Geometric Transformations in three dimensions

by

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Chapter I.

Introductory

Art. 1. nature of the elements.

Space is three-dimensional in the point considered as the primary element & also in the plane considered as primary element.

If we regard the point as the primary element & the plane as secondary element, the secondary is determined by three primaries. Similarly if the plane is primary & the point the secondary, three secondaries determine a primary.
Two primaries of either kind determine a line.
Thus to points planes & lines correspond planes points & lines.
Thus the lines of one space correspond to lines of another.

But besides a plane we may consider any surface as a secondary element, the point being the primary element & vice versa.
This can be done provided 3 secondaries intersect in one point (arbitrary).
A curved surface if it is to form a secondary element must pass thro' certain fixed points & lines (plane or curved).
For example a system of conicoids which has for base a fixed quartic curve may be taken as a system of secondaries.
In this case two secondaries intersect in a line (straight), a plane curve or a space curve.

2. Systems of elements.

We can then have the following systems:-

(1) the System of all points on a line (strt. ple curved, or space curve)

(2) the system of all surfaces intersecting in a common line (strt. ple-curve or space curve).

These are one-dimensional

(3) System of lines (ple-(strt. or curved) or space curves subject to two arbitrary conditions.

(4) System of surfaces subject to two arbitrary conditions.

These are two-dimensional.

An example of (3) is a system of lines thro' a point & of (4) a system of ples thro' a point.

(5) the system of all points

(6) " " " " surfaces subject to three arbitrary conditions.

These are 3-dimensional.

(1) will be called throughout a range of points
(2) " " " a <u>sheaf</u> of surfaces
(3) " " " a pencil of lines
(4) " " " " " " surfaces.

Art 3. Definition of correspondence between two systems

Given a law by virtue of which when an element of one system is given, an element of a system of the same order can be determined we establish a correspondence between the systems. We may also say that the law transforms the one system into the other.

The correspondence between the elements may or may not be (1-1).

We shall first consider only (1-1) correspondences leading to the birational transformations.

art. 4. Correspondence between one-dimensional systems.

We shall begin with the simplest case viz. the correspondence between two ranges of points on two lines (non-coplanar)

Let a & b be the lines & let c be a third line in any position.
Thro' any pt P on c we can draw *one* str. line to intersect a & b. Let points be P_a P_b.
Then as P moves on c P_a & P_b trace out ranges between which a (1-1) correspondence exists.

Correspondence between two ~~thin~~ sheaves of planes.
We shall call the common line of intersection of the ples the *base*.
If P denote a ple of one system we shall denote the ~~base~~ system by (P).
Let (P) & (Q) be the two systems & a & b any lines. If (P) generates a range on a & (Q) a range on b between which there

exists a $(1-1)$ correspondence then the sheaves are also related by a $(1-1)$ correspondence.

Correspondence between two sheaves of surfaces of degree n.

The bases will be curves of order n^2.
Let a & b be lines intersecting each ~~bea~~ base in $n-1$ pts respectively. Then the sheaves generate ranges on a & b. If between these ranges we establish a $(1-1)$ correspondence then the sheaves are also thereby related by a $(1-1)$ correspondence.

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Art. 5. Correspondence between two-dimensional systems.

An element of a two-dimensional system can be regarded as the intersection of two elements of two distinct one-dimensional systems.
Let (A) & (B) be the defining one-dimensional systems of first two-dimensional system (S_2) & (A') & (B') those of the system (S_2').
Then if we establish a $(1-1)$ relation between (A) & (A') & a $(1-1)$ relation between (B) & (B') we establish a $(1-1)$ relation " (S_2) & (S_2')..
For if two elements of (A) & (B) intersect in an element of (S_2) the corresponding elements of (A') & (B') intersect in the corresponding element of (S_2').
For example we establish a relation $(1-1)$ between two pencils of lines as follows.
Let V & V' be the vertices of the pencils

Let ∇A & ∇B be the bases for system (S_2)
& $\nabla' A'$ $\nabla' B'$ " " (S_2').

Suppose now that we establish a (1-1) corresp between the sheaves of bases ∇A & $\nabla' A'$ & a similar relation between the sheaves of bases ~~$\nabla' A'$ &~~ ∇B & $\nabla' B'$.

Then any line l in (S_2) which is the intersection of planes thro' ∇A & ∇B corresponds to a line l' the intersection of the corresp. planes thro' $\nabla' A'$ & $\nabla' B'$.

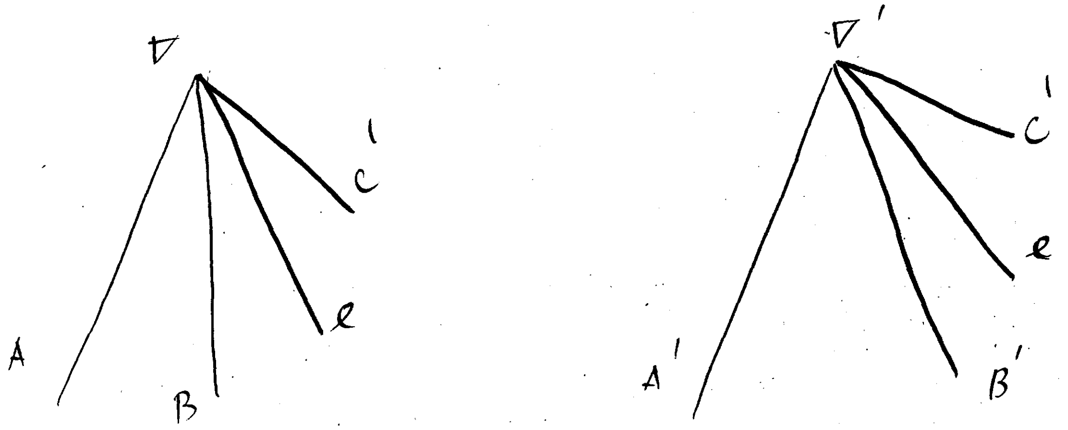
Correspondence between two pencils of surfaces vertices ∇ & ∇'.

We shall take the simplest case viz. that of two pencils of planes.

If the line l above describes a plane π then we must suppose the line l' to describe a corresponding plane π'.

This can be done provided the (1-1) relations are properly specified.

Fig (1).



In fig (1) let plane ∇AB of sheaf (∇A) corresp. to plane $\nabla' A' B'$ of sheaf $\nabla' A'$ & let same planes correspond to each other in sheaves (∇B) & $(\nabla' B')$. Then if l describes a plane sheaves (∇A) & (∇B) are in perspective. Consequently $(\nabla' A')$ & $(\nabla' B')$ are also in perspective hence l' describes a ple.

This relationship between pencils ∇ & ∇' may be called the <u>collinear</u> relationship.

If the above conditions do not hold then when l describes a line l' describes a quadric cone. Conversely when l' describes a ple l describes a quadric cone.

Thus we establish a (1-1) correspondence between a pencil of ples thro' ∇ & a pencil of cones thro' ∇'. The correspondence is reversible.

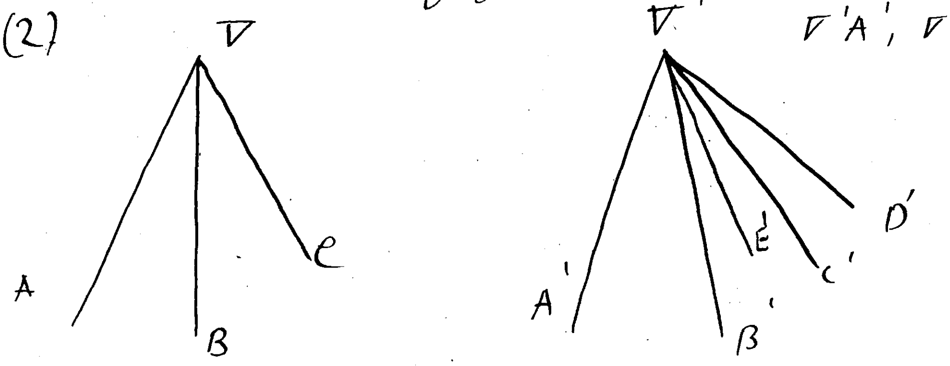
If ple ∇AB of sheaf(∇A) corresponds to ple ∇'A'C' of sheaf(∇'A') & if ple ∇'A'B' of sheaf (∇'A') corresp. to ple ∇AC of sheaf (∇A) with similar relations between the sheaves (∇B) & (∇'B') then the quadric cones pass thro' the lines ∇A, ∇B, ∇C & those of the other system thro' ∇'A', ∇'B', ∇'C'.

Thus the system of cones is two-dimensional.

Correspondence between a pencil of ples & a pencil of cubic cones.

Consider fig (2) a series of cones thro' ∇'A', ∇'B', ∇'C', & ∇'D'

fig(2)



& a series of ples thro' ∇'A'. These are both one-dimensional systems. Let a sheaf of ples be taken thro' ∇A having a (1-1) relation with sheaf(∇A') & let us take a sheaf (∇B) ~~[illegible]~~ having a (1-1) relation with the sheaf of cones. Then if the line of intersection of two ples of sheaves (∇A) & (∇B) describes a ple the curve of intersection of the sheaf of ples ($\nabla' A'$) & the sheaf of cones ($\nabla', A'B'C'D'$) describes a ruled cubic (~~surface~~ cone) having $\nabla' A'$ as a double line.

Thus we establish a correspondence between a ~~st~~ pencil of ples & a pencil of cubic cones.

To a line of pencil ∇ there corresponds in general one & only one line of pencil ∇'.
But there may be certain lines which have no unique correspondent.
Let us find e.g. the correspondent of line ∇A.
To ple ∇BA of sheaf (∇B) corresponds a cone ($\nabla' A'B'C'D'$) & ple thro' ∇A of sheaf (∇A) being indeterminate its corresp. ple in sheaf ($\nabla' A'$) is also indeterminate hence the corresp. of ∇A is a quadric cone.
Such a line is called an F-line (Fundamental line).
Again it is easily seen that correspondent of ∇B is a ple thro' $\nabla' A'$.
This ple $\nabla' A'$ meets the above cone in a line $\nabla' E'$.
All the cubic cones contain this line.
For since any ple of system ∇ meets ple ∇AB in a line & to this ple ∇AB there corresponds the single line $\nabla' E'$ it follows that every cubic cone must pass thro' $\nabla' E'$.
Thus each cone has a double gen. & 4 single generators. This is as it should be for two such cones must intersect in a line corresp. to the line of intersection of the corresp. ples.

We see also that ple ∇AB is an exceptional element & it is therefore called an F-ple.

The above relation between the pencils ∇ & ∇' is reversible.

Corresponding to a ple of system ∇' we get a cubic cone of ∇ having ∇A as double gen. & 4 fixed single generators including ∇B.

Let the line of intersection of $\nabla' A$ ple of sheaf $(\nabla' A')$ & cone of sheaf $(\nabla' A' B' C' D')$ describe a ple π'.

This ple meets a cone of sheaf $(\nabla' A' B' C' D')$ in two lines & hence corresp. to one cone we get 2 ples of sheaf ~~(∇B)~~ $(\nabla' A')$ & hence two of sheaf (∇A). Since there is a (1-1) relation between the ples of sheaf (∇B) & the sheaf of cones $(\nabla' A' B' C' D')$ it follows that the relation between the sheaves (∇A) & (∇B) is (2-1) i.e to one ple of (∇B) corresponds 2 ples of (∇A) & to one ple of (∇A), one ple of (∇B). The lines of intersection of these sheaves describe a cubic cone having ∇A as double generator & ∇B & 3 other fixed lines as single generators.

Corresponding to the lines $\nabla' C'$ $\nabla' D'$ $\nabla' E'$ & $\nabla' B'$ we get ples thro' ∇A.

Corresponding to the line $\nabla' A'$ we get a quadric cone thro' ∇A & ∇B. This meets the above ples in in 3 fixed lines ∇C ∇D ∇E which complete the F-system of lines of pencil ∇.

Thus the systems ∇ & ∇' are in every respect similar.

Proof of theorem that lines of intersection of sheaves (∇A) & (∇B) between which a (2-1) relation exists is a cubic cone.   Let a be any line & let O be a fixed origin in this line from which distances are measured.

The ples of sheaf (VB) cut a in a range of pts whose distances from O will be denoted by the variable s_2.

The ples of sheaf (VA) cut a in double range (involution) whose distances from O will be denoted by the variable s_1.

Since to a ple of (VA) there corresp. one & only one ple of (VB) & to a ple of (VB) two ples of (VA) we get the relation

$$\kappa s_1^2 s_2 + \rho s_1^2 + \lambda s_1 s_2 + \mu s_1 + \nu s_2 + \delta = 0 \ldots \text{(A)}$$

When $s_1 = s_2$ the surface traced out by intersection of sheaves meets line a.

If $s_1 = s_2 = s$ we get

$$\kappa s^3 + (\rho + \lambda) s^2 + (\mu + \nu) s + \delta = 0$$

a cubic equation giving 3 roots. Hence the surface must be a cubic one.

Relation (A) may be regarded as an equation of this cubic cone. It contains 5 independent constants & hence the cubic surface will be determined by 5 conditions. 3 of these will be taken up by considering that it must pass thro' the lines VC, VD, & VE & thus we are left with 2 arbitrary conditions. This is as it should be for since a ple is determined by two conditions so must the corresponding cubic cone.

Thus of the constants of (A) two only are arbitrary.

This method will be used frequently hereafter.

Correspondence between the pencils when they are superposed.

Let ∇ & ∇A ∇B etc ∇E coincide with ∇' & $\nabla' A'$, $\nabla' B'$ etc $\nabla' E'$.

Then the sheaves (∇A) have two double ples.

The sheaf of ples (∇B) & the sheaf of conicoids (∇B ∇A etc ∇D) trace out a cubic cone having ∇B^{+} as double gen. & ∇A ∇C ∇D ∇E as single gens.

The ples meet this surface in 4 generators. Also the gen. ∇A belongs to both systems hence there will be 5 double generators lines of pencils.

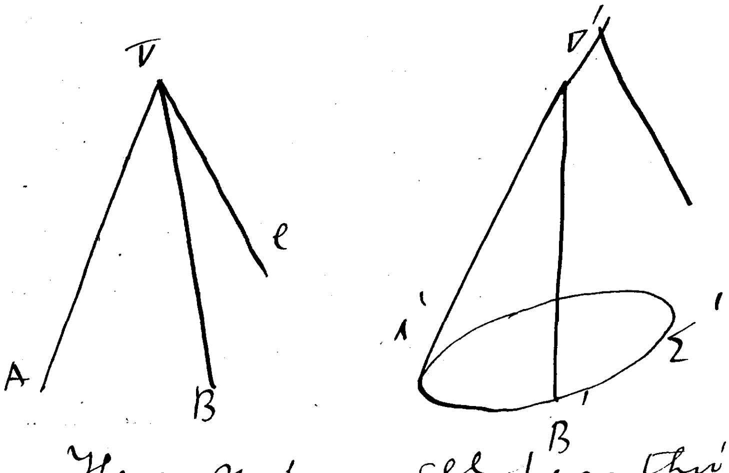
Another case of cubic correspondence between pencils which will be used afterwards is as follows.

Corresp. between a pencil of ples & a pencil of ruled cubics (not cones).

As before we begin with pencils vertices ∇ & ∇'.

For system ∇ we have again two sheaves of ples ∇A & ∇B. In (1-1) correspondence with these we have the sheaf ($\nabla' A'$) & the sheaf of conicoids ($\nabla' A'$, $\nabla' B'$, Σ') which contain $\nabla' A'$ & $\nabla' B'$ as generators & pass thro' the conic Σ'. Fig 3.

As before to a line of system ∇ corresponds a ruled surface of third order of system ∇' thro' $\nabla' B'$ $\nabla' A'$ & Σ' & having $\nabla' A'$ as a double line.



These surfaces all pass thro' another line which meets $\nabla' A'$ viz the line corresp to ple ∇AB.

As a special case this surface may degenerate into a conicoid. Thus if to ple ∇AB of sheaf (∇A) corresp $\nabla' A' B'$ of sheaf $(\nabla' A')$ to ple ∇AB of sheaf (∇B) the conicoid (degenerate) $(\nabla' A' B'$ & $\Sigma')$ then ple $\nabla' A' B'$ forms part of surface corresp. to ple. Hence the proper part of this surface must be a conicoid.

Geometric constructions for the above correspondences between pencils.

(a) The collinear correspondence.

Let π be any ple & let ∇ & ∇' be the vertices of the pencils. Any pt P on this ple if joined to ∇ & ∇' gives two lines ∇P & $\nabla' P$ which can be taken as corresp. rays of the pencils. If ∇P describes a ple $\nabla' P$ also describes a ple.

(b) The quadratic correspondence.

Take any conicoid thro' ∇ & ∇' & let a point P be taken on it. The rays ∇P & $\nabla' P$ again correspond. If ∇P describes a ple $\nabla' P'$ describes a cone & conversely.
The F-lines are the line $\nabla\nabla'$ & the generators of the conicoid thro' ∇ & ∇'.

(c) The cubic correspondence.

Take a cubic surface having conical pts at ∇ & ∇' & having 4 lines thro' ∇ & 4 thro' ∇' on surface. (We shall see later how to obtain such a surface). If P is any pt on it the rays ∇P & $\nabla' P$ corresp. If ∇P describes a ple $\nabla' P$ describes a cubic cone of above kind. The F-lines are ($\nabla\nabla'$ the double line of cones) & the 4 lines thro' ∇ & the 4 thro' ∇'.

We shall conclude the account of the correspondence between two-dimensional systems with the correspondence between a pencil of lines & a pencil of ples.
Let the pencils have vertices ∇ & ∇'.
Suppose that to sheaf of ples thro' ∇A corresponds a range of pts on line a & to sheaf of ples ∇B a range of points on a line b.
Then to the line which is the intersection of any two ples of sheaves (∇A) & (∇B) we get as correspondent the ple thro' ∇ & the pts of a & b corresp. to the ples thro' ∇A & ∇B.
If a & b are not coplanar then when l the line of intersn. of sheaves (∇A) & (∇B) describes a ple L the ple corresp. to l ~~describes~~ envelopes a cone. By projecting the lines a & b from ∇ as vertex on to any ple it is easily seen that this cone is a quadric cone. ~~Its~~ The ples ∇a & ∇b are tangent ples to it.
If the lines a & b are coplanar & if their pt of intersection ~~to~~ corresponds to the ple ∇AB considered as a ple of both sheaves (∇A) & (∇B) then if l describes a line L envelopes a line.
We can give a geometrical construction for this correspondence as follows.
Let π be a ple & Σ a conicoid. Let P be any pt on this ple & let its polar ple with respect to the conicoid be Q. Then to the ray ∇P corresponds the ple Q which passes thro' the pole of π which we may call ∇'. Thus corresponding to ray ∇P of pencil ∇ we get the ple $\nabla' Q$ of pencil ∇'.

When ∇P describes a ~~line~~ ple P describes a line & the ples Q envelope the conjugate line.

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If Σ be a sphere & ∇ its centre we get the lines of pencil ∇ at right angles to their corresponding ples in pencil ∇'.

Art 6. Correspondence between 3-dimensional spaces

Let two 3-dimensional spaces be called (S) & (S'). A point of (S) can be determined as the intersection (arbitrary) of three sheaves of surfaces. Suppose now that to each sheaf of surfaces of (S) we make to correspond by a (1-1) correspondence a sheaf of surfaces in (S') then corresp. to the 3 surfaces of S which intersect in a pt P we get 3 corresp. surfaces of (S') intersecting in the pt P'. P & P' are corresp. pts of the spaces. In this way we establish a correspondence between the spaces. If the three surfaces in (S') intersect only in one arbitrary pt then the correspondence is (1-1), always supposing that the three surfaces of (S) have a similar property.

In the following chapter we shall the simplest cases of this correspondence viz. those determined by 3 sheaves of ples. These correspondences are clearly (1-1). They give rise to birational transformations.

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Birational transformations.

Chapter II.

Art 7. The transformations that can be built up by means of 3 sheaves of ples in each space.

The collinear or homographic transformation.

In this transformation we shall find that to a line of (S) corresponds a line of (S') & to a ple of (S) a ple of (S') & conversely.

Let BC CD DB be three coplanar lines of space (S) intersecting in B C D & let B'C', C'D', D'B' be three coplanar lines of space S' intersecting in B'C' & D'.
Let us suppose that between the sheaves (BC) (CD) (DB) & (B'C') (C'D') (D'B') there exists a (1-1) correspondence each to each.
Further in the sheaves (BC) & (B'C') let ple BCD corresp. to ple B'C'D' & let there be corresp. ples for the other 2 pairs of sheaves also.
Then a pt M which is the intersection of ples BMC, CMD, DMB has for its correspondent the pt M' in which the corresp. ples B'M'C', ~~B'~~ C'M'D' & D'M'B' intersect. Thus there is a (1-1) corresp. between the pts of the two spaces.

Let M describe a line l in S - then the sheaves (BC) (CD) (DB) are in perspective in pairs & consequently the sheaves (B'C') (C'D') (D'B') have ~~the~~ same property.

Hence the locus M' of their intersection is likewise a line. Similarly if M' describes a line M describes a line also.

Let M describe a ple π in (S). M' will describe a surface π' in (S'). A line of (S) meets π in one pt hence the corresp. line meets π' in one pt viz the pt corresp. to pt of intsn. of line & π. Thus π' is of the first order & is therefore a ple.

Characteristic eqn. of transformation.
Let a be any line in (S) & π any ple.
If P be any pt a π ples BPC, CPD, DPB meet (a) in three points whose distances from a fixed origin may be denoted by $s_1\, s_2\, s_3$.
If two values s_1 & s_2 are given the value of s_3 is uniquely determined. For the ples corresp. to s_1 & s_2 intersect in a line which meets the ple π in a single pt P & the ple DPB meets (a) in the corresponding value of s_3.
Thus of the quantities $s_1\, s_2\, s_3$ any two determine the third. Hence they are related by an eqn of the form $K s_1 s_2 s_3 + \lambda s_2 s_3 + \mu s_3 s_1 + \nu s_1 s_2 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0$
In the case of the ple π however this relation takes a special form viz. $\alpha_1 s_1 + \beta_1 s_2 + \gamma_1 s_3 + \delta_1 = 0$.
For when $s_1 = s_2 = s_3$ the line (a) meets the ple & this it can only do in one point. The general relation given above corresponds to a cubic surface which in the special case of ple π consists of π & the ple BCD reckoned twice,
Let b be a line in (S') & let ples corresp. to BPC CPD, DPB meet (b) in pts whose abscissae are $t_1\, t_2\, t_3$ respectively then

$\alpha_1' t_1 + \beta_1' t_2 + \gamma_1' t_3 + \delta_1 = 0$

For of $t_1 \, t_2 \, t_3$ any two determine the third, hence they are related by the gen. relation given above.
But if K & L be two pts on line of intersection of π & BCD, corresp. to K & L we get ple B'C'D' counted twice. Hence in the general relation

$\kappa' t_1 t_2 t_3 + d' t_2 t_3 + \mu' t_3 t_1 + \cdot \quad + \lambda' t_1 + \cdot \quad + \delta' = 0$

when $t_1 = t_2 = t_3$ the resulting eqn has a double root which depends on choice of ~~fr~~ ple B'C'D'. This root does not count & consequently the relation between $t_1 \, t_2 \, t_3$ must reduce to the form $\alpha_1' t_1 + \beta_1' t_2 + \gamma_1' t_3 + \delta_1' = 0$
This relation may be looked upon as a form of the eqn of a ple.

The relation $K s_1 s_2 s_3 + d s_2 s_3 + \mu s_3 s_1 + \cdot + \lambda s_1 + \cdot - + \delta = 0$
transforms into the similar relation

$\kappa' t_1 t_2 t_3 + d' t_2 t_3 + \cdot - \quad + \lambda' t_1 + \cdot - + \delta' = 0$

This relation is characteristic of the transformations built up by 3 sheaves.

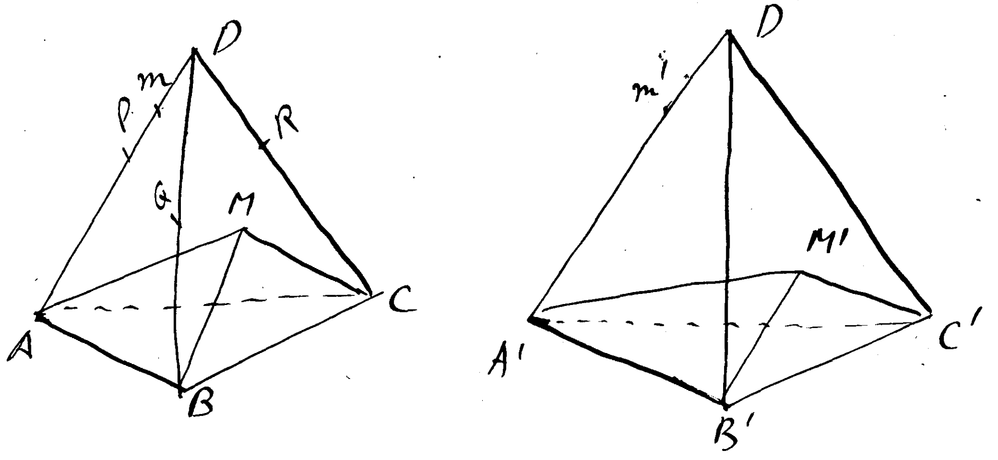
Effect of the Transformation on curves & surfaces.
It is easy to see that the degrees of curves & surfaces are unaltered by this transformation.
Thus to a conic corresponds a conic & to a conicoid a conicoid & so on.

The collinear transf. has the following properties
(1) The line PQ has for its corresp. the line P'Q' thro' the corresp. pts
(2) The pt of intersection of two lines p q has for its corresp. the pt of intsn. of the corresp. lines p'q'.

(3) A range of pts on a line q has for corresp.t a range of pts on line q' homographic with the first range.

(4) A pencil of lines V has for its corresp. a pencil of lines V' collinear with the first pencil.

Since a (1-1) relation between two sheaves of ples is completely determined when 3 pairs of correspondents are given the collinear transf. will be completely determined when 2 pairs of corresp. pts are given.



Thus let (fig) ples AMB, A'M'B', & ADB, A'D'B' corresp. in the sheaves (AB) (A'B') ~~&c~~ with similar suppositions for the other sheaves then (1-1) relations between the sheaves are completely determined. Thus when 2 pts M D
* & their correspondents M'D' are given the coll. transf is completely determined.

The lines vertices & faces of the polyhedron ABCMD correspond to the lines, vertices & faces of the polyhedron A'B'C'M'D', A to A' AB to A'B' & so on.

Geometrical cons. for the corresp. of a ple.

Let a ple Π meet ADB, DA & DC in the pts

PQ & R ~~then~~ the sheaves (BC, APD) (AB, DRC) & (AC, BQD) are completely determined & so are their correspondents. Since ranges in AD & A'D' are homographic we can construct the correspondent of P viz. P'. Similarly for the ranges on D'B' & D'C' which give pts Q' & R' corresp. to Q & R. The ple P'Q'R' corresponds to ple PQR.

The construction for the corresp^{c} of P is as follows. Let ple BMC meet AD in m & B'M'C' meet A'D' in m'. Join DD', mm' & AA' by lines. Choose any pt R on DD' & thro' R draw a line to meet mm' & AA' in S & T. Thro' P draw the ple (P, A'D') to meet RST in U. UP meets A'D' in P' which is the correspondent of P.

The vanishing planes.

Let I' be the pt on range A'D' corresp. to the pt at ∞^{y} on range AD & let J' & K' be similar pts on the ranges D'B' & D'C'.

Then the ple I'J'K' corresp. to the ple at ∞^{y} in (S). ~~Is~~ Similarly we can find a ple IJK in (S) which corresponds to the ple at infinity in (S'). These are the vanishing ples of the systems. If IJK & I'J'K' are both at ∞^{y} then the ples at infinity correspond to each other. In this case the systems are said to be in affinity.

Systems in affinity.

Since the pts at infinity on the ranges DA & D'A' corresp. these ranges are similar as are also the ranges DB, D'B' & DC D'C'.

Hence if P & P' are any two corresp. pts on DA & D'A' we get $\frac{DP}{DA} = \frac{D'P'}{D'A'}$.

Similarly for correspondents Q Q' & R R'

$$\frac{DQ}{DB} = \frac{D'Q'}{D'B'} \qquad \frac{DR}{DC} = \frac{D'R'}{D'C'}.$$

Let the ples BCA, B'C'A' be the ples at ∞?

Then if $DP = x$, $D'P' = x'$; $DQ = y$ $D'Q' = y'$ $DR = z$ $D'R' = z'$

$$x' = \frac{D'A'}{DA}x \qquad y' = \frac{D'B'}{DB}y \qquad z' = \frac{D'C'}{DC}z.$$

Then (xyz) & $(x'y'z')$ may be regarded as the Cartesian coords of two corresp. pts.

To a central conicoid in (S) corresponds a central conicoid in (S'). For a system of lines in S thro' the centre of the conicoid has the following property: each line is cut by the surface & the ple at ∞ in pts which with the centre form a harmonic system. Consequently the corresp. system in space (S') is harmonic & therefore they intersect in the centre of the corresp. conicoid.

If the ratios $\frac{D'A'}{DA}$, $\frac{D'B'}{DB}$, $\frac{D'C'}{DC}$ are equal the systems are *similar*.

If each of these ratios = 1 the systems are congruent & may be made to coincide on superposition.

Art 8. Special cases of collinear transf. Superposition of spaces (S) & (S').

When the two spaces are superposed there will generally be certain elements which coincide with their correspondents – the double elements.

Two collinear systems cannot have more than 4 double pts.

Suppose there are 5 viz A B C D & E. Let ABCD be taken as the base tetrahedra of both systems. Then 3 pairs of corresponding planes coincide in each sheaf (AB) (BC) (CA) hence the systems are identical.

Position of the double elements.

Corresp. ples of sheaves (AB) (A'B') trace out a hyperboloid as do also corresp. ples of sheaves (BC) (B'C') & (CA, C'A'). These hyds. intersect in 8 pts of which 4 have to be rejected. These hyds. have a common generator viz the line of intersection of ples ABC, A'B'C' these being homologous ples. This common gen^{n} counts as 4 pts of intersection & these are to be rejected since they depend on choice of base tetrahedra. Hence there are 4 proper d.ps. left.

Let two of the hyds. degenerate into ples.

Let the 3 surfaces be

(1) ple (AB, A'B') & a ple K_1

(2) " (BC, B'C') " K_2

& (3) a proper hyd. H.

Here AB, A'B' & BC, B'C' intersect in pts P & Q say. Ples K_1 & K_2 must intersect in line PQ which is a generator of H.

The 4 dps are the intersections

$\{AB, A'B'\}$, $\{BC, B'C'\}$, H (2 pts)
$\{AB, A'B'\}$, K_2, H (1 pt)
$\{BC, B'C'\}$, K_1, H (1 pt)

It may happen however that $\{AB, A'B'\}$ & K_2 intersect in a line which is a gen^n of H. In this case there is a line of dps & two isolated dps viz the intersection of $\{BC, B'C'\}$, K_1, H & the pt of intersection of the other gen^r in which $\{AB, A'B'\}$ meets the hyd. H, with the plane $\{BC, B'C'\}$.
This case is the *axial* correlation.
The line of intersection of $\{AB, A'B'\}$ & K_2 is the axis of the correlation. It passes thro' P & intersects BB' in a pt S say. It is a generator of H of same system as **A**C & **A**$'C'$.
(Consider a ~~generator of the hyd. H which meets AC $A'C'$ & PS. This gen^r transforms by the collineation into another line meeting AC, $A'C'$ & PS & the latter moreover cuts PS in the same pt since all pts on PS are dps. Hence the corresp^g of the gen^r must coincide with the gen^r itself. Hence the gens. of this hyd. transform into~~ themselves. Hence AA' & CC' must meet BB' in pt S.)

Again it may happen that $\{BC, B'C'\}$ & K_1 intersect in a line which is a generator of H. This gen^r is a line of dps & must pass thro' Q. It is also a gen^r of same system as AC, $A'C'$.
In this case the correlation has two lines of dps & no isolated dps.
It is the *skew* correlation.

Again let the three hyds. be degenerate & let them be the pairs of ples

$$\{AB, A'B'\}\ K_1\ ;\ \{BC, B'C'\}\ K_2,\ \{AC, A'C'\}\ K_3$$

The 4 dps are the intersections of the triads of ples

$$\{AB, A'B'\},\ \{BC, B'C'\},\ K_3 \qquad (1)$$
$$\{BC, B'C'\},\ \{CA, C'A'\},\ K_1 \qquad (2)$$
$$\{CA, C'A'\},\ \{AB, A'B'\}\ K_2 \qquad (3)$$
$$\{AB, A'B'\}\ \ \{BC\ ;\ B'C'\}\ \{CA, C'A'\} \qquad (4).$$

K_1 K_2 & K_3 must intersect in a line viz. the line of intersection of ples ABC, $A'B'C'$.

Now the ples K_1 K_2 K_3 may coincide. In this case there is a ple of dps. $AB, A'B'$; $BC, B'C'$ CA; $C'A'$; must intersect in 3 collinear pts.

This cases the <u>Central</u> collineation.

There is one dp which is isolated viz (4).

It is to be noted that to a pt P there correspond two pts P_1' & P_2' according as P is considered to belong to (S) or (S').

Coincident tetrahedra.

These transformations are most conveniently treated when they spaces are referred to a common base tetrahedron. Let this be $ABCD$. Suppose to the sheaf thro' AB corresp. another sheaf thro' AB & similarly for the sheaves thro' BC & CA. Since the sheaves thro' AB, BC, & CA are homographic with their correspondents it follows that thro' AB there will be two double ples & similarly for BC & CA. BCA is a d ple for each of the systems. Let the other d ples meet in D. Then clearly $ABCD$ are the dps of the transformation.

Thus one & only one tetrahedron can be found which will serve as base tet^n for (S) & (S').

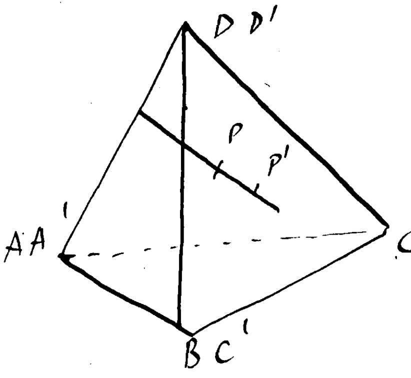
Invariants of the trans.

Let PP' & QQ' be two pairs of corresp. pts on a line which meets faces of tet^n ABCD in $\alpha\beta\gamma$ & δ. Then with the usual notation for cross ratios $\{\alpha\beta PQ\} = (\alpha\beta P'Q')$ hence $(\alpha\beta PP') = (\alpha\beta QQ')$. This follows from the fact that the faces of tet^n are double ples. Similarly $(\alpha\gamma PP') = \text{constant}$ & $(\beta\gamma PP') = \text{constant}$. Also $\dfrac{(\alpha\beta PP')}{(\alpha\gamma PP')} = (\beta\gamma PP')$ etc

hence we see that 6 constant cross-ratios can be obtained of which 3 are arbitrary. These are the invariants.

The axial ~~coaxl~~ collineation.

Suppose that to sheaf (AB) there corresponds a sheaf (AC) & to sheaf (AC) a sheaf (AB), to sheaf (BC) a sheaf (BC). In this case corresponding ples thro' AB & AC describe a ple thro' AD & corresp. pts P P' lie on a line which intersects AD. Fig ( ). AD is a line of dbl pts & is the axis of the collineation. We may imagine this case to arise thro' the coincidence of the tetrah. ABCD A'B'C'D' in the manner shown in the fig.



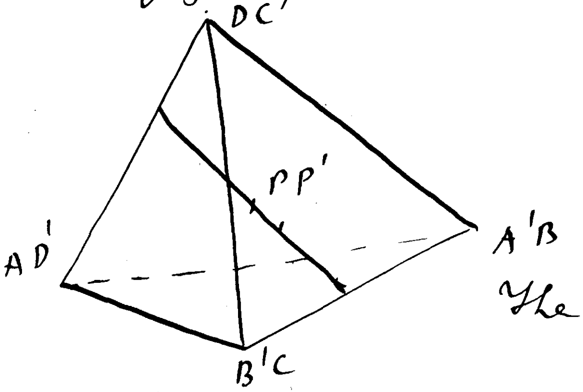
Invariants of S axial collineation. Two of pts $\alpha\beta\gamma\delta$ now coincide. Let these be $\alpha\beta$ then we can now only have out the 6 invariants 3 viz $(\alpha\gamma PP')$ $(\alpha\delta PP')$ & $(\gamma\delta PP')$.

The skew collineation.

Again suppose that to sheaf thro' AB corresp^g sheaf (BC) & to sheaf (AC) a sheaf (DB) to sheaf (BC) a sheaf (CD).

Corresp. ples thro AB & BC describe a ple containing BD & corresp. ples thro BC & CD describe a ple thro AC. AC & BD are lines of dps. & the line joining corresp. pts P & P' intersects both BD & AC. We may imagine this case to arise thro the coincidence of the tets ABCD, A'B'C'D' in the manner shown in fig ( ).

Invariants. The points α β γ δ reduce to 2 pairs say α & γ of coincident pts. There is one invariant viz (αγPP').

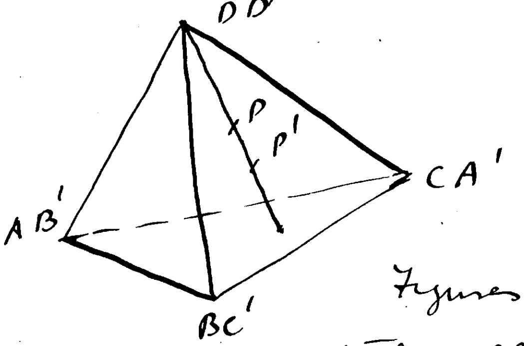


The _central_ collineation.

Let ~~the~~ a sheaf thro AB corresp. to sheaf (AC) & sheaf (BC) to sheaf (BA) & sheaf (AC) to a sheaf (BC). Then line joining corresp. pts must pass thro D which is an isolated dp of the transf. All pts in the ple ABC are dps. This case arises thro the coincidence of the tets as shown in fig ( ).

Invariants.

3 of pts αβγδ coincide at D say αβγ. There is again one invariant viz (αγPP').



Figures in homology are related by the central collinear transf.

Involutive transformations.

It is of interest to inquire under what circumstances these transf. can become involutive i.e. to a pt P there corresps. a single pt P' no matter which system P is considered to belong to.

Let P & P' be a pair of pts having this property. Then the line PP' transforms into the line PP'

& corresp. pts on this line form an involutive system which has two dps. These dps are consequently dps of transformation. Hence the line PP' must be one of the six edges of the tet.^{n} $ABCD$. Thus the general collineation has no involutive pt pairs outside of the base tet.^{n}

But if we take the skew coll. any pair of corresp. pts $P P'$ may be involutive. If ~~one~~ two pairs be known to be involutive then all pairs are involutive. Also $(\alpha\,\gamma\,PP') = -1$ since α & γ are dps of the involution (PP').

Again if we take the central collineation we can have likewise an involutive collineation. If any two pairs of corr: pts are involutive then the transformation is involutive. Also $(\alpha\,\gamma\,PP') = -1$

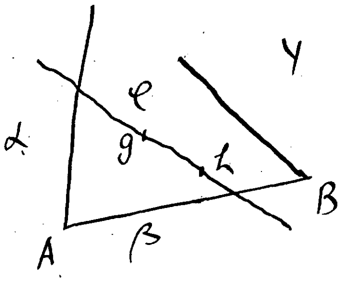
The vanishing ples of a central coll. are parallel to ABC the ple of dps. When the coll. is involutive the single vanishing ple bisects the distance between D & ABC.

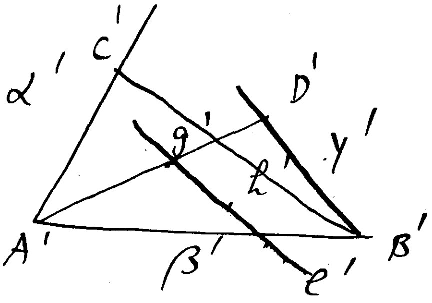
The special collineation which transforms a single infinity of conicoids each into itself.

In the general collineation it is in general impossible to find a conicoid which is transformed into itself. In the involutive skew & central collineations it is possible to find systems of conicoids each one of which transforms into itself. But there is a special form of collineation which possesses 4 dps & which transforms each one of a single infinity of conicoids into itself.

Let α & γ be non-coplanar lines intersected by a third β in space (S) & let α' & γ' be similar lines in (S′) intersected by β'.

Then as in the general collineation we can establish a pt-pt (1-1) relation between the spaces by means of sheaves thro' $\alpha\beta\gamma$ & $\alpha'\beta'\gamma'$. Suppose that to ple of sheaf α $(\alpha\beta)$ corresp. ple of α' $(\alpha'\beta')$ & to ple of sheaf β $(\alpha\beta)$ corresp ple $(\alpha'\beta')$; to ple γ $(\gamma\beta)$ ple γ' $(\gamma'\beta')$.





Let P describe a line l. Sheaves α β are in perspective consequently sheaves α' & β' are also in pers. Hence line of intersection of corresp. ples thro' α' & β' describes a ple & so does line of intersection of corr. ples thro' γ' & β'. Let P′ be the pt of intersection of these two lines. Then clearly it describes a line. Thus to a line corresponds a line & to a ple corresp. a ple.

If l meet ples $\alpha\beta$ & $\beta\gamma$ in g & h resp. its corresp. meets ples $(\alpha'\beta')$ & $(\beta'\gamma')$ in corr pts g' & h'.

Suppose the systems now to be superposed so that α coincides with α' β with β' & γ with γ'. Then sheaves thro' α' have two d ples of which $\alpha'\beta'$ will be one. Let the other meet γ' in D′. Then line A′D′ will be a double line of the trans.

Similarly we will get a double line thro' B′ meeting α' in C′. The pts A′B′C′D′ are the d pts.

Each of the lines of tet^n A'B'C'D' is a double line.
The lines α'γ' & A'D' B'C' determine a single infinity of conicoids α'γ' being gens. of one system & A'D' & B'C' gens of other system.
Let a line k be given intersecting A'D' & B'C'.
k with the above 4 lines will determine uniquely a conicoid. The line k will transform into another line k' also meeting A'D' & B'C'.
If k' be a gen^n of same conicoid ~~it will~~ the latter transforms into itself.
This can be effected as follows: consider sheaves thro' A'D' & B'C'. Let the lines A'C' k & B'D' be the lines of intersection of 3 corresp. ples thro' them. These are sufficient to completely determine a (1-1) corresp. between the sheaves. If k' be the line of inters^n of corresp. ples of these sheaves then the ~~sheaf~~ cross ratio A'D'(A'C', k, B'D', k') =
cross ratio of B'C'(B'D', k, A'C' k')
If this condition is satisfied then all conicoids thro' α'γ', A'D', B'C', transform into themselves.

Nature of correspondence.

It is clear that a generator transforms into another generator of same system. In each system there are two double generators viz α'. γ', A'D', B'C'.

Again suppose the systems to be superposed so that α' coincides with γ' & γ with α' while β coincides with β'.
The line of intersection of corresp. ples thro' α' & γ' traces out a hyd. There are two d ples thro' β' hence there are two conics which corresp. to each other on the hyd. But these conics are degenerate being made up of gens.. viz, α', A'D' &

y' $B'C'$. These lines corresp. to themselves & we get the preceding case again.

Case of an infinity of d.ps.

If the corresp. sheaves thro' x' are identical as are also those thro' y' then one set of generators transforms each into itself.

If there are two pairs of involutive points then the transformation is involutive.

— ′ —

We shall find later in the quadratic trans. that the generators of a conicoid can be made to transform into generators of the same system & the conicoid into itself.

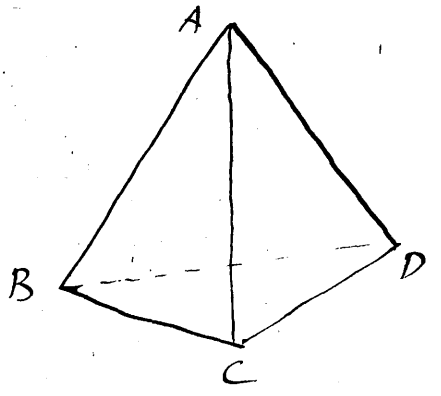
— ′ —

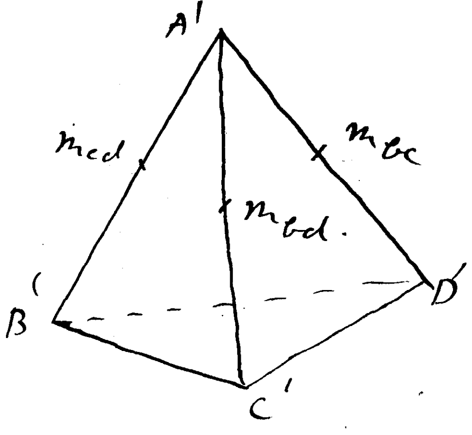
Art 9. The correlative transformation.

— ′ —

In the collinear transf. the pt as primary element of space (S) corresponded with the pt as primary element of (S'). In this transf. we shall make the pts of one space correspond to the planes of the other space.

Let the tetrahedra of reference be ABCD, A'B'C'D'.





Suppose that to a sheaf of planes thro' BC there corr. a range of pts on A'D'; to a sheaf CD a range on A'B' & to sheaf (BD) a range on A'C'.

Let each sheaf be homographic with its corresp. range. Let pt A' corresp. to ple BCD & let D' corr. to ple BAC, C' to BAD & B' to CAD.

Then to a pt M of space (1) determined by ples BMC, CMD, DMB corresponds the ple determined by the pts m_{bc}, m_{cd}, m_{db} which corresp. to these ples.

When M describes a line l ples BMC, CMD generate a homographic perspective system & consequently m_{cd} m_{db} generate two persp. ranges. Hence line $m_{cd} \cdot m_{db}$ passes thro' a fixed pt. Similarly for the lines (m_{db}, m_{bc}) etc. These 3 fixed pts lie on a line l' which corr. to l, for the ple $m_{cd} \cdot m_{db} \cdot m_{bc}$ passes thro' fixed pts & this it cannot do unless these lie on a line.

When M describes a ple π, two generating ples BMC, CMD being known the third is uniquely determined. Hence when two pts say m_{bc} m_{cd} are known the third is uniquely determined. Hence if $S_1 S_2 S_3$ be their distances from A' these must be connected by a relation of form

$$K S_1 S_2 S_3 + \lambda S_1 S_2 + \mu S_2 S_3 + \nu S_3 S_1 + \alpha S_1 + \beta S_2 + \gamma S_3 + \delta = 0$$

But if ple BMD is fixed it intersects π in a line & the correspondent of this line is given by putting $S_3 = \text{const.}$ If $S_3 = \text{const}$ the above relation must be a perspective homog. in which A' corresp. to A' ie $S_2 = S_1 = 0$ must satisfy it.

$\therefore \gamma S_3 + \delta = 0$ (for all S_3) $\therefore \gamma = \delta = 0$

∴ Similarly $\alpha = \delta = \beta = 0$

∴ reln is $K S_1 S_2 S_3 + \lambda S_1 S_2 + \mu S_2 S_3 + \nu S_3 S_1 = 0$

or $K + \frac{\lambda}{S_3} + \frac{\mu}{S_1} + \frac{\nu}{S_2} = 0$. (A)

But the eqn of the m_{bc} m_{cd} m_{bd} is

$\frac{x}{S_1} + \frac{y}{S_2} + \frac{z}{S_3} = \cancel{0}\,1$ hence the ple must pass thro' a fixed pt of coords $\left(-\frac{\mu}{K}, -\frac{\nu}{K}, -\frac{\lambda}{K}\right)$.
Eqn (A) involving 3 consts & 3 variables may be looked upon as the equation of a point when the ple is the primary element.

Thus to a pt of (S) corresponds a ple of (S')
to a line of (S) " line (S').
Conversely to a pt of (S') determined as the intersection of 3 ples corresponds the ple of (S) determined as the ple containing the 3 pts which correspond to the 3 ples of (S').

Superposition of the spaces.
Incident elements.

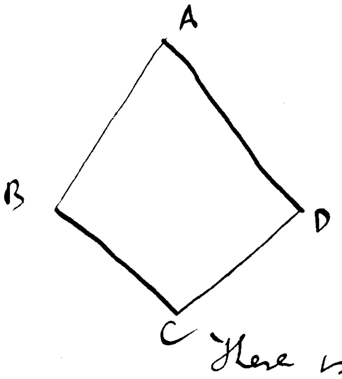
If a point lie on its corresp. ple it is an incident pt & if a ple contain its corresp. pt it is an incident ple.
To any pt there correspond two ples according as it is considered to belong to (S) or (S').
An incident pt lies on both its corr. ples & an incident ple contains both its corr. pts.

Let M describe a line l its corr. ple m describes a sheaf thro' l' the corr. of l. This sheaf describes a range on l homog. with the range described by M. This homog. system has 2 dps. Hence any line meets the surface of dps in 2 pts & the latter is therefore a conicoid. Similarly we find that envelope of incident ples is another conicoid.

Let these conicoids be denoted by C & C'.
Corr. to a pt P on C we have 2 tangent planes to C' both passing thro' P. Let the pts of contact of these planes be Q_1' & Q_2'. Then to pt P corr. two pts Q_1' Q_2' on C'.

Q_1' & Q_2' form a collinear system on C' for if P describe a gen^r of C, Q_1' ~~envelopes~~ describes a gen^r of C' & so does Q_2'.

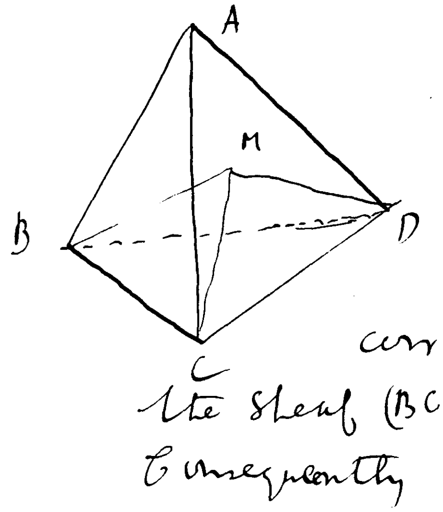
Now this collineation on C' has 4 dps. Let these be A B C D. To any pt Q_1 on AB there corresponds a pt Q_1' on AB. The pts ABCD are the vertices of a skew quadrilateral on C' whose sides are gens. But tangt. planes at Q_1 & Q_1' both pass thro' a pt on C. This pt must therefore lie also on AB. Similarly for pts on BC, CD & DA. Hence the skew quad. A B C D is the common curve of intersection of C & C'.



There is also a collineation of pts on C whose dps are also A B C D. The planes ABC ADC etc touch both conicoids at the vertices of the tetr^n.

Art. 10. Involutive correlation.

Spaces both referred to the tetrahedron A B C D. Let A B C D coincide respectively with A' B' C' D'. Then sheaf (B C) corresponds to range A D etc.



To a point on range AD corresponds a plane thro' ~~AD~~ BC etc. Since pt D corr. to plane BAC & pt A to plane BCD. the sheaf (BC) & the range AD are in involution. Consequently the trans. is involutive. To a pt M corresponds a single plane m.

Let M describe a line l. The planes m cut l in a range homographic with range described by M. These ranges have 2 dps hence surface of incident elements is as before a conicoid. These ranges are moreover involutive hence K & L are the dps on l & if plane m meets l in m the ratio $(m M K L) = -1$ & m & M are harmonic conjugates with respect to surface of ~~dps~~ incident elements. Thus a pt & its corresp. plane are pole & polar with respect to the conicoid of incident elements. Since to A corresponds plane BCD, BCD must be the polar plane of A with respect to conicoid. Similarly BAC is polar plane of D & so on. Hence the tetⁿ ABCD is self-polar with respect to the conicoid of incident elements. In this involutive transformation the two conicoids C & C′ reduce to one & we get the ordinary polar reciprocation.

Analytical formulae for the collinear & correlative transformations.

Let BCD be $w=0$ CAD be $y=0$
BAC " $x=0$ DAB " $z=0$

B′C′D′ " $w'=0$ C′A′D′ " $y'=0$
B′A′C′ " $x'=0$ D′A′B′ " $z'=0$

Let plane BMC be $x - k_1 w = 0$
Then plane B′M′C′ will be $x' - \frac{\alpha k_1+\beta}{\gamma k_1+\delta} w' = 0$

$$\therefore x' - \frac{\alpha x+\beta w}{\gamma x+\delta w} w' = 0$$

similarly $y' - \frac{\alpha_1 y+\beta_1 w}{\gamma_1 y+\delta_1 w} w' = 0$ etc

$$z' - \frac{\alpha_2 z+\beta_2 w}{\gamma_2 z+\delta_2 w} w' = 0$$

giving the relations

$$\gamma x x' + \delta x' w - \alpha x w' - \beta w w' = 0$$
$$\gamma_1 y y' + \delta_1 y' w - \alpha_1 y w' - \beta_1 w w' = 0$$
$$\gamma_2 z z' + \delta_2 z' w - \alpha_2 z w' - \beta_2 w w' = 0$$

now in collineanty $x = 0$ corr. to $x' = 0$ etc
$w = 0$ " $w' = 0$

$\therefore$ $\delta x' w - \alpha x w' = 0$ or $\frac{x'}{x} = \frac{\alpha}{\delta} \frac{w'}{w}$

$\delta_1 y' w - \alpha_1 y w' = 0$ or $\frac{y'}{y} = \frac{\alpha_1}{\delta_1} \frac{w'}{w}$

$\delta_2 z' w - \alpha_2 z w' = 0$ $\frac{z'}{z} = \frac{\alpha_2}{\delta_2} \frac{w'}{w}$

$\therefore x : y : z : w = a x' : b y' : c z' : d w'$
where a b c d are constants.

Special cases.

(1) Axial coll. $b = c$ for the plane $y - kz = 0$ is to be double. (BC axis).

(2) Skew coll. axes BC & AD
$a = d$ & $b = c$

(3) Central coll. A centre
$a = b = c$,

Spaces superposed & referred to same tetn which is not tetn of clps.

$x : y : z : w = \alpha_{11} x' + \beta_{12} y' + \gamma_{13} z' + \delta_{14} w' : \ldots$ etc.

Correlative transformation.

Let plane BMC be $x - \frac{k_1}{k_4} w = 0$

CMD " $y - \frac{k_2}{k_4} w = 0$

DMB " $z - \frac{k_3}{k_4} w = 0$

M is the pt $(k_1\ k_2\ k_3\ k_4)$

Then for M be $x' = \alpha k_4$ $w' = \delta k_1$

n_{cd}, $y' = \beta k_4$ $w' = \delta k_2$

n_{bd}, $z' = \gamma k_4$ $w' = \delta k_3$.

Let ple n_{bc}, n_{db}, n_{bc} be

$$\lambda x' + \mu y' + \nu z' + \rho w' = 0$$

then

$$\begin{vmatrix} x' & y' & z' & w' \\ \alpha k_4 & 0 & 0 & \delta k_1 \\ 0 & \beta k_4 & 0 & \delta k_2 \\ 0 & 0 & \gamma k_4 & \delta k_3 \end{vmatrix} = 0$$

giving $x'\beta\gamma\delta k_1 k_4^2 + y'\alpha\gamma\delta k_2 k_4^2 + z'\alpha\beta\delta k_3 k_4^2$

$+ w'\alpha\beta\gamma k_4^3 = 0$

thus corresp. to pt $(k_1\ k_2\ k_3\ k_4)$ or $(xyzw)$ generally there is the ple

$$\frac{x'xw^2}{\alpha} + \frac{y'yw^2}{\beta} + \frac{z'zw^2}{\gamma} + \frac{w'w^3}{\delta} = 0$$

or $\dfrac{x'x}{\alpha} + \dfrac{yy'}{\beta} + \dfrac{zz'}{\gamma} + \dfrac{ww'}{\delta} = 0$

Let $\frac{x}{\alpha} = \xi$ $\frac{y}{\beta} = \eta$ $\frac{z}{\gamma} = \zeta$ $\frac{w}{\delta} = \theta$

then $x = \xi\alpha$ $y = \eta\beta$ $z = \zeta\gamma$ $w = \delta\theta$

are the relations between the coords of M $(xyzw)$ & the coords of n $(\xi\ \eta\ \zeta\ \theta)$.

Coincident ~~tetrahedra~~.

M $(xyzw)$ lies on its corr. ple if

$\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} + \frac{w^2}{\delta} = 0$ showing that conicoid of incident elements has fund. tetⁿ

for self-polar tetⁿ

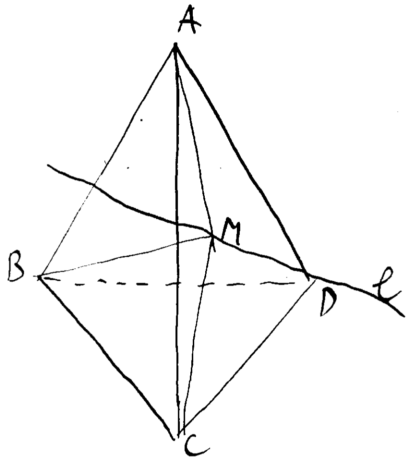
Ple $\xi x + \eta y + \zeta z + \theta w = 0$ contains its corresp. pt $(\xi\alpha, \eta\beta, \zeta\gamma, \delta\theta)$ if

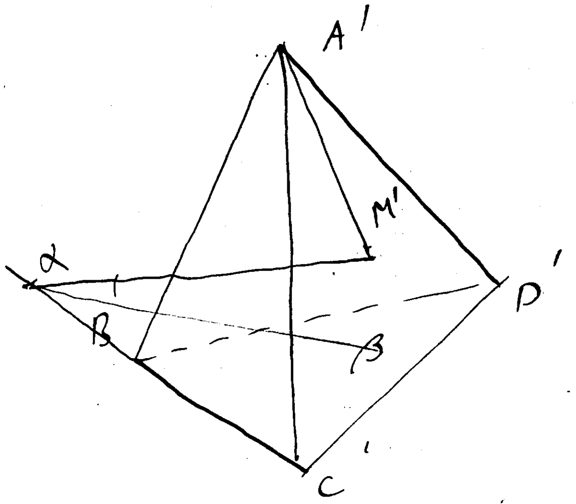
$\xi^2\alpha + \eta^2\beta + \zeta^2\gamma + \theta^2\delta = 0$. This is same as conicoid above but is now expressed in ple. coords.

Art. 11

The quadratic cono-cylindrical transf.

Let the spaces be referred to tet^s. ABCD & A'B'C'D'.





Let sheaves thro' AB, BC, & CA be related homog. with sheaves thro' A'B', B'C', C'A' resp., so that

to ple BAD (BA) corresp. ple B'A'D' (B'A')

& BAC (BA) " " B'A'C' (B'A')

" BAC (~~BA~~ AC) " " B'A'C' (A'C')

" CAD (AC) " " C'A'D' (A'C')

" BCD (BC) " " B'A'C' (B'C')

" BAC (BC) " " B'D'C' (B'C').

Let M be the intersection of 3 ples BMC, CMA & AMB then the 3 corresp. ples of (S') intersect in M' which is the correspondent of M & conversely.

Thus we establish a pt-pt (1-1) corresp. between the spaces.

Let now M describe a line. The line AM describes a ple & consequently A'M' also describes a ple.

The sheaf B'M'C' describes on this ple a pencil of lines thro' α the pt where the ple meets B'C'.

~~For an~~ The relation between the coplanar pencils A'M' & αM' is clearly homographic hence their pt of intersection describes a conic thro' A' & α.

Corresp. to the line $A'\alpha$ we get the line $\alpha\beta$ in which the ple described by $A'M'$ meets $B'C'D'$. Hence the conic touches the latter ple at α.
Thus to a line l corresponds a conic l' touching the ple $B'C'D'$ on $B'C'$ & passing thro' A'.
Conversely to line l' of (S') corresp. a conic thro' A & touching ple BCD in a pt on BC.

The corresp. of a plane of (S) will be a conicoid of (S') & conversely.
As before the generic eqn. of the conicoids of (S') will be

$$\kappa s_1 s_2 s_3 + d s_1 s_2 + \mu s_2 s_3 + \nu s_1 s_3 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0$$

Let s_3 correspond to ple $B'M'C'$. Then when s_3 is const the relation between s_1 & s_2 must be linear for when ple $B'M'C'$ is fixed, ples $B'A'M'$ & $C'A'M'$ generate a ple. Hence if $s_3 = c$ we must have $s_1 s_2 (\kappa c + d) = 0$ for all values of c

$\therefore \kappa = d = 0$ hence the relation is of the form

$$\mu s_2 s_3 + \nu s_3 s_1 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0 \quad (1)$$

When $s_1 = s_2 = s_3 = s$ this gives a quadratic eqn & hence the surfaces are conicoids.
They are not, however general conicoids for

$$s_3 = -\frac{\beta s_2 + \alpha s_1 + \delta}{\mu s_2 + \nu s_1 + \gamma}$$

The eqns $\beta s_2 + \alpha s_1 + \delta = 0$, $\mu s_2 + \nu s_1 + \gamma = 0$ give one value for s_1 & one for s_2. Let these be k_1 & k_2. Corresponding to k_1 & k_2 there is no unique correspondent on base line of relation (1). Hence the ples $A'B'k_1$, $A'C'k_2$ must intersect in a line which lies on the surface.
Thus thro' A' only one line can be drawn on the surface which must therefore be a cylinder or cone.

Exceptional elements of the transf.

To any pt on ple BAC corresp. the line B'C'
- - - BCD . pt A'
. . . . - line BC .. . a pt on line B'C'
or rather a line thro' A' meeting B'C'.

To any line thro A in ple BAC corresp. a pt on B'C',
& to any line thro' A' in ple B'A'C' .. .. .. BC.
Hence ple BAC is composed of exceptional elements
& so is ple BCD. The line BC is also
composed of exceptional elements.

These exceptional elements are the F-elements
of the transf.

The F-elements (fundamental elements) of
S are ples BAC BCD & line BC & also pt A;
of S' B'A'C', B'C'D', B'C', & A'.

Let a ple π of S meet ple BAC in a line p
& ple BCD in a line q these lines intersecting
in a pt α on BC. Then clearly the conicoid
into which the ple transforms will pass thro'
A' & contain the line B'C'.

Since a line thro' α transforms into a line
thro' a pt α' on B'C' we see that all the
lines on the ple thro' α transform into lines
on the conicoid thro' α'. α' is therefore a
conical pt & the conicoid is a cone having
its vertex on B'C' which is a generator.

All cones touch the ple B'C'D' for all ~~ple~~
sections thro' A' are conics touching ple B'C'D'.

Effect of transformation on higher curves &
surfaces.

A conic of (S) transforms in general into a quartic curve having a node at A' & intersecting the F-line $B'C'$ in two pts. This is evident since the conic meets the ples ABC & BCD each in 2 pts.
A cubic curve transforms into a sextic having a triple node at A' & intersecting $B'C'$ three & so on.

A conicoid in general transforms into a quartic having a conical pt of second order at A' & two conical pts of ~~second~~ third order on $B'C'$. $B'C'$ is a double line on such surfaces.
A cubic surface transforms in a sextic having a conical pt of 3rd order at A' & ~~two~~ three conical pts of ~~3rd~~ 4th order of on $B'C'$ which is a triple line & so on.

Particular cases arise if the curve or surface ~~intersect or~~ contain an F-element.
Thus a conicoid thro' A transforms into a cubic surface having 3 conical pts of second order one at A' & two on $B'C'$.
Let the conicoid meet BC in K & L. Then thro' K & L there are two pairs of gens. & there is another pair thro' A. Hence the cubic surface which corresponds to it will have 2 lines thro' each conical pt on $B'C'$ (besides $B'C'$) & two thro' A' besides ($A'F_1'$ & $A'F_2'$) where F_1' & F_2' are the conical pts on $B'C'$. These lines will intersect in pairs just as do the corr. generators on the conicoid. Thus there are 9 lines on this cubic surface.
This surface can be generated as follows.
Let the conicoid pass thro' BC. Then if pt M generate the conicoid thro' A B & C of the three ples AMB, BMC AMC any two determine the third uniquely

consequently the three corresp. planes $A'M'B'$, $B'M'C'$, $A'M'C'$ have the same property. If they cut of segments on any arbitrary line l which when referred to any origin on l are given by $s_1\ s_2\ s_3$ these quantities are related by the relation

$$\kappa s_1 s_2 s_3 + \lambda s_2 s_3 + \mu s_3 s_1 + \nu s_1 s_2 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0$$

This relation we shall call henceforth the <u>cubic</u> homography. Then 3 sheaves $(A'B')$ $(B'C')$ $(C'A')$ if they trace out ranges on any line which form a cubic homog. we shall say are cubic homographic sheaves.

Superposition of the two spaces.

When (S) & (S') are superposed there will be a limited number of d.ps. These as in the collinear transf. will be the intersections of 3 hyperboloids. There are thus apparently 8 d.ps. Two of the hyps. have a common generator viz. the line of intersection of planes ABC & $A'B'C'$. This line meets the third hyd. in 2 pts. These two pts depend on the ∞ choice of fund. tets & have therefore to be rejected. Hence there are 6 proper d.ps.

Let the tetrahedra be made to coincide, so that sheaf (AB) is superposed on sheaf $A'B'$ etc ie so that $A\ A'$, $B\ B'$, $C\ C'$, $D\ D'$ coincide respectively. The three lines AB, AC, AD are now double lines & hence the 6 d.ps lie on them two on each.

Art. 12. Important particular cases.

We shall now make the assumption that the sheaves thro' AB are identical as are those thro' AC.

In this case every line thro' A is a double line & corresponding pts are collinear with A.

The transformation is now involutive for to a pt M corresponds one pt M' no matter which space M is supposed to belong to.

In the sheaf BC ples BCD & BCA correspond to each other for each space hence the sheaf is involutive & consequently so is the transf.

Also if α & β be the dples thro' BC then corresp. pts M & M' which lie on a line thro' A meet α & β in two pts which with M & M' form a harmonic system.

The transformation is now a perspective one.

If π be any ple & π' its corresp. conicoid & if we take a ple thro' A meeting π in a line l & π' in a conic l' then l & l' correspond.

Representation of conicoid on its corresponding plane.

The conicoids which corresp. to a ple are either cones or cylinders.

Let the ple meet BC in α, then the ple transforms into a cone having its vertex at α.

If the ple meet BC in a pt at an infinite distance then it transforms into a cylinder.

Thus all ples parallel to BC transform into cylinders.

~~Then~~ the generator thro' A on the conicoid is the line Aα.

All conics thro' A project into lines on the ple

All generators project into concurrent or parallel lines according as conicoid is cone or cylinder.

All conics which do not pass thro' A project into conics on the ple.

A cubic curve thro' the vertex α but not thro' A projects into a cubic curve on ple having a dp at α.

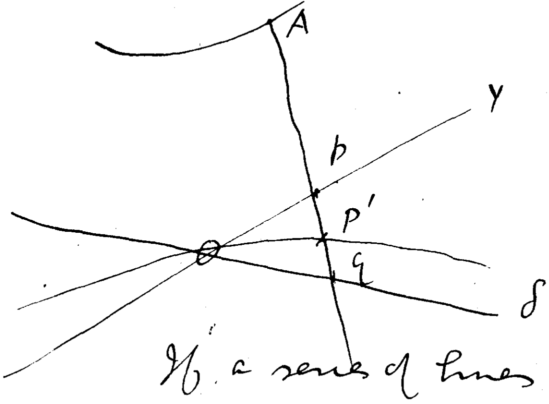
If it passes thro' A it projects into a conic thro' α.

A quartic curve having a dp at α projects into ple quartic having a triple pt at α.

A quartic which is the intersection of the cone with a conicoid projects into a quartic having a singular point at α. Let the tangent ple to cone thro' $A\alpha$ meet the ple π in a line k then two branches of the quartic touch k at α. Thus the quartic has a tac-node at α. It is of deficiency 1. The previous quartic is of deficiency 0.

The conicoid which corresponds to the ple at infinity.

(1) Let the dples of the sheaves thro' BC be real. Then the correspondent of the ple at infinity is a cylinder which is in this case hyperbolic. For let a section of the system be taken thro' A. This meets the real dples in lines $O\gamma$ & $O\delta$ where O is on BC. If P is a pt on ple at infinity



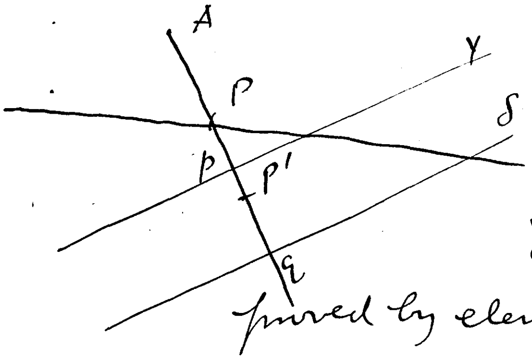
AP meets $O\gamma$ & $O\delta$ in pts p & q such that ~~A~~$(pqPP')$ is harmonic where P' corresp. to P. Hence P' is the mid pt of pq.

If a series of lines be drawn thro' A in the ple of the section to meet $O\gamma$ & $O\delta$ the locus of the middle pt of pq is a hyperbola thro' O & A whose asymptotes are parallel to $O\gamma$ & $O\delta$.

(2) Let the dples be imaginary. Then the conicoid is an elliptic cylinder thro' BC & A.

Again let the ple BCD be the ple at infinity. The conicoid which corresponds to a ple is in this case a parabolic cylinder since it meets the ple at infinity in 2 coincident generators. If the dples are real they are parallel.

Let a section thro' A meet them in the parallel lines γ & δ



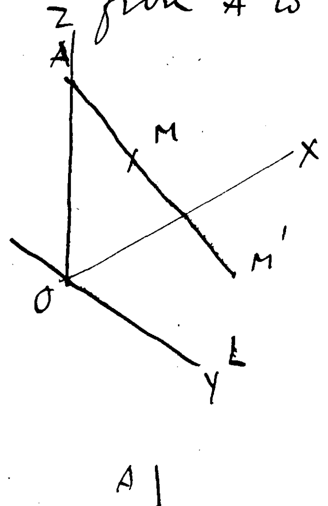
Then if the ple meets the section in the line l we get if P is any pt on l $(P P' p q)$ a harmonic system where P' corresponds to P.

The locus of P' is a parabola, as is easily proved by elementary geometry.

Geometrical construction for the cylindrical transf.

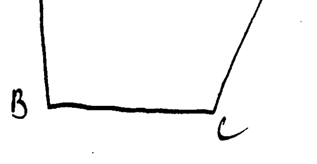
Let A be a fixed pt & L a fixed line (not thro' A). Let M be any pt. Construct the ple thro' LMα perp^r to ple LM & let it meet the line AM in M'. Then the pts M & M' correspond to each other & the transf. is a quadratic cylindrical transf of the above type. The F-system consists of A; L, & the ple thro' L perp^r to AL.

To a line corresponds a conic thro' A intersecting L & touching F-ple thro' L. To a ple corresponds a cylinder (elliptic or hyperbolic) thro' A & L & touching the F-ple. The double ples are the <u>isotropic</u> ples thro' L. Corresponding to the ple at infinity we get a <u>right circular cylinder</u> whose diameter is the perp^r from A to L.



Other quadratic cono-cylind. transf. Other varieties of this transf. can be got from three non-coplanar sheaves.

Thus if we take the lines in (S) AB, BC CD which are such that AB & CD do not intersect & three similar lines in (S') & take homog. sheaves



thro' $(AB)(A'B')$; (BC) $(B'C')$; $(CD)(C'D')$ we get if ple ABC corr for both sheaves $(AB)(BC)$ corresp. to ple $A'B'C'$ for both sheaves $(A'B')(B'C')$ a transf. in all respects similar to the above transformations.

If CD & $C'D'$ do not intersect any of the other lines the nature of the transf. is not altered; the lines CD & $C'D'$ take the place of BC $B'C'$ in the above transf. ie they are F-lines.

— ı —

General observations on the above transf.

Since two ples intersect in a line it follows that the conicoids into which they transform must intersect in a conic. Hence all such conicoids must pass thro' a fixed conic.
Thus the F-line must be considered as a conic ie as a line-pair.
We shall find later that this transf. is a particular case of a more general quadratic transf. in which the F-line is a proper conic or two distinct lines.
Such a transf. cannot however be built up by means of 3 sheaves.

— ı —

Analytical formulae of transf.
Let ples BAC, BAD, CAD, BCD be $x\ y\ z\ w$ resp^y
& " $B'A'C'$ etc – $x'y'z'w'$ "

Then the formulae are

$$xy' = by\,x' \qquad xz' = cz\,x'$$

$$\& \ xx' = a\,ww'$$

This gives $x : y : z : w = x'w' : \frac{y'w'}{b} : \frac{z'w'}{c} : \frac{x'^2}{a}$,
taking the case of the persp. transformation $b = c = 1$

Thus to the ple $\alpha x+\beta y+\gamma z+\delta w=0$ corresp.

$$\alpha x'w'+\beta y'w'+\gamma z'w'+\delta x'^2=0 \quad \text{(omitting accents)}$$

The line-pair is $x'^2=0 \quad w'=0$

In Cartesians when BCD is the ple at infinity

$$x : y : z = \frac{x'}{x'^2} : \frac{y'}{x'^2} : \frac{z'}{x'^2} , 1$$ & to ple

$\alpha x+\beta y+\gamma z=\delta$ corresp. $\alpha x'+\beta y'+\gamma z'=\delta x'^2$ a parabolic cylinder.

In the special cylindric transf. Let the perp AO from A to L be the axis of Z OY the axis of Y & a perp to OY in the F-ple the axis of X.

Then the formulae are

$$x' = \frac{axz}{az-(x^2+z^2)} \qquad y' = \frac{ayz}{az-(x^2+z^2)}$$

$$z' = -\frac{ax^2}{az-(x^2+z^2)} \qquad \text{where } a = AO.$$

corresp. to ple at infinity we get $az-(x^2+z^2)=0$ which is a <u>rt-circular</u> cylinder with AO as diam.

art, 13. The trinomial cubic transf.

Let the spaces (S) & (S') as before be referred to the tetrahedra ABCD & A'B'C'D'.

A pt M of (S) determined by three ples BMC, CMD, & DMB will have for its corresp. the pt. M' determined by ples B'M'C', C'M'D' & D'M'B' provided that between the pairs of sheaves (BC) (B'C'); (CD)(C'D'); (DB)(D'B') there exist homographic relations.

Let us make the following suppositions.

(1) In sheaf (BC) let ple BAC corresp. to B'A'C'
--- ---- BDC .... B'D'C'.

(2) In sheaf (CD) ... CAD " " C'B'D'

in sheaf CD – ple CBD corresp. to ple C'A'D'
,, DB ,, DAB – – C'B'D'
,, ,, DCB ,, ,, D'A'B'.

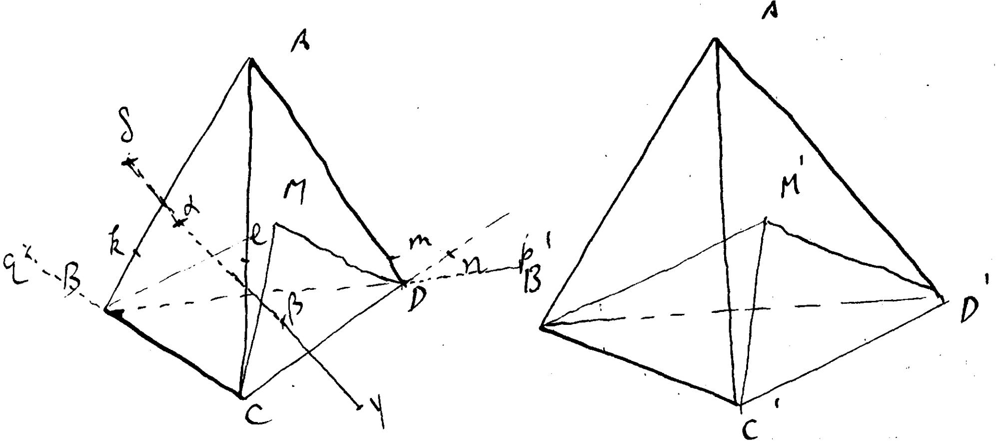
Let now M describe a line l ; M' describes a cubic curve for C'M' M'D' describe cones having a common gen.r hence M' their inters. describes a cubic curve.

The transf. is reversible & hence when M' describes a line M describes a cubic curve.

Hence when M describes a ple M' describes a cubic surface & conversely.

Exceptional elements.

To any pt M of (S) there corresponds in general one pt M' of (S'). But there are pts which have no unique correspondents.



Corresponding to any pt P on BC we get the line A'D'.
,, ,, ,, CD ...... pt D'
,, ,, ,, BD ,, ,, ,, D'.
,, ,, ,, AD ,, ,, line B'C'.
,, ,, ,, on ple BCD ,, ,, pt D'.
,, to pt B we get ple C'A'D'
,, ,, C ,, ,, B'A'D'
,, ,, D ,, ,, B'C'D'

The lines BC, CD, DB & DA are F-lines & the pts BCD F-pts of (S) & similarly for (S').

also ples BAD, CAD & BCD are F-ples.

Nature of the cubic curves & surfaces of (S') which correspond to lines & ples of (S).

Let a line l of (S) meet the faces of tet^{n} ABCD in pts $\alpha\beta\gamma\delta$ as in fig. α being on BAC, β on CAD γ on BCD & δ on BAD.

Corresp. to α we get pt α' on B'A'C'. Corresp. to β we get pt B' & to δ pt C'; to γ corresp. pt D'.

Thus all cubic curves pass thro' B'C' & D'. They intersect face B'A'C' in one pt not on the triangle A'B'C'. They also intersect the line A'D' in one pt not at D'.

Let a ple π meet the edges of tet^{n} ABCD in pts $k\ l\ m\ n\ p\ q$.

Corresp. to k & l we get C' & B'. Corresp. to n p we get D'. Corresp to m we get the line B'C'. Corresp. to q we get line A'D'.

Thus all cubic surfaces pass thro' B'C' & D' & contain the lines B'C' & D'A'.

Let M & M' be corresp-pts on ple & surface. Since BM, CM & DM meet the ple in 1 pt only B'M' C'M' & D'M' can meet surface in 1 pt only. Hence B'C' & D' are conical pts.

Let P be any pt on line km on face ABD. P transforms into C' & CP into a line touching surface at C'. As P traverses km lines thro' C' trace out a quadric cone which is the tangent cone at C'.

The tangent cones at C' & B' are proper tangent cones.

The conical pt D' is of an exceptional nature.

Let a ple thro' B'D' meet A'D'C' in a line r' thro D'.

Corresp. to r' we get the line BD. now BD meets ple in one pt p whose correspondent is D' hence line r' cannot meet cubic surface in any pt except D'.

Hence ple C'A'D' forms part of the tangent cone at D'. Similarly B'A'D' forms part of it. It is therefore two ples.. Each of these ples must meet the cubic surface in ple curves. Now ple C'A'D' meets surface in the two lines A'D' C'D' hence we must consider C'D' as reckoned twice. Similarly B'D' must count as two lines on surface. The lines joining conical pts must lie wholly in the surface.

Hence all cubic surfaces have 3 conical points at B' C' D' resp^{y} & contain six lines viz B'C' (C'D' twice) D'A' & (B'D' twice).

The latter system of lines is the F-system of lines of (S') & is of degree 6.

Since two ples of (S) intersect in a line it follows that two cubic surfaces of (S') must intersect in a cubic curve. This is the case for since their complete intersection is of deg. 9 the fixed F-system of lines which forms part of curve of intersection is of degree 6 & hence the remaining part is of deg 3.

Effect of transf. on curves & surfaces.

A curve of deg. n transf. into a curve of deg 3n
& a surface of deg n " " surface " 3n.

If the curve or surface contain F-elements the degree of its correspondent is lowered.

General formulae for the degrees will be given in connection with the next transf. (the tetraconical transf) of which the present transf. is a particular case.

A conic transforms into a sextic having nodes at ~~&~~ B'C'&D'. The sextic intersects the ple B'A'C' in 2 pts not on tri. B'A'C' & meets A'D' in 2 pts not at D'. Similar results hold for curves of higher order.

A conicoid transforms into a sextic having quadruple conical pts at B'C'D' & having the F-lines as double lines. The tangent cones at B' & C' are proper cones of the 4th order having 3 double sides. As in the case of the cubic surface the tangent cone at D' will be exceptional. The ples C'A'D' & B'A'D' will as before form part of it & the rest will be an ordinary quadric cone thro' C'D' & B'D'.

Similar results hold for surfaces of higher order.

Superposition of Spaces.

When the spaces are superposed there will be double elements. These are 8 in number.

Corr. ples thro' ~~A~~BD, ~~A'~~B'D', thro' CD, C'D'; & thro' BC, B'C', trace out hyds. by their intersections These hyds. intersect in <u>8</u> pts.

When the tetrahedra are superposed so that ABCD coincide with A'B'C'D' respy. 4 of these points will coincide with D' & 4 will lie in the ple B'A'C'.

It is interesting to note that a figure in the ple BAC transforms into a figure in the same ple of double the degree. In fact the transf. regarded as confined to ple BAC is a ple quadratic transf. of which BAC are the three F-pts. A line in this ple <u>transforms</u> into a conic thro' BAC & a conic into a trinodal quartic.

Analytical formulae.

Let ples BAC, BAD, CAD, BCD be $x \; y \; z \; w$ respy.

& -- B'A'C', B'A'D' etc — — $x' \; y' \; z' \; w'$ "

Then with the suppositions at first of p 53 we get

$$x : y : z : w = a x' : \frac{b w'^2}{y'} : \frac{c w'^2}{z'} : d w'$$

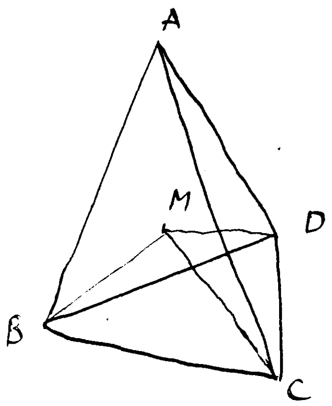
where a b c d are constants

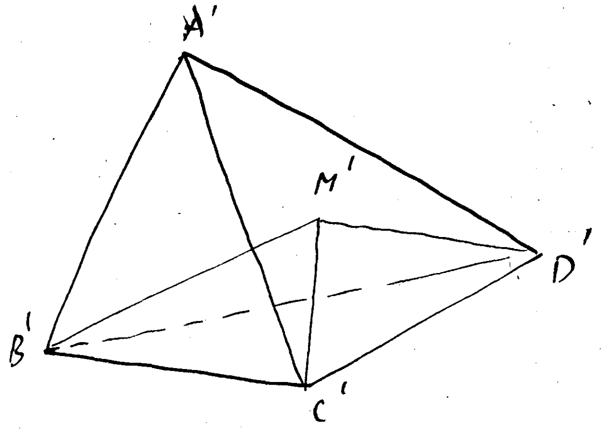
Conversely $x' : y' : z' : w' = \alpha x : \frac{\beta w^2}{y} : \frac{\gamma w^2}{z} : \delta w$

Corresp. to ple $lx+my+nz+rw=0$ we get
$la x'y'z'+mbw'^2z'+ncw'^2y'+ \cancel{x} rd w'y'z'=0$
From these formulae we can verify all the above conclusions.

Art. 14 The tetraconical transf (cubic).

This is the most general transf. which can be built up by means of 3 sheaves of ples whose bases are coplanar.





Let BC, CD, DB be the coplanar bases of (S) & B'C', C'D' D'B' the corresp. coplanar bases of (S').
Let ple BAC sheaf (BC) corresp. to ple (B'C'D') sheaf (B'C')
" BDC " " " (B'A'C') " "

with similar suppositions for the other sheaves.
Thus to ple BCD corresp. ples B'A'C', C'A'D', D'A'B' according as it is considered to belong to sheaf (BC) (CD) or (DB).

As before if 3 ples BCM, CMD, DMB meet in a pt M their correspondents B'M'C', C'M'D' & D'M'B' meet in the pt M' which is the correspondent of M.
Thus we establish a (1-1) pt-pt-correspondence between the spaces.

When M describes a line as in the previous transf. M' will describe a cubic curve & conversely.
When M describes a ~~line~~ ple M' will describe a

cubic surface. Thus the transf. is of degree 3 & is reversible.

Exceptional elements.

(1) Points. To pt A corresp. ple B'C'D', to B ple A'C'D', to C ple A'B'D' & to D ple A'B'C'. Thus A B C D are F-pts. Similarly A'B'C'D' are F-pts.

(2) Lines. Corresp. to pt P on A B we get line C'D'; corresp. to pt on AC we get line B'D' & so on. To a pt on an edge of tet^n ABCD corresponds an edge of tet^n A'B'C'D'. Hence the six edges of ABCD & the six edges of A'B'C'D' are F-lines.

(3) Planes. Clearly the four faces of ABCD & the four faces of A'B'C'D' are F-ples.

Nature of the cubic curves & surfaces.

Let a line l meet faces of ABCD in $\alpha\ \beta\ \gamma\ \delta$. Corresp. to these pts we get pts A'B'C'D' hence all cubic curves of (S') pass thro' A'B'C' & D'.
Since l is determined by 2 pts a cubic curve is determined by 2 pts. Thus 6 pts determine a cubic curve of (S') completely four of these being A'B'C'D'. Since such curves are the partial intersections of quadric cones we shall call them conicubics.

Let a ple π meet the edges of tet^n ABCD in $p\,q\,r\,l\,m\,n$ then corr. to each of these pts we get an edge of A'B'C'D' & corresponding to each of the 4 lines of intersection with the faces we get the 4 pts A'B'C'D'.
Thus the cubic surfaces contain the six edges of A'B'C'D' & have conical pts of second order at A'B'C'D'.
We shall call these surfaces conicubicoids.
Conicubics & conicubicoids are in many respects the simplest cubic curves & surfaces & have many interesting

properties which will be given later.

Effect of the transf. on curves & surfaces.

A curve of deg. m in general transforms into a curve of deg. $3m$.

A surface of deg n .. .. .. surface of deg. $3n$.

If a curve or surface contain F-elements its deg. will be lowered on transf.

Let a curve of the m^{th} degree in (S) have multiple pts at $ABCD$ of the i^{th}, j^{th}, k^{th} & l^{th} order resp. & let it meet the opposite faces in p q r & s pts respy.

It transforms into a curve of (S') having multiple pts. at $A'B'C'D'$ of the p^{th} q^{th} r^{th} & s^{th} order. & cutting the opp. faces in i, j, k & l pts respy.

Let degree of this curve be m'.

Then $m = p+j+k+l = q+i+k+l = r+i+j+l$
$= s+i+j+k$

& $m' = i+q+r+s = j+p+r+s = k+p+q+s$
$= l+p+q+r$

Hence $m' = i+m-(i+k+l)+m-(i+j+l)$
$+m-(i+j+k)$
$= 3m-2(i+j+k+l)$

Similarly $m = 3m'-2(p+q+r+s)$. (A)

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Let a surface of deg. n pass thro' A B C D & have conical pts of orders δ μ ν ρ at these pts respy. This surface meets BCD in a curve of n^{th} degree having sing. pts of order μ ν ρ at B C & D respy. The cone got by joining A to all pts of this curve transforms into the tangent cone of corresp. surface at A' & similarly for the other pts B C & D.

The cone thro' A transforms into a cone of order $2n-(\mu+\nu+\rho)$ or $2n+d-(d+\mu+\nu+\rho)$ thro' A'.

Since a line thro' A meets ~~curve~~ surface of order n in n−d pts distinct from A, the corresp. line thro' A' must meet corresp. surface in n−d pts. Hence the degree of the latter surface is

$$n' = 2n+d-(d+\mu+\nu+\rho)+n-d$$
$$= 3n-(d+\mu+\nu+\rho)$$

This result is symmetrical in $d\ \mu\ \nu\ \rho$ as it ought to be.

We have here assumed that surface of deg. n does not contain any of the F-lines of (S).

Let it contain the lines BC, CD, DB, BA, CA, DA $q\ p\ r\ s\ t\ u$ times respectively.

Curve of intersection with BCD is now of degree $n-(p+q+r)$. This curve passes thro' B C D $\mu-(q+r)$, $\nu-(p+q)$, $\rho-(p+r)$ times respectively.

Hence the cone vertex A & base this curve transforms into a cone of order

$$2n-2(p+q+r)-\{\mu-(q+r)+\nu-(p+q)+\rho-(p+r)\}$$
$$= 2n-2(\cancel{p+q+r})-(\mu+\nu+\rho)+2(\cancel{p+q+r})$$
$$= 2n-(\mu+\nu+\rho) \text{ as above}$$

Hence as before $n' = 3n-(d+\mu+\nu+\rho)$ (B).

& similarly $n = 3n'-(d'+\mu'+\nu'+\rho')$

By the aid of formulae (A) & (B) we can calculate the degrees of the correspondents of any curves & surfaces.

A conic not intersecting the F-system transforms into a sextic curve having nodes of second order at A'B'C'&D'.

A conicoid which does not contain any part of the F-system transforms into a sextic surface having quadruple conical pts at A'B'C'D' & having the 6 F-lines as double lines.

Similar results hold for curves & surfaces of higher order.

Superposition of the Spaces.

Exactly as in last transf. the number of dps will be 8.

Spaces referred to coincident tetrahedra

Let the pts ABCD be made to coincide with A'B'C'D' respectively & let the common tet^{n} be called ABCD.

The sheaves thro' BC will have two double ples & similarly for those thro' CD & DB.

The 8 dps are the intersections of these 3 pairs of ples.

Since ple BCD corresponds to plane BAC whether it belong to either of the sheaves (BC) the system of ples thro' BC is involutive. Similarly the other two systems are involutive.

Consequently the transformation is now involutive.

To a pt M corresponds a single pt M' whether M belong to (S) or (S').

The ples BCM & BCM' are harmonic conjugates with the dples thro' BC & similarly for the other systems.

Hence if the line MM' meet the dples thro' BC in α & β (MM' $\alpha\beta$) is harmonic).

Position of the dps.

Since the 8 dps are the intersections of 3 conicoids (proper or degenerate) they cannot all be chosen independently. 7 may be chosen arbitrarily & when this is done the 8th is fixed.

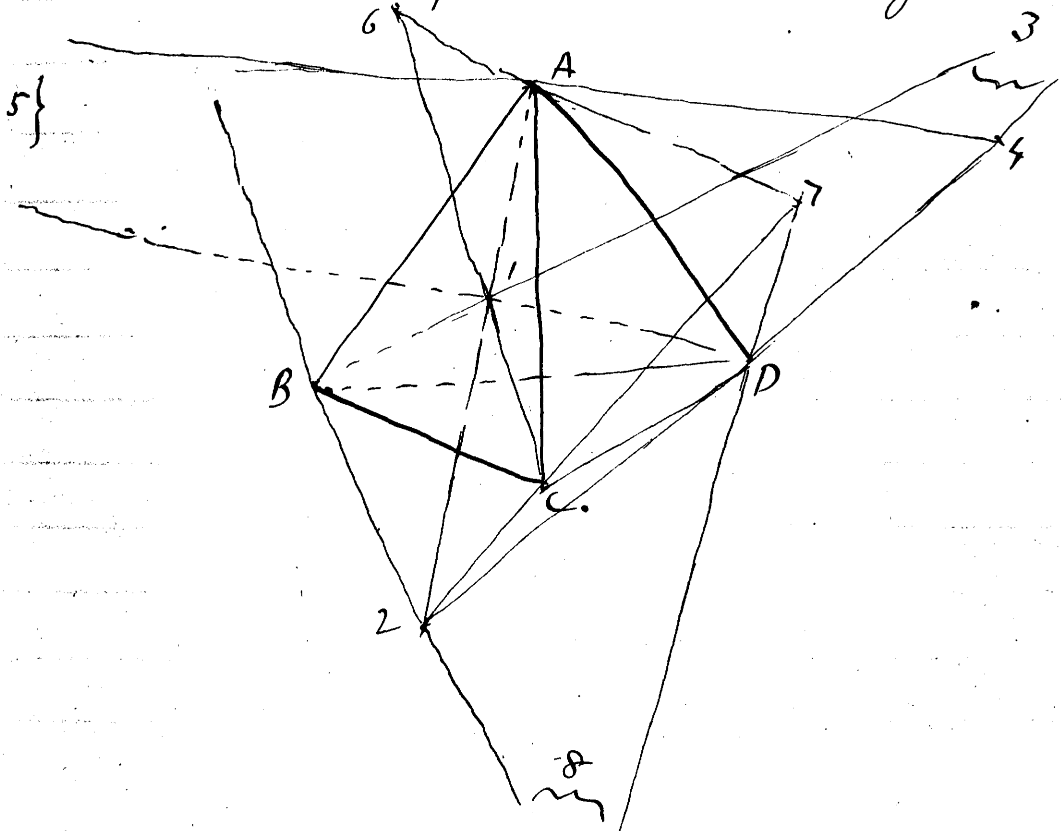
Let 1, 2, 3, 4, 5, 6, 7, 8 be the dps.

Let A1 meet BCD in O & ple BC2 in 2'.

Then 2' is the harm. conj. of 1 with respect to A & O.

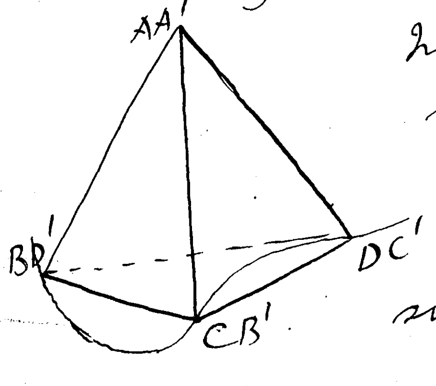
Similarly if A1 meet CD2 in 2'' 2'' is harm conj.

of 1 with respect to A & O. Hence $2''$ coincides with $2''$ & both with 2. Hence A1 passes thro' 2.
Similarly A6 passes thro' 7, A8 thro' 3 & A5 thro' 4.
Similar results hold for the other vertices B C & D.
Hence the pts are collinear in pairs with the F-pts.



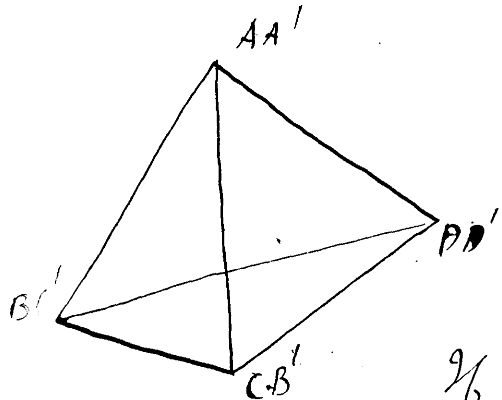
Any two corresp. pts are harmonic conjugates with respect to pair of deg. conicoids. They will also be harmonic conjugates with respect to any conicoid thro' 8 pts. ABCD is a self-conj. tet^n with respect to any conicoid thro' the 8 pts.

We may also make the tet^{ns} coincide as in fig.



In this case there is a curve of pts viz a cubic curve thro' B C & D.
If P & P' be corresponding pts the line PP' meets this curve in α & β such that (PP'αβ) is harmonic.

If the tet^{ns} be made to coincide as in fig.



There will be two cones of dps both passing thro' B & C. There are no other special cases.

Special involutive cubic transf. If the double ples are the internal & external bisectors of the dihedral base angles of ABCD then corresp. ples are symmetrical with respect to the internal bisectors. This special transf. is the tetrahedral inversion.

Art 15. The hyperboloidal cubic transf.

In this transf. the bases of the sheaves are non-coplanar & do not intersect. It is the most general transf. that can be built up with 3 sheaves.

Let P Q R & P'Q'R' be the bases of the sheaves in (S) & (S') respectively.
To a pt M determined by the ples (PM) (QM) (RM) corresponds the pt determined M' determined by the corresp. ples (P'M')(Q'M')(R'M'). Thus the transf. is (1-1) pt-pt transf.
When M describes a line M' describes a cubic curve which is the intersection (partial) of two hyperboloids. When M describes a ple M' describes a cubic surface. Thus the transf. is of the third order.
Exceptional elements.

Corresp. to any pt α on P we get the line of intersection of the two ples thro' Q' & R' which corresp to Qα & Rα. Hence the line P is an F-line.

Similarly Q & R are F-lines. Similar conclusions hold for P' Q' & R'.

Corresp. to P we get a conicoid thro' Q' & R' & so on.

Let β be a pt such that 3 ples corresp. to Pβ, Qβ, Rβ intersect in a common line l' which meets P' Q' & R'. As l' traces out the hyp. thro' P' Q' & R' β will trace out a cubic curve which is an F-curve. Let it be called Σ_3.

There is a corresp. F-curve in (S') Σ_3'.

Thus the F-system of lines in (S) consists of P, Q, R & Σ_3 a system of 6th degree. & similarly for (S').

Σ_3 intersects P Q R each in 2 pts.

The F-system of surfaces consists of three hyps. thro' PQ, QR & RP respectively & the hyp. (PQR).

Nature of the cubic curves & surfaces of (S').

We shall call these hypercubics & hypercubicoids resp^{ly}.

Let a line n meet the F-surfaces of (S) in the six pts ~~$\alpha\beta\gamma$~~ $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$. $\alpha\alpha'$ being on hyp. QR, $\beta\beta'$ on RP etc. Then corresp. to gens. of system PQR thro' $\alpha\alpha'$ $\beta\beta'$ & $\gamma\gamma'$ we get two pts on each of P' Q' & R'. Thus all cubic curves intersect P' Q' & R' in 2 pts each. This was evident from the mode of generation of such curves.

Let a ple π meet the F-system of surfaces in conics S_1 S_2 S_3 S_4. Then corresp. to these we get the lines P' Q' R' & the cubic Σ_3'.

Hence all cubic surfaces contain P' Q' R' & Σ_3'.

Effect of transf. on curves & surfaces.

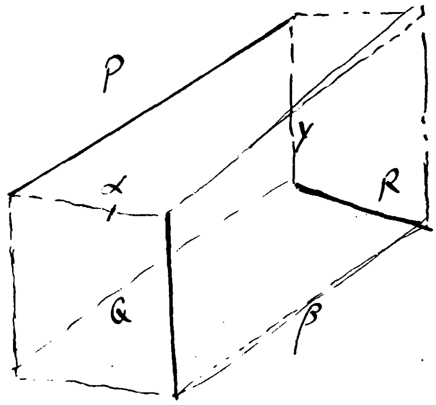
A curve of order n transforms in general into a curve of order 3n & a surface of order n into a surface of order 3n.

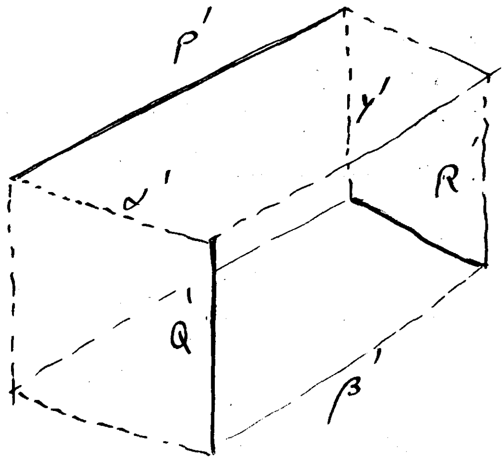
Calculation of degrees when the curve or surface contains F-elements.

Let the curve of deg m intersect $P\,Q\,R\,\Sigma_3$ in $p\;q\;r\;s$ pts respectively. Then clearly the degree of its corresp. is $3m-(p+q+r+s)$
Let a surface of deg. n contain $P\,Q\,R\,\Sigma_3$ $\lambda\;\mu\;\nu\;\rho$ times respectively then the degree of its corresp. is $3n-2(\lambda+\mu+\nu+\rho)$.

Superposition of ~~surface~~ spaces.

As before there will be 8 dps.





Particular transf.

Suppose that ples are drawn thro' P // Q & R & let ples be drawn Q // R & P & ples thro' R // P & Q. Such ples form a parallelepiped. Similar ples in (S') form a parallelepiped also. If we make corresponding ples thro' P symmetrical with respect to the bisector of the dihedral angle at P etc with similar suppositions for ples thro' Q & R, P'Q' & R' we get an involutive transformation. Such a transf. may be termed parallelepipedal inversion. The dps are the intersections of the ples bisecting the dihedral angles when the parallelepipeds are superposed.

All hyperbolic cubicoids pass thro' 3 fixed pts at infinity.

For to all pts on α corresponds a pt at infinity in (S') since ples $P\alpha$ $Q\alpha$ transform into ples $P'\gamma'$. $Q'\beta'$. Similarly for pts on β & γ. Since any ple meets α β γ in 1 pt each every hyperbolocubicoid passes thro' 3 fixed pts at infinity.

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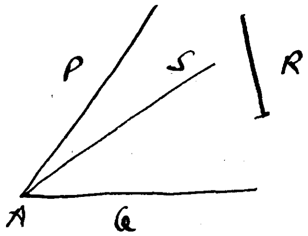
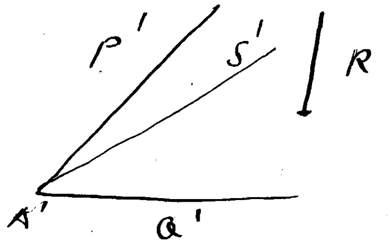
Art 16. Particular cases in which the bases of the sheaves have one or two pts ~~in common~~ of intersection.

(a) Let P & Q intersect & also P' & Q'.





We may make three distinct suppositions.

(1) We may suppose that ple PQ belonging to sheaves P & Q corresponds to ple $P'Q'$ " " P' & Q'. This case has been considered already. The transf. to which it gives rise is a quadratic trans. with R & R' as F-lines. (Cono-cyl. transf).

(2) Ple PQ belonging to sheaf P corresp. to ple $P'Q'$ belonging to sheaf P' but ple PQ sheaf Q does not corresp ~~to~~ ple $P'Q'$ sheaf Q'.

(3) Ple PQ does not corresp. to ple $P'Q'$ in either sheaf P, Q.

We shall treat (3) first.

The trans. is a cubic one.

The F-system.

Corr. to pts on P we get lines on a ple $Q'S'$ which corresp. to QP (sheaf Q). Corr. to pts on Q we get ~~pts~~ lines on ple $P'S'$. Thus to P corresponds a ple & to Q a ple.

Let QS & PS be the ples corresp. to ple $P'Q'$ S being their line of intersection. Corr to S we get the ple P'Q'. ~~& corr~~. The lines which meet P Q & R can be divided into 2 classes. (1) Those lying in ple PQ which pass thro' the pt of intersn. of R with that ple. Corr. to these lines we get the line S'.

(2) The lines joining A the pt of intersection of P & Q to pts on R. Corr. to these we get a conic lying in the ple thro' R' which corresp. to ple RA. Thus the complete F-systems of lines are P Q R S & conic just mentioned with similar systems in (S'). The F systems of ples are the ples PQ, PS, QS & the ple thro' R corr. to ple R'A' & a cone containing the lines P Q S with similar systems in (S').

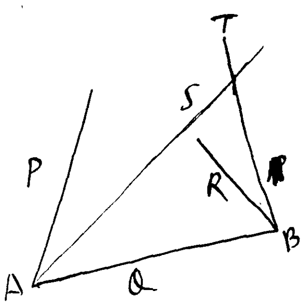
Thus the F-lines form again a system of the 6th order being composed of four lines & a conic.

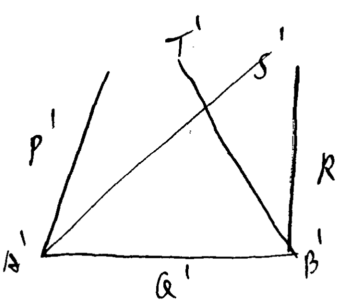
All cubic curves pass thro' A & A'.
All cubic surfaces have A & A' for conical pt.

(2) is a particular case of this transf. in which S coincides with Q & S' with Q'. These are double lines on their respective systems of surfaces.

— 1 —

(6) Let PQ intersect in A & QR in B with similar suppositions for (S').





We can make the following suppositions.

(1) ples PQ P'Q' corresp. for both systems of sheaves

PQ & P'Q'.

Ple QR corresp. to ple Q'R' under same conditions.
This case has been already given on pp 34, 35.
It gives rise to a collinear transf.

(2) Ple PQ corresp. to ple P'Q' for both pairs of sheaves but ple QR does not corresp. to ple Q'R' for both.
This case gives rise to a quadratic (cono-cyl.) transf

(3) Ple PQ corr. to ple P'Q' for sheaf P but not for Q. & so on.

(4) Ple PQ does not corr. to ple P'Q' for either sheaf. & neither does ple QR corr. to ple Q'R' for either sheaf.

We shall treat (4).

Let P'S' corr. to ple PQ sheaf P
" Q'S' " " " " Q
Let Q'T' " " QR " Q
" R'T' " " QR " R.

with similar suppositions for (S).

Then the F-system of lines is in (S) PQRST & the line of intersn. of ples PS & ST a line which corr. to Q'. There is a similar system in (S').
Thus the F-systems consist of 6 lines each.
The cubic curves will pass thro' AB & A'B' in their respective spaces & the cubic surfaces will have these points as conical pts. These are F-pts.
In (3) S or T may coincide with Q & so on.

— 1 —

No other transformations built up of 3 sheaves are possible.

We may now observe that the F-system in the case of the cubic transf. may consist either
(1) of 3 lines & a cubic curve.
(2) " 4 " conic
(3) " 6 lines which do not form a tet.

(4) of 6 lines which form a tet^n:

Later we shall treat the following cubic transf.

(5) The F-system is made up of two lines & a quartic.
(6) " " " conic "
(7) " " a line & a quintic
(8) " " sextic.
& also (9) " " three concurrent lines & a plane cubic.

The transf. (8) in which the F-curve is a proper sextic is the most general cubic transf. & all the others are special cases of it.

Art 17. The lines on the various cubic surfaces already obtained.

Any plane meets the F-system of (5) in 6 pts. Let these be $\alpha\beta\gamma\delta\epsilon\zeta$.

Corresponding to each of these points we get a line on the corresp. cubic surface in (5').

Corresp. to the sides of the hexagon $\alpha\beta\gamma\delta\epsilon\zeta$ & its diagonals we get lines also. For to $\alpha\beta$ corresponds a cubic curve made up partly of two lines belonging to the F-system of surfaces. Hence the remaining proper part of the cubic is a line.

The sides & diags. number 15 & hence give rise to 15 lines.

A conic thro' 5 of the pts $\alpha\beta\gamma\delta\epsilon\zeta$ transforms into a sextic containing 5 lines belonging to the F-system. Hence such a conic transforms also into a line.

There are 6 such conics & hence 6 lines.

Thus the total number of lines possible is $6+15+6$ i.e 27.

If the number of lines on the surface is 27 then it is clear

that (1) no three of the pts ~~can~~ $\alpha\beta\gamma\delta\epsilon\zeta$ can be collinear
(2) that the six pts " cannot lie on a conic.

(a) If three of the points lie on a line then the figure formed is a pentagon & the number of sides is diminished by 1 the number of diags. by 2 & the number of conics by 3. Thus the lines on the cubic surface are diminished by 6 & now number 21.
Such a cubic surface has a conical pt; for let the three coll. pts be $\alpha\beta\gamma$. Corresp. to the line $\alpha\beta\gamma$ we get a pt on the F-line system where the lines corr. to pts α β & γ meet. This pt is a conical pt.

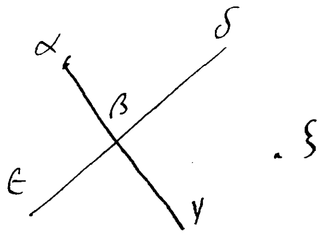
(b) If the 6 pts lie on a conic then clearly to this conic corresp. a pt & the six lines corr. to ~~&~~ $\alpha\beta\gamma\delta\epsilon\zeta$ meet in this pt which is also a conical pt.

Case (b) does not arise in any of the cubic trans. just given. It arises in a special cubic transf. which will be given later.

The hyperboloidal transf.
In this transf. the six pts $\alpha\beta\gamma\delta\epsilon\zeta$ may be generally placed & the cubic surface is a general cubic surface possessing 27 lines.
But the ple may cut PQR in 3 coll. pts i.e in a generator of the hyp. PQR. In this case the corr. cubic surface the hyperbocubicoid has a conical pt on its F-line Σ_3' & it possesses 21 lines. The curve Σ_3' is the locus of conical pts for surfaces of the system. There is no fixed conical pt for all surfaces.

Case (a) p. 69. In this case 3 points are always collinear viz the 3 which lie on the F-plane thro' R which contains the F-conic. In this case the surface has a conical pt which is fixed viz A'. The surface also possesses 21 lines. This transf. may be termed the monoconical cubic transf.

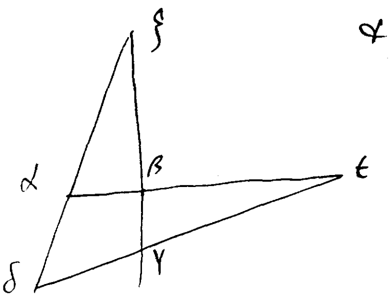
Case (b) p. 70. In this case two triads of pts are collinear for the sixth F-line meets P S T & R. There are two fixed conical pts viz A' & B' & the number of lines is 16. For let $\alpha\,\beta\,\gamma\,\delta\,\epsilon\,\xi$ be the pts on the plane in (5). $\alpha\beta\gamma$ are coll. & $\epsilon\beta\delta$.



There are 6 ~~pts~~ lines corr. to $\alpha\,\beta\,\gamma\,\delta\,\epsilon\,\xi$ resp; 5 lines corr. to the lines $\xi\gamma$, $\xi\epsilon$, $\xi\beta$, $\xi\alpha$ & $\xi\delta$. Four lines corr. to $\alpha\delta$, $\delta\gamma$, $\gamma\epsilon$ & $\epsilon\delta$. & a single line corr. to the conic $\alpha\delta\xi\gamma\epsilon$, in all 16. This transf. may be called the diconical cubic transf.

The tetraconical transf.

Here the six pts are the vertices of a complete quadrilateral. There are 4 triads of collinear pts & 4 fixed conical pts viz A'B'C'D'.



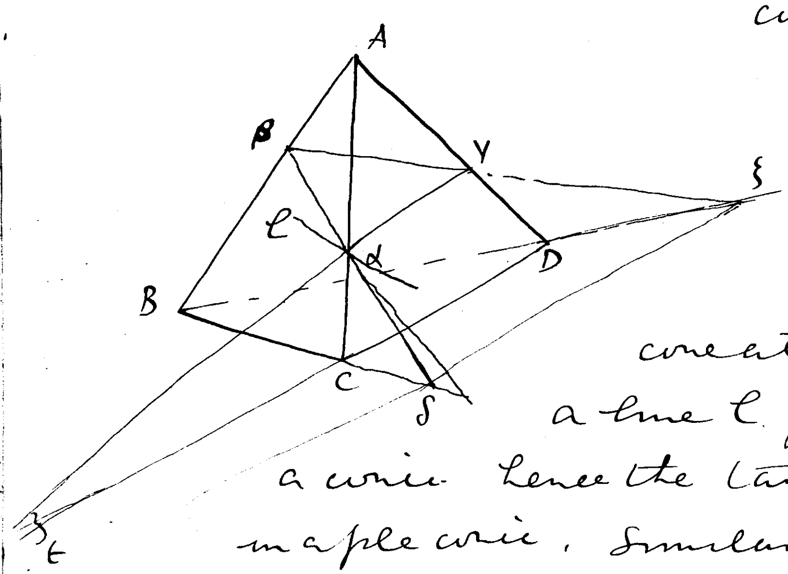
There are 9 lines viz the six corr. to $\alpha\beta\gamma\delta\epsilon\xi$ & the three corr. to $\alpha\gamma$, $\beta\delta$ & $\xi\epsilon$.

It may be mentioned here in passing that the Triconical cubic transf. Art. 13 is a particular tetraconical transf. in which 2 F-pts coincide viz A' & D'.

Art. 18. Some properties of cubic & higher surfaces obtained by the cubic transf.

The conicubicoid.

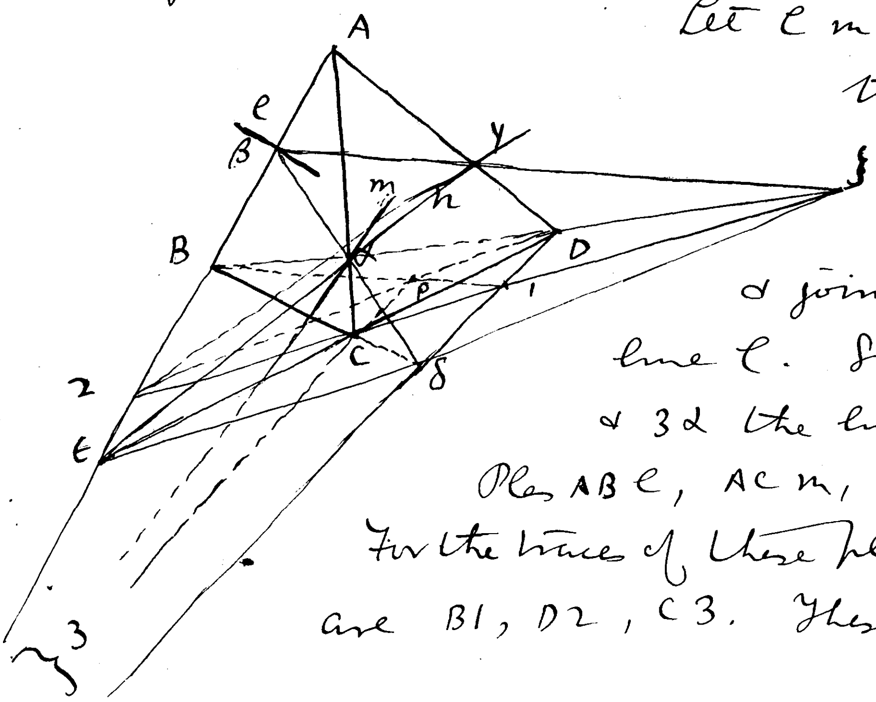
Let a ple meet the edges of the F-tet^n in (S) in the pts αβγδεξ. Corresp. to ple Dβαδ we get a cone in (S') namely the tangent cone at D' since all pts on the F-ple ABC corr. to D'.



Similarly corresp. to ple βγαε we get the tangent cone at B'. These ples intersect in a line l passing thro' α. To l corresponds a conic. Hence the tangent cones at B' & D' intersect in a ple conic. Similarly for the tangent cones at any other pair of vertices in (S'). Since l passes thro' a pt α on AC the ple of this common conic contains A'C'. Hence the tangent cones at any pair of vertices intersect in a ple conic thro' the opposite edge of the tet^n.

The ple thro' AC & l transforms into a common tangent ple to cones thro' B' & D'. Thus each pair of cones has a common tangt. ple.

The six ples of the common conics intersect in a point.

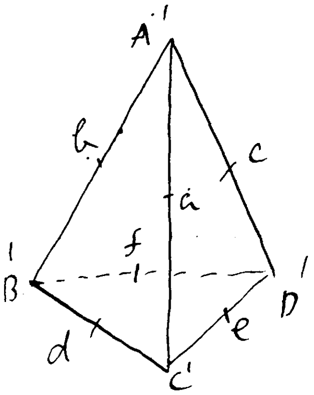


Let l m & n be the three lines thro' AB, AC, & AD resp^{ly}. l is got by joining Cξ & Dδ which intersect in 1 & joining pt 1 to β. 1β is the line l. Similarly 2γ is the line n & 3α the line m.

Ples ABl, ACm, ADn intersect in a line. For the traces of these ples on the face BCD are B1, D2, C3. These lines intersect in a pt

P. For $(D 1 \delta 3) = -1$ & $(3 \epsilon 2 B) = 1$
$\therefore B(D 1 \delta 3) = D(3 \epsilon 2 B) = -1$
& these pencils have common ray BD
hence B1, D2 intersect on line C3 & this pt is P.
Hence ples ABC ACn ADu intersect in a line AP.
Similarly the ples BCs, ACn, CDt intersect in a line which meets AP & ples BCS, BAC, BDu intersect in a line which meets AP & the line of intersection of ples BCS, BAC, BDu hence the 3 lines of intersection of the three sets of ples ~~thro'~~ which pass thro' A B & C intersect in a pt.
The corr. line thro' D must also pass thro' this pt hence the six ples ABC, ACn, ADn, BCS, ACm, CDt BDu intersect in a point. Hence their corresp. ples in (S') which are the ples of the common conics must intersect in a single point.

The four tangent cones at A'B'C'D' touch a conicoid which touches the six edges of the F-let^n in the pts where ples of the above conics meet the opposite edges.



Let a b c be the pts in which the common conics thro' B'D', D'C', C'B' meet the opp. edges. Then abc is the ple of contact of tangt. cone at A' with this conicoid. Let d e f be the other pts on B'C' C'D' & D'B' then from the last theorem it follows that eb, dc, & fa are concurrent. Hence a conicoid can be found to touch the six edges of A'B'C'D' at abcde & f. This conicoid is clearly touched by the four cones thro' A' B' C' & D' provided it is touched by 3 additional gen^s. of these cones. These conditions amount to 9 & are sufficient to determine the conicoid.

The characteristic eqn. for the cubic surfaces which can be built up by means of 3 sheaves.

Consider a pl. in (S) & let pls thro' BC & CD meet it in a pt P then ple DBP is determined. In space (S') when two ples of different sheaves are given the third is completely determined.
Let l be any line & let distances be measured from an arbitrary pt O in it. Then when the 3 ples of the sheaves in (S') describe a cubic surface their distances of their intersections with l from the pt O are related by an eqn of the form

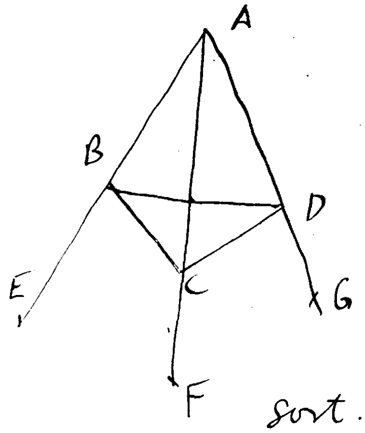
$$\kappa s_1 s_2 s_3 + \lambda s_2 s_3 + \mu s_3 s_1 + \nu s_1 s_2 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0 \quad (A)$$

where $\kappa\ \lambda\ \mu\ \nu$ etc are constants depending on the nature of the surface & $s_1\ s_2\ s_3$ variables determined by the three variable ples of the three sheaves. We shall call the relation (A) the cubic homography. The properties of this homography will be discussed later.

The tetraconical ~~transf.~~ & its effects.

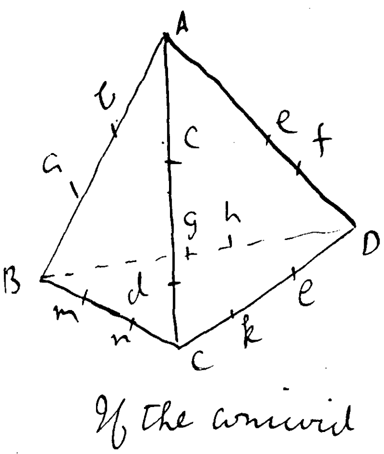
A conicoid thro' ABCD transforms into a conicoid thro' A'B'C'D'. Hence if 4 lines be drawn thro' ABCD respy which are generators of a hyd. these transform into 4 lines thro' A'B'C'D' resp. which are also gens. of a hyd. Such a group of lines we will call hyperboloidic.
A conicoid thro' BCD transforms into a cubic surface having 3 proper conical pts at B'C'D'. The gens. thro' BCD transform into lines on the cubic thro' B'C' & D' & intersecting 2 & 2 exactly as the gens. of the hyd. do.

This cubic surface has also the characteristic eqn (A).
To find the number of lines on it.



Let the conicoid pass thro' B C D & meet AB AC AD in E F & G respectively. Then corr. to pts E F G we get the lines ~~B'C'~~ C'D' D'B' & B'C' resp. Corr. to the gens thro' B C D we get six lines of same sort. Corr. to the conics CDE, BCG & BDF we get by formulae given previously, lines thus there are altogether 12 lines on this surface.

To a conicoid which does not pass thro' any of the F-pts corresponds a sextic having the edges of the F-tetn as double lines & having the F-pts as conical pts of the 4th order. The tangent cones at these pts have 3 double sides each.
The generators of the conicoid transform into two systems of conicubics. Conicubics belonging to same system do not intersect & those of opposite systems intersect. Let the conicoid meet the edges of the F-tetn ABCD in the ~~six~~ 12 pts a b.... n.



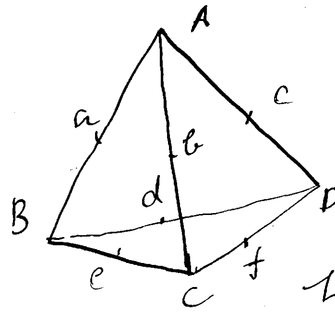
Corr. to the conic bcf there will be a cubic curve on the sextic (not a conicubic). Thus there will be altogether $\frac{3 \times 24}{3}$ i.e. 24 such cubic curves on the sextic.

If the conicoid touch the edges of the F-tetn the tangent cones at A'B'C'D' will have the edges as cuspidal sides.
Corresponding to this conicoid we can find in (S) a concubicoid whose 4 tangent cones at the F pts

touch the conicoid. This concubicoid transforms into a ple in (S') & its tangent cones into ~~coplanar~~ ples thro' the F-pts. These ples must consequently touch the sextic which corresp. to the conicoid ~~in~~ along a ple curve which corresp. to the conic of contact of the concubicoid with with the conicoid. This conic transforms into a ple cubic thro' an F-pt & having a node at the latter pt. Hence we infer in the case of the above sextic the existence of four singular tangent ples which touch the surface along ple unicursal cubics. The ples of these curves meet the opp. faces in coplanar lines.

If the conicoid touch the faces of the F-tet^n the tangent cones ~~to~~ at the F-pts of the corresp. sextic reduce to lines. These lines form a hyperboloidic system for their correspondents viz the lines joining the vertices of the F-tet^n in (S') to the pts of contact of the conicoid with the opp. faces are hyperboloidic. We may call such singular pts - cuspidal points & we get the theorem :- the four tangents at the four cuspidal pts of ~~a~~ the sextic are hyperboloidic. This is true of any sextic with four cuspidal pts.

There is another very elegant result on the sextic which corresponds to the conicoid which touches the 6 edges of the F-tet^n.



Let the conicoid touch the edges in the pts abcdef. Then the six ples CDa, BCc, BDb, ABf etc as we have already seen meet in a pt. These ples transform into the cuspidal tangt. ples thro' the edges of the F-tet^n in (S') to the tangent cones at the~~se~~ F-points. Hence the latter all meet in a point.

In fact this is a particular case of the following theorem on the general sextic.

The twelve pairs of tangent planes thro' the edges of the F-tetⁿ to the tangent cones at the F-pts touch a conicoid.

For with the fig on p 78 the twelve planes CDb, CDa, BCe, BCf, etc have the same property & this property is not altered by the transf. Hence since these planes transform into the tangent planes to the tangent cones thro' the edges of F-tetⁿ we get the above result.

The characteristic eqⁿ of the sextic which corresponds to a conicoid.

This is

$$S_1^2 S_2^2 S_3^2 + K_1 S_1^2 S_2^2 S_3 + K_2 S_2^2 S_3^2 S_1 + \cdots + K_4 S_1^2 S_2^2 + \cdots$$
$$+ K_7 S_1^2 S_2 + \cdots + K_{10} S_1 S_2 S_3 + K_{11} S_1^2 S_2 + \cdots$$
$$+ K_{17} S_1^2 + \cdots + K_{20} S_1 S_2 + \cdots$$
$$+ K_{23} S_2 + \cdots \quad + K_{26} = 0$$

It contains 26 indepᵗ constants.

A conic transforms in general into a sextic curve having nodes of second order at the F-pts.

If the conic touches the four faces of the tetⁿ the nodes become cusps.

If a surface or curve have a singular pt on a face of the F-tetⁿ the nature of the singularity will be completely altered by the transf.

For example let a cubic surface have a conical pt of second order on BCD with a real non-singular tangent cone. BCD meets the surface in a cubic curve having a dp. The conical pt at A' on the corresp. nonic surface has a tangent cone with 3 triple sides & a double side cor. to the dp.

Similarly let a curve have a node of second order on face BCD with real & distinct tangents not passing thro' A. This sing-pt transforms into a tac-node at A'. If one of the tangents pass thro' A there will be an inflexion on one of branches at A'.

art 19. Properties of the cubic homography & the cubic involution & their application to the theory of cubic curves & surfaces.

The eqn. of the homography is

$$\kappa s_1 s_2 s_3 + \lambda s_1 s_2 + \mu s_2 s_3 + \nu s_3 s_1 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0 \qquad (A)$$

Let us consider three ranges of pts on three str. lines & let them refered to fixed origins on their respective lines be characterised by the variables s_1 s_2 s_3. Then the relation (A) establishes a cubic homographic relation between these ranges. Let the bases be called l m & n then the distinctive property of the above ~~invol~~ homography is given two pts belonging to different bases there is one unique pt on the third base which corresponds to them. Such a homography is determined by 7 triads of correspondents.

Let I J K be the pts at ∞ on the bases l m n $(s_1\ s_2\ s_3)$.

Corr. to I J we get a value ~~a~~ c for s_3 — to J K a value a for s_1 & to I K a value b for s_2.

Substituting in (A) we get $\lambda = -\kappa c$ $\mu = -\kappa a$ & $\nu = -\kappa b$.

$$\kappa s_1 s_2 s_3 - \kappa c s_1 s_2 - \kappa a s_2 s_3 - \kappa b s_3 s_1 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0$$

Let O P Q be the origins $(s_1 = 0 \quad s_2 = 0 \quad s_3 = 0)$

& let $s_3 = d$ corr to pts I P. then $\alpha = \kappa bd$

Similarly let $s_2 = e$ " K O " $\beta = \kappa ce$

" $s_1 = f$ " J Q . . $\gamma = \kappa af$

& $s_1 s_2 s_3 - c s_1 s_2 - a s_2 s_3 - b s_3 s_1 + bd s_1 + ce s_2 + af s_3 + \frac{\delta}{k} = 0$

Thus meanings can be found for the coeffs.

$\lambda \ \mu \ \nu \ \alpha \ \beta \ \gamma$.

If the ranges have a common base the same results apply. In this case there will be 3 triple elements viz the roots of the cubic eqn

$$s^3 - (c+a+b)s^2 + (bd+ce+af)s + \frac{\delta}{k} = 0$$

One of these triple pts must always be real.

The fundamental pts of the cubic homography.

Let the homog be $k s_1 s_2 s_3 + \lambda s_2 s_3 + \mu s_3 s_1 + \nu s_1 s_2 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0$

we get $s_3 = -\dfrac{\nu s_1 s_2 + \alpha s_1 + \beta s_2 + \delta}{k s_1 s_2 + \lambda s_2 + \mu s_1 + \gamma}$

If $\begin{cases} \nu s_1 s_2 + \alpha s_1 + \beta s_2 + \delta = 0 \\ k s_1 s_2 + \lambda s_2 + \mu s_2 + \gamma = 0 \end{cases}$ simultaneously

we get on solving two values for s_1 & 2 for s_2. Let these be $(k_1 \ k_2)$ & $(l_1 \ l_2)$ respectively so that $k_1 \ l_1$ correspond & $k_2 \ l_2$ correspond. Then when $s_1 \ s_2$ have the values $(k_1 \ l_1)$ or $(k_2 \ l_2)$ there is no unique value of s_3 corresp. to them. Similarly we get two values of s_2 & two of s_3 which when combined in pairs have no unique correspondent. & so on. Thus we apparently get 12 exceptional pts. But we find on solving that these are equal in pairs so that there are only 6 distinct values.

~~These~~ Thus there are 6 exceptional or fundamental pts & two belong to each range.

Let $k_1 \ k_2$ $l_1 \ l_2$ & $m_1 \ m_2$ be the values then the correspondents are

$s_1 \ s_2 \rightarrow (k_1 \ l_2) \quad (k_2 \ l_1)$

$s_2 \ s_3 \rightarrow (l_2 \ m_1) \quad (l_1 \ m_2)$

$s_1 \ s_3 \rightarrow (k_1 \ m_2) \quad (k_2 \ m_1)$

a relation amongst the coefficients κ λ etc may lead to a diminution in this number.

We have seen already that three sheaves which generate a cubic surface trace out a cubic homography on any line.

The most general cubic surface thus traced out must therefore be determined by 7 points (besides the bases of the 3 sheaves).

This surface is obtained as follows. ~~As~~ In the tetrahedral transf. a cubic surface having dps at 3 of the F-pts transforms into a cubic surface of precisely the same nature having 3 dps at 3 F-pts. Since a dp counts as four conditions we see that these surfaces are determined by 7 ordinary pts.

From what we now know of the cubic homography it follows that such surfaces must have two lines thro' each dp other than the lines joining the dps. These 6 lines correspond to the six F-pts of the homography.

In the hyperboloidal transf. a cubic surface thro' P Q R transforms into a cubic surface thro' P' Q' R'. Consequently such surfaces will have 6 lines which intersect the F-lines each line intersecting two F-lines. Such surfaces are determined by 7 pts since P Q R count as 4 each. The six lines intersect in pairs.

In the cubic surfaces det^d by 7 pts having 3 conical pts the 6 lines thro' the conical pts (excluding the lines joining these pts) must ~~have~~ intersect in pairs.

For take any such surface with conical pts A B C. Take any other pt D on the surface. Take A B C D as

F-letⁿ & transform by the tetrahedral transf. We get a conicoid & thro' 3 fixed pts. The six lines of the cubic surface transform into 6 generators of the conicoid & these we know intersect in pairs. Hence the six lines on the cubic must intersect in pairs.

In this connection it is of interest to note that in connection with the involutive tetrahedral transf. there are 8 cubic surfaces having 3 conical pts which transform into themselves.
Take a cubic surface having conical pts at A B & C & passing thro' 7 of the dps of the transf.
This surface transf. into a similar cubic thro' the same 7 pts ie into itself.
Referring to the diagram on p 65 we see that the six lines thro' dps can be taken as B1, B2, C1, C2, D4, D7, for these intersect in pairs as required. They also include all the pts except 5.
Clearly there are 8 such cubic surfaces.

The cubic surfaces which correspond to planes in the various transformations are determined outside of F-system by 3 points each. Hence $\frac{3}{4}$ of the coefficients only of the homography (A) can be independent.
Consider the hyperboloidal transf.
A ~~hyperbocubicoid~~ cubic surface thro' P Q & R will meet the F-curve Σ_3 in 9 pts. It will contain that curve if it meet it in 10 pts. But since Σ_3 intersects each of P Q R in 2 pts we see that 6 of the pts of intersection lie on P Q & R. Hence if the surface meet the curve Σ_3 in 4 pts it contains the curve. Now the hyperbocubicoid contains Σ_3 & hence it is a cubic surface subject

to 4 conditions. It is therefore determined by 3 indep. pts.
In the tetraedral transf. let a cubic surface have conical pts at B C & D. ~~& pass thro' A~~ It meets the lines A B, A C, A D i.e. the remaining part of the F-system in 3 points. It will contain these 3 lines if it pass thro' A & 3 points one on each of A B, A C, A D. This amounts to 4 conditions. Hence the circumcubicoid is determined by 3 conditions. Thus in the case of the cubic surfaces which correspond to ples the absorption of the whole F-system accounts for 4 conditions leaving 3 to be satisfied. Such surfaces therefore are determined by 3 points. This must be the case since the ples which correspond to them are determined by 3 points.

The cubic homography on a cubic curve.
The three ranges of pts may also be arranged on a cubic curve eg a circumcubic.
There will be 3 triple-points & 6 F-pts on the curve.
The plane containing three corresponding points envelopes a surface. Let P & Q be two ~~cor~~ pts belonging to ranges S_1 & S_2. Thro' the line P Q there passes one tangent ple to surface. But P Q may also be considered as belonging to $S_2 S_3$ or $S_3 S_1$. Hence thro' P Q we can draw three tangent ples to surface. It is clearly of the third class.
Also if $k_1 l_2$ be two F-pts all ples thro' $k_1 l_2$ are tangent ples hence $k_1 l_2$ is a line on the surface. Similarly for the other ~~2~~ F-pts. Hence there are 6 lines on the surface.
Since the surface is of the third class it is the reciprocal of a cubic surface. It will be the

reciprocal of the general cubic surface generated by the cubic homographic system of planes. Since the 6 lines on the latter surface intersect in pairs it follows that the 6 lines which join the F-pts on the conicubic in pairs must be coplanar in pairs. Using the same notation as on p 82 the lines are $\{k_1 l_2\}$ $\{k_2 l_1\}$ $\{l_2 m_1\}$ $\{l_1 m_2\}$ $\{k_1 m_2\}$ $\{k_2 m_1\}$ where k_1 k_2 are F-pts of range S_1 l_1 l_2 of range S_2 & m_1 m_2 of range S_3. The lines $\{k_1 l_2\}$ $\{k_1 m_2\}$ are coplanar since they meet in k_1 & so on hence the above condition is satisfied.

In the case of the homography which generates the conicubicoid there are only 3 F-pts.

A cone passing thro' the pts BCD but not thro' A transforms by the tetrahedral transf into a cubic surface having conical pts at B'C'D' & another pt which corresponds to the vertex of the cone. The three gens of the cone thro' BCD transform into the 3 lines on the cubic surface thro' the fourth conical point. The cubic surface is a conicubicoid. The name conicubicoid has been chosen on account of the close relation of surface with the cone.

The 3 F-pts ~~correspond~~ of the homography which generates this surface correspond to the 3 lines thro' the 4th conical point.

Let P Q R be three pts of a general cubic homog. on a cubic curve P belonging to range S_1 Q to S_2 & R to S_3. Let X Y Z be any ~~other~~ three corr. pts of homog. X belonging to S_1, Y to S_2 & Z to S_3; planes PQZ, QRX, RPY meet in a pt M. As XYZ vary on the cubic the pt M traces out a conicoid. If $k_1 k_2$ $l_1 l_2$ $n_1 n_2$ be the F-pts. the planes PQn_1, RQl_2 meet on a line which is a gen^n of the conicoid thro' Q & so on. The six F pts give rise to 6 gens. of the conicoid. The conicoid passes thro' P Q & R & meets the curve in the triple points. Let a special coincubical homography be taken on the cubic & let P Q R be the 3 F-pts. Then in this case M traces out a plane which meets the curve in the triple pts of the homography. The proofs of these theorems follow readily from those already given. When the 6 F-pts of a homog. on a cubic curve & one triad are known the homog. is known.

The cubic involution.

The relation

$$s_1 s_2 s_3 + \partial(s_1 s_2 + s_2 s_3 + s_3 s_1) + \alpha(s_1 + s_2 + s_3) + \delta = 0 \quad (B)$$

will be called the cubic involution.

Its characteristic property is as follows.

If the ranges ~~are~~ have a given base then if P & Q be pts ~~on one~~ on different ~~bases~~ ranges they have a unique correspondent ~~in the third~~ no matter which ranges they are considered as belonging to.

Such an involution is determined by 3 triads of correspondents.

If a series of planes be taken thro' a fixed point O these planes trace out on any cubic curve a cubic involution. For if X Y Z be the pts of intersection of any plane when X Y are given Z is uniquely determined & the ranges traced out by X Y Z form a cubic involution.

Conversely the planes containing 3 corr. pts of a cubic involution on a cubic curve envelope a point. For three such planes meet in a point O & a system of planes thro' O determines an involution which has three triads in common with the given involution. The involutions must therefore be identical hence the ~~corresp~~. ples containing corr.-pts must pass thro' the fixed pt O.

The involution has 3 triple pts given by
$s^3 + ds^2 + \alpha s + \delta = 0$ hence from any pt O we can draw 3 osculating ples to a cubic curve.

Transforming these theorems by the cubic transf. we get the following: A series of cubic surfaces thro' a fixed pt (outside of F-system) meets a line in a cubic involution. Further three such surfaces can be found to osculate the line.

The F-pts of the involution.

Every cubic involution has 2 F-pts.
These are the roots of

$$\begin{cases} \lambda s_1 s_2 + \alpha(s_1 + s_2) + \delta = 0 \\ s_1 s_2 + d(s_1 + s_2) + \alpha = 0 \end{cases}$$

Let k l be the F.pts of an involution on a curve then k l have no unique correspondent. Any pt may be regarded as their correspondent.

The F-pts are the pts in which the chord of the curve thro' O the origin of the invol. meets the curve. Thus thro' any pt O we can draw a line to meet a cubic curve in 2 pts & one line only. An involution is known completely when its two F-pts & one triad are known.

Let P Q R be 3 corresp. pts of an involution on a cubic. & X Y Z any other triad.

Ples PQX, QRY, RPZ meet in pt M. as XYZ vary M traces out a cone. If k l are the F pts. the three gens. of the cone thro' P Q R are the intersns of (PQl, QRm) (PQm, RPl) $(PR\overset{m}{\cancel{k}}, QRl)$.

The one meets the curve in the triple pts of the involn.
If P Q be taken at K L then the locus of M is a ple meeting the curve in the triple points. R may be any other pt on curve.
As R varies this ple must remain fixed for it is determined completely by the three triple points on it. Hence if we can find a construction for the chord of the cubic thro' O we can find the triple pts & the osculating ples at these points.

Let P be any pt on the curve & O any pt in space. The sheaf of ples thro' PO meets the curve in quadratic involution. Let Q be any other pt on curve; the sheaf QO meets curve in another quadratic involution. These quad. involns. have two common elements which can be easily found. Let them be A & B. Then the ples POAB & QOAB have a common line of intersection viz OAB which is the required chord thro' O.

Polar ple of O with respect to the curve.

The ple which contains the three triple pts of the involution whose origin is O will be called the polar ple of O. If O describe a ple the polar ple will envelope a pt. For. let π be the ple described by O. & let P & Q be two pts on curve. The osculating ples at P & Q will intersect π in a pt F from which the third osculating ple can be at once drawn. Hence PQ determine a third pt on the curve uniquely & hence the ple containing the three triple pts passes thro' a fixed pt. Thus the relation between O & the ple containing the triple pts is exactly analogous to the pole & polar relation with respect to a conicoid.

Two involutions on a cubic curve.

Consider two involutions origins A & B & let their eqns. be

$$s_1 s_2 s_3 + \lambda_1 (s_1 s_2 + s_2 s_3 + s_3 s_1) + \alpha_1 (s_1 + s_2 + s_3) + \delta_1 = 0$$
$$s_1 s_2 s_3 + \lambda_2 (s_1 s_2 + s_2 s_3 + s_3 s_1) + \alpha_2 (s_1 + s_2 + s_3) + \delta_2 = 0$$

These relations have a single infinity of common solns. corresponding to the pts in which the sheaf of planes thro AB meets the curve. Solving for s_3 we get

$$\{\lambda_1 s_1 s_2 + \alpha_1 (s_1 + s_2) + \delta_1\} \{ s_1 s_2 + \lambda_2 (s_1 + s_2) + \alpha_2)$$
$$= \{\lambda_2 s_1 s_2 + \alpha_2 (s_1 + s_2) + \delta_2) \{ s_1 s_2 + \lambda_1 (s_1 + s_2) + \alpha_1 \}$$

When $s_1 = s_2 = s$ this gives a quartic eqn. & thus there are 4 planes thro' AB which touch the curve. This may be verified geometrically as follows. Projecting the curve from any pt O on a plane we get as projection a unicursal cubic i.e. having one d.p. Projecting from A we therefore get a unicursal cubic & line AB meets plane of projⁿ in a pt C. From C we can draw 4 tangents to curve since it is of class 4 hence thro AB we can draw 4 tangent planes to ^space^ curve. Transforming this theorem we see that we can find 4 cubic surfaces satisfying 6 conditions ~~to~~ to touch an arbitrary line.

Three involutions on a cubic,

Let ABC be the origins. Then the involutions will three common elements viz the pts in which the plane ABC meets the curve.

To find the number of chords of a cubic curve which intersect two arbitrary lines in space.

Let P & Q be the lines. The sheaf of planes thro' P generates the system of pts

$$\begin{cases} s_1s_2s_3+\lambda_1\Sigma s_1s_2+\mu_1\Sigma s_1+\nu_1=0 \\ s_1s_2s_3+\lambda_2\Sigma s_1s_2+\mu_2\Sigma s_2+\nu_2=0 \end{cases} \qquad (1)$$

& the sheaf thro' Q the system

$$\begin{cases} s_1s_2s_3+\lambda_3\Sigma s_1s_2+\mu_3\Sigma s_1+\nu_3=0 \\ s_1s_2s_3+\lambda_4\Sigma s_1s_2+\mu_4\Sigma s_1+\nu_4=0 \end{cases} \qquad (2)$$

Elim^n s_3 from (1) we get a relation.

$$(\lambda_1 s_1s_2+\alpha_1(s_1+s_2)+\nu_1)(s_1s_2+\lambda_2(s_1+s_2)+\nu_2) = (\lambda_2 s_1s_2+\alpha_2(s_1+s_2)+\nu_2)$$
$$(s_1s_2+\lambda_1(s_1+s_2)+\nu_1)$$

Let $s_1s_2=x$ & $s_1+s_2=y$ this gives

$$a_1x^2+b_1xy+c_1y^2+d_1x+e_1y+f_1=0$$

The relation (2) gives

$$a_2x^2+b_2xy+c_2y^2+d_2x+e_2y+f_2=0$$

These eqns have 4 solutions. Let X_1 Y_1 be a pair of solutions then s_1 & s_2 are the roots of

$$s^2-Y_1s+X_1=0$$

. Thus there are 4 chords of curve which intersect both P & Q.

If P Q each meet curve in 1 pt ~~ea~~ then the above relations in x & y reduce to linear relations & there is consequently only one ~~ch~~ chord.

If only one of lines P Q meets curve in 1 point then clearly one of relations will be linear & hence there will be two chords.

To find the nature of the surface generated by chords of the curve which intersect a fixed line.

Let P be the line; then thro' any pt on P we can draw one & only one chord of curve. Hence the surface is a skew surface. By the previous theorem 4 chords can be found to ~~to~~ intersect P & any other arbitrary line Q hence surface is of 4th order.

The cubic curve is a double curve on this surface for thro' any pt R on curve 2 generators of surface pass.

If P meet the curve in one point then by the previous theorem the surface generated is of the second order i.e. it is a conicoid. Thus we see that a cubic curve on a conicoid meets all the gens of one system in 2 pts & those of opp. system in ~~2~~ 1 pt.

Given a cubic curve & an arbitrary line which may be regarded as corresponding to each other in a cubic transf. Let ranges of pts be taken on the curve & the line respectively which are homog. with each other. The lines joining corresponding pts will trace out a surface.

To find the nature of this surface

Consider any chord P & let M & N be corr. pts on the curve & line then the sheaves (PM) & (PN) are homographic (~~[illegible]~~) ~~[illegible]~~. They have two double ples & hence two generators of the surface will meet P. But other two gens. of the surface will meet P in the pts where it meets the curve hence altogether 4 generators meet P. Consequently surface is of 4th order.

Any generator is intersected by 2 non consecutive gens. For let Y be any gen. a ple thro' Y & arbit. line meets curve in 2 pts & thro' each of these points there passes a genr which intersects Y, ~~as the ple thro' Y varies these pts of intersection with Y must remain fixed~~. Let pts of intersection of gens. with Y be a & b. Then locus of a b is a cubic curve ~~as~~ which is a double curve on the surface. For a ple thro' two intersecting gens. meets this locus in 3 points.

Thus this surface is of precisely the same nature as previous surface.

A particular case of this theorem is the following. Let P be any line & Σ_3 a cubic curve & let Q be a chord intersecting P. Let chord meet P in p & Σ_3 in a b. Let ~~p~~ q be the harmonic conjugate of p with respect to ab. Then locus of q is a cubic curve

If the sheaves (PM) & (PN) are congruent the quartic ruled surface is of a special nature. The chord P is a triple generator of the surface, & there is only one generator in any ple thro' P.

By means of these surfaces we can obtain the two varieties of quartic ple transformation.

(1) The variety which has 3 F-pts of second order & 3 F-pts of first order.

Let π & π' be two ples & Σ_3 any cubic curve. We establish a (1-1) corresp. between the pts in these ples as follows. Let P be a pt on π. Thro' P draw the chord of Σ_3 & let this meet π' in P'. P' corresponds to P. When P describes a line in π the chords of Σ_3 generate by above theorem a ruled quartic having Σ_3 as double curve. Hence the corresp. of the line of π is a ple quartic having 3 dps viz the pts where Σ_3 meets π'. Let the curve Σ_3 meet π in ABC & π' in A'B'C'. These are F-pts of second order. Corresp. to A we get a conic in π' etc. These conics intersect in A'B'C' & 3 other fixed pts D'E'F' which are the F-pts of first order on π'. We get three similar F-pts DEF on π. Clearly DEF & D'E'F' must lie on the line of intersection of π & π'. To a conic on π corresponds an octic curve on π' having quadruple pts at A'B'C' & dps at D'E'F'. Hence we infer the following theorem.

The chords of a cubic curve which intersect a fixed conic generate a ruled octic surface having the cubic curve as a quadruple curve.

(2) The variety which has one F-pt of third order & 6 F-pts of first order.

~~B~~ With the planes π & π' as before & a chord X of Σ_3 take any pt P on π & draw the plane XP cutting Σ_3 in L. The line PL meets π' in P' & P & P' are corresp. pts.

When P describes a line l the lines PL generate a ruled quartic having X as triple gen.^{r} This meets π' in a quartic curve. Let X meet π' in A' & let Σ_3 meet π' in $B'C'D'$. Then A' is the F-pt of 3rd order. If A be corr. pt on π we get for the corr. of A a cubic curve having a dp at A'. This curve meets the quartic in 6 pts at A' & in 6 other pts of which $B'C'D'$ are three. These 6 pts are the 6 F-pts of first order. They lie in line of intersn of π & π'.

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Analytical discussion of the cubic involution.

Parametric form of the involution.

This is $(\alpha_1 s^3+\beta_1 s^2+\gamma_1 s+\delta_1)+\lambda(\alpha_2 s^3+\beta_2 s^2+\gamma_2 s+\delta_2)+$
$\mu(\alpha_3 s^3+\beta_3 s^2+\gamma_3 s+\delta_3)=0$ (1)

Condition that four triads of points should be in involution.

Let these be

$$\alpha_i s^3+\beta_i s^2+\gamma_i s+\delta_i=0 \qquad i=1,2,3,4.$$

then $s_1 s_2 s_3 = -\dfrac{\delta_i}{\alpha_i}$ "

$\Sigma s_2 s_3 = \dfrac{\gamma_i}{\alpha_i}$,,

$\Sigma s_1 = -\dfrac{\beta_i}{\alpha_i}$,, Substituting in the form

$$\kappa s_1 s_2 s_3 + \lambda(s_1s_2 + s_2s_3 + s_3s_1) + \alpha(s_1+s_2+s_3) + \delta = 0$$

we get

$$-\kappa\delta_1 + \lambda\gamma_1 - \alpha\beta_1 + \delta\alpha_1 = 0$$
$$-\kappa\delta_2 + \lambda\gamma_2 - \alpha\beta_2 + \delta\alpha_2 = 0$$
$$-\kappa\delta_3 + \lambda\gamma_3 - \alpha\beta_3 + \delta\alpha_3 = 0$$
$$-\kappa\delta_4 + \lambda\gamma_4 - \alpha\beta_4 + \delta\alpha_4 = 0$$

giving

$$\begin{vmatrix} \delta_1 & \gamma_1 & \beta_1 & \alpha_1 \\ \delta_2 & \gamma_2 & \beta_2 & \alpha_2 \\ \delta_3 & \gamma_3 & \beta_3 & \alpha_3 \\ \delta_4 & \gamma_4 & \beta_4 & \alpha_4 \end{vmatrix} = 0$$

whence also $\delta_4 = \delta_1 + \lambda\delta_2 + \mu\delta_3$

$\gamma_4 = \gamma_1 + \lambda\gamma_2 + \mu\gamma_3$ etc

hence the 4th triad $\alpha_4 s^3 + \beta_4 s^2 + \gamma_4 s + \delta_4 = 0$ can be written in the form (1).

To find the triple points.

The values of λ & μ that give these points make the left side a perfect cube. The conditions for this are

$$\left[(1+\lambda+\mu)s + (\beta_1 + \lambda\beta_2 + \mu\beta_3)\right]^3 = 0$$
$$\left[(\alpha_1 + \lambda\alpha_2 + \mu\alpha_3)s + (\gamma_1 + \lambda\gamma_2 + \mu\gamma_3)\right]^3 = 0$$
$$\left[(\beta_1 + \lambda\beta_2 + \mu\beta_3)s + (\alpha_1 + \delta\alpha_2 + \mu\alpha_3)\right]^3 = 0$$

hence

$$\lambda(s+\beta_2) + \mu(\alpha_3 s + \gamma_3) + s + \beta_1 = 0$$
$$\lambda(\alpha_2 s + \beta_2) + \mu(\alpha_3 s + \gamma_3) + \alpha_1 s + \gamma_1 = 0$$
$$\lambda(\beta_2 s + \alpha_2) + \mu(\beta_3 s + \alpha_3) + \beta_1 s + \alpha_1 = 0$$

$$\therefore \quad \begin{vmatrix} s+\beta_2, & s+\beta_3, & s+\beta_1 \\ \alpha_2 s+\beta_2, & \alpha_3 s+\gamma_3, & \alpha_1 s+\gamma_1 \\ \beta_2 s+\alpha_2, & \beta_3 s+\alpha_3, & \beta_1 s+\alpha_1 \end{vmatrix} = 0$$

The roots of this eqn. give the triple pts.

Fundamental relation between the segments forming an involution. Let $(a_1 a_2 a_3)$ $(b_1 b_2 b_3)$ $(c_1 c_2 c_3)$ & $(d_1 d_2 d_3)$ be four triads in inv. on the same line & let them be given by

$$x^3 - 3\alpha_i x + 3\beta_i x - \delta_i = 0 \quad \text{etc} \qquad i = 1,2,3,4.$$

Then we must have
$$\begin{vmatrix} 1 & \alpha_1 & \beta_1 & \delta_1 \\ 1 & \alpha_2 & \beta_2 & \delta_2 \\ - & - & - & \\ 1 & \alpha_4 & \beta_4 & \delta_4 \end{vmatrix} = 0 .$$

If A is fixed origin on the line

$\alpha_1 = \Sigma Aa_1 \quad \beta_1 = \Sigma Aa_1 \cdot Aa_2$ etc

$\alpha_2 = \Sigma Ab_1 \quad \beta_2 = \Sigma Ab_1 \cdot Ab_2 .$

Let pts k l m n be chosen so that $3Ak = \Sigma Aa_1$, $3Al = \Sigma Ab_1$ etc.

then
$$\begin{vmatrix} 1 & Ak & \Sigma Aa_1 \cdot Aa_2 & \Pi(Aa_1) \\ 1 & Al & \Sigma Ab_1 \cdot Ab_2 & \Pi(Ab_1) \\ - & - & - & - \\ 1 & An & \Sigma Ad_1 \cdot Ad_2 & \Pi(Ad_1) \end{vmatrix} = 0$$

If we make A coincide in turn with each of the pts $a_1\ a_2 \ldots$ we get 12 relations of type
$$\begin{vmatrix} 1 & a_1a_2 + a_1a_3 & 0 & 0 \\ 1 & a_1 l & \Sigma a_1b_1 \cdot a_1b_2 & \Pi(a_1b_1) \\ 1 & a_1 m & \Sigma a_1c_1 \cdot a_1c_2 & \Pi(a_1c_1) \\ 1 & a_1 n & \Sigma a_1d_1 \cdot a_1d_2 & \Pi(a_1d_1) \end{vmatrix} = 0$$

This can be reduced to the form
$$\sum_{bcd} (a_1b_1 + a_2b_2 + a_3b_3)(a_1c_1 \cdot a_1c_2\, a_1d_1 \cdot a_1d_2 \cdot c_2d_2$$
$$+ \text{similar terms}) = 0$$
there being a cyclical interchange of the letters b c d.

The 'centre' of the involution.

The values of λ & μ which make two roots of ① infinite are given by

$$\alpha_1 + \lambda\alpha_2 + \mu\alpha_3 = 0 \qquad \text{ie} \qquad \lambda = -\begin{vmatrix} \alpha_3 & \alpha_1 \\ \beta_3 & \beta_1 \end{vmatrix} \Big/ \begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix}$$
$$\beta_1 + \lambda\beta_2 + \mu\beta_3 = 0$$

$\mu =$ sim. expn.

The remaining root is given by
$$\delta = -\frac{\delta_1 + \delta_2 + \delta_3}{\gamma_1 - \frac{\begin{vmatrix} \alpha_3 & \alpha_1 \\ \beta_3 & \beta_1 \end{vmatrix}}{\begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix}}\gamma_2 - \frac{\begin{vmatrix} \alpha_2 & \alpha_1 \\ \beta_2 & \beta_1 \end{vmatrix}}{\begin{vmatrix} & \end{vmatrix}}\gamma_3}$$

$$\text{or} = \cdot \frac{(\delta_1 + \delta_2 + \delta_3)\,(\alpha_2\beta_3 - \alpha_3\beta_2)}{(\alpha_3\beta_1 - \alpha_1\beta_3)\gamma_2 + (\alpha_2\beta_1 - \beta_2\alpha_1)\gamma_3 - (\alpha_2\beta_3 - \alpha_3\beta_2)\gamma_1}$$

This value gives a point which we may call the centre. It has ~~one~~ remarkable analogy with the centre of a quad. involution.

Referred to the centre the fund. relation becomes

$$\kappa S_1 S_2 S_3 + \alpha(S_1 + S_2 + S_3) + \delta = 0$$

From this it is evident that the sum of distances of the triple-points from the centre is zero.

The sexto-cubic homography. (p 80)

This is $S_1{}^2 S_2{}^2 S_3{}^2 + \kappa_1 S_1{}^2 S_2{}^2 S_3 + \kappa_2 S_2{}^2 S_3{}^2 S_1 + \cdots$

$$+ \kappa_{23} S_2 + \cdots + \kappa_{26} = 0$$

This is the relation characteristic of the sextic surfaces which correspond to conicoids.

It is completely determined by 26 triads of values of $S_1 S_2 S_3$.

(1) In the tetrahedral transf. the sextic which corresp. to a conicoid has 4 quadruple conical pts at the vertices of the F-tet^n. Each of these accounts for 20 conditions & hence the 4 are equiv. to 80 conditions. Now the general sextic is determined by 83 conditions hence apparently this special sextic will be determined by 3 conditions. ~~Now~~ But the conicoid which corresponds to it is determined by 9 conditions. Hence there is an apparent discrepancy. The discrepancy is explained by the fact that each sheaf is a double sheaf & that if we ~~are~~ take 3 pts on the conicoid we really take 9.

The most general sextic of the above type of relation is obtained by transforming a quintic surface having triple conical pts at 3 of the F-pts.

Such a surface is determined by 55 − 3×10 i.e 25 conditions since each conical pt counts as 10.

This transforms into a sextic having quadruple conical pts at 3 F-pts & passing thro' the 4^th F-pt.

A still more general form is obtained by transforming a sextic with 3 ~~4~~ quadruple conical pts at 3 F-pts. This transforms into sextic of same nature.

(2) The hyd. transf. A conicoid transforms into a sextic having each F-line as a double line. A sextic having the 3 lines P Q R as double lines transforms into a sextic of same nature. These sextics are det.^d by 26 conditions.

Sexto-cubic homography on a cubic curve. Let P Q R be corresp. pts of homog. — & let A B C be three corresp pts (P A belonging to range S, etc) then PQC, QRA, RPB generate by their intersection a quintic surface having triple conical pts at PQR. This surface meets the curve in 6 pts which are the 6 sextuple-pts of the homography. The investigation of the F-system of such a homography is a matter of considerable complexity. If however the above quintic surface have an additional quadruple conical pt then there will clearly be 3 F-pts of the homography viz the three obtained by joining P Q R to the 4^th conical point.

Other homographies intermediate between the cubic & the sexto-cubic are easily obtained. For example transforming a conicoid thro' 2 F-pts we get a quartic having two conical points at F-pts & ~~passing thro' the~~ having triple conical pts at the other F-pts.

This surface is characterised by the relation

$$S_1^2S_2S_3 + K_1S_1S_2S_3 + K_2S_1^2S_2 + K_3S_1^2S_3 + K_4S_1^2 + K_5S_2S_3$$
$$+ K_6S_3S_1 + K_7S_1S_2 + K_8S_1 + K_9S_2 + K_{10}S_3 + K_{11} = 0$$

Since there are two gen. thro' each of the F-pts on the conicoid we see that thro' each of the triple conical pts of this surface there will pass two lines (besides the F-lines). These lines will intersect in pairs. Thus we see that in this Homog. there is always an F-system of pts namely two pts on range S_2 & two on range S_3 & two on range S_1. The values of S_2 & S_3 however are not connected in any way.

Similarly in the hyperboloidal trans. if we transf a conicoid thro' one of the F-lines P Q R we get a quartic passing thro' one of F-lines & having the other two as double lines. This quartic possesses 4 lines in addition to the F-lines. If the conicoid pass thro' P these lines will correspond to the gens. which meet Q & R.

If we transform a quartic having 2 triple conical points & an ordinary conical pt at the F pts we get a quartic of same nature.

Thro' the triple pts. there pass 4 lines.

For taking the eqn ( ) we get

$$S_2 = -\frac{K_3 S_1^2 S_3 + K_4 S_1^2 + K_6 S_3 S_1 + K_8 S_1 + K_{10} S_3 + K_{11}}{S_1^2 S_3 + K_1 S_1 S_3 + K_2 S_1^2 + K_5 S_3 + K_7 S_1 + K_9}$$

Eliminating S_3 between

$$K_3 S_1^2 S_3 + K_4 S_1^2 + \quad - \quad + K_{11} = 0$$

$$\& \quad S_1^2 S_3 + K_1 S_1 S_3 + ~~etc~~ \quad + K_9 = 0$$

we get a quartic in S_1. Thus there are 4 values of S_1 & four of S_3 which when taken together have no unique corresp. in S_3. Similarly for S_1 & S_2

In the case of the particular quartic which corresponds to a conicoid these lines are halved in number

The sexto-cubic involution.

This is $S_1^2 S_2^2 S_3^2 + K_1 (S_1^2 S_2^2 S_3 + \text{--}) + K_2 (S_1^2 S_2^2 + \text{--}) + K_3 (S_1^2 S_2 S_3 + \text{--}$

$+ K_4 S_1 S_2 S_3 + K_5 (S_1^2 S_2 + \text{--}) + K_6 (S_1^2 + \text{--}) + K_7 (S_1 S_2 + \text{--}$

$+ K_8 (S_1 + - -) + K_9 = 0$. This involution is determined by 9 sets of values & hence will be closely related to the conicoid.

Consider a conicoid & a cubic curve. The tangent planes to the conicoid trace out a sexto-cubic involution on the curve. For if two pts on the curve are known say P Q their correspondent is got by the tangent plane thro' PQ to the conicoid. Since two tangt. planes can be drawn there will be 2 correspondents for P & Q. The involution is therefore sexto-cubic.

Since the conicoid is determined by 9 planes the involution will be determined by 9 triads. Conversely if a sexto-cubic involution be given on a cubic curve the planes joining corresp. pts will ~~trace out~~ envelope a conicoid. The conicoid will meet the curve ~~in the 6 sextuple pts of the involution~~ in 6 pts. Let A_1 be one of these. The tangt. plane thro' A_1 to conicoid will meet curve in B_1 & C_1. Corr. to $A_1 B_1$ we have a single pt not 2 pts as ordinarily. Similarly $A_1 C_1$ has a single corr. Hence there are 12 pairs of points which have single correspondents. The involution has 6 sextuple points & hence six osculating planes of curve can be found tangent to the conicoid.

Transforming this theorem we get the following; six cubic surfaces can be drawn to osculate a line & to touch a given sextic surface.

These theorems could be indefinitely extended by taking the homographic relations peculiar to surfaces of the 9th 12th 15th etc degrees.

The F-pts of the sexto-cubic involution.

Since a conicoid may have 3 generators which are chords of the cubic curve it follows that there may be at most 6 F-pts in this involution.

The fundamental theorem on the concubicoid.

In this theorem a relation between 4 given points (besides the 4 conical pts) is established.

The concubicoid is completely determined by its 4 conical pts & 3 ordinary pts. Hence we may expect that if 4 ordinary pts are given there will be a relation between the 8 pts.

Let $ABCD$ be the conical pts & let $1\ 2\ 3\ 4$ be the 4 ordinary points. Then the 4 triads of planes

$$\left\{\begin{array}{llll} (AC2) & (CB4) & (BA3) & (1) \\ (AC1) & (CD4) & (DA3) & (2) \\ (AB1) & (BD4) & (DA2) & (3) \\ (CB1) & (BD3) & (DC2) & (4) \end{array}\right.$$

intersect in pts which are coplanar.

The surface can be generated by sheaves thro' BC, CD & DB satisfying the relation

$$\kappa s_1 s_2 s_3 + \lambda s_1 s_2 + \mu s_2 s_3 + \nu s_3 s_1 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0$$

Consider the lines of intersection of planes $(AC2)(BA3)$, $(AC1)(DA3)$ & $(AB1)(DA2)$. Call these $L_1\ L_2\ L_3$ resp^ly^. These intersect in A.

The generating sheaves of the surface generate ranges on $L_1\ L_2\ L_3$ which referred to A as origin satisfy

$$\kappa s_1 s_2 s_3 + \lambda s_1 s_2 + \text{etc} + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0.$$

Since A is a pt on surface $s_1 = s_2 = s_3 = 0$ $\therefore \delta = 0$

Also since AD is a line on surface any pt P on it gives $s_1 = c$ $s_2 = s_3 = 0$ $\therefore \alpha c + \delta = 0$ i e $\alpha = 0$

Similarly $\beta = \gamma = 0$ hence

$$\kappa s_1 s_2 s_3 + \lambda s_1 s_2 + \mu s_2 s_3 + \nu s_3 s_1 = 0 \qquad \text{or}$$

$\kappa + \frac{\lambda}{s_3} + \frac{\mu}{s_1} + \frac{\nu}{s_2} = 0$ hence the ples joining corresp. pts of the ~~three~~ ranges on $L_1\ L_2\ L_3$ pass thro' a fixed pt.

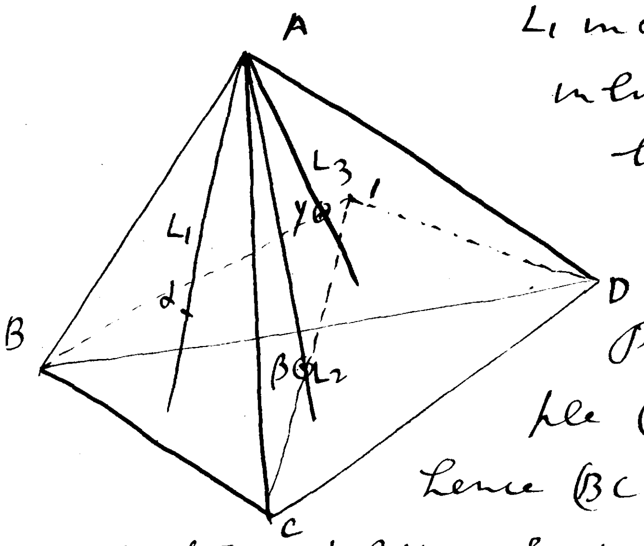
The above relation may be called the cubic perspective homography.

Determination of this fixed pt (P).

Consider the 3 triads of ples belonging to the generating system.

$$\left\{\begin{array}{l}(CB1)(BD1)(DC1)\\(BC2)(CD2)(DB2)\\(DC3)(CB3)(BD3)\end{array}\right.$$

These meet $L_1\ L_2\ L_3$ in 3 triads of pts & the ples of these triads determine P. Let ple CB1 meet L_1 in α; ples (CB1)(AB1) intersect in line B1 & ples (CB1)(AC1) in line C1 hence B1 meets L_3 & C1 meets L_2. Let pts of intersection be β, γ. Ple (CD1) meets L_2 in β & ple (BD1) " L_3 in γ. Hence (BC1) passes thro' P since it contains $\alpha\ \beta\ \gamma$.



Similarly (DC2) passes thro' P as does also (BD3). Hence P must be the pt of intersection of (CB1) (DC2) (BD3).

Consider now the triad (CB4)(CD4)(BD4). This triad intersects $L_1\ L_2\ L_3$ in the pts given by

$$\left\{\begin{array}{llll}(AC2) & (CB4) & (BA3) & (I)\\(AC1) & (CD4) & (DA3) & (II)\\(AB1) & (BD4) & (DA2) & (III)\end{array}\right.$$

The pts of intersection (I) (II) (III) of these triads must therefore be coplanar with P, hence the theorem is proved.

Number of these planes.

From the arrangement

$$\left\{\begin{array}{llll} (AC2) & (CB4) & (BA3) & (1) \\ (AC1) & (CD4) & (DA3) & (2) \\ (AB1) & (BD4) & (DA2) & (3) \\ (CB1) & (BD3) & (DC2) & (4) \end{array}\right.$$

we get by the interchange of (AC2) (AC1); (CB4) (CB1); (CD4) (DC2) etc 9 other arrangements making with the above 10 in all. Each of these gives rise to a plane. Thus corresponding to the grouping (AC1)(AB1)(CB1); (DA2) (DC2) (AC2); (BD3) (BA3) (AD3); etc we get 10 ples. But in this grouping the points 1234 can be permuted in 24 different ways hence we get on the whole 240 distinct ples.

Thro' any pt eg. (AB1) (BD4)(DA2) there pass 8 planes. viz the four given by

$$\left\{\begin{array}{lll} (AC2) & (CB4) & (BA3) \\ (AC1) & (CD4) & (DA3) \\ (AB1) & (BD4) & (DA2) \\ (CB1) & (BD3) & (DC2) \end{array}\right.$$

the different arrangements indicated by arrows.

& the four given by

$$\left\{\begin{array}{lll} (AC3) & (CB1) & (BA2) \\ (AC1) & (CD4) & (DA3) \\ (AB1) & (BD4) & (DA2) \\ (CB4) & (BD2) & (DC3) \end{array}\right.$$

the different arrangements being indicated as above.

This can be verified as follows.

The system of ples (AB1) (AB2) etc numbers 24 there being 4 thro' each edge of tet^{n}.

Any ple eg (AB1) is intersected by the other 23.

AB counts as the intersection of 3 of these viz (AB2) (AB3) (AB4) & A1 & B1 as the intersections of 2 each.

Hence outwith the Δ AB1 there will be 23-7 i.e 16 lines of intersection. These give rise to $\frac{16 \times 15}{2}$ ie 120 pts. Thro' each of these pts passes 8 planes hence the total number of ples is 8×120 but each of these is reckoned 4 times since each ple contains 4 pts hence the real number of ples is $\frac{8 \times 120}{4}$ ie 240.

— 1 —

This theorem supplies us with a mode of constructing the surface given 4 conical pts & 3 ordinary pts. For let the given pts be (ABCD 123). Construct the lines L_1 L_2 L_3 & the pt P as above. Let any ple thro' P meet L_1 L_2 L_3 in α β γ resp$^{\underline{ly}}$. Then the ples $(CB\alpha)$ $(DC\beta)$ $(BD\gamma)$ intersect in a point on the surface & by varying the ple thro' P we get as many pts on the surface as we please.

However it may happen that the three triads of ples

$$\left\{ \begin{array}{l} (CB1)\ (BD1)\ (DC1) \\ (BC2)\ (CD2)\ (DB2) \\ (DC3)\ (CB3)\ (BD3) \end{array} \right.$$

which intersect $L_1 L_2 L_3$ in 3 triads of pts do not give rise to a single pt P. That is the ples of the three triads on $L_1 L_2 L_3$ do not intersect in a single pt but in a line C. In this case it is evident that the above construction for the surface gives rise to a single infinity of points ie a curve. This covers a conicubic for out of 8 pts ~~we~~ ABCD 1234 we may take two groups of 7 viz ABCD 123 & ABCD 124 which have 6 in common. These determine two conicubicoids & in connection with these there are two pts P. All the ples thro' both these pts meeting L_1 L_2 & L_3 give rise to pts on both surfaces ie a conicubic. This conicubic passes thro' 6 pts ABCD 12.

—

Hence given 6 pts on a conicubic we can construct the curve.

Let the pts be (ABCD12). Choose any other two pts 3 & 4 & find the pts P_1 & P_2 which are the intersections of the triads $\left\{\begin{array}{l}(CB1)\,(DC2)\,(BD3)\\(CB1)\,(DC2)\,(BD4)\end{array}\right.$ (a)

Draw any ple thro' the line $P_1 P_2$ & let it meet $L_1 L_2 L_3$ in $\alpha \beta \gamma$. The ples $(CB\alpha)$ $(DC\beta)$ $(BD\gamma)$ intersect in a point on the curve & we can get as many pts as we please in this way.

In fact we can dispense with the pts 3 & 4 altogether for the triads (a) intersect in the line~~s~~ (CB1) (DC2). This line can clearly be constructed at once.

It must be observed that the pts 1 2 3 4 must not be coplanar & no one must lie on the faces of the tet^n ABCD.

For transforming the above theorem on the conicubicoid we get the following: If 1 2 3 4 be 4 pts on a ple the 4 pts of intersection of the triads

$$\left\{\begin{array}{l}(AC2)\,(CB4)\,(BA3)\\(AC1)\,(CD4)\,(DA3)\\(AB1)\,(BD4)\,(DA2)\\(CB1)\,(BD3)\,(DC2)\end{array}\right.$$

must lie on a conicubicoid.

Again let 4 lie on the face BCD. Then the lines $L_1 L_2 L_3$ must lie on a ple thro' A. In this case, as we might expect, the surfaces degenerate being made up of a cone thro' AB AC AD & containing 1 2 3 & the face BCD.

5-pt systems of conicubicoids.

If we take a pt O & a series of ples thro' it cutting $L_1 L_2 L_3$ in $\alpha\ \beta\ \gamma$ resp^ly then we have seen that the intersection of the ples $(BC\alpha)(CD\beta)$ & $(DB\gamma)$ describes a concubicoid whose concept pts are A B C & D.

Transforming this theorem we get the following: A system of 5-pt surfaces (4 of these being the concepts ABCD) meets $L_1 L_2 L_3$ in pts $\alpha\beta\gamma$ resp: the intersection of $(BC\alpha)(CD\beta)(DB\gamma)$ describes a ple. Thus if we ~~took~~ have 3 surfaces of the system given, these cut $L_1 L_2 L_3$ in 3 triads of pts from which we can construct 3 pts a b c on the above ple. Thus the ple is completely determined & consequently so is the system of surfaces. To every pt of the ple there corresponds a surface of the system.

Degenerate surfaces of the system.

The ple abc meets the tetⁿ ABCD in a complete quad. having six sides & six vertices. To any pt on this quad. there corresponds a degenerate surface. In particular the six surfaces corresponding to the six vertices are entirely composed of ples. For let the ple (abc) meet AB in k; the ples which determine the corresponding surface viz. $(BCk)(CDk)(DBk)$ intersect $L_1 L_2 L_3$ in A & a pt x on L_3. Hence the lines L_1 & L_3 must be tangent to this surface at A, since the latter meets each in 3 coincident pts at A. But in a surface determined by 5 pts this is not possible unless it be degenerate hence the surface must be the triad of ples (ABD)(ABC)(CDI) where I is the 5th pt common to all the surfaces of system. There will be 3 surfaces of this type corresponding to the pts in which (abc) meets AB, AC, AD.

Let (abc) meet BC in e; then (BCe)·(CDe) (DBe) meet L_1 L_2 L_3 resp. in 3 pts on BCD say x y z. x y z cannot lie on a concubicoid unless it be degenerate & contain as part of itself BCD. Hence there are 3 degen. surfaces of the type (ABC)(BCD)(ADE) Corresponding to any pt on a side of the complete quad. we get a surface made up of a cone & a ple. Thus corr. to a pt on face ABC we get the cone D(ABC) & the ple ABC etc.

Since any line meets a 5 pt system of surfaces in a cubic involution it follows that if a line meet the six degenerate surfaces composed of ples the pts of intersection grouped in 3's are in involution.

6 pt-systems of concubicoids.

It follows from the above theorem that if a system of 6 pt surfaces meets L_1 L_2 L_3 in α β γ resp. the ples $(BC\alpha)$ $(CD\beta)(DB\gamma)$ trace out by their intersection a line. To each pt of this line corresponds a surface of the system. & the system is consequently determined when two surfaces are given. Corresponding to the 4 pts in which the line meets the faces of ABCD we get degenerate surfaces made up of a cone & a face of ABCD.

Applications of these theorems.

Three concubicoids with the same 6 pts intersect in a single pt.

To find the pt of intersection of 3 concubicoids each given by 7 pts (4 of each being the pts ABCD).

Take 3 lines L_1 L_2 L_3 thro' A as before.

Let $a_1 b_1 c_1$, $a_2 b_2 c_2$, $a_3 b_3 c_3$ be the given pts of each surface.

now $(BCa_1)(CDa_1)(DBa_1)$; $(BCb_1)(CDb_1)(DBb_1)$ etc intersect $L_1 L_2 L_3$ in 3 triads of pts. The ples of these triads intersect in a pt P_1. Similarly the triads $(BCa_2)(CDa_2)(DBa_2)$; $(BCb_2)(CDb_2)(DBb_2)$ etc give a pt P_2 & the triads $(BCa_3)(CDa_3)(DBa_3)$ etc a pt P_3. The ple $P_1 P_2 P_3$ meets $L_1 L_2 L_3$ in pts x y & z say & $(BCx)(CDy)(DBz)$ intersect in a pt I. This pt is clearly the common pt of the system of surfaces.

To find the curve of intersection (concubic) of two concubicoids each given by 3 ordinary pts. Taking the two given triads of pts & proceeding as above we get the 2 pts P_1 & P_2. Any ple thro' P_1 & P_2 meets $L_1 L_2 L_3$ in x y z. (BCx) (CDy) (DBz) intersect in a pt. on curve & by varying the ple thro' $P_1 P_2$ we get as many pts on curve as we please.

To find the pts in which a line is cut by a concubicoid of which 3 ord. pts are given viz a_1, b_1, c_1. Thro' the 5 pts $ABCDa_1$ we have 3 degenerate surfaces of type $(ABD)(ABC)(CDa_1)$ & these determine a cubic invol. on the line. Similarly the groups $(ABCDb_1)$ $(ABCDc_1)$ give rise to involutions on the line. The three common elements of these invols. are the required 3 pts.

Note: the common elements may be found by transferring the invols. onto a cubic curve by means a chord of this curve & when the 3 common elements have been found, retransferring these back to the line.

To construct a cincubicoid to pass thro' the curve of intersn. of two cincubicoids given each by 3 ord. pts & to pass thro' a given pt Q.
Let the ord. pts be a_1, b_1, c_1 & a_2, b_2, c_2
Construct the pts P_1 & P_2 as before & let $(BCQ)(CDQ)(DBQ)$ meet $L_1 L_2 L_3$ in $\alpha\ \beta\ \gamma$.
Ple $\alpha\beta\gamma$ meets line $P_1 P_2$ in a pt R. A pencil of ples thro' R will give the required surface.

To find the pts in which a cincubic given by 2 cincubicoids of which 3 pts on each are given, meets a given ple.

It will be shown later in connection with the reciprocal cubic transf that if a pt P describe a ple & if we construct the triad $(BCP)(CDP)(DBP)$ meeting $L_1 L_2$ – in $x\ y\ z$ respy then the ple xyz envelopes a surface called a reciprocal cincubicoid. This surface has the property that thro' any line we can draw 3 tangent ples to it.
Let the pts P_1 P_2 be constructed for the cincubic as before. Take 3 pts $R_1 R_2 R_3$ on the given ple π & construct the triads $(BCR_1)(CDR_1)(DBR_1)$; (BCR_2) etc These meet $L_1 L_2 L_3$ resp. in 3 triads of pts (3 on L_1, 3 on L_2 etc) which form on $L_1 L_2 L_3$ a cubic homography. Let these pts be joined by ples with the line $P_1 P_2$ then the joining ples form a cubic homographic pencil which has 3 triple ples.
Let these 3 ples be $\alpha\beta\gamma$. They meet $L_1 L_2 L_3$ in 3 triads of pts $(a_1 b_1 c_1)(a_2 b_2 c_2)(a_3 b_3 c_3)$. The ples $(BCa_1)(CDb_1)(DBc_1)$ meet in a pt which lies both on the curve & on the ple π & hence is one of the pts of intersection. Similarly from β & γ we get the other two pts of intersection.

This construction requires a knowledge of the method of finding the 3 self-corr. pts of a cubic homography. But we have virtually in finding the pts in which a line meets a concubicoid formed a construction for the the 3 self corresponding or triple pts of a cubic homography on a line. This construction will also give the 3 triple ples of a cubic homographic pencil.

To const ~~a~~ the curve in which a concubicoid meets a given ple.

Let O be the base pt of the concubicoid as found above. Take a series of lines thro' O & find the pts by the above construction in which the concubics to which they give rise meet the ple. These pts trace out the required curve. If the given ple is the ple at infinity this practically gives the asymptotic cone of the surface.

1 [illegible] pt-systems of hyperbocubicoids.

A series of hyperbocubicoids thro' the F-system & one fixed pt is analogous to the series of 5 pt. concubicoids.

3 degenerate surfaces are possible viz

ple (PI) + hyd. (QR) ; ple (QI) & hyd (RP) etc.

where I is the common pt.

This system of surfaces meets any line in a cubic involution.

If a hyperbocubicoid be given by 3 pts a, b, c, we can as before find the pts in which a line meets the surface.

Determination of concubicoids to satisfy certain conditions.

(1) To describe a concubicoid to pass thro' a pt I & to osculate a line L. The method has already been given; there are 3 solutions corr. to the triple pts of the invol. on L.

(2) To describe a concubicoid to pass thro' 2 pts I J & to touch a line L.

Take any pt P on L. A concubicoid thro' I J & P meets L in 2 other pts. As P varies the pts in which the variable surface meet L trace out a cubic involution of the type generated by a ple thro' a line on a cubic curve. There are 4 dps in this invol. Hence 4 surfaces can be formed to pass thro' I J & touch L.

(3) To describe a concubicoid thro' ~~2 pts~~ 1 a pt I to touch 2 lines L & M. Take pt R on M. Then thro' I R. 4 surfaces can be formed to touch L. As R varies each of the touching surfaces traces out an invol. of the above kind on M. Hence there will be 4×4 or 16 surfaces touching both L & M & passing thro' I.

(4) To describe a concubicoid to touch 3 lines L M N. There will be 64 ie 4×4×4 surfaces.

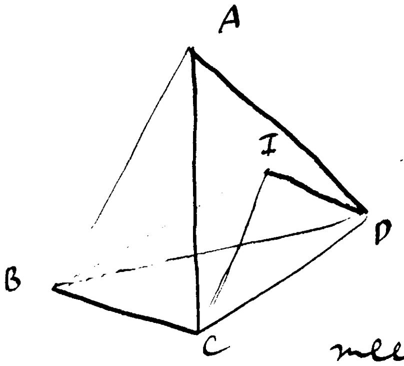
These theorems can be easily verified by transforming by the tetrahedral transf. Thus ~~th~~ transforming (3) we get: 'to find a ~~se~~ ple to touch ~~two~~ given cubic curves & to pass thro' a given pt I.

Clearly there are 16 such ples since the projection of a cubic curve on a ple is a ple cubic of class 4.

(5) To describe a concubicoid thro' ~~x~~ pts I & J to touch a given ple π.

We shall first prove that an infinite number of cubic curves thro' 5 pts can be drawn to touch a ple & that their pts of contact lie on a conic in the ple.

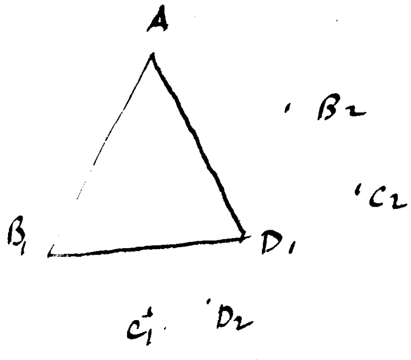
Let A B C D be the four F-pts of the curves. & I their common pt. The curves are the intersections of 3 cones of vertices B C D & ~~the~~ having the edges coterminous in these pts as generators.



Hence if any ple cut the tetⁿ A BCD the pts in which a cubic curve thro' I meets it are the pts of intersection of the conics in which 2 of these cones meet the ple.

Let us take the cones with vertices B C. Then the pt of intersection of B C with ple lies on both conics but is not on the cubic curve.

Thus we have two 4 pt conics on this ple having one pt in common. As the cubic curve varies these 4-pt conics vary & we have to find those conics which touch one another. If a conic of one system be fixed 4 conics of the other system can be found to touch it & the locus of the pts of contact is a conic. This can be shown by a quadratic ple transf. as follows.



Let $(A\ B_1\ C_1\ D_1)$ & $(A\ B_2\ C_2\ D_2)$ be the 4 pts of each system. Take $A\ B_1\ D_1$ as the F-triangle. The system thro' $A\ B_1\ C_1\ D_1$ gives a series of lines thro' C_1 on transf. & the system $A\ B_2\ C_2\ D_2$ gives a series of unicursal cubics having dps at A. If any cubic be fixed we can draw thro' C_1 four tangents to it & hence the theorem is proved.

Hence it follows that a series of cubics thro' I can be found to touch π & their pts of contact lie on a conic. Similarly a series of cubics can be found thro' J to touch π & their pts of contact also lie on a conic. These conics meet in 4 pts. Let these be $\alpha\ \beta\ \gamma\ \delta$.

Then 4 conicubicoids can be drawn viz one thro' I J α, one thro' I J β etc. These conicubicoids contain each 2 of the conicubics eg I J α contains the 2 conicubics thro' α & they each touch the ple, the pts of contact being α β γ δ.
Thus 4 surfaces can be drawn thro' I & J to touch π.

To describe a conicubicoid thro' a pt I to touch 2 ples π & ρ.
It will be shown later that the conicubicoid is of class 4.
In fact this follows from the previous theorem by transformation.
Hence thro' any pt 16 ples can be drawn ~~thro' a pt~~ to touch 2 conicubicoids; transforming ~~thus~~ this we get that 16 conicubicoids can be drawn thro' I to touch π & ρ.

To describe a conicubicoid to touch 3 ples π ρ σ.
From the previous theorem it follows that are 64 such surfaces.

To describe a conicubicoid to pass thro' 2 fixed pts I & J & touch an arbitrary conicoid.
Transforming this we have to find the number of ples thro' a line which will touch a sextic surface (corresp. to a conicoid). The following general method is applicable to this type of problem.
A ple meets such a sextic in a sextic curve with 6 dps where the edges of the F-tet^n meet the ple. Hence the class of this sextic is by Pluckers formulae $= 6 \times 5 - 2(6) = 18$. Hence thro' any pt in this ple we can draw 18 tangents to the sextic. A ple thro' an F-pt (ABCD) meets the sextic in a curve which has 3 dps & a singular pt (the F-pt) order 4; ie in all equivalent to 9 pts; hence the class of this curve is $30 - 2(9) = 12$. Hence 12 tangents can be drawn to this curve thro' an arbitrary pt.
From these results we see that the order of the tangent cone to the sextic thro' any pt O is 18 & that the curve of

contact on the sextic is a curve of order 18 having sextuple nodes at each of the 4 F-pts. This curve transforms into a curve of the 6th order on the conicoid. Now the curve of order 18 on the sextic lies on a quartic surface having 4 triple conical pts. This transforms into a cubic surface thro' ABCD. Hence the sextic on the conicoid is the intersection of that surface with a cubic surface. The sextic curve therefore meets each generator in 3 points. It is a curve of deficiency 4. Hence we have proved that an infinite number of cubic curves can be drawn to touch a conicoid & pass thro' a fixed pt I. Their pts of contact lie on a sextic curve of defy 4 on the conicoid. Similarly the ~~sex~~ cubic curves thro' J which touch the conicoid are infinite in number & their pts of contact lie on another sextic. Now it will be shewn in connection with the general quadratic space transf. that two such sextic curves on a conicoid intersect in 6×6 − 2(3×3) i.e. 18 pts.

Hence thro' I & J we can draw 18 concubicoids to touch a conicoid.

We can also draw 18^2 i.e. 324 concubicoids thro' a pt I to touch 2 conicoids.

We can find in an exactly similar manner the number of concubicoids thro' 2 pts I & J which touch an arbitrary cubic surface & so on.

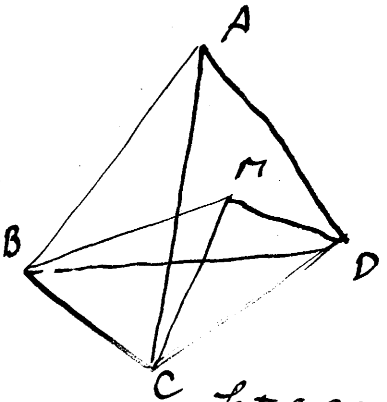
Art. 20.

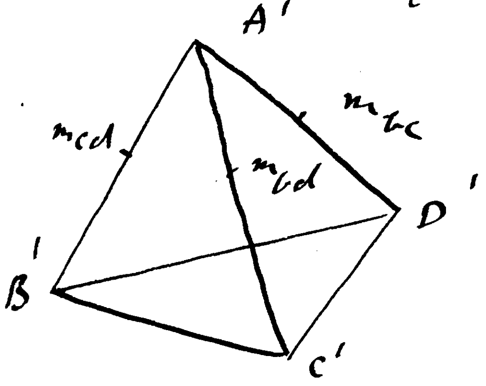
The correlatives of the foregoing transformations

(a) The correlative cono-cylindrical transf.

In space (S) the primary element is the pt & space (S') the primary element is the ple.

Let ABCD A'B'C'D' be the tetrah. of reference.





Let a range of pts be taken on A'D' homographic with the sheaf of ples (BC) & let similar ranges be taken on A'B' & A'C' homog. respectively with the sheaves (CD) & (BD). Let the following suppositions be made.

To ple	BDC	of sheaf	(BC)	corresponds	pt	D'	of range	A'D'
"	BAC	"	"	"	"	A'	"	"
"	BCD	"	(BD)	"	"	A'	"	A'C'
"	BAD	"	"	"	"	C'	"	"
"	CBD	"	(CD)	"	"	A'	"	A'B'
"	CAD	"	"	"	"	B'	"	"

Let the ples BMC, CMD, DMB intersect in M & the corr. pts of the ranges A'B', A'C', A'D' be m_{cd}, m_{bd} & m_{bc} then the ple (m_{cd} m_{bd} m_{bc}) or short (m) corr. to pt M.

Let now M describe a line l; then the pts m_{cd} m_{bd} describe perspective ranges having a common pt at A'. & hence line m_{cd} m_{bd} passes thro' a fixed pt.

The lines m_{bd} m_{bc} & m_{cd} m_{bc} envelope conics which touch A'C' A'D' & A'B' A'D' resp^{ly}.

Hence the ple m passes thro' a fixed pt &

envelopes a cone. The line A'D' is tangent to this cone as is also the ple A'B'C'. The vertex of the cone lies in ple A'B'C'.

Let now M describe a ple then the distances ($s_1\, s_2\, s_3$) of m_{bc} m_{bd} m_{cd} from A' are related by a relation of the form

$$\kappa s_1 s_2 s_3 + \lambda s_1 s_2 + \mu s_2 s_3 + \nu s_3 s_1 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0$$

but when $s_1 =$ const. the relation between $s_2\, s_3$ is of the form $a s_2 + b s_3 + c = 0$ hence

$\kappa = \mu = 0$ & relation becomes

$$\lambda s_1 s_2 + \nu s_3 s_1 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0 \qquad (A)$$

This may be regarded as the tangential eqn. of the surface enveloped by m. It is a conicoid since (A) is of order 2.

When $s_1 = a$ (const) relation (A) must reduce with the suppositions of the previous page to the form $k s_2 + l s_3 = 0$ since when $s_2 = 0$ $s_3 = 0$ also.

Hence $\alpha a + \delta = 0$ for all (a) hence $\alpha = \delta = 0$

Hence (A) takes the special form

$$\lambda s_1 s_2 + \nu s_3 s_1 + \beta s_2 + \gamma s_3 \cancel{+\delta} = 0 \qquad (A')$$

Exceptional elements of transf.

To any pt on ple BCD corresponds a ple thro' A'D'. This ple is indeterminate hence the pts of BCD are exceptional elements.

Corresp. to any pt on ple ACD we get ple A'C'D' hence all conicoids (A) must touch ple A'C'D'.

Just as in the cono-cyl. transf. there will be one value of s_2 & one of s_3 which have no unique corresp. s_1. The form (A') shows that these values are zero hence there will be a line thro' A' thro' which an infinite number of tangent ples can be drawn to the conicoid.

Since there is only one such line it follows that the conicoid must be degenerate. It must in fact be the ultimate form of a very flat conicoid i.e. a ple conic.

Form (A') therefore represents a ple conic & this conic must touch ple A'B'C' & the line A'D'.

Conversely if the ple m envelopes a point the pt M traces out a cone of generic eqn. $\lambda S_1 S_2 + \nu S_3 S_1 + \alpha S_1 + \beta S_2 + \gamma S_3 + \delta = 0$, & if ple m envelopes a line the pt M traces out a conic. The cone will contain the line BC & pass thro' the pt D. It will touch the ple BCD along the line BC.

— , —

Incident elements when spaces are superposed.

To find the locus of ples which contain their corresponding pts.

Let the ple m envelope a line. The corr. pt M traverses a conic. Let m meet A'B' A'C' A'D' in the pts m_{cd} m_{bd} m_{bc}. Then the lines joining M & m_{cd}, M & m_{bd} M & m_{bc} trace out hyperboloids.

The ple m meets the conic traced out by M in 2 pts which with M make up an involution of order 3. This has 3 double pts & hence thro' any line we can draw 3 ples which contain their corresp. pts. Such ples therefore envelope a surface of class 3.

To find the locus of pts which lie on their corresp. ples.

Let M describe a line the lines M m_{cd}, M m_{bd}, M m_{bc} trace out hyds. These hyds. have a common gen^r. viz the line described by M & hence they have 4 common tangent ples. One of these contains the line A'D' & so depends on choice of let^n. It therefore does not count.

Hence these hyps. have 3 proper common tangent ples hence there will be positions of pt M for which it lies in its corr. ple. The surface of incident pts is therefore of 3rd order.

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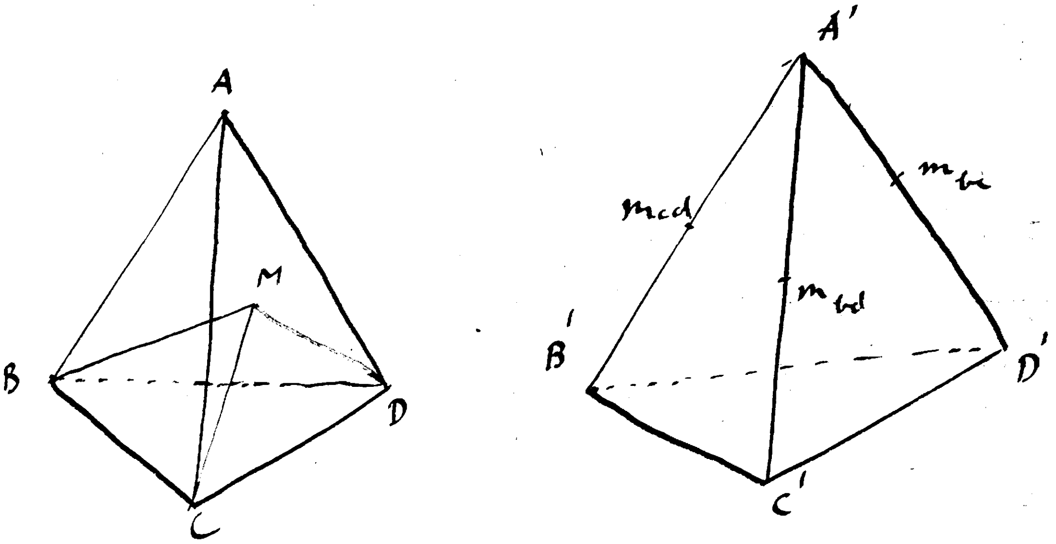
The correlative cubic transformations.

(a) The tetrahedral transf.



To ples of sheaf BMC let a range of pts on A'D' corresp. & to sheaf CMD a range on B'A' & to sheaf BMD a range on A'C'.

Let also ple BAC (BC) corr. to A' & ple BDC (BC) to D' with similar suppositions for the other sheaves. Corresp. to ples BMC, CMD, DMB we get the pts m_{cd} m_{bd} m_{bc} forming a ple m.

When M describes a line l lines m_{cd} m_{bd}, m_{bd} m_{bc} & m_{bc} m_{cd} envelope conics which touch the lines A'B' A'C' A'D'. The ple m touches these conics & traces out a developable surface. Thro' any pt 3 ples of this developable pass for thro' any pt. P of space we can draw three common tangent ples to these conics.

When M describes a ple m envelopes a surface. Let us before the distances of m_{cd}, m_{bd}, m_{bc} .

from A' be denoted by $S_2 S_3 S_1$ then the tangential eqn. of the surface is

$$\kappa S_1S_2S_3 + \lambda S_2S_3 + \mu S_3S_1 + \nu S_1S_2 + \alpha S_1 + \beta S_2 + \gamma S_3 + \delta = 0$$

The surface is therefore of the third class.
The developable is also of the third class.

Conversely if ple m envelopes a pt. M describes a surface of generic eqn.

$$\kappa S_1S_2S_3 + \lambda S_2S_3 + \quad etc \quad = 0$$

i.e. a cuncubicoid. If m envelopes a line M describes a cuncubic.

Hence to a pt P of space (S') there corresponds a cuncubicoid of (S). If P move along a line the cuncubicoids form a 6 pt-system having a common cuncubic of intersection. Since 4 such cuncubicoids can be drawn to touch an arbitrary ple it follows that the line which P describes meets the surface corresponding to a ple of (S) in 4 pts. Hence this surface is of the 4th order. We shall call it the reciprocal cuncubicoid.

When M describes a conicoid m envelopes a surface of 6th class. From the results given on p114 we see that this surface is of 18th order. When M describes a conic m envelopes a developable of 6th class also.

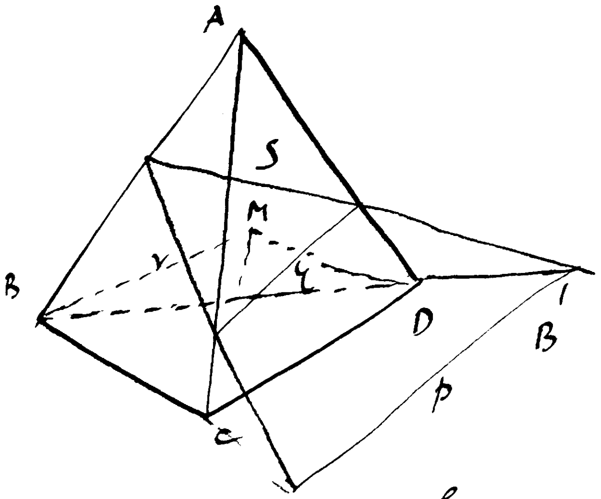
We shall call the developable which corresponds to a line the reciprocal cuncubic.
Properties of the recip. cuncubic & recip. cuncubicoid.

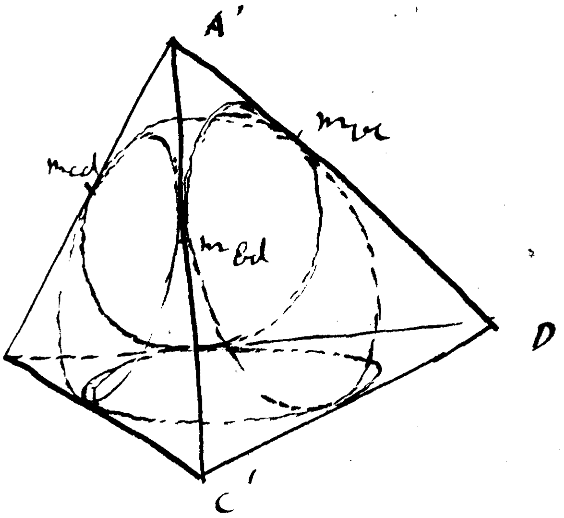
Consider any ple π in relation to the former.
Thro' any pt P on π there pass 3 ples. belonging to the

developable. These ples osculate the edge of regression, hence thro' a pt 3 osc. ples can be drawn to the edge of regression. The latter is a conicubic.

Consider the system of ples which make up the developable & which osculate its edge of regression. When any two ples of system are given their line of intersection meets π in a pt P say. Hence the third ple is known. Thus the system of ples belonging to the surface which meet in a given ple satisfies a relation which ~~is a~~ cubic involution. There will be three triple ples of system & these are clearly the osculating ples at the pts where the edge of regression meets π. Again there will be two osculating ples whose line of intersection lies in π. These are the F-ples of the system; they have no unique correspondent. The pts of contact of the osculating ples with the edge of regression clearly form a cubic involutive system on the edge of regression. This result we have already attained on p. 89.

The reciprocal concubicoid.





Let a ple π meet ~~edges~~ faces of F-tetⁿ in lines p q r & s.
Corr. to line p we have the single ple which must therefore be a singular tangent ple of surface. Similarly for lines q r & s.

Let M describe the line ξ. The ples BMC, BMD generate a perspective homog. system & hence (m_{bc} m_{bd}) generate homographic ranges. The line (m_{bc} m_{bd}) consequently envelopes a conic which must be the curve of contact of the singular tangt-ple C'A'D'. Similarly the curves of contact on the other faces are conics. These conics touch the edges of the faces in which they lie. The two pts of contact on any edge of A'B'C'D' are coincident. Thus these conics touch each other on the edges of the tet^n.

Any two of these conics are ple-sections of same cone & the six vertices of these cones lie on a ple.

Also the 4 conics of contact lie on a conicoid which touches the edges of the tet^n.

The fundamental theorem for the reciprocal conicuboid.

— / —

This surface is completely determined when when its 4 singular tangt-ples & 3 ord. tangt. ples are given.

Let four ples of the surface be π_1 π_2 π_3 π_4.

Let ple π_1 meet AB in X_{AB}, AC in X_{AC} etc

& " π_2 " AB " Y_{AB}, AC Y_{AC} etc

" π_3 " AB " Z_{AB} etc

" π_4 " " W_{AB}

then the 4 ples

$$\left\{\begin{array}{llll} (X_{AB}) & (W_{BD}) & (Y_{DA}) & (1) \\ (X_{CB}) & (Z_{BD}) & (Y_{DC}) & (2) \\ (Y_{AC}) & (W_{CB}) & (Z_{BA}) & (3) \\ (X_{AC}) & (W_{CD}) & (Z_{DA}) & (4) \end{array}\right.$$

are concurrent.

In connection with the 4 ples π_1 π_2 π_3 π_4 there are 240 such pts & these lie by groups of 8 on ples.

5 & 6 ple reciprocal cricubicoids.

Theorems analogous to those given on pp 106-107 can be given. For example with a system of 5 ple-surfaces the tangent ples which pass thro' an arbitrary line form a cubic involution pencil. This pencil has 3 triple ples & hence there will be 3 surfaces which have one of these ples as triple tangent ple.

Incident elements (spaces superposed).

Locus of pts which are united with their corresp. ples.

Let P describe an arbitrary line l. Lines Pm_{bd} Pm_{bc}, Pm_{cd} generate hyds. having a common generator l. 3 such hyds. have 4 common tangt. ples. These ples meet each hyd. in generators hence there will be four positions of P in which ple m passes thro' P. Thus the locus of incident pts is a surface of order 4. It clearly contains each of the edges of the tetⁿ ABCD. Also ABCD are conical pts of second order on it. for if l pass thro' A then A' is a common pt on the ranges generated by m_{bc} m_{bd} n_{cd} & the hyds. have another common genʳ viz AA'. The two common genˢ meet at A & hence the hyds. have now only 2 common tangent ples. Hence apart from A the line l meets surface in 2 pts. Hence ABCD are conical pts.

Locus of ples which contain their corresp. pts.

Consider a series of ples ~~in (2)~~ thro' an arbitrary line l. To this series corresponds a concubic thro' ABCD. The ples m cut this curve in 3 pts which with P the generating pt of the concubic constitute an involution of degree 4. This has 4 self-corr. pts & hence thro' l we can draw 4 ples which contain their corr. pts.

The envelope of these ples is therefore a surface of the 4th-class. The pts A' B' C' D' are conical pts on this surface. For if l pass thro' A' the ple l A'A contains its corr. ~~&~~ similarly for anyline thro' A'. Also only one ple can be drawn thro' l which contains its corr. viz A. The edges of the tetⁿ A'B'C'D' also lie on this surface.

Spaces referred to the same tetⁿ.
In this case the surfaces of incident elements coincide & are made up of the four faces of the common tetrahedron.

(b) The parallelepipedal transf.
The results for this transf. need only be stated.
If P describe a line m describes a developable of 3rd class as before. When P describes a ple m envelopes a surface of class 3 but in general of order 12. Conversely if m envelope a ~~pt~~ line P describes a cubic curve (a hyperbocubic) & if m envelope a pt P describes a hyperbocubicoid. The surfaces of incident elements are of the 4th order & 4th class as before but in general have no conical pts.

These transformations also result by combining with the foregoing pt-pt transformations, a correlative trans.
Thus let C_3 be a cubic pt-pt transf. e.g the tetrahedral & let C_1' be a correlation then $C_3 C_1'$ & $C_1' C_3$ denote the resulting correlative transf. $C_3 C_1'$ gives the transf in one direction e.g. to pt M corr. ple ~~x~~ m etc while $C_1' C_3$ gives the transf. in the opposite

director viz to a pt corresponds a cubic surface etc.

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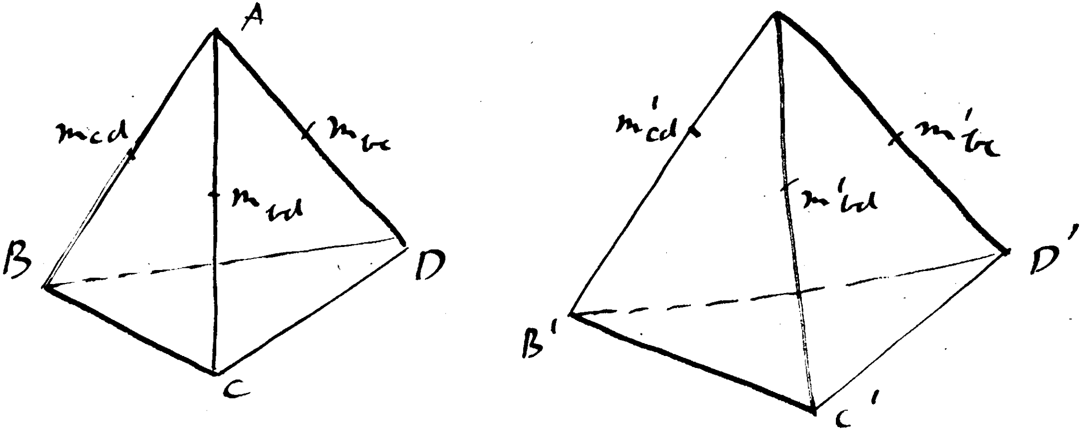
Art. 21. The planar transformations.

But we may also make the primary element in both spaces a ple. In this we get what we shall call a planar transf.

For example we shall give the cubic planar transf (tetrahedral).



To establish this transf. we take homographic ranges on (AB, A'B') (AC, A'C') (AD, A'D').

Let C corr. to A', D to A' & B to A' & conversely C' D' B' to A.

Then as in the previous transf. to the ple (m_{cd} m_{bd} m_{bc}) or m corr. the ple (m'_{cd}, m'_{bd} m'_{bc}) or m' which contains the corr. pts.

When m envelopes a line m' envelopes a reciprocal concubic & when m envelopes a pt, m' envelopes a reciprocal concubicoid & conversely.

These results follow as in the pt-pt correspondence.

Also if m envelopes a conicoid m' envelopes a reciprocal sextic & so on.

The transf. does not bring to light any new surfaces.

When the spaces are superimposed there are 8 double ples & the transf. is involutive.

The tetrahedra may be superimposed in two ways.

(a) A'B'C'D' may coincide with ABCD resp^ly. In this case there are 8 d ples.

(b) A'B'C'D' may coincide with ACDB resp^ly. In this case there is an envelope of double ples viz a reciprocal conicubic

Two corresp. ples intersect in a line thro which two planes of the surface pass & they are harmonic with respect to these ples.

This transf. may also be derived as follows. Let S_1 & S_2 be two spaces between which exists a cubic pt-pt corr. Let C_1' be correlative with S_1 & C_2' with S_2 then C_1' & C_2' are connected by a cubic planar correspondence.

— / —

Art 22. Analytical formulae for the foregoing transf.

(1) The cubic transformation. (tetrahedral).

Let ple BCD be $w=0$ & B'C'D' be $w'=0$
BAC " $x=0$ B'A'C' " $x'=0$
CAD " $y=0$ etc.
DAB " $z=0$

Let ple BCM be $k_4 x - k_1 w = 0$
then ple B'C'M' is $x' - \dfrac{\alpha k_1 + \beta k_4}{\gamma k_1 + \delta k_4} w' = 0$

Hence $\gamma x' x + \delta x' w - \alpha x w' - \beta w w' = 0$

now $x = 0$ corr. to $w' = 0$
& $w = 0$ " " $x' = 0$ $\therefore \delta = 0 = \alpha$

$\therefore \gamma x x' = \beta w w'$ ie $\dfrac{x x'}{\beta} = \dfrac{w w'}{\gamma}$

Similarly $\dfrac{y y'}{\beta_1} = \dfrac{w w'}{\gamma_1}$ $\dfrac{z z'}{\beta_2} = \dfrac{w w'}{\gamma_2}$

Hence $(x y z w)$ being coords of M & $(x' y' z' w')$ those of M' $\dfrac{x x'}{\lambda} = \dfrac{y y'}{\mu} = \dfrac{z z'}{\nu} = \dfrac{w w'}{\rho}$.

To the ple $\alpha x+\beta y+\gamma z+\delta w=0$ corresp.

$\frac{\alpha\lambda}{x'}+\frac{\beta\mu}{y'}+\frac{\gamma\nu}{z'}+\frac{\delta\rho}{w'}=0$ which is the eqn of a concubicoid.

Let a line be given by

$x=a_1s+a_2 \quad y=b_1s+b_2 \quad z=c_1s+c_2 \quad w=d_1s+d_2$

s being a variable parameter, the corresp. is

$x'=\frac{1}{a_1s+a_2} \quad . \quad y'=\frac{1}{b_1s+b_2} \quad z'=\frac{1}{c_1s+c_2}$ etc

which are the eqns of a concubic.

The fund. theorem on the concubicoid.

Let $(x_iy_iz_iw_i)$ $i=1,2,3,4$ be the coords of the 4 given pts (1234). Ple $(AC2)$ is $\frac{x}{x_2}=\frac{y}{y_2}$

$(CB4)$ $\frac{x}{x_4}=\frac{w}{w_4}$ & $(BA3)$ is $\frac{x}{x_3}=\frac{z}{z_3}$

hence their pt of intersn. $\left(1, \frac{y_2}{x_2}, \frac{y_3}{x_3}, \frac{w_4}{x_4}\right)$

Similarly pt of intersection of

$(AC1)(CD4)(DA3)$ is $\left(\frac{x_1}{y_1}, 1, \frac{z^3}{y_3}, \frac{w_4}{y_4}\right)$

& of $(AB1)(BD4)(DA2)$ is $\left(\frac{x_1}{z_1}, \frac{y_2}{z_2}, 1, \frac{w_4}{z_4}\right)$

& of $(CB1)(BD3)(DC2)$ is $\left(\frac{x_1}{w_1} \frac{y_2}{w_2}, \frac{z_3}{w_3} 1\right)$

But since (1234) lie on the surface whose eqn is

$\frac{l}{x}+\frac{m}{y}+\frac{n}{z}+\frac{r}{w}=0$ we get

$\frac{l}{x_1}+\frac{m}{y_1}+\frac{n}{z_1}+\frac{r}{w_1}=0$ etc

$\frac{l}{x_2}+\frac{m}{y_2}+\frac{n}{z_2}+\frac{r}{w_2}=0$

whence elimn $l\ m\ n\ r$ we get

$$\begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & \frac{1}{z_1} & \frac{1}{w_1} \\ \frac{1}{x_2} & \frac{1}{y_2} & \frac{1}{z_2} & \frac{1}{w_2} \\ \frac{1}{x_3} & \frac{1}{y_3} & \frac{1}{z_3} & \frac{1}{w_3} \\ \frac{1}{x_4} & \frac{1}{y_4} & \frac{1}{z_4} & \frac{1}{w_4} \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 1 & \frac{y_2}{x_2} & \frac{z_3}{x_3} & \frac{w_4}{x_4} \\ \frac{x_1}{y_1} & 1 & \frac{z_3}{y_3} & \frac{w_4}{y_4} \\ \frac{x_1}{z_1} & \frac{y_2}{z_2} & 1 & \frac{w_4}{z_4} \\ \frac{x_1}{w_1} & \frac{y_2}{w_2} & \frac{z_3}{w_3} & 1 \end{vmatrix} = 0$$

Hence the above pts of intersection are coplanar as proved on pp 101-2.

Parametric form of eqn. to crumbicoid.

A ple may be represented parametrically by

$x = \lambda_i u + \mu_i v + \nu_i w \qquad i = 1, 2, 3, 4.$

Hence the crumbicoid may be represented by

$$x' = \frac{1}{\lambda_1 u + \mu_1 v + \nu_1 w} \qquad y' = \frac{1}{\lambda_2 u + \mu_2 v + \nu_2 w} \quad \text{etc}. \qquad (A)$$

We may regard $(u\ v\ w)$ as the coords. of a pt in the ple. If pt $(u\ v\ w)$ trace out a curve of eqn. $f(u\ v\ w) = 0$ the eqns. $x' = \frac{1}{\lambda_1 u + \mu_1 v + \nu_1 w}$ etc

give with $f(u\ v\ w) = 0$ the eqn. of its corresp.

Thus let $f(u\ v\ w) = 0$ be the conic

$\alpha uv + \beta vw + \gamma wu = 0$ we get from (A)

$u = \frac{A_1}{x'} + \frac{B_1}{y'} + \frac{C_1}{z'} \qquad v = \frac{A_2}{x'} + \frac{B_2}{y'} + \frac{C_2}{z'}$ etc.

$$\therefore \alpha\left(\frac{A_1}{x'} + \frac{B_1}{y'} + \frac{C_1}{z'}\right)\left(\frac{A_2}{x'} + \frac{B_2}{y'} + \frac{C_2}{z'}\right) + \text{etc} = 0$$

This represents a cone thro' A'. The intersn. of this cone with the crumbicoid got by elimn. $u\ v\ w$ from (A) gives the corresp. of the conic $\alpha uv +$ etc $= 0$

(2) The parallelepipedal transf.

Let pts in space (S) be referred to axes thro' the centre of parallelepiped (PQR) // P, Q, R & similarly with space (S'). The formulae of transf. obtained in same manner are

$$\beta(y-c)(y'-c') = (x-a)(x'-a')$$
$$\beta_1(z+b)(z'+b') = (x+a)(x'+a')$$
$$\beta_2(y+c)(y'+c') = (z-b)(z'-b')$$

Hence $x = \dfrac{\varphi_1(x'y'z')}{\varphi_4(x'y'z')}, \qquad y = \dfrac{\varphi_2(x'y'z')}{\varphi_4(x'y'z')}$ etc

where $\varphi_1\ \varphi_2\ \varphi_3$ & φ_4 are cubic fns. of $(x'y'z')$.

(3) The correlative transf.

Let pl^e BMC be $k_4 x - k_1 w = 0$

CMD " $k_4 y - k_2 w = 0$

DMB " $k_4 z - k_3 w = 0$

& let n_{bc} be $x' = \alpha k_1 \quad w' = \delta k_4$

m_{cd} " $y' = \beta k_2 \quad w' = \delta k_4$

~~$z' =$~~ n_{bd} " $z' = \gamma k_4 \quad w' = \delta k_4$.

If $\lambda x' + \mu y' + \nu z' + \rho w' = 0$ be the eqn. of n we have

$$\begin{vmatrix} x' & y' & z' & w' \\ \alpha k_1 & 0 & 0 & \delta k_4 \\ 0 & \beta k_2 & 0 & \delta k_4 \\ 0 & 0 & \gamma k_3 & \delta k_4 \end{vmatrix} = 0$$

giving $$\frac{x'}{\alpha k_1} + \frac{y'}{\beta k_2} + \frac{z'}{\gamma k_3} + \frac{w'}{\delta k_4} = 0$$

Thus corresp. to the pt $(k_1 \, k_2 \, k_3 \, k_4)$ or $(x y z w)$ of space (S) we get

$$\frac{x'}{\alpha x} + \frac{y'}{\beta y} + \frac{z'}{\gamma z} + \frac{w'}{\delta w} = 0$$

the coords of this pl^e are

$$\xi' = \frac{1}{\alpha x} \quad \eta' = \frac{1}{\beta y} \quad \zeta' = \frac{1}{\gamma z} \cdot \theta' = \frac{1}{\delta w}.$$

Let $x = a_1 s + b_1$ $y = a_2 s + b_2$ $z = a_3 s + b_3$ etc be the eqn of a line its corr. is given by

$$\frac{x'}{\alpha(a_1 s + b_1)} + \frac{y'}{\beta(a_2 s + b_2)} + \cdots = 0$$

a developable of 3rd class.

Corr. to the pl^e $x = a_i u + b_i v + c_i w$ $i = 1\ 2\ 3\ 4$

we get $\dfrac{x'}{\alpha(a_1 u + b_1 v + c_1 w)} + \cdots = 0$

which gives an envelope of 3rd class.

Surfaces of incident elements.

Let the coords. of a pt $(x'y'z'w')$ referred to tet^n ABCD be $\overset{a}{\alpha_i} x + \overset{e}{\beta_i} y + \overset{c}{\gamma_i} z + \overset{d}{\delta_i} w$ $i = 1, 2, 3, 4$

then ple $\frac{x'}{\alpha x} + \frac{y'}{\beta y} + \frac{z'}{\gamma z} + \frac{w'}{\delta w} = 0$ contains its

corr. pt if $\frac{a_1x+b_1y+c_1z+d_1w}{\alpha x} + \frac{a_2x+b_2y+c_2z+d_2w}{\beta y} + \dots = 0$

this is the eqn of ~~pts~~ : surface of pts which lie in their corr. ples. Clearly it has conical pts of second order at A B C D.

Let coords of ple $\xi \quad \eta \quad \zeta \quad \omega$ $(= \frac{1}{\alpha x}$ etc $)$ be referred to tetn $A'B'C'D'$

$$\xi = \frac{1}{\alpha(A_1x'+B_1y'+C_1z'+D_1w')} \text{ etc.}$$

Hence $x' = \frac{p_1}{\xi} + \frac{q_1}{\eta} + \frac{r_1}{\zeta} + \frac{s_1}{\theta}$ etc.

The ple $\xi \frac{x'}{\cancel{\alpha x}} + \eta y' + \dots = 0$ contains its corr pt if

$$\xi\left(\frac{p_1}{\xi} + \frac{q_1}{\eta} + \frac{r_1}{\zeta} + \frac{s_1}{\theta}\right) + \dots = 0$$

giving a surface of 4th class.

(4) The planar correspondence.

Here we have the ple coords $(\xi\eta\zeta\theta)$ & $(\xi'\eta'\zeta'\theta')$

Let the eqn of pt n_{oc} be $k_4\xi - k_1\theta = 0$

.... n'_{oc} " $k_1\xi' - k_4\theta' = 0$

etc

then we get $\frac{-\xi\xi'}{\kappa} = \frac{\eta\eta'}{\lambda} = \frac{\zeta\zeta'}{\mu} = \frac{\theta\theta'}{\nu}$

To ple $(\xi\eta\zeta\theta)$ corr. the ple $(\frac{1}{\xi'}, \frac{1}{\eta'}, \frac{1}{\zeta'}, \frac{1}{\theta'})$.

To pt $\xi x + \eta y + \zeta z + \theta w = 0$ corr.

$\frac{x}{\xi'} + \frac{y}{\eta'} + \frac{z}{\zeta'} + \frac{w}{\theta'} = 0$ the eqn. of the recip conicubicoid in ple coords.

The point eqn of the recip. conicubicoid.

Let $\frac{x_1}{\xi} + \frac{y_1}{\eta} + \frac{z_1}{\zeta} + \frac{w_1}{\theta} \equiv \phi(\xi\eta\zeta\theta) = 0$

The eqn of pt of contact of ple $(\xi_1, \eta_1, \zeta_1, \theta_1)$ with ϕ is

$$\xi\frac{\partial\varphi}{\partial\xi_1}+\eta\frac{\partial\varphi}{\partial\eta_1}+\zeta\frac{\partial\varphi}{\partial\zeta_1}+\theta\frac{\partial\varphi}{\partial\theta_1}=0$$

ie $\xi\frac{x_1}{\xi_1^2}+\eta\frac{y_1}{\eta_1^2}+\zeta\frac{z_1}{\zeta_1^2}+\theta\frac{w_1}{\theta_1^2}=0$

∴ pt of contact is $x=\frac{x_1}{\xi_1^2}$ $y=\frac{y_1}{\eta_1^2}$ $z=\frac{z_1}{\zeta_1^2}$ etc

Hence locus of pt of contact is

$\frac{x_1}{\sqrt{\frac{x_1}{x}}}+\frac{y_1}{\sqrt{\frac{y_1}{y}}}+\frac{z_1}{\sqrt{\frac{z_1}{z}}}+\cdot=0$ or

$\sqrt{xx_1}+\sqrt{yy_1}+\sqrt{zz_1}+\sqrt{ww'}=0$ which is of 4th order.

Rationalised this reads

$$\left[\cancel{(x}\,(xx_1+yy_1-zz_1-ww_1)^2-4(xyx_1y_1+zwz_1w_1)\right]^2=64\,xyzw\,x_1y_1z_1w_1$$

the conics of contact of the singular tangent planes lie on the conicoid

$$(xx_1+yy_1-zz_1-ww_1)^2-4(xyx_1y_1+zwz_1w_1)=0$$

or $(xx_1)^2+(yy_1)^2+(zz_1)^2+(ww_1)^2-2xzx_1z_1$
$-2xwx_1w_1-2xyx_1y_1-2zwz_1w_1=0$

which clearly touches the edges of the F-tetn.

— 1 —

Note on the sector surface which corresponds to a conicoid in the tetrahedral transf.

If the eqn of the conicoid be

$$a_{11}x^2+a_{22}y^2+a_{33}z^2+a_{44}w^2+2a_{12}xy+2a_{23}yz+2a_{13}xz+2a_{14}xw+\quad=0$$

the eqn of the sector is

$$\frac{a_{11}}{x^2}+\frac{a_{22}}{y^2}+\frac{a_{33}}{z^2}+\frac{a_{44}}{w^2}+\frac{2a_{12}}{xy}+\cdot\qquad=0$$

omitting consts. of transf.

This can be written in the form

$$\left\{\sqrt{a_{11}}\,yzw+\sqrt{a_{22}}\,xzw+\dots+\sqrt{a_{44}}\,xyz\right\}^2$$
$$=2xyzw\left\{\sqrt{a_{11}a_{22}}\,zw+\sqrt{a_{22}a_{33}}\,xw+\cdot\ -\ -(a_{34}xy+a_{41}zx+\cdot\quad)\right\}$$

The quantities $\sqrt{a_{11}}$ etc may have either the + or - sign & considering the conicoid

$$\sqrt{a_{11}a_{22}}\,zw + \sqrt{a_{22}a_{33}}\,xw + \ldots - a_{34}\,xy - a_{41}\,zx - \text{etc} = 0$$

we get 15 different combinations of signs & hence 15 conicoids of above form.

Now these conicoids each meet the sextic in a curve of degree 12 but which is composed of two coincident curves of deg 6. The conicoids therefore touch the sextic along a sextic curve. Transforming this theorem we get the following. 15 conicoids thro' 4 pts A B C D can be drawn so as to have double contact along a conic with any given conicoid.

— 1 —

Art 23. The geometry of the tetrahedron in the light of these transformations.

(I) In tetrahedral inversion the d.ps are the centres of the escribed & inscribed spheres of the tet:

The mid-pts of the lines joining the centres of these spheres lie on a cono-cubicoid. Let 7 & 6 be the centres of two of these spheres. Corresp. pts which lie on the line 7 6 are harmonic conjugates with respect to 7 & 6. The mid pt of (7 6) & the pt at infinity on it are therefore corr. pts of transf. But the locus of the pts at infinity on all these lines is the pl. at infinity hence the mid pts of these lines lie on the cono-cubicoid which corresponds to the pl. at infinity.

(II) The feet of the perps. from any pt on this cono-cubicoid on the faces of the tet: are coplanar.

Let M & M' be corresp. pts. Their 8 projections on the faces of the tet: lie on a sphere. When one of the pts M' say is at infinity the sphere becomes a pl. for its centre is the mid pt of the line M M'.

This ple is perp^r to MM'. Hence the perps. from M on the faces of the tetr^n are coplanar. But M lies on the above circumcubicoid which has therefore the given property. It may consequently be called the pedal circumcubicoid.

(III) Let ABCD be a tetrah. & O any pt. Let a pencil of ples be described thro' O cutting the edges of the tetr^n in (ac) (ad) (ab).
The ples (BC, ad) (CD, ab) (BD, ac) meet in a pt P. The locus of P is a circumcubicoid with ABCD as conical pts. If we take a sheaf of ples thro' a line l the locus of P is a circumcubic thro' ABCD. Given 3 points on the circumcubicoid we can construct it as follows. The three ples ((ab) (ac) (ad)) constructed by means of the 3 given pts intersect in a pt O. Taking a pencil of ples thro' O we can get as many other pts as we please. Similarly we can construct the circumcubic which is given by 2 pts.

Take any conicoid C & let a tangent ple to it cut AB AC AD in the pts (ab) (ac) (ad): The locus of P is now a sextic surface having conical pts of 4th order at ABCD & the edges of tetr^n as double lines.
If we draw a tangent cone from a pt O to conicoid C the ples of this cone cut the edges AB AC AD in pts (ab) (ac) (ad) & the locus of P obtained from these pts is a sextic curve having cusps at ABCD.

Representation of a conicoid on a sextic surface. Each tangent ple to the conicoid gives rise to one pt on the sextic; hence we may regard the pt of contact of the tangent ple as corresponding to this pt on the sextic.

Thus we get a simple means of constructing a pt-pt corresp. between the cnicoid & the sextic. The conic of contact of tangent cone from A to cnicoid corresp. to the conical at A & so on. A conic on the cnicoid transforms into a sextic on the sextic.
When the pt O is at infinity the concubicoid is of special form & is det^{d} by 2 pts.
If O lies on a face of tet^{n} but not on an edge the concubicoid is degenerate & consists of face & a cone. If O lies on an edge the surface consists of 3 ples.

These theorems are examples of the correlative cubic transf.

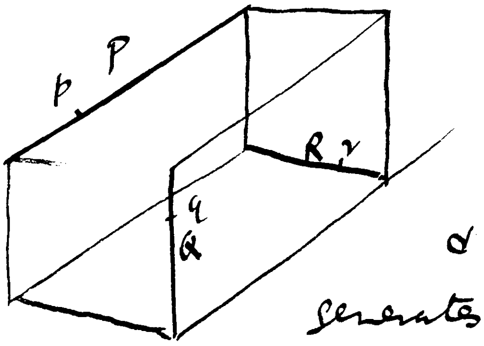
(IV) Let O be any pt as before & let its polar ple with respect to the tet^{n} be constructed. When O describes ~~a ple~~ a ple its polar ple envelopes a reciprocal concubicoid & when O describes a line its polar ple envelopes a reciprocal concubic.
If the polar ple envelope a fixed pt P its pole O generates a concubicoid with ABCD as conical pts. If it revolve about a fixed line, O generates a concubic. This is an example of the same transf.

(V) Let M be any pt & let AM, BM, CM & DM meet the opposite faces in M_1 M_2 M_3 M_4. Let the reciprocal of M_1 with respect to the triangle BCD be taken & so on. Let these reciprocals be m_1 m_2 m_3 m_4. Then Am_1 Bm_2, Cm_3 & Dm_4 meet in a pt M′ which we may call the ~~tetrahedral~~ reciprocal of M.
If M describes a ple M′ describes a concubicoid & conversely. If M describes a line M′ describes a concubic & conversely. This is an example of cubic pt-pt correspondence ABCD being the F pts.

(VI) If we take a ple cutting the 6 edges of the tet: ABCD in the pts (bc) (bd) (cd) etc & if we take the isotomic conjugates of these pts with respect to the edges in which they lie, the 6 conjugates lie in a ple which we may call the reciprocal ple of the given ple. Let the given ple be called m & its reciprocal m'; then when m envelopes a pt, m' envelopes a reciprocal circumbiroid. & when m envelopes a line m' envelopes a reciprocal circumbic.

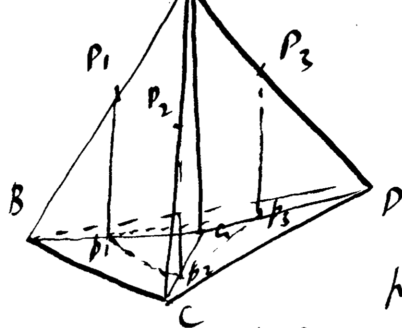
This is an example of the cubic planar transf.

(VII) Somewhat analogous theorems hold for the parallelepipedal transf.



Let a pencil of ples be described thro' O & let anyone cut the 3 fund. edges of the par^d PQR in p q & r, then the ples (Pr) (Rq) & (Qp) intersect in a pt X which generates a hyperbombiroid. If we take a sheaf of ples thro' a line l the pt X generates a hyperbombic. Other arrangements of the ples such as (Rp) (Pq) (Qr) also give rise to hyperbombiroids & hyperbombics.

(VIII)



Let pts P_1 P_2 P_3 on AB AC AD be projected on to BCD & let projections be p_1 p_2 p_3 & let proj. of A be a. Then if P_1 P_2 P_3 be variable so that area of Δ $p_1 p_2 p_3$ is constant ple $P_1 P_2 P_3$ envelopes a cubic surface of 3rd class.

The ples ABC, ACD, ABD are singular tangent ples & their curve of contact is a conic. The edges AB, AC & AD are asymptotes of these conics.

This surface is the reciprocal of a cubic surface having 3 conical pts.

It can be obtained by a correlative cubic transf. in which the 3 pts BCD are at infinity.

A conicoid thro' BCD at infinity transforms into a surface of 3rd class having 3 singular tangent ples & this is the kind of surface obtained above.

Chapter III.

Art 24. Derived transformations.

The transformations which have just been given maybe termed primary transformations.
By derived transf. we mean transf. made up by combining together primary transformations.

Transf. obtained by combining 3 correlations.
The general cubic transf.

Let a pt P of space (S) be related by 3 distinct correlations to ples π_1' π_2' π_3' of space (S'). These ples meet in a pt P' which is the correspondent of P. To a pt P corresp. one & only one pt P' & conversely to P' corresp. one & only one pt P.
Thus the transf. is (1-1).
Let P describe a line l. The ples π_1' π_2' π_3' each envelope a line & their intersection traces out a cubic curve.
Let P describe a ple p, & take any ple q' in (S'). The ples π_1' π_2' π_3' each envelop a pt & their lines of intersection with q' give rise to a triple linear collineation in q'. That is to any line in q' corresp. two other lines & when any line in q' is given its two correspondents are uniquely determined. When a line in q' envelopes a pt its correspondents also envelope pts.
The locus of the intersections of 3 concurrent corresponding lines in q' is a non-singular cubic curve.
For let P describe a line l in p the pt of intersection of π_1', π_2' π_3' describes a cubic curve which meets q' in 3 pts hence there will be 3 points on l for which the 3 corresp. lines on q' are concurrent. Hence the locus of pts on p

which give rise to concurrent lines on Σ' is a cubic curve. Further the locus of the pt of intersection of these concurrent triads must also be a curve of order 3 on Σ' by the reversible nature of the transf.

Hence any ple section of the surface corresp. to p is a cubic curve hence the surface must be a cubic surface.

Let p_1' p_2' p_3' be the pts corresp. to ple p. Then the cubic surface is the locus of the intersection of corresp. ples thro' p_1' p_2' p_3'. The ples thro' any one of these pts form a pencil & any two of the pencils are clearly related by a collinear relationship. Thus we have established that the cubic surface is the locus of the intersection of corresp. ples of 3 collinear pencils.

Exceptional elements of the transf.

For special positions of the pt P it may happen ~~that~~ the three ples π_1' π_2' π_3' ~~are~~ intersect in a line & not a pt.

Let P_1 be a pt whose corresp. ples π_1' π_2' π_3' intersect in a line L_1'. To any pt on L_1' corresponds the pt P_1. Since all ples of (S') meet L_1' it follows that the corresp. surfaces in (S) all pass thro' P_1. The lines L_1' in (S') generate a ruled surface & the corresp. pts P_1 in (S) a space curve. Similarly the lines L_1 in space (S) generate a ruled surface & the pts P_1' in " (S') " space curve.

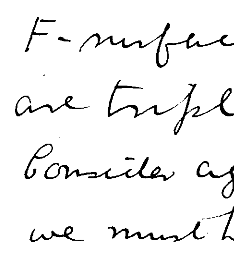
These space curves are F-curves for the space curve of (S) is common to all cubic surfaces in (S) & similarly for that of (S').

The degree of these space curves is 6. For any two cubic surfaces in (S') intersect in a curve of degree 9. Their corresp. ples in (S) intersect in a line & the correspondent of this line is a space cubic. Hence the proper (variable) intersection of two cubic surfaces is a cubic curve. All such surfaces must therefore contain a fixed sextic space curve. Hence the degree of the above F-curve is 6.

The F-surfaces.

The surfaces traced out by L_1 & L_1' respectively are F-surfaces. Their degree is 8 & the sextic F-curves are triple curves on them.

Consider again the line L_1'. Corresp. to L_1' in (S') we must have a cubic curve in (S). If a variable pt p_1' describe L_1' its corresp. ples in (S) envelope lines which meet in P_1. Hence the cubic curve consists of these 3 lines. There will be 3 points in L_1' for which the corresp. ples in (S) will intersect in ~~lines~~ a line. For if l_1' l_2 l_3 be the 3 lines meeting in P_1 which together corresp. to L_1' the line l_1 will corresp. to a certain pt on L_1' & similarly for l_2 & l_3. Hence the line L_1' meets the F-curve of (S') in 3 points.



Also l_1 l_2 l_3 are generators of the F-curve in (S) & hence each meets the curve in 3 pts one of these being P_1'. Hence l_1 l_2 l_3 meet the F-curve again in 2 pts each.

From this we conclude that thro' any pt on the F-curve we can draw 3 chords of ~~&~~ curve.

Hence the ruled surface traced out by triple chords of the curve contains the curve as a triple curve.

Any ple thro' a gen^r of this surface meets the surface in a curve which intersects ~~the~~ gen^r in 6 pts two at ~~&~~ each of pts where F-curve meets the gen^r & a 7th point at which the ple is tangent to the surface. Hence curve of intersection is of degree 7 & with the gen^{r} makes up a curve of degree 8.

Hence the F- surfaces are of the 8th order.

This result may be verified as follows.

A line meets a plane (S) in one point hence a cubic curve in (S') can only meet a cubic surface in (S') in one point (variable). The cubic curve must therefore meet the F-curve in 8 pts. Hence its corresp. line must meet the corresp. F-surface in 8 pts. The latter is therefore of the 8th order.

We have therefore established the following theorems

(1) to a line corresp. a cubic curve & conversely, the curve meeting a fixed sextic in 8 pts.

(2) to a ple corresp. a cubic surface containing a fixed sextic curve & conversely.

(3) the triple chords of the space sextic trace out a ruled surface of deg. 8 having the sextic as triple curve.

The transf. is therefore a cubic transf.

Effect of the transf. on curves & surfaces.

A surface of degree m not containing the F-curve in (S) meets the F-surface in (S) as follows. Each gen^{r} of the F-surface meets the surface in m pts & hence the corresp. surface contains the F-curve in (S') m times.

Thus a surface of degree m transforms into a surface of degree 3m containing the F-curve m times.

A curve of order n not intersecting the F-curve transforms into a curve of order 3n meeting the corresp. F curve in 8n pts.

If a surface of deg. m contain the F-curve d times it transforms clearly into a surface of order 3m - 8d.

This surface contains the corr. F-curve d' times we have $m' = 3n - 8\delta$

& $\quad n = 3m' - 8\delta'$ conversely

hence $n = 9n - 24\delta - 8\delta'$

or $\delta' = n - 3\delta$

A curve of order n meeting the F-curve in μ pts transforms clearly into a curve of order $3n - \mu$.

The cubic surfaces which correspond to planes.

Such surfaces are general cubic surfaces. For these are determined each by 19 pts. Now a sextic curve meets a cubic surface in 18 pts. Hence if we take 19 pts on the F-curve these determine uniquely a general cubic surface & this cubic will contain the F-curve. Hence the above cubic surfaces are general cubic surfaces. They possess no singularities. The 19 pts of course must be selected arbitrarily, e.g. no 3 of them must lie on a line.

The lines on the cubic surface.

Let a plane (S) meet its F-curve in the 6 pts $\alpha\beta\gamma\delta\epsilon\zeta$. In general these 6 pts are arbitrarily situated. Corresp. to $\alpha\beta\gamma\delta\epsilon\zeta$ we get six lines on the cubic surface. Corresponding to the fifteen diagonals of the hexagon $\alpha\beta\gamma\delta\epsilon\zeta$ we get other 15 lines. Corresp. to the six conics got by selecting 5 of these pts we get another group of 6 lines. Thus altogether we infer the existence of 27 lines on the general cubic.

All the ordinary properties of the lines on the surface can now be deduced e.g. the double sixes etc.

If 3 of the pts lie on a line or if the 6 lie on a conic we infer as in previous cubic transf. the existence of conical points on the surface. The locus of such conical pts is the F-curve.

Superposition of the spaces.

Number of double pts.

Let the ples corresp. to a pt P of the common space be π_1 π_2 π_3. If P lies on π_1 then from the correl. transf. we know that P lies on a certain conicoid. If P lies on π_2 it lies on another conicoid & so on.

Hence if P lies on π_1, π_2 π_3 ie if it coincides with its correspondent it ~~lies on~~ is one of the 8 pts of intersection of 3 conicoids. Thus there are 8 dps of which 7 are arbitrary.

The involutive transformation.

If the 3 correlations be involutive the transf. is involutive. If the corr. $(P\ \pi_1)$ is involutive the ple π_1 is the polar ple of P with respect to a certain conicoid. Similarly π_2 & π_3 are the polar ples of P with respect to two other conicoids. Hence the correspondent of P is the intersection of its polar ples with respect to 3 conicoids & this gives an involutive relation between P & its corresp. P'.

If the three conicoids have a common cubic curve of intersec. this curve will be a curve of dps. & as in previous cubic transf. the line joining corr. pts will be a chord of this curve & the corr. pts will be harmonic with respect to the pts in which their line intersects the curve.

In general the 8 dps are arbitrarily situated. If however they lie 4 by 4 on six ples the three conicoids are degenerate being pairs of ples. In this case the polar ples of P pass each thro' a fixed line viz the axis of the degenerate conicoid. Thus we get the parallelepipedal ~~transf~~. (involutive).

If further the 4 pts lie in pairs on lines which intersect by 3's in 4 fixed pts. the axes of the above degenerate conicoids are coplanar we get the involutive tetrahedral transf. The intermediate cases can be obtained without difficulty.

Determination of the cubic transformation.

We shall now investigate the number of pairs of corresponding pts. required to determine completely a cubic transf.

Any correlation is determined completely by 15 pairs of conjugate pts. By a pair of conjugate pts is meant any pt P & any pt P' which lies on the ple corresp. to P.

Since a corr. is determined by 5 pts & their corr. ples it follows that we require 5 × 3 or 15 conjugate pairs to determine it, 3 pts being required for each ple. This must be true when the 15 pairs of conjugates are chosen in any manner.

This is also evident analytically for the general correlation is $(\alpha_{11}x + \alpha_{12}y + \alpha_{13}z + \alpha_{14}w)x' + (\beta_{11}x + --)y'$

$+ (\gamma_{11}x + \gamma_{12}y + --)z' + (\delta_{11}x + \delta_{12}y + -)z' = 0$

containing 16 constants 15 of which are arbitrary.

Now if we are given 13 pairs of conjugates arbitrarily situated a two-fold infinity of correlations can be found to contain them.

Let C_1 C_2 C_3 be three such correlations.

Each pair of the 13 pairs is conjugate with respect to each of C_1 C_2 C_3 hence the 13 pairs are 13 pairs of corresponding points with respect to the cubic transf. obtained by combining C_1 C_2 C_3.

Further any three correlations belonging to this twofold infinity give rise to the same cubic transf.

~~Let~~ For let 3 different triads of correlations give rise to 3 different cubic transf. & let the corresp. of

P in these correlations be $P_1' P_2' P_3'$.
Then we have altogether 16 pairs of conjugates in each of the correlations which is impossible.
Hence the 3 ~~con~~ triads cannot give rise to different cubic trans. Two of the cubic transf. must be the same. Let $S_1\ S_1\ S_3$ be the three transf. S_1 & S_3 being different. In this case we have 15 pairs of conjugates common to the correlations which is also impossible for 15 pairs determine a corr. uniquely.
Hence all the triads of correlations having the 13 pairs of conjugates give rise to the same cubic transformation.
Thus a cubic transf. is uniquely determined by <u>13</u> pairs of corresponding pts.

If a pt P have two different conjugates $P_1' P_2'$ the pairs $(P P_1')(P P_2')$ belonging to the given 13 pairs ~~then~~ then P is an F-pt & $P_1' P_2'$ the corresp. F-line.
If we have therefore given an F-pt & its corr. line this counts as two pairs of conjugates.
Thus a cubic transf. is determined by an F-pt & line & 11 pairs of conjugates or 2 F pts & lines & 9 pairs & so on.
Again a pt P may have 3 different conjugates $P_1' P_2' P_3'$. In this case the ple $P_1' P_2' P_3'$ is an F-ple & P as before an F-pt. We may have in this cubic transf.

an F-pt & ple & 10 pairs of (conjugates) corr. pts
2 F-pts & ples & 7 " "
3 F . . . . " 4 " "
4 F " " " 1 " "

Thus in this case 4 F-pts & 4 F-ples are possible & no more.
This is the tetrahedral cubic ~~transf.~~
It is evident from our discussion of this transf. that when the F-systems are given one pair of corr. pts is sufficient to determine the transformation.
The other cubic transf. for example the parallelepipedal

is obtained when we are given 3 F-pts & the corr. F-lines. The cubic transf. is now determined by 3 F pts & their lines & 7 pairs of corresp. pts.

The characteristic eqn.

$$\kappa s_1 s_2 s_3 + \lambda s_1 s_2 + \mu s_2 s_3 + \nu s_3 s_1 + \alpha s_1 + \beta s_2 + \gamma s_3 + \delta = 0 \quad (A)$$

is determined by seven triads $s_1 s_2 s_3$. These seven triads are given by the 7 pairs of corresponding pts.

Thus we see that the eqn (A) is characteristic of cubic surfaces obtained by a transf. in which 3 F-pts & lines are given & 7 corresp. pts.

Consider any ple sections of the parallelepipedal trans. Let the ple π in (S) cut PQR in ABC & ple π' in (S') cut P'Q'R' in A'B'C'.

Then if we take ples thro' P & the 7 given pts in (S) & ples thro' P' & the 7 corr. pts in (S') these ~~pts~~ systems of ples give a pencil of 7 lines thro' A in π & a pencil of 7 lines thro' A' in π'.

Now these groups of 7 lines are projective in the par.e transf. Now it is known from the theory of the ple quadratic transf. that only 3 points can be found in each of the ples π & π' which have the property that the pencil of 7 lines thro' one is projective with the pencil of 7 lines thro' the other. These are the pts ABC A'B'C' above.

many of the theorems given in the previous cubic transf. apply to the general cubic transf.
Thus a series of cubics thro' the F-curve & one pt meet a line in a cubic involution & three cubic surfaces can be found to osculate the line. Also four surfaces can be found to pass thro' the F-curve & 2 fixed pts & to touch a line.

A ple section of the general cubic surface is a non-singular cubic. Thro' any pt in the ple of this cubic we can draw 6 lines to touch the cubic. Hence the tangent cone to the surface from an external pt is of the 6th order. The curve of contact is also of the 6th order & lies on a conicoid. Transforming this theorem we get:
An infinite series of cubic curves can be found to pass thro' a fixed pt ~~& 8~~ intersect a fixed sextic in 8 pts & touch a given ple. The locus of the pt of contact is a ple curve of order 6 having 6 dps at the pts where the ple meets the F-curve. Hence we infer that since the sextic on the cubic transforms into a sextic on the ple that the sextic on the cubic must meet the F-sextic on it in 12 pts. Further it must cut each of the 6 triple chords of the F-sextic lying on the cubic surface in 2 pts.
Similarly we can find another infinite series of cubic curves thro' another fixed pt to touch the ple in another sextic having the same 6 dps. These sextics intersect in 12 pts outside of the 6 dps & hence we can find 12 cubic surfaces to pass thro' 2 fixed pts & touch a ple.
This follows from the fact that two cubic curves each intersecting the F-curve in 8 pts & each passing thro' 3 pts namely the two fixed pts & a common pt of contact with the ple, determine a cubic surface.

Transforming this theorem again we see that we can find 12 ples thro' a line (2 fixed pts) to touch a cubic surface. The general cubic is therefore of class 12.

Again we can find 12^2 or 144 surfaces to pass thro' a fixed pt & touch 2 given ples.

Art. 25. The foregoing cubic transformations — the tetrahedral parallelepipedal etc are of course particular cases of the above general cubic transf.

But other particular cases can now be got as follows.

(a) The transf. (cubic) built up of one sheaf & two correlations.

Let P & P' be the bases of the sheaves in (S) & (S') resp^y. Any pt R of (S) ~~is determin~~ has for its corr. in (S') the intersection of ple P'R' corr to ple PR & the ples π_1' π_2' which corr. to R in the ~~two~~ correlations.

The transf. is a cubic one.

P & P' form part of the F-lines.

To any pt of P corresponds a gen^l of the conicoid determined by the intersection of ples thro' the lines which corr. to the line P in the ~~two~~ correlations.

The other parts of the F-system are quintic curves.

Let R describe any ple thro' P. The ples corr to R π_1' & π_2' describe in ~~the~~ ple corr. thro' P' a ~~ple~~ linear homographic ~~tran~~ system of lines.

To one line corr. a definite line & when a line ~~e~~ envelopes a point its corr. line also envelopes a pt. Such a homographic system has 3 double lines.

Hence in any ple thro' P' there exist 3 ~~do~~ F-lines & corr. to them in any ple thro' P there are 3 F-pts.

further two planes thro' P' can be drawn to touch the conicoid corr. to P. These planes meet the conicoid in two generators which each correspond to a pt on P. These gens. are F-lines & their corr. pts on P are F-pts.

Hence the locus of the F-pts in (S) meets P in 2 pts.

Hence any plane thro' P meets F-locus in 5 pts, two on P & 3 outside P. The F-system in (S) consists therefore of P & a quintic curve which has P for a chord.

Similarly for (S'). Consider again any F-line L_1' in (S'). If a moveable pt R describe this line its corr. pt traces out a cubic curve. But this cubic must consist of 3 lines. Let us first suppose that L_1' meets P'. In this case there are 3 pts on it which have for their corresp. lines in (S). These lines meet in a pt on a plane thro' P corr. to the plane thro' P' & L_1'. One of these pts is the pt in which L_1' meets P' & the corresp. line is a gen. of the F-conicoid in S. Hence the F-line L_1' is a chord of the F-curve. If L_1' does not meet P' it must be a gen. of the F-conicoid in (S'). To this gen. corresponds a cubic consisting of 3 lines thro' a point; hence this gen. must meet the ~~cubic~~ F-curve in 3 pts. The F-quintics therefore lie on the F-conicoids. An F-curve meets one set of gens. in 3 pts each & the other set in 2 pts each. They are therefore of deficiency 2.

(It is clear that in the above general cubic transf. the deficiency of the F-sextic is 3).

The F-surfaces.

Each system consists of two parts.

One part is the conicoid which corr. to P or P'. This is the surface generated by triple chords of the quintic. The other part is the surface generated by chords of the quintic which intersect P or P'.

Consider any pt Q on P. Thro' Q we can draw 4 chords of the quintic & including ~~I~~ P itself. Hence excluding P we can draw 3 chords. The line P is therefore a triple line on this surface. Further thro' any pt on curve we can draw 2 chords to meet P hence curve is a double curve on the surface. Any ple thro' a gen^r meets the curve in a curve of order 1+1+2+1 i.e 5. Hence complete curve of intersection including gen^r is of order 6. Hence surface is of order 6. This is evident at once since any ple thro' P meets surface in P a triple line & 3 other lines or a ~~curve~~ system of order 6. Thus the complete F-system of surfaces is of order 8 as before.

A line transforms into a cubic curve meeting P or P' in 2 pts & the cor. F-quintic in 6.

A surface containing P d times & of degree n not containing the F-quintic transforms into a surface of degree $3n-2d$ & so on.

(b) The cubic transf. built up of two sheaves & one correlation.

Let PQ & P'Q' be the bases of the sheaves. To a pt R corresponds the pt R' determined by the ples P'R' Q'R' corr. resp^{ly} to PR & QR & the ple π_1' corr. to R in the correlation.

The F-system of lines.

The lines PQ P'Q' form part of the F-systems. Corr. to each of P & Q we get conicoids which therefore form part of the F-system of surfaces.

Consider a ple thro' P. If a variable pt R describe this ple we get on the corresp. ple thro' P' a linear homog. system of lines as in the previous trans. This is now of a special kind for one system of lines passes thro' a fixed pt namely the pt where Q' intersects the plane thro' P'. In fact we have on the two planes thro' P & P' respectively two homographic pencils of lines & a collineation. The systems on plane thro' P' have as before 3 ~~double~~ F lines. Two of these the double lines will pass thro' the pt where Q' meets the plane. Hence in any plane thro' P' we have 3 F-lines. To the third F-line which does not meet Q' corresponds a pt on Q. Hence to the 3 F lines we get as corresp. on the plane thro' P 3 pts one of which is on Q. Hence the locus of F-pts (apart from P & Q) in (S) meets any plane thro. P in 2 pts.

Again thro' P' we can draw two tangent planes to the conicoid which corresponds to ~~Q~~ P & hence the F-locus must meet P in 2 pts as before. Hence altogether any plane thro' P meets F-locus in 4 pts & this locus is consequently a quartic curve. Similarly this quartic meets Q in 2 pts & like results hold for (S').

The F-lines in space (S') belong to 3 groups (1) those which intersect both P' & Q' (2) those which intersect Q' & (3) those which intersect P'.

Classes (2) & (3) clearly by what has been shown above belong to the conicoids which corresp. to P & Q.

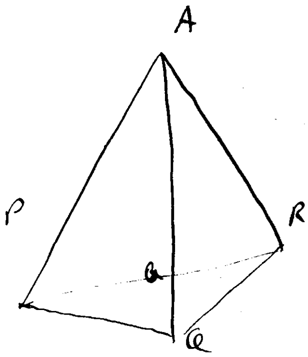
The surface generated by the F-lines which meet P' & Q' & intersect the F-quartic in 1 pt is a ruled surface of degree 4 having P' & Q' as double lines. Thus the complete F-system of surfaces is of degree 2+2+4 = 8.

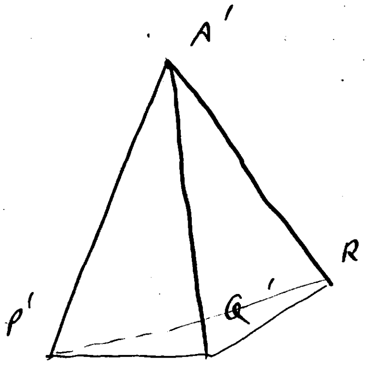
The surfaces generated by the systems of lines (2) & (3) give rise to conicoids & the F-quartic lies on each of these conicoids hence it is of deficiency 1.

This ~~transf.~~ may again give rise to several particular cases. ~~The~~ We have supposed above that P & Q P' & Q' occupy arbitrary positions & consequently do not intersect.

(c) We shall now treat the particular case in which PQ & P'Q' both intersect.

Let PQ meet in A & P'Q' in A'.





Suppose that to ple PAQ sheaf (PA) corr. ple P'A'R' sheaf (P'A')
& .... QAP " (QA) " " Q'A'R' " (Q'A')
" .... PAR " (PA) " " P'A'Q' " (P'A')
" " " QAR " (QA) " " Q'A'P' " (Q'A')

Then the lines AR & A'R' are F-lines for clearly to any pt on AR corr. a pt line on ple P'A'Q',

The F-system of lines in (S) contains AP AQ & AR
& .... (S') " A'P' A'Q' & A'R'

Clearly the ples PAQ, QAR, PAR form part of the F system of surfaces in (S) & P'A'Q' etc of the F system in (S')

In fact A & A' are the vertices of pencils between which a quadratic relation holds. To any ple thro' A corresps. a quadric cone thro' P'A', Q'A' & R'A'.

Corresponding to pt A in the correlation between the spaces we get a ple P'Q'R' & corr. to A' a ple PQR.

Also A & A' are F-pts for the transf. & P'Q'R', PQR are corresp. F-ples.

Considering any ple thro' AP & its corr. ple thro' A'P'.

(~~The latter ple meets Q'AR' & P'Q'R' in lines which are F lines on the ple. Thus a ple thro' A'P' contains as before 3 F-lines of which A'P' is o~~)

If as before a variable pt describe the ple thro' AP we get on the ple thro' A'P' two systems of lines one system thro' A' which form a linear homog. system.

as before this system has 3 ~~double~~ F lines of which (the double lines) 2 pass thro' A' & the third lies in the ple P'Q'R'.
Corresp. to each of the lines thro' A' we get a pt in the ple PQR. Thus outside of AP the ple thro' AP contains 2 F-pts both on the ple PQR & therefore lying on the line of intersection of ple thro' AP & PQR.
Also when a variable pt describes AP the ples of the correlation envelope a line in S' & the intersections of these ples with A'Q'R' give a series of F-lines.
One of these F-lines passes thro' A' & a ple thro' A'P' contains this line. Corresponding to this line we clearly get a pt on AP lying in PQR i.e. the pt P itself. Hence on the ple thro' AP there are 3 F-pts each on the line of intersection with PQR.
Hence the F-locus in (S) consists of a cubic curve lying in the ple PQR & passing thro' P Q & R.
The complete F-system in (S) therefore consists of AP, AQ AR & a ple cubic thro' PQR. A similar system exists in (S').
The F-system of surfaces in (S) consists of ples PAQ, QAR RAP & ple PQR (reckoned twice) & the cubic cone whose vertex is A & base the F-cubic in PQR.
The F-system in (S') is entirely similar.
Thus as before the F-system of curves is of degree 6 & the F-system of surfaces of degree 8.
In this transformation we get corresp. to a line in (S) a cubic curve in (S') thro' A' & meeting each of the lines A'P', A'Q', A'R' once & intersecting the F-cubic in 3 pts.
This follows on considering the intersections of the line with the F-system in (S).
Corresponding to a ple we get a cubic surface thro' the F-system in (S') & having a conical pt at A'.
Let the ple meet the F-system in (S) in the pts $\alpha\beta\gamma\delta\epsilon\zeta$.

Let δεζ be on the F-cubic. These pts are in a line. Corr. to them we get 3 lines on the cubic thro' A'. Thus 6 lines on the surface pass thro' A'. As in previous work we can show that the surface possesses 21 lines.

(d) A further particular case of great importance is obtained as follows.

Suppose that to ple PAQ sheaf (PA) corr ple P'A'Q' sheaf (P'A')
& " " PAQ " (QA) " " P'A'Q' " (Q'A')
" " PAR " (PA) " P'A'R' " (P'A')

etc

then the pencils whose vertices are A & A' are collinear pencils. They possess no F-lines.

Corr. to ple thro PA we get a ple thro' P'A'
& to pts on ple PA corr. two systems of lines on ple thro' P'A' one of these systems passing thro' A'.

As before these systems of lines have 2 double lines & hence in any ple thro' P'A' there are 2 F-lines.

Corr. to these F-lines we get two pts on the line of intersection of ple thro' PA & ple PQR.

~~There being no other F~~ Since ~~PA~~ there are no F-pts on PA with the exception of A it follows that the locus of F pts in (S) reduces in this case to a conic in the ple PQR. The locus of F-pts in (S') is similarly a conic in ple P'Q'R'. A & A' are as before F-pts & ples PQR P'Q'R' F-ples.

It should be noticed that these ples are not F-surfaces in the sense that they are ~~tra~~ traced out by F-lines.

The surface traced out by the F-lines in (S') is a quadric cone having A' as vertex & base the F-conic & there is a similar surface in (S).

The transformation is now a quadratic transf.

For if a pt R describe a line in (S) the line AR describes a ple thro' A. The corresp. line A'R' therefore describes a ple thro' A'. The ples of the correlation in (S') also envelope a line & their intersections with this ple thro' A' give a pencil of lines. Thus in the ple we have two pencils of lines homographically related & the locus of their intersections is a conic. Hence corresp. to the line described by R we get a conic in a ple thro' A'. This conic passes thro' A' & meets the F-conic in 2 pts. Similarly corr. to a line in (S') we get a conic thro' A meeting the F-curve in (S) in 2 pts. Hence the transf. is quadratic. Corr. to a ple we get a conicoid containing the F pt A or A' & the F conic.

This quadratic transf. is the general quad. transf. & the cono-cyl. transf. p 44 art 11 is merely a particular case of this transf. in which the F-conic is degenerate & consists of two coincident lines.

The general quad. transf. however falls most naturally under a special series of transf. which will be given later. Its principal properties will be discussed later.

The three correlations in space (S') of the general cubic transf. may be related to one another in various ways. In the general case already given there is supposed to be no special relation between them.

Let them be denoted by C_1 C_2 C_3.

Any two viz C_1 C_2 make up what we may call a planar homographic transf. between the planes of (S') as primary elements. For to a ple of C_1 corresp. a single ple of C_2 & to a line of C_1 a line of C_2 & to a pt of C_1 a pt of C_2. Now ~~as~~ exactly as in two superimposed correlations any two corresponding ples may have a line of intersection which intersects a fixed line. All ples thro' this line are double ples. Corr. to this line in (S) we have a line also. Now this line in (S) must be an F-line ~~hence~~ for corr. to any pt on it we get a line in (S′) hence in spaces (S) & (S′) there ~~are two~~ is an F-line. This is case (a) above but of a more general kind

Again C_2 & C_3 may have a ~~th~~ line such that all planes thro' it are double planes i.e belong to both systems.

In this case there will be two F-lines in each space & we get ~~space~~ case (b) in a more general form.

The properties of these general cases (a) & (b) are the same as those already given under a & (a) & (b).

Again the two correlations C_1 C_2 may be related in a manner analogous to a central collineation.

Any two corr. planes may have their line of intersection lying on a fixed plane. This plane is an F-plane. Corr. to it we get in (S) a pt ~~the~~ the F-pt. This is a more general form of the case (c) above.

Art 26. The transformation made up of 3 collineations.

This is the correlative of the general cubic trans.

Let a pt P of (S) be related by 3 distinct collineations to pts p_1' p_2' p_3' of (S′). Then to pt P corresponds the plane $p_1' p_2' p_3'$.

When P describes a line $(p_1' p_2' p_3')$ or briefly p' ~~the~~ envelopes a cubic developable whose edge of regression is a cubic curve.

When P describes a plane p' envelopes a surface of class 3.

Conversely if p' envelopes a pt P describes a general cubic surface & if p' envelopes a line P describes a cubic curve. The proofs of these theorems are similar to those given for the pt-pt-transf.

The common tangent ples of ~~a~~ two class ~~se~~ cubics in (S') generate a developable surface. The proper part of this is a cubic developable & the remainder the fixed points a developable of class 6. This developable is common to all class cubics in (S').
Corresp. to it in (S) there is a fixed sextic curve thro' which all cubics of (S) which corr. to pts of (S') pass.
Various ~~particular~~ cases may be derived as for the pt pt transf. by supposing one or more of the collineations to degenerate in a range of pts on a line.

Again we may have special relations between the 3 collineations. Let them be denoted by T_1 T_2 T_3. Corr. pts of T_1 T_2 when joined may give a line which intersects a fixed line. The fixed line is a line of clps for T_1 & T_2. Corr. to this line we get a line in (S). To any pt on this line of (S) there is no definite ple of (S') hence the line forms part of the F-sextic in (S). The remainder is of course a quintic curve. The corresp. part of the F-surface in (S') is clearly a conicoid. In the general transf. the F-surface in (S') consists of the surface generated by lines which contain three corr. pts p_1' p_2' p_3'. Any ~~line~~ ple thro' such a line is a ple of (S'). Corresp. to the ples thro' such a line L' we get in (S) a cubic curve which must be 3 straight lines thro' the pt corr. to L'. This pt must lie on the F-curve in (S). The F-surface in (S') must therefore as in the pt-pt transf. be of the 8th order.
Similarly if ~~the~~ line joining corr. pts of T_2 & T_3 intersect a fixed line we shall have 2 F-lines in (S) & two corresp. F-conicoids in (S') & so on.

The general cubic correlative transf. may be derived as follows. Combine with the cubic pt-pt transf a correlative transf, in ~~either~~ each of the orders S T or T S, where S is the cubic pt-pt trans & T the correlative transf.

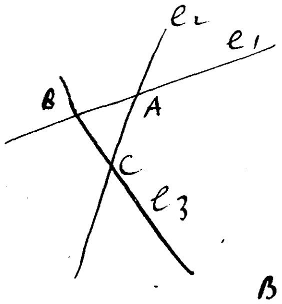
Art 27. The general planar cubic Transformation.

This transf. is obtained as follows. Let the ples of space (S) be related by 3 distinct correlations with the pts of space (S'). If π be a ple of (S) & $p_1' p_2' p_3'$ its corresp. pts of (S') then to ple π there corresponds ple $(p_1' p_2' p_3')$ or p'. Thus a (1-1) correspondence between the ∞^3 planes of the spaces (S) & (S') is established for clearly to a ple of (S') corresp. a ple of (S).
When π envelopes a line p' envelopes a cubic developable whose edge of regression is a cubic curve. When π envelopes a ~~ple~~ pt p' envelopes a surface of class 3 & conversely. The proofs of these theorems present no difficulty.

The F-systems.

When the 3 pts $(p_1' p_2' p_3')$ are in a line L' this line is an F-line & its corresp. ple in (S) is an F-ple.
The system of ples thro' L' gives rise to a cubic developable in (S) which in this case must consist of 3 coplanar lines lying in π which correspond to L'. There will be 3 ples thro' L' which are F-ples & which correspond each to a line in this F ple π. Hence thro' L' we can draw 3 tangent ples to the surface which is the envelope of the F-ples in (S'). This envelope is a sextic developable. Hence L' must meet the edge of regression of this sextic in 3 pts. The 3 F-ples thro' L' are then the osculating ples of the edge of regression at these pts. Hence the ruled surface generated by the lines L' in (S') consists of the triple chords of the edge of regression of the

the F sextic developable & similarly for (S).
This edge of regression is a sextic curve for any F-plane meets the ruled surface generated by its triple chords in 3 generators. Let these be l_1 l_2 l_3 in space (S)



Corr. to any plane thro' L' we get 3 pts one on each of l_1 l_2 l_3. Corr. ~~tt~~ to the 3 F planes thro' L' we get 3 sets of collinear pts on l_1 l_2 l_3. Two of each set are the pts A B, B C or C A in which l_1 l_2 l_3 intersect. Hence plane of l_1 l_2 l_3 meets the edge of regression in 6 pts three of which are A B & C. The edge of regression is therefore a sextic curve. Hence the F-surfaces are as before ruled surfaces of order 8.

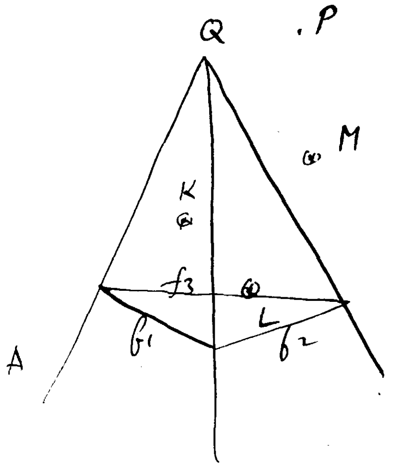
— 1 —

Note: The surfaces of class 3 of the two previous transf. are in general of order 12. For we have seen already that twelve cubic surfaces thro' a sextic curve & two fixed pts can be drawn to touch an arbitrary plane hence transforming this by the cubic pt-pl transf. we get that 12 tangent planes can be drawn thro' ~~to~~ a line to the general cubic surface. Hence by the cubic correlative transf. the order of the surface of class 3 is 12.

The relations between the systems of pts (p_1' p_2') (p_2' p_3') (p_1' p_3') are clearly collineations.
Special relations amongst these give rise to special planar cubic transformations. Thus one part of the edge of regression of the F sextic developable may be a line & the remainder a quintic curve & so on.
When the spaces are superposed there will be 8 double planes which are the common tangent planes of 3 conicoids.

Art 28. Generation of cubic surfaces.

The theorems of Art –   p135 on the generation of concubicoids can be extended as follows.



Let QA QB QC be any three lines K L M any three arbitrary pts & O a fourth arbitrary pt.

O Any plane thro' O meets QA, QB, QC in lines f1 f2 f3, f1 lying in AQB f2 in BQC etc. Let planes be described thro' f1 K, f2 L, & f3 M respectively meeting in a pt P. The locus of P for varying positions of the plane thro' O is a cubic surface having Q as a conical pt QA, QB, QC lines on it thro' this pt & passing thro' K L & M.

For the ~~p~~ systems of planes thro' K L & M resp. form 3 pencils such that when a plane of one pencil is given the planes of the others are uniquely determined. Further when a plane of one the pencils envelopes a line the planes of the others envelope lines. This follows by making the planes thro' O pass thro' an additional fixed pt. Hence the 3 pencils are of the kind treated in the general cubic transf. The locus of their intersections is therefore a cubic ~~curve~~. surface. This surface clearly contains QA, QB & QC & K L M. A plane thro' O & K L meets AQC in a line which lies wholly in the surface. Similarly a plane thro' OML meets AQB in a line lying wholly in the surface & so on.

If the planes thro' O pass thro' an additional fixed pt R the locus of P is a cubic curve passing thro' Q & intersecting QA QB & QC in one pt each. These pts are obtained by taking the planes thro' the fixed pts O & R & each of the pts K L & M.

Again let us take 3 non-coplanar & non-concurrent lines L_1 L_2 L_3 & 3 col. pts K L M & a fourth pt O as before. Let a ple thro' O meet L_1 L_2 L_3 in F_1 F_2 F_3 respectively. The ples $L_2 L_3 K$; $L_3 L_1 L$, & $L_1 L_2 M$ meet in a pt P. As the ple thro' O varies the locus of P will be again a cubic surface containing L_1 L_2 L_3 & K L M.

Again let us take a cubic curve S_3 & 3 pts K L M as before with the pt O. A similar construction gives a cubic surface thro' S_3 & K L M. The ples OKL, OLM, OKM give rise to 3 chords of the cubic lying on the surface. We may regard this cubic S_3 as the F cubic & these 3 chords as the 3 F-lines of a hyperboloid. Hence we can construct a hyperboloid given its F-cubic & its 3 F-lines & 3 pts on it.
Let K L M be the given pts F_1 F_2 F_3 the given chords of the cubic; the ples KLF_1, LMF_2, KMF_3 must meet in a pt O. Starting from O we can now find as many pts on the surface as we please. Here we have assumed that KLF_1, LMF_2, KMF_3 are coplanar. If this is not the case the construction does not apply.

Art 29. Analytical formulae for the general cubic transfs.

① The (pt-pt) transf.

Corresp. to the pt $(xyzw)$ of (S) we have the ples of (S') viz.

$$(a_{11}x+a_{12}y+a_{13}z+a_{14}w)x' + (b_{11}x+b_{12}y+b_{13}z+b_{14}w)y' + (c_{11}x + - -)z' + (d_{11}x + - -)w' = 0$$

$$(a_{21}x+a_{22}y+a_{23}z+a_{24}w)x' + \text{ etc } = 0$$

$$(a_{31}x+a_{32}y+a_{33}z+a_{34}w)x' + - - = 0$$

Solving these eqns. for x' y' z' w' we get

$$x' = \frac{f_3(xyzw)}{\varphi_3(xyzw)} \qquad y' = \frac{f_3'(xyzw)}{\varphi_3(xyzw)} \text{ . etc}$$

where f_3 f_3' f_3'' et φ_3 are cubic fns of $x\,y\,z\,w$.
Similarly on solving these eqns for $x^+ y^+ z^+ w^+$ we get

$$x' = \frac{F_3(x'y'z'w')}{\psi_3(x'y'z'w')} \qquad y = \frac{F_3'(x'y'z'w')}{\psi_3(x'y'z'w')} \quad \text{etc.}$$

the fns. being again cubic fns.
Hence the trans. is birational cubic.

(2) The general cubic correl. transf.
Corr. to pt $(xyzw)$ of (S) we get the pts of (S')

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}w \\ y' &= b_{21}x + b_{22}y + b_{23}z + b_{24}w \qquad (1) \\ z' &= c_{31}x + c_{32}y + c_{33}z + c_{34}w \\ w' &= d_{41}x + - \end{aligned}$$

$$\begin{aligned} x' &= \alpha_{11}x + -- \quad \text{et} \\ y' &= \beta_{21}x + -- \qquad (2) \\ z' &= \gamma_{31}x + \\ w' &= \delta_{41}x + - \end{aligned}$$

$$\begin{aligned} x' &= A_{11}x + - \\ y' &= B_{21}x + \qquad (3) \\ z' &= C_{31}x + - \\ w' &= D_{41}x + - \end{aligned}$$

The plane containing these pts is

$$\begin{vmatrix} x' & y' & z' & w' \\ K_{1x} & K_{1y} & K_{1z} & K_{1w} \\ L_{2x} & L_{2y} & L_{2z} & L_{2w} \\ M_{3x} & M_{3y} & M_{3z} & M_{3w} \end{vmatrix} = 0$$

where $K_{1x} \equiv a_{11}x + ---$ etc $\qquad L_{2x} = \alpha_{11}x +$ etc
Hence coords of plane of (S') corresp to. $(xyzw)$ are

$$\xi = \begin{vmatrix} K_{1y} & K_{1z} & K_{1w} \\ L_{2y} & L_{2z} & L_{2w} \\ M_{3y} & M_{3z} & M_{3w} \end{vmatrix} \quad \text{etc} —$$

(3) The gen. planar corresp.

Corr. to pt $(\xi, \eta, \zeta, \theta)$ of (S) we have the 3 pls of (S')

$$(a_{11}\xi + a_{12}\eta + \text{——})\,\xi' + (b_{11}\xi + b_{12}\eta + \text{—})\,\eta' + \text{etc} = 0$$
$$(a_{21}\xi + a_{22}\eta + \text{——})\,\xi' + (b_{21}\xi + \text{—})\,\eta' + \text{—} = 0$$
$$(a_{31}\xi + \qquad)\,\xi' + (\qquad)\,\eta' + \qquad = 0$$

Hence on solving for ξ' η' ζ' θ' we get

$$\xi' = \frac{\chi_3(\xi\eta\zeta\theta)}{\psi_3(\xi\eta\zeta\theta)} \quad \text{etc} -$$

$$\& \quad \xi = \frac{\chi_3(\xi'\eta'\zeta'\theta')}{\psi_3(\xi'\eta'\zeta'\theta')} \quad \text{etc} -$$

Chapter IV. Higher transformations.

Art. 30. The transf. which can be built up by two sheaves of ples & a sheaf of conicoids.

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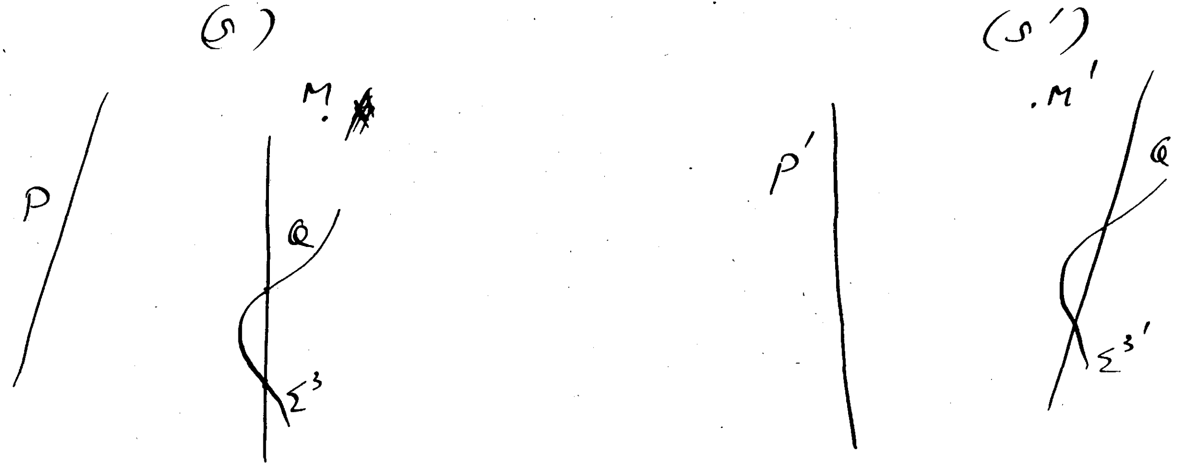
(1) The general transf. of this kind. (5-5).



Let us take in (S) two non-coplanar lines P & Q & a cubic curve Σ_3 which has Q for a chord & let a similar system be taken in (S'). Any pt M of space (S) is determined as the intersection of ples PM, QM & a conicoid thro' Q Σ^3 & M. Let us now suppose as in previous work that to sheaf base (P) corr. a sheaf base (P') & to the sheaf base (Q) ——— (Q')

& to sheaf of conicoids thro' $Q\Sigma^3$ & a sheaf of conicoids thro' $Q'\Sigma_3'$. Let us also suppose the relations between each of these pairs of sheaves ~~to~~ be (1-1) ie homographic. Corresponding to the pt M we now get a pt M' & the correspondence is clearly (1-1) in the pts of the two spaces.

Let now M describe a line. The ples P'M' Q'M' intersect in a line which describes a conicoid thro' P'&Q'. Also the ples P'M' & the conicoids thro' $Q'\Sigma_3'$ trace out by their intersections a ~~cubic~~ quartic surface containing P' Q' & Σ_3'. For corr. to a conicoid of (S) there are two positions of ple thro' P since a conicoid will meet the line traced out by M in 2 pts. Hence in (S') the relation between the sheaf (P') & sheaf of conicoids will be (2-1). Hence the locus of their intersection is a quartic surface having P' as double line. The conicoid thro' (P'Q') & this quartic surface thro' P'Q' & Σ_3' intersect in a curve of order 8 of which P' & Q' count as a system of order 2+1 ie 3. Hence the proper part of the intersection is a curve of order 8-3 ie 5. Thus corr. to a line of (S) we get a quintic curve of (S') & conversely. Hence corresp. to a ple of (S) we get a quintic surface of (S') & conversely. As in the tetrahedral tetraconical transf. we can deduce this directly as follows.

When two ples of sheaves P & Q are given their line of intersection meets the ple described by M in a single pt. & thro' this pt there passes one & only one conicoid. Again a conicoid & a ple say of sheaf P intersect in a conic which meets the ple in 2 pts. hence there will be two positions of ple thro' Q. If the ple be thro' Q, one ple thro' P will correspond to it & the conicoid.

Hence when two ples are known one conicoid corresp. to them is known; when a ple thro' P & a conicoid are known 2 ples thro' Q' are thereby determined; when a ple thro' Q & a conicoid are known one ple thro' P is determined. The same statements hold for the corresp. ples & conicoids in (S') hence the characteristic or intrinsic eqn. of of the surface corresp. to a ple in (S) is

$K_1 S_1 S_2^2 S_3^2 + K_2 S_1 S_2 S_3^2 + K_3 S_1 S_2^2 S_3 + K_4 S_2^2 S_3^2 + K_5 S_1 S_2 S_3 +$

$+ \cdots + K_{18} = 0$ in which highest powers of S_2 & S_3 are squares & highest power of S_1 the first. In this formula $S_1 S_2 S_3$ as before are the abscissae of pts of intersection of the sheaves thro' P' & Q' & Q'Σ_3' respectively with an arbitrary line in (S'). Hence when $S_1 = S_2 = S_3$ we get 5 pts of surface on an arbitrary line; ie surface is of 5th order. By taking the arbitrary line thro' Q' we see that above relation reduces to a quadratic relation between $S_1 S_2 S_3$ the variables - since the ples & conicoids can only meet this line in 2 variable pts. Hence it is evident that Q' is a triple line on the surface. P' is a single line on it & it also contains the base cubic Σ_3'.

The quintic curves which correspond to lines in (S) must have P' & Q' as quadruple chords; for a ple thro' P meets a line in (S) in 1 pt only hence corr. ple thro' P' meets quintic curve in 1 pt only.

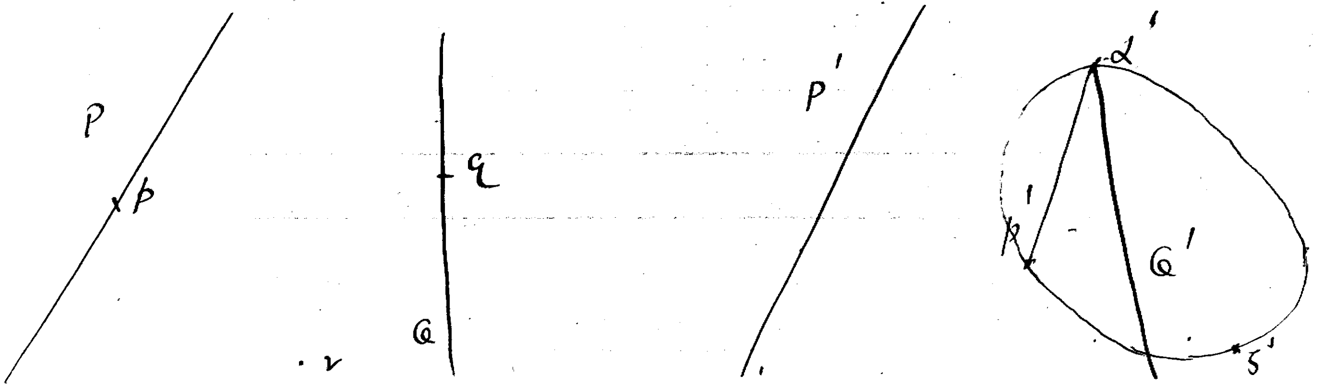
Exceptional or F-elements of the transf.

(a) Pts on P.

Corr. to a pt on P we get the line of intersection of a ple thro' Q' & a conicoid thro' Q'Σ_3'. Hence every pt. on P is an F-pt. & the locus of the corresp. F-lines in (S') is a ruled quartic having Q' as triple line & passing thro' Σ_3'. For corr. to any conicoid there are two positions of ple thro' Q. ~~Corr. to a pt on Σ_3 we~~

(b) Pts on Σ_3. Corr to a pt on Σ_3 we get a line viz the line of intersection of ples thro' P' & Q'. Hence all pts on Σ_3 are F pts. The locus of the corresp. F-lines in (S') is a ruled quartic for since any ple thro' P meets Σ_3 in 3 pts there will be 3 corresp. positions for ples thro' Q. Hence relation between the sheaves $(P'$ & $Q')$ is (1-3). This quartic has Q' as a triple line & P' as a single line.

(c) Pts on Q.



In this case the conicoids & ples thro' Q are both indeterminate, whereas in previous cases only one set of generating elements was indeterminate.

Any ple thro' Q meets P in a pt p. & base cubic in a pt r. Let ple be called ρ. Corresponding to any line in ρ we get a curve in corresp. ple ρ'. This curve is a conic for if ρ' meet Σ_3' in r' & P' in $\overset{p}{q}'$, it is the intersection of corresp. rays of the homographic pencils thro' p' & r'. Hence correspondent of Q itself is a conic in ρ'.

Again let q be a pt on Q & let the ple ρ vary. Let the ple Pq be called π. Corr to π we get a ple π'. Corr. to q for any ple ρ we get a single pt ~~a corresp~~ p' (~~ple p' & also~~) in π' namely the pt. of intersection of a conic & a line thro' α' where Q' meets π'. The conic passes thro' α' & the 3 pts of intersection of π' with Σ_3'. As the ple ρ varies this pt p' varies & describes a curve in π'. This curve is a unicursal ~~qx~~ cubic for the sheaf of ples ρ' & the sheaf of conicoids $(Q'r'\Sigma_3')$ generate a ruled cubic having Q' as double line. The intersection of π' with this surface is a unicursal cubic having a d.p. at on Q' viz at α'.

Hence corr. to pt q on Q we get a unicursal cubic lying in π'. As ~~q in ple~~ q varies on Q these cubics generate a surface which must be of the 4th order & have Q' as double line. For we have already seen that corr. to Q in a ple ρ we get a conic in ple ρ'. Hence section of surface thro' Q' consists of Q' & a conic. But since Q' is a double line it follows that the degree of this section is 4. Hence the surface corr. to line Q is a quartic. It has Q' as a double line & it also contains Σ_3'. Further as is easily seen by considering the conics in ρ' it contains P'.

The F-curves.

Consider a line which intersects P & Q & which meets Σ_3 in a pt R. Clearly corresp. to this line we get a pt in (S'). Hence the lines which intersect PQ & Σ_3 are F-lines. Their locus is a ruled surface of order 4 with Q as triple line & P & Σ_3 as single lines.

The locus of the corresp. pts in (S') is a septic curve. Any ple thro' P meets Σ_3 in 3 pts & thro' each of these pts passes a ple of sheaf Q. Again a conicoid meets P' in 2 pts & thro' each of these passes a gen^n of the ruled quartic. Corr. to each of these pts there is a ple thro' Q. Hence when the lines intersecting $PQ\Sigma_3$ traces out the ruled quartic the corr. ples P' of sheaves (P') & (Q') trace out a ruled quartic with Q' as triple line & P' as single line, while corr ples of sheaf (Q') & conicoids $(Q'\Sigma_3')$ trace ~~a~~ out a ruled quartic having Q' as triple line. The intersection of these quartics is of degree 16 but of this

Q' counts as a curve of degree 3×3 i.e 9. Hence the proper curve of intersection is of degree 7. This is the F-curve in (S'). Similarly there is a septic curve in (S) which is an F-curve & corr. to the lines which intersect $P'Q'$ & Σ_3'.

Hence the complete system of F-curves in (S) is P Q (triple) Σ_3 & the F septic Σ_7.
The complete system of F-surfaces in (S') consists of the ruled quartics corr. to P Σ_3 & Σ_7 & the quartic corresp. to Q. Similar results hold for the F systems of (S').
It is of interest to note that if π & π' be any corr. planes thro' P & P' respec. a line of π transforms into a unicursal quartic (having a triple pt) of π' & conversely. This follows from what has been already done. The sing. pt of the quartic in (S') lies on Q' & the sing. pt of the quartic in (S) -- Q & thus we verify that Q & Q' are triple lines on the quintic surfaces.
A line l of (S) meets each F-surface in 4 pts hence its corr. quintic curve in (S') meets P' Q' Σ_3' Σ_7' in 4 pts each. A plane σ of (S) meets each F-surface in plane quartics & hence ~~the~~ corr. quintic surface contains P' Q' (3 times) Σ_3' & Σ_7'.

Effect of the transf. on curves & surfaces.
Corr. to a conic of (S) we get a curve of order 10 in (S') meeting each constituent ~~pt~~ F-curve in 8 pts.
Corr. to a conicoid of (S) we get a surface of order 10 containing P' as double line Q' as sextuple line & Σ_3' Σ_7' each as double curves.
Generally a curve of order n transforms into a curve of order $5n$.

But if it meet P in λ pts. Q in μ pts Σ_3 in ν pts & Σ_7 in ρ pts its corresp. curve will be of order $m' = 3 \cdot 5m - \lambda - 3\mu - \nu - \rho$.

Generally a surface of order n transforms into a surface of order $5n$ having $P' \Sigma_3' \Sigma_7'$ as curves of n^{th} deg. of multiplicity & Q' as line of $3n^{th}$ degree of multiplicity. But if the surface contain P λ times Q μ times Σ_3 ν times & Σ_7 ρ times the degree of the corresp. surface will be

$n' = 5n - 4\lambda - 4\mu - 4\nu - 4\rho$ or $5n - 4(\lambda + \mu + \nu + \rho)$.

Correspondence between the plane of (S) & the quintic of (S').

Let a plane σ meet P Q Σ_3 Σ_7 in the pts p q $\alpha\beta\gamma$, $a\,b\,c\,d\,e\,f\,g$ respectively.

Corr. to pts $p\,\alpha\beta\gamma\,a\,b\,c\,d\,e\,f\,g$ we get lines on the quintic These are 11 in number. Corr. to the lines qp, $q\alpha$, $q\beta$, etc qg we also get lines on the quintic. These are again 11 in number. Hence in general there these pts & lines will give rise to 22 lines on the surface.

Any line thro' q gives rise to a conic & hence there will be an infinite number of conics on the quintic.

Again a conic thro' q & any other 4 of the pts gives rise to a cubic curve. The number of these cubics will clearly be ${}_{10}C_4$ ie $\frac{10!}{4!\,6!}$ or $\frac{\overset{5}{\cancel{10}} \cdot \overset{3}{\cancel{9}} \cdot \overset{2}{\cancel{8}} \cdot 7}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4}}$ or 210.

Again a unicursal cubic having its node at q & passing thro' 6 of the other pts is completely determined thereby. Its corr. is of degree $3 \times 5 - (3 \times 2 + 6)$ ie 3 & hence is a cubic curve. The number of these cubics is

${}_{10}C_6$ $\frac{10!}{6!\,4!}$ ie 210 hence there are altogether 420 cubic curves on the surface.

These two sets of cubics stand in the following relation to one another. Consider a conic thro q & abcd,

Consider also a unicursal cubic thro' q & $abcd$ $\alpha\beta$. These curves intersect in 6 pts at most & these are all taken up by q a b c d since q counts as 2. Hence the corresp. cubic curves on the quintic cannot intersect. Clearly there will be ${}_6C_2$ cubics of second set which do not intersect the cubic of first set corresp. to conic q $abcd$. This number is 15. Hence any cubic of one set is intersected by 210 − 15 i.e. 195 cubics of the other set. The number of pts of intersection may vary from 1 to 4.
Thus the cubic corresp. to the conic q $abcd$ & the cubic corresp. to the cubic q $p\alpha\beta efg$ intersect in 4 pts since q is the only pt the ple curves in (S) have in common. Since the arrangements q $abcd$ & q $p\alpha\beta efg$ exhaust all the F-pts on the ple σ it follows that any cubic of one set is intersected in 4 pts by only one cubic of the other set.
Again consider a unicursal quartic having q as triple pt & passing thro' 8 of the other pts. Such a curve is uniquely determined by these pts. Its corresp. is of degree $4 \times 5 - (3 \times 3 + 8)$ i.e. $20 - 17$ i.e. 3. It is therefore a cubic. The number of these cubics is ${}_{10}C_8$ i.e. 45.
Again the corresp. of a unicursal quintic having q as quadruple pt & ~~the~~ passing thro' the other 10 F-pts is a cubic also. There is only one such cubic. Let these sets of cubics be called A_i B_k C_l & D
$i = 1 2 \ldots 210$ $k = 1, 2, \ldots 210$ $l = 1, \ldots 45$.
Consider the sets B_k & C_l. Since a cubic & a quartic meet in 12 pts it follows that the cubics corr. to a cubic & quartic thro' q $\alpha\beta$ $abcd$ say do not intersect. A cubic of set B is not intersected by ${}_4C_2$ cubics i.e. by 6 cubics of set C. The maximum

number of pts of intersection of two cubics belonging to sets B & C resp. is 3. Again the cubic D cannot intersect any of the cubics C but it must intersect every cubic of set B in 1 pt & every cubic of set A in 2 pts.

There is still another set of cubics on the quintic. A line joining any two of the 10 F-pts $p\alpha\beta abcdefg$ on σ transforms into a cubic. There are clearly ${}_{10}C_2$ i.e. 45 such cubics. Call this set E_m $n=1..45$.

Any cubic of set E must intersect every cubic of the sets B C D. The maximum number of pts of intersection is 4. For any cubic of set E there is only one cubic of set C which intersects it in 4 pts. The cubic D must intersect every cubic of E in 3 pts. ~~Let~~ us now examine the intersections of cubics belonging to the same set.

In set A two cubics may intersect. In fact all cubics derived from conics which have no more than 3 F-pts common must intersect.

Again in set B since on σ two unicursal cubics intersect in 9 pts of which 4 are at Q it follows that if two plane cubics have not more than 4 F-pts common their corr. cubics must intersect & so on.

The relation between these 5 sets of cubics A B C D E is somewhat analogous to the double-sixer on a cubic surface but is much more complicated.

The cubic D meets the F-unicursal cubic corr. to Q in 4 pts; the cubics C meet this same curve in 3 pts each &, the cubics B meet it in 2 pts each & the cubics A in 1 pt each. The cubics E do not meet it.

Again any line on σ meets the unic.-quintic on σ in 5 pts & hence D must meet a quintic curve on σ' in 5 pts & so on.

Superposition of the spaces.

Double elements.

Let the spaces be superposed. Corr. plles thro' P & P' generate a conicoid as do also corresp. plles thro' Q & Q'. These conicoids intersect in a quartic curve which meets each of P P' Q Q' in 2 pts. Again corresp. conicoids thro' $Q\Sigma_3$ & $Q'\Sigma_3'$ intersect in a quartic surface thro' these bases. The quartic curve above intersects this quartic surface in 16 pts of which 4 lie on Q & Q'.
Hence there will be 12 proper d.ps left.
Let us now suppose P P' Q Q' Σ_3 Σ_3' ~~to [illegible]~~ coincide respectively. Corresponding pts on a plle & on a quintic surface can be formed as follows.
Let σ be a plle & R a pt on it. Thro' R we can draw one & only one line to intersect P & Q. This line meets the quintic in one pt R' which is the corresp. of R. When R describes a line on the plle the lines thro' R generate a conicoid thro' P & Q. This conicoid meets the quintic in a curve of order $2\times5-(3+1)$ i.e. 6.
Hence to a line on the plle corresp. a sextic curve on surface.

A slightly more general case of this transf. is obtained as follows.
(Let us replace the plles thro' P & the plles thro' P' by a correlation between (S) & (S').
A plle thro' Q meets a conicoid of sheaf $Q\Sigma_3$ in a line intersecting Q. The corr. plle & conicoid also intersect in a line meeting Q'. Let P be any pt on the line in (S) & let the plle corr. to it in the correlation meet the corr. line in P'. Then P & P' are corr. pts of the transf. The corr. of pts between (S) & (S') is clearly (1−1) & reversible & the transf. is as before a quintic transf.

The F-system of (S) now consists of a (triple line), Σ_3 & an octic curve Σ_8 & similarly for (S').
The F-system of surfaces consists in (S) of three ruled quartics & a quartic corr. to Q as before. The rest of the works as above.

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Art 31. Particular cases of the quartic transf.

A particular case of this transf. arises when the F-lines PQ & P'Q' intersect. Let A & A' be the pts of intersection of these pairs of lines respectively. Two further cases arise according as the pencils vertices A & A' are quadratic or collinear (Cap I.). (a) Case of quadratic pencils.

The transf. is still a quartic transf. but the F-systems of lines & surfaces are altered.

Let AP, AQ, AR, A'P', A'Q', A'R' be the F-lines of the quadratic pencils. AP, AQ, A'P' A'Q' are as before F-lines for the transf. Corr. to a pt on AR we get a line on ple P'A'Q' & corr. to pt on A'R' a line on ple PAQ. Hence AR & A'R' are also F-lines for the transf.

Corr. to pts on AP we get the ple A'Q'R'
& " " AQ " - - - A'P'R'.

Thus in (S') the ples A'Q'R' A'P'R' are F-ples as is also A'P'Q'.

If we take a series of ples thro' AQ & for a pt q on AQ we get for the corr. of Q a unicursal cubic in A'P'R' having a dp at A'. The locus of these unicursal cubics is now however for all pts q (except A) the ple A'P'R'.

Again consider the corresp. of A itself in each of these ples. The transf. between the ples AQ & A'Q' is quadratic & hence corr. to A we get in the corr. ples thro' A'Q' a line.

Consider now a series of ples thro' AP & the corr. ples A'P'. The transf. between these ples is a quartic one i.e. corr. to a line we get a unicursal quartic with triple pt at A'. Corr. to A in each of these ples we get a unicursal cubic having a d.p. at A'. The locus of these cubics will be a surface corr. to A & the locus of the lines in planes thro $A'Q'$ will also be the same surface corr. to A. Hence this surface must be a ruled cubic having $A'Q'$ as double line. Thus corr. to A which is clearly an F-pt we get in (S') a ruled cubic having $A'Q'$ as double line. Hence corr. to the complete line AQ we get the plane $P'A'R'$ & this cubic surface. Similarly we see that corr. to the complete line AP we get the plane $Q'A'R'$ & this cubic surface.

Corr. to pts. on Σ_3 we get a cone of order 4 with vertex A' having $A'Q'$ as triple line. Corr. to lines which meet AP, AQ & Σ_3 & which form the surface composed of plane PAQ & cone vertex A & base Σ_3 we get the line $A'R'$ & a curve of order 6 which is the intersection of a ruled cubic thro' $A'Q'$ & Σ_3' having $A'Q'$ as double line & a quartic cone vertex A' which has $A'Q'$ as triple line. This is altogether a curve of order 7.

Thus the number & order of the F-curves & surfaces is not altered but they are now degenerate to a certain extent.

A line l of (S) meets the planes PAQ, QAR, RAP & the F-surface corr. to A' on $\Sigma_3^{A'}$ in 1 1 1 & 3 pts resp. Hence the corr. quintic curve passes thro' A' three times & meets $A'P'$ $A'Q'$ $A'R'$ each once. Thus A' is a triple node on all these curves. The curve meets Σ_3' in 4 pts + Σ_6' in 3 pts.

Again a plane meets each of the F-quartics & in particular the F-cubic $\Sigma_3^{A'}$. ~~in a~~ It meets the latter in a unicursal cubic having d.p. on AQ. Let this curve be called S_3. Corr. to the cone vertex A & base S_3 we get a quartic cone vertex A' having $A'Q'$ as triple side. This quartic cone is the tangent cone to the corr. quintic surface at A'. All quintic surfaces of (S') corr. to planes have therefore a conical pt of 4th order at A'. They also contain the lines A'P' A'R' & A'Q' the latter 3 times & the curves Σ_3' & Σ_6'.

The involutive quartic transf.

Let the spaces (S) & (S') be now superposed & let the bases of the sheaves be made to coincide AP with A'P', AQ with A'Q' & Σ_3 with Σ_3'. Let the lines of the quadratic pencils be involutive. If the two systems of conicoids be now involutive the transf. as a whole will be involutive. Let AL be any line thro' the common vertex A. We shall say that the two sheaves of conicoids form an involutive system if the ranges they determine on AL are involutive.

If the pencils be superposed so that AP coincides with A'R' & AR with A'P' there will be a cone of double lines for them. The two homographic systems of conicoids will have two double conicoids & hence in the transf. there will be two curves of d.ps. namely two quartic curves on the cone of d.lines intersecting in 6 pts where the common base Σ_3 meets the cone.

(C) Case of collinear pencils.

The transf. is now clearly a quartic transf. For if l be a line in (S) its correspondent is a unicursal quartic having a d.p. of 3rd order at A' & lying in the ple thro' A' corresp. to Al. AQ & A'Q' must be corresp. lines in the collinear pencils. Consider a ple thro AQ & a line in it meeting AQ in q. Corresp. to this line we get a conic in the corresp. ple in (S'). Hence the quartic surfaces of (S') corresp. to ples in (S) must have A'Q' as double line. Corr. to any pt q on AQ we get the double line (conic) A'Q'. Corresp. to the F-pt A as before we get a unicursal ruled cubic surface having A'Q' as double line.

Corr. to pt on Σ_3 we get a line in (S') & these lines trace out a cubic cone vertex A' & base Σ_3'.
Corr. to the lines A Σ_3 we get the curve of intersection of above cone & a ruled cubic having A'Q' as double line.
This curve is of order 5 & is an F-curve in (S').
This F-curve has a multiple pt of order 3 at A'.
(The F-surfaces corresp. to A & A' in these transf. it should be noted contain all the F-curves in their spaces. Also the F-surfaces corr. to Q & ~~A~~ Q' in the gen^l transf of order 5 contain all the F-curves.
For taking the above transf. Let R' be a pt on the F-curve Σ_5' in (S'). Corr. to R' we get a line thro' A & therefore the pt A itself. Consequently R' must lie on the F-surface corresp. to A. Similarly for the other transf.)
The F-system of lines in (S) is therefore AQ (double) Σ_3 + Σ_5 & similarly for (S').
The F-surfaces in (S) are $\Sigma_3^{A'}$ (corresp. to A') & cone A Σ_3.
~~& ruled a~~ & similarly for (S').
A plane in (S) transforms into a quartic surface in (S') having A'Q' as double line containing the F-curves Σ_3' & Σ_5' & having a multiple pt of order 4 ~~on~~ at A'.

Superposition of spaces.

When the spaces are superposed there are 10 dps.
For the locus of the intersection of corresp. rays of the collinear pencils A & A' is a cubic curve thro' A & A' & this meets the quartic surface generated by the intersn. of corresp. conicoids in 12 pts two of which are at A & A'.
Hence there are 10 proper dps.
If the F-systems be superposed & if the pencils are congruent we get a surface of dps. For let AL be any line thro' the common vertex A. Corresp. conicoids trace out homog. ranges on AL which have 2 dps.

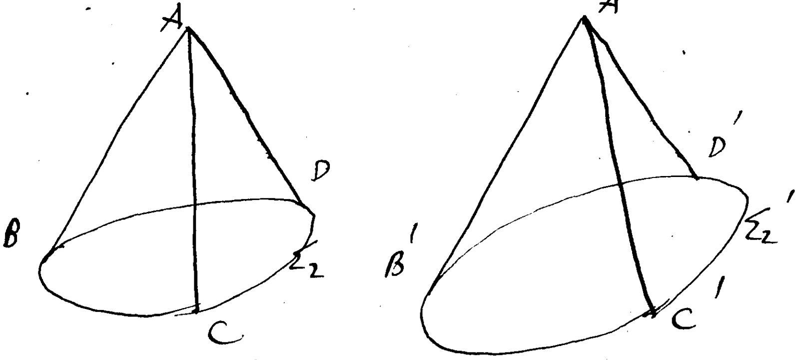
Hence any line thro' A meets the surface of dps in 2 pts. But since A corresp. also to itself twice (since its F-surface contains A twice) it follows that the surface is of order 4 & has a dp. at A. This may be verified thus a line l meets its corresp. quartic in 4 pts which must lie on the surface of dps hence a line meets the surface of dps in 4 pts & the latter is therefore a quartic.

This transf. can also be involutive if the systems of conicoids are involutive. Corresp. pts on any ray AL are now harmonic conjugates with respect to the two pts in which AL meets the quartic surface of dps.

This transf. is a particular case of a more general quartic transf. which will be given later. In the general transf. the F-curve is a single curve of order 12 having a sextuple node at A & the F-surface is a cone of order 6 vertex A & base the F-curve. The F-system of lines above is of order 12 but is degenerate.

Further particular cases.

We shall now give several important cases in which the base curve of the system of conicoids is degenerate & is composed of a line & a conic. We shall suppose also that the bases of the sheaves of ples are coplanar.



Let AB AC , A'B' A'C' be the bases of the ple sheaves & let AB, AC Σ_2 , A'B', A'C' Σ_2' be the bases of the sheaves of conicoids, Σ_2 & Σ_2' being conics intersecting AB AC &

AB A'C' respectively.

(c) Let the relation between the pencils A & A' be quadratic & let AB & A'D' be the lines corr. to ples B'A'C' & BAC respec.

(1) We shall make the following suppositions

(α) that to the deg. conicoid composed of ple BAC & conic Σ_2 corr. the deg conicoid ple B'A'C' & conic Σ_2'. & (β) that to cone $A\Sigma_2$ corr. the cone $A'\Sigma_2'$.

The transf. is again a quartic transf.

Corr. to a pt q on AB we get a conic in ple $\overset{C}{\cancel{B}}'A'D'$

& .. .. p .. AC -- -- -- -- B'A'D'

Corr. to A we get ~~in~~ for each ple thr' AB or AC a line thr' A' in the corr. ple thr' A'B' or A'C'. The locus of these lines is a cone ~~base~~ viz the cone (~~which corresponds to the cone~~) $A'\Sigma_2'$ in (S). AB & AC are F-lines & are double lines on the quartic surface which corr. to a ple in (S).

To a line in (S) corr. a quartic curve which meets A'B' & A'C' & A'D' once & has a node at A'.

To a ple of (S) corr. a quartic surface in (S') having a triple conical pt at A' & having A'B' A'C' as double lines. These surfaces pass thr' Σ_2' & thr' 2 gens. of the cone $A'\Sigma_2'$ viz the gens in which it is intersected by the cone corr. to the cone $A\Sigma_2$.

Let the cone corresp. to $A\Sigma_2$ be $A'\Sigma_2''$. & that corr. to $A'\Sigma_2'$ be $A\Sigma_2'''$. Corr. to any pt on Σ_2 we get a gen^n of $A'\Sigma_2''$ & vice versa. The gens. of these four cones are F-lines & enjoy the peculiarity of correspondence with each other.

The tangent cone at the conical pt A' on a quartic surface consists clearly of the ple B'A'C' & a proper quadric cone thr' B'A' & A'C' viz. the cone corr. to the F-cone $A\Sigma_2$.

If we suppose also that AD & A'D' meet the conics Σ_2 & Σ_2' resp. the F-surface corr. to A is ~~the~~ cone $A\Sigma_2$ & the tangent cones to the quartic surfaces in (S') at A' are now made up of the ple B'A'C' reckoned twice & the ple corr. to the cone $A\Sigma_2$.

(d) If the pencils are collinear we get a cubic transf.
Corr. to a line of (S) we get a cubic curve of (S') thro' A'.
Corr. to a ple of (S) we get a cubic surface thro' A'B' & A'C' & Σ_2' & having a conical pt at A'.
Corr. to the F-pt A we get the cone $A'\Sigma_2'$ in (S') & vice-versa.
The cubic surfaces have also two other F-lines viz the lines of intersection of the cone $A'\Sigma_2'$ & the cone corr. to $A\Sigma_2$. The cubic curves are ple curves & have a node at A'. This transf. is a particular case of a more general transf. to be given later in which the F-curve system is a proper sextic having a node at A'. In the above trans. the F-curve system consists of four lines thro' A' & the conic Σ_2'.
With coincident spaces we get 8 dps & with coincident F-curves a cubic surface of dps. thro' A'. Corr. pts are harmonic conjs. with resp. to the pts in which a line thro' A containing them meets the surface of dps. ~~Lastly~~ the transf. may be involutive.

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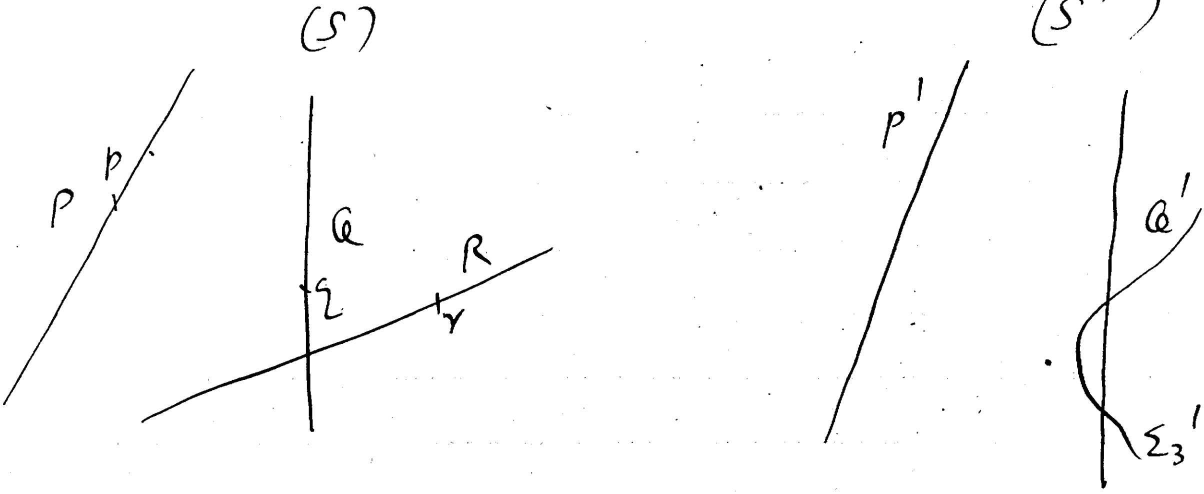
Art 32. The transformations built up by 3 sheaves of ples in (S) & two sheaves of ples in (S') & a sheaf of cuneoids.

We shall take the most general case to begin with. Let the 3 base lines in (S) be non-coplanar & non-concurrent.



The sheaves (P) & (P') are homographically related
" " (Q) (Q') " " "
" " (R) & the sheaf of cuneoids Q'Σ3' are also homographically related.

With these suppositions the corr. between the pts of the spaces (S) & (S') is clearly (1-1).

Let a pt M describe a line l in (S). The ples P'M' Q'M' ~~describe~~ generate a cuneoid & the sheaves (Q') & (Q'Σ3') generate a ruled cubic surface having Q' as double line. The intersection of the previous two surfaces is a curve of order 4.

Conversely let M' describe a line (l') in (S'). The ples PM QM generate a hyp. & the ples QM, RM generate a ruled ~~quartic~~ cubic having Q as double line. For since a cuneoid of system Q'Σ3' meets l' in 2 pts it follows that the relation between the sheaves (Q)(R) is (2, 1). These ~~cuneoids~~ surfaces meet in a curve of order 4. Hence the transf.n is reversible.

Hence corr. to a ple in (S) we get a quartic surface in (S') & viceversa.

The intrinsic eqn. of such a quartic surface is of the form $K_1 S_1 S_2 S_3^2 + K_2 S_1 S_3^2 + K_3 S_2 S_3^2 + K_4 S_1 S_2 S_3$
$+ K_5 S_1 S_2 + \cdots \qquad = 0$

The quartics in (S') have Q' as double line & P' as single line. They also contain Σ3'.

The quartics in (S) have Q as double line & P & R as single lines.

The F-systems.

Corr-to a pt. on P (p) we get a line in (S') & the locus of these lines as p moves on P is a ruled cubic surface having Q' as double line & containing Q'Σ3'.

Corr-to a pt q on Q we get a line in (S') & the locus of these lines as q varies is a cubic surface thro' P' & Q'Σ3'.

Corr-to a pt ~~on~~ r on R we get a line in (S') viz a line meeting P' & Q'. The locus of these lines is a hyp. having P' & Q' as generators.

Again consider a line meeting $P^+ Q^+$ & R^+ ie a gen^r of the hyp. PQR. Corr-to this line we get a pt viz the pt of intersection of corr-ples thro P & Q' & corr-conoid thro' Q'Σ3'. As the gen^r moves on the hyp PQR this pt describes an F-curve in S'. This F-curve is the intersection of a hyp. thro' P' & Q' & a ruled cubic thro' Q'Σ3' & having Q' as double line. ~~It is~~ therefore a quartic curve.

Again consider a line meeting P'Q' & Σ3'. The locus of these lines is a quartic ruled surface having Q' as triple line & P' as single line. Corr. to this surface we get in (S) a curve which is the inters. of a ~~conoid thro' PQ~~ surface of order 4 having Q' as triple line

& P as single line & a cubic having Q as double line & R a single line. These surfaces intersect in a curve of order 6 which is an F-curve in (S).

Thus the complete system of F-curves in (S) is P Q (double) R & the above curve Σ_6.

The complete system in (S′) is P′ Q′ (double) Σ_3' & Σ_4' (which corr. to the hyd. PQR).

Again corr. to a pt on P′ we get a line (S) & corr. to P′ as a whole we get a ruled cubic having Q^+ as double line & R as single line.

As in the Quintic transf. we can show that corr. to any pt q′ on Q′ we get a conic & the locus of these conics is a cubic ~~a~~ surface containing Q P R etc. Again corr. to a pt on Σ_3' we get a line & the surface generated by these lines is a ruled quartic having Q as double line.

The complete F system of surfaces in (S) consists therefore of a ruled cubic, a cubic surface, a ruled quartic & a conicoid.

The complete F-system in (S′) consists of a ruled cubic, a cubic surface, a ruled quartic & a hyd. Thus the F-surfaces are of the same nature in each space.

A ~~cubic~~ quartic curve in (S′) meets P′ Q′ each in 3 pts; it meets Σ_3' in 4 pts & Σ_4' in 2 pts.

A quartic curve in (S) meets P Q each in 3 pts R in 2 pts & Σ_6 in 4 pts.

A curve of order m in (S) transforms into a curve of order 4m in (S′) in general.

If curve meet P in α pts Q in β pts R in γ pts & Σ_6 in ρ pts its corresp. is of deg.

$m' = 4m - \delta - 2\mu - \nu - \rho$.

Again a surface of order n in general transforms into a surface of order $4n$ but if $\delta\ \mu\ \nu\ \rho$ be as before the degrees of multiplicity of $P\ Q\ R\ \Sigma_6$ respectively it transforms into a surface of order

$n' = 4n - \cancel{3(\delta} - 3\delta - 3\mu - \overset{2}{\cancel{4}}\nu - 4\rho$

$= 4n - 3\delta - 3\mu - 2\nu - 4\rho$.

Nature of these quartic surfaces..

Let a plane σ of (S) meet the F-system in $p\ q\ \alpha\ a\ b\ c\ d\ e\ f$, p being on P q on Q α on R & $a\ b\ c\ d\ e\ f$ on Σ_6.

Corr. to each of the pts $p\ \alpha\ a\ b\ c\ d\ e\ f$ we get a line & corr. to each of lines $q\alpha$, qa, qb etc we get a line. Thus there are altogether 16 lines on the surface excluding P' & Q'. The surface therefore possesses 17 single lines & a double line.

Corr. to the pt q we get a conic on the surface (E).

[It should have been mentioned in the quartic transf. that to the pt q there corr. a cubic curve on the quartic surface]

Corr. to each of the lines $p\alpha$, pa etc --- $8C_2$ in number we get a conic. The number of conics in this set is therefore $\frac{8\cdot 7}{1\cdot 2}$ i.e. 28. Call it A_i, $i = 1 \ldots 28$.

Corr. to each of the conics $q\ p\ \alpha\ a\ b$ etc we get a conic. The number of such conics is $8C_4$ or $\frac{\overset{2}{\cancel{8}}\cdot 7\cdot \cancel{6}\cdot 5}{1\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{4}}$ or 70. Call this set B_l, $l = 1 \ldots 70$.

Again a unicursal cubic having a node at q & passing thro' 6 of the 8 pts $p\ \alpha$ etc transforms into a conic. The number of such conics is $8C_6$ or 28. (C_k)

Lastly a unicursal quartic having a triple node at q & passing thro' the 8 pts $p\ \alpha\ a\ b\ c\ d$ etc transforms into a conic. The number of such conics is 1. (D).

The total number of conics on the surface is therefore $70 + 56 + 2$ i.e. 128. The intersections of the systems

ABCDE can be investigated as in the quintic transf. Thus the curve D cannot intersect any of the curves C. It must however intersect each of the conics B in 1 pt & each of the conics A in 2 pts & so on. Of course any section of the surface thro' Q' is a conic (excluding Q' itself.).

As in the quintic transf. we may give a slightly more general form to this quartic transf. by replacing the sheaves (P) & (P') by a correlation between the spaces (S) & (S'). The F-system of curves in (S) will then consist of Q (double), R & a septic curve Σ_7. In (S') it will consists of Q' (double) Σ_3' & a quintic curve Σ_5'.

Superposition of the spaces.
The number of dps is 10. We shall show this for the general transf. Corresp. ples thro' Q & Q' generate a hyp. & ples thro' R & corr. conicoids generate a cubic surface. The curve of intersection of these surfaces is of order 5 since Q' is a line on each.
In the correlation there is a conicoid of incident pts. The quintic curve meets this conicoid in 10 pts. These are the dps.

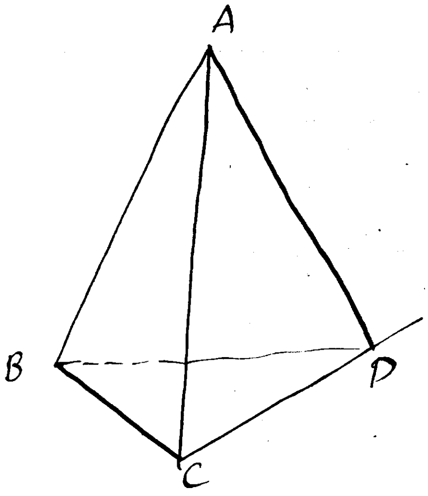
Art 33. Particular cases of the quartic transf.
As before we shall suppose now that the lines PQ & P'Q' are resp. coplanar. When this is the case the pencils A & A' may either be quadratic or collinear where A & A' are the pts of intersn. of PQ & P'Q' respectively.
(a) If the pencils are quadratic the transf. is still quartic & its main features are similar to those of the corr. quintic transf.

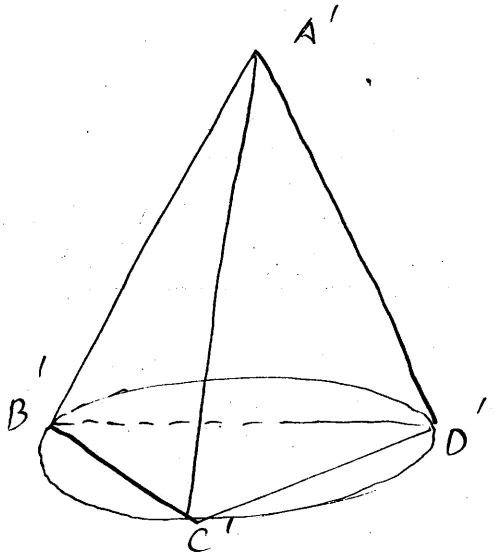
A & A' are triple conical pts on the quartic surfaces.

(b) If the pencils are collinear the order of the transf. is reduced & it becomes a cubic transf.
Corr. to a line we get a plane unicursal cubic & vice versa.
Corr. to a plane we get a cubic surface having a conical pt at the vertex of the pencil.
The F-line-system in (S') consists of $Q'\Sigma_3'$ & a pair of lines thro' A'. These lines corr. to the F-surface in (S) composed of lines thro' A meeting R.
The F-line-system in (S) consists of R, Q & the curve of intersection of a cubic cone thro' A (corr. to cone $A'\Sigma_3'$) & a conicoid thro' Q & R. These surfaces have a common gen. Q ~~& another gen. thro' A meeting R. The~~ which is a double gen.^{n} of the cubic cone thro' A. This curve of intersection is a quartic curve having a node at A. Hence we see that that in (S) the F line-system consists of Q & R & a quartic Σ_4 & in (S') of $Q'\Sigma_3'$ & Σ_2' (consisting of two lines thro' A').
The F surface system in (S) consists of the plane AR; ~~&~~ the cubic cone thro' A having AQ as double gen.^{n}, & the conicoid which corresponds to A' containing Q & R.
The F-surface system in (S') consists of the plane corr. to AR ~~&~~; the cubic cone vertex A' & base Σ_3', & the conicoid corr. to A which contains A'Q' & Σ_3'.
When the spaces are superposed we get Selps. With coincident bases the transf. ~~may~~ may become perspective. It is a particular case of the cubic transf before-mentioned.

Further particular cases of the general quartic transf. arise when the cubic base curve Σ_3 takes the degenerate form made up of a conic & a line.
One of these particular cases is of great importance in

that which gives the general quadratic transf.





In the following particular cases we shall make the supposition that the base lines in (S) are coplanar. Let them be AB AC & BC. In (S') we shall suppose that the base of the conicoid is made up A'B', A'C' & a conic meeting both of these lines. A'B' & A'C' will also be taken as the bases of the sheaves of ples in (S'). Six ~~Four~~ distinct cases can arise giving respectively transf. of order (3-3) (3-4). (2-2) (2-4). By (3-4) is meant that the transf. is cubic in one sense & quartic in the reverse sense & so on.

Case (1) (3-3) transf.

We make the following suppositions.

To ple BAC sheaf (BA) corr. ple B'A'C' sheaf (B'A')
– . CAB " (CA) " " C'A'D' " (C'A')
– BAC " (BC) " deg. conicoid (B'A'C', B'C'D')
– BDC " (BC) " " " $A'\Sigma_2'$
" BAD " (BA) " ple B'A'D' sheaf (B'A')
" CAD " (CA) " " C'A'B " (C'A').

In all cases if M be the pt of intersection of the ples ABM, BCM, ACM & the corr. pt M' in (S') is the intersection of ples B'A'M', C'A'M' & conicoid (B'A', A'C'Σ_2'M') which corr. respectively to ABM, ACM & BCM.

If M describe a line l in (S) ples B'A'M' & C'A'M' describe cone on which A'C' counts as two gens. Ple C'A'M' & conicoid corr. to BCM generate a ruled cubic thro' A'C' & A'B' A'C' being a double line on it. The intersection of these surfaces is a curve of order 3.

Conversely if M' describe a line l' in (S') ples BAM & CAM generate a cone. Now since a conicoid of (S') meets l' in 2 pts to any conicoid there corr. two positions of ple B'A'M'. The relation between the gen. sheaves (B'A') & (B'A'C'Σ_2'M') is (2-1). Hence the relation between the sheaves (BC) & (BA) is (1-2) & they generate a ruled cubic having ple BAC as part of surface. Hence proper surface is a conicoid thro' BA. The intersection of this conicoid & cone (BA, AC) is a cubic curve. Hence the transf. is (3-3).

F-systems.

Corr. to AB we get A'C' & C'D'.
" AC -- ple B'A'C'
.. BC .. line A'C'
.. CD .. A'B'

Hence F-line system in (S') is A'B' (A'C' reckoned twice) C'D' & Σ_2' altogether a system of 6th order.

F-system in (S) is AB, AC (reck. twice) BC & a conic in ple BCD thro' B & C corr. to gens. of F cone A'Σ_2' in (S').

Corr. to a ple in (S) we get a cubic surface in (S') having a conical pt at A' of which B'A'C' is tangent ple.

Corr. to ple in (S') we get a cubic surface in (S) having a conical pt at A of which BAC is tangent ple.

Case II. We make the same suppositions with respect to the sheaves (BA) (AC) (B'A') (A'C') as in Case I but with respect to (BC) & sheaf of conicoids we suppose that to ple BAC corr. cone A'Σ_2' &

that to plane BDC corr. deg. conicoid (B'A'C', B'C'D'). As before we see that corr. to a line of (S) we get a cubic curve of (S'). But corr. to a line of (S') we get in (S) a quartic curve the intersection of a ruled cubic & a cone the double gen^r of the cubic being common to both surfaces. Hence corr. to a plane of (S) we get a surface of order (4) in (S') & corr. to a plane of (S') we get a surface of order (3) in (S).

The transf. is therefore (3-4).

The F-line-system in (S'). Corr. to AB we get ~~AB'~~ A'C', A'D'. Corr. to CA we get A'B' & A'C'. Corr. to BC we get A'C'. Corr. to CD we get A'B'.

The complete F-system in (S') is A'C' (double line) A'B' (double line) Σ_2', A'D'

A'C' & A'B' are double lines on the quartic surfaces in (S') & in estimating the degree of the curve of intersection of two such surfaces A'C' must be considered as a c. of order 6.

In (S) the F-system is A B, AC, BC.

Case III Let us suppose that corr. to ple BAC (BA) there corr. ple B'A'D' (B'A') – & to ple CAB (CA) corr. ple (C'A'D') (C'A') – to ple BAC (BC) corr. deg. conicoid (B'A'C'; B'D'C') & to ple BDC (BC) corr. cone $A'\Sigma_2'$.

Cor. to a line in (S) we get the intersection of a cone thro' A'B', A'C', A'D' & a ruled cubic having A'C' as double line & A'B' as single line. The degree of this curve is 3.

Cor. to a line in (S') we get in (S) the intersection of a cone thro' AB, AC, AD with a ruled cubic (cone) on which AC is a double line. The degree of this curve is 4.

The transf. is again (3-4).

The F-system of lines in (S') is composed of A'B', A'C' A'D', B'D', C'D' & Σ_2'. The generatrix surfaces have triple conical pts. at A' & have A'B' A'C' as double lines.

The F-line system in (S) is composed of AB; AC; BC & a line EF in ple BDC which is the F-curve corr. to the F-cone in (S') viz $A'\Sigma_2'$.

Case IV. We suppose the sheaves of ples thro' AB, AC, A'B', A'C' to be as in case III. We also suppose that to ple BAC (BC) corr. cone $A'\Sigma_1'$ & that to ple BDC (BC) corr. deg. conicoid (B'A'C'; B'D'C').

The transf. is in this case (2-4).

Cor. to a line in (S) we get a conic in (S')
& cor. to a line in (S') a quartic in (S).

Cor. to a ple in (S) we get a quartic surface in (S') having A' as triple conical pt & A'B' A'C' A'D as double lines.

Cor. to a ple of (S') we get a conicoid which touches a fixed ple in (S) & which passes thro' 3 fixed pts viz B C D. The fixed tangent ple is the ple thro' A corr. in the quadratic rel. between the pencils to the cone $A'\Sigma_1'$.

The F-system in (S) consists of 4 pts A B C D in (S') of the lines A'B', A'C', A'D' & the conic Σ_2'.

Case V. In the following two cases the relation between the pencils A & A' is a collinear one.
Let us suppose that pencil A is coll. with pencil A'
& that to ple B A C (BC). corr. cone $A'\Sigma_2'$
& that to ple B D C (BC) .. deg. con. (B'A'C', B'D'C')
The trans. is again (3-3) but is of no particular interest.

Case VI. The pencils A & A' being again coll. let us suppose that to ple B A C (BC) corr. the deg. con. (B'A'C', B'D'C') & to ple B D C (BC) corr. cone $A'\Sigma_2'$.
This trans. is (2-2) & is the general quad. transf.

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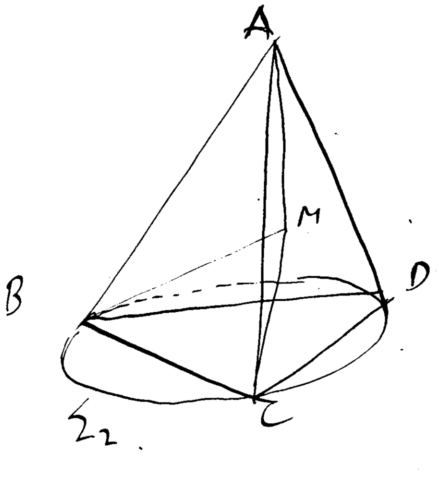
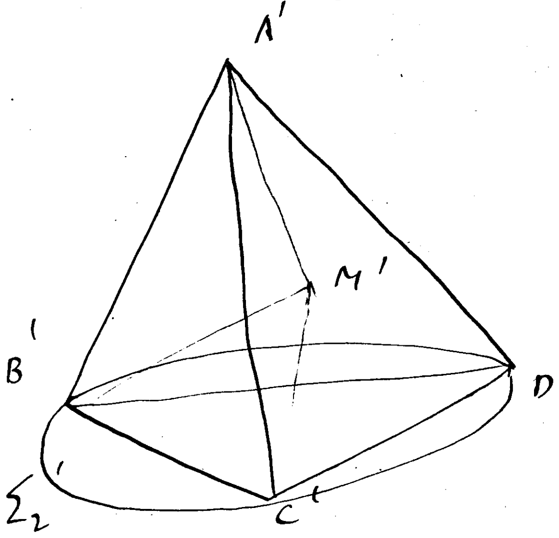
Art. 34. The general quad. transf. in detail.





Corr-to pt M which is the intersection of ples B A M, C A M, & B C M we get the pt M' which is the inters. of the corr-ples B'A'M' & C'A'M' & conicoid thro B'A'C'M' Σ_2'.

Exceptional elements of the transf.

Corr. to a pt P on ple BCD in (S) we get clearly the pt A' in (S'). Corr. to pt A we get pts on ple B'C'D' in S'. A & A' are F-pts of transf. & their corr. are the ples B'C'D' & BCD respy.

Corr. to any ~~pt~~ genr P' of Σ_2' we get a pt on ple BCD. As P' moves on Σ_2' ~~to~~ these pts generate a conic thro' BCD. For corr. to cone A'Σ_2' in (S') we get by the collinear relations between the pencils a cone AΣ_2 in (S). Let this conic be Σ_2. Corr. to a genr of cone AΣ_2 we get a pt on conic Σ_2'.

Thus the relation between the cones AΣ_2 A'Σ_2' is as follows. Let AP be a genr of AΣ. Corr. to AP as a whole we get the pt P' on Σ_2' where the corr. genr A'P' meets Σ_2': Corr. to A'P' we get the pt P.

The conics Σ_2 & Σ_2' are F-conics & the cones AΣ_2 A'Σ_2' F-surfaces.

The above F-elements give the complete F-system in both spaces.

Let M describe a line l in (S). AM describes a ple & so does A'M'. Ples C'A'M' & the conicoids of sheaf generate a cubic surface of which ple C'B'A' is a part. The other part is therefore a conicoid. This conicoid meets the ple described by A'M' in a conic.

This may also be shown as follows.

Let BC meet the ple Al in the pt α. As M describes l the lines AM & αM describe pencils in perspective. Let ple described by A'M' be called π'. The sheaf of conicoids determines a pencil of conics in this ple thro' A & the two pts in which π' meets Σ_2'. The pencil of lines A'M' & the pencil of conics are in perspective & hence the locus of their intersection is a conic thro' A' & the pts in which π' meets Σ_2'.

Conversely Let M' describe a line l' in (S'). AM describes a ple π in (S). The pencil of lines $A'M'$ & the pencil of conics in $A'l'$ are in perspective & the pencil of lines AM & the pencil αM (~~are in perspec~~) which corr. to these are homographic & also in perspective. But since a conic in $A'l'$ cuts l' in 2 pts. it follows that the relation between the pencil $A'M'$ & the pencil of conics is $(2-1)$. Hence the relation between the pencils AM & αM is $(2-1)$. The locus of their intersection is therefore a cubic having A as d.p. But this cubic is degenerate since the common ray αA belongs to both pencils & is therefore a part of the cubic. The remaining part is a conic thro' A. Since l' meets $A'\Sigma_2'$ in 2 pts, the conic must meet Σ_2 in 2 pts.

Thus we have established the quadratic nature of the pencils.

Corr. to a ple of (S) we get a conicoid of (S') thro' A' & containing Σ_2'. & corr. to a ple of (S') we get a conicoid of (S) thro' A & containing Σ_2.

Nature of the conicoids.

Let a ple of (S) meet Σ_2 in two real pts P & Q. Corr. to P & Q we get two real lines $A'P'$ $A'Q'$ on the conicoid which is therefore a ruled conicoid. If the ple. do not meet Σ_2 in real pts the conicoid of (S') will not be ruled.

Effect of the transf. on curves & surfaces.

A conic in (S) not meeting Σ_2 transforms into a quartic in (S') having a node of second order at A' & meeting Σ_2' in 4 pts.

A conicoid transforms into a quartic surface having a conical pt of second order at A' & having Σ_2' as a double curve.

Generally a curve of order m transf. into a curve of order 2m. But if it meet Σ_2 in d pts it transforms into a curve of order $2m-d$.

Generally a surface of order ~~n~~ transforms into a surface of order 2n but if it contain Σ_2 d times its corresp. is of degree $2n-2d$.

If the curve ~~meet~~ pass thro' A μ times & cut Σ_2 d times its corresp. is of deg. $2m-d-\mu$.

We can prove this as follows.. Since the curve passes thro' A μ times its correspondent in (S') must meet ple B'C'D' in μ pts. Also since the curve meets the cone $A\Sigma_2$ in 2m pts of which 2μ are at A & d lie on Σ_2 the remaining pts on $A\Sigma_2$ must be in number $2m-2\mu-d$. Corr. to these we get $2m-2\mu-d$ pts on Σ_2. Hence the total no of pts on ple B'C'D' is $\mu+2m-2\mu-d$ ie $2m-\mu-d$.

If a surface of order n have a μ-fold conical pt at A & contain Σ_2 d times it transforms into a surface of ~~the~~ order $n' = 2n-\mu-2d$ since the corresp. of A is the ple B'C'D'.

For example a conic meeting Σ_2 in 2 pts, but not passing thro' A transforms into a conic in (S').

A conicoid containing Σ_2 but not A transforms into a conicoid (not cont. A').

A conicoid thro' A but not containing Σ_2 transforms into a cubic surface having a conical pt at A' thro' which pass 6 lines. Two of these corr. to the two gens. of the conicoid & the other 4 to the 4 pts in which Σ_2 meets conicoid. Thus we establish the existence of a cubic passing thro' a fixed conic & having a conical pt at a fixed pt thro' which pass

6 pts. It has been mentioned in previous work that such a cubic surface contains 21 lines. We give the following elegant proof of this depending on the quad. transf.

Let the conicoid meet the conic Σ_2 in P Q R S.

Corr. to gens. thro' P, Q, R & S we get eight lines on the corr. cubic surface.

Again the ple PQA meets conicoid in a conic which passes thro' A & meets Σ_2 twice. This conic transforms into a line on the cubic. In all there are six ples like PQA & hence we get six additional lines on cubic. Again consider the cubic curve on the conicoid thro' A P Q R S. This cubic is uniquely determined by these conditions. By the above formulae p 209. it transforms into a line on the cubic surface. Hence with the six lines previously found we have 15 additional lines or 21 in all.

Another important example is as follows.

A quartic surface having Σ_2 as a double curve but not containing A transforms into a quartic surface having Σ_2' as double curve & not containing A'. For corr. to Σ_2 we get the cone $A'\Sigma_2'$. Now the intersection of quartic with the cone $A\Sigma_2$ is a quartic curve (excluding Σ_2) which meets every gen.^{r} of $A\Sigma_2$ in 2 pts. Hence the corr. quartic must contain Σ_2' twice.

Superposition of the spaces.

When the spaces are superposed the number of dps is 6. For the locus of the intersection of corresp. rays of the coll. pencils A & A' is a cubic curve. The intersection of ple BMC & corr. conicoid generate a cubic surface thro' BC & Σ_2' A'B' A'C'. The cubic curve intersects the line of intersection of ples BAC B'A'C' twice but this line also lies on the cubic surface since BAC & B'A'C' are corr. ples. Hence the cubic curve meets the cubic surface in 2 pts lying on the line of intersection of ples BAC B'A'C'. These pts depend on the choice of ~~fund.~~ tetrahedra of reference & therefore must be rejected (compare the corr. theorem on two superposed collineations). Again the cubic curve passes thro' A' which is also on the cubic surface. This being a ~~dp~~ F pt has to be rejected. Hence of the 9 pts of intersection of the cubic curve & cubic surface only 6 are proper dps.

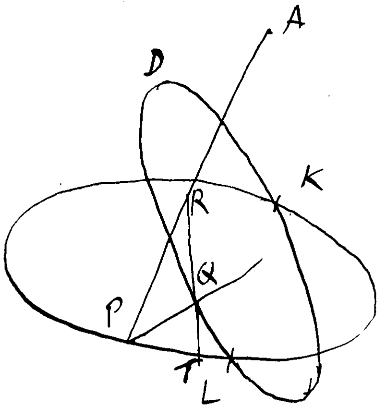
Let now the pencils be superposed so that A & A' coincide & let the pencils be congruent. The cones $A\Sigma_2$ & $A\Sigma_2'$ coincide but the conics Σ_2 Σ_2' do not necessarily coincide.

We shall consider first the case in which Σ_2 & Σ_2' do not coincide. Corr. pts are now collinear with A (the common vertex of the pencils). Consider a line AL thro' A. The ples BCM_i & the corr. conicoids trace out homog. ranges on this line which have 2 dps. On each line AL there are consequently 2 dps & hence there will be a surface of dps which is clearly a conicoid. This surface must contain Σ_2 & Σ_2. Again any conicoid thro' Σ_2 & Σ_2' must transform into another conicoid thro' Σ_2 & Σ_2'. Of the single infinity of such conicoids there is one viz the conicoid of dps which transforms into itself.

We shall now suppose that the conics Σ_2 & Σ_2' coincide. The transf. is now involutive, for let AL meet ple BCD (ple of Σ_2 & Σ_2') in L. Corr. to L we get the pt A for both homog ranges on AL hence these ranges are in involution. Corr. pts are now harmonic with respect to the points in which the line joining them meets the surface of dps. Corr. pts are in fact conjugate with respect to the conicoids of dps.

The cone AΣ (Σ common conic) is a tangent cone to the surface of dps. We may give a purely geometric form to this transf. as follows. Consider a fixed pt A & a fixed conicoid S. A line thro' A meets the conicoid in pts P & Q. Let M be any pt on this line & let its polar ple meet the line in M' then M & M' are corr. pts. The F-conic is the conic of contact of the tangent cone from A.
This involutive quadratic transf. may be called quadratic inversion. It is a perspective transf.

Any conicoid S_1 thro' Σ & not thro' A transforms into another conicoid S_1' thro' Σ. Let S_1 meet the conicoid of dps. in a conic D. D also lies on S_1'. Σ & D are the complete intersection of S_1 & S_1'.
Investigation of the conicoids which transform into themselves.
A single infinity of conicoids can be made to pass thro' Σ & D. Of these there is one which transf. into itself. A gen. of S_1 viz PQ transf. into a gen. of S_1' viz QR. If QR meet S_1 in another pt besides Q & T then the conicoids coincide completely since they have now 2 conics & a gen. common equiv. to 9 pts. PQ & QR are fixed for all conicoids of the series & there is clearly only one conicoid which will contain the line QR. This conicoid transforms into itself.

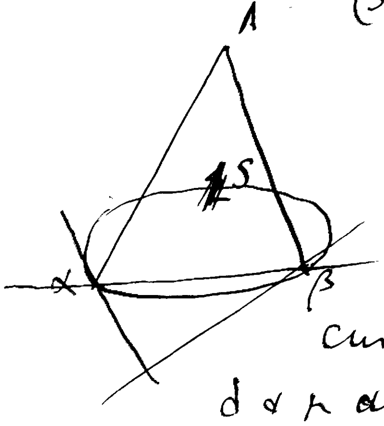


The gens. of one system transform into the gens. of the other system; corr. gens. intersect on the conic of dps D. The ~~transf.~~ of gens. on the conicoid is thus really a linear transf. & is the case referred to on page -- . The double lines of this linear transf. are the gens thro' K & L where D meets Σ. Since the conic D can be chosen on the conicoid of dps S in a triply infinite number of ways it follows that there ~~are~~ is a triply infinite system of conicoids which transform into themselves.

It is of interest to enquire whether there are any conicoids which by the quad. transf. transf. into themselves in such a way that gens. of ~~the same~~ one system transf. into gens. of the same system.

Let the pencils be not congruent but let be related by a central collineation so that there is a ~~line of~~ ple of double lines & a single double line. The conics Σ_2 & Σ_2' meet in 2 pts (on the double ple) but do not lie on a cone thro' A. There is now a conic of dps viz that lying on the ple of double lines, & a single pair of isolated dps on the single double line thro' A. Any conicoid thro' Σ_2 & the two isolated dps transforms into itself. For it meets the conic of dps in 4 pts (2 on Σ) & contains other 2 dps.
The gens. of one system transf. into gens. of same system. The double line of the pencil which meets Σ_2 & Σ_2' is a generator & corr. pts when joined by a line must give a line meeting this gen.ʳ

Properties of quadratic inversion.
Let the conicoid meet its corresp. ple in the conic S & let the gen.ˢ thro' A meet S in α & β.
Then we get the following results.
(1) The gens. of one system project on the ple into lines thro' α & the gens. of the other system into lines thro' β.
(2) Conics thro' A project into lines in ple which do not pass thro' either α or β.



(3) conics not thro' A project into conics on ple thro' α & β.

(4) A curve of degree m which meets the genr $A\alpha$ in λ pts & $A\beta$ in μ pts where $\lambda+\mu = m$ projects into a ple curve of degree m having nodes of order λ & μ at α & β respy. For it meets the conic Σ in m pts & hence its corr. is of degree $2m-m$ ie m,

Thus a cubic which meets $A\alpha$ in 2 pts & AB in 1 pt projects into a ple cubic with a node at α.

A quartic which meets $A\alpha$ in 2 pts & AB in 2 pts projects into a ple quartic having nodes at α & β.

Thus the intersection of two conicoids projects into a ple quartic of deficiency 1.

A quartic which meets $A\alpha$ in 3 pts & $A\beta$ in 1 pt projects into a quartic having a triple pt at α. This quartic is of deficiency 0. It is not the intersection of two conicoids. The surface of lowest degree which can be made to pass thro' it is a cubic for the line $A\alpha$ must meet this surface in 3 pts.

Similar results may be given for curves of higher order.

The geometry of the conicoid can be elegantly treated by this transf.

We may regard curves on the conicoid as traced out by the intersection of two generators which meet two fixed gens. which we may take as $A\alpha$ & $A\beta$. Consequently the corr. curves on the ple will be traced out by the intersection of lines thro' α & β. Let two homographic pencils be taken thro' α & β. The locus of their intersection is a conic thro' α & β.. The locus of the corr. gens. on the conicoid is therefore a conic on the surface. Hence if we take two homographic systems of gens. one system & intersecting

$A\alpha$ & the other $A\beta$ the locus of their pt of intersection is a conic. If the pencils be in perspective the locus of their intersection is a line & the locus of two systems of gens. in perspective is a conic thro' A.

Again if the relation between the pencils α β is (2-1) the locus of their intersections is a cubic curve having a dp. at α. The corr. curve described by the generators which are also related (2-1) is a cubic curve meeting $A\alpha$ in 2 pts & $A\beta$ in 1 pt.

Since a ple unicursal cubic is determined by 6 pts & its dp it follows that a unicursal cubic having a dp at α & passing thro' β is determined by 5 pts. The corr. cubic curve on the conicoid is therefore determined completely by 5 pts — a fact we have made use of already.

Again consider the case of two pencils α & β which are related (2-2). The locus of their pt of intersection is a quartic having dps. at α & β, a quartic with deficiency 1. ~~The g~~ The corr. curve on the conicoid is a quartic meeting $A\alpha$ & $A\beta$ each in 2 pts.

Since the ple quartic is completely determined by 7 pts (excluding α & β) we find that the quartic on the conicoid is completely determined also by 7 pts.

These theorems can be easily extended.

The number of pts of intersection of two curves on the conicoid is easily found by considering the number of pts of intersection of the corresp. ple. curves. For example let us find the number of pts of intersection of two sextic curves of deficiency 4 each meeting a gen^{r} in 3 pts. These project into ple sextics having 2 triple nodes. The ple sextics intersect in $6 \times 6 - 2(3 \times 3)$ ie 18 pts. Hence the space sextics intersect in 18 pts.

Lastly we give as a simple example the anharmonic property of the generators.
Four fixed gens. of one system cut any gen.ʳ of the opposite system in a range of constant cross-ratio. The four fixed gens. into 4 fixed lines thro' α say, & the gen.ʳ of other system into line thro' β. Since any line thro' β cuts the 4 lines thro' α in a range of constant cross ratio the property follows at once.

General theory of the transf. of a conicoid into itself by means of a quadratic transf.

Let A be a pt on a conicoid & let C be its centre. AC meets surface in B. Thro' B draw tangent ple to surface. The gens. thro' A meet this ple in 2 pts α & β. Choose any third pt γ on tangent ple & let αβγ be the F-pts of a ple quadratic transf. in the tangent ple. Corr. to any pt P on the conicoid we get a line AP meeting tangt ple in p. Corr. to p by the quad. transf. we get a pt p′ & the line Ap′ meets the conicoid in a pt P′ which corresp. to P. If a line thro' α in the ple ~~tran~~ corr. to a line thro' α & a line thro' β to a line thro' β it is clear that gens. of one system transform into generators of same system. Corr. to a conic we get a conic. The quad. transf. has 4 dps & hence there will be 4 dps on the conicoid. The linear transf which transforms the conicoid ~~into~~ itself has therefore 4 dps. Since the dps on the ple are collinear with the F-pts αβγ in pairs it follows that the lines joining the dps on the surface are generators & there are consequently 4 double gens. two belonging to each system.

It now follows that if the generators of one system transf. into the gens. of the other system there must be a conic of dps. For in this case the pencil (α) must corr. to the pencil β & there will be a conic of dps in the ple thro' α & β & consequently a conic of dps on the conicoid. Corr. gens. will intersect on this conic.

The linear transf. of the conicoid into itself is now involutive.

So far we have made no use of the fact that the ple of the quad. trans. is a tangent ple. parallel to the tangent ple. at A. For the demonstration of the above theorems this is not necessary.

The gens. thro' A will meet the tangent ple thro' B in two pts at infinity. Hence conics on the conicoid project into homothetic conics on the ple.

This is the stereographic projection.

If further A be an umbilic on the surface the generators thro' A are the lines to the circular pts at infinity in the tangent ple at A. The pts α & β in the ple thro' B are also in consequence the circular pts. Hence circles on the conicoid transform into circles on the ple.

Inversion.

Art 35. An important particular case of the general quad. transf. is the following.

Let the conic Σ in the involutive transf. be the imaginary circle at infinity. A ple now transforms into a sphere & a ~~cu~~ line into a circle. A circle since it meets Σ in 2 pts transforms into a circle & a sphere into a sphere. The generators of the cone $A\Sigma$ are now minimal lines & the conicoid of dps is a sphere. Corr. pts. are conjugate pts with respect to this sphere.

This is the well known inversion of elementary geometry. The sphere of c/ps is the sphere of inversion & the vertex A is the centre of inversion & is the centre of this sphere. A conicoid transforms into a surface of 4th order having a conical pt of second order at A & containing the imaginary circle at infinity as a double curve. These surfaces are known as ~~as~~ anallagmatic surfaces; if a centre of inversion be taken not lying on one it transforms into a surface of the same sort.

This transformation has the important property of being conformal. This can be proved as follows. Let P & P′ be corr. pts, & let p & p′ be two variable pts infinitely close to P & P′ respectively. Consider any plane thro' P P′. Pp describes a pencil thro' P & P′p′ a pencil thro' P′. The plane meets the imaginary circle in 2 pts K & L say. The line PK transforms into the line P′K & PL into P′L. Now the locus of the inters. of corr. lines of the pencils is a conic thro' K & L. This conic must be a circle. Hence the pencils are equiangular. The transf. is therefore conformal.

We saw in the general transf. that there was a triple infinity of conicoids thro' the F-conic which transform into themselves. Thus we see now that there will be a triple infinity of spheres which transform into themselves. Since the transf. is conformal it follows that such spheres must cut the sphere of c/ps (the sp. of inversion) orthogonally. From this it follows that we can find a double infinity of spheres passing thro' 2 corresp. pts of the transf. all orthog. to the sphere of inversion.

Also a single infinity ie a sheaf of spheres can be found to pass ~~to~~ thro' 2 pairs of corr-pts & cut the sp. of inv. orthog. Finally it follows that we can only get one sphere to pass thro' 3 pairs of corresp. pts & cut the sp. of inv. orthog.

Stereographic projection of sphere.

Consider a sphere & a diametral ple of pole P. Let the sphere be projected on the ple with P as centre of projection. This is a quad. transf. of the above kind. Great circles not thro' P project into circles on the ple cutting the circle of intersection of ple & sphere in 2 pts at the extremities of a diameter. Small circles project into ordinary circles on the ple. A great circle thro' P projects into a line on the ple. Transformation (linear) of sphere into itself by means of a plane quadratic inversion. The ple transf. possesses two dps which are real & two imaginary hence there will be two real dps on the sphere.

The cyclids.

If a sphere of variable radius move so that its centre lies on a curve ~~it~~ its envelope is a canal surface. Any generating sphere touches the envelope along a circle which is a circle of curvature. Transforming we get for the correspondent of the canal surface another canal surface which has also a series of circular lines of curvature viz the circles corr. to the circular lines of curv. on the original surface

As an example of such a surface consider a sphere which generates a right-circular cone or cylinder.

The cone or cylinder transforms into a quartic surface having the imaginary circle ($K\alpha$) as double line.

Corresponding to the circular lines of curv. on the cone or cyl. we get a series of circular lines of curv. on the quartic.

Corr. to the gens. of the cone or cylinder which are also lines of curvature we get again circles on the quartic.

Any ple thro' the axis of the cone meets the cone in two gens. These gens. are orthog. to the circles on the cone. Transforming, the ples thro' the axis become spheres which cut the quartic surface orthogonally along circles. Each sphere cuts the surface in two circles. These circles are also clearly lines of curvature & they will be orthogonal to the previous set. The surface will have a conical pt. at the centre of inversion & another conical pt corresp. to the & vertex of the cone. It will therefore consist of two distinct pieces which touch each other at the conical pts. This is the socalled horn cyclid.

If we transform this surface we get another cyclid for the surface contains K∞ as double curve.

If we choose the centre of inversion so that the surface transforms into itself it must clearly be the envelope of a sphere which cuts the circle of inversion orthogonally, for such spheres invert into themselves. The locus of its centre must be a conic for a ple which cuts the surface in two circles can only meet the centre locus in two pts.

Let O be a fixed pt & let a sphere with centre O be taken as sphere of inversion. Let P be any ple & let OP be the perp. from O on this ple. Also let S be a circle on the ple P centre P. Let a sphere be taken with on ~~this~~ circle & cutting the sphere of inversion orthogonally. The envelope of this sphere as its centre moves on the circle is an anchor-ring. This surface is a special form of cyclid where we define a cyclid as a surface of 4th order having ~~as~~ K∞ as double curve & which is the envelope of a sphere cutting the sp. of inversion orthogonally

& whose centre moves on a conic. That the anchor ring contains K & as a double curve is easily seen, for a ple section thro' the axis consists of two ~~curves~~ circles i.e. a curve of order 4 having 2 dps on the circle at infinity. If we ~~trans~~form the anchor ring with respect to O & given ~~sphere~~ radius of inversion it inverts into itself. With respect to an arbitrary centre ~~& radius~~ of inversion it inverts into another cyclid of more general type. Such surfaces must have two conical pts real, imaginary or coincident according as the generating sphere of the anchor ring cuts the axis in two real, imaginary or coincident pts. Since the lines of curvature on the anchor ring are determined by a sheaf of ples thro' the axis & by a system of spheres whose centres lie on the axis it follows that as in previous case that the systems of circles determined by these ples & spheres invert into circular lines of curvature on the cyclid belonging to orthogonal systems.

We shall now show that these cyclids are like the previous one the envelopes of a sphere whose centre moves on a conic & which cuts a fixed sphere orthogonally.

Let O, be the centre of inv. (arbitrary) & let P be the ple of centre-locus of the anchor ring, a ple which may be called the equatorial ple. Let r be the radius of the generating sphere & let R be the radius of an arbitrary sphere thro' O,. The position of the centre of this sphere is in the meantime arbitrary. Let the ple thro' O, ⊥r equat. ple meet the centre locus in Q & T Q being the pt of minimum distance from O,. With Q as centre let a circle be described in the ple ⊥r equat-ple & passing thro' O, of radius δ where $\delta^2 = R^2 + r^2$. Let a circle of radius R & centre O, also be described in this ple. These circles will intersect if $\delta + R > p$ where p is the length of O_1Q. These relations give $\sqrt{R^2+r^2} + R > \mu$ & since R is at our disposal we can choose it so that this relation is satisfied. When it is satisfied the two above circles intersect. Let S be a pt on the circle of radius R centre O_1 which lies within the circle of radius d. We now choose S as the centre of the sphere of radius R thro' O_1. Let X be a variable pt on the locus of centres. As X moves (~~starting~~ from Q) round the circle the length of XS increases until it reaches a max. at T. Since $(SQ)^2 < R^2+r^2$ by hyp. since S lies with circle of radius d where $d^2 = R^2+r^2$ it follows that there will be one position of X for which $(XS)^2 = R^2+r^2$ provided $(ST)^2 > R^2+r^2$. Let v be the distance O_1T this condition will be satisfied if $d + R < v$ ie if the circle centre O & radius R does not cut a circle of radius d centre T in the same ple. Hence the conditions determining R are $\sqrt{R^2+r^2}+R > \mu$ & $\sqrt{R^2+r^2}+R < v$. If any value of R be chosen satisfying these conditions if S the centre of the sphere of rad. R be then chosen as above we infer the existence of pt X such that $(XS)^2 = R^2+r^2$. Similarly we infer the existence of X_1 symmetrically situated with respect to X & diameter TQ such that $(X_1S)^2 = R^2+r^2$. The sp generating spheres whose centres are X & X_1 cut the sphere centre S orthogonally & there are only two such spheres. ~~(Transforming we get ~~that~~ the cyclid which is the envelope of a sphere cutting the sphere corr. to sphere centre O orthogonally, cuts also the ple corr. to sphere centre S orthog)~~

Transforming the anchor ring into a cyclid we see that there will be two generating spheres of the cyclid orthogonal to the ple into which the sphere centre S transforms. Hence the locus of the centre of the generating sphere of the cyclid is a conic.

If O_1 be inside the anchor ring two generating spheres will pass thro' it & hence two gen. spheres of the cyclid will have their centre at infinity. The conic is in this case a hyperbola. If O_1 lies on the ring, one sphere will have its centre at infinity & the conic will be a parabola. If O_1 lies outside the conic will be an ellipse. When O_1 lies on the ring surface the latter transforms into a cubic surface containing K_2 once. This surface is the parabolic cyclid.

The central surfaces of a cyclid.

Since both systems of lines of curvature on a cyclid are circles & in fact since the surface is a double canal surface it follows that the two surfaces of centres degenerate into curves. The above conic which is the locus of the centres of the generating spheres is one of these curves. Corr. to each of these curves we have a system of lines of curvature. If P be any pt on the surface there pass thro' P 2 circles of curvature one belonging to each system. The normal thro' P consequently must meet both central curves.

~~If~~ The other central curve is also a conic for let C be any pt on the locus of ~~the generating~~ centres of gen. spheres all the normals to the surface thro' C form a right circular cone which must contain the other central curve which being plane must be a conic.

That the second curve of centres must be a plane conic is easily seen. For if it were a space cubic &c it would ~~pass thro' the vertex~~ not lie on a right circular cone without passing thro' the vertex which is impossible.

Let the central conics be called C_1 & C_2. Let C_1 meet plane of C_2 in A & B. Then all lines thro' A meeting C_2 generate a flat circular cone which meets the surface in a circle. Similarly if C_2 meets ple of C_1 in E & F. cone thro' E & C_1 is a ple meeting the surface in a circle. The line EA which meets both central curves is a normal to the surface at some pt & the lines of curvature thro' this pt are perp^r. Hence the ples of C_1 & C_2 must be perp^r. These curves are therefore of precisely the same kind as the focal conics of a conicoid. One must be an ellipse & the other a hyperbola. Further the vertices of the ellipse are foci of the hyp. & vice versa.

The anallagmatic surfaces corr. to conicoids.
From the previous work it follows that a conicoid of revolution (excluding cone & cyl.) transforms into a quartic surface having K_∞ as double curve & one system of lines of curvature circles. One of the central ~~surfaces~~ will be degenerate & be a conic; the other will be a proper surface. ~~The~~ Other set of lines of curvature will be unicursal quartics having a dp at the centre of inversion which will be a conical pt on surface.

The general conicoid transforms into a quartic surface having a conical pt at centre of inversion & containing K_∞ as double curve. The central surfaces are proper surfaces. Since thro' any pt on the conicoid there pass two gens & two circular sections it follows that thro' any pt on the quartic there pass 4 circles.

Art 36. Analytical formulae for the quadratic transf.
Let BAC, BAD, CAD & BDC be the ples $x=0$ $y=0$ $z=0$ $w=0$ with similar suppositions for space (S').
By the collinearity of the pencils vertices A & A' we get

$$xy'=yx' \quad xz'=zx'$$

If ple BMC be $k_4x-k_1w=0$ the conicoid corr. to it will be $k_1(\Sigma')-k_4x'w'=0$ where $\Sigma'=0$ $w'=0$ is the curve Σ_2'. Thus we have the formulae

$$xy' = yx' \quad xz' = zx'$$

$$x(\Sigma') - \omega x'\omega' = 0$$

giving $\frac{x}{\omega} = \frac{x'\omega'}{\Sigma'}$ $\frac{y}{\omega} = \frac{y'\omega'}{\Sigma'}$ $\frac{z}{\omega} = \frac{z'\omega'}{\Sigma'}$

or $x : y : z : \omega = x'\omega' : y'\omega' : z'\omega' : \Sigma'$

Conversely from these we get easily

$$x' : y' : z' : \omega' = x\omega : y\omega : z\omega : \Sigma.$$

For the conformal transf. in which Σ' becomes $K\alpha$ we get $\frac{x}{\omega} = \frac{\frac{x'}{\omega'}}{\frac{\Sigma'}{\omega'^2}}$ or $x = \rho \frac{x'}{x^2+y^2+z^2}$ etc.

where $x^2+y^2+z^2=0$ is the eqn. of $K\alpha$.

Surface of double elements.

From the eqn $x\Sigma' - \omega x'\omega' = 0$ we get putting $x = x'$ etc

$\Sigma - \omega^2 = 0$ to which cone $A\Sigma$ is a tangent cone.

— / —

Analytical formulae for the particular cases p 199 et seq. are easily obtained in a brief & elegant form.

For example the formulae of the (2-4) transf. p 205 are $x : y : z : \omega = y'z'\Sigma' : z'x'\Sigma' : x'y'\Sigma' : x'y'z'\omega'$

& $x' : y' : z' : \omega' = yz : zx : xy : \omega L$

where L is a linear fn. of xyz.

— / —

Art 37. A particular case of the (4-4) transf. p 194 is of interest.

If the series of curves in (S') is composed of cones we can establish the corr. between the spaces as follows.

Let two pencils vertices A & A' be related as ex p 9 by a cubic transf. ie to a ple of (1) let there corr. a cubic cone of (2) & vice versa.

Let AB, AC, AD be the F-lines of the pencil (A) & let A'B', A'C', A'D', A'B'', A'C'' be the F-lines of (A'). Suppose that to a ple of (A) corr. a cubic cone of (A') thro' A'B', A'C' etc having A'D' as double side.

Let AL be any line of (A) & L a pt on it. Let us now establish a correl. between (S) & (S') so that to pt L corr. a pt L' where the ple corr. to L meets the corr. ray A'L'. Then L & L' are corr. pts in the quartic transf. between the spaces.

When L describes a line L' describes a quartic curve & when L describes a ple L' describes a quartic surface

The F-line systems.

The F-line system in (S) consists of the 5 F-lines of the pencil (A) & a ple quartic lying in the ple π which corresp. to A' in the correlation. There is a similar F-system in (S').

That the F-lines of the pencil (A) are F-lines for the transf. is obvious & we can deduce the existence of the ple F-quartic as follows. Let π' be the ple of (S') corr. to A. If P be any pt on π' its corr. in the correlation is a ple thro' A; its corr. in the pencil transf. will be the pt in which the ray AP corr. to A'P' meets π. If P describes a line on π' its corr. ple in the corr. describes a pencil of lines thro' a fixed pt on π & its corr. pt P' in the pencil transf. describes a unicursal cubic on π'. The pencil of lines generates a cubic involution on the cubic curve & the pts of this involution together with P' make up a quartic invol. on the cubic. This system has 4 dps. Hence there will be 4 positions on the line described by P' for which the pts P' lies on its corr. line in π. If P_1 be one such position it is clear that the correspondent of P_1 is a line viz the line of intersection of ple thro' A & a cubic cone belonging to the pencil (A). Hence the F-pts on π' lie on a quartic curve. Similarly for the F-pts on π. These F-curves pass thro' the intersections of the F-lines with the ples π & π'.

The F-systems of surfaces.

Cor. to the F-lines of pencil (A) we get 4 planes & a conicoid which are F-surfaces for the transf; we have also the cone vertex A & base the F-quartic & ple π, A similar F-system exists in (S').

Nature of the quartic curves & surfaces.

From the nature of the F-system each quartic curve corr. to a line will intersect in (S) the 4 F-lines once & the double line AD twice & pass thro' A. Similarly for S'. They will also meet the F-quartic in 4 pts.

The quartic surfaces in (S) will have A for triple conical pt (the tangent cone corr. to the line of int. of corr. ple in (S') with π') & will pass thro' the 4 single F lines & contain the double F-line AD (twice). ~~It~~ They will also contain the F-quartic. It may be observed that the F-quartics have each a dp at the pt where the double F-line meets their ple. They are therefore of deficiency 2.

Superposition of the spaces.

The locus of the pt of intersection of corr. rays of the pencil A & A' is a quartic curve thro' A & A'. The pts which lie on their corr. ples in the corr. lie on a conicoid. The quartic curve meets the conicoid in 10 pts. There are therefore 10 cpps. as before.

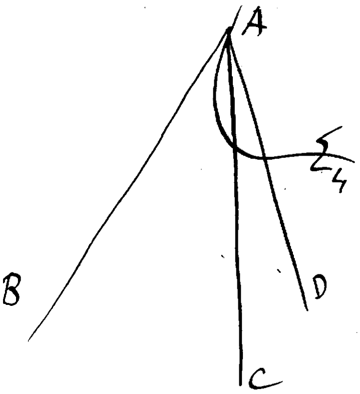
If the F-systems are superposed there will be 5 double lines of the common pencils & on each of these there will be an involution of corresp. pts.

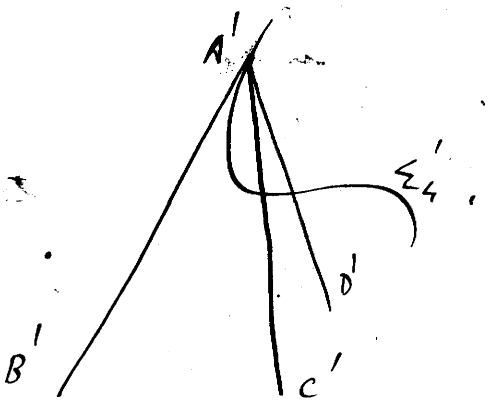
This particular (4-4) transf. belongs to a series of transf. beginning with the quadratic transf.

The second member of the series is a cubic transf. obtained by superposing a 2-dimensional quadratic transf with a correlation. The F-line system in each space consists of 3 lines (the F-lines of the pencil) & a ple cubic.

The series may clearly be continued indefinitely.

Art. 38. The most general transf. obtained by using 2 sheaves of pls & a sheaf of conicoids.
We shall now suppose the base of the sheaf of conicoids to be a quartic curve. Since two pls & a conicoid intersect in ~~x~~ 2 pts we must make the bases of the 3 sheaves intersect in a common pt.
Hence the bases of the sheaves of pls must be coplanar & their pt of intersection must lie on the quartic curve.





Let AB, AC & Σ_4 intersecting in (A) be the bases in (S) & A'B', A'C' & Σ_4' be the bases in (S') intersecting in A'.
The pl sheaves in each space determine two quadratic pencils.
Let AD & A'D' be the other F-lines of these pencils.
Let M be a pt of space (S). Corr. to A M we get a ray A'M' & corr. to the conicoid M Σ_4 we get a conicoid M'Σ_4' meeting A'M' in M'. M & M' are corr. pts.
The corr. is (1-1) between the spaces.
When M describes a line M' describes a sextic curve which is the intersection of a cone & a quartic surface having a double line which is also a gen. of the cone.
When M describes a pl M' describes a sextic surface having A'B' & A'C' as double lines & containing Σ_4'. & conversely.
Thus the transf. is a sextic transf.

The F-systems.

The F-lines of the pencils are F-lines of the transf.
Corr. to a pt on AB AC or AD we get a conic on their corr. ple in (S'). Each of these lines is a double line on the corresp. ~~quartic~~ sextic surface.
Corr. to a pt on Σ_4 we get a line in (S') & corr. the curve Σ_4 as a whole we get a sextic cone in (S') for which A'B', A'C', A'D' are triple sides.
Corr. to a line thro' A' intersecting Σ_4' we get a pt in (S). The locus of this pt is a curve of order 14 as we shall show presently. We shall call this curve Σ_{14}.
The pts A & A' are F-pts. Corresponding to A we get in (S') a quartic surface. For consider the pts on the ples BAC, CAD, DAB infinitely close to A.
To these pts corr. the lines A'D', A'C', & A'B' & a conic on each of the ples B'A'C', C'A'D', D'A'B'. Hence the ple B'A'C' meets the surface corresp. to A in two lines & a conic & hence the latter is of the 4th order. This is evident on other grounds for a ple thro' A transforms into a quadric cone thro' A'B', A'C', A'D' & hence corr. to A we must have a quartic surface to make up the order of the surface corr. to the ple to 6. The quartic surface corr. to A passes thro' A'B', A'C', A'D' & similarly the quartic corr. to A' passes thro' AB, AC, AD.
This quartic surface must also contain the F-curve Σ_{14}. For the pts on this curve corr. to lines thro' A' intersecting Σ_4'. To any pt on a line thro' A' intersecting Σ_4' we get the pt on Σ_{14} corr. to this line. But A' lies on the line & hence the pt on Σ_{14} must lie also on the F-quartic corr. to A'. The pts of Σ_{14} also lie on a sextic cone vertex A having AB, AC, AD for triple sides & hence Σ_{14} must be the intersection of the F quartic corr. to A' with this cone. The complete intersection

of these surfaces is of order 24 of which AB AC AD count of order 3 each. Hence the remaining part is of order 15. Consider now the tangent to Σ_4' at A'. An infinite number of curves of the sheaf have this line for tangent & hence corr. to the pt on it infinitely close to A' i.e. the pt consecutive to A' we get no definite pt on the F-surface corr. to A' but simply a line viz the line thro' A corr. to this tangent. This line also lies on the sextic cone thro' A corr. to pts on Σ_4' & hence it forms part of the above system of order 15. Hence the remaining curve which is the proper F-curve in (S) is of order 14. Similarly there is an F-curve of order 14 Σ_{14}' in (S').

Hence the complete F-system of curves in (S) is AB, AC, AD Σ_{14} & Σ_4 & similarly for (S').

The complete system of F-surfaces in (S) consists of pls BAC, CAD, DAB, F-quartic corr. to A', the sextic cone thro' A & the cubic cone composed of lines which pass thro' A & intersect Σ_4. Similarly for (S').

Nature of the quartic curves & surfaces in (S'). From the above investigation of the F-systems we see that corr. to a line in (S) we get a sextic curve in (S') having a quadruple node at A' & intersecting A'B', A'C', A'D' each once, intersecting Σ_{14} 3 times & intersecting Σ_4' 6 times. The 4 pts at A' corr. to the pts in which the line meets the F-quartic in (S).

Again corr. to a pl. of (S) we get a sextic surface in (S') having A' as a quintuple conical pt & having A'B', A'C' A'D' as double lines & also containing Σ_{14}' & Σ_4'.

The pl. of (S) meets the F-quartic of (S) in a quartic curve thro' inters. of AB AC AD. This transforms into a quartic cone thro' A' having A'B' A'C' A'D' as double sides & this is the tangent cone to the surface at A'. It is easily seen that the tangt. to Σ_4' at A' is a side of tangent cone at A'.

Effect of transf. on curves or surfaces

A curve of order m transforms into a curve of order 6m in general. But if the curve pass thro' A d times & intersect AB AC AD, Σ_{14}, Σ_4 $\mu\ \nu\ \rho\ \sigma\ \tau$ times resp. ~~so~~ the order of its corr. will be $6m - 5d - 2\mu - 2\nu - 2\rho - \sigma - \tau$. For example a ~~quartic~~ sextic curve having a quadruple node at A & intersecting AB, AC, AD each once &, Σ_{14} 3 times & Σ_4 6 times transforms into a curve of order $6\times6 - 5\times4 - 2 - 2 - 2 - 3 - 6$ ie $36 - 35$ ie 1. Such a curve transforms into a line as it ought to.

Again a surface of order n transforms in general into a surface of order 6n. But if it contain A^d as a conical pt of order d & the lines AB AC AD $\mu\ \nu\ \rho$ times; Σ_{14} σ times; Σ_4 τ times its corr. will be of order $6n - 4d - \mu - \nu - \rho - 3\sigma - 6\tau$.

For example a sextic surface having a quintuple conical pt at A containing AB, AC, AD 2 times each & containing Σ_{14} & Σ_4 transforms into a surface of order $6\times6 - 4\times5 - 2 - 2 - 2 - 3 - 6$ ie 1. Such a surface transforms into a plane as it ought to.

Number of lines on a sextic surface.

Let a plane π of (S) meet the F-system, It will meet it in $3 + 14 + 4$ ie 21 pts. Corr. to the pts on AB, AC, AD we get conics. Corr to the others we get lines thro' A'. Hence besides the double lines thro' A' the surface possesses 18 single-lines.

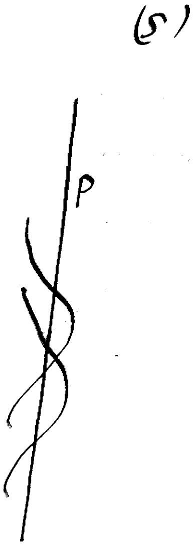
Superposition of the Spaces.

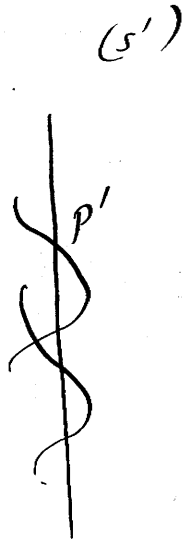
When the spaces are superposed there are 14 clps. For the locus of the intersection of corresp. lines of the quadratic pencils is a quartic curve thro' A & A'.

The locus of the intersection of corresp. conicoids is a quartic surface thro' A & A'. ~~These~~ The curve & surface intersect in 16 pts two of which are A & A'; hence there are 14 proper c.p.s.
If the F-lines are superposed we may have 4 lines on which an involutive system of corr-pts exists. These are the double lines of the pencil.

Art. 39. The transformation built up by two sheaves of conicoids & a sheaf of planes. (7-7) transf.

In order to obtain a (1-1) corr. between the pts of the two spaces - the base of the sheaf of planes must in each space form a part of the base of each system of conicoids.





Let P & P' be the bases of the sheaves of planes. The bases of the sheaves of conicoids in (S) must consist of P & two cubic curves each intersecting P in 2 pts. Similarly for (S'). Any two conicoids of (S) intersect in a cubic curve which meets P twice & hence a plane thro' P meets this curve in 1 pt only. Thus we establish a (1-1) corr - between the pts of the two spaces.
We shall denote the cubic bases by $_1\Sigma_3$ $_2\Sigma_3$ & $_1\Sigma_3'$ $_2\Sigma_3'$ resp.

Let a pt M of (S) describe a line l. The generating sheaves — a ple thro' P & a conicoid thro' $P_1 \Sigma_3$ are related by a (2-1) relation. Similarly the sheaves — ple P & conicoid $P_2 \Sigma_3$ are also related (2-1). Hence the corr. sheaves in (S') are similarly related.

The sheaf of ples (P') & sheaf of conicoids $P'_1 \Sigma'_3$ generate a quartic surface on which P' is a triple line & so do the sheaves (P') $(P'_2 \Sigma'_3)$. The intersection of these surfaces is of order 16 & P' forms a part of order 9. Hence their proper curve of intersection is a septic curve. Hence corr. to l we get a septic curve & conversely.

When M describes a ple M' its corr. pt in (S') describes a septic surface. The relation between the generating sheaves (P) $(P_1 \Sigma_3)$ $(P_2 \Sigma_3)$ of (S) is (3-1-1) & so is the relation between the corr. sheaves of (S'). Hence the intrinsic eqn of the septic surface is of form $K_1 S_1^3 S_2^2 S_3^2 + K_2 S_2^2 S_2^2 S_3^2 + \dots = 0$ where S_1 indicates the abscissa of the pt of intersection of ple P' with arbitrary line S_2 & S_3 the pts of intersection of ${}_1\Sigma'_3$ ${}_2\Sigma'_3$ with same line.

~~(The line P' is therefore a triple line on the septic surface)~~

Similar results hold for the reverse transf.

The transf. is a septic transf.

The line P' is a quintuple line on the septic surface as can be seen by considering the modification of the above intrinsic eqn for a line intersecting P'. The eqn in this case becomes $K_1 S_2 S_3 + K_2 S_1 + \dots = 0$ showing that the line can only meet the surface in 2 pts not on P'.

Consider a ple thro' P & a line l on it. Let M be a pt on l & let A & B be the pts of intersection of ${}_1\Sigma_3$ ${}_2\Sigma_3$ with the ple. The ple meets each conicoid ${}_1\Sigma_3$ ${}_2\Sigma_3$ in a line thro' A & B resp.

Let A' & B' be corr. pts on the corr. ple thro' P'.
Then corr. to l we get on the ple corr. a conic generated by the intersection of corr. lines thro' A' & B'.
If l meet P in p we see that corr. to p we must have a quintic curve.
Considered from the pt. of view of the ple transf. ~~Bela on~~ between the ples thro' P & P' ~~the~~ corr. of p is a pt on ple thro' P'. Hence the quintic curve corr. to it must intersect P' in 4 pts.
Again since a ple thro' P transf. into a ple thro' P' it follows that the corr. of line P is a sextic surface.
Hence as p moves on P the locus of the corr. quintic curves must be this sextic surface.
This sextic is an F-sextic in (S').
Again corr. to a pt on ${}_1\Sigma_3$ we get a line in (S').
Corr. to the curve ${}_1\Sigma_3$ we get a sextic surface in (S') having P' as ~~quadruple~~ quintuple line. For if R be a pt on ${}_1\Sigma_3$ the corr. between the sheaf of ples (P) & the sheaf of conicoids (P ${}_2\Sigma_3$) is (4–1) since a conicoid P ${}_2\Sigma_3$ meets ${}_1\Sigma_3$ in 4 pts. Hence the relation between the corresp. sheaves in (S') is (4–1) & therefore the locus of their intersection is a sextic having P' as quintuple line.
Similarly corr. to ${}_2\Sigma_3$ we get in (S') another sextic also having P' as quintuple line.
Again consider a line which intersects P & the cubics ${}_1\Sigma_3$ ${}_2\Sigma_3$. The locus of this line is a sextic surface having P as quintuple line.
This result is easily obtained by considering the number of pts of intersection of two unicursal cubics having coincident nodes. Or it may be obtained as follows. Transforming the line P & the curves ${}_1\Sigma_3$ ${}_2\Sigma_3$ by a cubic transfn we have to find the locus of a line which intersects two chords of

a cubic curve & the curve itself. This locus is a conicoid & transf. again we get for the above locus a sextic surface

Cor. to the generators of this sextic we get a locus of pts in (S'). Consider any gen. l. The ples (Pl) & the conicoid l, Σ_3 trace out a surface as l moves. The cor. ples (P') & conicoids $(P'{}_1\Sigma_3')$ trace out a sextic surface having P' as quintuple line.

Also the ples (P') & conicoids $(P'{}_2\Sigma_3')$ trace out another sextic having P' as quintuple line. The intersection of these surfaces gives a proper curve of order 36-25 i.e 11. This curve of order 11 is the curve cor. to the generators which meet $P, {}_1\Sigma_3\ {}_2\Sigma_3$. We shall call it Σ_{11}.

The complete system of F-curves in (S) is therefore $P, {}_1\Sigma_3, {}_2\Sigma_3$ & Σ_{11} & similarly for (S').

The complete system of F-surfaces in (S) consists of the F-sextic cor. to P' & the sextics cor. to ${}_1\Sigma_3'\ {}_2\Sigma_3'$ & Σ_{11}'.

Nature of the septic curves & surfaces.

To a line l of (S) corresponds a septic curve of (S') meeting P' in 6 pts & ${}_1\Sigma_3'\ {}_2\Sigma_3'\ \Sigma_{11}'$ in 6 pts each.

To a ple π of (S) corresponds a septic surface having P' as quintuple line & containing ${}_1\Sigma_3', {}_2\Sigma_3'$ & Σ_{11}'.

As in previous transf. we can show that the F-sextics cor. to P & P' contain Σ_{11}' & Σ_{11} respectively.

In fact Σ_{11} is the intersection of the F-sextic cor. to P' & the F-sextic cor. to the gens. which meet $P', {}_1\Sigma_3'\ {}_2\Sigma_3'$. ~~We infer consequently that the F-sextic cor. to P' has P for a quintuple line~~.

From the quadratic transf. between the pts of cor. ples thro' P & P' we see that the cor. of P considered from the pt of view of this ple transf. is a conic. Hence a ple ~~sect.~~ section thro' p' of the F-section corr. to P is a conic & hence P' must be a quadruple line on this section. Similarly for (S).

A curve of order m intersecting P, ${}_1\Sigma_3$ ${}_2\Sigma_3$ Σ_{11} d μ ν ρ times resp. transf. into a curve of order $m' = 7m - 5d - \mu - \nu - \rho$. A septic curve meeting P 6 times, ${}_1\Sigma_3$ ${}_2\Sigma_3$ & Σ_{11} 6 times each transf. into curve of order $= 7 \times 7 - 5 \times 6 - 3 \times 6$ i.e 1. Such a curve transf. into a line as it ought to.

A surface of order n containing P, ${}_1\Sigma_3$ ${}_2\Sigma_3$ Σ_{11} d μ ν ρ times resp. transf. into a surface of order $n' = 7n - 6d - 6\mu - 6\nu - 6\rho$

or $7n - 6(d + \mu + \nu + \rho)$.

— 1 —

Number of lines on a septic surface.

Let a ple π of (S) meet P in A & ${}_1\Sigma_3$, ${}_2\Sigma_3$ Σ_{11} in the pts $B_1 B_2 B_3$, $C_1 C_2 C_3$, $D_1 \ldots D_{11}$ resp. Corr. to the lines $AB_1 \ldots AC_1 \ldots AD_1 \ldots AD_{11}$ we get lines on the surface. Thus there are in all 17 lines on the surface over & above P'. These lines do not intersect one another.

Superposition of the spaces.

When the spaces are superposed there are 16 d.ps. The locus of the intersection of corr. ples of the sheaves P P' is a conicoid. The locus of the intersection of corr. conicoids of sheaves ${}_1\Sigma_3$, ${}_1\Sigma_3'$ is a quartic thro' P & P'. The locus of the intersection of corr. conicoids of sheaves ${}_2\Sigma_3$, ${}_2\Sigma_3'$ is a quartic also thro' P & P'. The number of pts of intersection of these surfaces is 32. Of these P counts as 8 pts of intersection & 80 does P'. Hence there are 16 proper elps left.

Various particular cases of this transf. may arise if we suppose the curves, Σ_3 etc to become degenerate. These can be investigated as before.

Art. 40. Transformations which can be built up by means of 3 sheaves of conicoids.

There are 3 main cases.

(1) The base curves may be quartics having 7 pts of intersection.

(2) The bases may have a common line & the remaining part is a cubic curve in each sheaf.

(3) The bases may have a common conic. The remaining part of each is a conic.

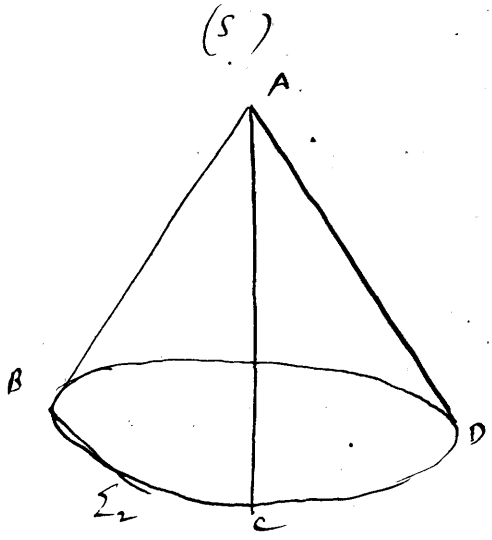
In (2) the 3 cubic curves must intersect in 3 points
" (3) " conics must intersect in 1 point.

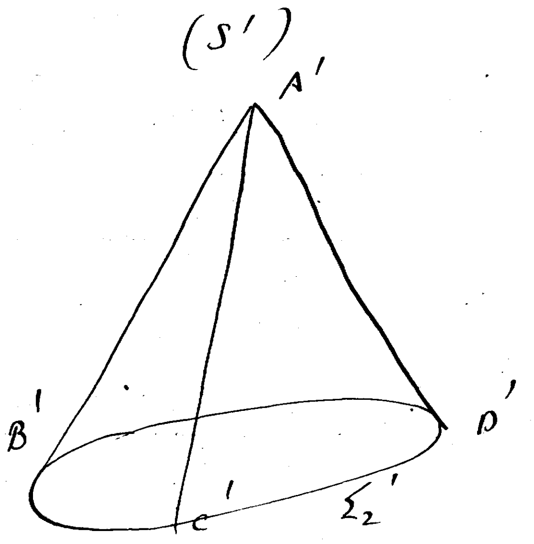
In all these cases 3 conicoids one belonging to each sheaf intersect in 8 pts of which 7 are on the base system; hence they have one variable pt of intersection. We can therefore by their means institute a (1-1) pt correspondence between the points of 2 spaces (S) & (S').

Case (1) is the most general case.
The transformation is of order 24 & is somewhat complicated. The seven pts of intersection of the base quartics are crunodal pts of order 12 on the surfaces of order 24. Each base curve is a curve of multiplicity 4 on the same surfaces. The F-system of curves outside of the base system is of order 360.

In case (2) the transf. is of order 18. The common line of the base system is a line of 9th degree of mult. on the surfaces of order 18 & the cubic base curves are curves of mult. 3 on same surfaces. The three pts of intersection of the base curves are conical pts of order 9.

In case (3) which is the simplest we may suppose that the remainder conics are degenerate & composed of two str. lines.





Let Σ_2 & Σ_2' be the common conics in (S) & (S') resp. The other conics are (AB, AC), (AC, AD), (AD, AB) intersecting in A* with similar system in (S').

In this case the transf. is of order 6 in general. The conics Σ_2 & Σ_2' are curves of multiplicity 3 on the sextic surfaces. Further the lines AB, AC, AD etc are double lines on these surfaces.

The complete F-system of curves in (S) is made up of Σ_2 & AB, AC, AD & similarly for (S').

A & A' are F-pts order 3. They are also conical pts of order 3 on their respective sextic surfaces. Corr. to each of these pts we get a cubic F-surface in the other space.

Consider the cone $A\Sigma_2$. Corr. to this conicoid which belongs to each of the sheaves in (S) we get the pt of

intersection of the three ~~cones~~ corr. conicoids in (S').
This pt of intersection is therefore an F-pt in (S') & it will be a conical pt of the second order on each sextic surface in (S'). Since a line of (S) meets the cone $A\Sigma_2$ in 2 pts each sextic curve in (S') will have a node at this F-pt. There is a similar F-pt in (S) corr. to the cone $A'\Sigma_2'$.

Thus we see that corr. to a line in (S) in general there corr. in (S') a sextic curve having a triple node at A' & a double node at K' the above F-pt in (S'). These sextic curves must clearly meet Σ_2' in 6 pts.
A line meeting Σ_2 transforms into a cubic curve & hence corr. to a pt on Σ_2 we get a cubic curve. This curve must pass thro' K'. Similarly corr. to a pt on AB AC or AD we get a conic thro' K'. Corr. to Σ_2 as a whole we get a sextic surface & corr. to each of AB AC AD we get conicoids. The complete F-system of surfaces in (S') consists of the Sextic corr. to Σ_2, the 3 conicoids corr. to AB, AC, AD the cubic corr. to A & the cone $A'\Sigma_2'$; there is a similar System in (S).

Superposition of the spaces.

When the spaces are superposed there are 14 d.ps. The sheaves $(ABC\Sigma_2)$ $(A'B'C'\Sigma_2')$ trace out by their intersection a quartic surface thro' the respective base systems; similarly $(ACD\Sigma_2)$ & $(A'C'D'\Sigma_2')$ trace out a quartic surface & so do $(ABD\Sigma_2)$ $(A'B'D'\Sigma_2')$. These 3 quartics meet in 64 pts. A certain number of these however lie on the F-surface systems.
The conics Σ_2 & Σ_2' which are common to all 3 surfaces count as 18 pts of ~~e~~ intersection each. A & A' are also pts of intersection. Again the cone $A\Sigma_2$ is met by the 3 surfaces corr. to it in 6 pts 2 for each surface (since 2 conicoids of (S') intersect in a line)

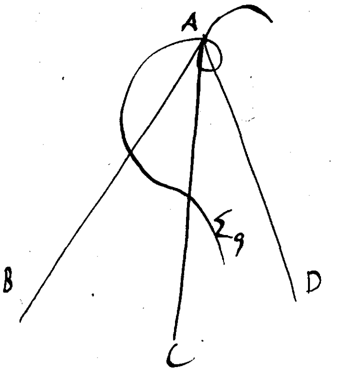
Similarly the arc $A'\Sigma_2'$ contains 6. pts of intersection. Thus the total number of pts of intersection lying on the F-surface systems is 18+18+2+6+6 ie 50 leaving 14 proper dps.
We may observe that as in previous transf. if the order of the transf. be n the number of dps is 2n+2. It seems to be a general theorem that for a reversible space transf. of order n the number of dps is 2n+2. We shall show later that this is true for all transf. built up by means of 3 sheaves of surfaces.

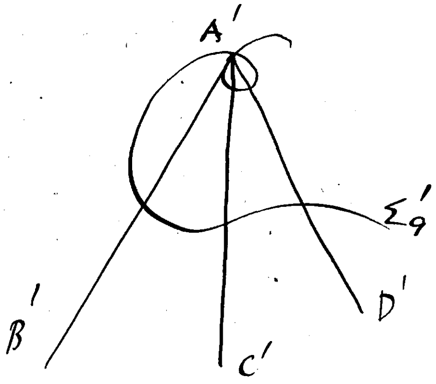
If the conics Σ_2 & Σ_2' be the imaginary circle at infinity & if we suppose the F-systems superposed we get for the corr. of a plane a sextic surface having the imag. circle as triple curve & having a conical pt of order 3 at A' & one of order 2 at K'.
These surfaces resemble the anallagmatic surfaces of the 6th order obtained by the quadratic inversion. A cubic surface with a conical pt inverts by that transf. into a sextic having K_∞ as triple curve & having another conical pt of second order. Thro' the triple conical pt at the centre of inversion there pass 6 lines on the surface. In the above transf. the 6 lines form 3 coincident pairs namely AB, AC, AD.

Art. 41.

The transf. which can be built up by means of two sheaves of planes & a sheaf of cubic surfaces.

The most general of these transf. is analogous to that given on p. 234.





Let $\left.\begin{matrix}AB \,\&\, AC\\ A'B' \;\; A'C'\end{matrix}\right\}$ be the bases of the sheaves of planes.
The cubic surfaces must have conical pts at A & A' respectively & the bases of the sheaves of cubic surfaces will be curves of order 9 having quadruple nodes at A & A'.
The corr. obtained between the pts. of the spaces (S) & (S') is clearly (1-1),
Let a pt M of (S) describe a line l. The line $A'M'$ will describe a quadric cone & the intersection of the sheaf of planes $(A'C')$ & the sheaf of cubics $(A'\Sigma_9)$ will describe a sextic surface on which $A'C'$ is triple line. The locus of M' is therefore a curve of order 9.
Consequently the transf. is of order 9.
When M describes a plane M' describes a nonic surface.

The F-line systems.

The F-lines of the pencils AB, AC, AD, $A'B'$, $A'C'$, $A'D'$ are F-lines for the transf.
Corr. to a pt. on any one of them we clearly get a plane cubic curve in the other space. Corr. to a pt on Σ_9 we get a line thro' A' in (S'). Corr. to Σ_9 as a whole we get a cone of order 10 (2×5) vertex A' in (S').
Corr. to the F-cone $(A\Sigma_9)$ which is of order 5 we get a

(S') a curve of order 36.

The complete F-curve system in (S') therefore consists of A'B', A'C', A'D', Σ_9' & the above curve Σ_{36}'.

Again since a ple thro' A transforms into a quadric ~~cone in~~ (S') it follows that corr. to A we must have a septic F-surface in (S'). The ples B'A'C', C'A'D' & D'A'B' are also clearly F-surfaces in (S'), as is also the cone of 5th order A'Σ_9'. The cone of 10th order corr. to ~~pts o~~ Σ_9 is the remaining F-surface in (S'). We shall now show that the F-septic in (S') corr. to A contains each of the F-lines A'B', A'C', A'D' as a double line.

Consider ~~any~~ the ple BAC thro' A. A cubic surface meets this ple in a unicursal ple cubic having a node at A. Consider the pts on the curve very close to A. These pts lie on one or other of the two tangents to the curve at A. Hence corr. to the two ~~consec~~ pts consec. to A one on either tangent we get a pt on A'D'. Hence if we consider all the ples thro' A we see that the surface corr. to A i.e. the locus of the correspondents of pts consecutive to A must contain A'B', A'C', A'D' as double lines. This surface must also contain the 4 lines thro' A' corr. to the 4 tangents to Σ_9 at A. It must also contain the curve Σ_{36} for A lies on all the generators of the cone AΣ_9.

We can now show that the F-curve of (S') Σ_{36}' is of order 36 exactly as in the transf. of p 234. For this curve is the intersection of the F-septic corr. to A & the cone of order 10 corr. to Σ_9. The lines A'B', A'C', A'D' are each of order 5 on this cone & hence on the whole these count as a system of order (5 × 2) × 3 i.e. 30 in the curve of inters.

The 4 lines thro' A' cor. to the tangents at A to Σ_9 are also a portion of the curve of intersection. Hence the order of the remaining curve of intersection is 70 − 34 i.e 36.

Nature of the nonic curves & surfaces.

A line l of (S) transforms now into a nonic curve in (S') having a node of order 7 at A' & intersecting A'B', A'C', A'D' each once. The septic must also intersect Σ_9' 10 times & Σ_{36}' 5 times.

A pl. of (S) will transform into a nonic surface in (S') having A' as a conical pt of 8th order lines A'B', A'C', A'D' as lines of multiplicity 3 containing Σ_{36}' & Σ_9'.

For the pl. meets the F-septic in (S) in a septic curve having 3 dps one on each of AB, AC, AD. ~~This curve transforms into a cone~~ The cone vertex A & base this septic curve transform into an octic cone in (S') having A'B', A'C', A'D' as triple sides. This is therefore the tangent cone to the nonic surface at A'.

A curve of order m transforms into a curve of order

$$m' = 9m - 8d - 3\mu - 3\nu - 3\rho - \sigma - \tau$$

where $d\ \mu\ \nu\ \rho\ \sigma\ \tau$ are the number of times the curve meets A, AB, AC, AD, Σ_{36}' Σ_9'.

A surface of order n transforms into a surface of order $n' = 9n - 7d - \mu - \nu - \rho - 10\sigma - 5\tau$.

$\lambda\ \mu\ \nu$ etc being the order of ~~mult.~~ of pt A, lines AB, AC, AD & curves Σ_{36} & Σ_9 respectively.

Number of lines on the nonic surface.

Since a pl. of (S) meets the curve Σ_{36} in 16 pts & the curve Σ_9 in 9 pts we will have besides the triple lines A'B', A'C', A'D', 45 single lines thro A'.

Superposition of spaces.

It is easily seen by the method of page 237 that the number of dps is 20.

Art 42. We can clearly extend this series of transformations indefinitely. The first of the series was the sextic transf. given on p 236. The next of the series will be of order 12 & the next of order 15 & so on. The general transf. made up of two sheaves of ples & a sheaf of surfaces of order n having each a conical pt of order n-1 is of order 3n & its characteristics are of precisely the same nature as those of the two transf. of the series already given.

The lines AB, AC AD etc will be lines of multip. n on the 3n-ic surfaces. The pts A & A' will be conical pts of order 3n-1.
Corr. to each of these we will have F-surfaces of order (3n-2) having AB, AC, AD as lines of mult. n-1. The base curves will be of order n^2 each & A & A' will be nodes of order ~~(n^2~~ ~~$(n-1)^2$ ie 2n-1,~~ $(n-1)^2$ on them.
The F-curves will be of order $5n^2-3n$ in each space.
The complete F-system of curves will therefore be AB, AC, AD, the base curve of order n^2, & the F curve of order $5n^2-3n$.
The complete F-system of surfaces will be ples, BAC, CAD, DAB, the F-surface corr to A' of order 3n-2, the cone $A\Sigma_{n^2}$ of order 2n-1, & the cone of order 2(2n-1) corr. to the curve Σ'_{n^2}.
A line of one space will transform into a curve of order 3n in the other having a node of order 3n-2 at A or A' & intersecting AB AC AD

in 1 pt each, Σ_{n^2} in 2(2n-1) pts & the F curve Σ_{5n^2-3n} in 2n-1 pts.

A-ple transforms into a surface of order 3n having a conical pt of order 3n-1 at A or A' & having the lines $\overset{AB,}{A'B',}$ $\overset{AC,}{A'C'}$ $\overset{AD}{A'D'}$ as lines of mult. n.

The surfaces also contain the curves Σ_{n^2} & Σ_{5n^2-3n}.

When the spaces are superposed the number of dps will 6n+2.

If the lines AB, AC, AD A'B', A'C', A'D' be superposed there will be 4 double lines in these quad. pencils. On each of these there will be an involutive system of corr. pts.

If however in the above transf. p 247 the relation between the pencils A & A' is not quadratic but collinear the order of the transf. becomes 6.

Corr. to a line of (S) we get a ple sextic curve in S' having a node of order 5 at A'.

Corr. to a ple of (S) we get a surface of order 6 in (S') having a conical pt. of order 5 at A'.

Corr. to A & A' we get F-surfaces of order 5.

Corr. to ~~pt on~~ Σ_9 we get a cone ~~A~~ of order 5 in (S').

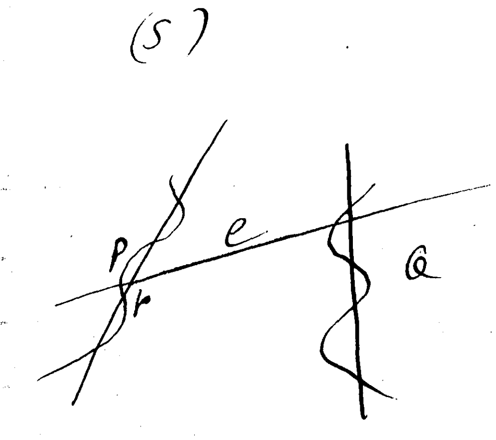
The intersection of F-quintic corr. to A & the cone corr. to Σ_9 is a curve of 25 of which the 4 lines corr. to the tangents at A form part. Hence the F-curve in (S') is of order 21.

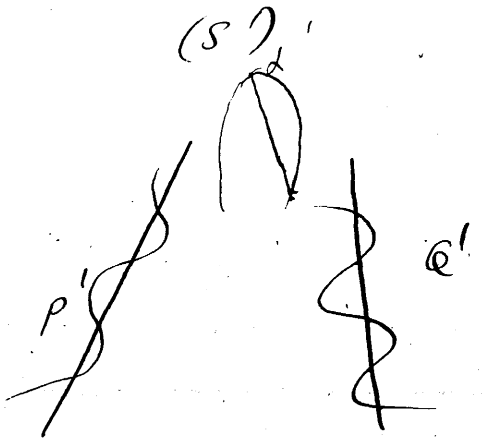
Hence the complete F-curve system in (S') is Σ_9' & Σ_{21}' altogether of order 30 as it ought to be.

The complete F-system of surfaces in (S') is the cone $A'\Sigma_9'$ & the cone of order 5 corr. to Σ_9, & the F-quintic corr. to A.

For collinear pencils just as above we get a series of transf. of which the general transf. is of order 2n. The F-system of curves consists of base curve Σ_{n^2} & the F-curve of order $n(3n-2)$.
The F-system of surfaces consists of surface cor. to A or A' of order (2n-1) ∞, the cone A Σ_{n^2} & the cone of order 2n-1 cor. to Σ_{n^2}.
A line transforms into a ple curve of order 2n having a node of order 2n-1 at A or A' & intersecting Σ_{n^2} in 2n-1 pts & $\Sigma_{n(3n-2)}$ in 2n-1 pts.
A ple transforms into a surface of order 2n having a conical pt of order 2n-1 at A or A' & containing Σ_{n^2} & $\Sigma_{n(3n-2)}$.
When the spaces are superposed the number of dps is 4n+2.
If the F-curve systems are superposed & if the pencils be congruent there will be a surface of dps of order 2n, for a line will meet its cor. curve of order 2n in 2n pts.
These transf. are however only special cases of a more general series which will be given shortly.

Art 43. A large number of particular cases of the nonic transf. can be deduced by taking various forms for the base curve.
One of the most interesting cases is the following. Let the base nonic curve split up into 2 lines & a septic curve the lines being used as bases for the two sheaves of ples.





Thus let PQ P'Q' be the lines ; the base septic must meet each of these lines in 4 pts.

Any two ples thro' P & Q meet a cubic surface in a pt M. The corr. ples thro' P' & Q' & the corr. cubic meet in M'. The cubic surfaces are in this case it may be noted non-singular cubics.

If M describe a line l in (S) ples (P') (Q') describe a hyd. & ples (Q') & sheaf of cubics describe a sextic having Q' as quadruple line & P' as single line. The intersection of this surface with the hyd. is a curve of order 7. Hence the transf. is of order 7.

The F-systems.

Let p be any pt in P & l a line thro' p lying in a ple π. Corr. to the ple Qp we get a ple Q' which we may call ρ'. The pt consecutive to p on the line l transforms into a pt on ρ' which is the intersection of a line thro' the pt & where P' meets ρ' & a conic in which the corr. cubic surface meets ρ'. The conic also contains α'. Let the ple π swing round P the the lines & conics thro' α are related (1-1) & the locus of their pt of intersection is a cubic curve having a node at α. Hence the corr. of the pt is a unicursal cubic having a dp on P'. As p moves on P these cubics clearly will generate a sextic surface having P' as double line & Q' as triple line & also containing the base septic Σ_7'. For the line l meets any cubic in 2 pts & hence the sheaf of planes (Q) & the sheaf of cubics in (S) generating C are related by a (2-1) relation. The corr. planes Q' & corr. cubics are also related by a (2-1) reln. & the locus of their intersection is consequently a quintic surface having Q' as triple line & P' a single line. The plane corr. to π viz π' meets this surface therefore in a unicursal quartic (excluding P') having a triple pt on Q'. Hence if we take a line in π parallel to P & infinitely close to it, its correspondent will be a unicursal quartic. Hence a plane π' thro' P' must meet the F-surface corr. to P' in a unicursal quartic & since P' is clearly a double line on surface (being locus of nodes of cubics) the order of the section of the F-surface by π' is 6. We also infer that since a section by a plane thro' Q' is a unicursal cubic that Q' must be a triple line on the surface.

Thus we have established that corr. to the line P we get a sextic surface having Q' as triple line & P' as ~~single~~ double line & containing Σ_7'. Similarly corr. to Q we get a sextic having P' as triple line & Q' as double line & also containing Σ_7'.

Corr. to a pt on Σ_7 we get a line in (S'). Corr. to Σ_7 as a whole we get a ruled surface of order 6 having P' & Q' as triple lines, for any plane thro' P meets Σ_7 in 3 pts & hence the relation between the sheaves (P')(Q') is (3-3).

Again consider a line which meets P Q & Σ_7. Corr. to this line we get a pt in (S'). (There are 8 exceptions however to this. For the lines which meet P Q & the curve in a pt on P or Q are 8 in number & they clearly transform into 8 lines meeting P' & Q'.)

The locus of these lines is a ruled surface of order 6 on which P' & Q' are triple lines. The locus corr. to this surface is of order 17.

For the locus must clearly lie on each the F-sextics corr. to P & Q. These surfaces intersect in a curve of total order 36 of which P' & Q' count as of order 6 each, whilst Σ_7' counts as of order 7. The remaining part curve which is of order $36-12-7$ ie 17 is therefore the above locus.

Hence the complete F-system of curves is : $P, Q; \Sigma_7, \Sigma_{17}$ & $P', Q', \Sigma_7', \Sigma_{17}'$.

The complete F-system of surfaces is P' sextic, Q' sextic Σ_7' sextic & the Σ_{17}' sextic & so on.

Effect of transf.

A line of (S) transforms into a septic curve meeting P' & Q' each in 6 pts, Σ_7' in 6, & Σ_{17}' also in 6 pts.

A plane of (S) transforms into a septic surface having P' & Q' as triple lines & containing Σ_{17}' & Σ_7'.

A curve of order m in (S) in our previous notation transforms into curve of order $7m - 3\lambda - 3\mu - \nu - \rho$ where $\lambda\ \mu\ \nu\ \rho$ are the nos. of pts of intersection with $P\ Q\ \Sigma_7\ \Sigma_{17}$. A surface of order n transforms into a surface of order

$$n' = 7n - 6\lambda - 6\mu - 6\nu - 6\rho$$
$$= 7n - 6(\lambda + \mu + \nu + \rho).$$

where $\lambda\ \mu\ \nu\ \rho$ are the degrees of mult. of $P\ Q\ \Sigma_7\ \Sigma_{17}$ on the surface.

Number of lines on a septic surface.

A plane π of (S) meets the F-line system in pts $A_1\ A_2$, $B_1 \ldots B_7$; $C_1 \ldots C_{17}$ respectively $A_1\ A_2$ being on P & Q $B_1 \ldots B_7$ on Σ_7 & $C_1 \ldots C_{17}$ on Σ_{17}.

Corr. to the line $A_1 A_2$ we get a line on septic & corr. to the 24 pts $B_1 \ldots C_{17}$ we also get lines on the surface. Thus there are 25 lines on the surface. All these lines intersect P' & Q' & 17 of them intersect Σ_7' in addition.

Cubic curves on the surface.

A line joining either A_1 or A_2 to any of the other F-pts on π transforms into cubic curve.

There are 2×24 i.e. 48 such cubic curves. They are clearly ple unicursal cubics. They are really sections of the surface by ples thro' P' or Q'.

When the spaces are superposed the number of dps is 16. If the F-systems are also superposed the number of proper dps is 8. If the sheaves base (P) & the sheaves base (Q) are congruent there will be two cubic surfaces of dps. If (P) & (Q) be interchanged there will be 2 quartic curves of dps.

Another case of interest is the following.

Let the cubic surfaces be ruled cubics having $\underset{Q'}{Q}$, as double lines the remaining base curves being of order 5.

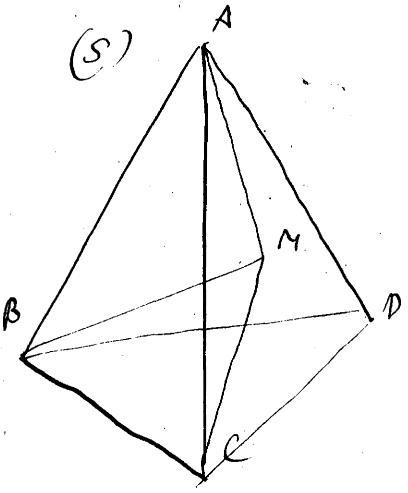
In this case Q & Q' are quintuple lines on the septic surfaces whilst P & P' are single lines.

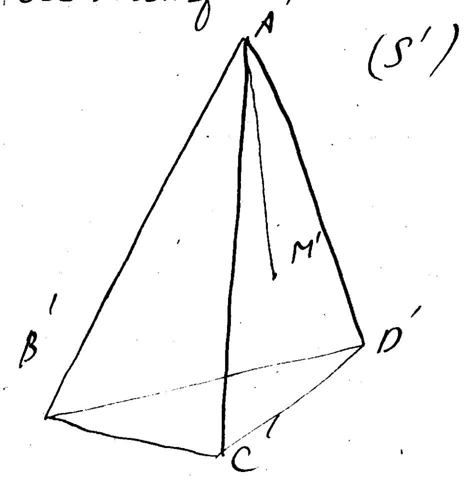
The F-curves are now of order 11.

A quintic transf. entirely analogous to the quartic transf. of p 194 can now be obtained & a further series of particular transf. in general non-reversible & analogous to the six cases of p 200 can also be obtained.

The most important of these particular cases is the following. It is the next of a series of which the quadratic transf. is the first.

Art. 44. The projective cubic transf.





Let A & A' be the vertices of two collinear pencils. Let BC be the base of a sheaf of ples in (S) & let us take as the corr. sheaf in (S') a sheaf of cubics passing thro' a sextic curve having a node at A' (of second order). This sextic is the intersection of a fixed conicoid with ~~a~~ a cubic surface having a conical pt of second order ~~A~~ at A'. All the cubic surfaces in (S') ~~are~~ have a conical pt at A'. The remaining base curve of the sheaf of cubics we shall take to be a unicursal cubic in the ple B'A'C' having a node at A'. Thus the base system is of order 9 & has a node of order 4 on the whole at A'. The transf. is therefore a particular case of the nonic transf. of p 247 in which the nonic base curve is degenerate & made up of a ple cubic & a sextic.

Corr. to the pt M in which the ples BAM, CAM, BMC intersect we get a pt M' in which the corr. surfaces B'A'M', C'A'M' & cubic intersect,

When M describes a line l in (S) A'M' describes a ple in (S') & the ples C'A'M' & cubics generate a quartic surface in general having a triple conical pt at A'. Hence corr. to a line in (S) we get a ple quartic curve in (S').

Conversely when M' describes a line it is easily proved that M describes a ple unicursal quartic.

Thus the transf. is in general reversible & (4-4).

Let us now make the following suppositions.

Suppose that to ple BAC of sheaf (BC) corr. the degenerate cubic made up of ple B'A'C' & the fixed conicoid mentioned above(~~& to ple BDC (BC)~~ ~~the~~). The ple B'A'C' now forms part of the quartic surface generated by the sheaf (A'C') & the sheaf of cubics. The proper part of the surface is therefore a cubic having a conical pt at A'. Hence corr. to the line l we get a unicursal cubic in (S') having its node at A'. Conversely corr. to a line (l') in (S') we get a unicursal cubic having its node at A.

Thus the transf. is now a cubic transf.

The F-system.

Corr. to the pt A we get the fixed conicoid in (S'). This conicoid we shall refer to as the F-conicoid. Corr. to A' we also get a conicoid in (S). For take a ple thro' A' & consider the pts on it infinitely close to A'. If we take a pencil of lines thro' A' & the consecutive pts to A' on these lines we get on the corr. ple thro' A a conic generated by the sheaf (BC) & the corr. pencil of
lines thro' A. Hence corr. to A' we get a F-conicoid in (S). A & A' are therefore F-pts & the corresp. F-surfaces are conicoids.

Again corr. to a pt on the F-sextic Σ_6' in (S') we get a line thro' A in (S) & corr. to Σ_6' as a whole we get a quartic cone thro' vertex A in (S). Again corr. the curve of intersection of this quartic cone in (S) with the F-conicoid corr. to A' which is a curve of order 6 as we shall see presently we get the cone vertex A' & base Σ_6'.

Thus corr. to a pt of one of these cones we get a generator of the other & corr. to an F-sextic we get the quartic F-cone of the other space.

To show that the F-curve in (S) which is the intersection of the F-cone & F-conicoid is of order 6 we proceed as follows. Consider the F-sextic Σ_6' in (S'). At A' there will be 2 tangents to the curve one to each branch. Corr. to these tangents we get 2 lines thro' A in (S). As in the general transf. p 247 each of these lines must lie on the quartic cone & conicoid & hence the curve of intersection is of order 8-2 i.e. 6. It must like Σ_6' in (S') have a node at A. In fact the tangents to the node at A' transform into the tangents to the node at A. Thus we have found out the nature of the complete F-systems.

The F-system in (S) consists of Σ_6 & the pt A together with the quartic cone A Σ_6 & there is a similar system in (S').

Effect of the transformation.

A line of (S) transf. into a ple unicursal cubic in (S') having its node at A' intersecting Σ_6' 4 times.

A ple of (S) transf. into a cubic surface having a conical pt at A' & passing thro' Σ_6'.

Number of lines on a cubic surface.

A ple π of (S) meets the F-sextic in 6 pts. These pts lie on the F-conicoid & therefore lie on a conic.

To each of these pts corresponds a line thro' A' on the surface. Corresponding to the 15 diagonals of the hexagon on π we get other 15 lines on the surface making in all 21 lines. No other lines are possible. Thus as we have already seen a cubic surface with a single conical pt possesses just 21 lines.

Superposition of the spaces.

The number of d.ps is 8 & this may be established in the same way as for the quad. transf.

If the pencils & F-curves are superposed & if the pencils are congruent we get a perspective cubic transf. like the persp. quad. transf.

In this case there will be a surface of d.ps of order 3. It will pass thro' A but A will be an ordinary pt on it. Let l be any line thro' A. This meets the surface of d.ps in pts P_1 & P_2. Any two corr. pts on l will be harmonic conjugates with respect to P_1 & P_2. The transf. is now involutive.

If the pencils be superposed but not the F-systems the F-sextics (for congruent pencils) will be on a common cone thro' A of 4th order & there will be a cubic surface of d.ps whose intersection with this cone is ~~the~~ made up of the two sextic curves but the transf. is not now involutive.

— , —

Art 45. This transf. is one of a series. The next of the series is a quartic transf. in which the F-surfaces corr. to A & A' are cubics – & the F-cones of 6th order. The F-curves are of order 12 & have nodes of order 6 at A & A'.

The general transf.

Starting from the general transf. on p 250 made up of two sheaves of planes & a sheaf of surfaces of order n we proceed as follows.

The base curve of order n^2 is to be degenerate & composed of a plane curve of order n (having node of order $n-1$) & a curve of order n^2-n having a node of order $(n-1)^2-(n-1)$ i.e. $(n-1)(n-2)$ at A'.

The F-surfaces corr. to A & A' will be of order $n-1$ & must have conical pts of order $n-2$ at A & A';

The F-curves will be of order n^2-n (above) or $n(n-1)$ & the F-cones will be of order $[-(n-1)(n-2)+n(n-1)]$ or ~~$(n-1)(n-2$~~ $(n-1)(n-n+2)$ i.e. $2(n-1)$.

Corr. to any pt on an F-cone we get a gen^r of the other F-cone.

The transf. is of order n.

Corr. to a line we get a ple unicursal curve of order n having a node of order $n-1$. The curve meets the F-curve in $2(n-1)$ pts.

Corr. to a ple we get a surface of order n having a conical pt of order $n-1$ & containing the F-curve.

These results can be established just as we have established the cubic transf.

That the F-surfaces corr. to A & A' must have conical pts of order $n-1$ at these pts is easily seen. The intersection of a surface of order n (having conical pt of order $n-1$) at A with the surface of order $n-1$ is to give a curve of with a node of order $(n-1)(n-2)$ at A. Hence x a ple section thro' A must meet these surfaces in curves of order $n-1$ & n. & the curve of order $^{n-1}$ must therefore have a node of order $n-2$ at A.

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Art 46. The formulae for the cubic transf. are

$$x : y : z : w = x'\varphi_2' : y'\varphi_2' : z'\varphi_2' : \varphi_3'$$

where φ_2' & φ_3' are the F-conicoid & a cubic surface of (S').

For the general transf. the formulae are

$$x : y : z : w = x'\varphi_{n-1}' : y'\varphi_{n-1}' : z'\varphi_{n-1}' : \varphi_n'$$

$$x' : y' : z' : w' = x\varphi_{n-1} : y\varphi_{n-1} : z\varphi_{n-1} : \varphi_n$$

These formulae are obtained as follows.

With proper choice of co-ord^s we get for the pencils A & A'

$$xy' = yx'$$
$$xz' = zx'$$

& for the sheaf (BC) & sheaf of surfaces φ_n' ~~φ_n'~~ : $x - \lambda w = 0$; $x'\varphi_{n-1}' - \lambda\,\varphi_n' = 0$

giving $\quad w\,x'\varphi_{n-1}' - x\,\varphi_n' = 0$

$$\therefore \frac{x}{x'} = \frac{w\varphi_{n-1}'}{\varphi_n'} = \frac{y}{y'} = \frac{z}{z'}$$

$$\therefore \frac{x}{w} = \frac{x'\varphi_{n-1}'}{\varphi_n'} \qquad \frac{y}{w} = \frac{y'\varphi_{n-1}'}{\varphi_n'} \qquad \frac{z}{w} = \frac{z'\varphi_{n-1}'}{\varphi_n'}$$

or $x : y : z : w = x'\varphi_{n-1}' : y'\varphi_{n-1}' : z'\varphi_{n-1}' : \varphi_n'$.

An interesting case of the perspective cubic transf. is the following.
Consider two conicoids Σ_1 & Σ_2 intersecting in a quartic curve & a fixed pt A.
Any line thro' A meets the conicoids ~~&~~ in pts P Q R S say. The pts P Q R S determine an involution (quadratic) on the line. Let M be any other pt on it. The corr. of M M' in this invol. may be taken as the corr. of M in the transf. Thus we establish a pt-pt (1-1) corresp. between the pts of the space.
When M describes a line M' will describe a ple cubic having a node at A. Corr. to A itself we get a conicoid the F-conicoid. Thro' A we can draw two chords of the quartic curve of intersection. These lines are clearly F-lines & together with the ~~P~~ quartic of intersection make up the F-sextic Σ_6.
Conversely if in a cubic transf. (proj. & invol.) we take the base sextic to be made up of a quartic (of def. 1) & two chords of curve thro A we infer that there will be two conicoids thro'

the F-quartic which transf. into themselves.

Art 47. The general theory of pt-pt (1-1) transformations.

We are now in a position to consider the general theory of a (1-1) pt-pt transf. between 2 spaces.

Suppose that to a plane of space (S) corresponds a surface of order n of space (S') & to a plane of (S') cor. a surface of order m in (S). In general m & n are different & if the transf. is reversible m = n. We shall suppose for simplicity that m = n & that the transf. is reversible.

The surfaces of order n in (S) or (S') must pass thro' certain fixed pts & lines the F-pts & F-curves. Further since 3 planes in (S) or (S') determine a single pt it follows that 3 surfaces of order n must also determine a single pt (variable).

Thus the system of surfaces thro' a fixed pt must be doubly infinite. The system of surfaces thro' 2 fixed pts must be singly infinite & the system of surfaces thro' ~~2~~3 fixed pts must reduce to a single surface. Hence each surface of order n must be completely determined by 3 fixed pts. These pts of course must be taken arbitrarily & must not lie on the F-system.

Again since two planes intersect in a line then cor. surfaces will intersect in a curve of order n & this curve must be completely fixed by 2 pts taken arbitrarily.

Since a single infinity of surfaces of deg. n pass thro' 2 arbitrary pts ~~are~~ this sheaf of surfaces must have for a common base a curve of order n. This base is of course not fixed & varies with the choice of pts.

Since the complete curve of intersection of two such surfaces is a curve of order n^2 it follows that the F-curve system (the fixed part of the base) must be of order n^2-n or $n(n-1)$. Again since a line can only meet a ple in 1 pt a curve of order n can only meet a surface of order n in 1 arbitrary pt. A curve of order n must therefore meet the F-curve system in n^2-1 pts. Two curves of order n on a surface of order n can only meet in 1 arbitrary pt.

Consider two fixed pts on a ple π. A & B & let homog. pencils of lines be taken thro' A & B. The locus of their intersections is a conic thro' A B. Corr. to these pencils on the surface of order n we get two homog. pencils of ~~curves~~ 'n-c' curves thro' 2 fixed pts A' & B'. The locus of the intersection of these pencils is a curve of order 2n on the surface meeting the F-system in $2(n^2-1)$ pts. Thus we see how the geometry of the ple transforms into the geometry of the surface.

The F-systems.

The F-systems may be of various kinds.

(1) There may be F-pts. All the surfaces of order n may have in common a conical pt of order δ. Let this pt be called A. at A the ordinary pt-pt corresp. between the spaces breaks down. We shall now show that the corr. of A is a surface of order δ.

Consider any one of the surfaces of order n having a conical pt of order δ at A. ~~(In the neighbourhood of A we may replace the surface by its tangent cone at A of order δ. Consider also a very small conicoid (ellips) surrounding A. The intersection of the cone & conicoid will be a curve of two parts & of order 2δ. This curve transforms into).~~ In the neighbourhood of A we may replace the surface by its ~~tangent~~ cone at A of order δ. Let a ple be taken cutting the surface very close to A. Its section with the surface is ultimately we may suppose a very small curve of order δ

& another curve of order $n-d$. Transforming into the other space we find that the plane transforms into a surface of order n & the surface of order n transforms into a ple. The section of ple & surface in the one space a curve of total order n therefore transforms into section of ple & surface in the other of precisely the same kind & of order n.

The vanishingly small curve of order d therefore transforms into a curve of order d the remainder curve of order $n-d$ into a curve of order $n-d$ of same nature (same deficiency etc). Hence we may say in the limit that a ple thro' A transforms into a surface of order $n-d$ since its section with any ple is a curve of order $n-d$ & that corr. to A we get a surface of order d since its section by any ple is a curve of order d. This surface is the locus of the correspondents of all the pts infinitely close to A.

There are however numerous exceptional cases. A F-pt of order d has for its corresp. in many cases a surface of order less than d. In fact the order cannot be greater than d but may be as little as $\frac{d}{2}$. In the transf. already given we have had numerous cases of this sort. As an example of the general case we may take the projective series of transf. beginning with the quadratic. These transf. possess an F-pt of order $n-1$ the order of the transf. being n. Corr. to these F-pts we get surfaces of order $n-1$ & a ple thro' such a pt transforms into a surface of order $n-(n-1)$ ie 1 viz a plane. Again taking the other series given on p 250. In the transf. of order $3n$ we have a ∞ F-pt of order $3n-1$ to which corresp. a surface of order $3n-2$. A ple thro' this pt transforms into a surface of order $3n-(3n-2)$ ie a conicoid (a cone in this series).

Again in the general transf. of order 24 built up by 3 sheaves of conicoids we have 7 pts of order 12. To each of these cor. a surface of order 6. A plane thro' one of these pts transforms into a surface of order 24 − 6 ie 18. In the particular case in which the 3 bases of order 4 hav~~ing~~ a common line we get a transf. of order 18 with 3 conical pts of order 9. Cor. to each of these pts we get a surface of order 6. Even in the comparatively simple case of the tetrahedral cubic transf. a case of the same kind occurs. Cor. to a vertex of the F-tetrahedron, an F-pt of the second order we get not a conicoid but a ple. These exceptional cases are to be explained by the fact that often the curve of order n−d does not transform into a curve of order n−d. Thus in the tet. cubic trans. we have a small conic & a line which transform into a line & a conic. In the nonic transf. of p 257 the small octic & a line transform into a ~~oct~~ septic & a conic respectively the deficiencies of the septic ~~be~~ & the octic being the same viz. 12. The octic has 3 triple pts & is therefore of def. 21 − 9 ie 12. The septic has 3 dps & is therefore of def. 15 − 3, ie 12 also. Thus the total deficiency is unaltered & this is the only essential condition to be observed in the transf. of these ple curves. Thus the deg. ple curve of made up of parts of order d & n − d & of deficiency p say transforms into a curve of parts of orders whose sum is n & whose total def. is p. The def. p & the orders of the transformed curves will clearly depend on the particular transf.

Again there may be an F-curve of order d & of multiplicity l in the transf.

Any line intersecting this curve meets the n-ic surface in n-l pts (arbitrary) & hence it must transform into a curve of order n-l. Consequently corr. to the pt of intersection with the F-line we must have a curve of order l. If a pt P moves on the F-line ~~we must have~~ the corr. curve of order l will generate a surface which corresponds to the F-curve as a whole. The order of this surface will depend on the nature of the transf.

F-curves & F-pts may occur simultaneously, & the F-pts may lie on the F-curve system. When an F-pt lies on an F-curve it must be the intersection of two or more branches of the F-system.

When there is however an isolated F-pt of order d i.e a F pt not lying on the F curve system it would seem that its correspondent must be a surface of order d. The section of the surface by a ple very near the F-pt consists of a small curve of order d (not intersecting the F-system) & a curve of order n-d. If any ple section of the n-ic is of def^y p the def^y of this curve of order n-d must also be of order p. The two curves of order d & n-d must transform into curves of order d & n-d resp. the latter being of def. p. Hence corr. to the F-pt we get a surface of order d. The curves of order d & n-d must by their intersections give the same system of multiple pts as any ple section of the n-ic. Let us take as an example of an isolated F pt the sextic transf. of page 264. This transf. possesses an F-pt of second order isolated whose corr. is a cone of second order. A ple section of the sextic surface is a sextic curve having 2 triple pts & 3 d.ps. i.e a curve of def. 1. A ple section very near the F-pt may be taken to be a small conic with a quartic of def. 1. Transforming we get a conic & a quartic of def. 1. The intersection of the conic & quartic gives rise to 2 triple pts & 3 d.ps as it ought to. In fact a ple thro' the F-pt transforms into a quartic having AB, AC, AD as single lines & Σ_2 as a double curve.

Again since a line meets the surface corr. to an F-pt of order d in d (or in the other cases a number $< d$ & $\geqq \frac{d}{2}$) pts the corr. n-ic curve must pass thro' the corr. F-pt d times (or in other cases etc).

The surfaces corr. to F-pts or F-lines are F-surfaces.

Nature of the n-ic curves & surfaces.

Since the coords. of a pt on a line can be expressed in terms of ~~one~~ arbitrary parameter it follows that the n-ic curves must have the same property. Such curves may be called rational curves. Let A be any vertex of projection & Π any ple then ~~at~~ on projecting the curve from A on to Π we get a unicursal ple curve. Since the pts of the n-ic & the pts of its corr. line are related by (1-1) relation it follows that the generators of the projecting cone & the pts of the line must be likewise related by a (1-1) relation. Hence the pts of intersection of the gens. of the cone with Π must also be connected with the pts on the line by means of a (1-1) relation. The ple curve of projection must therefore be unicursal.

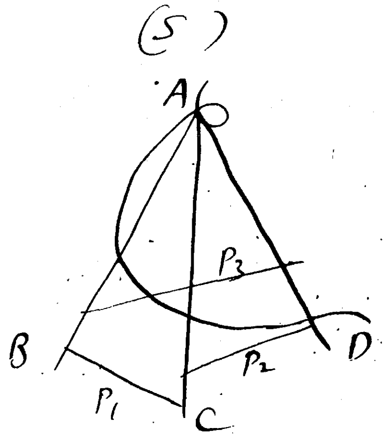
It will be in general possible to choose A so that the n-ic ple curve will have a node of order n-1. & hence we infer the existence of a chord of the space n-ic which meets it n-1 pts.

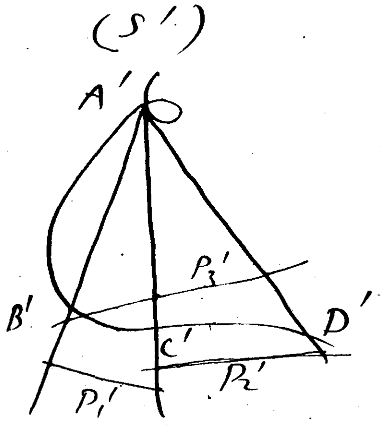
Since the coords. of a pt on a ple can be expressed in terms of two arbitrary parameters it follows that the like is true for the n-ic surfaces. Such surfaces may also be called rational. Their characteristic properties are that 3 of them must have one arbitrary pt of intersn. & that each is determined by 3 pts.

Art 48. The methods of the previous pages can clearly be extended indefinitely to the discovery of new transformations. Each transf. gives rise to new varieties of curves & surfaces some of great interest & possessing many peculiarities.

As a last example we shall give a transf. of degree 18 which belongs to a series of which the first is the transf. on p 244. This example will exemplify the preceding theory & will show that even when the order of the transf. is high there is no great difficulty in applying the foregoing methods.

This transf. is built up by means of 3 sheaves of cubic surfaces having a common sextic (nodal) as part of their base systems, each surface having one conical pt. Let Σ_6 & Σ_6' be the sextic curves (which are the intersections of a monoconical cubic & a conicoid) with nodes of the second order at A & A'.





Let three sheaves of monoconical cubics be taken with the conical pt at A & 3 similar sheaves in (S') with conical pts at A'. Let the remaining part of the base of each sheaf be made up of 3 coplanar strt. lines viz AB, AC, P_1, AC, AD, P_2; AD, AB, P_3, etc.

Two neighbouring sheaves have therefore also a line in common. We will suppose that the sextics intersect the common lines AB AC, AD, A'B', A'C', A'D' in one pt each (excluding A & A').

Any three cubics of (S) intersect in one arbitrary pt. Σ_6 accounting for 24 pts of intersection & A for 2. Similarly any 3 cubics of (S') intersect in one arbitrary pt. If as usual we establish a homographic relation between the different pairs of sheaves, we establish consequently a (1-1) relation between the pts of the two spaces. The transf. is of order 18.

Consider any ple π of (S). Two cubics of (S) belonging to different sheaves intersect ~~at~~ in a curve which meets π in 2 pts. Corr. to each of these we get a cubic of the 3rd sheaf. Hence when 2 generating cubics are given 2 cubics of the remaining sheaf are thereby determined. The same kind of relation will hold between the 3 generating sheaves in (S') & hence if we consider their intersections with an arbitrary line we get for the intrinsic eqn. of the surface

$$K_1(S_1{}^3)^2(S_2{}^3)^2(S_3{}^3)^2 + K_2(S_1{}^3)(S_2{}^3)^2(S_3{}^3)^2 + \text{etc} = 0$$

where $S_1{}^3$, $S_2{}^3$, $S_3{}^3$ refer symbolically to the intersection of ~~each~~ a cubic of each sheaf with the line. From this eqn. we infer that the order of the surface corr. to π is 18. By taking suitable positions of the arbitrary line we can at once deduce the following conclusions; (1) A & A' are conical pts on the 18ic of order 12; (2) Σ_6 & Σ_6' are multiple curves of order 6; (3) AB, AC, AD, A'B' etc are multiple lines of order 4; (4) P_1 P_2 P_3 P_1' etc are multiple lines of order 2.

The F-systems.

The base systems as in previous transfs. form part of the F-system of curves. Corr. to a pt on Σ_6 we get a curve of order 6. For a line thro' this pt meets an 18-ic surface in 12 arbitrary pts & hence it transforms into a curve of order 12. The pt on Σ_6 must

therefore have as its corr. a curve of order 6.
Corr. to a pt on AB, AC, AD A'B' etc we get a curve of order 4 & corr. to a pt on P_1 P_2 etc we get a conic. This last fact is evidently geometrically from the transf. for corr. to any pt on P_1 we get the intersection of 2 cubics in (S') i.e a conic.

Corr. to A we get a surface of order 9.
For take a ple thro' A. This meets ~~the cubic~~ two cubics of different sheaves in 2 unicursal cubics having their nodes at A. These cubics have besides A 4 pts in common viz the pts in which the ple meets the sextic Σ_6.
Hence the unicursal cubics will have one arbitrary pt of intersection. Hence with regard to this ple the relation between the generating sheaves of cubics is of the following kind – when two cubics belonging to different sheaves are given the third cubic belonging to the remaining sheaf is thereby determined. Hence the generating cubics in (S') being connected by the same relation will give a surface of ~~character~~ intrinsic eqn ~~kS_1S_2~~ $K_1(S_1^3)(S_2^3)(S_3^3)+K_2+=0$
Hence the surface corr. to the ple thro' A is of order 9 & hence the surface corr. to A must also be of order 9. There are also isolated conical pts of orders 3 & 2 respectively.

Consider the 3 surfaces composed respectively of ples BAC & conicoid Σ_2, CAD & Σ_2, DAB & Σ_2 (Σ_2 being the conicoid which generates Σ_6). To these surfaces corr. 3 cubics in (S') which meet in a pt K'. To this pt K corr. the conicoid Σ_2 & K is a F-pt of second order.
Again the sheaves in (S) have in general one single common cubic. [This can be seen by considering the projective cubic transf. p 261.]. To this surface corr. in (S') in general 3 different cubics which meet in a pt L'. L' is a F pt of 3rd order & its corr. is the common cubic in (S).

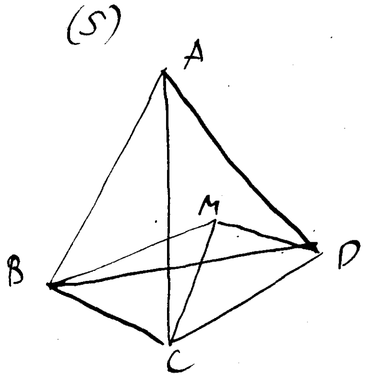
Similarly F-pts K & L of 2nd & 3rd orders respectively exist in (S).

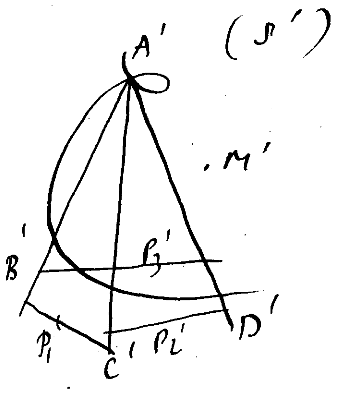
Again consider in (S) the cone whose vertex is A & base Σ_6. To any generator of this cone there corresponds a point in (S') & corr. to the cone as a whole we get a curve in (S'). Since the common cubic in (S) possesses 6 lines thro' A of which 3 lie outside the base system it follows that since these 3 lines are generators of the above cone, ~~that~~ the above curve must pass thro' L 3 times.

Similarly since the conicoid Σ_2 has 2 generators thro' A which are gen. of the cone, the curve must pass thro' K twice. The curve is of order 30.

As an example of a non-reversible transf. let us take the modification of the above transf. obtained as follows.

Art 49. In (S) let us take 3 sheaves of ples whose bases are 3 coplanar lines & in (S') let us take 3 homog. related sheaves of monoconical cubics of the above type having a common nodal sextic Σ_6'.





The base systems in (S') are the same as in the above 18-ic transf. To a pt M determined by the 3 ples BMC, CMD, DMB corr. the pt M' determined by the 3 corresp. cubics in (S').

The transf. is in general (6 - 9); ie to a line of (S) corr. a sextic curve in (S') & to a line of (S') corr. a nonic curve in (S); also to a ple of (S) corr. a nonic in (S') & to a ple of (S') a sextic of (S).

We shall give briefly the characteristics of these curves & surfaces.

In (S') the nonic surfaces have A' for a sextuple F-pt $A'B'$, $A'C'$, $A'D'$ as double lines, P_1' P_2' P_3' as single lines & Σ_6' as a triple curve. They possess also 12 other lines thro' A'; & 3 conics thro' A'. They have an F-pt of first order K' corr. to the ple BCD in (S), thro' which each of these conics passes.

In (S) the sextics have ~~4~~3 F-pts of 4th order at B C & D an F-pt of 3rd order at A corr. to the common cubic in (S') & an F-pt of second order at L corr. to the conicoid Σ_2'. BC, CD, DB are double lines & AB, AC, AD single lines on these sextics. They also possess an F-curve of 12th order corr. to the cone $A'\Sigma_6'$.

Corr. to A' we get a cubic F-surface in S & corr. to BCD we get cubic surfaces in (S') viz those corr. to the ples BCD for each sheaf.

The nonic curves in (S) have triple pts at A B C D & at pt L.

The sextic curves in (S') have triple pts at A', & pass thro' K'.

It may be noted that the nonic surface in (S') resembles closely the surface obtained by transf. a cubic surface by the projective cubic transf. A nonic of this type has a sextic triple curve, a conical pt of 6th order thro' which passes 18 lines.

Art 50. Particular suppositions for the correspondences between the sheaves of ples & the sheaves of cubics lead to other interesting transf.

For example if we suppose that to ple BCD (BC) corr. the deg-cubic made up of ple B'A'C' & conicoid Σ_2'; to ple CBD (CD) corr. cubic C'A'D', Σ_2' etc we get a (4-5) non reversible transf:

The quintic surfaces in (S') have Σ_2' as single line, $A'B'$, $A'C'$, $A'D'$ as double lines P_1' etc as single lines & 8 other lines thro' A'.
The quintics in (S) have a triple F-pt at A (corr. to the common cubic in (S')) & a F-curve of order 8. The pt A' is also a F-pt of order 4 on the quintics in (S').
We may again note that the quintics of (S') are analogous to the quintics obtained by transf. a ~~[illegible]~~ monoconical cubic having its conical pt at the F-pt by the projective cubic transf. Such quintics have a conical pt of 4th order & with 14 lines thro' it together with a sextic F-curve.

Let us now make the following suppositions.
Let us suppose that to ple BCD (BC) corr. common cubic of (S') & that to ples CBD (CD), BDC (BD) corr. the same cubic surface; whilst to ple BAC (BC) corr. the deg. cubic $(B'A'C', \Sigma_2')$, to ple BAD (BD) corr. the cubic $(B'A'D', \Sigma_2')$ & to ple CAD (CD) corr the cubic $(C'A'D', \Sigma_2')$.
In this case the transf. is (3-3) & is identical with the projective cubic transf. of p 261. The present form of this transf. is more symmetrical than that given on p 261. It is clear that this ~~projective~~ symmetrical form may be used for the complete series of projective transf.
For example the quadratic transf. may be constructed by means of 3 sheaves of ples in (S) & 3 sheaves of conicoids in (S') having a common conic & a common pt.

Surfaces representable on a ple.
When by a transf. we establish a corr. between a ple & surface which is (1-1) we shall say that the surface is representable on a ple. Again if by a transf. we can establish a corr. (1-1) between such a representable surface & another surface we can regard this second surface as representable also on a ple & there is

a transf. existing which enables us to carry out the correspondence directly, in many cases.
Thus a conicoid & a cubic surface of any form are always representable surfaces as is evident from the quadratic & cubic transf. Now if we transform a conicoid or a cubic by means of the quadratic or cubic projective transf. we get surfaces representable on these. But these surfaces are directly representable on a plane by transf. like those we have just considered. Examples of such surfaces are the sextics of p 244, the 18-ics; & the nonics & the quintics of the preceding transf.

<u>Art 51.</u> Number of dps of the transf. of order 18.
When the spaces are superposed the number of dps is 38. For the corr. sheaves of cubics generate sextic surfaces which have Σ_6 & Σ_6' in common. The number of pts of intersection of these surfaces is 216. Of these Σ_6 & Σ_6' count for 78 each. A & A' count for 2 each & 3 pts lie on each of the lines AB, AC, etc. Hence the number of intersections on the F-system is $2 \times 78 + 4 + 18$ leaving $216 - 178$ or 38 proper dps. This number $= 2 \times 18 + 2$

(1) For any reversible transf. of order $3lmn$ built up by means of 3 sheaves of surfaces of orders l m & n respectively the number of dps is $2(3lmn+2)$.
The base curves of the sheaves are of order l^2 m^2 n^2 respectively & must have $(lmn-1)$ pts in common. By general principles already used the order of the transf. is in general $3lmn$.
Corr. sheaves generate surfaces of order $2l$, $2m$, $2n$ resp.ly which have $8lmn$ pts in common. 1
Amongst these are the two groups of $lmn-1$ fixed pts on the base systems. Hence the number of proper

dps is $8lmn - 2(lmn-1)$ i.e. $6lmn+2$
or $2(3lmn)+2$.

Hence if d be the order of the transf. the number of dps is $2d+2$.

(2) Let us now suppose that the base curves have a common part of order rs which is the intersection of two surfaces of degrees r & s.

The curve rs counts as $rs(l+m+n-r-s)$ pts of intersection of 3 surfaces of deg. l m & n resp.

Hence the remaining parts of the base system must have $lmn-1-rs(l+m+n-r-s)$ pts in common.

The transf. is now of order $3lmn-rs(l+m+n)$.

Corr. sheaves generate as before surfaces of order $2l, 2m, 2n$ which have the curves of order rs in common. These curves count as

$rs(2l+2m+2n-r-s)$ pts of intersection each

Hence the ~~p~~ number of proper pts of intersection will be

$$8lmn - 2rs(2l+2m+2n-r-s) - 2[lmn-1 - rs(l+m+n-r-s)],$$

This reduces to $6lmn - 2rs(l+m+n)+2$
or $2[3lmn-rs(l+m+n)]+2$ i.e. $2d+2$ where d is the order of the transf.

(3) Let the base system of (S) be made up of 3 sheaves of surfaces of order p q r resp. whose base curves intersect in $pqr-1$ pts; & let the system in (S') be composed of 3 sheaves of surfaces of orders l m n resp. whose base curves have $lmn-1$ pts in common.

The order of the transf is

$d = lqr+mpr+npq$ & since it is to be reversible d must also be

$lmr+lnq+mnp$.

The 3 ~~sheaves~~ pairs of corr. sheaves generate surfaces of order $(p+l)$, $(q+m)$, $(r+n)$ resp^{ly}

The number of pts of intersection of these is

$$(p+l)(q+m)(r+n).$$

The number of pts of intersection on the F system is $pqr-1+lmn-1$ hence the number of proper d.ps is

$$(p+l)(q+m)(r+n)-(pqr-1)-(lmn-1)$$

$$= lqr+mpr+npq+lmr+lnq+mnp+2$$

$$= 2(lqr+mpr+npq)+2$$

$$= 2d+2.$$

Hence we infer that in any reversible transf. built up by 3 sheaves of surfaces the number of d.ps is $2d+2$ where d is the order of the transf.

Cap. V.

The non-birational transformations.

The methods already applied to the investigation of birational transf. are equally applicable to the investigation of ~~birational~~ non-birational transf.

We shall take the simplest case of non-birational systems as an example. The more complicated cases can be treated in precisely the same manner.

The (1-2) pt-pt-correspondence between two spaces.

When to a single pt of space (S) there corresponds two distinct pts of space (S') & to a single pt of space (S') a single pt of space (S) we shall say that the correspondence between the spaces is (1-2) with regard to the point as primary element.

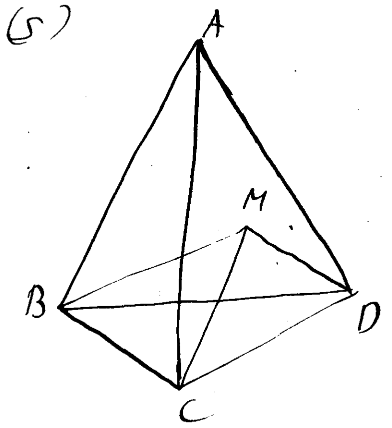
Transformations between 2 spaces related in this way are non-birational i.e. the formulae of transf. involve sq. root signs.

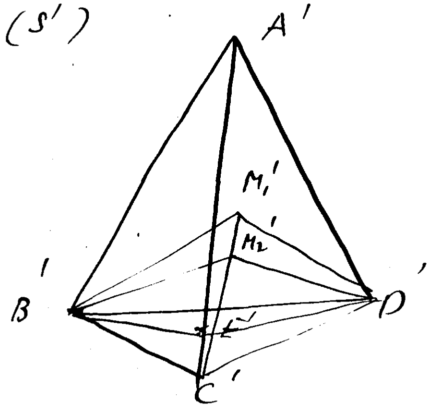
A series of transf. for spaces related in this way & analogous to the series of birational transf. already obtained can be constructed.

Art. 52.

(1) We shall begin with the transf. analogous to the collineation.

Let us take in (S) three coplanar lines as the bases of three sheaves of ples & let us take in (S') three coplanar lines also. Two of these we shall take as the bases of ordinary sheaves of ples whilst we shall take the third as the base of a double sheaf of ples – the ples of which we shall suppose related by means of a quadratic involution





Let the sheaves (BC) (B'C') & (CD) (C'D') be homog. related & let the sheaf (BD) be homog. related to the double sheaf (B'D') so that to ple BMD corr.. the ples B'M₁'D', B'M₂'D'. The correspondence between the sheaves (BD) & (B'D') may be constructed as follows. Let l & l' be lines in (S) & (S') respectively & let a range of pts be chosen on (l) & a range on (l') homog. with it. Let now another range be chosen on (l') ~~hom~~ in involution with the first range. If M be any pt on l & if M₁' M₂' be the corr. pts on l' then ple BMD corr. to ples B'M₁'D' & B'M₂'D'. Again to a pt M₀' on l' corresponds a pt M₀ on l & this pt must be the same no matter to which range M₀' belongs. Hence to ple B'M₀'D' corr. one ple BM₀D. In this way we establish a (1–2) relation between the pts of the spaces (S) & (S').

Corr. to the pt M which is the inter. of ples BMC, CMD, BMD we get the pts M₁' M₂' the inters. of the corr. ples B'M₁'M₂'C' C'M₁'M₂'D', B'M₁'D', B'M₂'D' in (S'). Corr. to pt M₀' in (S') the inter. of ples B'M₀'C', C'M₀'D' etc we get the pt M₀ the inter. of corr. ples BM₀C, CM₀D etc.

Let us now suppose that to ple BDC (BC) corr. ple $B'D'C'$ $(B'C')$; to ple CBD (CD) corr. ple $C'B'D'$ $(C'D')$;
to ple BDC (BD) corr. ple $B'C'D'$ $(B'D')$
" BAC (BC) --- $B'A'C'$ $(B'C')$
etc exactly as in the coll. transf. Cap II.

Corr. to a line of (S) described by M we get a conic in (S'); for $C'M_1'M_2'$ describes a ple & the inters. of ple $B'M_1'M_2'C'$ with the double sheaf $(B'D')$ generates a quadric cone. The inters. of the ple & cone is a conic which intersects $B'D'$.

Again to a ple in (S) corresponds a conicoid in (S'). For the generating sheaves give as the intrinsic eqn of the surface
$K_1 S_1^2 S_2 S_3 + --- = 0$ (A). where S_1^2 indicates the inters. of the sheaf $(B'D')$ with ~~the~~ an arbitrary line & S_2 S_3 the inters. of (BC) & (CD) with the same line.
As in the collinear transf. the relation between S_1 & S_2 S_3 must be linear hence the relation must be of the form
$K_1 S_1^2 + K_2 S_1 S_3 + K_3 S_1 S_2 + K_2 S_2 + -- = 0$
Hence the surface is a conicoid.

Reversal of transf.

Let M_1' describe a line. Then CM describes a ple & since the relation between the sheaves (CD) (BD) is $(2-1)$ there being two positions of CD for one of BD, their inters. describes a quadric cone vertex D one of whose generators is CD.
The intersection of the ple & cone is a conic thro' C.
Hence corr. to a line of (S') we get a conic in (S).
The pts M_1' M_2' give rise in (S') to a cono-cylindrical transf. p. 44 art 11. When M_1' describes a line M_2' describes a conic thro' C'. The corr. of the conic in (S) is really made up of this line & conic in (S').
If M_1' describes a ple we get since the relation between the sheaves (BC) (BD), (CD) (BD) is in each case $(2-1)$

for the intrinsic eqn. of the surface corr. to the plane in (S')

$K_1 S_1 S_2^2 S_3^2 + - - = 0$ (B)

where S_1 stands for the inters. of (BD) with an arbitrary line & $S_2^2 S_3^2$ for the inters. of (BC) & (CD) with same line,

As before the relation between $S_2^2 S_3^2$ & S_1 must be linear & hence the eqn. becomes

$K_1 S_2^2 + K_2 S_3^2 + - - = 0$

The corr. of the plane is therefore a conicoid

The relations (A) & (B) are analogous to the reln. $K_1 S_1 S_2 S_3 + K_2 S_2 S_3 + - - = 0$ which held for the simpler birational transf. of Cap II. They are characteristic for all the transf. of (1-2) correspondence which can be built up by sheaves of planes.

The F-systems.

Let the other plane of (S') which corr. to plane BDC (BC) be B'E'D' (E' being on A'C')

Corr. to pt C we get the plane B'E'D' & pt C' so that C is a F-pt of first order of the transf.

Corr. to any pt on plane BDC we get a pt on B'D' & a pt on plane B'D'C'. For all pts on plane BDC we see therefore that one corr. lies on B'D',

To any other pt K there corr. 2 distinct pts in (S') which are variable with K & which do not lie on any fixed line.

Let a plane of (S) meet BCD in a line l. Corr. to line l we get B'D' & another line in plane B'C'D'. Corr. to the pt of inters. of l with BD we get 2 coincident pts on B'D'. Indeed to any pt on BD we get 2 coincident pts on B'D'. Hence the surface (conicoid) which corr. to a plane of (S) must contain B'D' & must have a conical pt on that line. Hence it must be a cone whose vertex is on B'D'.

Consider now a plane of (S'). This meets B'E'D' in a line. Corr. to this line we get the pt C in (S)

Hence the conicoids in (S) all pass thro' C & meet the ple BCD in a line. Also the conics in (S) must all pass thro' C. Again it may be remarked that the cones of (S') must all touch the ple B'E'D'.

Effect of the transf. on curves & surfaces.

We may denote the transf. by the symbol $\left\{\begin{matrix}2-2\\2-2\end{matrix}\right\}$ indicating that with respect to curves the transf. is (2-2) & with respect to surfaces also (2-2).

A conic in (S) transforms into a quartic in (S') intersecting B'D' twice. Any ple thro' C' must meet this quartic in 4 pts which are collinear in pairs with C'.

A conicoid of (S) transforms into a quartic surface having B'D' as double line & having two conical pts of 3rd order on B'D' corr. to the pts of intersection of the conicoid with BD. This surface must also touch the ple B'E'D'. The ple B'E'D' plays the part of an F-ple in the transf. A ple thro' C' meeting B'D' in R' meets this quartic surface in a quartic curve having an osculation node at R'.

Corr. to a conic in (S') we get a quartic curve in (S) having a node at C & corr. to a conicoid in (S') we get a quartic surface in (S) having a conical pt of second order at C.

Superposition of the spaces.

When the spaces are superposed there will be certain pts which coincide with one or other of their correspondents. We may call such pts d.ps. There are in general 7 such d.ps. The sheaves (BC)(B'C'), (CD)(C'D'), (BD)(B'D') trace out 2 hyperboloids & a ruled cubic respectively. The line of inters. of ples BCD & B'C'D' lies on all

3 surfaces & counts as 5 pts of intersection. These pts depend on the choice of fund. - let^s & must be rejected. Hence there will be $2 \times 2 \times 3 - 5 = 7$ proper d.ps.

Here is a comparatively simple case simple case analogous to the usual collineation. Let the tetrahedra be superposed so that ABCD coincide with A'B'C'D' resp. Then the ple containing any pt & its two correspondents must pass thro' (AC)(A'C'). In this case the line B'D' is a line of d.ps. The 7 d.ps are situated as follows. B' & D' count as 2 d.ps each A' E' C' are the other 3.

Art 53. The transf. analogous to the concylindrical transf.

The only change which requires to be made in the suppositions of p 296 is the following.

To ple BAD (BD) corr. ple B'C'D' (B'D')

" BCD (") " " B'A'D' (")

This transf. has the symbol $\left\{ \begin{matrix} 3-3 \\ 3-3 \end{matrix} \right\}$.

To a line corresp. a cubic curve & conversely

" ple " " surface " .

Let M describe a line, C'M1' describes a ple & the intersection of the sheaves (B'C')(B'D') is a cubic cone having B'D' as double line. The intersection of the ple & cone is a unicursal cubic thro' C' its double pt lying on B'D'.

When M describes a ple we get for the intrinsic eqn. of the corr. surface

$K_1 S_1^2 S_2 + K_2 S_1^2 S_3 + \cdots = 0$ showing that the surface is a cubic. This surface passes thro' C'; B'D' is a double line on this surface & consequently they must be ruled cubics.

When M_1' describes a line, CM describes a ple & the inters. of the sheaves (CD)(DB) describes a cone with CT a double lines. The intersection of the ple & surface is a unicursal cubic having its node at C. When M_1' describes the line M_2' describes a conic thro' C'. The line & the conic are the complete corr. of the cubic in (S).

When M_1' describes a ple it is easily shown as in the previous transf. that M describes a cubic surface having a conical pt at C.

The F- systems.

Corr. to pt C we get 2 distinct ples in (S') thro' B'D' hence C must be regarded as a F-pt of second order & its corr-ples as an F-surface of second order.

Corr. to a pt on ple BCD we get the line B'D' taken twice.

Corr. to a pt on ple BAD we get the pt C' as one of the correspondents. Hence we may regard the ple BAD & its corr. pt C as F-elements.

From these correspondences we derive the following conclusions.

To a ple of (S) corr. a cubic surface having B'D' as double line & having a conical pt of 3rd order on B'D' corr. to the pt in which the ple meets BD. The cubic surfaces must therefore be cubic cones having a double side. [Compare the conocylindrical trans. in which the corr. of a ple was a cone]. These cubic cones must clearly pass thro' C'.

To a line of (S) corr. a cubic curve having a node on B'D' & passing thro' C'.

To a ple of (S') corr. a cubic surface of (S) having a conical pt of second order at C. If the ple meets the F-ples corr. to C viz B'A'D' & B'F'D' in 2 lines k & m corr. to k & m we get two tangent ples at C to the cubic surface. Hence the tangent cone at C reduces to two ples.

To a line of (S') corr. a cubic curve (ple) having a node at C.

Effect of the transf. on curves & surfaces.

A conic of (S) transforms into a sextic curve in (S') having two double nodes on B'D'.

A conicoid of (S) transforms into a sextic surface in (S') having B'D' as quadruple line & having two conical pts of 5th order on B'D' (corr. to the pts in which the conicoid meets BD. It has also a conical pt of second order at C'.

Generally a curve of order n transforms into a curve of order $3n$ which intersects B'D' n times & has double nodes at the pts of intersection.

A surface of order n transforms into a surface of order $3n$ having B'D' as a line of $2n$th order of mult. & having n conical pts on B'D' each of $2n+1$th order. It has also a conical pt of order n at C'.

A conic of (S') transforms into a ~~quartic~~ sextic in (S) having a node of 4th order at C.

A conicoid of (S') transforms into a sextic surface having a conical pt of 4th order at C, the tangent cone being degenerate & composed of two quadric cones.

A curve of nth order (S') transforms into a curve of order $3n$ in (S) having a node of order $2n$ at C.

A surface of mth order in (S') transforms into a surface of order $3m$ in (S) having a conical pt of order $2m$ at C at which the tangent cone consists of two cones of order m.

Superposition of the spaces.

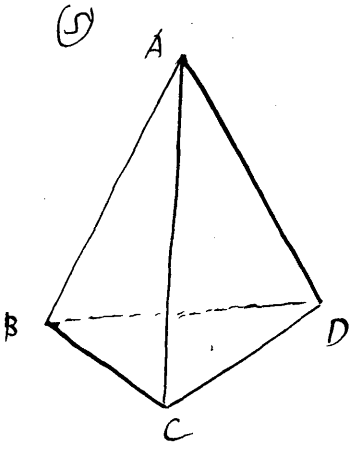
The number of dps is 9. For the hyds. & the cubic ruled surface of the last transf: intersect in 12 pts. The line of intersection of ples BCD, B'C'D' contains 3 of these pts & these have to be rejected since they depend on the choice of tet^{n}. Thus there are 9 proper dps.

Art 54.

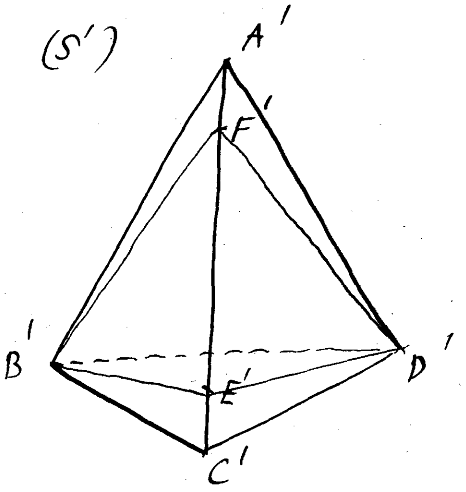
The transf. analogous to the tetrahedral cubic transf.

We make the following suppositions

(S)



(S')



To ple (BDC) (BC) corr - ple B'A'C' (B'C')

" (BAC) (BC) " " B'D'C' (B'C')

" (CBD) (CD) " " C'A'D' (C'D')

etc

" (BCD) (BD) " " B'A'D' (B'D')

" (BAD) (BD) " " B'C'D'. (B'D')

When M describes a line l of (S), sheaves (B'C') (C'D') generate a cone, whilst sheaves (B'C') (B'D') generate a ruled cubic surface having B'D' as double line.

The cone & cubic have a common line viz B'C' & hence their curve of intersection is of order 5. A line therefore transforms into a quintic curve.

When M describes a ple, M1'M2' generate a quartic surface. For the intrinsic eqn. of this surface is

$$K_1 S_1^2 S_2 S_3 + K_2 S_1 S_2 S_3 + K_3 S_1^2 S_2 + K_4 S_1^2 S_3 + K_5 S_1^2 + K_6 S_2 S_3$$
$$+ K_7 S_1 S_2 + K_8 S_1 S_3 + K_9 S_1 + K_{10} S_2 + K_{11} S_3 + K_{12} = 0 \qquad (A).$$

in our former notation. Hence corresp. to a ple in (S) we get a quartic surface in (S').

Reversal of the transf.

When M1' describes a line in (S') M describes a quartic curve in (S). For the sheaves (BC) (CD) generate a quadric cone whilst the sheaves (CD)(DB) between which a (2-1) relation exists generate a ruled cubic with CD as double line.

The cone & cubic have CD for a common line & this counts as of degree 2 in their intersection, hence their curve of intersection is of order 4.

Again when M_1' describes a ple M generates a quintic surface of intrinsic eqn.

$$K_1 S_1^2 S_2^2 S_3 + K_2 S_1^2 S_2 S_3 + K_3 S_2^2 S_1 S_3 + \quad - \qquad = 0 \quad (B)$$

Hence the symbol for this transf. is $\left\{ \begin{matrix} 5-4 \\ 4-5 \end{matrix} \right\}$.

Relation between the systems of pts M_1' M_2' in (S').

When M_1' describes a line M_2' describes a conic thro' C' lying in the ple containing the line & C'.

In fact as in the previous transformations the relation between the systems $M_1' M_2'$ is of the nature of a cono-cylindrical transf. We may call the curves described by M_1' & M_2' related curves & the surfaces related surfaces. Corr. to the quintic surface of (S) we have a ~~line~~ ple & its related conicoid (cone) & corr. to a quartic curve we have a line & its related conic. We may also call the pts $M_1' M_2'$ related pts in (S').

The F-systems.

Corr. to a pt on ple BCD we get the pts in which the line $A'C'$ meets the ples corr. to BCD (BD). One of these pts is A'. Let the other be F'. Corr. to a pt on BCD then we have a fixed pair of corr^ts^ A' F'. Corr. to a pt. on ple ~~BCD~~ BAC we have the pt D' at which the 2 corr^s^ coincide. Corr. to a pt on BAD we get 2 pts one of which is at C'. Corr. to a pt on CAD we get similarly two coincident pts at B'.

Thus each of the 4 ples BCD, BAC, CAD, BAD is composed of exceptional elements, one or more of whose corr^s^ occupies a fixed position.

All elements which one or more of their corr.s freed may be called F-elements. The 4 ples just mentioned are F-ples & their corresponding freed pts F-pts.

F-system in (S').

Corr. to a pt on B'C'D' we get the pt A.
Corr. .. .. B'A'C' - - — D
Corr. .. .. D'A'C' - - - B
Corr. .. . B'A'D' - - - - C
Corr .. .. B'F'D' - - —— C.

Hence in (S') the 5 ples B'C'D', B'A'C' etc B'F'D' are F-ples & A B C D are their corr. pts. C it may be noted is in the previous transf. a F-pt of second order.

The F-line system in (S).

Corr. to a pt on AC we get the line B'D' (double)
.. ... BC - - - - lines D'A', D'F'
.. .. CD .. - - - .. B'A', B'F'
.. - AB .. - - - line C'D' with a variable line in ple B'E'D' thro' D'
.. .. AD we get the line B'C' with a line thro' B' in the ple B'E'D'.
.. .. BD we get the line A'C'.

in (S').

Corr. to a pt on B'D' we get the line AC
.. - - - B'C' .. - - - AD
- - - B'E' .. pt D'
.. .. C'D' .. line AB
- - - D'E' — pt B'
.. .. A'B' .. line CD
.. - - B'F' .. .. CD
- - A'D' .. .. BC
.. .. D'F' .. .. BC
.. .. A'C' .. .. BD (~~double~~)

Nature of the quintic curves & the quartic surfaces in (S').
From the consideration of the F-system we deduce the following conclusions.

A line of (S) transforms into a quintic curve passing thro' A', E', B', D' (both of which are nodes of the second order) & C'.

A plane π of (S) transforms into a quartic surface of (S') having $B'D'$ as double line, B' & D' conical pts of 3rd order (the tangent cones at which correspond to the lines of inters. of π with BAC & CAD), $D'A'$, $D'F'$, $B'A'$, $B'F'$, $C'D'$, $B'C'$ & $A'C'$ as single lines. C' is a conical pt of second order (the tangent cone at which corresponds to line of inters. of π with BAD).
There are other two lines on the surface thro' B' & D' respectively.
The above system of lines on the quartic forms a complex of the 11th order. From this we can at once verify that two such surfaces must intersect in a quintic curve.

Nature of the quartic curves & quintic surfaces in (S).
A line of (S') transforms into a quartic curve in (S) passing thro' $ABCD$ & having a node of second order at C.
A plane of (S') transforms into a quintic surface having AD, AB, BC as single lines & CB, CA, CD as double lines. C is a conical pt of 4th order whose tangent cone is degenerate ~~& made up~~ of two quadric cones thro' CB, CA, CD. (These cones corr. to the lines of inters. of plane with $B'A'D'$ & $B'F'D'$). B A & D are conical pts of 3rd order. The tangent cones at these pts are cubic cones with a double side. This quintic surface is analogous to the quintic surface which is obtained by transforming a conicoid thro' an F-pt by the tetrahedral cubic transf. Two such surfaces intersect in a curve of order 10. Two of the above quintic surfaces must however intersect in a curve of order 4.

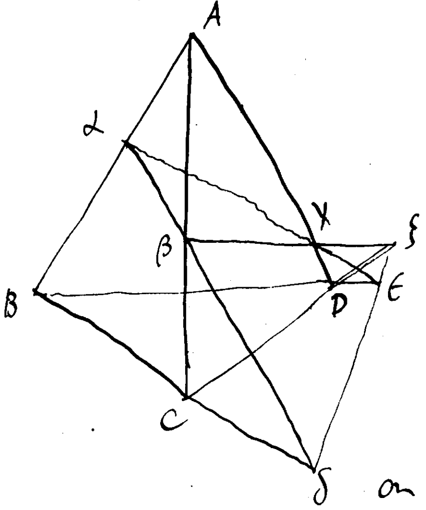
The alteration in the degree of the curve of intersection is due in the present instance to the exceptional nature of the conical pt at C. The line AC must count as of order 9 in the intersection of two quartic surfaces.

The F-pts.

The pts B A D are F-pts of 1st order & their cor. surfaces are faces of the F-tetⁿ in (S'). C is a F-pt of second order & its cor. surface consists of the ples $B'A'D'$, $B'F'D'$. The pts B' A' D' C' are likewise F-pts of 1st order whose cor. ples are the faces of the F-tetⁿ in (S).

The lines on the quartic surfaces in (S').

A ple π of (S) meets the F-tetⁿ ABCD in 6 pts. $\alpha\,\beta\,\gamma\,\delta\,\epsilon\,\xi$ which we may call the F-pts of the ple.

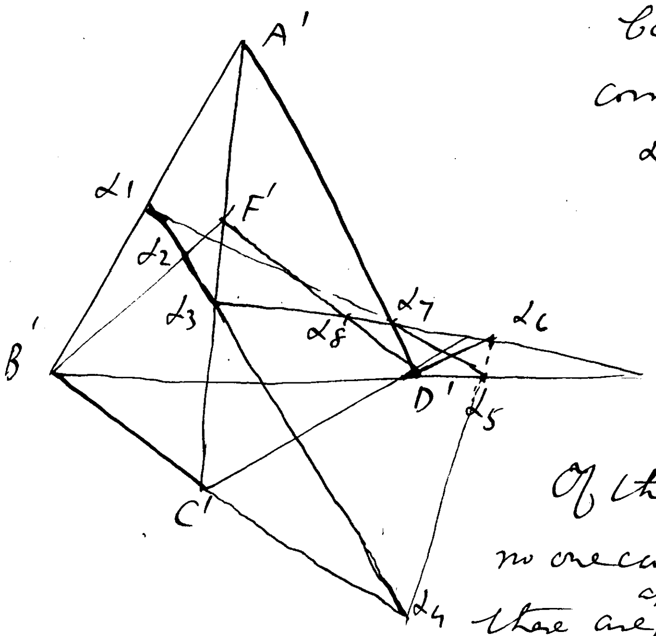


Cor. to α we get $C'D'$ & a line thro' D' in ple $B'E'D'$. Cor. to β we get the line $B'D'$ (double) Cor. to γ we get the $B'C'$ & a line thro' B' in ple $B'E'D'$. Cor. to δ we get $D'A'$, $D'F'$; cor. to ϵ we get $A'C'$ & cor. to ξ we get $B'A'$, $B'F'$! Cor. to line $\alpha\xi$ we get clearly another line on quartic since α & ξ account for 4 lines. Similarly cor. to $\gamma\delta$ we get a line on the quartic. Cor. to $\beta\epsilon$ we get a conic for β & ϵ account for 3 lines. Hence altogether there are 11 lines on the surface (one of them double) & a conic (special). Of course any section of the quartic surface thro' $B'D'$ gives a conic.

The lines on the quartic surface in (S).

A ple π' of (S') meets the F-system in 8 pts $\alpha_1 \ldots \alpha_8$. Let $\alpha_1 \ldots \alpha_8$ be as in the figure.

$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$ are collinear & so are $\alpha_8\ \alpha_7\ \alpha_6\ \alpha_3$; $\alpha_4\ \alpha_5\ \alpha_6$; & $\alpha_1\ \alpha_7\ \alpha_5$.



Corr. to α_1 & α_2 we get CD; corr. to α_3 we get BD; corr. to α_8 & α_7 we get BC; corr. to α_4 we get AD; corr. to α_5 we get AC; corr. to α_6 we get AB. There are consequently 8 F-pts on the ple π'.

Of the lines joining these pts in pairs no one can give rise to a line. Thus there are apparently just the six F-lines on the quintic surface. Consider the lines $\alpha_4\ \alpha_5\ \alpha_6$ & $\alpha_1\ \alpha_7\ \alpha_5$. These lines have for their correspondents the pts A & C respectively. But $\alpha_4\ \alpha_5\ \alpha_6$ etc account for only 3 lines & not 4 as they ought to. This apparent paradox is explained by the fact that α_5 on B'D' must be regarded as a double F-pt on the ple whose correspondent is AC reckoned twice. Regarding α_5 as a double F-pt we see that the lines $\alpha_5\alpha_8$, $\alpha_5\alpha_3$, $\alpha_5\alpha_2$ give rise to lines on the quintic surface. The line corr. to $\alpha_5\alpha_3$ must meet AC & BD & the lines corr. to $\alpha_5\alpha_2$ & $\alpha_5\alpha_8$ must pass thro' C.

Conics on the surface.

The lines $\alpha_8\alpha_2$, $\alpha_8\alpha_4$, $\alpha_7\alpha_2$, $\alpha_7\alpha_4$, $\alpha_1\alpha_8$, $\alpha_1\alpha_6$, $\alpha_6\alpha_2$ give rise to conics on the quintic. Thus there are 7 conics on each quintic surface.

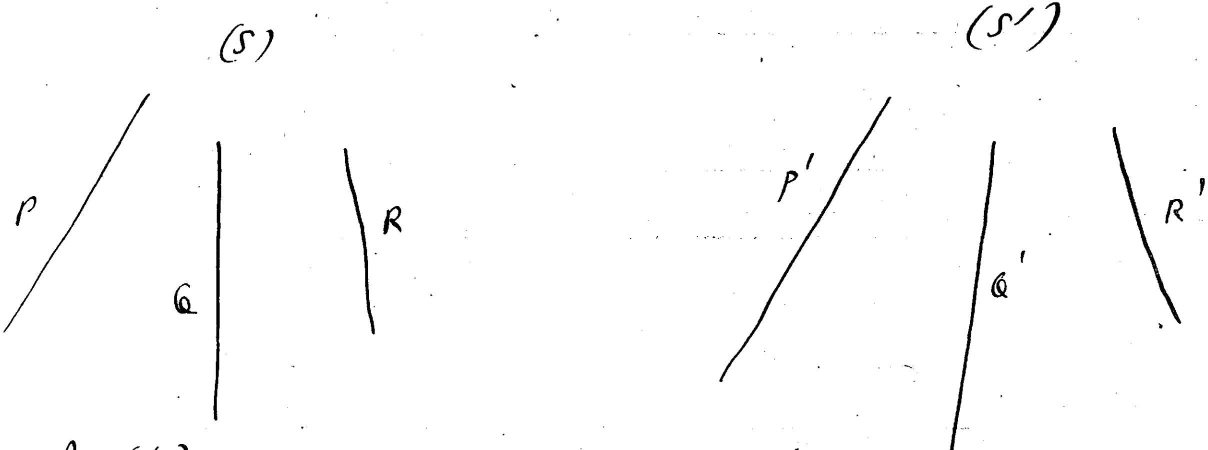
Superposition of the spaces.

There will be 12 dps for the hyds (BC, B'C') (CD, C'D') & the cubic surface (BD, B'D') will intersect in 12 pts.

Art 55.

The hyperboloidal $\left\{\begin{matrix}5-4\\4-5\end{matrix}\right\}$ transf.

The bases of the sheaves in (S) & (S') are in this case non-coplanar & non-concurrent lines.



In (S') we shall suppose that P' is the base of the double sheaf.

The characteristic properties of this transf. are the same as those of the previous transf. so that a brief treatment of it will be sufficient.

Corr. to a line in (S) we get a quintic curve in (S') of which Q' & R' are triple chords & P' a double chord. The ranges described by the sheaves (P') & (R') on Q' form a cubic involution which has 3 dps & hence Q' must be a triple chord of the quintic curve. The results for P' & R' are similarly obtained.

Corr. to a plane of (S) we get a quartic surface of (S') which has P' for double line & Q' & R' as single lines. Its intrinsic eqn. is the eqn (A) of the previous article.

Corr. to a line of (S') we get a quartic curve of (S) of which P is a ~~quadruple~~ double chord & Q' & R' triple chords.

Corr. to a plane of (S') we get a quintic surface in (S) which has Q & R as double lines & P as a single line.

The F-systems.

The base lines P Q R , P'Q'R' form part of the F-systems.

Corr. to the generator of the hyd. (PQR) we get 2 pts in (S'). Corr. to the hyd. (PQR) as a whole we get a quintic curve in (S') which like the quintics above is the intersection of a hyd. & a ruled cubic.

This F-quintic has Q' & R' for triple chords & P' as a double chord.

The F-system of lines in (S') is therefore composed of P'Q'R' & this F-quintic Σ_5'.

Corr. to the hyd. (P'Q'R') we get in (S) a F-cubic having PQ & R as double chords.

The F-system in (S) is composed of PQR & the cubic Σ_3.

Corr. to the line P we get a hyd. in (S').

" " " Q " " ruled cubic in (S') having P' as double line.

Corr. to R we get a ruled cubic having P' as double line.

The F-system of surfaces in (S') consists therefore of 2 hyds. & 2 ruled cubic surfaces.

Corr. to P' we get a hyd. in (S).

" " Q' " a ruled cubic [the relation between the generating sheaves (RP) being (2-1)] having P as double line.

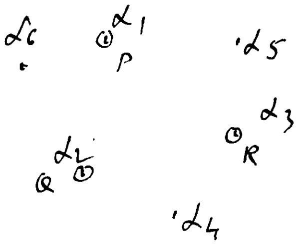
" " R' we get a ruled cubic having Q as double line

The F-system of surfaces in (S) consists of two hyds. & two ruled cubics as in (S').

The quartic surfaces in (S') contain the F-curve Σ_5' & the quintics in (S) the F-curve Σ_3.

Lines on the quartic & quintic surfaces.

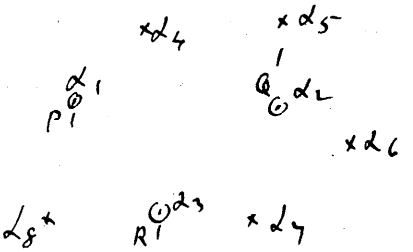
A plane π of (S) meets the F-line system in 6 pts $\alpha_1 \dots \alpha_6$. Let us suppose that $\alpha_1\ \alpha_2\ \alpha_3$ lie on $P\ Q\ R$ resp^y. Corr. to α_1, $\alpha_4\ \alpha_5\ \alpha_6$ we get single lines on the quartic surface. Corr. to α_2 & α_3 we get in each case a pair of lines. Corr. to the line $\alpha_2\alpha_3$ we get a line on the quartic. Corr. to the lines $\alpha_2\alpha_1$ & $\alpha_3\alpha_1$ we get degenerate conics each consisting of 2 lines.



Thus we get altogether 13 lines on the quartic. As an example of the arrangement of these lines we take the line $\alpha_2\alpha_1$. ~~to~~ Corr. to α_2 we get two lines each intersecting P' & R'. Corr. to α_1 we get a line intersecting ~~α_2~~ Q' & R'. Corr. to the line $\alpha_2\alpha_1$ we get two lines intersecting P' & R' & intersecting P' in the same two pts as the lines corr. to α_2. Thus the complete correspondent of the line $\alpha_2\alpha_1$ ~~corr~~ is a degenerate quintic composed of 5 lines which intersect Q' & R' in threes & two pairs of which intersect on P'.

Lines on the quintic surfaces in (S).

Let a plane π' meet the F-system in the 8 pts $\alpha_1 \dots \alpha_8$. $\alpha_1\ \alpha_2\ \alpha_3$ being on $P'\ Q'\ R'$ respectively. Corr. to $\alpha_1\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7\ \alpha_8$ we get single lines on the quintic. Corr. to α_2 we get two lines & corr. to α_3 we get two lines.



Corr. to $\alpha_3\alpha_1$, $\alpha_3\alpha_4$ etc we get lines on the surface & corr. to $\alpha_2\alpha_1$, $\alpha_2\alpha_4$ etc we get lines on the surface. Corr. to $\alpha_3\alpha_2$ we apparently get another line on the surface; but α_3 & α_2 account for 2 lines, hence the corr. of $\alpha_3\alpha_2$ ought to be a point, This paradox is explained by a consideration of the systems $M_1' M_2'$ in (S').

Let $M_1' M_2'$ be a pair of related points in (S').

The systems $M_1' M_2'$ form a hyperboloidal cubic involution transf. of a particular kind.

It is easily shown that corr. to a line described by M_1' we get for the locus of M_2' a cubic curve intersecting the line twice & corr. to a ple described by M_1' we get a cubic surface for the locus of M_2' which meets the ple in 3 lines. If the ple meets $P'Q'R'$ in $\alpha\beta\gamma$ the related cubic surface meets the ple in the ~~line~~ line $\beta\gamma$ & the intersections of the double ples of the sheaf (P') with the ple $\alpha\beta\gamma$.

If we are given the double ples of the involution (P') we can construct a pair of related pts as follows. Let M_1' be any pt & let the line be drawn thro' M_1' which meets Q' & R'. Let this line meet the double ples in two pts S & T. Then M_2' is the harmonic conjugate of M_1' with regard to S & T.

The line $\alpha_3 \alpha_2$ therefore belongs to ple & related cubic surface.

There are consequently 23 lines on the quintic surface.

Nature of the quintic & quartic curves.

From a consideration of its intersection with the F-system of surfaces in (S) we see that a line of (S) transforms into a quintic curve in (S') having 2 nodes on P' & having Q' & R' as triple chords.

A line of (S') transforms into a quartic ~~surface~~ curve having Q & R as triple chords & P as double chord.

Each of these curves meets its F-curve in 2 pts.

Let us examine the deficiency of each of these curves. The quartic curve is clearly of deficiency zero & this is as it ought to be for it related (1-1) with its corr-line. The quintic curve lies on a conicoid & meets one system of generators in 3 pts each & the other system in 2 pts each. Every quintic curve in (S') lies on a conicoid thro' Q' & R' & it meets the gens. of system Q' R' in 3 pts & the gens. of the opposite system in 2 pts each. Such a curve is by the quadratic transf. art 34. of deficiency 2. Thus a line of (S) transforms into a curve of def. 2.

Effect of transf. on curves & surfaces.

A curve of order m in (S) transforms into a curve of order 5m in (S') have 2m nodes on P' & meeting Q' & R' in 3m pts each.

A curve of order m' in (S') transforms into a curve of order 4m' in (S) having 2m' pts of intersection with P & 3m' pts of inters. with Q & R.

A surface of order n in (S) transforms into a surface of order 4n in (S') having P' as a line of 2n order of multiplicity & Q' R' as lines of mult. n.

A surface of order n' in (S') transforms into a surface of order 5n' in (S) having P as line of mult^y n' & Q' & R' lines of mult^y 2n'.

Formulae analogous to those of art. 15 may be given.

Let a surface have P Q R as lines of mult^y δ μ ν respectively & Σ_3 as a curve of mult^y ρ.

Then if n be its degree the deg. of its corr-surface is $5n - 2\delta - 3\mu - 3\nu - 2\rho$.

As an example let us take a quartic surface in (S).
By the above formula the degree of its cor. is
$$5 \times 4 - 2 - 6 - 6 - 2 = 4.$$
Its cor. surface in (S') consists of a ple + its related cubic surface altogether a system of degree 4.

Superposition of the spaces.
As before the number of dps is 12.

In all these transformations we have not specified the nature of the involution sheaf of ples.
Any involution system may be chosen. For example the involution of perpendicular ples may be chosen & the double ples will be the isotropic ples thro' the base. We then get a simple construction for the pairs of related points in the space (S').

Art 56. The general transformation of type $\left\{\begin{matrix}5-4\\4-5\end{matrix}\right\}$.

This transf. is analogous to the general cubic transf.

It is made up of two correlations & a quadratic correlation.
Let M' be a pt. of space (S') & let it be related to the ples π_1, π_2 of (S) by correlative transf. & with the ple π_3 by a quadratic correlative transf. The pt of intersection of π_1 π_2 π_3 · M is the correspondent of M'. Again corresponding to the pt M of (S) we get on reversing these transformations two ples π_1' π_2' in S' & a conicoid thro' a fixed conic & a fixed pt Σ_2'.
The line of intersection of π_1' π_2' meets Σ_2' in two pts M_1' M_2' which are the correspondents of M.

The transf. has the symbol $\left\{\begin{matrix}5-4\\4-5\end{matrix}\right\}$.
A line of (S) transforms into a quintic curve in (S').
Let a pt M describe a line l. The ples π_1' π_2' revolve about lines & their intersection traces out a hyp. which meets an arbitrary ple P in a conic. The conicoids Σ_2' form a sheaf which intersects P in a system of 4 pt conics. Let the line of inters. of π_1' π_2' meet P in the pt p'. p' describes a conic & the system of 4 pt-conics generates a quadruple range on this conic. Let $s_1 s_2 s_3 s_4$ be any quadruplet then the system of pts $s_1 s_2 s_3 s_4 p'$ constitute an involution of pts on the conic of the 5th order which has 5 d.ps. Hence the locus corr. to l is a curve of the 5th order lying on a conicoid. It is of precisely the same nature as the quintics of the previous article.

It meets one system of generators of the conicoid in 3 pts & the other system in 2.
A line l' of (S') transforms into a quartic curve in (S). For if M' describe a line l' the intersection of the planes π_1, π_2 traces out a hyd. which meets an arbitrary plane in a conic. Let p be the generating pt of this conic. The planes π_3 envelope a cone & the intersection with the arbitrary plane consists of a series of tangents to a conic.
These tangents generate a triple range on the conic described by p. As before we infer that the system of pts on the conic constitutes an involution of order 4 having 4 dps. Hence the locus corresp. to l is a curve of order 4 lying on a conicoid & of deficiency 0.
From this we infer by a proof similar to that given for the general cubic transf. on pp 140-1 that corr. to a plane in (S) we get a quartic surface in (S') & to a plane of (S') a quintic surface of (S).

The F- systems.
The F- system of lines in (S') must form a complex of order 11 & this must contain the conic thro' which all conicoids pass. Hence the remaining curve will be apparently of order 9.
The F-system in (S) will be a complex of order 21.
The complete investigation of the F-systems of this transf. I am at present engaged in.

art 57. Particular case.
Exactly as we deduced the quadratic transf. as a particular case of the general cubic transf., so in the present instance we can deduce the general transf. of symbol $\left\{\begin{matrix}3-3\\3-3\end{matrix}\right\}$ as follows.

Let two collinear pencils vertices A & A' be taken in S & S' respectively & let also a quadratic correlative corresp. be established between these spaces. Let AL be any line thro' A & L a pt on it. Corr. to L we get in (S') a pt L' which is the intersection of the corr. ray A'L' with the ple π' of the ~~corr.~~ quad. corr^n corr. to L.

This transf. has the symbol $\begin{Bmatrix} 3-3 \\ 3-3 \end{Bmatrix}$.
The proof of this proceeds on the same lines as that of previous article.

It will be noted that these methods can be adapted to the consideration of non-birational transf. between spaces between the pts of which higher correspondences such as (2-2) (3-3) (2-3) etc. exist. In fact they are of general application to all transformations.

Note on Transformations between two Surfaces & the Transformation of a surface into itself.

The collinear transf. of a conicoid into itself & the quadratic transf. of a conicoid into itself are particular cases of a type of transf. between two surfaces.
These transf. resemble plane transformations in their general characteristics.

I. The collinear transf. between two conicoids.

Let S_2 & S_2' be two conicoids & let A & A' be pts on S_2 & S_2' respectively. Suppose that A & A' are the vertices of collinear pencils. A ray of pencil A meets S_2 in a pt P & the corresp. ray of pencil A' meets S_2' in a pt P'. P & P' are now to be regarded as corresponding pts on the conicoidal surfaces.
If P describes a curve on S_2 P' will describe a curve on S_2' which is the correspondent of the curve on S_2. Exactly as in ple transf., there will be F-pts & lines on both surfaces. The correspondence between the pts of the two surfaces is clearly (1-1).

Thro' A there pass 2 gens. of S_2 & thro' A' two gens. of S_2'. Let the gens. AL_1, AL_2 of S_2 corr. to the gens. $A'L_1'$, $A'L_2'$ of S_2'. The tangent ple at A of pencil A therefore corresp. to the tangent ple at A' of pencil A'.

Corr. to pts very close to A on S_2 we get pts close to A' on S_2' ie corr. to A we get A'.
Corr. to a pt in either AL_1 or AL_2 we get the lines $A'L_1'$ or $A'L_2'$ & vice versa.
Thus the pts on the four gens. are exceptional pts in the transf. between the two surfaces.

Corr. to a gen^r of S_2 meeting AL_1 we get a gen^r of S_2' meeting $A'L_1'$ & similarly to a gen^r meeting AL_2 a gen meeting $A'L_2'$.
Corr. to a conic of S_2 we clearly get a conic on S_2'.
Corr. to a curve of order n meeting AL_1 & AL_2 in k & l pts respectively where $k + l = n$ we get a curve of order n which is the inters. of a cone of order n vertex A' & the conicoid S_2'. The cone & conicoid have as part of intersection the gens. $A'L_1'$ $A'L_2'$ & these account for a part of degree $k+l$ or n. The remaining part is therefore of degree n. It will intersect $A'L_2'$ & $A'L_2'$ in k & l pts respectively.
Thus the transf. between the pts. of the two surfaces is linear.

If the surfaces are superposed there will be 4 dps (excluding A & A'), namely the points in which the cubic curve which is the locus of the meets of corr. rays of the pencils, meets the common conicoid.
This agrees with the result of the collinear space transformation.

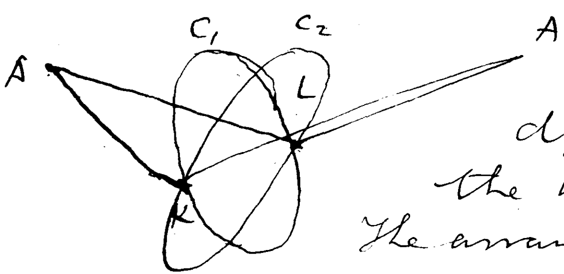
The study of this case is of special interest.
Let the gens. AL_1 & $A'L_1'$ belong to opposite systems
AL_2 $A'L_2'$.

then AL_1 $A'L_1'$ meet in a point K & AL_2 $A'L_2'$ meet in a pt L.

A gen^r of one system transforms now into a generator of the opposite system.

Consider the locus of the intersection of corresp. generators which meet AL_1 & $A'L_1'$ respectively. These lie in ples which belong to sheaves whose bases are AL_1 & $A'L_1'$ respectively. The locus of the intersection of the corr. ples of these sheaves is a ple for the ple AL_1A' corr. to the ple $A'L_1'A$ & these ples are identical. Hence the locus of ~~corr~~ gens the inters. of corr. gens. is a conic which clearly passes thro' K & L

Similarly the locus of the intersection of corr. gens. which meet AL_2 & $A'L_2'$ is another conic thro' K & L.



The pts K & L are clearly dps of the transf. & are the only dps. Thus in this case the transf. possesses only 2 dps. The arrangement is shown in diagram.

We can make use of these conics to find the corr. of a pt P on the conicoid. The gens. thro' P meet C_1 & C_2 in two pts one on each conic. The gens. thro' these pts intersect again in P' which is clearly the correspondent of P.

I clearly get two corresp. for P according to the pair of pts we select on C_1 & C_2.
Again the conics C_1 & C_2 may coincide & in this case there is a conic of d.ps. viz the common conic.
The transf. is now clearly involutive for corr. to P we can only get one pt P'.
Second let the gens. AL_1 $A'L_1'$ etc belong to same system then gens. of one system transform into gens. of the same system.

Let us now suppose further that the vertices A & A' & the gens AL_1 $A'L_1'$ AL_2 $A'L_2'$ coincide respectively. Then the ple AL_1L_2 is a d.ple of the coincident pencils & there are two other d.ples thro' AL_1 & AL_2 resp. These d.ples intersect the conicoid in 2 gens. one meeting AL_1 & the other AL_2.
The four gens – AL_1 AL_2 & the two just obtained are now double lines of the transf. They form the edges of a tet^h.
If AL_1 be made to coincide $A'L_2'$ & AL_2 with $A'L_1'$ the pencils will have a ple of double lines. This ple meets the conicoid in a conic which is a conic of d.ps. In the first case the gens. of one system transf. into gens. of the same system; in the second gens. of one system trans. into gens. of the opp. system these intersecting on the conic of d.ps.

These results we have already obtained.

The quadratic transf. between two conicoids.

Let the conicoids & pencils be chosen as in the collinear transf. but we no longer make the supposition that the gens. thro' A corr. to the gens. thro' A'. The pencils are still to be collinear.

The F- systems.

Corr. to pts very close to A we get pts on a conic thro' A'. Hence corr. to A we get ~~the~~ a conic thro' A' which is an F-conic & will be denoted by Σ_2'. A is a F-pt of second order on S_2. Similarly A' is a F-pt of second order on S_2' & its corr. is a conic thro' A the F-conic Σ_2.
Corr. to the gen^n AL_1 we get a pt on Σ_2', B'
& corr. to gen^n AL_2 - - - - Σ_2', C'
B' & C' are F-pts of first order & their corr. are the gens. AL_1 AL_2. Similarly there are two F-pts on Σ_2 B & C whose corr. are $A'L_1'$ & $A'L_2'$ resp.

Corr. to a gen^n of S_2 meeting AL_1 we get a conic on S_2' thro' A' & B'. Corr. to a gen^n meeting AL_2 we get a conic thro' A' & C'. The gens. of system AL_1 transform into conics thro' A' & C' & the gens. of system AL_2 into conics thro' A' & B'. A conic thro' A'C' & one thro' A'B' intersect in a pt, which corr. to the pt of inters. of their corr-gens. on S_1.

r. to a conic on S_2 which meets Σ_2 in 2 pts & AL_1 & AL_2 in one pt each we get a quartic curve on S_2' having a node at A' of second order & passing thro' B' & C'. This can be easily verified independently by the geometry of the systems.

Cor: To a curve of order n ~~having~~ intersecting Σ_2 in n pts (no one of which is A B or C) & ~~At~~ meeting AL_1, AL_2 in k & l pts resp. where $k + l = n$ we get a curve of order $2n$ having a node of order n at A' & nodes of order k & l at B' & C' respectively. Thus the curve meets Σ_2 in ~~R~~ $n + k + l$ i.e. $2n$ pts & its order is consequently $2n$.

The tangents to the various branches of this curve thro' A' must clearly lie on the tangent ple to S_2 at A'. The curve must also meet the gens. of each system in n pts each.

A curve of order n having a node of order δ at A one of order μ at B & one of order ν at C transforms into a curve of order

$n' = 2n - 2\delta - \mu - \nu$.

Since the curve of order n meets Σ_2 in n pts δ of which are at A μ at B & ν at C it follows that its cor. curve of order n' has a node of order $n - (\delta + \mu + \nu)$ at A'. Since the curve of order n meets a conic thro' A & B in n pts δ of which are at A & μ at B it follows that the curve of order n' must meet the cor. system of gens. of S_2' in $n - (\delta + \mu)$ pts & the gens. of the opp. system in $n - (\delta + \nu)$ pts.

ample a cubic curve not thro' A B or C & meeting he gens. of one system in 2 pts & the gens. of the other System in 1 pt transforms into a sextic having a node of order 3 at A', one of order 2 at B' or C' & an ordinary pt at C' or B'. This sextic meets the gens. of S_2' in 3 pts each.
This sextic is of def.y zero for projecting it from any pt on S_2' we get a ple curve having 3 triple nodes & a double pt equivalent altogether to 10 dps.

A quartic of def. 1 meeting each ge system of gens. on S_2 in 2 pts each transforms into an octic curve on S_2' having a node of order 4 at A' & nodes of order 2 at B' & C'. The octic also meets each System of gens. on S_2' in 4 pts. It is also of def.y 1 for projecting from any pt we get a ple curve with 3 quadruple nodes & 2 double nodes equivalent altogether to 20 dps. This is 1 short of the full number.

A curve of def.y k on S_2 transforms into a curve of def.y k on S_2'. The def.y of a curve is unaltered by this transf.. This can easily proved generally by considering the projections of a curve of order n & its corr. curve of order n' from A & A' respectively on any arbitrary ples. The ple curves to which they give rise are clearly of the same deficiency by the mechanism of the transf. & consequently the space curves are of the same deficiency. The ple curves have the same def.y because the projecting cones have

the same deficiency..
The transf. is a birational transf. between the pts of the two unicursal surfaces & is precisely analogous to the birational transf. between the pts of two ples.

Superposition of the conicoids.
If the conicoids be made to coincide there will be certain dps on the common surface.
Corr. lines of the pencils intersect in a curve of order 3 which intersects the conicoid in 4 pts besides A & A'. Hence there will be 4 dps for this transf. This is the same as the number of dps of a ple quadratic transf.
Here is a further analogy with the ple. quad. transf. In the above transf. the locus of the intersection of corr. gen. & conic is a cubic curve just as the locus of a line & its corr. conic in the ple is a cubic curve.

III. The quartic transf. between two unicursal surfaces.
Let now the pencils vertices A & A' be quadratic pencils with F-lines AD, AE AF A'D', A'E', A'F' resp.

The F-systems.
Corr. to the pt A on S_2 we get a unicursal quartic having a node at A' & passing thro' D'E'F'. The pt is a F-pt of order 4 & its corr. is a quartic F-curve Σ_4'. Corr. to the gens. thro' A, AL_1 & AL_2 we get two pts B' & C' on this quartic

ce on S_2 the complete F-system is composed of a quartic Σ_4 thro' A (dp) B, C, D, E, F ; the two gens. AL_1 AL_2 & 3 conics ADE, AEF, AF
The F-pts are A (of 4^{th} order) B C (of first order & D E F (of second order), for to the latter correspond conics on S_2' viz the conics A'E'F' etc.
A similar system exists on S_2'.
The triangles of conics DEF, D'E'F' we shall call the F-triangles.

A gen. of S_2 meeting AL_1, Σ_4 (in 2 pts) & the sides of the F-triangle in 1 pt each transforms into a quartic on S_2' having a node at A' & passing thro' B' D'E'F'.
A gen. of opp. system transforms into a similar quartic having a node at A' & passing thro' C'D'E'F'.

A conic transforms into an octic curve having a node of order 4 at A' which passes thro' B' & C' & has also nodes of second order at D'E' & F'.

Since two gens. of the same system cannot intersect it follows that two quartics having nodes at A' & passing thro' the same four pts B'D'E'F' say cannot intersect in any other pt. This conclusion can be easily verified from the quadratic transf. in which such curves corr. to conics on a ple.
Since two conics on S_2 intersect in 2 pts, it follows that 2 octics thro' A', B' C' D' E' F' must intersect in other 2 pts.

A conic on S_2 meets a quartic corr. to a gen. of in 4 pts it follows that an octic on S_2' must meet the gens. of each system in 4 pts.

Generally a curve of order n on S_2 meeting AL_1 & AL_2 in k & l pts resp. where $k+l=n$ & not containing any of the F-pts transforms into a curve of order $4n$ having a node of order $2n$ at A', nodes of order n at $D'E'F'$ & nodes of order k & l at B' & C' resp.
For a curve of order n meets the cone containing Σ_4 (quadric cone) in $2n$ pts & hence it can only meet the curve Σ_4 in $2n$ pts at most.

The transf. is therefore a quartic transf. It is reversible & is a birational transf. between the pts of the two conicoidal surfaces.
For reasons analogous to those given in the previous transf. it does not alter the deficiency of a curve.

Again corr. to a curve which has a node of order d at A. one of order μ at B, one of order ν at C one of order π at D, one of order ρ at E & one of order σ at E we get a curve of order

$$n' = 4n - 4d - \mu - \nu - 2(\pi + \rho + \sigma),$$

the curve being of order n.

Superposition of the conicoids.
When the two conicoids are superposed there will generally be 6 dps namely the pts in which the quartic curve of inters. of

orr. lines of the pencils meets the surface
excluding A & A'),

Particular cases of this transf.

Let the gen^r AL_1 be an F-line of the pencil vertex A, & let $A'L_1'$ be a F-line of the pencil A' corr. to the line AL_1.

The F-curves corr. to A & A' are now generators meeting AL_1 & $A'L_1'$ respectively.

Corr. to a pt. on AL_1 we get a conic thro' $A'D'E'$.

Corr. to pts DE we get gens. of S_2' meeting $A'L_1'$. Let these be called K_1' K_2'.

Let the corr. lines on S_2 be called K_1 K_2.

The complete F-system of curves on S_2 is the gen^r corr. to A' which we may call M the gens. K_1 K_2 & the conic ~~A~~ T_1 corr. to $A'L_1'$. There is a similar system on S_2'.

The F-pts on S_2 are ~~B where~~ A, D [illegible] E B & the F-pts on S_2' are A', D' & E' & B'

These F-pts are all of first order.

(B & B' corr. to gens. $A'L_2'$ & AL_2 resp.)

Corr. to gen^r meeting AL_1 we get a gen^r meeting $A'L_1'$ & corr. to a gen^r meeting AL_2 we get a cubic curve thro' $A'B'D'E'$ & intersecting $A'L_1'$.

Corr. to a conic we get a quartic thro' A' intersecting $A'L_1'$ twice (besides A') & passing thro' $D'E'B'$. This quartic meets the gens. of one system on S_2' in 3 pts & those of the opposite system in 1 pt. It has no singular pts.

· To a cubic curve meeting AL_1 in 2 pts AL_2 in 1 pt we get a quintic curve thro' A' $B'E'D'$ & meeting $A'L_1'$ in 3 pts (excluding A'). This quintic has no singular pts & meets the gens. of one system in 4 pts & those of opposite system in 1 pt.

Cor. to a cubic curve meeting AL_1 in 1 pt & AL_2 in 2 pts we get a septic curve having a node of order 2 at A' one of order 2 at B' E' D' also.
The septic curve meets the gens. of system $A'L_2'$ in 2 pts & those of set $A'L_1'$ in 5 pts.

Generally cor. to a curve of order n meeting AL_2 in $n-1$ pts & AL_1 in 1 pt (a unicursal curve without singular pts) we get a curve of order $3n-2$ ~~having~~ ~~meeting~~ having ~~on~~ a node of order $n-1$ at A' & nodes of order $n-1$ also at $B'E'D'$. It meets the gens. of set $A'L_2'$ in $n-1$ pts & those of opposite set in $2n-1$ pts.
Cor. to a curve of order n meeting AL_1 in $n-1$ pts & AL_2 in 1 pt we get a curve of order $n+2$ without singular pts & meeting the gens. of set $A'L_1'$ in $n+1$ pts & those of opp. set in 1 pt.

Thus this case is peculiar inasmuch as the order of the transformed curve depends on the nature of the intersections of the original curve with the

AL_1. It is a particular case of a de Jonquières transf
p 16 et seq.
both gens. ~~and~~ AL_1 AL_2 are F-lines of the pencil
A we get again a collinear transf. similar to
that of case I.

Higher transf. between the surfaces.

If we take two pencils which are related by a reversible cubic transf. (Cap II Thesis) we get a reversible transf. of order 6.

It is clear that in this way we can obtain an infinite series of transf. of even order generally.

General theory of trans. of order $2n$.

The F-systems.

Corr. to A & A' we have curves of order $2n$
hence A & A' are F-pts of $2n$th order.
Corr. to AL_1 & AL_2 ; $A'L_1'$ $A'L_2'$ we have
F-pts of first order $B'C'$, $B.C$.
Besides these we may have k_2 F-pts of second order, k_4 of 4th order & so on.
Corr. to these pts we have for ~~R~~ each of k_2 pts a conic, for each of k_4 pts a quartic & so on.

A genr. of S_2 transforms into a curve of order $2n$ having a node of order n at A', & passing thro' B' or C' according to its system.
It will have nodes of order i at each of the

F pts of order $2i$.

Two such curves of order $2n$ on S_2' must intersect in either 1 or 0 pts since their corr. gens. intersect either in 1 pt or 0 pts. Further a curve of order $2n$ must meet the gens. of each system on S_2' in n pts each. We consequently derive the following eqns.

$$n^2+n^2+n^2+\sum k_{2i}\, i^2 = 4n^2-1$$ ~~or $4n^2$~~

or $$n^2+n^2+n^2+1+\sum k_{2i}\, i^2 = 4n^2$$

The first case deals with the intersections of two curves corr. to two gens. of opp. systems. The second case deals with the intersections of two curves corr. to gens. of the same system, the extra term 1 denoting the pt B' or C' as the case may be

The eqns. are identical & both may be written

$$3n^2+\sum k_{2i}\, i^2 = 4n^2-1$$

or $$\sum k_{2i}\, i^2 = n^2-1 \qquad Ⓐ$$

Again a curve of order $2n$ meeting each ~~the~~ system of gens. in n pts is determined by

$$\frac{2n(2n+3)}{2} - \left[\frac{3n(n+1)}{2} + \underset{\text{or } 0}{1} + \sum k_{2i}\, \frac{i(i+1)}{2}\right] \text{pts}$$

This number must be 1 since a gen^n is determined by a single pt

hence

$$\frac{2n(2n+3)}{2} - \frac{3n(n+1)}{2} - \sum k_{2i}\, \frac{i(i+1)}{2} = 2 \text{ or } 1$$

or $$\sum k_{2i}\, \frac{i(i+1)}{2} = \frac{n(n+3)}{2} - \underset{\text{or } 1}{2} \qquad (B)$$

e formulae (A) & (B) are identical with the fundamental formulae for a ple Cremona transf. of order n.

The possibility of a transf. of order 2n of the type considered we may therefore regard as established.

The formulae of the transf. are in fact the same as the formulae of a Cremona transf. of order n with 3 F-pts of order n superadded.

If a gen^r thro' A or A' passes thro' a F-pt the character of the transf. is altered.
The curve which corr. to a gen^r varies with the system to which the gen^r belongs.
An example of this has been given in the case of the quartic transf.
In this case the order of the F-pts A & A' is depressed & becomes $<2n$.

It is easily established that a curve of order 2n on a conicoid cannot have a node of order $> n$.
For suppose it had a node of order $n+k$.
Let it meet the gen^r of each system in δ & μ pts where $\delta+\mu = 2n$ then one of δ μ must be $> n$ or both must be equal to n; hence projecting on to a ple we get a ple curve of order 2n having a node of order $k+n$ & one of order n or $> n$. But this is impossible hence the max order of a node is n.

also easily established that no curve of order 2n having a node of order n at A or A' meeting each system of gens. in n pts, can have another node of order n.

Hence the maximum value of i in the formulae (A)&(B) is n−1.

We therefore derive the following conclusion. If a curve of order 2n on a conicoid have a node of order n & meet each system of gens. in n pts then the order & number of its remaining singular pts supposing it to be a unicursal curve is the same as the number & order of the F-pts of a ~~oo~~ ple curve of order n which corresponds by a Cremona transf. to a line.

Finally we may remark that the formulae (A) & (B) may be written

$$\sum_{i=1}^{i=n} k_{2i}\, i^2 = 4n^2-1 \qquad (A)$$

where $k_{2n} = 3$

&

$$\sum_{i=1}^{i=n} k_{2i}\, \frac{i(i+1)}{2} = 2\frac{n(2n+3)}{2} - \underset{\text{or } 1}{2} \qquad (B)$$

where $k_{2n} = 3$

These are Cremona formulae for a transf. of order 2n.

Number of dps when the surfaces are superposed.

The locus of the intersections of corresp. rays of the pencils of order n is a curve of order n+2 thro A & A'.

curve meets the conicoid in $2(n+2)$ pts.
cluding A & A' which are not generally dps
we get $2n+2$ dps.
This is 2 more than the order of the transf.
& in this the transf. agrees with the
Cremona ple transf. of order $2n$.

Involutive point pairs.

Again the number of involutive point pairs is $\frac{2n(2n-1)}{2}$ or $n(2n-1)$. This number can be established exactly as in theory of the ple. transf.

The De Jonquières transf.

Let the system of F-pts $k_2 \dots k_{2(n-1)}$ be that of a de Jonquières transf ie let there be $2(n-1)$ F-pts of first order & a F-pt of order $n-1$.

Further let the gen^r AL_1 pass thro' the F-pt of order $n-1$ & similarly let $A'L_1'$ pass thro' the F-pt of order $n-1$ on S_2'.

Hence corr. to the $2(n-1)$ F pts on S_2 we get $2(n-1)$ gens. of S_2' meeting $A'L_1'$ & vice versa.
Corr. to a pt. on AL_1 we get a F-curve of order n meeting AL_1' in $n-1$ pts & gens. of opp. system in 1 pt. ~~& conversely~~ vice versa. $[\Sigma_n' \ \& \ \Sigma_n]$.
Corr. to a pt on AL_2 we get a pt B' on S_2'.
Corr. to pt A we get a F-gen^r of S_2' K_0' which meets $A'L_1'$ & there is a similar F-gen^r on S_2.
There is a further F-pt on each surface

mely the pts in which K_0 K_0' meet Σ_n Σ_n' espectively. Let these be called C & C'.

Thus on the whole we have a system of 2n F-pts & the F-pts A & A' & a system of 2n F-gens & a F-curve of order n on each surface.

Corr. to a gen^r meeting AL_1 we get a gen^r meeting $A'L_1'$.

Corr. to a gen^r meeting AL_2 we get a curve of of order n+1 meeting the gen^r $A'L_1'$ in n pts (since the corr. gen^r meets F curve corr. to $A'L_1'$ in n pts). & passing thro' the 2n F-pts.
Each curve of order n+1 meets the gens. of system of $A'L_1'$ in n pts & those of other system in 1 pt. It is unicursal & has no singular pts. Since a gen^r is det^d by 1 pt each curve will be determined by 1 pt & hence we infer that a curve, unicursal & without singular pts, on a conicoid, which meets the gens. in n & 1 pts resp. is determined completely by 2n+1 pts, if its order is n+1.
For a cubic n = 2 & 2n+1 = 5 hence a cubic curve is completely determined by 5 pts.
A unicursal quartic of type (3-1) is determined by 7 pts & so on.
A conic transforms into a unicursal non-singular curve curve of order n+2, passing thro' the 2n F-pts & of type (n+1, 1)

may therefore look on the transf. as de Jonquières transf. of order $n+1$ aving $2n$ F-pts of 1st order & one of the n^{th} order.

A curve meeting AL_1 (corr. to F-pt of n^{th} order) in μ pts & of order m transforms into a curve of order $m(n+1) - \mu n$ each pt of intersection with the genˡ. AL_1 having for correspondent the F-curve Σ_n' of order n.

Generally therefore a curve of order m meeting the genˡ. AL_1 in μ pts & AL_2 in d pts where $d + \mu = m$ transforms into a curve of order $m(n+1) - \mu n$ or $(m-\mu)n + m$ having a node of order d at B' & nodes of of order d at each of the other F-pts including A'. It has therefore $2n+1$ nodes of order d.

Again a unicursal curve of order m on S_2 meeting AL_1 in $m-1$ pts transforms into a unicursal curve of order $m+n$ meeting AL_1' in $m+n-1$ pts & passing thro' the $2n$ F-pts.

—,—

These methods can be applied to surfaces of higher order.

As an example we shall establish a transf. between the surface of a conicoid S_2 & the surface of a cubic having a ~~node~~ conical pt with 6 lines thro' this conical pt.

t take A' at the conical pt on the ic S_3',

Let the pencils vertices A & A' be collinear pencils.

The F-systems.

Corr. to A on S_2 we get a ple unicursal cubic on S_3' having a node at A' (Σ_3')

Corr. to the gens. AL_1 AL_2 we get the pts B' & C' on this cubic.

Corr. to A' we get a quartic curve on S_2 having a node at A & meeting each of AL_1 AL_2 in 2 pts.

Corr. to each of the six lines thro' A' we get 6 pts $K_1 \dots K_6$ on this quartic Σ_4.

A gen^l of S_2 transf. into a ple cubic curve on S_3' having its node at A' & passing thro' B' or C'.

A line of S_3' (there are 15 such) transforms into a conic thro' A & two of the pts $K_1 \dots K_6$.

A conic on S_2 transf. into a sextic on S_3' having a node of order 4 at A' & passing thro' B' & C'. This sextic meets each of the 15 lines on S_3' in 2 pts.

A conic on S_3' transf into a quartic on S_2 having a node at A & passing thro' 4 of the pts $K_1 \dots K_6$

A curve of order n on S_2 meeting AL_1 AL_2 in d & μ pts resp. where $d + \mu = n$

forms into a curve of order 3n having a node order 2n at A' & nodes of order d & μ at B' & C'. It also meets each of the 15 lines in n pts.

A curve of order m on S_3' transf. into a curve of order 2m on S_2 having a node of order m at A & nodes of order d at $K_1 \dots K_6$ where d is the number of pts in which the curve of order n meets each of the six lines thro' A'.
Thus the transf. is of order (3-2).

II. The case of quadratic pencils & the case of higher pencils can be similarly investigated.

The De Jonquières Transf. deduced geometrically.

Although the transfns. between the two conicoidal surfaces S_2 & S_2' have been deduced by means of two-dimensional pencils, they can also be deduced in a purely geometrical fashion.

Consider a singly infinite system of non-singular unicursal curves of order n, thro' 2(n-1) fixed pts on S_2, & meeting the gens. of one system in n-1 pts & those of the other system in 1 pt.
Let AL_1 be as before a gen^n meeting each curve of the pencil in n-1 pts.
We can construct a sheaf of ples

AL_1 which give rise to a singly infinite system of gens. which meet each curve in 1 pt.

The system of curves of order n can be obtained as the intersection with the conicoid S_2 of a sheaf of ruled surfaces of order $n-1$ having the gen.^{r} AL_1 as a line of multiplicity $n-2$.

Let us take on S_2' a gen.^{r} $A'L_1'$ & thro' it a sheaf of planes homographic with the above sheaf & a singly infinite system of curves of order n, unicursal without singularities, which is the intersection of S_2' with a sheaf of ruled surfaces of order $n-1$ as before.

Further let the two sheaves of surfaces of order $n-1$ be homographically related.

If S & S' are gens. meeting AL_1 & $A'L_1'$ respectively this can be secured if the two systems of curves describe homographic ranges on S & S' respectively.

~~The~~ Each sheaf of surfaces must pass thro' a fixed curve of order $(n-1)^2 - (n-2)^2$ ie $2n-3$. This fixed curve meets a conicoid in $2(2n-3)-(2n-4)$ fixed pts not on AL_1 or $A'L_1'$. This gives the $2(n-1)$ fixed pts above.

Then corr- to a pt P on S_2 which is the intersection of a gen. meeting AL_1 & a curve of system we get a corresp. pt P' on S_2' the inters. of corr. gen.^{r} & curve.

The F-systems.

~~plane thro' A~~

Corr. to each of the $2(n-1)$ base pts we get a gen.^{r} of S_2' meeting $A'L_1'$ hence these pts are F-pts of 1st order & their corr. lines F-lines.

Let them be K_0 -- K_{2n-2}.

Consider a gen.^{r} thro' K_0 meeting AL_1. No curve of the system can meet this gen. unless it be deg. & hence corr. to this gen.^{r} we get a pt on S_2' d_0'.... which corr. to the pt where the degenerate curve of order $n-1$ meets the gen.^{r} corr. to one thro' K_0. This degenerate curve is completely determined by the remaining $2n-3$ F-pts K_1 -- K_{2n-2}.

Hence we get on S_2' an additional $2(n-1)$ F-pts corr. to the $2(n-1)$ gens. thro' K_0 -- K_{2n-2}.

There are similarly on S_2 $2(n-1)$ additional F-pts. d_0 ... d_{2n-2}.

Corr. to a pt P on AL_1 we get no def. correspondent. We shall see presently what is the order of the curve corr. to a pt. on AL_1

Let a variable pt P describe a gen.^{r} of S_2 not intersecting AL_1. Corr. to any gen. thro' P of system opp. to AL_1 we get a single curve of system & corr. to a curve we get $n-1$ positions of gen.^{r} hence on S_2 the corr. gens. & curves generate on a gen.^{r} not meeting $A'L_1'$ an involution of order $n-1+n-1$ i.e. $2n-2$. This invol. has $2n-2$ dps & hence the locus of the inters. of corr. gens & curves is a curve of order $2n-1$ meeting gens. of system $A'L_1'$ in

2 pts & the gens. of opposite system in 1 pt.
This curve clearly passes thro' each of the $4(n-1)$ F-pts.

Corr. to a genⁿ meeting AL_1 we get a generator meeting $A'L_1'$.

Corr. to a conic on S_2 we get a curve of order $2n$ meeting gens. of system $A'L_1'$ in $2n$ pts & those of opp. system in 1 pt..
It also passes thro' each of the $4(n-1)$ F-pts

Corr. to a cubic meeting AL_1^* in 2 pts we get a unicursal non-sing. curve of order $2n+1$ thro' the $4(n-1)$ F-pts.

Corr. to a unicursal curve of order m meeting AL_1 in $m-1$ pts we get a curve of order $2n+m-2$ unicursal & non-sing.

From these results we infer that corr. to a pt. on AL_1 we get a curve of order $2n-2$ & that consequently any pt on AL_1 is a F-pt of order $2n-2$. For since the curve of order $2n-1$ above meets $A'L_1'$ in $2n-2$ pts the corr. genⁿ on S_2 must meet the curve corr. to $A'L_1'$ in $2n-2$ pts
Hence we may consider the transf. as a de Jonquières transf. of order $2n-1$
This is the most general transf. of this kind.
The previous transf. given on p 16 is a particular case in which the curves of order n pass thro' n fixed

on AL_1 & a fixed pt on AL_2. The surfaces of
der $n-1$ are in this case cones.

Again we may deduce a large number of special cases by making special suppositions concerning the correspondence between the sheaves of planes & the systems of curves.
For example if we suppose that to the deg. curve fixd pt on the genⁿ thro' K_0 corr. a deg. curve of same nature & to the genⁿ thro' K_0 the genⁿ thro' K_0', we get a diminution in the order of transf. of 1. The no of F-pts is also diminished by 2. If we make $n-2$ suppositions of this kind the order of the transf. is reduced to $n+1$ & the no. of F-pts to $4(n-1)-2(n-2)$ or $2n$. This is the case already given p. 16 et sq.

When the conoids are superposed there will be in general only 4 dps which do not lie on F-systems. For let AL_1 & $A'L_1'$ belong to same system. There will be two double gens. of the sheaves AL_1 & $A'L_1'$ & on each of these there be 2 dps.
There may also be a curve of dps.
For if AL_1 & $A'L_1'$ coincide & if the sheaves thro' AL_1 be congruent there will be a curve of order $2n$ meeting AL_1 in $2n-2$ pts & the gens. of opp. system in 2 pts. In this case the transf. is involutive.

J. F. Grant

Addition
The transformations of odd order between two conicoidal surfaces.

Suppose as in the general transf. of order $2n$ that we have two pencils vertices A & A' on each conicoid, related by a Cremona transf of order n having k_2 F-lines ~~pts~~ of 1st order k_4 of second order k_{2i} of order i & so on.
Let us suppose also that the gens. AL_1, AL_2 of conicoid (S_2) correspond to the gens $A'L_1'$, $A'L_2'$ of conicoid (S_2').
By means of the pencils we establish a $(1-1)$ corr. between the pts of the conicoids.

The F-system.

Corr. to A we get on (S_2') the intersection with (S_2') of a cone of order n of which $A'L_1'$ & $A'L_2'$ are generators. This curve is of order $2n-2$ & has a node of order $i-2$ at A where i depends on the F-system of the pencils. It meets each generator in $n-1$ pts & in addition it passes thro' the k_2 F-pts of 1st order k_4 of second order & so on.
Thus A & A' are F-pts of order $2n-2$.
The other F-pts are the k_2 pts of 1st order k_4 pts of second order & so on. The curves corr. to these pts are respectively conics, quartics etc. This we indicate by the suffixes 2, 4 etc.
Corresp. to a gen^l of (S_2) we get a curve of order $2n-1$ having a node of order $n-1$ at A & meeting the gens. $A'L_1'$

L_2' in n & $n-1$ pts or $n-1$ & n pts respectively according to the system to which it belongs. If the gen^n of S_2 meet AL_1 its corr. curve meets $A'L_1'$ in n pts; if it meet AL_2 its corr. curve meets $A'L_1'$ in $n-1$ pts. In addition these curves contain the Cremona F system of pts on (S_2').

Corresp. to a conic on (S_2) we get a curve of order $4n-2$ & so on & hence the transf. is of order $\underline{2n-1}$.

The Cremona formulae given for the transf. of order $2n$ are easily modified. For the F-system of the Cremona transf. of order n we have

$$\sum_1^{n-1} k_{2i} i^2 = n^2 - 1$$

& hence ~~$\sum k_{2i} i^2 + 2(n-1)^2 + (n^2$ or $n(n-1))$~~

$= (2n-1)^2$ ~~$+(0$ or $-$~~ $1)$.

$$\sum k_{2i} i^2 + (n-1)^2 + 2n(n-1)$$

$$= 4n^2 - 4n = (2n-1)^2 - 1$$

or $\sum k_{2i} i^2 + 2(n-1)^2 + n^2$

$$= (2n-1)^2$$

according as two curves of order $2n-1$ belong to opposite or the same systems.

Again

$$\sum k_{2i} \frac{i(i+1)}{2} = \frac{n(n+3)}{2} - 2$$

& hence

$$\sum k_{2i} \frac{i(i+1)}{2} + 2\frac{n(n-1)}{2} + \frac{n(n+1)}{2} =$$

$$= \frac{(2n-1)(2n+2)}{2} - 1 .$$

Appended are tables for all the transf. up to order 8.

$n=1$ $\quad k_2 = 0$

$n=2$ $\quad k_2 = 1$

$n=3$ $\quad k_2 = 4$

$n=4$ $\quad k_2 = 3$, $k_4 = 1$

$n=5$ $\quad k_2 = 4$, $k_4 = 2$

$n=6$ $\quad k_2 = 4$, $k_4 = 1$, $k_6 = 1$

$n=7$

$k_2 =$	6	3
$k_4 =$	0	3
$k_6 =$	2	1

$n=8$

$k_2 =$	6	3
$k_4 =$	0	3
$k_6 =$	1	0
$k_8 =$	1	1

For the odd transf. the number of dps when the surfaces are superposed is the same as the number of dps of the even transf. immediately above it.

The cubic transf.

The case of $n=3$ is closely associated with the tetrahedral cubic space transf.

In this case the pencils A & A' are quadratic pencils. Let B C D B'C'D' be the other F-pts. Corr. to each of A B C D we get conics thro' B'C'D' A'C'D' etc respectively. Corr to a gen^l of S_2 we get a cubic curve

S_2' thro' A'B'C'D'. Cor. To a conic of S_2 we get a sextic on S_2' having nodes of order 2 at A'B'C'D'.

This is the case of the space-cubic transf. of a conicoid thro' the 4 F-pts ABCD into another conicoid thro' A'B'C'D'.
With superposed spaces & superposed F-systems we get of a conicoid transf. into itself – a conicoidal transf. with 6 clps.

We may remark in conclusion that transf.^{ns} between the surfaces of two conicoids fall into two classes.
In the first class we have transf. analogous to ple Cremona transf. in which the F-pts are isolated from one to another.
In the second class (to which belong the De Jonquières transf.) we find transf. possessing F-pts which lie on a curve.
To each of pt of this curve there corresponds a definite F-curve of determinate order.
Such transf. resemble space transf. rather than ple transf.
As a simple example consider the transf. built up as follows.

Let K & L be two gens. of S_2 of same system & let K' & L' be gens of (S_2') of same nature.
Let us pass thro' $\underset{K'}{K}$ a sheaf of ples & thro'

L L' a sheaf of conicoids. The sheaf of conicoids also pass thro' a fixed cubic curve which intersects $\genfrac{}{}{0pt}{}{S_2,}{S_2}$ in 4 pts.

Let the sheaves of ples & the sheaves of conicoids ~~by~~ be now homographically related.

Let ~~If~~ P be a pt on S_2. Thro' P there pass a single ple of system K & a single conicoid of system L, π & C say. Corresp. to π & C we get a ple π' & a conicoid C' which intersect S_2' in a pt P'. For C' meets S_2' in a cubic curve of which K' & L' are chords & hence the ple π' will meet this cubic in one other pt P'. Thus we establish a (1-1) corresp. between the pts of the conicoids S_2 S_2'.

The gen. K & L are entirely composed of F-pts.

For corr. to a pt on K we get a cubic curve on S_2'.

Corr. to a pt on L we get a gen^r on S_2'.

The transf. is of order 4.

Instead of a fixed cubic we might have chosen a conic on $\genfrac{}{}{0pt}{}{S_2,}{S_2'}$ & a fixed line (not on $\genfrac{}{}{0pt}{}{S_2}{S_2'}$). In this case we would have a conic of F-pts.

Such lines or curves of F-pts we may call singular lines. The corresp. of any curve on the surface will have a degree modified by the number of its intersections with the singular lines or curves. In the above case an intersection with K means a loss of 3 in the degree of the corr. curve.

Supplementary note. (1)

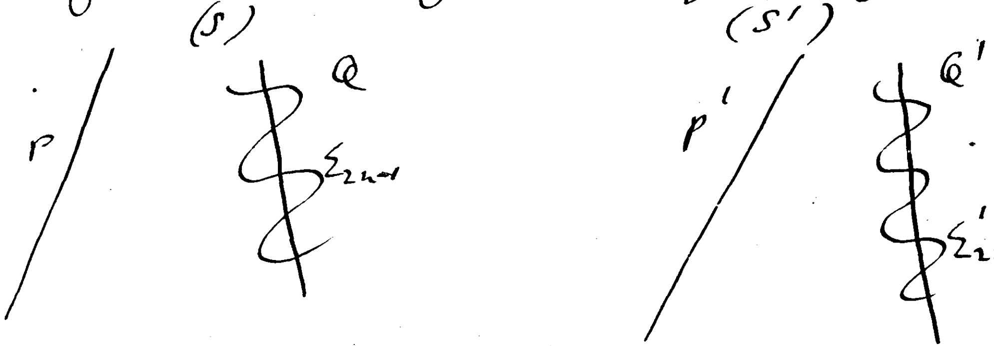
— 1 —

Generalisation of the transf. built up by two sheaves of ples & a sheaf of monoids (Cap IV).

The quintic transf. of Cap IV can be generalised as follows.

Consider two sheaves of ples & a sheaf of surfaces of order n having a line of multiplicity n-1, & a base curve of order 2n-1 of which the line is a chord of order 2n-2. Let this base curve be denoted by Σ_{2n-1}.

Let p q p' q' be the bases of the ple sheaves & let q q' be the base lines of mult. n-1 of the surfaces of order n.



By homographically relating the sheaves p p' q q' & the sets of surfaces of order n we establish a (1-1) corresp. between the pts of the spaces.
The transf. is of order 2n+1.
Corr. to a line of (S) we get the intersection of a hypl. (p'q') & a surface of order 2n having q' as line of mult. 2n-1. This is a curve of order 2n+1 of which p' & q' are

ds of order 2n. It is a unicursal curve.
esp. to a ple of (S) we get a surface of
rder 2n+1 in (S') having P' as a single line
Q' as a line of mult. 2n−1 & containing Σ'_{2n-1}.

The F-system.

The lines P Q Σ_{2n-1} of (S) are F-lines.
Corr. to a pt on P we get a line in (S').
meeting Q' & intersecting Σ'_{2n-1}.
Corr. to a pt on Q we get a curve of order 2n−1.
For consider a ple π thro' P meeting Q in pt A.
The pts in this ple transform into the pts of the
corr. ple by a De Jonquières transf. of order 2n.
Consider a small closed curve surrounding A.
Thro' any pt on it there pass a line thro' A &
a curve of order n having A as node of order
n−1. Corr. to any direction thro' A we get a
single curve of order n & corr. to a curve of order
n we get n−1 directions thro' A viz the
directions of the n−1 tangents at the node.
Hence the corresp. between the lines thro' A &
the curves is (n−1, 1). Hence corr. to the
small curve enclosing A we get a curve
generated by the intersection of a pencil of order
n−1 & a series of unicursal curves of order n,
This gives a curve of order 2n−1, which is in
the limit the corresp. of A.
Corresp. to a pt on Σ_{2n-1} we get a line
in (S') intersecting P' & Q'.
We may note that the curves of order 2n−1
corresp. to pts on Q have nodes of order 2n−2
on Q'.

am cor. To a line meeting P Q & Σ_{2n-1} we get pt in (S') & cor. to the ruled surface in (S) generated by such lines we get a F-curve in (S').

The ruled surface is of order 2n & the corr. F-curve is of order 4n-1. for it is the intersection of a ruled surface of order 2n generated by sheaves thro' P' & Q' between which there is a relation (2n-1, 1) & a ruled generated by a sheaf of planes thro' Q' & a sheaf of surfaces of order n between which there is a relation (n, 1). The latter surface is also of order 2n & both have the line Q' as a line of mult. (2n-1). Their curve of intersection is therefore of order

$$4n^2 - (2n-1)^2 = 4n-1$$

We shall call this F-curve Σ'_{4n-1}. There is a similar curve in (S) Σ_{4n-1}.

P P' are chords of order $4n-1-(2n-1)$ i.e. 2n & Q & Q' are chords of order 4n-2 for these curves.

The complete system of F-curves is therefore composed of

$$\begin{matrix} P, & Q, & \Sigma_{2n-1} & \Sigma_{4n-1}. \\ P' & Q' & \Sigma'_{2n-1} & \Sigma'_{4n-1} \end{matrix}$$

The F-system of surfaces.

Cor. to P we get a surface of order 2n having Q' as line of mult^y 2n-1.

Cor. to Σ_{2n-1} we get a surface of order 2n also having Q' as line of mult^y 2n-1.

Each of these surfaces clearly contains the F-curve Σ'_{4n-1} & the second surface also contains P' & Σ'_{2n-1}.

Cor. to Σ_{4n-1} we get a ruled surface of order 2n having Q' as line of mult. 2n-1 & containing

~~Σ'_{4n-1}~~ & P'!

Corr. to Q^F we get a surface of order $2n$ having Q' as a line of mult. $2n-2$ & containing P', Σ'_{4n-1} & Σ'_{2n-1}. For consider any ple ρ thro' Q; a line in this ple transforms into a conic in the corr. ple ρ' & hence a line parallel to Q & very close to it transforms into a conic in ρ'. Hence the section of the surface corr. to Q by ρ' is a conic & hence the order of the surface must be $2n$.

The complete system of F-surfaces therefore consists of four surfaces of order $2n$ each.

A line of (S) therefore transforms into a curve of order $2n+1$ meeting P' Q' Σ'_{2n-1} & Σ'_{4n-1} in $2n$ pts each.

A ple of (S) transforms into a surface of order $2n+1$ containing P' Q' ($2n-2$ times) Σ'_{2n-1} & Σ'_{4n-1}.

Number of lines on this surface.

A ple π of (S) meets the F-system in α β $\gamma_1 \ldots \gamma_{2n-1}$ & $\delta_1 \ldots \delta_{4n-1}$. α & β lying on P & Q resp.^{ly}, Hence there will be besides P' & Q' $2n-1+1+4n-1+(2n+4n-1)$ ~~lines~~ or $12n-2$ lines. The total number of single lines is therefore $12n-1$. Of these $6n-1$ intersect both P' & Q'. The lines excluding P' can be arranged in pairs each pair lying in a ple thro' Q'. Hence thro' Q' we can draw $6n-1$ ples to intersect the surface in line conics & these ples are tangent ples to the surface.

eneralisation of the transf. built up by sheaf of ples & two sheaves of conicoids. (Cap 10).

Consider two sheaves of ruled surfaces of order n having a common line of mult.^{y} n-1 & separate base curves of order 2n-1 each. Consider also a sheaf of ples thro' the common line of mult.^{y} n-1. Let these lines be called p & p'. By homographically relating the sheaves we build up a (1-1) corresp. between the pts of the spaces.

The order of the transf. is 4n-1.
Corr. to a line of (S) we get the intersection of a surface of order 2n having p' as line of mult.^{y} 2n-1 & another surface of order 2n also having p' as line of mult.^{y} 2n-1. The order of their curve of intersection is therefore $4n^2-(2n-1)^2 = 4n-1$.
Both of these surfaces are ruled surfaces & their curve of intersection has p' for a chord of order 4n-2.
Corr. to a ple of (S) we get a surface of order 4n-1 having p' as line of mult.^{y} 4n-3 & containing both base curves which we shall denote by ${}_1\Sigma'_{2n-1}$, ${}_2\Sigma'_{2n-1}$.

The F-line system.

Corr. to a pt on either of the base curves ${}_1\Sigma_{2n-1}$, ${}_2\Sigma_{2n-1}$ we get a line in (S'). These curves are F-curves.

to a pt on P we get a curve of order $4n-3$ having P' for a chord of order $4n-4$.
This can be shown in a perfectly general manner applicable to all transf. as follows.
Let any ple π cut P in pt A & the curves $_1\Sigma_{2n-1}$, $_2\Sigma_{2n-1}$ in $\alpha_1 \ldots \alpha_{2n-1}$, $\beta_1 \ldots \beta_{2n-1}$ & let another ple π' be taken in (S') intersecting the base system in A' $\alpha'_1 \ldots \alpha'_{2n-1}$, $\beta'_1 \ldots \beta'_{2n-1}$
Let P be any pt on π. Thro' P there pass a line thro' A & a curve of order n thro' $\alpha_1 \ldots \alpha_{2n-1}$. Cor. to P we get a pt P' on π' which is the intersection of the cor. line thro' A' & the cor. curve thro' $\alpha'_1 \ldots \alpha'_{2n-1}$. This cor. gives rise to a ple De Jonquières transf. of order 2n. Similarly by taking a curve of order n thro' $\beta_1 \ldots \beta_{2n-1}$ we get another De Jonq. transf. of order 2n.
Cor. to A in each of these transf. we get a curve of order $2n-1$ having a node of order $2n-2$ at A'. The number of intersections of these two curves is $(2n-1)^2-(2n-2)^2$ ie $4n-3$.
Hence the curve cor. to the pt A meets the ple π' in $4n-3$ pts. It is therefore of order $4n-3$.
Again if ρ be a ple thro' P a line in ρ parallel & very close to P transforms into a conic in ρ'. This conic along with the curve of order $4n-3$ must make up a curve of order $4n-1$ having P' as chord of order $4n-2$; hence the curve of order $4n-3$ must meet P' in $4n-4$ pts.

Again corresp. to a line intersecting $_1\Sigma_{2n-1}$ $_2\Sigma_{2n-1}$ & P we get a pt in (S'). This line generates a

face of order $4n-\overset{2}{\cancel{3}}$. For if we project the curves ${}_1\Sigma_{2n-1}$, ${}_1\Sigma_{2n-2}$ from any pt of P onto an arbitrary ple the projections intersect in $(2n-1)^2-(2n-2)^2$ i.e. $4n-3$ pts. Hence P is a line of order $4n-3$ on this surface. The surface is therefore of order $4n-2$.
Cor. to this surface we get a curve of order $8n-5$.

Any surface of the sheaf thro' ${}_2\Sigma_{2n-1}$ meets ${}_1\Sigma_{2n-1}$ in $3n-2$ pts. Thro' each of these pts let us pass a ple ch of sheaf P. This ple meets the surface thro' ${}_2\Sigma_{2n-2}$ in a single line & since this line must meet ${}_2\Sigma_{2n-2}$ in 1 pt only we get cor. to the surface thro' ${}_2\Sigma_{2n-2}$ $3n-2$ ples thro' P. Similarly corresp. to any surface thro' ${}_1\Sigma_{2n-1}$ we get $3n-2$ ples thro' P each ple containing a line which intersects ${}_1\Sigma_{2n-1}$ ${}_2\Sigma_{2n-1}$ & P. Hence the generating surfaces of the F-curve in (S') are each of order $4n-2$ & have P' as a line of mult^y $4n-3$. Hence the order of their curve of intersection is $(4n-2)^2-(4n-3)^2$ i.e. $8n-5$.
We shall denote these F-curves by Σ_{8n-5}, Σ'_{8n-5}.
The complete F-system of lines is therefore composed of

$$\begin{cases} P, \ {}_1\Sigma_{2n-1} \ \ {}_2\Sigma_{2n-1} \ \ \Sigma_{8n-5} \ \ (S) \\ P' \ {}_1\Sigma'_{2n-1} \ \ {}_2\Sigma'_{2n-1} \ \ \Sigma'_{8n-5} \ \ (S'). \end{cases}$$

The F-surfaces.

Cor. to ${}_1\Sigma_{2n-1}$ we get a surface of order $4n-2$ thro' ${}_2\Sigma'_{2n-2}$ P' & Σ'_{8n-5}.
Cor. to ${}_2\Sigma_{2n-2}$ we get a surface of order $4n-2$ thro' ${}_1\Sigma'_{n-2}$ P' & Σ_{8n-5}.

ch of these surfaces P' is a line of mult^y
4n-3.

Cor. to P we get a surface of order 4n-2 thro' ${}_1\Sigma'_{2n-1}$ ${}_2\Sigma'_{2n-1}$ Σ'_{8n-5} & P' on which P' is [illegible] a line of mult. 4n-4.

The F-system of surfaces therefore consists of four surfaces of order 4n-2.

Cor. to a line of (S) we get a curve of order 4n-1 in (S') intersecting ${}_1\Sigma'_{2n-1}$ ${}_2\Sigma'_{2n-1}$ & P' in 4n-2 pts each.

Cor. to a ple of (S) we get a surface of (S') having P' as line of multiplicity 4n-3 & containing ${}_1\Sigma'_{2n-1}$ ${}_2\Sigma'_{2n-1}$ & Σ'_{8n-5}.

Number of lines on this surface.

A ple π of (S) meets the F-system in
α $\beta_1 \dots \beta_{2n-1}$ $\gamma_1 \dots \gamma_{2n-1}$ $\delta_1 \dots \delta_{8n-5}$ α lying on P. Hence we get besides P' a number of lines on surface cor. to π equal to
$2(2n-1+2n-1+8n-5) = 24n-14$.
Hence we infer the existence of 24n-14 single lines on this surface.

As in the previous transf. these lines can be arranged in pairs each pair lying in a ple thro' P'. There are 12n-7 ples thro' P' which meet the surface in line conics & these are tangent ples to the surface.

This result & the corresp. result of the last transf. are particular cases of a general theorem on surfaces which I have deduced as follows.

urface of order n have a line of mult^y i.e. the number of tangent ples which can e drawn thro' the multiple line is 3n−4. Tangent ples whose pts of contact lie on the multiple line are excluded.

Let the axis of z be the multiple line then the eqn. of the surface is
$u \equiv x^{n-2} f_2 + y^{n-2} \phi_2 = 0$ where
f_2 & ϕ_2 are quadratic fns of xyz.
A ple $\frac{x}{\lambda} = \frac{y}{\mu} = s$ meets the surface in

the conic $\lambda^{n-2} f_2' + \mu^{n-2} \phi_2' = 0$
where f_2' ϕ_2' are quad. fns of z only.
The discriminant of the left side is of degree 3n−4 in λ & μ & hence we get 3n−4 positions of the ple $\frac{x}{\lambda} + \frac{y}{\mu} = s$ in which it is tangent to the surface.
If instead of n we write 4n−1 we get 3(4n−1)−4 i.e. 12n−7 : if instead of n we write 2n+1 we get 3(2n+1)−4 i.e. 6n−1.

The above series of transf^{ns} overlap. For example when n=1 we get the hyperboloidic cubic transf. in each case. When n=3 in 2n+1 & n=2 in 4n−1 we get a septic transf. These septic transf. do not differ essentially from one another.

The transf. of even order.

The transf. of even order are particular cases. For example to obtain the transf. of order $2n$ in the first series we take in (S) instead of a sheaf of surfaces of order n, a sheaf of order $n-1$ having Q as line of mult. $n-2$ & passing thro' a base curve of order $2n-3$. The base system of (S') is taken the same as before.

The transf. is now of order $2n$.
The F-curve in (S) is of order $4n-2$ & the F-curve in (S') is of order $4n-4$.
The lines Q & Q' are lines of mult. $2n-2$ on the surface of order $2n$.
The number of lines on the surface of order $2n$ is $12n-8$ & these lie in pairs on $6n-4$ tangent ples thro' Q or Q'.
Putting $n = 2n$ in the formula we get $3(2n)-4 = 6n-4$ as it should be.

On the whole therefore with the exception of the case $n=2$ we get a complete series of surfaces of integral order from $2 \rightarrow n$ which are rational & representable on a ple. The hyperboloid belongs to this series.

J. F. Tinto