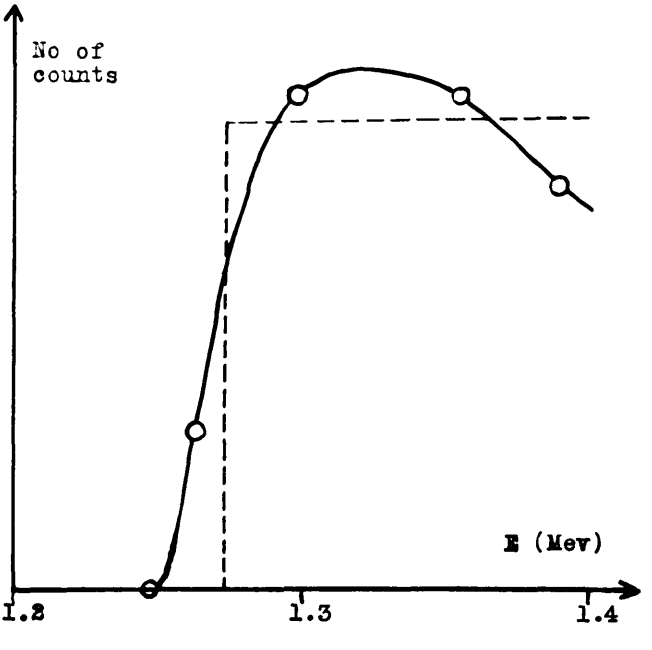
On the other hand there might be a few technical reasons which might alleviate the discrepancy from the experimental side. Wiedenbecks experiments were carried out with a constant current but it seems hard to assess what a constant current really means in terms of numbers of electrons actually hitting the target. A practically constant excitation curve could be turned into a curve showing a maximum near the threshold if the actual number of electrons would decrease sufficiently fast with increasing energy. This would also tend to make the higher excitation levels show more pronounced maxima - in agreement with Wiedenbecks measurement shown in figure 3. This is due to the fact that the negative slopes of the apparent excitation curves would add, so that the third excitation level should show a peak approximately one

Figure 3.



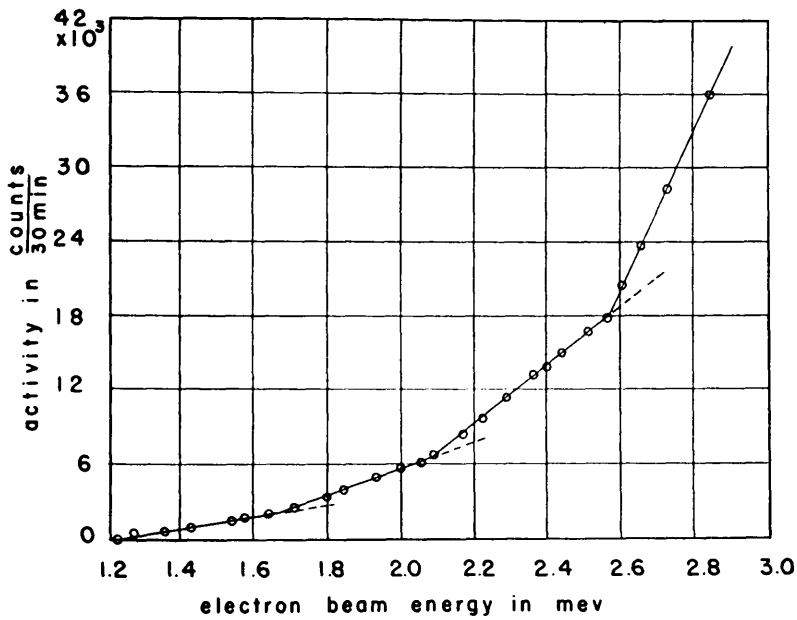
Wiedenbeck's electron excitation curve.

Figure 4.



The contribution of the first level of the excitation curve shown in figure 3. The dotted curve indicates the 'theoretical' curve discussed in paragraph 11.

Figure 5.



A thick target γ-excitation curve measured by Wiedenbeck. The kinks correspond to the excited levels of the target nucleus.

third the width of the first. The general trend of the excitation curve to rise with increasing energy could then be explained by the factor κ^2 occurring in the expression (50) for the threshold behaviour in the quadripole case.(It seems rather plausible to assume that the excited states represent quadripole transitions from the ground state, since the long lifetime of 50 min at an excitation energy of 195 kev in Cd suggests a transition with $\Delta l \sim 2$. Assuming that the spins of the excited states (at 1.25 and 1.68 Mev) lie halfway between the spins of the ground state and the metastable state, this gives a quadripole transition from the groundstate to the excited state.)

The order of magnitude of the predicted cross section may be checked against an estimate given by Collins and Waldmann.[10] A first check of the quality of the liquid drop model is obtained by comparing the energy of 1.9 Mev calculated from equation (92) for $Z = 48$ (Cd) with the value 1.2 Mev observed in Wiedenbecks experiments. For the same value of Z equation (93) gives a value of approximately $0.85\ 10^{-3}$ for the quadripole moment. This is very large - nearly of the

order of the nuclear cross section. From figure 2. we find F to be approximately 10 in the quadripole case and for an energy of about 100 kev above the threshold. This has tonbe multiplied by a factor of the order 5 to account for the rather strong Coulomb interaction between the nucleus and both the outgoing and incoming electron. We should therefore expect a cross section of the order of 10^{-29} cm^2.

This is in excess of the value 10^{-32} estimated by Collins and Waldmann for indium. However, considering the limitations of the liquid drop model and the assumptions which Collins and Waldmann had to introduce about the efficiency of their counters the discrepancy is not so large. If we use their estimate we could calculate back the quadripole moment required to give their value. It comes out of the order 0.4 10^{-25} which is small but within the possible range of this quantity. There is no obvious reason to believe that Collins and Waldmann have actually observed a quadripole transition - an octupole would give excellent agreement with the estimated value of 10^{-32} cm^2 for the cross section. It is to be hoped that with the development of the synchrotron more experimental data will be forthcoming to make a quantitative check of the theory possible.

14)

The Production of Mesons by Electrons.

The general formalism developed in paragraphs 2-4 may be used to calculate the cross section for the production of mesons by electrons. The production of a meson in a collision between an electron and an atomic nucleus, i.e. a process of the type

$$e + (Z)^A \rightarrow (Z \mp 1)^A + \pi^{\pm} + e' \qquad (94)$$

can be derived exhaustively from the matrixelement of the current corresponding to the production of a meson. Recent investigations have shown that the problem of the production of mesons cannot be satisfactorily treated under neglect of radiation damping.[15] However, the same considerations show that the effect of radiation damping on the cross section for the production of a (vector or pseudoscalar) meson is not excessively large for energies up to approximately 600 Mev. This is equally true for the production of mesons by γ- rays and the production of mesons in nucleon-nucleon collisions. In the following we shall

neglect radiation damping altogether. The results are therefore inapplicable to cosmic radiation problems but should give a reasonably good approximation in the 300 Mev range. The limitation to 'small' energies allows us to neglect the influence of heavy particle recoil - and the neglect of this effect will involve an error of the same order of magnitude (namely up ton50%) as the possible error due to the neglect of radiation damping.

The analysis will be carried out for pseudoscalar charged mesons the mass of which we assume to be approximately 300 electron masses.

The simplest interpretation of the process seems to be the following. The electron with momentum $\underline{p}$ produces an electromagnetic field $(\underline{p}'|A|\underline{p})$ (defined in equation (4)) in a transition to a state with momentum $\underline{p}'$. The field carrying a momentum $\underline{k} = \underline{p} - \underline{p}'$ interacts with the nucleus, which under the influence of the field makes a transition from a state with 'bound' or no mesons to a state with a free meson of momentum $\underline{q}$. This transition can be described in terms of a current $(f|j_\nu|0)$, where 0 signifies the initial state of the

nuclear system (consisting of the nucleons and the meson field) and f its final state. The matrix-element corresponding to this process can then be assumed to have the form (11) for which - by using the equations of continuity-we may write

$$H' = \int d\underline{r} \left\{ \underline{A}\,\underline{j} - \frac{1}{\kappa^2}(\underline{A}\,\underline{k})(\underline{k}\,\underline{j}) \right\} e^{i\kappa t} \qquad (95)$$

This expression may-in the extreme relativistic region of electron energies- be simplified considerably by means of the considerations of paragraph 2 . Of the quantities involved in (95) $\underline{A}$ is the only one which depends strongly on $\underline{k}$. As a function of $\underline{k}$ $\underline{A}$ shows a pronounced maximum when $\underline{k}$ is parallel to $\underline{p}$ in which case $k \cong \kappa$ the energy transferred from the electrons to the nuclear system. Whenever $\underline{k}$ occurs in a quantity other than $\underline{A}$ we may therefore write $\underline{k} = \kappa\,\underline{p}/p$. With this approximation involving an error of the order $1/p'$ the right hand side of equation (95) reduces to the transverse part of the interaction between the vector potential and the transition current. Equation (95) may therefore be replaced by

$$H' = \int d\underline{r}\ (\underline{A}_\perp \underline{j}_\perp) e^{i\kappa t} \qquad (96)$$

where $\underline{F}_\perp$ denotes the transverse part of the vector field $\underline{F}$: $\underline{F}_\perp = \underline{F} - (\underline{F}k)/k$. With this simplification the whole calculation reduces to the determination of the matrix element of j corresponding to the production of a free meson.

With the assumption of infinitely heavy nucleons (i.e. under neglect of recoil effects) the interactions between the nucleons and the mesons can be described in terms of the large sourcefunction alone. This, in the case of pseudo scalar meson has axial vector character. The Lagrangian function of the system consisting of nucleons and mesons may then be written

$$\mathcal{L} = \mathcal{L}_0 - \{(\nabla\varphi^+)(\nabla\varphi) - \dot{\varphi}^+\dot{\varphi} + \mu^2\varphi^+\varphi + (\underline{P}\nabla\varphi^+) + (\underline{P}^+\nabla\varphi)\} \qquad (97)$$

L_0 describes the bare nucleons, φ^+ and φ are the meson operators, φ^+ corresponding to the production of a positive or the annihilation of a negative meson. $\underline{P}$ is the vector source function. To ensure gauge-invariance it has to be assumed that under a gauge transformation $\underline{P}$ transforms in the same way as φ . The only non vanishing matrixelements of $\underline{P}$ then represent transitions from a proton- into a neutron state. The

expression (97) represents the Lagrangian in the absence of an electromagnetic field. In the presence of such a field we have to replace the operators $\frac{\partial}{\partial t}$ and ∇ in (97) by $\frac{\partial}{\partial t} - ieA_0$ and $\nabla - ie\underline{A}$ when acting on φ and by their complex conjugates when acting on φ^+. With the usual procedure we then obtain for the current vector $\underline{j}$:

$$\underline{j} = \frac{\partial \mathcal{L}}{\partial \underline{A}} = ie\{(\nabla\varphi^+)\varphi - \varphi^+(\nabla\varphi) + \underline{P}^+\varphi - \underline{P}\varphi^+\} \qquad (98)$$

This expression for the current vector has an obvious physical interpretation: In a model in which a proton is considered as a mixture of a bare proton state and a state consisting of a neutron and a 'bound' positive meson the production of a free meson may arise in either of two ways:-

(i) The meson is produced spontaneously by the source, i.e. the bare proton is transmuted to a bare neutron and a free positive meson; or

(ii) the meson bound to the neutron is liberated by the action of the electromagnetic field.

Whereas the process (i) has no counterpart in a classical theory, (ii) bears a close resemblance to an ionisation process.

The commutation relations for the meson operators φ^+ and φ can be satisfied by putting

$$\varphi = \varphi_{free} = \sum_{\underline{q}} \sqrt{\frac{1}{2\kappa}} \left\{ a(\underline{q}) \Phi_{\underline{q}}(\underline{r}) + b^+(\underline{q}) \Phi^*_{\underline{q}}(\underline{r}) \right\} \qquad (99)$$

$$\kappa^2 = \mu^2 + q^2$$

where the $(\underline{r})$ are wave functions with the asymptotic behaviour

$$\Phi_{\underline{q}}(\underline{r}) \sim e^{i(\underline{q}\,\underline{r})}$$

Near the origin this simple form will be distorted as the result of the electrostatic interaction between the meson and the nucleus. This will give rise to a Coulomb factor which, in the first instance, we shall assume to be unity and which we shall reintroduce at a later stage of the calculation. The operators a and b correspond to the annihilation of positive and negative mesons respectively and for such transitions they have the value unity. Although they satisfy the commutation relations the expressions (99) do not, in the presence of sources, lead to a diagonal form for the Hamiltonian matrix. If we neglect the reaction of the meson field on the sources (i.e. if the source function $\underline{P}$ is assumed to be known) the general ~~solution~~ solution of the problem may be written in the form

$$\varphi = \varphi_0 + \varphi_{free} \tag{100}$$

where φ_0 is the solution of the static equation

$$(\nabla^2 - \mu^2)\varphi_0 = -(\nabla \underline{P}) \tag{101}$$

It is then easily verified that equation (100) defines a matrix φ which satisfies the necessary commutation relations and makes the Hamiltonian diagonal.

To obtain a solution of equation (101) we have to specify the source function $\underline{P}$. We have

$$\underline{P} = \frac{\sqrt{4\pi}\cdot f}{\mu} \underline{\sigma} \frac{i}{2}(\tau_1 - i\tau_2) \tag{102}$$

where $\underline{\sigma}$ is the spin operator acting on the nucleons and the τ's are the Pauli isotopic spin matrices. The constant f is the coupling parameter which, in our system of units is of the order unity. We now have to determine the matrix element of (102) corresponding to a transition in which one of the nuclear protons is transformed into a neutron and a momentum $\underline{\xi}$ is transferred to the nuclear system. We may write for this matrix element

$$\underline{P} = \sqrt{4\pi}\, \frac{f}{\mu}\, e^{i(\underline{\xi}\,\underline{r})}\, \underline{\sigma} \tag{103}$$

in which is the expectation value of the matrix element $\frac{1}{2}\underline{\sigma}\,(\tau_1 - i\,\tau_2)$, that is

$$\underline{\sigma} = \frac{1}{2} \sum_i \int d\xi_1 \cdots d\xi_A \, \omega_f^* (\xi_1 \cdots \xi_A) \underline{\sigma}^{(i)} (\vec{\tau}_1^{(i)} - i \vec{\tau}_2^{(i)}) \omega_0 (\xi_1 \cdots \xi_A)$$

with the notation used in appendix 2. The solution of equation (101) can now be written down immediately:

$$\varphi_0 = i \frac{\sqrt{4\pi}}{\mu \epsilon^2} f e^{i(\underline{\xi} \underline{r})} \qquad (104)$$

where $\epsilon^2 = \mu^2 + \xi^2$. This method of building up solutions based on the solutions of the static equation is justifyable only if the recoil energy of the nucleons can be treated as negligible.

Substituting from equations (99),(100) and (104) into the expression (98) for the matrix element of the current we obtain

$$\underline{j} = \sqrt{\frac{2\pi}{\kappa}} e \frac{f}{\mu} \left\{ \frac{1}{\epsilon^2} (\underline{q} + \underline{\xi})(\underline{\xi} \underline{\sigma}) - \underline{\sigma} \right\} \qquad (105)$$

According to equation (96) we have to combine this matrix element with the vector potential corresponding to a transition in which the momentum $\underline{k}$ is transferred from the electron to the system of mesons and nucleons. From the conservation of momentum it follows

$$\underline{q} - \underline{\xi} = \underline{k}$$

so that we obtain for the transverse component of the current:

$$j_\perp = \sqrt{\frac{2\pi}{\kappa}}\, e\, \frac{f}{\mu}\left\{\frac{2}{\epsilon^2}\,\underline{q}\,(\underline{\xi}\,\underline{\sigma}) - \underline{\sigma}\right\} \qquad (106)$$

Here again the first term is the ionsiation term and it is seen that it gives rise to an anisotropic distribution of mesons with very high energy with a marked preference for the forward direction. The second term due to the direct conversion of a quantum into a meson gives an isotropic distribution of mesons. When added together the two terms tend to cancel each other in the forward direction, though this cancelation is far from being complete. It will be seen later that the major contribution at all meson energies is given by the second term in equation (106).

Inserting from equation (106) into equation (96) we obtain an expression for the matrix element H' which in Born's approximation is related to the differential cross section by the formula (17).

From equations (96), (106) and (17) we therefore obtain as a result of carrying out the summation over the orientations of the nucleus

$$d\Phi = \frac{1}{4\pi^3}\sigma_x^2\,\frac{1}{2341}\cdot\frac{f^2}{(137\mu)^2}\,d\kappa\, q\, p'^2\, d\Omega\, d\Omega'(k^2-\kappa^2)$$
$$\left\{\underline{a}_\perp\left(\frac{2}{\epsilon^2}\,\underline{q}\,\underline{\xi} - I\right)\left(\frac{2}{\epsilon^2}\,\underline{q}\,\underline{\xi} - I\right)\underline{a}_\perp\right\}$$

where the spin summation over the tensor $\underline{a}_\perp\,\underline{a}_\perp$ has yet

to be carried out. σ_x denotes an arbitrary component of the vector $\underline{\sigma}$. It is observed from this expression that the ionisation term tends to cancel the leading term for mesons in the forward direction.

In the determination of the cross section we first carry out the integration over the directions of the meson and obtain

$$\int_\Omega d\Phi = \frac{\sigma_x^2}{\pi^2(2J+1)} \frac{f^2}{(137\mu)^2} d\kappa q p'^2 d\Omega' (k^2-\kappa^2)^{-2} \left\{1 - \frac{2\mu^2}{\kappa q}\left(\log\frac{\kappa+q}{\mu} - \frac{q}{\kappa}\right)\right\}(a_\parallel a_\perp) \qquad (107)$$

where the dot now denotes a scalar product. In performing this integration it was assumed tahat in the meson terms $k \sim \kappa$ with the same degree of approximation as is used in the derivation of equation (96). The first term in the curly bracket is due to the process (i), while the second comes from (ii) and the interference between the processes (i) and (ii). The second term is usually small. For non relativistic mesons $q \ll \mu$ and this term behaves like q^2/μ^2 while in the relativistic region it is of the order $2\mu^2/\kappa^2$ times a slowly increasing logarithmic factor. Even in the case $\kappa = 2\mu$ it amounts to only 26% of the leading term.

The scalar product ($\alpha_\perp \cdot \alpha_\perp$) has been discussed earlier. It is simply equal to (S) where S is the tensor defined in equation (21). (S) has been evaluated in equation (38). Inserting from this expression into equation (107), the integrations over the angles can immediately be carried out and give

$$\int_\Omega \int_{\Omega'} d\phi = \frac{4\sigma_x^2 f^2}{\pi (2J'+1)(137\mu)^2} S(\kappa) d\kappa \qquad (108)$$

in which $S(\kappa)d\kappa$, the spectral distribution function of the mesons is given by

$$S(\kappa)d\kappa = \frac{q\, d\kappa}{E\, \kappa} \left(\frac{E^2+E'^2}{E^2}\right)\left\{\log\frac{2EE'}{\kappa} - \frac{2EE'}{E^2+E'^2}\right\} \left\{1 - \frac{2\mu^2}{\kappa q}\left(\log\frac{\kappa+q}{\mu} - \frac{q}{\kappa}\right)\right\} \qquad (109)$$

The total cross section is obtained from this expression by integrating over $d\kappa$ from μ to E.

14)

Numerical Evaluation of the Meson-Spectrum and the Cross Section for Meson Production; Discussion of the Results.

In the derivation of equations (108) and (109)

the effect of the electrostatic field of the nucleus has been neglected. We can correct this by multiplying $S(\kappa)$ by N^2 - a 'Gamow factor' for the meson wave. This factor is defined by $|N|^2 = |\phi_q(0)|^2 / |\phi_q(\infty)|^2$ where ϕ_q is a regular solution of the Schrödinger Gordon equation of a free meson in the electrostatic field of the nucleus. The nearly isotropic distribution of the mesons in the final state suggests that there is no appreciable error involved in taking s-waves for the meson functions. For light nuclei ($Z \ll 20$), for both positive and negative mesons N^2 can be taken to be unity. For large values of Z we can make use of the W.B.K.-method to obtain an estimate of the function $\phi_s(r)$ and hence of N^2. In the case of a negative meson it is easily shown that

$$N^2 = N_-^2 = q_s^{(-)}/q \qquad (110)$$

where $q_s^{(\pm)}$ denotes the momentum of the meson at the surface of the nucleus, so that

$$q_s^{(\pm)} = \{(\kappa \mp \alpha Z/R)^2 - \mu^2\}^{1/2}$$

When the meson has positive charge we must distinguish between two cases. In the first case the energy of the meson may be less than the Coulomb barrier: $\kappa - \mu < \alpha Z/R$

in which case N^2 is very small so that in first approximation we may put $N = 0$, since the short wavelength of the meson will make the barrier practically impenetrable. For higher energies $\kappa - \mu > \alpha Z/R$ we may take

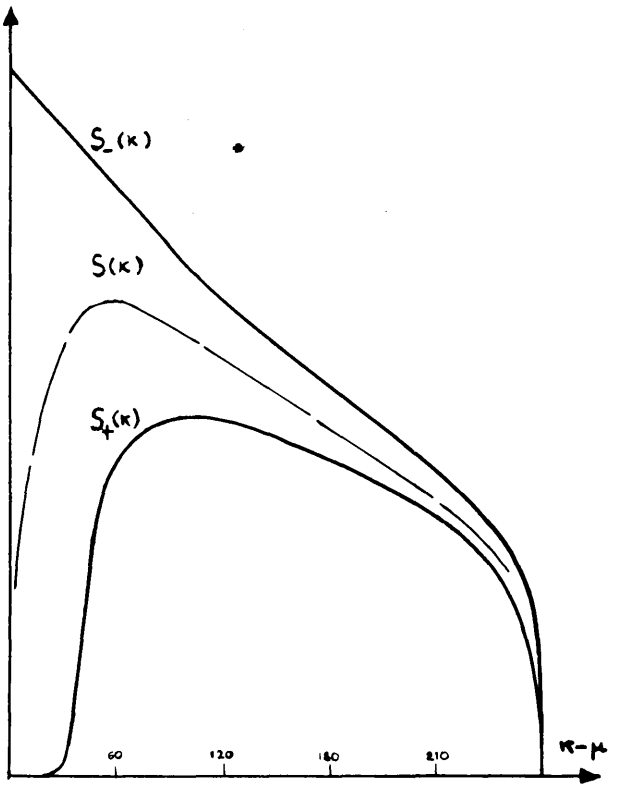
$$N^2 = N_+^2 = q_s^{(+)}/q \qquad (111)$$

with $q_s^{(+)}$ defined as above. The spectral distributions for mesons - positive and negative - $S_\pm(\kappa) = N_\pm^2 S(\kappa)$ obtained by multiplying the expression (109) by the factors (110), (111) respectively are shown graphically in figure 6, for the case in which $\alpha Z/R \simeq 30$ (i.e. for a very heavy nucleus) and a maximum energy of 600. The meson mass has been assumed to be 300. If we denote by n_+ and n_- respectively the total number of positive and negative mesons produced under these conditions we find by a numerical integration that

$$n_-/n_+ = 1.74 \; ; \; n_-/n_0 = 1.27 \; ; \; n_+/n_0 = 0.73$$

n_0 being proportional to the area under the dotted curve. The dotted curve gives the spectral distribution $S(\kappa)$ defined by equation (109). Even in the case of this extremely heavy nucleus the error involved in taking $N^2 = 1$ is less than 30% for this energy. For

Figure 6.



The energy distribution of pseudoscalar mesons produced in a collision between an electron and a nucleus. S_+ and S_- represent the spectral distribution functions of positive and negative mesons for a very heavy nucleus. S is for a very light nucleus. $\kappa-\mu$ is the kinetic energy of the mesons in units of mc^2 . The maximum energy was assumed to be $300mc^2$.

lower energies it is greater.

With the assumption $N^2 = 1$ we may now proceed to estimate the total cross section for which we find from equations (108) and (109)

$$\Phi = \Phi_0 \, g(E) \, \sigma_x^2 / 2J+1 \tag{112}$$

where

$$\Phi_0 = \frac{4 f^2}{\pi (137\mu)^2} \tag{113}$$

and

$$g(E) = \int_{\mu}^{E} S(\kappa) \, d\kappa \tag{114}$$

With $\mu = 300$ anf $f^2 \sim 1$ we find that $\Phi_0 = 1.1 \; 10^{-30} cm^2$. If the energy of both the electron and the meson in the final state is large compared to some average nuclear excitation energy (of the order of 10 Mev) we may consider the possible final states of the nucleus to be energetically equivalent. Then by a process similar to that used in paragraph - i.e. by using the completeness of the set of nuclear eigenfunctions we may evaluate $\sum_f \sigma_x^2 / 2J+1$ and obtain:

$$\sum_{f} \frac{\sigma_x^2}{2J+1} = \begin{cases} Z, \text{ for positive mesons} \\ A-Z, \text{ for negative mesons} \end{cases} \qquad (115)$$

The integral g(E) can easily be estimated in the non-relativistic and extreme relativistic range of meson energies. From what has been said previously we may neglect the contribution to this integral made by the ionisation term. Replacing the logarithmic term in the integrand by a suitably chosen mean value ($E' \sim \frac{1}{2}(E-\mu)$) we find

$$g(E) \sim \frac{2\sqrt{2}}{3} \left(\frac{E-\mu}{\mu}\right)^{3/2} \log (E-\mu) \qquad (116)$$

in non relativistic approximation and

$$g(E) \sim \frac{4}{3} \log (2E) \qquad (117)$$

in extreme relativistic approximation. As a result of a numerical integration of the spectral distribution function defined in equation (109) we find that $g(600) \simeq 1.9$. This is a factor 4 less than the value given by either equation (116) or (117) but we would not expect better agreement in this case since the energy is outside the range of validity of both approximations. In deriving these approximations we have

neglected terms which would tend to decrease the total cross section, so that equations (116) and (117) should be considered to define upper limits.

It follows from these considerations that the cross section is proportional to the number of nucleons in the target nucleus, a result which is immediately evident by considering the small wavelengths of all particles occurring in this process. The absolute value of the cross section is very small; at 300 Mev it is approximately 2×10^{-30} cm^2 per nucleon. From equation (117) it is seen that the cross section increases slowly with energy but the result is unreliable in the high energy reagion because we have neglected to take into account the effects of nuclear recoil and radiation damping. However it follows from recent calculations reviewed by Heitler [15] that up to energies of 600 Mev the influence of radiation damping is not large in the case of pseudoscalar mesons.

The cross section for the production of mesons by nucleons is only about one hundredth part of the cross section for the production of mesons by γ-rays,

as determined by Chang (without radiation damping) [29] and Peng. [30] It should be observed however that the values for the cross section given by Morette and Peng are rather high owing to the small meson mass assumed in the calculations. In order to observe the production of mesons by electrons it appears therefore that very thin targets (of the order of one hundredth of a radiation length) must be used. In the case of lead this is of the order 0.05mm.

The shape of the meson spectrum produced by electrons is roughly the same as that produced by a thin target Brems spectrum of γ-rays. The reason for this is that the nucleus radiates its mesons under the influence of the Maxwell field of an electron transition. The spectrum of this equivalent Maxwell field is approximately the same as the spectrum derived from thin target radiation. Even quantitatively we should expect the resemblance to be close considering the equivalence of electrons and photons revealed in the Weizsäcker-Williams method.

Appendix 1.

The Determination of Nuclear Matrix Elements by Means of the Becker Leibfried Method [31].

The Becker Leibfried method of 'second quantisation' represents a very convenient way for the determination of matrix elements defined in terms of field operators. It is in a way a mathematical precisation of the ideas of Hartree and Fock and hardly adds anything substantially new. In field theory however it helps one to avoid the cumbersome momentum representation and enables one to give a meaning to operators without any restrictions as to the choice of suitable coordinate systems in momentum space. In this capacity it has been recently used by Schwinger in his theory of the Lamb-shift.

Since no account of this method - in particular in its application to Fermi-systems - has been published - we shall outline the basic idea of the Becker and Leibfried approach in the following pages. The formulation given here is based on a discussion of

this method given by Heisenberg [33] in his colloquium on the quantum theory of radiation and to many informal talks with Prof. Becker and Prof. Rellich in Göttingen.

In quantum theory a nucleon can be described in terms of five coordinates: the 3 space coordinates x,y,z a spin coordinate s (capable of the two values ± 1) and a coordinate of isotopic spin t. The isotopic spin-variable is also capable of two eigenvalues ± 1, the first of which describes the proton state. In the following we shall treat the nucleons in non relativistic approximation. It is well known that Fermi-fields can be described in terms of two wave-operators $\psi^+(\xi)$ and $\psi(\xi)$, the first of which symbolises the creation of a particle ξ, that is the creation of a particle with spins s,t at the point $\underline{r}$ of ordinary space. A movement of a particle in this representation is a sequence of creation and annihilation processes along the points in the neighbourhood of its classical trajectory. The quantum properties of the field are expressed in terms of the commutation relations

$$\{\psi(\xi)\psi(\xi')\} = \{\psi^+(\xi)\psi^+(\xi')\} = 0$$
$$\{\psi^+(\xi)\psi(\xi')\} = \delta(\xi - \xi') \qquad (1)$$

where the curly brackets denote the + commtator : $\{AB\} = AB + BA$. $\delta(\xi - \xi')$ is used as an abbreviation for $\delta(x-x')\,\delta(y-y')\,\delta(z-z')\,\delta_{ss'}\,\delta_{tt'}$. The δ's are Dirac functions when they act on a continuous variable and Kronnecker symbols when they act on a discrete variable. The operators $\Psi^+(\xi)$ and $\Psi(\xi)$ have to be such that that the operator representing the energy obtains the form of a diagonal matrix. In non relativistic nuclear theory this operator - the Hamiltonian - has the form

$$H = \int d\xi \frac{1}{2M} |(\nabla\Psi)|^2 + \int d\xi \int d\xi' \Psi^+(\xi)\Psi^+(\xi')U(\xi-\xi'/\Psi(\xi')\Psi(\xi) \quad (2)$$
$$+ \ldots$$

Here M is the mass of a nucleon (the trivial mass-difference between neutron and proton has been neglected in equation (2)). The first term represents the kinetic energy of the nucleons, the second interaction between pairs of nucleons and the ... indicate 3- and more particle interactions. The order of the operators in the second term has been chosen in such a way that no selfenergy terms will occur.

It follows immediately from equations (1) and

(2) that the operator

$$A = \int \psi^+(\xi)\psi(\xi)\,d\xi \qquad (3)$$

commutes with H and therefore represents a constant of motion. The operator A has the following properties:

(i) the eigenvalues of A are real and positive,

(ii) One of the eigenvalues of A is 0.

(iii) the eigenvalues of A are integers.

The proof of (i) is trivial and follows from the positive definite form of (3). (ii) follows from the commutation relations (1):

$$\begin{aligned} A\psi - \psi A &= -\psi \\ A\psi^+ - \psi^+ A &= +\psi^+ \end{aligned} \qquad (4)$$

If $\chi_{A'}$ is an eigen-vector of A to the eigenvalue A' then because of the first equation (4) $\psi\chi_{A'}$ is an eigen-vector of A to A'-1 . Applying the operator ψ sufficiently often to the vector $\chi_{A'}$ we would therefore arrive at negative eigenvalues which is in contradiction to (i). It therefore follows that there must be an eigenvector χ_0 with the property $\psi\chi_0 = 0$ and hence

$A\chi_0 = 0$ and that all the eigenvalues of A must be integers. The physical interpretation of the operator A is obvious: It represents the number of Fermi particles. The eigenvector χ_0 corresponds to the vacuum state.

The operator A gains its importance from the fact that in most theories it commutes - or commutes approximately - with the Hamiltonian of the total system. It is therefore useful to divide the Hilbert space - on which the operators $\psi, \psi^\dagger$ are allowed to act into subspaces each corresponding to a certain eigenvalue A' of A, i.e. to a certain number of particles. In the case of the Hamiltonian (2) these different subspaces do not communicate: there are no matrix elements of the type $(A'|M|A'') \neq 0$ for $A' \neq A''$, provided that the quantity M is observable. It should be noted that the operators ψ and $\psi^\dagger$ - the amplitudes of a Fermi-field-do• not represent observable quantities.

Starting from the eigenvector χ_0 which we assume to be normalised $(\chi_0\chi_0) = 1$, we may now proceed to construct the eigenvectors χ_A by using equation (4). We obtain in this way:

$$\chi_1(\xi_1) = \frac{1}{\sqrt{1!}} \psi^+(\xi_1)\chi_0$$
$$\chi_2(\xi_2\xi_1) = \frac{1}{\sqrt{2!}} \psi^+(\xi_2)\psi^+(\xi_1)\chi_0$$
$$\cdots \qquad (5)$$
$$\chi_A(\xi_A \cdots \xi_1) = \frac{1}{\sqrt{A!}} \psi^+(\xi_A) \cdots \psi^+(\xi_1)\chi_0$$

The Hilbert vectors defined in this way are functions of the coordinates of the particles. There is one Hilbert vector to each configuration of the particles in coordinate space. The normalisation employed in equation (5) gives the following orthogonality relation:

$$(\chi_A \chi'_A) = \frac{1}{A!} \sum_P (-)^P \prod_{j=1}^{A} \delta(\xi_j - \xi_{j'}) \qquad (6)$$

which follows immediately from (5) by applying the commutation relations (1). The scalar product between two vectors $\chi_{A'}$ and $\chi_{A''}$ is nothing for $A' \neq A''$. The eigenvectors χ_A defined in equation (5) are antisymmetric in all their variables: $\chi_A(\cdots \xi_i \cdots \xi_\kappa \cdots) = -\chi_A(\cdots \xi_\kappa \cdots \xi_i \cdots)$.

The right hand side of equation (6) has the same property. (The summation has to be carried out over all permutations of the indices of ξ'_i, $(-)^P$ being +1 for an even permutation and -1 for an odd permutation).

In the following we shall use the abbreviation $\delta(\xi - \xi')$ for the right hand side of equation (6). This antisymmetrised δ -function can then be replaced by a normal Dirac- δ -function whenever it acts on functions which are antisymmetric in all coordinates.

The vectors χ_A defined in equation (5) can be interpreted as basisvectors of the Hilbertspace on which the operators ψ and ψ^+ act. The quantities $\xi_1, \xi_2, \ldots \xi_A$ correspond to the indices of the unit-vectors in an ordinary vector space.

The most general Hilbert vector U in the subspace A will have the form

$$U = \int d\xi_1 \ldots d\xi_A \; u(\xi_1 \ldots \xi_A) \chi_A(\xi_1 \ldots \xi_A) \qquad (7)$$

the functions $u(\xi_1, \ldots \xi_A)$ representing the 'coordinates' of the vector U in the frame of reference defined by the basisvectors (5). Without loss of generality we can assume that u is antisymmetric in the same way as χ - the symmetric part would not contribute to the integration (7). The normalisation chosen for the basis vectors χ now guarantees that U is a unit

vector $(UU) = 1$, provided that

$$\int d\xi_1 \ldots d\xi_A \, |u(\xi_1 \ldots \xi_A)|^2 = 1 \qquad (8)$$

This normalisation rule and the fact that u has to be antisymmetric suggests the interpretation of u as the Schrödinger function of the A particle problem.

The analysis now lets us determine unambiguously any operator built up in the conventional way out of the Fermi-amplitudes ψ and ψ^+. Let a denote a state with A particles and b a state with B particles. The state A can then be described in terms of a Schrödinger function $u_a(\xi_1, \ldots \xi_A)$, the state b by $u_b(\xi_1, \ldots \xi_B)$. We can now determine the matrix-element of ψ^+, say corresponding to the transition a b. This matrix-element is simply $(bB|\psi^+(\xi)|aA) = (U_B \psi^+ U_A)$ where U_A and U_B are defined in equation (8). Now, by using (5) and (6) and the symmetry property of the function we have

$$(bB|\psi^+(\xi)|aA) = \delta_{A+1} \sqrt{B} \int d\xi_1 \ldots d\xi_A \, u_b^*(\xi_1 \ldots \xi_A, \xi) u_a(\xi_1 \ldots \xi_A) \qquad (9)$$

We therefore see that the operator ψ^+ corresponds to the creation of a particle - the number B of particles

is one larger than the number A of particles in the initial state. Taking the Hermitian adjoint of equation (9) we can determine the matrix-element of ψ :

$$(b\,B|\psi(\xi)|a\,A) = \delta_{B+1,A}\sqrt{A}\int d\xi_1 \ldots d\xi_B\, u_a(\xi_1 \ldots \xi_B, \xi)\, u_b^*(\xi_1 \ldots \xi_B) \quad (10)$$

and it is therefore obvious that the operator ψ represents the annihilation of particles.

The matrix-elements of the operators (10) and (9) may be very useful in the discussion of β-decay phenomena for which the number of electrons does not remain constant. Whereas the original procedure in Fermi's theory is to replace ψ^+ by the conjugate of the wavefunction of the created electron we see from (9) that this procedure might lead to erroneous results - since even the one particle electron eigenfunctions are not orthogonal in the initial and final state (Change of Z!). This gives rise to exchange phenomena - introducing an exceedingly small correction in the shape of β -spectra - and to internal ionisation processes.

However, in the applications given in this paper we are mainly concerned with operators which leave the

the number A of particles unaltered. By means of the equations (9) and (10) and the completeness relations for the Schrödinger functions

$$\sum_c u_c(\xi_1 \cdots \xi_A)\, u_c^*(\eta_1 \cdots \eta_A) = \delta(\xi - \eta) \qquad (11)$$

the matrix-element of the simplest operator of this type

$$\bar{Q}(\xi) = \psi^+(\xi)\, Q_\xi\, \psi(\xi) \qquad (12)$$

may be readily evaluated. Q_ξ is an operator acting on the coordinate ξ. With the general rule for matrix multiplication we have

$$(bB|\bar{Q}(\xi)|a\,A) = \sum_{cC} (bB|\psi^+(\xi)|\,cC)\, Q_\xi\, (cC|\psi(\xi)|aA)$$

and this relation can be immediately evaluated by means of equations(9),(10),(11). The result is

$$(bB|\bar{Q}(\xi)|aA) = A\,\delta_{A,B} \int d\xi_1 \cdots d\xi_{A-1}\, u_b^*(\xi_1 \cdots \xi)\, Q_\xi\, u_a(\xi_1 \cdots \xi) \qquad (13)$$

The δ-functions implies that the operator $\bar{Q}$ leaves the number of particles invariant. The factor A is not immediately obvious but it can be understood that for $Q = I$ - the unit operator - Q represents the particle

density. In this case the expectation value of $\int d\xi \bar{Q}(\xi)$ should be equal to the number of particles. (13) shows immediately that this is the case for $a = b$ - i.e. for the expectation value.

In some cases average values of $\bar{Q}$ over the whole space will occur. Because of (13) - and keeping in mind the symmetry property of the wave functions - we may write for these

$$(bB|\int \bar{Q} d\xi |aA) = \sum_i \int d\xi_1 \ldots d\xi_A \, u_b^* Q_{\xi_i} u_a \qquad (14)$$

It will be seen that the method outlined above gives an unambiguous procedure for the determination of matrix-elements without reference to a transformation in momentum space. It therefore enables to deal with matrices of the type given in equation (12) as if they were one particle matrices and make the transformation to the many particle situation at a very late stage of the calculation. This of course is one of the basic ideas of the Hartree Fock method but it seems to us to come out in a much more tangible form in the Becker Leibfried version of the theory.

Appendix 2.

Formal Remarks on the Magnetic Dipole Moment of Nuclear Transitions.

It is well known that the proton as well as the neutron has a magnetic moment. The proton moment is 2.79 nuclear magnetons (1/2M in our units), that of the neutron -1.91.[34] One therefore has to assume that it is not only the charge of the nuclear particles which interacts with an electromagnetic field but their magnetic moments as well.

Throughout this paper it has been assumed that a 4-vector j_ν - subject to an equation of continuity - is a sufficient description of a nuclear transition, but the verification of this assumption is only obvious in the case where the charge of the nuclear particles is alone responsible for the interaction with the electromagnetic field. We want to show in this paragraph that the 'magnetic' interaction can also be described in the form of an equivalent current, such that the total current used previously has to be interpreted as the sum of an 'orbital' current

$j^{(e)}$ and a 'magnetic' current $j^{(m)}$. A formula will be given for the determination of the 'magnetic' current in terms of the magnetic moments and spins of the nuclear particles.

We start from a Lagrangian for the system of nuclear particles in an electromagnetic field. We assume that this Lagrangian has the form

$$\mathcal{L} = \mathcal{L}_0(\psi_i^+\psi_j(\partial_\nu + ieqA_\nu)\psi_j^+(\partial_\nu - ieqA_\nu)\psi) + \frac{1}{2} M_{ik}(\partial_i A_k - \partial_k A_i) \quad (1)$$

Here ψ^+ and ψ are the operators corresponding to the creation or annihiltation of a nucleon respectively and q is an operator acting on the isotopic spin with the eigenvalues 1 for a proton and 0 for a neutron. ($q = \frac{1}{2}(1+\tau_3)$). M_{ik} is an antisymmetric tensor (i,k 1,2,3) representing the magnetic part of the interaction. We may write

$$M_{ik} = \frac{1}{2M}\psi^+\{2.79q - 1.91(1-q)\}\sigma_{ik}\psi \quad (2)$$

where σ_{ik} is the spin operator. In a relativistically invariant theory the tensor M_{ik} will have i-4-components as well, so that (1) has to be interpreted as a non-relativistic approximation. The current j may now be defined - by general principles of the

quantum theory of wave fields - as the functional derivative of the Lagrangian with respect to the components of the vector-potential. (25) The current defined in this way satisfies the equation of continuity provided that the Lagrangian is gauge-invariant, i.e. invariant against the gauge transformation

$$A_\nu \to A_\nu + \partial_\nu \Phi$$

$$\psi \to e^{ieq\Phi}\psi \qquad \psi^+ \to \psi^+ e^{-ieq\Phi}$$

This is certainly the case for the Lagrangian leading to the Hamiltonian (2) of appendix 1, and for the additional term M_{ik} introduced in equation (1). With this definition of the current we obtain

$$j_\nu = \frac{\delta \mathcal{L}}{\delta A_\nu} = \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \qquad \begin{aligned} j_i &= j_i^{(0)} + \partial_k M_{ik} \\ j_4 &= j_4^{(0)} \end{aligned} \qquad (3)$$

the first term of which represents the 'orbital' current the second the current due to the magnetic moment of the nucleons. It is easily seen that both components of the current separately satisfy an equation of continuity. In the case of the magnetic component this is due to the absence of a four-component and the antisymmetry of M_{ik}.

The magnetic dipole moment has been defined in

paragraph , equations (33) and (35). Considering the antisymmetry of T, equation (33) may be written

$$\kappa T_{ik} = \frac{1}{2}\int \underline{dr}\,\{j_i x_k - j_k x_i\} \qquad (4)$$

Inserting from equation (3) we then have

$$\kappa T_{ik} = \kappa \overset{(0)}{T}_{ik} + \int \underline{dr}\, M_{ik} \qquad (5)$$

where $\overset{(0)}{T}_{ik}$ is that part of the magnetic dipole moment which is due to the orbital components of the current. The result (5) is obtained after a partial integration.

It is therefore seen that our definition (33) of the magnetic moment, namely $\overset{(1)}{\mu} = \frac{1}{\kappa}\int M \underline{dr}$ differs from the one commonly used by a factor $1/\kappa$. This difference is clearly trivial and due to the more convenient 'electric' interpretation of the magnetic dipole moment used in this paper. As a result of this interpretian our formulae show a close resemblance between the electric quadripole case and the magnetic dipole case, whereas commonly the resemblance lies between electric and magnetic dipoles. It appears however that for our problem the interpretation used is preferable, since the field produced by a moving electron does not show symmetry between the electric and magnetic field com-

ponents - in contrast to the electromagnetic field of a quantum.

Owing to the magnitude of the magnetic moments and to the fact that the neutrons may also contribute in the case of magnetic dipole transitions of the type defined by $j^{(1)}$ the orbital contribution of magnetic dipole transitions will generally be small. In the discussion in paragraph we have therefore neglected the orbital current altogether. This will be the more justified since there are a great number of selection rules forbidding orbital magnetic dipole transitions and it will be remembered that in the theory of atomic spectra such transitions are altogether impossible for optical frequencies. If orbital magnetic dipole transitions are allowed they will in general behave very similar to ~~magnet~~ electric quadripoles.

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