THE ABSOLUTE SUMMABILITY OF SERIES WITH APPLICATIONS TO FOURIER SERIES.

By

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PREFACE

This thesis contains a fairly complete account of the modern development of the theory of absolute summability and its applications to Fourier Series. It is necessary to assume a knowledge of the definitions and some elementary properties of the Lebesgue and the Stieltjes integrals. I have endeavoured, however, in the introductory chapter to state briefly some results in the theory of integration which are of frequent application. Chapter 2 contains the definitions of the Cesàro, Riesz and Abel methods of absolute summability, and Chapter 3 some fundamental theorems, including the consistency theorem for each method and a Tauberian theorem for the Abel method. The equivalence theorem for absolute Cesàro and absolute Rieszian summability is proved in Chapter 4.

The remainder of the thesis is devoted to the absolute summability of Fourier Series. Chapter 5 consists largely of introductory exposition while in Chapters 6 and 7 very general theorems are obtained for a Fourier Series and its allied series respectively. In Chapter 8 the behaviour of the Fourier series of a function satisfying a Lipschitz condition is discussed.

I have endeavoured whenever possible to indicate, by means of footnotes, the sources from which the various theorems of the thesis have been derived. The following results I claim to be original: Theorems 12, 13, 14, 15, 21, 22, 23, 30 and 31, and Lemmas 44, 45, 46, 47, 48, and 49. In addition the proofs of Theorems 16 and 17 are new, although the theorems themselves were originally proved by Dr. L.S. Bosanquet, using different methods from those which I have employed. Theorems 21, 22 and 23 have been extracted from a paper which I wrote in collaboration with Dr. Bosanquet. The proofs of these theorems were criticized and improved by him and, in consequence, are not completely original. For the sake of unity I have found it advisable, in the case of some of the well-known results, to include proofs of my own. Where these occur I have inserted an explanatory footnote.

Neither part nor the whole of this thesis has been submitted previously by me for a degree at a University.

JAMES M. HYSLOP.

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CHAPTER 1.

INTRODUCTION.

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1. 1. General Remarks. In this introductory chapter we state some properties of Lebesgue and Stieltjes integrals, since integrals of these types will be used constantly in the thesis. We assume the definitions of these integrals as well as all relevant knowledge of measurable functions and sets of points, and simply give, in a form convenient for reference purposes, these properties of the integrals which are required in the sequel. It is to be understood that no attempt has been made to state the results in their most general forms. In each case, however, the result as enunciated is sufficient for all desired applications.

1. 2. Functions[1] of Bounded Variation. Suppose that the function $f(x)$ is defined in the interval $a \leq x \leq b$. Take any points $x_0, x_1, \ldots, x_{n-1}$ in the range (a,b) such that

$$a = x_0 < x_1 < x_2 \cdots < x_{n-1} < x_n = b,$$

and form the sum

$$\sum_{r=1}^{n} | f(x_r) - f(x_{r-1}) | .$$

If, for all possible subdivisions of (a,b), this sum is less than some fixed positive number, then $f(x)$ is said to be of bounded variation in (a,b). The upper bound of this sum is called the total variation of $f(x)$ in (a,b), and will

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[1] Titchmarsh, 34. Hobson, 19.

be denoted by $V_f(a,b)$.

The total variation of a function $f(x)$ over the range (a,∞) is defined by the relation

$$(1.21) \qquad V_f(a,\infty) = \lim_{X\to\infty} V_f(a,X).$$

The same type of definition also applies to the case when $f(x)$ is not defined at one of the end points of a finite interval.

It is known that a function $f(x)$ of bounded variation can be expressed in the form

$$f(a) + P(x) - N(x),$$

where $P(x)$ and $N(x)$ are bounded, monotonic increasing functions. Conversely the difference of two bounded, monotonic increasing functions is a function of bounded variation. In particular if $f(x)$ is of bounded variation in (a,b) then $V_f(a,x)$ is also of bounded variation in (a,b). In fact,

$$(1.22) \qquad V_f(a,x) = P(x)+N(x).$$

LEMMA 1. <u>If $f(x)$ is of bounded variation in (a,b) and if c is any point in (a,b), then $f(c+0)$ and $f(c-0)$ are finite.</u>

LEMMA 2. <u>If $f(x)$ is of bounded variation in (a,b), then $f(x)$ possesses a finite derivative almost everywhere in (a,b).</u>

1.3. <u>Integrals</u>.[1] If the function $f(x)$ is integrable in

[1] Hobson, <u>19</u>. Titchmarsh, <u>34</u>.

the Lebesgue sense over (a,b), then the function $F(x)$, where

$$F(x) = \int_a^x f(t)\,dt + F(a),$$

is defined, except for an additive constant, for $a \leq x \leq b$. It is called an integral in (a,b). The following properties of $F(x)$ are important.

LEMMA 3. <u>The function $F(x)$ is continuous and of bounded variation for $a \leq x \leq b$</u>.

LEMMA 4. <u>For almost all values of x in (a,b), we have</u>

$$F'(x) = f(x).$$

<u>If $f(x)$ is continuous for $a \leq x \leq b$ then this relation holds for all values of x in (a,b)</u>.

LEMMA 5. <u>If $\varphi(x)$ is an integral in (a,b) and if $f(x)$ is integrable in the Lebesgue sense over (a,b) with integral $F(x)$, then</u>

$$\int_a^b f(x)\varphi(x)\,dx = \Big[F(x)\varphi(x)\Big]_{x=a}^{x=b} - \int_a^b F(x)\varphi'(x)\,dx.$$

LEMMA 6. <u>If $F(x)$ is an integral in (a,b), then the total variation of $F(x)$ over (a,b) is given by</u>

$$V_F(a,b) = \int_a^b |F'(x)|\,dx.$$

It should be noted in passing that the symbol $\int_a^\infty dx$, whether it occurs in connection with a Lebesgue or a Stieltjes integral, is to be taken to mean

$$\lim_{X\to\infty} \int_a^X dx.$$

1.4. Further Properties of the Lebesgue Integral. We now state some results pertaining to the subject of integration rather than to the integral itself.

LEMMA 7. If two functions $f(x)$ and $g(x)$ are integrable in the Lebesgue sense over the interval (a,b), then their sum and product are integrable over (a,b). Moreover $|f(x)|$ is integrable over (a,b) and

$$(1.41) \qquad \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

This lemma does not necessarily hold when the interval is infinite. In this case the first integral in (1.41) may exist while the second may not.

LEMMA 8. If $f(x) = g(x)$ for almost all values of x in (a,b) and if $f(x)$ is integrable in the Lebesgue sense over (a,b), then $g(x)$ is also integrable over (a,b) and

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

From Lemmas 4 and 8 we at once have the following result.

LEMMA 9. If $F(x)$ is an integral in (a,b), then

$$F(x) = \int_a^x F'(t)\,dt + F(a).$$

LEMMA 10. If the function $f(x)$ is positive, bounded and decreasing in the range (a,b), and if $\varphi(x)$ is integrable in the Lebesgue sense over (a,b), then

$$\int_a^b f(x)\,\varphi(x)\,dx = f(a+0)\int_a^\xi \varphi(x)\,dx,$$

where $a \leq \xi \leq b$.

LEMMA 11. If one of the integrals

$$\int_a^b dx \int_c^d |f(x,y)|\, dy\,, \quad \int_c^d dy \int_a^b |f(x,y)|\, dx,$$

is finite, then

$$\int_a^b dx \int_c^d f(x,y)\, dy = \int_c^d dy \int_a^b f(x,y)\, dx,$$

where b and d may be finite or infinite.

If one of the expressions

$$\int_a^b \left\{ \sum_{n=0}^{\infty} |u_n(x)| \right\} dx\,, \quad \sum_{n=0}^{\infty} \int_a^b |u_n(x)|\, dx,$$

is finite, then

$$\int_a^b \left\{ \sum_{n=0}^{\infty} u_n(x) \right\} dx = \sum_{n=0}^{\infty} \int_a^b u_n(x)\, dx.$$

where b may be finite or infinite.

LEMMA 12. If[1] $\int_0^1 f(x,y)dx$ exists for $y \geqslant b$ and if, for all values of x, the limit $f(x,\infty)$ exists, then, in order that

$$\lim_{y\to\infty} \int_0^1 f(x,y)\, dx = \int_0^1 f(x,\infty)\, dx,$$

it is sufficient that, for $0 \leq x \leq 1$, $y \geqslant b$,

$$|f(x,y)| \leq \varphi(x),$$

where $\varphi(x)$ is integrable over $(0,1)$.

1.5. The Stieltjes Integral.[2] We shall require to use certain properties of the Stieltjes integral

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[1] Hobson, 20, 323.

[2] For an account of the Stieltjes integral see Hobson 19, Lebesgue, 27, Saks, 32 and Pollard, 29, 30.

$$(1.51) \qquad \int_a^b f(x)\, d\,\varphi(x),$$

where $f(x)$ is continuous and $\varphi(x)$ is of bounded variation in (a,b). In these circumstances (1.51) certainly exists.

LEMMA 13. <u>If $f(x)$ is continuous in (a,b) and $\varphi_1(x)$ is the integral of $\varphi(x)$ in (a,b), then</u>

$$\int_a^b f(x)\, d\,\varphi_1(x) = \int_a^b f(x)\,\varphi(x)\,dx.$$

LEMMA 14. <u>If $f(x)$ and $\varphi(x)$ are continuous and of bounded variation in (a,b), then</u>

$$\int_a^b f(x)\,d\varphi(x) = \Big[f(x)\,\varphi(x)\Big]_{x=a}^{x=b} - \int_a^b \varphi(x)\,d f(x).$$

Suppose that we take points $x_1, x_2, \cdots, x_{n-1}$ in (a,b) such that

$$a = x_0 < x_1 < x_2 < \cdots\cdots < x_n = b.$$

Let $\xi_1, \xi_2, \cdots\cdots \xi_n$ be any points in the intervals (x_0, x_1), (x_1, x_2), ... (x_{n-1}, x_n) respectively. Let

$$\lambda = \max_{r=1,2,\cdots,n} (x_r - x_{r-1}).$$

Then we define the integral

$$\int_a^b f(x)\,|d\varphi(x)|$$

to be the limit

$$\lim_{\lambda \to 0} \sum_{r=1}^{n} f(\xi_r)\,|\varphi(x_r) - \varphi(x_{r-1})|,$$

if this limit exists.

It follows at once from this definition that

$$\int_a^b |d\varphi(x)| = V_\varphi(a,b),$$

and that

$$\left| \int_a^b f(x)\,d\varphi(x) \right| \leq \int_a^b |f(x)|\,|d\varphi(x)|,$$

if the integral on the right exists.

LEMMA 15. <u>If $f(x)$ is continuous in (a,b), and if $\varphi_1(x)$ is the integral of a function $\varphi(x)$ in (a,b), then</u>

$$\int_a^b |f(x)|\,|d\varphi_1(x)| = \int_a^b |f(x)|\,|\varphi(x)|\,dx.$$

LEMMA 16. <u>If $f(x)$ and $\varphi(x)$ are of bounded variation in (a,b) and one of these functions is continuous in (a,b), then</u>

$$\int_a^b |d\{f(x)\varphi(x)\}| \leq \int_a^b |f(x)|\,|d\varphi(x)| + \int_a^b |\varphi(x)|\,|df(x)|,$$

$$\int_a^b |f(x)|\,|d\varphi(x)| \leq \int_a^b |d\{f(x)\varphi(x)\}| + \int_a^b |\varphi(x)|\,|df(x)|.$$

LEMMA 17. <u>If one of the integrals</u>

$$\int_a^b dx \int_c^d |f(x,y)|\,|d\varphi(y)|, \quad \int_c^d |d\varphi(y)| \int_a^b |f(x,y)|\,dx,$$

<u>exists, then</u>

$$\int_a^b dx \int_c^d f(x,y)\,d\varphi(y) = \int_c^d d\varphi(y) \int_a^b f(x,y)\,dx,$$

where b and d may be finite or infinite.

CHAPTER 2.

Definitions of Methods of Absolute Summability.

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2.1. The Cesàro Method. Suppose that $\sum a_n$ is a given series and that

$$(2.11) \quad A_n = a_0 + a_1 + a_2 + \cdots + a_n,$$

$$(2.12) \quad A_n^{(1)} = A_0 + A_1 + A_2 + \cdots + A_n,$$

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$$(2.13) \quad A_n^{(k)} = A_0^{(k-1)} + A_1^{(k-1)} + \cdots + A_n^{(k-1)},$$

where, of course, k is a positive integer. Let $E_n^{(k)}$ denote $A_n^{(k)}$ for the particular series $1+0+0+\cdots$. Then[1] the series $\sum a_n$ is said to be summable (C, k) to the sum s if, as n tends to infinity,

$$(2.14) \quad \frac{A_n^{(k)}}{E_n^{(k)}} \to s.$$

From (2.13) we have, formally,

$$\sum_{n=0}^{\infty} A_n^{(k)} x^n = (1-x)^{-1} \sum_{n=0}^{\infty} A_n^{(k-1)} x^n$$
$$= (1-x)^{-2} \sum_{n=0}^{\infty} A_n^{(k-2)} x^n,$$

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so that

$$(2.15) \quad \sum_{n=0}^{\infty} A_n^{(k)} x^n = (1-x)^{-k-1} \sum_{n=0}^{\infty} a_n x^n.$$

Equating coefficients of x^n we then obtain

$$(2.16) \quad A_n^{(k)} = \sum_{\nu=0}^{n} \binom{k+n-\nu}{n-\nu} a_\nu,$$

and, in particular,

$$(2.17) \quad E_n^{(k)} = \binom{k+n}{n},$$

which is the coefficient of x^n in the expansion of $(1-x)^{-k-1}$.

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[1] Cesàro, 10.

We have already remarked that the above definition of summability (C,k) is valid only when k is a positive integer. By means of the relations (2.16) and (2.14), however, the definition may be extended[1] to other values of k. We say, in fact, that the series $\sum a_n$ is summable (C,k), where $k>-1$, to the sum s if, as n tends to infinity.

$$(2.18) \qquad c_n^{(k)} = \frac{A_n^{(k)}}{E_n^{(k)}} = \frac{1}{E_n^{(k)}} \sum_{\nu=0}^{n} E_{n-\nu}^{(k)} a_\nu \to s,$$

where $E_n^{(k)}$ is the coefficient of x^n in the formal expansion of $(1-x)^{-k-1}$. The restriction $k>-1$ is imposed since, when k is a negative integer, $E_n^{(k)}$ is zero on and after some value of n. The expressions $A_n^{(k)}$ and $c_n^{(k)}$ are called repectively the n-th Cesaro sum and the n-th Cesaro mean of the series $\sum a_n$ of order k.

If[2]

$$(2.19) \qquad a_n^{(k)} = c_n^{(k)} - c_{n-1}^{(k)},$$

and if $\sum |a_n^{(k)}|$ is convergent, the series $\sum a_n$ is said to be absolutely summable (C,k) or summable $|C,k|$. Altematively the sequence A_n is said to be absolutely summable (C,k) or summable $|C,k|$.

It follows at once from these definitions that a series which is absolutely summable (C,k) is also summable (C,k), and that summability $(C,0)$ and summability $|C,0|$ are respectively

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1) Knopp, 24. Chapman, 11
2) Fekete, 12. Kogbetliantz, 25.

equivalent to convergence and absolute convergence.

2.2. The Rieszian Method. Suppose that λ_n is a sequence such that

(2.21) $$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \cdots, \quad \lambda_n \to \infty.$$

Let

(2.22) $$A_k(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^k a_n, \qquad = 0 \quad , \omega \leq \lambda_0,$$

where $k > -1$. Then[1] the series $\sum a_n$ is said to be summable (R, λ_n, k) to the sum s if, as ω tends to infinity continuously,

$$C_k(\omega) = \omega^{-k} A_k(\omega) \to s.$$

The expressions $A_k(\omega)$ and $C_k(\omega)$ are called respectively the Rieszian sum and the Rieszian mean of the series $\sum a_n$ of order k and type λ_n.

Throughout the thesis we shall be concerned only with Riezian summability when $k \geq 0$. When $k < 0$ a complication arises owing to the possible presence of a large term on the right of (2.22). We note in passing that summability $(R, \lambda_n, 0)$ is equivalent to convergence.

The following Lemma regarding Rieszian sums is fundamental.

LEMMA 18. If $k > -1$, $\delta > 0$, we have

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[1] Riesz, 31.
[2] Hardy and Riesz, 18.

$$(2.23) \qquad A_{k+\delta}(\omega) = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta)} \int_0^\omega (\omega-u)^{\delta-1} A_k(u)\,du .$$

Putting

$$u = \lambda_n + (\omega-\lambda_n)t$$

in the integral

$$\int_{\lambda_n}^{\omega} (\omega-u)^{\delta-1} (u-\lambda_n)^k\,du,$$

it becomes

$$(\omega-\lambda_n)^{k+\delta} \int_0^1 (1-t)^{\delta-1} t^k\,dt = \frac{\Gamma(k+1)\Gamma(\delta)}{\Gamma(k+\delta+1)} (\omega-\lambda_n)^{k+\delta}$$

Hence

$$\begin{aligned} A_{k+\delta}(\omega) &= \sum_{\lambda_n<\omega} a_n(\omega-\lambda_n)^{k+\delta} \\ &= \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta)} \sum_{\lambda_n<\omega} a_n \int_{\lambda_n}^{\omega} (\omega-u)^{\delta-1}(u-\lambda_n)^k\,du \\ &= \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta)} \int_0^\omega (\omega-u)^{\delta-1} \Big\{ \sum_{\lambda_n<u} (u-\lambda_n)^k a_n \Big\}\,du \\ &= \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta)} \int_0^\omega (\omega-u)^{\delta-1} A_k(u)\,du. \end{aligned}$$

By putting $k-1$ for k and $\delta=1$ we obtain the important particular case:

$$(2.24) \qquad A_k(\omega) = k\int_0^\omega A_{k-1}(u)\,du, \qquad k>0.$$

Hence, when $\omega \geq 0$, $k>0$, $A_k(\omega)$ is an integral. By Lemma 4 its derivative is almost everywhere equal to $kA_{k-1}(\omega)$, and, when $k>1$, its derivative is everywhere equal to the continuous function $kA_{k-1}(\omega)$. Since ω^{-k} is an integral for $\omega>0$ we see that, when $k>0, a>0$, $C_k(\omega)$ is an integral for any finite range (a, X).

The series $\sum a_n$ is said to be absolutely[1] summable (R,λ_n,k) or summable $|R,\lambda_n,k|$, for $k>0$, if $C_k(\omega)$ is of bounded variation in any range (a,∞), $a>0$; that is, if

$$(2.25) \qquad \int_a^\infty |d\,C_k(\omega)| < \infty .$$

It is clear that summability $|R,\lambda_n,0|$ is equivalent to absolute convergence. Also, by Lemma 1, summability $|R,\lambda_n,k|$ implies summability (R,λ_n,k) .

Throughout the thesis we only have occasion to use summability $|R,n,k|$; that is, the particular case of absolute Rieszian summability when $\lambda_n = n$. In future therefore, it is to be understood that the symbols $A_k(\omega)$ and $C_k(\omega)$ refer to this particular type of summability.

It is convenient to state here a result which we shall require later on.

LEMMA 19. <u>If[2] the series $\sum a_n$ is summable (R,n,k) then</u>

$$(2.26) \qquad A_\ell(\omega) = \underline{o}(\omega^k),$$

<u>where ℓ is any positive integer less than k</u>.

2.3. <u>The Abel Method</u>. The series $\sum a_n$ is said to be summable (A) to the sum s if (i) the series $\sum_{n=0}^\infty a_n x^n$ converges, for $0 \le x < 1$, to a function $f(x)$ and (ii) $f(x) \to s$ as $x \to 1$.

The[3] series $\sum a_n$ issaid to be absolutely summable (A) , or summable $|A|$, if (i) the series $\sum_{n=0}^\infty a_n x^n$ converges,

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[1] Obreschkoff, <u>28</u>.
[2] Hardy and Riesz, <u>18</u>.
[3] Whittaker, <u>35</u>.

for $0 \leq x < 1$, to a function $f(x)$ and (ii) $f(x)$ is of bounded variation in $(0,1)$.

It follows at once from Lemma 1 that summability $|A|$ implies summability (A) .

In dealing with summability $|A|$ we shall find it convenient on occasion to use a slightly different but completely equivalent[1] definition. We shall say that the series $\sum a_n$ is summable $|A|$ if the series $\sum_{n=0}^{\infty} a_n e^{-ns}$ converges, for $s > 0$, to a function $g(s)$ which is such that

$$(2.31) \qquad \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} |g'(s)|\, ds < \infty .$$

2.4. General Remarks. These three definitions are the only ones with which we shall be concerned here. Although the definitions of absolute summability follow very naturally from those of ordinary summability, it is somewhat surprising to record that these absolute summability definitions have been given, at least in the form stated, only within recent years. Fekete,[2] however, as far back as 1911 had stated Kogbetliantz's $|C,k|$ definition in the case when k was a positive integer. Historically, the earliest method of absolute summability was due to Borel, who included an account of it in a book[3] on divergent series which he published in 1901. At that time Borel laid much greater stress on his definition of absolute summability than

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[1] See Lemma 6.
[2] Fekete, 12.
[3] Borel, 2.

on his definition of ordinary summability. The latter, however, attracted almost immediately the attention of mathematicians, whereas the former was all but neglected. Recently there has been a certain revival of interest in the theory of absolute summability, although not in the case of Borel's method, and this has led to interesting results in connection with particular series such as Fourier and Dirichlet series.

CHAPTER 3.

Some Fundamental Theorems on Absolute Summability.

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3.1. Introductory Remark. When a method of absolute summability has been defined it is essential to know if the new definition really constitutes a generalization of the idea of absolute convergence. It should be possible to prove, for example, that every absolutely convergent series is also summable $|C,k|$, $|R,n,k|$ and $|A|$, where, in the case of the first two, k is positive. In this chapter these results are obtained as particular cases of more general results which will be required later on.

3.2. The Consistency Theorem for Summability $|C,k|$.

Before proceeding to the statement and proof of the theorem we obtain some necessary Lemmas.

LEMMA 20. We have

$$A_n^{(k+\delta)} = \sum_{\nu=0}^{n} E_{n-\nu}^{(k-1)} A_\nu^{(\delta)}, \tag{3.21}$$

$$E_n^{(k+\delta)} = \sum_{\nu=0}^{n} E_{n-\nu}^{(k-1)} E_\nu^{(\delta)}, \tag{3.22}$$

$$E_n^{(k)} \sim \frac{n^k}{\Gamma(k+1)}. \tag{3.23}$$

The first two results are true for all values of k and δ, while the last is true provided that k is not a negative integer.

From (2.15) we have

$$\sum_{n=0}^{\infty} A_n^{(k+\delta)} x^n = (1-x)^{-k-\delta-1} \sum_{n=0}^{\infty} a_n x^n$$
$$= (1-x)^{-k} \sum_{n=0}^{\infty} A_n^{(\delta)} x^n,$$

so that (3.21) and (3.22) at once follow. The third relation is merely the statement of a well-known limit.

LEMMA 21. <u>If $k \geqslant 0$, $\delta > 0$ we have</u>[1]

(3.24) $$a_n^{(k+\delta)} = \frac{1}{n E_n^{(k+\delta)}} \sum_{\nu=0}^{n} E_{n-\nu}^{(\delta-1)} \nu E_\nu^{(k)} a_\nu^{(k)}.$$

From (2.18), (2.19) and (3.21) we have

$$\sum_{\nu=0}^{n} E_{n-\nu}^{(\delta-1)} \nu E_\nu^{(k)} a_\nu^{(k)} = \sum_{\nu=1}^{n} E_{n-\nu}^{(\delta-1)} \{ \nu A_\nu^{(k)} - (k+\nu) A_{\nu-1}^{(k)} \}$$
$$= \sum_{\nu=0}^{n} E_{n-\nu}^{(\delta-1)} \nu A_\nu^{(k-1)} - k \sum_{\nu=1}^{n} E_{n-\nu}^{(\delta-1)} A_{\nu-1}^{(k)}$$
$$= -k A_{n-1}^{(k+\delta)} + n A_n^{(k+\delta-1)} - \sum_{\nu=0}^{n-1} E_{n-\nu}^{(\delta-1)} (n-\nu) A_\nu^{(k-1)}$$
$$= -k A_{n-1}^{(k+\delta)} + n A_n^{(k+\delta)} - n A_{n-1}^{(k+\delta)} - \delta \sum_{\nu=0}^{n-1} E_{n-1-\nu}^{(\delta)} A_\nu^{(k-1)}$$
$$= n A_n^{(k+\delta)} - (k+n+\delta) A_{n-1}^{(k+\delta)}$$
$$= n E_n^{(k+\delta)} a_n^{(k+\delta)}.$$

From this lemma we have at once the formal relation

(3.25) $$\sum_{n=0}^{\infty} n E_n^{(k+\delta)} a_n^{(k+\delta)} x^n = (1-x)^{-\delta} \sum_{n=0}^{\infty} n E_n^{(k)} a_n^{(k)} x^n,$$

and an important particular case of this is the following relation:-

(3.26) $$\sum_{n=0}^{\infty} n a_n x^n = (1-x)^k \sum_{n=0}^{\infty} n E_n^{(k)} a_n^{(k)} x^n.$$

THEOREM 1. <u>If[2] the series $\sum a_n$ is summable $|C,k|$ it is also summable $|C, k+\delta|$, for $k \geqslant 0, \delta > 0$.</u>

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1) Kogbetliantz, <u>25</u>.
2) Kogbetliantz, <u>25</u>.

From Lemma 21 we have

$$\sum_{n=1}^{N} |a_n^{(k+\delta)}| = \sum_{n=1}^{N} \frac{1}{nE_n^{(k+\delta)}} \left| \sum_{\nu=1}^{n} E_{n-\nu}^{(\delta-1)} \nu E_\nu^{(k)} a_\nu^{(k)} \right|$$

$$\leq \sum_{\nu=1}^{N} |a_\nu^{(k)}| \sum_{n=\nu}^{N} \frac{\nu E_\nu^{(k)} E_{n-\nu}^{(\delta-1)}}{n E_n^{(k+\delta)}}$$

$$= \sum_{\nu=1}^{N} \lambda_\nu(N) |a_\nu^{(k)}|,$$

where

$$\lambda_\nu(N) = \sum_{n=\nu}^{N} \nu E_\nu^{(k)} E_{n-\nu}^{(\delta-1)} \frac{\Gamma(n)\Gamma(k+\delta+1)}{\Gamma(k+\delta+1+n)}$$

$$= \sum_{n=\nu}^{N} \nu E_\nu^{(k)} E_{n-\nu}^{(\delta-1)} \int_0^1 x^{n-1}(1-x)^{k+\delta} dx$$

$$\leq \nu E_\nu^{(k)} \int_0^1 x^{\nu-1}(1-x)^{k+\delta} \left\{ \sum_{n=\nu}^{\infty} E_{n-\nu}^{(\delta-1)} x^{n-\nu} \right\} dx$$

$$= \nu E_\nu^{(k)} \int_0^1 x^{\nu-1}(1-x)^{k+\delta}(1-x)^{-\delta} dx$$

$$= \nu E_\nu^{(k)} \frac{\Gamma(\nu)\Gamma(k+1)}{\Gamma(k+\nu+1)} = 1.$$

The theorem now follows at once.

Putting $k=0$ in this theorem we see that every absolutely convergent series is summable $|C,\delta|$ for $\delta>0$. The direct converse of this theorem is not true for it is easy to see that the non-absolutely convergent series $1-\frac{1}{2}+\frac{1}{3}-\cdots$ is summable $|C,1|$.

3. 3. A Necessary Condition for Summability $|C,k|$.

We shall now prove a theorem of a slightly different character.

THEOREM 2. If[1] the series $\sum a_n$ is summable $|C,k|$, then the series $\sum n^{-k} a_n$ is absolutely convergent.

In this proof, and elsewhere in the thesis, A denotes some positive constant which has not necessarily the same value each time it occurs.

Let $\rho = [k+1]$. Then, by (3.26),

$$\frac{a_n}{E_n^{(k)}} = \frac{1}{n E_n^{(k)}} \sum_{\nu=1}^{n}{}' E_{n-\nu}^{(-k-1)} \nu E_\nu^{(k)} a_\nu^{(k)},$$

so that

$$\left| \frac{a_n}{E_n^{(k)}} \right| \leq \frac{(-1)^\rho}{n E_n^{(k)}} \sum_{\nu=1}^{n-\rho}{}' E_{n-\nu}^{(-k-1)} \nu E_\nu^{(k)} |a_\nu^{(k)}| + A \sum_{\nu=n-\rho+1}^{n}{}' |a_\nu^{(k)}| .$$

Hence

$$\sum_{n=\rho+1}^{N+\rho}{}' \left| \frac{a_n}{E_n^{(k)}} \right| \leq \sum_{\tau=1}^{N} \frac{(-1)^\rho}{(\tau+\rho) E_{\rho+\tau}^{(k)}} \sum_{\nu=1}^{\tau}{}' E_{\rho+\tau-\nu}^{(-k-1)} \nu E_\nu^{(k)} |a_\nu^{(k)}| + A \sum_{n=\rho+1}^{N+\rho}{}' \sum_{\nu=n-\rho+1}^{n}{}' |a_\nu^{(k)}|$$

$$= S_1(N) + S_2(N),$$

where

$$S_2(N) < A \sum_{\nu=0}^{N+\rho}{}' |a_\nu^{(k)}| \sum_{n=\nu}^{\nu+\rho-1}{}' 1. < A \sum_{\nu=0}^{N+\rho}{}' |a_\nu^{(k)}| = O(1),$$

and

$$S_1(N) = \sum_{\nu=1}^{N}{}' \nu E_\nu^{(k)} |a_\nu^{(k)}| \sum_{\tau=\nu}^{N} \frac{(-1)^\rho E_{\rho+\tau-\nu}^{(-k-1)}}{(\tau+\rho) E_{\rho+\tau}^{(k)}}$$

$$= \sum_{\nu=1}^{N}{}' \nu E_\nu^{(k)} |a_\nu^{(k)}| T(\nu, N),$$

where

$$T(\nu,N) = \sum_{\tau=\nu}^{N}{}' \frac{(-1)^\rho E_{\rho+\tau-\nu}^{(-k-1)}}{(\tau+\rho) E_{\tau+\rho}^{(k)}} .$$

Writing $\rho+\tau-\nu = \mu$ we then have

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[1] Kogbetliantz, 25.

$$T(\nu,N) = \sum_{\mu=\rho}^{N+\rho-\nu} \frac{(-1)^\rho E_\mu^{(-k-1)}}{(\mu+\nu) E_{\mu+\nu}^{(k)}} < \sum_{\mu=\rho}^{\infty} \frac{(-1)^\rho E_\mu^{(-k-1)}}{(\mu+\nu) E_{\mu+\nu}^{(k)}}$$

$$\leq T_1(\nu) + T_2(\nu),$$

where

$$T_1(\nu) = \left| \sum_{\mu=0}^{\infty} \frac{(-1)^\rho E_\mu^{(-k-1)}}{(\mu+\nu) E_{\mu+\nu}^{(k)}} \right|$$

$$= \left| \sum_{\mu=0}^{\infty} E_\mu^{(-k-1)} \int_0^1 x^{\mu+\nu-1} (1-x)^k \, dx \right|$$

$$= \left| \int_0^1 x^{\nu-1} (1-x)^k (1-x)^k \, dx \right| < \frac{A}{\nu E_\nu^{(2k)}},$$

and

$$T_2(\nu) < A \sum_{\mu=0}^{\rho-1} \frac{|E_\mu^{(-k-1)}|}{\nu E_\nu^{(k)}} < \frac{A}{\nu E_\nu^{(k)}}.$$

It follows that

$$S_1(N) < A \sum_{\nu=1}^{N} |a_\nu^{(k)}| = O(1),$$

so that the series $\sum \left| \frac{a_n}{E_n^{(k)}} \right|$ is convergent. The convergence of the series $\sum n^{-k} |a_n|$ then follows from (3.23).

This theorem is a particular case of a more general theorem which was also proved by Kogbetliantz[1]. The hypothesis implies, in fact, the summability $|C, k-\ell|$ of the series $\sum n^{-\ell} a_n$ where $0 \leq \ell \leq k$.

3.4. <u>The Consistency Theorem for Summability $|R, n, k|$.</u>

We first prove a lemma[2] which is fundamental in the theory of absolute Rieszian summability.

LEMMA 22. <u>If $B_k(\omega)$ denotes the Rieszian sum of order k for the series $\sum b_n$, where $b_n = n a_n$, then</u>

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[1] Kogbetliantz, <u>25</u>.
[2] Obreschkoff, <u>28</u>. Basanquet and Hyslop, <u>8</u>.

$$(3.41)\quad \frac{d}{d\omega} C_k(\omega) = k\omega^{-k-1} B_{k-1}(\omega) = k\omega^{-1}\{C_{k-1}(\omega) - C_k(\omega)\}.$$

<u>The result holds for all positive values of ω when $k>1$, and for all values of ω except the positive integers when $0<k\leq 1$.</u>

If k is positive, and ω is not an integer

$$\frac{d}{d\omega}\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^k a_n = k\omega^{-2}\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{k-1} na_n = k\omega^{-k-1}B_{k-1}(\omega)$$

$$= k\omega^{-1}\Big\{\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{k-1}a_n - \sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^k a_n\Big\}$$

$$= k\omega^{-1}\{C_{k-1}(\omega) - C_k(\omega)\}.$$

If $k>1$, $C_k(\omega)$ is the integral of a continuous function so that the formula holds for all positive values of ω.

It is not difficult to see that, when $0<k\leq 1$ the left hand derivative of $C_k(\omega)$ exists for all positive values of ω while the right hand derivative is infinite at the integer points.

THEOREM 3. <u>If[1] $\sum a_n$ is absolutely convergent then it is summable $|R, n, k|$ for any positive k</u>.

By Lemma 22 we have

$$\int_0^X \left|\frac{d}{d\omega}C_k(\omega)\right| d\omega = k\int_0^X \omega^{-k-1}|B_{k-1}(\omega)|\,d\omega$$

$$\leq k\int_0^X \omega^{-k-1}\Big\{\sum_{n<\omega}(\omega-n)^{k-1} n|a_n|\Big\}d\omega$$

$$= k\sum_{n<X} n|a_n|\int_n^X \omega^{-k-1}(\omega-n)^{k-1}\,d\omega$$

$$\leq k\sum_{n<X}|a_n|\int_1^\infty u^{-k-1}(u-1)^{k-1}\,du = \underline{O}(1).$$

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[1] Obreschkoff, <u>28</u>.

The theorem therefore follows.

THEOREM 4. <u>If[1] the series $\sum a_n$ is summable $|R,n,k|$, then it is also summable $|R,n,k+\delta|$, where $k>0$, $\delta>0$.</u>

By (3.41) and (2.23) we have

$$\int_0^x \left|\frac{d}{d\omega} C_{k+\delta}(\omega)\right| d\omega = (k+\delta)\int_0^x \omega^{-k-\delta-1}\,|B_{k+\delta-1}(\omega)|\,d\omega$$

$$= \frac{\Gamma(k+\delta+1)}{\Gamma(k)\Gamma(\delta)}\int_0^x \omega^{-k-\delta-1}d\omega\,\left|\int_0^\omega (\omega-u)^{\delta-1} B_{k-1}(u)\,du\right|$$

$$\leq \frac{\Gamma(k+\delta+1)}{\Gamma(k)\Gamma(\delta)}\int_0^x |B_{k-1}(u)|\,du\int_u^x \omega^{-k-\delta-1}(\omega-u)^{\delta-1}\,d\omega$$

$$\leq \frac{\Gamma(k+\delta+1)}{\Gamma(k)\Gamma(\delta)}\int_0^x u^{-k-1}|B_{k-1}(u)|\,du\int_1^\infty v^{-k-\delta-1}(v-1)^{\delta-1}\,dv = O(1).$$

The theorem therefore follows.

3.5. <u>A Relation between Summabilities $|C,k|$ and $|A|$.</u>

THEOREM 5. <u>If[2] the series $\sum a_n$ is summable $|C,k|$, $k>0$ then it is also summable $|A|$.</u>

By hypothesis and Theorem 2 the series $\sum n^{-k}|a_n|$ is convergent. In particular $a_n = o(n^k)$ so that the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent for $0 \leq x < 1$. Let the sum function be $f(x)$. Then we have to prove that $f(x)$ is of bounded variation in $(0,1)$.

When $0 \leq x < 1$ we have, by (3.26),

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

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1) Obreschkoff, 28.
2) Proved by Fekete, 13, for the case when k is a positive integer.

$$= (1-x)^k \sum_{n=1}^{\infty} n E_n^{(k)} a_n^{(k)} x^{n-1} .$$

Thus, by Lemma 11,

$$\int_0^{1-\varepsilon} |f'(x)|\,dx \le \sum_{n=1}^{\infty} n E_n^{(k)} |a_n^{(k)}| \int_0^1 x^{n-1}(1-x)^k\,dx$$
$$= \sum_{n=1}^{\infty} n E_n^{(k)} \frac{\Gamma(n)\Gamma(k+1)}{\Gamma(k+n+1)} |a_n^{(k)}|$$
$$\le \sum_{n=1}^{\infty} |a_n^{(k)}| < \infty .$$

Hence,

$$\lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} |f'(x)|\,dx$$

is finite and the theorem follows.

An important particular case of the theorem provides us with the consistency Theorem[1] for summability $|A|$.

THEOREM 6. <u>If the series $\sum a_n$ is absolutely convergent then it is summable $|A|$.</u>

We shall now show that summability $|A|$ is more general than summability $|C,k|$ for any positive k . We shall show, in effect, that the converse of Theorem 5 is false.

THEOREM 7. <u>There exists a series which is summable $|A|$ but which is not summable $|C,k|$ for any positive value of k.</u>

Consider the series $\sum a_n$ where[2]

$$a_n = (-1)^n \sum_{\nu=0}^{\infty} \frac{E_n^{(\nu-1)}}{\nu!} .$$

We have

1) Whittaker, 35.
2) This series is due to H. Bohr. See Landau 26, 51.

$$n^{-k}|a_n| > An^{-k}\sum_{\nu=0}^{\infty}\frac{n^{\nu-1}}{\nu!} = An^{-k-1}e^n,$$

so that the series $\sum n^{-k}|a_n|$ diverges for every positive value of k. Hence, by Theorem 2, the series $\sum a_n$ cannot be summable $|C,k|$ for any positive value of k.

On the other hand, when $0 \le x < 1$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(-1)^n x^n \sum_{\nu=0}^{\infty}\frac{E_n^{(\nu-1)}}{\nu!}$$
$$= \sum_{\nu=0}^{\infty}\frac{1}{\nu!}\sum_{n=0}^{\infty}(-1)^n E_n^{(\nu-1)} x^n,$$

the interchange in the order of summation being justified by absolute convergence. It follows that

$$f(x) = \sum_{\nu=0}^{\infty}\frac{1}{\nu!}\frac{1}{(1+x)^\nu} = e^{\frac{1}{1+x}}.$$

Clearly $f(x)$ is of bounded variation $(0,1)$, so that the series is summable $|A|$.

3.6. <u>A Tauberian Theorem for Summability $|C,k|$.</u> We have seen that a series which is summable $|C,k|$ is summable $|C,k'|$ where k' is any number greater than k, and that the converse is, in general, not true. In this section we shall find a condition which, when satisfied along with the hypothesis that $\sum a_n$ is absolutely summable by some Cesàro method, ensures that the series is also summable $|C,k|$. This type of condition is known as a Tauberian condition, so called, because, for the case of ordinary summability, Tauber[1] was the first to investigate

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[1] Tauber, <u>33</u>.

theorems of this kind. Throughout the subsequent paragraphs we shall write,

$$b_n = na_n\,,\quad \Delta b_n = b_n - b_{n-1}\,,$$

and denote by $B_n^{(k)}$ the n-th Cesàro sum of order k for the series $\sum b_n$. We shall also denote by $d_n^{(k)}$ and $d_n'^{(k)}$ the n-th Cesàro means of order k for the series $\sum b_n$ and $\sum \Delta b_n$ respectively.

LEMMA 23. <u>When[1] $k>0$ we have</u>

$$(3.61)\qquad d_n'^{(k)} = na_n^{(k)}.$$

From (3.26) we at once have

$$\sum_{n=0}^{\infty} nE_n^{(k)} a_n^{(k)} x^n = (1-x)^{-k} \sum_{n=0}^{\infty} na_n x^n,$$

whence

$$nE_n^{(k)} a_n^{(k)} = B_n^{(k-1)}.$$

The result now follows since

$$d_n'^{(k)} = \frac{B_n^{(k-1)}}{E_n^{(k)}}.$$

LEMMA 24. <u>When[2] $k>-1$ we have</u>

$$(3.62)\qquad d_n'^{(k+1)} = (k+1)\{c_n^{(k)} - c_n^{(k+1)}\}.$$

From (3.61) we have

$$d_n'^{(k+1)} = n\,\frac{A_n^{(k+1)}}{E_n^{(k+1)}} - n\,\frac{A_{n-1}^{(k+1)}}{E_{n-1}^{(k+1)}}$$

$$= \frac{n}{E_n^{(k+1)}}A_n^{(k+1)} - \frac{n}{E_{n-1}^{(k+1)}}\{A_n^{(k+1)} - A_n^{(k)}\}$$

$$= \frac{nE_n^{(k)}}{E_{n-1}^{(k+1)}}\,\frac{A_n^{(k)}}{E_n^{(k)}} - \left\{\frac{nE_n^{(k+1)}}{E_{n-1}^{(k+1)}} - n\right\}\frac{A_n^{(k+1)}}{E_n^{(k+1)}}$$

$$= (k+1)\{c_n^{(k)} - c_n^{(k+1)}\}.$$

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[1] Kogbetliantz, <u>25</u>.
[2] Hardy, <u>14</u>.

From these lemmas we can at once deduce some straighforward theorems.[1]

THEOREM 8. <u>If the series $\sum a_n$ is summable $|C,k|$, $k \geq 0$ then the series $\sum \Delta b_n$ or the sequence b_n, is summable $|C,k+1|$.</u>

By (3.62) we have

$$\sum_{n=1}^{\infty} |d_n'^{(k+1)} - d_{n-1}'^{(k+1)}| \leq (k+1)\sum_{n=1}^{\infty} |a_n^{(k)}| + (k+1)\sum_{n=1}^{\infty} |a_n^{(k+1)}| < \infty,$$

by hypothesis and Theorem 1.

THEOREM 9. <u>If the series $\sum a_n$ is absolutely summable by Cesàro's method of some order, and if the sequence b_n is summable $|C,k+1|$, then $\sum a_n$ is summable $|C,k|$.</u>

Suppose that $\sum a_n$ is summable $|C,\ell|$ where $\ell > k \geq 0$. If $\ell < k$ the theorem merely reduces to Theorem 1 the hypothesis regarding the sequence b_n being superfluous.

Write $\ell = k+m-\delta$ where $0 \leq \delta < 1$ and m is a positive integer. Then, by Theorem 1, the series $\sum a_n$ is summable $|C,k+m|$ and the sequence b_n is summable $|C,k+r|$ for $r = 1, 2, \ldots, m, \ldots$. From (3.62)

$$(k+m)\sum_{n=1}^{\infty} |a_n^{(k+m-1)}| \leq (k+m)\sum_{n=1}^{\infty} |a_n^{(k+m)}| + \sum_{n=1}^{\infty} |d_n'^{(k+m)} - d_{n-1}'^{(k+m)}|$$
$$< \infty.$$

Thus $\sum a_n$ is summable $|C,k+m-1|$. Repeating this argument other $m-1$ times we clearly obtain the desired result.

An important particular case of Theorem 9 is the following:-

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[1] Bosanquet and Hyslop, <u>8</u>.

THEOREM 10. If the series $\sum a_n$ is absolutely summable by Cesàro's method of some order, and if the sequence b_n is summable $|C,1|$, then $\sum a_n$ is absolutely convergent.

We may combine the enunciations of Theorems 8 and 9 as follows:-

THEOREM 11. If the series $\sum a_n$ is absolutely summable by Cesàro's method of some order, then a necessary and sufficient condition for it to be summable $|C,k|$, $k \geq 0$ is that the sequence na_n should be summable $|C,k+1|$.

3. 7. A Tauberian Theorem for Summability $|A|$.

We turn now to the question of a Tauberian condition for summability $|A|$. It will be shown that the Tauberian condition which was sufficient for absolute Cesàro summability is also sufficient for absolute Abel summability. Theorem 9, in fact, may be replaced by the following more general[1] theorem.

THEOREM 12. If[2] the series $\sum a_n$ is summable $|A|$, and if the sequence na_n is summable $|C,k+1|$, $k \geq 0$ then $\sum a_n$ is summable $|C,k|$.

Throughout the proof we shall suppose that N is a positive integer, $m = [\omega]$, $\omega = s^{-1}$ and that

$$J(\omega) = \frac{1}{\omega(1-e^{-1/\omega})}, \quad \omega \geq 1.$$

Clearly we can find positive constants J_1 and J_2 such that

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1) By Theorem 5.
2) Hyslop, 22.

$$J_1 < J(\omega) < J_2.$$

It is sufficient to prove that

$$(3.41)\quad \varphi_N = \int_1^N |d_m'^{(k+1)}|\, \omega^{-1} J(\omega)\, d\omega = O(1),$$

for, by (3.61),

$$\varphi_N = \sum_{n=1}^{N-1} \int_n^{n+1} |d_m'^{(k+1)}|\, \omega^{-1} J(\omega) d\omega$$

$$= \sum_{n=1}^{N-1} n^{-1} |d_n'^{(k+1)}| \int_n^{n+1} n\omega^{-1} J(\omega) d\omega$$

$$> \tfrac{1}{2} J_1 \sum_{n=1}^{N-1} |a_n^{(k+1)}|.$$

The series $\sum a_n$ will then be summable $|C, k+1|$ and its summability $|C, k|$ will follow from hypothesis and Theorem 9.

We proceed therefore to establish (3.41). Write

$$\varphi_N \leq S_1 + S_2,$$

where

$$S_1 = \int_1^N \omega^{-2} |g'(\tfrac{1}{\omega})|\, d\omega,$$

$$S_2 = \int_1^N |\omega^{-2} g'(\tfrac{1}{\omega}) + d_m'^{(k+1)} \omega^{-1} J(\omega)|\, d\omega,$$

the function $g(s)$ being defined as in §2.3. Now

$$S_1 = \int_{1/N}^1 |g'(s)|\, ds = O(1),$$

since $\sum a_n$ is summable $|A|$. Also, by (2.15) and (3.61),

$$g'(\tfrac{1}{\omega}) = -\sum_{n=0}^{\infty} b_n e^{-n/\omega}$$

$$= -(1 - e^{-1/\omega})^{k+1} \sum_{n=0}^{\infty} E_n^{(k+1)} d_n'^{(k+1)} e^{-n/\omega},$$

each series being convergent for $1 \leq \omega \leq N$. It follows that

$$S_2 = \int_1^N \left| \omega^{-2} \{\omega J(\omega)\}^{-k-1} \sum_{n=0}^{\infty} E_n^{(k+1)} e^{-n/\omega} \{d_m'^{(k+1)} - d_n'^{(k+1)}\} \right| d\omega$$

$$\leq S_{2,1} + S_{2,2},$$

where

$$S_{2,1} = \int_1^N \omega^{-2} \{\omega J(\omega)\}^{-k-1} d\omega \sum_{n=0}^{m} E_n^{(k+1)} e^{-n/\omega} \sum_{\tau=n+1}^{m} |\Delta d_\tau'^{(k+1)}|,$$

$$S_{2,2} = \int_1^N \omega^{-2} \{\omega J(\omega)\}^{-k-1} d\omega \sum_{n=m+1}^{\infty} E_n^{(k+1)} e^{-n/\omega} \sum_{\tau=m+1}^{n} |\Delta d_\tau'^{(k+1)}|.$$

We then write

$$S_{2,1} \leq S_{2,1}^{(1)} + S_{2,1}^{(2)},$$

where

$$S_{2,1}^{(1)} = \int_1^N \omega^{-k-3} \{J(\omega)\}^{-k-1} d\omega \sum_{\tau=1}^{m} |\Delta d_\tau'^{(k+1)}|$$

$$= O\left\{ \sum_{\tau=1}^{N} |\Delta d_\tau'^{(k+1)}| \int_\tau^\infty \omega^{-k-3} d\omega \right\}$$

$$= O\left\{ \sum_{\tau=1}^{N} |\Delta d_\tau'^{(k+1)}| \right\}$$

$$= O(1),$$

and

$$S_{2,1}^{(2)} \leq \int_1^N \omega^{-2} \{\omega J(\omega)\}^{-k-1} d\omega \sum_{\tau=1}^{m} |\Delta d_\tau'^{(k+1)}| \sum_{n=1}^{\tau} E_n^{(k+1)} e^{-n/\omega}$$

$$= O\left[\sum_{\tau=1}^{N} |\Delta d_\tau'^{(k+1)}| \sum_{n=1}^{\tau} E_n^{(k+1)} \int_\tau^\infty \omega^{-2} e^{-n/\omega} \{\omega J(\omega)\}^{-k-1} d\omega \right]$$

$$= \underline{O}\Big[\sum_{\tau=1}^{N} |\Delta d_\tau'^{(k+1)}| \{\tau J(\tau)\}^{-k-1} \sum_{n=1}^{\tau} n^{-1} E_n^{(k+1)}\Big]$$

$$= \underline{O}\Big[\sum_{\tau=1}^{N} |\Delta d_\tau'^{(k+1)}| \tau^{-k-1} \sum_{n=1}^{\tau} n^{k}\Big]$$

$$= \underline{O}\Big\{\sum_{\tau=1}^{N} |\Delta d_\tau'^{(k+1)}|\Big\}$$

$$= \underline{O}(1).$$

Also[1)]

$$S_{2,2} = \underline{O}\Big[\sum_{\tau=1}^{N} |\Delta d_\tau'^{(k+1)}| \int_1^{\tau} \omega^{-2}\{\omega J(\omega)\}^{-k-1} d\omega \sum_{n=\tau}^{\infty} E_n^{(k+1)} e^{-n/\omega}\Big]$$

$$= \underline{O}\Big[\sum_{\tau=1}^{N} |\Delta d_\tau'^{(k+1)}| \int_1^{\tau} \omega^{-2}\{\omega J(\omega)\}^{-k-1} e^{-\tau/2\omega} d\omega \sum_{n=0}^{\infty} E_n^{(k+1)} e^{-n/2\omega}\Big]$$

$$= \underline{O}\Big\{\sum_{\tau=1}^{N} |\Delta d_\tau'^{(k+1)}|\, 2\tau J(2\tau) \int_1^{\tau} \omega^{-2} e^{-\tau/2\omega} d\omega\Big\}$$

$$= \underline{O}\Big\{\sum_{\tau=1}^{N} |\Delta d_\tau'^{(k+1)}|\Big\}$$

$$= \underline{O}(1).$$

Thus (3.41) follows and the theorem is proved.

As with Cesàro summability the following particular case is of interest.

THEOREM 13. If the series $\sum a_n$ is summable $|A|$ and if the sequence na_n is summable $|C,1|$, then $\sum a_n$ is absolutely convergent.

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1) The summation term $\sum_{\tau=N+1}^{\infty} \int_1^{N} d\omega \sum_{n=\tau}^{\infty}$ has been omitted. It is easy to see, however, that it is $\underline{O}(1)$.

CHAPTER 4.

The Equivalence of Summability $|C,k|$ and Summability $|R,n,k|$.

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4.1. General Remarks. When discussing the summability (C,k) of special series such as Fourier Series or Dirichlet series it has often been found convenient to deal with the Rieszian mean rather than with the Cesàro mean. It is permissible to do so in virtue of the well known equivalence theorem[1] concerning the methods (C,k) and (R,n,k). In this chapter it will be shown that summability $|R,n,k|$ is equivalent to summability $|C,k|$, in the sense that a series which is summable by one of these methods is also summable by the other. Later we shall make extensive use of this theorem when considering the absolute summability of Fourier series.

4.2. Introductory Lemmas. For the proofs of the theorems three lemmas[2] are required and it is convenient to state and prove them here.

LEMMA 25. If k is any real number except a negative integer, and if q is any positive integer, there exists a sequence of polynomials $p_0(\theta), p_1(\theta) \dots p_n(\theta)$, such that, for $n \geqslant 1$,

$$(4.21) \qquad (n+\theta)^k = \sum_{r=0}^{q} p_r(\theta)\, E_n^{(k-r)} + O\left(n^{k-q-1}\right),$$

uniformly in $0 \leqslant \theta \leqslant 1$.

Suppose that k is not an integer. By Taylor's Theorem we have

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1) Hobson, 20, 90-93.
2) The first two of these lemmas were proved by Mr. A.E. Ingham in a course of lectures which he delivered in 1930-31. See Hyslop 21, 48.

$$(n+\Theta)^k = \sum_{s=0}^{q} (-1)^s E_s^{(-k-1)} \Theta^s n^{k-s} + \underline{O}(n^{k-q-1}),$$

uniformly in $0 \leq \Theta \leq 1$.

Employing Stirling's Theorem[1] we have

$$E_n^{(k-r)} = \frac{(k-r+n)(k-r+n-1)\,\dots\,(k-r+1)}{n(n-1)\,\dots\,3.2.1}$$

$$= \sum_{s=r}^{q} \delta_{r,s}\, n^{k-s} + \underline{O}(n^{k-q-1}),$$

where $r = 0,1,2,\dots q$, $\delta_{r,s}$ is a constant and

$$\delta_{r,r} = \frac{1}{\Gamma(k-r+1)} \neq 0,$$

since k is not an integer.

It follows that

$$\sum_{r=0}^{q} p_r E_n^{(k-r)} = \sum_{r=0}^{q} \sum_{s=r}^{q} p_r \delta_{r,s}\, n^{k-s} + \underline{O}(n^{k-q-1})$$

$$= \sum_{s=0}^{q} n^{k-s} \sum_{r=0}^{s} p_r \delta_{r,s} + \underline{O}(n^{k-q-1}).$$

Clearly we can now determine the polynomials $p_r(\Theta)$ from the equations

$$\sum_{r=0}^{s} p_r \delta_{r,s} = (-1)^s E_s^{(-k-1)} \Theta^s \quad , \quad s = 0,1,2,\dots,q.$$

If k is zero or a positive integer the same argument gives an exact formula without the $\underline{O}$ term if we take $q=k$. If $q > k$ the lemma is still true provided $p_r(\Theta)=0$ for $r > k$.

LEMMA 26. <u>If $0<\Theta \leq 1$, $k>0$, q is any positive integer or zero, and</u>

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[1] Bromwich, $\underline{9}$.

$$(4.22) \qquad \gamma_n(\theta) = \sum_{\nu=0}^{n} (n+\theta-\nu)^{k-1} E_\nu^{(-k-1)},$$

then

$$(4.23) \qquad \gamma_n(\theta) = \delta(\theta) E_n^{(-k-1)} + O\left\{ \sum_{\nu=0}^{n-1} (\nu+1)^{-k-1} (n-\nu)^{k-q-2} \right\},$$

where

$$\delta(\theta) = \theta^{k-1} + \sum_{\tau=0}^{q} e_\tau \theta^\tau,$$

and e_τ is a constant.

From (4.22) we see at once that, for $0 \leq x < 1$,

$$\sum_{n=0}^{\infty} \gamma_n(\theta) x^n = (1-x)^k \sum_{n=0}^{\infty} (n+\theta)^{k-1} x^n.$$

Now, by Lemma 25,

$$(n+\theta)^{k-1} = \sum_{\tau=0}^{q} p_\tau(\theta) E_n^{(k-1-\tau)} + \beta_n(\theta),$$

where, for $n \geqslant 1$,

$$\beta_n(\theta) = O(n^{k-2-q}).$$

Let e_τ be defined by the relation

$$\sum_{\tau=0}^{q} e_\tau \theta^\tau = -\sum_{\tau=0}^{q} p_\tau(\theta),$$

and let $\beta_0(\theta) = \delta(\theta)$. Then

$$\sum_{n=0}^{\infty} \gamma_n(\theta) x^n = (1-x)^k \left\{ \sum_{n=0}^{\infty} \sum_{\tau=0}^{q} p_\tau E_n^{(k-1-\tau)} x^n + \sum_{n=0}^{\infty} \beta_n x^n \right\}$$

$$= \sum_{\tau=0}^{q} p_\tau (1-x)^\tau + (1-x)^k \sum_{n=0}^{\infty} \beta_n x^n,$$

and therefore, for $n > q$,

$$\gamma_n(\theta) = \sum_{\nu=0}^{n} E_\nu^{(-k-1)} \beta_{n-\nu}.$$

Since

$$E_n^{(-k-1)} = O\{(n+1)^{-k-1}\},$$

the result follows. If n is less than q the lemma is obviously true.

It should be noted that, when $k > 1$, the $S(\theta)$ term in (4.23) can be incorporated in the summation term, giving,

$$(4.24) \quad \gamma_n(\theta) = O\left\{ \sum_{\nu=0}^{n} (\nu+1)^{-k-1}(n-\nu+1)^{k-q-2} \right\}.$$

We now obtain a Lemma of a different type.

LEMMA 27. <u>If[1] p is a positive integer or zero, $A_n^{(p)}$ can be expressed in the form</u>

$$\sum_{\rho=0}^{p} d_\rho \, A_p\left(n + \frac{\rho}{p}\right),$$

<u>where d_ρ is a constant</u>.

We have

$$A_n^{(p)} = \sum_{\nu=0}^{n} E_{n-\nu}^{(p)} a_\nu$$

$$= \sum_{\nu=0}^{n} \frac{(p+n-\nu)(p-1+n-\nu)\cdots(1+n-\nu)}{1.2.\cdots p} a_\nu,$$

and this may be expressed in the form

$$(4.25) \quad A_n^{(p)} = \sum_{\tau=0}^{p} c_\tau \sum_{\nu=0}^{n} (n-\nu)^{p-\tau} a_\nu = \sum_{\tau=0}^{p} c_\tau A_{p-\tau}(n),$$

where c_τ is a constant.

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[1] Hobson, <u>20</u>, 93.

Let $F(n)$ denote the expression

$$\sum_{\mu=0}^{p-\rho}(-1)^{\mu}\binom{p-\rho}{\mu} A_p\left(n+\frac{\mu}{p}\right),$$

where $0 \leq \rho \leq p-1$. Then

$$F(n) = \sum_{\mu=0}^{p-\rho}(-1)^{\mu}\binom{p-\rho}{\mu}\sum_{\tau=0}^{n}\left(n+\frac{\mu}{p}-\tau\right)^{p} a_{\tau}$$

$$= \sum_{\tau=0}^{n} a_{\tau}\sum_{\mu=0}^{p-\rho}(-1)^{\mu}\binom{p-\rho}{\mu}\left(n+\frac{\mu}{p}-\tau\right)^{p}$$

$$= \sum_{\tau=0}^{n} a_{\tau} E_{n,p-\rho},$$

where $E_{n,p-\rho}$ is the coefficient of $\frac{x^p}{p!}$ in the expansion of

$$\sum_{\mu=0}^{p-\rho}(-1)^{\mu}\binom{p-\rho}{\mu} e^{(n-\tau+\frac{\mu}{p})x},$$

that is, in the expansion of

$$(-1)^{p-\rho} e^{(n-\tau)x}\left(e^{\frac{x}{p}}-1\right)^{p-\rho}.$$

It follows that $E_{n,p-\rho}$ is of the form

$$e_0(n-\tau)^{\rho} + e_1(n-\tau)^{\rho-1} + \cdots + e_{\rho},$$

where $e_0, e_1, \ldots, e_{\rho}$ depend only on p and ρ and $e_0 \neq 0$.

Hence we have, for $\rho = 0, 1, 2, \ldots, p-1$,

$$\sum_{\mu=0}^{p-\rho}(-1)^{\mu}\binom{p-\rho}{\mu} A_p\left(n+\frac{\mu}{p}\right) = \sum_{\tau=0}^{n} a_{\tau}\sum_{\sigma=0}^{\rho} e_{\rho-\sigma}(n-\tau)^{\sigma}$$

$$= \sum_{\sigma=0}^{\rho} e_{\rho-\sigma} A_{\sigma}(n).$$

On giving ρ in turn the values $0, 1, \dots, p-1$ we see at once that $A_0(n), A_1(n), \dots, A_{p-1}(n)$ can each be expressed as a linear function of $A_p(n), A_p(n+\frac{1}{p}), \dots, A_p(n+1)$. It follows from (4.25) that $A_n^{(p)}$ is expressible as a linear function of $A_p(n), A_p(n+\frac{1}{p}), \dots, A_p(n+1)$, and the lemma is therefore proved.

4.3. <u>Summability $|C,k|$ implies Summability $|R,n,k|$</u>. It will sometimes be found convenient to use, in the proofs of the theorems which follow, symbols such as $\sum_{n=0}^{X}$ where X is a continuous variable. This is to be taken to mean $\sum_{n=0}^{\alpha}$ where $\alpha = X-1$ or $[X]$ according as X is a positive integer or not. A similar meaning is to be attached to $\sum_{n=X}^{\infty}$.

THEOREM 14. <u>If[1] $k \geq 0$, and if the series $\sum a_n$ is summable $|C,k|$, then it is also summable $|R,n,k|$.</u>

The theorem is true when $k=0$ since summability $|R,n,0|$ and summability $|C,0|$ are each equivalent to absolute convergence. We shall therefore assume that k is positive.

By (3.41) and (3.26) we have, for almost all values of ω,

$$\frac{d}{d\omega} C_k(\omega) = k\omega^{-k-1} B_{k-1}(\omega)$$

$$= k\omega^{-k-1} \sum_{n=1}^{\omega} (\omega-n)^{k-1} n a_n$$

$$= k\omega^{-k-1} \sum_{n=1}^{\omega} (\omega-n)^{k-1} \sum_{\nu=1}^{n} E_{n-\nu}^{(-k-1)} \nu E_\nu^{(k)} a_\nu^{(k)}.$$

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[1] Hyslop, <u>21</u>.

Let $\omega = N+\Theta$, $0<\Theta\leq 1$ and let $n-\nu=\mu$. Then, interchanging the orders of summation, we obtain, for almost all values of ω,

$$\frac{d}{d\omega}C_k(\omega) = k\omega^{-k-1}\sum_{\nu=1}^{\omega}\nu E_\nu^{(k)}a_\nu^{(k)}\sum_{\mu=0}^{N-\nu}(N+\Theta-\nu-\mu)^{k-1}E_\mu^{(-k-1)},$$

and, using the notation of Lemma 26.

$$\int_1^x\left|\frac{d}{d\omega}C_k(\omega)\right|d\omega = \underline{O}\left\{\int_1^x\omega^{-k-1}d\omega\sum_{\nu=1}^{\omega}\nu E_\nu^{(k)}|a_\nu^{(k)}||\gamma_{N-\nu}(\Theta)|\right\}$$

$$= I_1 + I_2,$$

where

$$I_1 = \underline{O}\left\{\int_1^x\omega^{-k-1}d\omega\sum_{\nu=1}^{\omega}\nu E_\nu^{(k)}|a_\nu^{(k)}|\sum_{\mu=1}^{N-\nu}\mu^{-k-1}(N+1-\nu-\mu)^{k-q-2}\right\},$$

$$I_2 = \underline{O}\left\{\int_1^x\omega^{-k-1}d\omega\sum_{\nu=1}^{\omega}\nu E_\nu^{(k)}|a_\nu^{(k)}||\delta(\Theta)||E_{N-\nu}^{(-k-1)}|\right\}.$$

Rearranging the orders of summation and integration, and putting $\rho-\nu+1=\mu$, we obtain

$$I_1 = \underline{O}\left\{\sum_{\nu=1}^{x}\nu E_\nu^{(k)}|a_\nu^{(k)}|\sum_{\rho=\nu}^{x}(\rho-\nu+1)^{-k-1}\int_{\rho+1}^{x}\omega^{-k-1}(N-\rho)^{k-q-2}d\omega\right\}$$

$$= \underline{O}\left\{\sum_{\nu=1}^{x}\nu E_\nu^{(k)}|a_\nu^{(k)}|\sum_{\rho=\nu}^{x}(\rho-\nu+1)^{-k-1}\rho^{-k-1}\sum_{\sigma=1}^{\infty}\int_{\rho+\sigma}^{\rho+\sigma+1}(N-\rho)^{k-q-2}d\omega\right\}$$

$$= \underline{O}\left\{\sum_{\nu=1}^{x}\nu E_\nu^{(k)}|a_\nu^{(k)}|\sum_{\rho=\nu}^{x}\rho^{-k-1}(\rho-\nu+1)^{-k-1}\sum_{\sigma=1}^{\infty}\sigma^{k-q-2}\right\}.$$

Choose q greater than $k-1$. Then

$$I_1 = \underline{O}\left\{ \sum_{\nu=1}^{x} \nu E_\nu^{(k)} |a_\nu^{(k)}| \sum_{\rho=\nu}^{x}{}' \rho^{-k-1} (\rho-\nu+1)^{-k-1} \right\}$$

$$= \underline{O}\left\{ \sum_{\nu=1}^{x} \nu^{-k} E_\nu^{(k)} |a_\nu^{(k)}| \sum_{\rho=\nu}^{\infty}{}' (\rho-\nu+1)^{-k-1} \right\}$$

$$= \underline{O}\left\{ \sum_{\nu=1}^{x} |a_\nu^{(k)}| \right\}$$

$$= \underline{O}(1).$$

Also, from Lemma 26,

$$I_2 = \underline{O}\left\{ \sum_{\nu=1}^{x} \nu E_\nu^{(k)} |a_\nu^{(k)}| \int_\nu^x \omega^{-k-1} (N-\nu+1)^{-k-1} |S(\theta)| d\omega \right\}$$

$$= \underline{O}\left\{ \sum_{\nu=1}^{x} |a_\nu^{(k)}| \int_\nu^x (\omega-N)^{k-1} (N-\nu+1)^{-k-1} d\omega \right\}$$

$$= \underline{O}\left\{ \sum_{\nu=1}^{x}{}' |a_\nu^{(k)}| \sum_{\rho=0}^{\infty} \int_{\nu+\rho}^{\nu+\rho+1} (\omega-N)^{k-1} (N-\nu+1)^{-k-1} d\omega \right\}$$

$$= \underline{O}\left\{ \sum_{\nu=1}^{x} |a_\nu^{(k)}| \sum_{\rho=0}^{\infty}{}' (\rho+1)^{-k-1} \int_{\nu+\rho}^{\nu+\rho+1} (\omega-\nu-\rho)^{k-1} d\omega \right\}$$

$$= \underline{O}\left\{ \sum_{\nu=1}^{x}{}' |a_\nu^{(k)}| \sum_{\rho=0}^{\infty}{}' (\rho+1)^{-k-1} \right\}$$

$$= \underline{O}\left\{ \sum_{\nu=1}^{x} |a_\nu^{(k)}| \right\}$$

$$= \underline{O}(1).$$

The Theorem is therefore proved.

4.4. Summability $|R, n, k|$ implies Summability $|C, k|$. We now proceed to prove the converse of Theorem 14.

THEOREM 15. If[1] $k \geqslant 0$ and if the series $\sum a_n$ is summable $|R, n, k|$, then it is also summable $|C, k|$.

As in the case of Theorem 14 we may take k to be positive. By (3.26) and (2.15) we have

$$n E_n^{(k)} a_n^{(k)} = \sum_{\nu=0}^{n} E_{n-\nu}^{(k-1)} b_\nu$$

$$= \sum_{\nu=0}^{n} E_{n-\nu}^{(k-1)} \sum_{\mu=0}^{\nu} E_{\nu-\mu}^{(-i-2)} B_\mu^{(i)},$$

where i is an integer greater than k. Let

$$\varphi = \varphi(\rho) = \rho / i,$$

and let

$$D_k = \frac{\Gamma(i+1)}{\Gamma(k+1)\Gamma(1+i-k)}.$$

Then, by Lemma 27 and (2.23) we have

$$n E_n^{(k)} a_n^{(k)} = \sum_{\rho=0}^{i} d_\rho \sum_{\nu=0}^{n} E_{n-\nu}^{(k-1)} \sum_{\mu=0}^{\nu} E_{\nu-\mu}^{(-i-2)} B_i(\mu+\varphi)$$

$$= D_k \sum_{\rho=0}^{i} d_\rho \sum_{\nu=0}^{n} E_{n-\nu}^{(k-1)} \sum_{\mu=0}^{\nu} E_{\nu-\mu}^{(-i-2)} \int_0^{\mu+\varphi} k B_{k-1}(u) (\mu+\varphi-u)^{i-k} du.$$

Using (3.41) and interchanging the orders of summation

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[1] Hyslop, 21.

and integration we obtain

$$n E_n^{(k)} a_n^{(k)} = D_k \sum_{\rho=0}^{i} d_\rho \int_0^n u^{k+1} \frac{d}{du}\{C_k(u)\}\,du \quad \times$$

$$\times \sum_{\mu=u-\varphi}^{n} (\mu+\varphi-u)^{i-k} \sum_{\nu=\mu}^{n} E_{n-\nu}^{(k-1)} E_{\nu-\mu}^{(-i-2)} .$$

Now, by (3.22),

$$\sum_{\nu=\mu}^{n} E_{n-\nu}^{(k-1)} E_{\nu-\mu}^{(-i-2)} = \sum_{\tau=0}^{n-\mu} E_{n-\mu-\tau}^{(k-1)} E_\tau^{(-i-2)}$$
$$= E_{n-\mu}^{(k-i-2)} .$$

Hence

$$n E_n^{(k)} a_n^{(k)} = D_k \sum_{\rho=0}^{i} d_\rho \int_0^n u^{k+1} \frac{d}{du}\{C_k(u)\}\,du \quad \times$$

$$\times \sum_{\mu=u-\varphi}^{n} (\mu+\varphi-u)^{i-k} E_{n-\mu}^{(k-i-2)} .$$

Divide by $n E_n^{(k)}$, take absolute values, sum from zero to N, and apply (4.24). Then, since $i > k$,

$$\sum_{n=0}^{N} |a_n^{(k)}| = \underline{O}\left\{ \sum_{\rho=0}^{i} |d_\rho| \sum_{n=0}^{N} (n+1)^{-k-1} \int_0^n u^{k+1} \left|\frac{d}{du}\{C_k(u)\}\right| du \quad \times \right.$$

$$\left. \times \sum_{\mu=u-\varphi}^{n} (n-\mu+1)^{k-i-2} (\mu+\varphi+1-u)^{i-k-q-1} \right\}.$$

Taking $q = i$ and interchanging the order of the summations and integration we obtain

$$\sum_{n=0}^{N}{}' |a_n^{(k)}| = \underline{O}\left\{ \sum_{\rho=0}^{i}{}' |d_\rho| \int_0^N u^{k+1} \left|\frac{d}{du} C_k(u)\right| du \quad \times \right.$$

$$\left. \times \sum_{\mu=u-\varphi}^{N}{}' (\mu+\varphi+1-u)^{-k-1} \sum_{n=\mu}^{N}{}'(n+1)^{-k-1}(n-\mu+1)^{k-i-2} \right\}$$

$$= \underline{O}\left\{ \sum_{\rho=0}^{i}{}' |d_\rho| \int_0^N u^{k+1} \left|\frac{d}{du} C_k(u)\right| du \sum_{\mu=u-\varphi}^{N}{}' (\mu+\varphi+1-u)^{-k-1} (\mu+1)^{-k-1} \right\}$$

$$= \underline{O}\left\{ \sum_{\rho=0}^{i} |d_\rho| \int_0^N u^{k+1} \left|\frac{d}{du} C_k(u)\right| (u+1-\varphi)^{-k-1} du \right\}$$

$$= \underline{O}\left\{ \int_0^N \left|\frac{d}{du} C_k(u)\right| du \right\}$$

$$= \underline{O}(1).$$

The Theorem is therefore proved.

CHAPTER 5.

Introduction to the Absolute Summability of Fourier Series.

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5. 1. General Remarks. In this chapter some attempt has been made to arrange in a compact form certain definitions, and deductions therefrom, which will be required repeatedly in the two subsequent chapters. Moreover, in order to simplify the proofs of the principal theorems in the next two chapters, certain results have been included here as lemmas which are virtually constituent parts of these proofs. These lemmas occur at the end of the chapter.

5. 2. Definitions[1] relating to Fourier Series. If the function $f(x)$ is periodic, with period 2π, integrable in the sense of Lebesgue over $(-\pi,\pi)$ and the constants α_n and β_n are defined by the relations

$$(5.21) \qquad \alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$

$$(5.22) \qquad \beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$$

where $n=0,1,2,\ldots$, then the Fourier series of $f(x)$ is defined to be

$$(5.23) \qquad \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}(\alpha_n\cos nx + \beta_n\sin nx).$$

The Allied Series of the Fourier Series of $f(x)$ is defined to be

$$(5.24) \qquad \sum_{n=1}^{\infty}(\beta_n\cos nx - \alpha_n\sin nx).$$

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[1] See for example Titchmarsh, 34. Hobson, 20, Zygmund, 38.

The constants α_n and β_n are called the Fourier constants of the function $f(x)$.

It may easily be proved from these definitions that the Fourier series of the even function

$$(5.25) \qquad \varphi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}$$

is

$$(5.26) \qquad \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx) \cos nt,$$

and that the Fourier series of the odd function

$$(5.27) \qquad \psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}$$

is

$$(5.28) \qquad \sum_{n=1}^{\infty} (\beta_n \cos nx - \alpha_n \sin nx) \sin nt.$$

For the sake of definiteness we suppose throughout the subsequent pages that $t \geqslant 0$. It follows at once from (5.26) that the Fourier series of $\varphi(t)$ at the point $t=0$ is the Fourier series of $f(t)$ at the point $t=x$.

We shall write (5.26) and (5.28) in the form

$$(5.29) \qquad \sum_{n=0}^{\infty} a_n \cos nt, \quad \sum_{n=1}^{\infty} \bar{a}_n \sin nt,$$

so that

$$a_0 = \tfrac{1}{2}\alpha_0,$$

$$a_n = \alpha_n \cos nx + \beta_n \sin nx,$$

$$\bar{a}_n = \beta_n \cos nx - \alpha_n \sin nx.$$

5.3. The Function $\Theta(t)$. We shall have occasion to refer to a function related to the function $\psi(t)$, and its definition depends on an elementary lemma which we now proceed to prove.

LEMMA 28. The integral

$$I(t) = \int_t^\infty \frac{\psi(u)}{u}\,du,$$

exists for every positive value of t.

Let

$$I_X(t) = \int_t^X \frac{\psi(u)}{u}\,du, \quad t > 0,$$

and let m and N be integers such that

$$m\pi \leq t < (m+1)\pi, \quad N\pi \leq X < (N+1)\pi.$$

Then

$$I_X(t) = \int_t^{(m+1)\pi} \frac{\psi(u)}{u}\,du + \sum_{\nu=m+1}^{N-1} \int_{\nu\pi}^{(\nu+1)\pi} \frac{\psi(u)}{u}\,du + \int_{N\pi}^{X} \frac{\psi(u)}{u}\,du$$

$$= \int_t^{(m+1)\pi} \frac{\psi(u)}{u}\,du + \sum_{\nu=m+1}^{N-1} \int_0^{\pi} \frac{\psi(u+\nu\pi)}{u+\nu\pi}\,du + O\left(\frac{1}{N}\right),$$

since $|\psi(u)|$ is integrable over any finite range. Now $\psi(u)$ is odd and periodic so that, as $X \to \infty$,

$$I_X(t) \to \int_t^{(m+1)\pi} \frac{\psi(u)}{u}\,du + \int_0^{\pi} \psi(u)\chi(u)\,du,$$

where

$$\chi(u) = \sum_{\nu=m+1}^{\infty} (-1)^\nu \frac{1}{u+\nu\pi}.$$

The function $\chi(u)$ is continuous for $0 \leq u \leq \pi$ since the series is uniformly convergent in this range. The result therefore follows.

When $t>0$ the function $\Theta(t)$ is defined by the relation

$$\Theta(t) = \frac{2}{\pi}\int_t^\infty \frac{\psi(u)}{u}\,du. \tag{5.31}$$

LEMMA 29. The[1] function $\Theta(t)$ is integrable in the sense of Lebesgue over any range $(0,a)$ where a is finite and positive. In fact

$$\int_0^a \Theta(t)\,dt = a\Theta(a) + \frac{2}{\pi}\int_0^a \psi(t)\,dt. \tag{5.32}$$

It is clear, from Lemma 28 and the fact that $\psi(u)$ is integrable over any finite range, that the two integrals

$$I_1 = \int_0^a \frac{\psi(u)}{u}\,du\int_0^u dt \quad , \quad I_2 = \int_a^\infty \frac{\psi(u)}{u}\,du\int_0^a dt$$

exist.

Now

$$I_2 = a\int_a^\infty \frac{\psi(u)}{u}\,du = \int_0^a dt\int_a^\infty \frac{\psi(u)}{u}\,du,$$

and by Lemma 11,

$$I_1 = \int_0^a dt\int_t^a \frac{\psi(u)}{u}\,du.$$

Hence

$$I_1+I_2 = \int_0^a dt\left\{\int_t^a + \int_a^\infty\right\}\frac{\psi(u)}{u}\,du = \frac{\pi}{2}\int_0^a \Theta(t)\,dt,$$

so that (5.32) follows.

5.4. Some Functions related to $\Theta(t)$, $\varphi(t)$ and $\psi(t)$. We define the functions $\Phi_\alpha(t)$, $\varphi_\alpha(t)$ by means of the relations

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[1] Hardy, 15.

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\varphi(u)\,du\,,\ \alpha > 0,$$

(5.41) $$\Phi_0(t) = \varphi(t),$$

$$\varphi_\alpha(t) = \Gamma(\alpha+1)\,t^{-\alpha}\,\Phi_\alpha(t)\,,\ \alpha \geqslant 0,$$

and the functions $\Psi_\alpha(t)$, $\psi_\alpha(t)$, $\Theta_\alpha(t)$, $\theta_\alpha(t)$ are defined similarly. The function $\Phi_\alpha(t)$ is called the Riemann-Liouville integral of order α for the function $\varphi(t)$. It should be observed that $\varphi_\alpha(t)$ is a kind of average of the function $\varphi(t)$.

We now prove some important results concerning these functions.

LEMMA 30. If[1] $\beta > \alpha \geqslant 0$ we have

(5.42) $$\Phi_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha-1}\,\Phi_\alpha(u)\,du\,.$$

A similar result holds for the functions Θ and Ψ.

We have

$$\frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha-1}\Phi_\alpha(u)\,du = \frac{1}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha-1}du\int_0^u (u-v)^{\alpha-1}\varphi(v)\,dv$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^t \varphi(v)\,dv\int_v^t (t-u)^{\beta-\alpha-1}(u-v)^{\alpha-1}du,$$

the inversion of the order of integration being justified by Lemma 11. Let $u = v+(t-v)x$. Then

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[1] Bosanquet, 3.

$$\frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha-1}\Phi_\alpha(u)du = \frac{1}{\Gamma(\beta)}\int_0^t (t-v)^{\beta-1}\varphi(v)dv$$
$$= \Phi_\beta(t).$$

From this lemma it follows at once that, if $\alpha \geqslant 1$,

$$\Phi_\alpha(t) = \int_0^t \Phi_{\alpha-1}(u)du,$$

so that $\Phi_\alpha(t)$ is an integral for $t \geqslant 0$ and, for almost all positive values of t,

$$\Phi_\alpha'(t) = \Phi_{\alpha-1}(t).$$

Another Lemma of the same type is the following.

LEMMA 31. <u>If[1] $\beta > \alpha > 0$, $\Phi_\alpha(t)$ is of bounded variation in $(0, a)$, where $a > 0$, and $\Phi_\alpha(+0) = 0$, then</u>

$$(5.43)\quad \Phi_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)}\int_0^t dv \int_0^v (v-u)^{\beta-\alpha-1} d\Phi_\alpha(u),$$

<u>in $(0, a)$. A similar relation holds for the function Ψ</u>.

By Lemmas 30 and 14 we have

$$\Phi_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha-1}\Phi_\alpha(u)du$$
$$= \frac{1}{\Gamma(\beta-\alpha)}\left[-\frac{1}{\beta-\alpha}(t-u)^{\beta-\alpha}\Phi_\alpha(u)\right]_{u\to 0}^{u=t}$$
$$+ \frac{1}{\Gamma(\beta+1-\alpha)}\int_0^t (t-u)^{\beta-\alpha} d\Phi_\alpha(u)$$

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[1] Bosanquet, <u>4</u>.

$$= \frac{1}{\Gamma(\beta-\alpha)}\frac{1}{(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha}\, d\Phi_\alpha(u)$$

$$= \frac{1}{\Gamma(\beta-\alpha)}\int_0^t d\Phi_\alpha(u)\int_u^t (v-u)^{\beta-\alpha-1}\, dv$$

$$= \frac{1}{\Gamma(\beta-\alpha)}\int_0^t dv\int_0^v (v-u)^{\beta-\alpha-1}\, d\Phi_\alpha(u),$$

the inversion of the order of integration being justified by Lemma 17.

An immediate corollary from this lemma is the following:

LEMMA 32. <u>If[1] $\beta > \alpha > 0$, $\Phi_\alpha(t)$ is of bounded variation in $(0,a)$, where $a > 0$, and $\Phi_\alpha(+0) = 0$, then $\Phi_\beta(t)$ is an integral in $(0,a)$ and, for almost all values of t in $(0,a)$</u>,

$$(5.44) \qquad \Phi_\beta'(t) = \frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha-1}\, d\Phi_\alpha(u).$$

<u>A similar relation holds for the function Ψ</u>.

It should be observed that, when $\beta > \alpha \geq 1$, the relation (5.44) reduces to (5.32) with $\beta-1$ for β and $\alpha-1$ for α.

LEMMA 33. <u>If[2] $\alpha \geq 0, t > 0$ we have</u>

$$(5.45) \qquad \Psi_{\alpha+1}(t) = \tfrac{1}{2}\pi(\alpha+1)\left\{\Theta_{\alpha+1}(t) - \Theta_\alpha(t)\right\}.$$

We have, from (5.32), for $t > 0$,

$$\Psi_1(t) = \tfrac{1}{2}\pi\left\{\Theta_1(t) - t\,\Theta_0(t)\right\},$$

whence, by (5.42),

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[1] Bosanquet, <u>4</u>.
[2] Cf. Bosanquet and Hyslop, <u>8</u>.

$$\psi_{\alpha+1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\,\Psi_1(u)\,du$$

$$= \frac{\pi}{2\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\,\Theta_1(u)\,du - \frac{\pi}{2\Gamma(\alpha)}\int_0^t u(t-u)^{\alpha-1}\,\Theta_0(u)\,du$$

$$= \frac{\pi}{2}\,\Theta_{\alpha+1}(t) + \frac{\pi}{2\Gamma(\alpha)}\int_0^t (t-u)^{\alpha}\,\Theta_0(u) - \frac{\pi t}{2\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\,\Theta_0(u)\,du$$

Thus

$$(5.46)\qquad \Psi_{\alpha+1}(t) = \frac{\pi}{2}\left\{(\alpha+1)\,\Theta_{\alpha+1}(t) - t\,\Theta_\alpha(t)\right\},$$

and

$$\psi_{\alpha+1}(t) = \Gamma(\alpha+2)\,t^{-\alpha-1}\,\frac{\pi}{2}\left\{(\alpha+1)\,\Theta_{\alpha+1}(t) - t\,\Theta_\alpha(t)\right\}$$

$$= \frac{\pi}{2}(\alpha+1)\left\{\theta_{\alpha+1}(t) - \theta_\alpha(t)\right\}.$$

LEMMA 34. <u>If[1] $\alpha \geq 0,\ t > 0$ we have</u>

$$(5.47)\qquad \theta_\alpha(t) = \frac{2}{\pi}\int_t^\infty \frac{\psi_\alpha(u)}{u}\,du.$$

From (5.45) and (5.46) we see that, for $\alpha \geq 0$ and $t > 0$ the functions $\theta_\alpha(t)$ and $\Theta_\alpha(t)$ are integrals. Thus, by differentiating (5.46) we obtain, for $\alpha \geq 0$ and almost all positive values of t,

$$\Psi_\alpha(t) = \frac{\pi}{2}\left\{(\alpha+1)\,\Theta_\alpha(t) - \Theta_\alpha(t) - t\,\Theta_\alpha'(t)\right\}$$

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[1] Cf. Bosanquet and Hyslop, <u>8</u>.

$$= \frac{\pi}{2}\left\{\alpha\,\Theta_\alpha(t) - t\,\Theta_\alpha'(t)\right\},$$

whence, for $\alpha \geq 0$ and almost all positive values of t,

$$\frac{\psi_\alpha(t)}{t} = \frac{\pi}{2}\Gamma(\alpha+1)\left\{\alpha t^{-\alpha-1}\Theta_\alpha(t) - t^{-\alpha}\Theta_\alpha'(t)\right\}$$

$$= -\frac{\pi}{2}\frac{d}{dt}\left\{\Gamma(\alpha+1)\,t^{-\alpha}\,\Theta_\alpha(t)\right\}$$

$$= -\frac{\pi}{2}\,\theta_\alpha'(t).$$

It follows that

$$\frac{2}{\pi}\int_t^X \frac{\psi_\alpha(u)}{u}\,du = \theta_\alpha(t) - \theta_\alpha(X).$$

As $X \to \infty$, $\Theta(X) \to 0$ from its definition. Hence, from (5.42), $\theta_\alpha(X) \to 0$, and the result follows.

LEMMA 35. <u>If[1] $\varphi_\alpha(t)$ is of bounded variation in an interval $(0,a)$, where $a > 0$, then $\varphi_\beta(t)$ is of bounded variation in $(0,a)$ in the following cases:- (i) $\beta > \alpha \geq 1$, (ii) $\beta = \alpha+1$, $\alpha \geq 0$, (iii) $\beta > \alpha = 0$</u>.

<u>Case (i), $\beta > \alpha \geq 1$</u>. In this case $\varphi_\alpha(t)$ and $\varphi_\beta(t)$ are integrals for $t > 0$. Hence, by (5.42) we have, for almost all positive values of t,

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[1] This lemma has been proved for $\beta > \alpha \geq 0$ by Bosanquet, 5. In his proof he uses the function $\Phi_\alpha(t)$ for $\alpha < 0$, and the definition of this function seems open to criticism. In any event the three cases enunciated above are sufficient for our present purpose.

$$\varphi_\beta'(t) = \Gamma(\beta+1)\frac{d}{dt}\{t^{-\beta}\Phi_\beta(t)\}$$

$$= \Gamma(\beta+1)\{t^{-\beta}\Phi_{\beta-1}(t) - \beta t^{-\beta-1}\Phi_\beta(t)\}$$

$$= \frac{\Gamma(\beta+1)}{t^{\beta+1}\Gamma(\beta-\alpha)}\Big\{\int_0^t (t-u)^{\beta-\alpha-1}\, t\,\Phi_\alpha'(u)\,du - \beta\int_0^t (t-u)^{\beta-\alpha-1}\Phi_\alpha(u)\,du\Big\}$$

$$= \frac{\Gamma(\beta+1)}{t^{\beta+1}\Gamma(\beta-\alpha)}\Big\{\int_0^t (t-u)^{\beta-\alpha}\Phi_\alpha'(u)\,du + \int_0^t (t-u)^{\beta-\alpha-1}u\,\Phi_\alpha'(u)\,du - \beta\int_0^t (t-u)^{\beta-\alpha-1}\Phi_\alpha(u)\,du\Big\}.$$

Integrating the first integral by parts and observing that $\Phi_\alpha(+0)=0$, we obtain

$$\varphi_\beta'(t) = \frac{\Gamma(\beta+1)}{t^{\beta+1}\Gamma(\beta-\alpha)}\Big\{\int_0^t (t-u)^{\beta-\alpha-1}u\,\Phi_\alpha'(u)\,du - \alpha\int_0^t (t-u)^{\beta-\alpha-1}\Phi_\alpha(u)\,du\Big\}$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha)}\, t^{-\beta-1}\Big\{\int_0^t u^{\alpha+1}(t-u)^{\beta-\alpha-1}\varphi_\alpha'(u)\,du\Big\}$$

Thus,

$$\int_0^a |\varphi_\beta'(t)|\,dt \le \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha)}\int_0^a t^{-\beta-1}dt\int_0^t u^{\alpha+1}(t-u)^{\beta-\alpha-1}|\varphi_\alpha'(u)|\,du$$

$$\le \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha)}\int_0^a u^{\alpha+1}|\varphi_\alpha'(u)|\,du\int_u^\infty t^{-\beta-1}(t-u)^{\beta-\alpha-1}dt$$

$$= \int_0^a |\varphi_\alpha'(u)|\,du < \infty.$$

Case (ii); $\beta = \alpha + 1,\ \alpha \geq 0$: Since $\varphi_{\alpha+1}(t)$ is an integral for $t > 0$ we have, for almost all positive values of t,

$$\varphi_{\alpha+1}'(t) = \Gamma(\alpha+2)\{t^{-\alpha-1}\Phi_\alpha(t) - (\alpha+1)t^{-\alpha-2}\Phi_{\alpha+1}(t)\}$$

$$= (\alpha+1)t^{-\alpha-2}\{t^{\alpha+1}\varphi_\alpha(t) - (\alpha+1)\int_0^t u^\alpha \varphi_\alpha(u)\,du\}$$

$$= (\alpha+1)t^{-\alpha-2}\{t^{\alpha+1}\varphi_\alpha(t) - \Big[u^{\alpha+1}\varphi_\alpha(u)\Big]_{u\to 0}^{u=t} + \int_0^t u^{\alpha+1}\,d\varphi_\alpha(u)\}$$

$$= (\alpha+1)t^{-\alpha-2}\int_0^t u^{\alpha+1}\,d\varphi_\alpha(u).$$

Hence,

$$\int_0^a |d\varphi_{\alpha+1}(t)| \leq (\alpha+1)\int_0^a t^{-\alpha-2}\,dt\int_0^t u^{\alpha+1}|d\varphi_\alpha(u)|$$

$$\leq (\alpha+1)\int_0^a u^{\alpha+1}|d\varphi_\alpha(u)|\int_u^\infty t^{-\alpha-2}\,dt$$

$$\leq \int_0^a |d\varphi_\alpha(u)|.$$

Case (iii); $\beta > \alpha = 0$:

We have

$$\varphi_\beta(t) = \Gamma(\beta+1)\, t^{-\beta}\, \Phi_\beta(t)$$
$$= \beta t^{-\beta} \int_0^t (t-u)^{\beta-1} \varphi(u)\,du$$
$$= \beta \int_0^1 (1-v)^{\beta-1} \varphi(vt)\,dv.$$

Since $\varphi(t)$ is of bounded variation in $(0,a)$ it can be expressed in the form $\varphi^*(t)-\varphi^{**}(t)$, where $\varphi^*(t)$ and $\varphi^{**}(t)$ are positive, bounded, monotonic increasing functions. Hence we may write

$$\varphi_\beta(t) = \varphi_\beta^*(t) - \varphi_\beta^{**}(t),$$

where

$$\varphi_\beta^*(t) = \beta \int_0^1 (1-v)^{\beta-1} \varphi^*(vt)\,dv,$$

$$\varphi_\beta^{**}(t) = \beta \int_0^1 (1-v)^{\beta-1} \varphi^{**}(vt)\,dv.$$

Clearly the functions $\varphi_\beta^*(t)$ and $\varphi_\beta^{**}(t)$ are positive, bounded, monotonic increasing functions. The results therefore follows.

It should be noted that the proof of Case (i) of this lemma when translated directly to the function θ is valid for $\beta > \alpha \geqslant 0$. This follows since $\theta_\alpha(t)$ is an integral for $t > 0$ and $\alpha \geqslant 0$ whereas $\varphi_\alpha(t)$ and $\psi_\alpha(t)$ are known to be integrals only when $t > 0$ and $\alpha \geqslant 1$.

LEMMA 36. [1] If $\alpha \geq 0$, necessary and sufficient conditions that $\Theta_\alpha(t)$ should be of bounded variation in an interval $(0, a)$, where $a > 0$, are that $\Psi_{\alpha+1}(t)$ and $\Theta_\lambda(t)$ should be of bounded variation in $(0, a)$ for some $\lambda\ (> \alpha)$.

The conditions are necessary for, if $\Theta_\alpha(t)$ is of bounded variation in an interval $(0, a)$, so also are $\Theta_\lambda(t)$ and $\Theta_{\alpha+1}(t)$, by Lemma 35. From Lemma 33 it at once follows that $\Psi_{\alpha+1}(t)$ is of bounded variation in $(0, a)$.

The conditions are sufficient for, if $\lambda = \alpha + m + \rho$, where $0 \leq \rho < 1$ and m is a positive integer, the function $\Theta_{\alpha+m+1}(t)$ is of bounded variation in $(0, a)$. Also by Lemma 35 the function $\Psi_{\alpha+m+1}(t)$ is of bounded variation in $(0, a)$. Hence, by Lemma 33, the function $\Theta_{\alpha+m}(t)$ is of bounded variation in $(0, a)$. Repeating this argument we see in turn that $\Theta_{\alpha+m-1}(t), \ldots, \Theta_\alpha(t)$ are each of bounded variation in $(0, a)$.

We now prove a lemma similar in type to Lemma 35.

LEMMA 37. If [2] $\beta > \alpha > 0$, $\Psi_\alpha(t)$ is of bounded variation in $(0, a)$ where $a > 0$, $\Psi_\alpha(+0) = 0$ and

$$\Gamma(\alpha) \int_0^a u^{-\alpha} |d\,\Psi_\alpha(u)| < A, \tag{5.48}$$

then

$$\Gamma(\beta) \int_0^a u^{-\beta} |d\,\Psi_\beta(u)| < A.$$

---

[1] Bosanquet and Hyslop, 8.
[2] Cf. Bosanquet, 4.

From Lemmas 31, 32 and 15, we have

$$\Gamma(\beta)\int_0^a u^{-\beta}|d\bar{\Psi}_\beta(u)| = \Gamma(\beta)\int_0^a u^{-\beta}|\bar{\Psi}_\beta'(u)|du$$

$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\int_0^a u^{-\beta}du\left|\int_0^u (u-v)^{\beta-\alpha-1} d\bar{\Psi}_\alpha(v)\right|$$

$$\leq \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\int_0^a |d\bar{\Psi}_\alpha(v)|\int_v^\infty u^{-\beta}(u-v)^{\beta-\alpha-1}du$$

$$= \Gamma(\alpha)\int_0^a v^{-\alpha}|d\bar{\Psi}_\alpha(v)| < A.$$

LEMMA 38. <u>If[1] $\alpha>0, a>0$ necessary and sufficient conditions that $\bar{\Psi}_\alpha(t)$ be of bounded variation in $(0,a)$,</u>

$$(5.49)\quad \int_0^a u^{-\alpha}|d\bar{\Psi}_\alpha(u)| < \infty,$$

<u>and $\bar{\Psi}_\alpha(+0)=0$ are that $\psi_\alpha(t)$ and $\Theta_\lambda(t)$ should be of bounded variation in $(0,a)$ for some $\lambda(>\alpha)$.</u>

If $\alpha = 1+\gamma$, where $\gamma \geq 0$, then $\bar{\Psi}_{1+\gamma}(+0)=0$ since $\bar{\Psi}_{1+\gamma}(t)$ is an integral for $t\geq 0$ and $\bar{\Psi}_{1+\gamma}(0)=0$. Also the left hand side of (5.49) becomes

$$\int_0^a u^{-1-\gamma}|\bar{\Psi}_\gamma(u)|du = \frac{1}{\Gamma(1+\gamma)}\int_0^a u^{-1}|\psi_\gamma(u)|du = \frac{1}{\Gamma(1+\gamma)}\int_0^a |\Theta_\gamma'(u)|du$$

Thus, when $\alpha \geq 1$, the lemma reduces simply to Lemma 36.

---

[1] Bosanquet and Hyslop, 8.

We shall therefore suppose that $0<\alpha<1$.

The conditions are necessary, for, if $\overline{\Psi}_\alpha(+0)=0$ and (5.49) holds, it follows from Lemma 37 that

$$\int_0^a u^{-\alpha-1}\,|d\,\overline{\Psi}_{\alpha+1}(u)| < \infty,$$

that is,

$$\int_0^a \frac{|\psi_\alpha(u)|}{u}\,du < \infty.$$

In other words $\Theta_\alpha(t)$, and therefore $\Theta_\lambda(t)$, is of bounded variation in $(0,a)$. Again, by Lemma 16,

$$\int_0^a |d\,\psi_\alpha(u)| \leq \Gamma(\alpha+1)\int_0^a u^{-\alpha}\,|d\,\overline{\Psi}_\alpha(u)| + \alpha\int_0^a \frac{|\psi_\alpha(u)|}{u}\,du,$$

so that $\psi_\alpha(t)$ is of bounded variation in $(0,a)$.

The conditions are sufficient, for if $\psi_\alpha(t)$ and $\Theta_\lambda(t)$ are of bounded variation in $(0,a)$, it follows, as in the proof of Lemma 36, that $\Theta_\alpha(t)$ is of bounded variation in $(0,a)$; that is,

$$\int_0^a \frac{|\psi_\alpha(u)|}{u}\,du < \infty.$$

Thus, by Lemmas 15 and 16,

$$\Gamma(\alpha+1)\int_0^a u^{-\alpha}\,|d\,\overline{\Psi}_\alpha(u)| \leq \int_0^a |d\,\psi_\alpha(u)| + \alpha\int_0^a \frac{|\psi_\alpha(u)|}{u}\,du$$
$$< \infty.$$

Finally, since $\psi_\alpha(t)$ is of bounded variation in $(0,a)$, $\psi_\alpha(+0)$ is finite. Hence $\overline{\Psi}_\alpha(+0)=0$.

5.5. <u>The Functions $\gamma_\alpha(t)$ and $\overline{\gamma}_\alpha(t)$</u>. We now consider in some detail the particular cases of the function $\Phi_\alpha(t)$ when $\varphi(t)$ is $\cos t$ and when $\varphi(t)$ is $\sin t$. The functions

$\Gamma_\alpha(t), \bar{\Gamma}_\alpha(t), \gamma_\alpha(t)$ and $\bar{\gamma}_\alpha(t)$ are defined [1] by means of the relations,

$$\Gamma_\alpha(t) + i\bar{\Gamma}_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} e^{iu}\,du, \quad \alpha > 0,$$

$$(5.51) \qquad \Gamma_0(t) + i\bar{\Gamma}_0(t) = e^{it},$$

$$\gamma_\alpha(t) + i\bar{\gamma}_\alpha(t) = \Gamma(\alpha)\,t^{-\alpha}\{\Gamma_\alpha(t) + i\bar{\Gamma}_\alpha(t)\}, \quad \alpha \geq 0.$$

It should be noted that $\gamma_\alpha(t), \bar{\gamma}_\alpha(t)$ do not quite correspond to $\varphi_\alpha(t)$ when $\varphi(t)$ is $\cos t$ or $\sin t$, since $\Gamma(\alpha+1)$, which appears on the right hand side of (5.41), has been replaced by $\Gamma(\alpha)$.

It is clear from these definitions that

$$(5.52) \qquad \gamma_\alpha(t) + i\bar{\gamma}_\alpha(t) = \int_0^1 (1-u)^{\alpha-1} e^{itu}\,du, \quad \alpha > 0,$$
$$= e^{it}, \qquad \alpha = 0,$$

and that, for $\alpha > 0$, $\gamma_\alpha(0) = \alpha^{-1}$, $\bar{\gamma}_\alpha(0) = 0$.

We now obtain some important results concerning these functions.

LEMMA 39. [2] <u>If $\alpha > 0, t \geq 0$ we have</u>

$$(5.53) \qquad \gamma_{\alpha+1}(t) - \gamma_\alpha(t) = -\bar{\gamma}_\alpha'(t),$$
$$\bar{\gamma}_{\alpha+1}(t) - \bar{\gamma}_\alpha(t) = \gamma_\alpha'(t).$$

It is only necessary to prove the second of these

---

1) These functions were first considered by Young, 36.
2) Bosanquet and Hyslop, 8.

since the proof of the first is similar.

We have

$$\gamma_\alpha'(t) = -\int_0^1 (1-u)^{\alpha-1} u \sin tu \, du$$

$$= \int_0^1 (1-u)^{\alpha} \sin tu \, du - \int_0^1 (1-u)^{\alpha-1} \sin tu \, du$$

$$= \bar{\gamma}_{\alpha+1}(t) - \bar{\gamma}_\alpha(t),$$

which is the required result.

LEMMA 40. If [1] $t \geqslant 0$ we have

$$(5.54) \quad \begin{aligned} |\gamma_\alpha^{(h)}(t)| &< A(1+t)^{-\lambda}, \\ |\bar{\gamma}_\alpha^{(h)}(t)| &< A(1+t)^{-\mu}, \end{aligned}$$

where h is a positive integer or zero, $\alpha \geqslant 0$ and

$$\lambda = \operatorname{Min}(\alpha, h+2), \quad \mu = \operatorname{Min}(\alpha, h+1).$$

It should be noted in the first place that all the derivatives of $\gamma_\alpha(t)$ and $\bar{\gamma}_\alpha(t)$ are bounded for $\alpha \geqslant 0$ and finite values of $t \geqslant 0$. We need only prove therefore that, for large values of t,

$$|\gamma_\alpha^{(h)}(t)| < At^{-\lambda}, \quad |\bar{\gamma}_\alpha^{(h)}(t)| < At^{-\mu}.$$

The proof is divided into several parts.

Case (i); $\alpha = 0, h \geqslant 0$. In this case the result is obvious.

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[1] The proof of this lemma has been constructed from the proofs of a particular case; see Hobson, 20, 565.

Case (ii); $0<\alpha\leq 1, h\geq 0$. We have

$$\gamma_\alpha^{(h)}(t) + i\,\bar{\gamma}_\alpha^{(h)}(t) = i^h\int_0^1 (1-u)^{\alpha-1} u^h e^{itu}\,du$$

$$= i^h\int_0^1 u^{\alpha-1}(1-u)^h e^{it(1-u)}\,du$$

$$= i^h \sum_{\nu=0}^{h}(-1)^\nu \binom{h}{\nu}\int_0^1 u^{\alpha+\nu-1} e^{it} e^{-itu}\,du$$

$$= i^h \sum_{\nu=0}^{h}(-1)^\nu \binom{h}{\nu} e^{it} t^{-\nu-\alpha}\int_0^t v^{\alpha+\nu-1} e^{-iv}\,dv$$

$$= i^h \sum_{\nu=0}^{h}(-1)^\nu \binom{h}{\nu} e^{it} t^{-\alpha} I(t),$$

where

$$I(t) = t^{-\nu}\int_0^t v^{\alpha+\nu-1} e^{-iv}\,dv$$

$$= t^{-\nu}\left\{\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^t\right\} v^{\alpha+\nu-1} e^{-iv}\,dv$$

$$= I_1(t) + I_2(t),$$

say. Now clearly

$$|I_1(t)| < A,$$

and by Lemma 10,

$$|I_2(t)| = \left|\int_{\zeta}^t v^{\alpha-1} e^{-iv}\,dv\right| = \zeta^{\alpha-1}\left|\int_{\zeta}^{\xi} e^{-iv}\,dv\right|,$$

where $\frac{\pi}{2}\leq\zeta\leq\xi\leq t$. It follows that

$$|I_2(t)| < A,$$

so that Case (ii) is proved.

Case (iii); $1<\alpha\leq 2, h=0$. We have

$$\gamma_\alpha(t) = \int_0^1 (1-u)^{\alpha-1}\cos tu\,du$$

$$= \Big[t^{-1}\sin tu\,(1-u)^{\alpha-1}\Big]_{u=0}^{u=1} + (\alpha-1)t^{-1}\int_0^1 (1-u)^{\alpha-2}\sin tu\,du$$

$$= (\alpha-1)\,t^{-1}\,\bar{\gamma}_{\alpha-1}(t),$$

so that

$$|\gamma_\alpha(t)| \leq A t^{-1} t^{1-\alpha} \leq A t^{-\alpha}.$$

Case (iv); $\alpha>2, h=0$. We have, on integration by parts,

$$\gamma_\alpha(t) = \Big[t^{-1}\sin tu\,(1-u)^{\alpha-1}\Big]_{u=0}^{u=1} - (\alpha-1)t^{-1}\Big[t^{-1}\cos tu\,(1-u)^{\alpha-2}\Big]_{u=0}^{u=1}$$

$$-(\alpha-1)(\alpha-2)t^{-2}\int_0^1 (1-u)^{\alpha-3}\cos tu\,du,$$

whence

$$|\gamma_\alpha(t)| \leq A t^{-2}.$$

Similarly it may be shown that, if $\alpha>1$,

$$|\bar{\gamma}_\alpha(t)| \leq A t^{-1}.$$

Case (v); $\alpha>1, h\geq a$. From Lemma 30 we have

$$\gamma_\alpha'(t) = \Gamma(\alpha)\frac{d}{dt}\{t^{-\alpha}\,\Gamma_\alpha(t)\}$$

$$= \Gamma(\alpha)\{-\alpha t^{-\alpha-1}\Gamma_\alpha(t) + t^{-\alpha}\Gamma_{\alpha-1}(t)\}$$

$$= -t^{-1}\{\alpha\, \gamma_\alpha(t) - (\alpha-1)\gamma_{\alpha-1}(t)\},$$

and, by repeated differentiation, it is clear that $\gamma_\alpha^{(h)}(t)$ is of the form

$$c t^{-h}\gamma_\alpha(t) + \sum_{\mu=0}^{h-1} t^{-h+\mu} c_\mu\, \gamma_{\alpha-1}^{(\mu)}(t),$$

where c are c_μ are definite constants independent of t.

Suppose that the Lemma is true for $\gamma_\alpha(t), \gamma_\alpha'(t), \dots, \gamma_\alpha^{(h-1)}(t)$. Then

$$|\gamma_\alpha^{(h)}(t)| < At^{-\alpha-h} + At^{-h-2} + A\sum_{\mu=0}^{h-1} t^{-h+\mu} t^{1-\alpha} + A\sum_{\mu=0}^{h-1} t^{-h+\mu} t^{-\mu-2}$$

$$< At^{-\alpha} + At^{-h-2}$$

The results of Case (v) for $\gamma_\alpha^{(h)}(t)$ now follows by induction and it is clear that a similar proof holds also for $\bar{\gamma}_\alpha^{(h)}(t)$.

Combining Cases (i), (ii) and (v), we see that the lemma is completely established.

LEMMA 41. <u>If $\alpha > 1$ then $|\gamma_\alpha(t)|$ is integrable over $(0,\infty)$, and $\gamma_\alpha(t)$, $\bar{\gamma}_\alpha(t)$ are of bounded variation over $(0,\infty)$.</u>

By Lemma 40.

$$|\gamma_\alpha(t)| < A(1+t)^{-\alpha} + A(1+t)^{-2},$$

$$|\gamma_\alpha'(t)| < A(1+t)^{-\alpha} + A(1+t)^{-3},$$

$$|\bar{\gamma}_\alpha'(t)| < A(1+t)^{-\alpha} + A(1+t)^{-2},$$

from which the results follow at once.

LEMMA 42. If[1] $0<\alpha<1$, $0<\varepsilon<1$, and

$$(5.55)\qquad \bar{\gamma}_{\alpha,\varepsilon}(t) = \int_0^{1-\varepsilon}(1-u)^{\alpha-1}\sin tu\,du,$$

then, for all values of $t\geqslant 0$, $|\bar{\gamma}_\alpha(t)-\bar{\gamma}_{\alpha,\varepsilon}(t)|<A\varepsilon^\alpha$, and, for $t\geqslant\varepsilon^{-1}$,

$$|\bar{\gamma}_\alpha(t)-\bar{\gamma}_{\alpha,\varepsilon}(t)|<At^{-\alpha}.$$

We have, for $t\geqslant 0$,

$$|\bar{\gamma}_\alpha(t)-\bar{\gamma}_{\alpha,\varepsilon}(t)|\leqslant\int_{1-\varepsilon}^1(1-u)^{\alpha-1}du<A\varepsilon^\alpha,$$

while, if $t\geqslant\varepsilon^{-1}$, we have by Lemma 10,

$$|\bar{\gamma}_\alpha(t)-\bar{\gamma}_{\alpha,\varepsilon}(t)|\leqslant\left|\int_{1-\varepsilon}^{1-t^{-1}}(1-u)^{\alpha-1}\sin tu\,du\right|+\left|\int_{1-t^{-1}}^1(1-u)^{\alpha-1}\sin tu\,du\right|$$

$$\leqslant t^{1-\alpha}\max_{t^{-1}<\xi<\varepsilon}\left|\int_{1-\xi}^{1-t^{-1}}\sin tu\,du\right|+\int_{1-t^{-1}}^1(1-u)^{\alpha-1}du$$

$$\leqslant At^{-\alpha}.$$

LEMMA 43. If[2] $\alpha>1$, we have

$$(5.56)\qquad \int_0^\infty\gamma_\alpha(t)\cos xt\,dt=\frac{\pi}{2}(1-x)^{\alpha-1},\quad 0<x\leqslant 1,$$
$$=0\qquad,\quad x\geqslant 1,$$

and

$$(5.57)\qquad \int_0^\infty\bar{\gamma}_\alpha(t)\sin xt\,dt=\frac{\pi}{2}(1-x)^{\alpha-1},\quad 0<x\leqslant 1,$$
$$=0\qquad,\quad x\geqslant 1.$$

On integration by parts we obtain

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1) Bosanquet and Hyslop, 8.
2) Hobson, 20, 566.

$$\gamma_\alpha(t) = \left[t^{-1}\sin tu\,(1-u)^{\alpha-1}\right]_{u=0}^{u=1} + (\alpha-1)t^{-1}\int_0^1 (1-u)^{\alpha-2}\sin tu\,du$$

$$= (\alpha-1)t^{-1}\int_0^1 (1-u)^{\alpha-2}\sin tu\,du.$$

Hence

$$\int_0^\infty \gamma_\alpha(t)\cos xt\,dt = (\alpha-1)\int_0^1 (1-u)^{\alpha-2}du\int_0^\infty \frac{\sin tu\cos xt}{t}dt,$$

the inversion of the order of integration being justified by Lemma 12. Now

$$\int_0^\infty \frac{\sin tu\cos xt}{t}dt = \tfrac{1}{2}\int_0^\infty \frac{\sin(u+x)t}{t}dt + \tfrac{1}{2}\int_0^\infty \frac{\sin(u-x)t}{t}dt,$$

which is equal to $\frac{1}{2}\pi$ if $x<u$ and zero if $x>u$. Thus

$$\int_0^\infty \gamma_\alpha(t)\cos xt\,dt = \frac{(\alpha-1)\pi}{2}\int_x^1 (1-u)^{\alpha-2}, \quad x \leq 1,$$

$$= 0 \qquad\qquad , \quad x \geqslant 1.$$

Relation (5.56) therefore follows.

To prove (5.54) we have

$$\bar{\gamma}_\alpha(t) = \left[-t^{-1}\cos tu\,(1-u)^{\alpha-1}\right]_{u=0}^{u=1} - t^{-1}(\alpha-1)\int_0^1 (1-u)^{\alpha-2}\cos tu\,du,$$

whence

$$\int_0^\infty \bar{\gamma}_\alpha(t)\sin xt\,dt = \int_0^\infty \frac{\sin xt}{t}dt - (\alpha-1)\int_0^\infty \frac{\sin xt}{t}dt\int_0^1 (1-u)^{\alpha-2}\cos tu\,du$$

$$= \frac{\pi}{2} - (\alpha-1)\int_0^1 (1-u)^{\alpha-2}du\int_0^\infty \frac{\sin xt\cos tu}{t}dt$$

and the results follow as in the case of (5.56).

5.6. <u>Excerpts from the Proofs of Subsequent Theorems.</u>

LEMMA 44. <u>If ρ is positive, finite or infinite, and if</u>

$$L(\omega,u,\rho) = \frac{(-1)^{h+1}\omega^{h+1}u^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)}\int_u^\rho (t-u)^{h-\alpha}\gamma_\beta^{(h+2)}(\omega t)\,dt,$$

<u>where</u>

$$0 \leq h = [\alpha] \leq \alpha < \beta - 1 < h+1,$$

<u>then, for $\omega > 0$, $u > 0$</u>,

$$|L(\omega,u,\rho)| \leq A\omega^\alpha u^{\alpha+1}(1+\omega u)^{-\beta}.$$

We have, if $u+\omega^{-1} < \rho$,

$$|L(\omega,u,\rho)| \leq L_1 + L_2,$$

where

$$L_1 = \frac{\omega^{h+1}u^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)}\left|\int_u^{u+\omega^{-1}} (t-u)^{h-\alpha}\gamma_\beta^{(h+2)}(\omega t)\,dt\right|,$$

$$L_2 = \frac{\omega^{h+1}u^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)}\left|\int_{u+\omega^{-1}}^\rho (t-u)^{h-\alpha}\gamma_\beta^{(h+2)}(\omega t)\,dt\right|.$$

Now by Lemma 40,

$$L_1 < A\omega^{h+1}u^{\alpha+1}(1+\omega u)^{-\beta}\int_u^{u+\omega^{-1}} (t-u)^{h-\alpha}\,dt$$

$$< A\omega^\alpha u^{\alpha+1}(1+\omega u)^{-\beta},$$

while, by Lemmas 10 and 40,

$$L_2 = \frac{\omega^{\alpha+1}u^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)} \max_{u+\omega^{-1}<\xi<\rho}\left|\int_{u+\omega^{-1}}^\xi \gamma_\beta^{(h+2)}(\omega t)\,dt\right|$$

$$< A\omega^\alpha u^{\alpha+1}(1+\omega u)^{-\beta}.$$

If $u+\omega^{-1} > \rho$ the integral need not be split up and the argument is simpler.

LEMMA 45. <u>If ρ is positive, finite or infinite, and if</u>

$$D(\omega,u,\rho) = \frac{(-1)^h \omega^h}{\Gamma(h+1-\alpha)} \int_u^\rho (t-u)^{h-\alpha}\, \bar{\gamma}_\rho^{(h+1)}(\omega t)\,dt,$$

<u>where</u>

$$0 \le h = [\alpha] \le \alpha < \beta < h+1, \quad \alpha > 0,$$

<u>then, for $\omega > 0$, $0 < u \le \pi$,</u>

$$|D(\omega,u,\rho)| < A\,\omega^{\alpha-1}(1+\omega u)^{-\beta}.$$

The proof of this lemma is precisely the same as that of Lemma 44 except that we use the inequality for $\bar{\gamma}_\rho^{(h+1)}(t)$ instead of the inequality for $\gamma_\rho^{(h+2)}(t)$.

LEMMA 46. <u>If ρ, α, h, β are defined as in Lemma 45 and</u>

$$E(\omega,u,\rho) = \frac{1}{\Gamma(\alpha+1)} \int_0^u v^\alpha \frac{\partial}{\partial v} D(\omega,v,\rho)\,dv,$$

<u>then, for $0 < u \le \pi$, $\omega > 0$,</u>

$$|E(\omega,u,\rho)| < A\,\omega^{\alpha-1} u^\alpha (1+\omega u)^{-\beta}.$$

Let

$$D^*(\omega,u,\rho) = \int_0^u v^{\alpha-1} D(\omega,v,\rho)\,dv.$$

Then, on integration by parts, we have

$$E(\omega,u,\rho) = \frac{1}{\Gamma(\alpha+1)}\Big[v^\alpha D(\omega,v,\rho)\Big]_{v=0}^{v=u} - \frac{1}{\Gamma(\alpha)} D^*(\omega,u,\rho),$$

whence

$$|E(\omega,u,\rho)| < A u^\alpha |D(\omega,u,\rho)| + A\,|D^*(\omega,u,\rho)|$$

$$< A\omega^{\alpha-1} u^{\alpha} (1+\omega u)^{-\beta} + A\,|D^{*}(\omega, u, \rho)|,$$

by Lemma 45. The result will follow if we show that

$$|D^{*}(\omega,u,\rho)| < A\,\omega^{\alpha-1} u^{\alpha} (1+\omega u)^{-\beta}.$$

If $0<\omega u \leq 1$ we have, from Lemma 45,

$$|D^{*}(\omega,u,\rho)| < A\int_0^u v^{\alpha-1}\,\omega^{\alpha-1}\,dv < A\omega^{\alpha-1}u^{\alpha}.$$

Hence it remains to show that, if $\omega u \geq 1$,

$$|D^{*}(\omega,u,\rho)| < A\,\omega^{\alpha-1-\beta}\,u^{\alpha-\beta}.$$

Now

$$D^{*}(\omega,\rho,\rho) = \frac{(-1)^h\omega^h}{\Gamma(h+1-\alpha)}\int_0^{\rho} v^{\alpha-1}dv\int_v^{\rho}(t-v)^{h-\alpha}\,\bar{\gamma}_{\beta}^{(h+1)}(\omega t)\,dt$$

$$= \frac{(-1)^h\omega^h}{\Gamma(h+1-\alpha)}\int_0^{\rho}\bar{\gamma}_{\beta}^{(h+1)}(\omega t)\,dt\int_0^{t} v^{\alpha-1}(t-v)^{h-\alpha}\,dv,$$

provided that the inversion of the order of integration is permissible. When ρ is finite this presents no difficulty, the justification following from Lemma 11. When ρ is infinite the interchange will be justified if we show that, as $X\to\infty$,

$$I(X) = \int_0^X v^{\alpha-1}dv\int_X^{\infty}(t-v)^{h-\alpha}\,\bar{\gamma}_{\beta}^{(h+1)}(\omega t)\,dt \to 0,$$

for each fixed positive value of ω. Write

$$I(X) = I_1(X) + I_2(X),$$

where

$$|I_1(X)| = \left| \int_0^X v^{\alpha-1} dv \int_X^{X+1} (t-v)^{h-\alpha}\, \bar{\gamma}_\beta^{(h+1)}(\omega t)\, dt \right|$$

$$\leq \int_0^X v^{\alpha-1} dv \int_X^{X+1} (t-v)^{h-\alpha}\, t^{-\beta}\, dt$$

$$\leq A X^{-\beta} \int_0^X v^{\alpha-1} (X-v)^{h-\alpha} dv$$

$$= A X^{h-\beta},$$

and

$$|I_2(X)| = \left| \int_0^X v^{\alpha-1} dv \int_{X+1}^{\infty} (t-v)^{h-\alpha}\, \bar{\gamma}_\beta^{(h+1)}(\omega t)\, dt \right|$$

$$\leq \int_0^X v^{\alpha-1} (X-v)^{h-\alpha} dv \; \underset{\xi > X+1}{\operatorname{Max}} \left| \int_{X+1}^{\xi} \bar{\gamma}_\beta^{(h+1)}(\omega t)\, dt \right|$$

$$< A X^{h-\beta}.$$

Thus $I(X) \to 0$, and the interchange is completely justified.

Returning to the expression for $D^*(\omega, \rho, \rho)$, we have

$$D^*(\omega,\rho,\rho) = \frac{(-1)^h \omega^h \Gamma(\alpha)}{\Gamma(h+1)} \int_0^\rho t^h\, \bar{\gamma}_\beta^{(h+1)}(\omega t)\, dt$$

$$= \frac{(-1)^h \omega^h \Gamma(\alpha)}{\Gamma(h+1)} \left[\sum_{\nu=0}^{h} (-1)^\nu h(h-1)\cdots(h-\nu+1)\, t^{h-\nu}\, \bar{\gamma}_\beta^{(h-\nu)}(\omega t)\, \omega^{-\nu-1} \right]_{t=0}^{t=\rho}.$$

The terms all vanish at $t=0$, while, for fixed positive t and large ω,

$$\bar{\gamma}_\beta^{(h-\nu)}(\omega t) = O\{(\omega t)^{\nu-h-1}\}, \quad \nu = 1,2,\ldots,h,$$

$$\gamma_\beta^{(h)}(\omega t) = O\{(\omega t)^{-\beta}\}.$$

Thus, if ρ is finite

$$D^*(\omega,\rho,\rho) = \underline{O}(\omega^{h-1-\beta}) = \underline{O}(\omega^{\alpha-1-\beta}),$$

while, if ρ is infinite,

$$D^*(\omega,\rho,\rho) = 0 .$$

It now follows from the relation,

$$D^*(\omega,u,\rho) = D^*(\omega,\rho,\rho) - \int_u^\rho v^{\alpha-1} D(\omega,v,\rho)\,dv,$$

and Lemma 45 that, if $\omega u \geqslant 1$,

$$|D^*(\omega,u,\rho)| < A\omega^{\alpha-1-\beta} + A\int_u^\rho v^{\alpha-1-\beta}\omega^{\alpha-1-\beta}dv$$

$$< A\omega^{\alpha-1-\beta} + A\omega^{\alpha-1-\beta}u^{\alpha-\beta}$$

$$< A u^{\alpha-\beta}\omega^{\alpha-1-\beta}.$$

The lemma is therefore proved.

LEMMA 47. <u>If $0<\alpha<1$, and</u>

$$\mathcal{G}_\alpha(\omega,t) + i\,\bar{\mathcal{G}}_\alpha(\omega,t) = \sum_{n<\omega-1}' (\omega-n)^{\alpha-1} e^{int}$$

<u>then, for $\omega \geqslant 1$, $0<t<\pi$ we have</u>

$$|\mathcal{G}_\alpha(\omega,t)| \leq A\omega^\alpha(1+\omega t)^{-\alpha}, \quad |\bar{\mathcal{G}}_\alpha(\omega,t)| \leq A\omega^{\alpha+1}t(1+\omega t)^{-\alpha-1},$$

$$\left|\frac{\partial}{\partial t}\mathcal{G}_\alpha(\omega,t)\right| \leq A\omega^{\alpha+2}t(1+\omega t)^{-\alpha-1}, \quad \left|\frac{\partial}{\partial t}\bar{\mathcal{G}}_\alpha(\omega,t)\right| \leq A\omega^{\alpha+1}(1+\omega t)^{-\alpha}.$$

The proof of each of these relations is similar. We shall therefore prove one of them, say the third. If $0<\omega t<1$ we have

$$\left|\frac{\partial}{\partial t} G_{\alpha}(\omega,t)\right| = \left|\sum_{n<\omega-1}' (\omega-n)^{\alpha-1} n \sin nt\right|$$

$$\leq t \sum_{n<\omega-1} n^2 (\omega-n)^{\alpha-1}$$

$$< t \int_0^{\omega} x^2 (\omega-x)^{\alpha-1} dx < A\omega^{\alpha+2} t,$$

while, if $\omega t \geq 1$,

$$\left|\frac{\partial}{\partial t} G_{\alpha}(\omega,t)\right| \leq \left|\sum_{n<\omega-t^{-1}} (\omega-n)^{\alpha-1} n \sin nt\right| + \left|\sum_{\omega-t^{-1}<n<\omega-1} (\omega-n)^{\alpha-1} n \sin nt\right|$$

$$< \omega t^{1-\alpha} \operatorname*{Max}_{N,N'} \left|\sum_{n=N}^{N'} \sin nt\right| + \omega \int_{\omega-t^{-1}}^{\omega} (\omega-x)^{\alpha-1} dx$$

$$< A\omega t^{-\alpha}.$$

The result then follows.

LEMMA 48. <u>If $0<\alpha<\beta<1$ and</u>

$$J(\omega,u) + i\bar{J}(\omega,u) = \frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)} \int_u^{\pi} (t-u)^{-\alpha} \left\{\sum_{n<\omega}' (\omega-n)^{\beta-1} \frac{d}{dt} e^{int}\right\} dt,$$

<u>then, for $0<u<\pi$, $\omega>1$</u>,

$$|J(\omega,u)| + |\bar{J}(\omega,u)| < A\omega^{\alpha-1}(1+\omega u)^{-\beta} + A[\omega]^{\alpha-\beta-1}(\omega-[\omega])^{\beta-1}.$$

If $N = [\omega]$ we have

$$\bar{J}(\omega, u) = \frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)} \int_u^\pi (t-u)^{-\alpha} \left\{ \sum_{n<\omega} (\omega-n)^{\beta-1} n \cos nt \right\} dt$$

$$= \frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)} \int_u^\pi (t-u)^{-\alpha} \frac{\partial}{\partial t} \bar{\zeta}_\beta(\omega, t)\, dt$$

$$+ \frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)} \int_u^\pi (t-u)^{-\alpha} (\omega-N)^{\beta-1} N \cos Nt\, dt$$

$$= \bar{J}_1 + \bar{J}_2,$$

say. Now, by Lemmas 10 and 47, we have

$$|\bar{J}_1| \leq \frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)} \left\{ \int_u^{u+\omega^{-1}} (t-u)^{-\alpha} \left| \frac{\partial}{\partial t} \bar{\zeta}_\beta(\omega, t) \right| dt + \left| \int_{u+\omega^{-1}}^\pi (t-u)^{-\alpha} \frac{\partial}{\partial t} \bar{\zeta}_\beta(\omega, t)\, dt \right| \right\}$$

$$\leq A(1+\omega u)^{-\beta} \int_u^{u+\omega^{-1}} (t-u)^{-\alpha} dt + A\omega^{\alpha-\beta-1} \max_{u+\omega^{-1} < \xi < \pi} \left| \int_{u+\omega^{-1}}^{\xi} \frac{\partial}{\partial t} \bar{\zeta}_\beta(\omega, t)\, dt \right|$$

$$< A\omega^{\alpha-1}(1+\omega u)^{-\beta} + A\omega^{\alpha} u (1+\omega u)^{-\beta-1}$$

$$< A\omega^{\alpha-1}(1+\omega u)^{-\beta}.$$

Also, by Lemma 10,

$$|\bar{J}_2| \leq \frac{\omega^{-\beta-1} N}{\Gamma(1-\alpha)} \int_u^{u+N^{-1}} (t-u)^{-\alpha} (\omega-N)^{\beta-1} |\cos Nt|\, dt$$

$$+ \frac{\omega^{-\beta-1} N (\omega-N)^{\beta-1}}{\Gamma(1-\alpha)} \left| \int_{u+N^{-1}}^\pi (t-u)^{-\alpha} \cos Nt\, dt \right|$$

$$\leq AN^{\alpha} \omega^{-\beta-1} (\omega-N)^{\beta-1} + AN^{\alpha+1} (\omega-N)^{\beta-1} \max_{u+N^{-1} < \xi < \pi} \left| \int_{u+N^{-1}}^{\xi} \cos Nt\, dt \right|$$

$$\leqslant A N^{\alpha} \omega^{-\beta-1}(\omega-N)^{\beta-1}$$

$$< A[\omega]^{\alpha-\beta-1}(\omega-[\omega])^{\beta-1}.$$

Similar results hold also for $J(\omega,u)$. The lemma is therefore established.

LEMMA 49. <u>If $0<\alpha<\beta<1$, and</u>

$$\overline{K}(\omega,u) = \frac{1}{\Gamma(\alpha+1)}\int_0^u x^{\alpha} \frac{\partial}{\partial x}\overline{J}(\omega,x)\,dx,$$

<u>then, for $0<u\leqslant\pi$, $\omega\geqslant 1$,</u>

$$|\overline{K}(\omega,u)| \leqslant A\omega^{\alpha-1}u^{\alpha}(1+\omega u)^{-\beta} + A[\omega]^{\alpha-\beta-1}(\omega-[\omega])^{\beta-1}.$$

We have, on integration by parts,

$$\overline{K}(\omega,u) = \frac{1}{\Gamma(\alpha+1)}\Big[x^{\alpha}\overline{J}(\omega,x)\Big]_{x=0}^{x=u} - \frac{1}{\Gamma(\alpha)}\int_0^u x^{\alpha-1}\overline{J}(\omega,x)\,dx$$

$$= \frac{1}{\Gamma(\alpha+1)}u^{\alpha}\overline{J}(\omega,u) - \frac{1}{\Gamma(\alpha)}\overline{K}^{*}(\omega,u),$$

and the lemma will be proved if we show that

$$|\overline{K}^{*}(\omega,u)| = \left|\int_0^u x^{\alpha-1}\overline{J}(\omega,x)\,dx\right|$$

$$< A\omega^{\alpha-1}u^{\alpha}(1+\omega u)^{-\beta} + A[\omega]^{\alpha-\beta-1}(\omega-[\omega])^{\beta-1}.$$

If $0<\omega u\leqslant 1$ we have, from Lemma 48,

$$|\overline{K}^{*}(\omega,u)| < A\int_0^u x^{\alpha-1}\omega^{\alpha-1}\,dx + A[\omega]^{\alpha-\beta-1}(\omega-[\omega])^{\beta-1}\int_0^u x^{\alpha-1}\,dx$$

$$< Au^{\alpha}\omega^{\alpha-1} + A[\omega]^{\alpha-\beta-1}(\omega-[\omega])^{\beta-1}.$$

Also

$$\bar{K}^*(\omega,\pi) = \int_0^\pi x^{\alpha-1}dx\,\frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)}\int_x^\pi (t-x)^{-\alpha}\{\sum_{n<\omega}(\omega-n)^{\beta-1}n\cos nt\}dt$$

$$= \frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)}\int_0^\pi\{\sum_{n<\omega}(\omega-n)^{\beta-1}n\cos nt\}dt\int_0^t x^{\alpha-1}(t-x)^{-\alpha}dx,$$

the inversion of the order of integration being justified by Lemma 11. Thus

$$\bar{K}^*(\omega,\pi) = \omega^{-\beta-1}\Gamma(\alpha)\int_0^\pi\{\sum_{n<\omega}(\omega-n)^{\beta-1}n\cos nt\}dt$$

$$= 0.$$

Hence

$$\bar{K}^*(\omega,u) = -\int_u^\pi x^{\alpha-1}\bar{J}(\omega,x)dx,$$

and, if $\omega u \geq 1$ we have, by Lemma 48,

$$|\bar{K}^*(\omega,u)| < A\int_u^\pi x^{\alpha-1-\beta}\omega^{\alpha-1-\beta}dx + A\int_u^\pi x^{\alpha-1}[\omega]^{\alpha-\beta-1}(\omega-[\omega])^{\beta-1}dx$$

$$\leq A\,u^{\alpha-\beta}\omega^{\alpha-1-\beta} + A[\omega]^{\alpha-\beta-1}(\omega-[\omega])^{\beta-1}.$$

The result now follows at once.

CHAPTER 6.

The Absolute Summability of Fourier Series.

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6.1. General Remarks. We are now in a position to discuss the absolute summability[1] of a Fourier series. Theorems 16 and 17 below were proved by Bosanquet by the use of Cèsaro means. Throughout we shall employ Rieszian means, and it will appear that, while the proof of Theorem 16 is not any improvement on Bosanquet's proof, that of Theorem 17 is slightly simpler. We first state two classical theorems on Fourier series which we require in the proofs of the theorems.

6.2. Two Classical Results.

LEMMMA 50. If[2] the function $f(x)$ has period 2π and is integrable in the sense of Lebesgue over $(0, 2\pi)$, and if $g(x)$ is of bounded variation and $|g(x)|$ is integrable over $(0,\infty)$, then we may evaluate

$$\int_0^\infty f(x)\, g(x)\, dx$$

by substituting for $f(x)$ its Fourier Series and integrating term by term.

The same result is true for a finite range of integration (α,β) if $g(x)$ is of bounded variation in (α,β).

LEMMA 51. If[3] $f(x)$ is periodic and integrable in the sense of Lebesgue over $(0,2\pi)$ then α_n and β_n are $o(1)$. If $f(x)$ is, in addition, of bounded variation in $(0,2\pi)$ we have

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1) The question of the ordinary Cesàro summability of a Fourier series and its Allied series has been exhaustively studied by many writers. For references, see Bosanquet and Hyslop 8. 2) Hobson 20, 582-584. 3) Hobson, 20, 514-516.

$$\alpha_n = O\left(\frac{1}{n}\right), \quad \beta_n = O\left(\frac{1}{n}\right).$$

6. 3. Deduction from Function to Series.

THEOREM 16. If[1] $\varphi_\alpha(t)$ is of bounded variation in $(0,\pi)$, then the Fourier Series of $f(t)$, at the point $t=x$, is summable $|C,\beta|$ where $\beta>\alpha\geqslant 0$.

By Theorem 1 there will be no loss in generality if we suppose that

$$0 \leq h = [\alpha] \leqslant \alpha < \beta < h+1.$$

We divide the proof into two parts.

Case (i); $\beta>\alpha\geqslant 1$. By Lemmas 50, 43 and 41, we have

$$\int_0^\infty \gamma_{1+\beta}(\omega t)\,\varphi(t)\,dt = \sum_{\mu=0}^{\infty} a_\mu \int_0^\infty \gamma_{1+\beta}(\omega t)\cos \mu t\, dt$$

$$\int_0^\infty \gamma_{1+\beta}(\omega t)\,\varphi(t)\,dt = \sum_{\mu=0}^{\infty} a_\mu \int_0^\infty \gamma_{1+\beta}(\omega t)\cos \mu t\, dt$$

$$= \frac{\pi}{2\omega} \sum_{\mu<\omega} a_\mu \left(1-\frac{\mu}{\omega}\right)^\beta.$$

Thus

$$C_\beta(\omega) = \frac{2\omega}{\pi}\int_0^\infty \gamma_{1+\beta}(\omega t)\,\varphi(t)\,dt,$$

and, from Lemmas 22 and 39, it follows that, for $\omega>0$,

$$\tfrac{1}{2}\pi\beta^{-1}\frac{d}{d\omega}C_\beta(\omega) = \frac{\pi}{2\omega}\left\{C_{\beta-1}(\omega) - C_\beta(\omega)\right\}$$

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[1] Bosanquet, 6 & 7.

$$= \int_0^\infty \{ \gamma_\beta(\omega t) - \gamma_{1+\beta}(\omega t) \} \varphi(t)\,dt$$

$$= \int_0^\infty \bar{\gamma}_\beta'(\omega t)\varphi(t)\,dt.$$

Denote this integral by $I(\omega)$ and let

$$I_1(\omega) = \int_0^\pi \bar{\gamma}_\beta'(\omega t)\,\varphi(t)\,dt,$$

$$I_2(\omega) = \int_\pi^\infty \bar{\gamma}_\beta'(\omega t)\,\varphi(t)\,dt.$$

If $\rho = \min(\beta, 2)$ and $\omega \geq 1$ we have, by Lemma 40,

$$|I_2(\omega)| < A \sum_{s=1}^\infty \int_{(2s-1)\pi}^{(2s+1)\pi} (\omega t)^{-\rho} |\varphi(t)|\,dt$$

$$< A\,\omega^{-\rho} \sum_{s=1}^\infty \{(2s-1)\pi\}^{-\rho} \int_{-\pi}^{\pi} |\varphi(t)|\,dt,$$

since $\varphi(t)$ has period 2π. It follows that

$$|I_2(\omega)| < A\,\omega^{-\rho},$$

and, since $\rho > 1$,

$$\int_1^\infty |I_2(\omega)|\,d\omega < \infty.$$

We must now show that the same is true of $I_1(\omega)$. Integrate $I_1(\omega)$ by parts h times. We then obtain

$$I_1(\omega) = \Big[\sum_{\nu=1}^{h} (-1)^{\nu-1} \omega^{\nu-1} \Phi_\nu(t)\, \bar{\gamma}_\beta^{(\nu)}(\omega t) \Big]_{t=0}^{t=\pi}$$

$$+ (-1)^h \omega^h \int_0^\pi \bar{\gamma}_\beta^{(h+1)}(\omega t)\, \Phi_h(t)\,dt$$

$$= I_{1,1}(\omega) + I_{1,2}(\omega),$$

say. Now $\Phi_\nu(0)=0$ and $\Phi_\nu(\pi)$ is finite so that, by Lemma 40,

$$I_{1,1}(\omega) = \underline{O}(\omega^{h-1-\beta}) + \underline{O}(\omega^{-2}).$$

It follows that

$$\int_1^\infty |I_{1,1}(\omega)|\, d\omega < \infty.$$

By Lemma 30, we have

$$I_{1,2}(\omega) = \frac{(-1)^h \omega^h}{\Gamma(h+1-\alpha)} \int_0^\pi \bar{\gamma}_\beta^{(h+1)}(\omega t) \int_0^t (t-u)^{h-\alpha}\, \Phi_{\alpha-1}(u)\, du$$

$$= \int_0^\pi \Phi_{\alpha-1}(u)\, D(\omega, u, \pi),$$

by Lemma 45, the change in the order of integration being justified by Lemma 11. Integration by parts gives

$$I_{1,2}(\omega) = \Big[\Phi_\alpha(u)\, D(\omega,u,\pi)\Big]_{u=0}^{u=\pi} - \int_0^\pi \frac{1}{\Gamma(\alpha+1)} u^\alpha \varphi_\alpha(u) \frac{\partial}{\partial u} D(\omega,u,\pi)\, du$$

$$= -\Big[E(\omega,u,\pi)\, \varphi_\alpha(u)\Big]_{u\to 0}^{u=\pi} + \int_0^\pi E(\omega,u,\pi)\, \varphi_\alpha'(u)\, du.$$

Since $\varphi_\alpha(u)$ is of bounded variation in $(0,\pi)$, the limit $\varphi_\alpha(+0)$ is finite. Also $E(\omega,0,\pi)=0$. Hence, by Lemma 46,

$$I_{1,2}(\omega) = \underline{O}(\omega^{\alpha-1-\beta}) + \int_0^\pi E(\omega,u,\pi)\, \varphi_\alpha'(u)\, du,$$

and,

$$\int_1^\infty |I_{1,2}(\omega)|\, d\omega < A\int_1^\infty \omega^{\alpha-1-\beta} d\omega + A\int_0^\pi |\varphi_\alpha'(u)|\, du \left\{\int_1^{u^{-1}} u^\alpha \omega^{\alpha-1} d\omega + \int_{u^{-1}}^\infty u^{\alpha-\beta} \omega^{\alpha-1-\beta} d\omega\right\}$$

$$< A + A\int_0^\pi |\varphi_\alpha'(u)|\, du < \infty.$$

The first case of the theorem is therefore proved.

Case (ii); $0 \leq \alpha < 1$. The formula for $C'_\beta(\omega)$ which formed the basis of the previous proof was only valid for $\beta > 1$. Hence we require a separate examination for this case.

We have

$$\tfrac{1}{2}\pi B_{\beta-1}(\omega) = \int_0^\pi \varphi(t)\Big\{\sum_{n<\omega}(\omega-n)^{\beta-1} n \cos nt\Big\}dt.$$

Since $\varphi_\alpha(t)$ is of bounded variation in $(0,\pi)$ the limit $\varphi_\alpha(+0)$ is finite and therefore $\Phi_\alpha(+0)=0$. It follows,[1] by Lemmas 22 and 32, that, when ω is not a positive integer,

$$\tfrac{1}{2}\pi\beta^{-1}C'_\beta(\omega) = \frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)}\int_0^\pi\Big\{\sum_{n<\omega}(\omega-n)^{\beta-1} n\cos nt\Big\}dt\int_0^t (t-u)^{-\alpha}\,d\,\Phi_\alpha(u)$$

$$= \frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)}\int_0^\pi d\,\Phi_\alpha(u)\int_u^\pi (t-u)^{-\alpha}\Big\{\sum_{n<\omega}(\omega-n)^{\beta-1} n\cos nt\Big\}dt,$$

the interchange in the order of integration being justified by Lemma 17. Hence, when ω is not a positive integer.

$$\tfrac{1}{2}\pi\beta^{-1}C'_\beta(\omega) = \int_0^\pi \bar{J}(\omega,u)\,d\,\Phi_\alpha(u)$$

$$= \Big[\Phi_\alpha(u)\,\bar{J}(\omega,u)\Big]_{u\to 0}^{u=\pi} - \int_0^\pi \frac{u^\alpha\varphi_\alpha(u)}{\Gamma(\alpha+1)}\frac{\partial}{\partial u}\bar{J}(\omega,u)\,du$$

$$= -\Big[\bar{K}(\omega,u)\,\varphi_\alpha(u)\Big]_{u\to 0}^{u=\pi} + \int_0^\pi \bar{K}(\omega,u)\,d\,\varphi_\alpha(u)$$

$$= -\bar{K}(\omega,\pi)\,\varphi_\alpha(\pi) + \int_0^\pi \bar{K}(\omega,u)\,d\,\varphi_\alpha(u).$$

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[1] This transformation is unnecessary when $\alpha=0$.

Now, by Lemma 49,

$$\int_1^\infty |\bar{K}(\omega,u)|\,d\omega < A\int_1^{u^{-1}} u^\alpha \omega^{\alpha-1}\,d\omega + A\int_{u^{-1}}^\infty u^{\alpha-\beta}\omega^{\alpha-1-\beta}\,d\omega$$
$$+ A\sum_{n=1}^{\infty}\int_n^{n+1} n^{\alpha-\beta-1}(\omega-n)^{\beta-1}\,d\omega$$
$$= O(1),$$

uniformly for $0<u\leq\pi$. It at once follows that

$$\int_1^\infty |C_\beta'(\omega)|\,d\omega < A + A\int_0^\pi |d\varphi_\alpha(u)| < \infty.$$

The proof of the theorem is therefore completed.

The most interesting case of the theorem occurs when $\alpha=0$ and, although the proof in this case is included in that of Case (ii) above, it is perhaps advisable to treat it separately.

We have, if $N=[\omega]$ and $0<\beta<1$,

$$\tfrac{1}{2}\pi\, B_{\beta-1}(\omega) = \int_0^\pi \varphi(t)\,\{\sum_{n<\omega}(\omega-n)^{\beta-1}\, n\cos nt\}\,dt$$
$$= -\int_0^\pi \{\sum_{n<\omega}(\omega-n)^{\beta-1}\sin nt\}\,d\varphi(t)$$
$$= -\int_0^\pi \bar{G}_\beta(\omega,t)\,d\varphi(t) - \int_0^\pi (\omega-N)^{\beta-1}\sin Nt\,d\varphi(t),$$

whence, by Lemma 47,

$$\int_1^\infty |C_\beta'(\omega)|\,d\omega \leq A\int_0^\pi |d\varphi(t)|\,\Big\{\int_1^{t^{-1}}\omega^{-\beta-1}\omega^{\beta+1}t\,d\omega + \int_{t^{-1}}^\infty \omega^{-\beta-1}t^{-\beta}\,d\omega$$
$$+ \sum_{n=1}^\infty\int_n^{n+1}(\omega-n)^{\beta-1}\omega^{-\beta-1}\sin nt\,d\omega\Big\}$$
$$< \infty.$$

6.4. Deduction from Series to Function. We now consider the converse problem.

THEOREM 17. If[1] the Fourier series of the function $f(t)$, at the point $t=x$, is summable $|C,\alpha|$, then $\varphi_\beta(t)$ is of bounded variation in $(0,\infty)$ where $\beta-1>\alpha\geq 0$.

Since the Fourier series of $\varphi(t)$ is

$$\sum_{n=0}^{\infty} a_n \cos nt,$$

and since $(1-u)^{\beta-1}$ is, for $\beta>1$, of bounded variation in $(0,1)$, we have, by Lemma 50 and the definitions of $\varphi_\beta(t)$ and $\gamma_\beta(t)$,

$$\beta^{-1}\varphi_\beta(t) = \int_0^1 (1-u)^{\beta-1}\varphi(tu)\,du$$

$$= \sum_{n=0}^{\infty} a_n \int_0^1 (1-u)^{\beta-1}\cos ntu\,du$$

$$= \sum_{n=0}^{\infty} a_n \gamma_\beta(nt).$$

The series obtained by formally differentiating the right hand side is

$$\sum_{n=1}^{\infty} b_n \gamma_\beta'(nt)$$

This series is uniformly convergent for $t\geq\varepsilon>0$ for, if $\beta<3$, we have, by Lemma 40 and Theorem 2,

$$\sum_{n=1}^{\infty}|b_n||\gamma_\beta'(nt)| < A\varepsilon^{-\beta}\sum_{n=1}^{\infty} n^{1-\beta}|a_n| < A\sum_{n=1}^{\infty} n^{-\alpha}|a_n| < \infty,$$

[1] Bosanquet, 6, 7.

while, if $\beta > 3$ we have, by Lemmas 40 and 51,

$$\sum_{n=1}^{\infty} |b_n| |\gamma_\beta'(nt)| < A\varepsilon^{-3} \sum_{n=1}^{\infty} n^{-2} < \infty .$$

It follows that, for $t > 0$,

$$(6.41) \quad \beta^{-1} \varphi_\beta'(t) = \sum_{n=1}^{\infty} b_n \gamma_\beta'(nt)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} b_n \gamma_\beta'(nt)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N-1} B_n \left[\gamma_\beta'(nt) - \gamma_\beta'\{(n+1)t\}\right]$$

$$+ \lim_{N\to\infty} B_N \gamma_\beta'(Nt).$$

Now, if $\beta < 3$, $t \geqslant \varepsilon > 0$,

$$\gamma_\beta'(Nt) = O(N^{-\beta}),$$

and by Theorem 2,

$$|B_N| \leqslant \sum_{n=1}^{N} n|a_n| \leqslant \sum_{n=1}^{N} n^{1+\alpha} n^{-\alpha} |a_n| = O(N^{1+\alpha}),$$

while, if $\beta \geqslant 3$, $t \geqslant \varepsilon > 0$,

$$|B_N| |\gamma_\beta'(Nt)| = O(N^2) . O(N^{-3}) = O(N^{-1}),$$

by Lemmas 40 and 51. Thus, since $\beta > \alpha+1$, we have, for $t > 0$,

$$\beta^{-1} \varphi_\beta'(t) = \sum_{n=1}^{\infty} B_n \left[\gamma_\beta'(nt) - \gamma_\beta'\{(n+1)t\}\right]$$

$$= -t \sum_{n=1}^{\infty} B_n \int_n^{n+1} \gamma_\beta''(ut)\,du$$

$$= -t \int_0^{\infty} B(u)\, \gamma_\beta''(ut)\,du.$$

By Lemma 35 there is no loss in generality in supposing that

$$h = [\alpha] \leqslant \alpha < \beta - 1 < h+1.$$

We shall also suppose in this proof that $\alpha > 0$. The interesting case $\alpha = 0$ will be considered separately.

Integrating by parts h times we have, for $t > 0$,

$$\beta^{-1}\varphi_\beta'(t) = \Big[\sum_{\nu=1}^{h} \frac{(-1)^\nu}{\nu!} B_\nu(u)\, t^\nu \gamma_\beta^{(\nu+1)}(ut)\Big]_{u=0}^{u\to\infty}$$

$$+ (-1)^{h+1}\frac{t^{h+1}}{\Gamma(h+1)}\int_0^\infty B_h(u)\,\gamma_\beta^{(h+2)}(ut)\,du.$$

The integrated terms vanish when $u=0$ and, as $u\to\infty$, we have by Lemma 51,

$$|B_\nu(u)| = \Big|\sum_{n<u}(u-n)^\nu n a_n\Big| = o(u^{\nu+2}),$$

and, by Lemmas 22 and 19,

$$|B_h(u)| = o(u^{\alpha+1}),$$

while, by Lemma 40,

$$\gamma_\beta^{(\nu+1)}(ut) = O(u^{-\nu-3}), \quad \nu = 1, 2, \cdots, h-2,$$

$$= O(u^{-\beta}), \quad \nu = h-1, h.$$

It therefore follows that

$$\beta^{-1}\varphi_\beta'(t) = \frac{(-1)^{h+1}t^{h+1}}{\Gamma(h+1)}\int_0^\infty B_h(u)\,\gamma_\beta^{(h+2)}(ut)\,du,$$

whence, by Lemma 18,

$$\beta^{-1}\varphi_\beta'(t) = \frac{(-1)^{h+1}t^{h+1}}{\Gamma(\alpha)\Gamma(h+1-\alpha)}\int_0^\infty \gamma_\beta^{(h+2)}(ut)\,du\int_0^u (u-v)^{h-\alpha} B_{\alpha-1}(v)\,dv$$

$$\int_0^\infty |\varphi_\beta'(t)|dt < A\sum_{n=0}^\infty |a_n| \left\{\int_0^{n^{-1}} n|\gamma_\beta'(nt)|dt + \int_{n^{-1}}^\infty n|\gamma_\beta'(nt)|dt\right\}$$

$$< A\sum_{n=0}^\infty |a_n| \left\{\int_0^{n^{-1}} n\,dt + \int_{n^{-1}}^\infty n^{1-\beta}t^{-\beta}dt\right\},$$

$$= \frac{(-1)^{h+1}t^{h+1}}{\Gamma(\alpha)\Gamma(h+1-\alpha)}\int_0^\infty B_{\alpha-1}(v)\,dv\int_v^\infty (u-v)^{h-\alpha}\gamma_\beta^{(h+2)}(ut)\,du$$

$$= \int_0^\infty \alpha v^{-\alpha-1}B_{\alpha-1}(v)\,L(t,v,\infty)\,dv,$$

the inversion of the order of integration being justified by Lemmas 11 and 22.

It now follows by Lemma 44 that

$$\int_0^\infty |\varphi_\beta'(t)|dt < A\int_0^\infty \left|\frac{d}{dv}C_\alpha(v)\right|dv\left\{\int_0^{v^{-1}} t^\alpha v^{\alpha+1}dt + \int_{v^{-1}}^\infty t^{\alpha-\beta}v^{\alpha+1-\beta}dt\right\}$$

$$< \infty .$$

The theorem is therefore proved when $\alpha > 0$. The case $\alpha = 0$ deserves special consideration and we therefore give a separate proof. From (6.41) we have

$$\int_0^\infty |\varphi_\beta'(t)|dt < A\sum_{n=0}^\infty |a_n| \left\{\int_0^{n^{-1}} n|\gamma_\beta'(nt)|dt + \int_{n^{-1}}^\infty n|\gamma_\beta'(nt)|dt\right\}$$

$$< A\sum_{n=0}^\infty |a_n| \left\{\int_0^{n^{-1}} n\,dt + \int_{n^{-1}}^\infty n^{1-\beta}t^{-\beta}dt\right\},$$

by Lemma 40. Thus

$$\int_0^\infty |\varphi_\beta'(t)|\,dt < A\sum_{n=0}^\infty |a_n| < \infty.$$

6.5. A General Statement of the Preceding Results.

We may summarise the results of these two theorems as follows.

THEOREM 18. A necessary and sufficient condition that the Fourier series of $f(t)$ be summable $|C,k|$, at the point $t=x$, for some k is that the function $\varphi_\lambda(t)$ be of bounded variation in $(0,\pi)$ for some λ.

6.6. A Particular Case of the Preceding Theorems. We now show that Theorems 16 and 17 are 'best possible' when $\alpha = 0$ in the sense that they are not necessarily true for $\delta = 0$.

THEOREM 19. There[1] exists a function of bounded variation in $(0,\pi)$ whose Fourier series is not absolutely convergent at the point $t = 0$.

Consider the even function,

$$\varphi(t) = \tfrac{1}{4}\pi \operatorname{sgn}\left(\tfrac{1}{2}\pi - |t|\right),$$

where $-\pi \leq t \leq \pi$. Clearly $\varphi(t)$ is of bounded variation in $(0,\pi)$, and if it's Fourier series is

$$\sum_{n=0}^{\infty} a_n \cos nt,$$

we have

$$\begin{aligned} a_m &= \frac{2}{\pi}\cdot\frac{\pi}{4}\int_0^{\pi} \operatorname{sgn}\left(\tfrac{1}{2}\pi - |t|\right)\cos mt\,dt \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos mt\,dt - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \cos mt\,dt \\ &= \frac{1}{m}\sin\frac{m\pi}{2}. \end{aligned}$$

Thus

$$a_{2n-1} = (-1)^{n-1}\frac{1}{2n-1}, \quad a_{2n} = 0,$$

so that the Fourier series of $\varphi(t)$ at $t = 0$ is

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{2n-1},$$

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[1] Bosanquet, 6.

and this series is not absolutely convergent.

To obtain a similar result for Theorem 17 we require two elementary lemmas.

LEMMA 52. <u>If $a>b>0$, then</u>

$$(6.61) \qquad \left| \int_0^{\pi} \frac{\sin at \sin bt}{t}\, dt \right| < \frac{b}{a-b}.$$

Denote the left hand side of (6.61) by I. Then

$$I = \left| \tfrac{1}{2} \int_0^{\pi} \frac{\cos(a-b)t - \cos(a+b)t}{t}\, dt \right|$$

$$= \tfrac{1}{2} \left| \int_0^{\pi} dt \int_{a-b}^{a+b} \sin xt\, dx \right|$$

$$= \tfrac{1}{2} \left| \int_{a-b}^{a+b} \left(\frac{1}{x} - \frac{1}{x}\cos \pi x \right) dx \right| < \frac{b}{a-b}.$$

LEMMA 53. <u>If m is a positive integer then, as $m \to \infty$,</u>

$$\int_0^{\pi} \frac{\sin^2 mt}{t}\, dt \sim \tfrac{1}{2}\log m.$$

We have

$$\int_0^{\pi} \frac{\sin^2 mt}{t}\, dt = \int_0^{m\pi} \frac{\sin^2 t}{t}\, dt$$

$$= \sum_{\nu=1}^{m} \int_{(\nu-1)\pi}^{\nu\pi} \frac{\sin^2 t}{t}\, dt$$

$$= \sum_{\nu=1}^{m} \int_0^{\pi} \frac{\sin^2 t}{(\nu-1)\pi + t}\, dt$$

$$= \int_0^{\pi} \frac{\sin^2 t}{t}\, dt + \int_0^{\pi} \sin^2 t \left\{ \sum_{\nu=1}^{m-1} \frac{1}{\nu\pi + t} \right\} dt.$$

Now

$$\sum_{\nu=1}^{m-1} \frac{1}{t+\nu\pi} = \int_1^{m-1} \frac{dx}{t+x\pi} + \lambda,$$

where

$$0 < \lambda < \frac{1}{t+\pi} < \frac{1}{\pi},$$

and

$$\int_1^{m-1} \frac{dx}{t+x\pi} = \frac{1}{\pi}\Big[\log(t+x\pi)\Big]_{x=1}^{x=m-1}$$

$$= \frac{1}{\pi}\log m + O(1),$$

uniformly for $0 \leq t \leq \pi$. Hence

$$\int_0^{\pi} \frac{\sin^2 mt}{t}\, dt \sim \frac{1}{\pi}\log m \int_0^{\pi} \sin^2 t\, dt = \frac{1}{2}\log m.$$

THEOREM 20. <u>There[1] exists a function $\varphi(t)$ whose Fourier series is absolutely convergent for $t=0$, but which is such that $\varphi_1(t)$ is not of bounded variation in $(0,\pi)$.</u>

Let

$$\varphi(t) = \sum_{m=0}^{\infty} \alpha_m \cos \lambda_m t,$$

where the series $\sum |\alpha_m|$ is convergent and λ_m is an increasing sequence of positive integers satisfying the relations

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[1] A brief sketch of this proof was given by Bosanquet, 6.

(6.62) $\qquad \lambda_{m+1} > (1+\theta)\lambda_m, \ \theta > 0,$

(6.63) $\qquad d_m \log \lambda_m \to \infty.$

For example, we may take $d_m = m^{-2}$, $\lambda_m = 2^{m^3}$. In these circumstances the Fourier series of $\varphi(t)$ at any point in $(-\pi,\pi)$ is absolutely convergent.

Now

$$\varphi_1(t) = t^{-1}\int_0^t \varphi(u)\,du = \sum_{m=0}^{\infty} \frac{d_m}{\lambda_m t}\sin\lambda_m t \quad , \ 0 < t \leq \pi,$$

and, if $\varphi_1(t)$ is defined in $(-\pi,0)$ so as to be an odd function, the Fourier series of $\varphi_1(t)$ is $\sum b_n \sin nt$ where

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nt \sum_{m=0}^{\infty} \frac{d_m}{\lambda_m t}\sin\lambda_m t$$

$$= \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{d_m}{\lambda_m}\int_0^{\pi}\frac{\sin nt \sin\lambda_m t}{t}\,dt.$$

The change in the order of integration and summation will be justified if we show that, for each fixed value of n,

$$\lim_{\varepsilon \to 0}\sum_{m=0}^{\infty}\frac{d_m}{\lambda_m}\int_0^{\varepsilon}\frac{\sin nt \sin\lambda_m t}{t}\,dt = 0.$$

Choose ε such that $0 < 2n\varepsilon < \pi$. Then

$$\left|\sum_{m=0}^{\infty}\frac{d_m}{\lambda_m}\int_0^{\varepsilon}\frac{\sin nt \sin\lambda_m t}{t}\,dt\right| \leq \sum_{m=0}^{\infty}\frac{|d_m|}{\lambda_m}\int_0^{\varepsilon} n\,|\sin\lambda_m t|\,dt$$

$$< n\varepsilon\sum_{m=0}^{\infty}\frac{|d_m|}{\lambda_m} < An\varepsilon.$$

Hence the interchange is justified.

Returning to the expression for b_n we have

$$b_{\lambda_n} = \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{d_m}{\lambda_m} \int_0^{\pi} \frac{\sin \lambda_n t \sin \lambda_m t}{t} dt$$

$$= \frac{2}{\pi} \frac{d_n}{\lambda_n} \int_0^{\pi} \frac{\sin^2 \lambda_n t}{t} dt + E_1 + E_2,$$

where

$$E_1 = \frac{2}{\pi} \sum_{m=n+1}^{\infty} \frac{d_m}{\lambda_m} \int_0^{\pi} \frac{\sin \lambda_n t \sin \lambda_m t}{t} dt,$$

$$E_2 = \frac{2}{\pi} \sum_{m=0}^{n-1} \frac{d_m}{\lambda_m} \int_0^{\pi} \frac{\sin \lambda_n t \sin \lambda_m t}{t} dt.$$

From Lemma 52 and (6.62) we have

$$|E_1| \leq \frac{2}{\pi} \sum_{m=n+1}^{\infty} \frac{|d_m|}{\lambda_m} \frac{\lambda_n}{\lambda_m - \lambda_n}$$

$$< \frac{2}{\pi} \sum_{m=n+1}^{\infty} \frac{\lambda_n}{(1+\theta)\lambda_n - \lambda_n} \frac{|d_m|}{\lambda_n} < \frac{A}{\lambda_n},$$

and

$$|E_2| \leq \frac{2}{\pi} \sum_{m=0}^{n-1} \frac{|d_m|}{\lambda_m} \frac{\lambda_m}{\lambda_n - \lambda_m}$$

$$= \frac{2}{\pi \lambda_n} \sum_{m=0}^{n-1} \frac{|d_m|}{\left(1 - \frac{\lambda_m}{\lambda_n}\right)}$$

$$< \frac{4(1+\theta)}{\theta \lambda_n} \sum_{m=0}^{n-1} |d_m| < \frac{A}{\lambda_n}$$

It therefore follows from Lemma 53 that

$$b_{\lambda_n} \sim \frac{1}{\pi} \frac{d_n}{\lambda_n} \log \lambda_n \neq O\left(\frac{1}{\lambda_n}\right),$$

by (6.63) . It follows from Lemma 51 that $\varphi_1(t)$ is not of bounded variation in $(0, \pi)$.

CHAPTER 7.

The Absolute Summability of the Allied Series.

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7.1. General Remarks. In this chapter we obtain results analogous to these of the preceding chapter for the Allied Series. Some slight additional complications arise in this case. First we require an extension of Lemma 50.

LEMMA 54. If[1] $f(x)$ has period 2π and is integrable in the sense of Lebesgue over $(0, 2\pi)$ and if $g(x)$ is of bounded variation in $(0, \infty)$ and tends to zero as x tends to infinity, then

$$\int_0^\infty f(x)\, g(x)\, dx$$

may be evaluated by substituting for $f(x)$ its Fourier series and integrating term by term, provided that the Fourier series of $f(x)$ has no constant term.

7.2. Deduction from Function to Series.

THEOREM 21. If[2]

$$\int_0^\pi \frac{|\psi_\alpha(t)|}{t}\, dt < \infty,$$

then the Allied series is summable $|C, \beta|$, at the point $t = x$, for $\beta - 1 > \alpha \geq 0$.

If $\beta > 0$ we have, by Lemmas 54 and 43,

$$\int_0^\infty \bar{\gamma}_{1+\beta}(\omega t)\, \psi(t)\, dt = \sum_{\mu=1}^\infty \bar{a}_\mu \int_0^\infty \bar{\gamma}_{1+\beta}(\omega t) \sin \mu t\, dt$$

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1) Hobson, 20, 583.
2) Bosanquet and Hyslop, 8.

$$= \omega^{-1} \sum_{\mu=1}^{\infty} \bar{a}_\mu \int_0^\infty \bar{\gamma}_{1+\beta}(t) \sin \frac{\mu}{\omega} t \, dt$$

$$= \frac{\pi}{2\omega} \sum_{\mu<\omega} \left(1 - \frac{\mu}{\omega}\right)^\beta \bar{a}_\mu$$

$$= \frac{\pi}{2\omega} \bar{C}_\beta(\omega).$$

Hence, by Lemmas 22 and 39, if $\beta > 1$,

$$-\tfrac{1}{2}\pi \beta^{-1} \bar{C}_\beta'(\omega) = \int_0^\infty \{\bar{\gamma}_{1+\beta}(\omega t) - \bar{\gamma}_\beta(\omega t)\} \psi(t) \, dt$$

$$= \int_0^\infty \gamma_\beta'(\omega t) \psi(t) \, dt$$

$$= I_1(\omega) + I_2(\omega),$$

where

$$I_1(\omega) = \int_0^\pi \gamma_\beta'(\omega t) \psi(t) \, dt,$$

$$I_2(\omega) = \int_\pi^\infty \gamma_\beta'(\omega t) \psi(t) \, dt.$$

Now, if $\rho = \min(\beta, 3)$,

$$|I_2(\omega)| \leq A \sum_{s=1}^{\infty} \int_{(2s-1)\pi}^{(2s+1)\pi} (\omega t)^{-\rho} |\psi(t)| \, dt$$

$$< A \omega^{-\rho} \sum_{s=1}^{\infty} \{(2s-1)\pi\}^{-\rho} \int_{-\pi}^{\pi} |\psi(t)| \, dt$$

$$< A\omega^{-\beta},$$

so that, if $\beta > 1$,

$$\int_1^\infty |I_2(\omega)|\, d\omega < \infty.$$

By Theorem 1 there is no loss in generality in supposing that

$$0 \leq h = [\alpha] \leq \alpha < \beta - 1 < h + 1.$$

Integrating $I_1(\omega)$ by parts $h+1$ times we obtain

$$I_1(\omega) = \Big[\sum_{\nu=1}^{h+1} (-1)^{\nu-1} \omega^{\nu-1}\, \bar{\Psi}_\nu(t)\, \gamma_\beta^{(\nu)}(\omega t) \Big]_{t=0}^{t=\pi}$$

$$+ (-1)^{h+1} \omega^{h+1} \int_0^\pi \gamma_\beta^{(h+2)}(\omega t)\, \bar{\Psi}_{h+1}(t)\, dt$$

$$= I_{1,1}(\omega) + I_{1,2}(\omega),$$

say,

Now

$$I_{1,1}(\omega) = \underline{O}(\omega^{h-\beta}),$$

and therefore

$$\int_1^\infty |I_{1,1}(\omega)|\, d\omega < \infty,$$

since

$$\beta > h+1.$$

Also, by Lemmas 30 and 11.

$$I_{1,2}(\omega) = \frac{(-1)^{h+1}\omega^{h+1}}{\Gamma(h+1-\alpha)}\int_0^\pi \gamma_\beta^{(h+2)}(\omega t)\,dt\int_0^t (t-u)^{h-\alpha}\,\Psi_\alpha(u)\,du$$

$$= \frac{(-1)^{h+1}\omega^{h+1}}{\Gamma(h+1-\alpha)}\int_0^\pi \Psi_\alpha(u)\,du\int_u^\pi (t-u)^{h-\alpha}\,\gamma_\beta^{(h+2)}(\omega t)\,dt$$

$$= \int_0^\pi \frac{\psi_\alpha(u)}{u}\,L(\omega,u,\pi)\,du,$$

the inversion of the order of integration being justified by Lemma 11. It follows from Lemma 44 that

$$\int_1^\infty |I_{1,2}(\omega)|\,d\omega \le \int_1^\infty d\omega\int_0^\pi \frac{|\psi_\alpha(u)|}{u}\,|L(\omega,u,\pi)|\,du$$

$$= \int_0^\pi \frac{|\psi_\alpha(u)|}{u}\,du\int_1^\infty |L(\omega,u,\pi)|\,d\omega$$

$$< A\int_0^\pi \frac{|\psi_\alpha(u)|}{u}\,du\left\{\int_1^{u^{-1}} u^{\alpha+1}\omega^\alpha\,d\omega + \int_{u^{-1}}^\infty u^{\alpha+1-\beta}\omega^{\alpha-\beta}\,d\omega\right\}$$

$$< \infty.$$

The result therefore follows.

As in Chapter 6 it is worth while to examine separately the theorems of this chapter when $\alpha = 0$.

We have, from the preceding proof,

$$\tfrac{1}{2}\pi\,\beta^{-1}\,\bar{C}_\beta'(\omega) = I_1(\omega) + I_2(\omega),$$

where

$$\int_1^\infty |I_2(\omega)|\,d\omega < \infty,$$

and

$$I_1(\omega) = \int_0^\pi \gamma_\beta'(\omega t)\,\psi(t)\,dt.$$

If we suppose, as we may without loss of generality, that $1<\beta<2$, we have

$$\int_1^\infty |I_1(\omega)|\,d\omega \leq A\int_0^\pi |\psi(t)|\,dt\left\{\int_1^{t^{-1}} d\omega + \int_{t^{-1}}^\infty \omega^{-\beta}t^{-\beta}\,d\omega\right\}$$

$$< A\int_0^\pi \frac{|\psi(t)|}{t}\,dt < \infty.$$

The particular case of the theorem therefore follows.

We now prove a theorem similar in type to Theorem 21 but which implies as conclusion the summability of the Allied series when $0<\beta\leq 1$.

THEOREM 22. If[1] $0<\alpha<1$, $\Psi_\alpha(+0)=0$, $\Psi_\alpha(t)$ is of bounded variation in $(0,\pi)$ and

$$\int_0^\pi t^{-\alpha}\,|d\,\Psi_\alpha(t)| < \infty,$$

then the Allied series is summable $|C,\beta|$, at the point $t=x$, for $\beta>\alpha$.

By Theorem 1, there will be no loss in generality if we suppose that $0<\alpha<\beta<1$.

We have

$$\bar{a}_n = \frac{2}{\pi}\int_0^\pi \psi(t)\sin nt\,dt,$$

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1) Bosanquet and Hyslop, 8.

so that

$$\tfrac{1}{2}\pi \bar{B}_{\beta-1} = \int_0^\pi \psi(t) \{ \sum_{n<\omega} (\omega-n)^{\beta-1} n \sin nt \} dt.$$

Thus, if ω is not an integer, we have, by Lemma 32,

$$\tfrac{1}{2}\pi \beta^{-1} \bar{C}'_\beta(\omega) = \frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)} \int_0^\pi \{ \sum_{n<\omega} (\omega-n)^{\beta-1} n \sin nt \} dt \int_0^t (t-u)^{-\alpha} d\,\Psi_\alpha(u)$$

$$= \frac{\omega^{-\beta-1}}{\Gamma(1-\alpha)} \int_0^\pi d\,\Psi_\alpha(u) \int_u^\pi (t-u)^{-\alpha} \{ \sum_{n<\omega} (\omega-n)^{\beta-1} n \sin nt \} dt,$$

the interchange in the order of integration being justified by Lemma 17. It follows that

$$\tfrac{1}{2}\pi \beta^{-1} \bar{C}'_\beta(\omega) = -\int_0^\pi u^\alpha J(\omega,u)\, u^{-\alpha} d\,\Psi_\alpha(u),$$

and

$$\int_1^\infty |\bar{C}'_\beta(\omega)| d\omega \leq A \int_1^\infty d\omega \int_0^\pi u^\alpha |J(\omega,u)|\, u^{-\alpha} |d\,\Psi_\alpha(u)|$$

$$\leq A \int_0^\pi u^{-\alpha} |d\,\Psi_\alpha(u)| \int_1^\infty u^\alpha |J(\omega,u)| d\omega.$$

Now, by Lemma 48,

$$\int_1^\infty u^\alpha |J(\omega,u)| d\omega = \underline{O}\{ \int_1^{u^{-1}} u^\alpha \omega^{\alpha-1} d\omega \} + \underline{O}\{ \int_{u^{-1}}^\infty u^{\alpha-\beta} \omega^{\alpha-1-\beta} d\omega \}$$

$$+ \underline{O}\{ \int_1^\infty [\omega]^{\alpha-\beta-1} (\omega-[\omega])^{\beta-1} d\omega \}$$

$$= \underline{O}(1) + \underline{O}(1) + \underline{O}\{ \sum_{N=1}^\infty N^{\alpha-\beta-1} \int_N^{N+1} (\omega-N)^{\beta-1} d\omega \}$$

$$= \underline{O}(1),$$

uniformly for $0<u<\pi$.

The theorem now follows at once.

It will be shown later on that this theorem is false when $\alpha=0$.

4.3. <u>Deduction from Series to Function</u>. We now consider the converse problem.

THEOREM 23. <u>If [1] the Allied series is summable $|C,\alpha|$ at the point $t=x$, then</u>

$$\int_0^\infty \frac{|\psi_\beta(t)|}{t}\,dt<\infty,$$

<u>for $\beta>\alpha\geq 0$</u>.

Since the Fourier series of $\psi(t)$ is $\sum_1^\infty \bar{a}_n \sin nt$, and since $(1-u)^{\beta-1}$ is of bounded variation in $(0,1-\varepsilon), 0<\varepsilon<1$, we have, by Lemma 50,

$$\begin{aligned}\psi_{\beta,\varepsilon}(t) &= \beta\int_0^{1-\varepsilon}(1-u)^{\beta-1}\psi(tu)\,du\\ &= \beta\sum_{n=1}^\infty \bar{a}_n\int_0^{1-\varepsilon}(1-u)^{\beta-1}\sin ntu\,du\\ &= \beta\sum_{n=1}^\infty \bar{a}_n\,\bar{\gamma}_{\beta,\varepsilon}(nt).\end{aligned}$$

If $\beta\geq 1$ the same is true with $\varepsilon=0$, and we then have

$$\psi_\beta(t)=\beta\sum_{n=1}^\infty \bar{a}_n\,\bar{\gamma}_\beta(nt).$$

If $0<\beta<1$, we have, by Lemma 42,

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[1] Bosanquet and Hyslop, <u>8</u>.

$$\lim_{\varepsilon\to 0} \sum_{n=1}^{\infty} \bar{a}_n \{ \bar{\gamma}_\beta(nt) - \bar{\gamma}_{\beta,\varepsilon}(nt) \} = \lim_{\varepsilon\to 0} \{ \underline{O}(\sum_{n=1}^{N} |\bar{a}_n| \varepsilon^\beta)$$

$$+ \underline{O}(\sum_{n=N+1}^{\infty} |\bar{a}_n| n^{-\beta})$$

$$= \underline{O}(\sum_{n=N+1}^{\infty} |\bar{a}_n| n^{-\beta}).$$

The left hand side is independent of N and, by Theorem 2, the series $\sum |\bar{a}_n| n^{-\alpha}$ is convergent. Hence, for $0 \leq \alpha < \beta < 1$,

$$\lim_{\varepsilon\to 0} \psi_{\beta,\varepsilon}(t) = \beta \sum_{n=1}^{\infty} \bar{a}_n \bar{\gamma}_\beta(nt).$$

It follows that, for $\beta > \alpha \geq 0$,

$$\psi_\beta(t) = \beta \sum_{n=1}^{\infty} \bar{a}_n \bar{\gamma}_\beta(nt),$$

provided that the integral for $\psi_\beta(t)$ is interpreted in the Cauchy sense.

By Lemma 35 there will be no loss in generality in supposing that

$$0 \leq h = [\alpha] \leq \alpha < \beta < h+1.$$

For convenience we shall also suppose that $\alpha > 0$ and prove separately the case $\alpha = 0$.

Since $\bar{A}_n = \underline{O}(n^\sigma)$, where $\sigma = \min(\alpha, 1)$ and, for every fixed positive t, $\bar{\gamma}_\beta(nt) = O(n^{-\tau})$, where $\tau = \min(\beta, 1)$, we have, by partial summation,

$$\beta^{-1}\psi_\beta(t) = -t\sum_{n=1}^{\infty}\bar{A}_n\int_n^{n+1}\bar{\gamma}_\beta'(ut)\,du$$
$$= -t\int_0^\infty \bar{\gamma}_\beta'(ut)\,\bar{A}(u)\,du.$$

Integrating by parts h times we obtain

$$-\beta^{-1}t^{-1}\psi_\beta(t) = \Big[\sum_{\nu=1}^{h}(-1)^{\nu-1}t^{\nu-1}\{\Gamma(\nu+1)\}^{-1}\bar{A}_\nu(u)\bar{\gamma}_\beta^{(\nu)}(ut)\Big]_{u=0}^{u\to\infty}$$
$$+\frac{(-1)^h t^h}{\Gamma(h+1)}\int_0^\infty \bar{\gamma}_\beta^{(h+1)}(ut)\,\bar{A}_h(u)\,du.$$

Now $\bar{A}_\nu(0)=0$ and, for each fixed positive t, as $u\to\infty$, we have, by Lemmas 51 and 40,

$$\bar{A}_\nu(u)\,\bar{\gamma}_\beta^{(\nu)}(ut) = o(u^{\nu+1})\,O(u^{-\nu-1}) = o(1),$$

for $\nu=1,2,\dots,h-1$. Also, by Lemmas 19 and 40,

$$\bar{A}_h(u)\,\bar{\gamma}_\beta^{(h)}(ut) = o(u^\alpha)\,O(u^{-\beta}) = o(1).$$

Hence the integration terms vanish. We then have, by Lemma 18,

$$-\beta^{-1}t^{-1}\psi_\beta(t) = \frac{(-1)^h t^h}{\Gamma(h+1)}\int_0^\infty \bar{\gamma}_\beta^{(h+1)}(ut)\,\bar{A}_h(u)\,du$$
$$= \frac{(-1)^h t^h}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)}\int_0^\infty \bar{\gamma}_\beta^{(h+1)}(ut)\,du\int_0^u (u-v)^{h-\alpha}\,d\bar{A}_\alpha(v)$$

$$= \frac{(-1)^h t^h}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)} \int_0^\infty d\bar{A}_\alpha(v) \int_v^\infty (u-v)^{h-\alpha} \bar{\gamma}_\beta^{(h+1)}(ut)\,du$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty D(t,v,\infty)\, d\bar{A}_\alpha(v),$$

provided that the inversion of the order of integration can be justified.

To justify the inversion it will be sufficient to show that, as X tends to infinity.

$$\int_0^X d\bar{A}_\alpha(v) \int_X^\infty (u-v)^{h-\alpha} \bar{\gamma}_\beta^{(h+1)}(ut)\,du \to 0,$$

for every fixed positive t. Writing

$$\int_X^\infty (u-v)^{h-\alpha} \bar{\gamma}_\beta^{(h+1)}(ut)\,du = I_1 + I_2,$$

where

$$I_1 = \int_X^{X+1} (u-v)^{h-\alpha} \bar{\gamma}_\beta^{(h+1)}(ut)\,du,$$

$$I_2 = \int_{X+1}^\infty (u-v)^{h-\alpha} \bar{\gamma}_\beta^{(h+1)}(ut)\,du,$$

we have, for every fixed positive t, if $v<X$,

$$|I_1| = O\left\{ \int_X^{X+1} (u-v)^{h-\alpha} u^{-\beta}\,du \right\} = O(X^{-\beta}),$$

and, by Lemma 10,

$$|I_2| \leq (X+1-v)^{h-\alpha} \max_{X'>X+1} \left| \int_{X+1}^{X'} \bar{\gamma}_\beta^{(h+1)}(ut)\,du \right| = O(X^{-\beta}),$$

uniformly for $0<v<X$. Also

$$X^{-\beta}\int_0^X |d\bar{A}_\alpha(v)| = X^{-\beta}\underline{O}\left\{\int_0^X v^\alpha |d\bar{C}_\alpha(v)|\right\} + X^{-\beta}\underline{O}\left\{\int_0^X \frac{|\bar{A}_\alpha(v)|}{v}dv\right\}$$

$$= \underline{O}(X^{\alpha-\beta})\int_0^\infty |d\bar{C}_\alpha(v)| + \underline{O}(X^{-\beta})\int_0^X v^{\alpha-1}dv$$

$$= \underline{O}(X^{\alpha-\beta}) = \underline{o}(1),$$

since $\sum' \bar{u}_n$ is summable $|R,n,\alpha|$ and $\beta>\alpha$.

Returning to the expression for $\psi_\beta(t)/t$ we obtain, on integration by parts,

$$\beta^{-1}t^{-1}\psi_\beta(t) = \left[-\frac{1}{\Gamma(\alpha+1)}\bar{A}_\alpha(v)\,D(t,v,\infty)\right]_{v=0}^{v\to\infty} + \frac{1}{\Gamma(\alpha+1)}\int_0^\infty v^\alpha \bar{C}_\alpha(v)\frac{\partial D}{\partial v}dv.$$

The integrated term vanishes since $\bar{A}_\alpha(0)=0$ and since, for fixed positive t, we have, by Lemma 45,

$$\bar{A}_\alpha(v)\,D(t,v,\infty) = \underline{O}(v^\alpha).\,\underline{O}(v^{-\beta}) = \underline{o}(1).$$

Integrating by parts again we have

$$\beta^{-1}t^{-1}\psi_\beta(t) = -\int_0^\infty E(t,v,\infty)\,d\bar{C}_\alpha(v),$$

the integrated term vanishing by Lemma 46.

It follows from Lemma 46 that

$$\int_0^\infty \frac{|\psi_\beta(t)|}{t}dt < A\int_0^\infty |d\bar{C}_\alpha(v)|\int_0^\infty |E(t,v,\infty)|\,dt$$

$$< A\int_0^\infty |d\bar{C}_\alpha(v)|\left\{\int_0^{v^{-1}} t^{\alpha-1}v^\alpha dt + \int_{v^{-1}}^\infty t^{\alpha-1-\beta}v^{\alpha-\beta}dt\right\}$$

$$< \infty.$$

In the case $\alpha=0$ we have, as in the previous proof,

$$\int_0^\infty \frac{|\psi_\beta(t)|}{t}\,dt \leq \beta\int_0^\infty t^{-1}dt \sum_{n=1}^\infty |\bar{a}_n|\,|\bar{\gamma}_\beta(nt)|$$

$$= \beta\sum_{n=1}^\infty |\bar{a}_n| \int_0^\infty |\bar{\gamma}_\beta(nt)|\,t^{-1}dt$$

$$= \beta\sum_{n=1}^\infty |\bar{a}_n| \left\{ \underline{O}\left(\int_0^{n^{-1}} n\,dt\right) + \underline{O}\left(\int_{n^{-1}}^\infty n^{-\beta}t^{-1-\beta}dt\right)\right\}$$

$$= \sum_{n=1}^\infty |\bar{a}_n| . \underline{O}(1) \quad < \infty.$$

The theorem is therefore completely proved.

At this stage it is convenient to show that Theorem 22 is false when $\alpha=0$.

Consider the function [1]

$$\psi(t) = \left(\log\frac{2\pi}{t}\right)^{-1}.$$

Clearly $\psi(t)$ is an integral for $\varepsilon \leq t \leq \pi$,

$$\psi'(t) = t^{-1}\left(\log\frac{2\pi}{t}\right)^{-2},$$

and

$$\lim_{\varepsilon\to 0}\int_\varepsilon^\pi |\psi'(t)|\,dt$$

is finite. It follows that $\psi(t)$ is of bounded variation in $(0,\pi)$. Moreover $\psi(+0)=0$, so that the hypothesis of Theorem 22 are satisfied. The function $t^{-1}|\psi_1(t)|$ however, is not integrable over $(0,\pi)$, for

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[1] Bosanquet and Hyslop, <u>8</u>.

$$\Psi_1(t) = \int_0^t \left(\log \frac{2\pi}{u}\right)^{-1} du,$$

$$\psi_1(t) = t^{-1}\int_0^t \left(\log \frac{2\pi}{u}\right)^{-1} du = \int_0^1 \left(\log \frac{2\pi}{ut}\right)^{-1} du,$$

and

$$\begin{aligned}\int_\varepsilon^\pi t^{-1}|\psi_1(t)|\,dt &= \int_\varepsilon^\pi t^{-1}\,dt \int_0^1 \left(\log \frac{2\pi}{ut}\right)^{-1} du \\ &= \int_0^1 du \int_\varepsilon^\pi \left(t \log \frac{2\pi}{ut}\right)^{-1} dt \\ &> \int_{\frac{1}{2}}^1 du \int_\varepsilon^\pi \left(t \log \frac{2\pi}{ut}\right)^{-1} dt \\ &> \frac{1}{2}\int_\varepsilon^\pi \left(t \log \frac{4\pi}{t}\right)^{-1} dt \\ &\to \infty,\end{aligned}$$

as $\varepsilon \to 0$. It follows, by Theorem 23, that $\sum \bar{a}_n$ cannot be summable $|C, \beta|$ for $0 < \beta < 1$.

4.4. <u>General Consideration of the Preceding Theorems.</u> We now consider [1] these theorems in the light of some lemmas which were proved in Chapter 5.

By Lemma 34 we see that Theorem 21 may be written in the following form.

THEOREM 24. <u>If $\Theta_\alpha(t)$ is of bounded variation in $(0, \pi)$, then the Allied series of $f(t)$, at the point $t = x$, is summable $|C, \alpha+\delta+1|$, where $\alpha \geq 0, \delta > 0$.</u>

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1) Bosanquet and Hyslop, <u>8</u>.

We may also rewrite Theorem 23 as follows.

THEOREM 25. If the Allied series of $f(t)$, at the point $t=x$ is summable $|C,\alpha|$, then $\Theta_{\alpha+\delta}(t)$ is of bounded variation in $(0,\infty)$, for $\alpha \geq 0, \delta > 0$.

We at once obtain the analogue for the Allied series of Theorem 18.

THEOREM 26. A necessary and sufficient condition for the Allied series of $f(t)$ to be summable $|C,\mu|$ for some μ, at the point $t=x$, is that $\Theta_\lambda(t)$ be of bounded variation in $(0,\pi)$, for some λ.

By Lemmas 36 and 38 we see that the statements of Theorems 21 and 22 may be combined as follows.

THEOREM 27. If $\alpha \geq 0, \delta > 0$ and $\psi_\alpha(t), \Theta_\lambda(t)$ are of bounded variation in $(0,\pi)$ for some λ, then the Allied series of $f(t)$, at the point $t=x$, is summable $|C,\alpha+\delta|$.

Theorem 21 is the particular case of this theorem when $\alpha \geq 1$ and Theorem 22 the case $0<\alpha<1$. We have seen that Theorem 22 is false with $\alpha=0$. Theorem 27, however, is true when $\alpha=0$ and its truth is at once deducible from Lemma 35 and the case of the theorem when $\alpha>0$.

7.5. A Particular Case of the Preceding Theorems. We conclude this chapter by considering the case $\delta=0$ of Theorems 25 and 27. It will be shown that, as with Fourier series, these theorems are not true for $\alpha=\delta=0$.

THEOREM 28. There[1] exists a function $f(t)$ for which $\psi(t)$ and $\theta(t)$ are of bounded variation in $(0,\pi)$, but whose Allied series at the point $t=0$ is not absolutely convergent.

Let $f(t)=\frac{1}{2}t$. Then $\psi(t)=\frac{1}{2}t$, $\theta'(t)=-\frac{1}{2}$, so that $\psi(t)$ and $\theta(t)$ are of bounded variation in $(0,\pi)$. Also the Fourier series of $f(t)$ is $\sum_1 \beta_n \sin nt$, where

$$\beta_n = \frac{1}{\pi}\int_0^\pi t \sin nt = (-1)^{n-1} n^{-1}.$$

Thus the Allied series of $f(t)$ is

$$\sum_1 (-1)^{n-1} n^{-1} \cos nt,$$

which is not absolutely convergent at $t=0$.

For the proof of the next theorem we require two elementary lemmas.

LEMMA 55. If $a > b > 0$, then

$$(7.51) \qquad \left| \int_0^\pi \frac{\cos at - \cos bt}{t} dt \right| < \frac{a-b}{b}.$$

This result is a simple corollary from (6.61).

LEMMA 56. If m is a positive integer then, as m tends to infinity,

$$\int_0^\pi \frac{\cos mt(1-\cos mt)}{t} dt \sim -\frac{1}{2}\log m.$$

Suppose that m is even and equal to 2μ. Then

$$\int_0^\pi \frac{\cos mt(1-\cos mt)}{t} dt = \int_0^{2\mu\pi} \frac{\cos t(1-\cos t)}{t} dt$$

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[1] Bosanquet and Hyslop, 8.

$$= \int_0^{2\pi} \cos t(1-\cos t)\left\{ \frac{1}{t} + \frac{1}{t+2\pi} + \cdots + \frac{1}{t+2(\mu-1)\pi} \right\} dt$$

$$= \int_0^{2\pi} \frac{\cos t(1-\cos t)}{t}\, dt + \int_0^{2\pi} \cos t(1-\cos t)\left\{ \sum_{\nu=1}^{\mu-1} \frac{1}{t+2\nu\pi} \right\} dt .$$

Now

$$\sum_{\nu=1}^{\mu-1} \frac{1}{2\nu\pi+t} = \int_1^{\mu-1} \frac{dx}{t+2\pi x} + \lambda,$$

where

$$0 < \lambda < \frac{1}{2\pi+t} < \frac{1}{2\pi},$$

and

$$\int_1^{\mu-1} \frac{dx}{t+2\pi x} = \frac{1}{2\pi}\Big[\log(t+2\pi x)\Big]_{x=1}^{x=\mu-1} = \frac{1}{2\pi}\log m + \underline{O}(1),$$

uniformly for $0 \leq t \leq \pi$. Hence

$$\int_0^{\pi} \frac{\cos mt(1-\cos mt)}{t}\, dt \sim \frac{1}{2\pi}\log m \int_0^{2\pi} (\cos t - \cos^2 t)\, dt$$

$$\sim -\tfrac{1}{2}\log m.$$

A similar proof also holds when m is odd.

THEOREM 29. <u>There[1] exists a function $f(t)$ whose allied series, at the point $t=0$, is absolutely convergent, but for which $\Theta(t)$ is not of bounded variation in $(0,\pi)$</u> .

Let Σd_m be an absolutely convergent series and λ_m

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[1] For a brief sketch of the proof of this theorem see Bosanquet and Hyslop, <u>8</u>.

a steadily increasing sequence of positive integers satisfying the relations

(7.52) $\qquad \lambda_{m+1} > (2+\theta)\lambda_m, \quad 0<\theta<1,$

(7.53) $\qquad \alpha_m \log \lambda_m \to \infty,$

as m tends to infinity. For example, we may take

$$\alpha_m = m^{-2}, \quad \lambda_m = 2^{m^3}.$$

Let

$$f(x) = \sum_{m=1}^{\infty} \alpha_m \sin \lambda_m x.$$

Then the Allied series of $f(x)$ at $x=0$ is $\sum \alpha_m$ which is absolutely convergent. Also, at $x=0$,

$$\psi(t) = \sum_{m=1}^{\infty} \alpha_m \sin \lambda_m t,$$

so that, if $0<t\leq\pi$,

$$\psi_1(t) = t^{-1}\int_0^t \psi(u)\,du = \sum_{m=1}^{\infty} \frac{\alpha_m}{\lambda_m t}(1-\cos\lambda_m t).$$

Let $\psi_1(-t) = \psi_1(t)$. Then the Fourier series of $\psi_1(t)$ is

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt,$$

where, for $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^\pi \cos nt\,dt \sum_{m=1}^{\infty} \frac{\alpha_m}{\lambda_m t}(1-\cos\lambda_m t)\,dt$$

$$= \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{\alpha_m}{\lambda_m}\int_0^\pi \frac{\cos nt\,(1-\cos\lambda_m t)}{t}\,dt.$$

The change in the order of integration and summation will be justified if we show that, for each fixed value of n,

$$\lim_{\varepsilon\to 0}\sum_{m=1}^{\infty}\frac{d_m}{\lambda_m}\int_0^{\varepsilon}\frac{\cos nt\,(1-\cos\lambda_m t)}{t}\,dt = 0.$$

Now

$$\left|\sum_{m=1}^{\infty}\frac{d_m}{\lambda_m}\int_0^{\varepsilon}\frac{\cos nt\,(1-\cos\lambda_m t)}{t}\,dt\right| < 2\sum_{m=1}^{\infty}\frac{|d_m|}{\lambda_m}\int_0^{\varepsilon}\frac{\lambda_m}{2}\left|\sin\frac{\lambda_m t}{2}\right|dt$$

$$\leq \varepsilon\sum_{m=1}^{\infty}|d_m| \to 0.$$

Returning to the expression for a_n, we have

$$a_{\lambda_n} = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{d_m}{\lambda_m}\int_0^{\pi}\frac{\cos\lambda_n t\,(1-\cos\lambda_m t)}{t}\,dt$$

$$= \frac{2}{\pi}\frac{d_n}{\lambda_n}\int_0^{\pi}\frac{\cos\lambda_n t\,(1-\cos\lambda_n t)}{t}\,dt + E_1 + E_2,$$

where

$$E_1 = \frac{2}{\pi}\sum_{m=1}^{n-1}\frac{d_m}{\lambda_m}\int_0^{\pi}\frac{\cos\lambda_n t\,(1-\cos\lambda_m t)}{t}\,dt,$$

$$E_2 = \frac{2}{\pi}\sum_{m=n+1}^{\infty}\frac{d_m}{\lambda_m}\int_0^{\pi}\frac{\cos\lambda_n t\,(1-\cos\lambda_m t)}{t}\,dt.$$

Now, from Lemma 55,

$$|E_1| \leq \frac{1}{\pi}\sum_{m=1}^{n-1}\frac{|d_m|}{\lambda_m}\left|\int_0^{\pi}\frac{\cos\lambda_n t-\cos(\lambda_m+\lambda_n)t}{t}\,dt\right|$$

$$+\frac{1}{\pi}\sum_{m=1}^{n-1}\frac{|d_m|}{\lambda_m}\left|\int_0^{\pi}\frac{\cos\lambda_n t-\cos(\lambda_n-\lambda_m)t}{t}\,dt\right|$$

$$< \frac{1}{\pi}\sum_{m=1}^{n-1}\frac{|d_m|}{\lambda_m}\frac{\lambda_m}{\lambda_n} + \frac{1}{\pi}\sum_{m=1}^{n-1}\frac{|d_m|}{\lambda_m}\frac{\lambda_m}{\lambda_n-\lambda_m}$$

$$< \frac{1}{\pi\lambda_n}\sum_{m=1}^{n-1}|d_m| + \frac{1}{\pi\lambda_n}\frac{2+\theta}{1+\theta}\sum_{m=1}^{n-1}|d_m|$$

$$< \frac{A}{\lambda_n},$$

and

$$|E_2| \leq \frac{1}{\pi}\sum_{m=n+1}^{\infty}\frac{|d_m|}{\lambda_m}\left|\int_0^\pi \frac{\cos\lambda_n t - \cos(\lambda_m+\lambda_n)t}{t}\,dt\right|$$

$$+\frac{1}{\pi}\sum_{m=n+1}^{\infty}\frac{|d_m|}{\lambda_m}\left|\int_0^\pi \frac{\cos\lambda_n t - \cos(\lambda_m-\lambda_n)t}{t}\,dt\right|$$

$$< \frac{1}{\pi}\sum_{m=n+1}^{\infty}\frac{|d_m|}{\lambda_m}\frac{\lambda_m}{\lambda_n} + \frac{1}{\pi}\sum_{m=n+1}^{\infty}\frac{|d_m|}{\lambda_m}\frac{\lambda_m-2\lambda_n}{\lambda_n}$$

$$< \frac{1}{\pi\lambda_n}\sum_{m=n+1}^{\infty}|d_m| + \frac{1}{\pi\lambda_n}\frac{\theta}{2+\theta}\sum_{m=n+1}^{\infty}|d_m|$$

$$= \underline{o}\left(\frac{1}{\lambda_n}\right).$$

It therefore follows from Lemma 56 that

$$a_{\lambda_n} \sim -\frac{1}{\pi}\frac{d_n}{\lambda_n}\log\lambda_n,$$

and this is not equal to $\underline{O}(\lambda_n^{-1})$ by (7.53). It follows from Lemma 51 that $\psi_1(t)$ is not of bounded variation in $(0,\pi)$ and therefore, by Lemma 36, $\theta(t)$ is not of bounded variation in $(0,\pi)$.

CHAPTER 8.

The Absolute Summability of the Fourier Series of a Function satisfying a Lipschitz Condition.

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8. 1. Preliminary Remarks. In Chapter 6 we investigated the summability of the Fourier Series of $f(t)$ at the point $t=x$ when $f(t)$ was of bounded variation in $(0,\pi)$. Instead of taking as hypothesis the fact that $f(t)$ is of bounded variation we now suppose that $f(t)$ satisfies a Lipschitz condition of order α in $(0,\pi)$. The function $f(t)$ is said to satisfy a Lipschitz condition of order α, or, more briefly, $f(t)$ is said to belong to Lipα, in $(0,\pi)$, if

$$(8.11) \qquad |f(t+h)-f(t)| = O(|h|^{\alpha}),$$

uniformly for $0 \leq t \leq \pi$. It is clear that, if $f(t)$ belongs to Lipα, $\alpha > 0$ in $(0,\pi)$, it is continuous in $(0,\pi)$.

It has been proved by Bernstein[1] that, if $f(t)$ belongs to Lipα in $(0,\pi)$, its Fourier series is absolutely convergent for all values of x in $(0,\pi)$ when $\alpha > \frac{1}{2}$ but is not necessarily absolutely convergent when $\alpha \leq \frac{1}{2}$. The principal theorem of this chapter constitutes an extension of Bernstein's results for the case $\alpha \leq \frac{1}{2}$.

8. 2. Preliminary Lemmas. Before proceeding to the proof of the theorem we state some well known results which will be required.

LEMMA 57. If[2] the function $f(x)$ is such that its square

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[1] Bernstein, 1.
[2] Hobson, 20, 575.

is integrable in the sense of Lebesgue over $(-\pi,\pi)$, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\{f(x)\}^2dx = \tfrac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}(\alpha_n^2+\beta_n^2),$$

where α_n, β_n are the Fourier coefficients of $f(x)$.

LEMMA 58. We have the inequality [1)]

$$\left|\sum_{\nu=m}^{\nu=n} u_\nu v_\nu\right| \leqslant \left\{\sum_{\nu=m}^{\nu=n} u_\nu^2\right\}^{\frac{1}{2}}\left\{\sum_{\nu=m}^{\nu=n} v_\nu^2\right\}^{\frac{1}{2}}.$$

LEMMA 59. If [2)] $0<\alpha<1$, $a>0$, the series

$$(8.21)\qquad \sum_{n=2}^{\infty} n^{-\frac{1}{2}-\alpha}\cos(an\log n + nx)$$

converges uniformly for $0\leqslant x\leqslant 2\pi$ to a function $f(x)$ which belongs to Lipα in $(0,2\pi)$.

8.3. The Principal Theorem. We proceed to the statement and proof of the main result.

THEOREM 30. If [3)] the function $f(x)$ is periodic and belongs to Lipα, $0<\alpha\leqslant\frac{1}{2}$ in $(0,\pi)$, then the Fourier series of $f(x)$ is summable $|C,\delta|$, for all values of x, when $\delta>\frac{1}{2}-\alpha$.

We have

$$a_n = \alpha_n\cos nx + \beta_n\sin nx = \frac{2}{\pi}\int_0^{\pi}\varphi(t)\cos nt\,dt,$$

whence

$$na_n = \frac{2}{\pi}\int_0^{\pi}\varphi(t)\frac{d}{dt}\sin nt\,dt.$$

It follows that

$$d_n^{\prime(\delta)} = \frac{2}{\pi}\int_0^{\pi}\varphi(t)\frac{d}{dt}\left\{\frac{1}{E_n^{(\delta)}}\sum_{k=0}^{n}E_k^{(\delta-1)}\sin(n-k)t\right\}dt,$$

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1) Titchmarsh, 34, 381.
2) Hardy and Littlewood, 16. See also Zygmund, 38, 116-9.
3) Hyslop, 23.

where $d_n^{\prime(\delta)}$ is defined as in 3.6, and, by Lemma 23, it is sufficient to prove that the series

$$\sum_{n=1}^{\infty} n^{-1} |d_n^{\prime(\delta)}|$$

is convergent.

Now, by (3.23),

$$\sum_{n=1}^{\infty} n^{-1} |d_n^{\prime(\delta)}| < A \sum_{n=1}^{\infty} n^{-1-\delta} \left| \int_0^{\pi} \varphi(t) \left\{ \sum_{k=0}^{n} (n-k) E_k^{(\delta-1)} \cos(n-k)t \right\} dt \right|$$

$$\leq A \left\{ \sum_{n=1}^{\infty} S_1(n) + \sum_{n=1}^{\infty} S_2(n) \right\},$$

where

$$S_1(n) = n^{-\delta} \left| \int_0^{\pi} \varphi(t) \left\{ \sum_{k=0}^{\infty} E_k^{(\delta-1)} \cos(n-k)t \right\} dt \right|,$$

$$S_2(n) = n^{-1-\delta} \left| \int_0^{\pi} \varphi(t) \left\{ \sum_{k=n+1}^{\infty} n E_k^{(\delta-1)} \cos(n-k)t + \sum_{k=0}^{n} k E_k^{(\delta-1)} \cos(n-k)t \right\} dt \right|.$$

We now write

$$S_2(n) \leq S_{2,1}(n) + S_{2,2}(n),$$

where the integral in $S_{2,1}(n)$ extends over $(0, n^{-1})$ and that in $S_{2,2}(n)$ over (n^{-1}, π). Since $E_k^{(\delta-1)}$ steadily decreases and $kE_k^{(\delta-1)}$ steadily increases as k increases, the absolute value of each of the sums in $S_{2,1}(n)$ is less than $An^{\delta}t^{-1}$.

Hence, by hypothesis,

$$\sum_{n=1}^{\infty} S_{2,1}(n) < A \sum_{n=1}^{\infty} n^{-1-\delta} \int_0^{n^{-1}} n^{\delta} t^{-1} |\varphi(t)| dt$$

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1) There is no loss of generality in supposing, as we have here, that $0 < \delta < 1$.

$$< A\sum_{n=1}^{\infty} n^{-1}\int_0^{n^{-1}} t^{\alpha-1}dt$$

$$< A\sum_{n=1}^{\infty} n^{-1-\alpha} < A.$$

We now show that $S_{2,2}(n)$ behaves in a similar way. We have

$$S_{2,2}(n) = n^{-1-\delta}\left|\int_{n^{-1}}^{\pi}\varphi(t)\,(2\sin\tfrac{1}{2}t)^{-1}\right. \times$$

$$\times\Big[\sum_{k=n+1}^{\infty} n E_k^{(\delta-1)}\{\sin(k-n+\tfrac{1}{2})t-\sin(k-n-\tfrac{1}{2})t\}$$

$$\left.+\sum_{k=0}^{n} kE_k^{(\delta-1)}\{\sin(n-k+\tfrac{1}{2})t-\sin(n-k-\tfrac{1}{2})t\}\Big]dt\right|,$$

and summation by parts given

$$S_{2,2}(n) = n^{-1-\delta}\left|\int_{n^{-1}}^{\pi}\varphi(t)\,(2\sin\tfrac{1}{2}t)^{-1}\right. \times$$

$$\times\Big[-nE_{n+1}^{(\delta-1)}\sin\tfrac{1}{2}t+n\sum_{k=n+1}^{\infty}\{E_k^{(\delta-1)}-E_{k+1}^{(\delta-1)}\}\sin(k-n+\tfrac{1}{2})t$$

$$\left.+nE_n^{(\delta-1)}\sin\tfrac{1}{2}t+\sum_{k=0}^{n-1}\{(k+1)E_{k+1}^{(\delta-1)}-kE_k^{(\delta-1)}\}\sin(n-k-\tfrac{1}{2})t\Big]dt\right|$$

$$\leq S_{2,2,1}(n). + S_{2,2,2}(n) + S_{2,2,3}(n),$$

where

$$S_{2,2,1}(n) = n^{-\delta}\left|\int_{n^{-1}}^{\pi}\tfrac{1}{2}\varphi(t)\{E_n^{(\delta-1)} - E_{n+1}^{(\delta-1)}\}dt\right|,$$

$$S_{2,2,2}(n) = n^{-\delta}\left|\int_{n^{-1}}^{\pi}\varphi(t)\,(2\sin\tfrac{1}{2}t)^{-1}\ \times\right.$$

$$\left.\times\left[\sum_{k=n+1}^{\infty}\{E_k^{(\delta-1)} - E_{k+1}^{(\delta-1)}\}\sin(k-n+\tfrac{1}{2})t\right]dt\right|,$$

$$S_{2,2,3}(n) = n^{-1-\delta}\left|\int_{n^{-1}}^{\pi}\varphi(t)\,(2\sin\tfrac{1}{2}t)^{-1}\ \times\right.$$

$$\left.\times\left[\sum_{k=1}^{n}\{kE_k^{(\delta-1)} - (k-1)E_{k-1}^{(\delta-1)}\}\sin(n-k+\tfrac{1}{2})t\right]dt\right|.$$

Since

$$E_k^{(\delta-1)} - E_{k+1}^{(\delta-1)} = \underline{O}(k^{\delta-2}),$$

we have

$$\sum_{n=1}^{\infty} S_{2,2,1}(n) < A\sum_{n=1}^{\infty} n^{-2}\int_0^{\pi}|\varphi(t)|\,dt < A.$$

Since the expression

$$E_k^{(\delta-1)} - E_{k+1}^{(\delta-1)}$$

steadily decreases as k increases, the absolute value of the sum in $S_{2,2,2}(n)$ is less than $An^{\delta-2}t^{-1}$. Hence

$$\sum_{n=1}^{\infty} S_{2,2,2}(n) < A\sum_{n=1}^{\infty} n^{-2}\int_{n^{-1}}^{\pi} t^{-2}\,|\varphi(t)|\,dt$$

$$< A\sum_{n=1}^{\infty} n^{-2}\int_{n^{-1}}^{\pi} t^{\alpha-2}\,dt$$

$$< A\sum_{n=1}^{\infty} n^{-1-\alpha} + A\sum_{n=1}^{\infty} n^{-2} < A.$$

Using the relation

$$k E_k^{(\delta-1)} - (k-1)E_{k-1}^{(\delta-1)} = \delta E_{k-1}^{(\delta)},$$

we may write

$$S_{2,2,3}(n) \leq S_{2,2,3,1}(n) + S_{2,2,3,2}(n),$$

where

$$S_{2,2,3,1}(n) = \delta n^{-1-\delta}\left|\int_{n^{-1}}^{\pi}\varphi(t)(2\sin\tfrac{1}{2}t)^{-1}\left\{\sum_{k=0}^{\infty} E_k^{(\delta-1)}\sin(n-k-\tfrac{1}{2})t\right\}dt\right|,$$

$$S_{2,2,3,2}(n) = \delta n^{-1-\delta}\left|\int_{n^{-1}}^{\pi}\varphi(t)(2\sin\tfrac{1}{2}t)^{-1}\left\{\sum_{k=n}^{\infty} E_k^{(\delta-1)}\sin(n-k-\tfrac{1}{2})t\right\}dt\right|.$$

Arguing in much the same way as in the case of $S_{2,2,2}(n)$, we at once obtain

$$\sum_{n=1}^{\infty} S_{2,2,3,2}(n) < A.$$

Also

$$\left|\sum_{k=0}^{\infty} E_k^{(\delta-1)}\sin(n-k-\tfrac{1}{2})t\right| = \left|\mathfrak{I}\left\{\sum_{k=0}^{\infty} E_k^{(\delta-1)} e^{(n-\frac{1}{2})it}e^{-kit}\right\}\right|$$

$$= \left|\mathfrak{I}\left\{e^{(n-\frac{1}{2})it}(1-e^{-it})^{-\delta}\right\}\right|$$

$$\leq |1-e^{-it}|^{-\delta} < At^{-\delta},$$

for $0<t<\pi$. It follows that

$$\sum_{n=1}^{\infty} S_{2,2,3,1}(n) < A\sum_{n=1}^{\infty} n^{-1-\delta}\int_{n^{-1}}^{\pi} t^{-1-\delta}|\varphi(t)|\,dt$$

$$< A\sum_{n=1}^{\infty} n^{-1-\delta}\int_{n^{-1}}^{\pi} t^{\alpha-1-\delta}\,dt$$

$$< A\sum_{n=1}^{\infty} n^{-1-\alpha} + A\sum_{n=1}^{\infty} n^{-1-\delta} < A.$$

It has been proved that the series $\sum S_2(n)$ is convergent. We proceed to prove that the same is true for the series $\sum S_1(n)$. We shall assume now, as we may, without loss of generality, that $\delta < \frac{1}{2}$.

We write

$$\int_0^{\pi}\varphi(t)\left\{\sum_{k=0}^{\infty} E_k^{(\delta-1)}\cos(n-k)t\right\}dt = \int_0^{\pi}\varphi(t)p(t)\cos nt\,dt + \int_0^{\pi}\varphi(t)q(t)\sin nt\,dt$$

$$= p_n + q_n,$$

where

$$p(t) = \sum_{k=0}^{\infty} E_k^{(\delta-1)}\cos kt, \quad q(t) = \sum_{k=0}^{\infty} E_k^{(\delta-1)}\sin kt.$$

Now $p(t)$ and $q(t)$ are continuous for $\eta\leq t\leq\pi$, and their absolute values when $0<t\leq\eta$ are each less than $At^{-\delta}$. The constants p_n and q_n are thus the Fourier coefficients

of an even and an odd function respectively, and each of these functions has its square integrable. It follows from Lemma 57 and (5.28) that

$$\sum_{n=1}^{\infty} p_n^2 \sin^2 nh < A\int_0^{\pi} |\varphi(t+h)\,p(t+h) - \varphi(t-h)\,p(t-h)|^2 dt$$

$$\leqslant 2A\{I_1(h) + I_2(h)\},$$

where

$$I_1(h) = \int_0^{\pi} \{p(t+h)\}^2 \{\varphi(t+h) - \varphi(t-h)\}^2 dt,$$

$$I_2(h) = \int_0^{\pi} \{\varphi(t-h)\}^2 \{p(t+h) - p(t-h)\}^2 dt.$$

Now, taking h to be positive, we have

$$|\varphi(t+h) - \varphi(t-h)| = \tfrac{1}{2}\Big|\big[\{f(x+t+h) - f(x+t-h)\}$$
$$- \{f(x-t+h) - f(x-t-h)\}\big]\Big|$$

$$< A h^{\alpha},$$

and therefore

$$I_1(h) < A h^{2\alpha}\int_0^{\pi} t^{-2\delta} dt = O(h^{2\alpha}),$$

since $0 < \delta < \frac{1}{2}$. It should be observed in passing that only in this part of the proof is it necessary to use the full hypothesis.

We now split up $I_2(h)$ into two parts $I_{2,1}(h)$ and $I_{2,2}(h)$, where

$$I_{2,1}(h) = \int^{h} \{\varphi(t)\}^2 \{p(t+2h) - p(t)\}^2 dt,$$

$$I_{2,2}(h) = \int_{h}^{\pi-h} \{\varphi(t)\}^2 \{p(t+2h) - p(t)\}^2 dt.$$

$$I_{2,2}(h) = \int_{h} \{\varphi(t)\} \{p(t+2h) - p(t)\} dt.$$

We have then

$$I_{2,1}(h) \leq 2\int_{-h}^{h} \{\varphi(t)\}^2 \left[\{p(t+2h)\}^2 + \{p(t)\}^2\right] dt$$

$$= \underline{O}\left\{\int_{-h}^{h} t^{2\alpha}(t+2h)^{-2\delta} dt\right\} + \underline{O}\left\{\int_{-h}^{h} t^{2\alpha-2\delta} dt\right\}$$

$$= \underline{O}(h^{2\alpha-2\delta+1}),$$

and

$$I_{2,2}(h) = 4h^2\int_{h}^{\pi-h} \{\varphi(t)\}^2 \{p'(t+2\theta h)\}^2 dt, \quad 0<\theta<1,$$

$$= \underline{O}\left\{h^2\int_{h}^{\pi} t^{2\alpha} (\sin\tfrac{1}{2}t)^{-2\delta-2} dt\right\}$$

$$= \underline{O}\left\{h^2\int_{h}^{\pi} t^{2\alpha-2\delta-2} dt\right\}$$

$$= \underline{O}(h^{2\alpha-2\delta+1}) + \underline{O}(h^2).$$

It follows that, as h tends to zero,

$$\sum_{n=1}^{\infty} p_n^2 \sin^2 nh = O(h^{2\alpha}),$$

and it may be proved similarly that

$$\sum_{n=1}^{\infty} q_n^2 \sin^2 nh = O(h^{2\alpha}).$$

Let $h = \pi/2N$. Then we obtain

$$\sum_{n=1}^{N} p_n^2 \sin^2 \frac{n\pi}{2N} = O(N^{-2\alpha}),$$

and, writing $N = 2^\nu$, we at once deduce that

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} p_n^2 = O(2^{-2\nu\alpha}).$$

Applying Lemma 58, we have

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} n^{-\delta} |p_n| \leq \Big\{ \sum_{n=2^{\nu-1}+1}^{2^\nu} p_n^2 \Big\}^{\frac{1}{2}} \Big\{ \sum_{n=2^{\nu-1}+1}^{2^\nu} n^{-2\delta} \Big\}^{\frac{1}{2}}$$

$$= O\{2^{-\nu\alpha}\} . O\{2^{\nu(\frac{1}{2}-\delta)}\}$$

$$= O\{2^{-\nu(\alpha+\delta-\frac{1}{2})}\}.$$

A similar relation also holds in the case of q_n .

It now follows that

$$\sum_{n=1}^{\infty} S_1(n) \leq \sum_{n=1}^{\infty} n^{-\delta} |p_n| + \sum_{n=1}^{\infty} n^{-\delta} |q_n|$$

$$= \sum_{\nu=1}^{\infty} \left\{ \sum_{n=2^{\nu-1}+1}^{2^{\nu}} (n^{-\delta}|p_n| + n^{-\delta}|q_n|) \right\}$$

$$< A \sum_{\nu=1}^{\infty} 2^{-\nu(\alpha+\delta-\frac{1}{2})} < A,$$

since $\delta > \frac{1}{2} - \alpha$.

The theorem is therefore proved.

8.4. Proof that the Preceding Theorem is a 'Best Possible'' Result.

THEOREM 31. There[1] exists a function $f(x)$ belonging to Lipα, where $0<\alpha<1$, whose Fourier series, for any values of x, is not summable $|C, \frac{1}{2}-\alpha|$.

For the series (8, 21), when $\delta = \frac{1}{2} - \alpha$, we have

$$\sum_{n=2}^{\infty} n^{-\delta} |a_n| = \sum_{n=2}^{\infty} n^{-1} |\cos(an \log n + n \log x)|$$

$$\geqslant \sum_{n=2}^{\infty} n^{-1} \cos^2 (an \log n + nx)$$

$$= \frac{1}{2} \sum_{n=2}^{\infty} n^{-1} \{1 + \cos(2an \log n + 2nx)\}$$

$$\geqslant \frac{1}{2} \sum_{n=2}^{\infty} n^{-1} - \frac{1}{2} \left| \sum_{n=2}^{\infty} n^{-1} \cos(2an \log n + 2nx) \right|$$

$$= \infty,$$

---

[1] This proof is due to Bosanquet. See Hyslop, 23.

since the last series is (8.21), with $\alpha = \frac{1}{2}$, a replaced by $2a$ and x by $2x$, and so it converges for every value of x. Since the series $\sum n^{-\delta} |a_n|$ is not convergent it follows, from Theorem 2, that $\sum a_n$ cannot be summable $|C, \delta|$. The theorem is therefore proved.

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