"SWITCHING SURGES"

"A Study of Transient Phenomena in Electro-Magnetic Machinery, with Particular Reference to the Use of the Heaviside Operational Calculus."

THESIS

for

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Submitted to the University of Glasgow by

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FOREWORD

The Thesis is divided primarily into three parts.

In the first of these three parts is considered the standard treatment of transient phenomena by the methods of the Newtonian Calculus. Various problems are worked out by these methods; Part I,C, refers to the original article "Ignition Coils", published by the author in the "Royal Technical College Journal", January, 1934, a copy of which article forms a part of the Thesis; Part I,D, contains a method of dealing with the transformer magnetising current surge by regarding the magnetisation curve as made up of two straight lines, - a method which the author believes is original; Part I,E, contains a summary of two methods of direct mathematical attack on the problem of alternator short-circuit, due respectively to Shimidzu and Ito, and to Rüdenberg. Part I also shows wherein the difficulties of normal mathematical methods usually lie, - viz., in the evaluation of the arbitrary constants.

Part II is devoted mainly to the exposition of the principles of the Operational Calculus invented by Heaviside. One or two simple problems are worked out, and the solutions shown to be identical with those obtained by the standard methods of Part I. The last section of Part II is given over to a general survey of problems arising in connection with the Operational Calculus; more particularly, various formulae necessary for the work of Part III are established. Of these, those relating to the solution of simultaneous operational equations having a cyclically symmetrical matrix determinant, and applied forces symmetrically disposed in the complex plane, are original.

Part III forms the main body of the Thesis. It is devoted exclusively to the application of operational methods to the case of transients in rotating machinery,- in particular, to the case of alternator short-circuit. In general, two methods are used in this treatment, - the "Method of Reflections", and the "Method of Equivalent Circuits", - both of which are derived from the Heaviside Calculus. So far as the author is aware, the Heaviside Calculus has not before been adapted in this particular manner, so that Part III is in the main an original treatment.

It is shown that Shimidzu and Ito, and Rüdenberg, obtained slightly different results (Part I,E,), on account of the fact that the internal constructions of the machines they treated were somewhat different - Rüdenberg having taken a polyphase field circuit, and Shimidzu and Ito, a single-phase field circuit: It is shown that the operational methods of Part III may be extended to cover both arrangements, giving identical results in each case to those obtained by direct mathematics, (Part III, A & C). A case is also considered in which the short-circuit exists in the stator phases before the field current is switched on,(Part III, B).

The latter portion of Part III deals with the short-circuit of a machine having a third, or damper, winding, single-phase in section D, and polyphase in section E; it is shown that even such involved circuits yield readily to operational methods of treatment.

Part III,F, deals with a question which arises from the previous section, namely, what value is to be used for the resistance of the "Equivalent" phase - (as used in the "Method of Equivalent Circuits") - when the polyphase winding concerned is in the form of a cage?

The concluding note shows further possible applications of the method, and also points out its limitations.

While Part III is original throughout, Parts I and II have in several places been taken over from other articles and books. Reference has already been made to the methods of Shimidzu and Ito, and of Rüdenberg. In addition to these, an outline of ignition coil theory, and a theory of the current changes in a contact-breaker arc are included in Part I,C, and I,D, respectively, both these being due to A.E.Watson, and reference is made in Part I,C, to Prof. E.Taylor-Jones' work on the induction coil; further, Part I,D, contains a graphical method of calculating the transformer switching surge, which method is due to C.P.Steinmetz. Part II is in the main taken from E.J.Berg's book - "Heaviside Operational Calculus" - though in section D the author has inserted the physical explanations in connection with the interruption of a current in a network.

A word is required in connection with the numbering of the equations. The numbers start from 1 at the commencement of each part of the Thesis; when reference is made to an equation, it is understood that unless some other part is specifically named, the equation referred to belongs to the same part as the reference.

The author wishes to acknowledge with gratitude the assistance and kindly comment given to him in the prosecution of his studies by the supervisor of his research, Dr. S. Parker-Smith, Professor of Electrical Engineering at the Royal Technical College, on whose advice it was that he first took up the study of Operational methods. Thanks must also be returned to Professor Street and Professor Muir, for guidance and correction on the mathematical side.

"A Study of Transient Phenomena in Electro-Magnetic Machinery, with Particular Reference to the Use of the Heaviside Operational Calculus."

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PART I

"The Standard Treatment of Transients"

Transients - i.e., those current and voltage surges which always accompany a change of circuit conditions in a network, - have been in the past in general evaluated by the application of formal Differential Equations. Any simple electrical circuit may be represented by a single Differential Equation, and any compound circuit, - i.e., a circuit involving several simple circuits coupled together- by a series of simultaneous Differential Equations, either of which arrangements may be solved more or less readily by standard Mathematical methods.

The solution of these equations involves always two parts, - the Particular Integral, and the Complementary Function.

1. The Particular Integral.

This is the particular solution of the equation, or set of equations, corresponding to the particular applied force. It gives the normal steady state of the circuit.

2. The Complementary Function.

This is quite independant of the applied force, and is, in general, obtained by equating the applied force to zero. It thus depends (for its mathematical form) on the circuit constants, - resistance, inductance & c. - and occurs in the same form, though with different amplitudes, for all changes of the circuit conditions. Its amplitude depends on the circuit conditions, - current, voltage, & c., - immediately prior to the act of switching, and the final steady value of these variables; in mathematical form it is invariably of a gradually diminishing nature. It is therefore known as the "Transient".

A. Illustration of Normal Calculus Method.

To illustrate the method, the well-known simple case of a steady voltage E suddenly switched-in to a circuit containing inductance L in series with resistance R may be taken.

If the current at any instant t is given by i, then at that instant the voltage drop across the resistance is iR, and the voltage drop across the inductance is L.di/dt, (since in the inductance is generated a reverse E.M.F. L.di/dt). The applied force is of course E.

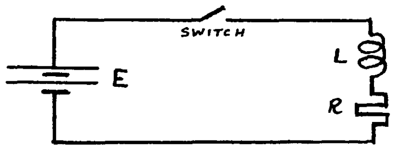

Fig 1.

Thus the differential equation of the circuit is :-

$$Ri + L\frac{di}{dt} = E \qquad \text{--(1)}$$

1. Particular Integral.

A solution of the particular equation is obviously :-

$$i = \frac{E}{R} \; ; \qquad \frac{di}{dt} = 0 \qquad \text{--(2)}$$

This is the "Steady condition" of normal circuit analysis.

2. Complementary Function.

Put $E = 0$.

Then $$Ri + L\frac{di}{dt} = 0$$

i.e., $$\frac{di}{dt} + \frac{R}{L}i = 0 \qquad \text{--(3)}$$

The left-hand side of eqn.(3) may be written :-

$$\varepsilon^{-\frac{Rt}{L}} \frac{d}{dt}\left\{ i\,\varepsilon^{\frac{Rt}{L}} \right\}$$

Hence, by integration, after dividing both sides by $\varepsilon^{-\frac{Rt}{L}}$:-

$$i\,\varepsilon^{\frac{Rt}{L}} = \text{Const., say A.}$$

i.e., $$i = A\varepsilon^{-\frac{Rt}{L}} \qquad \text{--(4)}$$

Hence the total current is given by :-

$$i = \frac{E}{R} + A\varepsilon^{-\frac{Rt}{L}} \qquad \text{--(5)}$$

where A is arbitrary, and is fixed by the conditions obtaining in the circuit immediately prior to switching.

NOTE :-

The whole equation might have been derived in one step, as follows :-

$$L\frac{di}{dt} + Ri = E$$

$$\therefore \varepsilon^{\frac{Rt}{L}} \cdot \frac{di}{dt} + \varepsilon^{\frac{Rt}{L}} \cdot \frac{R}{L} \cdot i = E \cdot \frac{1}{L} \cdot \varepsilon^{\frac{Rt}{L}}$$

$$\therefore \frac{d}{dt}\left\{ i\,\varepsilon^{\frac{Rt}{L}} \right\} = \frac{E}{L} \cdot \varepsilon^{\frac{Rt}{L}}$$

By integration :- $$i = \frac{E}{R} + A\varepsilon^{-\frac{Rt}{L}}.$$

At the moment $t = 0$, $\varepsilon^{-\frac{Rt}{L}} = 1$;

Hence $$i_0 = \frac{E}{R} + A \qquad \text{--(6)}$$

where i_0 is the current at $t = 0$.

Since there is inductance in the circuit, the current cannot be instantaneously established. Hence the current at $t = 0$ is the same as the current immediately before switching. That is, $i_0 = 0$.

Hence, $$0 = \frac{E}{R} + A \quad \text{or} \quad A = -\frac{E}{R} \qquad \text{--(7)}$$

Thus the full equation for the current is :-

$$i = \frac{E}{R} - \frac{E}{R} \cdot \varepsilon^{-\frac{Rt}{L}} = \frac{E}{R}\left\{ 1 - \varepsilon^{-\frac{Rt}{L}} \right\} \qquad \text{--(8)}$$

This simple illustration shows the basic method of procedure for circuits of greater complexity. No difficulty attaches to the method for such circuits; only the working out of the arbitrary constants becomes rather laborious.

Various other simple circuits of greater or less complication are worked out in Appendix A and B, and serve to indicate wherein lie the main difficulties of the straightforward application of the Calculus.

B. Application of Normal Calculus to the Case of Coupled Circuits.

Consider a simple coupled circuit of the type shown in the diagram, Fig 2, - the steady E.M.F. E being applied suddenly in circuit 1 at the time $t = 0$.

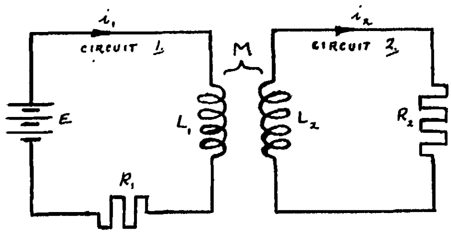

Fig 2.

Let M be the mutual inductance between the two circuits, i.e., the E.M.F. induced in circuit 1 or 2 by unit rate of change of current in circuit 2 or 1.

Then the differential equation of circuit 1 is :-

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} = E. \qquad \text{--(9)}$$

and of circuit 2 is :-

$$R_2 i_2 + L_2 \frac{di_2}{dt} + M\frac{di_1}{dt} = 0 \qquad \text{--(10)}$$

These are a pair of simultaneous differential equations.

From eqn.(9) :-

$$\frac{di_2}{dt} = \frac{1}{M}\left\{E - R_1 i_1 - L_1\frac{di_1}{dt}\right\} \qquad \text{--(11)}$$

Differentiating (10) and substituting (11) therein :-

$$R_2\cdot\frac{1}{M}\left\{E - R_1 i_1 - L_1\frac{di_1}{dt}\right\} + L_2\cdot\frac{1}{M}\cdot\left\{-R_1\frac{di_1}{dt} - L_1\frac{d^2i_1}{dt^2}\right\} + M\cdot\frac{d^2i_1}{dt^2} = 0$$

or $$(L_1L_2 - M^2)\frac{d^2i_1}{dt^2} + (R_1L_2 + R_2L_1)\frac{di_1}{dt} + R_1R_2 i_1 = ER_2 \qquad \text{--(12)}$$

and similarly :-

$$(L_1L_2 - M^2)\frac{d^2i_2}{dt^2} + (R_1L_2 + R_2L_1)\frac{di_2}{dt} + R_1R_2 i_2 = 0 \qquad \text{--(13)}$$

The solutions of these equations involve the roots of the auxiliary equation :-

$$(L_1L_2 - M^2)\lambda^2 + (R_1L_2 + R_2L_1)\lambda + R_1R_2 = 0$$

i.e., $$\lambda = \frac{-(R_1L_2 + R_2L_1) \pm \sqrt{(R_1L_2+R_2L_1)^2 - 4R_1R_2(L_1L_2-M^2)}}{2(L_1L_2-M^2)} = \alpha \text{ and } \beta.$$

The solutions for the currents are then :-

$$i_1 = \frac{E}{R_1} + A\varepsilon^{\alpha t} + B\varepsilon^{\beta t} \qquad \text{--(14)}$$

and as elimination between eqns.(9) and (10) gives :-

$$i_2 = \frac{1}{MR_2}\left\{(L_1L_2-M^2)\frac{di_1}{dt} + R_1L_2 i_1 - EL_2\right\}$$

hence :-

$$i_2 = \frac{1}{MR_2}\left\{A[R_1L_2 + \alpha(L_1L_2-M^2)]\varepsilon^{\alpha t} + B[R_1L_2 + \beta(L_1L_2-M^2)]\varepsilon^{\beta t}\right\} \qquad \text{--(15)}$$

In order to evaluate the arbitrary constants A and B, we have now to substitute the initial conditions, which are:-

$$i_1 = i_2 = 0 \qquad \text{at} \qquad t = 0$$

Hence, from eqn.(14) :-

$$0 = \frac{E}{R_1} + A + B \qquad \text{--(16)}$$

and, from eqn.(15) :- $$0 = A[R_1L_2 + \alpha(L_1L_2-M^2)] + B[R_1L_2 + \beta(L_1L_2-M^2)]$$

i.e., $$0 = -EL_2 + [A\alpha + B\beta][L_1L_2 - M^2] \qquad \text{--(17)}$$

Hence $$0 = \frac{E}{R_1}\cdot(L_1L_2 - M^2)\cdot\alpha + EL_2 + B(L_1L_2 - M^2)\alpha - B(L_1L_2 - M^2)\beta$$

or $$B = -\frac{L_2 + \frac{\alpha}{R_1}(L_1L_2 - M^2)}{(\alpha-\beta)(L_1L_2 - M^2)} \qquad \text{--(18)}$$

and $$A = -\frac{L_2 + \frac{\beta}{R_1}(L_1L_2 - M^2)}{(\beta-\alpha)(L_1L_2 - M^2)} \qquad \text{--(19)}$$

i.e., $$i_1 = \frac{E}{R_1}\left\{1 + \left[\frac{L_2R_1 + \beta(L_1L_2 - M^2)}{(\alpha-\beta)(L_1L_2 - M^2)}\right]\varepsilon^{\alpha t} - \left[\frac{L_2R_1 + \alpha(L_1L_2 - M^2)}{(\alpha-\beta)(L_1L_2 - M^2)}\right]\varepsilon^{\beta t}\right\} \qquad \text{--(20)}$$

and $$i_2 = \frac{E}{MR_1R_2}\cdot\frac{[R_1L_2 + \alpha(L_1L_2 - M^2)][R_1L_2 + \beta(L_1L_2 - M^2)]}{[(\alpha-\beta)(L_1L_2 - M^2)]}\cdot\left\{\varepsilon^{\alpha t} - \varepsilon^{\beta t}\right\} \qquad \text{--(21)}$$

In this, the simplest possible coupled circuit, it is seen that the total expressions for the current in each circuit are already becoming extremely complicated; it must be remembered, moreover, that α and β are themselves fairly complex.

It is obviously better, in any specific case, to solve the original equations directly from the supplied data, rather than to substitute in the final symbolic equations.

Since every circuit in the network under consideration represents an extra equation, and each inductance or capacitance an extra arbitrary constant in the penultimate solution, which constant must be evaluated by substitution of the conditions at $t = 0$, it follows that with any degree of complication, the above method of treating the problem becomes unmanageable. A simpler method is required; this is provided by the Heaviside Operational Calculus, which is dealt with in Part II of the Thesis.

In the present section, certain practical cases will be investigated, wherein the methods of the Newtonian Calculus will be employed.

C. The Automobile Ignition Coil:-

It has been shown that an Internal Combustion Engine requires for the ignition of the compressed gases a voltage of at least 7·5 K.V., if the spark-plug points are to be sufficiently open to avoid being short-circuited by carbon particles. A six-cylinder automobile engine running at 4,600 r.p.m. requires 3x4,600 sparks per minute, i.e. one spark per cylinder per two revs. in a four-stroke engine. This corresponds to 230 sparks per sec., each of at least 7·5 K.V. Since the ignition apparatus must not be too weighty, costly, or bulky, it is apparent that no ordinary Alternator would suffice. Use is therefore made of a convenient modification of the laboratory Induction Coil, an instrument which depends for its working entirely on transients.

In essence, both induction and ignition coils consist of two coupled circuits, as in Fig. 3. The first of these

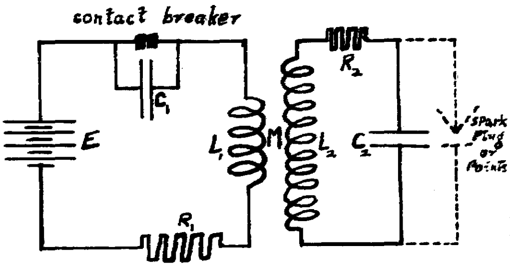

Fig. 3

circuits, the primary, contains a source of low direct voltage E, in series with which is a coil having resistance R_1 and inductance L_1, and a pair of contact breakers shunted by a capacitance C_1. The contact breakers are actuated in the laboratory by the magnetisation of the iron core of L_1, and thus cannot be altered in frequency of break. In the ignition coil, they are actuated by a cam run from the engine shaft, and therefore give a number of sparks per minute corresponding to the engine speed. The second circuit, or secondary, consists of an inductance L_2 coupled to the primary L_1 with a degree of coupling M, in series with a resistance R_2,-which really represents eddy current and hysteresis losses in the core rather than true ohmic loss,- and an equivalent condenser C_2, which is made up of self-capacitance of secondary winding, shunted by any capacitance due to, say, high tension cable. The spark plug is connected across this secondary condenser.

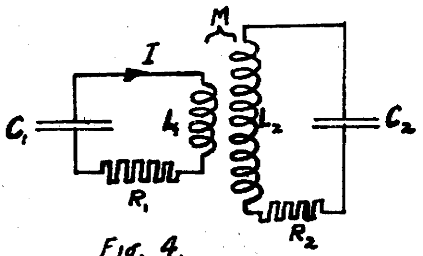

Fig. 4.

At the moment of break, we have, in effect, a circuit as shown in the diagram, Fig. 4, the equations for which would be:-

$$L_1\frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1\,dt + M\frac{di_2}{dt} = 0 \qquad \text{--(22)}$$

$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt + M\frac{di_1}{dt} = 0 \qquad \text{--(23)}$$

and at $t = 0$;-

$i_1 = I$, $i_2 = 0$, Volts at C_1 = Volts at $C_2 = 0$,

these last relationships giving the values of the Arbitrary Constants.

The full solutions of these equations by classical methods are complicated, and it would be pointless to reproduce them here. The theory for the laboratory coil may be found in Prof. E. Taylor-Jones' book "The Theory of the Induction Coil", and the modified theory used in present-day Automobile practice in A.E.Watson's paper "Notes on the Ignition Coil". (See bibliography; C1, C2, & C3).

The following results emerge from the pure theory;-

At the moment of "Break" there is energy to the extent of $\frac{1}{2}L_1I^2$ units stored in the primary inductance, but there is no energy in the secondary inductance, or in the primary or secondary condensers. Immediately after "Break", all four variables - (current in primary, current in secondary, volts at primary condenser and volts at secondary condenser) - commence to oscillate with two fundamental frequencies of oscillation, which are functions of the natural frequencies of each circuit separately and the coupling between the circuits. That is to say, the energy which was originally stored in the primary inductance scatters itself throughout the whole network, and is passed back and forth between the various "sinks" of energy - inductive and capacitive - until it is finally completely dissipated by the resistive branches. In general, at any time after "Break", the original energy is distributed between the various storage points in a continually changing manner, and is not likely to occur at any one point in its entirety, unless it be deliberately arranged that this shall happen. Prof. Taylor-Jones has shown, in the book referred to above, that by suitably arranging the circuit constants - and particularly the degree of coupling - this may be accomplished; i.e. it can be arranged that a short time after "Break", the whole of the original energy, except for the small portion dissipated in resistance, appears at the secondary terminals. That is, it becomes stored electrostatically in the secondary condenser.

Then, neglecting losses, if e_2 is the "Peak" voltage attained by the secondary;-

$$\tfrac{1}{2} L_1 I_1^2 = \tfrac{1}{2} C_2 e_2^2 \qquad \text{--(24)}$$

$$\text{or} \qquad e_2 = I_1 \sqrt{L_1/C_2} \qquad \text{--(25)}$$

This gives the best possible conditions of secondary voltage for a given primary current at "Break".

The Commercial Coil.

The coil described above is not in general regarded with favour by the automobile manufacturer, for the following reason. The coil as described depends for its excellence on the circuit constants for which it was designed. Were a different value of primary condenser put in, or an extra piece of secondary cable, it is quite possible that so far from giving the best possible, the coil would give very poor conditions of secondary voltage. Since it does not follow that all makes of engine will require the same lengths of distributor cable, & c., this postulates that a different coil be designed for each and every make of engine, which is undesirable, for obvious reasons.

The type of coil preferred is one having close coupling between the primary and the secondary. In such a coil, the primary and secondary voltages must rise and fall together, so that it is quite impossible to have all the available energy at one instant in the secondary condenser. At best, it is shared between the primary and secondary condensers in approximately equal parts, so that the best possible secondary voltage is about $\frac{3}{4}$ of that in the theoretical perfect coil described above. i.e.-

$$\text{Secondary energy} = \tfrac{1}{2}(\text{available energy})$$

$$\text{Secondary voltage} = \sqrt{\tfrac{1}{2}}(\text{best possible voltage})$$

$$\doteq \tfrac{3}{4}(\text{best possible voltage})$$

Although $\frac{1}{4}$ of the available voltage is lost, the benefit is reaped in consistancy, for the addition of extra secondary cable, or slight alteration of the primary condenser, do not fundamentally alter the ratio;- (max. secy. volts.)/(primary current broken). Furthermore, it is very much simpler to manufacture a coil which has simply to be as tightly coupled as possible, than to manufacture one with a specific degree of coupling.

A.E.Watson develops and illustrates the design of a coil on these lines in the paper mentioned above. He shows that such a coil gives perfectly satisfactory performance for all normal working. It may however fail when applied to an extremely high-speed engine, on account of the inductive lag of the current during "Make". When the time of "Make" drops below about 4 millisecs., the current growth may be too slow, so that the energy stored in the primary is insufficient to produce the requisite 7·5 K.V. in the secondary.

With this limitation in view, the Author set out to try whether a modification in the design would increase the rate of growth of primary current without exceeding the permissible heating of the coil at slow speeds. It was found that such an improvement was not possible.

The results of this research were published in the article "Ignition Coils" in the Royal Technical College Journal for 1934, a copy of which forms part of this thesis.

D. Certain Peculiar Transients due to Non-linear Circuit Characteristics.

In working with transient phenomena, it is generally assumed that the characteristics of the circuit are linear - that is, saturation, hysteresis (both magnetic and dielectric) and skin effect are neglected. In the majority of cases, such approximation gives good results, and appears thus to be justified - it is practically impossible to give a rigid mathematical solution taking such unknown functions into account. There are some cases, however, where the nature of the characteristic itself gives rise to a marked transient, and we shall now proceed to consider one or two such cases.

1. The Surge of Current on Switching-in a Transformer.

When an inductive circuit without iron or of constant L is switched in to an A.C. supply, there is a transient which under the worst conditions causes a rise of current to almost twice normal value (Fig. 5). When an iron-clad circuit is considered, the same physical causes may produce much more than double the normal current.

Suppose the switching-in takes place at that point of the E.M.F. wave where the current and flux in the core under steady conditions would be a negative maximum, as shown in the diagram, Fig. 5. Then the flux immediately commences to change at a rate such that the E.M.F. induced by it exactly balances the applied voltage - and if resistance drop be neglected, will continue varying in this manner. Under steady conditions, half a period

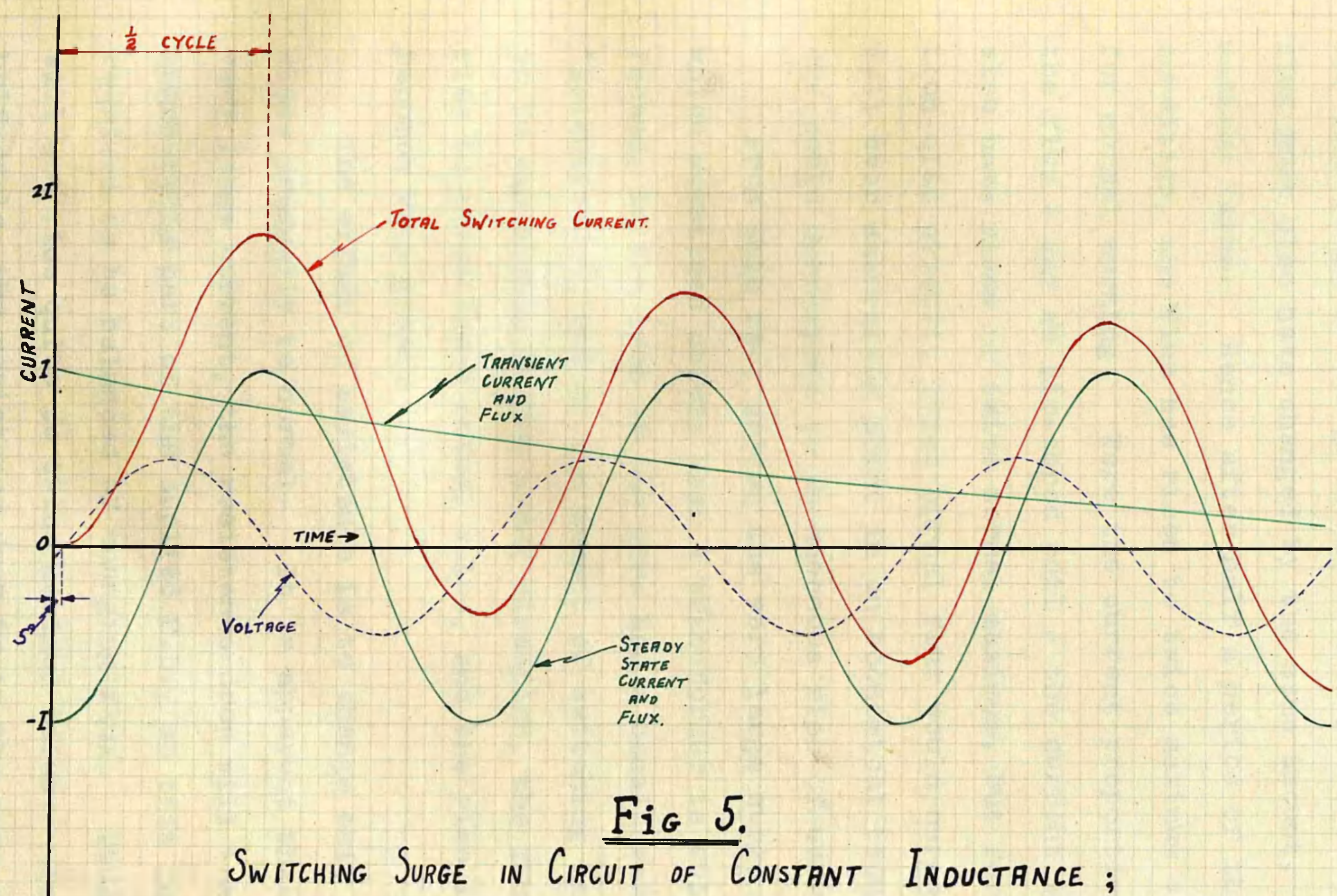

Fig 5.

Switching Surge in Circuit of Constant Inductance ;
Worst Conditions.

$\frac{R}{L} = 23{\cdot}5$; Applied Volts (at 50 ~) $= E \sin(\omega t - 5°)$; $\phi = \tan^{-1}\frac{\omega L}{R} = 85°$:

This gives worst conditions, for at $t = 0$, normal current would be $I \sin(0 - 5° - 85°) = I \sin(-90°)$.

i.e. normal current would be negative max.

Then current is $I\left\{\sin(\omega t - 90°) + \varepsilon^{-\frac{Rt}{L}}\right\}$

(For full development, see Appendix A, 2.)

later the flux would have changed from the negative max. to the positive max. , this change of twice maximum flux having been necessitated by the applied voltage. Now the only difference between the steady and switching conditions is that the flux and current start from zero under switching conditions, and from neg. max. in the steady state case; the E.M.F. has varied over precisely the same range in the given half-cycle, and hence the flux must also have changed by the same amount, i.e. twice maximum value. Hence after half a period of the transient condition, the flux has risen to twice maximum value for normal working. Were the current proportional to the flux - say an air-cored coil - the current would also have risen to twice normal maximum, but for an iron-clad circuit, twice normal flux would certainly be well over saturation point in an economical transformer, and would correspond to an enormous rise of current.

From this fact arises the very large surge of current which sometimes occurs when a transformer is switched in. Indeed, if the iron of the core has a considerable "Retentivity", and at the moment of switching happens to be magnetised in the wrong direction, the flux may rise to three times normal value, and the rise in current becomes very great.

The effect of resistance is to change this abnormal state gradually to normal. It is apparent that with a very large current, the resistance drop will take up an appreciable part of the applied P.D., so that less is required to be balanced by changing flux. Thus the flux will not rise quite to twice maximum value. In the second half-cycle, the resistance drop is in the same direction as before for the most part, but the applied voltage is changed in direction; thus the change of flux must be more than twice normal maximum. This process continues, the centre line of the flux wave gradually approaching the time axis until the flux is varying between symmetrical positive and negative values. This approach is obviously very much speeded up in the case of the iron-clad circuit, wherein it has been shown that the current maxima are very large. Thus the transformer switching surge is of much larger amplitude but much shorter duration than that in an air-cored circuit.

While the rigourous mathematical treatment of a circuit containing such an involved variable as a magnetisation curve is practically impossible, an approximate solution may be obtained by various methods.

a. Using approximate straight-line formulae for the saturation curve; hysteresis neglected.

The magnetisation curve is regarded as made up of straight lines as shown in the diagram, Fig. 6. It is here assumed that under steady conditions the upper and lower limits of flux density are 12,000 lines/sq.cm., so that under these conditions, only the line $B = 12{,}000 \times i$ affects the current and flux changes. This arrangement may be brought about by suitable adjustment of the applied voltage, as will be shown.

Under all conditions, the differential equation of the circuit is given by;-

$$Ri + T\frac{d\phi}{dt}10^{-8} = E\sin(\omega t + \theta) \qquad \text{--(26)}$$

where T = no. of turns of primary,
ϕ = total flux at instant t,
R = resistance of primary, and so on.

This may be written;-

$$Ri + T\frac{d\phi}{di}\cdot\frac{di}{dt}\cdot 10^{-8} = E\sin(\omega t + \theta) \qquad \text{--(27)}$$

If A is the total area of the coil, then ϕ, as a function of i, may be written;-

	$\phi = A \times 12{,}000\,i$	for first part of curve,
&	$\phi = A\{11{,}750 + 250i\}$	for second part of curve.
i.e.	$\frac{d\phi}{di} = 12{,}000\,A$	for 1st part,
and	$\frac{d\phi}{di} = 250\,A$	for 2nd part.

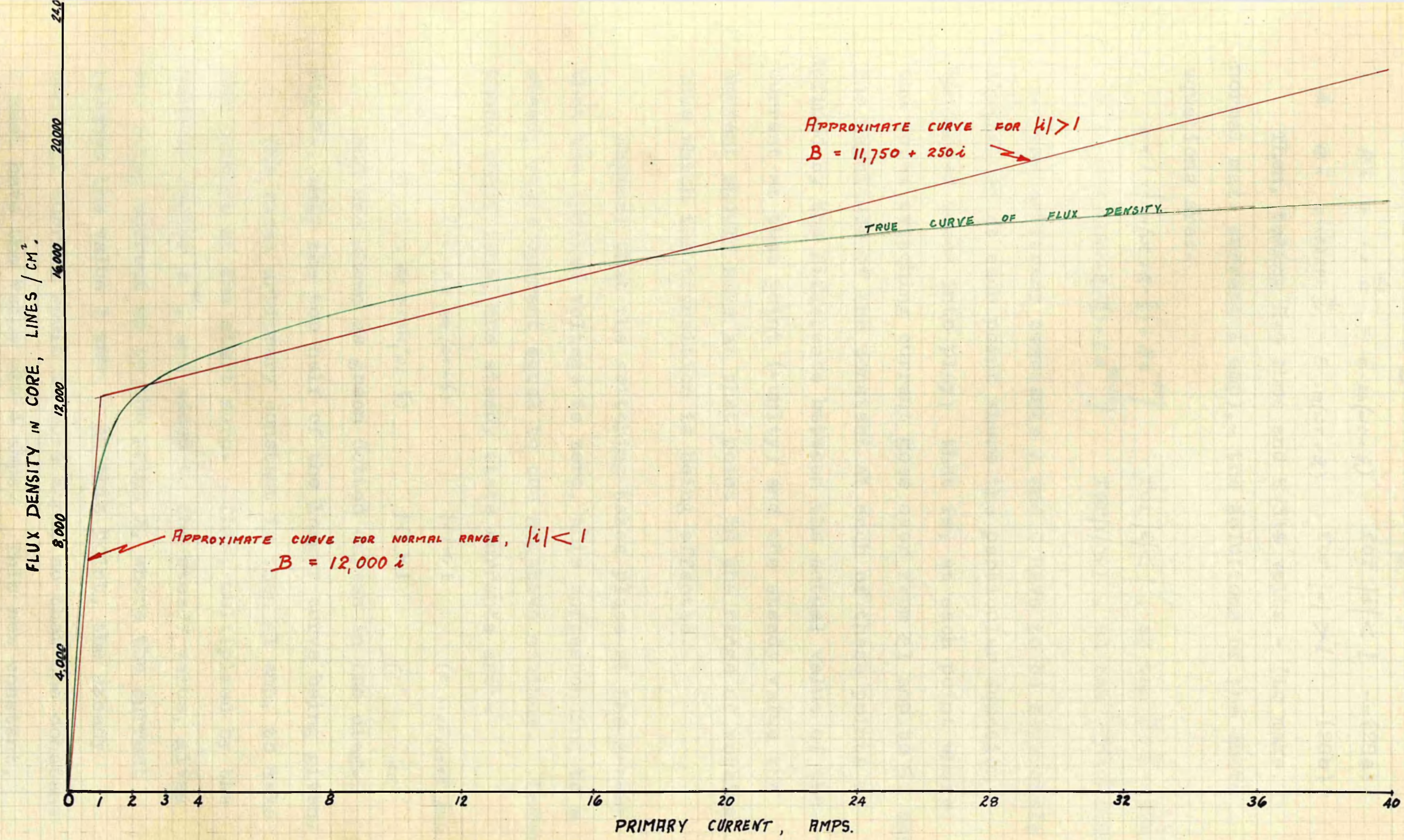
APPROXIMATE CURVE FOR |i|>1
B = 11,750 + 250i
TRUE CURVE OF FLUX DENSITY.
APPROXIMATE CURVE FOR NORMAL RANGE, |i|<1
B = 12,000 i
FLUX DENSITY IN CORE, LINES / CM².
PRIMARY CURRENT, AMPS.
4,000
8,000
12,000
16,000
20,000
24,0
0
1
2
3
4
8
12
16
20
24
28
32
36
40

Fig. 6.

Assuming now that $\underline{A}$=83·3 cm^2 and $\underline{T}$=100, we have:-

$$Ri + 1 \times \frac{di}{dt} = E\sin(\omega t+\theta_1) \quad \text{for } |i|<1 \quad \text{--(28a)}$$

$$\& \quad Ri + \cdot 0209 \times \frac{di}{dt} = E\sin(\omega t+\theta_2) \quad \text{for } |i|>1 \quad \text{--(28b)}$$

Then, taking R=4 ohms and E=314 volts - (to make normal max. current 1 amp),- the solutions of the above equations are:-

$$i = 1 \times \left\{ \sin(\omega t+\theta_1-\tfrac{\pi}{2}) + A\varepsilon^{-4t} \right\} \quad \text{for } |i|<1, \text{ as } \tan^{-1}\tfrac{314}{4} \doteq \tfrac{\pi}{2} \text{ --(29a)}$$

$$\& \quad i = 40{\cdot}8 \times \left\{ \sin(\omega t+\theta_2-\tfrac{\pi}{3}) + C\varepsilon^{-191\cdot5t} \right\} \quad \text{for } |i|>1, \text{ as } \tan^{-1}\tfrac{1\cdot0209\times314}{4} \doteq \tfrac{\pi}{3} \text{ (29b)}$$

The arbitrary constants A and C have to be separately determined at each point where the particular equation concerned comes into play; that is, at each point where the total switching current goes over from $\gtrless$1 amp to $\lessgtr$1 amp. The magnitude of the constant at each of these points is defined by the difference between the actual value of the current at that point (=unity) and the steady value the current should have at that point in the range of working into which the transition is being effected.

Suppose that the switching takes place at the moment that the applied voltage is zero, this corresponding to a steady state current equal to the negative maximum. Under these conditions, the steady state currents are:-

$$i = 1 \times \sin(\omega t - \tfrac{\pi}{2}) \qquad |i|<1 \qquad (t \text{ measured from start})$$

$$\& \quad i = 40{\cdot}8 \times \sin(\omega t - \tfrac{\pi}{3}) \qquad |i|>1 \qquad (\quad \text{do.} \quad)$$

and are shown as green dotted curves in the diag., Fig.7., only the top half of the larger curve being given.

The first arbitrary constant is then +1 amp., to make the current at the start zero. This, multiplied by the damping factor ε^{-4t}, and added to the steady value, gives the total current up to the point X, where the current reaches the value 1 amp. At this point, the steady abnormal current would be 20·4 amps, so that the constant C must have the value -19·4 amps. This new constant, multiplied by the damping factor $\varepsilon^{-191\cdot5\times t}$,- where $\underline{t}$ is measured from the point X - and added to the abnormal current

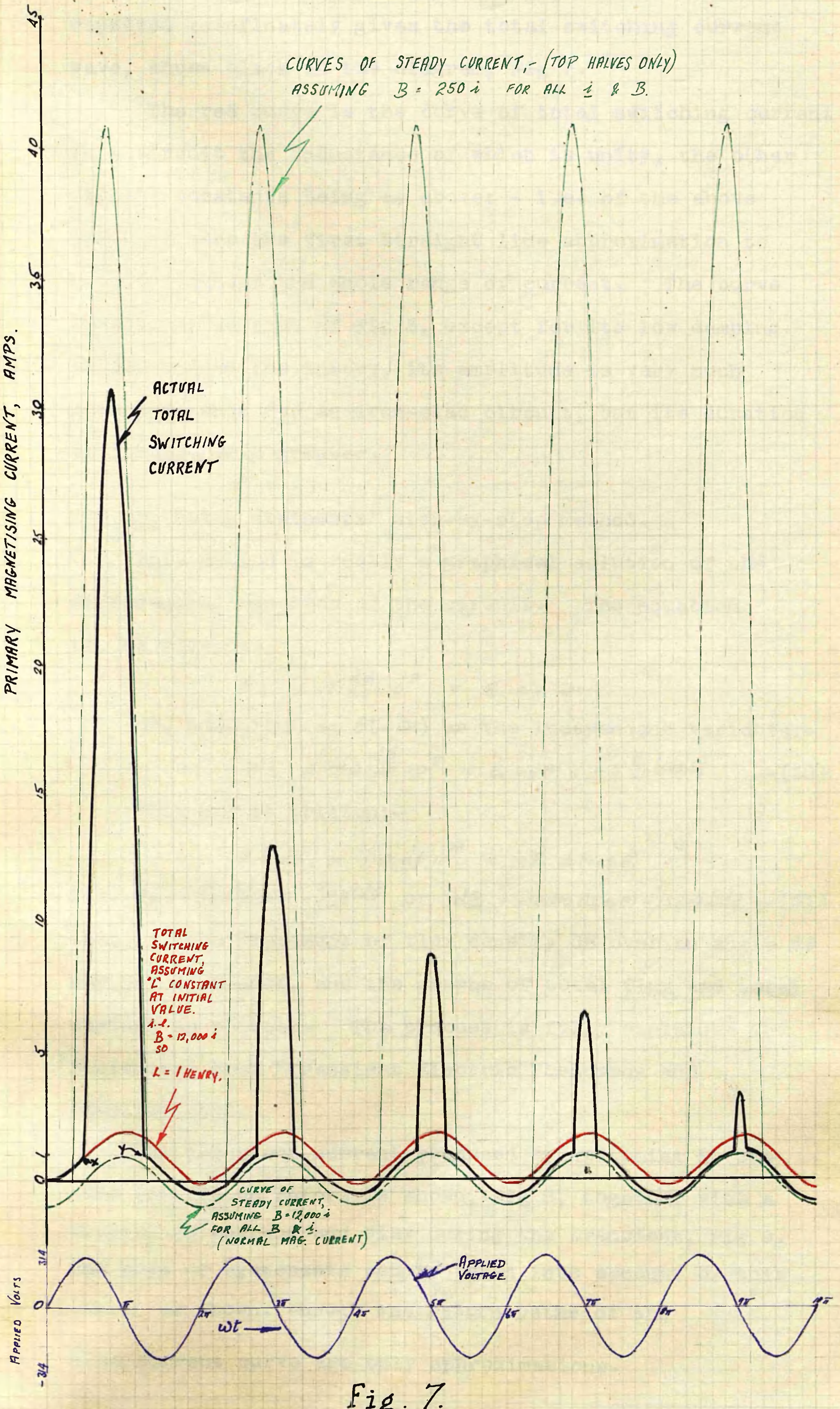

Fig. 7.

SWITCHING SURGE OF TRANSFORMER — STRAIGHT-LINE APPROXIMATION METHOD.

steady state value, gives the total current up to the point Y, where the current is again unity. This procedure repeated indefinately gives the total switching current wave, shown black in the diagram, Fig 7.

The red curve is the curve of total switching current in a circuit the inductance of which is unity, the other circuit constants being as above; - i.e. of the above circuit, were the first straight line approximation to hold throughout the whole range of current. The curve is similar to that of Fig 5, except for its low damping. As deduced in the theory, its amplitude is very much less than that for an iron-clad circuit, but its duration is considerably greater.

b. Using Steinmetz' step-by-step method.

This method is really a graphical solution of the differential equation of the circuit. The equation is, as before:-

$$Ri + T\frac{d\phi}{dt}10^{-8} = E \sin \omega t$$

or, substituting $\theta (= \omega t)$ as the independant variable:-

$$Ri + T\omega\frac{d\phi}{d\theta}10^{-8} = E \sin\theta = -E\frac{d}{d\theta}(\cos\theta) \qquad \text{--(30)}$$

This may be written:-

$$Ri\,d\theta + T\omega\,d\phi\,10^{-8} = -E\,d(\cos\theta)$$

i.e. $d\phi = -\frac{E}{T\omega}10^{8}d(\cos\theta) - \frac{Ri}{T\omega}10^{8}d\theta$ or $dB = -12{,}000\,d(\cos\theta) - 152{\cdot}9\,i\,d\theta$ --(31)

that is, the increment of flux density dB over an angle $d\theta$ may be calculated, and the graphs of total flux and total current so plotted. The process is fully explained in *Steinmetz' book "Transient Electric Phenomena and Oscillations".

The transient current produced by switching in the same circuit as before is shown, Fig 8, together with a diagram of current and flux during the transient, Fig 9. The loop of hysteresis can be taken into account by this method, as shown, though the return paths of the flux/current curve are only approximations.

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*Bibliography, C,8.

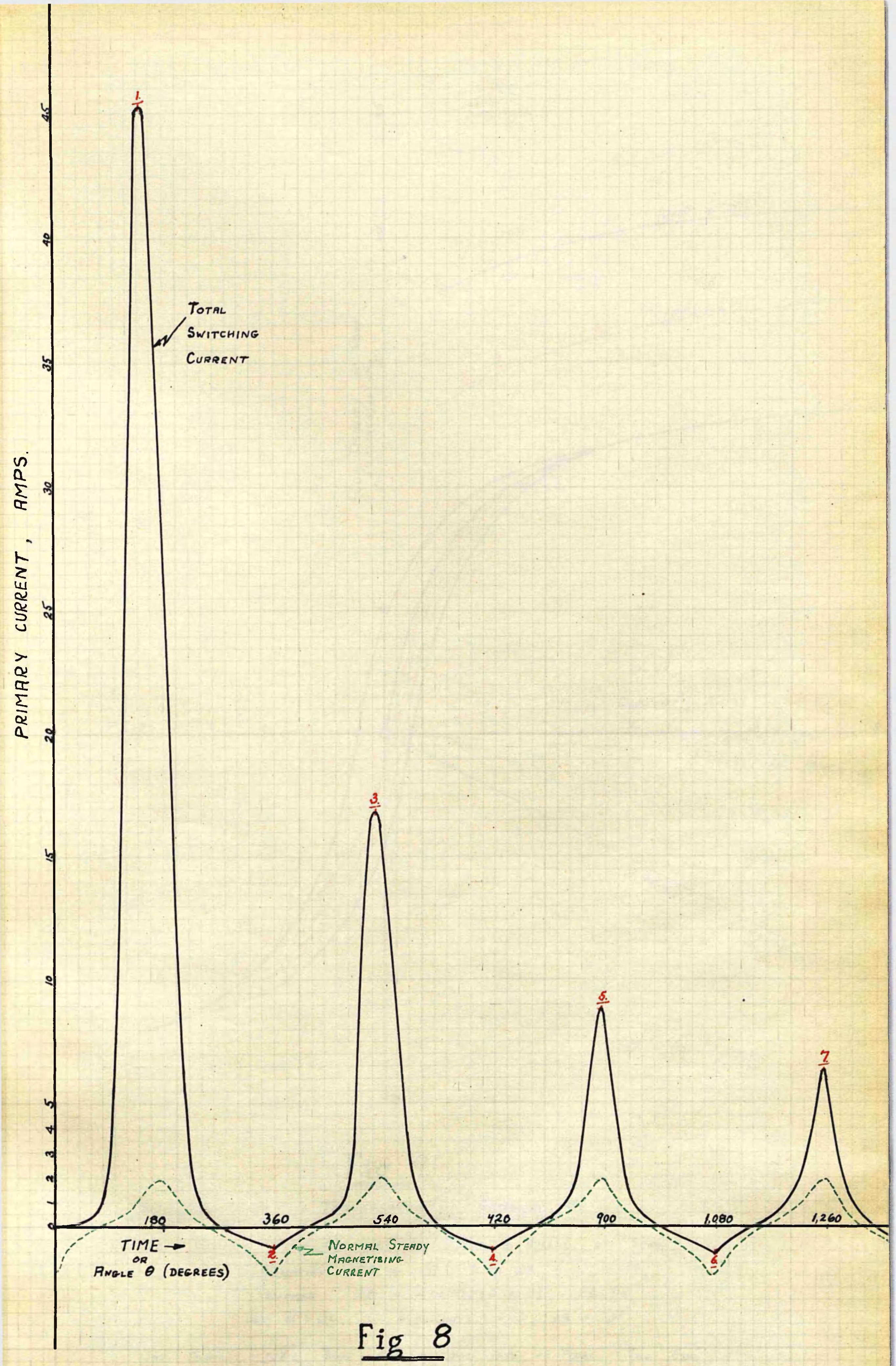


Fig 8

SWITCHING SURGE OF TRANSFORMER ; GRAPHICAL METHOD.

THE PEAK CURRENT POINTS, 1.2.3.&c., ARE ALSO SHOWN IN Fig 9.

2. The Transient due to the Presence of an Arc.

The special case of the ignition coil will be considered, the arc being that occuring at the moment of "Break" at the primary contact points.

The simple theory of the ignition coil assumes that at "Break", the whole current of the primary circuit is transferred from the contact point circuit to the local condenser circuit, as shown below, Fig 10, a & b:-

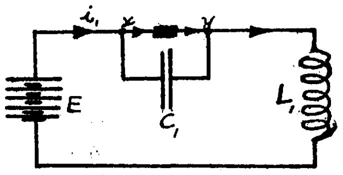


Fig 10 a.
Before break.

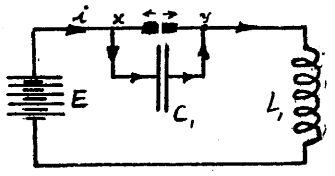


Fig 10 b.
After break.

The criterion for "Sparkless break" on this assumption would be that the contact points separated at a rate sufficient to ensure that the voltage from x to y built up by the current in charging the condenser was below the sparking voltage for the gap.

In point of fact, it is impossible to transfer the current instantaneously from the one circuit to the other, since both circuits contain inductance, however small. Further, a true spark cannot take place below a voltage of about 300 volts in air, so that if the condenser voltage stayed below 300, there should never be a spark on this theory, whereas in practice sparks may be obtained below this voltage, if the current is sufficiently large. Following up these main points and others, Dr N.J.Campbell showed conclusively that the so-called spark at the points was in reality a metallic arc, the characteristics of which depend more on the heating, material, and configuration of the points, and the current at "Break", than on the distance of separation of the points. (Philosophical Magazine, May, 1919.)

Reconsidering the phenomenom in the light of this knowledge, we see that the local circuit at "Break"

(consists of the)

parts as shown in the diagram, Fig 11, "A" being a branch having the voltage-current characteristics of the arc between the points - (which are opening out) - L being the inductance of the local circuit, and C the condenser capacitance. I is the current at "Break".

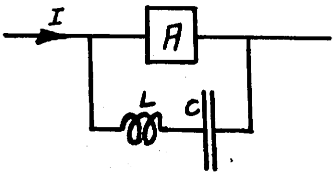


Fig 11.

Now the transient due to this arc is of very short duration. In addition, the voltage-current characteristic of an arc does not vary much with a small change of arcing distance; hence it may be assumed that for the duration of the transient, the characteristic of A remains constant at the value for a very short arc, so that the speed of opening of the points may be neglected. Such characteristics have been worked out by H.E.Ives, (Journal of the Franklin Institute, Oct 1924) by a method involving the plotting of the curve on a series of its tangents, the curve being called by him the "Minimal Arc Characteristic", the characteristic for an indefinitely short arc.

As the transient time is very short, and the primary circuit contains very high inductance, the total current for the duration of the phenomena may be taken as constant. The equations for the network then become:-

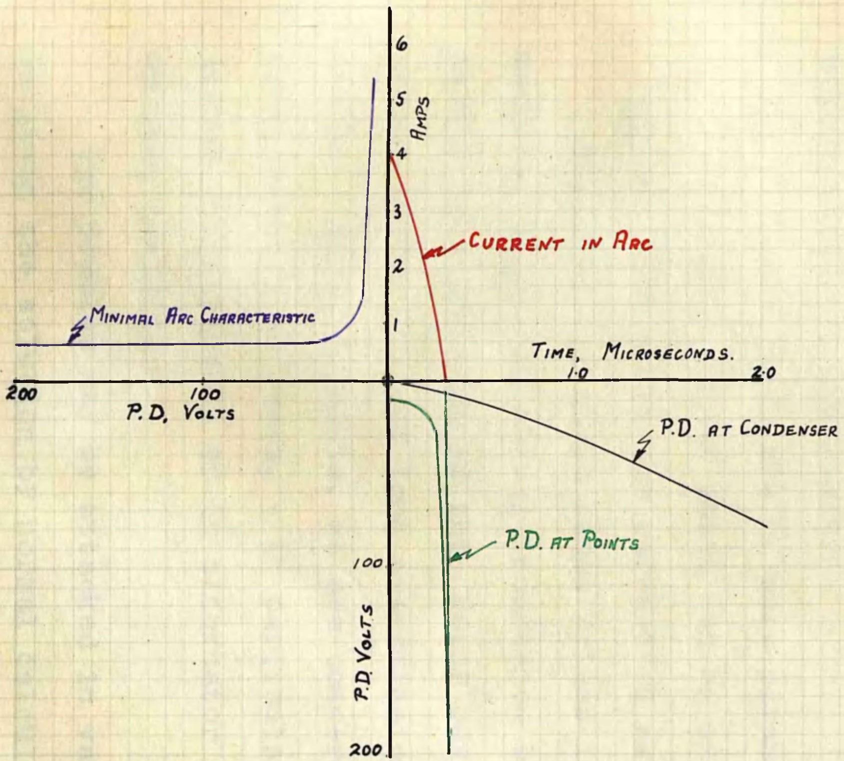
Condenser circuit,- $$V = L\frac{d}{dt}\{I - i\} + \frac{1}{C}\int\{I-i\}\,dt \qquad --(32)$$

Arc circuit,- $$V = f(i) \qquad --(33)$$

where V is the voltage across the arc,
i is the current in the arc,
& $V = f(i)$ is the "Minimal Arc Characteristic".

Since equation (33) is not of mathematical form, the equations cannot be solved by normal methods. They may, however, be solved by a step-by-step process similar to that employed in the case of the transformer switching surge.

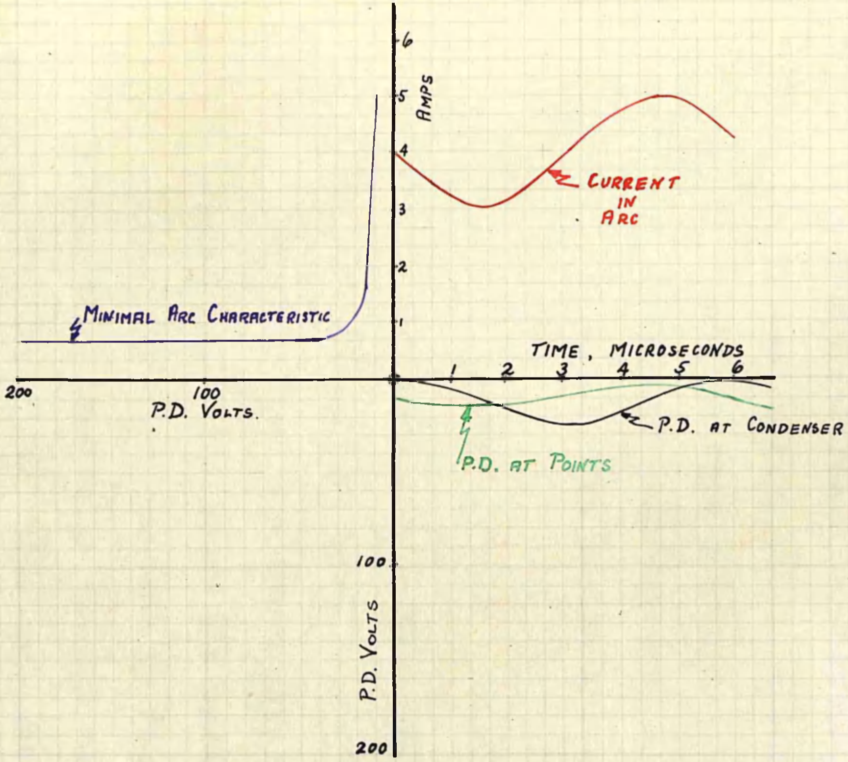
A.E.Watson, in his paper "The Effect of Hydrocarbon Vapour on the Contact Points of Ignition Apparatus" -



Calculated Break Curve

$C = 10^{-1}$ mFd., $L = 10^{-6}$ H., $I_o = 4$ Amps.

Platinum Contacts



Calculated Break Curve

$C = 10^{-1}$ m.Fd., $L = 10^{-5}$ H., $I_o = 4$ Amps.

Platinum Contacts.

Fig 12.

[Reproduced from the paper "The Effect of Hydrocarbon Vapour on the Contact-points of Ignition Apparatus," contributed by A.E. Watson to the Institution of Automobile Engineers].

contributed to the Institution of Automobile Engineers, elaborates the method, and develops curves showing the effect of the inductance L of the local circuit. He shows there exists a critical value of this inductance, such that - at values higher than this - the current in the arc is sustained for a considerable time (until the separation of the contact points appreciably alters the arc characteristic), this corresponding to a loss of energy, and a loss of peak voltage, as well as wear and tear on the points. His curves are reproduced herewith, Fig 12.

E. The Transients Introduced by Switching, in Circuits Containing Rotating Machinery.

So far have been considered only those transients which arise in circuits where the parameters - inductance, capacitance & c. - are constant in value, or else vary in a manner which cannot be exactly represented by any known mathematical law. The constants have moreover been regarded as "Lumped". There are two other very important cases, however:-

1. The case where one or more parameters vary with time in a manner which can be exactly represented.
2. The case where the parameters are distributed in space.

The first of these cases arises in the rotating machine, where the mutual inductance between rotor and stator is a function of the rotor position; the second arises in the case of long transmission or telegraph wires. In this Thesis, we are concerned only with the first of the two cases, - in particular, the case of the sudden short-circuit of an alternator previously running on open circuit.

The problem may be attacked in various ways. Below is given the solution by normal calculus methods.

1. The Direct Mathematical Method.

Shimidzu and Ito have shown that by making certain feasible assumptions, the case is amenable to direct mathematics. (See bibliography; C,10). It is assumed that the flux in the machine is sinusoidal - permissible in modern alternators - so that the variation of Mutual Inductance with time is a sine wave. Further, saturation of the iron is neglected; as the flux does not rise above normal max. value, this is permissible. The normal angular speed of the machine is assumed to remain constant during the transient; a reasonable premise, since the mechanical vibration of such a mass as the rotor is of very long period compared with the electric transient.

It will be assumed that the normal field current I_f acts continually, and that on it the transient field current is superposed. There is therefore an alternating voltage of maximum value $cI_f\omega$ continually impressed on all three stator phases, where c is the maximum value of the mutual inductance between rotor and stator. If $cI\omega = d$, then the voltages impressed in phases 1,2,and 3, are $d.\sin(\omega t)$, $d.\sin(\omega t-\frac{2\pi}{3})$, and $d.\sin(\omega t-\frac{4\pi}{3})$ respectively. The initial displacement of the rotor at the moment of short-circuit is taken into account later.

If x,y, & z are the currents in the three stator phases,

a & b′ the resistance and inductance of a stator phase,

w the transient current in the rotor,

a_1 & b_1 the resistance and inductance of the rotor,

and c′ the mutual inductance between any two of the three stator phases,

then the differential equations for the four coupled circuits are:-

$$\text{Phase 1} \quad ax + b'\frac{dx}{dt} + c\frac{d}{dt}\{w.\cos(\omega t)\} + c'\frac{d}{dt}\{y+z\} = d.\sin\omega t \quad \text{--(34a)}$$

$$\text{Phase 2} \quad ay + b'\frac{dy}{dt} + c\frac{d}{dt}\{w.\cos(\omega t-\tfrac{2\pi}{3})\} + c'\frac{d}{dt}\{z+x\} = d.\sin(\omega t-\tfrac{2\pi}{3}) \quad \text{--(34b)}$$

$$\text{Phase 3} \quad az + b'\frac{dz}{dt} + c\frac{d}{dt}\{w.\cos(\omega t-\tfrac{4\pi}{3})\} + c'\frac{d}{dt}\{x+y\} = d.\sin(\omega t-\tfrac{4\pi}{3}) \quad \text{--(34c)}$$

$$\text{Rotor.} \quad a_1 w + b_1\frac{dw}{dt} + c\frac{d}{dt}\{x.\cos\omega t + y.\cos(\omega t-\tfrac{2\pi}{3}) + z.\cos(\omega t-\tfrac{4\pi}{3})\} = 0 \quad \text{--(35)}$$

since the mutual inductance between rotor and stator is a sinusoidal function of rotor position.

By making the substitutions:-

$$x + y + z = 0$$

$$b' - c' = b$$

and $\quad x = \dfrac{d\xi}{dt}, \quad y = \dfrac{d\eta}{dt}, \quad z = \dfrac{d\zeta}{dt}, \quad$ and $\quad w = \dfrac{d\chi}{dt},$

the equations may be rewritten as below:-

$$a\xi + b\frac{d\xi}{dt} + c\frac{d\chi}{dt}\cdot\cos\omega t = -\frac{d}{\omega}\cos\omega t$$

$$a\eta + b\frac{d\eta}{dt} + c\frac{d\chi}{dt}\cdot\cos\left(\omega t - \frac{2\pi}{3}\right) = -\frac{d}{\omega}\cos\left(\omega t - \frac{2\pi}{3}\right)$$

$$a\zeta + b\frac{d\zeta}{dt} + c\frac{d\chi}{dt}\cdot\cos\left(\omega t - \frac{4\pi}{3}\right) = -\frac{d}{\omega}\cos\left(\omega t - \frac{4\pi}{3}\right)$$

and $$a_1\chi + b_1\frac{d\chi}{dt} + c\left\{\frac{d\xi}{dt}\cdot\cos\omega t + \frac{d\eta}{dt}\cdot\cos\left(\omega t - \frac{2\pi}{3}\right) + \frac{d\zeta}{dt}\cdot\cos\left(\omega t - \frac{4\pi}{3}\right)\right\} = 0$$

These may easily be simplified, giving the following equation for w, the field current:-

$$b\left(bb_1 - \tfrac{3}{2}c^2\right)\frac{d^3w}{dt^3} + \left\{a\left(bb_1 - \tfrac{3}{2}c^2\right) + b(ab_1 + a_1b)\right\}\frac{d^2w}{dt^2} + \left\{b\omega^2\left(bb_1 - \tfrac{3}{2}c^2\right) + 2aa_1b + a^2b_1\right\}\frac{dw}{dt} + a_1\left(a^2 + b^2\omega^2\right)w = 0 \quad \text{-- (36)}$$

This is a linear differential equation of the 3rd order, which is therefore amenable to straightforward mathematics. The expression for the rotor current may be substituted in the equations for the stator currents. The resulting expressions all contain three arbitrary constants, which may be evaluated from the conditions existing immediately prior to the short circuit.

Shimidzu and Ito show that the solutions for all four variables may be written:-

$$x = -\frac{d}{b\omega}\left\{\cos\omega t + \frac{1-\sigma'}{\sigma'}\varepsilon^{\alpha(t-t_0)}\cos\omega t - \frac{1-\sigma'}{2\sigma'}\varepsilon^{\mu(t-t_0)}\cos(2\omega t - \omega t_0) - \frac{1+\sigma'}{2\sigma'}\varepsilon^{\mu(t-t_0)}\cos\omega t_0\right\} \quad \text{-- (37a)}$$

$$y = -\frac{d}{b\omega}\left\{\cos\left(\omega t - \frac{2\pi}{3}\right) + \frac{1-\sigma'}{\sigma'}\varepsilon^{\alpha(t-t_0)}\cos\left(\omega t - \frac{2\pi}{3}\right) - \frac{1-\sigma'}{2\sigma'}\varepsilon^{\mu(t-t_0)}\cos\left(2\omega t - \frac{2\pi}{3} - \omega t_0\right) - \frac{1+\sigma'}{2\sigma'}\varepsilon^{\mu(t-t_0)}\cos\left(\omega t_0 - \frac{2\pi}{3}\right)\right\} \quad \text{-- (37b)}$$

$$z = -\frac{d}{b\omega}\left\{\cos\left(\omega t - \frac{4\pi}{3}\right) + \frac{1-\sigma'}{\sigma'}\varepsilon^{\alpha(t-t_0)}\cos\left(\omega t - \frac{4\pi}{3}\right) - \frac{1-\sigma'}{2\sigma'}\varepsilon^{\mu(t-t_0)}\cos\left(2\omega t - \frac{4\pi}{3} - \omega t_0\right) - \frac{1+\sigma'}{2\sigma'}\varepsilon^{\mu(t-t_0)}\cos\left(\omega t_0 - \frac{4\pi}{3}\right)\right\} \quad \text{-- (37c)}$$

and $$w = \frac{1-\sigma'}{\sigma'}I_f\left\{\varepsilon^{\alpha(t-t_0)} - \varepsilon^{\mu(t-t_0)}\cos(\omega t - \omega t_0)\right. \quad \text{-- (38)}$$

provided that $\frac{a_1}{b_1\omega}$ and $\frac{a}{b\omega}$ are vanishingly small, which holds for short-circuit conditions.

In these equations,

$$\alpha \doteqdot -\frac{a_1}{b_1\omega}\cdot\frac{1}{\sigma'}\cdot\omega$$

$$\mu \doteqdot -\tfrac{1}{2}(1+\sigma')\cdot\frac{a}{b\omega}\cdot\frac{1}{\sigma'}\cdot\omega$$

and $$\sigma' = 1 - \tfrac{3}{2}\cdot\frac{c^2}{bb_1}$$

and the short-circuit is regarded as taking place at the instant $\underline{t} = \underline{t}_0$.

The Term σ' $\left(= 1 - \frac{3}{2}\cdot\frac{c^2}{bb_1}\right)$.

The main flux produced by a single stator phase may be represented as in the diagram, Fig 13. That is, the flux density at a point "θ" is given by:-

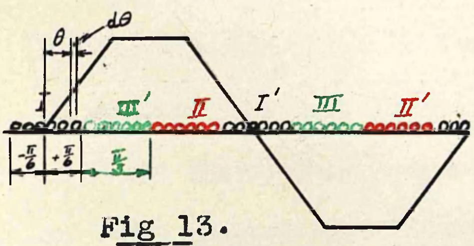


Fig 13.

$$B = B_1 \sin\theta + B_3 \sin 3\theta + B_5 \sin 5\theta + \&c.$$

Hence the total flux embraced by a single turn of the phase at θ is proportional to $\int_\theta^{\theta+\pi}\left\{B_1\sin\theta + B_3\sin 3\theta + B_5 \sin 5\theta + -\right\}d\theta$

i.e. propl. to $2\left\{B_1\cos\theta + \frac{1}{3}B_3\cos 3\theta + \frac{1}{5}\cdot B_5\cos 5\theta + -\right\}$

The number of turns in a length $d\theta$ is propl. to $d\theta$.

Hence the inductance of a phase is propl. to:-

$$2\int_{-\frac{\pi}{6}}^{+\frac{\pi}{6}}\left\{B_1\cos\theta + \frac{1}{3}B_3\cos 3\theta + \frac{1}{5}B_5\cos 5\theta + \&c\right\}d\theta$$

i.e., propl. to $2\times 2\left\{B_1\sin\frac{\pi}{6} + \frac{1}{9}B_3\sin 3\cdot\frac{\pi}{6} + \frac{1}{25}\cdot B_5\sin 5\cdot\frac{\pi}{6} + - \&c\right\}$

The mutual inductance due to phase 3 is propl. to:-

$$-2\times\int_{\frac{\pi}{6}}^{\frac{\pi}{3}+\frac{\pi}{6}}\left\{B_1\cos\theta + \frac{1}{3}B_3\cos 3\theta + \frac{1}{5}B_5\cos 5\theta + \&c\right\}d\theta$$

i.e., propl. to $-2\left\{B_1\left[\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right] + \frac{1}{9}B_3\left[\sin 3\left(\frac{\pi}{3}+\frac{\pi}{6}\right) - \sin 3\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right] + \&c\right\}$

i.e., propl. to $-2\left\{2B_1\cos\frac{\pi}{3}\sin\frac{\pi}{6} + 2\frac{1}{9}B_3\cos 3\cdot\frac{\pi}{3}\sin 3\cdot\frac{\pi}{6} + 2\cdot\frac{1}{25}\cdot B_5\cos 5\cdot\frac{\pi}{3}\sin 5\cdot\frac{\pi}{6} + \&c\right\}$

i.e., propl. to $-4\left\{B_1\sin\frac{\pi}{6}\cos\frac{\pi}{3} + \frac{1}{9}B_3\sin\frac{\pi}{2}\cos\pi + \frac{1}{25}B_5\cdot\sin\frac{5\pi}{6}\cos\frac{5\pi}{3} + \&c\right\}$

If the higher harmonics are neglected, these expressions become:-

$4B_1\sin\frac{\pi}{6}$ for the self-inductance,

and $-2B_1\sin\frac{\pi}{6}$ for the mutual inductance.

Actually, to take account of overhang leakage, the self-inductance should be raised a little - say by the factor ν.

Then $b' \propto 4B_1\nu\sin\frac{\pi}{6}$

and $c' \propto -2B_1\sin\frac{\pi}{6}$

$$\therefore\quad b'-c' = b = 4B_1\nu\sin\frac{\pi}{6}\left\{1+\frac{1}{2\nu}\right\} = b'\left\{1+\frac{1}{2\nu}\right\} \doteqdot \frac{3}{2}b'$$

Then $\frac{3}{2}\cdot\frac{c^2}{bb_1} \doteqdot \frac{c^2}{b'b_1} \doteqdot \frac{1}{\nu_f}$, where ν_f is the field leakage coefficient.

$$i.e.\quad \sigma' \doteqdot 1 - \frac{1}{\nu_f}.$$

The above results will be referred to later, when they will be compared with the results obtained by Rüdenberg's method, and by a method involving Heaviside's Operational Calculus.

2. Rüdenberg's Method.

In the 2nd edition of "Electrische Schaltvorgänge", *Rüdenberg develops a very ingenious and simple method of attack for the problem of alternator short-circuit.

He considers a generator with polyphase windings on both stator and rotor, and replaces the actual phase currents by sinusoidally distributed current systems.

These current systems move across the rotor and stator phases and so vanish. If the absolute angular velocity of the stator system is α, and the angular velocity of the rotor is ω, then the velocity of the stator system relative to the rotor is $\beta = \alpha - \omega$, which must be the angular velocity of the rotor system relative to the rotor, in order that the points of maximum and minimum current in stator and rotor may remain opposite one another.

For a sinusoidal distribution of current round the periphery, the stator and rotor currents at a particular point ϕ electrical radians from the point of zero current may be represented by the expressions $I_1 \varepsilon^{j\phi}$ and $I_2 \varepsilon^{j\phi}$. The point of zero current travels in common with the rest of the wave at speed α for the stator, and β for the rotor -(relative to the rotor). Hence the current at a certain point of the stator wave at a time t may be represented by the expression $I_1 \varepsilon^{i(\alpha t + \phi)}$ and the current at the corresponding point of the rotor wave by $I_2 \varepsilon^{i(\beta t + \phi)}$ relative to the rotor, or $I_2 \varepsilon^{i(\omega t + \beta t + \phi)}$ relative to the stator.

Hence E.M.F. induced in stator due to rotor system is given by $-M \frac{d}{dt}\left\{I_2 \varepsilon^{i(\omega t + \beta t + \phi)}\right\} = -MI_2 j(\omega + \beta)\varepsilon^{i(\omega t + \beta t + \phi)} = -MI_2 j\alpha \varepsilon^{j(\alpha t + \phi)}$.

and E.M.F. induced in rotor due to stator current system is given by $-M \frac{d}{dt}\left\{I_1 \varepsilon^{i[(\alpha - \omega)t + \phi]}\right\} = -MI_1 j(\alpha - \omega)\varepsilon^{i[(\alpha - \omega)t + \phi]} = -MI_1 j\beta \varepsilon^{i(\beta t + \phi)}$.

if M is the coupling between stator and rotor.

Further, if R_1 and L_1 are stator resistance and inductance, and R_2 and L_2 are rotor resistance and inductance, then total voltage induced in stator due to both systems is:-

*Bibliography, C,7.

$$R_1 I_1 \varepsilon^{j(\alpha t+\phi)} + L_1 j\alpha I_1 \varepsilon^{j(\alpha t+\phi)} + M j\alpha I_2 \varepsilon^{j(\alpha t+\phi)} = 0 \qquad \text{--(39a)}$$

and in rotor is:-

$$R_2 I_2 \varepsilon^{j(\beta t+\phi)} + L_2 j\beta I_2 \varepsilon^{j(\beta t+\phi)} + M j\beta I_1 \varepsilon^{j(\beta t+\phi)} = 0 \qquad \text{--(39b)}$$

whence:- $\quad j\alpha L_1 I_1 + R_1 I_1 + j\alpha M I_2 = 0$

and $\quad j\beta L_2 I_2 + R_2 I_2 + j\beta M I_1 = 0$

i.e., $$\frac{I_1}{I_2} = -\frac{j\alpha M}{j\alpha L_1 + R_1} = -\frac{j\beta L_2 + R_2}{j\beta M} \qquad \text{--(40)}$$

From these, by using the identity $\omega + \beta = \alpha$, both α and β may be evaluated.

If $\quad 1 - \frac{M^2}{L_1 L_2} = \sigma$

$\frac{R_1}{L_1 \sigma} = \rho'$

and $\quad \frac{R_2}{L_2 \sigma} = \rho''$

then α is given by:-

$$\alpha^2 - \alpha\left[\omega - j(\rho' + \rho'')\right] = \sigma\rho'\rho'' - j\omega\rho' \qquad \text{--(41)}$$

and β is given by the substitution of the two roots of equation (41) in the identity $\alpha - \omega = \beta$.

It is seen that α and β have each two values, which are complex, and hence may be written:-

$$\alpha_1 = \nu_1 + j\rho_1 \qquad \alpha_2 = \nu_2 + j\rho_2 \qquad \text{--(42a)}$$

$$\beta_1 = (\nu_1 - \omega) + j\rho_1 \qquad \beta_2 = (\nu_2 - \omega) + j\rho_2 \qquad \text{--(42b)}$$

It is apparent that the transient current is a compound wave in both cases, consisting of two waves of different frequency and different damping.

The solutions are of the form:-

Stator current $= i_1 = K_1 \varepsilon^{-\rho_1 t} \varepsilon^{j\nu_1 t} + K_2 \varepsilon^{-\rho_2 t} \varepsilon^{j\nu_2 t}$ --(43a)

Rotor current $= i_2 = K_3 \varepsilon^{-\rho_1 t} \varepsilon^{j(\nu_1 - \omega)t} + K_4 \varepsilon^{-\rho_2 t} \varepsilon^{j(\nu_2 - \omega)t}$ --(43b)

By expanding the equation for α, it may be shown that one root - say α_2 - is large, and the other root, α_1, is small. Actually, ν_1 and ν_2 sum to ω, ν_2 being slightly less than ω and ν_1 about zero. Thus $(\nu_2 - \omega)$ is small and negative, while $(\nu_1 - \omega)$ is large and negative.

It is apparent that the K_1 term of $\underline{i}_1$ and the K_3 term of $\underline{i}_2$ correspond, since they have the same frequency in space, - which also holds for K_2 and K_4. Thus the first terms and the second terms of equations (43) may be substituted in equation (40), giving:-

$$\frac{I_1'}{I_2'} = \frac{K_1}{K_3} = -\frac{j\beta_1 L_2 + R_2}{j\beta_1 M} \doteq -\frac{L_2}{M}$$

as $\beta_1 L_2 (\beta_1 = \alpha_1 - \omega)$ is much bigger than R_2,

and $$\frac{I_1''}{I_2''} = \frac{K_2}{K_4} = -\frac{j\alpha_2 M}{j\alpha_2 L_1 + R_1} \doteq -\frac{M}{L_1}$$

as $\alpha_2 L_1$ is much bigger than R_1.

Thus the equations become:-

$$i_1 = K_1 \varepsilon^{-\rho_1 t} \varepsilon^{j\vartheta_1 t} + K_2 \varepsilon^{-\rho_2 t} \varepsilon^{j\vartheta_2 t} \quad \text{--(44a)}$$

$$\& \quad i_2 = -\left\{ \frac{M}{L_2} \cdot K_1 \cdot \varepsilon^{-\rho_1 t} \varepsilon^{j(\vartheta_1 - \omega)t} + \frac{L_1}{M} \cdot K_2 \cdot \varepsilon^{-\rho_2 t} \varepsilon^{j(\vartheta_2 - \omega)t} \right\} \quad \text{--(44b)}$$

These equations refer to a sinusoidally distributed current system, (a) in the stator, and (b) in the rotor. Any fixed point on either stator or rotor will experience a variation of current which is a damped sine wave as the system passes over it. For the maximum transients, and worst conditions, we must consider that part of the periphery where the transient is a maximum at the instant of switching, i.e., the peak of the current system. Thus we substitute cosines for the rotating vectors of these equations. - (44).

The solutions then become:-

$$i_1 = K_1 \varepsilon^{-\rho_1 t} \cos \vartheta_1 t + K_2 \varepsilon^{-\rho_2 t} \cos \vartheta_2 t \quad \text{--(45a)}$$

$$\& \quad i_2 = -\left\{ \frac{M}{L_2} \cdot K_1 \cdot \varepsilon^{-\rho_1 t} \cos(\vartheta_1 - \omega)t + \frac{L_1}{M} \cdot K_2 \cdot \varepsilon^{-\rho_2 t} \cos(\vartheta_2 - \omega)t \right\} \quad \text{--(45b)}$$

for the worst conditions.

Application of foregoing to case of alternator short-circuit.

These equations may be applied to an alternator provided the rotor may truly be regarded as polyphase; such cases as that of a machine with a cylindrical solid rotor, for instance, or even that of a machine with salient poles and a continuous low-resistance damper winding may be treated by them. A non-polyphase field winding produces stationary fluctuations of flux and current, - (relative to the rotor) - and rather alters the problem.

If such a machine is running on open circuit, and is suddenly short-circuited, then the transients follow the given equations. If I_K is the steady short-circuit current:-

$$i_{10} = K_1 + K_2 = -I_K$$

where i_{10} = transient current at switching, and

$$i_{20} = -\frac{M}{L_2}K_1 - \frac{L_1}{M}K_2 = 0$$

as I_μ = field current before and after transient.

Hence $K_1 = -\frac{I_K}{\sigma}$

and $K_2 = \frac{1-\sigma}{\sigma} I_K$

i.e., total short-circuit current is given by:-

$$i_1 = I_K \left\{ \cos \omega t - \frac{1}{\sigma}\left[\varepsilon^{-\rho_1 t} \cos \nu_1 t - (1-\sigma)\, \varepsilon^{-\rho_2 t} \cos \nu_2 t\right]\right\} \qquad \text{--(46a)}$$

and total field current by:-

$$i_2 = I_\mu + \frac{I_K}{\sigma} \cdot \left\{ \frac{M}{L_2} \cdot \varepsilon^{-\rho_1 t} \cos(\nu_1 - \omega)t - (1-\sigma) \cdot \frac{L_1}{M} \varepsilon^{-\rho_2 t} \cos(\nu_2 - \omega)t \right\}$$

which may be reduced to:-

$$i_2 = I_\mu \left\{ 1 - \frac{1-\sigma}{\sigma}\left[\varepsilon^{-\rho_1 t} \cos(\nu_1 - \omega)t - \varepsilon^{-\rho_2 t} \cos(\nu_2 - \omega)t\right]\right\} \qquad \text{--(46b)}$$

as $I_K \doteq \frac{M}{L_1} \cdot I_\mu$ for negligible stator resistance.

The above results, and also Shimidzu's and Ito's, will later be compared with those obtained by the Author using Heaviside Operational Calculus.

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"A Study of Transient Phenomena in Electro-Magnetic Machinery, with Particular Reference to the Use of the Heaviside Operational Calculus."

P A R T II

"The Heaviside Operational Calculus."

As was explained at the beginning of Part I, there corresponds to every electric network a differential equation, which by its complete solution, gives both the steady and transient current and voltage, for all times after switching, and in the case of a distributed network, for all points in the network.

The complete solution of the equation is however not always easy to obtain, particularly in the cases involving distributed constants, and in those involving a considerable number of coupled circuits. It was shown in Part I,B, that even with but two circuits, involving only two arbitrary constants, the form of the constants was cumbersome. For a large number of circuits, with a corresponding number of arbitrary constants, the difficulties of solution may be well-nigh insurmountable. These arbitrary constants, which have to be evaluated from the "Boundary Conditions", form the stumbling block, and it is in the establishment of them by fairly simple algebraic methods that the Heaviside Calculus is most important. By this Calculus, the two steps of solution and substitution of boundary conditions are condensed into one, and are made to involve only algebraic processes and simple differentiation.

Before applying this method, its fundamentals will be considered.

A. The Fundamentals of the Method.

1. The Unit Function, 1.

In order to bring in the boundary conditions in the fundamental equation, it is necessary to have a notation for discontinuous functions. (e.g., on switching in a simple D.C. circuit, the voltage is zero from time $-\infty$ to time 0, rises to the value E at time 0, and remains at value E from time 0 to time $+\infty$. It is therefore discontinuous at time $t = 0$.)

A convenient function of this type is that which is zero for $-\infty < t < 0$, and unity for $0 < t < +\infty$. This function, designated by the symbol 1, is discontinuous at $t = 0$, and has the useful property, that all other functions which are discontinuous at $t = 0$ may be expressed in terms of it.

2. The Operator "p".

In order to avoid the use of calculus as far as possible, and make the method algebraic, the operation of differentiating with respect to t is denoted by the operator "p". i.e., dx/dt is written "px".

Thus the fundamental equation for a simple D.C. circuit involving resistance and inductance, suddenly switched to a voltage E, is:-

$$Ri + L\frac{di}{dt} = E \qquad \text{where } i = 0 \text{ at } t = 0 \qquad \text{--(1)}$$

in the formal mathematical method, but becomes:-

$$Ri + Lpi = E\,1 \qquad \text{--(2)}$$

in the Operational method, the initial conditions being included in the equation.

Equation (2) may be written:-

$$i = \frac{E}{R + pL} \cdot 1 \qquad \text{--(3)}$$

which should give as solution the complete value of i, including steady state and transient, without any later substitution of initial conditions, provided the rules governing such peculiar equations can be established.

3. The Solution of Operational Equations.

The actual solving of equations of this type may be accomplished in various ways. The two chief ways are the method of "Algebrizing", as Heaviside calls it, and the "Expansion Theorem".

a. Method of "Algebrizing".

Heaviside shows that we may write:-

$$\frac{1}{p^n}1 = \frac{t^n}{n!}1 \qquad \text{--(4)}$$

If equation (3) is expanded by the Binomial Theorem, an equation in the form of a series results, viz:-

$$i = \frac{E}{L}\left\{\frac{1}{p} - \frac{R}{L}\cdot\frac{1}{p^2} + \frac{R^2}{L^2}\cdot\frac{1}{p^3} - \frac{R^3}{L^3}\cdot\frac{1}{p^4} + --\right\}1 \qquad \text{--(5)}$$

Substituting the variable t for p, we get:-

$$\begin{aligned} i &= \frac{E}{L}\left[t - \frac{R}{L}\cdot\frac{t^2}{2!} + \frac{R^2}{L^2}\cdot\frac{t^3}{3!} - \frac{R^3}{L^3}\cdot\frac{t^4}{4!} + --\right\}1 \\ &= \frac{E}{L}\left\{\frac{L}{R} - \frac{L}{R}\left[1 - \left(\frac{R}{L}\right)t + \left(\frac{R}{L}\right)^2\frac{t^2}{2!} - \left(\frac{R}{L}\right)^3\cdot\frac{t^3}{3!} + --\right]\right\}1 \\ &= \frac{E}{L}\cdot\frac{L}{R}\left[1 - \varepsilon^{-\frac{Rt}{L}}\right]1 \\ &= \frac{E}{R}\cdot\left[1 - \varepsilon^{-\frac{Rt}{L}}\right]1 \qquad \text{--(6)} \end{aligned}$$

which is the result that was obtained by normal treatment. In this case, the series obtained was easy to interpret, but this is by no means always so.

b. The Expansion Theorem.

The result of the simple case - i.e., that $\frac{1}{p+\alpha}1$ is equivalent to $\frac{1}{\alpha}\left[1-\varepsilon^{-\alpha t}\right]1$ - may be used to solve more complex operational equations.

Suppose the operational solution of a problem has been found in the form:-

$$i = E\cdot\frac{Y(p)}{Z(p)}\cdot 1 \qquad \text{--(7)}$$

where $Y(p)$ and $Z(p)$ are certain functions of p, - which may take into account not only impedance, but also variation of applied voltage, as for A.C. working, & c. $Y(p)$ is considered to be of lower power in p than $Z(p)$. If this is not the case, the fraction should be divided out, and changed to a sum of several terms, one of which is of lower power in the numerator than in the denominator.

If the various roots of the equation $Z(p) = 0$ are p_1, p_2, p_3, &c., we may write, by splitting up into partial fractions:-

$$\frac{Y(p)}{Z(p)} = \frac{A}{p-p_1} + \frac{B}{p-p_2} + \frac{C}{p-p_3} + -- \qquad --(8)$$

in which A, B, C, & c., are constants, and are given in fact by:-

$$A = \frac{Y(p_1)}{Z'(p_1)}, \quad B = \frac{Y(p_2)}{Z'(p_2)}, \quad C = \frac{Y(p_3)}{Z'(p_3)}, \quad \text{where } Z'(p) = \frac{dZ(p)}{dp}.$$

(See Bibliography; $A, 1.$)

Thus:-

$$\frac{Y(p)}{Z(p)}1 = A\cdot\frac{1}{p-p_1}1 + B\cdot\frac{1}{p-p_2}1 + C\cdot\frac{1}{p-p_3}1 + -- \qquad --(9)$$

Now as $\frac{1}{p+\alpha}1 = \frac{1}{\alpha}[1-\varepsilon^{-\alpha t}]1$, we see that:-

$$\frac{Y(p)}{Z(p)}1 = \frac{A}{-p_1}[1-\varepsilon^{p_1 t}]1 + \frac{B}{-p_2}[1-\varepsilon^{p_2 t}]1 + \frac{C}{-p_3}[1-\varepsilon^{p_3 t}]1 + --$$

$$= \left[\frac{A}{-p_1} + \frac{B}{-p_2} + \frac{C}{-p_3} + --\right]1 + \left[\frac{A}{p_1}\varepsilon^{p_1 t} + \frac{B}{p_2}\varepsilon^{p_2 t} + \frac{C}{p_3}\varepsilon^{p_3 t} + --\right]1 \qquad --(10)$$

The first term is apparently $\frac{Y(p)}{Z(p)}1$ for $p = 0$, which is usually written $\frac{Y(0)}{Z(0)}1$.

The second, substituting for A, B, C, & c., is:-

$$\left[\frac{Y(p)}{pZ'(p)}\varepsilon^{pt}\right]_{p=p_1}1 + \left[\frac{Y(p)}{pZ'(p)}\varepsilon^{pt}\right]_{p=p_2}1 + --$$

i.e., $$i = E\frac{Y(p)}{Z(p)}1 = E\frac{Y(0)}{Z(0)}1 + E\sum_{p_1 p_2 p_3}\frac{Y(p)}{pZ'(p)}\varepsilon^{pt}1 \qquad --(11)$$

This is the Heaviside Expansion Theorem, which holds if all the roots p_1, p_2, p_3, & c., are unequal, and none are zero. Modified forms may be worked out for these special cases.

The beauty of the treatment lies in the extremely simple operations involved in finding the arbitrary constants which specify the magnitudes of the transient currents. The roots p_1, p_2, p_3, & c., are found from the equation $Z(p) = 0$, which is in fact the "Auxiliary Equation" of pure mathematics, but the constants are arrived at by a process involving only differentiation.

These are the two principal methods involved in the Operational Calculus, and we shall be concerned mainly with the second - i.e., the Expansion Theorem - applied to the case of several coupled circuits, with "lumped" circuit parameters.

B. The Straightforward Application of the Expansion Theorem.

As $$\frac{1}{\alpha}\left\{1-\varepsilon^{-\alpha t}\right\}1 = \frac{1}{p+\alpha}1$$

$$\therefore \quad \left\{1-\varepsilon^{-\alpha t}\right\}1 = \frac{\alpha}{p+\alpha}1$$

$$\therefore \quad \left\{1-\varepsilon^{-\alpha t}-1\right\}1 = \left\{\frac{\alpha}{p+\alpha}-1\right\}1$$

or $$\varepsilon^{-\alpha t}1 = \frac{p}{p+\alpha}1 \qquad \text{--(12)}$$

and $$\varepsilon^{\alpha t}1 = \frac{p}{p-\alpha}1 \qquad \text{--(13)}$$

Hence a sine wave may be expressed as a function of p, for:-

$$\left\{\sin \omega t\right\}1 = \frac{1}{2j}\left\{\varepsilon^{j\omega t}-\varepsilon^{-j\omega t}\right\}1$$
$$= \frac{1}{2j}\left\{\frac{p}{p-j\omega}-\frac{p}{p+j\omega}\right\}1$$
$$= \left\{\frac{p\omega}{p^2+\omega^2}\right\}1 \qquad \text{--(14)}$$

and similarly,

$$\left\{\sin(\omega t+\theta)\right\}1 = \frac{1}{2j}\left\{\frac{p}{p-j\omega}\varepsilon^{j\theta}-\frac{p}{p+j\omega}\varepsilon^{-j\theta}\right\}1$$
$$= \left\{\frac{p^2\sin\theta + p\omega\cos\theta}{p^2+\omega^2}\right\}1 \qquad \text{--(15)}$$

Other formulae may be developed in a similar manner.

Thus if the problem is that of a sinusoidal E.M.F., $E\sin(\omega t+\theta)$, suddenly applied at time $t = 0$ to a circuit containing resistance and inductance, we may write:-

$$Ri + L\frac{di}{dt} = E\sin(\omega t+\theta) \quad \text{where } i = 0 \text{ at } t = 0. \qquad \text{--(16)}$$

or, operationally:-

$$\left\{R+pL\right\}i = E\left\{\frac{p^2\sin\theta + p\omega\cos\theta}{p^2+\omega^2}\right\}1 \qquad \text{--(17)}$$

i.e., $$i = E\left\{\frac{p^2\sin\theta + p\omega\cos\theta}{(R+pL)(p^2+\omega^2)}\right\}1 = E\frac{Y(p)}{Z(p)}1 \qquad \text{--(18)}$$

The expansion theorem may be applied directly to this expression. It is, however, shorter to consider the E.M.F. as made up of two parts:-

i.e., $$\left\{R+pL\right\}i = E\cdot\frac{1}{2j}\left\{\frac{p}{p-j\omega}\varepsilon^{j\theta}-\frac{p}{p+j\omega}\varepsilon^{-j\theta}\right\}1 \qquad \text{--(19)}$$

or $$i = E\frac{1}{2j}\frac{p\varepsilon^{j\theta}}{(R+pL)(p-j\omega)}1 + E\frac{1}{-2j}\frac{p\varepsilon^{-j\theta}}{(R+pL)(p+j\omega)}1 \qquad \text{--(20)}$$

These two expressions are complex and complementary. If, therefore, one is solved, the complete current solution may be obtained by doubling the real part of the answer. This is the normal procedure in the vector treatment of A.C. circuits for steady state conditions, where a sinusoidal variation is represented by the real part of a single rotating vector, instead of by the sum of two equal but oppositely rotating vectors. Considerable use will be made of this simplification in Part III of this Thesis, where Rotating Machinery transients will be considered.

Taking then the first of the two terms of eqn. (20);-

$$i = \frac{E}{2j}\frac{p\varepsilon^{i\theta}}{(R+pL)(p-j\omega)}1 = \frac{Y(p)}{Z(p)}1 \qquad --(21)$$

By the expansion theorem:-

$$i = \frac{Y(0)}{Z(0)}1 + \sum_{p_1 p_2 p_3 \ldots}\frac{Y(p)}{pZ'(p)}\varepsilon^{pt}1$$

1. Here, $\frac{Y(0)}{Z(0)} = 0$ --(22)

2. For the root $p = -\frac{R}{L}$, $Y(p) = \frac{E}{2j}\left(-\frac{R}{L}\right)\varepsilon^{i\theta}$ --(23a)

and $pZ'(p) = -\frac{R}{L}\cdot L\cdot\left\{-\frac{R}{L}-j\omega\right\}$ --(23b)

Thus

$$\begin{aligned} i &= -\frac{E}{2j}\cdot\frac{1}{L(\frac{R}{L}+j\omega)}\cdot\varepsilon^{i\theta}\varepsilon^{-\frac{Rt}{L}}1 \\ &= \frac{E}{2}\cdot\frac{R-j\omega L}{(R^2+\omega^2L^2)}\cdot\varepsilon^{i\theta}\varepsilon^{i\frac{\pi}{2}}\varepsilon^{-\frac{Rt}{L}}1 \\ &= \frac{E}{2}\cdot\frac{1}{Z}\cdot\varepsilon^{-\frac{Rt}{L}}\varepsilon^{i(\frac{\pi}{2}+\theta-\phi)}1 \quad \text{where } Z = \sqrt{R^2+\omega^2L^2},\ \&\ \phi = \tan^{-1}\frac{\omega L}{R} \qquad --(24)\end{aligned}$$

Hence, taking real parts and doubling:-

$$i = \frac{E}{Z}\varepsilon^{-\frac{Rt}{L}}\cos\left(\frac{\pi}{2}+\theta-\phi\right)1 = -\frac{E}{Z}\varepsilon^{-\frac{Rt}{L}}\sin(\theta-\phi)1 \qquad --(25)$$

3. For the root $p = j\omega$, $Y(p) = \frac{E}{2j}\cdot j\omega\cdot\varepsilon^{i\theta}$ --(26a)

and $pZ'(p) = j\omega\cdot L\cdot\left\{j\omega+\frac{R}{L}\right\}$ --(26b)

Thus

$$\begin{aligned} i &= \frac{E}{2j}\cdot\frac{1}{R+j\omega L}\varepsilon^{j(\theta+\omega t)}1 \\ &= -\frac{E}{2}\frac{1}{Z}\varepsilon^{j(\frac{\pi}{2}+\theta+\omega t-\phi)}1 \qquad --(27)\end{aligned}$$

Hence, taking real parts and doubling:-

$$i = -\frac{E}{Z}\cos\left(\frac{\pi}{2}+\omega t+\theta-\phi\right)1 = \frac{E}{Z}\sin(\omega t+\theta-\phi)1 \qquad --(28)$$

Hence the total current, given by the sum of the currents in eqns. (22), (25), and (28); is:-

$$i = \frac{E}{Z}\left\{\sin(\omega t+\theta-\phi) - \varepsilon^{-\frac{Rt}{L}}\sin(\theta-\phi)\right\}1 \qquad --(29)$$

This is the formula as obtained by direct mathematics, (Appendix, A,2.).

C. The Operational Calculus Applied to the Case of Coupled Circuits.

It has been remarked that the Operational Calculus is of great value in dealing with the case of coupled circuits. The reason is that the simultaneous equations involved may be solved by ordinary Determinantal methods, to give immediately the operational solution for any particular current, this solution being attacked in its turn by the expansion theorem, - all comparatively simple operations, involving algebraic transformations only.

1. The General Case of Coupled Circuits.

Let there be n circuits. Let the self-impedance of circuit r be represented by Z_{rr}, where Z_{rr} is a function of p depending on self-inductance, resistance & c. Let the mutual impedance between circuits h and k be Z_{hk}, where Z_{hk} is a function of p depending on mutual resistance, inductance, & c., between circuits h and k.

Thus, volts induced in circuit h by current in circuit k is $Z_{hk} \times i_k$, and volts induced in circuit k by current in circuit h is $Z_{kh} \times i_h$.

Actually, for a passive network, $Z_{hk} = Z_{kh}$.

Let the voltages applied in the various circuits at time $t = 0$ be e_1, e_2, e_3, & c. Then, adding up the induced volts, and equating them to the applied volts for each circuit:-

$$\begin{array}{l} Z_{11} i_1 + Z_{12} i_2 + Z_{13} i_3 + Z_{14} i_4 + - + Z_{1n} i_n = e_1 1 \\ Z_{21} i_1 + Z_{22} i_2 + Z_{23} i_3 + Z_{24} i_4 + - + Z_{2n} i_n = e_2 1 \\ \text{— — — — — — — — — — — —} \\ Z_{n1} i_1 + Z_{n2} i_2 + Z_{n3} i_3 + Z_{n4} i_4 + - + Z_{nn} i_n = e_n 1 \end{array} \qquad \text{--(30)}$$

For the current in circuit r, we may write:-

$$i_r = \frac{M_{1r}}{D} e_1 1 + \frac{M_{2r}}{D} e_2 1 + \frac{M_{3r}}{D} e_3 1 + \frac{M_{4r}}{D} e_4 1 + \text{---} + \frac{M_{nr}}{D} e_n 1 \qquad \text{--(31)}$$

where D is the determinant formed by the coefficients of the currents. i.e.:-

$$D = \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & - & - & Z_{1n} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} & - & - & Z_{2n} \\ - & - & - & - & - & - & - \\ Z_{n1} & Z_{n2} & Z_{n3} & Z_{n4} & - & - & Z_{nn} \end{vmatrix}$$

and M_{sr} is the principal minor of s^{th} row and r^{th} column.

Alternatively,

$$i_r = \frac{M_{er}}{D} 1 \qquad \text{--(32)}$$

where M_{er} is the determinant formed by substituting the values e_1, e_2, e_3, & c., for the impedances of the r^{th} column of the determinant D.

This operational expression is equivalent to the expression :-

$$i = \frac{Y(p)}{Z(p)} 1$$

and may therefore be solved by the expansion theorem. The operator D is in general of the $2n^{th}$ power in p, though it may be less.

In specific cases, considerable simplification of the expressions may be made. For instance, there may be symmetry in the determinant D, - as occurs in the case of alternator short-circuit. Similarly, many of the voltages e_1, e_2, & c., may be zero.

2. The Case of two Coupled Circuits having Resistance and Inductance only.

This case has already been treated by formal methods. It will now be shown how much simpler is the treatment by the operational calculus.

The operational equations are :-

$$[R_1 + pL_1] i_1 + pM i_2 = E 1 \qquad \text{--(33a)}$$

$$pM i_1 + [R_2 + pL_2] i_2 = 0 \qquad \text{--(33b)}$$

Hence

$$i = \frac{E[R_2 + pL_2]}{[R_1 + pL_1][R_2 + pL_2] - p^2M^2} 1 = \frac{E[R_2 + pL_2]}{[L_1L_2 - M^2]p^2 + [R_1L_2 + R_2L_1]p + R_1R_2} 1$$

$$= \frac{E}{(L_1L_2 - M^2)} \cdot \frac{R_2 + pL_2}{(p-\alpha)(p-\beta)} 1 \qquad \text{--(34)}$$

where α and β are the roots of the equation:-

$$p^2 + \frac{R_1L_2 + R_2L_1}{(L_1L_2 - M^2)} p + \frac{R_1R_2}{(L_1L_2 - M^2)} = 0 \qquad \text{--(35)}$$

Applying the expansion theorem :-

$$\frac{Y(0)}{Z(0)} = \frac{E}{(L_1L_2 - M^2)} \cdot \frac{R_2}{\alpha\beta} = \frac{E}{(L_1L_2 - M^2)} \cdot \frac{R_2}{R_1R_2} \cdot (L_1L_2 - M^2) = \frac{E}{R_1}$$

$$\frac{Y(\alpha)}{\alpha Z'(\alpha)} = \frac{E}{(L_1L_2 - M^2)} \cdot \frac{R_2 + \alpha L_2}{\alpha(\alpha - \beta)}$$

$$\frac{Y(\beta)}{\beta Z'(\beta)} = \frac{E}{(L_1L_2 - M^2)} \cdot \frac{R_2 + \beta L_2}{\beta(\beta - \alpha)}$$

Hence, total solution is :-

$$i_1 = E\left\{\frac{1}{R_1} + \frac{1}{(\alpha-\beta)(L_1L_2 - M^2)}\left[\frac{R_2 + \alpha L_2}{\alpha}\varepsilon^{\alpha t} - \frac{R_2 + \beta L_2}{\beta}\varepsilon^{\beta t}\right]\right\} 1 \qquad \text{--(36)}$$

This equation is identical with that obtained by normal methods in Part I, though of slightly different form. The identity may be shown as below :-

Equation (36) may be written :-

$$i_1 = \frac{E}{R_1}\left\{1 + \frac{1}{(\alpha-\beta)(L_1L_2 - M^2)}\left[\left(R_1L_2 + \frac{R_1R_2}{\alpha}\right)\varepsilon^{\alpha t} - \left(R_1L_2 + \frac{R_1R_2}{\beta}\right)\varepsilon^{\beta t}\right]\right\} 1$$

or as $\alpha\beta = \frac{R_1R_2}{(L_1L_2 - M^2)}$

$$i_1 = \frac{E}{R_1}\left\{1 + \frac{1}{(\alpha-\beta)(L_1L_2 - M^2)}\left[\left\{R_1L_2 + (L_1L_2 - M^2)\beta\right\}\varepsilon^{\alpha t} - \left\{R_1L_2 + (L_1L_2 - M^2)\alpha\right\}\varepsilon^{\beta t}\right]\right\} 1$$

which is identical with eqn. (20), Pt. I .

The operational method is of most use when dealing with numerical examples, as is the case with the vector method of analysis of A.C. circuits. In both methods, the complete symbolic solutions for current are very involved in appearance, and the use of the method lies not so much in obtaining a general solution, as in obtaining a particular solution for a particular case.

D. The Operational Treatment of Phenomena arising from the Sudden Interruption of a Current.

1. Interruption of Current Flowing in a Coil, by Insertion of a Condenser.

This is the case of an ignition coil, with no secondary winding.

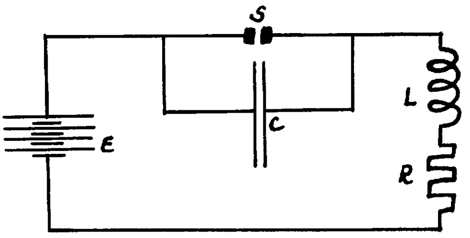


Fig 14.

It is apparent that at the moment immediately prior to the switching, the switch S carries the steady current I amps. Thereafter, it carries zero current. Hence the effect of the switching may be simulated by imagining a peculiar E:M.F. inserted in the switch circuit, which produces the current -I for all times after switching. The circuit may be redrawn, as shown in Fig 15.

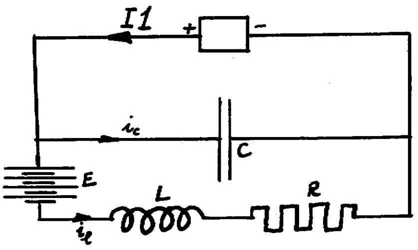


Fig 15.

This current I divides up in the branches containing C and (L and R) in the ratio of their admittances. If the resultant current in each circuit is now added to the previous steady current, the full switching current may be obtained.

The voltage E may be neglected when the transient is considered. Then the current in the condenser circuit is given by the ratio :-

$$\frac{i_c}{I1} = \frac{pC}{pC + \frac{1}{R+pL}} \qquad \text{--(37)}$$

Hence $$i_c = I\,\frac{pC(R+pL)}{p^2LC + pRC + 1}\,1 = \frac{E}{R}\,\frac{pC(R+pL)}{p^2LC + pRC + 1}\,1 \qquad \text{--(38)}$$

The denominator of the operational expression in eqn.(38) is the natural impedance of the condenser and coil circuit. The curious expression $(R + pL)$ in the numerator is of interest, and its origin may be shown by an alternative method of deriving the formula. This method is given below.

Let us assume for the moment that the current i_c may be used to represent the inductance current as well as the condenser current, which is tantamount to putting $i_c = i_\ell$. At all times after $t = 0$, this is evidently valid, since the only return path then available for i_c is through the coil. At $t = 0$, the supposition is invalid, for it implies a surge of current $I1$ in the inductance, which cannot in fact exist.

The E.M.F. in the condenser and coil circuit is given of course by the expression $\left\{(R + pL)i_\ell + \frac{1}{pC} i_c\right\}$, and it is zero for all "t", since the circuit is closed. With the assumption $i_c = i_\ell$, this expression becomes $\left\{pL + R + \frac{1}{pC}\right\} i_c$; this latter expression may not be equated to zero, however, for by operating on i_c with the operator $(R + pL)$, we are operating not only on i_ℓ, but on the additional surge $I1$, (introduced by the invalidity of the assumption $i_c = i_\ell$ at $t = 0$), as well. Thus the operational expression formed by taking $i_c = i_\ell$ is equal to zero plus the term introduced by this "spurious" surge, which term is $(R + pL)I1$. The full current equation is therefore :-

$$(pL + R + 1/pC)i_c = (R + pL)I1 \qquad \text{--(39)}$$

$$\text{or} \quad (p^2 LC + pRC + 1)i_c = pC(R + pL)I1 \qquad \text{--(40)}$$

which equation is seen to be identical with eqn.(38).

This irregularity due to a "spurious" discontinuity has been investigated in some detail because a similar phenomenon arises in the treatment of alternator short-circuit by operational methods, as will be shown (Part III).

The variable i_c is identical with the coil current for all times after $t = 0$, and may therefore be evaluated in place of i_ℓ. The evaluation may be done by the usual method - i.e., by the expansion theorem.

2. The Complete Ignition Coil Case.

Once again it is considered that a steady current $-I1$ is impressed in the switch circuit.

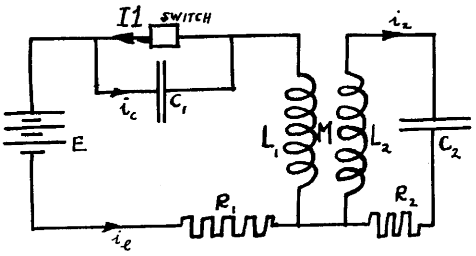


Fig 16.

Then the voltage across the condenser circuit is i_c/pC_1, and the voltage across the coil is $\left\{i_\ell(R_1+pL_1) - pMi_2\right\}$. (The -ve sign is taken for the voltage due to i_2 since the +ve direction of i_ℓ due to the switch current is opposite to that usually taken, the +ve direction of the initial current.)

Thus the current I of the switch divides in the ratio of the admittances, or :-

$$\frac{i_c}{I1} = \frac{pC_1}{pC_1 + \left\{\frac{1}{R_1 + pL_1 - pM\frac{i_2}{i_\ell}}\right\}}$$

$$\text{or} \qquad i_c = \frac{pC_1}{pC_1 + \left\{\frac{1}{R_1 + pL_1 - pM\frac{i_2}{i_\ell}}\right\}} I1 \qquad \text{--(41)}$$

$$\text{i.e.,} \quad pC_1(R_1+pL_1)i_c + i_c - p^2C_1M\frac{i_2}{i_\ell}i_c = pC_1(R_1+pL_1-pM\frac{i_2}{i_\ell})I1$$

$$\text{or} \quad (p^2L_1C_1+pC_1R_1+1)i - p^2C_1M\frac{i_2}{i_\ell}(i_c-I1) = pC_1(R_1+pL_1)I1 \qquad \text{--(42)}$$

But as $i_c = I1 - i_\ell$, then $i_c - I1 = -i_\ell$.

Thus we get :-

$$(p^2L_1C_1 + pC_1R_1 + 1)i_c + p^2C_1Mi_2 = pC_1(R_1 + pL_1)I1 \qquad \text{--(43)}$$

Consider now the voltage induced in the secondary winding.

The +ve current flowing in the primary coil is $(I - i_\ell)$, since the current I flows at all times before and after switching. Thus the volts induced in the secondary is $pM(I - i_\ell)$, or as $i_\ell = I1 - i_c$, volts induced is $pM(I - I1 + i_c)$.

Hence for the secondary :-

$$(R_2 + pL_2 + 1/pC_2)i_2 + pMi_c = -pM(I - I1) = pMI1 \qquad \text{--(44)}$$

since pI is zero for all values of t.

Once again the extra terms have appeared on the right-hand sides of the equations, due to the "spurious" voltage surge introduced by taking i_c as the coil current. The terms are $(R_1+pL_1)I1$, as before, and $pMI1$; remembering that i_c is the same as i_ℓ with a surge of current I1 at $t = 0$, these terms are what we should expect.

Collecting these results, we have :-

$$(L_1C_1p + R_1C_1p + 1)i_c + p^2MC_1i_2 = pC_1(R_1 + pL_1)I1$$

$$\& \quad p^2MC_2i_c + (L_2C_2p^2 + R_2C_2p + 1)i_2 = p^2MC_2\,I1$$

Thus the operational solutions are :-

$$i_c = \frac{\begin{vmatrix} pC_1(R_1 + pL_1) & p^2MC_1 \\ p^2MC_2 & (L_2C_2p^2 + R_2C_2p + 1) \end{vmatrix}}{\begin{vmatrix} (L_1C_1p^2 + R_1C_1p + 1) & p^2MC_1 \\ p^2MC_2 & (L_2C_2p^2 + R_2C_2p + 1) \end{vmatrix}} I1 \qquad \text{--(45)}$$

$$\text{and } i_2 = \frac{\begin{vmatrix} (L_1C_1p^2 + R_1C_1p + 1) & pC_1(R_1 + pL_1) \\ p^2MC_2 & p^2MC_2 \end{vmatrix}}{\begin{vmatrix} (L_1C_1p^2 + R_1C_1p + 1) & p^2MC_1 \\ p^2MC_2 & (L_2C_2p^2 + R_2C_2p + 1) \end{vmatrix}} I1 \qquad \text{--(46)}$$

or, simplified :-

$$i_c = \frac{p\left\{(L_1L_2 - M^2)p^3 + (R_1L_2 + R_2L_1)p^2 + (R_1R_2 + \frac{L_1}{C_2})p + (\frac{R_1}{C_2})\right\} \quad I1}{\left\{(L_1L_2 - M^2)p^4 + (R_1L_2 + R_2L_1)p^3 + (R_1R_2 + \frac{L_1}{C_2} + \frac{L_2}{C_1})p^2 + (\frac{R_1}{C_2} + \frac{R_2}{C_1})p + (\frac{1}{C_1C_2})\right\}} \text{-(47)}$$

$$\& \quad i_2 = \frac{p^2\frac{M}{C_1}\,I1}{\left\{(L_1L_2 - M^2)p^4 + (R_1L_2 + R_2L_1)p^3 + (R_1R_2 + \frac{L_1}{C_2} + \frac{L_2}{C_1})p^2 + (\frac{R_1}{C_2} + \frac{R_2}{C_1})p + (\frac{1}{C_1C_2})\right\}} \text{-(48)}$$

which may be solved by the expansion theorem.

There is here a danger of overlooking the voltage terms due to taking $\underline{i_c}$ as the whole primary current. It must be remembered that $\underline{i_c}$ has a surge at $\underline{t} = 0$, whereas the true current is $+I$ at $\underline{t} = 0$, and follows the $\underline{i_c}$ curve thereafter. As has been remarked, the point arises in the treatment of rotating machine transients, and must be kept carefully in mind.

E. Development of Various Formulae Necessary for the Work of Part III.

1. On Heaviside's "Shifting".

If a function of $\underline{p}$ operates on a function of time which includes an exponential, certain simplification may be made, as follows :-

Let the expression be :-

$$f(p)\left\{u\,\varepsilon^{\alpha t}\right\}$$

where u is any function of t, - possibly a sine wave, or the unit function 1, & c.

Then the expression may be written :-

$$\varepsilon^{\alpha t}\cdot f(p+\alpha)\left\{u\right\}$$

This is called the operation of "Shifting", since the exponential is moved outside the operator. The expression $f(p+\alpha)\{u\}$ may be evaluated, and the t function forming the solution multiplied by the function $\varepsilon^{\alpha t}$.

The relation may readily be proved :-

Suppose $f(p) = p.$

Then $$p\left\{u\,\varepsilon^{\alpha t}\right\} = \frac{d}{dt}\left\{u\,\varepsilon^{\alpha t}\right\} = u\alpha\,\varepsilon^{\alpha t} + \varepsilon^{\alpha t}\cdot\frac{du}{dt}$$
$$= \varepsilon^{\alpha t}\left\{p+\alpha\right\}\left\{u\right\} \qquad \text{--(49)}$$

Then if $f(p) = p^2,$

$$p^2\left\{u\,\varepsilon^{\alpha t}\right\} = \frac{d^2}{dt^2}\left\{u\,\varepsilon^{\alpha t}\right\} = \frac{d}{dt}\left\{u\alpha\,\varepsilon^{\alpha t} + \varepsilon^{\alpha t}\cdot\frac{du}{dt}\right\}$$
$$= u\,\alpha^2\varepsilon^{\alpha t} + \alpha\,\varepsilon^{\alpha t}\frac{du}{dt} + \alpha\,\varepsilon^{\alpha t}\cdot\frac{du}{dt} + \varepsilon^{\alpha t}\cdot\frac{d^2u}{dt^2}$$
$$= \varepsilon^{\alpha t}\left\{p^2 + 2\alpha p + \alpha^2\right\}\left\{u\right\} = \varepsilon^{\alpha t}\left\{p+\alpha\right\}^2\left\{u\right\} \qquad \text{--(50)}$$

and similarly for $f(p) = p^3$ & c.

Thus if $f(p) = p^n,$

$$p^n\left\{u\,\varepsilon^{\alpha t}\right\} = \varepsilon^{\alpha t}\left\{p+\alpha\right\}^n\left\{u\right\} \qquad \text{--(51)}$$

Thus the relation holds for any function of p which can be expanded in a power series. It also holds for the reciprocal $\frac{1}{p}$.

Considerable use will be made of this relation in Part III.

2. On Determinants.

a. General.

In the case of n coupled circuits, it has been shown that the impedance operator is a determinant of the n^{th} order, in general of the $2n^{th}$ degree in p.

In the general case, this determinant is not amenable to direct symbolic solution; the numerical values of the elements must be substituted, and the result worked out arithmetically.

On occasion, however, a determinant turns up which may readily be factorised. We shall go on to consider some such determinants.

b. Axi-symmetric determinants.

In any passive network - i.e., a network in which there is no relative motion between the various parts, - the following relation holds :-

E.M.F. in circuit s due to current i in circuit r

= E.M.F. in circuit r due to current i in circuit s.

or, if A_{rs} is the element of the r^{th} row and s^{th} column of the impedance determinant :-

$$A_{rs}\, i = A_{sr}\, i$$

so that $$A_{rs} = A_{sr} \qquad \text{--(52)}$$

A determinant having this relation between its elements is called axi-symmetric, since it is symmetrical about its major axis.

Such a determinant is not of itself very much simplified, but the property of axi-symmetry combined with other special forms may lead to easy factorisation.

c. Centri-symmetric determinants.

Centri-symmetry is the property possessed by a determinant which is symmetrical about its centre. Thus the relation holds between its elements, that :-

$$a_{rs} = a_{(n+1-r)(n+1-s)} \qquad \text{--(53)}$$

where the determinant is of the n^{th} order, having a_{rs} as the element of the r^{th} row and s^{th} column.

Such a determinant may be expressed as a product of two simpler determinants, of order $\frac{n\pm1}{2}$ if n is odd, and each of order $\frac{n}{2}$ if n is even. (See bibliography; A,2.).

A circuit having such an impedance operator would require to have the following physical characteristics :-

Self-impedance of circuit r = self-impedance of circuit $(n+1-r)$

Mutual impedance between circuit $\underline{r}$ & circuit $(\underline{r} + \underline{s})$

= mutual impedance between circuit $(\underline{n}+1-\underline{r})$ & circuit $(\underline{n}+1-\underline{r}-\underline{s})$

Apparently, a symmetrical polyphase winding has this degree of symmetry, since all circuits have the same self-impedance, and any two circuits separated by the same distance have the same mutual impedances. It is obvious also, that a polyphase winding has considerably more symmetry than may be represented by an ordinary centri-symmetric determinant.

<u>d.</u> Cyclically symmetric determinants.

In this case, the determinant is of the form :-

$$D = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & - & - & a_n \\ a_n & a_1 & a_2 & a_3 & - & - & a_{n-1} \\ a_{n-1} & a_n & a_1 & a_2 & - & - & a_{n-2} \\ a_{n-2} & a_{n-1} & a_n & a_1 & - & - & a_{n-3} \\ - & - & - & - & - & - & - \\ a_2 & a_3 & a_4 & a_5 & - & - & a_1 \end{vmatrix} \qquad \text{--(54)}$$

That is, each row contains the elements of the preceding row, displaced one space to the right in cyclical order. Such a determinant may be factorised with great ease, as follows :-

Let ω_1, ω_2, ω_3, & c. be the $\underline{n}$ primitive roots of unity. Let P be a determinant such that :-

$$P = \begin{vmatrix} 1 & \omega_1 & \omega_1^2 & \omega_1^3 & - & - & \omega_1^{n-1} \\ 1 & \omega_2 & \omega_2^2 & \omega_2^3 & - & - & \omega_2^{n-1} \\ 1 & \omega_3 & \omega_3^2 & \omega_3^3 & - & - & \omega_3^{n-1} \\ - & - & - & - & - & - & - \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & - & - & \omega_n^{n-1} \end{vmatrix} \qquad \text{--(55)}$$

Let the expression :-

$$(a_1 + a_2\omega + a_3\omega^2 + a_4\omega^3 + \;-\;-\; + a_n\omega^{n-1})$$

be denoted by $\phi(\omega)$.

If D and P be multiplied together, then by the ordinary rules for the multiplication of determinants :-

(over)

$$D \times P = \begin{vmatrix} \phi(\omega_1) & \phi(\omega_2) & - & - & \phi(\omega_n) \\ \omega_1\phi(\omega_1) & \omega_2\phi(\omega_2) & - & - & \omega_n\phi(\omega_n) \\ \omega_1^2\phi(\omega_1) & \omega_2^2\phi(\omega_2) & - & - & \omega_n^2\phi(\omega_n) \\ - & - & - & - & - \\ \omega_1^{n-1}\phi(\omega_1) & \omega_2^{n-1}\phi(\omega_2) & - & - & \omega_n^{n-1}\phi(\omega_n) \end{vmatrix} = \phi(\omega_1) \times \phi(\omega_2) \times \phi(\omega_3) \times - - \times P \qquad \text{--(56)}$$

remembering that $\omega^n = 1$, and that rows and columns of a determinant may be interchanged.

Thus $D = \prod\{\phi(\omega)\}$

$$\text{or } D = \prod_{\omega_1\omega_2\omega_3 - - \omega_n}\{a_1 + a_2\omega + a_3\omega^2 + - - + a_n\omega^{n-1}\} \qquad \text{--(57)}$$

The determinant is thus split up into n factors, none of which is of higher degree than the second in p; it can easily be reduced to a product of linear functions of p.

Since the roots of unity are given by :-

$$\omega = \varepsilon^{j\frac{2k\pi}{n}} \quad \text{where} \quad k = 1, 2, 3, - - (n-1),$$

we may write :-

$$D = \prod_{k=0}^{k=(n-1)}\{a_1 + a_2\varepsilon^{j\frac{2k\pi}{n}} + a_3\varepsilon^{j\frac{4k\pi}{n}} + - - + a_n\varepsilon^{j\frac{2(n-1)k\pi}{n}}\}$$

$$= \prod_{k=0}^{k=(n-1)}\{\sum_{r=1}^{r=n} a_r\,\varepsilon^{j\frac{2(r-1)k\pi}{n}}\} \qquad \text{--(58)}$$

Now since the determinant must also be axi-symmetric for a passive network, it is apparent by inspection that $a_n = a_2$, $a_{n-1} = a_3$, & c. Thus, by inspection, the determinant is also centrisymmetric. This is the type of function which turns up in the consideration of a symmetrical polyphase winding, as will be shown in Part III.

In this case :-

$$D = \prod_{k=0}^{k=(n-1)}\{a_1 + a_2\left[\cos\frac{2k\pi}{n} + j\sin\frac{2k\pi}{n}\right] + - - + a_n\left[\cos\frac{2(n-1)k\pi}{n} + j\sin\frac{2(n-1)k\pi}{n}\right]\}$$

and as $\cos\frac{2(n-1)k\pi}{n} = \cos\left(2k\pi - \frac{2k\pi}{n}\right) = \cos\frac{2k\pi}{n}$

and $\sin\frac{2(n-1)k\pi}{n} = -\sin\frac{2k\pi}{n}$

then if n is odd,

$$D = \prod_{k=0}^{k=(n-1)}\{a_1 + 2a_2\cos\frac{2k\pi}{n} + - - - + 2a_{\frac{n+1}{2}}\cos\frac{n-1}{2}\frac{2k\pi}{n}\} \qquad \text{--(59)}$$

and if n is even,

$$D = \prod_{k=0}^{k=(n-1)}\{a_1 + 2a_2\cos\frac{2k\pi}{n} + - - - + a_{\frac{n+2}{2}}\left[\cos\frac{n}{2}\frac{2k\pi}{n} + j\sin\frac{n}{2}\frac{2k\pi}{n}\right]\}$$

$$= \prod_{k=0}^{k=(n-1)}\{a_1 + 2a_2\cos\frac{2k\pi}{n} + - - - + a_{\frac{n+2}{2}}\cos\frac{n}{2}\frac{2k\pi}{n}\} \qquad \text{--(60)}$$

neither of which terms contains imaginaries. Note that as $\sin\frac{n}{2}\frac{2k\pi}{n} = 0$, both equations may be expressed similarly, viz:-

$$D = \prod_{k=0}^{k=(n-1)}\{a_1 + a_2\cos\frac{2k\pi}{n} + a_3\cos\frac{4k\pi}{n} + - - + a_n\cos\frac{(n-1)2k\pi}{n}\} \qquad \text{--(61)}$$

e. Theorem on the minors of a cyclic determinant.

A special form of this determinant and its minors turns up in the consideration of alternator short-circuit. This form includes the determinant shown below :-

$$Y_{(\pm)} = \begin{vmatrix} 1 & a_2 & a_3 & - & a_n \\ \varepsilon^{\pm j\frac{2\pi}{n}} & a_1 & a_2 & - & a_{n-1} \\ \varepsilon^{\pm j\frac{4\pi}{n}} & a_n & a_1 & - & a_{n-2} \\ - & - & - & - & - \\ \varepsilon^{\pm j\frac{(n-1)2\pi}{n}} & a_3 & a_4 & - & a_1 \end{vmatrix} \qquad \text{--(62)}$$

in which all the elements of one column are removed, and the factors :-

$$1, \quad \varepsilon^{\pm j\frac{2\pi}{n}}, \quad \varepsilon^{\pm j\frac{4\pi}{n}}, \quad - - \quad \varepsilon^{\pm j\frac{(n-1)2\pi}{n}},$$

are substituted in their stead.

This determinant always occurs in the form $\frac{Y}{D}$, which may be evaluated as below :-

If M_{rs} is the principal minor of the $\underline{r}^{th}$ row and $\underline{s}^{th}$ column belonging to the determinant D, then :-

$$Y_{(\pm)} = M_{11} + \varepsilon^{\pm j\frac{2\pi}{n}} M_{21} + \varepsilon^{\pm j\frac{4\pi}{n}} M_{31} + - + \varepsilon^{\pm j\frac{(n-1)2\pi}{n}} M_{n1}$$

Thus we may write the product

$$\left\{a_1 + a_n\,\varepsilon^{\mp j\frac{2\pi}{n}} + a_{n-1}\,\varepsilon^{\mp j\frac{4\pi}{n}} + - + a_2\,\varepsilon^{\mp j\frac{(n-1)2\pi}{n}}\right\} \times Y_{(\pm)}$$

as below :-

$$\text{Prod.} = \left\{a_1 + a_n\,\varepsilon^{\mp j\frac{2\pi}{n}} + a_{n-1}\,\varepsilon^{\mp j\frac{4\pi}{n}} + - + a_2\,\varepsilon^{\mp j\frac{(n-1)2\pi}{n}}\right\}\left\{M_{11} + M_{21}\,\varepsilon^{\pm j\frac{2\pi}{n}} + M_{31}\,\varepsilon^{\pm j\frac{4\pi}{n}} + - + M_{n1}\,\varepsilon^{\pm j\frac{(n-1)2\pi}{n}}\right\}$$

$$= \left\{a_1 + a_n\,\varepsilon^{\mp j\frac{2\pi}{n}} + a_{n-1}\,\varepsilon^{\mp j\frac{4\pi}{n}} + - + a_2\,\varepsilon^{\mp j\frac{(n-1)2\pi}{n}}\right\}M_{11} + \left\{a_1\,\varepsilon^{\mp j\frac{(n-1)2\pi}{n}} + a_n + a_{n-1}\,\varepsilon^{\mp j\frac{2\pi}{n}} + - + a_2\,\varepsilon^{\mp j\frac{(n-2)2\pi}{n}}\right\}M_{21}$$

$$+ \left\{a_1\,\varepsilon^{\mp j\frac{(n-2)2\pi}{n}} + a_n\,\varepsilon^{\mp j\frac{(n-1)2\pi}{n}} + a_{n-1} + a_{n-2}\,\varepsilon^{\mp j\frac{2\pi}{n}} + -\right\}M_{31} + \left\{a_1\,\varepsilon^{\mp j\frac{(n-3)2\pi}{n}} + a_n\,\varepsilon^{\mp j\frac{(n-2)2\pi}{n}} + a_{n-1}\,\varepsilon^{\mp j\frac{(n-1)2\pi}{n}} + a_{n-2} + -\right\}M_{41}$$

and so on,

since $\varepsilon^{\pm j\frac{2\pi}{n}} = \varepsilon^{\mp j\frac{(n-1)2\pi}{n}}$ & c.

Collecting terms having the same exponential coefficient:-

$$\text{Prod.} = \left\{a_1 M_{11} + a_n M_{21} + a_{n-1} M_{31} + -\right\} + \varepsilon^{\mp j\frac{2\pi}{n}}\left\{a_n M_{11} + a_{n-1} M_{21} + a_{n-2} M_{31} + -\right\}$$

$$+ \varepsilon^{\mp j\frac{4\pi}{n}}\left\{a_{n-1} M_{11} + a_{n-2} M_{21} + --\right\} \quad \& \text{ c.}$$

The first term in brackets is the determinant D. The second term in brackets is the same determinant, with the elements of the first column moved one place up in cyclical order, so that the first column becomes identical with the $\underline{n}^{th}$. But any determinant with two identical rows or columns has zero value. Thus the second bracketed term is zero.

For the third term, it is seen that the resultant determinant is similar to D, with the first and the $(n-1)$th columns identical, and so on for all the other terms after the first.

Thus all the terms vanish except the first, and we may write :-

$$\left\{a_1 + a_n \varepsilon^{\mp j\frac{2\pi}{n}} + a_{n-1}\varepsilon^{\mp j\frac{4\pi}{n}} + - - - + a_2 \varepsilon^{\mp j\frac{2(n-1)\pi}{n}}\right\} \cdot Y_{(\pm)} = D \qquad \text{--(63)}$$

i.e., $$\frac{Y_{(\pm)}}{D} = \frac{1}{\left\{a_1 + a_n \varepsilon^{\mp j\frac{2\pi}{n}} + a_{n-1}\varepsilon^{\mp j\frac{4\pi}{n}} + - - - + a_2 \varepsilon^{\mp j\frac{2(n-1)\pi}{n}}\right\}}$$

or $$\frac{Y_{(\pm)}}{D} = \frac{1}{\left\{a_1 + a_2 \varepsilon^{\pm j\frac{2\pi}{n}} + a_3\varepsilon^{\pm j\frac{4\pi}{n}} + - - - + a_n \varepsilon^{\pm j\frac{2(n-1)\pi}{n}}\right\}} \qquad \text{--(64)}$$

When the determinant D is axi-symmetric, it follows that :-

$$\frac{Y}{D} = \frac{1}{\left\{a_1 + a_2 \cos\frac{2\pi}{n} + a_3 \cos\frac{4\pi}{n} + - - - + a_n \cos\frac{2(n-1)\pi}{n}\right\}} \qquad \text{--(65)}$$

whether the positive or the negative sign of the exponentials in $Y_{(\pm)}$ is considered; i.e., $Y_{(+)} = Y_{(-)}$.

3. On the Solution of Equations of Higher Power than the Second.

It generally happens that the equation $Z_{(p)} = 0$, the solution of which is necessary for the application of the expansion theorem, cannot be factorised. In these circumstances, a numerical method of extracting the roots is required; several such methods are available.

E.J.Berg, in his book "Heaviside Operational Calculus" refers to a method originally invented by Germinal Dandelin, (usually known as the "Graeffe" method). This method consists in constructing an equation, the roots of which are a high power - say the *n*th - of the roots of the given equation, and thus are widely separated. Such an equation may be factorised by inspection. The roots of the original equation are then evaluated by taking the *n*th roots of this second equation's roots, the correct sign being found by trial.

The full development, with proof and procedure for special cases such as equal roots & c. is given in the above-mentioned book.

"A Study of Transient Phenomena in Electro-Magnetic Machinery, with Particular Reference to the use of the Heaviside Operational Calculus."

PART III

"The Heaviside Calculus Applied to the Case of Rotating Machinery."

Preliminary :-

General Introduction to the Method.

1. Definitions.

In the ensuing discussion, certain of the circuit constants will turn up continually, and it is therefore preferable that these be denoted by definite symbols. The fundamental symbols are as below.

The suffices s, r, and d, represent the stator, rotor, and damper circuits respectively, an additional numeral suffix denoting the particular phase under consideration. The stator circuit will be the name given to that circuit in which the alternating current flows during normal working, and the rotor circuit will be the field circuit. The fact that occasionally the field is stationary, and the A.C. winding rotates, makes no difference to the mathematical development, since only relative velocities influence the induced E.M.F.

R, with appropriate suffices, is the resistance of a Phase,

L' is the self-inductance of a Phase,

L is the equivalent self-inductance of a Phase, taking account of the mutual inductance between it and the other Phases belonging to the same circuit, rotor, damper, or stator. L is ,of course, a more or less complex function of the circuit constants, and a formula for it will be developed in due course.

$Z_{(p)}$ $(=R + pL)$ is the operational impedance of an "equivalent" Phase, the nature of which will be explained, and the circumstances,under which it exists, developed.

As the mutual inductances between the stator and either the rotor or the damper vary with rotor position, the symbols denoting such inductance must be taken as holding for some specified position of the rotor; in this case, the position of maximum linkage will be taken.

Hence :-

C is the mutual inductance between a stator and a rotor Phase in the position of maximum linkage between the Phases,

Q is the mutual inductance between a damper and a stator Phase in the position of maximum linkage between the Phases,

K is the mutual inductance between a rotor Phase and that damper Phase which has maximum linkage with it; i.e., that damper Phase which has the same axis as the rotor Phase. If no such damper Phase exists, K is the mutual inductance which would exist between such a damper Phase and the rotor Phase.

The definition of K is slightly different to that of C and Q , since there is no relative motion between rotor and damper, so that the relative positions of a Phase in the rotor and a Phase in the damper cannot be altered. Fundamentally, however, the definitions are the same. It should be noted that the theory is worked out in general for a damper winding consisting of n separate phases symmetrically disposed round the rotor, and rotating with it, coupled together only through mutual inductance, and not through resistance. That is, the damper is considered as a short-circuited symmetrical n-phase winding, and not as a squirrel-cage winding. The theory may be extended to cover the cage winding, as will be shown, (Part III, F.)

2. Assumptions.

Various assumptions have to be made to reduce the equations to workable size.

In the first place, such non-linear phenomena as saturation and hysteresis will be neglected. This will not invalidate the subsequent reasoning, since the transients depend largely on the leakage fields, which, being in air, are linear functions of the current.

It will also be assumed that the entire iron path is laminated. It is virtually impossible to take account of solid poles when calculating the transients.

Further, it will be assumed that the mutual inductance between two phases belonging to two of the three fundamental circuits is a sinusoidal function of the angle between them; - i.e., that the flux due to any phase is sinusoidal relative to the air-gap. This assumption is a good approximation for modern alternators, with distributed windings.

Thus if the mutual inductance between a certain rotor phase and a certain stator phase is of max. value C, then after rotation through θ electrical radians, the mutual inductance is $C \cos \theta$,

Similarly, if the mutual inductance between a rotor phase and the damper phase with which it has max. linkage is K, then the mutual inductance between that rotor phase and the damper phase displaced by angle θ is $K \cos \theta$.

It is assumed throughout that the switching conditions, whether short-circuiting or switching-in, the load conditions, and the internal structure of the machine are symmetrical.

3. The General Methods.

Two fundamental methods will be employed in the working. The first of these - the method of "Reflections" - is mainly useful in that it illustrates the principles employed in both systems, while being more easily grasped than the second, to which it leads as a natural corollary. Of itself, it is of but limited applicability, being practicable only in the simplest case of all, viz., that of a machine with no damper winding.

a. The method of Reflections.

This method consists in regarding the phenomenon as being due to repeated reflections of current from one system to another. The current in the stator due to the field current at short-circuit is evaluated; then is discovered the secondary field current due to this stator current; then the secondary stator current, then the tertiary field current, and so on. The resulting series of terms for rotor and stator currents is summed up to give the total switching currents.

b. The method of Equivalent Circuits.

In this method, all the phases of one winding are initially combined into one equivalent winding, which is then divided into two mutually independant parts. These parts have imaginary characteristics and currents, - the word "Imaginary" being used both in its literal and its mathematical sense, - but the parts may be regarded as perfectly real for the purpose of insertion into an "Equivalent Determinant" which gives the solution.

This method is perfectly general, being applicable to cases with a large number of independant systems, though the arithmetical side of the working soon becomes rather laborious.

A. The Simplest Case of Alternator Short-circuit.

Single phase rotor, polyphase stator, no damper.

D.C. excitation.

Consider two static circuits, coupled inductively together. Then if the flux produced in circuit 2 by a current i in circuit 1 is ϕ, it is apparent that the E.M.F. in circuit 2 due to a change of current in circuit 1 is given by the relation :-

$$\left| E.M.F. \right| = T \frac{d\phi}{dt} 10^{-8} = T\phi' \frac{di}{dt} 10^{-8} = M \frac{di}{dt} \qquad \text{--(1)}$$

where T is the number of turns in the secondary winding, ϕ' is the flux in 2 due to unit current in 1 (assumed constant) and M, called the mutual inductance, is $T\phi' 10^{-8}$.

If, however, the flux in circuit 2 is not directly proportional to the current in circuit 1, equation (1) cannot in general be simplified as above.

Suppose this flux is a function both of current in circuit 1 and position of circuit 1 relative to circuit 2, as is the case when the rotor and stator windings of an alternator are considered. Suppose, moreover, this flux is given by the equation :-

Flux in 2 due to i in 1 $= \phi' i \cos \omega t$, where ϕ' is constant, a relation which may be reasonably be supposed to hold for a modern alternator.

Then the numerical value of the E.M.F. in the second circuit due to the change of current in the first is :-

$$\left| \text{E.M.F.} \right| = T \frac{d\phi}{dt} 10^{-8} = T\phi' 10^{-8} \frac{d[i \cos \omega t]}{dt} \qquad \text{--(2)}$$

ϕ' is here the maximum flux which can link the second circuit due to unit current in the first, so that $T\phi' 10^{-8} = C$ will be the maximum possible mutual inductance between the circuits, - i.e., the mutual inductance when $\omega t = 0$, or when the axes of the coils are parallel.

Thus we may write :-

$$\left| \text{E.M.F.} \right| = C \frac{di'}{dt} \qquad \text{--(3)}$$

where C is the maximum mutual inductance between the circuits, and $i' (= i \cos \omega t)$ will be called the current in circuit 1 relative to circuit 2.

This relation will always be used in going from rotor to stator and vice-versa.

1. The Method of Reflections.

As was explained in the introduction to Part III, this method consists in summing the various current current terms caused by successive "Reflections" of current between stator and rotor.

If such a method be attempted by straightforward mathematics, the various terms of the series have no obvious relationship, the one to the other, and the sum of the series is therefore impossible to come by. By operational methods, however, the series takes an easily workable mathematical form, as we shall now proceed to show.

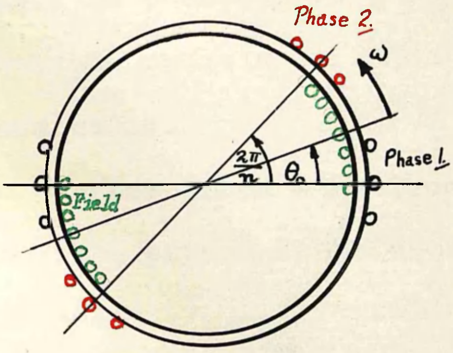


Fig 17.

Suppose the short-circuit takes place at the instant when the rotor coil axis has passed the axis of a stator phase - called phase 1 - and is inclined to it at an angle θ_0 electrical radians. The angular speed of the rotor, ω elec. rads./sec, will be assumed constant for the duration of the transient. Then phase 2 will be in the position shown, relative to the rotor, and similarly for phase 3 & c. (Fig 17)

Now if the rule governing the terms in the series of "Reflected" currents in rotor and stator is simple, it should be ascertainable by taking any term in the series and finding the next.

We shall consider therefore one of the terms in the series for the rotor current, which takes the form $F(p)1$, $F(p)$ being a function of p. This assumption of discontinuous current will be justified later.

The E.M.F. induced in the stator phases by this term may be evaluated as shown in equation(3), by taking the rate of change of this current relative to each phase, and multiplying by the maximum mutual inductance, which is a constant.

The current relative to each phase may be obtained by operating on $F(p)1$ with the cosine of the angle between the rotor axis and the phase axis. Thus, at any time t after the short-circuit the angle between phase 1 and the rotor is $(\omega t + \theta_0)$, so that :-

Rotor current relative to phase 1 $= \cos(\omega t + \theta_0)\, F(p)1$ --(4)

Expressing the cosine as a sum of exponentials,

Rotor current rel. to ph.1 $= \frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} F(p)1$ --(5a)

Rotor current rel. to ph.2 $= \frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0-\frac{2\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2\pi}{n})}\right\} F(p)1$ --(5b)

and so on for phase 3 & c.

Then the rate of change of the rotor current relative to the stator phases is given by :-

Rel. to phase 1 $= \frac{1}{2}\, p\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} F(p)\, 1$ --(6a)

Rel. to phase 2 $= \frac{1}{2}\, p\left\{\varepsilon^{j(\omega t+\theta_0-\frac{2\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2\pi}{n})}\right\} F(p)\, 1$ --(6b)

and so on.

From these equations, it follows that, if C is the maximum mutual inductance between the rotor and a stator phase :-

E.M.F. induced in ph. 1 $= -\frac{1}{2}\, C\, p\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} F(p)\, 1$ --(7a)

E.M.F. induced in ph. 2 $= -\frac{1}{2}\, C\, p\left\{\varepsilon^{j(\omega t+\theta_0-\frac{2\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2\pi}{n})}\right\} F(p)\, 1$ --(7b)

and so on for phase 3, & c.

Considering the polyphase stator system, it is apparent that we are concerned with n similar circuits, the various E.M.F.'s being equally spaced in time-phase.

Obviously, the mutual inductance between any two phases is dependant only on the distance apart in space-angle of the phases, and is independant of their absolute position on the stator. That is, the mutual inductance between phases s and s±r is quite independant of s, and is fixed for any given r. To illustrate this, a table of mutual inductances is drawn up below, Fig 18.

	Ph 1	Ph 2	Ph 3	Ph 4	--	Ph n
Ph 1		M_1	M_2	M_3	--	M_{n-1}
Ph 2	M_{n-1}		M_1	M_2	--	M_{n-2}
Ph 3	M_{n-2}	M_{n-1}		M_1	--	M_{n-3}
Ph 4	M_{n-3}	M_{n-2}	M_{n-1}		--	M_{n-4}
--	--	--	--	--		--
Ph n	M_1	M_2	M_3	M_4	--	

Fig 18.

Thus, the mutual inductance between phases 1 and 2 is M_1 (if we move counter-clockwise), between phases 1 and 3 is M_2, and so on. To move from phase 2 to phase 1 requires the passage of (n — 1) phases, so the mutual inductance is M_{n-1}; it is obvious that $M_r = M_{n-r}$, as mutual inductance must be the same in both directions for a passive network, so that $M_1 = M_{n-1}$, $M_2 = M_{n-2}$, and so on.

Denoting the mutual inductance between a phase and another phase displaced <u>r</u> phases counter-clockwise from it by M_r, we may write the operational equations of the <u>n</u> circuits as below :-

$$
\begin{array}{l}
(R_s + pL_s')i_{s1} + pM_1 i_{s2} + ---- + pM_{n-1} i_{sn} = -\tfrac{1}{2}Cp\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} F(p)\,1 \\
pM_{n-1} i_{s1} + (R_s + pL_s')i_{s2} + ---- + pM_{n-2} i_{sn} = -\tfrac{1}{2}Cp\left\{\varepsilon^{j(\omega t+\theta_0-\frac{2\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2\pi}{n})}\right\} F(p)\,1 \\
---- \quad ---- \quad ---- \quad ---- \quad ---- \quad ---- \\
pM_1 i_{s1} + pM_2 i_{s2} + ---- + (R_s + pL_s')i_{sn} = -\tfrac{1}{2}Cp\left\{\varepsilon^{j(\omega t+\theta_0-\frac{2(n-1)\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2(n-1)\pi}{n})}\right\} F(p)\,1
\end{array}
\qquad -(8)
$$

where $M_r = M_{n-r}$, as explained above.

In evaluating the operational expressions for the current, care must be taken to see that the functions of <u>t</u> on the right-hand side turn up in the correct positions. This point may be illustrated by the consideration of a simple circuit, having a function of <u>t</u> in the applied force operator. Let such a circuit have the operational equation :-

$$Z(p) = \varepsilon^{\alpha t} 1 \qquad --(9)$$

Then the equation for the current is given by :-

$$i = \frac{1}{Z(p)}\left\{\varepsilon^{\alpha t} 1\right\} \qquad --(10)$$

since the **operator** $1/Z(p)$ operates on the whole function $(\varepsilon^{\alpha t} 1)$. It would not be correct to write $i = \varepsilon^{\alpha t}\left\{\frac{1}{Z(p)} 1\right\}$, since by applying Heaviside's "Shifting" to equation (10) we see that :-

$$i = \varepsilon^{\alpha t}\left\{\frac{1}{Z(p+\alpha)} 1\right\} \qquad --(11)$$

which is obviously different from $i = \varepsilon^{\alpha t}\left\{\frac{1}{Z(p)} 1\right\}$.

This point is important, for the operational solution of equations (8) has the expressions involving <u>t</u> occurring as one column of a determinant, - and a determinant takes no account of the order of the factors of any single term, once the sign of the term has been fixed.

Bearing this in mind, we may write the solution for the current in phase 1 as below :-

(over)

$$i_{s_1} = -\tfrac{1}{2}Cp \frac{\begin{vmatrix} \left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} & pM_1 & pM_2 & -- & pM_{n-1} \\ \left\{\varepsilon^{j(\omega t+\theta_0-\frac{2\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2\pi}{n})}\right\} & Z_s' & pM_1 & -- & pM_{n-2} \\ -- & -- & -- & -- & -- \\ \left\{\varepsilon^{j(\omega t+\theta_0-\frac{2(n-1)\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2(n-1)\pi}{n})}\right\} & pM_2 & pM_3 & -- & Z_s' \end{vmatrix}}{\begin{vmatrix} Z_s' & pM_1 & pM_2 & -- & pM_{n-1} \\ pM_{n-1} & Z_s' & pM_1 & -- & pM_{n-2} \\ -- & -- & -- & -- & -- \\ pM_1 & pM_2 & pM_3 & -- & Z_s' \end{vmatrix}} F(p)\,1 \quad --(12)$$

where $Z_s' = (R_s + pL_s')$.

Applying the ordinary rules for the addition of determinants, we may write :-

$$i_{s_1} = -\tfrac{1}{2}Cp \frac{\begin{vmatrix} \varepsilon^{j(\omega t+\theta_0)} & pM_1 & pM_2 & -- & pM_{n-1} \\ \varepsilon^{j(\omega t+\theta_0-\frac{2\pi}{n})} & Z_s' & pM_1 & -- & pM_{n-2} \\ -- & -- & -- & -- & -- \\ \varepsilon^{j(\omega t+\theta_0-\frac{2(n-1)\pi}{n})} & pM_1 & pM_2 & -- & Z_s' \end{vmatrix}}{\begin{vmatrix} Z_s' & pM_1 & pM_2 & -- & pM_{n-1} \\ pM_{n-1} & Z_s' & pM_1 & -- & pM_{n-2} \\ -- & -- & -- & -- & -- \\ pM_1 & pM_2 & pM_3 & -- & Z_s' \end{vmatrix}} F(p)\,1 - \tfrac{1}{2}Cp \frac{\begin{vmatrix} \varepsilon^{-j(\omega t+\theta_0)} & pM_1 & pM_2 & -- & pM_{n-1} \\ \varepsilon^{-j(\omega t+\theta_0-\frac{2\pi}{n})} & Z_s' & pM_1 & -- & pM_{n-2} \\ -- & -- & -- & -- & -- \\ \varepsilon^{-j(\omega t+\theta_0-\frac{2(n-1)\pi}{n})} & pM_2 & pM_3 & -- & Z_s' \end{vmatrix}}{\begin{vmatrix} Z_s' & pM_1 & pM_2 & -- & pM_{n-1} \\ pM_{n-1} & Z_s' & pM_1 & -- & pM_{n-2} \\ -- & -- & -- & -- & -- \\ pM_1 & pM_2 & pM_3 & -- & Z_s' \end{vmatrix}} F(p)\,1 \quad --(13)$$

Each numerator has a common factor which may be removed. This factor is a function of <u>t</u> , and the question of its positioning after removal at once arises. From equation (10) , we see that the impedance operator must be regarded as operating on the <u>t</u> function, (and anything that goes with it, such as $F(p)$), so that the common factors must be placed <u>after</u> the determinants. Thus we get :-

$$i_{s_1} = -\tfrac{1}{2}Cp\varepsilon^{j\theta_0} \frac{\begin{vmatrix} 1 & pM_1 & pM_2 & -- & pM_{n-1} \\ \varepsilon^{-j\frac{2\pi}{n}} & Z_s' & pM_1 & -- & pM_{n-2} \\ -- & -- & -- & -- & -- \\ \varepsilon^{-j\frac{2(n-1)\pi}{n}} & pM_2 & pM_3 & -- & Z_s' \end{vmatrix}}{\begin{vmatrix} Z_s' & pM_1 & pM_2 & -- & pM_{n-1} \\ pM_{n-1} & Z_s' & pM_1 & -- & pM_{n-2} \\ -- & -- & -- & -- & -- \\ pM_1 & pM_2 & pM_3 & -- & Z_s' \end{vmatrix}} \varepsilon^{j\omega t} F(p)\,1 - \tfrac{1}{2}Cp\varepsilon^{-j\theta_0} \frac{\begin{vmatrix} 1 & pM_1 & pM_2 & -- & pM_{n-1} \\ \varepsilon^{j\frac{2\pi}{n}} & Z_s' & pM_1 & -- & pM_{n-2} \\ -- & -- & -- & -- & -- \\ \varepsilon^{j\frac{2(n-1)\pi}{n}} & pM_2 & pM_3 & -- & Z_s' \end{vmatrix}}{\begin{vmatrix} Z_s' & pM_1 & pM_2 & -- & pM_{n-1} \\ pM_{n-1} & Z_s' & pM_1 & -- & pM_{n-2} \\ -- & -- & -- & -- & -- \\ pM_1 & pM_2 & pM_3 & -- & Z_s' \end{vmatrix}} \varepsilon^{-j\omega t} F(p)\,1 \quad --(14)$$

where the expressions $\varepsilon^{j\theta_0}$, $\varepsilon^{-j\theta_0}$, not being functions of <u>t</u>, may be placed anywhere.

Each term of equation (14) is of the form described in Part II, E2; that is, the denominator is a cyclically symmetrical determinant, and the numerator is similar, with one column removed, and the factors 1, $\varepsilon^{\pm j\frac{2\pi}{n}}$, $\varepsilon^{\pm j\frac{4\pi}{n}}$, & c. substituted. Applying the rules obtained for such cases, we obtain the equation :-

$$i_{s1} = -\tfrac{1}{2}Cp\varepsilon^{j\theta_0}\left\{\frac{1}{Z_s' + \varepsilon^{j\frac{2\pi}{n}}pM_1 + \varepsilon^{j\frac{4\pi}{n}}pM_2 + --- + \varepsilon^{j\left[\frac{2(n-1)\pi}{n}\right]}pM_{n-1}}\right\}\varepsilon^{j\omega t}F_{(p)}1 - \tfrac{1}{2}Cp\varepsilon^{-j\theta_0}\left\{\frac{1}{Z_s' + \varepsilon^{-j\frac{2\pi}{n}}pM_1 + \varepsilon^{-j\frac{4\pi}{n}}pM_2 + --- + \varepsilon^{-j\left[\frac{2(n-1)\pi}{n}\right]}pM_{n-1}}\right\}\varepsilon^{-j\omega t}F_{(p)}1 \quad (15)$$

Consider the denominators of these terms:-

$$Z_s' + \varepsilon^{\pm j\frac{2\pi}{n}}pM_1 + \varepsilon^{\pm j\frac{4\pi}{n}}pM_2 + --- + \varepsilon^{\pm j\left[\frac{2(n-1)\pi}{n}\right]}pM_{n-1}$$

They may be written :-

$$R_s + p\left\{L_s' + \varepsilon^{\pm j\frac{2\pi}{n}}M_1 + \varepsilon^{\pm j\frac{4\pi}{n}}M_2 + --- + \varepsilon^{\pm j\left[\frac{2(n-1)\pi}{n}\right]}M_{n-1}\right\}$$

or $$R_s + p\left\{L_s' + M_1\left[\cos\tfrac{2\pi}{n} \pm j\sin\tfrac{2\pi}{n}\right] + M_2\left[\cos\tfrac{4\pi}{n} \pm j\sin\tfrac{4\pi}{n}\right] + --- + M_{n-1}\left[\cos\tfrac{2(n-1)\pi}{n} \pm j\sin\tfrac{2(n-1)\pi}{n}\right]\right\}$$

Now, for the polyphase circuit, it has been explained that $M_r = M_{n-r}$. Thus, as $\sin(\frac{2r\pi}{n}) = -\sin(\frac{2(n-r)\pi}{n})$, all the imaginary terms vanish, and the above expression reduces to:-

$$R_s + p\left\{L_s' + M_1\cos\tfrac{2\pi}{n} + M_2\cos\tfrac{4\pi}{n} + --- + M_{n-1}\cos\tfrac{2(n-1)\pi}{n}\right\}$$

<u>NOTE :-</u>

If <u>n</u> is even, there is a sine term which has no corresponding term of opposite sign. This is the term $M_{\frac{n}{2}}\,j\sin(\frac{n}{2}\cdot\frac{2\pi}{n})$. However, as $\sin(\frac{n}{2}\cdot\frac{2\pi}{n}) = \sin(\pi) = 0$, this term vanishes, and the rule is unaltered.

If we write :-

L_s = equiv. self-ind. of a phase $= L_s' + M_1\cos\frac{2\pi}{n} + M_2\cos\frac{4\pi}{n} + -- + M_{n-1}\cos\frac{2(n-1)\pi}{n}$ – (16)

and

$Z_{s(p)}$=equiv. impedance of a phase = $(R_s + pL_s)$ –(17)

then the current in phase 1 may be written :-

$$i_{s1} = -\tfrac{1}{2}Cp\frac{\varepsilon^{j\theta_0}}{Z_{s(p)}}\varepsilon^{j\omega t}F_{(p)}1 - \tfrac{1}{2}Cp\frac{\varepsilon^{-j\theta_0}}{Z_{s(p)}}\varepsilon^{-j\omega t}F_{(p)}1$$

$$= -\tfrac{1}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0}\varepsilon^{j\omega t}F_{(p)}1 + \varepsilon^{-j\theta_0}\varepsilon^{-j\omega t}F_{(p)}1\right\} \quad --(18)$$

For the current in phase 2, the same expression holds, except that $\varepsilon^{\pm j\theta_0}$ is replaced by $\varepsilon^{\pm j(\theta_0 - \frac{2\pi}{n})}$. This may be seen by putting the first of equations (8) to the last position, and moving the first column, (involving operations on <u>i_{s1}</u>), to the last column. The equations then become :-

(over)

$$
\left.\begin{aligned}
&(R_s + pL_s')i_{s2} + pM_1 i_{s3} + --- + pM_{n-2} i_{sn} + pM_{n-1} i_{s1} = -\tfrac{1}{2}Cp\left\{\varepsilon^{j(\omega t+\theta_0-\frac{2\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2\pi}{n})}\right\}F_{(p)}1 \\
&pM_{n-1} i_{s2} + (R_s + pL_s')i_{s3} + -- + pM_{n-3} i_{sn} + pM_{n-2} i_{s1} = -\tfrac{1}{2}Cp\left\{\varepsilon^{j(\omega t+\theta_0-\frac{4\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{4\pi}{n})}\right\}F_{(p)}1 \\
&-- \quad -- \quad -- \quad -- \quad -- \quad -- \quad -- \\
&pM_2 i_{s2} + pM_3 i_{s3} + -- + (R_s + pL_s')i_{sn} + pM_1 i_{s1} = -\tfrac{1}{2}Cp\left\{\varepsilon^{j(\omega t+\theta_0-\frac{2(n-1)\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2(n-1)\pi}{n})}\right\}F_{(p)}1 \\
&pM_1 i_{s2} + pM_2 i_{s3} + -- + pM_{n-1} i_{sn} + (R_s + pL_s')i_{s1} = -\tfrac{1}{2}Cp\left\{\varepsilon^{j(\omega t+\theta_0-\frac{2n\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2n\pi}{n})}\right\}F_{(p)}1
\end{aligned}\right\} \quad --(19)
$$

These equations are obviously identical with eqns (8) as far as the operational expression goes, except that for θ_0 in eqns (8), we have here $(\theta_0 - \frac{2\pi}{n})$. The operational solution for the variable of the first column - viz. $\underline{i}_{s2}$ - will therefore be identical with that previously obtained for $\underline{i}_{s1}$, except that $(\theta_0 - \frac{2\pi}{n})$ must be substituted for θ_0. We have, therefore :-

$$
\left.\begin{aligned}
i_{s1} &= -\tfrac{1}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0}\varepsilon^{j\omega t}F_{(p)}1 + \varepsilon^{-j\theta_0}\varepsilon^{-j\omega t}F_{(p)}1\right\} \\
i_{s2} &= -\tfrac{1}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j(\theta_0-\frac{2\pi}{n})}\varepsilon^{j\omega t}F_{(p)}1 + \varepsilon^{-j(\theta_0-\frac{2\pi}{n})}\varepsilon^{-j\omega t}F_{(p)}1\right\} \\
&-- \quad -- \quad -- \quad -- \quad -- \\
i_{sn} &= -\tfrac{1}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j(\theta_0-\frac{2(n-1)\pi}{n})}\varepsilon^{j\omega t}F_{(p)}1 + \varepsilon^{-j(\theta_0-\frac{2(n-1)\pi}{n})}\varepsilon^{-j\omega t}F_{(p)}1\right\}
\end{aligned}\right\} \quad --(20)
$$

Consider now the E.M.F. induced in the rotor winding by each of these currents. The currents relative to the rotor are obtained by operating on the true currents with the cosine of the angle between the rotor winding and the particular phase under consideration. Thus we get :-

$\underline{i}_{s1}$ rel. to rotor $= \frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} \times -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0}\varepsilon^{j\omega t}F_{(p)}1 + \varepsilon^{-j\theta_0}\varepsilon^{-j\omega t}F_{(p)}1\right\}$

Applying Heaviside's "Shifting" - (Part II, E1) - the exponential operators on the right may be moved to the left, past the term $-\frac{1}{2}(Cp/Z_{s(p)})$, and united with the exponentials on the left. In full, the steps are :-

$\underline{i}_{s1}$ rel. to rotor

$$
\begin{aligned}
&= \tfrac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} \times -\tfrac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{j\theta_0}\varepsilon^{j\omega t}F_{(p)}1 + \tfrac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} \times -\tfrac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{-j\theta_0}\varepsilon^{-j\omega t}F_{(p)}1 \\
&= \tfrac{1}{2}\left\{\varepsilon^{2j(\omega t+\theta_0)} + 1\right\} \times -\tfrac{1}{2}\frac{C(p+j\omega)}{Z_{s(p+j\omega)}}F_{(p)}1 + \tfrac{1}{2}\left\{\varepsilon^{-2j(\omega t+\theta_0)} + 1\right\} \times -\tfrac{1}{2}\frac{C(p-j\omega)}{Z_{s(p-j\omega)}}F_{(p)}1 \\
&= -\tfrac{1}{4}\left\{\varepsilon^{2j(\omega t+\theta_0)}\frac{C(p+j\omega)}{Z_{s(p+j\omega)}}F_{(p)}1 + \varepsilon^{-2j(\omega t+\theta_0)}\frac{C(p-j\omega)}{Z_{s(p-j\omega)}}F_{(p)}1 + \left[\frac{C(p+j\omega)}{Z_{s(p+j\omega)}} + \frac{C(p-j\omega)}{Z_{s(p-j\omega)}}\right]F_{(p)}1\right\} \quad --(21a)
\end{aligned}
$$

similarly, $\underline{i}_{s2}$ rel. to rotor

$$
= -\tfrac{1}{4}\left\{\varepsilon^{2j(\omega t+\theta_0-\frac{2\pi}{n})}\frac{C(p+j\omega)}{Z_{s(p+j\omega)}}F_{(p)}1 + \varepsilon^{-2j(\omega t+\theta_0-\frac{2\pi}{n})}\frac{C(p-j\omega)}{Z_{s(p-j\omega)}}F_{(p)}1 + \left[\frac{C(p+j\omega)}{Z_{s(p+j\omega)}} + \frac{C(p-j\omega)}{Z_{s(p-j\omega)}}\right]F_{(p)}1\right\} \quad --(21b)
$$

and so on for $\underline{i}_{s3}$, $\underline{i}_{s4}$, & c., where $Z_s(p \pm j\omega) = R_s + (p \pm j\omega)L_s$.

Hence the total stator current relative to the rotor, given by the sum of such equations as (21a), (21b), is :-

$$\sum_{r=1}^{r=n} \underline{i}_{sr}(\text{rel. to rotor}) = -\tfrac{1}{4}\left\{\varepsilon^{2j(\omega t+\theta_0)} + \varepsilon^{2j\left(\omega t+\theta_0-\frac{2\pi}{n}\right)} + \varepsilon^{2j\left(\omega t+\theta_0-\frac{4\pi}{n}\right)} + --\right\} \frac{C_{(p+j\omega)}}{Z_{s(p+j\omega)}} F_{(p)} 1$$
$$-\tfrac{1}{4}\left\{\varepsilon^{-2j(\omega t+\theta_0)} + \varepsilon^{-2j\left(\omega t+\theta_0-\frac{2\pi}{n}\right)} + \varepsilon^{-2j\left(\omega t+\theta_0-\frac{4\pi}{n}\right)} + --\right\} \frac{C_{(p-j\omega)}}{Z_{s(p-j\omega)}} F_{(p)} 1$$
$$-\tfrac{1}{4}\left\{1 + 1 + 1 + --\right\} \frac{C_{(p+j\omega)}}{Z_{s(p+j\omega)}} F_{(p)} 1$$
$$-\tfrac{1}{4}\left\{1 + 1 + 1 + --\right\} \frac{C_{(p-j\omega)}}{Z_{s(p-j\omega)}} F_{(p)} 1 \qquad \text{--(22)}$$

Each of the first two lines of eqn. (22) obviously sums to zero. Accordingly, we may write :-

$$\sum_{r=1}^{r=n} \underline{i}_{sr}(\text{rel. to rotor}) = -\tfrac{1}{4} n \frac{C_{(p+j\omega)}}{Z_{s(p+j\omega)}} F_{(p)} 1 - \tfrac{1}{4} n \frac{C_{(p-j\omega)}}{Z_{s(p-j\omega)}} F_{(p)} 1$$
$$= -\tfrac{1}{4} n C \left\{\frac{(p+j\omega) Z_{s(p-j\omega)} + (p-j\omega) Z_{s(p+j\omega)}}{Z_{s(p+j\omega)} Z_{s(p-j\omega)}}\right\} F_{(p)} 1 \qquad \text{--(23)}$$

Thus the E.M.F. induced in the rotor is :-

$$\text{E.M.F.} = +\tfrac{1}{4} n C p \left\{\frac{(p+j\omega) Z_{s(p-j\omega)} + (p-j\omega) Z_{s(p+j\omega)}}{Z_{s(p+j\omega)} Z_{s(p-j\omega)}}\right\} F_{(p)} 1 \qquad \text{--(24)}$$

and the rotor current term following F in the series is :-

$$i_r = +\tfrac{1}{4} n \frac{C^2 p}{Z_{r(p)}} \left\{\frac{(p+j\omega) Z_{s(p-j\omega)} + (p-j\omega) Z_{s(p+j\omega)}}{Z_{s(p+j\omega)} Z_{s(p-j\omega)}}\right\} F_{(p)} 1 \qquad \text{--(25)}$$

there being only one phase on the rotor. In eqn.(25), the impedance functions are :-

$Z_{r(p)} = R_r + pL_r$,

and $Z_{s(p\pm j\omega)} = R_s + (p \pm j\omega)L_s$.

Equation (25) may be simplified by considering the ratio ; $\left\{\frac{(p+j\omega) Z_{s(p-j\omega)} + (p-j\omega) Z_{s(p+j\omega)}}{Z_{s(p+j\omega)} Z_{s(p-j\omega)}}\right\}$.

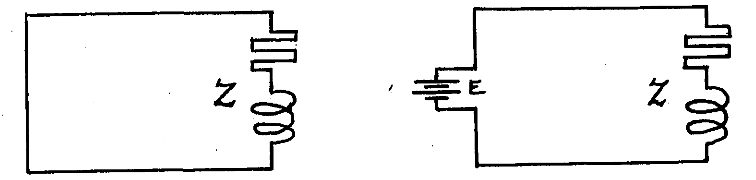
$$\text{Ratio} = \frac{(p+j\omega)\left[R_s + (p-j\omega)L_s\right] + (p-j\omega)\left[R_s + (p+j\omega)L_s\right]}{\left[R_s + (p+j\omega)L_s\right]\left[R_s + (p-j\omega)L_s\right]} = 2 \cdot \frac{pR_s + p^2L_s + \omega^2 L_s}{(R_s + pL_s)^2 + \omega^2 L_s^2} = 2 \cdot \frac{pR_s + p^2L_s + \omega^2 L_s}{Z_s^2(p) + \omega^2 L_s^2}$$

Therefore :-

$$i_r = +\tfrac{1}{2} n C^2 \frac{p}{Z_{r(p)}} \left\{\frac{pR_s + p^2 L_s + \omega^2 L_s}{Z_s^2(p) + \omega^2 L_s^2}\right\} F_{(p)} 1 \qquad \text{--(26)}$$

Fundamental to the development of this relation was the assumption that the initial term of rotor current was $F_{(p)} 1$, We shall now investigate this assumption, and see what conditions are implied by its adoption.

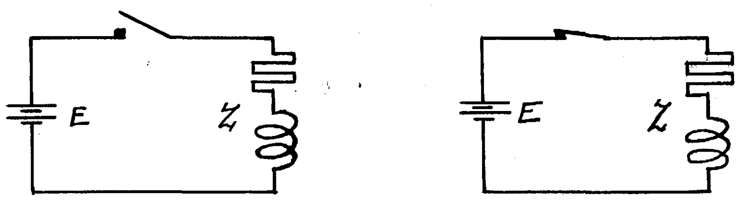
When we say that a voltage E1 is applied to a circuit, we mean that a circuit, originally short-circuited on itself, has a voltage E suddenly impressed in it at the time $\underline{t} = 0$. In actual practice, however, the mechanical arrangements are rather different. Generally the phenomenon is brought about, not by the sudden insertion of voltage in the short-circuited network, but by the sudden short-circuiting of a network which contains a voltage E, this voltage being inoperative until the network is closed. Figs 19 & 20 show the different arrangements.



a. Before $t = 0$. b. After $t = 0$.

Fig 19

Showing switching as assumed in mathematical development.



a. Before $t = 0$. b. After $t = 0$.

Fig 20

Showing switching as it usually takes place.

In this case, the resulting current in the circuit is identical for both arrangements, so that no error is introduced by considering a voltage $E1$.

In the case of the rotating machine, however, there may be an error due to a similar assumption, which may be seen as below.

By finding the stator current due to a current $F_{(p)}1$ in the rotor, we have actually assumed the stator to be short-circuited permanently on itself, and the current $F_{(p)}$ to be suddenly, by some extraneous source, impressed into the rotor circuit. This assumption is true for all the rotor current terms except the first, since the stator must be short-circuited before the second, third & c. terms of rotor current can come into existence. The first term of rotor current is however an exception to the general rule.

For consider the implication of the statement that the initial rotor current is $I_f 1$, where I_f is the normal steady field current. It would be, that up to time $t = 0$, the stator and rotor were both short-circuited, and carried no currents, but that at $t = 0$, the rotor current instantaneously assumed the value I_f, this producing an infinite surge of voltage in the stator windings. No such surge actually takes place, for the true state of affairs is that I_f exists in the rotor both before and after $t = 0$. An exact equivalent to the phenomenon could be obtained by saying that the rotor current was I_f, the stator permanently short-circuited, and the rotor velocity impulsive, - equal to $\omega 1$, so that its displacement would be $(\omega 1 t + \theta_0)$ at time t. Such a statement, however, is of little use, for we cannot deal with the function turning up in the index of an exponential.

The best way to get the true state of affairs is to consider the field current as $I_f 1$, and then to make allowance for the surge of voltage at $t = 0$ implied in the statement. For the values of $t > 0$, the assumption of $I_f 1$ as the first term of the field current series is obviously true. A similar phenomenon occurred in the operational treatment of the interruption of a network, which was shown in Part II, D1. Here the condenser current was equal to the whole current, except that it was impulsive at $t = 0$, and account had to be taken of that impulse when writing the equations.

The first term of rotor current may thus be said to be $I_f 1$. Hence :-

i_r rel. to stator ph. 1 $= \frac{1}{2}\left\{\varepsilon^{i(\omega t+\theta_0)} + \varepsilon^{-i(\omega t+\theta_0)}\right\} I_f 1$ --(27)

E.M.F. apparently induced in stator ph. 1 :-

$$\text{E.M.F.} = -\tfrac{1}{2} C p \left\{\varepsilon^{i(\omega t+\theta_0)} + \varepsilon^{-i(\omega t+\theta_0)}\right\} I_f 1 \qquad \text{--(28)}$$

This E.M.F. however involves a surge of voltage and current at $t = 0$, as explained above, which has no existence in fact.

This surge of current is given by $I_f 1$ in the rotor. This, relative to stator ph. 1, is $\frac{1}{2}\left\{\varepsilon^{i\theta_0} + \varepsilon^{-i\theta_0}\right\} I_f 1$, as $t = 0$ for the duration of the surge. Thus the E.M.F. induced in the stator by the surge is given by :-

$$\text{E.M.F.} = -\tfrac{1}{2} C p \left\{\varepsilon^{i\theta_0} + \varepsilon^{-i\theta_0}\right\} I_f 1 \qquad \text{--(29)}$$

The true E.M.F. induced in stator ph. 1 is that given by eqn.(28) diminished by that given by eqn.(29). Hence, true E.M.F. in stator ph. 1 is :-

(over)

True E.M.F. in ph. 1 $= -\frac{1}{2} C_p \left\{ \varepsilon^{j(\omega t+\theta_0)} - \varepsilon^{j\theta_0} + \varepsilon^{-j(\omega t+\theta_0)} - \varepsilon^{-j\theta_0} \right\} I_t 1$ --(30)

Thus, $\underline{i}_{s1} = -\frac{1}{2} \frac{C_p}{Z_{s(p)}} \left\{ \varepsilon^{j(\omega t+\theta_0)} - \varepsilon^{j\theta_0} + \varepsilon^{-j(\omega t+\theta_0)} - \varepsilon^{-j\theta_0} \right\} I_t 1$ --(31)

where $Z_{s(p)}$ is derived from the characteristics of the stator windings by the determinantal methods already discussed, and is, of course, given by equation (17).

By expanding $\varepsilon^{\pm j\omega t} 1$ as an operational expression, eqn.(31) may be written :-

$$i_{s1} = -\frac{1}{2} C_p \frac{1}{Z_{s(p)}} \left\{ \frac{p}{p-j\omega} \varepsilon^{j\theta_0} - \varepsilon^{j\theta_0} + \frac{p}{p+j\omega} \varepsilon^{-j\theta_0} - \varepsilon^{-j\theta_0} \right\} I_t 1$$

$$= -\frac{1}{2} \frac{C_p}{Z_{s(p)}} \left\{ \varepsilon^{j\theta_0} \frac{j\omega}{p-j\omega} + \varepsilon^{-j\theta_0} \frac{-j\omega}{p+j\omega} \right\} I_t 1 \quad \text{--(32)}$$

Similar expressions hold for $\underline{i}_{s2}$, $\underline{i}_{s3}$, & c.

For $\underline{i}_{s1}$ relative to the rotor, the cosine operator may be applied directly to this expression, since there is now no spurious surge to take account of. Thus :-

$\underline{i}_{s1}$ rel. to rotor $= \frac{1}{2} \left\{ \varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)} \right\} \times -\frac{1}{2} \frac{C_p}{Z_{s(p)}} \left\{ \varepsilon^{j(\omega t+\theta_0)} - \varepsilon^{j\theta_0} + \varepsilon^{-j(\omega t+\theta_0)} - \varepsilon^{-j\theta_0} \right\} I_t 1$ --(33)

By "Shifting" the left-hand exponentials to the right :-

$\underline{i}_{s1}$ rel. to rotor $= \frac{1}{2} \left\{ -\frac{1}{2} \frac{C(p-j\omega)}{Z_s(p-j\omega)} \right\} \left\{ \varepsilon^{2j(\omega t+\theta_0)} - \varepsilon^{j(\omega t+2\theta_0)} + 1 - \varepsilon^{j\omega t} \right\} I_t 1$

$+ \frac{1}{2} \left\{ -\frac{1}{2} \frac{C(p+j\omega)}{Z_s(p+j\omega)} \right\} \left\{ \varepsilon^{-2j(\omega t+\theta_0)} - \varepsilon^{-j(\omega t+2\theta_0)} + 1 - \varepsilon^{-j\omega t} \right\} I_t 1$ --(34)

As before, when the total effect of all the stator currents is taken, terms involving θ_0 vanish. Thus:-

Total stator current rel. to rotor

$$= -\frac{nC}{4} \left\{ \frac{p-j\omega}{Z_s(p-j\omega)} \right\} \left\{ 1 - \varepsilon^{j\omega t} \right\} I_t 1 - \frac{nC}{4} \left\{ \frac{p+j\omega}{Z_s(p+j\omega)} \right\} \left\{ 1 - \varepsilon^{-j\omega t} \right\} I_t 1 \quad \text{--(35)}$$

Simplifying this expression as for equation (32):-

$\sum_{r=1}^{r=n} \underline{i}_{sr}$ rel. to rotor $= -\frac{nC}{4} \left\{ \frac{p-j\omega}{Z_s(p-j\omega)} \left[1 - \frac{p}{p-j\omega} \right] I_t 1 + \frac{p+j\omega}{Z_s(p+j\omega)} \left[1 - \frac{p}{p+j\omega} \right] I_t 1 \right\}$

$$= -\frac{nC}{4} \left\{ \frac{-j\omega}{Z_s(p-j\omega)} + \frac{j\omega}{Z_s(p+j\omega)} \right\} I_t 1 \quad \text{--(36)}$$

Hence, current induced in rotor :-

$$i_r = +\frac{1}{4} \frac{nC^2 p}{Z_{r(p)}} \left\{ \frac{-j\omega}{Z_s(p-j\omega)} + \frac{j\omega}{Z_s(p+j\omega)} \right\} I_t 1 \quad \text{--(37)}$$

or $$i_r = +\frac{1}{4} \frac{nC^2 p}{Z_r(p)} \left\{ \frac{-j\omega Z_s(p+j\omega) + j\omega Z_s(p-j\omega)}{Z_s(p-j\omega) Z_s(p+j\omega)} \right\} I_t 1$$

$$= +\frac{n}{2} \frac{C^2 p \omega^2 L_s}{Z_r(p)} \left\{ \frac{1}{(R_s + pL_s)^2 + \omega^2 L_s^2} \right\} I_t 1 \quad \text{--(38)}$$

For all other rotor current terms after this, (which is the second of the series), the rotor current is truly impulsive at $\underline{t} = 0$, for all terms contain the unit function. Thus the assumption $\underline{i}_r = F(p) 1$ is valid for all remaining terms.

We have therefore derived the complete series of terms for the rotor switching current. In order to see the series and the connection between its various terms more clearly, it is put out in detail below :-

Rotor Current Series.

-1st Term $= I_f$

2nd Term $= \frac{n}{2}\frac{C^2 p\,\omega^2 L_s}{Z_r(p)}\left\{\frac{1}{Z_s^2(p)+\omega^2 L_s^2}\right\} I_f 1$

3rd Term $= \frac{n}{2}\frac{C^2 p}{Z_r(p)}\left\{\frac{pR_s+(p^2+\omega^2)L_s}{Z_s^2(p)+\omega^2 L_s^2}\right\} \times \frac{n}{2}\frac{C^2 p\,\omega^2 L_s}{Z_r(p)}\left\{\frac{1}{Z_s^2(p)+\omega^2 L_s^2}\right\} I_f 1$

(Obtained by operating on 2nd term with the operator of eqn(26))

4th Term $= \left\{\frac{n}{2}\frac{C^2 p}{Z_r(p)}\left[\frac{pR_s+(p^2+\omega^2)L_s}{Z_s^2(p)+\omega^2 L_s^2}\right]\right\}^2 \times \frac{n}{2}\frac{C^2 p\,\omega^2 L_s}{Z_s(p)}\left\{\frac{1}{Z_s^2(p)+\omega^2 L_s^2}\right\} I_f 1$

(Obtained by operating on 3rd term.)

and so on.

If the first term, I_f, be for the moment neglected, it is apparent that the series for the rotor current is a geometric progression, having its first term equal to $\frac{n}{2}\frac{C^2 p\,\omega^2 L_s}{Z_r(p)}\left[\frac{1}{Z_s^2(p)+\omega^2 L_s^2}\right] I_f 1$ and its common ratio equal to $\frac{n}{2}\frac{C^2 p}{Z_r(p)}\left[\frac{pR_s+(p^2+\omega^2)L_s}{Z_s^2(p)+\omega^2 L_s^2}\right]$.

Expanding the common ratio, we see that it may be written :-

$$\text{Ratio} = \frac{n}{2}\cdot\frac{C^2 p}{(R_r+pL_r)}\cdot\frac{(p^2+\omega^2)L_s+pR_s}{L_s\left\{(p^2+\omega^2)L_s+2pR_s+\frac{R_s^2}{L_s^2}\right\}} = \frac{\frac{n}{2}C^2 p}{(R_r+pL_r)L_s}\cdot\frac{(p^2+\omega^2)L_s+pR_s}{[(p^2+\omega^2)L_s+pR_s]+pR_s+\frac{R_s^2}{L_s^2}} \quad \text{--(39)}$$

Obviously the second factor is less than unity. Considering the first, we see that it is less than $\frac{\frac{n}{2}C^2}{L_r L_s}$.

Now L_s has been given as :-

$$L_s = L_s' + M_1\cos\frac{2\pi}{n} + M_2\cos\frac{4\pi}{n} - - - - \quad \text{--(16)}$$

For an approximately sinusoidal distribution of the flux due to a phase :-

$$M_1 \doteq L_s'\cos\frac{2\pi}{n}, \quad M_2 \doteq L_s'\cos\frac{4\pi}{n}, \quad \&\text{c}.$$

Hence, $L_s \doteq L_s' + L_s'\cos^2\frac{2\pi}{n} + L_s'\cos^2\frac{4\pi}{n} + - - -$

$$\doteq L_s'\left\{1 + \tfrac{1}{2}\left(1+\cos\frac{4\pi}{n}\right) + \tfrac{1}{2}\left(1+\cos\frac{8\pi}{n}\right) + - - -\right\}$$

or $L_s \doteq \frac{n}{2}L_s'$ --(40)

Thus, $\frac{\frac{n}{2}C^2}{L_r L_s} \doteq \frac{C^2}{L_r L_s'} < 1$ --(41)

(Note that the ratio L_s/L_s' for 3-phase working is $\frac{3}{2}$, as was established in Part I, E1.)

Thus the common ratio is less than unity.* This might have been said at once, since for the rotor current to be finite, higher terms in the series must tend to vanish.

---

*Appendix C.

If a G.P. is given by :-

$$a + ar + ar^2 + ar^3 + - - - + ar^{n-1}$$

then the sum to $\underline{n}$ terms is given by :-

$$S_n = a(1 - r^n)/(1 - r) \qquad \text{--(42)}$$

or, if $\underline{r}$ is less than unity, and $\underline{n} \rightarrow \infty$:-

$$S_n = a/(1 - r) - ar^{\infty}/(1 - r) = a/(1 - r) \qquad \text{--(43)}$$

Thus, summing the rotor terms to infinity, we may write for the total rotor current :-

$$\underline{i}_r = \frac{\frac{n}{2}\frac{C^2 p\omega^2 L_s}{Z_{r(p)}}\left\{\frac{1}{(R_s+pL_s)^2+\omega^2 L_s^2}\right\} I_f 1}{1 - \frac{n}{2}\frac{C^2 p}{Z_{r(p)}}\left\{\frac{pR_s + (p^2+\omega^2)L_s}{(R_s+pL_s)^2+\omega^2 L_s^2}\right\}} \qquad \text{--(44)}$$

to which must be added the first term, the normal field current, I_f.

Simplifying the above expression :-

$$\underline{i}_r = \frac{\frac{n}{2} C^2 p\omega^2 L_s I_f 1}{Z_{r(p)}\left\{(R_s+pL_s)^2+\omega^2 L_s^2\right\} - \frac{n}{2} C^2 p\left\{pR_s + (p^2+\omega^2)L_s\right\}} \qquad \text{--(45)}$$

or

$$\underline{i}_r = \frac{\frac{n}{2} C^2 p\omega^2 L_s I_f 1}{p^3 L_s(L_s L_r - \frac{n}{2}C^2) + p^2\left\{R_s(L_s L_r - \frac{n}{2}C^2) + L_s(R_s L_r + R_r L_s)\right\} + p\left\{\omega^2 L_s(L_s L_r - \frac{n}{2}C^2) + L_r R_s^2 + 2L_s R_r R_s\right\} + R_r(R_s^2+\omega^2 L_s^2)} \qquad \text{--(46)}$$

$$= \frac{Y_{(p)}}{Z_{(p)}} 1$$

It is of interest that the equation $Z_{(p)} = 0$ is in fact the auxiliary equation for the rotor current as developed by Shimidzu and Ito, (Part I, eqn. 36,) with the number of phases made $\underline{n}$ instead of 3. The symbol $\underline{b}$ used in Part 1 is equivalent to the symbol L_s as used above, but for a 3-phase case; i.e., $\underline{b}$ was defined as $\underline{b}' - \underline{c}'$, where $\underline{b}'$ was self-inductance of a phase, and $\underline{c}'$ was mutual inductance between two phases. L_s for a 3-phase case would be given by :-

$$L_s = L_s' + M_1 \cos\frac{2\pi}{n} + M_2 \cos\frac{4\pi}{n} = L_s' + M_1 \times -\tfrac{1}{2} + M_2 \times -\tfrac{1}{2}$$

$$= L_s' - M_1 \quad , \quad \text{as } M_1 = M_2 \text{ for 3-phase case.}$$

Since the equation $Z_{(p)} = 0$ gives the natural frequencies of the transient, this relationship is what we should expect. This development, however, gives a full operational expression for the current, which may be evaluated by the Heaviside Expansion Theorem. This will be done later. For the moment, the above method will be reconsidered, and simplified.

It seems at first sight, that the method is anything but simple, and certainly no improvement on the direct mathematical method. This is, however, on account of the fact that every step was analysed completely as it was made; particularly were the solution of the n simultaneous equations for the stator current, and the irregularity of the first reflected term of rotor current considered in detail. If we bear these two peculiarities in mind from the start, not only will the method be revealed as very simple, but also the connection between it and the next method will be apparent.

Proceeding on this principle, the establishment of the operational solution for the rotor current may be accomplished as below :-

First term of i_r $= I_f 1$

Rel. to ph.1, this is $\frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} I_f 1$

$$= \frac{1}{2}\left\{\varepsilon^{j\theta_0}\frac{p}{p-j\omega} + \varepsilon^{-j\theta_0}\frac{p}{p+j\omega}\right\} I_f 1 \qquad \text{--(47)}$$

E.M.F. induced in ph.1 $= -\frac{1}{2} C p \left\{\varepsilon^{j\theta_0}\frac{p}{p-j\omega} + \varepsilon^{-j\theta_0}\frac{p}{p+j\omega} - \varepsilon^{j\theta_0} - \varepsilon^{-j\theta_0}\right\} I_f 1$ --(48)

(Where the last two terms take account of the implied surge.)

Thus, E.M.F. in ph.1 $= -\frac{1}{2} C p \left\{\varepsilon^{j\theta_0}\left[\frac{p}{p-j\omega} - \frac{p-j\omega}{p-j\omega}\right] + \varepsilon^{-j\theta_0}\left[\frac{p}{p+j\omega} - \frac{p+j\omega}{p+j\omega}\right]\right\} I_f 1$

$$= -\frac{1}{2} C p \left\{\varepsilon^{j\theta_0}\frac{j\omega}{p-j\omega} + \varepsilon^{-j\theta_0}\frac{-j\omega}{p+j\omega}\right\} I_f 1 \qquad \text{--(49)}$$

Hence, $i_{s1} = -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0}\frac{j\omega}{p-j\omega} + \varepsilon^{-j\theta_0}\frac{-j\omega}{p+j\omega}\right\} I_f 1$ --(32)

(Where $Z_{s(p)}$ is given by equation (17).)

i_{s1} rel. to rotor $= \frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} x - \frac{1}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0}\frac{j\omega}{p-j\omega} + \varepsilon^{-j\theta_0}\frac{-j\omega}{p+j\omega}\right\} I_f 1$ --(50)

Taking the total stator current relative to the rotor, and remembering that when all the phases are summed, terms involving θ_0 vanish, and the others are increased n-fold :-

$\sum_{r=1}^{r=n} i_{sr}$ rel. to rotor $= \frac{n}{2}\varepsilon^{j(\omega t+\theta_0)} x - \frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{-j\theta_0}\frac{-j\omega}{p+j\omega} I_f 1 + \frac{n}{2}\varepsilon^{-j(\omega t+\theta_0)} x - \frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{j\theta_0}\frac{j\omega}{p-j\omega} I_f 1$

or, by shifting :-

$\sum_{r=1}^{r=n} i_{sr}$ rel. to rotor $= \frac{n}{2} x - \frac{1}{2}\frac{C(p-j\omega)}{Z_{s(p-j\omega)}}\cdot\frac{-j\omega}{(p+j\omega-j\omega)}\frac{p}{(p-j\omega)} I_f 1 + \frac{n}{2} x - \frac{1}{2}\frac{C(p+j\omega)}{Z_{s(p+j\omega)}}\frac{j\omega}{(p-j\omega+j\omega)}\frac{p}{(p+j\omega)} I_f 1$

$$= -\frac{nC}{4}\left\{\frac{-j\omega}{Z_{s(p-j\omega)}} + \frac{j\omega}{Z_{s(p+j\omega)}}\right\} I_f 1 \qquad \text{--(36)}$$

Hence, E.M.F. in rotor $= +\frac{nC^2p}{4}\left\{\frac{-j\omega}{Z_{s(p-j\omega)}} + \frac{j\omega}{Z_{s(p+j\omega)}}\right\} I_f 1$ --(51)

Hence, 2nd term of rotor current $= \frac{n}{4}\frac{C^2 p I_f}{Z_{r(p)}}\left\{\frac{-j\omega}{Z_{s(p-j\omega)}} + \frac{j\omega}{Z_{s(p+j\omega)}}\right\} 1$

$$= \frac{n}{2}\frac{C^2 p\omega^2 L_s I_f}{Z_{r(p)}}\left\{\frac{1}{Z^2_{s(p)} + \omega^2 L_s^2}\right\} 1 \qquad \text{--(38)}$$

For all subsequent terms of rotor current :-

Rotor current $= F_{(p)}1$

$\underline{i}_r$ rel. to ph.1 $= \frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} F_{(p)}1$

$$= \frac{1}{2}\left\{\varepsilon^{j\theta_0} F_{(p-j\omega)} \frac{p}{p-j\omega}1 + \varepsilon^{-j\theta_0} F_{(p+j\omega)} \frac{p}{p+j\omega}1\right\} \qquad \text{--(52)}$$

Hence, $i_{s_1} = -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0} F_{(p-j\omega)} \frac{p}{p-j\omega}1 + \varepsilon^{-j\theta_0} F_{(p+j\omega)} \frac{p}{p+j\omega}1\right\}$ --(53)

i_{s_1} rel. to rotor $= \frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} \times -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0} F_{(p-j\omega)} \frac{p}{p-j\omega}1 + \varepsilon^{-j\theta_0} F_{(p+j\omega)} \frac{p}{p+j\omega}1\right\}$

Hence, $\sum_{r=1}^{r=n} i_{sr}$ rel. to rotor $= \frac{n}{2} \times -\frac{1}{2}\frac{C(p+j\omega)}{Z_{s(p+j\omega)}} F_{(p)} \frac{p+j\omega}{p}\frac{p}{p+j\omega}1 + \frac{n}{2} \times -\frac{1}{2}\frac{C(p-j\omega)}{Z_{s(p-j\omega)}} F_{(p)} \frac{p-j\omega}{p}\frac{p}{p-j\omega}1$

$$= -\frac{nC}{4}\left\{\frac{p+j\omega}{Z_{s(p+j\omega)}} + \frac{p-j\omega}{Z_{s(p-j\omega)}}\right\} F_{(p)}1 \qquad \text{--(54)}$$

Hence, $i_r = +\frac{n}{4}\frac{C^2 p}{Z_{r(p)}}\left\{\frac{p+j\omega}{Z_{s(p+j\omega)}} + \frac{p-j\omega}{Z_{s(p-j\omega)}}\right\} F_{(p)}1$

$$= +\frac{n}{2}\frac{C^2 p}{Z_{r(p)}}\left\{\frac{R_s p + (p^2+\omega^2)L_s}{(R_s + pL_s)^2 + \omega^2 L_s^2}\right\} F_{(p)}1 \qquad \text{--(26)}$$

Thus, the sum of the G.P. for the rotor current is given by :-

$$i_r = \frac{\frac{n}{2}C^2 p\,\omega^2 L_s I_f 1}{\left\{Z_{s(p)}^2 + \omega^2 L_s^2\right\} Z_{r(p)} - \frac{n}{2}C^2 p\left\{R_s p + (p^2+\omega^2)L_s\right\}} \qquad \text{--(45)}$$

which equation was shown to simplify to equation (46).

The development may viewed in toto when condensed into this brief compass. The advantage of this "Bird's eye view" will become apparent when the next method of solving the problem is considered.

A similar series for the stator current could be developed. However, the next method is shorter both as regards rotor and stator currents, so the development of of the stator current expression will be left until that method is considered.

Consider now the operational expression already developed for $\underline{i}_r$, given by equation (46). This equation is reproduced below :-

$$i_r = \frac{\frac{n}{2}C^2 p\,\omega^2 L_s I_f 1}{p^3 L_s(L_s L_r - \frac{n}{2}C^2) + p^2\left\{R_s(L_s L_r - \frac{n}{2}C^2) + L_s(R_s L_r + R_r L_s)\right\} + p\left\{\omega^2 L_s(L_s L_r - \frac{n}{2}C^2) + L_r R_s^2 + 2L_s R_s R_r\right\} + R_r(R_s^2 + \omega^2 L_s^2)} \qquad \text{--(46)}$$

If $\frac{L_s L_r - \frac{n}{2}C^2}{L_s L_r} = \sigma'$, as was taken by Shimidzu and Ito, so that $\frac{n}{2}C^2 = L_s L_r(1-\sigma')$, and if $\frac{R_s}{L_s} = S$, and $\frac{R_r}{L_r} = r$, then the equation becomes :-

$$i_r = \frac{\frac{1-\sigma'}{\sigma'}\omega^2 I_f p 1}{p^3 + p^2\left\{S + \frac{S+r}{\sigma'}\right\} + p\left\{\omega^2 + S\frac{S+r}{\sigma'} + \frac{rS}{\sigma'}\right\} + r\frac{S^2+\omega^2}{\sigma'}} \qquad \text{--(55)}$$

If $Z_{(p)}$ is the denominator of the expression in equation (55), let the auxiliary equation, $Z_{(p)} = 0$, have the three roots, α, $\mu + j\vartheta$, $\mu - j\vartheta$. This assumption of a pair of imaginary roots is generally valid when short-circuit phenomena are being considered - i.e., when r and s are small, - as shown by Shimidzu. (Theory of Distortionless Alternators, see Bibliography.) If all the roots are real, the solution of equation (55) is however quite straightforward.

Equation (55) may be written :-

$$i_r = \frac{\frac{1-\sigma'}{\sigma'}\,\omega^2 I_f\,p}{(p-\alpha)\{p-(\mu+j\vartheta)\}\{p-(\mu-j\vartheta)\}}\,1 \qquad \text{--(56)}$$

Applying the expansion theorem :-

$$\frac{Y_{(0)}}{Z_{(0)}} = 0 \qquad \text{--(57)}$$

$$\frac{Y_{(\alpha)}}{\alpha Z'_{(\alpha)}}\,\varepsilon^{\alpha t} = \frac{\frac{1-\sigma'}{\sigma'}\,\omega^2 I_f}{(\alpha-\mu)^2+\vartheta^2}\,\varepsilon^{\alpha t} \qquad \text{--(58)}$$

$$\frac{Y_{(\mu+j\vartheta)}}{(\mu+j\vartheta)Z'_{(\mu+j\vartheta)}}\,\varepsilon^{(\mu+j\vartheta)t} = \frac{\frac{1-\sigma'}{\sigma'}\,\omega^2 I_f}{\{(\mu-\alpha)+j\vartheta\}\,2j\vartheta}\,\varepsilon^{(\mu+j\vartheta)t} \qquad \text{--(59)}$$

and the substitution of the third root obviously gives the complement of eqn. (59).

The term of rotor current in eqn. (59) may be written :-

$$i_r = -\frac{1-\sigma'}{\sigma'}\,\frac{\omega^2 I_f}{2\vartheta}\left\{\frac{\vartheta+j(\mu-\alpha)}{(\mu-\alpha)^2+\vartheta^2}\right\}\varepsilon^{(\mu+j\vartheta)t}\,1$$

or $$i_r = -\frac{1-\sigma'}{\sigma'}\,\frac{\omega^2 I_f}{2\vartheta}\,\frac{1}{\sqrt{(\mu-\alpha)^2+\vartheta^2}}\,\varepsilon^{j\phi}\,\varepsilon^{(\mu+j\vartheta)t}\,1 \qquad \text{--(60)}$$

where $\phi = \tan^{-1}\frac{\mu-\alpha}{\vartheta}$.

Combining this term with its complement :-

$$i_r = -\frac{1-\sigma'}{\sigma'}\,\omega^2 I_f\,\frac{1}{\vartheta\sqrt{(\mu-\alpha)^2+\vartheta^2}}\,\varepsilon^{\mu t}\cos(\vartheta t+\phi)\,1 \qquad \text{--(61)}$$

Thus the full expression for the rotor current is :-

$$i_r = \frac{1-\sigma'}{\sigma'}\,\omega^2 I_f\left\{\frac{1}{(\alpha-\mu)^2+\vartheta^2}\,\varepsilon^{\alpha t} - \frac{1}{\vartheta\sqrt{(\mu-\alpha)^2+\vartheta^2}}\,\varepsilon^{\mu t}\cos(\vartheta t+\phi)\right\}1 \qquad \text{--(62)}$$

to which, of course, must be added the constant term of current, I_f.

If r and s in eqn. (55) are small compared with ω, and σ' is not too small, then the roots of the auxiliary equation may be written :-

$$\alpha \doteqdot -\frac{r}{\sigma'}\,, \qquad \mu \pm j\vartheta \doteqdot -\frac{1}{2}\cdot\frac{1+\sigma'}{\sigma'}\cdot s \pm j\omega\,. \qquad \text{--(63)}$$

For proof of these statements, see "Theory of Distortionless Alternators", referred to above.

Thus α^2 and μ^2 may be neglected when compared with ω^2, and $\phi = \tan^{-1}\frac{\mu-\alpha}{\omega} \doteq 0$.

Hence, equation (62) reduces to :-

$$i_r = \frac{1-\sigma'}{\sigma'} I_f \left\{ \varepsilon^{\alpha t} - \varepsilon^{\mu t} \cos \omega t \right\} 1 \qquad \text{--(64)}$$

as a first approximation.

This equation is identical with that obtained by Shimidzu and Ito by direct mathematical methods, - Part 1, eqn. (38). The symbol t in their equation, it should be remembered, represents time after the rotor has passed zero position, the short-circuit taking place at $t = t_0$. Thus the t of the equation given above - (64) - is really equivalent to $(t - t_0)$ in the equation of Shimidzu and Ito.

The establishment of the arbitrary constants is very simple by the above method, - it is not even necessary to evaluate the stator currents. The first approximation given by eqn.(64) is sufficiently accurate for all practical purposes, since absolute accuracy is in any case impossible owing to the initial assumptions, which neglect saturation and so on.

The stator currents have not been evaluated, since the next method of attack to be described provides a much easier way of establishing their magnitudes than expanding in a series. The method of reflections, as has been stated, has its main use in the fact that it leads on to this next method, which will now be considered.

2. Method of Equivalent Circuits.

It was shown by the method of reflections that a current of the form $F_{(p)}1$ in the rotor produces a current in the stator, phase 1, given by the equation :-

$$i_{s_1} = -\frac{1}{2}\frac{C_p}{Z_{s(p)}} \left\{ \varepsilon^{j\theta} F_{(p-j\omega)} \frac{p}{p-j\omega} 1 + \varepsilon^{-j\theta_0} F_{(p+j\omega)} \frac{p}{p+j\omega} 1 \right\} \qquad \text{--(52)}$$

$Z_{s(p)}$ being a function of p taking account of the various mutual impedances between the stator phases.

Further, this stator current when combined with the currents in the other stator phases produces a current in the rotor given by :-

$$i_r = + \frac{\pi}{4} \frac{pC^2}{Z_{r(p)}} \left\{ \frac{p+j\omega}{Z_{s(p+j\omega)}} F_{(p)} 1 + \frac{p-j\omega}{Z_{s(p-j\omega)}} F_{(p)} 1 \right\} \qquad \text{--(65)}$$

where $Z_{r(p)}$ is given by $(R_r + pL_r)$.

These equations hold whatever the value of $F_{(p)}1$, provided only that the expression truly denotes an operation on the unit function, 1. The equations thus hold for all terms of the rotor current series except the first, which is the normal field current, I_f.

If the whole rotor transient current is defined by $F_{(p)}1$, then $F_{(p)}1$ is the sum of all the rotor current terms but the first, and the same relations must hold.

Thus the whole stator current,-except for the first term, due to I_f,- would be given by the relation in eqn.(52).

Suppose the operation of obtaining the stator current from the rotor current could be expressed by a single operator, say $\phi_{(p)}$, and that of obtaining the rotor current from the stator current by another single operator, say $\psi_{(p)}$. This operator $\psi_{(p)}$ would hold for all the stator current terms, since all are impulsive at $t = 0$, and we could write two simultaneous operational equations for the ee whole arrangement :-

$$\begin{aligned} i_{s1} &= \phi_{(p)}\, i_r \;+\; (\text{Stator current phase 1 due to } I_f) \\ i_r &= \psi_{(p)}\, i_{s1} \end{aligned} \qquad \text{-(66)}$$

i_r being understood to indicate only the transient part of the rotor current.

These equations could then be solved by ordinary algebraic methods. The crux of the matter lies in the nature of the operators $\phi_{(p)}$ and $\psi_{(p)}$, which we shall now proceed to investigate.

The operator $\phi_{(p)}$ is obvious by inspection. It is apparently given by :-

$$\phi_{(p)} = -\frac{1}{2} \frac{Cp}{Z_{s(p)}} \left\{ \varepsilon^{i(\omega t+\theta_0)} + \varepsilon^{-i(\omega t+\theta_0)} \right\} \qquad \text{--(67)}$$

for this operator, applied to the rotor current, gives an expression for the stator current equal to that given in equation (52).

The operator $\psi_{(p)}$ is not by any means so obvious. It is an operator which, applied to an expression such as the right-hand side of equation (52) will give an expression such as the right-hand side of equation (65). The difficulty is that this "Reflection" is really the combined effect of three - or n - phases, this combining causing all terms involving θ_0 to disappear. With any single operator, terms such as $\varepsilon^{j2\theta_0}$ turn up; there is not at first glance any way to get rid of them if we try to treat the stator winding as an equivalent single phase.

Actually, however, the desired expression for the rotor current may be obtained by operating on the two halves of the stator current expression with two different operators. e.g.,

$$\text{Stator current} = -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0}F_{(p-j\omega)}\frac{p}{p-j\omega}1 + \varepsilon^{-j\theta_0}F_{(p+j\omega)}\frac{p}{p+j\omega}1\right\} \qquad \text{--(52)}$$

Operate on the first half of this expression with the operator $-\frac{n}{2}Cp\frac{1}{Z_{r(p)}}\varepsilon^{-j(\omega t+\theta_0)}$, and on the second half with the operator $-\frac{n}{2}Cp\frac{1}{Z_{r(p)}}\varepsilon^{j(\omega t+\theta_0)}$. Then we get :-

$$i_r = -\frac{n}{2}\frac{Cp}{Z_{r(p)}}\varepsilon^{-j(\omega t+\theta_0)}\left\{-\frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{j\theta_0}F_{(p-j\omega)}\frac{p}{p-j\omega}1\right\} - \frac{n}{2}\frac{Cp}{Z_{r(p)}}\varepsilon^{j(\omega t+\theta_0)}\left\{-\frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{-j\theta_0}F_{(p+j\omega)}\frac{p}{p+j\omega}1\right\} \qquad \text{--(68)}$$

or, by "Shifting",

$$i_r = +\frac{n}{4}\frac{C^2p}{Z_{r(p)}}F_{(p)}\frac{p+j\omega}{Z_{s(p+j\omega)}}\frac{p+j\omega}{p}\frac{p}{p+j\omega}1 + \frac{n}{4}\frac{C^2p}{Z_{r(p)}}F_{(p)}\frac{p-j\omega}{Z_{s(p-j\omega)}}\frac{p-j\omega}{p}\frac{p}{p-j\omega}1$$

$$= +\frac{n}{4}\frac{C^2p}{Z_{r(p)}}\left\{\frac{p+j\omega}{Z_{s(p+j\omega)}} + \frac{p-j\omega}{Z_{s(p-j\omega)}}\right\}F_{(p)}1$$

which, being identical with equation (65), is the desired expression.

This indicates that the desired transformations may be made, not by regarding the whole stator system as a single circuit, but by first reducing to this single circuit, and then splitting it up into two parts, the current in each part being given by the two halves of the expression in equation (52); these currents are actually imaginary, since the operators for obtaining them from the rotor current are imaginary, but their sum is real, and is the true stator current, phase 1. Further, equation (52) gives the full stator current term, and yet only involves terms produced in the two hypothetical stator circuits by the rotor current. Thus it appears that the two stator circuits must be regarded as mutually independant, since the current terms of each contain no terms produced by the other - that is, their mutual impedance is zero. The diagram, Fig 21 , shows two complete reflections of rotor current, and serves to illustrate the above reasoning.

Note that the two terms of rotor current resulting from the two stator currents are condensed into one term. This is necessary, for although the stator circuits, a and b, do not influence each other directly, nevertheless the term of rotor current due to one stator circuit affects the other stator circuit - i.e., they affect each other indirectly. This may be seen from the diagram, Fig 21. If the two rotor current terms were not combined, but were tabulated under the stator circuit terms which caused them, we would, for instance, lose the term due to the rotor current $\frac{nC^2}{4}\frac{p}{Z_{r(p)}}\frac{p-i\omega}{Z_{s(p-i\omega)}}F_{(p)}1$ (itself resulting from circuit b) in the stator circuit a, this term being $-\frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{i(\omega t+\theta_0)}\left\{\frac{nC^2}{4}\frac{p}{Z_{r(p)}}\frac{p-i\omega}{Z_{s(p-i\omega)}}F_{(p)}1\right\}$. Thus the stator current terms must be kept separate with their imaginary circuits, but the rotor terms at each stage must be summed up to give a single expression.

In connection with this note, it is worth mentioning at this stage that when the rotor circuit is itself polyphase, then the rotor terms must be kept separate. In this case, the rotor system is also regarded as two equivalent imaginary circuits, and we obtain the condition that rotor a and stator a are coupled, and rotor b and stator b are coupled, but rotor a and stator a are quite independant of rotor b and stator b. In fact, all symmetrical polyphase systems may be split up into two imaginary parts as above. The question will be treated in detail later.

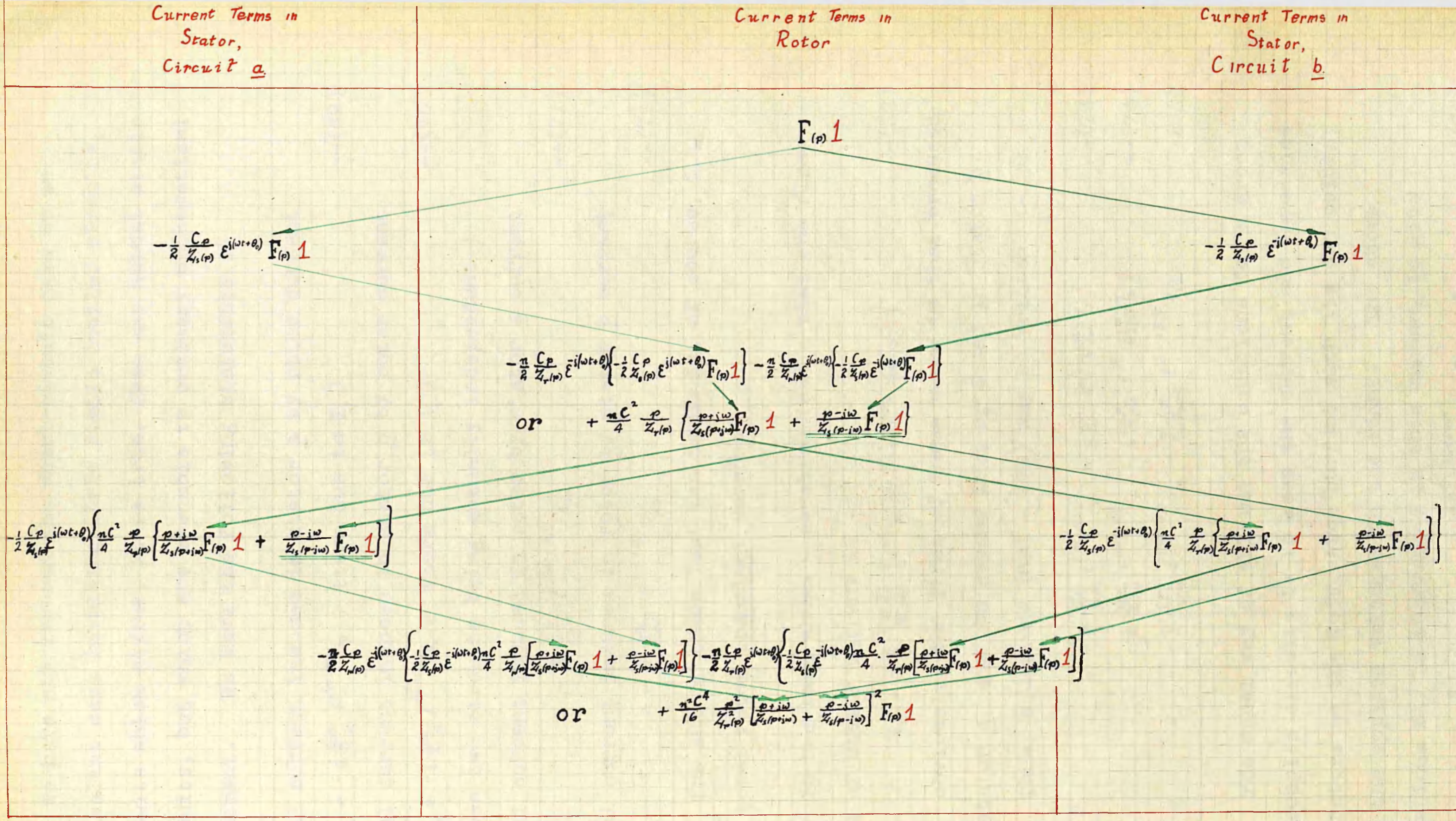


Fig 21.

SHOWING TWO COMPLETE REFLECTIONS OF ROTOR CURRENT.

THE TERMS REFERRED TO IN THE TEXT ARE UNDERLINED IN GREEN.

We have now reduced the short-circuit case to an equivalent case involving three static coupled circuits, circuits which differ, it is true, from any normal static circuits, but which are amenable to ordinary mathematical treatment. We have the following statements :-

Total current induced in stator a by rotor current

$= i_{sa} = -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\,\varepsilon^{j(\omega t+\theta_0)}\,i_r$ + (term due to I_f) --(69)

Total current induced in stator b by rotor current

$= i_{sb} = -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\,\varepsilon^{-j(\omega t+\theta_0)}\,i_r$ + (term due to I_f) --(70)

(these two circuits being mutually independant.)

Total current induced in rotor by stator a current

$$= -\frac{n}{2}\frac{Cp}{Z_{r(p)}}\,\varepsilon^{-j(\omega t+\theta_0)}\,i_{sa} \qquad \text{--(71)}$$

Total current induced in rotor by stator b current

$$= -\frac{n}{2}\frac{Cp}{Z_{r(p)}}\,\varepsilon^{j(\omega t+\theta_0)}\,i_{sb} \qquad \text{--(72)}$$

The sum of these last two terms is, of course, i_r, the total rotor transient current.

It has been shown - equation (32) - that the first term of stator current is :-

$$i_{s1} = -\frac{1}{2}\frac{Cp\,I_f}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0}\frac{j\omega}{p-j\omega}1 + \varepsilon^{-j\theta_0}\frac{-j\omega}{p+j\omega}1\right\} \qquad \text{--(32)}$$

Obviously, the first of these terms is that produced in stator a, and the second that produced in stator b.

Thus, equations (69) - (72) may be written :-

$$i_{sa} = -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\,\varepsilon^{j(\omega t+\theta_0)}\,i_r - \frac{1}{2}\frac{Cp}{Z_{s(p)}}\,\varepsilon^{j\theta_0}\frac{j\omega}{p-j\omega}I_f 1 \qquad \text{--(73)}$$

$$i_{sb} = -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\,\varepsilon^{-j(\omega t+\theta_0)}\,i_r - \frac{1}{2}\frac{Cp}{Z_{s(p)}}\,\varepsilon^{-j\theta_0}\frac{-j\omega}{p+j\omega}I_f 1 \qquad \text{--(74)}$$

$$i_r = -\frac{n}{2}\frac{Cp}{Z_{r(p)}}\,\varepsilon^{-j(\omega t+\theta_0)}\,i_{sa} - \frac{n}{2}\frac{Cp}{Z_{r(p)}}\,\varepsilon^{j(\omega t+\theta_0)}\,i_{sb} \qquad \text{--(75)}$$

The usual way of writing the equations for a group of coupled circuits is to keep the current and impedance operators on the left, each current with its own column, and the applied voltages on the right. The above equations may be rearranged in this manner, to give :-

$$Z_{r(p)}\,i_r + \frac{n}{2}Cp\,\varepsilon^{-j(\omega t+\theta_0)}i_{sa} + \frac{n}{2}Cp\,\varepsilon^{j(\omega t+\theta_0)}i_{sb} = 0 \qquad \text{--(76)}$$

$$\frac{1}{2}Cp\,\varepsilon^{j(\omega t+\theta_0)}i_r + Z_{s(p)}\,i_{sa} + 0 = -\frac{1}{2}Cp\,\varepsilon^{j\theta_0}\frac{j\omega I_f}{p-j\omega}1 \qquad \text{--(77)}$$

$$\frac{1}{2}Cp\,\varepsilon^{-j(\omega t+\theta_0)}i_r + 0 + Z_{s(p)}\,i_{sb} = -\frac{1}{2}Cp\,\varepsilon^{-j\theta_0}\frac{-j\omega I_f}{p+j\omega}1 \qquad \text{--(78)}$$

Put in this way, it is obvious what the equations represent. Eqn.(76) has on its left-hand side the sum of all the induced voltages in the rotor circuit, the right-hand side (=zero) being the total applied voltage. This is zero, and not $I_f R$, because the first term of rotor current, I_f, is not included in $\underline{i_r}$: (cf. eqn.(35), Part I.) Eqn. (77) has its left-hand side equal to the total induced voltages in stator circuit $\underline{a}$, while its right-hand side is the total voltage in this circuit due to the rotor current I_f: (cf. eqns. (34), Part I.) Eqn. (78) is similar, but for stator circuit $\underline{b}$. The phenomenon is, in fact, regarded as starting with the sudden application of discontinuous voltage from an outside source to the stator, which is exactly equivalent to the true state of affairs, provided we remember to add I_f to the field current so obtained. This change of starting point is of course necessitated by the irregular induction conditions holding for the first stator current term, which have already been discussed. Conditions thereafter are uniform, and these equations hold.

The equations above differ from those obtained with ordinary static analysis in various ways. In the first place, the mutual impedances between the rotor circuit and either of the two stator equivalent phases are not the same in both directions; this is partly on account of the rotation of the machine, the sign of the exponential being reversed, and partly on account of the fact that the two stator circuits are equivalent to the $\underline{n}$ stator phases, so that their effect is increased $\underline{n}$-fold. Further, the various operational expressions on the left-hand side contain functions of $\underline{t}$ which cannot be removed without altering the dependant variables, i_r , i_{sa}, and i_{sb}. Since the position of these functions in the resulting expression for any of the currents is important, ordinary determinantal methods may not be used, and the equations must be solved by other methods. For instance, let us consider the simple case of two simultaneous equations, first of the normal form, and then of the form of eqns. (76) - (78): i.e., involving "mixed" operators. The normal form would be :-

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned} \qquad \text{--(79)}$$

The determinantal solution of these equations would be obtained by reasoning as below :-

Multiplying the 1st line by b_2, and the 2nd by b_1 :-

$$(a_1 b_2 + a_2 b_1)x + (b_1 b_2 - b_2 b_1)y = (c_1 b_2 - c_2 b_1)$$

$$\text{or} \quad x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad \text{--(80)}$$

as $(b_1 b_2 - b_2 b_1) = 0$

If, however, the equations are of the second form, this method of attack breaks down. The equations would be of the form :-

$$\begin{aligned} \phi_{1(p)}\, x + \psi_{1(p)}\, \varepsilon^{j\omega t} y &= \chi_{1(p)}\, 1 \\ \phi_{2(p)}\, \varepsilon^{-j\omega t} x + \psi_{2(p)}\, y &= \chi_{2(p)}\, 1 \end{aligned} \qquad \text{--(81)}$$

Applying the method outlined above :-

$$\left\{\psi_{2(p)}\phi_{1(p)} - \psi_{1(p)}\varepsilon^{j\omega t}\phi_{2(p)}\varepsilon^{-j\omega t}\right\}x + \left\{\psi_{2(p)}\psi_{1(p)}\varepsilon^{j\omega t} - \psi_{1(p)}\varepsilon^{j\omega t}\psi_{2(p)}\right\}y = \left\{\psi_{2(p)}\chi_{1(p)} - \psi_{1(p)}\varepsilon^{j\omega t}\chi_{2(p)}\right\}1 \quad \text{--(82)}$$

Here the coefficient of <u>y</u> does not vanish, for though its two halves contain the same factors, these factors are not in the same order; if shifting be employed, the equation becomes :-

$$\left\{\psi_{2(p)}\phi_{1(p)} - \psi_{1(p)}\phi_{2(p-j\omega)}\right\}x + \left\{\psi_{2(p)}\psi_{1(p)}\varepsilon^{j\omega t} - \psi_{1(p)}\psi_{2(p-j\omega)}\varepsilon^{j\omega t}\right\}y = \left\{\psi_{2(p)}\chi_{1(p)} - \psi_{1(p)}\chi_{2(p-j\omega)}\frac{p}{p-j\omega}\right\}1$$

and <u>x</u> cannot therefore be expressed as a simple ratio of determinants, as in equation (80). The correct solution for <u>x</u> may be obtained by first dividing the second of the two original equations by $\psi_{2(p)}$, and then operating on each term in it with the operator $\psi_{1(p)}\varepsilon^{j\omega t}$; viz. -

Second equation becomes :-

$$\frac{\phi_{2(p)}}{\psi_{2(p)}}\varepsilon^{-j\omega t}\cdot x + y = \frac{\chi_{2(p)}}{\psi_{2(p)}}1$$

and then $\quad \psi_{1(p)}\varepsilon^{j\omega t}\dfrac{\phi_{2(p)}}{\psi_{2(p)}}\varepsilon^{-j\omega t}x + \psi_{1(p)}\varepsilon^{j\omega t}y = \psi_{1(p)}\varepsilon^{j\omega t}\dfrac{\chi_{2(p)}}{\psi_{2(p)}}1$

Subtracting from first :-

$$\left\{\phi_{1(p)} - \psi_{1(p)}\frac{\phi_{2(p-j\omega)}}{\psi_{2(p-j\omega)}}\right\}x = \left\{\chi_{1(p)} - \psi_{1(p)}\frac{\chi_{2(p-j\omega)}}{\psi_{2(p-j\omega)}}\cdot\frac{p}{p-j\omega}\right\}1$$

This last equation may be rearranged to give the correct solution, viz. -

$$x = \frac{(p-j\omega).\psi_2(p-j\omega).\chi_1(p) - \psi_1(p).\chi_2(p-j\omega).p}{(p-j\omega).\psi_2(p-j\omega).\phi_1(p) - (p-j\omega).\psi_1(p).\phi_2(p-j\omega)}\,1 \qquad \text{--(83)}$$

It is thus apparent that if the equations contain "mixed" operators, determinantal solution is invalid, and (other) algebraic methods of eliminating unwanted variables must be employed. The method of Equivalent Circuits always results in a set of such equations, as (76) - (78) above.

Bearing these points in mind, we may proceed to evaluate the currents in the case under consideration.

a. The rotor current.

Eqns. (77) and (78) may be rewritten :-

$$i_{sa} = -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{j\theta_0}\frac{j\omega}{p-j\omega}I_t 1 - \frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{j(\omega t+\theta_0)}i_r \qquad \text{--(84)}$$

$$i_{sb} = -\frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{-j\theta_0}\frac{-j\omega}{p+j\omega}I_t 1 - \frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{-j(\omega t+\theta_0)}i_r \qquad \text{--(85)}$$

Substituting these values in eqn. (76), we have :-

$$Z_{r(p)}i_r + \frac{n}{2}Cp\varepsilon^{-j(\omega t+\theta_0)}\left\{-\frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{j\theta_0}\frac{j\omega}{p-j\omega}I_t 1 - \frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{j(\omega t+\theta_0)}i_r\right\} + \frac{n}{2}Cp\varepsilon^{j(\omega t+\theta_0)}\left\{-\frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{-j\theta_0}\frac{-j\omega}{p+j\omega}I_t 1 - \frac{1}{2}\frac{Cp}{Z_{s(p)}}\varepsilon^{-j(\omega t+\theta_0)}i_r\right\} = 0 \qquad \text{-(86)}$$

Applying "Shifting" throughout, and collecting terms involving $\underline{i_r}$:-

$$\left\{Z_{r(p)} - \frac{n}{4}C^2p\frac{p+j\omega}{Z_{s(p+j\omega)}} - \frac{n}{4}C^2p\frac{p-j\omega}{Z_{s(p-j\omega)}}\right\}i_r - \frac{n}{4}C^2p\frac{p+j\omega}{Z_{s(p+j\omega)}}\frac{j\omega}{p}\frac{p}{p+j\omega}I_t 1 + \frac{n}{4}C^2p\frac{p-j\omega}{Z_{s(p-j\omega)}}\frac{j\omega}{p}\frac{p}{p-j\omega}I_t 1 = 0$$

i.e.,
$$i_r = \frac{\frac{n}{4}C^2pj\omega\left\{\frac{1}{Z_{s(p+j\omega)}} + \frac{1}{Z_{s(p-j\omega)}}\right\}}{Z_{r(p)} - \frac{n}{4}C^2p\left\{\frac{p+j\omega}{Z_{s(p+j\omega)}} + \frac{p-j\omega}{Z_{s(p-j\omega)}}\right\}}I_t 1$$

or
$$i_r = \frac{\frac{n}{2}C^2p\,I_s\,\omega^2\,1}{Z_{r(p)}Z_{s(p-j\omega)}Z_{s(p+j\omega)} - \frac{n}{2}C^2p\left\{R_sp + (p^2+\omega^2)L_s\right\}} \qquad \text{--(87)}$$

which equation, being identical with that obtained before, (eqn. 45), gives the correct solution for $\underline{i_r}$.

Once again, the method appears long, but in reality is very simple. When the fundamentals are grasped, and the mode of reasoning understood, the problem may be commenced with the writing of eqns. (76) - (78), the only reasoning thereafter being shown in the above simple solution of these equations. The final operational expression for the current is solved by the expansion theorem, as before.

b. The stator current.

The transient current in the stator may be calculated very simply by the substitution of $\underline{i}_r$ in the fundamental equations, (77) and (78). The procedure is as follows :-

It is apparent that $\underline{i}_{sa}$ and $\underline{i}_{sb}$ are conjugate complex quantities. It is therefore only necessary to evaluate one of them, say $\underline{i}_{sa}$, and then double its real part to get their sum, the true phase 1 stator current.

Thus we may write :-

$$i_{sa} = -\frac{1}{2}\frac{C_p}{Z_{s(p)}}\varepsilon^{i\theta_0}\frac{j\omega}{p-i\omega}I_f 1 - \frac{1}{2}\frac{C_p}{Z_{s(p)}}\varepsilon^{i(\omega t+\theta_0)}i_r \qquad \text{--(84)}$$

Substituting the operational solution of $\underline{i}_r$ already found, eqn. (55), :-

$$i_{sa} = -\frac{1}{2}C\varepsilon^{i\theta_0}\frac{pj\omega}{(p-j\omega)Z_{s(p)}}I_f 1 - \frac{1}{2}\frac{C_p}{Z_{s(p)}}\varepsilon^{j(\omega t+\theta_0)}\left\{\frac{\frac{1-\sigma'}{\sigma'}\cdot\omega^2 I_f p}{p^3+\left[s+\frac{s+r}{\sigma'}\right]p^2+\left[\omega^2+s\frac{s+r}{\sigma'}+\frac{sr}{\sigma'}\right]p+\left[r\frac{s^2+\omega^2}{\sigma'}\right]}\right\}1 \qquad \text{--(88)}$$

or, by "Shifting" :-

$$i_{sa} = -\frac{1}{2}C\varepsilon^{i\theta_0}\frac{pj\omega}{(p-j\omega)Z_{s(p)}}I_f 1 - \frac{1}{2}C\varepsilon^{i\theta_0}\left\{\frac{\frac{1-\sigma'}{\sigma'}\ \omega^2 I_f\, p(p-j\omega)\frac{p}{p-j\omega}}{\left\{R_s+pL_s\right\}\left\{(p-j\omega)^3+\left[s+\frac{s+r}{\sigma'}\right](p-j\omega)^2+\left[\omega^2+s\frac{s+r}{\sigma'}+\frac{sr}{\sigma'}\right](p-j\omega)+\left[r\frac{s^2+\omega^2}{\sigma'}\right]\right\}}\right\}1 \qquad \text{-(89)}$$

This operational expression may obviously be considered in two parts.

i. First part.

For this part, the equation is :-

$$i_{sa} = -\frac{1}{2}C\varepsilon^{i\theta_0}\frac{pj\omega}{(p-j\omega)Z_{s(p)}}I_f 1 = -\frac{1}{2}C\varepsilon^{i\theta_0}\cdot\frac{1}{L_s}\cdot\frac{pj\omega}{(p+\frac{R_s}{L_s})(p-j\omega)}I_f 1 \qquad \text{--(90)}$$

and this equation obviously gives the current in the stator due to the first term of rotor current, the normal field current, I_f.

By the expansion theorem :-

$$\frac{Y(0)}{Z(0)} = 0$$

$$\frac{Y(-\frac{R_s}{L_s})}{-\frac{R_s}{L_s}Z'(-\frac{R_s}{L_s})} = -\frac{1}{2}C\varepsilon^{i\theta_0}\frac{1}{L_s}I_f\frac{j\omega}{-\frac{R_s}{L_s}-j\omega}$$

$$= -CI_f\,\omega\frac{1}{\sqrt{R_s^2+\omega^2L_s^2}}\frac{1}{2j}\varepsilon^{j(\theta_0-\phi)} \qquad \text{where } \phi = \tan^{-1}\frac{\omega L_s}{R_s}$$

Full stator current due to this root, obtained by doubling and taking real parts :-

$$i_{s1} \doteqdot -\frac{CI_f}{L_s}\varepsilon^{-\frac{R_s t}{L_s}}\sin(\theta_0-\tfrac{\pi}{2})\,1 \qquad \text{as } R_s \text{ is small rel. to } \omega L_s,$$

$$\text{or} \quad i_{s1} \doteqdot +\frac{CI_f}{L_s}\varepsilon^{-\frac{R_s t}{L_s}}\cos\theta_0\,1 \qquad \text{--(91)}$$

$$\frac{Y(j\omega)}{j\omega Z'(j\omega)} = -\frac{1}{2} C \varepsilon^{i\theta} \frac{1}{L_s} I_f \frac{j\omega}{j\omega + \frac{R_s}{L_s}}$$

$$= + \frac{C I_f \omega}{\sqrt{R_s^2 + \omega^2 L_s^2}} \frac{1}{2j} \varepsilon^{j(\theta_0 - \phi)}$$

Full stator current due to this root :-

$i_{s1} \doteqdot + \frac{C I_f}{L_s} \sin(\omega t + \theta_0 - \frac{\pi}{2}) 1$ as above,

or $i_{s1} \doteqdot - \frac{C I_f}{L_s} \cos(\omega t + \theta_0) 1$ --(92)

Thus the full stator current, phase 1, due to the first expression in eqn. (89), and thus due to the normal field current, I_f, is given by :-

$$i_{s1} = -\frac{C I_f}{L_s} \left\{ \cos(\omega t + \theta_0) - \varepsilon^{-\frac{R_s t}{L_s}} \cos\theta_0 \right\} 1 \qquad \text{--(93)}$$

ii. Second part.

For this part, the equation is :-

$$i_{sa} = -\frac{1}{2} C p \varepsilon^{j\theta_0} \frac{\frac{1-\sigma'}{\sigma'} \omega^2 I_f (p-j\omega) \cdot \frac{p}{(p-j\omega)} 1}{Z_s(p) \left\{ (p-j\omega)^3 + \left[s + \frac{s+r}{\sigma'}\right](p-j\omega)^2 + \left[\omega^2 + s\frac{s+r}{\sigma'} + \frac{sr}{\sigma'}\right](p-j\omega) + \left[r\frac{s^2+\omega^2}{\sigma'}\right] \right\}} \qquad \text{--(94)}$$

As the auxiliary equation for i_r (cf. eqn. 55) has the roots α, $\mu + j\nu$, and $\mu - j\nu$, the above equation may be written :-

$$i_{sa} = -\frac{1}{2} C \varepsilon^{j\theta_0} \frac{1}{L_s} \frac{\frac{1-\sigma'}{\sigma'} \omega^2 I_f p^2}{[p+s][(p-j\omega)-\alpha][(p-j\omega)-(\mu+j\nu)][(p-j\omega)-(\mu-j\nu)]} 1 \qquad \text{--(95)}$$

A full symbolic solution of this equation would be unwieldy, though the solution for a numerical case may readily be obtained. The values of α, μ, and ν given before - eqns. (63) - will be taken as giving a sufficiently good approximation. i.e.,

$$\alpha \doteqdot -\frac{r}{\sigma'}$$
$$\mu \pm j\nu \doteqdot -\frac{1}{2} \frac{1+\sigma'}{\sigma'} s \pm j\omega \qquad \text{--(63)}$$

With these values, eqn. (95) reduces to :-

$$i_{sa} = -\frac{1}{2} \frac{C \omega^2 I_f}{L_s} \varepsilon^{j\theta_0} \frac{1-\sigma'}{\sigma'} \frac{p^2}{[p+s][p-(\alpha+j\omega)][p-(\mu+2j\omega)][p-\mu]} 1 \qquad \text{--(96)}$$

Applying the expansion theorem :-

$$\frac{Y(0)}{Z(0)} = 0$$

$$\frac{Y(-s)}{-s Z'(-s)} = -\frac{1}{2} \frac{C \omega^2 I_f}{L_s} \varepsilon^{j\theta_0} \frac{1-\sigma'}{\sigma'} \frac{-s}{[(-s-\alpha)-j\omega][(-s-\mu)-2j\omega][-s-\mu]}$$

$$= -\frac{1}{2} \frac{C \omega^2 I_f}{L_s} \varepsilon^{j\theta_0} \frac{1-\sigma'}{\sigma'} \frac{s}{[(s+\alpha)+j\omega][(s+\mu)+2j\omega][s+\mu]}$$

(over)

Hence $\quad \frac{Y(-s)}{-sZ'(-s)} \doteqdot -\frac{1}{2}\,\frac{C\omega^2 I_t}{L_s}\,\frac{1-\sigma'}{\sigma'}\,\varepsilon^{j\theta_0}\, s\,\frac{1}{\omega}\varepsilon^{-j\frac{\pi}{2}}\,\frac{1}{2\omega}\varepsilon^{-j\frac{\pi}{2}}\,\frac{1}{s\{1-\frac{1}{2}\frac{1+\sigma'}{\sigma'}\}}$

as ω is large compared with s, α, or μ.

$$\therefore\quad \frac{Y(-s)}{-sZ'(-s)} \doteqdot -\frac{1}{2}\,\frac{C\omega^2 I_t}{L_s}\,\frac{1-\sigma'}{\sigma'}\,\varepsilon^{j\theta_0}\,\frac{1}{2\omega^2}\,\varepsilon^{-j\pi}\,\frac{s}{s\{-\frac{1-\sigma'}{2\sigma'}\}}$$

$$\doteqdot -\frac{1}{2}\,\frac{CI_t}{L_s}\,\varepsilon^{j\theta_0}$$

Full stator current corresponding to this root :-

$$i_{s_1} = -\frac{CI_t}{L_s}\,\varepsilon^{-\frac{R_s t}{L_s}}\cos\theta_0\ 1 \qquad\text{--(97)}$$

which therefore cancels out with that given in eqn. (91).

$$\frac{Y(\alpha+j\omega)}{(\alpha+j\omega)Z'(\alpha+j\omega)} = -\frac{1}{2}C\varepsilon^{j\theta_0}\,\frac{1-\sigma'}{\sigma'}\,\frac{I_t\omega^2}{L_s}\,\frac{(\alpha+j\omega)}{[(\alpha+s)+j\omega][(\alpha-\mu)-j\omega][(\alpha-\mu)+j\omega]}$$

$$\doteqdot -\frac{1}{2}C\varepsilon^{j\theta_0}\,\frac{1-\sigma'}{\sigma'}\,\frac{I_t\omega^2}{L_s}\,\omega\,\varepsilon^{j\frac{\pi}{2}}\,\frac{1}{\omega}\,\varepsilon^{-j\frac{\pi}{2}}\cdot\frac{1}{\omega^2}$$

$$\doteqdot -\frac{1}{2}C\,\frac{1-\sigma'}{\sigma'}\,\frac{I_t}{L_s}\,\varepsilon^{j\theta_0}$$

Full stator current corresponding to this root :-

$$i_{s_1} = -\frac{I_t C}{L_s}\,\frac{1-\sigma'}{\sigma'}\,\varepsilon^{\alpha t}\cos(\omega t+\theta_0)\ 1 \qquad\text{--(98)}$$

$$\frac{Y(\mu+2j\omega)}{(\mu+2j\omega)Z'(\mu+2j\omega)} = -\frac{1}{2}C\varepsilon^{j\theta_0}\,\frac{1-\sigma'}{\sigma'}\,\frac{I_t\omega^2}{L_s}\,\frac{(\mu+2j\omega)}{[(s+\mu)+2j\omega][(\mu-\alpha)+j\omega][2j\omega]}$$

$$\doteqdot -\frac{1}{2}C\varepsilon^{j\theta_0}\,\frac{1-\sigma'}{\sigma'}\,\frac{I_t\omega^2}{L_s}\,2\omega\,\varepsilon^{j\frac{\pi}{2}}\,\frac{1}{2\omega}\varepsilon^{-j\frac{\pi}{2}}\,\frac{1}{\omega}\varepsilon^{-j\frac{\pi}{2}}\,\frac{1}{2\omega}\varepsilon^{-j\frac{\pi}{2}}$$

$$\doteqdot -\frac{1}{2}\,\frac{CI_t}{L_s}\,\frac{1-\sigma'}{\sigma'}\,\frac{1}{2}\,\varepsilon^{-j\pi}\varepsilon^{j\theta_0}$$

Full stator current corresponding to this root :-

$$i_{s_1} = +\frac{CI_t}{L_s}\,\frac{1-\sigma'}{2\sigma'}\,\varepsilon^{\mu t}\cos(2\omega t+\theta_0)\ 1 \qquad\text{--(99)}$$

$$\frac{Y(\mu)}{\mu Z'(\mu)} = -\frac{1}{2}C\varepsilon^{j\theta_0}\,\frac{1-\sigma'}{\sigma'}\,\frac{I_t}{L_s}\,\omega^2\,\frac{\mu}{[s+\mu][(\mu-\alpha)-j\omega][-2j\omega]}$$

$$\doteqdot -\frac{1}{2}C\varepsilon^{j\theta_0}\,\frac{1-\sigma'}{\sigma'}\,\frac{I_t\omega^2}{L_s}\left(-\frac{1}{2}\frac{1+\sigma'}{\sigma'}\right)s\,\frac{1}{[1-\frac{1}{2}\frac{1+\sigma'}{\sigma'}]s}\,\frac{1}{\omega}\varepsilon^{j\frac{\pi}{2}}\,\frac{1}{2\omega}\,\varepsilon^{j\frac{\pi}{2}}$$

$$\doteqdot -\frac{1}{2}C\varepsilon^{j\theta_0}\,\frac{1-\sigma'}{\sigma'}\,\frac{I_t}{L_s}\,\frac{1+\sigma'}{\sigma'}\,\frac{1}{-[\frac{1-\sigma'}{2\sigma'}]}\left(-\frac{1}{2}\right)\left(+\frac{1}{2}\right)\varepsilon^{j\pi}$$

$$\doteqdot +\frac{1}{2}\,\frac{CI_t}{L_s}\,\frac{1+\sigma'}{2\sigma'}\,\varepsilon^{j\theta_0}$$

Full stator current corresponding to this root :-

$$i_{s_1} = +\frac{CI_t}{L_s}\,\frac{1+\sigma'}{2\sigma'}\,\varepsilon^{\mu t}\cos\theta_0\ 1 \qquad\text{--(100)}$$

The whole stator phase 1 current is given by the sum of the currents in eqns. 91, 92, 97, 98, 99, and 100. Eqns. 91 and 97 cancel out, so that the total stator short-circuit current is given by :-

$$i_{s_1} = -\frac{CI_t}{L_s}\left\{\cos(\omega t+\theta_0) + \frac{1-\sigma'}{\sigma'}\varepsilon^{\alpha t}\cos(\omega t+\theta_0) - \frac{1-\sigma'}{2\sigma'}\varepsilon^{\mu t}\cos(2\omega t+\theta_0) - \frac{1+\sigma'}{2\sigma'}\varepsilon^{\mu t}\cos\theta_0\right\}1 \qquad\text{--(101)}$$

the currents in phases 2, 3 & c., being obtained by substituting the values $(\theta_0 - \frac{2\pi}{\pi})$, $(\theta_0 - \frac{4\pi}{\pi})$, & c. for θ_0 in the above equation.

This equation is identical with that obtained by Shimidzu and Ito - (Eqn. 37a, Part 1) - bearing in mind that the variable t in their equation represents time after the rotor coil has passed zero position, and not after short-circuit, thus corresponding to the variable $(t + \frac{\theta_0}{\omega})$ in eqn. (101). Hence, $(t - t_0)$ in eqn. 37a, part 1, corresponds with t above, and $(2\omega t - \omega t_0)$ in the same equation corresponds with $\left\{2\omega(t + \frac{\theta_0}{\omega}) - \theta_0\right\}$ or $(2\omega t + \theta_0)$ above, and so on.

This method is very simple, for nothing but ordinary algebraic processes has been used throughout. Further, as has been explained, once the principles of the method are grasped, it may be considerably shortened. It is of interest that simplification of analysis is obtained, not by reducing to the smallest number of equivalent circuits, but by reducing to this smallest number first - (two, in the case considered above) - and then substituting two independant and imaginary circuits for each system of distributed windings. The word "Imaginary" is particularly apt in this connection, for the two circuits are imaginary in the ordinary sense of the word, in that they have no real existence, and imaginary in the mathematical sense of the word, in that the currents flowing in each are complex functions.

The method may be extended indefinitely, for any number of circuits - damper windings & c. - the difficulties lying only in the arithmetic, and not in the solutions of the differential equations, per se. Before showing these extensions, however, the results shown above will be worked out for a practical alternator.

3. Numerical Application of Results to a Specific Case.

a. Maximum values and worst conditions.

The results of Part III, A2, show that the full equations for rotor and stator currents at time t after short-circuit are :-

$$i_r = \frac{1-\sigma'}{\sigma'} I_s \left\{ \varepsilon^{\alpha t} - \varepsilon^{\mu t} \cos \omega t \right\} 1 \qquad \text{--(64)}$$

$$i_{s_1} = -\frac{C I_t}{L_s} \left\{ \cos(\omega t + \theta_0) + \frac{1-\sigma'}{\sigma'} \varepsilon^{\alpha t} \cos(\omega t + \theta_0) - \frac{1-\sigma'}{2\sigma'} \varepsilon^{\mu t} \cos(2\omega t + \theta_0) - \frac{1+\sigma'}{2\sigma'} \varepsilon^{\mu t} \cos\theta_0 \right\} 1 \qquad \text{--(101)}$$

where σ' is given by :-

$$\sigma' = \frac{L_s L_r - \frac{\pi}{2} C^2}{L_s L_r}$$

It is of interest that the rotor current transient is quite independant of the point at which short-circuit takes place, since it contains no function of θ_o.

To calculate the maximum values of the short-circuit currents to a first approximation, it will be assumed that $\alpha \doteqdot \mu \doteqdot 0$, so that the equations become :-

$$i_r = \frac{1-\sigma'}{\sigma'} I_f \left\{ 1 - \cos \omega t \right\} \qquad \text{--(102)}$$

and
$$i_{s1} = -I_k \left\{ \cos(\omega t + \theta_o) + \frac{1-\sigma'}{\sigma'} \cos(\omega t + \theta_o) - \frac{1-\sigma'}{2\sigma'} \cos(2\omega t + \theta_o) - \frac{1+\sigma'}{2\sigma'} \cos \theta_o \right\}$$
$$= -I_k \frac{1}{\sigma'} \left\{ \cos(\omega t + \theta_o) - \frac{1-\sigma'}{2} \cos(2\omega t + \theta_o) - \frac{1+\sigma'}{2} \cos \theta_o \right\} \qquad \text{--(103)}$$

where $I_k = \{C/L_s\} I_f$ is the maximum steady short-circuit current.

It is obvious by inspection that the max. value of i_r occurs when $\omega t = \pi$ for all θ_o, and in magnitude is given by :-

$$i_{r(max)} = 2 \frac{1-\sigma'}{\sigma'} I_f \qquad \text{--(104)}$$

The stator current, i_{s1}, is a function of two variables, t and θ_o. To find its maximum possible value, it must be treated as such, so that its two partial derivatives with respect to t and θ_o must separately vanish. i.e.,

$$\frac{\partial i_{s1}}{\partial \theta_o} = -I_k \frac{1}{\sigma'} \left\{ -\sin(\omega t + \theta_o) + \frac{1-\sigma'}{2} \sin(2\omega t + \theta_o) + \frac{1+\sigma'}{2} \sin \theta_o \right\} = 0 \qquad \text{--(105)}$$

and
$$\frac{\partial i_{s1}}{\partial (\omega t)} = -I_k \frac{1}{\sigma'} \left\{ -\sin(\omega t + \theta_o) + (1-\sigma') \sin(2\omega t + \theta_o) \right\} = 0 \qquad \text{--(106)}$$

Hence, by subtraction :-

$$-\sin(\omega t + \theta_o) + (1+\sigma') \sin \theta_o = 0$$

or
$$\sin(\omega t + \theta_o) = (1+\sigma') \sin \theta_o \qquad \text{--(107)}$$

Substituting eqn.(107) in eqn.(105), -

$$(1+\sigma') \sin \theta_o - \frac{1-\sigma'}{2} \left\{ \sin \left[2(\omega t + \theta_o) - \theta_o \right] \right\} - \frac{1+\sigma'}{2} \sin \theta_o = 0$$

Putting $\frac{1}{2}(1 + \sigma') = N$, and $\frac{1}{2}(1 - \sigma') = M$, this becomes:-

$$N \times \sin \theta_o - M \times \left\{ \sin 2(\omega t + \theta_o) \cos \theta_o - \cos 2(\omega t + \theta_o) \sin \theta_o \right\} = 0$$

or
$$N \cdot \sin \theta_o - M \cdot \left\{ 4N \sin \theta_o \cos \theta_o \cos(\omega t + \theta_o) - (1 - 8N^2 \sin^2 \theta_o) \sin \theta_o \right\} = 0$$

From these equations, it follows that either $\sin \theta_o = 0$, or:-

$$N - M \cdot \left\{ 4N \sqrt{\left\{ 1 - (2N \sin \theta_o)^2 \right\} (1 - \sin^2 \theta_o)} - 1 + 8N^2 \sin^2 \theta_o \right\} = 0$$

or
$$1 - 8MN^2 \sin^2 \theta_o = 4MN \sqrt{(1 - 4N^2 \sin^2 \theta_o)(1 - \sin^2 \theta_o)}$$

as $(M + N) = \frac{1}{2}(1 - \sigma') + \frac{1}{2}(1 + \sigma') = 1$.

This last equation reduces to :-

$$1 - 16MN^2\sin^2\theta_0 + 64M^2N^4\sin^4\theta_0 = 16MN^2\left\{1 - (1+4N^2)\sin^2\theta_0 + 4N^2\sin^4\theta_0\right\}$$

or
$$\sin^2\theta_0\left\{16M^2N^2 + 64M^2N^4 - 16MN^2\right\} = 16M^2N^2 - 1$$

i.e.,
$$\sin^2\theta_0 = \frac{(4MN+1)(4MN-1)}{16MN^2\{M + 4MN^2 - 1\}} = \frac{(4MN+1)(4MN-1)}{16MN^2(4MN-1)N}$$
$$= \frac{4MN+1}{16MN^3}$$

i.e.,
$$\sin\theta_0 = \frac{1}{1+\sigma'}\sqrt{\frac{1+(1-\sigma'^2)}{1-\sigma'^2}}$$
$$= \frac{1}{1+\sigma'}\sqrt{\frac{2-\sigma'^2}{1-\sigma'^2}} \qquad \text{--(108)}$$

Eqn.(108) gives, however, by substitution in eqn.(107):-

$$\sin(\omega t + \theta_0) = \sqrt{\frac{2-\sigma'^2}{1-\sigma'^2}} \quad > 1$$

which is impossible.

Thus the only possible solution is that given by $\sin\theta_0 = 0$, which gives $\sin(\omega t + \theta_0) = 0$ also, by eqn.(107).

These two equations give of course several values for θ_0 and ωt , some of which are redundant; it is however apparent by inspection of the graphs and consideration of the phenomenon that the true maximum values, as opposed to the turning values, are given by such solutions as :-

$$\theta_0 = 0,\ \pi,\ 2\pi,\ --\ r\pi \quad , \quad \omega t = \pi,\ 3\pi,\ --\ (2n-1)\pi,$$

which solutions we shall proceed to consider.

With these values for θ_0 and ωt , we have :-

$$i_{s_1} = -I_k\frac{1}{\sigma'}\left\{\cos\{r\pi + (2n-1)\pi\} - \frac{1-\sigma'}{2}\cos\{(2n+2)\pi + r\pi\} - \frac{1+\sigma'}{2}\cos(r\pi)\right\}$$
$$= -I_k\frac{1}{\sigma'}\left\{(-1)^{r+1} - \frac{1-\sigma'}{2}(-1)^r - \frac{1+\sigma'}{2}(-1)^r\right\}$$
$$= (-1)^r I_k\frac{2}{\sigma'} \qquad \text{--(109)}$$

The worst conditions thus occur at $\theta_0 = 0, \pi, -- r\pi$, & c., and $\omega t = \pi$, the total maximum values of the currents being then :-

$$i_r = I_f + 2\frac{1-\sigma'}{\sigma'}I_f = \frac{2-\sigma'}{\sigma'}I_f \qquad \text{--(110)}$$
$$|i_{s_1}| = \frac{2}{\sigma'}I_f \qquad \text{--(109)}$$

<u>b.</u> Evaluation of the circuit constants.

No difficulty attaches to the evaluation of the resistance constants, R_s and R_r, it being remembered that R_s is <u>phase</u>, and not terminal-to-terminal resistance.

L_r may be measured by the application of alternating volts to the field winding, the stator being open-circuited.

The mutual inductance C may be deduced from the open-circuit characteristic of the machine, for on open-circuit, the max. voltage per phase is given by :-

$$\hat{e} = C I_f \omega$$

Thus, if E is the effective, or root-mean-square voltage per phase at field current I_f :-

$$C = \frac{\sqrt{2}\,E}{I_f\,\omega} \qquad \text{--(111)}$$

The inductance L_s is of interest. It was given by the formula :-

$$L_s = L_s' + M_1 \cos\frac{2\pi}{n} + M_2 \cos\frac{4\pi}{n} + -- \qquad \text{--(16)}$$

In spite of its complex mathematical form, however, it is simple to evaluate, for it is the inductance corresponding to the so-called "Synchronous Impedance" of the machine. For if we write the equations for the stator phases for the steady short-circuit state, using the "j" notation, i.e.,

$$\begin{array}{ccccc}
(R_s + j\omega L_s')I_{s1} + & j\omega M_1 I_{s2} + & -- + & j\omega M_{n-1} I_{sn} = & E \\
j\omega M_{n-1} I_{s1} + & (R_s + j\omega L_s')I_{s2} + & -- + & j\omega M_{n-2} I_{sn} = & E\,\varepsilon^{j\frac{2\pi}{n}} \\
-- & -- & -- & -- & -- \\
j\omega M_1 I_{s1} + & j\omega M_2 I_{s2} + & -- + & (R_s + j\omega L_s')I_{sn} = & E\,\varepsilon^{j\frac{2(n-1)\pi}{n}}
\end{array}$$

- we see that, by the rules for cyclically symmetrical determinants already developed, I_{s1} is given by:-

$$\begin{aligned}
I_{s1} &= \frac{E}{\{(R_s + j\omega L_s') + j\omega M_1 \cos\frac{2\pi}{n} + j\omega M_2 \cos\frac{4\pi}{n} + --\}} \\
&= \frac{E}{\{R_s + j\omega[L_s' + M_1 \cos\frac{2\pi}{n} + M_2 \cos\frac{4\pi}{n} + --]\}} \\
&= \frac{E}{R_s + j\omega L_s}
\end{aligned}$$

Thus L_s may be obtained from the open- and short-circuit characteristics of the machine.

c. Numerical example.

For convenience, a hypothetical alternator has been taken in the following example, the object of which is to show the effect of varying the coupling between rotor and stator.

In the two cases considered, the circuit constants other than the mutual inductance are identical. The absolute values of the constants are unimportant, the quantities r, s, and σ', (i.e., R_r/L_r , R_s/L_s , and $\{1 - \frac{1}{2}nC^2/L_sL_r\}$) being sufficient to establish the shapes of the curves by themselves.

The values taken are :-

i. $r = 5$; $s = 10$; $\sigma' = \cdot 1$; $\omega = 2\pi f = 314$.

ii. $r = 5$; $s = 10$; $\sigma' = \cdot 3$; $\omega = 2\pi f = 314$.

The values i give by the approximations of eqns.(63) the following equations for i_r and i_{s_1} ; ωt ($= \theta$) is taken as a more convenient variable than t , and θ_0 is taken as zero, to obtain the worst conditions :-

$$i_r = I_f + 9 I_f \left\{ \varepsilon^{-\frac{1}{2\pi}\theta} - \varepsilon^{-\frac{1\cdot 1}{2\pi}\theta} \cos\theta \right\} 1 \qquad \text{--(112)}$$

$$i_{s_1} = -I_k \left\{ \cos\theta + 9\varepsilon^{-\frac{1}{2\pi}\theta} \cos\theta - 4\cdot 5\, \varepsilon^{-\frac{1\cdot 1}{2\pi}\theta} \cos 2\theta - 5\cdot 5\, \varepsilon^{-\frac{1\cdot 1}{2\pi}\theta} \right\} 1 \qquad \text{--(113)}$$

where I_K ($= \{C/L_s\} I_f$) is the steady stator short-circuit current.(max)

The values ii give, under the same conditions :-

$$i_r = I_f + 2\cdot 33\, I_f \left\{ \varepsilon^{-\frac{\cdot 333}{2\pi}\theta} - \varepsilon^{-\frac{\cdot 434}{2\pi}\theta} \cos\theta \right\} 1 \qquad \text{--(114)}$$

$$i_{s_1} = -I_k \left\{ \cos\theta + 2\cdot 33\, \varepsilon^{-\frac{\cdot 333}{2\pi}\theta} \cos\theta - 1\cdot 167\, \varepsilon^{-\frac{\cdot 434}{2\pi}\theta} \cos 2\theta - 2\cdot 167\, \varepsilon^{-\frac{\cdot 434}{2\pi}\theta} \right\} 1 \qquad \text{--(115)}$$

These curves are illustrated in the diagrams, Figs 22 and 23. The maxima are not as great as would appear from eqns.(109) and (110) on account of the damping effect of resistance, which was neglected in these last equations.

It is of interest to note the effect of increasing the number of stator phases, without disturbing the symmetry of the arrangement, or altering the other circuit constants. For approximately sinusoidal distribution of flux, L_s is approximately equal to $\frac{1}{2}nL_s'$ - eqn.(40) - so that the ratio $\frac{1}{2}nC^2/L_sL_r \doteqdot C^2/L_s'L_r$ is unchanged by the increase; thus σ' is not affected. The only two symbols of eqn.(101) numerically affected by the change of n are in fact L_s (in the term $[C/L_s]I_f$, or I_k), and the exponential index coefficient, μ. If we neglect the small difference due to the change in μ - which is only an increase of duration,

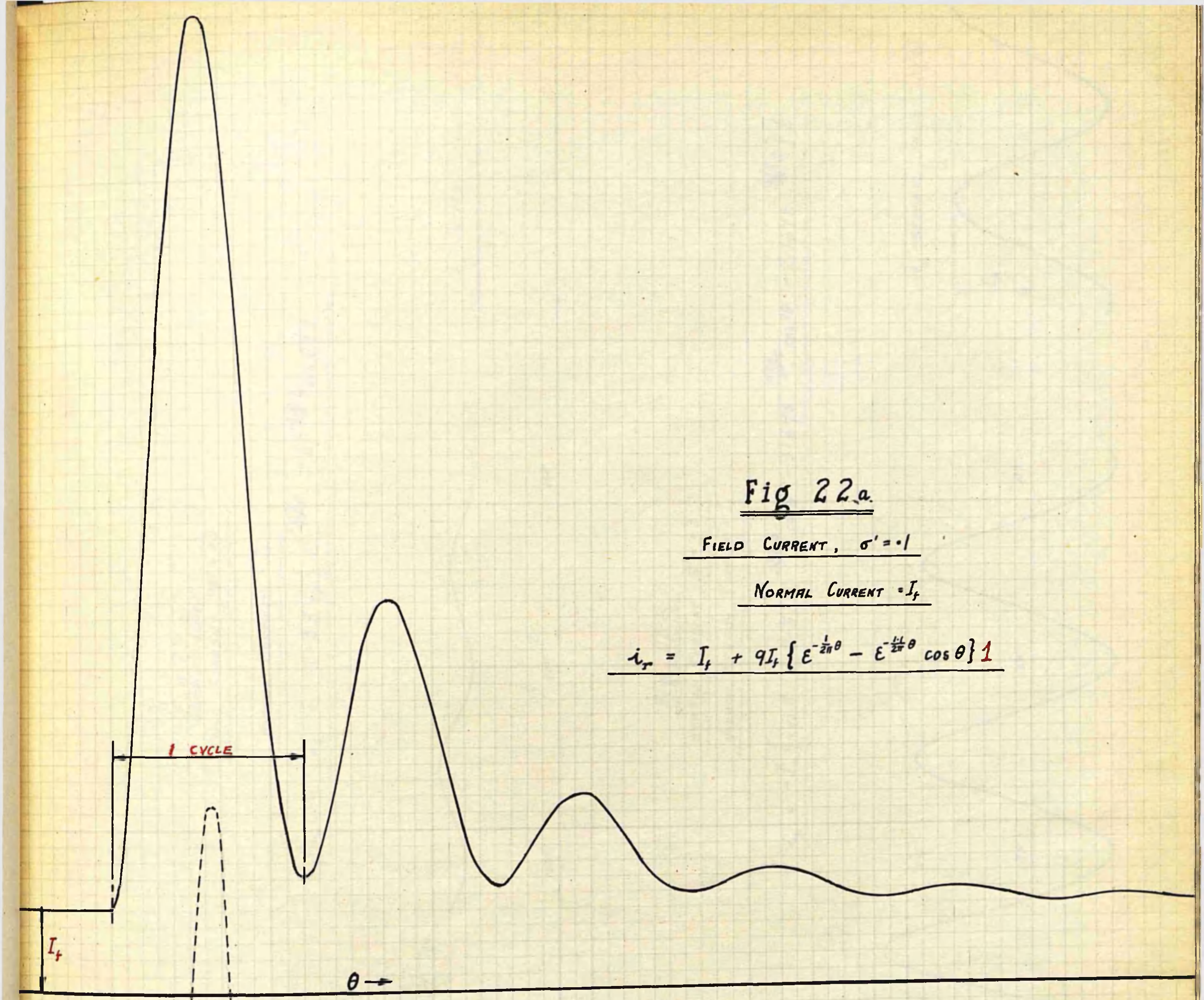


Fig 22a.

FIELD CURRENT, $\sigma' = \cdot 1$

NORMAL CURRENT $= I_f$

$$i_r = I_f + 9I_f\left\{\varepsilon^{-\frac{1}{2\pi}\theta} - \varepsilon^{-\frac{1 \cdot 1}{2\pi}\theta}\cos\theta\right\} \quad 1$$

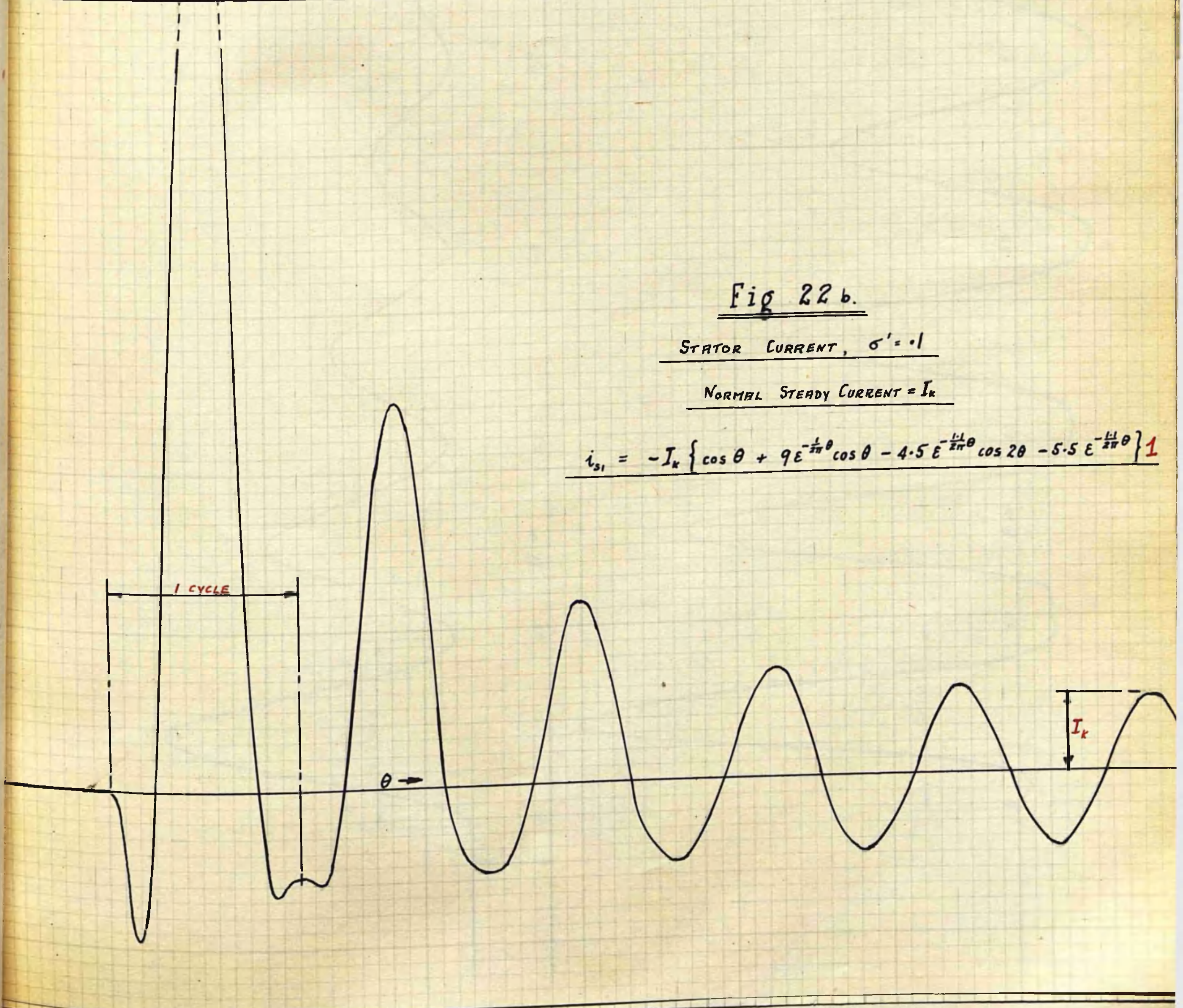


Fig 22b.

STATOR CURRENT, $\sigma' = \cdot 1$

NORMAL STEADY CURRENT $= I_k$

$$i_{s_1} = -I_k\left\{\cos\theta + 9\varepsilon^{-\frac{1}{2\pi}\theta}\cos\theta - 4\cdot 5\,\varepsilon^{-\frac{1\cdot 1}{2\pi}\theta}\cos 2\theta - 5\cdot 5\,\varepsilon^{-\frac{1\cdot 1}{2\pi}\theta}\right\} \quad 1$$

Fig 23 a.

FIELD CURRENT, $\sigma' = \cdot 3$

NORMAL CURRENT $= I_f$

$$i_r = I_f + 2\cdot 33\, I_f \left\{ \varepsilon^{-\frac{\cdot 333}{2\pi}\theta} - \varepsilon^{-\frac{\cdot 434}{2\pi}\theta} \cos\theta \right\} \quad 1$$

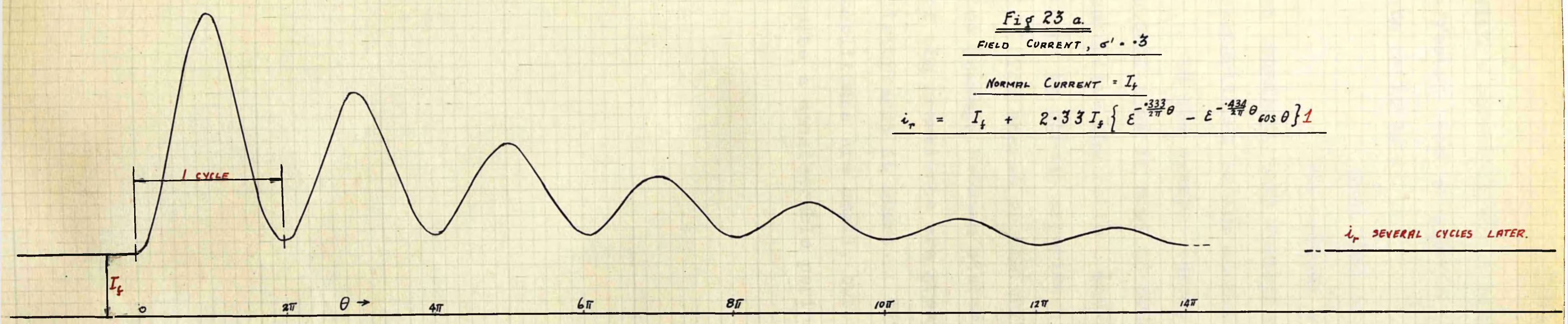
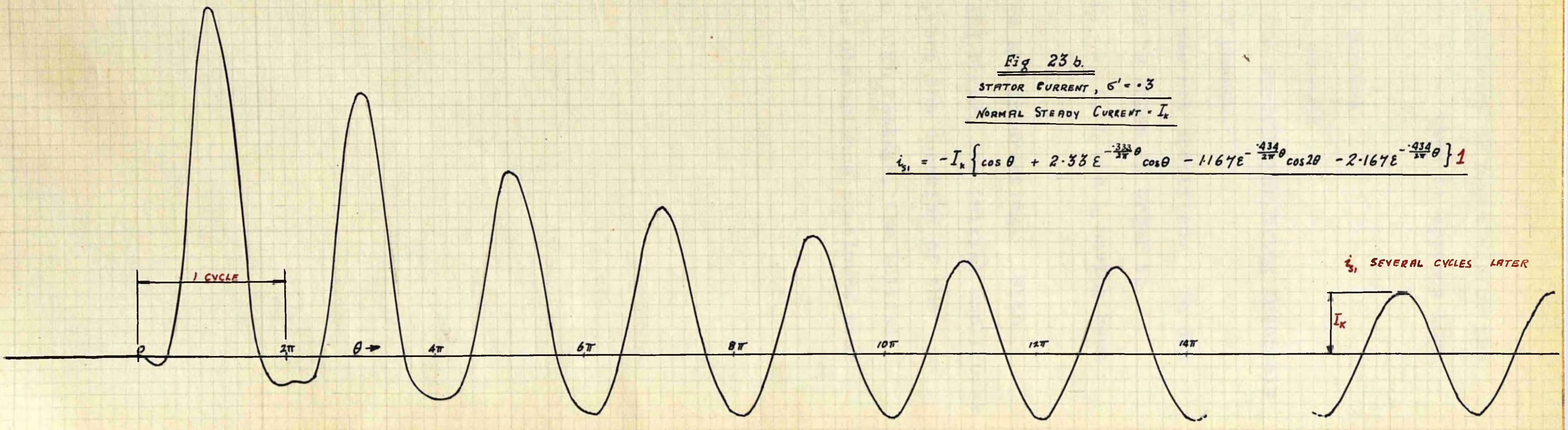


Fig 23 b.

STATOR CURRENT, $\sigma' = \cdot 3$

NORMAL STEADY CURRENT $= I_k$

$$i_{s_1} = -I_k \left\{ \cos\theta + 2\cdot 33\, \varepsilon^{-\frac{\cdot 333}{2\pi}\theta} \cos\theta - 1\cdot 167\, \varepsilon^{-\frac{\cdot 434}{2\pi}\theta} \cos 2\theta - 2\cdot 167\, \varepsilon^{-\frac{\cdot 434}{2\pi}\theta} \right\} \quad 1$$



with a corresponding slight increase of magnitude of the affected term at a given time t - then it appears that the ratio :-

$$\frac{\text{max. switching current}}{\text{max. steady s.c. current}}$$

is constant, both numerator and denominator being inversely proportional to the number of phases.

In the example taken, n was not specified. In any actual case it must of course be known in order to calculate the ratio $\frac{1}{2}nC^2/L_sL_r$, - C, L_s and L_r having been found by direct experiment.

The shapes of the curves are interesting. With low leakage between stator and rotor, the initial amplitudes of the transients are great, but the duration of the disturbance is short. With high leakage, the initial amplitudes are small, but the disturbance continues over quite a considerable period.

B. The Simple Case Under Different Initial Conditions.

It may be that the short-circuiting of the machine takes place under different conditions from those treated in Part III, A. Suppose, for instance, the stator circuit is initially short-circuited on itself, but that no field current flows, and that at $t = 0$, the full voltage is suddenly switched into the field circuit. What forms do the rotor and stator current waves take up?

1. By "Reflections".

It is apparent that the first term of rotor current is $\frac{E}{Z_{r(p)}}1$, where $E1$ is the field voltage. Since this is of the form $F_{(p)}1$, the second term of rotor current is obtainable by operating on it with the operator $\frac{n}{4}\frac{C^2p}{Z_{r(p)}}\left\{\frac{p-j\omega}{Z_{s(p-j\omega)}} + \frac{p+j\omega}{Z_{s(p+j\omega)}}\right\}$ as was established in A,1, and so on for all the other terms. In fact, the whole rotor series is a G.P., the first term being $\frac{E}{Z_{r(p)}}1$, and the common ratio $\frac{n}{4}\frac{C^2p}{Z_{r(p)}}\left\{\frac{p-j\omega}{Z_{s(p-j\omega)}} + \frac{p+j\omega}{Z_{s(p+j\omega)}}\right\}$ Herein lies the difference between this case and the case already discussed in A.1. In that case, the rotor series from the second term onwards was a G.P., but the relation between the first and second terms was irregular, and had to be specially established, since the first term was not discontinuous at $t = 0$.

Applying the rule for the G.P. summed to infinity :-

$$i_r = \frac{\frac{E}{Z_{r(p)}}1}{1 - \frac{n}{4}\frac{C^2p}{Z_{r(p)}}\left\{\frac{p-j\omega}{Z_{s(p-j\omega)}} + \frac{p+j\omega}{Z_{s(p+j\omega)}}\right\}} \qquad \text{--(116)}$$

or

$$i_r = \frac{E\left\{(R_s+pL_s)^2 + \omega^2L_s^2\right\}}{Z_{r(p)}\,Z_{s(p-j\omega)}\,Z_{s(p+j\omega)} - \frac{n}{2}C^2p\left\{R_sp + p^2L_s + \omega^2L_s\right\}}\,1 \qquad \text{--(117)}$$

2. By "Equivalent Circuits".

In the argument of Part III, A2, it was shown that since the first "Reflection" was irregular, it was necessary to regard the short-circuit phenomena as commencing one "Reflection" later than was strictly the case, - i.e., as commencing with the application of discontinuous voltage to the stator, - the initial field current term being added at the end to get the full equations. Since all the reflections are now regular, there is no longer any need to perform this "Shift" of the applied force, so the equations may be written straight away, with the discontinuous applied force E1 in the rotor equation, to which it truly belongs. i.e., -

$$Z_{r(p)}\, i_r + \tfrac{n}{2} C p\, \varepsilon^{-i(\omega t+\theta_0)} i_{sa} + \tfrac{n}{2} C p\, \varepsilon^{i(\omega t+\theta_0)} i_{sb} = E1 \qquad \text{--(118a)}$$

$$\tfrac{1}{2} C p\, \varepsilon^{i(\omega t+\theta_0)} i_r + Z_{s(p)}\, i_{sa} + 0 = 0 \qquad \text{--(118b)}$$

$$\tfrac{1}{2} C p\, \varepsilon^{-i(\omega t+\theta_0)} i_r + 0 + Z_{s(p)}\, i_{sb} = 0 \qquad \text{--(118c)}$$

Substituting the values of $\underline{i}_{sa}$ and $\underline{i}_{sb}$ from (b) and (c) in (a), and performing the operation of "Shifting" :-

$$\left\{ Z_{r(p)} - \frac{n}{4} C^2 p \frac{p+j\omega}{Z_{s(p+j\omega)}} - \frac{n}{4} C^2 p \frac{p-j\omega}{Z_{s(p-j\omega)}} \right\} i_r = E1$$

or $$i_r = \frac{E\left\{ (R_s + pL_s)^2 + \omega^2 L_s^2 \right\} 1}{Z_{r(p)} Z_{s(p+j\omega)} Z_{s(p-j\omega)} - \frac{n}{2} C^2 p \left\{ R_s p + p^2 L_s + \omega^2 L_s \right\}} \qquad \text{--(117)}$$

Continuing the solution for $\underline{i}_r$ and $\underline{i}_s$, we may write, following eqns.(46) and (56), -

$$i_r = \frac{E\left\{ (R_s + pL_s)^2 + \omega^2 L_s^2 \right\} 1}{p^3 L_s (L_s L_r - \frac{n}{2} C^2) + p^2 \left\{ R_s (L_s L_r - \frac{n}{2} C^2) + L_s (R_s L_r + L_s R_r) \right\} + p \left\{ \omega^2 L_s (L_s L_r - \frac{n}{2} C^2) + L_r R_s^2 + 2 L_s R_s R_r \right\} + R_r (R_s^2 + \omega^2 L_s^2)} \qquad \text{--(119)}$$

or $$i_r = \frac{I_f \frac{r}{\sigma'} \left\{ (s+p)^2 + \omega^2 \right\} 1}{(p-\alpha)\left\{ p - (\mu + j\nu) \right\}\left\{ p - (\mu - j\nu) \right\}} \qquad \text{--(120)}$$

where $I_f = E/R_r$ = final steady field current, and $\underline{s}$, $\underline{r}$, & c., are as used in eqn.(55).

By the approximations of eqns.(63), eqn.(120) may be written :-

$$i_r = \frac{I_f \frac{r}{\sigma'} \left\{ p^2 + 2sp + (s^2 + \omega^2) \right\}}{(p-\alpha)\left\{ p - (\mu + j\omega) \right\}\left\{ p - (\mu - j\omega) \right\}} 1 = \frac{Y(p)}{Z(p)} 1 \qquad \text{--(121)}$$

By the expansion theorem :-

$$\frac{Y(0)}{Z(0)} = \frac{E\left\{ R_s^2 + \omega^2 L_s^2 \right\}}{R_s^2 R_r + \omega^2 L_s^2 R_r} = \frac{E}{R_r} = I_f$$

(Eqn.119 being used for this term, rather than eqn.121.)

$$\frac{Y(\alpha)}{\alpha Z'(\alpha)} = \frac{I_f \frac{r}{\sigma'} \left\{ \alpha^2 + 2\alpha s + (s^2 + \omega^2) \right\}}{\alpha \left\{ (\alpha - \mu)^2 + \omega^2 \right\}}$$

$\doteqdot -I_f$, as $-\frac{r}{\sigma'} \doteqdot \alpha$, and α, μ, and s are small rel. to ω.

$$\frac{Y(\mu + j\omega)}{(\mu + j\omega) Z'(\mu + j\omega)} = \frac{I_f \frac{r}{\sigma'} \left\{ \mu^2 + 2\mu s + s^2 + 2j\omega(\mu + s) - \omega^2 + \omega^2 \right\}}{\left\{ (\mu - \alpha) + j\omega \right\} \left\{ 2j\omega \right\}}$$

$$\doteqdot \frac{I_f \frac{r}{\sigma'} \left\{ \mu + s \right\}}{j\omega}$$

$\doteqdot 0$ as above.

The solution for the root $(\mu - j\omega)$, being the complement of this last, is also zero.

Thus the complete solution for i_r is :-

$$i_r = I_f 1 - I_f \varepsilon^{\alpha t} 1 + 0 + 0 = I_f\left[1 - \varepsilon^{-\frac{rt}{\sigma'}}\right]1 \qquad \text{--(122)}$$

This would seem to indicate that the original equation - (121) - could be expressed more simply. If s and μ are small relative to ω, then the expressions $\{p^2 - 2\mu p + (\mu^2+\omega^2)\}$ and $\{p^2 + 2sp + (s^2+\omega^2)\}$ cancel out, and eqn.(121) becomes :-

$$i_r = \frac{I_f \frac{r}{\sigma'}}{(p-\alpha)} 1 = \frac{(-\alpha) I_f}{(p-\alpha)} 1 \qquad \text{--(123)}$$

which is known to solve to :-

$$i_r = I_f\left[1 - \varepsilon^{\alpha t}\right] 1$$

Using this simplified eqn.(123) for i_r to get i_{s1}, then by substitution in eqn.(118b) :-

$$i_{sa} = -\frac{1}{2}\frac{C p}{Z_s(p)} \varepsilon^{i\theta_0} \frac{(-\alpha) I_f}{\{p-(\alpha+j\omega)\}} \frac{p}{p-j\omega} 1$$

or $$i_{sa} = -\frac{1}{2}\frac{C I_f}{L_s} \varepsilon^{j\theta_0} \frac{(-\alpha)\, p^2}{(p+s)\{p-(\alpha+j\omega)\}(p-j\omega)} 1 = \frac{Y(p)}{Z(p)} 1 \qquad \text{--(124)}$$

By expansion theorem :-

$$\frac{Y(0)}{Z(0)} = 0$$

$$\frac{Y(-s)}{(-s)Z'(-s)} = -\frac{1}{2}\frac{C I_f}{L_s}\varepsilon^{i\theta_0}\frac{(-\alpha)(-s)}{[-s-\alpha-j\omega](-s-j\omega)}$$

$$\doteqdot 0$$

$$\frac{Y(j\omega)}{(j\omega)Z'(j\omega)} = -\frac{1}{2}\frac{C I_f}{L_s}\varepsilon^{i\theta_0}\frac{(-\alpha)(j\omega)}{[s+j\omega][-\alpha]}$$

$$\doteqdot -\frac{1}{2}\frac{C I_f}{L_s}\varepsilon^{i\theta_0}$$

Total i_{s1} corresponding to this term is :-

$$i_{s1} = -\frac{C I_f}{L_s}\cos(\omega t + \theta_0)\, 1$$

$$\frac{Y(\alpha+j\omega)}{(\alpha+j\omega)Z'(\alpha+j\omega)} = -\frac{1}{2}\frac{C I_f}{L_s}\varepsilon^{i\theta_0}\frac{(-\alpha)(\alpha+j\omega)}{[s+\alpha+j\omega][\alpha]}$$

$$\doteqdot \frac{1}{2}\frac{C I_f}{L_s}\varepsilon^{i\theta_0}$$

Total i_{s1} corresponding to this term is :-

$$i_{s1} = \frac{C I_f}{L_s}\varepsilon^{\alpha t}\cos(\omega t + \theta_0)\, 1$$

Hence, total stator current phase 1 is :-

$$i_{s1} = 0 + 0 - \frac{C I_f}{L_s}\cos(\omega t+\theta_0)\, 1 + \frac{C I_f}{L_s}\varepsilon^{\alpha t}\cos(\omega t+\theta_0) 1$$

$$= -I_k\left[1 - \varepsilon^{-\frac{rt}{\sigma'}}\right]\cos(\omega t+\theta_0)\, 1 \qquad \text{--(125)}$$

Thus the full equations for i_r and i_{s1} are :-

$$i_r = I_f\left[1 - \varepsilon^{-\frac{rt}{\sigma'}}\right] 1 \qquad \text{--(122)}$$

and $$i_{s1} = -I_k\left[1 - \varepsilon^{-\frac{rt}{\sigma'}}\right]\cos(\omega t+\theta_0)\, 1 \qquad \text{--(125)}$$

3. Numerical Example.

For convenience, the same alternators will be considered as were treated in A.3.

It is apparent that, since neither the field current nor the stator current rises above its normal steady value during switching (vide eqns.(122) and (125)), no question of "Worst conditions" or "Maxima" arises. Any value of θ_0 may therefore be taken when plotting the curves. The value $\theta_0 = 0$ will be taken.

The full equations for rotor and stator currents then become :-

For the values of s, r & c. as in A.3.c.i. :-

$$i_r = I_f \left\{ 1 - \varepsilon^{-\frac{1}{2\pi}\theta} \right\} 1 \qquad \text{--(126)}$$

$$i_{s_1} = -I_k \left\{ 1 - \varepsilon^{-\frac{1}{2\pi}\theta} \right\} \cos\theta \, 1 \qquad \text{--(127)}$$

For the values of s, r & c. as in A.3.c.ii. :-

$$i_r = I_f \left\{ 1 - \varepsilon^{-\frac{.333}{2\pi}\theta} \right\} 1 \qquad \text{--(128)}$$

$$i_{s_1} = -I_k \left\{ 1 - \varepsilon^{-\frac{.333}{2\pi}\theta} \right\} \cos\theta \, 1 \qquad \text{--(129)}$$

These curves are illustrated in Figs (24) and (25). The red curve in these figures represents the normal growth of field current - i.e., with the stator open-circuited - this curve being given by the formula :-

$$i_r = I_f \left\{ 1 - \varepsilon^{-rt} \right\} 1$$

or

$$i_r = I_f \left\{ 1 - \varepsilon^{-\frac{.1}{2\pi}\theta} \right\} 1 \qquad \text{--(130)}$$

in this case, for both values of σ'.

No abnormal currents occur under these conditions of switching, and the curves are not of very great interest in themselves. The main interest in the case lies in the development of the equations, for the fact that the first term of rotor current is truly a function of 1 changes the final current equations for both stator and rotor completely. That is, the example provides the answer to the question which naturally arises from the treatment of normal short-circuit by operational methods, namely, how is it that the

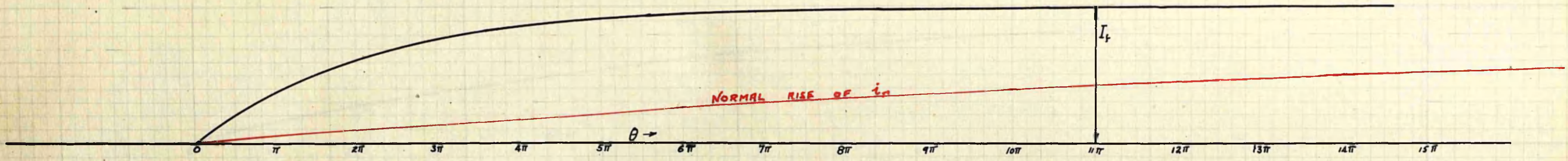


Fig 24a

ROTOR CURRENT. $\sigma' = \cdot 1$

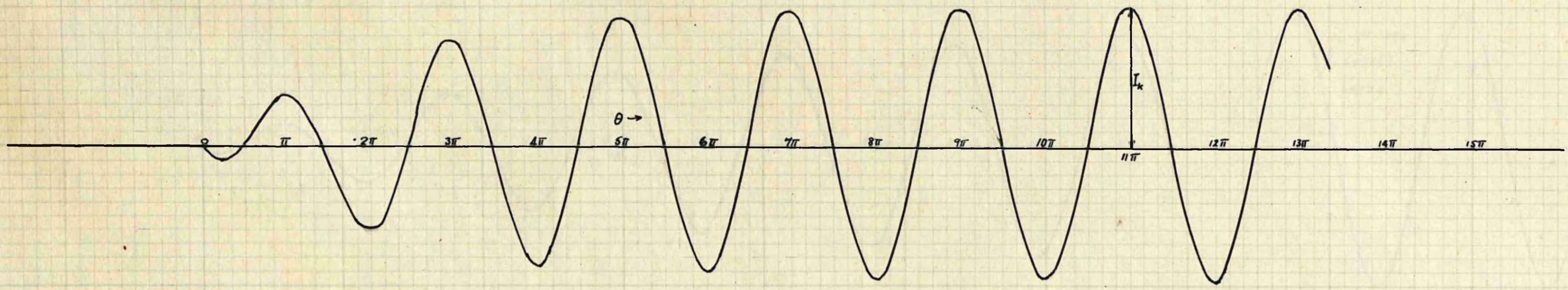


Fig 24b

STATOR CURRENT. $\sigma' = \cdot 1$

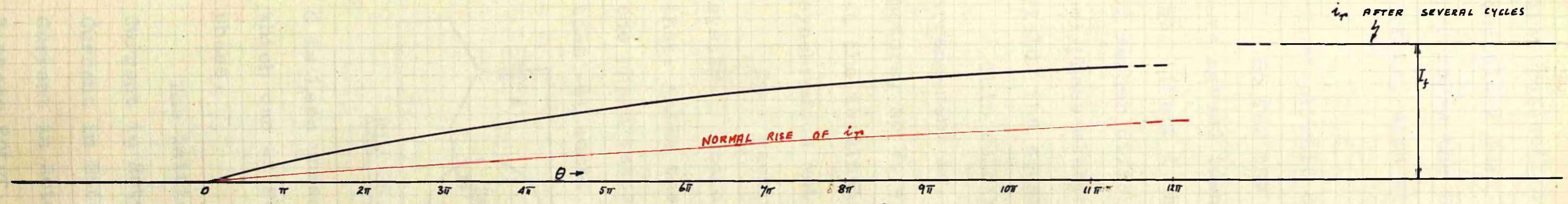


Fig 25a

ROTOR CURRENT. $\sigma' = \cdot 3$

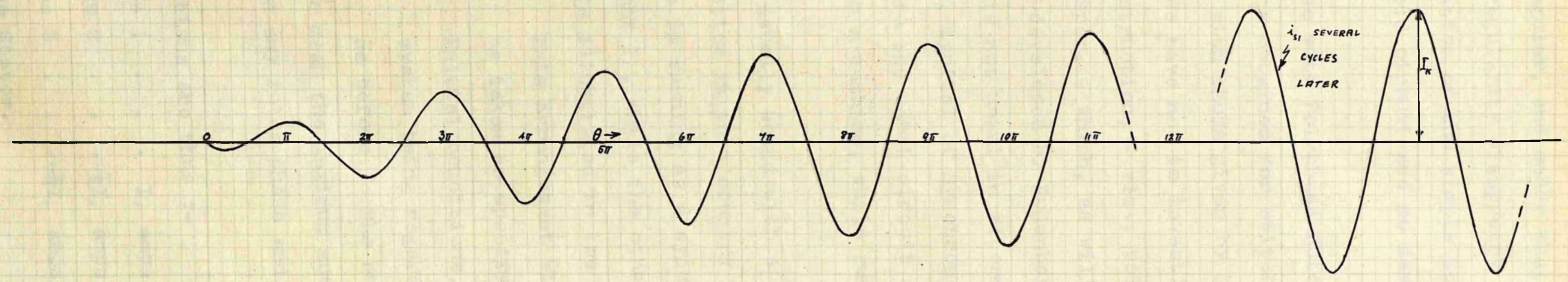


Fig 25b.

STATOR CURRENT. $\sigma' = \cdot 3$

first reflection is irregular, and under what physical conditions does that irregularity disappear? The example thus makes the investigation of the simple polyphase machine, with single field circuit and no damper, complete.

C. Short-circuit of Machine with Polyphase Rotor.

So far the equations obtained for switching current have agreed identically with those given by Shimidzu and Ito; (Part I, E.1.) We have now to investigate a machine of rather different construction, - viz., polyphase rotor, polyphase stator, no damper, - which, as will be shown, yields the equations as developed by Rudenberg; (Part I, E.2.)

This construction is not usual in alternators. The arrangement is, however, typical of Synchronous Induction motors, so we will regard the short-circuit as occurring at the terminals of such a machine, while running as a synchronous motor.

Such machines are usually wound with a 3-phase rotor system. Standard practice in this country is to arrange that, while the machine is running as a synchronous motor, one phase takes a current I_f, while the other two phases take $-\frac{1}{2}I_f$ each. This is illustrated in the diagram, Fig (26).

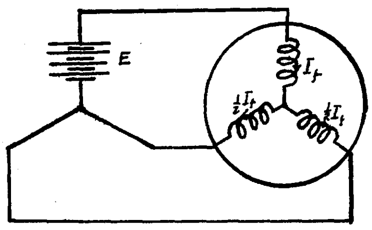


Fig 26.

This arrangement will therefore be taken as specifying the initial conditions, the stator system being regarded as n-phase, as before. The mutual inductance C is here understood to mean the maximum mutual inductance which can exist between any rotor phase and any stator phase.

The initial conditions are thus :-

Current in Rotor, phase 1 . . . I_f amps.
Current in Rotor, phase 2 . . . $-\frac{1}{2}I_f$ amps.
Current in Rotor, phase 3 . . . $-\frac{1}{2}I_f$ amps.
3-phase Rotor, n-phase Stator.

1. By "Reflections".

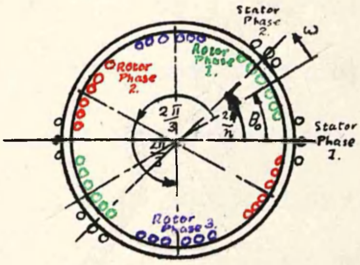


Fig 27.

The diagram, Fig 27, shows the arrangement of the phases on both stator and rotor, the rotation of the machine being counter-clockwise. The short-circuit takes place when the rotor axis is inclined at θ_0 radians, as before. The reflection formulae may then be established as below :-

First term of rotor current, phase 1, $= \quad I_f$ --(131)

i_{r1} rel. to stator 1 is $\frac{1}{2}\left\{\varepsilon^{i(\omega t+\theta_0)} + \varepsilon^{-i(\omega t+\theta_0)}\right\} I_f$

i_{r2} rel. to stator 1 is $\frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0+\frac{2\pi}{3})} + \varepsilon^{-i(\omega t+\theta_0+\frac{2\pi}{3})}\right\} \times -\frac{1}{2} I_f$

i_{r3} rel. to stator 1 is $\frac{1}{2}\left\{\varepsilon^{i(\omega t+\theta_0+\frac{4\pi}{3})} + \varepsilon^{-i(\omega t+\theta_0+\frac{4\pi}{3})}\right\} \times -\frac{1}{2} I_f$

Hence, total i_r rel. to stator 1 is $\frac{3}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-i(\omega t+\theta_0)}\right\} I_f$

E.M.F. induced in stator 1 $= -\frac{3}{2} Cp \left\{\varepsilon^{i(\omega t+\theta_0)} + \varepsilon^{-i(\omega t+\theta_0)} - \varepsilon^{i\theta_0} - \varepsilon^{-i\theta_0}\right\} I_f 1$

$$= -\frac{3}{2} Cp \left\{\varepsilon^{j\theta_0}\frac{j\omega}{p-j\omega} I_f 1 + \varepsilon^{-i\theta_0}\frac{-i\omega}{p+j\omega} I_f 1\right\}$$

as previously treated in A.1.

Current in stator 1 $= -\frac{3}{2}\frac{Cp}{Z_s(p)}\left\{\varepsilon^{i\theta_0}\frac{i\omega}{p-j\omega} I_f 1 + \varepsilon^{-i\theta_0}\frac{-i\omega}{p+j\omega} I_f 1\right\}$

and similarly for stator 2 & c.

i_{s1} rel. to rotor 1 $= \frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-i(\omega t+\theta_0)}\right\} \times -\frac{3}{2}\frac{Cp}{Z_s(p)}\left\{\varepsilon^{i\theta_0}\frac{i\omega}{p-j\omega} I_f 1 + \varepsilon^{-i\theta_0}\frac{-j\omega}{p+j\omega} I_f 1\right\}$

Hence, as terms involving θ_0 cancel out :-

Total i_s rel. to rotor 1 $= -\frac{3\pi}{4} C\left\{\frac{p+j\omega}{Z_s(p+j\omega)}\cdot\frac{j\omega}{p}\cdot\frac{p}{p+j\omega} I_f 1 + \frac{p-j\omega}{Z_s(p-j\omega)}\frac{-j\omega}{p}\frac{p}{p-j\omega} I_f 1\right\}$

$$= -\frac{3\pi}{4} C\left\{\frac{j\omega}{Z_s(p+j\omega)} I_f 1 + \frac{-j\omega}{Z_s(p-j\omega)} I_f 1\right\}$$

E.M.F. induced in rotor 1 $= +\frac{3\pi}{4} C^2 p\left\{\frac{j\omega}{Z_s(p+j\omega)} I_f 1 + \frac{-j\omega}{Z_s(p-j\omega)} I_f 1\right\}$

Current in rotor 1 $= \frac{3\pi}{4}\frac{C^2 p}{Z_r(p)}\left\{\frac{j\omega}{Z_s(p+j\omega)} I_f 1 + \frac{-j\omega}{Z_s(p-j\omega)} I_f 1\right\}$ --(132)

Current in rotor 2 $= \frac{3\pi}{4}\frac{C^2 p}{Z_r(p)}\left\{\varepsilon^{-j\frac{2\pi}{3}}\frac{j\omega}{Z_s(p+j\omega)} I_f 1 + \varepsilon^{i\frac{2\pi}{3}}\frac{-i\omega}{Z_s(p-j\omega)} I_f 1\right\}$

and so on. $Z_r(p)$ in these equations is given by :-

$$Z_r(p) = R_r + pL_r$$

$$= R_r + p(L_r' + M_{r1}\cos\tfrac{2\pi}{3} + M_{r2}\cos\tfrac{4\pi}{3})$$

with the usual definitions of L_r', L_r, M_{r1}, and M_{r2}.

It will be noticed that the two terms in eqn.(132) have not been combined, as was done before,—cf. eqn.(38). The reason for this will become apparent as we proceed;

it is on account of the polyphase nature of the rotor winding, for if we were to take the general term of phase 1 rotor current as $F_{(p)}1$, we should not know what to take as the corresponding phase 2 current term. Moreover, the simple ratio between successive terms for which we are looking is found to exist not between whole terms, but between half terms, as will be shown.

Considering now the general term of phase 1 rotor current, we may suppose it to be of the form :-

$$i_{r1} = F_{(p)}\left\{\xi_{(p+j\omega)}1 + \xi_{(p-j\omega)}1\right\} \qquad \text{--(133)}$$

$$i_{r2} = F_{(p)}\left\{\varepsilon^{-j\frac{2\pi}{3}}\xi_{(p+j\omega)}1 + \varepsilon^{j\frac{2\pi}{3}}\xi_{(p-j\omega)}1\right\} \qquad \text{\& c.}$$

(This is certainly the form of the second term, so if the term subsequent to it is also of this form, it follows that all terms but the first are of this form, so that equation (133) gives the true general term.)

Using this general term :-

$\underline{i}_{r1}$ rel. to stator 1 $= \frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} \times F_{(p)}\left\{\xi_{(p+j\omega)}1 + \xi_{(p-j\omega)}1\right\}$

$\underline{i}_{r2}$ rel. to stator 1 $= \frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0+\frac{2\pi}{3})} + \varepsilon^{-j(\omega t+\theta_0+\frac{2\pi}{3})}\right\} \times F_{(p)}\left\{\varepsilon^{-j\frac{2\pi}{3}}\xi_{(p+j\omega)}1 + \varepsilon^{j\frac{2\pi}{3}}\xi_{(p-j\omega)}1\right\}$

and so on.

Thus, total $\underline{i}_{r}$ rel. to stator 1 $= \frac{3}{2}\left\{\varepsilon^{j\theta_0}F_{(p-j\omega)}\xi_{(p)}\frac{p}{p-j\omega}1 + \varepsilon^{-j\theta_0}F_{(p+j\omega)}\xi_{(p)}\frac{p}{p+j\omega}1\right\}$

Hence, current in stator 1 $= -\frac{3}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0}F_{(p-j\omega)}\xi_{(p)}\frac{p}{p-j\omega}1 + \varepsilon^{-j\theta_0}F_{(p+j\omega)}\xi_{(p)}\frac{p}{p+j\omega}1\right\}$

$\underline{i}_{s1}$ rel. to rotor 1 $= \frac{1}{2}\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} \times -\frac{3}{2}\frac{Cp}{Z_{s(p)}}\left\{\varepsilon^{j\theta_0}F_{(p-j\omega)}\xi_{(p)}\frac{p}{p-j\omega}1 + \varepsilon^{-j\theta_0}F_{(p+j\omega)}\xi_{(p)}\frac{p}{p+j\omega}1\right\}$

Hence, current in rotor 1 $= +\frac{3n}{4}\frac{C^2p}{Z_{r(p)}}\left\{\frac{p+j\omega}{Z_{s(p+j\omega)}}F_{(p)}\xi_{(p+j\omega)}1 + \frac{p-j\omega}{Z_{s(p-j\omega)}}F_{(p)}\xi_{(p-j\omega)}1\right\}$

$$= \frac{3n}{4}\frac{C^2p}{Z_{r(p)}}F_{(p)}\left\{\frac{p+j\omega}{Z_{s(p+j\omega)}}\xi_{(p+j\omega)}1 + \frac{p-j\omega}{Z_{s(p-j\omega)}}\xi_{(p-j\omega)}1\right\} \qquad \text{--(134)}$$

Equation (134) is of the form of equation(133), so it follows that the reflection rule given above covers all terms of rotor current after the first.

Considering now equations (133) and (134), we see that there is no simple ratio connecting them as there was when a machine having a single phase rotor was considered. The case can, however, be reduced to a G.P. by consideration of the two parts of the current expression separately, the two parts being conjugate functions. The equations (131) (132), (133) and (134) then become :-

First rotor current, ph.1 = $\frac{1}{2} I_f$ --(131a)

Second rotor current, ph.1 = $\frac{3\pi}{4} \frac{C^2 p}{Z_{r(p)}} \frac{j\omega}{Z_{s(p+j\omega)}} I_f 1$ --(132a)

General term rotor current, ph.1 = $F_{(p)} \; \xi_{(p+j\omega)} 1$ --(133a)

Next term rotor current, ph.1 = $\frac{3\pi}{4} \frac{C^2 p}{Z_{r(p)}} \frac{p+j\omega}{Z_{s(p+j\omega)}} F_{(p)} \xi_{(p+j\omega)} 1$ --(134a)

Considering the last three of these equations, it is apparent that the case becomes that of a G.P. having its first term $\frac{3\pi}{4} \frac{C^2 p}{Z_{r(p)}} \frac{j\omega}{Z_{s(p+j\omega)}} I_f 1$ and common ratio $\frac{3\pi}{4} \frac{C^2 p}{Z_{r(p)}} \frac{p+j\omega}{Z_{s(p+j\omega)}}$. The steady current I_f must be added to the final expression, as before.

Thus the full rotor current, phase 1, is obtained by doubling the real part of the equation :-

$$i_r = \frac{\frac{3\pi}{4} \frac{C^2 p}{Z_{r(p)}} \frac{j\omega}{Z_{s(p+j\omega)}} I_f 1}{1 - \frac{3\pi}{4} \frac{C^2 p}{Z_{r(p)}} \frac{p+j\omega}{Z_{s(p+j\omega)}}} = \frac{\frac{3\pi}{4} C^2 p \, j\omega I_f 1}{Z_{r(p)} Z_{s(p+j\omega)} - \frac{3\pi}{4} C^2 p (p+j\omega)} \qquad \text{--(135)}$$

Once again, the method of reflections necessitates rather complicated working, but indicates the method to be adopted in solving the problem by "Equivalent Circuits". We shall proceed to apply this latter method.

2. By "Equivalent Circuits".

To make the process clear, two complete reflections of rotor current are shown in the diagram, Fig 28, the rules for the reflections being those established above.

Comparing Fig 28 with Fig 21, we perceive immediately the effect of the polyphase rotor. Whereas, with the single phase rotor, all the rotor terms were collected into one term after each complete reflection, with the polyphase rotor the terms are kept separate; further, only one of these terms affects each stator circuit at the next reflection - i.e., the term $\frac{3\pi}{4} \frac{C^2 p}{Z_{r(p)}} \frac{j\omega}{Z_{s(p+j\omega)}} 1$ induces current in stator circuit a, but no current in stator circuit b, while the term $\frac{3\pi}{4} \frac{C^2 p}{Z_{r(p)}} \frac{-j\omega}{Z_{s(p-j\omega)}} 1$ affects stator b without affecting stator a. When the rotor with only one phase was considered, it was pointed out that the rotor terms should be combined at each stage so that no stator terms should be lost (cf. page 66); for the polyphase case, the terms have to be kept separate in

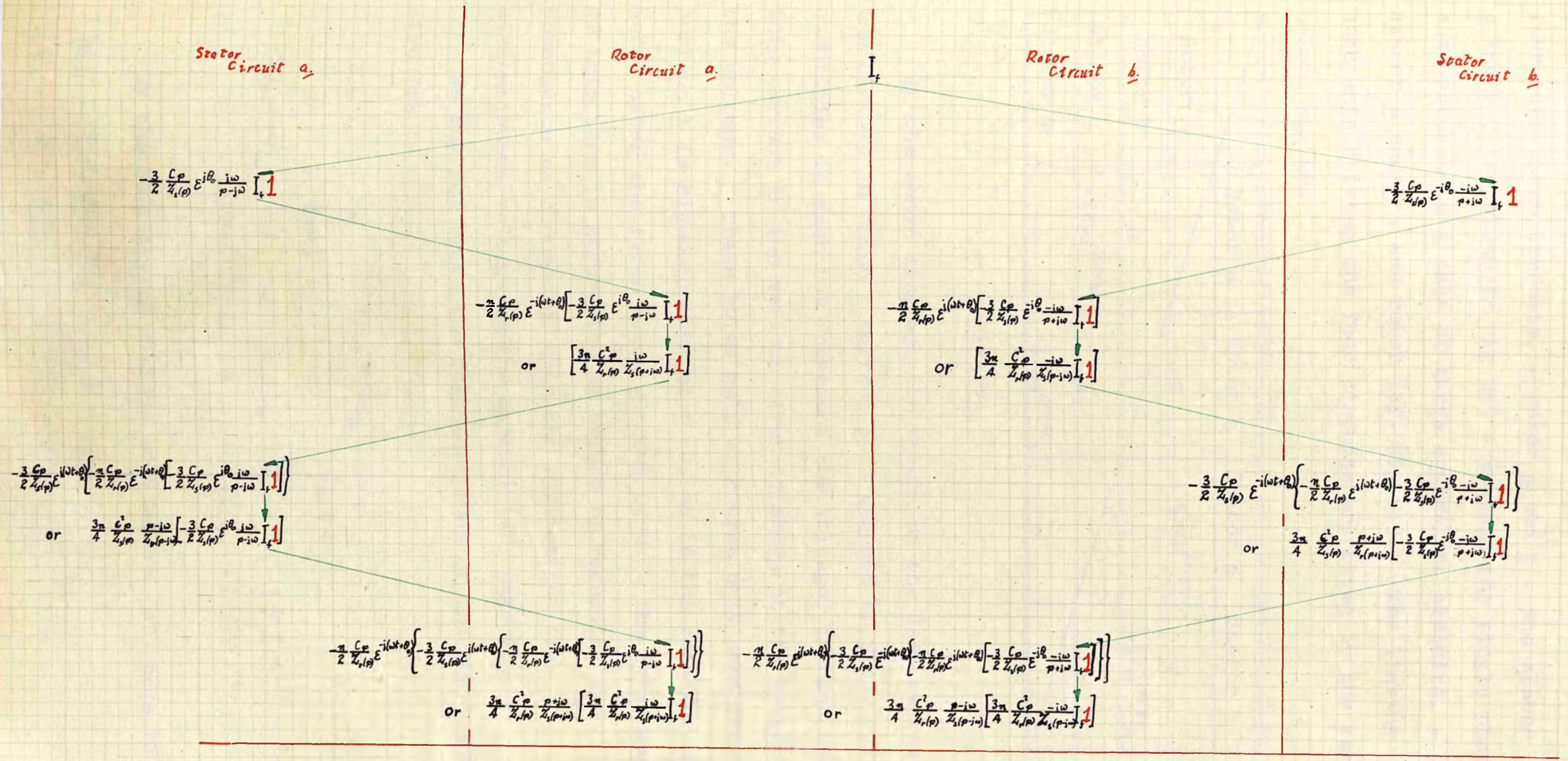


Fig 28

SHORT-CIRCUIT OF MACHINE WITH POLYPHASE ROTOR.

SHOWING TWO COMPLETE REFLECTIONS OF ROTOR CURRENT: THE TERMS ARE GIVEN BOTH AS FUNCTIONS OF "p" AND AS FUNCTIONS OF "t" AND "p".

order that no spurious terms shall be introduced.

In simple language, the case reduces to a case of four equivalent circuits, - rotor a and b, and stator a and b, (as shown in Fig 28) - whereas the single phase case reduced to a case of three equivalent circuits, - stator a and b, and rotor (cf. Fig 21). The linking constants between the four circuits are given in the table below, Fig 29.

	Rotor a	Rotor b	Stator a	Stator b
Rotor a		0	$-\frac{3}{2}Cp\varepsilon^{j(\omega t+\theta_0)}i$	0
Rotor b	0		0	$-\frac{3}{2}Cp\varepsilon^{-j(\omega t+\theta_0)}i$
Stator a	$-\frac{\pi}{2}Cp\varepsilon^{-j(\omega t+\theta_0)}i$	0		0
Stator b	0	$-\frac{\pi}{2}Cp\varepsilon^{j(\omega t+\theta_0)}i$	0	

Fig 29.

In the above diagram, the column on the left represents the circuit in which a current i is assumed to flow. The quantities in each row represent the voltages induced by this current in the various circuits shown at the heads of the columns. These rules hold of course only when the current i is discontinuous at t = 0, as is the case for all terms after the first rotor current term. The irregularity of this term is surmounted as before by regarding the phenomenon as commencing with the application of discontinuous voltage to the stator; (cf. page 68).

We may now write the full equations of the equivalent circuits :-

$$
\begin{aligned}
&Z_{r(p)}\, i_{ra} + 0 + \tfrac{\pi}{2} Cp\, \varepsilon^{-j(\omega t+\theta_0)} i_{sa} + 0 = 0 \\
&0 + Z_{r(p)}\, i_{rb} + 0 + \tfrac{\pi}{2} Cp\, \varepsilon^{+j(\omega t+\theta_0)} i_{sb} = 0 \\
&\tfrac{3}{2} Cp\, \varepsilon^{j(\omega t+\theta_0)} i_{ra} + 0 + Z_{s(p)}\, i_{sa} + 0 = -\tfrac{3}{2}\frac{Cpj\omega}{p-j\omega}\varepsilon^{j\theta_0} I_f 1 \\
&0 + \tfrac{3}{2} Cp\, \varepsilon^{-j(\omega t+\theta_0)} i_{rb} + 0 + Z_{s(p)}\, i_{sb} = -\tfrac{3}{2}\frac{Cp(-j\omega)}{p+j\omega}\varepsilon^{-j\theta_0} I_f 1
\end{aligned}
\qquad \text{--(136)}
$$

By the simplifying methods used before (cf. page 70):-

$$
\begin{aligned}
&\left\{Z_{r(p)} - \tfrac{3\pi}{4} C^2 p \frac{p+j\omega}{Z_{s(p+j\omega)}}\right\} i_{ra} = +\tfrac{3\pi}{4} C^2 p \frac{j\omega}{Z_{s(p+j\omega)}} I_f 1 \\
&\left\{Z_{r(p)} - \tfrac{3\pi}{4} C^2 p \frac{p-j\omega}{Z_{s(p-j\omega)}}\right\} i_{rb} = +\tfrac{3\pi}{4} C^2 p \frac{-j\omega}{Z_{s(p-j\omega)}} I_f 1
\end{aligned}
\qquad \text{--(137)}
$$

The two equations (137) obviously give two conjugate functions as solutions for i_{ra} and i_{rb}. It is necessary therefore to evaluate only one of them. We shall take the first, since this will give the solution already obtained by the method of "Reflections".

The equation becomes :-

$$i_{ra} = \frac{\frac{3n}{4} c^2 p j\omega I_t 1}{Z_{r(p)} Z_{s(p+j\omega)} - \frac{3n}{4} c^2 p (p+j\omega)}$$

which is identical with equation (135).

Expanding eqn. (135) :-

$$i_{ra} = \frac{\frac{3n}{4} c^2 p j\omega I_t 1}{p^2\{L_s L_r - \frac{3n}{4}c^2\} + p\{j\omega(L_s L_r - \frac{3n}{4}c^2) + R_r L_s + L_r R_s\} + R_r\{R_s + j\omega L_s\}}$$

$$= \frac{\frac{1-\sigma'}{\sigma'} p j\omega I_t 1}{p^2 + p\{j\omega + \frac{r+s}{\sigma'}\} + r\{\frac{s+j\omega}{\sigma'}\}} = \frac{Y_{(p)}}{Z_{(p)}} 1 \qquad \text{--(138)}$$

using the usual notation, as $L_s \doteq \frac{n}{2} L'_s$ and $L_r \doteq \frac{3}{2} L'_r$. (cf page 58)

Consider the equation :-

$$Z_{(p)} = p^2 + p\{j\omega + \frac{r+s}{\sigma'}\} + r\{\frac{s+j\omega}{\sigma'}\} = 0 \qquad \text{--(139)}$$

as compared with the equation :-

$$\alpha^2 - \alpha\{\omega + j\frac{r+s}{\sigma'}\} + s\{\frac{j\omega - r}{\sigma'}\} = 0 \qquad \text{--(140)}$$

this latter being the auxiliary equation of Rüdenberg, - (Eqn. 41, Part I.),- arranged in our present notation.

We might have expected these two equations to be identical, since both give the natural frequencies of the circuits. Various corrections have to be made, however, before the identity is apparent. In the first place, eqn.(140) gives the frequencies (α_1 and α_2) of the stator circuit, while eqn.(139) gives those of the rotor. Since the rotor frequencies β_1 and β_2 are given by $\alpha - \omega = \beta$, (cf page 21), eqn.(140) corrected to give rotor frequencies becomes :-

$$(\beta+\omega)^2 - (\beta+\omega)\{\omega + j\frac{r+s}{\sigma'}\} + s\{\frac{j\omega - r}{\sigma'}\} = 0$$

or $$\beta^2 + \beta\{\omega - j\frac{r+s}{\sigma'}\} - r\{\frac{s+j\omega}{\sigma'}\} = 0 \qquad \text{--(140a)}$$

In the second place, the current was defined in Part I, E2, as $I\varepsilon^{j\alpha t}$, where α was a root of eqn.(140). Eqn.(139) should thus give not β, but $j\beta$, as its roots. If eqn.(140a) be rearranged to give $j\beta$ instead of β, it becomes :-

$$\beta^2 + \beta\left\{j\omega + \frac{r+s}{\sigma'}\right\} + r\left\{\frac{s+j\omega}{\sigma'}\right\} = 0 \qquad \text{--(140b)}$$

whereupon the identity of Rüdenberg's equation with equation (139) becomes apparent.

Had $\underline{i}_{rb}$ been taken instead of $\underline{i}_{ra}$, eqn.(139) would have been slightly different, all the real coefficients being the same, but all the imaginaries taking the opposite sign, so that the roots obtained would have been the complements of $j\beta_1$ and $j\beta_2$. Since real parts only are taken, this would have made no difference to the final solution.

Equation (139) has thus been shown to have the roots $j\beta_1$ and $j\beta_2$ as given by Rüdenberg. i.e., -

$$p^2 + p\left\{j\omega + \frac{r+s}{\sigma'}\right\} + r\left\{\frac{s+j\omega}{\sigma'}\right\} = \{p - j\beta_1\}\{p - j\beta_2\}$$

$$= \left\{p - \left[-\rho_1 + j(\vartheta_1 - \omega)\right]\right\}\left\{p - \left[-\rho_2 + j(\vartheta_2 - \omega)\right]\right\}$$

where
$$\rho_1 \doteq \frac{s}{\sigma'} + \frac{r}{\sigma'}\frac{\left(\frac{s}{\sigma'}\right)^2(1-\sigma')}{\omega^2 + \left(\frac{s}{\sigma'}\right)^2} \doteqdot \frac{s}{\sigma'}$$

$$\rho_2 = \frac{r}{\sigma'} - \frac{r}{\sigma'}\frac{\left(\frac{s}{\sigma'}\right)^2(1-\sigma')}{\omega^2 + \left(\frac{s}{\sigma'}\right)^2} \doteqdot \frac{r}{\sigma'}$$

$$\vartheta_1 = \omega\left\{\frac{\left(\frac{rs}{\sigma'^2}\right)(1-\sigma')}{\omega^2 + \left(\frac{s}{\sigma'}\right)^2}\right\} \doteqdot 0$$

and
$$\vartheta_2 = \omega\left\{1 - \frac{\left(\frac{rs}{\sigma'^2}\right)(1-\sigma')}{\omega^2 + \left(\frac{s}{\sigma'}\right)^2}\right\} \doteqdot \omega$$

so that $\vartheta_2 + \vartheta_1 = \omega$, $\quad \vartheta_1 - \omega = -\vartheta_2$, $\quad$ & $\quad \vartheta_2 - \omega = -\vartheta_1$.

<u>a.</u> The rotor current.

Applying the expansion theorem to eqn.(138) :-

$$\frac{Y(0)}{Z(0)} = 0$$

$$\frac{Y(-\rho_1 - j\vartheta_2)}{(-\rho_1 - j\vartheta_2)Z'(-\rho_1 - j\vartheta_2)} = \frac{\frac{1-\sigma'}{\sigma'}j\omega I_f}{\left[(-\rho_1 - j\vartheta_2) - (-\rho_2 - j\vartheta_1)\right]} \doteqdot \frac{\frac{1-\sigma'}{\sigma'}j\omega I_f}{\left[(-\rho_1 + \rho_2) - j(\omega - 0)\right]} \doteqdot -\frac{1-\sigma'}{\sigma'}I_f$$

Full current corresponding to this root is :-

$$i_{r_1} = -\frac{1-\sigma'}{\sigma'} I_f\, \varepsilon^{-\rho_1 t} \cos \vartheta_2 t\, 1$$

$$\frac{Y(-\rho_2 - j\vartheta_1)}{(-\rho_1 - j\vartheta_1)Z'(-\rho_2 - j\vartheta_1)} = \frac{\frac{1-\sigma'}{\sigma'}j\omega I_f}{\left[(-\rho_2 - j\vartheta_1) - (-\rho_1 - j\vartheta_2)\right]} \doteqdot \frac{\frac{1-\sigma'}{\sigma'}j\omega I_f}{\left[(-\rho_2 + \rho_1) + j(\omega - 0)\right]} \doteqdot \frac{1-\sigma'}{\sigma'}I_f$$

Full current corresponding to this root is :-

$$i_{r_1} = \frac{1-\sigma'}{\sigma'} I_f\, \varepsilon^{-\rho_2 t} \cos \vartheta_1 t\, 1$$

Thus, full transient rotor current is given by :-

$$i_{r_1} = \frac{1-\sigma'}{\sigma'} I_f\left\{\varepsilon^{-\rho_2 t}\cos\vartheta_1 t - \varepsilon^{-\rho_1 t}\cos\vartheta_2 t\right\}1 \qquad \text{--(141)}$$

b. The stator current.

To calculate the stator current, eqn.(138) must be substituted in the requisite eqn.(136), - viz., the third. This gives the equation below :-

$$i_{sa} = -\frac{3}{2}\frac{Cpj\omega}{Z_1(p)(p-j\omega)}\varepsilon^{j\theta_0}I_t 1 - \frac{3}{2}\frac{Cp}{Z_1(p)}\varepsilon^{j(\omega t+\theta_0)} i_{ra}$$

$$= -\frac{3}{2}\frac{Cpj\omega\varepsilon^{j\theta_0}I_t}{L_s(p+s)(p-j\omega)} 1 - \frac{\frac{3}{2}\frac{Cp}{L_s}\varepsilon^{j\theta}\frac{1-\sigma'}{\sigma'}j\omega I_t(p-j\omega)}{(p+s)\{(p-j\omega)^2+(p-j\omega)[j\omega+\frac{r+s}{\sigma'}]+r[\frac{s+j\omega}{\sigma'}]\}}\frac{p}{(p-j\omega)}1$$

$$= -\frac{3}{2}\frac{CI_t}{L_s}\varepsilon^{j\theta_0}\frac{pj\omega}{(p+s)(p-j\omega)}1 - \frac{\frac{3}{2}\frac{CI_t}{L_s}\varepsilon^{j\theta_0}\frac{1-\sigma'}{\sigma'}j\omega p^2}{(p+s)\{p-(-\rho_1+j\vartheta_1)\}\{p-(-\rho_2+j\vartheta_2)\}}1 \qquad \text{--(142)}$$

Consider the first of these two terms, viz., -

$$i_{sa} = -\frac{3}{2}\frac{CI_t}{L_s}\varepsilon^{j\theta_0}\frac{pj\omega}{(p+s)(p-j\omega)}1$$

By the expansion theorem :-

$$\frac{Y(0)}{Z(0)} = 0$$

$$\frac{Y(-s)}{(-s)Z'(-s)} = -\frac{3}{2}\frac{CI_t}{L_s}\varepsilon^{j\theta_0}\frac{j\omega}{(-s-j\omega)} \doteqdot \frac{3}{2}\frac{CI_t}{L_s}\varepsilon^{j\theta_0}$$

Full current corresponding to this root is :-

$$i_{s_1} = \frac{3}{2}\frac{CI_t}{L_s}\varepsilon^{-st}\cos\theta_0 1$$

$$\frac{Y(j\omega)}{(j\omega)Z'(j\omega)} = -\frac{3}{2}\frac{CI_t}{L_s}\varepsilon^{j\theta_0}\frac{j\omega}{(j\omega+s)} \doteqdot -\frac{3}{2}\frac{CI_t}{L_s}\varepsilon^{j\theta_0}$$

Full current corresponding to this root is.:-

$$i_{s_1} = -\frac{3}{2}\frac{CI_t}{L_s}\cos(\omega t+\theta_0) 1$$

Thus, full stator current corresponding to first term is :-

$$i_{s_1} = -I_k\{\cos(\omega t+\theta_0) - \varepsilon^{-st}\cos\theta_0\} 1$$

Where $I_k = \frac{3}{2}\frac{CI_t}{L_s}$ = steady stator short-circuit current (max).

Consider the second term of equation (142), viz., -

$$i_{sa} = -I_k\varepsilon^{j\theta_0}\frac{\frac{1-\sigma'}{\sigma'}j\omega p^2}{(p+s)\{p-(-\rho_1+j\vartheta_1)\}\{p-(-\rho_2+j\vartheta_2)\}}1$$

By the expansion theorem :-

$$\frac{Y(0)}{Z(0)} = 0$$

$$\frac{Y(-s)}{(-s)Z'(-s)} = -I_k\varepsilon^{j\theta_0}\frac{\frac{1-\sigma'}{\sigma'}j\omega(-s)}{\{-s-(-\rho_1+j\vartheta_1)\}\{-s-(-\rho_2+j\vartheta_2)\}}$$

$$\doteqdot +I_k\varepsilon^{j\theta_0}\frac{sj\omega[\frac{1-\sigma'}{\sigma'}]}{\{\rho_1-s-0\}\{-j\omega\}}$$

$$\doteqdot -I_k\varepsilon^{j\theta_0}\frac{s[\frac{1-\sigma'}{\sigma'}]}{s[\frac{1}{\sigma'}-1]} \doteqdot -I_k\varepsilon^{j\theta_0}$$

Full current corresponding to this root is :-

$$i_{s_1} = -I_k\varepsilon^{-st}\cos\theta_0 1$$

$$\frac{Y(-\rho_1+j\vartheta_1)}{(-\rho_1+j\vartheta_1)Z'(-\rho_1+j\vartheta_1)} = -I_k\,\varepsilon^{j\theta_0}\,\frac{\frac{1-\sigma'}{\sigma'}(-\rho_1+j\vartheta_1)\,j\omega}{\{(-\rho_1+j\vartheta_1)+s\}\{(-\rho_1+j\vartheta_1)-(-\rho_2+j\vartheta_2)\}}$$

$$\doteqdot +I_k\,\varepsilon^{j\theta_0}\,\frac{1-\sigma'}{\sigma'}\,\frac{-\frac{s}{\sigma'}}{s\left[1-\frac{1}{\sigma'}\right]}$$

$$\doteqdot +I_k\,\frac{1}{\sigma'}\,\varepsilon^{j\theta_0}$$

Full current corresponding to this root is :-

$$i_{s_1} = I_k\,\frac{1}{\sigma'}\,\varepsilon^{-\rho_1 t}\cos(\vartheta_1 t+\theta_0)\,1$$

$$\frac{Y(-\rho_2+j\vartheta_2)}{(-\rho_2+j\vartheta_2)Z'(-\rho_2+j\vartheta_2)} = -I_k\,\varepsilon^{j\theta_0}\,\frac{\frac{1-\sigma'}{\sigma'}(-\rho_2+j\vartheta_2)\,j\omega}{\{(-\rho_2+j\vartheta_2)+s\}\{(-\rho_2+j\vartheta_2)-(-\rho_1+j\vartheta_1)\}}$$

$$\doteqdot -I_k\,\varepsilon^{j\theta_0}\,\frac{1-\sigma'}{\sigma'}\,\frac{j\omega}{j\omega}\,\frac{j\omega}{j\omega} \doteqdot -I_k\,\frac{1-\sigma'}{\sigma'}\,\varepsilon^{j\theta_0}$$

Full current corresponding to this root is :-

$$i_{s_1} = -I_k\,\frac{1-\sigma'}{\sigma'}\,\varepsilon^{-\rho_2 t}\cos(\vartheta_2 t+\theta_0)\,1$$

Thus, full current corresponding to second term is :-

$$i_{s_1} = -I_k\left\{\varepsilon^{-st}\cos\theta_0 - \frac{1}{\sigma'}\varepsilon^{-\rho_1 t}\cos(\vartheta_1 t+\theta_0) + \frac{1-\sigma'}{\sigma'}\varepsilon^{-\rho_2 t}\cos(\vartheta_2 t+\theta_0)\right\}1$$

Hence, full stator current is given by :-

$$i_{s_1} = -I_k\left\{\cos(\omega t+\theta_0) - \frac{1}{\sigma'}\varepsilon^{-\rho_1 t}\cos(\vartheta_1 t+\theta_0) + \frac{1-\sigma'}{\sigma'}\varepsilon^{-\rho_2 t}\cos(\vartheta_2 t+\theta_0)\right\}1 \quad \text{--(143)}$$

The equations given above, - (Nos. 141 and 143) - for the rotor and stator currents respectively, are virtually identical with those obtained by Rüdenberg; (Eqns. 46b and 46a, part I.)

3. Numerical Example.

a. Maxima and worst conditions.

The maximum values of the switching currents are apparent by inspection. For the rotor, the surge is independant of θ_0, and has maximum value $\left\{I_t + 2\frac{1-\sigma'}{\sigma'}I_t\right\}$, occurring at $\omega t = \pi$. For the stator, as ϑ_1 is very small, and $\vartheta_2 \doteqdot \omega$, the equation may be written :-

$$i_{s_1} = -I_k\left\{\cos(\omega t+\theta_0) - \frac{1}{\sigma'}\varepsilon^{-\rho_1 t}\cos\theta_0 + \frac{1-\sigma'}{\sigma'}\varepsilon^{-\rho_2 t}\cos(\omega t+\theta_0)\right\}1$$

For small damping, this current has obviously maximum value when $\theta_0 = 0$, and $(\omega t+\theta_0) = \pi$. The max. value is thus :-

$$i_{s_1} = -I_k\left\{-1-\frac{1}{\sigma'}-\frac{1-\sigma'}{\sigma'}\right\} \doteqdot \frac{2}{\sigma'}I_k$$

b. Specific case.

The values used for the circuit constants are the same as those previously used, though of course their physical significance is somewhat altered. That is, the example should not be regarded as referring to the same machine, with the addition of two coils to the rotor, for while $\underline{s}$ and σ' would be unaffected by this operation, $\underline{r}$ ($= R/L_r$) would be diminished, as L_r would be increased by 50% over the original single phase value ($= L_r'$), since $L_r = L_r' + M_{r1}\cos\frac{2\pi}{3} + M_{r2}\cos\frac{4\pi}{3} \doteqdot \frac{3}{2}L_r'$.

The equations then become :- ($\omega t = \theta$)

$$i_{r1} = I_f + 9I_f\left\{\varepsilon^{-\frac{.92}{2\pi}\theta}\cos\tfrac{1}{25}\theta - \varepsilon^{-\frac{2.08}{2\pi}\theta}\cos\tfrac{24}{25}\theta\right\}1 \qquad \text{--(144)}$$

$$i_{s1} = -I_k\left\{\cos\theta - 10\,\varepsilon^{-\frac{2.08}{2\pi}\theta}\cos\tfrac{1}{25}\theta + 9\,\varepsilon^{-\frac{.92}{2\pi}}\cos\tfrac{24}{25}\theta\right\}1 \qquad \text{--(145)}$$

for $\underline{s} = 10$, $\underline{r} = 5$, $\omega = 2\pi \times 50$, and $\sigma' = .1$.

$$i_{r1} = I_f + 2.33\,I_f\left\{\varepsilon^{-\frac{.332}{2\pi}\theta}\cos\tfrac{1}{250}\theta - \varepsilon^{-\frac{.668}{2\pi}\theta}\cos\tfrac{249}{250}\theta\right\}1 \qquad \text{--(146)}$$

$$i_{s1} = -I_k\left\{\cos\theta - 3.33\,\varepsilon^{-\frac{.668}{2\pi}\theta}\cos\tfrac{1}{250}\theta + 2.33\,\varepsilon^{-\frac{.332}{2\pi}\theta}\cos\tfrac{249}{250}\theta\right\}1 \qquad \text{--(147)}$$

for $\underline{s} = 10$, $\underline{r} = 5$, $\omega = 2\pi \times 50$, and $\sigma' = .3$.

The diagrams, Figs. (30) and (31) show these curves. They should be compared with Figs. (22) and (23), which show the currents in a machine with a single-phase rotor.

The double-frequency component of the stator current in Figs. (22) and (23) is missing in the polyphase rotor case. Apart from that, the curves are very similar, except in the speed of damping. For the curves of current for $\sigma' = .1$, the damping is so rapid as to alter quite considerably the shapes of the curves, as compared with those of Fig. (22) ; for $\sigma' = .3$, the difference of shape due to damping is much less marked.

The ratio of maximum possible short-circuit current to normal maximum current is exactly the same for both rotor and stator currents as it was in the single-phase rotor case; (cf. eqns. 109 and 110). The remarks made in connection with the surges in this latter case apply equally well to the case considered above.

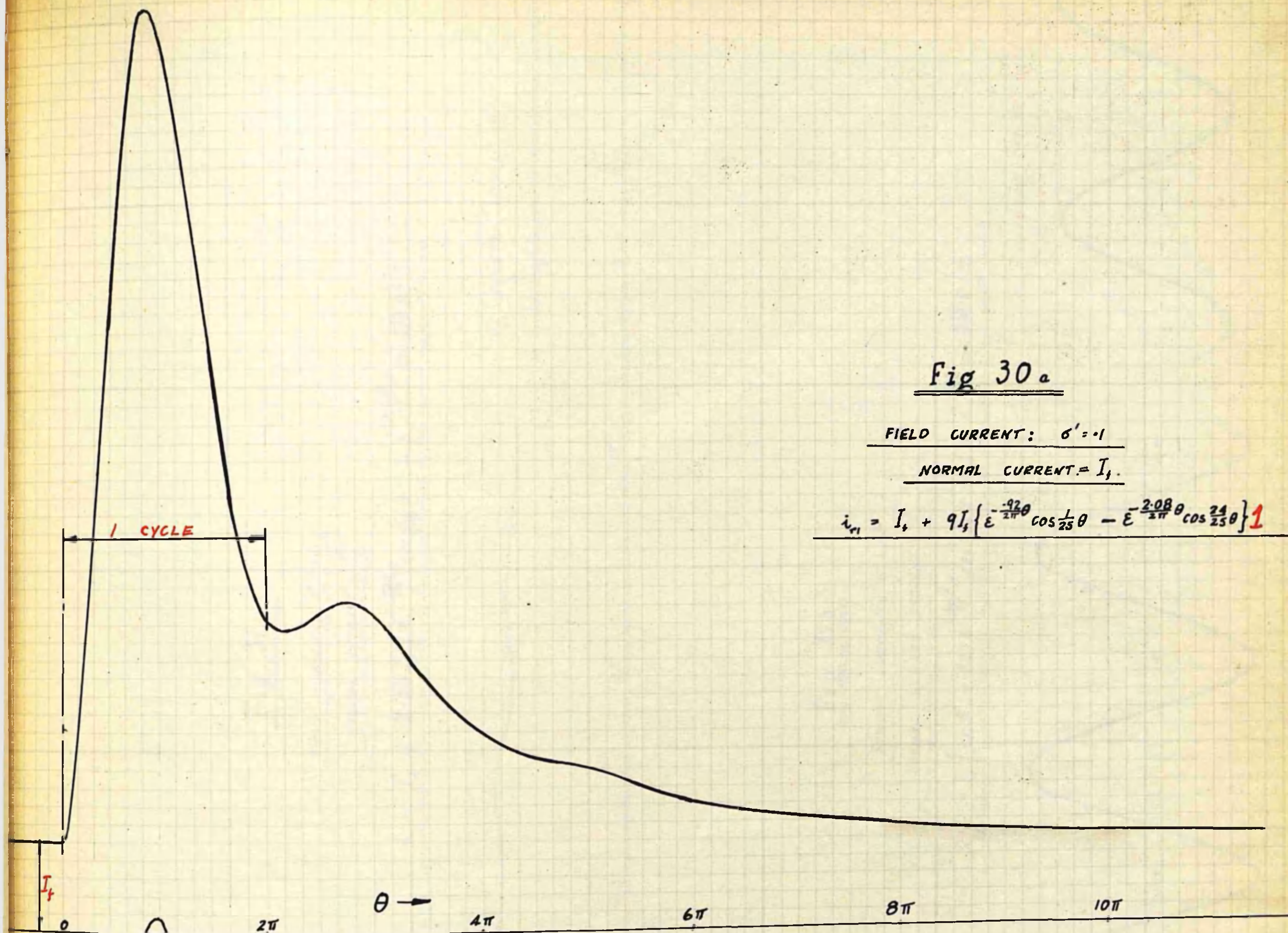


Fig 30 a

FIELD CURRENT: $\sigma' = \cdot 1$

NORMAL CURRENT = I_f.

$$i_{r1} = I_f + 9I_f\left\{\varepsilon^{-\frac{\cdot 92}{2\pi}\theta}\cos\tfrac{1}{25}\theta - \varepsilon^{-\frac{2\cdot 08}{2\pi}\theta}\cos\tfrac{24}{25}\theta\right\} \quad 1$$

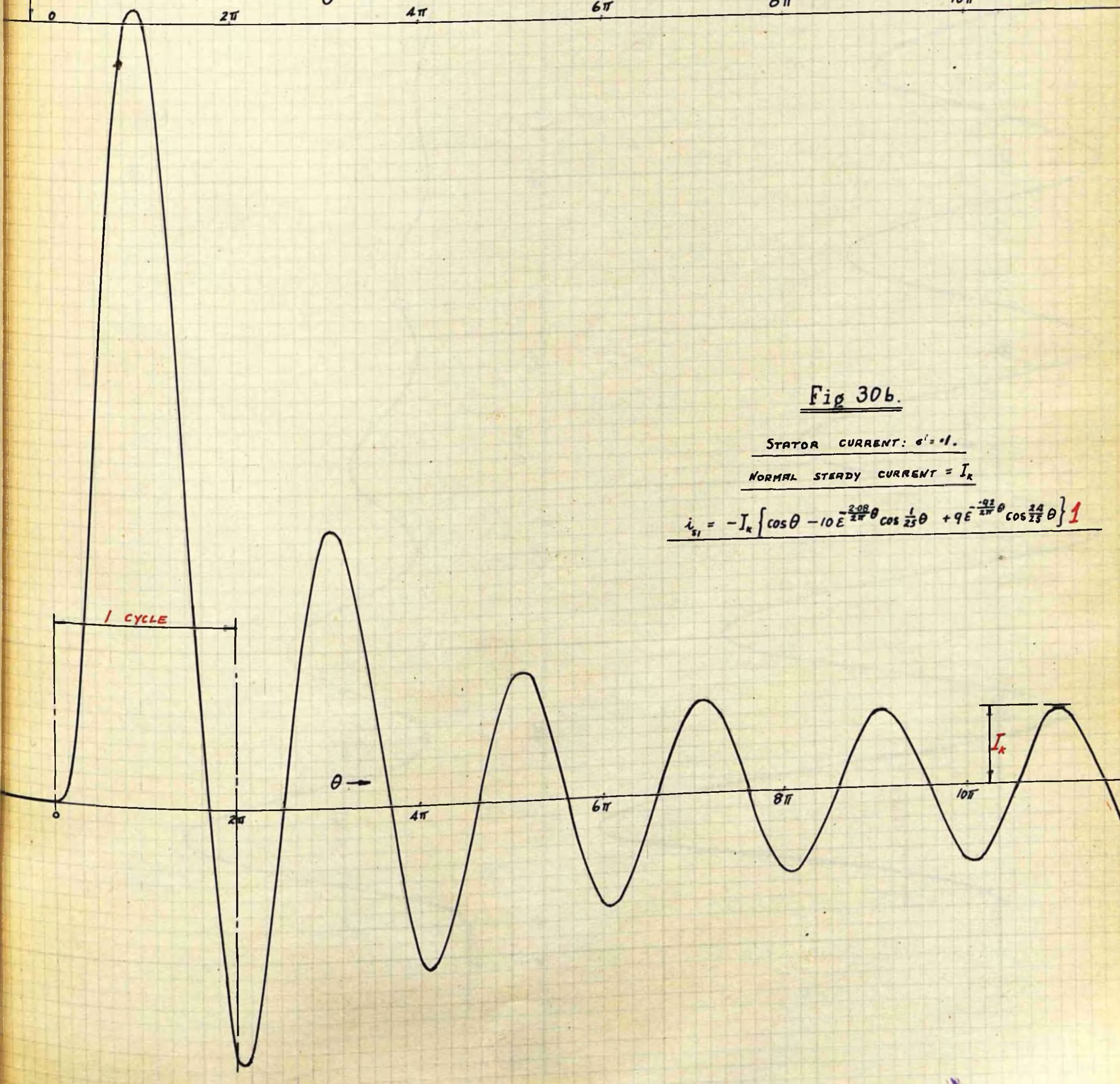


Fig 30b.

STATOR CURRENT: $\sigma' = \cdot 1$.

NORMAL STEADY CURRENT = I_k

$$i_{s1} = -I_k\left\{\cos\theta - 10\varepsilon^{-\frac{2\cdot 08}{2\pi}\theta}\cos\tfrac{1}{25}\theta + 9\varepsilon^{-\frac{\cdot 92}{2\pi}\theta}\cos\tfrac{24}{25}\theta\right\} \quad 1$$

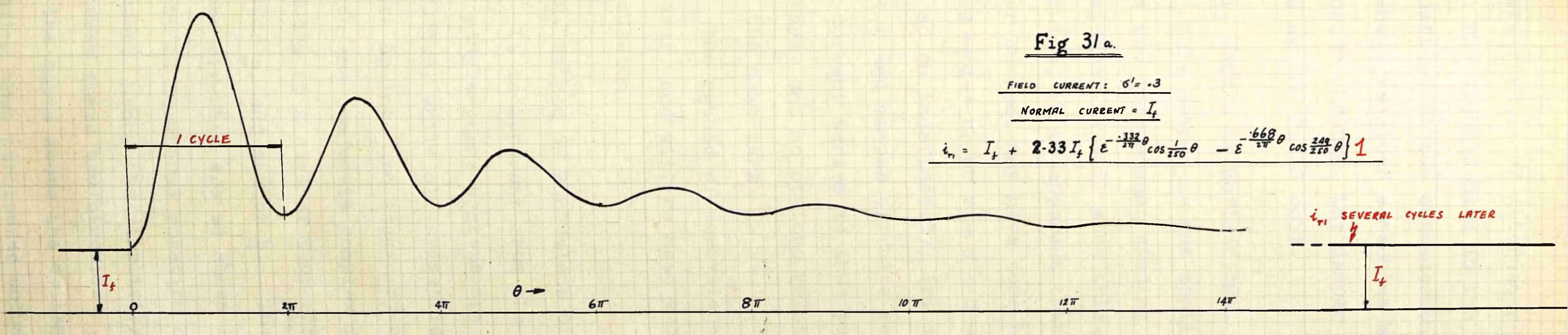


Fig 31a.

FIELD CURRENT: $\sigma' = .3$

NORMAL CURRENT = I_f

$$i_{r1} = I_f + 2{\cdot}33\, I_f \left\{ \varepsilon^{-\frac{.332}{2\pi}\theta} \cos \tfrac{1}{250}\theta - \varepsilon^{-\frac{.668}{2\pi}\theta} \cos \tfrac{249}{250}\theta \right\} \ 1$$

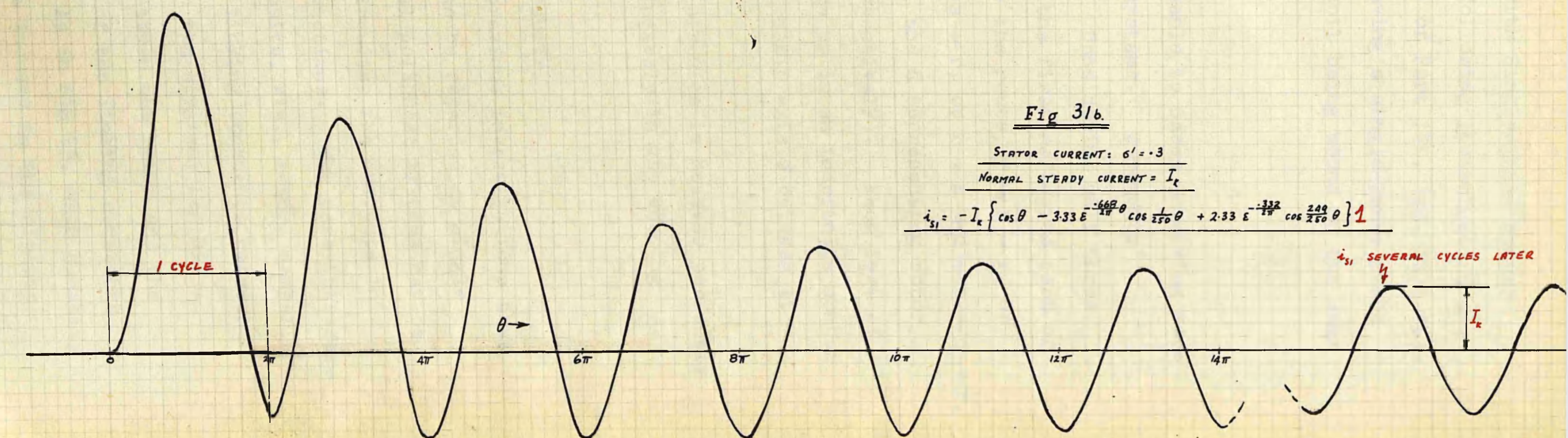


Fig 31b.

STATOR CURRENT: $\sigma' = \cdot 3$

NORMAL STEADY CURRENT = I_k

$$i_{s1} = -I_k \left\{ \cos\theta - 3{\cdot}33\, \varepsilon^{-\frac{.668}{2\pi}\theta} \cos \tfrac{1}{250}\theta + 2{\cdot}33\, \varepsilon^{-\frac{.332}{2\pi}} \cos \tfrac{249}{250}\theta \right\} \ 1$$

D. Short-circuit of Alternator with Single-Phase Damper.

The symbols used in the following investigation are as defined at the commencement of Part III, (pp. 43 & 44). We shall consider a machine having a single-phase field or rotor circuit, the damper coil being wound on the same axis as the field coil.

1. Method of Reflections.

This method is not applicable to cases involving more than two independant winding systems. A moments' consideration will show this. Considering only three circuits, it is apparent that the first reflected term of rotor current may have come by the route, $r \rightarrow s \rightarrow r$, or $r \rightarrow s \rightarrow d \rightarrow r$, or $r \rightarrow s \rightarrow d \rightarrow s \rightarrow r$, or $r \rightarrow s \rightarrow d \rightarrow s \rightarrow d \rightarrow r$, and so on indefinitely, (r, s, and d representing rotor, stator and damper circuits respectively). Each term of the current series is thus made up in an indefinite number of ways, and no obvious relationship exists between it and the next term.

The method of equivalent circuits is however perfectly general, and will be used exclusively in the ensuing discussion.

2. Method of Equivalent Circuits.

It is apparent from the previous investigations that all single-phase systems are to be regarded as single circuits, while all symmetrical polyphase systems may be split up into two imaginary circuits, the currents in which are conjugate complex functions. The case therefore reduces to a case of four circuits, viz., rotor, damper, stator a, and stator b. The linking constants between these circuits are as shown in the diagram, Fig 32. There is no mutual rotation between the rotor and the damper; hence the simplicity of the linking constants between these two circuits. As in Fig 29, each quantity in the table shows the voltage induced in the circuit heading that particular column, by a current i - a function of t - in the circuit belonging to that particular row.

	Rotor	Damper	Stator a	Stator b
Rotor		$-pKi$	$-\frac{1}{2}pC\varepsilon^{i(\omega t+\theta_0)}i$	$-\frac{1}{2}pC\varepsilon^{-i(\omega t+\theta_0)}i$
Damper	$-pKi$		$-\frac{1}{2}pQ\varepsilon^{i(\omega t+\theta_0)}i$	$-\frac{1}{2}pQ\varepsilon^{-i(\omega t+\theta_0)}i$
Stator a	$-\frac{1}{2}pC\varepsilon^{-i(\omega t+\theta_0)}i$	$-\frac{1}{2}pQ\varepsilon^{-i(\omega t+\theta_0)}i$		0
Stator b	$-\frac{1}{2}pC\varepsilon^{i(\omega t+\theta_0)}i$	$-\frac{1}{2}pQ\varepsilon^{i(\omega t+\theta_0)}i$	0	

Fig 32.

The fundamental equations for the arrangement may thus be written immediately; i.e., -

$$
\begin{aligned}
&Z_{r(p)}\, i_r + pK\, i_d + \tfrac{1}{2}pC\varepsilon^{-i(\omega t+\theta_0)} i_{sa} + \tfrac{1}{2}pC\varepsilon^{i(\omega t+\theta_0)} i_{sb} = 0\\
&pK\, i_r + Z_{d(p)}\, i_d + \tfrac{1}{2}pQ\varepsilon^{-i(\omega t+\theta_0)} i_{sa} + \tfrac{1}{2}pQ\varepsilon^{i(\omega t+\theta_0)} i_{sb} = 0\\
&\tfrac{1}{2}pC\varepsilon^{i(\omega t+\theta_0)} i_r + \tfrac{1}{2}pQ\varepsilon^{i(\omega t+\theta_0)} i_d + Z_{s(p)}\, i_{sa} + 0 = -\tfrac{1}{2}\frac{Cpj\omega}{p-j\omega}\varepsilon^{i\theta_0} I_t 1\\
&\tfrac{1}{2}pC\varepsilon^{-i(\omega t+\theta_0)} i_r + \tfrac{1}{2}pQ\varepsilon^{-i(\omega t+\theta_0)} i_d + 0 + Z_{s(p)}\, i_{sb} = -\tfrac{1}{2}\frac{Cp(-j\omega)}{p+j\omega}\varepsilon^{-i\theta_0} I_t 1
\end{aligned}
\qquad \text{--(148)}
$$

These equations cannot, of course, be solved by determinantal methods. Expanding in the usual way, - (cf page 70),- and writing $Z_{(p)} = Z$, and $Z_{(p\pm j\omega)} = Z_{(\pm)}$, for brevity, we obtain :-

$$
\begin{aligned}
&\left\{Z_r - \tfrac{n}{4}C^2p\left[\frac{p+j\omega}{Z_{s(+)}} + \frac{p-j\omega}{Z_{s(-)}}\right]\right\} i_r + \left\{Kp - \tfrac{n}{4}CQp\left[\frac{p+j\omega}{Z_{s(+)}} + \frac{p-j\omega}{Z_{s(-)}}\right]\right\} i_d = \tfrac{n}{4}C^2p\left[\frac{j\omega}{Z_{s(+)}} + \frac{-j\omega}{Z_{s(-)}}\right] I_t 1\\
&\left\{Kp - \tfrac{n}{4}CQp\left[\frac{p+j\omega}{Z_{s(+)}} + \frac{p-j\omega}{Z_{s(-)}}\right]\right\} i_r + \left\{Z_d - \tfrac{n}{4}Q^2p\left[\frac{p+j\omega}{Z_{s(+)}} + \frac{p-j\omega}{Z_{s(-)}}\right]\right\} i_d = \tfrac{n}{4}CQp\left[\frac{j\omega}{Z_{s(+)}} + \frac{-j\omega}{Z_{s(-)}}\right] I_t 1
\end{aligned}
\qquad \text{--(149)}
$$

or, writing $\left[\frac{p+j\omega}{Z_{s(+)}} + \frac{p-j\omega}{Z_{s(-)}}\right] = \mathcal{G}_{(p)}$, and $\left[\frac{j\omega}{Z_{s(+)}} + \frac{-j\omega}{Z_{s(-)}}\right] = F_{(p)}$, :-

$$
\begin{aligned}
&\left[Z_r - \tfrac{n}{4}C^2p\,\mathcal{G}_{(p)}\right] i_r + \left[Kp - \tfrac{n}{4}CQp\,\mathcal{G}_{(p)}\right] i_d = \tfrac{n}{4}C^2p\,F_{(p)} I_t 1\\
&\left[Kp - \tfrac{n}{4}CQp\,\mathcal{G}_{(p)}\right] i_r + \left[Z_d - \tfrac{n}{4}Q^2p\,\mathcal{G}_{(p)}\right] i_d = \tfrac{n}{4}CQp\,F_{(p)} I_t 1
\end{aligned}
\qquad \text{--(150)}
$$

The equations (150) contain all functions of p, and no function of t; they are therefore amenable to direct determinantal solution. Hence :-

$$
i_r = \frac{n}{4}Cp\,\frac{\begin{vmatrix} C & \left[Kp - \tfrac{n}{4}CQp\,\mathcal{G}_{(p)}\right]\\ Q & \left[Z_d - \tfrac{n}{4}Q^2p\,\mathcal{G}_{(p)}\right]\end{vmatrix}}{\begin{vmatrix} \left[Z_r - \tfrac{n}{4}C^2p\,\mathcal{G}_{(p)}\right] & \left[Kp - \tfrac{n}{4}CQp\,\mathcal{G}_{(p)}\right]\\ \left[Kp - \tfrac{n}{4}CQp\,\mathcal{G}_{(p)}\right] & \left[Z_d - \tfrac{n}{4}Q^2p\,\mathcal{G}_{(p)}\right]\end{vmatrix}}\,F_{(p)} I_t 1 \qquad \text{--(151)}
$$

$$
i_d = \frac{n}{4}Cp\,\frac{\begin{vmatrix} \left[Z_r - \tfrac{n}{4}C^2p\,\mathcal{G}_{(p)}\right] & C\\ \left[Kp - \tfrac{n}{4}CQp\,\mathcal{G}_{(p)}\right] & Q\end{vmatrix}}{\begin{vmatrix} \left[Z_r - \tfrac{n}{4}C^2p\,\mathcal{G}_{(p)}\right] & \left[Kp - \tfrac{n}{4}CQp\,\mathcal{G}_{(p)}\right]\\ \left[Kp - \tfrac{n}{4}CQp\,\mathcal{G}_{(p)}\right] & \left[Z_d - \tfrac{n}{4}Q^2p\,\mathcal{G}_{(p)}\right]\end{vmatrix}}\,F_{(p)} I_t 1 \qquad \text{--(152)}
$$

a. The rotor current.

The numerator of the expression in eqn.(151) reduces to :-

$$\text{Numerator} = \tfrac{n}{4} C p \left\{ C Z_d - QKp \right\} F_{(p)} I_f \, 1$$

as the function $\xi_{(p)}$ vanishes.

The denominator simplifies to :-

$$\text{Denominator} = Z_r Z_d - \tfrac{n}{4} C^2 p \, Z_d \, \xi_{(p)} - \tfrac{n}{4} Q^2 p Z_r \, \xi_{(p)} - K^2 p^2 + \tfrac{n}{2} CKQ p^2 \xi_{(p)}$$

$$= \left[Z_r Z_d - K p^2 \right] + \xi_{(p)} \left[\tfrac{n}{2} CQK p^2 - \tfrac{n}{4} C^2 p Z_d - \tfrac{n}{4} Q^2 p Z_r \right]$$

as the function $\xi^2_{(p)}$ vanishes.

If the values of $F_{(p)}$ and $\xi_{(p)}$ are substituted, the equation may be written :-

$$i_r = \frac{\tfrac{n}{2} C p \left[C Z_d - QKp \right] \omega^2 L_s I_f \, 1}{Z_{s(+)} Z_{s(-)} \left[Z_r Z_d - K^2 p \right] + \left[R_s p + (p^2 + \omega^2) L_s \right] \left[n CQKp^2 - \tfrac{n}{2} C^2 p Z_d - \tfrac{n}{2} Q^2 p Z_r \right]} \qquad \text{--(153)}$$

This expression is rather cumbersome. It may be expressed more simply, however, if certain abbreviations are used. These are as below :-

$\tfrac{1}{2} n C^2 / L_s L_r = c^2$; $\tfrac{1}{2} n Q^2 / L_s L_d = q^2$; $K^2 / L_r L_d = k^2$:

$R_r / L_r = r$; $R_s / L_s = s$; $R_d / L_d = d$:

The equation then becomes, by ordinary arithmetic processes :-

$$i_r = \frac{p c^2 \left\{ d + p\left(1 - \tfrac{q k}{c}\right) \right\} \omega^2 I_f \, 1}{\left\{ A_1 p^4 + A_2 p^3 + A_3 p^2 + A_4 p + A_5 \right\}} \qquad \text{--(154)}$$

where $A_1 = \left\{ 1 - c^2 - k^2 - q^2 + 2ckq \right\}$

$A_2 = \left\{ s\left[1 - c^2 - k^2 - q^2 + 2ckq\right] + d(1 - c^2) + s(1 - k^2) + r(1 - q^2) \right\}$

$A_3 = \left\{ \omega^2 \left[1 - c^2 - k^2 - q^2 + 2ckq\right] + s\left[d(1 - c^2) + s(1 - k^2) + r(1 - q^2)\right] + sr + rd + ds \right\}$

$A_4 = \left\{ \omega^2 \left[d(1 - c^2) + r(1 - q^2) \right] + s\left[sr + rd + ds \right] + rds \right\}$

$A_5 = \left\{ rd(\omega^2 + s) \right\}$

It does not seem possible to get a general symbolic solution for the above equation, for the denominator cannot be factorised. We may, however, obtain an approximate solution by neglecting all resistance, as will be shown later.

b. The damper current.

By precisely similar methods to those shown above for the rotor current, the damper current equation may be expressed as below :-

$$i_d = \frac{\frac{C}{Q} I_t \, p q^2 \{ r + p(1 - \frac{ck}{q}) \} \omega^2 \, 1}{\{ A_1 p^4 + A_2 p^3 + A_3 p^2 + A_4 p + A_5 \}} \qquad \text{--(155)}$$

c. The stator current.

Substituting eqns.(154) and (155) in the requisite eqn.(148), we obtain for i_{sa} :-

$$i_{sa} = -\frac{1}{2} \frac{Cpj\omega}{(p-j\omega) Z_s} I_t \varepsilon^{j\theta_0} 1 - \frac{1}{2} \frac{Cp}{Z_s} \varepsilon^{j(\omega t + \theta_0)} i_r - \frac{1}{2} \frac{Qp}{Z_s} \varepsilon^{j(\omega t + \theta_0)} i_d$$

$$= -\frac{1}{2} \frac{CI_t}{L_s} \frac{pj\omega}{(p+s)(p-j\omega)} \varepsilon^{j\theta_0} 1 - \frac{1}{2} \frac{CI_t}{L_s} \varepsilon^{j\theta_0} \left\{ \frac{[c^2 d + q^2 r] p^2 \omega^2 + (p-j\omega)[c^2 + q^2 - 2ckq] p^2 \omega^2}{(p+s)[A_1 (p-j\omega)^4 + A_2 (p-j\omega)^3 + A_3 (p-j\omega)^2 + A_4 (p-j\omega) + A_5]} \right\} 1$$

$$= -\frac{1}{2} I_k \varepsilon^{j\theta_0} \frac{pj\omega}{(p+s)(p-j\omega)} 1 - \frac{1}{2} I_k \varepsilon^{j\theta_0} \left\{ \frac{\{ (c^2 d + q^2 r) + (p-j\omega)[c^2 + q^2 - 2ckq] \} p^2 \omega^2}{(p+s)[A_1 (p-j\omega)^4 + A_2 (p-j\omega)^3 + A_3 (p-j\omega)^2 + A_4 (p-j\omega) + A_5]} \right\} 1 \qquad \text{--(156)}$$

3. The Complete Solutions for the Currents.

Let us first of all solve for the currents for one or two numerical cases, and follow that up by attempting a very approximate general solution.

Two numerical cases are considered below. In both, the coupling between rotor and stator is 0·95 — i.e., $c^2 = 0{\cdot}9$ — this being the value taken before, for the first of the problems involving single-phase rotor, no damper, and polyphase stator (cf page 78). Further, the quantities r, s and d are respectively 5, 10 and 2 for both cases, r and s being as taken in the problem already referred to. The difference between the two cases lies in the values of q and k. In the first case, the rotor and damper are very closely coupled; in the second, the stator and damper are very closely coupled. The actual values are thus :-

i. $s = 10$; $r = 5$; $d = 2$; $c^2 = 0{\cdot}9$; $q^2 = 0{\cdot}92$; $k^2 = 0{\cdot}97$; $\omega = 2\pi \times 50 = 314$.

ii. $s = 10$; $r = 5$; $d = 2$; $c^2 = 0{\cdot}9$; $q^2 = 0{\cdot}97$; $k^2 = 0{\cdot}92$; $\omega = 2\pi \times 50 = 314$.

If these values are substituted in the operational expressions for the currents, the denominators may be factorised by the "Graeffe" method (vide. Part II, E3) and the expression solved by the Expansion Theorem. The expansion is rather laborious, but the difficulties lie only in the arithmetic. The following expressions result for the currents :-

For the values as in i. :-

$$\begin{aligned} i_r &= I_f + I_f\left\{3.41\,\varepsilon^{-18t} - 1.48\,\varepsilon^{-232t} - 2.42\,\varepsilon^{-67t}\cos(306t - 35^\circ)\right\}1 \\ i_d &= \frac{C}{Q} I_f\left\{8.35\,\varepsilon^{-18t} + 1.49\,\varepsilon^{-232t} - 9.9\,\varepsilon^{-67t}\cos(306t - 7^\circ)\right\}1 \\ i_{s1} &= -I_k\left\{\cos(314t+\theta_0) + 11.82\,\varepsilon^{-18t}\cos(314t+\theta_0) - 7\,\varepsilon^{-67t}\cos(8t+\theta_0+9^\circ) - 6\,\varepsilon^{-67t}\cos(620t+\theta_0-9^\circ)\right\}1 \end{aligned} \qquad \text{--(157)}$$

These curves are shown in Figs (33a), (34a), & (35a).

For the values as in ii :-

$$\begin{aligned} i_r &= I_f + I_f\left\{11.7\,\varepsilon^{-42t} - 11.91\,\varepsilon^{-124t} - 4.42\,\varepsilon^{-140t}\cos(227t - 93^\circ)\right\}1 \\ i_d &= \frac{C}{Q} I_f\left\{12.5\,\varepsilon^{-42t} + 41.1\,\varepsilon^{-124t} - 55.7\,\varepsilon^{-140t}\cos(227t - 15^\circ)\right\}1 \\ i_{s1} &= -I_k\left\{\cos(314t+\theta_0) + 24.2\,\varepsilon^{-42t}\cos(314t+\theta_0) + 29.2\,\varepsilon^{-124t}\cos(314t+\theta_0) - 29.2\,\varepsilon^{-170t}\cos(87t+\theta_0+21^\circ) - 23.4\,\varepsilon^{-170t}\cos(541t+\theta_0-20^\circ)\right\}1 \end{aligned} \qquad \text{--(158)}$$

These curves are shown in Figs (33b), (34b), & (35b).

When the maxima and worst conditions for the alternator without damping winding were considered, it was shown that an approximate solution could be obtained by neglecting all resistance; this solution was the more accurate, the more nearly σ' approached to unity, for with σ' large, terms like $\frac{r}{\sigma'}$ & c. could be neglected in comparison with ω, so that the approximate amplitudes were more nearly accurate, and the damping coefficients $\varepsilon^{-\frac{r}{\sigma'}t}$ & c. could be regarded as unity for all t. (cf pp. 62, 71 et seq., 75.)

A similar approximation may be made in the case now being considered, except that here the conditions of accuracy are rather more complex. For consider the various operational equations for the currents, with all resistance neglected; they are :-

$$i_r = \frac{p c^2\{p(1-\frac{kq}{c})\} I_f \omega^2}{[1-c^2-k^2-q^2+2ckq]p^4 + \omega^2[1-c^2-k^2-q^2+2ckq]p^2}\,1 = \frac{c^2(1-\frac{kq}{c})}{(1-c^2-k^2-q^2+2ckq)} I_f \frac{\omega^2}{p^2+\omega^2}\,1 \qquad \text{--(159)}$$

and similarly :-

$$i_d = \frac{C}{Q} I_f \frac{q^2(1-\frac{ck}{q})}{(1-c^2-k^2-q^2+2ckq)} \frac{\omega^2}{p^2+\omega^2}\,1 \qquad \text{--(160)}$$

and :-

$$\begin{aligned} i_{sa} &= -\tfrac{1}{2} I_k \varepsilon^{j\theta_0} \frac{j\omega}{p-j\omega}\,1 - \tfrac{1}{2} I_k \varepsilon^{j\theta_0}\left\{\frac{(p-j\omega)[c^2+q^2-2ckq]p^2\omega^2}{p[(p-j\omega)^2+\omega^2][1-c^2-k^2-q^2+2ckq](p-j\omega)^2}\right\}1 \\ &= -\tfrac{1}{2} I_k \varepsilon^{j\theta_0} \frac{j\omega}{p-j\omega}\,1 - \tfrac{1}{2} I_k \varepsilon^{j\theta_0}\left\{\frac{c^2+q^2-2ckq}{1-c^2-k^2-q^2+2ckq}\right\}\frac{\omega^2}{(p-2j\omega)(p-j\omega)}\,1 \end{aligned} \qquad \text{--(161)}$$

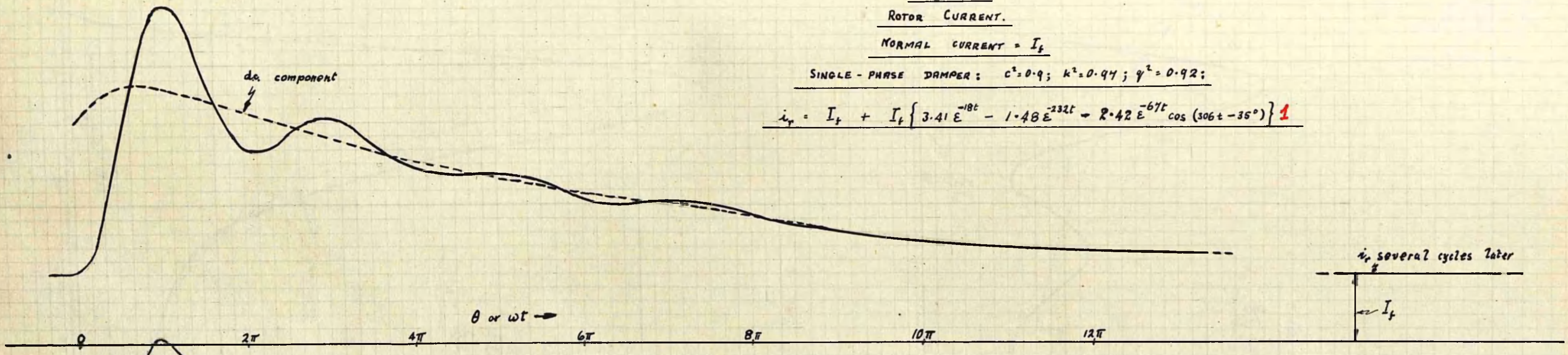


Fig 33a.

ROTOR CURRENT.

NORMAL CURRENT = I_f

SINGLE-PHASE DAMPER: $c^2 = 0{\cdot}9$; $k^2 = 0{\cdot}97$; $q^2 = 0{\cdot}92$;

$$i_r = I_f + I_f\left\{3{\cdot}41\,\varepsilon^{-18t} - 1{\cdot}48\,\varepsilon^{-232t} - 2{\cdot}42\,\varepsilon^{-67t}\cos(306t - 35^\circ)\right\} 1$$

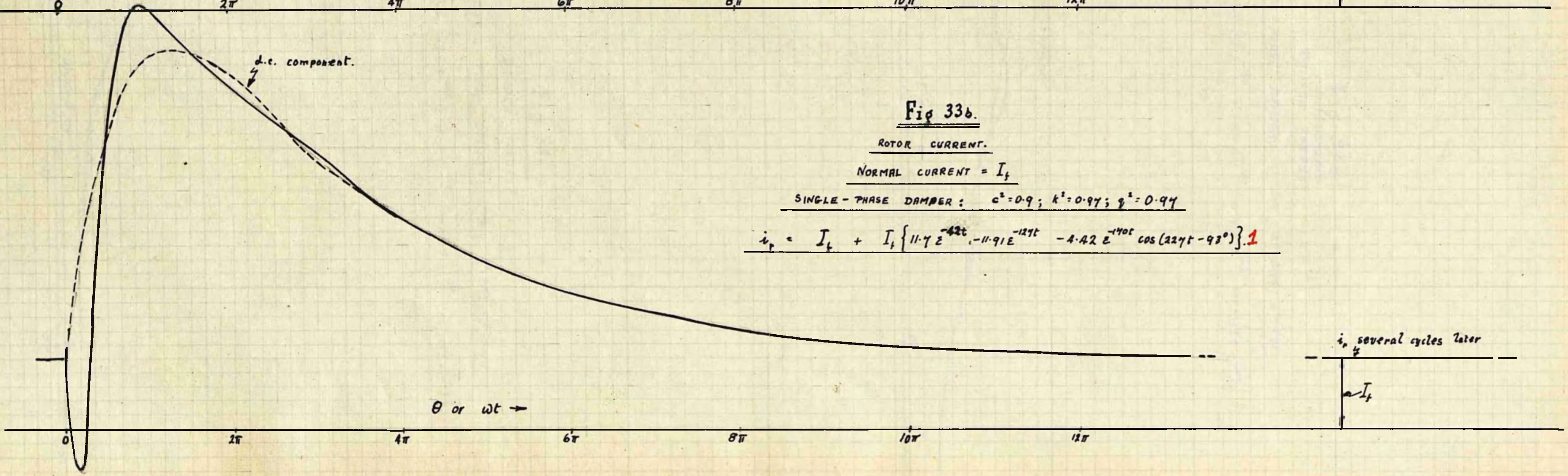


Fig 33b.

ROTOR CURRENT.

NORMAL CURRENT = I_f

SINGLE-PHASE DAMPER: $c^2 = 0{\cdot}9$; $k^2 = 0{\cdot}97$; $q^2 = 0{\cdot}97$

$$i_r = I_f + I_f\left\{11{\cdot}7\,\varepsilon^{-42t}, -11{\cdot}91\,\varepsilon^{-127t} - 4{\cdot}42\,\varepsilon^{-140t}\cos(227t - 93^\circ)\right\}.1$$

Fig 34a

DAMPER CURRENT

SINGLE-PHASE DAMPER: $c^2 = 0{\cdot}9$; $k^2 = 0{\cdot}97$; $q^2 = 0{\cdot}92$.

$$i_d = \frac{C}{Q} I_f \left\{ 8{\cdot}35\, \varepsilon^{-18t} + 1{\cdot}49\, \varepsilon^{-232t} - 9{\cdot}9\, \varepsilon^{-64t} \cos(306t - 7^\circ) \right\} \quad 1$$

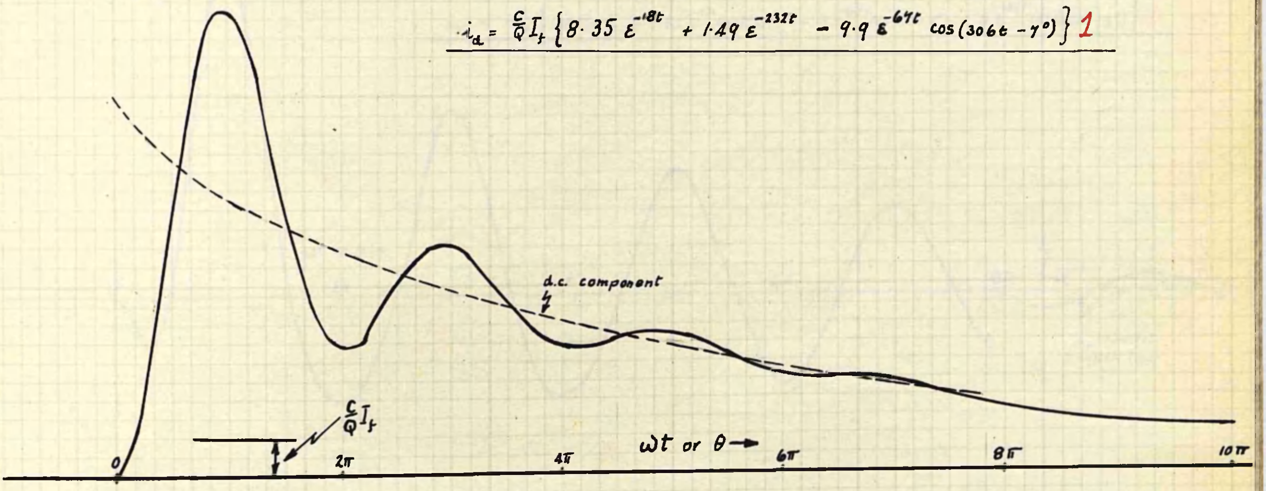


Fig 34b

DAMPER CURRENT

SINGLE-PHASE DAMPER: $c^2 = 0{\cdot}9$; $k^2 = 0{\cdot}92$; $q^2 = 0{\cdot}98$.

$$i_d = \frac{C}{Q} I_f \left\{ 12{\cdot}5\, \varepsilon^{-42t} + 41{\cdot}1\, \varepsilon^{-124t} - 55{\cdot}7\, \varepsilon^{-170t} \cos(227t - 15^\circ) \right\} \quad 1$$

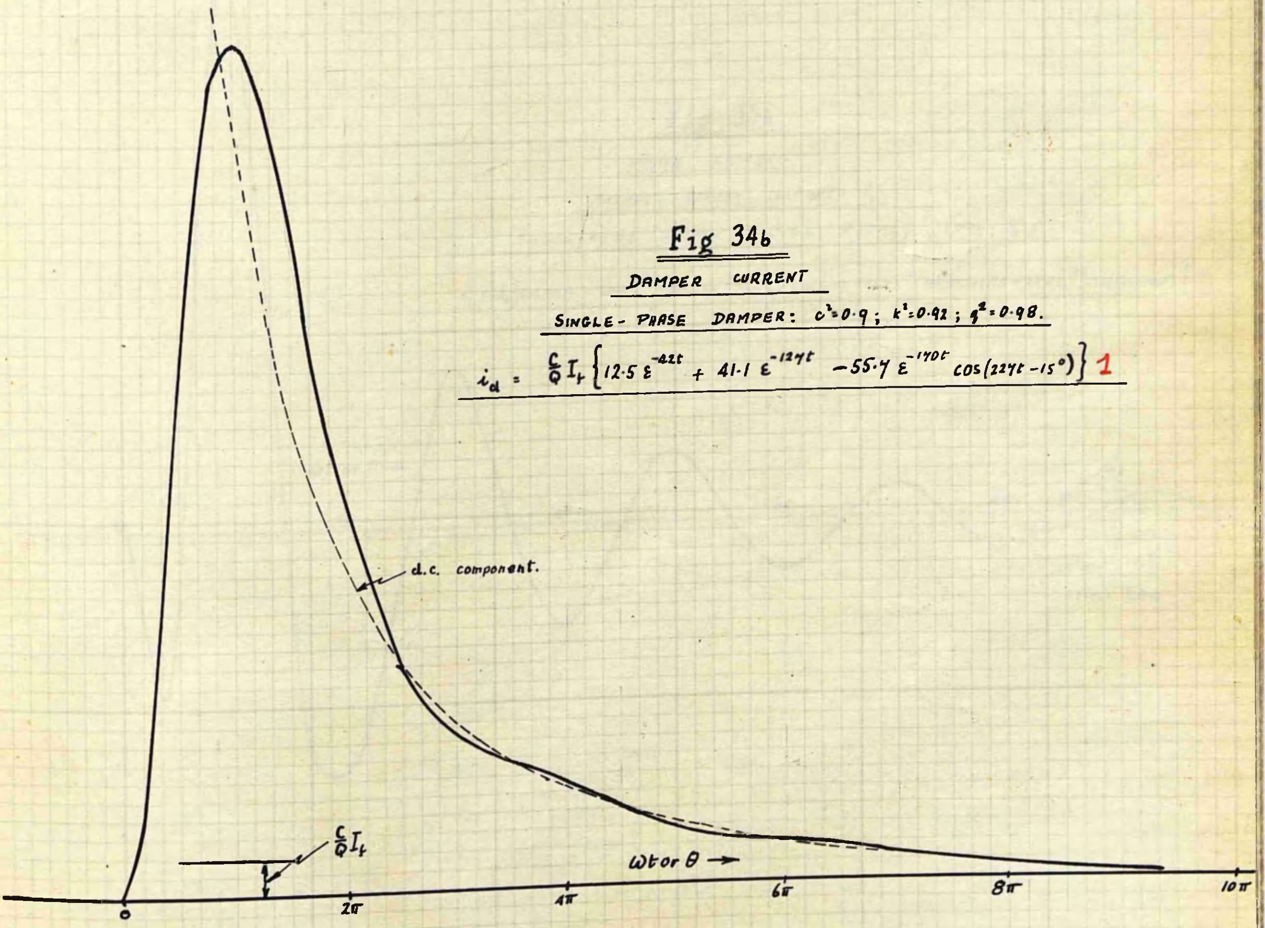


Fig 35a.

STATOR CURRENT.

NORMAL STEADY CURRENT = I_k.

SINGLE-PHASE DAMPER: $c^2 = 0{\cdot}9$; $k^2 = 0{\cdot}97$; $q^2 = 0{\cdot}92$; $\theta_0 = 0$:

$$i_{s1} = -I_k\left\{\cos(314t) + 11{\cdot}82\,\varepsilon^{-18t}\cos(314t) - 7\,\varepsilon^{-67t}\cos(8t+9^\circ) - 6\,\varepsilon^{-67t}\cos(620t-9^\circ)\right\} \quad 1$$

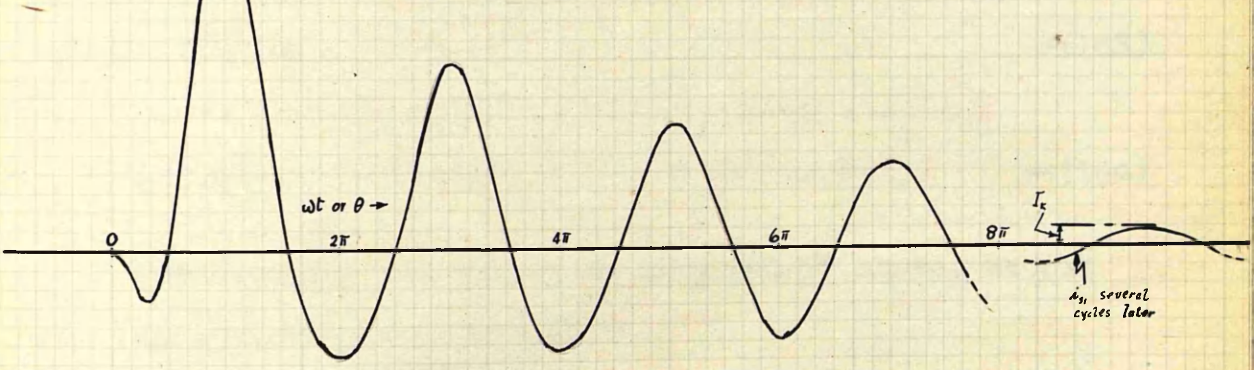


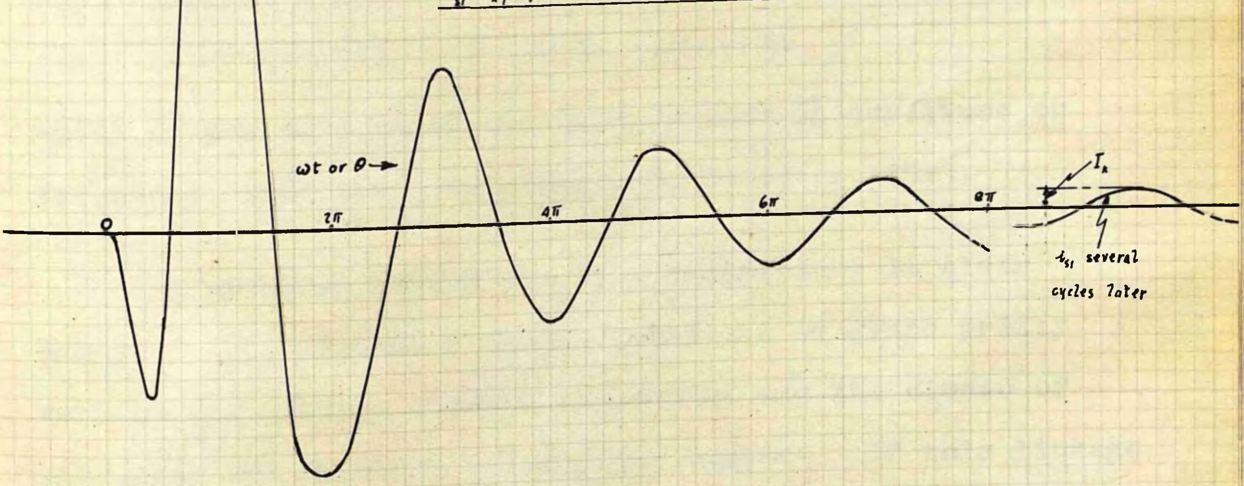
Fig 35b.

STATOR CURRENT.

NORMAL STEADY CURRENT = I_k

SINGLE-PHASE DAMPER: $c^2 = 0{\cdot}9$; $k^2 = 0{\cdot}92$; $q^2 = 0{\cdot}97$; $\theta_0 = 0$:

$$i_{s1} = -I_k\left\{\cos(314t) + 24{\cdot}2\,\varepsilon^{-42t}\cos(314t) + 29{\cdot}2\,\varepsilon^{-117t}\cos(314t) - 29{\cdot}2\,\varepsilon^{-110t}\cos(87t+21^\circ) - 23{\cdot}4\,\varepsilon^{-110t}\cos(541t-20^\circ)\right\} \quad 1$$



These last equations will be reasonably accurate only if the quantities $(1-\frac{kq}{c})$, $(1-\frac{kc}{q})$, and $(1-c^2-k^2-q^2+2ckq)$ are not too small, so that resistance may truly be neglected, and the roots of the auxiliary equation are truly 0, 0, & $\pm j\omega$.

The full current equations corresponding to the above formulae are, by the expansion theorem :-

$$i_r = \frac{c^2[1-\frac{kq}{c}]}{[1-c^2-k^2-q^2+2ckq]} I_f \left\{1-\cos\omega t\right\} 1 \qquad + I_f \qquad \text{--(162)}$$

$$i_d = \frac{C}{Q}\,\frac{q^2(1-\frac{kc}{q})}{[1-c^2-k^2-q^2+2ckq]} I_f \left\{1-\cos\omega t\right\} 1 \qquad \text{--(163)}$$

$$\begin{aligned} i_{s1} &= -I_k\left\{-\cos\theta_0 + \cos(\omega t+\theta_0)\right\}1 - I_k\left\{\frac{c^2+q^2-2ckq}{1-c^2-k^2-q^2+2ckq}\right\}\left\{-\tfrac{1}{2}\cos\theta_0 - \tfrac{1}{2}\cos(2\omega t+\theta_0) + \cos(\omega t+\theta_0)\right\}1 \\ &= -I_k\left\{\frac{[1-k^2]}{[1-c^2-k^2-q^2+2ckq]}\cos(\omega t+\theta_0) - \tfrac{1}{2}\left[\frac{1-k^2}{1-c^2-k^2-q^2+2ckq}-1\right]\cos(2\omega t+\theta_0) - \tfrac{1}{2}\left[\frac{1-k^2}{1-c^2-k^2-q^2+2ckq}+1\right]\cos\theta_0\right\}1 \end{aligned} \qquad \text{--(164)}$$

Substituting the values as in i and ii above, we get for the approximate solutions :-

For the values as in i :-

$$\begin{aligned} i_r &= 1{\cdot}59\, I_f\left\{1-\cos\omega t\right\}1 \quad + I_f \\ i_d &= 9{\cdot}95\,\tfrac{C}{Q} I_f\left\{1-\cos\omega t\right\}1 \\ i_{s1} &= -I_k\left\{12{\cdot}5\cos(\omega t+\theta_0) - 5{\cdot}75\cos(2\omega t+\theta_0) - 6{\cdot}75\cos\theta_0\right\}1 \end{aligned} \qquad \text{--(165)}$$

which compare very well in amplitude and frequency with the true solutions, in eqns. (157).

For the values as in ii :-

$$\begin{aligned} i_r &= 1{\cdot}59\, I_f\left\{1-\cos\omega t\right\}1 \quad + I_f \\ i_d &= 30{\cdot}8\,\tfrac{C}{Q} I_f\left\{1-\cos\omega t\right\}1 \\ i_{s1} &= -I_k\left\{33{\cdot}4\cos(\omega t+\theta_0) - 16{\cdot}2\cos(2\omega t+\theta_0) - 17{\cdot}2\cos\theta_0\right\}1 \end{aligned} \qquad \text{--(166)}$$

which do not compare at all well, either in amplitude or frequency, with the true solutions, in eqns. (158).

The physical reason for this discrepancy is clear enough. The frequency of the transient is fixed pretty well by the linkage between the stator and the closer of the other two windings - i.e., the damper. If this linkage is not too great, - not more than about 0·95 - then the transient frequency is approximately ω, the frequency of the machine. If, however, the linkage is tight, then the transient frequency falls far short of ω. This frequency turns up in the denominators, and materially affects the amplitudes.

For the values i, the linkages between the various windings are all high, but that between the stator and the damper is not the highest. The frequency of the transient is thus almost ω - actually 306 - so that the effect of neglecting the resistance, and thereby assuming the value ω for this frequency, is not very great. For the values ii, however, the damper and stator are very closely coupled, and the natural frequency of the transient is only 227, so that the amplitudes of the transient curves are in reality some 50% greater than those given by the approximate solution. The rotor current i_r is affected still more in this latter case, being quite unlike the approximate curve. Consideration of the numerator of the rotor current expression shows why this should be. The numerator contains the factor $(1-\frac{kg}{c})$, and c is always the smallest of the three constants, c, k, & g, (unless the damper circuit is external to the machine, as say the field coil of the exciter dynamo, in which case it is not, of course, a true damper winding at all). Thus, if all three constants are large, $(1-\frac{kg}{c})$ becomes almost zero; it is not therefore correct to neglect the resistance terms in the numerator. This error, added to that due to the small natural frequency, produces the very large discrepancy.

It is, however, hardly possible that the coupling constants should be so high. It is certainly unlikely that the stator and damper should be so tightly coupled, and the damper will probably be linked more tightly with the rotor than with the stator. This being so, we may take the approximate solutions as being substantially correct, provided g, and therefore of course c, are not too high. The solutions will at least be sufficiently accurate for the establishment of worst conditions and maximum currents.

4. Maxima and Worst Conditions.

It is apparent by inspection of equations (162), (163) and (164) that worst conditions again occur at $\theta_0 = 0$ and $\omega t = \pi$. Under these circumstances, the max. currents are given by :-

$$i_{r(max)} = I_f + 2\frac{c^2[1-\frac{kq}{c}]}{[1-c^2-k^2-q^2+2ckq]} I_f \qquad \text{--(167)}$$

$$i_{d(max)} = 2\frac{c}{Q} I_f \frac{q^2[1-\frac{ck}{q}]}{[1-c^2-k^2-q^2+2ckq]} \qquad \text{--(168)}$$

and
$$i_{s1(max)(max)} = I_k \left\{ \frac{[1-k^2]}{[1-c^2-k^2-q^2+2ckq]} + \frac{1}{2}\left[\frac{[1-k^2]}{[1-c^2-k^2-q^2+2ckq]} - 1\right] + \frac{1}{2}\left[\frac{[1-k^2]}{[1-c^2-k^2-q^2+2ckq]} + 1\right]\right\}$$

$$= 2\frac{[1-k_2]}{[1-c^2-k^2-q^2+2ckq]} I_k \qquad \text{--(169)}$$

These equations are to be compared with eqns(110) and (109), which give the maxima under similar conditions in a machine without a damper winding. These last two equations are reproduced below :-

$$i_{r(max)} = I_f + 2\frac{1-\sigma'}{\sigma'} I_f = I_f + 2\frac{c^2}{1-c^2} I_f \qquad \text{--(110)}$$

$$i_{s1(max)(max)} = \frac{2}{\sigma'} I_k = \frac{2}{1-c^2} I_k \qquad \text{--(109)}$$

changing from the σ' notation to the present c notation.

It is, in general, true that q and k are each individually greater than c, if the damper coil is between the rotor and stator coils. It is also true that the product kq is in general less than c, as may be seen from the diagram, Fig 36. For let s and d be close-coupled. Then c = k, and q = 1, so that kq = c. If, now, d moves away from s, q diminishes, and k increases, until finally q = c, and k = 1. But the flux density at s due to s is considerably greater than the flux density at s due to r, so that for the first half of the motion, q diminishes at a more rapid rate than k increases. At the half-way point, q = k, and from then onwards, k increases more rapidly than q diminishes. Thus the product qk is always less than c, reaching a minimum value half-way between s and r.

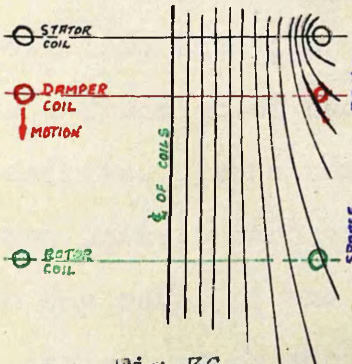


Fig 36.

SHOWING HALF OF FLUX DUE TO S.

The rule above was in point of fact tacitly observed when $\underline{c}^2$ was taken as 0·9, and $\underline{k}^2$ and $\underline{q}^2$ as 0·97 and 0·92.

The factor $(1-c^2-k^2-q^2+2ckq)$ may thus be written :-

$$
\begin{aligned}
(1-c^2-k^2-q^2+2ckq) &= (1-c^2) - (k^2+q^2-2kq) - 2kq(1-c) \\
&> (1-c^2) - (k-q)^2 - 2c(1-c) \\
&> (1-c)^2 - (k-q)^2
\end{aligned}
$$

Now since $1 > [k \text{ or } q] > c$, $(1-c)$ is in general greater than $(k-q)$. Hence the factor $(1-c^2-k^2-q^2+2ckq)$ is always positive for these conditions.

We may now go on to compare the short-circuit currents with and without damper windings. We shall consider the ratio :-

$$\frac{\text{Max. current with single-phase damper winding}}{\text{Max. current without damper winding}}$$

a. Rotor current.

The ratio is, considering only transient current, :-

$$
\begin{aligned}
\text{Ratio} &= \frac{\left(1-\frac{kq}{c}\right)(1-c^2)}{\{1-c^2-k^2-q^2+2ckq\}} \\
&= 1 - \frac{\{ckq+\frac{kq}{c}-k^2-q^2\}}{\{1-c^2-k^2-q^2+2ckq\}} \\
&= 1 - \frac{1}{c}\,\frac{(k-cq)(q-ck)}{\{1-c^2-k^2-q^2+2ckq\}}
\end{aligned}
$$

Now, $\underline{qc}$ and $\underline{kc}$ are both less than $\underline{c}$, and are thus both less than $\underline{q}$ or $\underline{k}$. Hence the right-hand term above is a fraction of two positive quantities, and is itself positive. Thus the ratio of the currents is always less than unity; moreover, $\left(1-\frac{kq}{c}\right)$ and $(1-c^2)$ are both positive, so the ratio of the currents is positive. Thus the transient rotor current is always decreased by the addition of a damper. It is of interest that $\underline{k}$ and $\underline{q}$ are interchangeable in the above expression, so that given certain values for the two, it makes no difference which value pertains to which. This is also shown by eqns. (165) and (166), the first equations of each group being identical.

Before commenting further, we shall go on to consider the stator current ratio.

b. The stator current.

The ratio for the stator currents is :-

$$\text{Ratio} = \frac{(1-k^2)(1-c^2)}{\{1-c^2-k^2-q^2+2ckq\}}$$

$$= 1 + \frac{(c^2k^2+q^2-2ckq)}{\{1-c^2-k^2-q^2+2ckq\}}$$

$$= 1 + \frac{(q-ck)^2}{\{1-c^2-k^2-q^2+2ckq\}}$$

The ratio on the right is here again positive, as the numerator is a square. Thus the stator current is always greater with a damper than without. It appears from this equation that the term (q-ck) should not be allowed to become too great if heavy switching currents are to be prevented; this involves a large k and a small q , which causes the damper to lose some of its effectiveness in counteracting "Hunting", so the stator current cannot be very much lowered. If q is made large, the rotor current ratio tends to become unity.

We shall not spend more time on this case, but go on to consider the polyphase damper machine, which is of more general interest.

E. Short-circuit of Machine with Polyphase Damper.

We shall consider a machine having a single-phase rotor circuit, three-phase stator, and n-phase damper, the damper being regarded as a symmetrical system of n coils linked by inductance only. It will be shown later that the theory holds for a damper winding of the more usual squirrel-cage type.

1. The Solution of the Problem, by Equivalent Circuits.

Consider a current $F_{(p)}1$ flowing in the rotor coil. This current will react on the damper as follows :-

Volts induced in damper, phase 1 $= -\frac{1}{2}K_p\{\varepsilon^{i0^\circ} + \varepsilon^{-i0^\circ}\}F_{(p)}1$

Volts induced in damper, phase 2 $= -\frac{1}{2}K_p\{\varepsilon^{i\frac{2\pi}{n}} + \varepsilon^{-i\frac{2\pi}{n}}\}F_{(p)}1$

and so on.

Hence, current in damper, phase 1 $= -\frac{1}{2}\frac{K_p}{Z_{d(p)}}\{\varepsilon^{i0^\circ} + \varepsilon^{-i0^\circ}\}F_{(p)}1$

and current in damper, phase 2 $= -\frac{1}{2}\frac{K_p}{Z_{d(p)}}\{\varepsilon^{i\frac{2\pi}{n}} + \varepsilon^{-i\frac{2\pi}{n}}\}F_{(p)}1$

and so on, $Z_{d(p)}$ having the usual physical interpretation.

Volts induced in rotor by $\underline{i}_{d1} = -\frac{1}{2}Kp\{\varepsilon^{j0^\circ} + \varepsilon^{-j0^\circ}\} \times -\frac{1}{2}\frac{Kp}{Z_d}\{\varepsilon^{j0^\circ} + \varepsilon^{-j0^\circ}\} F_{(p)} 1$

Volts induced in rotor by $\underline{i}_{d2} = -\frac{1}{2}Kp\{\varepsilon^{j\frac{2\pi}{n}} + \varepsilon^{-j\frac{2\pi}{n}}\} \times -\frac{1}{2}\frac{Kp}{Z_d}\{\varepsilon^{j\frac{2\pi}{n}} + \varepsilon^{-j\frac{2\pi}{n}}\} F_{(p)} 1$

and so on.

Thus total current induced in rotor $= \frac{n}{4}\frac{K^2p^2}{Z_r Z_d}\{\varepsilon^{j0^\circ} + \varepsilon^{-j0^\circ}\} F_{(p)} 1$

$= \frac{n}{2}\frac{K^2p^2}{Z_r Z_d} F_{(p)} 1$

With a single-phase damper, this last expression would have been $\frac{K^2p^2}{Z_r Z_d} F_{(p)} 1$. Apparently, then, we must regard the damper as consisting of two equivalent imaginary circuits in the usual way, so that although the damper current, phase 1, is given by $-\frac{1}{2}\frac{Kp}{Z_d}\{\varepsilon^{j0^\circ} + \varepsilon^{-j0^\circ}\}F_{(p)}1 = -\frac{Kp}{Z_d}F_{(p)}1$, we may not say that the linkage between rotor and damper is -Kp, but that the linkage, rotor to damper a is $-\frac{1}{2}$Kp, and the linkage, rotor to damper b is also $-\frac{1}{2}$Kp.

We may draw up a table showing the various linkages in the usual manner :-

	Rotor	Damper a	Damper b	Stator a	Stator b
Rotor		$-\frac{1}{2}Kp\,i$	$-\frac{1}{2}Kp\,i$	$-\frac{1}{2}Cp\varepsilon^{-i(\omega t+\theta_0)}i$	$-\frac{1}{2}Cp\varepsilon^{i(\omega t+\theta_0)}i$
Damper a	$-\frac{n}{2}Kp\,i$		0	$-\frac{n}{2}Qp\varepsilon^{i(\omega t+\theta_0)}i$	0
Damper b	$-\frac{n}{2}Kp\,i$	0		0	$-\frac{n}{2}Qp\varepsilon^{-i(\omega t+\theta_0)}i$
Stator a	$-\frac{3}{2}Cp\varepsilon^{-i(\omega t+\theta_0)}i$	$-\frac{3}{2}Qp\varepsilon^{-i(\omega t+\theta_0)}i$	0		0
Stator b	$-\frac{3}{2}Cp\varepsilon^{i(\omega t+\theta_0)}i$	0	$-\frac{3}{2}Qp\varepsilon^{i(\omega t+\theta_0)}i$	0	

Fig 37.

The fundamental equations may thus be written :-

$$
\begin{aligned}
&Z_r i_r + \tfrac{n}{2}Kp\,i_{da} + \tfrac{n}{2}Kp\,i_{db} + \tfrac{3}{2}Cp\varepsilon^{-i(\omega t+\theta_0)}i_{sa} + \tfrac{3}{2}Cp\varepsilon^{i(\omega t+\theta_0)}i_{sb} = 0\\
&\tfrac{1}{2}Kp\,i_r + Z_d i_{da} + 0 + \tfrac{3}{2}Qp\varepsilon^{-i(\omega t+\theta_0)}i_{sa} + 0 = 0\\
&\tfrac{1}{2}Kp\,i_r + 0 + Z_d i_{db} + 0 + \tfrac{3}{2}Qp\varepsilon^{i(\omega t+\theta_0)}i_{sb} = 0 \qquad \text{--(170)}\\
&\tfrac{1}{2}Cp\varepsilon^{i(\omega t+\theta_0)}i_r + \tfrac{n}{2}Qp\varepsilon^{i(\omega t+\theta_0)}i_{da} + 0 + Z_s i_{sa} + 0 = -\tfrac{1}{2}Cp\varepsilon^{i\theta_0}\tfrac{i\omega}{p-j\omega}I_t 1\\
&\tfrac{1}{2}Cp\varepsilon^{-i(\omega t+\theta_0)}i_r + 0 + \tfrac{n}{2}Qp\varepsilon^{-i(\omega t+\theta_0)}i_{db} + 0 + Z_s i_{sb} = -\tfrac{1}{2}Cp\varepsilon^{-i\theta_0}\tfrac{-i\omega}{p+j\omega}I_t 1
\end{aligned}
$$

By the usual method of expansion (vide page 70), these equations may be cleared of all "t" functions, to give the following three equations :-

$$
\begin{aligned}
&\left[Z_r - \tfrac{3}{4}C^2p\,F_{(p)}\right]i_r + \left[\tfrac{n}{2}Kp - \tfrac{3n}{4}CQp\tfrac{p+j\omega}{Z_s(+)}\right]i_{da} + \left[\tfrac{n}{2}Kp - \tfrac{3n}{4}CQp\tfrac{p-j\omega}{Z_s(-)}\right]i_{db} = \tfrac{3}{4}C^2p\,F_{(p)}I_t 1\\
&\left[\tfrac{1}{2}Kp - \tfrac{3}{4}CQp\tfrac{p+j\omega}{Z_s(+)}\right]i_r + \left[Z_d - \tfrac{3n}{4}Q^2p\tfrac{p+j\omega}{Z_s(+)}\right]i_{da} + 0 = \tfrac{3}{4}CQp\tfrac{j\omega}{p+j\omega}I_t 1 \qquad \text{--(171)}\\
&\left[\tfrac{1}{2}Kp - \tfrac{3}{4}CQp\tfrac{p-j\omega}{Z_s(-)}\right]i_r + 0 + \left[Z_d - \tfrac{3n}{4}Q^2p\tfrac{p-j\omega}{Z_s(-)}\right]i_{db} = \tfrac{3}{4}CQp\tfrac{-j\omega}{p-j\omega}I_t 1
\end{aligned}
$$

In the equations (171), $F_{(p)}$ and $\xi_{(p)}$ are as defined previously, viz :-

$$F_{(p)} = \left\{ \frac{j\omega}{Z_{s(+)}} + \frac{-j\omega}{Z_{s(-)}} \right\}$$

$$\xi_{(p)} = \left\{ \frac{p+j\omega}{Z_{s(+)}} + \frac{p-j\omega}{Z_{s(-)}} \right\}$$

The equations (171) may be solved by ordinary determinantal methods. The resulting operational expressions are however rather complicated, and each current possesses its own points of interest; it is therefore advisable to keep the investigations of the different circuits separate. A separate section of the Thesis will be devoted to each circuit and its current.

In the ensuing discussion, the currents will first of all be worked out for definite values of the circuit constants, these values being as in D3,i, viz :-

$s = 10$; $r = 5$; $d = 2$; $c^2 = 0{\cdot}9$; $k^2 = 0{\cdot}97$; $q^2 = 0{\cdot}92$; $\omega = 2\pi . 50 = 314$.

Subsequently, for the purpose of obtaining an approximate general solution, the currents will be worked out with all resistance neglected.

Since the matrix determinant of equations (171) is obviously of importance, being in fact the denominator of the rotor and damper current operational expressions, a separate section will be devoted to it.

2. The Matrix Determinant.

This is the determinant :-

$$\begin{vmatrix} \left[Z_r - \frac{3}{4}C^2 p\,\xi_{(p)}\right] & \left[\frac{\pi}{2}K_p - \frac{3\pi}{4}CQp\frac{p+j\omega}{Z_{s(+)}}\right] & \left[\frac{\pi}{2}K_p - \frac{3\pi}{4}CQp\frac{p-j\omega}{Z_{s(-)}}\right] \\ \left[\frac{1}{2}K_p - \frac{3}{4}CQp\frac{p+j\omega}{Z_{s(+)}}\right] & \left[Z_d - \frac{3\pi}{4}Q^2 p\frac{p+j\omega}{Z_{s(+)}}\right] & 0 \\ \left[\frac{1}{2}K_p - \frac{3}{4}CQp\frac{p-j\omega}{Z_{s(-)}}\right] & 0 & \left[Z_d - \frac{3\pi}{4}Q^2 p\frac{p-j\omega}{Z_{s(-)}}\right] \end{vmatrix}$$

Its full expanded form is very unwieldly. By making the abbreviations as before, however, (cf page 96), it may be somewhat simplified. It becomes :-

$$D = \frac{L_c^2 L_d^2 L_r}{Z_{s(+)}Z_{s(-)}} \cdot \left\{ A_1 p^5 + A_2 p^4 + A_3 p^3 + A_4 p^2 + A_5 p + A_6 \right\} \qquad \text{--(172)}$$

where the constants A_1, A_2, & c. are as overleaf.

In eqn. (172) :-

$$A_1 = [1-c^2-k^2-q^2+2ckq](1-q^2)$$

$$A_2 = \{[1-c^2-k^2-q^2+2ckq](s+d) + (1-q^2)[r(1-q^2) + s(1-k^2) + d(1-c^2)]\}$$

$$A_3 = \{[1-c^2-k^2-q^2+2ckq][\omega^2(1-q^2)+sd] + (s+d)[r(1-q^2) + s(1-k^2) + d(1-c^2)] + (1-q^2)[rd+ds+sr]\}$$

$$A_4 = \{[1-c^2-k^2-q^2+2ckq]d\omega^2 + \omega^2(1-q^2)[r(1-q^2)+d(1-c^2)] + sd[r(1-q^2)+s(1-k^2)+d(1-c^2)] + rds(1-q^2) + (d+s)[rd+ds+sr]\}$$

$$A_5 = \{[r(1-q^2)+d(1-c^2)]d\omega^2 + rd\omega^2(1-q^2) + sd[rd+ds+sr] + (d+s)rds\}$$

$$A_6 = \{[s^2+\omega^2]rd^2\}$$

<u>a.</u> With the above values for the constants.

With the numerical values as explained, the determinant takes on the form -(disregarding the function $\frac{L_s^2 L_d^2 L_r}{Z_{s(+)} Z_{s(-)}}$, which cancels with the same function in the numerator) :-

$$D = 0{\cdot}000192\,\{p^5 + (5{\cdot}25\times10^2)p^4 + (1{\cdot}882\times10^5)p^3 + (3{\cdot}22\times10^7)p^2 + (1{\cdot}041\times10^9)p + 1{\cdot}029\times10^{10}\}$$

or, by the "Graeffe" method of root extraction :-

$$D = 0{\cdot}000192\,\{[p+(127+j306)][p+(127-j306)][p+(18+j8)][p+(18-j8)][p+232]\} \qquad \text{--(173)}$$

<u>b.</u> With all resistance neglected.

The approximate value of the matrix is apparently, (disregarding the function $\frac{L_s^2 L_d^2 L_r}{Z_{s(+)} Z_{s(-)}}$) :-

$$D' = [1-c^2-k^2-q^2+2ckq](1-q^2)\,[p^2+\omega^2]p^3$$

$$= \{[1-c^2-k^2-q^2+2ckq](1-q^2)\}\{p^3(p+j\omega)(p-j\omega)\} \qquad \text{--(174)}$$

<u>3.</u> The Rotor Current.

<u>a.</u> The numerator.

This is the determinant formed by substituting the right-hand members of eqns. (171) for the members of the first column of the matrix determinant. With the usual abbreviations, it becomes :-

$$M = \frac{L_s^2 L_d^2 L_r}{Z_{s(+)} Z_{s(-)}} \times I_f\, c^2\omega^2 p\,\{p(1-q^2)+d\}\{p(1-\tfrac{kq}{c})+d\}\,1 \qquad \text{--(175)}$$

<u>b.</u> With numerical values.

Disregarding the function $\frac{L_s^2 L_d^2 L_r}{Z_{s(+)} Z_{s(-)}}$, which cancels with the denominator, the numerator becomes :-

$$M = I_f \times 88{,}900\,p\,\{0{\cdot}08p+2\}\{0{\cdot}00424p+2\}\,1$$

$$= I_f \times 30{\cdot}5\,p\,\{p+25\}\{p+473\}\,1 \qquad \text{--(176)}$$

Thus the full numerical operational solution for the current is :- (over)

$$i_r = I_f \frac{156{,}900\,p\,[p+25][p+473]}{[p+232][p+(18\pm j8)][p+(127\pm j306)]}\,1 \qquad \text{--(177)}$$

Applying the expansion theorem to eqn. (177) produces the following full numerical solution for the rotor current:-

$$i_r = I_f + I_f\left\{-1{\cdot}63\,\varepsilon^{-232t} + 4{\cdot}19\,\varepsilon^{-18t}\cos(8t-45^\circ) - 2{\cdot}23\,\varepsilon^{-124t}\cos(306t-51^\circ)\right\}1 \qquad \text{--(178)}$$

This curve is shown in Fig 38. It should be compared with the rotor current without damper - (Fig 30a) - and with single-phase damper - (Fig 33a). The rotor current with a polyphase damper is obviously much less than that with no damper at all, but is very similar to that with a single-phase damper; except for the rapid damping of the alternating component in Fig (38) the curves are almost identical. It should be remembered that this machine is not the same machine as before with extra damper coils added, for this operation would alter the ratio d ($= R_d/L_d$) by increasing L in the ratio $\frac{1}{2}$n; thus by taking d at the same value as before, viz., 2 , we have imagined the resistance of a damper coil increased in the same ratio. This probably explains why the damping in the polyphase case has come out greater than in the single-phase case, a view which will receive confirmation when we consider the approximate general case. More will be said on this subject when the damper current is evaluated.

c. Approximate general solution.

With all resistance neglected, the numerator of the operational expression becomes :-

$$M = I_f\,c^2\omega^2\,p^3\,[1-q^2]\left[1-\frac{qk}{c}\right]1 \qquad \text{--(179)}$$

Thus the full operational solution for the current is:-

$$i_r = I_f\,\frac{c^2\left(1-\frac{kq}{c}\right)}{(1-c^2-k^2-q^2+2ckq)}\cdot\frac{\omega^2}{p^2+\omega^2}\,1 \qquad \text{--(180)}$$

Applying the expansion theorem to this equation :-

$$i_r = I_f + I_f\,\frac{c^2\left(1-\frac{kq}{c}\right)}{(1-c^2-k^2-q^2+2ckq)}\left\{1-\cos\omega t\right\}1 \qquad \text{--(181)}$$

With the values of c, k and q as above, this becomes:-

$$i_r = I_f + 1{\cdot}59\,I_f\left\{1-\cos\omega t\right\}1 \qquad \text{--(182)}$$

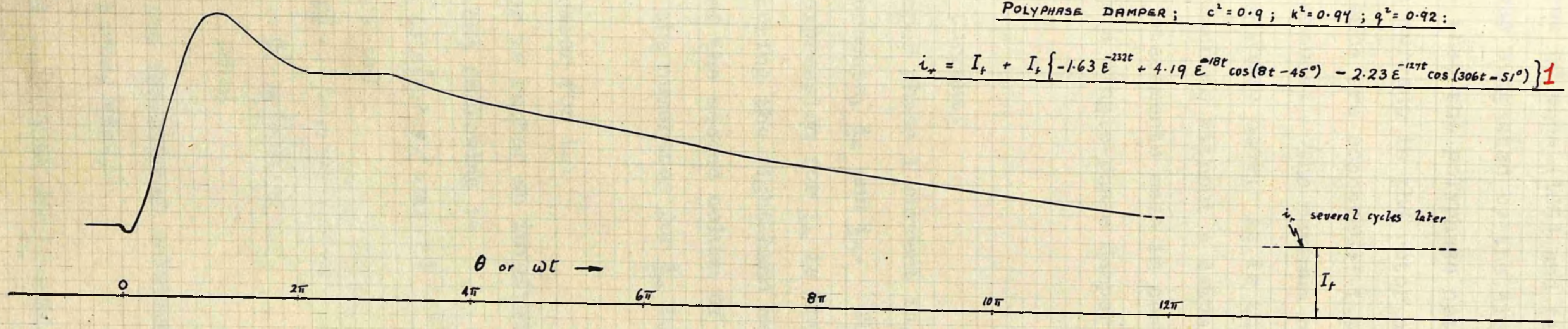
Fig 38.

ROTOR CURRENT.

NORMAL CURRENT = I_r

POLYPHASE DAMPER ; $c^2 = 0{\cdot}9$; $k^2 = 0{\cdot}97$; $q^2 = 0{\cdot}92$:

$$i_r = I_r + I_r\left\{-1{\cdot}63\,\varepsilon^{-232t} + 4{\cdot}19\,\varepsilon^{-18t}\cos(8t-45^\circ) - 2{\cdot}23\,\varepsilon^{-127t}\cos(306t-51^\circ)\right\} \quad 1$$



It is apparent that eqn.(182) is identical with those obtained for the rotor current with a single-phase damper - the first of eqns.(165) and (166). This confirms the view which was suggested by the similarity of the true equations - that the polyphase nature of the damper only affects the current in the rotor as far as terms involving resistance are concerned; i.e., in terms of second order magnitude. The maximum rotor current is as for the single-phase case; it is thus always less than that for a machine without a damper winding.

In general, statements made in connection with the rotor current for a single-phase damper apply also to this case.

4. The Damper Current.

The full damper phase 1 current is, of course, the sum of the two currents, $\underline{i}_{da}$ and $\underline{i}_{db}$. The numerator of the operational expression for $\underline{i}_{da}$ is the determinant formed by substituting the right-hand members of eqns.(171) for the members of the second column of the matrix determinant, and the numerator for $\underline{i}_{db}$ is the complement of this.

a. The numerator for i_{da}.

The numerator is rather an involved function, but it simplifies to the following :-

$$M = \frac{L_s^2 L_d^2 L_r}{Z_{s(+)} Z_{s(-)}} \cdot \frac{C}{nQ} I_s q^2 p \left\{ B_1 p^3 + B_2 p^2 + B_3 p + B_4 \right\} 1 \qquad \text{--(183)}$$

where $B_1 = +j\omega \left[1 - c^2 - k^2 - q^2 + 2ckq \right]$

$B_2 = \omega^2 \left[(1-q^2)(1 - \frac{ck}{q}) \right] + j\omega \left[r(1-q^2) + s(1-k^2) + d(1-c^2) \right]$

$B_3 = \omega^2 \left[r(1-q^2) + d(1 - \frac{ck}{q}) \right] + j\omega \left[rd + ds + sr \right]$

and $B_4 = \omega^2 [rd] + j\omega [rds]$

Apparently, the operational solution for $\underline{i}_{da}$ may be divided into two parts, viz.,-

$$i_{da} = \frac{C}{nQ} I_s q^2 p \left\{ \frac{\omega^2 \left[(1-q^2)p + d \right] \left[(1 - \frac{ck}{q})p + r \right]}{Z_{(p)}} + j\omega \frac{\left[1 - c^2 - k^2 - q^2 + 2ckq \right] p^3 + \left[r(1-q^2) + s(1-k^2) + d(1-c^2) \right] p^2 + \left[rd + ds + sr \right] p + rds}{Z_{(p)}} \right\} 1 \qquad \text{--(184)}$$

where $Z_{(p)}$ is the matrix determinant D without the function $\frac{L_s^2 L_d^2 L_r}{Z_{s(+)} Z_{s(-)}}$.

b. The full damper phase 1 current, i_{d_1}.

i. Operational solution.

The full current i_{d_1} is the sum of i_{da} and i_{db}, which are complementary functions. Thus i_{d_1} is given by twice the real part of i_{da}, or :-

$$i_{d_1} = \frac{C}{\frac{n}{2}Q} I_f \frac{q^2\omega^2 p\left[(1-q^2)p+d\right]\left[(1-\frac{ck}{q})p+r\right]}{Z(p)} 1 \qquad \text{--(185)}$$

ii. Numerical solution.

With the numerical values as before, eqn.(185) becomes:-

$$i_{d_1} = \frac{C}{\frac{n}{2}Q} I_f \frac{980,000\,[p+25][p+193]}{(p+232)\left[p+(18\pm j8)\right]\left[p+(127\pm j306)\right]} 1 \qquad \text{--(186)}$$

By the expansion theorem :-

$$i_{d_1} = \frac{C}{\frac{n}{2}Q} I_f \left\{1{\cdot}65\,\varepsilon^{-232t} + 9{\cdot}85\,\varepsilon^{-18t}\cos(8t-40^\circ) - 9{\cdot}47\,\varepsilon^{-127t}\cos(306t-14^\circ)\right\} 1 \qquad \text{--(187)}$$

This current is shown in the diagram, Fig 39a.

iii. Approximate solution.

With all resistance neglected, the operational equation (185) becomes :-

$$i_{d_1} = \frac{C}{\frac{n}{2}Q} I_f \frac{q^2(1-\frac{ck}{q})}{(1-c^2-k^2-q^2+2ckq)} \cdot \frac{\omega^2}{p^2+\omega^2} 1 \qquad \text{--(188)}$$

By the expansion theorem :-

$$i_{d_1} = \frac{C}{\frac{n}{2}Q} I_f \frac{q^2(1-\frac{ck}{q})}{(1-c^2-k^2-q^2+2ckq)} \left[1-\cos\omega t\right] 1 \qquad \text{--(189)}$$

With the values of c, k and q as above, this becomes:-

$$i_{d_1} = \frac{C}{\frac{n}{2}Q} I_f \cdot 9{\cdot}95 \left[1-\cos\omega t\right] 1 \qquad \text{--(190)}$$

which is a good approximation to equation (187) above.

c. The damper current in the other phases.

i. The general solution.

Remembering the method employed to get the phase 2 stator current - (cf page 73) - we might be inclined to insert the angles $-\frac{2\pi}{n}$, $-\frac{4\pi}{n}$, & c. in eqn.(187) in order to get the damper currents in the other phases. This is incorrect, however, for the following reason. When considering the stator phase 2, we say simply, "Let us assume that the short-circuit takes place when the rotor

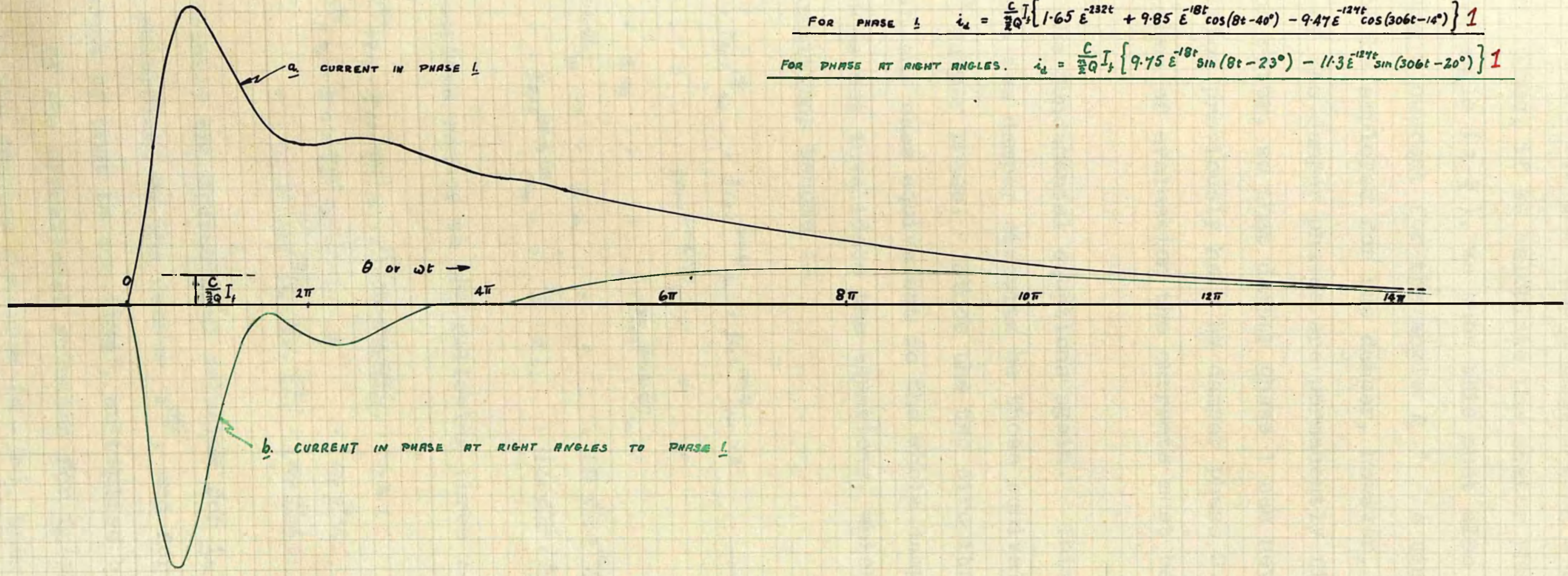
Fig 39.

Damper current.

Polyphase damper : $c^2 = 0.9$; $k^2 = 0.94$; $q^2 = 0.92$:

For phase 1 $\quad i_d = \frac{C}{2Q} I_f \left\{ 1.65\, \varepsilon^{-232t} + 9.85\, \varepsilon^{-18t} \cos(8t - 40^\circ) - 9.47\, \varepsilon^{-124t} \cos(306t - 14^\circ) \right\} 1$

For phase at right angles. $\quad i_d = \frac{C}{2Q} I_f \left\{ 9.75\, \varepsilon^{-18t} \sin(8t - 23^\circ) - 11.3\, \varepsilon^{-124t} \sin(306t - 20^\circ) \right\} 1$



is at angle $(\theta_0 - \frac{2\pi}{3})$; that is, when the stator phase 1 is in the position relative to the rotor which the stator phase 2 would possess, were the short-circuit to take place at θ_0. (We are for the moment considering a three-phase stator.) Thus, if we calculate the stator phase 1 current for the angle $(\theta_0 - \frac{2\pi}{3})$, we know this must also be the stator phase 2 current for the angle θ_0." A similar argument cannot be employed for the damper, however, for the positions of the damper phases are permanently fixed relative to the rotor, so that damper phase 1 can never take up a position previously held by damper phase 2. A different method of evaluation the currents must be used.

Consider the fundamental equations again. This time, however, let the two damper circuits be those equivalent to the $(r+1)^{th}$ damper phase. Since the two circuits together are in any case equivalent to the whole damper system, this does not invalidate the equations, which under these conditions become :-

$$\begin{array}{llllll}
Z_r i_r & + \frac{n}{2} Kp\, \varepsilon^{-j\frac{2\pi r}{n}} i_{da} & + \frac{n}{2} Kp\, \varepsilon^{j\frac{2\pi r}{n}} i_{db} & + \frac{3}{2} Cp\, \varepsilon^{-j(\omega t+\theta_0)} i_{sa} & + \frac{3}{2} Cp\, \varepsilon^{j(\omega t+\theta_0)} i_{sb} & = 0 \\
\frac{1}{2} Kp\, \varepsilon^{j\frac{2\pi r}{n}} i_r & + Z_d i_{da} & + 0 & + \frac{3}{2} Qp\, \varepsilon^{-j(\omega t+\theta_0-\frac{2\pi r}{n})} i_{sa} & + 0 & = 0 \\
\frac{1}{2} Kp\, \varepsilon^{-j\frac{2\pi r}{n}} i_r & + 0 & + Z_d i_{db} & + 0 & + \frac{3}{2} Qp\, \varepsilon^{j(\omega t+\theta_0-\frac{2\pi r}{n})} i_{sb} & = 0 \\
\frac{1}{2} Cp\, \varepsilon^{j(\omega t+\theta_0)} & + \frac{n}{2} Qp\, \varepsilon^{j(\omega t+\theta_0-\frac{2\pi r}{n})} i_{da} & + 0 & + Z_s i_{sa} & + 0 & = -\frac{1}{2} Cp \frac{j\omega}{p-j\omega} \varepsilon^{j\theta_0} I_f 1 \\
\frac{1}{2} Cp\, \varepsilon^{-j(\omega t+\theta_0)} & + 0 & + \frac{n}{2} Qp\, \varepsilon^{-j(\omega t+\theta_0-\frac{2\pi r}{n})} i_{db} & + 0 & + Z_s i_{sb} & = -\frac{1}{2} Cp \frac{-j\omega}{p+j\omega} \varepsilon^{-j\theta_0} I_f 1
\end{array} \qquad \text{--(191)}$$

These equations reduce to the following three :-

$$\begin{array}{llll}
\left[Z_r - \frac{3}{4} C^2 p\, S_{(p)}\right] i_r & + \varepsilon^{-j\frac{2\pi r}{n}} \left[\frac{n}{2} Kp - \frac{3n}{4} CQp \frac{p+j\omega}{Z_s(+)}\right] i_{da} & + \varepsilon^{j\frac{2\pi r}{n}} \left[\frac{n}{2} Kp - \frac{3n}{4} CQp \frac{p-j\omega}{Z_s(-)}\right] i_{db} & = \frac{3}{4} C^2 p\, F_{(p)} I_f 1 \\
\left[\frac{1}{2} Kp - \frac{3}{4} CQp \frac{p+j\omega}{Z_s(+)}\right] i_r & + \varepsilon^{-j\frac{2\pi r}{n}} \left[Z_d - \frac{3n}{4} Q^2 p \frac{p+j\omega}{Z_s(+)}\right] i_{da} & + 0 & = \frac{3}{4} CQp \frac{j\omega}{Z_s(+)} I_f 1 \\
\left[\frac{1}{2} Kp - \frac{3}{4} CQp \frac{p-j\omega}{Z_s(-)}\right] i_r & + 0 & + \varepsilon^{j\frac{2\pi r}{n}} \left[Z_d - \frac{3n}{4} Q^2 p \frac{p-j\omega}{Z_s(-)}\right] & = \frac{3}{4} CQp \frac{-j\omega}{Z_s(-)} I_f 1
\end{array} \qquad \text{--(192)}$$

Thus apparently the operational solution for $\underline{i_{da}}$ is as eqn.(184), multiplied by the factor $\varepsilon^{j\frac{2\pi r}{n}}$, and for $\underline{i_{db}}$ is the complement of that in eqn.(184), multiplied by the factor $\varepsilon^{-j\frac{2\pi r}{n}}$. If the operational solution for $\underline{i_{da}}$ (phase 1) be represented by the complex function $(a + jb)$, then apparently the operational solutions for $\underline{i_{da}}$ and $\underline{i_{db}}$ (phase r+1) are given by :-

$$i_{da} = (a+jb)\,\varepsilon^{j\frac{2\pi r}{n}}$$

$$i_{db} = (a-jb)\,\varepsilon^{-j\frac{2\pi r}{n}}$$

and the total damper current, phase (r + 1), by :-

$$i_{d(r+1)} = 2a\cos\frac{2\pi r}{n} - 2b\sin\frac{2\pi r}{n}. \qquad \text{--(193)}$$

Thus, the full operational solution for the current in damper phase (r + 1) is :-

$$i_{d(r+1)} = \frac{C}{\frac{n}{2}Q}I_f\left\{\frac{q^2\omega^2 p\left[(1-q^2)p+d\right]\left[(1-\frac{ck}{q})p+r\right]}{Z(p)}\cos\frac{2\pi r}{n}\right\}1$$

$$-\frac{C}{\frac{n}{2}Q}I_f\left\{\frac{q^2\omega p\left\{\left[1-c^2-k^2-q^2+2ckq\right]p^3+\left[r(1-q^2)+s(1-k^2)+d(1-c^2)\right]p^2+\left[rs+sd+dr\right]p+\left[rds\right]\right\}}{Z(p)}\sin\frac{2\pi r}{n}\right\}1 \qquad \text{--(194)}$$

ii. Numerical solution.

With the values for the constants as before, the operational solution for the current becomes :-

$$i_{d(r+1)} = \frac{C}{\frac{3}{2}Q}I_f\left\{\frac{980000\,[p+25][p+193]}{(p+232)\left[p+(18\pm j8)\right]\left[p+(124\pm j306)\right]}\cos\frac{2\pi r}{n} - \frac{3610\,[p+1{\cdot}24][p+121][p+232]}{(p+232)\left[p+(18\pm j8)\right]\left[p+(124\pm j306)\right]}\sin\frac{2\pi r}{n}\right\}1 \qquad \text{--(195)}$$

By the expansion theorem, this becomes :-

$$i_{d(r+1)} = \frac{C}{\frac{3}{2}Q}I_f\left\{1{\cdot}65\,\varepsilon^{-232t} + 9{\cdot}85\,\varepsilon^{-18t}\cos(8t-40^\circ) - 9{\cdot}47\,\varepsilon^{-124t}\cos(306t-14^\circ)\right\}\cos\frac{2\pi r}{n}\,1$$

$$+\frac{C}{\frac{3}{2}Q}I_f\left\{9{\cdot}75\,\varepsilon^{-18t}\sin(8t-23^\circ) - 11{\cdot}3\,\varepsilon^{-124t}\sin(306t-20^\circ)\right\}\sin\frac{2\pi r}{n}\,1 \qquad \text{--(196)}$$

For $r = 0$, i.e., phase 1, eqn.(196) is identical with eqn.(187), as it should be. The diagram, Fig 39b, shows the current in a phase such that $\frac{2\pi r}{n} = \frac{\pi}{2}$; i.e., a damper phase at right angles to damper phase 1.

iii. Approximate solution.

With all resistance neglected, eqn.(194) becomes :-

$$i_{d(r+1)} = \frac{C}{\frac{3}{2}Q}I_f\left\{\frac{q^2(1-\frac{ck}{q})}{(1-c^2-k^2-q^2+2ckq)}\cdot\frac{\omega^2}{p^2+\omega^2}\cdot\cos\frac{2\pi r}{n} - \frac{q^2}{(1-q^2)}\cdot\frac{p\omega}{p^2+\omega^2}\cdot\sin\frac{2\pi r}{n}\right\}1 \qquad \text{--(197)}$$

By the expansion theorem :-

$$i_{d(r+1)} = \frac{C}{\frac{3}{2}Q}I_f\left\{\left\{\frac{q^2(1-\frac{ck}{q})}{1-c^2-k^2-q^2+2ckq}\right\}\left\{1-\cos\omega t\right\}\cos\frac{2\pi r}{n} - \left\{\frac{q^2}{1-q^2}\right\}\sin\omega t\,\sin\frac{2\pi r}{n}\right\}1 \qquad \text{--(198)}$$

With the values of c, k and q as above, this becomes:-

$$i_{d(r+1)} = \frac{C}{\frac{3}{2}Q}I_f\left\{9{\cdot}95\,[1-\cos\omega t]\cos\frac{2\pi r}{n} - 11{\cdot}5\sin\omega t\,\sin\frac{2\pi r}{n}\right\}1 \qquad \text{--(199)}$$

which is a good approximation to eqn.(196).

iv. Maximum damper current.

The damper current is independant of θ_o, the angle at which short-circuit takes place. It is, however, a function of the two variables "t" and "r", and this must be born in mind when calculating its max. value, for it does not necessarily follow that the max. current occurs in phase 1. Thus, differentiating the current in eqn.(198), and leaving out the constant term, we may write :-

$$\frac{\partial(i_d)}{\partial(\omega t)} = \left\{\frac{q^2(1-\frac{ck}{q})}{1-c^2-k^2-q^2+2ckq}\right\} \sin\omega t \cdot \cos\frac{2\pi r}{n} - \left\{\frac{q^2}{1-q^2}\right\} \cos\omega t \;\sin\frac{2\pi r}{n}$$

$$= 0 \quad \text{if} \quad \tan\omega t = \frac{(1-c^2-k^2-q^2+2ckq)}{(1-q^2)(1-\frac{ck}{q})}\tan\frac{2\pi r}{n} = (\text{say})\ \frac{b}{a}\tan\frac{2\pi r}{n} \qquad \text{--(200)}$$

$$\frac{\partial(i_d)}{\partial(\frac{2\pi r}{n})} = -\left\{\frac{q^2(1-\frac{ck}{q})}{1-c^2-k^2-q^2+2ckq}\right\}\left[1-\cos\omega t\right]\sin\frac{2\pi r}{n} - \left\{\frac{q^2}{1-q^2}\right\} \sin\omega t \;\cos\frac{2\pi r}{n}$$

$$= 0 \quad \text{if} \quad \tan\frac{2\pi r}{n} = -\frac{b}{a}\frac{\sin\omega t}{[1-\cos\omega t]} \qquad \text{--(201)}$$

Thus, for max. damper current :-

$$\frac{\sin\omega t}{\cos\omega t} = -\frac{b^2}{a^2}\frac{\sin\omega t}{[1-\cos\omega t]} \qquad \text{--(202)}$$

A solution of eqn.(202) is $\sin\omega t = 0$, provided that $\cos\omega t \neq 1$. That is, $\omega t = \pi$.

The solutions of the trigonometric functions of $\frac{2\pi r}{n}$ corresponding to this root are :-

$\sin\frac{2\pi r}{n} = 0$ and $\cos\frac{2\pi r}{n} = \pm 1$. (i.e., $\frac{2\pi r}{n} = 0$ or π)

Substituting in eqn.(198) :-

$$i_{d(max)(max)} = \frac{C}{\frac{3}{2}Q} I_f \left\{\frac{2q^2(1-\frac{ck}{q})}{1-c^2-k^2-q^2+2ckq}\right\} \qquad \text{--(203)}$$

There is another possible solution for eqn.(202) if $\sin\omega t \neq 0$; i.e., -

$$\cos\omega t = -\frac{1}{\frac{b^2}{a^2}-1} = -\frac{a^2}{b^2-a^2}$$

If this is truly a solution - i.e., if $a^2 < (b^2-a^2)$, or $b^2 > 2a^2$, - then the other trigonometric functions corresponding to it are :-

$\sin\omega t = \frac{b}{b^2-a^2}\sqrt{b^2-2a^2}$; $\sin\frac{2\pi r}{n} = \pm\sqrt{\frac{b^2-2a^2}{b^2-a^2}}$; $\cos\frac{2\pi r}{n} = \mp\frac{a}{\sqrt{b^2-a^2}}$;

Substituting these values in eqn.(198) :-

$$i_{d(max)(max)} = \pm\frac{C}{\frac{3}{2}Q} I_f \left\{\frac{q^2(1-\frac{ck}{q})}{1-c^2-k^2-q^2+2ckq}\right\} \frac{1}{a}\frac{b^2}{\sqrt{b^2-a^2}}$$

Consider the ratio b/a $= \frac{(1 - c^2 - k^2 - q^2 + 2ckq)}{(1-q^2)(1-\frac{ck}{q})}$.

All three variables c, k and q may vary subject only to the conditions :-

$$1 \nless (k \text{ or } q) \nless c \nless kq$$

It is impossible to form any idea of the manner in which b/a varies by analytical methods alone, for the ratio is far too involved. It is possible to show the variations by graphs, however, and this is done in Fig 40. A word of explanation is required in connection with these graphs.

Each group of curves shows the variation of **b/a** for a specific value of c, these specific values being 0·6, 0·7, 0·8, 0·9, 0·95, and 0·98. Each curve of each group corresponds to a particular value of k, these values being as far as possible the same as above, - viz., 0·6, 0·7, 0·8, 0·9, 0·95, and 0·98,- though of course in the groups of curves corresponding to the higher values of c, the possible range of k, and therefore the number of curves, becomes smaller, since $1 \nless k \nless c$. (In the last two groups, extra values of k are taken.) The abscissae of the curves are in each case graduated in values of q, and the ordinates of course in values of b/a. The curves of each group stop short on the left-hand side at the point where $q = c$, since q may not be less than c; on the right, the limit of q for any particular k-curve is that value which gives $kq = c$, since kq may not be greater than c. When $kq = c$, the ratio b/a becomes :-

$$\frac{b}{a} = \frac{1 - k^2q^2 - k^2 - q^2 + 2k^2q^2}{(1-q^2)(1-k^2)} = \frac{1 + k^2q^2 - k^2 - q^2}{(1-q^2)\ (1-k^2)} = 1$$

so that the curves always stop on the right at the value $[b/a] = 1$.

It is seen that the ratio b/a is always greater than unity, but never seems to rise higher than 2, only reaching this value in exceptional cases. In general, increase of k always results in increase of the maximum

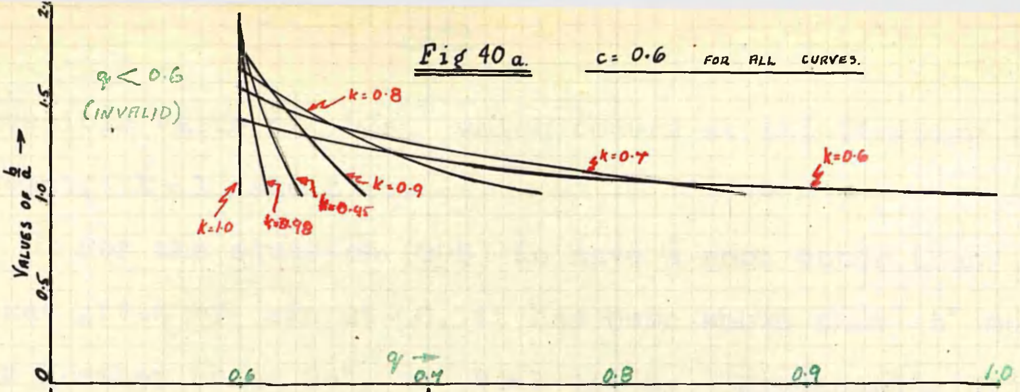


Fig 40a. c = 0.6 FOR ALL CURVES.

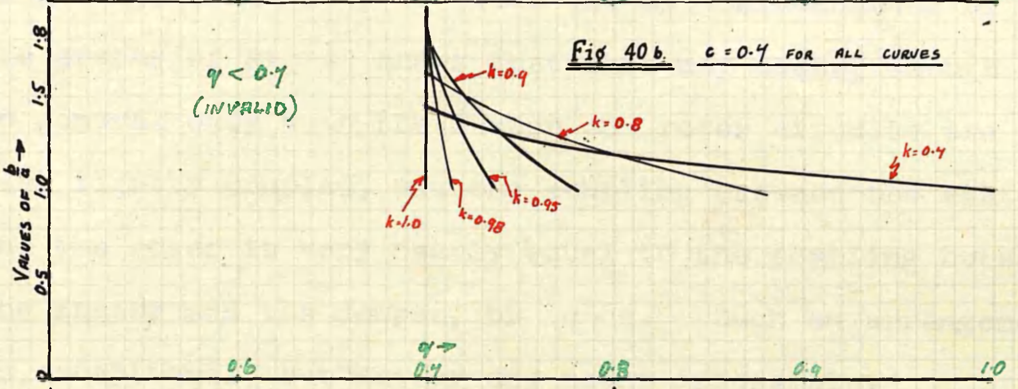


Fig 40b. c = 0.7 FOR ALL CURVES

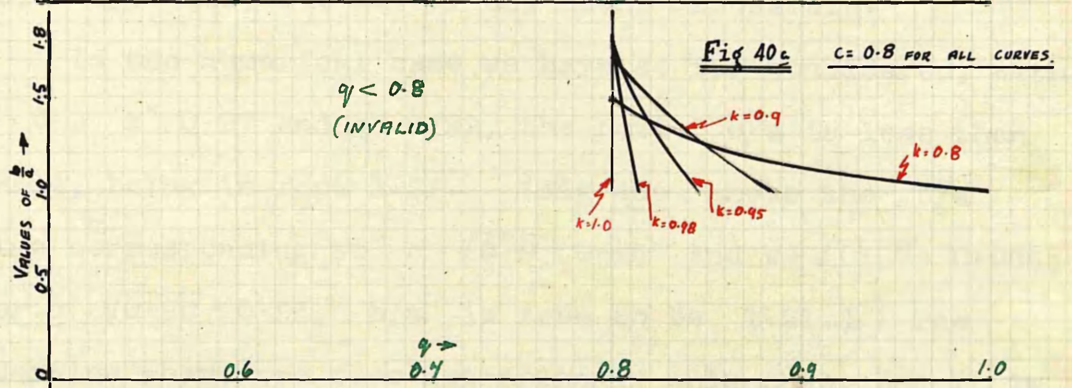


Fig 40c. C = 0.8 FOR ALL CURVES.

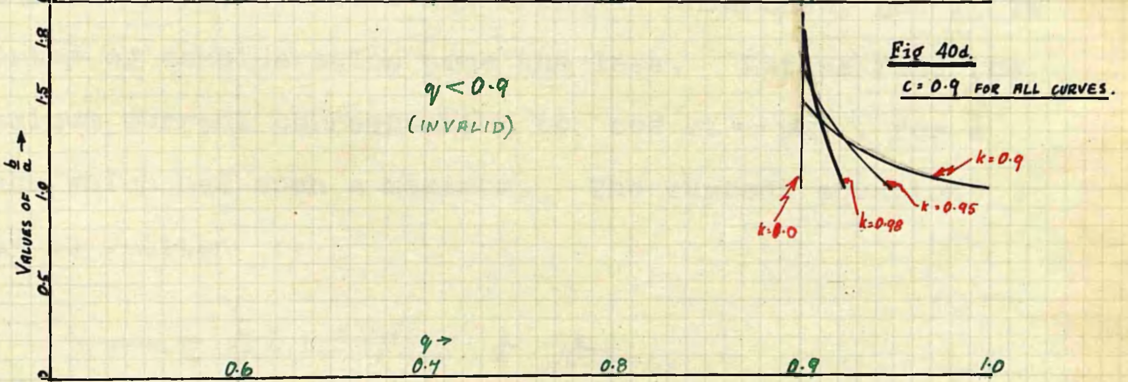


Fig 40d. C = 0.9 FOR ALL CURVES.

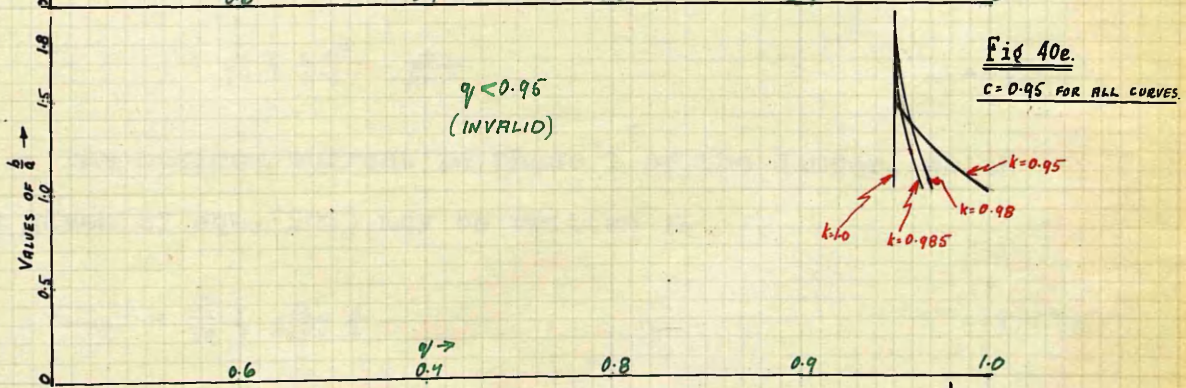


Fig 40e. C = 0.95 FOR ALL CURVES.

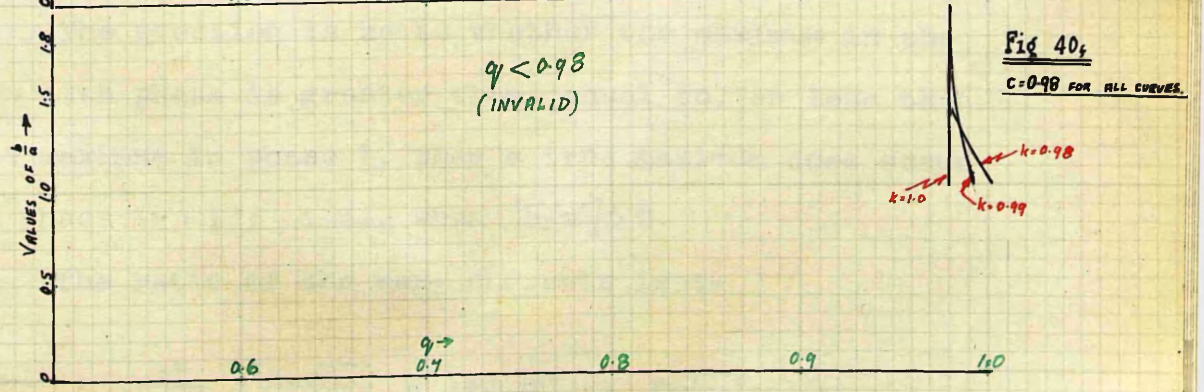


Fig 40f. C = 0.98 FOR ALL CURVES.

possible value of b/a, which occurs at the limiting point, $k = 1$ and $c = q$.

For the equation (202) to have a root other than that given by $\sin \omega t = 0$, it has been shown that b^2 must be greater than $2a^2$, or $b > 1{\cdot}414\ a$. Examination of the graphs of Fig 40 shows that this may occur, though in general only when the damper and rotor circuits are very tightly coupled, and the coupling between the stator and the rotor is very nearly equal to the coupling between the stator and the damper, or $c = q$. Such an arrangement is exceptional, but not by any means impossible.

In the numerical case we have so far considered, with $c^2 = 0{\cdot}9$, $k^2 = 0{\cdot}97$ and $q^2 = 0{\cdot}92$, the ratio b/a is less than $1{\cdot}414$, being in fact $1{\cdot}21$. (Fig 40e shows the b/a line corresponding to $c = \sqrt{0{\cdot}9} = 0{\cdot}95$ and $k = \sqrt{0{\cdot}97} = 0{\cdot}985$; for $q = \sqrt{0{\cdot}92} = 0{\cdot}96$, b/a is seen to be $1{\cdot}21$.) The question therefore does not arise in this case, but it is worthy of consideration none the less. Let us find the maximum current corresponding to $\cos \omega t = -\frac{a^2}{b^2 - a^2}$, for a case which has such a maximum. The current equation may be written :-

$$i_{d(r+1)(max)} = \frac{C}{\frac{\pi}{2}Q} I_f \frac{q^2\left(1 - \frac{ck}{q}\right)}{\left(1 - c^2 - k^2 - q^2 + 2ckq\right)} \cdot \frac{b}{a} \cdot \frac{b}{\sqrt{b^2 - a^2}}$$

$$= \frac{C}{\frac{\pi}{2}Q} I_f \frac{q^2}{(1 - q^2)} \cdot \frac{b}{\sqrt{b^2 - a^2}} \qquad \text{--(204)}$$

The maximum current in phase 1 of the damper, which is given by eqn.(203), may be written :-

$$i_{d1(max)} = \frac{C}{\frac{\pi}{2}Q} I_f \frac{2q^2}{(1 - q^2)} \frac{a}{b} \qquad \text{--(203a)}$$

The question is as to whether the maximum in the $(r + 1)$th phase is greater than, equal to, or less than the maximum in phase 1, when a true maximum does occur in phase $(r + 1)$; i.e., when $\left[b^2/a^2\right] > 2$.

The ratio of the max. currents is :-

$$\frac{i_{d1(max)}}{i_{d(r+1)(max)}} = \frac{2\frac{a}{b}}{\frac{b}{\sqrt{b^2 - a^2}}} = \frac{2a\sqrt{b^2 - a^2}}{b^2} = 2\frac{a}{b}\sqrt{1 - \left(\frac{a}{b}\right)^2} = 2x\sqrt{1 - x^2}$$

To find the range of the function $2x\sqrt{1-x^2}$, first differentiate with respect to $\underline{x}$:-

$$d(F_{(x)})/dx = 2\left\{x\tfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) + (1-x^2)^{\frac{1}{2}}\right\} = 2(1-2x^2)/(1-x^2)^{\frac{1}{2}}$$

As $x^2 (= \frac{a^2}{b^2}) < \frac{1}{2}$, the derivative of $F_{(x)}$ is always positive for our working range. Further, at $x^2 = \frac{1}{2}$, or $b^2 = 2a^2$, $F_{(x)}$ is $2\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}}$, which is unity. Hence, for values of $\underline{x}^2$ less than $\frac{1}{2}$, $F_{(x)}$ is always less than unity. In fact, as b^2/a^2 goes from 2 to 4, - our working range, - $F_{(x)}$ goes from 1 to 0·866.

Thus, if b/a is such that a true current maximum occurs in a phase other than phase 1, then this maximum is always greater than the phase 1 maximum, though at most not more than 1/0·866 (= 1·15) times as great. In any case, it is unlikely that the "External" maximum exists, for b/a is generally less than 1·414 for a practical alternator, as may be seen from the graphs of Fig 40. With any specific case, it may be seen from this figure whether the true current maximum occurs in phase 1 or in some other phase, and the damper may be designed accordingly.

Let us consider the usual case, where $[b/a] < 1{\cdot}414$, so that the true current maximum occurs in phase 1. The current is then given by eqn.(203), which should be compared with eqn.(168), this latter equation giving the max. damper current for a single-phase damper machine. That part of the expression dealing with the proportionate linking between the various circuits is the same in each case, but the first term is slightly different, being $\frac{C}{Q} I_f$ for single-phase damping, and $\frac{C}{2Q} I_f$ for polyphase damping. Thus, if a single-phase damper is designed for a current I, then if the damper is made polyphase, and of $\underline{n}$ phases, each coil need only be designed for a current $\frac{2}{n}$I, or each coil's resistance may be made $\frac{1}{2}$n times that of the single coil. This was assumed when the numerical case was considered, for $\underline{d}$ ($= R/L_d$) was kept at the same value as for the single-phase case, viz., 2, although L_d was of course increased in the ratio $\frac{1}{2}n$.

5. The Stator Current.

The full operational equation for the stator current $\underline{i}_{sa}$ is as follows :-

$$i_{sa} = -\frac{1}{2} I_k \varepsilon^{j\theta_0} \left\{ \frac{pj\omega}{(p+s)(p-j\omega)} \right\} 1 - \frac{1}{2} I_k \varepsilon^{j\theta_0} \left\{ \frac{p^2\omega^2 [(p-j\omega)(1-q^2)+d][(p-j\omega)(c^2+q^2-2ckq)+q^2r+c^2d]}{(p+s)\, Z_{(p-j\omega)}} \right\} 1$$

$$-\frac{1}{2} I_k \varepsilon^{j\theta_0} \left\{ j \frac{p^2 q^2 \omega \{ [1-c^2-k^2-q^2+2ckq](p-j\omega)^3 + [(1-q^2)r + (1-k^2)s + (1-c^2)d](p-j\omega)^2 + [rd+ds+sr](p-j\omega) + [rds] \}}{(p+s)\, Z_{(p-j\omega)}} \right\} 1 \quad \text{--(205)}$$

this equation being obtained in the usual way by substituting the operational values of $\underline{i}_r$ and $\underline{i}_{da}$ in the requisite equation (170). $Z_{(p-j\omega)}$ is again the matrix determinant of the equations (171), without the term $\frac{L_s^2 L_d^2 L_r}{Z_s(p) Z_r(p)}$, and with $(p - j\omega)$ substituted for $\underline{p}$ throughout; $\underline{i}_{s1}$ is of course twice the real part of $\underline{i}_{sa}$.

a. Numerical solution.

With the usual values for the circuit constants, eqn. (205) becomes :-

$$i_{sa} = -\frac{1}{2} I_k \varepsilon^{j\theta_0} \left\{ \frac{pj\omega}{(p+10)(p-j\omega)} \right\} 1 - \frac{1}{2} I_k \varepsilon^{j\theta_0} \left\{ \frac{1{,}135{,}000\, p^2 [(p-j\omega)+25][(p-j\omega)+232]}{(p+10)[(p-j\omega)+(18\pm j8)][(p-j\omega)+232][(p-j\omega)+(124\pm j306)]} \right\} 1$$

$$-\frac{1}{2} I_k \varepsilon^{j\theta_0} \left\{ j \cdot \frac{3{,}610\, [(p-j\omega)+1{\cdot}24][(p-j\omega)+141][(p-j\omega)+232]}{(p+10)[(p-j\omega)+(18\pm j8)][(p-j\omega)+232][(p-j\omega)+(124\pm j306)]} \right\} 1 \quad \text{--(206)}$$

which equation can apparently be split up into three parts, each part being investigated separately.

i. First part.

Solving by the expansion theorem, and doubling real parts in order to obtain $\underline{i}_{s1}$:-

$$i_{s1} = -I_k \left\{ \cos(\omega t + \theta_0) - \varepsilon^{-10t} \cos\theta_0 \right\} 1 \quad \text{--(207)}$$

ii. Second part.

Solving by the expansion theorem, and doubling reals:-

$$i_{s1} = -I_k \left\{ 0{\cdot}491\, \varepsilon^{-10t} \cos(\theta_0 - 7^\circ) + 4{\cdot}12\, \varepsilon^{-18t} [\cos(322t + \theta_0 - 40^\circ) + \cos(306t + \theta_0 + 44^\circ)] - 5{\cdot}65\, \varepsilon^{-111t} \cos(620t + \theta_0 - 20^\circ) - 6{\cdot}11\, \varepsilon^{-117t} \cos(8t + \theta_0 - 21^\circ) \right\} 1 \quad \text{--(208)}$$

iii. Third part.

Solving by the expansion theorem, and doubling reals:-

$$i_{s1} = -I_k \left\{ 0{\cdot}598\, \varepsilon^{-10t} \cos(\theta_0 + 11^\circ) - 4{\cdot}84\, \varepsilon^{-18t} [\cos(322t + \theta_0 - 21^\circ) - \cos(306t + \theta_0 + 25^\circ)] + 5{\cdot}63\, \varepsilon^{-111t} \cos(620t + \theta_0 - 20^\circ) - 6{\cdot}1\, \varepsilon^{-117t} \cos(8t + \theta_0 + 21^\circ) \right\} 1 \quad \text{--(209)}$$

Adding the various currents of eqns.(207), (208) and (209) together (graphically where necessary) the following equation emerges for the total stator current:-

$$i_{s_1} = -I_k \left\{ \cos(\omega t + \theta_0) + 3\,\varepsilon^{-18t} \cos(322t + \theta_0 - 72^\circ) + 11{\cdot}87\,\varepsilon^{-18t} \cos(306t + \theta_0 + 38^\circ) - 12{\cdot}21\,\varepsilon^{-124t} \cos(8t + \theta_0 + 21^\circ) \right\} 1 \qquad \text{--(210)}$$

This curve is shown (for $\theta_0 = 0$) in the diagram, Fig 41. It should be compared with the corresponding curves for a single-phase damper (Fig 34a) and for a machine without a damper (Fig 22b).

<u>b.</u> Approximate solution.

With all resistance neglected, eqn.(205) becomes :-

$$i_{sa} = -\tfrac{1}{2} I_k\, \varepsilon^{j\theta_0} \left\{ \frac{j\omega}{p - j\omega} + \frac{(c^2 + k^2 - 2ckq)}{(1 - c^2 - k^2 - q^2 + 2ckq)} \cdot \frac{\omega^2}{(p - 2j\omega)(p - j\omega)} + \frac{q^2}{1 - q^2} \cdot \frac{j\omega}{p - 2j\omega} \right\} 1 \qquad \text{--(211)}$$

This equation may also be split up into three parts.

<u>i.</u> First part.

By expansion theorem, and doubling real parts :-

$$i_{s_1} = -I_k \left\{ \cos(\omega t + \theta_0) - \cos\theta_0 \right\} 1 \qquad \text{--(212)}$$

<u>ii.</u> Second part.

By expansion theorem, and doubling real parts :-

$$i_{s_1} = -I_k \left\{ \frac{c^2 + k^2 - 2ckq}{1 - c^2 - k^2 - q^2 + 2ckq} \right\} \left\{ -\tfrac{1}{2}\cos\theta_0 - \tfrac{1}{2}\cos(2\omega t + \theta_0) + \cos(\omega t + \theta_0) \right\} 1 \qquad \text{--(213)}$$

<u>iii.</u> Third part.

By expansion theorem, and doubling real parts :-

$$i_{s_1} = -I_k \left\{ \frac{q^2}{1 - q^2} \right\} \left\{ -\tfrac{1}{2}\cos\theta_0 + \tfrac{1}{2}\cos(2\omega t + \theta_0) \right\} 1 \qquad \text{--(214)}$$

The total stator current is the sum of those given by eqns.(212), (213) and (214), and is thus :-

$$i_{s_1} = -I_k \left\{ -\tfrac{1}{2}\left[\frac{1 - k^2}{1 - c^2 - k^2 - q^2 + 2ckq} + \frac{1}{1 - q^2} \right] \cos\theta_0 + \left[\frac{1 - k^2}{1 - c^2 - k^2 - q^2 + 2ckq} \right] \cos(\omega t + \theta_0) - \tfrac{1}{2}\left[\frac{1 - k^2}{1 - c^2 - k^2 - q^2 + 2ckq} - \frac{1}{1 - q^2} \right] \cos(2\omega t + \theta_0) \right\} 1 \qquad \text{--(215)}$$

With the values of <u>c</u>, <u>k</u> and <u>q</u> as above, this is :-

$$\begin{aligned} i_{s_1} &= -I_k \left\{ -\tfrac{1}{2}[12{\cdot}5 + 12{\cdot}5]\cos\theta_0 + [12{\cdot}5]\cos(\omega t + \theta_0) + \tfrac{1}{2}[12{\cdot}5 - 12{\cdot}5]\cos(2\omega t + \theta_0) \right\} 1 \\ &= +I_k \,.\, 12{\cdot}5 \left\{ \cos\theta_0 - \cos(\omega t + \theta_0) \right\} 1 \qquad \text{--(216)} \end{aligned}$$

which is a good approximation to eqn.(210).

It is interesting to see that the disappearance of the double frequency component of the stator current in

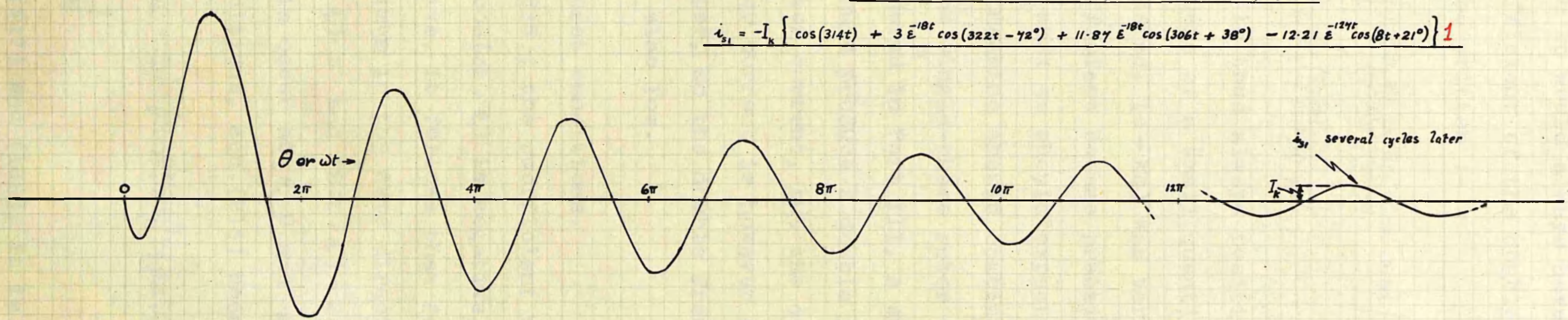
Fig 41.

STATOR CURRENT.

NORMAL STEADY CURRENT = I_k

POLYPHASE DAMPER: $c^2 = 0{\cdot}9$; $k^2 = 0{\cdot}97$; $q^2 = 0{\cdot}92$; $\theta_0 = 0$.

$$i_{s1} = -I_k \left\{ \cos(314t) + 3\,\varepsilon^{-18t}\cos(322t - 72^\circ) + 11{\cdot}87\,\varepsilon^{-18t}\cos(306t + 38^\circ) - 12{\cdot}21\,\varepsilon^{-124t}\cos(8t + 21^\circ) \right\} \quad 1$$



the numerical example is merely due to coincidence. Had other values been taken for the constants, it would not have vanished, though it would always have been small. For consider the coefficient of the double-frequency component. It may be written :-

$$\text{Coefficient} = \frac{1}{2}\left\{\frac{(1-q^2)(1-k^2) - (1-c^2-k^2-q^2+2ckq)}{(1-c^2-k^2-q^2+2ckq)(1-q^2)}\right\}$$

$$= \frac{1}{2}\left\{\frac{(c-kq)^2}{(1-c^2-k^2-q^2+2ckq)(1-q^2)}\right\}$$

The coefficient is thus always positive, and has for numerator the square of a small quantity, viz., $(c-kq)$. In the example considered, $(c-kq)$ was very small, (0·008) so naturally the coefficient became practically zero.

This is exactly what we should expect, for with a polyphase damper the machine should combine the characteristics of the single-phase rotor and polyphase rotor machines considered in Part III, A and C. Thus, the rotor circuit should produce a double frequency component in the stator current, but the damper circuit should not; the rotor current is however always low in a machine with a damper, so the double frequency component of stator current is also low.

c. Maxima and worst conditions.

Since, in eqn.(215), the coefficient of $\cos\theta_0$ is always positive, of $\cos(\omega t+\theta_0)$ is negative, and of $\cos(2\omega t+\theta_0)$ is positive, it follows that for all three terms to reach a maximum in the same direction together, θ_0 must be zero, $(\omega t+\theta_0)=\pi$ and $(2\omega t+\theta_0)=2\pi$. That is, worst conditions again occur at $\theta_0=0$ and $\omega t=\pi$.

Under these conditions, eqn.(215) becomes :-

$$i_{s_1(max)(max)} = -I_k\left\{-\frac{1}{2}\left[\frac{1-k^2}{1-c^2-k^2-q^2+2ckq}+\frac{1}{1-q^2}\right]-\left[\frac{1-k^2}{1-c^2-k^2-q^2+2ckq}\right]-\frac{1}{2}\left[\frac{1-k^2}{1-c^2-k^2-q^2+2ckq}-\frac{1}{1-q^2}\right]\right\}$$

$$= I_k\left\{\frac{2(1-k^2)}{1-c^2-k^2-q^2+2ckq}\right\} \qquad \text{--(217)}$$

Comparing eqns.(217) and (169), it is seen that the maximum stator current current is the same, whether the

the damper is polyphase or single-phase, though the shapes of the current curves are different in the two cases. Even the shapes of the curves are not very different when resistance is taken into account, as may be seen by comparing Figs (35a) and (41), though when resistance is included, the stator current for the polyphase case does not rise as high as that for the single-phase case; this is again due to the higher rate of damping in the polyphase machine, caused by the higher phase resistance implicit in the assumption that the constant $\underline{d}$ was the same in each machine, as has been explained.

The stator maximum current with a polyphase damper is always greater than that with no damper, the ratio of the maxima being as for the single-phase damper machine.

It is worthy of note that except for the somewhat different rates of damping, and the constant $\frac{1}{2}n$ in the damper current term, there is no difference between the maximum currents with single-phase and with polyphase damping, in any of the three circuits, stator, rotor or damper.

F. On The Symmetrical Cage Winding.

So far, the damping winding has been regarded as consisting of $\underline{n}$ phases, linked only by inductance, and not by resistance. The object of this concluding section of the Thesis is to show that the cage winding, in which the ends of all the coils are made into one conductor, the end-ring, may with suitable modification of the circuit constants be regarded as equivalent to a symmetrical winding consisting of $\underline{n}$ such separate phases. There are really two questions; first, as to whether the two arrangements are truly equivalent, and second, as to what we are to understand by the "Resistance of a phase" in the cage winding, if the answer to the first question is in the affirmative.

The term "Symmetrical" applied to a cage winding will be understood to denote a cage having all bars similar and equally spaced, uniform homogeneous end-rings, and a whole number of bars per pole.

1. On the Identity of a Cage Winding and an n-phase Symmetrical Winding.

The question arises from the fact that the various phases of the cage winding are connected by common resistances at more than one point; i.e., the phases have a common resistance at each end of the machine. For consider two coils having a single common resistance,

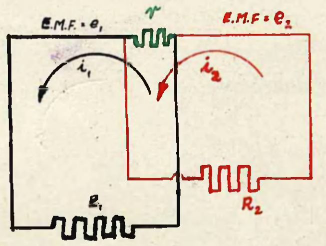


Fig 42.

as shown in the diagram, Fig 42. (The common resistance is shown green). Considering the currents as direct, for simplicity, the equations for the two circuits become :-

$$\begin{aligned} [R_1 + r] i_1 \quad + \quad r i_2 &= e_1 \\ r i_1 \quad + [R_2 + r] i_2 &= e_2 \end{aligned} \qquad \text{--(218)}$$

These two equations are valid, since the current i_1 must exist at all points of circuit 1 - shown black - and the current i_2 at all points of circuit 2 - shown red; by elimination of one of the currents, the arrangement may be reduced to that of a single equivalent coil.

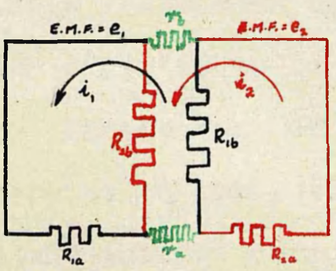


Fig 43.

If the coils have two separate resistances in common, as shown in Fig 43, then the equations would apparently be, following eqns.(218) :-

$$\begin{aligned} [R_{1a} + R_{1b} + r_a + r_b] i_1 \quad + \quad [r_a + r_b] i_2 &= e_1 \\ [r_a + r_b] i_1 \quad + [R_{2a} + R_{2b} + r_a + r_b] i_2 &= e_2 \end{aligned} \qquad \text{--(219)}$$

These equations are however not necessarily valid, for we have no means of knowing that the current is truly i_1 at all points of the circuit which we have called circuit 1 - shown black - and similarly for circuit 2.

Indeed, with an asymmetrical arrangement such as that shown in the figure, the same current almost certainly does not exist at all points in the "Circuits" taken. Thus the arrangement is not primarily equivalent to two circuits coupled by a single resistance. If, however, we could prove that the current in the branch R_{1b} was i_1 (upwards), and that that in the branch R_{1b} was i_2 (downwards), then it would follow that i_1 existed at all points of circuit 1, and i_2 at all points of circuit 2, and the equations would be valid, so that the arrangement would truly be equivalent to two separate circuits having the resistance $(r_a + r_b)$ in common. If the number of circuits be increased to make a "Ladder" arrangement as shown in Fig 44, the same criterion holds, that if the currents in the return paths can be proved to be equal to the currents in the initial paths, then although the various coils have two resistances in common, they may nevertheless be regarded as separate coils with simple resistance coupling. In Fig 44, the number of such circuits would be four.

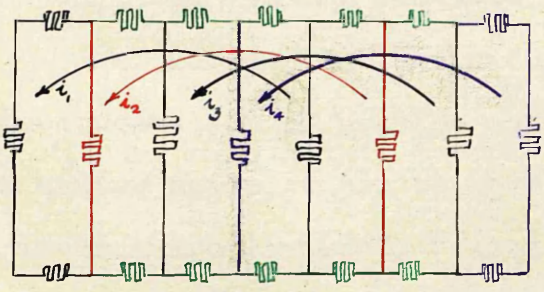


Fig 44.

In the case of a cage winding, we do know that the currents in the return bars are the same as the currents in the initial bars, for the following reason. Since the winding is symmetrical, - i.e., with a whole number of bars per pole, - the bar having the return current for each phase is diametrically opposite to the initial bar, and thus is governed at all times by precisely the same arithmetic conditions of flux and velocity as the initial bar, though the flux is reversed in direction; thus the return bar must contain the same current as the initial bar, reversed in direction.

The case is complicated by the fact that there are two possible paths for the current from each bar to its corresponding return bar; that is, it may go either way round the end ring. The principles are however precisely the same.

2. The Equivalent Resistance of the Cage Winding.

Having shown that the cage may be regarded as a polyphase system of coils, with resistive as well as inductive coupling, we have now to consider the physical interpretation of the term R_d in the expression $Z_{d(p)} = R_d + pL_d$. When the phases were regarded as linked only by inductance, the term R_d signified the ohmic resistance of any one phase; its meaning must now be extended.

Let us consider again the equations which gave the interpretation of L_s, the equivalent self-inductance of a stator phase, these being eqns.(8), Part III. If the circuits are linked by resistance as well as inductance, then each of the terms pM_1, pM_2, & c. must be extended to cover these mutual resistances. They become in fact, $(R_1 + pM_1)$, $(R_2 + pM_2)$, & c., where R_r is the mutual resistance between two phases displaced r phases from each other. Thus the equations become :-

$$\left.\begin{aligned}
(R_s' + pL_s')i_{s1} + (R_1 + pM_1)i_{s2} + \cdots + (R_{n-1} + pM_{n-1})i_{sn} &= -\tfrac{1}{2}Cp\left\{\varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)}\right\} F_{(p)}1 \\
(R_{n-1} + pM_{n-1})i_{s1} + (R_s' + pL_s')i_{s2} + \cdots + (R_{n-2} + pM_{n-2})i_{sn} &= -\tfrac{1}{2}Cp\left\{\varepsilon^{j(\omega t+\theta_0-\frac{2\pi}{n})} + \varepsilon^{-j(\omega t+\theta_0-\frac{2\pi}{n})}\right\} F_{(p)}1 \\
\text{-----} \quad \text{-----} \quad \text{-----} \quad & \text{-----} \quad \text{-----} \quad \text{-----} \\
(R_1 + pM_1)i_{s1} + (R_2 + pM_2)i_{s2} + \cdots + (R_s' + pL_s')i_{sn} &= -\tfrac{1}{2}Cp\left\{\varepsilon^{j\left[\omega t+\theta_0-\frac{2(n-1)\pi}{n}\right]} + \varepsilon^{-j\left[\omega t+\theta_0-\frac{2(n-1)\pi}{n}\right]}\right\} F_{(p)}1
\end{aligned}\right\} \quad -(220)$$

where $M_r = M_{n-r}$, $R_r = R_{n-r}$, and R_s' = self-resistance of a phase.

Eliminating all currents except i_{s1}, we obtain an equation similar to eqn.(15), but with the following expressions in the denominators :-

$$(R_s' + pL_s') + (R_1 + pM_1)\left[\cos\tfrac{2\pi}{n} \pm j\sin\tfrac{2\pi}{n}\right] + (R_2 + pM_2)\left[\cos\tfrac{4\pi}{n} \pm j\sin\tfrac{4\pi}{n}\right] + \cdots + (R_{n-1} + pM_{n-1})\left[\cos\tfrac{2(n-1)\pi}{n} \pm j\sin\tfrac{2(n-1)\pi}{n}\right]$$

Since $\sin\left(\frac{2r\pi}{n}\right) = -\sin\left(\frac{2(n-r)\pi}{n}\right)$, all the imaginaries vanish, and this expression becomes :-

$$\left[R_s' + R_1\cos\tfrac{2\pi}{n} + R_2\cos\tfrac{4\pi}{n} + \cdots + R_{n-1}\cos\tfrac{2(n-1)\pi}{n}\right] + p\left[L_s' + M_1\cos\tfrac{2\pi}{n} + M_2\cos\tfrac{4\pi}{n} + \cdots + M_{n-1}\cos\tfrac{2(n-1)\pi}{n}\right]$$

Thus the term R_s, the equivalent self-resistance of a stator phase when the phases have resistive as well as inductive coupling, is to be understood to have the physical interpretation :-

$$R_s = R_s' + R_1 \cos\frac{2\pi}{n} + R_2 \cos\frac{4\pi}{n} + ---- + R_{n-1} \cos\frac{2(n-1)\pi}{n} \qquad --(221)$$

This interpretation is naturally the same when the damper winding is under consideration.

Consider now a symmetrical cage winding, for simplicity in a two-pole machine, there being n bars per pole. We must first consider how many phases this represents in the equivalent resistance-coupled winding. The tendency is to say immediately that 2n bars are equivalent to n phases, the angles between the successive phases being then $\frac{2\pi}{n}$ radians. If n is odd, this is valid, the arrangement of initial and return bars being then as shown in Fig 45, which gives a "Splayed-out" view of one damper ring and its bars; the return bars of the various phases are shown red. If n is even, however, this mode of arrangement breaks down, for as shown in Fig 46, (where n = 6), under these conditions half the damper bars are not used at all. We will therefore regard the n bars per pole as being equivalent to 2n resistance-coupled phases, in which case every bar fulfils two functions; that of initial bar for a phase, and that of return bar for the phase diametrically opposite. The angle between successive phases is then $\frac{\pi}{n}$ radians, the development is equally valid for n odd or even, and the arrangement becomes as in Fig 47. The red

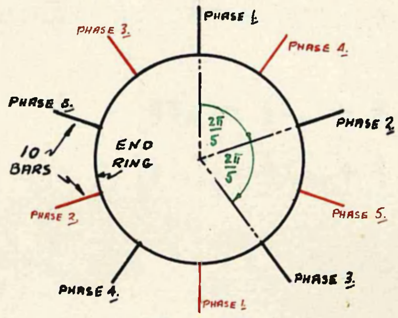


Fig 45.

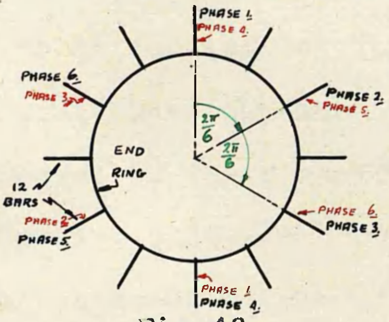


Fig 46.

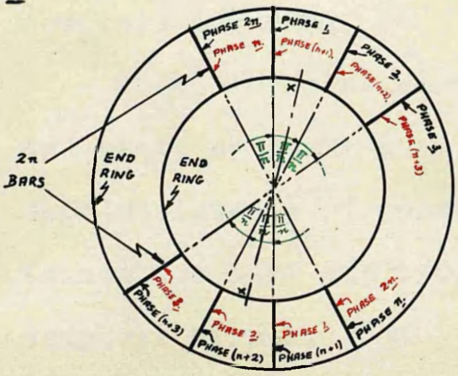


Fig 47.

lettering in Figs (46) and (47) denotes the phase for which the bar concerned is the return path.

We may make a further simplification before proceeding with the evaluation of R_d. The phase 1 current, $\underline{i}_{d_1}$, enters the end ring at $\underline{a}$, and splits into two halves, taking the two possible routes round the ring to its return bar at $\underline{b}$; similarly for the phase 2 current, $\underline{i}_{d_2}$, and so on. Thus, the current in the portion of end ring $\underline{ac}$ is given by :-

$$\begin{aligned} i_{ac} = & \left[\tfrac{1}{2}i_{d_1} - \tfrac{1}{2}i_{d_2} - \tfrac{1}{2}i_{d_3} - - - - - \tfrac{1}{2}i_{d(n+1)}\right] \\ & + \left[\tfrac{1}{2}i_{d(n+2)} + \tfrac{1}{2}i_{d(n+3)} + \tfrac{1}{2}i_{d(n+4)} + - - - + \tfrac{1}{2}i_{d(2n)}\right] \end{aligned} \qquad \text{--(222)}$$

But $\sum_{r=1}^{r=2n} i_{d_r} = 0$, by Kirchhoff's first law. Hence:-

$$i_{ac} = i_{d_1} + i_{d(n+2)} + i_{d(n+3)} + - - - - + i_{d(2n)} \qquad \text{--(223)}$$

That is, the current in $\underline{ac}$ is the sum of all the currents peculiar to the phases the initial bars of which are on the left of the line XX. This is exactly what we should have obtained, had we imagined all the currents to travel to their various return bars by one route only, namely, clockwise round the end ring; since this latter arrangement is simpler to work with, it will be used throughout in the subsequent evaluation of R_d.

Let the resistance of a single bar be $\underline{r}$, and of a complete end ring, R.

Then, for clockwise travel of current round the end ring, as assumed :-

Self-resistance of phase 1 $= 2r + 2\cdot\tfrac{1}{2}R = 2r + R$

(since two bars and two semi end rings are involved in the complete circuit.)

Mutual resis., phase 1 to phase 2 = 2(resis. between c & b)

$$= 2\cdot\tfrac{1}{2}R\left[\frac{\pi - \frac{\pi}{n}}{\pi}\right]$$

$$= R(1 - 1/n)$$

Mutual resis., phase 1 to phase 3 $= R(1 - 2/n)$

and so on, until we reach the $\underline{n}$th phase, for which the mutual resistance is $R\left\{1 - (n-1)/n\right\}$.

For the $(n+1)$th phase, there is no mutual resistance due to the end ring, for the current of this phase travels from c to a by the left-hand route, and the phase 1 current of course from a to c by the right-hand route. There is, however, mutual resistance due to the fact that both currents occupy the same bars; this mutual resistance is $-2r$, the negative sign being on account of the fact that the positive direction of i_{d_1} in each bar is the negative direction of $i_{d_{(n+1)}}$, so that the resistive voltage induced in phase 1 by a positive current i in phase $(n+1)$ is $-2ri$.

$$\text{Mutual resis., phase 1 to phase } (n+2) = 2\cdot\tfrac{1}{2}R\left\{\frac{\frac{(n+1)}{n}\pi-\pi}{\pi}\right\} = R\left\{\frac{n+1}{n}-1\right\}$$

and so on, until we reach the 2nth phase, for which the mutual resistance is $R\left\{\frac{2n-1}{n}-1\right\}$.

Thus the total equivalent resistance of damper, phase 1, is, by eqn.(221) :-

(N.B.,- the number of phases is 2n, and not n, as in eqn.221)

$$\begin{aligned}R_d = {} & [2r+R] + R\left[1-\tfrac{1}{n}\right]\cos\tfrac{\pi}{n} + R\left[1-\tfrac{2}{n}\right]\cos\tfrac{2\pi}{n} + \text{----} + R\left[1-\tfrac{n-2}{n}\right]\cos\tfrac{n-2}{n}\pi + R\left[1-\tfrac{n-1}{n}\right]\cos\tfrac{n-1}{n}\pi \\ & + [-2r]\cos\tfrac{n\pi}{n} \qquad\qquad \text{--(224)} \\ & + R\left[\tfrac{n+1}{n}-1\right]\cos\tfrac{(n+1)\pi}{n} + R\left[\tfrac{n+2}{n}-1\right]\cos\tfrac{(n+2)\pi}{n} + \text{----} + R\left[\tfrac{2n-2}{n}-1\right]\cos\tfrac{2n-2}{n}\pi + R\left[\tfrac{2n-1}{n}-1\right]\cos\tfrac{2n-1}{n}\pi\end{aligned}$$

Now the coefficient of $\cos\frac{\pi}{n}$ in eqn.(224) is identical with the coefficient of $\cos\frac{2n-1}{n}\pi$; that of $\cos\frac{2\pi}{n}$ with that of $\cos\frac{2n-2}{n}\pi$; - - - - that of $\cos\frac{n-1}{n}\pi$ with that of $\cos\frac{n+1}{n}\pi$, and the trigonometric functions with identical coefficients are themselves identical, since $\cos\theta = \cos(2\pi-\theta)$. In addition, $\cos\frac{n\pi}{n} = \cos\pi = -1$. Thus the equation may be written :-

$$R_d = [4r+R] + 2R\left\{\left[1-\tfrac{1}{n}\right]\cos\tfrac{\pi}{n} + \left[1-\tfrac{2}{n}\right]\cos\tfrac{2\pi}{n} + \text{-----} + \left[1-\tfrac{n-1}{n}\right]\cos\tfrac{n-1}{n}\pi\right\} \qquad \text{--(225)}$$

The various terms involving trigonometric functions may be summed as follows :-

$$\begin{aligned}\sum_{p=1}^{p=(n-1)}\left\{\left[1-\tfrac{p}{n}\right]\cos\tfrac{p\pi}{n}\right\} = {} & 1\left\{\cos\tfrac{\pi}{n} + \cos\tfrac{2\pi}{n} + \cos\tfrac{3\pi}{n} + \text{---} + \cos\tfrac{(n-1)\pi}{n}\right\} \\ & -\tfrac{1}{n}\left\{\cos\tfrac{\pi}{n} + \cos\tfrac{2\pi}{n} + \text{---} + \cos\tfrac{(n-1)\pi}{n}\right\} - \tfrac{1}{n}\left\{\cos\tfrac{2\pi}{n} + \cos\tfrac{3\pi}{n} + \text{---} + \cos\tfrac{n-1}{n}\pi\right\} \\ & -\tfrac{1}{n}\left\{\cos\tfrac{3\pi}{n} + \cos\tfrac{4\pi}{n} + \text{---} + \cos\tfrac{n-1}{n}\pi\right\} - \tfrac{1}{n}\left\{\text{--------------}\right\} \\ & -\tfrac{1}{n}\left\{\cos\tfrac{n-1}{n}\pi\right\}\end{aligned}$$

In this last rearrangement of the terms to be summed, each separate group of terms is a simple sum of cosines of angles in arithmetic progression; the groups thus become :-

$$1\left\{\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{n-1}{n}\pi\right\} = \frac{\cos\frac{\pi}{2}\,\sin\frac{n-1}{2}\frac{\pi}{n}}{\sin\frac{\pi}{2n}} = 0\,.$$

$$-\frac{1}{n}\left\{\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{n-1}{n}\pi\right\} = -\frac{1}{n}\left\{\frac{\cos(\frac{\pi}{2}+0)\sin(\frac{\pi}{2}-\frac{\pi}{2n})}{\sin\frac{\pi}{2n}}\right\} = \frac{1}{n}\left\{\frac{\sin 0\,\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\right\} = \frac{1}{2n}\left\{\frac{\sin\frac{\pi}{2n} - \sin\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\right\}$$

$$-\frac{1}{n}\left\{\cos\frac{2\pi}{n} + \cos\frac{3\pi}{n} + \cdots + \cos\frac{n-1}{n}\pi\right\} = -\frac{1}{n}\left\{\frac{\cos(\frac{\pi}{2}+\frac{\pi}{2n})\sin(\frac{\pi}{2}-\frac{\pi}{n})}{\sin\frac{\pi}{2n}}\right\} = \frac{1}{n}\left\{\frac{\sin\frac{\pi}{2n}\cos\frac{\pi}{n}}{\sin\frac{\pi}{2n}}\right\} = \frac{1}{2n}\left\{\frac{\sin\frac{3\pi}{2n} - \sin\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\right\}$$

$$-\frac{1}{n}\left\{\cos\frac{3\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{n-1}{n}\pi\right\} = -\frac{1}{n}\left\{\frac{\cos(\frac{\pi}{2}+\frac{\pi}{n})\sin(\frac{\pi}{2}-\frac{3\pi}{2n})}{\sin\frac{\pi}{2n}}\right\} = \frac{1}{n}\left\{\frac{\sin\frac{\pi}{n}\cos\frac{3\pi}{2n}}{\sin\frac{\pi}{2n}}\right\} = \frac{1}{2n}\left\{\frac{\sin\frac{5\pi}{2n} - \sin\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\right\}$$

- -

$$-\frac{1}{n}\left\{\cos\frac{n-1}{n}\pi\right\} = -\frac{1}{n}\left\{\frac{\cos(\frac{\pi}{2}+\frac{n-2}{2n}\pi)\sin(\frac{\pi}{2}-\frac{n-1}{2n}\pi)}{\sin\frac{\pi}{2n}}\right\} = \frac{1}{n}\left\{\frac{\sin(\frac{n-2}{2n}\pi)\cos(\frac{n-1}{2n}\pi)}{\sin\frac{\pi}{2n}}\right\} = \frac{1}{2n}\left\{\frac{\sin(\frac{2n-3}{2n}\pi) - \sin\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\right\}$$

Thus the sum of the trigonometric terms is given by:-

$$S = 0 + \frac{1}{2n}\left\{\frac{\sin\frac{\pi}{2n} + \sin\frac{3\pi}{2n} + \sin\frac{5\pi}{2n} + \cdots + \sin\frac{2n-3}{2n}\pi}{\sin\frac{\pi}{2n}}\right\} - \frac{1}{2n}\left\{n-1\right\}$$

$$= \frac{1}{2n\sin\frac{\pi}{2n}}\left\{\frac{\sin(\frac{\pi}{2}-\frac{\pi}{2n})\,\sin(\frac{\pi}{2}-\frac{\pi}{2n})}{\sin\frac{\pi}{2n}}\right\} - \frac{1}{2n}\left\{n-1\right\}$$

$$= \frac{1}{2n}\cdot\frac{\cos^2\frac{\pi}{2n}}{\sin^2\frac{\pi}{2n}} - \frac{1}{2n}(n-1) \qquad \text{--(226)}$$

Hence, equivalent resistance of damper, phase 1, is:-

$$R_d = 4r + R\left\{1 + \frac{1}{n}\cdot\frac{\cos^2\frac{\pi}{2n}}{\sin^2\frac{\pi}{2n}} - 1 + \frac{1}{n}\right\} = 4r + R\,\frac{1}{n}\left\{\operatorname{cosec}^2\frac{\pi}{2n}\right\}$$

$$= 4r + R\,\frac{1}{n\,\sin^2\frac{\pi}{2n}} \qquad \text{--(227)}$$

It is of interest to see what would be the effect of working out the problem with the arrangement as in Fig 45, n being then necessarily odd. Under these conditions, the resistance R_d turns out to be exactly one-half of that given by eqn.(227). A moment's consideration will show that this value for R_d gives precisely the same solution of the general short-circuit problem, for with the arrangement as in Fig 45, the number of phases is n, and is thus also halved. Hence the equivalent damper inductance, which varies directly as the number of phases, (cf eqn. 40), is halved, and the ratio of resistance to inductance, which is the term d, is unaltered. Further, although the current in each damper phase is doubled by halving the number of phases, (cf eqn. 194), yet the total bar current is unaltered, for when the number of phases is considered as 2n, the current in any one bar consists of the difference between two currents in diametrically opposite phases; these are equal and opposite, so that the total current is double the value of each.

Concluding Note :-

The object of the Thesis has been the setting-out of a new method, involving an adaptation of the Operational Calculus invented by Heaviside, for calculating switching surges in rotating machinery. To this end, the actual working out of numerical examples has been kept subordinate to the general treatment.

The "Method of Equivalent Circuits", which is made possible by the fact that all polyphase symmetrical circuits may be split up into two imaginary circuits, the currents in which are conjugate complex functions, is a very powerful tool in the treatment of switching phenomena in rotating machinery. Its possibilities have by no means been exhausted in this Thesis; for instance, A.C. commutator motors of the more complicated type have several sets of windings, and would yield much more easily to this form of attack than to straightforward calculus, if indeed their complete analysis by classical methods is possible. Again, although the tertiary winding in the treatment in the Thesis has been called a damper winding, and has been regarded as lying between the rotor and stator coils, this is not the only possibility. The tertiary winding might be the field coil of the exciter dynamo, in which case it would be linked inductively and resistively with the rotor coil, and not linked at all with the stator system of coils.

Limitations :-

The main drawback to the method appears to lie in the fact that it cannot be applied to the case of single-phase short-circuit. This is on account of the fact that with only a single phase, the "t" and "θ_0" functions do not disappear when the stator current is eliminated, so that the resultant operational expression is not of workable form. Take, for instance, a simple single-phase case, with one rotor coil, one stator coil, and no damper. With the usual notation, the equations are :-

$$\begin{aligned} Z_r i_r + \tfrac{1}{2} C p \left\{ \varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)} \right\} i_s &= 0 \\ \tfrac{1}{2} C p \left\{ \varepsilon^{j(\omega t+\theta_0)} + \varepsilon^{-j(\omega t+\theta_0)} \right\} + Z_s i_s &= -\tfrac{1}{2} C p \left\{ \varepsilon^{j\theta_0} \frac{j\omega}{p-j\omega} + \varepsilon^{-j\theta_0} \frac{-j\omega}{p+j\omega} \right\} I_f 1 \end{aligned} \quad \text{--(228)}$$

Eliminating $\underline{i_s}$ by the usual methods :-

$$\left\{ Z_r - \tfrac{1}{4} C^2 p\, \xi_{(p)} - \tfrac{1}{4} C^2 p \left[\frac{p+j\omega}{Z_s(+)} \varepsilon^{-2j(\omega t+\theta_0)} + \frac{p-j\omega}{Z_s(-)} \varepsilon^{2j(\omega t+\theta_0)} \right] \right\} i_r = \tfrac{1}{4} C^2 p \left\{ \frac{j\omega}{Z_s(+)} + \frac{-j\omega}{Z_s(-)} + \varepsilon^{2j\theta_0} \frac{j\omega}{Z_s(-)} \frac{p}{(p-2j\omega)} + \varepsilon^{-2j\theta_0} \frac{-j\omega}{Z_s(+)} \frac{p}{(p+2j\omega)} \right\} I_f 1 \quad \text{--(229)}$$

The coefficient of $\underline{i_r}$ here contains "t" functions, which cannot apparently be removed. Thus we are unable to obtain an expression for $\underline{i_r}$ of the form $Y_{(p)}/Z_{(p)}$, and so we cannot apply the expansion theorem.

Physically, this limitation exists for the following reasons. As is well known, the sum of $\underline{n}$ pulsating fluxes which are symmetrically disposed in time-phase and space-angle is a single rotating flux, of magnitude equal to $\frac{1}{2}n$ times the maximum magnitude of one of the fluxes. This is on account of the fact that all pulsating fluxes may be regarded as consisting of two equal and oppositely rotating fluxes, each of magnitude one half times the max. of the pulsating flux; when the disposition of the pulsating fluxes is symmetrical in time and space, reverse rotating components cancel out, and leave only the forward rotating parts. Thus, terms involving 2ω and $2\theta_0$ vanish in the rotor current operational equation for a symmetrical polyphase machine, since only the forward rotating components of the stator flux (i.e., those stationary relative to the rotor) exist, while the reverse components (i.e., those travelling at -2ω elec. rad. per sec. relative to the rotor) all vanish. For a single-phase machine, however, the reverse as well as the forward flux exists, and so, as in eqn.(229), terms involving $2\theta_0$

and $2\omega t$ occur in the rotor current equation.

This limitation to the method is unfortunate, though it may possibly be removed by further research, for eqn.(229) must have some physical interpretation. For symmetrical polyphase machines, the method does all that is necessary.

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"A Study of Transient Phenomena in Electro-Magnetic Machinery, with Particular Reference to the Use of the Heaviside Operational Calculus."

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"A Study of Transient Phenomena in Electro-Magnetic Machinery, with Particular Reference to the Use of the Heaviside Operational Calculus".

APPENDIX

A. Transients in Circuits Having Resistance and Inductance.

1. On Switching-in Direct Voltage.

This case is worked out in the Thesis, Part I,A, and the total current is there shown to be :-

$$i = \frac{E}{R}\left\{1 - \varepsilon^{-\frac{Rt}{L}}\right\} \qquad --(1)$$

2. On Switching-in Alternating Voltage.

The fundamental equation for this case is :-

$$Ri + L\frac{di}{dt} = E\sin(\omega t + \theta) \qquad --(2)$$

Calling the final steady current - (the Particular Integral) - i', and the transient current - (the Complementary Function) - i'', the equations for each of these currents become :-

$$i' = I'\sin(\omega t + \theta - \phi) \qquad --(3)$$

where $I' = \frac{E}{\sqrt{R^2+\omega^2L^2}}$ and $\phi = \tan^{-1}\frac{\omega L}{R}$ (this being the normal steady state equation), and :-

$$Ri'' + L\frac{di''}{dt} = 0 \qquad --(4)$$

for the transient current. By integration, eqn.(4) becomes :-

$$i'' = I''\varepsilon^{-\frac{Rt}{L}} \qquad --(5)$$

where I'' is arbitrary, and is defined by the initial conditions. At time $t = 0$, the sum of the steady and transient currents is zero, from which :-

$$I'\sin(\theta-\phi) + I'' = 0$$

Thus the full equation for the current is :-

$$i = I'\left\{\sin(\omega t + \theta - \phi) - \varepsilon^{-\frac{Rt}{L}}\sin(\theta - \phi)\right\} \qquad --(6)$$

Fig(5) of the Thesis shows this current for the values $(R/L) = 23{\cdot}5$ and $\theta = -5°$.

B. Transients in Circuits Having Resistance, Inductance and Capacitance in Series.

1. On Switching-in Direct Voltage.

The fundamental equation for such a circuit is :-

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i.dt = E \qquad \text{--(7)}$$

or

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0 \qquad \text{--(8)}$$

The steady state current in the circuit is zero, so that $i' = 0$ for all values of t. The transient current i'' is given by the equation :-

$$\frac{d^2i''}{dt^2} + \frac{R}{L}\cdot\frac{di''}{dt} + \frac{1}{LC}\cdot i'' = 0 \qquad \text{--(9)}$$

Provided (1/LC) is greater than $R^2/4L^2$, the solution of this differential equation of the second order is :-

$$i'' = I''\varepsilon^{-\alpha t}\cos(\nu t + \gamma) \qquad \text{--(10)}$$

where $\alpha = \frac{R}{2L}$ and $\nu = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$, I'' and γ being arbitrary.

This transient current produces a transient voltage at the condenser terminals, given by :-

$$e_c'' = -L\frac{di''}{dt} - Ri'' \qquad \text{--(11)}$$

Substituting for i'' in eqn.(11), we may write :-

$$e_c'' = I''\sqrt{\frac{L}{C}}\,\varepsilon^{-\alpha t}\sin(\nu t + \gamma - \delta) \qquad \text{--(12)}$$

where $\delta = \tan^{-1}\frac{R}{2L\nu}$.

At the time $t = 0$, the current and the voltage at the condenser terminals are both zero. Thus the sum of the transient and steady values are zero for both current and voltage at $t = 0$. As the steady value for the condenser voltage is E, the applied voltage, we may write :-

$$i_0'' + i_0' = I''\cos\gamma + 0 = I''\cos\gamma = 0 \qquad \text{--(13)}$$

and

$$e_{c0}'' + e_{c0}' = I''\sqrt{\frac{L}{C}}\sin(\gamma - \delta) + E = 0 \qquad \text{--(14)}$$

Thus $\gamma = \frac{\pi}{2}$, and $I'' = -\frac{E}{\cos\delta}\sqrt{\frac{C}{L}}$

Hence the full current and voltage equations are :-

$$i = i' + i'' = \frac{E}{\cos\delta}\sqrt{\frac{C}{L}}\,\varepsilon^{-\frac{Rt}{2L}}\sin\nu t \qquad \text{--(15)}$$

$$e_c = e_c' + e_c'' = E\left\{1 - \frac{1}{\cos\delta}\varepsilon^{-\frac{Rt}{2L}}\cos(\nu t - \delta)\right\} \qquad \text{--(16)}$$

Now it is well known that such electrical circuits as these have mechanical analogues, in which inductance is replaced by mass (or inertia), capacitance by elasticity, resistance by fluid friction and voltage by applied force. For R very small, so that $(R/2L) \doteqdot 0$ and $\delta \doteqdot 0$, the maximum value of e_c in eqn.(16) occurs at $t = \pi$, and is in magnitude 2E. In the corresponding mechanical system, this maximum also occurs, its physical interpretation being that the force in the elastic member is at this moment double the external force impressed into the system - this force having been applied suddenly. In other words, we have here the electrical equivalent for the well-known mechanical rule, that "A live load is twice as destructive as a dead load".

2. On Switching-in Alternating Voltage.

Let the applied voltage be $E \sin(\omega t + \theta)$.

The transient current and condenser voltage are given as before by eqns.(10) and (12).

The normal steady current - the Particular Integral - is given by the usual equation for such a circuit, viz.:-

$$i' = I' \sin(\omega t + \theta - \phi) \qquad \text{--(17)}$$

where $I' = \frac{E}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$, and $\phi = \tan^{-1} \frac{\omega L - \frac{1}{\omega C}}{R}$.

The steady condenser voltage is thus :-

$$e_c' = \frac{I'}{\omega C} \sin(\omega t + \theta - \phi - \tfrac{\pi}{2}) = -\frac{I'}{\omega C} \cos(\omega t + \theta - \phi) \qquad \text{--(18)}$$

At the time $t = 0$, the sums of steady and transient components for both current and condenser voltage are zero. Thus we may write :-

$$I'' \cos\gamma + I' \sin(\theta - \phi) = 0 \qquad \text{--(19)}$$

and $$I''\sqrt{\tfrac{L}{C}} \sin(\gamma - \delta) - \frac{I'}{\omega C} \cos(\theta - \phi) = 0 \qquad \text{--(20)}$$

in which equations, the constants to be determined are I'' & γ.

These two constants are obviously complicated in form. simplification may however be made by assuming R very small, as is usually the case in practical circuits. Then $\delta \doteqdot 0$, and eqn.(20) becomes :-

$$I'' \sin\gamma = \frac{I}{\omega\sqrt{LC}} \cos(\theta - \phi) \qquad \text{--(21)}$$

From this, it follows that :-

$$\tan\gamma = -\frac{1}{\omega\sqrt{LC}} \cot(\theta - \phi)$$

or more correctly, $$\frac{\sin\gamma}{\cos\gamma} = \frac{1}{\omega\sqrt{LC}} \cdot \frac{\cos(\theta - \phi)}{-\sin(\theta - \phi)} \qquad \text{--(22)}$$

(over)

since eqn.(22) shows that if $(\theta-\phi)$ is (say) in the first quadrant, then γ is in the second.

Substituting eqn.(22) in eqn.(19), and bearing in mind to which quadrant γ belongs :-

$$I'' = -I'\frac{\sin(\theta-\phi)}{\cos\gamma} = -I' \times \pm\left\{1+\tan^2\gamma\right\}^{\frac{1}{2}}\sin(\theta-\phi)$$

$$= -I' \times \pm\left\{\sin^2(\theta-\phi) + \frac{1}{\omega^2 LC}\cos^2(\theta-\phi)\right\}^{\frac{1}{2}} \qquad --(23)$$

For the sign of the root, we see that since $\sin(\theta-\phi)$ and $\cos\gamma$ are opposite in sign (eqn. 22), then I'' must be positive. Thus the negative sign of the root is taken, and we may write :-

$$I'' = I'\left\{\sin^2(\theta-\phi) + \frac{1}{\omega^2 LC}\cos^2(\theta-\phi)\right\}^{\frac{1}{2}} \qquad --(24)$$

Thus the full current and voltage equations are :-

$$i = I'\left\{\sin(\omega t+\theta-\phi) + \varepsilon^{-\frac{Rt}{2L}}\left\{\sin^2(\theta-\phi)+\frac{1}{\omega^2 LC}\cos^2(\theta-\phi)\right\}^{\frac{1}{2}}\cos(\nu t+\gamma)\right\} \qquad --(25)$$

and $$e_c = \frac{I'}{\omega C}\left\{-\cos(\omega t+\theta-\phi) + \varepsilon^{-\frac{Rt}{2L}}\left[\omega^2 LC\sin^2(\theta-\phi)+\cos^2(\theta-\phi)\right]^{\frac{1}{2}}\sin(\nu t+\gamma)\right\} \qquad --(26)$$

where $\gamma = \tan^{-1}\left[\frac{1}{\omega\sqrt{LC}}\cdot\frac{\cos(\theta-\phi)}{-\sin(\theta-\phi)}\right]$.

These examples are sufficient to show the usual method of working out transient currents.

C. On the Common Ratio for the G.P. in the "Method of Reflections".

On page 58 of the Thesis, the statement was made that the common ratio, which was a function of p, was less than unity; this statement was not however mathematically rigid, for we do not know whether p is positive, negative or imaginary, - indeed, we do not know what interpretation to put on the operator when it appears as here without the unit function, 1.

The statement can probably not be proved without delving deep into the realms of higher mathematics, which are outwith the range of this Thesis. The best proof in such a case is the one used by Heaviside, - "Does it work?". It does, and gives results in the simple case which are known to be correct; thus the assumption appears to be justified.

We can perhaps be a little more rigid, as follows.

The series we are summing is not really a series of "p" functions, but this series after the expansion theorem has been applied. Thus the general term is not really $[Y_{(p)}/Z_{(p)}]1$, but $\left[\frac{Y_{(0)}}{Z_{(0)}} + \sum_{p_1, p_2, p_3 \cdots} \frac{Y_{(p)}}{pZ'_{(p)}} \varepsilon^{pt}\right]1$. Now, for the rotor current to be finite, the <u>n</u>th term of the series must tend to vanish as <u>n</u> tends towards infinity. This means that the "p" function corresponding to the <u>n</u>th term must have no effect on the current - i.e., that the expansion theorem applied to the <u>n</u>th "p" function must give zero current, but not necessarily that the <u>n</u>th "p" function is itself zero. But the criterion that the series may be summed to infinity may be said to be that successive terms have a diminishing effect on the current; thus the G.P. may be summed to infinity, even though the later terms may not tend towards zero, simply because the later terms do not effect the current.

Even this proof is not by any means rigid - (one is reminded of the series $[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots]$, in which successive terms have a diminishing effect, and yet the sum to infinity is not finite) - but it might be extended by rigid mathematics. It is better than the proof given in the Thesis, since it gives a better physical picture of the phenomenon, and thus gets a little nearer to the heart of things.

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