THE GEOMETRY OF THE CONIC AND TRIANGLE

A Thesis presented for the Doctorate of Science
in the University of Glasgow

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Edinburgh:
PRINTED BY JOHN LINDSAY, 304 LAWNMARKET
1909

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THE GEOMETRY OF THE CONIC AND TRIANGLE

INTRODUCTION.

If a triangle with its associated system of lines and circles be orthogonally projected on a second plane, a second triangle is derived with an associated system of lines and homothetic ellipses. Pairs of perpendicular lines project into pairs of lines parallel to conjugate diameters; such lines in the sequel will be called conjugate lines. Where a relation before projection exists between the lengths of lines we have only to replace a length by the ratio of the projection to the parallel radius of one of the homothetic ellipses.

The question then arises if there is a corresponding geometry with hyperbolae instead of ellipses. By general projection pairs of perpendicular lines do not project into pairs of conjugate lines since parallel lines in general project into concurrent lines. Also all the members of a system of circles do not yield hyperbolae, much less homothetic hyperbolae. In fact, as will be seen later, the only points with general projective properties are the symmedian centre (with an extended definition) and the poles of the sides. It is hardly allowable then to assume that what has been derived by a special kind of projection from the circle can be at once generalised further. Perhaps an argument from the principle of continuity might be advanced, stating that what is true for the ellipse should be true for the hyperbola. But that is not satisfactory; it would only be so, perhaps, if the results were built up for the ellipse, without the mediation of the circle, and the continuity in the *proofs* asserted. The results in the geometry of the circle and triangle are, however, generally derived from the consideration of angles, and such a method of proof is not applicable to the ellipse. Besides we have a case of failure of corresponding points for the hyperbola; when the vertices of a triangle do not all lie on the same branch of a hyperbola, there are no homothetic hyperbolae touching the sides. The general correspondence, however, for the ellipse and hyperbola does exist, and the first aim of the present paper is to build up a geometry of the central conic and triangle corresponding to the parts of the geometry of the circle and triangle more generally known. The matter at our disposal is of course very great, so that just so much will be done as to form a connected whole, and to convince one that the remaining results for the circle have their real analogues for the conic, the necessary modifications in their statements being evident from the theorems already proved. The paper by *Dr A. Emmerich*, "*Die Brocardschen Gebilde und ihre Beziehungen zu den verwandten merkwürdigen Punkten und Kreisen des Dreiecks*, 1891," contains a useful list of original papers.

Of not less importance is the method of proof we have adopted. We have intentionally discarded the use of coordinates as we wished to show as far as possible the continuity from the circle to the more general conic, not only in results, but in the means used to derive them. The properties of conics required are not many, and it will be seen in the detailed working that the consideration of angles in the case of the circle is replaced by an appeal to principles from which equality of angles and similarity of triangles for the particular conic, the circle, arise. We proceed to a short discussion of the fundamental reasons of the extension of the geometry; from these reasons we derive indications of the theorems in conics to be generally used in the proofs.

All circles, as is well known, have two common (imaginary) points at infinity. Let us consider two different points, real or imaginary, at infinity. The set of conics having these points common we shall call homothetic conics. If the points are real, so that the system consists of hyperbolae, we shall include in the system a hyperbola, its conjugate, and the asymptotes. By this means, as in the case of a circle, every three non-collinear points in the finite part of the plane determine a conic of the system. The common points at infinity determine the involution of conjugate diameters of each of the conics. Two such involutions would coincide on a motion of translation without rotation. Thus when the common points are imaginary the conics are similar, and similarly situated ellipses. When the common points are real the system consists of similar and similarly situated hyperbolae, with the conjugate hyperbolae and their asymptotes. Therefore parallel radii of two conics of the system have the same ratio. This ratio in the case of a hyperbola and its conjugate is imaginary, but the results derived are still real.

The ratio of the products of the segments of two chords or secants of a conic through a given point is equal to the ratio of the squares of the parallel radii. This theorem will be continually employed. For the circle the ratio is unity, and we have similar triangles aud equal angles. To this also we may attribute the fact that angles in the same segment of a circle are equal, for if ACB, ADB are two such angles, CB and AD cutting in P, we have AP·PD and BP·PC equal; from this and the equality of the opposite angles APC, BPD the angles C and D are equal as P moves on BC or AD. For the constancy of the angle in the segment of a circle will also sometimes be used the constancy of the anharmonic ratio formed by the pencil from a variable point to four fixed points on a conic. The involution of the diameters of a circle is that formed by the rotation of two perpendicular lines so that equal angles occur. For proofs involving angles we shall have therefore sometimes proofs using involutions.

In the case of real points at infinity, when the conic is a pair of straight lines through them, the term "radius" ceases to have a definite meaning, but still the ratio of the products of the segments of two secants through a point is equal to the ratio of the squares of the parallel radii of a hyperbola of the system.

When in a circle the two chords are a diameter and a perpendicular chord we get the theorem of Pythagoras. Corresponding to this we have a theorem for a triangle, two of whose sides are conjugate lines; this theorem is really the central equation $x^2/a^2 + y^2/b^2 = 1$ of a conic of the system.

Other theorems used are mainly *direct* consequences of the preceding. Such is Carnot's, and even the theorem of Menelaus, sometimes employed, is a direct result of Carnot's when the conic is two lines, one of which is at infinity.

In the *Proceedings of the Edinburgh Mathematical Society*, 1905–6, Mr Pinkerton gave as an isolated result the extension of the nine point circle to a nine point conic, without suggesting the possibility of the extension of the whole geometry, or laying down any principles for further advance. Much of course has been written on the nine point conic, but not in the same association; see the paper of Professor Allardice in the *Proceedings* of the above Society for 1900–1.

On my giving an excerpt of the present paper to the same Society in the beginning of 1908, Dr Muirhead kindly referred me to a paper by L. Ripert (*La Dualité et l'Homographie dans le Triangle et le Tétrahèdre*, Gauthier-Villars et Fils, Paris, 1898). In this short generalising paper M. Ripert has anticipated some of my fundamental ideas. His paper, however, is mainly an application of barycentric coordinates, and, besides being admittedly only an outline or introduction, is of quite a different character from my own. A few remarks on this paper are contained in a final short section on the algebraic continuity.

When once this geometry is established it is easily seen to be capable of further extension. We might make the two (real) points at infinity coincide when we have a system of parabolas with parallel axes. The conjugates to all lines being parallel to the axis many results degenerate, and are therefore not of much interest. It is not difficult to find the results which do not degenerate.

The most important extension will be that made by taking the two common points X, Y, real or imaginary, not at infinity. We have only to make a projection of the present geometry from any point, and we have a whole series of new results in the geometry of the triangle and conic. For parallel lines we shall have lines meeting in the same point on the real line XY. For homothetic conics we have conics through the points X, Y. Corresponding to the centres of the former are the intersections (real) of the tangents at X and Y of the latter. Instead of the middle point of a line BC we shall have the harmonic conjugate with respect to B and C of its intersection D with XY. For the conjugate from a point A to a line BC, we shall have the harmonic conjugate with respect to AX and AY of the "parallel" through A to BC.

The analogues of the centroid, orthocentre, symmedian centre, etc, are then easily seen, and corresponding theorems for this new geometry can be immediately stated.

Some further points of interest are also discussed in the paper. When a triangle is given, any point in general in its plane can be taken to represent one of the well known points in the geometry of the circle and triangle, and the remaining points and lines can be determined by the aid of compasses and ruler only; such problems, however, are of little interest. Of more interest, however, are the loci of the points for a system of circumconics passing through a fixed point. These are almost all conics or straight lines. These loci are investigated in the paper by the methods chiefly of projective geometry. The results give some interesting theorems in conics. The envelopes, which are points or conics, of the well known lines are also discussed.

As is well known, Brocard's first triangle is triply in perspective with the original triangle. With the ∞^2 circumconics of a given triangle we have therefore ∞^2 triangle triply in perspective with a given triangle. As the discussion of such triangles arises so naturally, and as the nature of the centres of perspective is so evident, we have been tempted to work out again by the methods of the present paper the theory of triangles triply and quadruply in perspective. For the addition of more theorems we have been indebted to the paper by Dr Third in the *Proceedings of the Edinburgh Mathematical Society*, 1900–1.

The application of the principle of reciprocation to the geometry of the triangle is of course not new, and many theorems derived by this are found scattered in the exercises in books on coordinate geometry. Reciprocation brings in conics, and the results often do not appear striking; to this is perhaps due a failure of systematised results. More perhaps might be done in examining further the points, lines, and conics thus obtained. To show this a short section is devoted to some further interesting results obtained by such examination.

We have closed the paper with a few short notes, and a table for reference of the coordinates and equations of the chief points, lines, and conics.

The proofs in the paper are sometimes, as is to be expected, of a tedious nature. They might be shortened, but where they are specially tedious, as in the point O theorem, algebraic proofs obtaining the required metric relations would, we think, be hardly less fatiguing.

Since the present paper was written we have found a paper in the *Archiv der Mathematik und Physik, Dritte Reihe, Band* 11, 1907, by Herr Gustav Berkhan, entitled "Zur projektivischen Behandlung der Dreiecksgeometrie." In this, Herr Berkhan takes up the discussion of the further extended geometry already mentioned. His work is algebraic, so that we still think there is room for approaching the subject in the purely geometric manner of the present paper.

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November **1908.**

NOTATION.

The analogues of the well-known points in the geometry of the circle and the triangle will be denoted for convenience by the same names. As the circle will not be specially mentioned, no ambiguity will be found. The following notation will be generally used, additions or temporary deviations being expressly mentioned in the text:—

A, B, C, the vertices of the original triangle and a, b, c the opposite sides.

a_1, b_1, c_1 the radii of the circumconic parallel to a, b, c.

AD, BE, CF, conjugates to a, b, c.

H, the orthocentre.

S, the circumcentre.

d, e, f, the radii conjugate to a, b, c.

P, Q, R, the poles of the sides a, b, c.

p, q, r, the radii conjugate to SA, SB, SC or parallel to QR, RP, PQ.

G, the centroid.

N, the nine point conic centre.

U, V, W, the mid points of a, b, c.

K, the symmedian centre.

Ω, Ω', the first and second Brocard points.

I, I′, I″, I‴, the centres of the in- and ex-conics.

A_1, B_1, C_1, the vertices of Brocard's first triangle.

A_2, B_2, C_2, the vertices of Brocard's second triangle.

A′, B′, C′, the vertices of the triangle formed by producing AK, BK, CK to cut the circumconic again.

P_1, P_2, P_3, the feet of the conjugates from any point P to a, b, c.

P_a, P_b, P_c, the intersections with the opposite sides of AP, BP, CP.

p_1, p_2, p_3, the radii of the circumconic parallel to AP, BP, CP.

(ABCD) denotes $\dfrac{AC}{BC} : \dfrac{AD}{BD}$ according to Cremona's notation.

{AB, CD} denotes a harmonic range so that $(ABCD) = -1$.

von Staudt's symbol $\barwedge$ will be used to denote projectivity.

{P...} will denote the range traced out by a moving point P on a given line.

SECTION I.

The Orthocentre. The Nine Point Conic.

§1. Since BF, CF and BE, CE are pairs of conjugates, the conic on BC as diameter homothetic to the circumconic passes through F, E (Fig. 1). The ratio of any pair of radii of a conic is equal to the ratio of the parallel pair of radii of a homothetic conic.

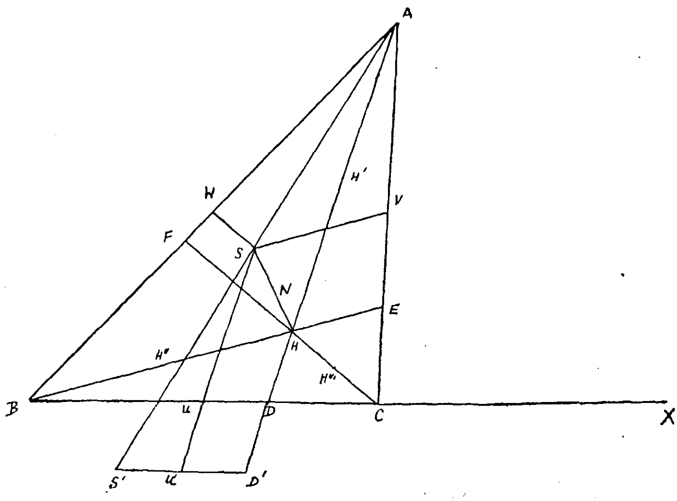


Fig 1

$$\therefore \frac{c\,.\,\text{AF}}{b\,.\,\text{AE}}=\frac{c_1^2}{b_1^2} \text{ and similarly } \frac{a\,.\,\text{BD}}{c\,.\,\text{BF}}=\frac{a_1^2}{c_1^2},\ \frac{b\,.\,\text{CE}}{a\,.\,\text{CD}}=\frac{b_1^2}{a_1^2}.$$

By Ceva's theorem AD, BE, CF are concurrent in H.

The first equation gives

$$\frac{c(c-\text{BF})}{c_1^2}=\frac{b(b-\text{CE})}{b_1^2} \text{ or } \frac{b\,.\,\text{CE}}{b_1^2}-\frac{c\,.\,\text{BF}}{c_1^2}=\frac{b^2}{b_1^2}-\frac{c^2}{c_1^2}.$$

The second two give

$$\frac{a\,.\,\text{BD}}{a_1^2}=\frac{c\,.\,\text{BF}}{c_1^2},\ \frac{a\,.\,\text{CD}}{a_1^2}=\frac{b\,.\,\text{CE}}{b_1^2}$$

and on addition

$$\frac{b\,.\,\mathrm{CE}}{b_1^2}+\frac{c\,.\,\mathrm{BF}}{c_1^2}=\frac{a^2}{a_1^2}.$$

$$\therefore\quad \frac{2b\,.\,\mathrm{CE}}{b_1^2}=\frac{2a\,.\,\mathrm{CD}}{a_1^2}=\frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}-\frac{c^2}{c_1^2},$$

$$\frac{2c\,.\,\mathrm{BF}}{c_1^2}=\frac{2a\,.\,\mathrm{BD}}{a_1^2}=\frac{a^2}{a_1^2}-\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2},$$

and similarly $$\frac{2c\,.\,\mathrm{AF}}{c_1^2}=\frac{2b\,.\,\mathrm{AE}}{b_1^2}=\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}.$$

§2. We give a second simple proof of the concurrency.

Let parallels through S to a, b, c meet respectively (i) b, c in Y′, Z, (ii) c, a in Z′, X, (iii) a, b in X′, Y.

$\frac{\mathrm{X'U}}{\mathrm{UX}}=\frac{\mathrm{BD}}{\mathrm{DC}}$. We have therefore to prove $\frac{\mathrm{X'U}}{\mathrm{UX}}\cdot\frac{\mathrm{Y'V}}{\mathrm{VY}}\cdot\frac{\mathrm{Z'W}}{\mathrm{WZ}}=1.$

Now $$\frac{\mathrm{AZ}}{\frac{c}{2}}=\frac{\mathrm{AY'}}{\frac{b}{2}}\quad\text{or}\quad\frac{\frac{c}{2}+\mathrm{WZ}}{\frac{c}{2}}=\frac{\frac{b}{2}+\mathrm{VY'}}{\frac{b}{2}}.$$

$\therefore\quad \frac{\mathrm{WZ}}{\mathrm{VY'}}=\frac{c}{b}$. From the two corresponding results the concurrency follows.

§3. If AS, AD cut the circumconic again in S′, D′, S′D′ is parallel to a, for it is conjugate to AD′. If SU which is parallel to AD′ cuts S′D′ in U′ (Fig. 1), U′ is the mid point of S′D′ and AD′ = 2SU′. The line joining the mid points of CH, AH is parallel and equal to UW. Therefore AH = 2SU, HD′ = 2UU′ and HD = DD′.

Let λ be the ratio of similarity of the homothetic conic on AB as diameter to the circumconic.

$$\frac{\mathrm{AD}^2}{\lambda^2 d^2}+\frac{\mathrm{BD}^2}{\lambda^2 a_1^2}=4.$$

$$\therefore\quad \frac{\mathrm{AD}^3}{d^2}=4\lambda^2-\frac{\mathrm{BD}^2}{a_1^2}=\frac{c^2}{c_1^2}-\left(\frac{a^2}{a_1^2}-\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right)^2\Big/4\frac{a^2}{a_1^2}.$$

$$\therefore\quad \frac{\mathrm{AD}^2}{d^2}=\left(\Sigma 2\frac{a^2b^2}{a_1^2b_1^2}-\Sigma\frac{a^4}{a_1^4}\right)\Big/4\frac{a^2}{a_1^2}=\frac{\Delta^2}{4\frac{a^2}{a_1^2}}=\frac{4s\left(s-\frac{a}{a_1}\right)\left(s-\frac{b}{b_1}\right)\left(s-\frac{c}{c_1}\right)}{\frac{a^2}{a_1^2}},$$

where $$\Delta^2 = 2\Sigma\frac{a^2b^2}{a_1^2b_1^2} - \Sigma\frac{a^4}{a_1^4} \text{ and } 2s = \frac{a}{a_1} + \frac{b}{b_1} + \frac{c}{c_1}.$$

$$\therefore\ \frac{2a\,.\,\mathrm{AD}}{a_1d} = \frac{2b\,.\,\mathrm{BE}}{b_1e} = \frac{2c\,.\,\mathrm{CF}}{c_1f} = \Delta.$$

$\frac{2\mathrm{AH}\,.\,\mathrm{AD}}{d^2} = \frac{2b\,.\,\mathrm{AE}}{b_1^2}$ since a homothetic conic on CH as diameter passes through D, E.

$$\therefore\ \frac{\mathrm{AH}}{d} = \frac{2\mathrm{SU}}{d} = \frac{a}{a_1}\left(\frac{b^2}{b_1^2} + \frac{c^2}{c_1^2} - \frac{a^2}{a_1^2}\right)\Big/\Delta,$$

$$\frac{\mathrm{BH}}{e} = \frac{2\mathrm{SV}}{d} = \frac{b}{b_1}\left(\frac{c^2}{c_1^2} + \frac{a^2}{a_1^2} - \frac{b^2}{b_1^2}\right)\Big/\Delta,$$

$$\frac{\mathrm{CH}}{f} = \frac{2\mathrm{SW}}{f} = \frac{c}{c_1}\left(\frac{a^2}{a_1^2} + \frac{b^2}{b_1^2} - \frac{c^2}{c_1^2}\right)\Big/\Delta.$$

$$\frac{\mathrm{HD}}{d} = \frac{\mathrm{AD} - \mathrm{AH}}{d} = \left(\frac{a^2}{a_1^2} - \frac{b^2}{b_1^2} + \frac{c^2}{c_1^2}\right)\left(\frac{a^2}{a_1^2} + \frac{b^2}{b_1^2} - \frac{c^2}{c_1^2}\right)\Big/\frac{2a\Delta}{a_1}.$$

§4. Let H′, H″, H‴ (Fig. 1) be the middle points of AH, BH, CH, and N the mid point of SH. Then $\mathrm{NH}' = \frac{1}{2}\mathrm{SA}$, $\mathrm{NH}'' = \frac{1}{2}\mathrm{SB}$, $\mathrm{NH}''' = \frac{1}{2}\mathrm{SC}$. A homothetic conic with centre N and ratio of similarity $\frac{1}{2}$ passes through H′, H″, H‴. Since UH′, VH″, WH‴ are bisected at N this conic passes through U, V, W, and because UH′ is a diameter and UD, H′D are conjugate it passes through D and similarly through E and F. Again, because SU = AH′, G lies on SH and SG:GN:NH = 2:1:3. If O be any point on the circumconic the intersection M of OH with the nine point conic is the mid point of OH, for the parallel through N to SO bisects HO and is half SO.

Section II.

The Wallace Line.

§5. If O is a point on the circumconic, O_1, O_2, O_3 are collinear. Let OO_1, OO_2, OO_3 cut the circumconic again in X, Y, Z (Fig. 2). The homothetic conic on OB as diameter passes through O_1, O_3 for OO_1, BO_1 and OO_3, BO_3 are pairs of conjugates. The circumconic, the conic $\mathrm{OBO}_1\mathrm{O}_3$ and the pair of lines OO_1, BO_3 have a common chord OB. Their other three chords then meet in a point. But the second chord of the two conics is at infinity, and therefore AX, O_1O_3 are parallel. By taking the conic OO_3, BO_1 we have O_1O_3, CZ

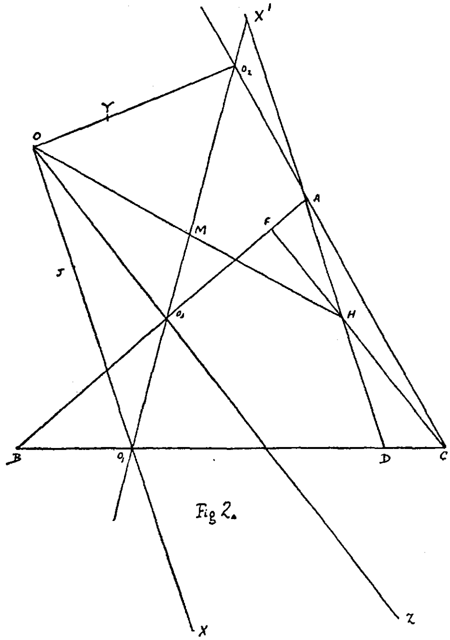


Fig 2.

parallel. Again, by taking the conic OO_2AO_3 we have in the same way BY and CZ parallel to O_2O_3. O_1, O_2, O_3 are therefore in a line parallel to AX, BY, CZ.

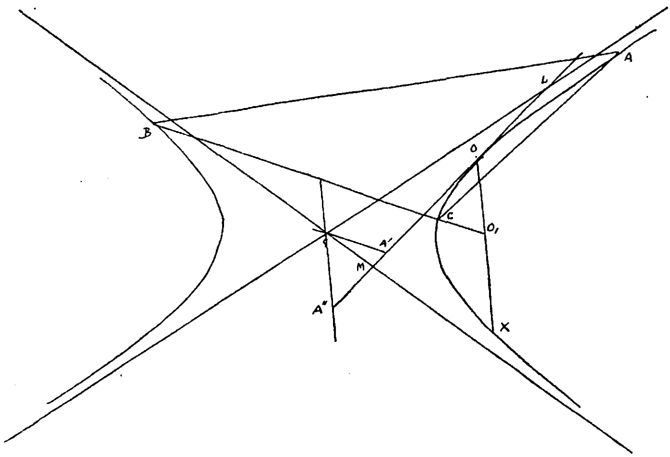
Let O_1 be *any* point on BC. Describe the homothetic conic OBO_1 cutting c in O_3, and the homothetic conic OAO_3 cutting b in O_2. Then in exactly the same way it can be proved that O_1, O_2, O_3 are collinear. O, C, O_1, O_2 lie on a homothetic conic.

§6. We add a second proof which shows that the polars of O_1, O_2, O_3 are concurrent.

Let the tangent at O meet the asymptotes in L, M, the diameter parallel to a in A′ and the diameter conjugate to a in A″ (Fig. 3). Let B′, B″ and C′, C″ be the corresponding points for the other sides. Then A′A″, B′B″, C′C″, LM form an involution with the double points L, M,

O_1 is on the polar of P and on the polar of A′ since BC is the polar of P and OX is the polar of A′, X being the point of intersection of OO with the circumconic. Therefore PA′ is the polar of O_1. Similarly QB′, RC′ are the polars of O_2 and O_3.

Fig. 3.



If two triangles circumscribe a conic their vertices lie on a second conic.

P, Q, R, S, L, M then lie on a conic.

PA″, QB″, RC″ are concurrent in S.

Let O′ be the intersection of PA′, QB′.

Since {A′A″, B′B″, C′C″, LM} is a range in involution with L, M as double points $(LMA'B') \barwedge (LMA''B'')$ and $O'(LMA'B')$, $S(LMA''B'')$ are projective pencils.

Consider the conic generated by the homographic pencils with centres O′, S and O′L, SL; O′M, SM; O′A′, SA″ as pairs of corresponding rays. This conic passes through O′, S, L, M, P and Q, and is therefore the conic through the six vertices P, Q, R, S, L, M.

O′R and SR or SC″ are corresponding rays.

$\therefore\ O'(LMPR) \barwedge S(LMPR) \barwedge (LMA''C'')$.

Let O′R meet LM in C_1'. Then $(LMA'C_1') \barwedge (LMA''C'')$.

But $(LMA'C') \barwedge (LMA''C'')$. Therefore C′ and C_1' coincide, and RC′ passes through O′.

§7. Let J be taken on O_1O so that $O_1J = XO_1$. J is then the orthocentre of the triangle OBC, and $OJ = AH$. If the Wallace line meets AD in X′ (Fig. 2) O_1XAX' is a parallelogram, and $XO_1 = AX'$.

Therefore $O_1O = HX'$, and $O_1OX'H$ is a parallelogram with O_1X' and OH as diagonals. The Wallace line then bisects OH in M at a point on the nine point conic.

Let GO cut SM in G′ and GG″, parallel to NM, cut SM in G″ (Fig. 4). Since $SG = \frac{2}{3}SN$, $GG'' = \frac{2}{3}NM$.

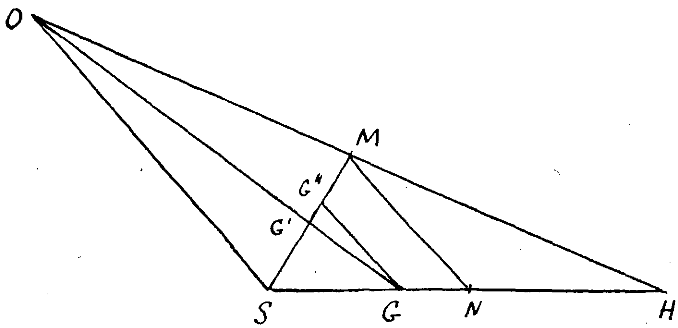


Fig 4.

$$\frac{OG'}{G'G} = \frac{OS}{G''G} = \frac{2MN}{\frac{2}{3}MN} = 3.$$

G′ is the centre of mean position of the points A, B, C, O.

Also $$\frac{SG'}{G'G''} = \frac{OS}{G''G} = 3 \text{ and } SG'' = \tfrac{2}{3}SM.$$

$\therefore\ SG' = 3(SG'' - SG') = 2SM - 3SG'.$

$\therefore\ 2SG' = SM.$

M is then on the line joining the centre S of the conic to the mean position G′ of the four points A, B, C, O and G′ bisects SM. The symmetry shows that the Wallace line of any one of the four points A, B, C, O with respect to the triangle formed by the remaining three passes through a fixed point.

Let OT be a diameter of the circumconic and let TT_1 meet the circumconic again in T_1'. Since OX, TT_1' are parallel and OT is a diameter, XT_1' is a diameter. AX, AT_1' are parallel to the Wallace lines of O and T. Therefore the Wallace lines of the ends of a diameter are parallel to a pair of conjugate diameters. If M′ is the mid point of TH, MM′ is a diameter of the nine point conic, and the Wallace lines intersect on this conic.

§8. Any transversal $O_1O_2O_3$ can be considered a Wallace line in an infinite variety of ways.

Draw *any* line O_1O and let AD, parallel to OO_1 cut $O_1O_2O_3$ in X′ (Fig. 2). Let M be the mid point of O_1X'. Then U, V, W, D, M are five points on the nine point conic. These determine it, and so the homothetic circumconic can be drawn and the point O found. Otherwise we can cut off $MY' = O_2M$ and $MZ' = O_3M$. Thus the conjugates Y′BE, Z′CF to the sides *b*, *c* and the orthocentre are determined, etc.

The locus of O for a given line $O_1O_2O_3$ is a straight line. Complete the parallelogram O_1BO_3X (Fig. 5) and the parallelogram

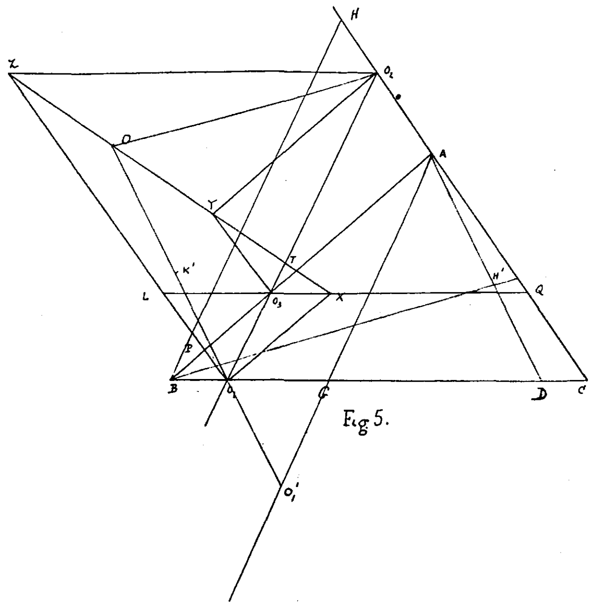


Fig 5.

O_2AO_3Y. Let XY meet the parallel O_1Z to *b* in Z, and let XO_3 meet O_1Z in L and AC in Q.

Then $$\frac{ZL}{YO_3}=\frac{LX}{O_3X}=\frac{LO_3}{O_3X}+1.$$

$$\therefore \frac{ZL}{O_2A}=\frac{LO_3}{BO_1}+1.$$ Let LO_1 meet BO_3 in P.

$$\frac{ZL}{O_2A}=\frac{LP}{PO_1}+1=\frac{LO_1}{PO_1}=\frac{O_2Q}{O_2A}.$$

$\therefore$ $ZL=O_2Q$ and ZO_2 is parallel to BC.

The line XYZ is the locus of O. Take any point O in it and draw AD, BE, CF parallel to OO_1, OO_2, OO_3. Let AG, BH, CK be parallel to $O_1O_2O_3$ and let $O_1O_2O_3$ cut XYZ in T.

Then $(BGDC)\barwedge(XTOZ)$ since they are the intersections on a and XYZ of two pencils with parallel rays and centres A, O_1. Similarly $(CHEA)\barwedge(ZTOY)$ and $(AKFB)\barwedge(YTOX)$.

$$\therefore \frac{BD\,.\,GC}{DC\,.\,BG}=\frac{XO\,.\,TZ}{OZ\,.\,XT},$$

$$\frac{CE\,.\,HA}{EA\,.\,CH}=\frac{ZO\,.\,TY}{OY\,.\,ZT},$$

$$\frac{AF\,.\,KB}{FB\,.\,AK}=\frac{YO\,.\,TX}{OX\,.\,YT}.$$

By multiplying

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}\cdot\frac{GC}{BG}\cdot\frac{HA}{CH}\cdot\frac{KB}{AK}=1.$$

But $$\frac{HA}{CH}=\frac{BG}{CB} \text{ and } \frac{KB}{AK}=\frac{CB}{GC}.$$

$$\therefore \frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$$ and AD, BE, CF are concurrent.

Therefore the lines through U, V, W parallel to AD, BE, CF are concurrent in a point S, and if this is taken as the centre of a circumconic, the conic will pass through O. For if it does not let O_1O cut the conic in O' and let $O'O_1$ cut the conic again in ξ. Then $A\xi$ is parallel to the Wallace line, and this must be $O_1O_2O_3$. Hence OO_3, $O'O_3$ are both conjugate to c, which is impossible.

The locus of the orthocentre is a hyperbola through A, B, C with asymptotes parallel to $O_1O_2O_3$, XYZ. We shall denote the orthocentre here by H′. Let OO_1 cut the circumconic again in O_1' (Fig. 5) and from O_1O cut off $O_1K' = O_1'O_1$. Then $AH' = OK'$.

Now if lines be drawn from a fixed point to cut two fixed lines, and if through another fixed point lines be drawn parallel to, equal to, and in the same or opposite sense as, the intercept on the first varying line, the locus of the end of this second varying line is a hyperbola with asymptotes parallel to the fixed lines. The line through K′ parallel to $O_1O_2O_3$ will be a fixed line since $O_1O_2O_3$ will be midway between it and AO_1'. Hence K′O will be the intercept on the secant from O_1 to the lines XYZ and this third fixed line. The locus of H′ is therefore a hyperbola with asymptotes parallel to XYZ and $O_1O_2O_3$, and it evidently passes through A. Similarly it passes through B and C.

The locus of the circumcentre S is a hyperbola with asymptotes parallel to the same lines; it passes through U, V, W. US is parallel and equal to $\frac{1}{2}$H′A. The locus of the mid point of H′A will also be a hyperbola similar to the locus of H′. Now the line joining this point to U is bisected by SH′ in N, the nine point centre. Therefore the locus of N will also be a hyperbola with asymptotes in the same directions.

Suppose O given; then, as is well known, the locus of S is a conic which passes through U, V, W. OO_1 is parallel to SU. Therefore $O\{O_1\ldots\} \barwedge U\{S\ldots\}$. Similarly $O\{O_2\ldots\} \barwedge V\{S\ldots\}$. But $U\{S\ldots\} \barwedge V\{S\ldots\}$ since U, V are on the conic, the locus of S. $\therefore O\{O_1\ldots\} \barwedge O\{O_2\ldots\}$ and $\{O_1\ldots\} \barwedge \{O_2\ldots\}$.

Therefore O_1O_2 envelopes a conic which touches *a* and *b* and similarly touches *c*. To find where this conic touches *a* suppose this line to be a particular Wallace line. Complete the parallelogram BOCZ. Then Z is the corresponding orthocentre. Join AZ and draw OQ parallel to AZ to cut *a* in Q. The conic touches *a* in Q. The points of contact with CA and AB can be found in the same way, and the conic is thus determined.

The locus of M the mid point of OH, where H is the orthocentre, is a conic. If G′ is the centroid of the points A, B, C, O, then $SG' = G'M$, and the locus of M is the image of the locus of S with respect to G′. Since $GH = 2SG$ and $SG = 2GN$ the loci of H and N are conics, the images of the locus of S with respect to G but with dimensions altered in the ratio 2 : 1 and 1 : 2. The locus of S passes through U; and since GA = 2UG, the locus of H passes through A; similarly it passes through B and C. Suppose H is at A. Then OO_2 and OO_3 are parallel to BA and CA, BA being conjugate to CA. S is at the mid point of BC and N bisects SA. The locus of N therefore passes through the mid points of the medians.

Suppose S is at the intersection of AB and OC. The circumconic is now the pairs of lines AB, OC (Fig. 6). The Wallace line

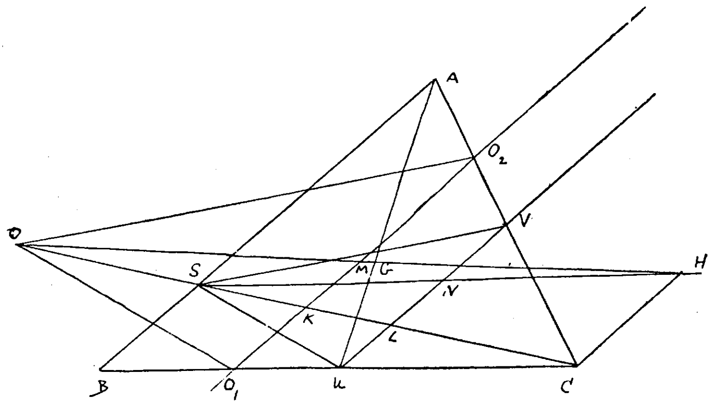


Fig 6.

of O is parallel to AB since OO_2, OO_1 are parallel to SV, SU. Join SG and produce to N so that SG = 2GN. Then since SG : GN = AG : GU, N is on UV, and since SN = NH, H is on the parallel through C to AB. Let O_1O_2 cut OH in M, OC in K, and UV cut OC in L. Then $\frac{CL}{CK} = \frac{CV}{CO_2} = \frac{CS}{CO}$. But $CL = \frac{1}{2}CS$; therefore $CK = \frac{1}{2}CO$ as was to be expected. M lies then on the parallel through N to OC.

The Wallace line of C with respect to the triangle ABO and the circumconic CO, AB is the same line O_1O_2.

§9. Let (Fig. 1) FE meet BC in X, DF meet CA in Y, DE meet AB in Z. Then since D, X are harmonically conjugate with respect to B, C for a system of circumconics through O, $\{D\ldots\} \barwedge \{X\ldots\}$. Similarly $\{E\ldots\} \barwedge \{Y\ldots\}$. But $\{D\ldots\} \barwedge \{E\ldots\}$ since $\{D\ldots\} \barwedge A\{H\ldots\}$, $\{E\ldots\} \barwedge B\{H\ldots\}$ and $A\{H\ldots\} \barwedge B\{H\ldots\}$ because H describes a conic through A, B, C. Both ranges $\{D\ldots\}$, $\{E\ldots\}$ have a common correspondent C. Therefore XYZ passes through a fixed point. XYZ is the polar of H with respect to the triangle ABC, the axis of perspective of ABC, DEF and the radical axis of the circum and nine point conic. The fixed point can be found easily. Let OC intersect AB in S (Fig. 6). H can be found as before and XYZ. Here CZ is CW. Similarly by joining O to B a second position of XYZ is found.

If also instead of O a Wallace line is given, XYZ still passes through a fixed point. Let O be at T (Fig. 5); then AD, BE, CF are AG, BH, CK. Draw the harmonic conjugates of these lines with respect to the sides, and one position of the radical axis XYZ is obtained. These conjugates are the lines through A, B, C bisecting O_2O_3, O_3O_1, O_1O_2. Suppose the point O is at infinity on the locus of O (Fig. 5). Then AD, BE, CF are parallel to the locus of O. Let this locus cut the sides a, b, c in ξ_1, ξ_2, ξ_3. Then the harmonic conjugates of AD, BE, CF are the lines joining ABC to the mid points of $\xi_2\xi_3$, $\xi_3\xi_1$, $\xi_1\xi_2$ and a second position of the radical axis is obtained.

Section III.

In- and Ex-conics.

§10. Four conics homothetic to the circumconic can be drawn to touch the sides unless the circumconic is a hyperbola and the vertices of the triangle are not all on the same branch; in this case the tangent conics are imaginary. Draw the diameters of the circumconic conjugate to the sides a, b, c, and draw tangents at their ends. Thus eight triangles in all are formed each similar to ABC; four of them are equal to the remaining four, but oppositely placed. (See Fig. 7 *a*, *b*.) In these figures $A_1B_1C_1$ and the conic correspond to the triangle ABC and the inscribed conic; $A_1B_2C_2$ (in Fig. 7*b* $A_1'B_3C_3$) and the conic to ABC and the conic touching BC externally; $A_2B_1C_3$ and the conic to ABC and the conic touching CA externally; $A_3B_3C_1$ and the conic to ABC and the conic touching AB externally.

The centres I, I′, I″, I‴ can now be easily found, for S say lies with regard to A_1, B_1, C_1, as I with regard to A, B, C.

A, I, I′ are collinear, and A, I″, I‴ are collinear.

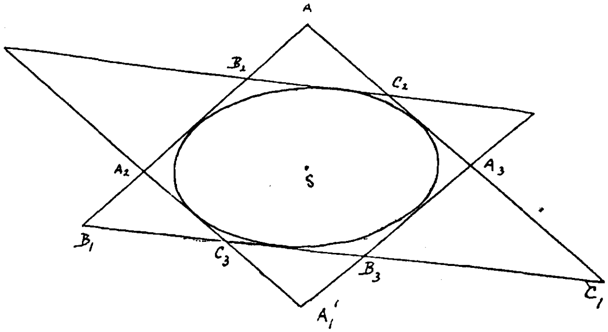
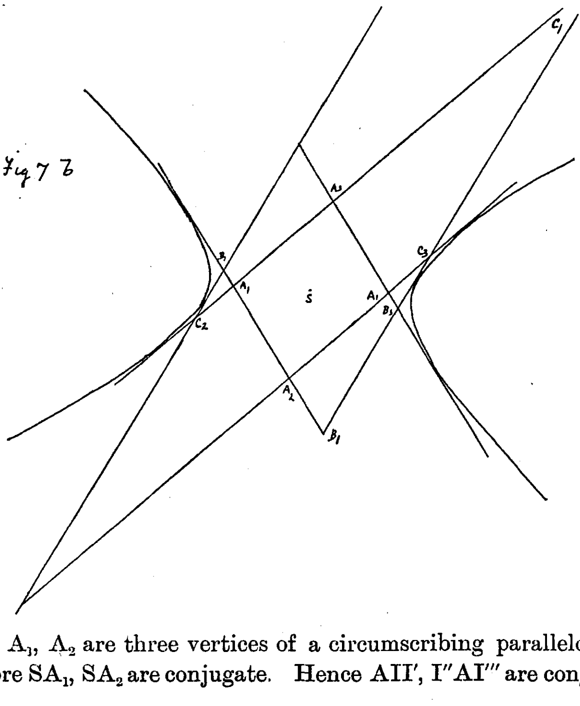


Fig 7a.

Fig 7 b



A_3, A_1, A_2 are three vertices of a circumscribing parallelogram; therefore SA_1, SA_2 are conjugate. Hence AII′, I″AI‴ are conjugate.

Similarly BII″, I′BI‴ and CII‴ and I″CI′ are pairs of conjugates. ABC is therefore the pedal triangle of I′I″I‴ and I is the orthocentre. The circumconic of ABC is the nine point conic of I′I″I‴ and the points, H, G, say, where it meets I″I‴ and II′ are the mid points of I″I‴ and II′. Since GA, HA are conjugate, GH is a diameter. Also since the mid points of the diagonals of the quadrangle BI′CI are collinear GH bisects BC, I″I‴ being the third diagonal of BICI′. Therefore GH is conjugate to BC, and is parallel to the radii r, r_1, r_2, r_3 drawn from I, I′, I″, I‴ to the points of contact of the conics with a. Let GH = 2R.

Then $r_2 + r_3 = 2\text{UH}$, $r_1 - r = 2\text{GU}$.

Therefore if r, r_1, r_2, r_3, R be *any* five parallel radii

$$r_1 + r_2 + r_3 - r = 4\text{R}.$$

The preceding results give a method of inscribing in a conic a triangle whose sides shall be parallel to those of a given triangle. Draw three chords parallel to the sides and find their diameters. Through three of the ends draw tangents giving a circumscribing triangle similar to the given triangle. Through the point of contact of a side draw a parallel to the line joining the opposite vertex to the centre. The three lines thus drawn meet the conic again in the vertices of the required triangle. If the conic is a hyperbola sometimes no such triangle can be inscribed.

§11. If the inscribed ellipse or corresponding hyperbola touch a, b, c in I_1, I_2, I_3, then

$$\frac{\text{AI}_3^2}{c_1^2} = \frac{\text{AI}_2^2}{b_1^2}, \quad \frac{\text{BI}_1^2}{a_1^2} = \frac{\text{BI}_3^2}{c_1^2}, \quad \frac{\text{CI}_1^2}{a_1^2} = \frac{\text{CI}_2^2}{b_1^2}.$$

$$\therefore \quad \frac{\text{AI}_3}{c_1} = \frac{\text{AI}_2}{b_1} = \tfrac{1}{2}\left(\frac{b}{b_1} + \frac{c}{c_1} - \frac{a}{a_1}\right)$$

with corresponding results for the other segments and the other conics.

Let r, R be the radii of the inconic and circumconic along AI.

$$\text{Then } \frac{(\text{AI} - r)(\text{AI} + r)}{\text{R}^2} = \frac{\text{AI}_3^2}{c_1^2} = \left(s - \frac{a}{a_1}\right)^2.$$

$$\therefore \quad \frac{\text{AI}^2}{\text{R}^2} = \left(s - \frac{a}{a_1}\right)^2 + \frac{r^2}{\text{R}^2}. \quad \text{Similarly } \frac{\text{AI}'^2}{\text{R}^2} = s^2 + \frac{r_1^2}{\text{R}^2}. \quad \text{Also } \frac{\text{AI}}{\text{AI}'} = \frac{s - \dfrac{a}{a_1}}{s}$$

and $\frac{r_1}{r}=\frac{4R}{r}+1-\frac{r_2}{r}-\frac{r_3}{r}=\frac{4R}{r}+1-\frac{s}{s-\frac{b}{b_1}}-\frac{s}{s-\frac{c}{c_1}}$. From these equations $\lambda=\frac{r}{R}=\frac{4\left(s-\frac{a}{a_1}\right)\left(s-\frac{b}{b_1}\right)\left(s-\frac{c}{c_1}\right)}{\frac{abc}{a_1b_1c_1}}$. The ratios of similarity for the remaining conics can be immediately written down.

Let r be the conjugate from I to BC. $\frac{BIC}{BAC}=\frac{r}{AD}=\lambda\Big/\frac{AD}{d}=\lambda\Big/\frac{\triangle}{2\frac{a}{a_1}}$.

Adding this to the two corresponding equations, we have $\triangle=\lambda\left(\frac{2a}{a_1}+\frac{2b}{b_1}+\frac{2c}{c_1}\right)=4\lambda s=\frac{\triangle^2}{\frac{abc}{a_1b_1c_1}}$.

$\therefore\ \triangle=\frac{abc}{a_1b_1c_1}$ and $\lambda=\frac{\triangle}{4s}$. For a hyperbola $\triangle^2$ is negative, and for an ellipse positive. Modifications of Casey's and M'Cay's proofs of Feuerbach's theorem will be given in the sections on Inversion and Coaxal Conics. The loci of the in- and ex-centres for a four point system are given in the section on the Cosine Conic.

Section IV.

Isogonal Conjugates.

§12. If O, O′ are two points such that $\frac{OO_2\,.\,O'O_2'}{e^2}=\frac{OO_3\,.\,O'O_3'}{f^2}$, then AO, AO′ will be called isogonally conjugate with respect to the angle A. O_2, O_2', O_3, O_3' lie on a homothetic conic (Fig. 8).

Let P be the mid point of OO′, and consider the homothetic conic with centre P passing through O_3. It passes also through O_3' since the conjugate from P to AB bisects O_3O_3'. Join $O_3'P$ and produce to cut OO_3 in L; $OL=O_3'O'$, and L is a point on the conic. Suppose this conic does not pass through O_2, O_2'. Let OO_2 cut the conic in M and O_2O produced cut it in N, and let $O'O_2'$ cut NP in K. K is then a point on the conic, and NO = O′K.

Also $\frac{OO_3.O'O_3'}{f^2}=-\frac{OO_3.OL}{f^2}=-\frac{ON.OM}{e^2}=\frac{OM.O'K}{e^2}=\frac{OO_2.O'O_2'}{e^2}$

$\therefore\ OM\,.\,O'K=OO_2\,.\,O'O_2'$.

Therefore if $OM > OO_2$, $O'K < O'O_2'$ and *vice versâ*.

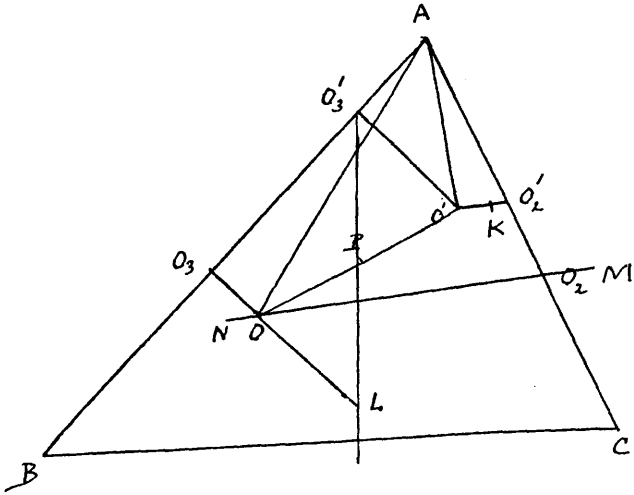


Fig 8.

In the trapezium $OO_2O_2'O'$ the radius, through P the mid point of OO', parallel to OO_2 or $O'O_2'$ bisects MK and therefore is conjugate to it. MK is then parallel to AC and the above inequality could not hold. The conic therefore passes through O_2, O_2'. If AO, BO, CO are concurrent their isogonal conjugates with respect to the three angles are concurrent in a point O' the isogonal conjugate of O. The feet of the six conjugates to the sides from O, O' lie by the above theorem on a homothetic conic, whose centre is the mid point of OO'.

If the conic is a hyperbola and the three vertices are on one branch, O and O' lie both inside or both outside the triangle, as in the case of the circle and ellipse. If A is on a different branch from B and C, O and O' are on the same side of AB or AC but on different sides of BC, because then e and f are then imaginary and their squares negative.

§13. Pairs of isogonal conjugates with respect to AB, AC form a pencil in involution with AII', $I''AI'''$ as double lines.

The in- and ex-centres are evidently their own isogonal conjugates. Let two isogonal conjugates AM, AN cut a in M and N and AI, AI'' cut a in X, Y.

Then $\frac{XX_2}{e}=\frac{XX_3}{f}$, $\frac{YY_2}{-e}=\frac{YY_3}{f}$,

$$\frac{MM_2 . NN_2}{e^2}=\frac{MM_3 . NN_3}{f^2}.$$

$$\therefore \quad \frac{MM_2 . NN_2}{XX_2 . YY_2}+\frac{MM_3 . NN_3}{XX_3 . YY_3}=0.$$

$$\therefore \quad \frac{MC . NC}{XC . YC}+\frac{MB . NB}{XB . YB}=0.$$

Let P be the mid point of XY.

$$(PC-PM)(PC-PN)(PB-PX)(PB+PX)$$
$$+(PB-PM)(PB-PN)(PC-PX)(PC+PX)=0.$$

$$\therefore \quad PM . PN(PB^2-2PX^2+PC^2)+2PB^2 . PC^2-PX^2(PB^2+PC^2)$$
$$=(PM+PN)(PC . PB^2-PC . PX^2+PB . PC^2-PB . PX^2).$$

But since AB, AC are tangents to the conic with centre I, and AX is a diameter and AY is parallel to the diameter conjugate to AX, A{BC, XY} is a harmonic pencil.

$$\therefore \quad PB . PC=PX^2.$$

$$\therefore \quad PM . PN(PB-PC)^2=PX^2(PB-PC)^2,$$

or $PM . PN=PX^2$, and

A{MN, XY} is a harmonic pencil. This proves the theorem. In the case where I, I′, I″, I‴ are imaginary two of the three involutions are elliptic. Without considering imaginary double rays we could prove that A{BC, GK, MN} is an involution where K is the symmedian centre which, as will be proved in a subsequent section, is the isogonal conjugate of G.

§14. If a parallel to BC cut the circumconic in O, O′, then AO, AO′ are isogonally conjugate.

Let OO′ cut AB, AC in G and H.

$$\frac{OO_3 . O'O_3'}{f^2}=\frac{CF^2}{f^2}.\frac{GO.GO'}{a^2}=\frac{CF^2}{f^2}.\frac{GB.GA}{c_1^2}.\frac{a_1^2}{a^2}$$

$$=\frac{\Delta^2}{4c^2}. GB . GA . \frac{a_1^2}{a^2} \quad (§4)$$

$$=\frac{\Delta^2}{4}.\frac{HC . HA}{b^2}.\frac{a_1^2}{a^2}$$

$$=\frac{OO_2 . O'O_2'}{e^2}.$$

The double rays are therefore the lines joining A to the ends of the diameter bisecting BC. From this a second construction for finding the in- and ex-centres may be derived.

§15. O_2O_3 is conjugate to AO′ and $O_2'O_3'$ conjugate to AO. Let O_3O, O_2O cut AC, AB in P and Q (Fig. 9).

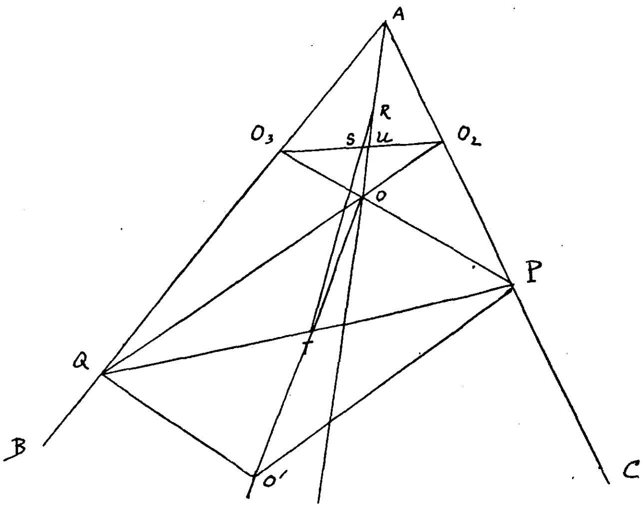


Fig 9

AO, O_2O_3, PQ are the diagonals of the quadrangle AO_2OO_3, and their mid points R, S, T are collinear. Since QO_2, PO_2 and QO_3, PO_3 are pairs of conjugates, the homothetic conic on PQ as diameter passes through O_2, O_3.

$$\therefore \quad \frac{QO.OO_2}{e^2}=\frac{PO.OO_3}{f^2}.$$

Complete the parallelogram POQO′.

$$\text{Then} \quad \frac{O'Q.OO_3}{f^2}=\frac{O'P.OO_2}{e^2}.$$

AO′ is then isogonally conjugate to AO, and we have a simple construction for finding AO′ when AO is given. Let U be the intersection of AO and O_2O_3. U is the pole of PQ with respect to the homothetic conic on AO as diameter and passing through O_2, O_3.

Therefore AO joining U to the centre is conjugate to PQ. RS bisecting O_2O_3 is conjugate to it, and the parallel AO′ to RST is therefore conjugate to O_2O_3.

§16. The isogonal curve of a line is a circumconic.

Let A′ be any point on the circumconic. Draw A′B′, A′C′ parallel to AB and AC. BA′, BC′ are isogonally conjugate, as also CA′, CB′. The hexagon ABC′A′B′C has two pairs of parallel opposite sides; therefore CB′, BC′ are parallel and the isogonal conjugate of A′, that is, any point on the circumconic is at infinity.

Let AI, AI″ cut a given line in X and Y. Let P be any point in this line and AQ be isogonally conjugate to AP. A{XY, PQ} is a harmonic pencil and $\{P...\} \barwedge A\{Q...\}$. If BI, BI″ cut the same line in X′, Y′ and BR is isogonally conjugate to BP, {X′Y′, PR} is harmonic and $\{P...\} \barwedge B\{R...\}$.

$\therefore$ $A\{Q...\} \barwedge B\{R...\}$. The intersection of AQ and BR, which is the isogonal conjugate of P, therefore describes a conic passing through A, B and similarly through C. According as the line cuts, touches, or does not cut the circumconic, the isogonal curve is a hyperbola, parabola, or ellipse.

§17. S and H are isogonal conjugates (Fig. 1).

Let BH cut the circumconic in E′.

$$\frac{BH.HE'}{e^2} = \frac{AH.HD'}{d^2}.$$

$$\therefore \quad \frac{2SV.2HE}{e^2} = \frac{2SU.2HD}{d^2}.$$

The result follows from a second corresponding equation.

Section V.

Antiparallels.

§18. Let MN parallel to the tangent to the circumconic at A cut AC, AB in M and N; MN is said to be antiparallel to BC. A homothetic conic will pass through B, C, M, N.

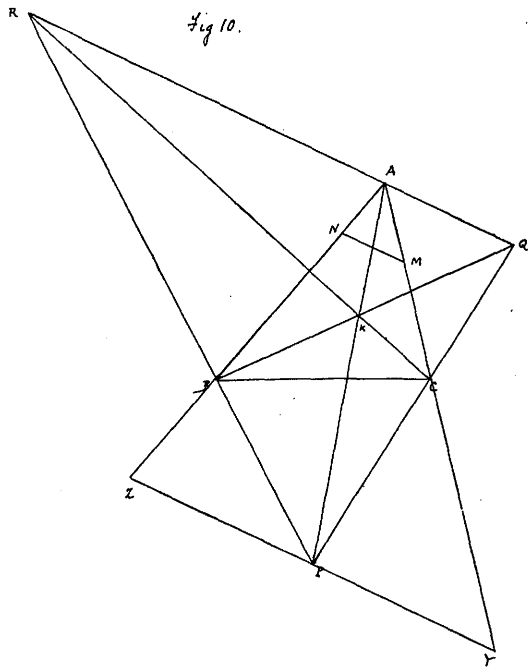
The homothetic conic through B, C, M, the circumconic and the lines BA, CA have the common chord BC. Their other three chords are therefore concurrent. The second chord of the two conics is at infinity, and the second chord of the circumconic and the two lines is the tangent at A. The second chord therefore of the conic BCM and the lines is parallel to the tangent at A and is therefore MN, and N lies on the conic.

The sides EF, FD, DE (Fig. 1) of the pedal triangle are antiparallel to a, b, c and conjugate to SA, SB, SC. The diameter therefore of the nine point conic parallel to SA bisects EF.

The lines through A, B, C bisecting the three sets of antiparallels are concurrent.

Draw the antiparallel through P to BC cutting AC, AB in Y, Z (Fig. 10). The triangles PCY, QCA are similar, and the triangles PBZ, RBA.

Fig 10.



$$\therefore \frac{PY}{PC} = \frac{QA}{QC} \text{ and } \frac{ZP}{PB} = \frac{AR}{RB}.$$

$$\therefore \frac{ZP}{PY} = \frac{RA}{AQ} \cdot \frac{QC}{CP} \cdot \frac{PB}{BR} = 1$$ since PA, QB, CR are concurrent in a point K,

Then ZP = PY and the three lines PA, QB, CR, called the symmedians, bisect the antiparallels and are concurrent in the symmedian centre K.

Locus of K.

§19. The locus of K for a system of circumconics passing through a fixed point O is the polar of O with respect to the triangle ABC. Let OK (Fig. 11) cut a, b, c in X, Y, Z, and let the tangents at A and B cut OK in L and M.

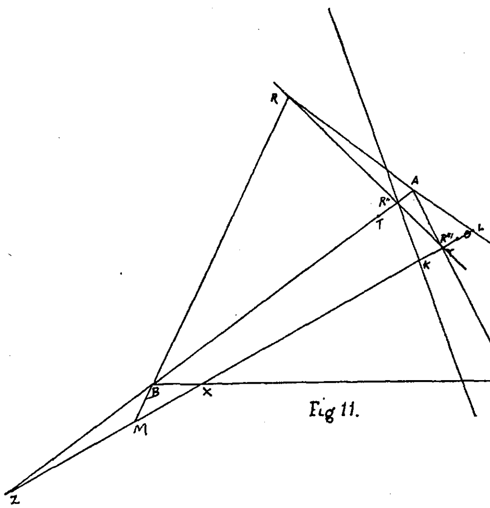


Fig 11.

$$A\{ABCO\} \barwedge B\{ABCO\}.$$

$$\therefore \{LZYO\} \barwedge \{ZMXO\}.$$

$$\therefore \frac{LZ}{LY}\cdot\frac{OY}{OZ}=\frac{ZM}{ZX}\cdot\frac{OX}{OM}. \quad \text{(i)}.$$

Also by a well-known theorem A{BC, KL}, B{CA, KM} are harmonic pencils.

$\therefore$ {ZY, KL} and {XZ, KM} are harmonic.

$$\therefore \frac{LZ}{LY}=\frac{KZ}{YK} \text{ (ii), and } \frac{ZM}{MX}=\frac{ZK}{XK} \text{ (iii).}$$

By multiplying (i) and (ii)

$$\frac{OY}{OZ}\cdot\frac{KZ}{YK}\cdot\frac{ZX}{OX}=\frac{ZM}{OM}=\frac{ZM}{ZM-ZO}.$$

But from (iii) $\frac{ZM}{ZX}=\frac{ZK}{ZK+XK}$.

$$\therefore \frac{OY.KZ.ZX}{OZ.YK.OX}=\frac{ZM}{ZM-ZO}=\frac{ZK.ZX}{ZK.ZX-ZO(ZK+XK)}.$$

$$\therefore OY(KZ.ZX+ZO.ZK+ZO.XK)=OZ.YK.OX.$$

After division by OZ . OX . OK . OY

$$\frac{(OZ-OK)(OX-OZ)}{OZ.OX.OK}+\frac{OZ-OK}{OK.OX}+\frac{OX-OK}{OK.OX}=\frac{OK-OY}{OY.OK}.$$

$\therefore \frac{3}{OK}=\frac{1}{OX}+\frac{1}{OY}+\frac{1}{OZ}$ and K is on the polar of O with respect to the triangle.

If O is any point on a circumconic and K is the symmedian centre, $\frac{3}{OK}=\frac{1}{OX}+\frac{1}{OY}+\frac{1}{OZ}$.

Loci of P, Q, R.

§20. Let CKR cut AB in T (Fig. 11). Then {CT, KR} is a harmonic range since B{CA, KR} is a harmonic pencil. Because K describes a straight line, R also describes a straight line which passes through the intersection of AB and the locus of K at a point R″. C{AB, OR″} is then a harmonic pencil, or CR″ is the harmonic conjugate of CO with respect to CA, CB. Let R‴ be the intersection of CA and the locus of R; K now lies on CA, so that {CA, KR‴} is harmonic. BR‴ therefore passes through O, and the locus is determined. The loci of P and Q may be similarly found.

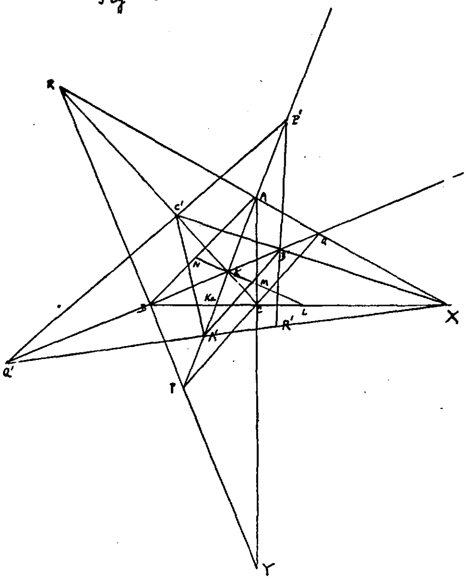
Cosymmedian Triangles.

§21. The results of this paragraph may be derived by projection from the circle. Let AK, BK, CK meet the conic again in A′, B′, C′ (Fig. 12) and let P′, Q′, R′ be the poles of B′C′, C′A′, A′B′.

In the quadrangle BCB′C′ by a well-known theorem the line joining the poles of BC and B′C′ passes through the intersection of the diagonals BB′, CC′. PP′ then passes through K, but KP passes through A, A′. P′, A, A′, P are therefore collinear; similarly Q′, B, B′, Q and R′, C, C′, R. K is then the symmedian centre of A′B′C′. P, Q, R, P′, Q′, R′ lie on a conic.

Let the tangents at A, B, C cut the opposite sides in X, Y, Z. X, Y, Z are collinear, $\{BC, K_aX\}$ is harmonic; therefore K_a is on the polar of X.

Fig. 12



The tangents at B and C meet in P; therefore P is also on the polar of X. PA is the polar of X and as XA is a tangent, XA′ is also a tangent. BC, QR, Q′R′ meet in X. Similarly B′C′ passes through X.

The four triangles ABC, A′B′C′, PQR, P′Q′R′ are in perspective with a common centre and axis of perspective. XYZ is the polar of K, with respect to the conic and the two triangles ABC, A′B′C′.

Let AA′, BB′, CC′ cut XYZ in X′, Y′, Z′. B′C, BC′ intersect in X′. K is the intersection of the diagonals BB′, CC′ of the quadrangle BCB′C′, and therefore the opposite sides BC′, B′C meet on the polar of K. If AA′ cut B′C′ in K_a', $\{B'C', K_a'X\}$ is harmonic, and therefore AA′ must pass through the intersection of BC′ and B′C. ABC, A′B′C′ are then quadruply in perspective.

The intersections of non-corresponding sides of ABC, A′B′C′ determine a hexastigm whose fifteen connectors pass through

But AF . BD . CE = AE . BF . CD.

By Carnot's theorem then the six points X, X′, Y, Y′, Z, Z′ lie on a conic.

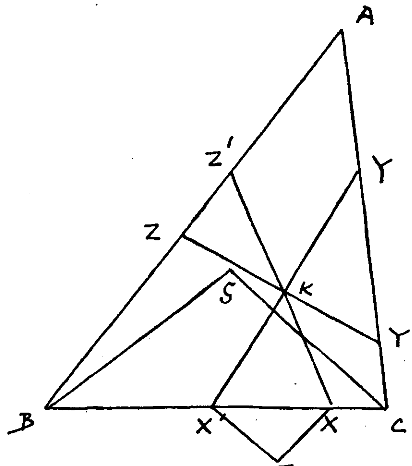


Fig. 13.

Let a_2, b_2, c_2, be the radii of this conic parallel to a, b, c.

Then $\dfrac{AZ'.AZ}{AY.AY'}=\dfrac{c_2^2}{b_2^2}$.

But $\dfrac{AZ'.AZ}{AY.AY'}=\dfrac{c.AF}{b.AE}=\dfrac{c_1^2}{b_1^2}$.

$\therefore\ \dfrac{b_2^2}{b_1^2}=\dfrac{c_2^2}{c_1^2}=\dfrac{a_2^2}{a_1^2}$ and the conic is homothetic with the circum-conic. The homothetic conic on YZ (Fig. 10) as diameter passes through B, C; this is an ex-cosine conic.

$$BX'+X'X+XC=a \quad \text{(Fig. 13)}.$$

Taking BC, CA, AB as positive directions we have

$$\frac{BX'}{a}=\frac{X'Z}{b}=\frac{Y'Y}{b} \text{ and } \frac{XC}{a}=\frac{Y'X}{c}=\frac{Z'Z}{c}.$$

$$\therefore\ \frac{X'X}{a}+\frac{Y'Y}{b}+\frac{Z'Z}{c}=1 \quad \text{(i)}.$$

Also $\dfrac{CX(CX+XX')}{a_1^2}=\dfrac{CY'(CY'+Y'Y)}{b_1^2}.$

$$\therefore \left(\frac{X'X}{a}+\frac{Z'Z}{c}\right)\frac{a^2}{a_1^2}=\left(\frac{Y'Y}{b}+\frac{Z'Z}{c}\right)\frac{b^2}{b_1^2} \text{ (ii).}$$

Again $\dfrac{BX'(BX'+X'X)}{a_1^2}=\dfrac{BZ(BZ+ZZ')}{c_1^2}.$

$$\therefore \left(\frac{X'X}{a}+\frac{Y'Y}{b}\right)\frac{a^2}{a_1^2}=\left(\frac{Z'Z}{c}+\frac{Y'Y}{b}\right)\frac{c^2}{c_1^2} \text{ (iii).}$$

From these three equations we obtain

$$\frac{\dfrac{X'X}{a}}{\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}-\dfrac{a^2}{a_1^2}}=\frac{\dfrac{Y'Y}{b}}{\dfrac{c^2}{c_1^2}+\dfrac{a^2}{a_1^2}-\dfrac{b^2}{b_1^2}}=\frac{\dfrac{Z'Z}{c}}{\dfrac{a^2}{a_1^2}+\dfrac{b^2}{b_1^2}-\dfrac{c^2}{c_1^2}}=\frac{1}{\Sigma\dfrac{a^2}{a_1^2}}.$$

YX is conjugate to X'X. Therefore, if S is the centre of the circumconic and P the pole of BC, YX is parallel to SP; also ZX is parallel to SR and ZY to SQ. XY, X'Z'; ZY, X'Y'; Y'Z', XZ are pairs of parallel lines. Since XY is conjugate to BC, YZ to CA and ZX to AB, we see that, if through the vertices of a triangle conjugates to the sides be drawn, a new triangle is formed whose symmedian centre is the circumcentre of the original triangle. By considering X'Y'Z' we see that two such triangles can be drawn, the conjugates being taken in the reverse order, and that both triangles have as symmedian centre the original circumcentre. Similarly within a given triangle two triangles can be inscribed whose sides (in reverse order) are conjugate to those of the first triangle, and both have the same circumcentre, namely the symmedian centre of the original triangle. XYZ and X'Y'Z' are similar, their sides being parallel.

$KK_2=\frac{1}{2}X'Y'.$ $\quad \dfrac{X'Y'}{BE}=\dfrac{X'C}{a}=\dfrac{X'X}{a}+\dfrac{XC}{a}=\dfrac{X'X}{a}+\dfrac{Z'Z}{c}=\dfrac{\dfrac{2b^2}{b_1^2}}{\Sigma\dfrac{a^2}{a_1^2}}.$

$$\therefore \frac{KK_2}{e}=\frac{\dfrac{b^2}{b_1^2}\cdot\dfrac{BE}{e}}{\Sigma\dfrac{a^2}{a_1^2}}=\frac{\dfrac{b}{b_1}\Delta}{2\Sigma\dfrac{a^2}{a_1^2}}.$$

$$\therefore \frac{KK_1}{d}\Big/\frac{a}{a_1}=\frac{KK_2}{e}\Big/\frac{b}{b_1}=\frac{KK_3}{f}\Big/\frac{c}{c_1}=\frac{\Delta}{2\Sigma\frac{a^2}{a_1^2}}.$$

$$\therefore \frac{X'Y'}{e}\Big/\frac{b}{b_1}=\frac{Y'Z'}{f}\Big/\frac{c}{c_1}=\frac{Z'X'}{d}\Big/\frac{a}{a_1}.$$

Hence in an extended sense the triangles ABC, Y′Z′X′, ZXY are similar. This is a particular case of a theorem which will be taken up in the section on the point O theorem.

If the tangents at X and X′ meet in T, the triangle TXX′ is similar to SBC for TX is conjugate to Z′X, which is conjugate to SB.

$$\therefore \frac{TK_1}{SU}=\frac{X'X}{a}=\frac{\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}}{\Sigma\frac{a^2}{a_1^2}}.$$

$$\therefore \frac{TK_1}{d}=\frac{\frac{a}{a_1}\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}\right)^2}{2\Delta\Sigma\frac{a^2}{a_1^2}} \quad (\S3).$$

$$\therefore KT=KK_1+K_1T=\frac{2ab^2c^2d}{a_1b_1^2c_1^2}\Big/\Delta\Sigma\frac{a^2}{a_1^2}.$$

$KT \,.\, KK_1=d_1^2$ where d_1 is the radius of the cosine conic conjugate to a.

The ratio of similarity $\frac{d_1}{d}$ is therefore $\frac{abc}{a_1b_1c_1}\Big/\Sigma\frac{a^2}{a_1^2}$.

Let UU_2, UU_3 be conjugate to b, c.

$$\frac{KK_2 \,.\, UU_2}{e^2}=\frac{KK_2 \,.\, BE}{2e^2}=\frac{\Delta^2}{8\Sigma\frac{a^2}{a_1^2}}=\frac{KK_3 \,.\, UU_3}{f^2}.$$

From this and the corresponding equations G and K are isogonal conjugates.

§24. K is the centroid of the triangle $K_1K_2K_3$.

If through G a parallel GL be drawn to BC, the pencil G{BC, UL} or G{VW, UL} is harmonic. Let GV′, GW′, GU′, GL′

be conjugate to GV, GW, GU, GL. Then G{VV′, WW′, UU′, LL′} is a pencil in involution. Therefore G{V′W′, U′L′} is harmonic. Since G and K are isogonally conjugate K_2K_3, K_3K_1, K_1K_2 are conjugate to AG, BG, CG or GU, GV, GW (§15). If then K_1J be drawn parallel to K_2K_3, $K_1\{K_3K_2, JK\}$ is harmonic and K_1K bisects K_2K_3. Similarly K_2K, K_3K bisect K_3K_1 and K_1K_2.

§25. *Loci of in- and ex-centres.*

From §10 we see that A, B, C are the intersections of the diagonals of the quadrangle II′I″I‴, so that ABC is self conjugate with respect to all conics circumscribing the quadrangle.

A{BC, II″}, B{CA, II‴}, C{AB, II′} are harmonic pencils. Consider a system of circumconics of ABC through a fixed point O, and let J, J′, J″, J‴ be a second set of centres. Let a conic be drawn through I, I′, I″, I‴, J. This conic will also pass through J′, J″, J‴, for since A is the pole of BC with respect to the conic, and since C{AB, JJ′} is a harmonic pencil, J′ must be the second intersection of AJ with the conic. Similarly J‴ and J″ are the second intersections of CJ and CJ″. All the sets of in- and ex-centres must lie therefore on a certain definite conic with regard to which ABC is a self conjugate triangle.

Consider now the circumconic OABC in which I″, I‴, say, lie on AO. AII′ is then determined as the harmonic conjugate of AO with respect to AB, AC. AK is determined as the harmonic conjugate of AG with respect to AO and AI, and as K lies on the polar of O with respect to the triangle, K is found. CI, CI′ are determined as the double rays of the pencil in involution C{AB, GK}. Thus I, I′, I″, I‴ are found for this circumconic. Now I being considered the orthocentre of the triangle I′I″I‴, the circumconic of ABC thus considered is the corresponding nine point conic of I′I″I‴. It therefore bisects I″I‴, and as it cuts it in O, this point bisects I″I‴. If we take a second circumconic OABC in which the new points I′, I″ lie on CO, we see similarly that O is the mid point of this chord I′I″. Since O bisects the chords, the locus of the in- and ex-centres is the conic with centre O with respect to which ABC is a self conjugate triangle. This conic only cuts two sides AB, BC, say, of ABC. Produce AB to M and CB to N. Then ABM, CBN divide the curve into four parts, each of which is the locus of one of the centres.

SECTION VII.

The Lemoine Conic.

§26. Through K draw Z′Y parallel to BC cutting AB, AC in Z′, Y, XY′ parallel to AB cutting CB, CA in X, Y′, and ZX′ parallel to CA cutting BC, BA in X′, Z.

X, X′, Y, Y′, Z, Z′ lie on a homothetic conic.

AZ = Y′K, CY = X′K, BX = Z′K, AY′ = ZK, CX′ = YK, BZ′ = XK (Fig. 14).

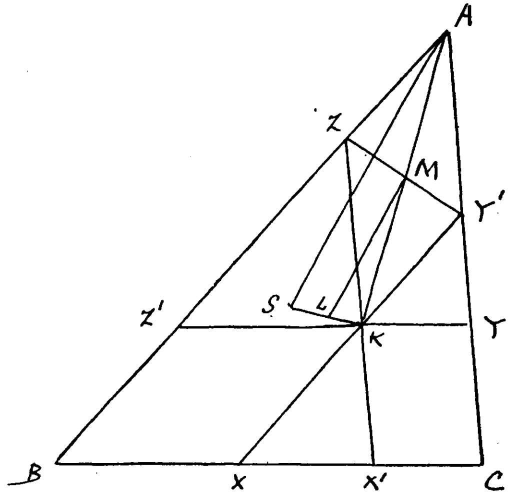


Fig. 14.

$$\therefore \frac{AZ \cdot AZ' \cdot BX \cdot BX' \cdot CY \cdot CY'}{AY' \cdot AY \cdot CX' \cdot CX \cdot BZ' \cdot BZ} = \frac{Y'K}{ZK} \cdot \frac{c}{b} \cdot \frac{Z'K}{YK} \cdot \frac{a}{c} \cdot \frac{X'K}{XK} \cdot \frac{b}{a}$$

$= \frac{KY'}{KY} \cdot \frac{Z'K}{ZK} \cdot \frac{X'K}{XK} = \frac{c}{a} \cdot \frac{a}{b} \cdot \frac{b}{c} = 1.$ Therefore the points lie on a conic.

Let a_2, b_2, c_2 be the radii of this conic parallel to a, b, c.

ZY′ is bisected by AK since AY′KZ is a parallelogram. ZY′ is therefore antiparallel to BC and parallel to EF. The triangles AZY′, AFE are similar.

$$\therefore \frac{AZ}{AY'} = \frac{AF}{AE}. \quad \therefore \frac{AZ \cdot AZ'}{AY \cdot AY'} = \frac{AF \cdot c}{AE \cdot b} = \frac{c_1^2}{b_1^2}.$$

$\therefore \ \dfrac{c_2^2}{c_1^2} = \dfrac{b_2^2}{b_1^2}$ and similarly each ratio is equal to $\dfrac{a_2^2}{a_1^2}$ and the conics are homothetic.

SA is conjugate to ZY′. Hence the parallel through L, the mid point of SK, is conjugate to ZY′, and it bisects ZY′; L is then the centre of the conic.

$$\frac{CY.CY'}{CX'.CX} = \frac{b_1^2}{a_1^2}. \quad \therefore \quad \frac{CY}{CX'} \cdot \frac{b}{a} = \frac{b_1^2}{a_1^2}.$$

$$CY = X'K \text{ and } \frac{X'K}{XX'} = \frac{b}{a}. \quad \therefore \quad \frac{XX'}{CX'} \cdot \frac{b^2}{a^2} = \frac{b_1^2}{a_1^2}.$$

$$\therefore \ CX' \Big/ \frac{b^2}{b_1^2} = X'X \Big/ \frac{a^2}{a_1^2} = XB \Big/ \frac{c^2}{c_1^2} = a \Big/ \Sigma \frac{a^2}{a_1^2}.$$

$$\frac{Z'K}{KY} = \frac{BX}{X'C} = \frac{c^2}{c_1^2} \Big/ \frac{b^2}{b_1^2}.$$

Let AK cut BC in K_a. Then $BK_a \Big/ \dfrac{c^2}{c_1^2} = K_aC \Big/ \dfrac{b^2}{b_1^2} = a \Big/ \left(\dfrac{b^2}{b_1^2} + \dfrac{c^2}{c_1^2}\right)$.

Since $\dfrac{XX'}{\dfrac{a^2}{a_1^2}} = \dfrac{a}{\Sigma \dfrac{a^2}{a_1^2}}$ we have

$$\frac{XX'}{a} \cdot \frac{a_1^2}{a^2} = \frac{YY'}{b} \cdot \frac{b_1^2}{b^2} = \frac{ZZ'}{c} \cdot \frac{c_1^2}{c^2} = \frac{1}{\Sigma \dfrac{a^2}{a_1^2}}.$$

Let λ be the ratio of similarity of this conic to the circumconic and let M be the mid point of ZY′.

$$\text{Then } \frac{ZM^2}{p^2} + \frac{LM^2}{SA^2} = 4\lambda^2 \text{ or } \frac{ZY'^2}{p^2} = 16\lambda^2 - 1.$$

$$\therefore \ \frac{Y'Z}{p} = \frac{Z'X}{q} = \frac{X'Y}{r} = \sqrt{(16\lambda^2 - 1)}.$$

Now Y′Z is equal to the radius of the cosine conic parallel to Y′Z. If p'' be the radius of the Lemoine conic parallel to Y′Z

$$16\lambda^2 - 1 = \frac{Y'Z^2}{p^2} = \frac{a^2b^2c^2}{a_1^2b_1^2c_1^2} \Big/ \left\{\Sigma \frac{a^2}{a_1^2}\right\}^2 \quad (\S 23).$$

$$\therefore \ \lambda = \frac{\sqrt{\left[\left\{\Sigma \dfrac{a^2}{a_1^2}\right\}^2 + \dfrac{a^2b^2c^2}{a_1^2b_1^2c_1^2}\right]^2}}{4\Sigma \dfrac{a^2}{a_1^2}}.$$

§27. If ZY′ meet a in α (Fig. 14), XZ′ meet b in β, and YX′ meet c in γ, then α, β, γ are collinear, and the line is the radical axis of the circumconic and Lemoine conic.

Let ZY′ cut the circumconic in Z_1, Y_1'.

Then $\dfrac{\alpha Z_1 \,.\, \alpha Y_1'}{p^2} = \dfrac{\alpha C \,.\, \alpha B}{a_1^2}$. Also $\dfrac{\alpha Z \,.\, \alpha Y'}{p^2} = \dfrac{\alpha C \,.\, \alpha B}{a_1^2}$ for a homothetic conic passes through B, C, Y′, Z. Hence $\alpha Z \,.\, \alpha Y' = \alpha Z_1 \,.\, \alpha Y_1'$, so that α and similarly β, γ are on the radical axis of the conics. Since the sides ZY′, XX′ of the quadrangle ZY′X′X meet in α and the diagonals ZX′, XY′ meet in K, the polar of α with respect to the Lemoine conic passes through K. Similarly K is on the polars of β and γ, so that K is the pole of $\alpha\beta\gamma$ with respect to the Lemoine conic. $\alpha\beta\gamma$ is also the Pascal line of the hexagon ZY′YX′XZ′.

If ZY′ meet X′Y, Z′X in Q and R and X′Y, Z′X meet in P, the triangles ABC, PQR are in perspective with K as centre, and $\alpha\beta\gamma$ as axis of perspective.

It is already proved that the corresponding sides meet in $\alpha\beta\gamma$. The conjugates from Q and K to a are in the ratio of the conjugates from Q and Y to a, since Z′Y is parallel to a. This ratio is QX′ : YX′ = QZ : Y′Z since ZX′ and b are parallel. Hence the ratios of the conjugates from Q and K to a and c are equal, so that B, K, Q are collinear. Similarly C, K, R and A, K, P are collinear.

It is easily seen that L is the centre of a homothetic conic touching the sides of the triangle PQR and that this conic is equal to the nine point conic.

§28. Let ZY, Z′Y′ meet in 1, ZY, Y′X′ in 2, Y′X′, YX in 3, YX, X′Z′ in 4, X′Z′, XZ in 5, and XZ, Z′Y′ in 6 (Fig. 15).

Consider the hexagon ZYXY′X′Z′. The sides ZZ′, XY′ are parallel; the line joining 2, 4 is therefore parallel to c. Similarly the lines 46 and 62 are parallel to b and a.

The triangle 624 is therefore similar to ABC. Since Z′Y, XX′ are parallel, the polar of 4 with respect to the Lemoine conic is parallel to XX′. Therefore L4 bisects XX′ and is conjugate to 62. L is then the orthocentre of 246 and L, H and the centre of similitude of the triangles ABC, 246 are collinear. Consider the hexagon Y′Z′X′ZYX. Y′Z′, ZY meet in 1, Z′X′, YX in 4, X′Z, XY′ in K. Therefore 1, 4, K are collinear, and the triangles 135, 462 are in perspective with K as centre of perspective.

The triangles $L_1L_2L_3$, 426 are in perspective, the centre being L.

Consider the hexagon ZY′XZ′YX′. If the sides ZY′, Z′Y meet in α_1, Y′X, YX′ in γ_1, XZ′, X′Z in β_1, then by Pascal's theorem α_1, β_1, γ_1, are collinear.

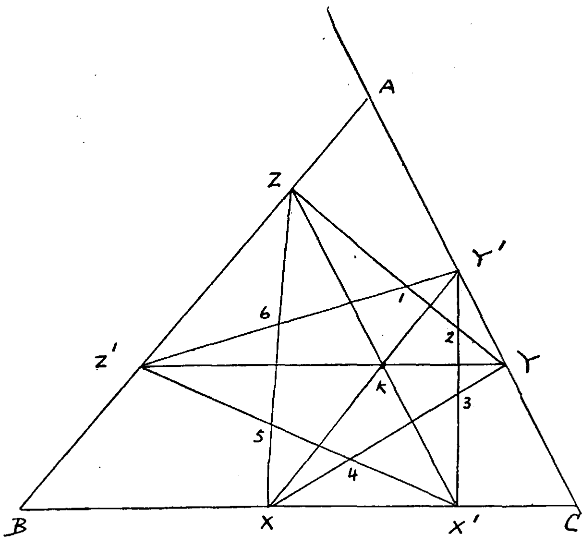


Fig. 15.

$A\alpha_1$ being the third diagonal of the quadrangle ZZ′YY′, α_1 is the pole of A1 with respect to the Lemoine conic. Similarly β_1 and γ_1 are the poles of 5B and 3C. Therefore A1, B5, C3 are concurrent in the pole of the Pascal line of ZY′XZ′YX′.

Further properties will be given in the section on the Brocard points.

Section VIII.

Tucker's Conics.

§29. Divide KA, KB, KC at L, M, N in the ratio $\mu : 1$. From LK, MK, NK cut off LL′, MM′, NN′ equal to AL, BM, CN (Fig. 16).

Let L′M′ cut a, b in X, Y′, M′N′ cut b, c in Y, Z′, N′L′ cut a, c in X′, Z. L′M′, M′N′, N′L′ are parallel to c, a, b.

Since AZL′Y′ is a parallelogram, ZL = LY′ and ZY′ being bisected by AK is antiparallel to BC. Similarly Z′X, X′Y are antiparallel to b, c. Taking AB, BC, CA as positive directions

$$-\frac{AZ'}{AY}=\frac{c}{b};\quad -\frac{BZ}{BX'}=\frac{c}{a};\quad -\frac{CX}{CY'}=\frac{a}{b}.$$

$$\therefore \frac{AZ.AZ'.BX.BX'.CY.CY'}{AY'.AY.CX'.CX.BZ'.BZ}=-\frac{AZ.BX.CY}{AY'.CX'.BZ'}$$

$$=-\frac{Y'L'.Z'M'.X'N'}{ZL'.YN'.XM'}$$

$$=-\frac{Z'M'}{ZL'}\cdot\frac{Y'L'}{YN'}\cdot\frac{X'N'}{XM'}$$

$$=-\frac{M'N'}{L'N'}\cdot\frac{L'M'}{N'M'}\cdot\frac{N'L'}{M'L'}=1.$$

Therefore the six points X, X′, Y, Y′, Z, Z′ lie on a conic.

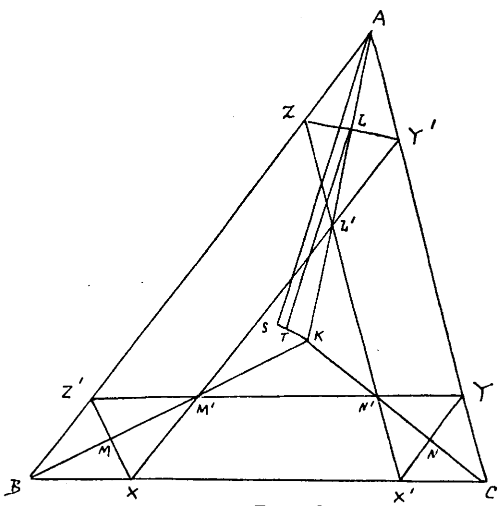


Fig 16.

Let a_2, b_2, c_2 be the radii of this conic parallel to a, b, c.

Then $\frac{AZ.AZ'}{AY'.AY}=\frac{c_2^2}{b_2^2}$. But $-\frac{AZ'}{AY}=\frac{c}{b}$ and since ZY′ is antiparallel to BC, $\frac{AZ}{AY'}=\frac{AF}{AE}$. $\therefore\ \frac{c_2^2}{b_2^2}=\frac{c_1^2}{b_1^2}$. $\therefore\ \frac{a_1^2}{a_2^2}=\frac{b_1^2}{b_2^2}=\frac{c_1^2}{c_2^2}$

and the conic is homothetic with the circumconic. The centre T is on KS, and KT : TS $= \mu : 1$.

If p' is the radius of the cosine conic parallel to ZY', $\frac{\text{ZL}}{p'} = \frac{1}{\mu+1}$.

Let p'' and x be the radii of the Tucker conic parallel to ZY' and TL.

Then $\frac{\text{ZL}^2}{p''^2} + \frac{\text{TL}^2}{x^2} = 1$ and $\frac{\text{TL}}{\text{SA}} = \frac{\mu}{\mu+1}$.

$$\therefore \frac{p'^2}{p''^2} + \mu^2\frac{\text{SA}^2}{x^2} = (\mu+1)^2.$$

Let $\lambda = \frac{x}{\text{SA}} = \frac{p''}{p} =$ the ratio of similarity.

Then $\frac{p'^2}{p^2} + \mu^2 = \lambda^2(\mu+1)^2$. But from §23, $\frac{p'}{p} = \frac{abc}{a_1b_1c_1}\Big/\Sigma\frac{a^2}{a_1^2}$.

$$\therefore \lambda = \frac{\sqrt{\left\{\frac{a^2b^2c^2}{a_1^2b_1^2c_1^2} + \mu^2\left(\Sigma\frac{a^2}{a_1^2}\right)^2\right\}}}{(\mu+1)\Sigma\frac{a^2}{a_1^2}} = \frac{\sqrt{\left\{\Delta^2 + \mu^2\left(\Sigma\frac{a^2}{a_1^2}\right)^2\right\}}}{(\mu+1)\Sigma\frac{a^2}{a_1^2}}$$

$$= \frac{\sqrt{\left\{2(1+\mu^2)\Sigma\frac{a^2b^2}{a_1^2b_1^2} - (1-\mu^2)\Sigma\frac{a^4}{a_1^4}\right\}}}{(1+\mu)\Sigma\frac{a^2}{a_1^2}}.$$

If Y'Z, XZ' meet in R', then $\frac{\text{ZY}'}{\text{Z}'\text{X}} = \frac{\text{R}'\text{Z}}{\text{R}'\text{Z}'}$.

If R is the pole of c with respect to the circumconic, the triangles RAB, R'ZZ' are similar.

$\therefore \frac{\text{ZY}'}{\text{Z}'\text{X}} = \frac{\text{RA}}{\text{RB}}$. But $\frac{\text{RA}}{\text{RB}} = \frac{p}{q}$.

$$\therefore \frac{\text{ZY}'}{p} = \frac{\text{Z}'\text{X}}{q} = \frac{\text{X}'\text{Y}}{r} = \frac{2}{\mu+1}\,\frac{\frac{abc}{a_1b_1c_1}}{\Sigma\frac{a^2}{a_1^2}} = \frac{2\Delta}{(\mu+1)\Sigma\frac{a^2}{a_1^2}}.$$

$\text{M}'\text{N}' = \frac{\mu-1}{\mu+1}a$; $\text{N}'\text{L}' = \frac{\mu-1}{\mu+1}b$; $\text{L}'\text{M}' = \frac{\mu-1}{\mu+1}c$.

$\frac{\text{XX}'}{\text{M}'\text{N}'} = \frac{\text{XL}'}{\text{M}'\text{L}'} = \frac{\text{X}'\text{L}'}{\text{N}'\text{L}'}$.

$$\therefore \quad \frac{XX'}{a}=\frac{BZ'+\frac{\mu-1}{\mu+1}c}{c}=\frac{CY+\frac{\mu-1}{\mu+1}b}{b}.$$

But $\frac{BZ'.c}{c_1^2}=\frac{BX.a}{a_1^2}$ and $\frac{CY.b}{b_1^2}=\frac{CX'.a}{a_1^2}$.

$$\therefore \quad \frac{XX'}{a}=\frac{BX.a}{a_1^2}\cdot\frac{c_1^2}{c^2}+\frac{\mu-1}{\mu+1}=\frac{CX'.a}{a_1^2}\cdot\frac{b_1^2}{b^2}+\frac{\mu-1}{\mu+1}.$$

$$\therefore \quad \frac{XX'}{a}\left\{\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right\}=\frac{a}{a_1^2}\left\{a-XX'\right\}+\frac{\mu-1}{\mu+1}\left\{\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right\}.$$

$$\therefore \quad \frac{XX'}{a}\cdot\Sigma\frac{a^2}{a_1^2}=\frac{a^2}{a_1^2}+\frac{\mu-1}{\mu+1}\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right).$$

$$\frac{BX}{a}\cdot\Sigma\frac{a^2}{a_1^2}=\frac{2c^2}{(\mu+1)c_1^2} \quad \text{and} \quad \frac{CX'}{a}\cdot\Sigma\frac{a^2}{a_1^2}=\frac{2b^2}{(\mu+1)b_1^2}.$$

$\mu=0,\ 1,\ \infty$ gives the cosine, Lemoine and circum-conic.
Let U′ be the mid point of XX′.

Then $\frac{XX'^2}{a_1^2}+\frac{4TU'^2}{d^2}=4\lambda^2$ where λ is the ratio of similarity.

$$\therefore \frac{4TU'^2}{d^2}=\frac{8(1+\mu^2)\Sigma\frac{a^2b^2}{a_1^2b_1^2}-4(1-\mu^2)\Sigma\frac{a^4}{a_1^4}}{(1+\mu)^2\left\{\Sigma\frac{a^2}{a_1^2}\right\}^2}-\frac{\frac{a^2}{a_1^2}\left[\frac{a^2}{a_1^2}+\frac{\mu-1}{\mu+1}\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right)\right]^2}{\left\{\Sigma\frac{a^2}{a_1^2}\right\}^2}.$$

§30. If ZY′, a meet in α, Z′X, b in β, X′Y, c in γ, then, as in §27, α, β, γ are collinear, the line being the radical axis of the Tucker conic and the circumconic.

If XZ, X′Y′ meet in A′, Z′Y′, XY in B′, Z′X′, ZY in C′, then A′L′, B′M′, C′N′ meet in the pole of $\alpha\beta\gamma$ with respect to the Tucker conic.

Let ZY′ meet X′Y, XZ′ in Q and R, and let X′Y, XZ′ meet in P; the triangles PQR, ABC, L′M′N′ are in perspective.

$$\begin{aligned} QQ_1 : M'M_1' &= QQ_1 : YY_1 \\ &= QX' : YX' \\ &= QZ \ : Y'Z \\ &= QQ_3 : Y'Y_3 \\ &= QQ_3 : M'M_3'. \end{aligned}$$

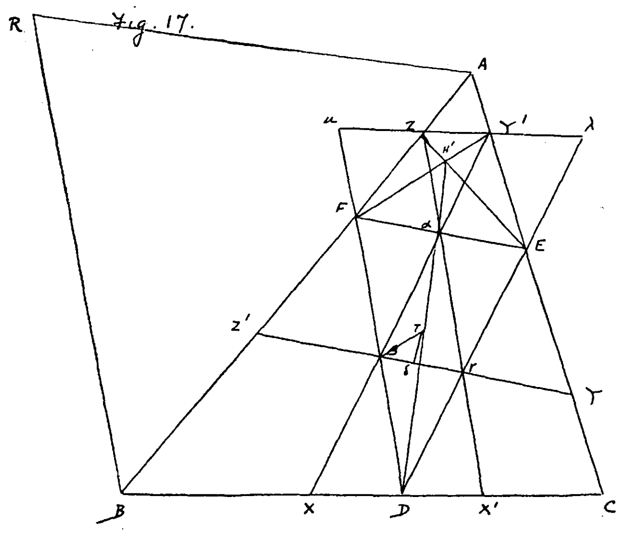
Therefore B, M′, K, Q are collinear. Similarly C, N′, K, R and A, L′, K, P are collinear. The axis of perspective of ABC, PQR is $\alpha\beta\gamma$.

If the points 1, 2, 3, 4, 5, 6 be found as in §28, then in exactly the same way it can be proved that 624 is similar to ABC and has T for orthocentre, and that ABC, 153 are in perspective. Further properties will be given in the section on the point O theorem.

SECTION IX.

Taylor's Conic.

§31. Let α, β, γ be the mid points of the sides EF, FD, DE of the pedal triangle. If $\alpha\beta$ meet a, b in XY′, $\beta\gamma$ meet b, c in Y, Z′, $\gamma\alpha$ meet c, a in Z, X′ (Fig. 17), then X, X′, Y, Y′, Z, Z′ lie on a Tucker's conic.



Since the sides of the pedal triangle are antiparallel to the sides ABC, Z′Y, X′Z, Y′X are antiparallel to the sides of ABC because they are parallel to the sides of DEF.

Let P, Q, R be the poles of a, b, c. The triangles βFZ′, RBA are similar, their sides being parallel.

$$\therefore \frac{\beta Z'}{RA}=\frac{\beta F}{RB}. \quad \text{Similarly } \frac{\gamma Y}{QA}=\frac{\gamma E}{QC}.$$

But $\frac{RA}{RB} = \frac{p}{q}$ and $\frac{QA}{QC} = \frac{p}{r}$.

$$\therefore \frac{\beta Z'}{\beta F} = \frac{p}{q} \text{ and } \frac{\gamma Y}{\gamma E} = \frac{p}{r}.$$

$$\therefore \frac{Z'Y}{p} = \tfrac{1}{2}\left(\frac{EF}{p} + \frac{FD}{q} + \frac{DE}{r}\right).$$

$$\therefore \frac{Z'Y}{p} = \frac{X'Z}{q} = \frac{Y'Z}{r} = \tfrac{1}{2}\left(\frac{EF}{p} + \frac{FD}{q} + \frac{DE}{r}\right).$$

Therefore X, X′, Y, Y′, Z, Z′ lie on a Tucker's conic (§29).

Since $\frac{\beta F}{q} = \frac{\beta D}{q} = \frac{\beta Z'}{p}$, a homothetic conic on FD as diameter passes through Z′; DZ′, FZ′ are therefore conjugate.

Hence DZ′, EZ are conjugate to c, EX′, FX to a, and DY, FY′ to b.

Let T be the centre of the homothetic inconic of $\alpha\beta\gamma$, and δ the point of contact with $\beta\gamma$.

$$\frac{\beta\delta}{p} = \tfrac{1}{2}\left(\frac{\alpha\beta}{r} + \frac{\beta\gamma}{p} - \frac{\gamma\alpha}{q}\right) = \tfrac{1}{4}\left(\frac{DE}{r} + \frac{EF}{p} - \frac{FD}{q}\right).$$

$$\therefore \frac{Z'\delta}{p} = \frac{Z'\beta}{p} + \frac{\beta\delta}{p} = \tfrac{1}{4}\left\{\frac{EF}{p} + \frac{FD}{q} + \frac{DE}{r}\right\} = \frac{Z'Y}{2p}.$$

T is then the centre of Taylor's conic.

The conjugate from T to XX′ bisects it, and as this conjugate is parallel to FX, EX′ it passes through α the mid point of EF. The conjugates from α, β, γ to a, b, c then meet in T.

Let H′, H″, H‴ be the orthocentre of the triangles AEF, BFD, CDE. H′F and Tβ are parallel, both being conjugate to b; also EH′ and Tγ are parallel. The quadrangles H′FDE, TβDγ are therefore homothetic with the ratio of similarity 2 : 1.

H′D, H″E, H‴F therefore pass through T and are there bisected.

Hence H″H‴ is parallel to EF, H‴H′ to FD, and H′H″ to DE.

Therefore AH′ is conjugate to H″H‴, BH″ to H‴H′, and CH‴ to H′H″. But AH′ being conjugate to EF, an antiparallel to BC, passes through S. S is then the orthocentre of H′H″H‴ and as DEF, H′H″H‴ are similar with T as centre of similarity and 1 : 1 as ratio, T bisects the line joining S and the orthocentre of DEF.

H, A, B, C are collinear in two pairs with each of the vertices of the triangle DEF, and the pair of lines through each vertex are conjugate and form a harmonic pencil with the sides. H, A, B, C are therefore the in- and ex-centres of the triangle DEF.

Let DE, DF meet ZY′ in λ, μ. The ex-conic with centre A will touch DE, DF in λ and μ.

Because $\lambda\mu$ is parallel to BC, $\dfrac{D\lambda}{D\gamma}=\dfrac{X'Z}{X'\gamma}$.

Also $\dfrac{D\gamma}{r}=\dfrac{X'\gamma}{q}$ and $\dfrac{X'Z}{q}=\dfrac{Z'Y}{p}=\tfrac{1}{2}\left(\dfrac{EF}{p}+\dfrac{FD}{q}+\dfrac{DE}{r}\right)$.

Therefore $\dfrac{D\lambda}{r}=\tfrac{1}{2}\left(\dfrac{EF}{p}+\dfrac{FD}{q}+\dfrac{DE}{r}\right)$ and the conic touches DE at λ and similarly touches DF at μ. Therefore ZY′ is the polar of D with respect to the conic with centre A touching the sides of the pedal triangle.

Let X′Y, XZ′ meet in ξ, X′Y, ZY′ in η and ZY′, XZ′ in ζ. Since DZ′ is conjugate to X′Y, which is parallel to c, and DY is conjugate to XZ′, D is the orthocentre of $Z'Y\xi$. But Z′Y is parallel to EF. Therefore ξD passes through the orthocentre H_0 of DEF. Hence ξD, ηE, ζF meet in H_0. The triangle $\xi\eta\zeta$ is similar to ABC and FE antiparallel to BC is antiparallel to $\eta\zeta$. $H_0\xi$ is conjugate to EF, and therefore conjugate to the tangent of the homothetic circumconic of $\xi\eta\zeta$ at ξ. Hence H_0 is the centre of the homothetic circumconic of $\xi\eta\zeta$.

Suppose AH′ meets FE in θ. Since T bisects SH_0 and DH′, $H_0D=H'S$. Since $Y\xi$ is parallel to AZ′ and $Z'\xi$ is parallel to AY, Z′Y bisects $A\xi$, and this point δ of bisection is the centre of similarity of the similar and equal triangles $Z'Y\xi$, YZ′A. But D is the orthocentre of $Z'Y\xi$. Therefore the orthocentre of YZ′A must lie on $D\delta$ at a distance from δ equal to $D\delta$. But it also lies on AH′ and so θ is the orthocentre of AZ′Y.

$\therefore\ A\theta = D\xi. \quad \therefore\ H_0\xi = H_0D + D\xi = H'S + A\theta = AS + H'\theta.$

$\therefore\ H_0\xi =$ parallel diameter of nine point conic $+\,H'\theta$.

Let N_0 or H be the centre of the inconic of DEF and N_0D' be conjugate to EF, FN_0 is parallel to EH′ being conjugate to AB. Therefore $FH'EN_0$ is a parallelogram.

$\therefore\ H'\theta = D'N_0 =$ parallel radius of inconic of DEF.

§32. Consider the homothetic conic round B, C, E, F whose centre is U. Uα is conjugate to EF.

Let UE_1, UF_1 be conjugate to DF, DE (Fig. 18).

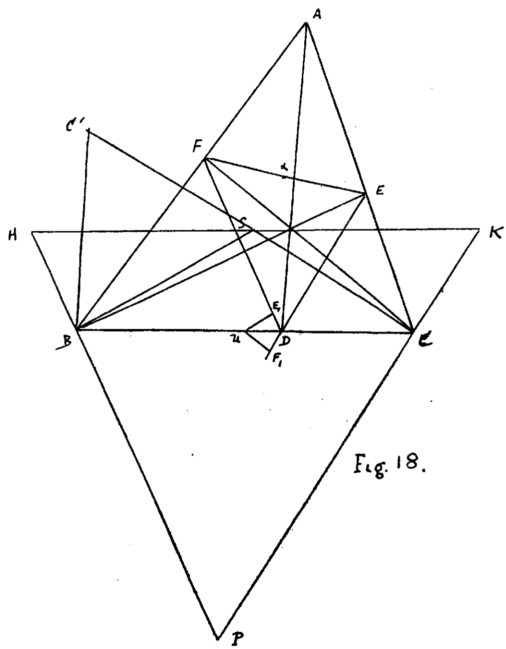


Fig. 18.

Then UE_1, UF_1 are parallel to SB, SC. Let the parallel through S to BC meet PB, PC in H and K. The triangles UDE_1, SHB are similar as also UDF_1, SKC.

$$\therefore \frac{UE_1}{SB} = \frac{UD}{SH} = \frac{UD}{SK} = \frac{UF_1}{SC}.$$

$$\therefore \frac{UE_1}{UF_1} = \frac{SB}{SC}.$$

Let CS meet the conic again in C′. Then BC, BC′ are conjugate.

But since $\dfrac{UE_1}{UF_1} = \dfrac{SB}{SC'}$ and the contained angles are equal, E_1F_1 is parallel to BC′, and is therefore conjugate to BC. E_1F_1 is the Wallace line of U with respect to the triangle DEF and the nine point conic, and therefore passes through α as $U\alpha$ is conjugate to EF. E_1F_1 therefore passes through T. The Wallace lines of the mid points of the sides of a triangle with reference to the pedal triangle all pass through the centre of the Taylor conic.

Let AD cut the nine point conic again in D′. DD′ is conjugate to VW. Therefore UD′ is parallel to the Wallace line of D with respect to the triangle UVW. If AD cut VW in A_0, A_0 is the mid point of AD. The Wallace line of D is then parallel to SA for SUD′A is a parallelogram, SU being equal to AD′. Now H′, the orthocentre of AFE, lies on AS. Therefore the Wallace line of D bisects H′D and so passes through T. Hence the Wallace lines of D, E, F with respect to the triangle UVW all pass through the centre of Taylor's conic.

Section X.

The Brocard Points.

§33. Suppose the homothetic conics drawn

(i) through A, C and touching AB in A,
(ii) through A, B and touching BC in B,
(iii) through B, C and touching CA in C.

These will be called auxiliary conics. Let Ω be the intersection of (i) and (ii).

$$\frac{\Omega_a\Omega\,.\,\Omega_aA}{\omega_1^2}=\frac{\Omega_aB^2}{a_1^2}=\frac{\Omega_aA'\,.\,\Omega_aC}{a_1^2}$$ where A′ is the second intersection of (i) with a (Fig. 19).

$$\text{Also } \frac{a\,.\,BA'}{a_1^2}=\frac{c^2}{c_1^2}.\quad\therefore\quad \frac{\Omega_aA'}{\Omega_aB}=\frac{\Omega_aB}{\Omega_aC}\ \text{ or }\ \frac{BA'}{\Omega_aB}=\frac{a}{\Omega_aC}.$$

$$\therefore\quad \frac{B\Omega_a}{\frac{c^2}{c_1^2}}=\frac{\Omega_aC}{\frac{a^2}{a_1^2}}=\frac{a}{\frac{c^2}{c_1^2}+\frac{a^2}{a_1^2}}.\quad\therefore\quad \frac{B\Omega_a}{\Omega_aC}=\frac{AK_b}{K_bC}.$$

Similarly if through B and the intersection of (ii) and (iii) a line be drawn cutting CA in a point Ω_b

$$\frac{C\Omega_b}{\frac{a^2}{a_1^2}}=\frac{\Omega_bA}{\frac{b^2}{b_1^2}}=\frac{b}{\frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}}\quad\text{and}\quad \frac{C\Omega_b}{\Omega_bA}=\frac{BK_c}{K_cA}.$$

If through C and the intersection of (iii) and (i) a line be drawn cutting AB in Ω_c

$$\frac{A\Omega_c}{\frac{b^2}{b_1^2}} = \frac{\Omega_c B}{\frac{c^2}{c_1^2}} = \frac{c}{\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}} \quad \text{and} \quad \frac{A\Omega_c}{\Omega_c B} = \frac{CK_a}{K_a B}.$$

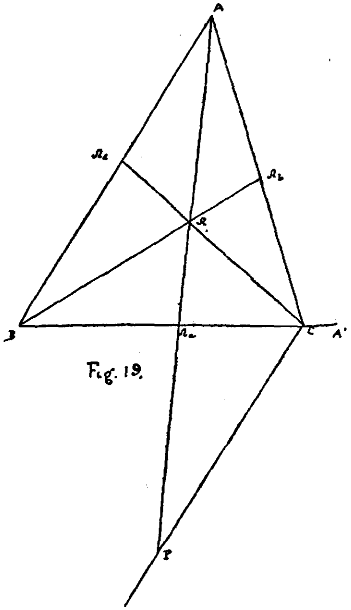


Fig. 19.

By Ceva's theorem the three conics meet in a point Ω.

$$\frac{\Omega_a\Omega \,.\, \Omega_a A}{\omega_1^2} = \frac{B\Omega_a^2}{a_1^2} = \frac{\frac{c^4a^2}{c_1^4a_1^2}}{\left(\frac{c^2}{c_1^2}+\frac{a^2}{a_1^2}\right)^2}.$$

Let $A\Omega$ meet the conic (iii) again in P.

Then $$\frac{\Omega\Omega_a \,.\, \Omega_a P}{\omega_1^2} = \frac{B\Omega_a \,.\, \Omega_a C}{a_1^2} = \frac{\frac{c^2a^4}{c_1^2a_1^4}}{\left(\frac{c^2}{c_1^2}+\frac{a^2}{a_1^2}\right)^2}.$$

$$\therefore \frac{A\Omega_a}{\Omega_a P} = \frac{\frac{c^2}{c_1^2}}{\frac{a^2}{a_1^2}} = \frac{B\Omega_a}{\Omega_a C}.$$

Therefore CP is parallel to AB. If BΩ meet (i) in Q and CΩ meet (ii) in R, AQ and BR are parallel to a and b.

$$\frac{\text{A}\Omega\,.\,\text{AP}}{\omega_1^2}=\frac{b^2}{b_1^2} \quad \text{and} \quad \frac{\text{AP}}{\text{A}\Omega_a}=\frac{\dfrac{c^2}{c_1^2}+\dfrac{a^2}{a_1^2}}{\dfrac{c^2}{c_1^2}}.$$

$$\therefore \frac{\text{A}\Omega\,.\,\text{A}\Omega_a}{\omega_1^2}=\frac{\dfrac{b^2c^2}{b_1^2c_1^2}}{\dfrac{c^2}{c_1^2}+\dfrac{a^2}{a_1^2}}.$$

$$\therefore \frac{\text{A}\Omega_a}{\omega_1^2}\left\{\text{A}\Omega_a-\frac{\dfrac{c^4a^2\omega_1^2}{c_1^4a_1^2}}{\text{A}\Omega_a\left(\dfrac{c^2}{c_1^2}+\dfrac{a^2}{a_1^2}\right)^2}\right\}=\frac{\dfrac{b^2c^2}{b_1^2c_1^2}}{\dfrac{c^2}{c_1^2}+\dfrac{a^2}{a_1^2}}.$$

$$\therefore \frac{\text{A}\Omega_a^2}{\omega_1^2}=\frac{\dfrac{c^2}{c_1^2}\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}}{\left(\dfrac{c^2}{c_1^2}+\dfrac{a^2}{a_1^2}\right)^2}.$$

$$\therefore \frac{\text{A}\Omega_a}{\omega_1}=\frac{\dfrac{c}{c_1}\sqrt{\left\{\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}\right\}}}{\left(\dfrac{c^2}{c_1^2}+\dfrac{a^2}{a_1^2}\right)},\quad \frac{\Omega\Omega_a}{\omega_1}=\frac{\dfrac{c^3a^2}{c_1^3a_1^2}}{\left(\dfrac{c^2}{c_1^2}+\dfrac{a^2}{a_1^2}\right)\sqrt{\left\{\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}\right\}}},\quad \frac{\text{A}\Omega}{\omega_1}=\frac{\dfrac{b^2c}{b_1^2c_1}}{\sqrt{\left\{\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}\right\}}}.$$

The corresponding equations are got by cyclical permutation of letters.

§34. Let the homothetic conics be drawn

(i) through A, B and touching b in A,

(ii) through B, C and touching c in B,

(iii) through C, A and touching a in C.

These auxiliary conics intersect in the second Brocard point Ω'.

The following results are proved in a similar way.

$$\frac{B\Omega'_a}{\dfrac{a^2}{a_1^2}} = \frac{\Omega_a'C}{\dfrac{b^2}{b_1^2}} = \frac{a}{\dfrac{a^2}{a_1^2}+\dfrac{b^2}{b_1^2}}, \quad \frac{B\Omega_a'}{\Omega_a'C} = \frac{BK_c}{K_cA},$$

$$\frac{C\Omega_b'}{\dfrac{b^2}{b_1^2}} = \frac{\Omega_b'A}{\dfrac{c^2}{c_1^2}} = \frac{b}{\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}}, \quad \frac{C\Omega_b'}{\Omega_b'A} = \frac{CK_a}{K_aB},$$

$$\frac{A\Omega_c'}{\dfrac{c^2}{c_1^2}} = \frac{\Omega_c'B}{\dfrac{a^2}{a_1^2}} = \frac{c}{\dfrac{c^2}{c_1^2}+\dfrac{a^2}{a_1^2}}, \quad \frac{A\Omega_c'}{\Omega_c'B} = \frac{AK_b}{K_bC}.$$

If $A\Omega'$ cut (ii) in P', BP' is parallel to CA.

$$\frac{A\Omega_a'}{\omega_1'} = \frac{\dfrac{b}{b_1}\sqrt{\left(\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}\right)}}{\dfrac{a^2}{a_1^2}+\dfrac{b^2}{b_1^2}}, \quad \frac{\Omega'\Omega_a'}{\omega_1'} = \frac{\dfrac{a^2b^3}{a_1^2b_1^3}}{\left(\dfrac{a^2}{a_1^2}+\dfrac{b^2}{b_1^2}\right)\sqrt{\left(\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}\right)}}, \quad \frac{A\Omega'}{\omega_1'} = \frac{\dfrac{bc^2}{b_1c_1^2}}{\sqrt{\left(\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}\right)}}.$$

§35. Let S_1 be the centre of the auxiliary conic $A\Omega B$ so that S_1B and S_1W are parallel to AD and CF. The triangles S_1BW, HAF are similar.

$$\therefore \frac{S_1B}{\dfrac{c}{2}} = \frac{AH}{AF}. \quad \therefore \frac{S_1B}{d} = \frac{\dfrac{ac^2}{a_1c_1^2}}{\Delta} = \frac{\dfrac{c}{c_1}}{\dfrac{b}{b_1}}. \quad (\S\S3,\ 11).$$

This is the ratio of similarity of the conic $A\Omega B$ to the circumconic.

§36. Ω and Ω' are isogonally conjugate.

Let CF, Ω_aF' be conjugate to c; BE, Ω_aE' conjugate to b.

$$\text{Then } \frac{\Omega_aF'}{CF} = \frac{B\Omega_a}{a}, \quad \frac{\Omega_aE'}{BE} = \frac{\Omega_aC}{a}.$$

$$\therefore \frac{\Omega_aF'}{\Omega_aE'} = \frac{B\Omega_a}{\Omega_aC}\cdot\frac{CF}{BE} = \frac{\dfrac{c^2}{c_1^2}}{\dfrac{a^2}{a_1^2}}\cdot\frac{\dfrac{bf}{b_1}}{\dfrac{ce}{c_1}} = \frac{\dfrac{bcf}{b_1c_1}}{\dfrac{a^2e}{a_1^2}}.$$

Similarly if $\Omega_a'F_1'$, $\Omega_a'E'$ be conjugate to c and b,

$$\frac{\Omega_a'F_1'}{\Omega_a'E_1'}=\frac{\dfrac{a^2f}{a_1^2}}{\dfrac{bce}{b_1c_1}}. \quad \therefore \frac{\Omega_a F'.\Omega_a'F_1'}{f^2}=\frac{\Omega_a E'.\Omega_a'E_1'}{e^2}.$$

From this and the two corresponding results Ω and Ω' are two isogonally conjugate points, and a homothetic conic passes through the feet of the conjugates to the sides.

Loci of Ω, Ω'.

§37. $$\frac{BK_a}{\dfrac{c^2}{c_1^2}}=\frac{K_aC}{\dfrac{b^2}{b_1^2}}=\frac{a}{\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}} \text{ (§26) and}$$

$$\frac{A\Omega_c}{\dfrac{b^2}{b_1^2}}=\frac{\Omega_cB}{\dfrac{c^2}{c_1^2}}=\frac{c}{\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}}. \quad \therefore \frac{K_aC}{a}=\frac{A\Omega_c}{c}.$$

For a system of circumconics passing through a fixed point O the points K_a, Ω_c form two similar projective ranges on a and c. $K_a\Omega_c$ is parallel to b, and if K′ be the isotomic conjugate of K, Ω_cK_a' envelopes one of Artzt's parabolas.

Similarly the points K_b form a range on b projective with the range Ω_a on a. But $\{K_a\ldots\}\barwedge\{K_b\ldots\}\barwedge\{K\ldots\}$.

$$\therefore \{\Omega_c\ldots\}\barwedge\{\Omega_a\ldots\}. \quad \therefore C\{\Omega_c\ldots\}\barwedge A\{\Omega_a\ldots\}.$$

Ω then describes a conic through A and C, and as B is a corresponding point on both ranges the conic passes through B. Let the locus of K cut a, b, c in L, M, N. As K passes from N to M or K_a from B to C, Ω_c passes from B to A. Also K_b passes from A to M and Ω_a from B to the point in its range corresponding to M. This point on BC corresponds to A in the range Ω_c and therefore the conic, the locus of Ω, touches the ray from A to this point at A. Hence if α, β, γ be the three points on a, b, c such that $\frac{B\alpha}{a}=\frac{AM}{b}$, $\frac{C\beta}{b}=\frac{BN}{c}$ and $\frac{A\gamma}{c}=\frac{CL}{a}$, the conic touches $A\alpha$, $B\beta$, $C\gamma$ at A, B and C.

Similarly it may be proved that Ω' describes a second conic through A, B, C. If α', β', γ' be the three points on a, b, c such that $\frac{B\alpha'}{a}=\frac{BN}{c}$, $\frac{C\beta'}{b}=\frac{CL}{a}$, $\frac{A\gamma'}{c}=\frac{AM}{b}$, the conic touches $A\alpha'$, $B\beta'$, $C\gamma'$ at A, B, C.

Neuberg's Conic.

§38. Another locus is that given by Neuberg's conic.

Suppose the base BC of a triangle be given in magnitude and position, and the direction BΩ of the first Brocard point be given ; the locus of the vertex A is a conic.

Let a_1 be the radius parallel to BC of one of the given system of homothetic conics. Draw the conjugate BV to BC and the harmonic conjugate BL of BΩ with respect to BV and BC (Fig. 20).

Suppose V, L to lie on AΩ. We first prove that the homothetic conic through A, B, Ω_a touches BL at B.

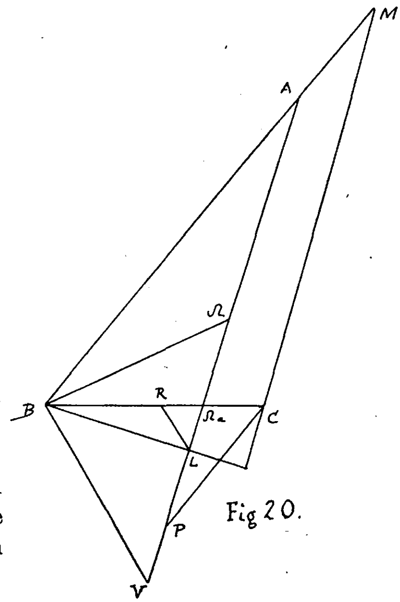


Fig 20.

Draw LR conjugate to BC and let l, d, ω_1 be the radii of the initial conic parallel to BL, RL, AΩ.

$$\text{Then } \frac{BL^2}{l^2}=\frac{BR^2}{a_1^{\,2}}+\frac{RL^2}{d^2}$$

$$=\frac{(B\Omega_a+\Omega_a R)^2}{a_1^{\,2}}+\frac{L\Omega_a^{\,2}}{\omega_1^{\,2}}-\frac{R\Omega_a^{\,2}}{a_1^{\,2}}$$

$$=\frac{B\Omega_a^{\,2}}{a_1^{\,2}}-\frac{2B\Omega_a\,.\,R\Omega_a}{a_1^{\,2}}+\frac{L\Omega_a^{\,2}}{\omega_1^{\,2}}.$$

$$\text{But } \frac{B\Omega_a^{\,2}}{a_1^{\,2}}=\frac{\Omega_a\Omega\,.\,\Omega_a A}{\omega_1^{\,2}} \quad\text{and}\quad \frac{R\Omega_a}{B\Omega_a}=\frac{\Omega_a L}{\Omega_a V}.$$

Also since $\{V\Omega_a, L\Omega\}$ is a harmonic range,

$$\frac{2}{\Omega_a V}=\frac{\Omega_a L+\Omega_a\Omega}{\Omega_a L\,.\,\Omega_a\Omega}.$$

$$\therefore \frac{BL^2}{l^2} = \frac{\Omega_a\Omega \,.\, \Omega_a A}{\omega_1^2} - \frac{2B\Omega_a^2 \,.\, \Omega_a L}{a_1^2\Omega_a V} + \frac{L\Omega_a^2}{\omega_1^2}$$
$$= \frac{\Omega_a\Omega \,.\, \Omega_a A}{\omega_1^2} - \frac{\Omega_a A(\Omega_a L + \Omega_a\Omega)}{\omega_1^2} + \frac{L\Omega_a^2}{\omega_1^2}$$
$$= \frac{L\Omega_a \,.\, LA}{\omega_1^2}$$

and the conic touches BL at B.

From this it is seen that the homothetic conic through B and C and touching BL at B will cut BA at M, so that CM will be parallel to Ω_aA. If AΩ meet the conic BΩC again in P, then as has been proved (§33) CP is parallel to AB; therefore CP = MA.

$$\text{Also } \frac{BA}{PC} = \frac{B\Omega_a}{\Omega_a C} = \frac{\dfrac{c^2}{c_1^2}}{\dfrac{a^2}{a_1^2}}.$$

$$\therefore AM = PC = \frac{a^2}{a_1^2} \Big/ \frac{c}{c_1^2}.$$

$$\therefore \frac{BA \,.\, AM}{c_1^2} = \frac{a^2}{a_1^2}.$$

Let O be the centre of the fixed conic BCM and OT be conjugate to BM, so that BT = TM.

Then $\dfrac{BA \,.\, AM}{c_1^2} = \dfrac{TM^2 - TA^2}{c_1^2} = \dfrac{OM^2}{m^2} - \dfrac{OA^2}{n^2}$, where m, n are the radii of the initial conic parallel to OM, OA.

If λ be the ratio of similarity of the conic BCM to the initial conic $\dfrac{OA^2}{n^2} = \lambda^2 - \dfrac{a^2}{a_1^2} = \text{constant}.$

The locus of A is therefore a homothetic conic with centre O. The locus of G is the locus of a point dividing UA in the ratio 1 : 2, and is therefore a second homothetic conic (M'Cay's). The two have U as a centre of similarity.

§39. Let AΩ, BΩ, CΩ meet the circumconic again in B′, C′, A′. Then Ω is the second Brocard point of A′B′C′. Let the sides and parallel radii of A′B′C′ be denoted by a', b', c'; a_1', b_1', c_1'.

If A′C′ meet BC in L, $\dfrac{LC \,.\, LB}{a_1^2} = \dfrac{LC' \,.\, LA'}{b_1'^2}$ (i).

Since A′C′L is a transversal of BΩC

$$\frac{BL}{LC}\cdot\frac{CA'}{A'\Omega}\cdot\frac{\Omega C'}{C'B}=-1,$$

or $(a+CL)CA'.\Omega C'=CL.A'\Omega.C'B.$

$$\therefore\ CL=\frac{a.CA'.\Omega C'}{A'\Omega.C'B-CA'.\Omega C'}\text{ and }BL=\frac{a.A'\Omega.C'B}{A'\Omega.C'B-CA'.\Omega C'}.$$

$$\therefore\ CL=\frac{a.CA'.\Omega C'}{A'\Omega.\Omega B-C\Omega.\Omega C'}\text{ and }BL=\frac{a.A'\Omega.C'B}{A'\Omega.\Omega B-C\Omega.\Omega C'}.$$

Also since BCL is a transversal of A′ΩC′ we have in a similar way

$$LC'=\frac{b'.C'B.\Omega C}{B\Omega.\Omega A'-C'\Omega.\Omega C}\text{ and }LA'=\frac{b'.B\Omega.CA'}{B\Omega.\Omega A'-C'\Omega.\Omega C}.$$

Using these in (i) we have

$$\left.\begin{aligned}&\frac{b'^2}{b^2}B\Omega.\Omega C=\frac{a^2}{a_1^{\,2}}A'\Omega.\Omega C'.\\ \text{Similarly}\quad&\frac{c'^2}{c^2}C\Omega.\Omega A=\frac{b^2}{b_1^{\,2}}B'\Omega.\Omega A'.\\ \text{and}\quad&\frac{a'^2}{a^2}A\Omega.\Omega B=\frac{c^2}{c_1^{\,2}}C'\Omega.\Omega B'.\end{aligned}\right\}\quad\text{(ii)}.$$

From §33 $\quad\dfrac{A\Omega^2}{\omega_1^{\,2}}\Big/\dfrac{B\Omega^2}{\omega_2^{\,2}}=\dfrac{b^4}{b_1^{\,4}}\Big/\dfrac{c^2a^2}{c_1^{\,2}a_1^{\,2}}.$

Divide the square of the second of equations (ii) by the product of the first and third;

$$\frac{\dfrac{c'^4}{c_1'^4}}{\dfrac{a'^2b'^2}{a_1'^2b_1'^2}}\cdot\frac{\Omega C}{\omega_2^{\,2}}=\frac{\Omega A}{\omega_1^{\,2}}\cdot\frac{\Omega B'.\Omega A'}{\Omega C'^2}.\quad\text{(iii)}.$$

Now $\quad\dfrac{B\Omega.\Omega C'}{\omega_2^{\,2}}=\dfrac{C\Omega.\Omega A'}{\omega_3^{\,2}}=\dfrac{A\Omega.\Omega B'}{\omega_1^{\,2}}.$

Inserting in (iii) the ratio $\Omega C:\Omega A$ we have

$$\frac{\Omega A'^2}{\omega_3^{\,2}}\Big/\frac{\Omega C'^2}{\omega_2^{\,2}}=\frac{c'^4}{c_1'^4}\Big/\frac{a'^2b'^2}{a_1'^2b_1'^2}.$$

In the same way we can obtain the two other corresponding equations. Comparing these with those derived from §34 we see that Ω is the second Brocard point of A′B C′.

$$\frac{B\Omega_a \,.\, \Omega_a C}{a_1^2} = \frac{A\Omega_a \,.\, \Omega_a B'}{\omega_1^2}.$$

$$\therefore \quad \frac{\Omega_a B'}{\omega_1} = \frac{\dfrac{ca^4}{c_1 a_1^4}}{\left(\dfrac{c^2}{c_1^2} + \dfrac{a^2}{a_1^2}\right)\sqrt{\left\{\Sigma \dfrac{a^2 b^2}{a_1^2 b_1^2}\right\}}} \qquad (\S 33).$$

$$\therefore \quad \frac{\Omega B'}{\omega_1} = \frac{\Omega\Omega_a + \Omega_a B'}{\omega_1} = \frac{\dfrac{ca^2}{c_1 a_1^2}}{\sqrt{\left\{\Sigma \dfrac{a^2 b^2}{a_1^2 b_1^2}\right\}}}.$$

$$\therefore \quad \frac{A\Omega \,.\, \Omega B'}{\omega_1^2} = \frac{\dfrac{a^2 b^2 c^2}{a_1^2 b_1^2 c_1^2}}{\sqrt{\left(\Sigma \dfrac{a^2 b^2}{a_1^2 b_1^2}\right)}} = \frac{A\Omega' \,.\, \Omega' B''}{\omega_1^2}$$ where $A\Omega'$ cuts the circumconic again in B''.

If therefore ω and ω' are the radii parallel to $S\Omega$ and $S\Omega'$, $\dfrac{S\Omega}{\omega} = \dfrac{S\Omega'}{\omega'}$.

§40. Some further properties of the Lemoine conic will now be given.

Suppose in Fig. 14 that AP is drawn parallel to ZX to cut BC in P. Then $\dfrac{BP}{BX} = \dfrac{BA}{BZ}$.

But $BZ' \Big/ \dfrac{a^2}{a_1^2} = Z'Z \Big/ \dfrac{c^2}{c_1^2} = ZA \Big/ \dfrac{b^2}{b_1^2} = c \Big/ \Sigma \dfrac{a^2}{a_1^2} = BZ \Big/ \left(\dfrac{c^2}{c_1^2} + \dfrac{a^2}{a_1^2}\right)$ (§26),

and $BX \Big/ \dfrac{c^2}{c_1^2} = a \Big/ \Sigma \dfrac{a^2}{a_1^2}$.

$$\therefore \quad \frac{BP}{c} = \frac{BX}{BZ} = \frac{\dfrac{ac}{c_1^2}}{\dfrac{c^2}{c_1^2} + \dfrac{a^2}{a_1^2}}.$$

Therefore by §33, P coincides with Ω_a, or ZX is parallel to $A\Omega$. Hence ZX, XY, YZ are parallel to $A\Omega$, $B\Omega$, $C\Omega$; similarly $Z'X'$, $X'Y'$, $Y'Z'$ are parallel to $C\Omega'$, $A\Omega'$, $B\Omega'$.

Let ZX cut BΩ in U.

$$\frac{ZU}{UX}=\frac{A\Omega}{\Omega\Omega_a}. \quad \therefore\ ZU\,.\,UX=\frac{A\Omega}{\Omega\Omega_a}UX^2.$$

Also $$\frac{UX}{\Omega\Omega_a}=\frac{BX}{B\Omega_a}. \quad \therefore\ ZU\,.\,UX=A\Omega\,.\,\Omega\Omega_a\,.\,\frac{BX^2}{B\Omega_a{}^2}.$$

$$\therefore\ \frac{ZU\,.\,UX}{\omega_1{}^2}=\frac{\dfrac{a^2b^2c^4}{a_1{}^2b_1{}^2c_1{}^4}\left(\dfrac{c^2}{c_1{}^2}+\dfrac{a^2}{a_1{}^2}\right)}{\Sigma\left(\dfrac{a^2b^2}{a_1{}^2b_1{}^2}\right)\left\{\Sigma\dfrac{a^2}{a_1{}^2}\right\}^2}.$$

$$\frac{U\Omega}{BU}=\frac{X\Omega_a}{BX}$$

$$\therefore\ BU\,.\,U\Omega=\frac{BU^2\,.\,X\Omega_a}{BX}=B\Omega^2\,.\,\frac{BX\,.\,X\Omega_a}{B\Omega_a{}^2}.$$

$$\therefore\ \frac{ZU\,.\,UX}{\omega_1{}^2}\div\frac{BU\,.\,U\Omega}{\omega_2{}^2}=\frac{A\Omega\,.\,\Omega\Omega_a}{\omega_1{}^2}\,.\,\frac{BX}{X\Omega_a}\div\frac{B\Omega^2}{\omega_2{}^2}=1.$$

$$\therefore\ \frac{ZU\,.\,UX}{\omega_1{}^2}=\frac{BU\,.\,U\Omega}{\omega_2{}^2}.$$

The homothetic conic then through B, X, Z passes through Ω. Therefore the homothetic conics BXZ, XCY, YAZ cut in Ω. Similarly the homothetic conics BZ′X′, X′CY′, Y′AZ′ cut in Ω′.

Let R be the radius of the auxiliary conic AΩB along S_1X, where S_1 is the centre of the conic AΩB, and a_2 the radius parallel to BX.

Then $$\frac{BX^2}{a_2{}^2}=\frac{S_1X^2-R^2}{R^2}.$$

$$\therefore\ BX^2\,.\frac{c_1{}^4\Delta^2}{a^2c^4}=\frac{S_1X^2}{R^2}-1\ \ (\S 35).$$

$$\therefore\ \frac{S_1X^2}{R^2}=1+\frac{\Delta^2}{\left(\Sigma\dfrac{a^2}{a_1{}^2}\right)^2}=\frac{4\Sigma\dfrac{a^2b^2}{a_1{}^2b_1{}^2}}{\left(\Sigma\dfrac{a^2}{a_1{}^2}\right)^2}.$$

Let BN be drawn conjugate to S_1X and let n, n_1 be the radii of the circumconic and the conic AΩB parallel to BN.

Then $\dfrac{BN^2}{n_1^{\ 2}} = 1 - \dfrac{S_1N^2}{R^2}$. But $S_1N \,.\, S_1X = R^2$.

$$\therefore \frac{BN^2}{n^2} \cdot \frac{\Delta^2}{\dfrac{c^4a^2}{c_1^{\ 4}a_1^{\ 2}}} = 1 - \frac{R^2}{S_1X^2} = 1 - \frac{\left(\Sigma\dfrac{a^2}{a_1^{\ 2}}\right)^2}{4\Sigma\dfrac{a^2b^2}{a_1^{\ 2}b_1^{\ 2}}} = \frac{\Delta^2}{4\Sigma\dfrac{a^2b^2}{a_1^{\ 2}b_1^{\ 2}}}.$$

$$\therefore \frac{BN^2}{n^2} = \frac{\dfrac{c^4a^2}{c_1^{\ 4}a_1^{\ 2}}}{4\Sigma\dfrac{a^2b^2}{a_1^{\ 2}b_1^{\ 2}}} = \frac{B\Omega^2}{4\omega_2^{\ 2}}.$$

Therefore BN lies on BΩ and is equal to half of it since Ω is a point on the conic. ΩX is then a tangent to the conic. X, Y, Z are on the conjugates to BΩ, CΩ, AΩ through their middle points. Similar results hold for X′, Y′, Z′ with respect to Ω′.

Consider the conics BXZ, AΩB and the lines BX, ΩX. They have a common chord BΩ which is a double chord of AΩB and BX, ΩX. The second chord of the two conics being at infinity, the second chord of BXZ and BX, ΩX must be parallel to BΩ. XY is therefore a tangent to the conic BXZ at X. From this and the two corresponding results it follows that Ω is the first Brocard point of the triangle XYZ. Similarly Ω′ is the second Brocard point of X′Y′Z′. The two triangles XYZ, X′Y′Z′ correspond in the case of the circle to the triangles inscribed in ABC with sides making an angle with those of ABC equal to the Brocard angle.

Section XI.

Brocard's First Conic.

§41. Let US (where U is the mid point of BC) meet BΩ in A_1. Then $\dfrac{A_1U}{\frac{a}{2}} = \dfrac{\Omega\Omega_1}{B\Omega_1}$ and $\dfrac{\Omega\Omega_1}{AD} = \dfrac{\Omega\Omega_a}{A\Omega_a}$ (Fig. 21).

$$\therefore \frac{A_1U}{\frac{a}{2}} = \frac{\Omega\Omega_a}{A\Omega_a} \cdot \frac{AD}{B\Omega_1}.$$

$B\Omega_1 = B\Omega_a + \Omega_a\Omega_1$ and $\dfrac{\Omega_a\Omega_1}{\Omega_aD} = \dfrac{\Omega\Omega_a}{A\Omega_a}$.

$$\therefore\ A\Omega_a \,.\, B\Omega_1 = B\Omega_a \,.\, A\Omega_a + \Omega_a D \,.\, \Omega\Omega_a.$$

$$\therefore\ \frac{A_1U}{\frac{a}{2}} = \frac{AD}{B\Omega_a \,.\, \frac{A\Omega_a}{\Omega\Omega_a} + \Omega_a D} = \frac{AD}{BD + B\Omega_a \,.\, \frac{A\Omega}{\Omega\Omega_a}} = \frac{d\Delta}{a_1 \Sigma \frac{a^2}{a_1^2}}.$$

$$\therefore\ \frac{A_1U}{d} \Big/ \frac{a}{a_1} = \Delta \Big/ 2\Sigma \frac{a^2}{a_1^2} = \frac{KK_1}{d} \Big/ \frac{a}{a_1}\ (\S 23).$$

A_1U is then equal to KK_1 and KA_1 is parallel to BC. The homothetic conic on SK as diameter therefore passes through A_1.

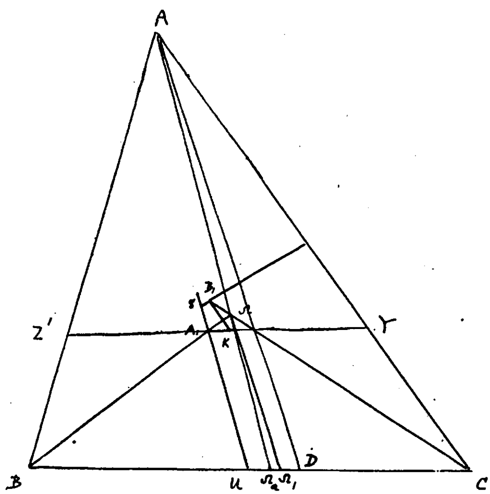


Fig. 21.

Let SU meet $C\Omega'$ in A_1'.

Then $A_1'U \Big/ \frac{a}{2} = \Omega'\Omega_1' / \Omega_1' C.$

Also $\frac{\Omega'\Omega_1'}{AD} = \frac{\Omega'\Omega_a'}{A\Omega_a'}.\quad \therefore\ \frac{A_1'U}{\frac{a}{2}} = \frac{\Omega'\Omega_a'}{A\Omega_a'} \cdot \frac{AD}{\Omega_1'C}.$

$$\Omega_1'C = \Omega_1'\,\Omega_a' + \Omega_a'\,C$$
$$= D\Omega_a' \cdot \frac{\Omega'\Omega_a'}{A\Omega_a'} + \Omega_a'\,C.$$

$$\therefore \frac{A_1'U}{\frac{a}{2}} = \frac{\Omega'\Omega_a'.AD}{D\Omega_a'.\Omega'\Omega_a' + A\Omega_a'.\Omega_a'C}$$

$$= \frac{AD.\Omega'\Omega_a'}{DC.\Omega'\Omega_a' + A\Omega'.\Omega_a'C} = \frac{AD}{DC + \Omega_a'C.\frac{A\Omega'}{\Omega'\Omega_a'}}$$

$$= \frac{d\Delta}{a_1\Sigma\frac{a^2}{a_1^2}} = \frac{KK_1}{\frac{a}{2}}.$$

A_1' therefore is also equal to the conjugate from K to BC. $B\Omega$, $C\Omega'$ then meet in A_1 where A_1K is parallel to BC and A_1, S, U are collinear. Similarly $C\Omega$, $A\Omega'$ meet in B_1 where KB_1 is parallel to CA and B_1, S, V are collinear and $A\Omega$, $B\Omega'$ meet in C_1, KC_1 being parallel to AB and C_1, S, W collinear. B_1, C_1 lie like A_1 on the homothetic conic having SK as diameter. From §39, $\Omega\Omega'$ is conjugate to SK.

$$\frac{BA_1}{B\Omega} = \frac{A_1U}{\Omega\Omega_1}. \quad \therefore \frac{BA_1.B\Omega}{\omega_2^2} = \frac{A_1U}{\Omega\Omega_1}.\frac{B\Omega^2}{\omega_2^2}.$$

Similarly $\dfrac{BC_1.B\Omega'}{\omega_2'^2} = \dfrac{C_1W}{\Omega'\Omega_3'}.\dfrac{B\Omega'^2}{\omega_2'^2}.$

$$\therefore \frac{\frac{BA_1.B\Omega}{\omega_2^2}}{\frac{BC_1.B\Omega'}{\omega_2'^2}} = \frac{A_1U.\Omega'\Omega_3'}{C_1W.\Omega\Omega_1}.\frac{\frac{B\Omega^2}{\omega_2^2}}{\frac{B\Omega'^2}{\omega_2'^2}}$$

$$= \frac{\frac{da}{a_1}}{\frac{fc}{c_1}}.\frac{\Omega'\Omega_3'}{\Omega\Omega_1}.\frac{\frac{c^2}{c_1^2}}{\frac{a^2}{a_1^2}} \quad (\S\S 23, 33, 34).$$

$$\frac{\Omega'\Omega_3'}{CF} = \frac{\Omega'\Omega_c'}{C\Omega_c'}, \quad \frac{\Omega\Omega_1}{AD} = \frac{\Omega\Omega_a}{A\Omega_a}.$$

$$\therefore \frac{\Omega'\Omega_3'}{\Omega\Omega_1} = \frac{CF}{AD}.\frac{\Omega'\Omega_c'}{C\Omega_c'}.\frac{A\Omega_a}{\Omega\Omega_a} = \frac{f}{d}.\frac{\frac{a}{a_1}}{\frac{c}{c_1}}.$$

$$\therefore \frac{BA_1.B\Omega}{\omega_2} = \frac{BC_1.B\Omega'}{\omega_2'^2}$$

A homothetic conic therefore passes through Ω, Ω', A_1, C_1. Similarly a homothetic conic passes through Ω, Ω', B_1, A_1, so that A_1, B_1, C_1, Ω, Ω' lie on the homothetic conic having SK as diameter.

§42. Ω is the first Brocard point of $\Omega_1\Omega_2\Omega_3$.

We have to prove that the homothetic conic $\Omega_2\Omega_3\Omega$ (which passes through A) touches $\Omega_3\Omega_1$. Hence if P (Fig. 22) is the mid

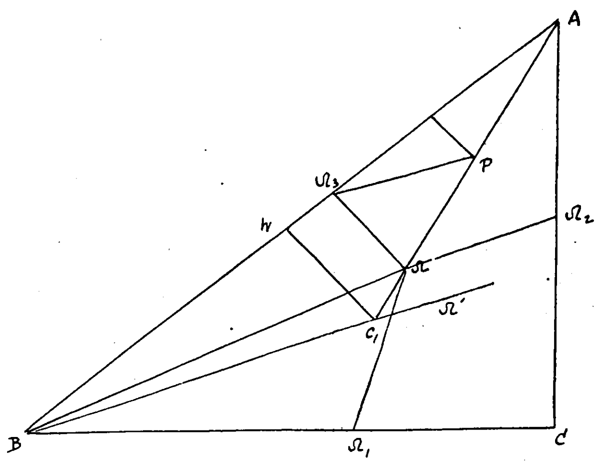


Fig 22.

point of $A\Omega$, $P\Omega_3$ is to be proved conjugate to $\Omega_3\Omega_1$. Now $\Omega_1\Omega_3$ is conjugate to $B\Omega'$ (§15). Therefore $P\Omega_3$ is to be proved parallel to BC_1, C_1 being the intersection of $A\Omega$, $B\Omega'$. C_1W is parallel to $\Omega\Omega_3$.

$$\therefore \frac{AP}{AC_1} = \frac{A\Omega}{2AC_1} = \frac{A\Omega_3}{2AW} = \frac{A\Omega_3}{c}.$$

Therefore Ω_3P is parallel to BC_1.

Similarly Ω' is the second Brocard point of $\Omega_1'\Omega_2'\Omega_3'$.

Brocard's First Triangle.

§43. $A_1B_1C_1$, ABC have the same centroid.

Let G_1 be the centroid of $A_1B_1C_1$.

Then $3G_1BC = A_1BC + B_1BC + C_1BC$.

$A_1BC = KBC$ since KA_1 is parallel to BC.

$$\frac{B_1BC}{B_1CA}=\frac{B\Omega_c}{\Omega_cA}=\frac{c^2}{c_1^2}\Big/\frac{b^2}{b_1^2} \text{ and } B_1CA=KCA.$$

$$\frac{KCA}{KAB}=\frac{CK_a}{K_aB}=\frac{b^2}{b_1^2}\Big/\frac{c^2}{c_1^2}.$$

$\therefore$ $B_1BC=KAB$.

Also $\dfrac{C_1BC}{C_1BA}=\dfrac{C\Omega_b'}{\Omega_b'A}=\dfrac{b^2}{b_1^2}\Big/\dfrac{c^2}{c_1^2}$ and $C_1BA=KBA$.

$$\frac{KBA}{KCA}=\frac{BK_a}{K_aC}=\frac{c^2}{c_1^2}\Big/\frac{b^2}{b_1^2}. \quad \therefore C_1BC=KCA.$$

$\therefore$ $3G_1BC=KBC+KCA+KAB=ABC=3GBC$.

Similarly $3G_1CA=3GCA$. Therefore G and G_1 coincide.

§44. Let KA_1 cut AB in Z′, AC in Y (Fig. 21). Z′, Y are points on the Lemoine conic. The conjugate from the mid point of SK, the centre of Lemoine's and Brocard's first conic, bisects Z′Y and A_1K. AK and AA_1 are therefore isotomically conjugate. AA_1, BB_1, CC_1 then meet in a point K′, the isotomic conjugate of K. The triangles ABC, $A_1B_1C_1$ are triply in perspective, the centres of perspective being Ω, Ω′, K′.

For a system of circumconics passing through a fixed point O, the points K_a, K_c generate two projective ranges on BC, AB with C corresponding to A. Their isotomic conjugates K_a', K_c' therefore generate two projective ranges on BC, AB with B in BC corresponding to B in AB. The rays AK_a', CK_c' intersect on a conic passing through A, B, C. If the locus of K cuts BC, CA, AB in L, M, N, the isotomic of LCB is CA. The conic therefore touches at A the ray AL′ the isotomic of AL. Similarly it touches at C the ray CN′ the isotomic of CN with respect to AB, and at B the ray BM′ the isotomic of BM′ with respect to CA.

§45. The loci of A_1, B_1, C_1 are straight lines.

If P, Q, R are the poles of a, b, c with respect to the circumconic, $\{AK_a, KP\}$ is a harmomic range, because C{AB, KP} is a harmonic pencil. Let PU meet the parallel to BC through K in A_1 and the parallel to BC through A in T (Fig. 23). Then $\{PA_1, UT\}$ is a harmonic range. U is fixed, and T, P describe two straight lines (§20). A_1 therefore describes a straight line which passes through the intersection of AT and the locus of P.

Hence AU_1, $A\Omega''$ are isogonally conjugate with respect to $A\Omega$, $A\Omega'$, and therefore with respect AB, AC. Similar results hold for BV_1, $B\Omega''$ and CW_1, $C\Omega''$. AU_1, BV_1, CW_1 then meet in the isogonal conjugate O of Ω''.

Since UU_1 is parallel to and half AA_1, U_1 is equally distant from the conjugate to BC from A_1 and A, that is from S and A. Similar results hold for V_1 and W_1. A_1U_1 is then divided internally at G and externally by AH in the ratio 2 : 1. Therefore $A\{A_1O, GH\}$ is harmonic. The pencil then formed by the isogonal conjugates AA_1', $A\Omega''$, AK, AS of AA_1, AO, AG, AH is harmonic (§13). Similarly $C\{C_1'\Omega'', KS\}$, $B\{B_1'\Omega'', KS\}$, where A_1', B_1', C_1' are the isogonally conjugate points of A_1, B_1, C_1, are harmonic. AA_1', BB_1', CC_1' therefore meet in a point D' on SK such that $\{D'\Omega'', KS\}$ is harmonic or D' is the pole of $\Omega\Omega'$ with respect to the Brocard conic. A_1' is the intersection of BC_1, CB_1; B_1' of AC_1, CA_1; C_1' of AB_1, BA_1. Hence $A_1'B_1'C_1'$ is also in perspective with $A_1B_1C_1$ as well as with ABC. The three triangles have the same axis of perspective, and the centre of perspective of $A_1B_1C_1$ and $A_1'B_1'C_1'$ lies on D'K'.

Brocard's Second Triangle.

§49. Let AK, BK, CK meet Brocard's first conic in A_2, B_2, C_2 and the circumconic again in A', B', C'. Since SK is a diameter of Brocard's conic SA_2, SB_2, SC_2 are conjugate to A_2K, B_2K, C_2K, and therefore A_2, B_2, C_2 are the mid points of AA', BB', CC'. Let AK cut the auxiliary conic $CA\Omega$ in A_2' and BC cut the same conic again in M.

$$\frac{AK_a^2}{k_1^2}=\frac{AD^2}{d^2}+\frac{K_aD^2}{a_1^2}=\frac{AD^2}{d^2}+\frac{(K_aC-DC)^2}{a_1^2}$$

$$=\frac{\Delta^2}{4\dfrac{a^2}{a_1^2}}+\left\{\frac{\dfrac{ab^2}{a_1b_1^2}}{\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}}-\frac{\dfrac{a^2}{a_1^2}+\dfrac{b^2}{b_1^2}-\dfrac{c^2}{c_1^2}}{2\dfrac{a}{a_1}}\right\}^2.$$

$$\therefore\ \frac{AK_a}{k_1}=\frac{\dfrac{bc}{b_1c_1}\sqrt{\left(\dfrac{2b^2}{b_1^2}+\dfrac{2c^2}{c_1^2}-\dfrac{a^2}{a_1^2}\right)}}{\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}}.$$

Again $\frac{c^2}{c_1^2}=\frac{a\,.\,\mathrm{BM}}{a_1^2}=\frac{a}{a_1}\left\{\frac{\mathrm{BK}_a}{a_1}+\frac{\mathrm{K}_a\mathrm{M}}{a_1}\right\}$.

$$\therefore\ \frac{a\mathrm{K}_a\mathrm{M}}{a_1^2}=\frac{\frac{c^2}{c_1^2}\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}\right)}{\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}}.$$

Also $\frac{\mathrm{K}_a\mathrm{C}\,.\,\mathrm{K}_a\mathrm{M}}{a_1^2}=\frac{\mathrm{K}_a\mathrm{A}'_2\,.\,\mathrm{K}_a\mathrm{A}}{k_1^2}$.

$$\therefore\ \frac{\mathrm{A}_2'\mathrm{K}_a}{k_1}=\frac{\frac{bc}{b_1c_1}\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}\right)}{\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right)\sqrt{\left(\frac{2b^2}{b_1^2}+\frac{2c^2}{c_1^2}-\frac{a^2}{a_1^2}\right)}}.$$

Also $\frac{\mathrm{BK}_a\,.\,\mathrm{K}_a\mathrm{C}}{a_1^2}=\frac{\mathrm{AK}_a\,.\,\mathrm{K}_a\mathrm{A}'}{k_1^2}$.

$$\therefore\ \frac{\mathrm{K}_a\mathrm{A}'}{k_1}=\frac{\frac{a^2bc}{a_1^2b_1c_1}}{\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right)\sqrt{\left(\frac{2b^2}{b_1^2}+\frac{2c^2}{c_1^2}-\frac{a^2}{a_1^2}\right)}}.$$

$$\therefore\ \frac{\mathrm{A}_2'\mathrm{A}'}{k_1}=\frac{\mathrm{A}_2'\mathrm{K}_a}{k_1}+\frac{\mathrm{K}_a\mathrm{A}'}{k_1}=\frac{\frac{bc}{b_1c_1}}{\sqrt{\left(\frac{2b^2}{b_1^2}+\frac{2c^2}{c_1^2}-\frac{a^2}{a_1^2}\right)}}.$$

$$\therefore\ \frac{\mathrm{AA}'}{k_1}=\frac{\mathrm{AK}_a}{k_1}+\frac{\mathrm{K}_a\mathrm{A}'}{k_1}=\frac{\frac{2bc}{b_1c_1}}{\sqrt{\left(\frac{2b^2}{b_1^2}+\frac{2c^2}{c_1^2}-\frac{a^2}{a_1^2}\right)}}=\frac{2\mathrm{A}_2'\,\mathrm{A}'}{k_1}.$$

A_2' then coincides with A_2. Similarly it may be proved that AK cuts the auxiliary conic $\mathrm{AB}\Omega'$ in A_2. Like results hold for B_2 and C_2.

Loci of A_2, B_2, C_2

§50. $\frac{\mathrm{AK}_a}{\mathrm{KK}_a}=\frac{\mathrm{AD}}{\mathrm{KK}_1}=\frac{\Sigma\frac{a^2}{a_1^2}}{\frac{a^2}{a_1^2}}.\quad\therefore\ \frac{\mathrm{AK}}{\mathrm{KK}_a}=\frac{\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}}{\frac{a^2}{a_1^2}}.$

$$\therefore \frac{AK}{AK_a}=\frac{\frac{b^2}{b_1^{\,2}}+\frac{c^2}{c_1^{\,2}}}{\Sigma\frac{a^2}{a_1^{\,2}}}. \quad \therefore AK=\frac{\frac{bc}{b_1c_1}\sqrt{\left(\frac{2b^2}{b_1^{\,2}}+\frac{2c^2}{c_1^{\,2}}-\frac{a^2}{a_1^{\,2}}\right)}}{\Sigma\frac{a^2}{a_1^{\,2}}}.$$

$$\frac{AA_2}{k_1}=\frac{\frac{bc}{b_1c_1}}{\sqrt{\left(\frac{2b^2}{b_1^{\,2}}+\frac{2c^2}{c_1^{\,2}}-\frac{a^2}{a_1^{\,2}}\right)}}.$$

$$\therefore \frac{k_1}{AA_2}+\frac{k_1}{AK}=\frac{3\left(\frac{b^2}{b_1^{\,2}}+\frac{c^2}{c_1^{\,2}}\right)}{\frac{bc}{b_1c_1}\sqrt{\left(\frac{2b^2}{b_1^{\,2}}+\frac{2c^2}{c_1^{\,2}}-\frac{a^2}{a_1^{\,2}}\right)}}=\frac{3k_1}{AK_a}.$$

Draw the parallel through G to BC meeting AK in Q say.

Then $\frac{1}{AA_2}+\frac{1}{AK}=\frac{2}{AQ}$ or $\{AQ, KA_2\}$ is a harmonic range.

The locus of A_2 for a system of circumconics passing through a given fixed point is a straight line through the intersection of the locus of K with the parallel to BC through the centroid, this parallel being the polar of A with respect to the line and the locus of K.

§51. A_2 lies on the homothetic conic BSC.

SU, AK both pass through P the pole of BC. The homothetic conic on SP as diameter passes through A_2, B, C since SA_2 is conjugate to A_2P, SB to BP and SC to CP. Let the tangent at A to the circumconic meet BC in X. $\{BC, K_aX\}$ is a harmonic range. The tangent at A_1' therefore, where AK cuts the circumconic, passes through X. SA_2 the diameter to AA' then also passes through X, and $A_2\{BC, AS\}$ is a harmonic pencil. If x be the radius of the circumconic parallel to A_2X, $\frac{XA_2 . XS}{x^2}=\frac{XB . XC}{a_1^{\,2}}$. X is then on the radical axis of the circumconic and Brocard's conic. XYZ is therefore the radical axis of the two conics.

§52. A_1A_2, B_1B_2, C_1C_2 bisect B_1C_1, C_1A_1, A_1B_1.

$$\frac{A_1BC}{ABC}=\frac{KBC}{ABC}=\frac{KK_a}{AK_a}=\frac{\frac{a^2}{a_1^{\,2}}}{\Sigma\frac{a^2}{a_1^{\,2}}}.$$

$$\frac{A_2BC}{ABC}=\frac{A_2K_a}{AK_a}=\frac{\dfrac{b^2}{b_1{}^2}+\dfrac{c^2}{c_1{}^2}-\dfrac{a^2}{a_1{}^2}}{\dfrac{2b^2}{b_1{}^2}+\dfrac{2c^2}{c_1{}^2}-\dfrac{a^2}{a_1{}^2}}.$$

Taking a point G′ in A_1A_2 dividing A_1A_2 in the ratio

$$\left(\frac{2b^2}{b_1{}^2}+\frac{2c^2}{c_1{}^2}-\frac{a^2}{a_1{}^2}\right):\Sigma\frac{a^2}{a_1{}^2}\text{ we have }\frac{G'BC}{ABC}=\tfrac{1}{3}.$$

$$\frac{A_1AB}{A_1BC}=\frac{A\Omega_b}{\Omega_bC}=\frac{b^2}{b_1{}^2}\Big/\frac{a^2}{a_1{}^2}.\quad\therefore\ \frac{A_1AB}{ABC}=\frac{\dfrac{b^2}{b_1{}^2}}{\Sigma\dfrac{a^2}{a_1{}^2}}.$$

$$\frac{A_2AB}{A_2AC}=\frac{BK_a}{K_aC}=\frac{c^2}{c_1{}^2}\Big/\frac{b^2}{b_1{}^2}.\quad\therefore\ \frac{A_2AB}{ABC-A_2BC}=\frac{\dfrac{c^2}{c_1{}^2}}{\dfrac{b^2}{b_1{}^2}+\dfrac{c^2}{c_1{}^2}}.$$

$$\frac{ABC}{ABC-A_2BC}=\frac{\dfrac{2b^2}{b_1{}^2}+\dfrac{2c^2}{c_1{}^2}-\dfrac{a^2}{a_1{}^2}}{\dfrac{b^2}{b_1{}^2}+\dfrac{c^2}{c_1{}^2}}.$$

$$\therefore\ \frac{A_2AB}{ABC}=\frac{\dfrac{c^2}{c_1{}^2}}{\dfrac{2b^2}{b_1{}^2}+\dfrac{2c^2}{c_1{}^2}-\dfrac{a^2}{a_1{}^2}}.\quad\therefore\ G'AB=\tfrac{1}{3}ABC.$$

G′ is therefore the centroid of ABC; but it is also the centroid of $A_1B_1C_1$, which proves the theorem. G is then the centre of perspective of the triangles $A_1B_1C_1$, $A_2B_2C_2$. The axis of perspective is the polar of G with respect to the Brocard conic, and is therefore conjugate to S′G.

§53. The ratio of similarity of Brocard's first conic.

$$\frac{A_1K}{K_a'K_a}=\frac{AK}{AK_a}=\frac{\dfrac{b^2}{b_1{}^2}+\dfrac{c^2}{c_1{}^2}}{\Sigma\dfrac{a^2}{a_1{}^2}}\quad(\S 49).$$

$$K_a' K_a = a - 2K_aC = a - \frac{2ab^2}{b_1^2} \Big/ \left(\frac{b^2}{b_1^2} + \frac{c^2}{c_1^2}\right) = \frac{a\left(\frac{c^2}{c_1^2} - \frac{b^2}{b_1^2}\right)}{\frac{b^2}{b_1^2} + \frac{c^2}{c_1^2}}.$$

$$\therefore \frac{A_1K}{a_1} = \frac{\frac{a}{a_1}\left(\frac{c^2}{c_1^2} - \frac{b^2}{b_1^2}\right)}{\Sigma\frac{a^2}{a_1^2}}.$$

$$\frac{A_1S}{d} = \frac{KK_1}{d} - \frac{SU}{d} = \frac{\frac{a}{a_1}\Delta}{2\Sigma\frac{a^2}{a_1^2}} - \frac{\frac{a}{a_1}\left(\frac{b^2}{b_1^2} + \frac{c^2}{c_1^2} - \frac{a^2}{a_1^2}\right)}{2\Delta}$$

$$= \frac{\frac{a}{a_1}\left(\frac{c^2a^2}{c_1^2a_1^2} + \frac{a^2b^2}{a_1^2b_1^2} - \frac{b^4}{b_1^4} - \frac{c^4}{c_1^4}\right)}{\Delta\Sigma\frac{a^2}{a_1^2}}.$$

If k be the radius of the circumconic parallel to SK.

$$\frac{SK^2}{k^2} = \frac{A_1K^2}{a_1^2} + \frac{A_1S^2}{d^2} = \frac{4a^2b^2c^2}{a_1^2b_1^2c_1^2} \frac{\left\{\Sigma\frac{a^4}{a_1^4} - \Sigma\frac{b^2c^2}{b_1^2c_1^2}\right\}}{\Delta^2\left(\Sigma\frac{a^2}{a_1^2}\right)^2}.$$

Therefore the ratio of similarity $\frac{S'K}{k}$ is $\dfrac{\sqrt{\left\{\Sigma\frac{a^4}{a_1^4} - \Sigma\frac{b^2c^2}{b_1^2c_1^2}\right\}}}{\Sigma\frac{a^2}{a_1^2}}$

since $\frac{abc}{a_1b_1c_1} = \Delta$ (§11).

Cobrocardal Triangles.

§54. Let $\Omega\Omega'$ cut SK in Ω''. $2\Omega''\Omega_1'' = \Omega\Omega_1 + \Omega'\Omega_1'$.

$$\frac{\Omega\Omega_1}{AD} = \frac{\Omega\Omega_a}{A\Omega_a} = \frac{\frac{c^2a^2}{c_1^2a_1^2}}{\Sigma\frac{a^2b^2}{a_1^2b_1^2}}. \quad \frac{\Omega\Omega_1'}{AD} = \frac{\Omega'\Omega_a'}{A\Omega_a'} = \frac{\frac{a^2b^2}{a_1^2b_1^2}}{\Sigma\frac{a^2b^2}{a_1^2b_1^2}}.$$

$$\therefore\ \frac{\Omega''\Omega_1''}{d}=\frac{\dfrac{a^2}{a_1^2}\left(\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}\right)}{2\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}}\cdot\frac{\mathrm{AD}}{d}=\frac{\dfrac{a}{a_1}\left(\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}\right)\triangle}{4\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}}.$$

$$\frac{\mathrm{S}\Omega''}{\Omega''\mathrm{K}}=\frac{\Omega''\Omega_1''-\mathrm{SU}}{\mathrm{KK}_1-\Omega''\Omega_1''}.$$

$$\frac{\Omega''\Omega_1''}{d}-\frac{\mathrm{SU}}{d}=\frac{\dfrac{a}{a_1}\left(\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}\right)\triangle}{4\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}}-\frac{\dfrac{a}{a_1}\left(\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}-\dfrac{a^2}{a_1^2}\right)}{2\triangle}$$

$$=\frac{\dfrac{a}{a_1}\Sigma\dfrac{a^2}{a_1^2}\left\{\dfrac{a^2b^2}{a_1^2b_1^2}+\dfrac{c^2a^2}{c_1^2a_1^2}-\dfrac{b^4}{b_1^4}-\dfrac{c^4}{c_1^4}\right\}}{4\triangle\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}}.$$

$$\frac{\mathrm{KK}_1-\Omega''\Omega_1''}{d}=\frac{\dfrac{a}{a_1}\triangle}{2\Sigma\dfrac{a^2}{a_1^2}}-\frac{\dfrac{a}{a_1}\left(\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}\right)\triangle}{4\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}}$$

$$=\frac{\dfrac{a}{a_1}\triangle\left\{\dfrac{a^2b^2}{a_1^2b_1^2}+\dfrac{c^2a^2}{c_1^2a_1^2}-\dfrac{b^4}{b_1^4}-\dfrac{c^4}{c_1^4}\right\}}{4\Sigma\dfrac{a^2}{a_1^2}\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}}.$$

$$\therefore\ \frac{\mathrm{S}\Omega''}{\Omega''\mathrm{K}}=\frac{\left(\Sigma\dfrac{a^2}{a_1^2}\right)^2}{\triangle^2}.$$

Also $$\frac{\mathrm{SK}^2}{k^2}=\frac{4\left\{\Sigma\dfrac{a^4}{a_1^4}-\Sigma\dfrac{b^2c^2}{b_1^2c_1^2}\right\}}{\left(\Sigma\dfrac{a^2}{a_1^2}\right)^2}.$$

Let XYZ the polar of K with respect to the circumconic cut SK in Q. Then $\mathrm{SQ}\,.\,\mathrm{SK}=k^2$.

$$\therefore\ \frac{\mathrm{SK}}{\mathrm{SQ}}=\frac{4\left\{\Sigma\dfrac{a^4}{a_1^4}-\Sigma\dfrac{b^2c^2}{b_1^2c_1^2}\right\}}{\left(\Sigma\dfrac{a^2}{a_1^2}\right)^2}.$$

$$\therefore \quad \frac{KQ}{SK} = \frac{3\Delta^2}{4\left\{\Sigma\frac{a^4}{a_1^4} - \Sigma\frac{b^2c^2}{b_1^2c_1^2}\right\}}.$$

$$\therefore \quad \frac{SQ}{KQ} = \frac{\left(\Sigma\frac{a^2}{a_1^2}\right)^2}{3\Delta^2}. \qquad \therefore \quad \frac{S\Omega''}{\Omega''K} = 3\frac{SQ}{KQ}.$$

All triangles therefore having the same circumconic and the same symmedian centre have the same Brocard points. The triangle A′B′C′ of §21 is such a triangle, but there is an infinity of other such triangles.

Consider Brocard's second conic, that is the conic touching the sides BC, CA, AB at K_a, K_b, K_c. If any chord B″C″ of the circumconic be drawn touching this second conic in K_a'' and the tangents from B″, C″ to this conic meet in A″, A″ lies on the circumconic for A, B, C, A″, B″, C″ lie on a conic. An infinite number of triangles can therefore be inscribed in the circumconic and circumscribed about Brocard's second conic. The two conics are in perspective, K being the centre and XYZ the axis of perspective and A, K_a; B, K_b; C, K_c pairs of corresponding points. A, K, K_a are collinear as also B, K, K_b and C, K, K_c. The tangents at A and K_a meet in X, at B and B_b in Y, and at C, K_c in Z. K_bK_c passes through X, K_cK_a through Y, and K_aK_b through Z. The sides of A′B′C′ are the second tangents from X, Y, Z to Brocard's second conic. Corresponding to every point A we have a triangle with K as symmedian centre. Take the corresponding K_a and the tangent at K_a cuts the circumconic in the other two vertices. The results follow easily by projection from the circle. For the circle we have an infinite number of triangles in the same circle with the same K, and with their sides touching Brocard's ellipse at the feet of the symmedians. If we project this system on any plane, as K has projective properties, we have a system of triangles inscribed in the same conic, having the same K and touching the same conic, Brocard's second conic, at the feet of the symmedians for the new circumconic. For the circle, Ω, Ω′ are the foci of Brocard's ellipse. The new Brocard points are not the projections of the old Brocard points, but in an extended sense they may be called foci of Brocard's second conic. We have

$$\frac{\Omega\Omega_1 \,.\, \Omega'\Omega_1'}{d^2} = \frac{\Omega\Omega_2 \,.\, \Omega'\Omega_2'}{e^2} = \frac{\Omega\Omega_3 \,.\, \Omega'\Omega_3'}{f^2} = \frac{\dfrac{a^4b^4c^4}{a_1^4b_1^4c_1^4}}{4\Sigma\dfrac{a^2b^2}{a_1^2b_1^2}}.$$

It may also be proved that the centre of the conic is the mid point of $\Omega\Omega'$.

The system of Tucker's conics is the same for all the cobrocardal triangles as the centres lie on SK, and the ratio of similarity depends only on the ratio into which SK is divided by the centre. Of course the Taylor's conics do not correspond.

Section XII.

Steiner's Ellipses. Steiner's and Tarry's point.

§55. If the symmedian centre K coincide with the centroid G the circumconic is Steiner's first ellipse. The orthocentre, nine point centre and circumcentre coincide with G, and Brocard's first conic degenerates to the point G. The incentre I also is at G, and the inconic and nine point conic are the same, namely, Steiner's second or inscribed ellipse. The excentres coincide with the poles P, Q, R, and these are found by producing AU, BV, CW their own lengths. The inconic or nine point conic touches the exconics at the mid points of the sides. The polar of any point on the circumconic with respect to the triangle passes through G.

For any circumconic the isogonal curve of the polar XYZ of K is Steiner's circumellipse.

Produce AU, BV, CW to G′, G″, G‴ so that GU = UG′, GV = VG″, GW = WG‴.

$$\frac{G'G_1'}{d} = -\frac{GG_1}{d} = -\frac{AD}{3d}.$$

If CK cuts the circumconic in C′, $\dfrac{C'C_1'}{d} = \dfrac{CC'}{CK} \cdot \dfrac{KK_1}{d}$.

$$\therefore \frac{G'G_1'.\, C'C_1'}{d^2} = -\frac{CC'}{3CK} \cdot \frac{AD \,.\, KK_1}{d^2} = -\frac{CC' \,.\, KK_a}{3CK \,.\, AK_a} \cdot \frac{AD^2}{d^2} \quad \text{(i)}.$$

$$\frac{G'G_3'}{f} = \frac{2GG_3}{f} = \frac{2CF}{3f} \text{ and } \frac{C'C_3'}{f} = -\frac{K_cC'}{CK_c} \cdot \frac{CF}{f}.$$

$$\therefore \frac{G'G_3'.\, C'C_3'}{f^2} = -\frac{2K_cC'}{3CK_c} \cdot \frac{CF^2}{f^2} \quad \text{(ii)}.$$

The ratio of the left sides of (i) and (ii) is therefore

$$\frac{\dfrac{AD^2}{d^2}}{\dfrac{CF^2}{f^2}}\cdot\frac{CC'}{2CK}\cdot\frac{KK_a}{AK_a}\cdot\frac{CK_c}{K_cC'}=\frac{\dfrac{AD^2}{d^2}}{\dfrac{CF^2}{f^2}}\cdot\frac{\dfrac{a^2}{a_1^{\,2}}}{\dfrac{c^2}{c_1^{\,2}}}=1.$$

BC′, BG′ are then isogonally conjugate; similarly CB′, CG′ are isogonally conjugate. BC′, CB′ meet in X′ on XYZ (§21). G′ is therefore the isogonal conjugate of X′. Similarly G″ is the isogonal conjugate of Y′. The conic therefore through A, B, C, G′, G″ is the isogonal curve of XYZ. This conic is Steiner's circumellipse.

§56. CG bisects AB. Since for any circumconic CC_1 passes through K′ the isotomic conjugate of K and C_1K is parallel to AB, CG also bisects C_1K; similarly AG, BG bisect A_1K, B_1K. A conic with centre G homothetic to Steiner's circumellipse and passing through K passes also through A_1, B_1, C_1. As G is the centroid also of $A_1B_1C_1$ this conic is the Steiner circumellipse of $A_1B_1C_1$. For a given triangle ABC this ellipse is fixed by the position of K or the position of any one of the points A_1, B_1, C_1. One of these four points determines the three others. By letting K move on this ellipse we have a system of Brocard's first triangles inscribed in the same ellipse. Hence, conversely, to every triangle $A_1B_1C_1$ inscribed in this ellipse and having the same centroid as ABC we have a point K on this ellipse such that A_1K, B_1K, C_1K are parallel to BC, CA, AB. We may now invert the rôles of the two ellipses and consider $A_1B_1C_1$ as the original triangle and ABC as its Brocard triangle. Then there is a point R on the Steiner ellipse of ABC such that RA, RB, RC are parallel to B_1C_1, C_1A_1, A_1B_1. If we consider the Steiner circumellipse of ABC as a circumconic, G is the symmedian centre and the polar of R with respect to the triangle passes through G. If R were the intersection of the Steiner ellipse and the circumconic giving the symmedian centre K, the polar of R with respect to the triangle would pass through K. R will then be on the circumconic if GK is the polar of R with respect to the triangle.

Let GK cut BC in X and AR cut BC in R_a. We have to prove that $\{BC, R_aX\}$ is harmonic (Fig. 24).

From §53, $B_1K=b\left(\dfrac{c^2}{c_1^{\,2}}-\dfrac{a^2}{a_1^{\,2}}\right)\Big/\Sigma\dfrac{a^2}{a_1^{\,2}}$ and $C_1K=c\left(\dfrac{a^2}{a_1^{\,2}}-\dfrac{b^2}{b_1^{\,2}}\right)\Big/\Sigma\dfrac{a^2}{a_1^{\,2}}$.

Since AR is parallel to B_1C_1, if the triangle B_1KC_1 have a motion of translation parallel and equal to B_1A, C_1 and K will take up two positions L, M on AR and AC. Draw R_aN parallel to C_1K or AB meeting AC in N and let

$$\frac{BR_a}{R_aC}=\frac{AN}{NC}=\lambda. \quad \text{Then } AN=\frac{b\lambda}{\lambda+1}.$$

$$\frac{R_aN}{c}=\frac{NC}{b}=\frac{1}{\lambda+1} \text{ or } R_aN=\frac{c}{\lambda+1}.$$

$$\therefore \frac{AN}{R_aN}=\frac{b\lambda}{c}.$$

$$\text{But } \frac{AN}{R_aN}=\frac{AM}{LM}=\frac{B_1K}{C_1K}=\frac{b\left(\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}\right)}{c\left(\frac{a^2}{a_1^2}-\frac{b^2}{b_1^2}\right)}.$$

$$\therefore \lambda=\frac{\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}}{\frac{a^2}{a_1^2}=\frac{b^2}{b_1^2}}.$$

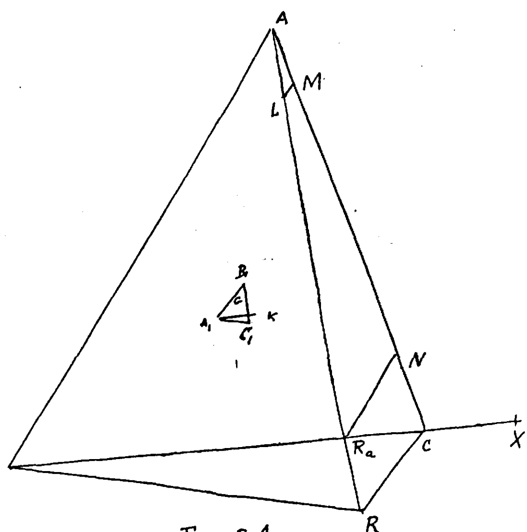


Fig 24.

Draw GG′ parallel to AK to cut BC in G′.
Since UG : UA = 1 : 3, UG′ : UK_a = 1 : 3.

$$\text{Now } K_aC = \frac{ab^2}{b_1^2}\Big/\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right). \quad \therefore\ UK_a = \frac{a}{2} - K_aC = \frac{a\left(\frac{c^2}{c_1^2}-\frac{b^2}{b_1^2}\right)}{2\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right)}.$$

$$\therefore\ G'K_a = \frac{a\left(\frac{c^2}{c_1^2}-\frac{b^2}{b_1^2}\right)}{3\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right)}.$$

$$\text{Also } \frac{GG'}{KK_a} = \frac{G'X}{K_aX} \text{ and } \frac{AK_a}{KK_a} = \frac{\Sigma\frac{a^2}{a_1^2}}{\frac{a^2}{a_1^2}}.$$

Therefore, since GG′ = $\frac{1}{3}AK_a$,

$$\frac{G'X}{K_aX} = \frac{\Sigma\frac{a^2}{a_1^2}}{3\frac{a^2}{a_1^2}}. \quad \therefore\ \frac{G'K_a}{K_aX} = \frac{\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{2a^2}{a_1^2}}{3\frac{a^2}{a_1^2}}.$$

$$\therefore\ K_aX = \frac{3\frac{a^2}{a_1^2}G'K_a}{\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{2a^2}{a_1^2}} = \frac{\frac{a^3}{a_1^2}\left(\frac{c^2}{c_1^2}-\frac{b^2}{b_1^2}\right)}{\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right)\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{2a^2}{a_1^2}\right)}.$$

$$\therefore\ CX = \frac{a\left\{\frac{c^2a^2}{c_1^2a_1^2}+\frac{a^2b^2}{a_1^2b_1^2}-\frac{b^2c^2}{b_1^2c_1^2}-\frac{b^4}{b_1^4}\right\}}{\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right)\left(\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{2a^2}{a_1^2}\right)} = \frac{a\left(\frac{a^2}{a_1^2}-\frac{b^2}{b_1^2}\right)}{\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{2a^2}{a_1^2}}.$$

$$\therefore\ BX = \frac{a\left(\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}\right)}{\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{2a^2}{a_1^2}}.$$

$$\therefore\ \frac{BX}{CX} = \frac{\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}}{\frac{a^2}{a_1^2}-\frac{b^2}{b_1^2}} = \frac{BR_a}{R_aC}.$$

In the same way it may be proved that if GK meet CA in Y, $\{CA, YR_b\}$ is harmonic. GK is therefore the polar of R with respect to the triangle, and R is the intersection of the circumconic and Steiner's first ellipse. From the reciprocal relation of ABC and $A_1B_1C_1$, K is the Steiner point of $A_1B_1C_1$ the circumconic being Brocard's first conic. Knowing that K was on this conic we might at once have said that R was on the circumconic.

The locus of Steiner's point for all circumconics of ABC is Steiner's first ellipse. If we take a point R on this ellipse and consider the system of circumconics passing through this point we see that the Brocard first triangles form a system of similar triangles with their corresponding sides parallel.

§57. The parallels through A, B, C to B_1C_1, C_1A, A_1B_1 meeting in a point R on the circumconic, the conjugates through A, B, C to these lines will also meet in a point T on the circumconic which is the other end of the diameter through R. For a system of circumconics through R a fixed point on the Steiner circumellipse, the locus of T is a conic homothetic with the locus of S, R being the centre of similarity and the ratio being 2 : 1. S is the Tarry point of $A_1B_1C_1$ as A_1S, B_1S are conjugate to a, b.

Section XIII.

The Point O Theorem.

§58. If X, Y, Z be three points on the sides a, b, c, the homothetic conics AYZ, BZX, CXY meet in a point O.

Let the conics AYZ, CXY meet in O, ZO meet a, b in L, M and the conic CXY again in N (Fig. 25). If l, m, n are the radii of the circumconic parallel to OX, OY, OZ, then

$$\frac{MC \cdot MY}{b_1^2} = \frac{MN \cdot MO}{n^2} \quad \text{and} \quad \frac{MY \cdot MA}{b_1^2} = \frac{MO \cdot MZ}{n^2}.$$

$$\therefore \frac{MC}{MA} = \frac{MN}{MZ} \text{ and CN is parallel to AB.}$$

$$\therefore \frac{CL}{LB} = \frac{NL}{LZ}.$$

Again $\dfrac{NL \cdot LO}{n^2} = \dfrac{CL \cdot LX}{a_1^2}$. $\therefore \dfrac{LZ \cdot LO}{n^2} = \dfrac{LB \cdot LX}{a_1^2}$.

The homothetic conic XBZ therefore passes through O.

Conversely if O is any point not on the sides and an arbitrary point Z be taken on AB giving a particular homothetic conic cutting CA in Y, the homothetic conics BZO, CYO cut BC in the same point X. We shall confine ourselves to the particular and most important cases when O is Ω or Ω′.

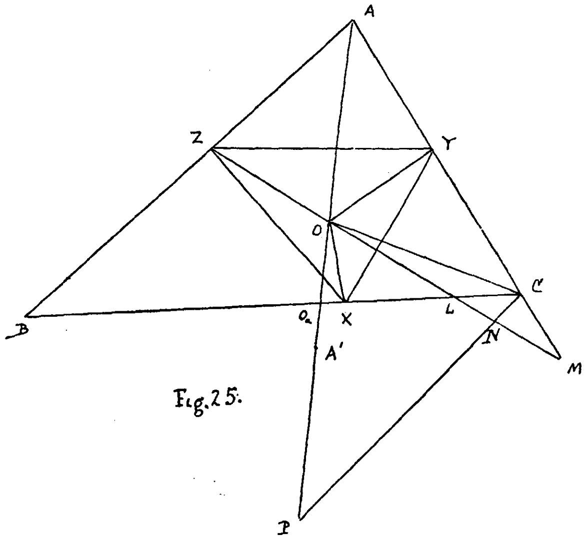


Fig. 25.

§59. In Fig. 25 replace O by Ω, let CN cut AΩ in P and let AΩ cut the conic CXY again in A′.

$$\text{Then } \frac{\text{PA}'.\text{P}\Omega}{\omega_1^2}=\frac{\text{PN}.\text{PC}}{c_1^2}.$$

$$\text{But } \frac{\text{PN}}{\text{P}\Omega}=\frac{\text{AZ}}{\text{A}\Omega} \text{ and } \frac{\text{PC}}{\text{A}\Omega_c}=\frac{\text{C}\Omega}{\Omega_c\Omega}.$$

$$\therefore \frac{\text{A}'\text{P}}{\omega_1}=\frac{\frac{\text{AZ}}{\text{A}\Omega}}{\omega_1}\cdot\frac{\text{A}\Omega_c}{c_1^2}\cdot\frac{\text{C}\Omega}{\Omega\Omega_c}=\frac{\frac{a^2u}{a_1^2c_1}\sqrt{\Sigma\frac{a^2b^2}{a_1^2b_1^2}}}{\frac{b^2c^2}{b_1c_1^2}} \text{ where } u=\text{AZ}.$$

$$\frac{\text{AP}}{\text{A}\Omega}=\frac{\text{C}\Omega_c}{\Omega\Omega_c}. \quad \therefore \frac{\text{AP}}{\omega_1}=\frac{\sqrt{\Sigma\frac{a^2b^2}{a_1^2b_1^2}}}{\frac{c}{c_1}}.$$

$$\therefore\ \frac{AA'}{\omega_1}=\frac{AP-A'P}{\omega_1}=\frac{\left(\frac{b^2c}{b_1{}^2c_1}-\frac{a^2u}{a_1{}^2c_1}\right)\sqrt{\Sigma\frac{a^2b^2}{a_1{}^2b_1{}^2}}}{\frac{b^2c^2}{b_1{}^2c_1{}^2}}.$$

$$\frac{AY\,.\,b}{b_1{}^2}=\frac{A\Omega\,.\,AA'}{\omega_1{}^2}.\qquad \therefore\ \frac{AY}{b_1}=\frac{\left(\frac{b^2c}{b_1{}^2c_1}-\frac{a^2u}{a_1{}^2c_1}\right)}{\frac{bc}{b_1c_1}}.$$

Also $ZB=c-u,\ \dfrac{YC}{b_1}=\dfrac{\frac{a^2u}{a_1{}^2c_1}}{\frac{bc}{b_1c_1}}.$

Similarly if we denote BX by v we shall find that

$$\frac{BZ}{c_1}=\frac{\left(\frac{c^2a}{c_1{}^2a_1}-\frac{b^2v}{b_1{}^2a_1}\right)}{\frac{ca}{c_1a_1}}.$$

Equating the two values of BZ we have $\dfrac{v}{a_1}=\dfrac{\frac{uca}{c_1{}^2a_1}}{\frac{b^2}{b_1{}^2}}.$

$$\therefore\ \frac{BX}{a_1}=\frac{\frac{uca}{c_1{}^2a_1}}{\frac{b^2}{b_1{}^2}}\ \text{ and }\ \frac{XC}{a_1}=\frac{\frac{a}{a_1}\left(\frac{b^2}{b_1{}^2}-\frac{uc}{c_1{}^2}\right)}{\frac{b^2}{b_1{}^2}}.$$

Let YY_3 be conjugate to AB.

$$\frac{AY_3}{c_1}=\frac{AF}{c_1}\cdot\frac{AY}{AC}=\frac{\left(\frac{b^2}{b_1{}^2}+\frac{c^2}{c_1{}^2}-\frac{a^2}{a_1{}^2}\right)\left(\frac{b^2c}{b_1{}^2c_1}-\frac{a^2u}{a_1{}^2c_1}\right)}{\frac{2b^2c^2}{b_1{}^2c_1{}^2}}.$$

$$\therefore\ \frac{Y_3Z}{c_1}=\frac{u}{c_1}-\frac{AY_3}{c_1}=\frac{\frac{u}{c_1}\left(\frac{a^2b^2}{a_1{}^2b_1{}^2}+\frac{c^2a^2}{c_1{}^2a_1{}^2}+\frac{2b^2c^2}{b_1{}^2c_1{}^2}-\frac{a^4}{a_1{}^4}\right)-\frac{b^2c}{b_1{}^2c_1}\left(\frac{b^2}{b_1{}^2}+\frac{c^2}{c_1{}^2}-\frac{a^2}{a_1{}^2}\right)}{\frac{2b^2c^2}{b_1{}^2c_1{}^2}}.$$

$$\frac{Y_3Y}{AY}=\frac{FC}{AC}. \quad \therefore \quad \frac{YY_3}{f}=\frac{\Delta\left(\frac{b^2c}{b_1^2c_1}-\frac{a^2u}{a_1^2c_1}\right)}{\frac{2b^2c^2}{b_1^2c_1^2}}.$$

If a, β, γ be the radii of the circumconic parallel to ZY, XZ, XY,

$$\frac{ZY^2}{a^2}=\frac{ZY_3^2}{c_1^2}+\frac{Y_3Y^2}{f^2}$$

$$=\left\{\frac{b^4c^2}{b_1^4c_1^2}-\frac{ub^2c}{b_1^2c_1^2}\Sigma\frac{a^2}{a_1^2}+\frac{u^2}{c_1^2}\Sigma\frac{a^2b^2}{a_1^2b_1^2}\right\}\div\frac{b^2c^2}{b_1^2c_1^2}.$$

Let YY_1 be conjugate to BC.

$$\frac{CY_1}{CD}=\frac{CY}{CA}. \quad \therefore \quad \frac{CY_1}{a_1}=\frac{\frac{ua}{c_1a_1}\left(\frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}-\frac{c^2}{c_1^2}\right)}{\frac{2b^2c}{b_1^2c_1}}.$$

$$\frac{XY_1}{c_1}=\frac{XC-Y_1C}{c_1}=\frac{\frac{a}{a_1}\left\{\frac{2b^2c}{b_1^2c_1}-\frac{u}{c_1}\Sigma\frac{a^2}{a_1^2}\right\}}{\frac{2b^2c}{b_1^2c_1}}.$$

$$\frac{CY}{CA}=\frac{YY_1}{AD}. \quad \therefore \quad \frac{YY_1}{d}=\frac{\frac{ua}{c_1a_1}\Delta}{\frac{2b^2c}{b_1^2c_1}}.$$

$$\frac{XY^2}{\gamma^2}=\frac{XY_1^2}{a_1^2}+\frac{YY_1^2}{d^2}$$

$$=\frac{a^2}{a_1^2}\left\{\frac{b^4c^2}{b_1^4c_1^2}-\frac{ub^2c}{b_1^2c_1^2}\Sigma\frac{a^2}{a_1^2}+\frac{u^2}{c}\Sigma\frac{a^2b^2}{a_1^2b_1^2}\right\}\div\frac{b^4c^2}{b_1^4c_1^2}$$

$$=\frac{\frac{a^2}{a_1^2}}{\frac{b^2}{b_1^2}}\cdot\frac{ZY^2}{a^2}.$$

From this and the remaining corresponding equation

$$\frac{\frac{XY}{\gamma}}{\frac{a}{a_1}}=\frac{\frac{ZY}{a}}{\frac{b}{b_1}}=\frac{\frac{ZX}{\beta}}{\frac{c}{c_1}}.$$

In an extended sense then the triangles ZXY, ABC are similar. If O be the point Ω' and Z' be any point in AB, it can be proved in a similar fashion that the corresponding triangle Y'Z'X' is, in the same sense, similar to ABC.

Further, Ω and Ω' are the first and second Brocard points of the triangles XYZ, X'Y'Z' respectively.

$$\frac{ZN}{Z\Omega}=\frac{C\Omega_c}{\Omega\Omega_c}. \quad \therefore \quad \frac{Z\Omega \,.\, ZN}{n^2}=\frac{\Omega\Omega_c}{C\Omega_c}\cdot\frac{ZN^2}{n^2}.$$

Let CF, NN_3 be conjugate to AB.

$$\frac{ZN^2}{n^2}=\frac{ZN_3^{\,2}}{c_1^{\,2}}+\frac{NN_3^{\,2}}{f^2}.$$

But $NN_3=CF$ and $FN_3=CN$.

$$\therefore \quad \frac{ZN^2}{n^2}=\frac{CF^2}{f^2}+\frac{(AF+CN-AZ)^2}{c_1^{\,2}}.$$

Also $\dfrac{CN}{Z\Omega_c}=\dfrac{C\Omega}{\Omega\Omega_c}$. $\quad\therefore\quad CN=(A\Omega_c-u)\dfrac{C\Omega}{\Omega\Omega_c}$.

$$\therefore \quad \frac{ZN^2}{n^2}=\frac{\Delta^2}{\frac{4c^2}{c_1^{\,2}}}+\left\{\frac{\frac{b^2}{b_1^{\,2}}+\frac{c^2}{c_1^{\,2}}-\frac{a^2}{a_1^{\,2}}}{\frac{2c}{c_1}}+\frac{A\Omega_c\,.\,C\Omega}{c_1\Omega\Omega_c}-\frac{uC\Omega_c}{c_1\Omega\Omega_c}\right\}^2$$

$$=\frac{\Delta^2}{\frac{4c^2}{c_1^{\,2}}}+\left\{\frac{\Sigma\frac{a^2}{a_1^{\,2}}}{\frac{2c}{c_1}}-\frac{\frac{u}{c}\Sigma\frac{a^2b^2}{a_1^{\,2}b_1^{\,2}}}{\frac{b^2c^2}{b_1^{\,2}c_1^{\,2}}}\right\}^2$$

$$\therefore \quad \frac{Z\Omega\,.\,ZN}{n^2}=\left\{\frac{b^4c^2}{b_1^{\,4}c_1^{\,2}}-\frac{ub^2c}{b_1^{\,2}c_1^{\,2}}\Sigma\frac{a^2}{a_1^{\,2}}+\frac{u^2}{c_1^{\,2}}\Sigma\frac{a^2b^2}{a_1^{\,2}b_1^{\,2}}\right\}\div\frac{b^2c^2}{b_1^{\,2}c_1^{\,2}}$$

$$=\frac{ZY^2}{a^2}.$$

ZY therefore touches the conic $X\Omega Y$ at Y. Similarly XY touches the conic $Z\Omega X$ at X so that Ω is the first Brocard point of ZXY. In the same way it may be proved that Ω' is the second Brocard point of X'Y'Z'.

Suppose XY′ to be drawn parallel to AB to cut AC in Y′. Then

$$\frac{AY'}{b}=\frac{BX}{a}=\frac{\frac{uc}{c_1^2}}{\frac{b^2}{b_1^2}}.$$

$$\therefore\ \frac{AY'}{b_1}=\frac{\frac{uc}{c_1^2}}{\frac{b}{b_1}}.$$

This point Y′ will determine a triangle X′Y′Z′.

$$\frac{AY'\,.\,b}{b_1^2}=\frac{uc}{c_1^2}=\frac{AZ\,.\,c}{c_1^2}.$$

Therefore Y′Z is antiparallel to BC. Similarly it may be proved that Z′X is antiparallel to CA and X′Y to AB. X, Y, Z, X′, Y′, Z′ then lie on a Tucker's conic.

Section XIV.

Inversion. Feuerbach's Theorem.

§60. If P and Q be two points on a radius OA of a central conic such that $OP\,.\,OQ=OA^2$, Q is the inverse of P with respect to the conic. If P describes a curve, Q also describes a curve, and the inverses of two curves which touch are evidently in contact at the inverse of the point of contact of the original curves.

The inverse of a straight line is a conic homothetic with the conic of inversion.

Let OP be conjugate to the given line of which A is any point. If Q and B are the inverses of P and A and p, a the radii of the conic of inversion along OP, OA,

$$\frac{OP\,.\,OQ}{p^2}=\frac{OA\,.\,OB}{a^2}.$$

A homothetic conic therefore passes through P, Q, B, A, and since AP, PQ are conjugate, AQ is a diameter. AB, BQ are then conjugate, and a homothetic conic on OQ as diameter passes through B. This conic is then the locus of B; it passes through O. Conversely, the inverse of a homothetic conic passing through the centre of inversion is a straight line.

The inverse of a homothetic conic not passing through the centre of inversion is a homothetic conic.

Let C be the centre of the conic to be inverted, and P any point in it, the inverse of P being Q. Let OP cut the conic again in P′ and OT be a tangent.

$\frac{OP \,.\, OP'}{p^2} = \frac{OT^2}{t^2}$ where p and t are the radii of the conic of inversion along OP, OT. Also $OP \,.\, OQ = p^2$.

$\therefore \frac{OQ}{OP'} = \frac{t^2}{OT^2} = \text{constant}.$

Take C_1 on OC so that $\frac{OC_1}{OC} = \frac{t^2}{OT^2}$. Then $\frac{OQ}{OP'} = \frac{OC_1}{OC}$.

CP′ is then parallel to C_1Q and $\frac{C_1Q}{CP'} = \frac{OC_1}{OC} = \text{constant}$. Q therefore describes a homothetic conic having C_1 as centre, O being a centre of similarity of the conic and its inverse.

§61. We now give a modification of Casey's proof of Feuerbach's theorem.

$$UD = UC - DC = \frac{a}{2} - \frac{\frac{a^2}{a_1^{\,2}} + \frac{b^2}{b_1^{\,2}} - \frac{c^2}{c_1^{\,2}}}{\frac{2a}{a_1^{\,2}}} = \frac{\frac{c^2}{c_1^{\,2}} - \frac{b^2}{b_1^{\,2}}}{\frac{2a}{a_1^{\,2}}} \quad \text{(Fig. 26).}$$

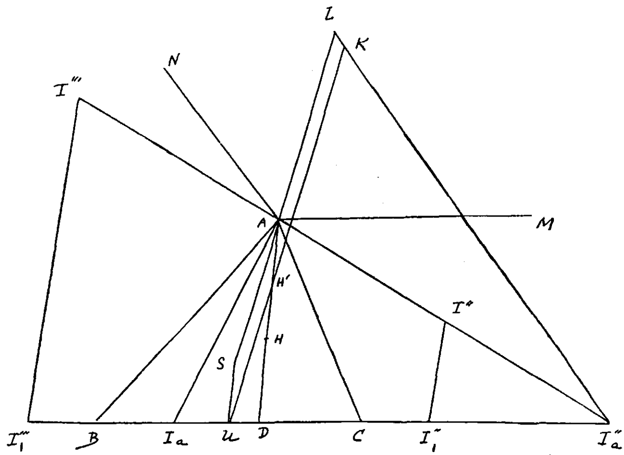


Fig. 26.

I_1''', I_1'' being the feet of the conjugates from I''', to I'' to a,
$\frac{I_1'''C}{a_1}=s$ and $\frac{CI_1''}{a_1}=s-\frac{a}{a_1}$.

$$\therefore\ \frac{I_1'''I_1''}{a_1}=2s-\frac{a}{a_1}=\frac{b}{b_1}+\frac{c}{c_1}.$$

Also, since $I_1'''B=CI_1''$, U is the mid point of $I_1'''I_1''$.

$$\frac{I_1''I_a''}{r_2}=\frac{I_1'''I_1''}{r_3-r_2}.$$

$$\therefore\ \frac{I_1''I_a''}{a_1}=\frac{\frac{b}{b_1}+\frac{c}{c_1}}{\frac{r_3}{r_2}-1}=\frac{\frac{b}{b_1}+\frac{c}{c_1}}{\frac{s-\frac{b}{b_1}}{s-\frac{c}{c_1}}-1}=\frac{\left(s-\frac{c}{c_1}\right)\left(\frac{b}{b_1}+\frac{c}{c_1}\right)}{\frac{c}{c_1}-\frac{b}{b_1}}.$$

$$\frac{UI_1''}{a_1}=\tfrac{1}{2}\left(\frac{b}{b_1}+\frac{c}{c_1}\right).\quad\therefore\ \frac{UI_a''}{a_1}=\frac{UI_1''+I_1''I_a''}{a_1}=\frac{\frac{a}{a_1}\left(\frac{b}{b_1}+\frac{c}{c_1}\right)}{2\left(\frac{c}{c_1}-\frac{b}{b_1}\right)}.$$

$$\therefore\ UD\,.\,UI_a''=\frac{a_1^2}{4}\left(\frac{b}{b_1}+\frac{c}{c_1}\right)^2=UI_1''^2.$$

I_a'' is therefore the inverse of D with respect to the homothetic conic whose centre is U, and whose radius along BC is UI_1''.

Let the fourth common tangent to the ex-conics with centres I'', I''', that is the second common tangent through I_a'', be $I_a''L$, L being the point of contact with the conic with centre I'''. $I_a''I_1''$, $I_a''L$, the two tangents, are harmonic conjugates to the diameter $I_a''A$ and the parallel through I_a'' to the conjugate AII_a to AI''. AL, LI_a'' are conjugate as also AD, DI_a''. Draw AM, AN parallel to BC, LI_a''. Then since AI_a, AI_a''; AL, AN; AD, AM are parallel to pairs of conjugate diameters $A\{I_aI_a'', DM, LN\}$ is a pencil in involution.

But $A\{I_aI_a'', MN\}$ is harmonic; therefore $A\{I_aI_a'', DL\}$ is also harmonic. Also $A\{I_aI_a'', DS\}$ where S is the circumcentre is harmonic (§17). Therefore L, A, S are collinear. If H is the orthocentre and H′ the mid point of AH, H′ is a point on the nine point conic and UH′ is parallel to SA and therefore conjugate to LI_a''. Let UH' cut LI_a'' in K. Since $H'K$ is conjugate to KI_a'' and $H'D$ to DI_a'', a homothetic conic on $H'I_a''$ as diameter passes through D and K.

$$\therefore \frac{UH'.UK}{SA^2}=\frac{UD.UI_a''}{a_1^2}.$$

$$\therefore UH'.UK=\frac{SA^2}{a_1^2}.UI_1''^2=\lambda^2 SA^2$$ where λ is the ratio of similarity of the conic of inversion to the circumconic.

Therefore $UH'.UK$ = square of the radius of the conic of inversion along UH' and K is the inverse of H'.

The exconics with centres I'', I''' invert into themselves and, since the line inverted touches these conics, its inverse the nine point conic through UDH' touches the conics with centres I'', I'''.

Let the in- and ex-conics with centres I and I' touch BC in I_1, I_1'. U is the mid point of I_1I_1' for $\frac{CI_1}{a_1}=s-\frac{c}{c_1}$ and $\frac{BI_1'}{a_1}=s-\frac{c}{c_1}$.

$$\therefore UI_1=\frac{a}{2}-\frac{a_1}{2}\left(\frac{a}{a_1}+\frac{b}{b_1}-\frac{c}{c_1}\right)=\frac{a_1}{2}\left(\frac{c}{c_1}-\frac{b}{b_1}\right).$$

$$BI_a''=BU+UI_a''=\frac{a}{2}+\frac{a\left(\frac{b}{b_1}+\frac{c}{c_1}\right)}{2\left(\frac{c}{c_1}-\frac{b}{b_1}\right)}$$

$$=\frac{\frac{ac}{c_1}}{\frac{c}{c_1}-\frac{b}{b_1}}.$$

$$(BCI_aI_a'')=-1. \quad \therefore \frac{2}{BC}=\frac{1}{BI_a}+\frac{1}{BI_a''}=\frac{1}{BI_a}+\frac{\frac{c}{c_1}-\frac{b}{b_1}}{\frac{ac}{c_1}}.$$

$$\therefore BI_a=\frac{\frac{ac}{c_1}}{\frac{b}{b_1}+\frac{c}{c_1}}. \quad \therefore UI_a=BI_a-\frac{a}{2}=\frac{a\left(\frac{c}{c_1}-\frac{b}{b_1}\right)}{2\left(\frac{c}{c_1}+\frac{b}{b_1}\right)}.$$

$$\therefore UI_a.UD=\frac{a_1^2}{4}\left(\frac{c}{c_1}-\frac{b}{b_1}\right)^2=UI_1^2.$$

The fourth common tangent to the conics with centres I, I′ passes through I_a. This and BC form with AI_a and the conjugate to AI_a through I_a a harmonic pencil. This common tangent is therefore parallel to the fourth common tangent to the ex-conics with centres I″, I‴. As before then SA and UH′ are conjugate to the fourth common tangent and the inverse of the nine point conic with respect to the homothetic conic on I_1I_1' as diameter touches the conics with centres I and I′. The tangent to the nine point conic at D is parallel to the fourth common tangent. Let this common tangent touch the inconic at Z. Then the points D, Z are corresponding points on the two homothetic conics, the nine point conic and the inconic. DZ must therefore pass through the point of contact. The other points of contact may be similarly found.

§62. Let A′, B′ be the inverses of A, B with respect to a conic of centre O. Let x, y, l, m be the radii parallel to OA, OB, BA, B′A′, and suppose BA, B′A′ to meet in P.

$OA \cdot OA' = x^2$ and $OB \cdot OB' = y^2$. A homothetic conic passes through A, B, A′, B′ and $\dfrac{PA \cdot PB}{l^2} = \dfrac{PA' \cdot PB'}{m^2}$.

Since OAA′ is a transversal of the triangle PBB′

$$PA \cdot BO \cdot B'A' = -AB \cdot OB' \cdot A'P,$$

and as OBB′ is a transversal of PAA′.

$$PB \cdot AO \cdot A'B' = -BA \cdot OA' \cdot B'P.$$

Multiplying the last two equations together we obtain

$$\frac{OB \cdot OA}{OB' \cdot OA'} \cdot \frac{PA \cdot PB}{PA' \cdot PB'} = \frac{AB^2}{A'B'^2}.$$

$$\therefore \ \frac{AB^2}{l^2} \Big/ \frac{A'B'^2}{m^2} = \frac{OB \cdot OA}{OB' \cdot OA'} = \frac{OB^2 \cdot OA^2}{x^2 y^2} \text{ or } \frac{AB}{l} \Big/ \frac{A'B'}{m} = \pm \frac{OB \cdot OA}{xy}.$$

If A, B, C, D be four collinear points

$$AB \cdot CD + BC \cdot AD + CA \cdot BD = 0.$$

Invert the line with respect to a point O outside it and we get, by using the preceding relation, the analogue of Ptolemy's theorem.

$$\frac{A'B' \cdot C'D'}{pq} \pm \frac{B'C' \cdot A'D'}{rs} \pm \frac{C'A' \cdot B'D'}{tu} = 0$$

where p, q, r, s, t, u are the radii parallel to the chords. If the conic is a hyperbola a simple discussion of the cases where the points are not all on the same branch shows that the result is real; the denominators are all real or all imaginary together.

Section XV.

Coaxal Homothetic Conics. Feuerbach's Theorem.

§63. Let O_1, O_2 be the centres of two homothetic conics, λ_1, λ_2 their ratios of similarity to a given homothetic conic, r_1 and r_2 their radii along O_1O_2. Divide O_1O_2 at L so that $O_1L^2 - O_2L^2 = r_1^2 - r_2^2$ and draw LM conjugate to O_1O_2. LM is their radical axis. If the conics intersect in A, B, then LM is the common chord. Let AB cut O_1O_2 in L′. AB is conjugate to O_1O_2 and if p, x, y, z are the radii of the initial conic parallel to O_1O_2, O_1A, O_2A, AB

$$\frac{O_1A^2}{x^2} = \frac{O_1L'^2}{p^2} + \frac{L'A^2}{z^2} \text{ and } \frac{O_2L'^2}{p^2} + \frac{L'A^2}{z^2} = \frac{O_2A^2}{y^2}.$$

$$\therefore \lambda_1^2 - \frac{O_1L'^2}{p^2} = \lambda_2^2 - \frac{O_2L'^2}{p^2}.$$

$\therefore O_1L'^2 - O_2L'^2 = r_1^2 - r_2^2$ and L′ coincides with L.

Let P be any point in the plane and PM, PN be conjugate to LM and O_1O_2. If PT_1, PT_2 be tangents to the two conics and t_1, t_2 the radii of the initial conic parallel to the tangents

$$\frac{PT_2^2}{t_2^2} - \frac{PT_1^2}{t_1^2} = \left(\frac{O_1P^2}{p_1^2} - \lambda_1^2\right) - \left(\frac{O_2P^2}{p_2^2} - \lambda_2^2\right)$$

where p_1, p_2 are the radii of the initial conic parallel to OP_1, O_2P.

$$\therefore \frac{PT_2^2}{t_2^2} - \frac{PT_1^2}{t_1^2} = \left(\frac{O_1N^2}{p^2} + \frac{NP^2}{z^2} - \lambda_1^2\right) - \left(\frac{O_2N^2}{p^2} + \frac{NP^2}{z^2} - \lambda_2^2\right)$$

$$= \frac{1}{p^2}(O_1N^2 - O_2N^2 - r_1^2 + r_2^2)$$

$$= \frac{1}{p^2}(2O_1O_2 \,.\, M_1N - O_1L^2 + O_2L^2)$$

where M_1 is the mid point of O_1O_2.

$$\therefore \frac{PT_2^2}{t_2^2} - \frac{PT_1^2}{t_1^2} = \frac{2}{p^2}O_1O_2(M_1N - M_1L)$$

$$= \frac{2O_1O_2 \,.\, LN}{p^2} = \frac{2O_1O_2 \,.\, MP}{p^2}.$$

If P is on LM, $\dfrac{PT_1^2}{t_1^2} = \dfrac{PT_2^2}{t_2^2}$.

If P is on the conic with centre O_1, $\dfrac{PT_1^2}{t_1^2} = \dfrac{2O_1O_2 \,.\, MP}{p^2}$.

For a system of homothetic conics with the same radical axis

$$O_1L^2 - r_1^2 = O_2L^2 - r_2^2 = O_3L^2 - r_3^2 = \ldots = \pm k^2.$$

Therefore if the radius along O_1O_2 be given of any such conic the centre can be found. When r_3, say, is zero, we obtain the limiting points L_1, L_2, which when real are point ellipses for a system of ellipses or the intersections of two pairs of lines parallel to the asymptotes for a system of hyperbolas. When two conics touch, their limiting points coincide at the point of contact.

Let P be a point on the conic with centre O_3; then

$$\frac{PT_1^2}{t_1^2}=\frac{2O_3O_1 . MP}{p^2} \text{ and } \frac{PT_2^2}{t_2^2}=\frac{2O_3O_2 . MP}{p^2}.$$

$$\therefore \frac{PT_1^2}{t_1^2}\bigg/\frac{PT_2^2}{t_2^2}=\frac{O_1O_3}{O_2O_3} \text{ and in particular } \frac{PL_1^2}{l_1^2}\bigg/\frac{PT_2^2}{t_2^2}=\frac{L_1O_3}{O_2O_3},$$

where l_1 is the radius of the initial conic parallel to PL_1.

§64. A modification of M'Cay's proof of Feuerbach's theorem can now be easily given.

Let SI cut the circumconic in X and Y. The Wallace lines of X and Y are conjugate and meet, in R say, on the nine point conic (§7). Let them cut BC in X_1, Y_1. The homothetic conic on X_1Y_1 as diameter passes through R and U is its centre for RX_1, RY_1 are conjugate, and XX_1, YY_1 are parallel to SU and S is the mid point of XY.

$$\therefore \frac{XS}{SI}=\frac{X_1U}{UI_1}=\frac{a_1}{x}\cdot\frac{x}{a_1}\cdot\frac{X_1U}{UI_1}\cdot\frac{UR}{UR} \text{ where } x \text{ is the radius of the}$$

circumconic parallel to UR.

$$\text{But } \frac{x}{UR}=\frac{a_1}{X_1U}. \quad \therefore \frac{XS}{SI}=\frac{\dfrac{UR}{x}}{\dfrac{UI_1}{a_1}}.$$

From the two corresponding equations we obtain

$$\frac{UR}{x}\bigg/\frac{UI_1}{a_1}=\frac{VR}{y}\bigg/\frac{VI_2}{b_1}=\frac{WR}{z}\bigg/\frac{WI_3}{c_1},$$

y and z being the radii of the circumconic parallel to VR, WR. Therefore R is a limiting point of the nine point conic and the inconic, and since it lies on the nine point conic, the two conics touch at R.

Section XVI.

Triangles Triply and Quadruply in Perspective.

§65. Since a first Brocard triangle $A_1B_1C_1$ of ABC is triply in perspective with ABC the theory of triangles triply in perspective follows very naturally. Many of the following results were worked out before I found Dr Third's interesting paper in the *Proceedings of the Edinburgh Mathematical Society* 1900-1. The present section only claims to be a reworking of the greater part of the results of that paper by the preceding methods and results. The intimate connection with the extended geometry of the triangle will perhaps be sufficient excuse for the following pages.

§66. The Steiner circumellipse of $A_1B_1C_1$ is concentric and homothetic with that of ABC and the symmedian centre K lies on it (§56). To each point K on this ellipse belongs one Brocard triangle so that there is a singly infinite set in this conic. K may also describe an infinity of such conics so that there is a doubly infinite set of Brocard triangles triply in perspective with ABC. Considering K as the Steiner point of $A_1B_1C_1$ with respect to ABC we see that the rôles of ABC and $A_1B_1C_1$ may be interchanged and that there is a doubly infinite set of triangles triply in perspective with $A_1B_1C_1$. The two systems of triangles are the same, for a point taken on a conic of the system as a vertex fixes the triangle. All the triangles of the system are therefore triply in perspective with each other. The tangents at the vertices of one of the triangles to the circumscribed Steiner ellipse are parallel to the opposite sides. Each set in an ellipse of the system is circumscribed to another ellipse of the system, the lines joining the vertices to the points of contact of the opposite sides passing through the common centre of gravity G. All members of the system which have their sides parallel are quadruply in perspective, the fourth centre and axis of perspective being G and the line at infinity. Those with their sides parallel to AB, BC, CA have their vertices on three lines AG, BG, CG.

When K describes a straight line we have seen that A_1, B_1, C_1 describe straight lines so that we have an infinite set triply in perspective with each other and with the vertices on three lines.

The centres of prespective of ABC and $A_1B_1C_1$ are K′, Ω, Ω', (§§44, 47), and $K'\Omega\Omega'$ is one of the set in triple perspective with

ABC, the centres of perspective being A_1, B_1, C_1. If K is on the Steiner circumellipse of ABC, K′ and therefore Ω, Ω' are at infinity because for this ellipse as circumconic isogonal and isotomic conjugates coincide (§16). For a set then of triangles inscribed in the same Steiner ellipse the mutual centres of perspective are on the line at infinity. Reciprocating with respect to this ellipse we have a system circumscribed about and therefore inscribed in an ellipse of the system and it is seen that the mutual axes of perspective pass through G.

We can consider $A_1B_1C_1$ as a Brocard triangle of ABC in three ways, the corresponding symmedian centres being K and the isotomic conjugates M, L of Ω, Ω'. These are on the Steiner ellipse of A_1, B_1, C_1. KML can be considered in three ways KML, LKM, MLK as a Brocard triangle of ABC, the symmedian centres being A_1, B_1, C_1. Therefore KML is one of the system of triangles and the three centres of perspective with ABC are the isotomic conjugates of A_1, B_1, C_1.

The conic $A_1B_1C_1\Omega\Omega'$ cuts the Steiner ellipse of $A_1B_1C_1$ in K. By considering $A_1B_1C_1$ in the other two ways we get the corresponding points M, L. The three conics therefore through the vertices of a first triangle of the system and two of the vertices of a second triangle intersect the Steiner ellipse of the first in three points forming a third triangle of the system. Reciprocating with respect to a Steiner ellipse we see that the three conics inscribed in one triangle of the system and touching two sides of a second have as fourth common tangents with the inscribed Steiner ellipse of the first the sides of a third triangle of the system.

K being the symmedian centre of ABC let BC_1, CB_1 meet in A_1' the isogonal conjugate of A_1 (§48) and meet AA_1 in B_3, C_3. The vertices of $A_1B_1C_1$ lie on the sides of $A_1'B_3C_3$ so that these two triangles are triply in perspective. $A'B_3C_3$ is also triply in perspective with ABC for AC_3, BA_1', CB_3 meet in B_3, AB_3, CA_1', BC_3 meet in C_3 and BB_3, CC_3, AA_1' meet in A_1'.

$K'\Omega\Omega'$ as shown belongs to the system. On reciprocation with respect to a Steiner ellipse it is seen that the axes of perspective of two triangles of the system form a triangle of the system.

Let BC, B_1C_1 meet in D, CA, C_1A_1 meet in E and AB, A_1B_1 meet in F. AC_1 passes through Ω.

$$\therefore\ \frac{AC_1^2}{\omega_1^2}=\frac{c^2}{4c_1^2}+\frac{KK_3^2}{f^2}=\frac{\frac{c^2}{c_1^2}\Sigma\frac{a^2b^2}{a_1^2b_1^2}}{\left(\Sigma\frac{a^2}{a_1^2}\right)^2}\quad(\S 23).$$

$$\therefore\ \frac{AC_1}{A\Omega_a}=\frac{\frac{c^2}{c_1^2}+\frac{a^2}{a_1^2}}{\Sigma\frac{a^2}{a_1^2}}\quad(\S 33).$$

If R is the Steiner point of ABC (Fig. 24)

$$\frac{\Omega_a R_a}{DR_a}=\frac{A\Omega_a}{AC_1}=\frac{\Sigma\frac{a^2}{a_1^2}}{\frac{c^2}{c_1^2}+\frac{a^2}{a_1^2}}.$$

Also $\dfrac{BR_a}{R_aC}=\dfrac{\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}}{\frac{a^2}{a_1^2}-\frac{b^2}{b_1^2}}$ (§56).

$$\therefore\ \frac{R_aC}{a}=\frac{\frac{a^2}{a_1^2}-\frac{b^2}{b_1^2}}{\frac{c^2}{c_1^2}-\frac{b^2}{b_1^2}}\ \text{and}\ \frac{\Omega_aC}{a}=\frac{\frac{a^2}{a_1^2}}{\frac{c^2}{c_1^2}+\frac{a^2}{a_1^2}}\quad(\S 33).$$

$$\therefore\ \frac{\Omega_aR_a}{a}=\frac{\Omega_aC-R_aC}{a}=\frac{\frac{b^2c^2}{b_1^2c_1^2}-\frac{a^4}{a_1^4}}{\left(\frac{c^2}{c_1^2}+\frac{a^2}{a_1^2}\right)\left(\frac{c^2}{c_1^2}-\frac{b^2}{b_1^2}\right)}.$$

$$\therefore\ \frac{DR_a}{a}=\frac{\frac{b^2c^2}{b_1^2c_1^2}-\frac{a^4}{a_1^4}}{\left(\frac{c^2}{c_1^2}-\frac{b^2}{b_1^2}\right)\Sigma\frac{a^2}{a_1^2}}.$$

$$\therefore\ \frac{DC}{a}=\frac{DR_a+R_aC}{a}=\frac{\frac{c^2a^2}{c_1^2a_1^2}-\frac{b^4}{b_1^4}}{\left(\frac{c^2}{c_1^2}-\frac{b^2}{b_1^2}\right)\Sigma\frac{a^2}{a_1^2}}.$$

The remaining results can be transformed easily in the same way. We shall only consider the triangle $A_1' B_3C_3$ of the preceding paragraph. From this it is seen that if two triangles be triply in perspective the joins of their vertices in pairs when not concurrent determine another triangle of the system and on reciprocating, the intersections of the sides in pairs when not concurrent determine a triangle of the system.

As p can assume ∞^2 positions we have ∞^4 triangles in triple perspective with ABC and ∞^2 triangles quadruply in perspective with ABC.

Of course the results derived from the projective transformation are only those obtained from the extended geometry mentioned in the introduction in which the two points at infinity are replaced by two points (real or imaginary) not at infinity.

SECTION XVII.

On Duality in the Geometry of the Triangle.

§68. Dual theorems can be derived in an infinite variety of ways by different reciprocations. We shall discuss only the reciprocation of the geometry of a triangle A'B'C' with respect to a circumconic of centre S confining our attention mainly to a circle on account of some not uninteresting results that hence arise. A triangle ABC is derived circumscribed about the conic of centre S. Homothetic conics become conics having a pair of common tangents and, when the circumconic is a circle, all circles as is well known become conics with S as one focus and the centres reciprocate into the corresponding directrices. Pairs of conjugate lines become pairs of points on two conjugate radii of the inscribed conic of centre S. Thus for an inscribed circle of centre S the perpendiculars SX, SY, SZ to AS, BS, CS cut the opposite sides in three points on a line XYZ which is the reciprocal of the orthocentre of A'B'C'.

S{AX, BY, CZ} is an involution, in the general case of a conic when for perpendiculars we substitute conjugates; of this involution the double lines are the asymptotes of the conic inscribed in ABC with S as centre.

If a line in the original figure be divided in a given ratio, by including the point at infinity on the line, we obtain, on reciprocating, a pencil with a given anharmonic ratio, one of the rays passing through S. Thus the centroid of any triangle reciprocates into the trilinear polar of S with respect to the reciprocal triangle and for ABC and a circle it is the line joining the intersections X′, Y′, Z′ with the opposite sides of the external bisectors of its angles. For a system of conics inscribed in ABC and touching a given line, the envelope of the trilinear polars of S is an inscribed conic (§22).

The centre of the nine point circle of A′B′C′ reciprocates into the directrix of a corresponding nine line ellipse. This line and the lines XYZ, X′Y′Z′ are parallel. If a perpendicular through S to these lines cut them respectively in S′, H and G, then

$$SH = HS' \text{ and } 2SH = HG.$$

§69. Let S be the centre of a circle inscribed in ABC and touching the sides in A′, B′, C′. Let AS, BS, CS cut the opposite sides in D, E, F. Consider the involution on BC {BC, A′D...}. This is the reciprocal of the involution of the isogonal conjugates through A′ in the triangle A′B′C′. The tangents from points on SA cut BC in pairs of corresponding points so that the double points L, M are the intersections with BC of the tangents at the ends of the diameter on SA. Let L be internal and M external on

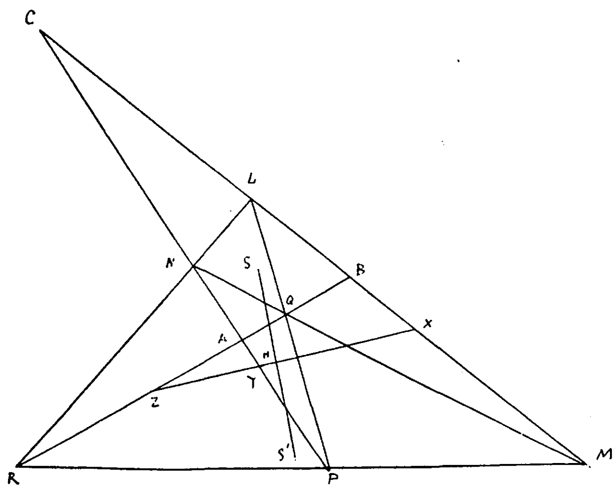


Fig 27.

is $\sqrt{2Rr}$; the semi-latus rectum is $2r$. The auxiliary circle of this conic is then the circumcircle of ABC. Consider the infinite set (α) of triangles inscribed in the circle with centre O and radius R and circumscribed about the circle with centre S and radius r. These have O, S, S′, H and the lines XYZ, X′Y′Z′ in common. Thus the perpendiculars from S to the lines joining S to the vertices cut the opposite sides on the same line, and the external bisectors of their angles cut the opposite sides on another line. The set (α) have the same nine line ellipse, which is therefore the envelope of the external bisectors of their angles. It is also the envelope of the lines AX, etc. Since the distance between S and the centres of the nine point circles of the set (α) is $\frac{1}{2}R - r$, the nine point centres lie on a circle with centre S, and the envelope of the nine point circles is two circles with centre S. The loci of the centroids and orthocentres of the set (α) are two circles with centres on OS and radii $\frac{1}{3}(R-2r)$, $(R-2r)$. Reciprocating this set (α) with respect to S we see that the infinite set of triangles inscribed in the same circle of centre S and having the same orthocentre P have their sides enveloping a conic with S as focus and directrix perpendicular to SP.

Let o any tangent to the inscribed circle of ABC cut SX, SY, SZ in O_x, O_y, O_z; then AO_x, BO_y, CO_z meet in a (Wallace) point W. The locus of W for all tangents is a cubic of class four (the reciprocal of the three cusped hypocycloid) with three real points of inflexion on the directrix of the nine line ellipse through S′. It passes through A, B, C, X, Y, Z and has triple contact with this ellipse. The Wallace points of two parallel tangents lie on a tangent to the ellipse and subtend a right angle at S.

Section XVIII.

The Continuity through Stereographic Projection.
The Algebraic Continuity. Duality. Further Extension.

§70. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ be the equation of any central conicoid referred to any three conjugate axes; we shall assume c to be real. The equation of the cone with (o, o, c) as vertex passing through the intersection of the conicoid with the plane $lx+my+nz=1$ is

$$(1-cn)^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\left(lx+my+\frac{z}{c}-1\right)^2=(lx+my+nz-nc)^2.$$

The intersection with the plane $z=o$ is given by

$$(1-cn)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-2lx-2my+1+nc=0.$$

If $n=\frac{1}{c}$ the plane passes through $(o,\ o,\ c)$ and the "stereographic" projection is a line. For all planes not passing through $(o,\ o,\ c)$ the projections are homothetic conics which are ellipses or hyperbolae according to the reality of a and b.

Let ABC be a triangle inscribed in the circle $x^2+y^2=r^2$. If P, P′ be the spherical poles of the circle $z=o$ on the sphere $x^2+y^2+z^2=r^2$, the geometry of the triangle, as is well known, is transformed into a geometry on the sphere by stereographic projection from P. This sphere can by a homographic transformation, in which points and planes correspond to points and planes, be transformed into a conicoid. If Q on the conicoid corresponds to P on the sphere, then by "stereographic" projection from Q on the diametral plane parallel to the tangent plane at Q a geometry is obtained with homothetic conics. It is clear at any rate that the conics obtained touching the sides are the homothetic in- and ex-conics already discussed.

If the sides of ABC are arcs of circles, a triangle is obtained whose sides are arcs of homothetic conics and Dr Hart's extension of Feuerbach's theorem can be at once extended to such a triangle.

§71. When the circumconic is a circle the barycentric coordinates of the incentre are a, b, c. The equation of the circumcircle is $a^2yz+b^2zx+c^2xy=o$ and the coordinates of the symmedian centre are a^2, b^2, c^2.

M. Ripert generalises the geometry of the triangle by replacing the coordinates of the incentre by three arbitrary quantities α, β, γ. It seems to me preferable to start with the symmedian centre and replace a^2, b^2, c^2 by α, β, γ, because in the first place the in- and ex-conics may be imaginary, a fact which he does not indicate. K, if defined as the centre of perspective of ABC and its polar triangle, and the vertices P, Q, R at once derived from it are the only points with general projective properties. The equation $\alpha yz+\beta zx+\gamma xy=0$ of the circumconic at once gives K and *vice versâ* and the well known points are very easily derived with the aid of the ruler and compasses alone from K. AQ is the harmonic conjugate of AK with respect to AB, AC. S is the intersection of

PU, QV, RW. The parallels through A, B, C to PU, QV, RW meet in H. The parallels through K to BC, CA, AB meet PU, QV, RW in A_1, B_1, C_1. BA_1, CB_1, AC_1 meet in Ω, BC_1, CA_1, AB_1 meet in Ω'. Five points S, K, A_1, B_1, C_1 of Brocard's first conic being known the intersection A_2 of AK with the conic is easily found, etc.

§72. In the coordinates of the points and the equations of the lines and circles in the geometry of the circle and triangle we have only to express the trigonometric ratios, wherever they occur, in terms of the sides, and replace a^2, b^2, c^2, by α, β, γ to get the correspondents. A list is given for reference. Many questions on loci will occur by adding an arbitrary relation between α, β, and γ.

$$K(\alpha, \beta, \gamma),\ P(-\alpha, \beta, \gamma),\ Q(\alpha, -\beta, \gamma),\ R(\alpha, \beta, -\gamma).$$

$$S\{\alpha(\beta+\gamma-\alpha),\ \beta(\gamma+\alpha-\beta),\ \gamma(\alpha+\beta-\gamma)\},\ H\left(\frac{1}{\beta+\gamma-\alpha}, \frac{1}{\gamma+\alpha-\beta}, \frac{1}{\alpha+\beta-\gamma}\right),$$

$$SH,\ (\beta-\gamma)(\beta+\gamma-\alpha)x+(\gamma-\alpha)(\gamma+\alpha-\beta)y+(\alpha-\beta)(\alpha+\beta-\gamma)z=0.$$

$$SK,\ \frac{x}{\alpha}(\beta-\gamma)+\frac{y}{\beta}(\gamma-\alpha)+\frac{z}{\gamma}(\alpha-\beta)=0.$$

Nine point conic,

$$(\beta+\gamma-\alpha)x^2+(\gamma+\alpha-\beta)y^2+(\alpha+\beta-\gamma)z^2=2\gamma xy+2\alpha yz+2\beta zx.$$

Wallace line of $\left(\frac{\alpha}{p-q}, \frac{\beta}{q-r}, \frac{\gamma}{r-p}\right)$,

$$\frac{x(p-q)}{\alpha(p+q-2r)+(\beta-\gamma)(p-q)}+\frac{y(q-r)}{\beta(q+r-2p)+(\gamma-\alpha)(q-r)}+\frac{z(r-p)}{\gamma(r+p-2q)+(\alpha-\beta)(r-p)}=0.$$

$$A_1(\alpha,\gamma,\beta),\ B_1(\gamma, \beta, \alpha),\ C_1(\beta, \alpha, \gamma).$$

$$\Omega\left(\frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\alpha}\right), \Omega'\left(\frac{1}{\gamma}, \frac{1}{\alpha}, \frac{1}{\beta}\right), K'\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right).$$

Conic $CA\Omega$, $\alpha yz+\beta zx=\gamma y(y+z)$.

Conic $AB\Omega'$, $\alpha yz+\gamma xy=\beta z(y+z)$.

Brocard's first conic,

$$(\alpha+\beta+\gamma)(\alpha yz+\beta zx+\gamma xy)=(x+y+z)(\beta\gamma x+\gamma\alpha y+\alpha\beta z)$$

$$\text{or } \frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z^2}{\gamma}=\frac{\alpha yz}{\beta\gamma}+\frac{\beta zx}{\gamma\alpha}+\frac{\gamma xy}{\alpha\beta}.$$

$$A_2(\beta+\gamma-\alpha, \beta, \gamma),\ B_2(\alpha, \gamma+\alpha-\beta, \gamma),\ C_2(\alpha, \beta, \alpha+\beta-\gamma).$$

Tucker's conics are given by varying the value of μ in

$$(\mu+1)(ayz+\beta zx+\gamma xy)(\alpha+\beta+\gamma)=2(x+y+z)$$
$$\times\left\{\beta\gamma x+\gamma\alpha y+\alpha\beta z-\frac{2\alpha\beta\gamma(x+y+z)}{(\mu+1)(\alpha+\beta+\gamma)}\right\}$$

$\mu=0$ gives the cosine conic, and $\mu=1$ the Lemoine conic.

$$\mathrm{I}(\sqrt{\alpha},\ \sqrt{\beta},\ \sqrt{\gamma}),\ \mathrm{I}'(-\sqrt{\alpha},\ \sqrt{\beta},\ \sqrt{\gamma}),\ \text{etc.}$$

Inconic,

$$\sqrt{(\sqrt{\beta}+\sqrt{\gamma}-\sqrt{\alpha})x}+\sqrt{(\sqrt{\alpha}-\sqrt{\beta}+\sqrt{\gamma})y}+\sqrt{(\sqrt{\alpha}+\sqrt{\beta}-\sqrt{\gamma})z}=0.$$

Exconic with centre I′,

$$\sqrt{-(\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma})x}+\sqrt{(\sqrt{\alpha}+\sqrt{\beta}-\sqrt{\gamma})y}+\sqrt{(\sqrt{\alpha}-\sqrt{\beta}+\sqrt{\gamma})z}=0.$$

The equation of a circumconic whose radii parallel to a, b, c are a_1, b_1, c_1 is

$$\frac{a^2}{a_1^2}yz+\frac{b^2}{b_1^2}zx+\frac{c^2}{c_1^2}xy=0.$$

$$\therefore\ \frac{\alpha}{\frac{a^2}{a_1^2}}=\frac{\beta}{\frac{b^2}{b_1^2}}=\frac{\gamma}{\frac{c^2}{c_1^2}}.$$

From these equations all the segments of lines in the earlier sections are easily derived.

§73. Let $yz+zx+xy=0$ be the equation of the Steiner circum-ellipse of a triangle A′B′C′, the mid points of whose sides are A, B, C.

The polar of (x', y', z') with respect to this ellipse is

$$x'(y+z)+y'(z+x)+z'(x+y)=0.$$

If $(\xi,\ \eta,\ \zeta)$ be the complementary of $(x,\ y,\ z)$ (See Casey's Analytical Geometry, Second Edition, page 81) the equation of the transformation of this line with ABC as triangle of reference is $x'\xi+y'\eta+z'\zeta=0$.

For all points in the preceding list we may then take lines whose barycentric tangential coordinates are the coordinates of these points. Similarly for all lines we may take points whose barycentric tangential equations are the equations of these lines. Thus the previous list of coordinates of points and equations of lines and conics may be taken as the tangential coordinates of lines and equations of points and conics. Let, for example $(x,\ y,\ z)$, be a point on the circumconic $\alpha yz + \beta zx + \gamma xy = 0$. Its polar with respect to $xy + yz + zx = 0$ is $x'(y+z) + y'(z+x) + z'(x+y) = 0$. Eliminating x' we have

$$\gamma y'^2(z+x) + y'z'\{\beta(z+x) + \gamma(x+y) - \alpha(y+z)\} + \beta z'^2(x+y) = 0.$$

The reciprocal of the conic $\alpha yz + \beta zx + \gamma xy = 0$ is therefore

$$\beta^2(z+x)^2 + \gamma^2(x+y)^2 + \alpha^2(y+z)^2 - 2\beta\gamma(z+x)(x+y) - 2\alpha\beta(z+x)(y+z) - 2\gamma\alpha(x+y)(y+z) = 0.$$

The complementary transformation of this is

$\sqrt{(\alpha\xi)} + \sqrt{(\beta\eta)} + \sqrt{(\gamma\zeta)} = 0$, a conic inscribed in ABC.

The tangential barycentric equation of this is

$$\alpha yz + \beta zx + \gamma xy = 0$$

where x, y, z are the tangential coordinates of a line.

The detailed interpretation of the results would carry me too far, and I refer to Ripert's paper for a somewhat analogous discussion and the interpretation of a few results.

§74. If now in paragraphs 71 and 72 we make the transformation $x = l\xi$, $y = m\eta$, $z = n\zeta$, we obtain the further extended geometry mentioned in the introduction. Thus all the conics now cut the line $l\xi + m\eta + n\zeta = 0$ in the same two points X, Y. The circumconic is $\frac{\alpha}{l}\eta\zeta + \frac{\beta}{m}\zeta\xi + \frac{\gamma}{n}\xi\eta = 0$ and the symmedian centre is $\left(\frac{\alpha}{l}, \frac{\beta}{m}, \frac{\gamma}{n}\right)$. The new "centroid" is $\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}\right)$; the new "orthocentre" is $\left\{\frac{1}{l(\beta+\gamma-\alpha)}, \frac{1}{m(\gamma+\alpha-\beta)}, \frac{1}{n(\alpha+\beta-\gamma)}\right\}$, etc., and the results can be stated and interpreted as mentioned in the introduction.