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Essays on Decision Making under Variable Information

by

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To my beloved wife, *Yuanmei*, and
my dear daughter, *Yuxi*.

Abstract

This thesis is composed of three chapters that invoke axiomatic approaches to study models of decision making under objective and variable information.

In Chapter 1, we propose a model of choice from choice architectures that refer to environments where alternatives are presented with objective and observable choice-relevant information. We identify choice architectures by directed graphs on sets of alternatives where directed edges represent choice-relevant information about the alternatives. In this domain, a choice function hence singles out a vertex from each given directed graph, whereas a choice correspondence assigns to every directed graph a set of vertices. A choice function and choice correspondence are respectively characterized by a choice procedure that separates the role of information processing from that of preferences. Notably, both choice procedures suggest the same machinery of information processing that is hinged on properties of directed graphs, hence being objective and predictable. We then explore its implications on the formation mechanism of consideration sets and the sources of the stochasticity of choice. Later in the chapter, we also study the applications in terms of demand shaping and revealing equilibrium, respectively.

Chapter 2 considers decision making under uncertainty with objective and variable information structures. We take as primitive a family of information-dependent preferences over subjective acts indexed by partitions of the state space. Each partition corresponds to an information structure. We characterize a utility representation that comprises an affine utility index over simple lotteries, a unique capacity over the state space, and for each partition, a probability measure on the σ-algebra generated by the partition. We find that such a representation is equivalent to the Choquet expected utility representation with specific machinery of non-additive belief formation. We then connect the utility representation to the definition of comparative uncertainty aversion to explore the characteristic conditions related to the translatability of uncertainty attitude among variable information structures.

In Chapter 3, we explicitly incorporate framing of information into decision making under uncertainty. As in Chapter 2, we also study a family of partition-indexed preferences over subjective acts, where we interpret each partition as a frame of information.

Under a modest set of axioms, we characterize a general utility representation, which we call frame-adaptive expected utility. Having the general utility representation, we focus on two parameterized forms of frame-adaptive expected utility featuring attitude towards informativeness and degree of salience, respectively. We then apply the frame-adaptive models to the definition of comparative uncertainty aversion and that of definitive uncertainty-aversion to study the translatability of uncertainty attitude among variable frames of information. We also conduct a comparative analysis and find that the decision maker's reaction to information frames plays a role in modifying the degree of uncertainty attitude revealed from choices. Later in the chapter, we relate frame-adaptivity to ambiguity-aversion and argue that the latter can be viewed as a manifestation of the decision-maker performing frame-adaptive reasoning.

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Acknowledgment

As I rewind the memories, a lot has happened during my journey of Ph.D. It is indeed a tortuous yet memorable path, and it was possible to reach the end only with the support of my family and of many fantastic people whom I met during these years.

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Faithfully,

Glasgow
Dec 2020

Affidavit

I declare that, except where explicit reference is made to the contribution of others, that this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Printed Name: Hualin Li

Signature:

CHAPTER 1

Choosing from Graphical Choice Architectures

1.1. Introduction

The standard revealed preference theory considers choice from *sets*.[1] It evinces a parsimonious stance on what should be observed in the model, yet it has led us to experience identification problems when we try to understand bounded rationality, procedural rationality, and attention structures (e.g., Kahneman (2003); Manzini and Mariotti (2012a); Masatlioglu et al. (2012)). For example, once attention structures are employed to explain choice data, we often fail to separate preference and the strength of attention-grabbing. Even when the choice data satisfy the Weak Axiom of Revealed Preference (WARP), we still have at least two explanations: (i) the choice is obtained by a preference maximization (Richter (1966)), or (ii) an exogenous order on alternatives describing their strength of attention-grabbing determines the choice while preference is empty. Such identification problems seem to be inevitable when we insist on choice from sets and introduce subjective concepts to explain choice data. Moreover, since choice problems are entirely described by the given sets of alternatives (i.e., menus), the standard theory suggests only one-dimensional identification of choice problems. Hence, the standard framework might become uninformative about whether and how the behavioral properties that we observed in a choice problem translate to the relevant applications. For instance, in the game-theoretical context, it is often considered that the rules of a game influence players' choices solely through determining the way of strategic interaction among players, while the underlying preference of each player is fixed. However, when multiple games are under consideration, we are unable to tell if games with different rules also modify players' preferences or solely alter the form of strategic interactions.

In the real world, choices are often made in environments where many objective and observable features or information can influence them. Thaler et al. (2010) referred to those environments as *choice architectures*. For example, a DM might browse items through hyperlinks on a shopping site. In task scheduling, the designated rules in which one task necessitates another constrain the arrangements. Choice architectures give us

[1]For a comprehensive summary, see Chambers and Echenique (2016).

various choice observations even with a fixed set of alternatives, hence providing rich choice data that allow us to isolate more accurate descriptions of choice than what we used to in the domain of sets. That is to say, if we equip choice architectures with suitable analytical structures and take them as primitives, then in this rich domain, we might be able to separate the unconditional choices and explicate how others depend on exogenous information given by choice architectures. Moreover, once we unveil such an architecture dependency, it becomes possible to *lead the choice* via shaping the choice architectures to achieve desirable outcomes. Thus, why not utilize the richness?

We propose a model of choice from a class of choice architectures and study both choice functions and correspondences. Our model specifies choice architectures as directed graphs (digraphs) on sets of alternatives. This is because, apart from those architectures that have the structure of digraphs by nature (hyperlink connections, store layout, etc.), we might find that some architectures suggest specific orders over the alternatives (e.g., material conditionals, eligibility in task scheduling, rules of games), and such orders can be represented by directed graphs. A choice function hence singles out a vertex from every digraph, whereas a choice correspondence assigns a set of vertices to every digraph.

In the spirits of the Monotonicity axiom and the Independence of Irrelevant Alternatives (IIA) axiom in the context of choice from sets, we impose two sets of analogous axioms on our choice function and correspondence. They jointly characterize the choice function and correspondence as two specific procedures, both of which involve *sorting* alternatives. Such sorting obeys the acyclic part of the order induced by each given digraph and coincides with *topological sorting* of each digraph with a specific transformation. Intuitively, the choice function is a position-based selection from the sorted lists that admits the choice function from lists studied by Rubinstein and Salant (2006) (RS).[2] That is, given a choice architecture, the DM sorts the alternatives following the order induced by the architecture. She then picks the first or last most preferred alternative from the sorted list according to her preference. Meanwhile, the choice correspondence is described by the union of alternatives that survive a position-based elimination from each possible sorted list. The elimination is specified by a pair of transitive binary relations $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$ and incorporates the maximization of a partial order $\succ^*$ given by $\succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}}$. Given a choice architecture, the DM considers all possible sorted lists that obey the order induced by the digraph. From each sorted list, she eliminates the alternatives that are weakly dominated either by $\succsim_{\mathbf{0}}$ from the front or by $\succsim_{\mathbf{1}}$ from behind. The potential choices are then obtained by gathering the remaining alternatives from each sorted list. Notably, since the topological sorting of digraphs represents the sorting procedure, the sorted lists depend only on each given choice architecture,

[2]For a related model, see also Tversky and Kahneman (1991).

hence being objective and independent of the preference. Thus, our model achieves a physical separation of the preference maximization and the architecture dependency, and we subsequently explore the significance of such a separation.

In Section 1.5, we discuss the implications of our model. It is proved that the selection procedure of our choice function is compatible with the elimination procedure of our choice correspondence, which suggests two translation properties that link our choice function and correspondence. Moreover, our model implies a mechanism of shortlisting, which is closely related to the one suggested by the iterative search in Masatlioglu and Nakajima (2013). We also identify a possible source of the stochastic description of choice that can unify the approaches of preference maximization and stochastic choice (Manzini and Mariotti (2014)). We then present two strands of applications in Section 1.6, in terms of shaping demand and revealing equilibrium. We show that (i) the interested party can lead individual choice via shaping choice architectures, and (ii) our model can be introduced as an alternative formalization of games with discrete payoffs where our choice correspondence reveals the pure-strategy Nash equilibria as in Chambers et al. (2017).

The rest of the chapter is organized as follows. Section 1.2 includes the formal notations. In Section 1.3, we present the axioms and the characterization of the choice function. Section 1.4 provides the full characterization of the choice correspondence and investigates the rationalization of the induced choice correspondence on sets. The implications and applications are studied in Section 1.5 and Section 1.6. Section 1.7 discusses related literature. Proofs are concluded in Appendix 1.A.

1.2. Preliminaries

Let X be a finite set of alternatives and 2^X denote its power set. For a given positive integer $n \leq \#X$, the set of all n-element subsets of X is denoted by $[X]^n$. Let $D = D(V, E, \iota, \tau)$, simply $D = D(V, E)$, be the typical digraph on vertices $V \in 2^X \setminus \{\emptyset\}$ with edges $E \subset V \times V$, where E defines an irreflexive relation on V.[3] Whenever $E \neq \emptyset$, the mapping $\iota : E \to V$ maps each $e \in E$ to its first coordinate, while τ maps to the second coordinate. Let $\mathcal{D}$ be the set of all digraphs on the nonempty subsets of X. Given a D, the set of vertices and that of edges are denoted $V(D)$ and $E(D)$, respectively. Moreover, let $\mathcal{C}(D)$ and $\mathcal{P}(D)$ denote the sets of all cycles and paths in D. We write $uPv \subseteq D$ when the path $P \in \mathcal{P}(D)$ goes through u, v following the directions of every $e \in P$. Unless otherwise stated, we do not distinguish a set V from the digraph $D(V, \emptyset)$, and simply write V. Similarly, given $u, v \in X$, simply write

[3] That is, $D(V, E, \iota, \tau)$ does not allow any loops or multiple edges.

(u, v) to denote $D(\{u, v\}, \{(u, v)\})$. Given a D, its induced subgraph on $A \subseteq V(D)$ is denoted by $D[A]$.[4] Let $T : \mathcal{D} \to \mathcal{D}$ be the transitive closure operator. Denote by $\mathcal{A} : \mathcal{D} \to \mathcal{D}$ the mapping which, from every $D \in \mathcal{D}$, deletes all the edges contained in cycles in D. That is,

$$\mathcal{A}(D) := (V(D),\, E(D) \setminus \{e \mid \exists C \in \mathcal{C}(D), e \in E(C)\}) .$$

Unless otherwise stated, the induced subgraph operator always acts last for any combination of operators. For instance, $\mathcal{A} \circ TD[V] = (\mathcal{A} \circ TD)[V]$.

A digraph D is a *directed acyclic graph* (DAG) if $\mathcal{C}(D) = \emptyset$. A DAG S is a *string* if S is connected and satisfies $\#\{e \mid \iota(e) = v\} \leq 1$ and $\#\{e \mid \tau(e) = v\} \leq 1$ for all $v \in V(S)$. The set of all strings in $\mathcal{D}$ is denoted by $\mathcal{S}$. For a DAG D, a string $\varphi(D)$ on $V(D)$ is called a *topological sorting* of D if $\iota(e) P \tau(e) \subseteq \varphi(D)$ for all $e \in E(D)$. We identify each topological sorting by a mapping $\varphi : \{D \in \mathcal{D} \mid \mathcal{C}(D) = \emptyset\} \to \mathcal{S}$, and denote by $\Phi(D)$ the set of all topological sorting of a given DAG D (see Figure 1.1).

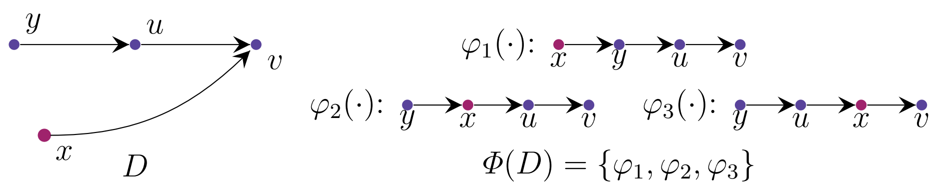

FIGURE 1.1 – Topological Sorting

1.3. CHOICE FUNCTION

In a collection of non-repeated observations, choice behavior is described by the choice function $z : \mathcal{D} \to X$ such that $z(D) \in V(D)$ for all $D \in \mathcal{D}$. We impose the following two axioms on z.

AXIOM 1.1 (Dominance of Acyclic Connectivity, DAC). *For any $V \in 2^X \setminus \{\emptyset\}$ and $v \in V$, if $z(D) \neq v$ for any connected DAG D on V, then $z(\tilde{D}) \neq v$ for all $\tilde{D}$ with $V \subseteq V(\tilde{D})$.*

We refer to the connected DAGs as "simple" architectures, since every connected DAG represents a nonempty partial order on the underlying set of vertices. DAC states that, given a menu, if an alternative is not chosen from all possible simple

[4] $D[A]$ contains all the edges of D which connects the vertices in A.

architectures on the menu, it would never be chosen from extensive menus regardless of the choice architectures. Conversely, if an alternative is chosen from a particular choice architecture, then for any of its sub-menus, it is possible to design an appropriate architecture from which this alternative would be chosen. Hence, DAC implies that the DM's choice must reveal her preference if it exists.

Given a D, we call a collection of digraphs $\{D_i\}_i$ an *induced partition* of D if (i) $\{V(D_i)\}_i$ partitions $V(D)$, and (ii) $D_j = TD[V(D_j)]$ for every $D_j \in \{D_i\}_i$.[5] Note that every member of an induced partition ($D_j \in \{D_i\}_i$) preserves all the information given in D since, for any $u, v \in V(D_j)$, if there is a $P \in \mathcal{P}(D)$ such that $uPv \subset D$, then $(u, v) \in E(D_j)$.

Axiom 1.2 (Independence of Induced Partition, IIP). *For any $D \in \mathcal{D}$ and any of its induced partitions $\{D_i\}_i$, if $v = z(D) \neq z\left(TD[\{z(D_i)\}_i]\right) = u$, then*
(i) $v \in \{z(D_i)\}_i \implies [(u, v) \in E(TD) \Leftrightarrow (v, u) \in E(TD)]$;
(ii) $v \in D_j \setminus \{z(D_j)\} \implies [(z(D_j), v) \in E(TD) \Leftrightarrow (v, z(D_j)) \in E(TD)]$.

IIP considers a compound choice that divides a choice architecture into several sub-architectures then chooses from the alternatives which are chosen from each sub-architecture and compares such a compound choice with that from the original architecture. It states that, *without loss of information*, the choice reversal is allowed only among those pairs of alternatives that are not simply intervened ($(u, v) \in E(TD)$ or $(v, u) \in E(TD)$ exclusively). Intuitively, for some pairs of alternatives, when a given choice architecture is restrictive or stimulative to neither of the alternatives ($(u, v), (v, u) \notin E(TD)$), then the choice might depend on how they are related to the others in the architecture and what is available on the menu. On the other hand, when the given choice architecture is stimulative (resp., restrictive) to both alternatives ($(u, v), (v, u) \in E(TD)$), then the DM should be free to implement either of those interventions in her choice. In connection with the IIA axiom, IIP requires that omitting the alternatives that are rejected under simple interventions from a choice architecture does not alter the choice. The following example gives a simple illustration of IIP.

Example 1.1 (IIP). Consider the digraph D in Figure 1.2, where TD is given by including the dashed edges and $\{C_1, S_1\}, \{\{y\}, D_1\}$ define two induced partitions of D. (i) Suppose $z(C_1) = y$ and $z(S_1) = v$ in $\{C_1, S_1\}$. Then IIP requires that $z(D) \neq x$ since x is not chosen from $\{v, x\}$ under a single edge. While it does allow $z(D) = u$

[5] Caution should be exercised in the definition of induced partition. In particular, each $D_j \in \{D_i\}_i$ is not necessarily the induced subgraph of D, as the induced subgraph operation is taken after the transitive closure. Hence, D_j becomes the induced subgraph only if $TD = D$.

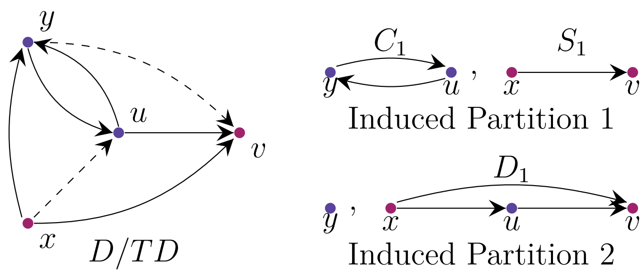

FIGURE 1.2 – Independence of Induced Partition

because the choice from cycles is not necessarily consistent. (ii) Suppose $z(D_1) = v$, then it must hold that $z(D) = z((y, v))$ for the induced partition 2.

Let $\succsim \subset X \times X$ be a connex and transitive binary relation and let $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ be an indicator function such that $\delta(x) = \delta(y)$ whenever $x \sim y$. Denote by $z_{\succsim,\delta} : \mathcal{S} \to X$ the choice function that picks the first or the last $\succsim$-maximal vertex from every $S \in \mathcal{S}$, when $\delta = \mathbf{0}$ or $\mathbf{1}$, respectively (Rubinstein and Salant (2006)).

THEOREM 1.1. *A choice function $z : \mathcal{D} \to X$ satisfies DAC and IIP if and only if there exist a unique connex and transitive binary relation $\succsim \subset X \times X$, a unique function $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$, and for each $D \in \mathcal{D}$, there is a topological sorting $\varphi_D \in \Phi(\mathcal{A} \circ TD)$ such that, for every $D \in \mathcal{D}$, $z(D) = z_{\succsim,\delta}(\varphi_D \circ \mathcal{A} \circ TD)$. Moreover, given $\succsim \subset X \times X$, this choice procedure is unique.*

PROOF. See Appendix 1.A.2.

REMARK 1.1. In the theorem, the binary relation $\succsim$ and the function $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ are given in accordance with RS. That is, $\succsim := \succ \cup \sim$ and $\sim := \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$, where

$$\succ := \{(u, v) \mid z((u, v)) = z((v, u)) = u\};$$

$$\sim_{\mathbf{0}} := \{(u, v) \mid z((u, v)) = u \wedge z((v, u)) = v\};$$

$$\sim_{\mathbf{1}} := \{(u, v) \mid z((u, v)) = v \wedge z((v, u)) = u\}.$$

Accordingly, for any $v \in X$, we assign $\delta(v) = \mathbf{1}$ if a $u \in X$ exists such that $v \sim_{\mathbf{1}} u$, and $\delta(v) = \mathbf{0}$, otherwise. Then, $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ is well-defined under IIP.

DAC and IIP characterize the choice function by the procedure that comprises the maximization of $\succsim \subset X \times X$ and the resolution of indifference. Concretely, the selection of topological sorting φ_D and the priority indicator δ govern the resolution of indifference. Given a choice architecture, the DM sorts the alternatives in a linear order that is compatible with the information given by the architecture. She then picks the most preferred alternative, which presents in the first or last in the sorted list, according to the preference and priority indicator. The following example provides a simple demonstration of this procedure.

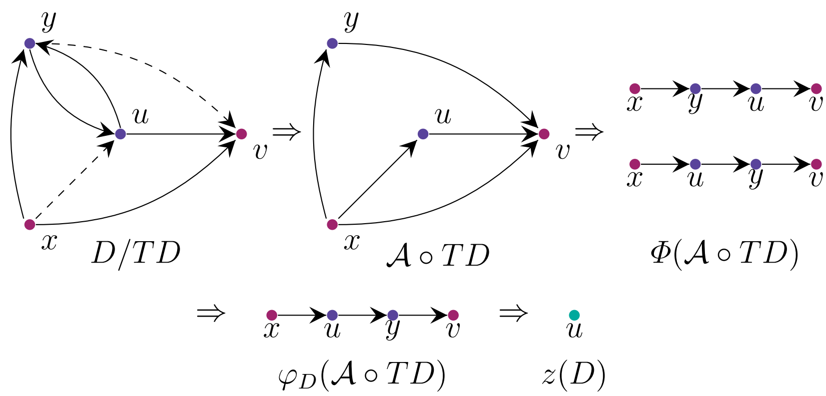

FIGURE 1.3 – Choice Function from Architectures

EXAMPLE 1.2 (Choice Function). Suppose $y \sim_{\mathbf{0}} u \succ v \sim_{\mathbf{1}} x$ and consider the choice architecture D in Figure 1.3. Since D induces two partial orders, the DM sorts $\{x, y, u, v\}$ into φ_D or $\overrightarrow{xyuv}$. Then, from one of the sorted lists, say φ_D, the DM picks the first $\succsim$-maximal alternative u as the indifferent class $\{u, y\}$ is endowed with $\delta(\cdot) = \mathbf{0}$.

For a given D, $\Phi(\mathcal{A} \circ TD)$ is predetermined yet non-singleton in general. As the choice procedure is not informative about the realization of topological sorting $\varphi_D \in \Phi(\cdot)$, it involves a systematic indeterminacy. The implication of such indeterminacy is discussed in Section 1.5.3.

1.4. CHOICE CORRESPONDENCE

In this section, we consider the collective observation in which choice behavior is described by a choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ such that $\emptyset \neq Z(D) \subseteq V(D)$ for any $D \in \mathcal{D}$. We postulate the following axioms on the choice correspondence.

AXIOM 1.3 (Dominance of Acyclic Connectivity*, DAC*). *For any $V \in 2^X \setminus \{\emptyset\}$ and any $u \in V$, if $u \notin Z(D)$ for all connected DAG D on V, then $u \notin Z(\tilde{D})$ for all $\tilde{D} \in \mathcal{D}$ such that $V \subseteq V(\tilde{D})$.*

Axiom 1.4 (Difference in Acyclic Difference, DAD). *For any $V \in 2^X \setminus \{\emptyset\}$ and any connected DAGs D and $\tilde{D}$ on V, $u \in Z(D), v \in Z(\tilde{D})$ implies $u, v \in Z(D \cap \tilde{D})$.*

The first two axioms regard the variation of choice architectures on a fixed menu, where DAC* is clearly the natural extension of DAC. Meanwhile, if we suppose that two different simple choice architectures $(D, \tilde{D})$ on a fixed menu (V) yield different choices $(u \in Z(D), v \in Z(\tilde{D})$ and $u \neq v)$, then it is reasonable to attribute the cause to the difference between those architectures. DAD hence states that, if we remove such difference and preserve only the interventions that those architectures have in common, then the new choice architecture $(D \cap \tilde{D})$ must admit all the alternatives that are chosen separately $(u, v \in Z(D \cap \tilde{D}))$. That is because, a plainer choice architecture must not contradict the reasons that lead to the choices from finer architectures, meaning that choices from a plainer architecture should be as flexible as those from finer architectures.

Axiom 1.5 (Independence of Induced Partition*, IIP*). *For any $D \in \mathcal{D}$, (i) $Z(D) = Z\left(TD\left[\bigcup_i Z(D_i)\right]\right)$ for all induced partition $\{D_i\}_i$; (ii) if $\#V(D) > 2$, and $\bigcup_i Z(D_i) = V(D)$ for every induced partition $\{D_i\}_i$, then $Z(D) = V(D)$.*

IIP* regards the variation of the induced subgraphs of a fixed choice architecture. Clearly, IIP*-(i) is the natural extension of IIP, while IIP*-(ii) rules out a specific situation in which some alternatives are eliminated from a choice architecture, with it being chosen from every possible member of induced partition, only if the architecture is presented as a whole.

The following theorem and proposition provide two equivalent characterizations of choice correspondence from architectures with different primitives. In Theorem 1.2, a unique pair of transitive binary relations $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$ and a unique filtration mapping γ characterize the choice correspondence. Meanwhile, in Proposition 1.1, the choice correspondence is characterized by the unique filtration mapping γ and a unique choice profile $\mathit{\Delta}$ that assigns the chosen alternatives directly to every potential edge.

Given a D, $\mathcal{Y}(D) \subseteq [V(D)]^2$ denotes the set of all pairs of vertices contained in the same cycles in D. That is, $\mathcal{Y}(D) := \{\{u, v\} \mid \exists C \in \mathcal{C}(D), u, v \in V(C)\}$. Let $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$ be a pair of partial orders on X. Denote by $Z_{\succsim_\delta} : \mathcal{S} \twoheadrightarrow X$ the choice correspondence that picks, from each string, all the vertices which are not dominated under $\succsim_{\mathbf{0}}$ from the front, nor under $\succsim_{\mathbf{1}}$ from behind. Formally, for any $S \in \mathcal{S}$,

$$Z_{\succsim_\delta}(S) := \left\{ v \in V(S) \middle| \forall u \in V(S), \begin{bmatrix} uPv \subseteq S \Rightarrow \neg(u \succsim_{\mathbf{0}} v) \wedge \\ vPu \subseteq S \Rightarrow \neg(u \succsim_{\mathbf{1}} v) \end{bmatrix} \right\}.$$

THEOREM 1.2. *A choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, and IIP* if and only if there exist a connex quasi-transitive binary relation $\mathcal{R} \subset X \times X$ with a unique decomposition $\mathcal{R} = \succsim_{\mathbf{0}} \cup \succsim_{\mathbf{1}} \cup \sim^*$ and a unique mapping $\gamma : [X]^2 \to [X]^1 \cup \{\emptyset\}$ such that, for all $D \in \mathcal{D}$,*

$$Z(D) = \left(\bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} Z_{\succsim_\delta} (\varphi \circ \mathcal{A} \circ TD) \right) \setminus \left(\bigcup_{\{u,v\} \in \mathcal{Y}(D)} \gamma(\{u, v\}) \right).$$

Moreover, $\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}}$ are transitive, and given $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$, the expression of Z is unique.

PROOF. See Appendix 1.A.3.

REMARK 1.2. The binary relations $\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}}$ and $\sim^*$ are specified as follows:

$$\succsim_{\mathbf{0}} := \{(u, v) \mid v \notin Z((u, v))\}\,; \quad \succsim_{\mathbf{1}} := \{(u, v) \mid v \notin Z((v, u))\}\,;$$
$$\sim^* := \{(u, v) \mid \{u, v\} = Z((u, v)) = Z((v, u))\}\,.$$

Then, $\sim^*$ is symmetric yet not necessarily transitive, while $\succsim_{\mathbf{0}}$ and $\succsim_{\mathbf{1}}$ are transitive. When $\succ^* := (\succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}}) \neq \emptyset$, it characterizes the *unconditional choice* in the sense that, if $u \succ^* v$, then v would never be chosen whenever u is available.

The theorem suggests that "selection" and "elimination" are behaviorally compatible when we consider the cumulative choice data. That is, given a choice architecture, the potential choice can be described by the collective outcome of a specific elimination. From every sorted list that obeys the order induced by the given digraph, the DM rejects an alternative if another alternative is listed in front (resp., listed behind) that weakly dominates it under $\succsim_{\mathbf{0}}$ (resp., $\succsim_{\mathbf{1}}$), or it is excluded by $\Gamma(D)$.

Note that the mapping $\gamma : [X]^2 \to [X]^1 \cup \{\emptyset\}$ assigns to each choice architecture a set of alternatives that would never be chosen from the architecture even if some of them might survive the elimination. That is, for particular choice architectures, γ systematically rules out some alternatives, as if the DM ignores them from those choice architectures. This observation is compatible with the notions of limited attention and limited consideration (e.g., Manzini and Mariotti (2014); Lleras et al. (2017)).

Let $\mathcal{E} := (X \times X) \setminus \{(x, x) \mid x \in X\}$ be the set of all potential edges on X. Let $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ be a nonempty correspondence. Denote by $Z_\Delta : \mathcal{S} \twoheadrightarrow X$ the choice correspondence that picks, from every string, the vertices which are neither the initials of edges with $\Delta(e) = \{\mathbf{1}\}$ nor the terminals of edges with $\Delta(e) = \{\mathbf{0}\}$ in TS. Formally, for any $S \in \mathcal{S}$,

$$Z_\Delta(S) := \left\{ v \in V(S) \,\middle|\, \forall e \in E(TS), \begin{bmatrix} \iota(e) = v \Rightarrow \Delta(e) \neq \{\mathbf{1}\} \wedge \\ \tau(e) = v \Rightarrow \Delta(e) \neq \{\mathbf{0}\} \end{bmatrix} \right\}.$$

Proposition 1.1. *A choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, and IIP* if and only if there exist a unique nonempty correspondence $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ and a unique mapping $\gamma : [X]^2 \to [X]^1 \cup \{\emptyset\}$ such that, for all $D \in \mathcal{D}$,*

$$Z(D) = \left(\bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} Z_\Delta (\varphi \circ \mathcal{A} \circ TD) \right) \setminus \left(\bigcup_{\{u,v\} \in \mathcal{Y}(D)} \gamma(\{u, v\}) \right).$$

Moreover, given $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$, the expression of Z is unique.

Proof. See Appendix 1.A.4.1.

Example 1.3 (Choice Correspondence). This example gives a demonstration of these equivalent choice procedures. Consider D given in Figure 1.4, where $\Gamma(D) = V(D) = \{x, y, u, v\}$. (i) Suppose $\Delta((x, u)) = \Delta((u, v)) = \Delta((y, v)) = \{\mathbf{0}, \mathbf{1}\}$ (blue edges), $\Delta((u, y)) = \Delta((y, u)) = \{\mathbf{0}\}$ (black edges) and $\Delta((x, y)) = \Delta((x, v)) = \{\mathbf{1}\}$ (red edges). Clearly, $\mathcal{A} \circ TD$ has topological sorting S_1, S_2. The DM eliminates x, u from S_1 since x is the initial of the edges labeled $\Delta(\cdot) = \{\mathbf{1}\}$ ($(x, y), (x, v)$), while u is the terminal of the edge labeled $\Delta(\cdot) = \{\mathbf{0}\}$ ((y, u)) in TS_1. Analogously, x, y are eliminated from S_2. As a result, $Z(D) = \{y, u, v\}$. (ii) Suppose $v \succsim_{\mathbf{1}} x, v \sim^* u, v \sim^* y, y \sim_{\mathbf{0}} u$ and $u \succsim_{\mathbf{0}} x$ in Figure 1.4. Again, the DM rejects x, u from S_1 as v is listed after x, satisfying $v \succsim_{\mathbf{1}} x$, while y appears before u, satisfying $y \succsim_{\mathbf{0}} u$. Similarly, the DM rejects x, y from S_2. Hence, $Z(D) = \{y, u, v\}$.

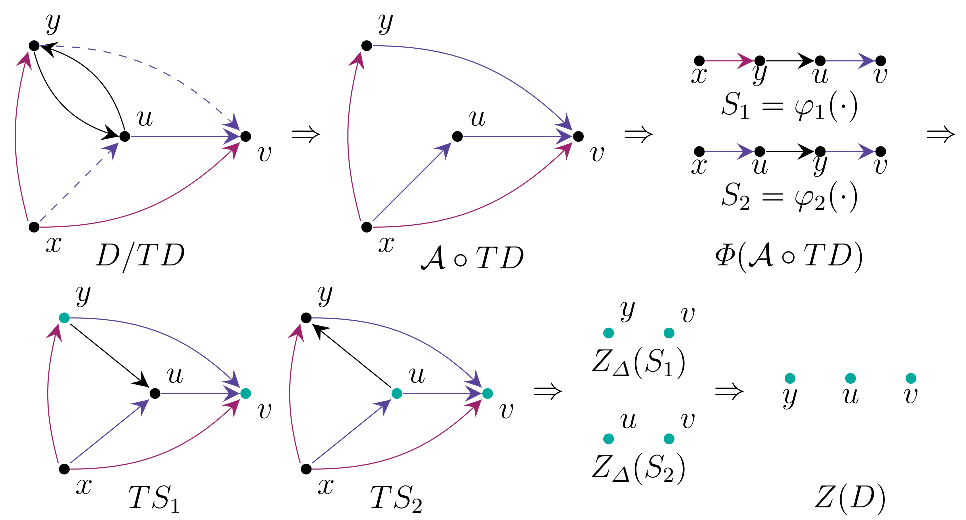

Figure 1.4 – Choice Correspondence from Architectures

In what follows, we establish a connection between the properties of choice from the architectures and those of choice from sets. To this end, we explicitly distinguish a nonempty subset $V \subseteq X$ from the digraph $(V, \emptyset)$ and consider the induced choice correspondence $Z^*(V) := Z((V, \emptyset))$ on nonempty subsets $V \subseteq X$. Then, $Z^* : 2^X \setminus$

$\{\emptyset\} \twoheadrightarrow X$ represents the choice from sets of alternatives without explicit structures. The next corollary shows that, under DAC*, DAD, and IIP*, Z^* is rationalized by a transitive binary relation.

Corollary 1.2.1. *If a choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD and IIP*, then a unique transitive binary relation $\succ^* \subset X \times X$ exists such that $Z^*(V) = \{v \in V \mid \nexists u \in V, u \succ^* v\}$ for all $V \in 2^X \setminus \{\emptyset\}$.*

Proof. See Appendix 1.A.5.

Note that the rationale $\succ^* \subset X \times X$ might lack explanatory power or normative implication as a $v \in X$ might exists such that $\neg(u \succ^* v)$ and $\neg(v \succ^* u)$ for all $u \in X$ even when $\# \succ^*$ is sufficiently large. The following property provides a sufficient condition under which every alternative is related to some alternatives regarding the rationale. We show that, under the following property, Z^* can be described by the maximization of a semiorder.[6]

Axiom 1.6 (Relevance). *For any $u, v \in X$, $Z((u, v)) = \{u, v\}$ implies $Z((v, u)) \neq \{u, v\}$.*

Relevance requires that, in binomial choices, the DM should be sensitive or sophisticated to either the alternatives or architectures, such that there is a simple architecture (in this case, a digraph on two vertices with a single edge) under which she consistently prefers one alternative over another. In terms of preference, the implication of Relevance is restrictive in the sense that, even if some alternatives are indifferent or incomparable, ties must be broken under some interventions.

Corollary 1.2.2. *If a choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, IIP*, and Relevance, then there is a unique semiorder $\succ^S \subset X \times X$ such that $Z^*(V) = \{v \in V \mid \nexists u \in V, u \succ^S v\}$ for all $V \in 2^X \setminus \{\emptyset\}$.*

Proof. See Appendix 1.A.5.

We now establish a connection between the properties of choice from architectures and WARP of classical choice correspondences. In the context of choice from sets, a choice correspondence $Z : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$ satisfies WARP if for any $A, B \subseteq X$ and any $x, y \in A \cap B$, $x \in Z(A)$ and $y \in Z(B)$ imply $x \in Z(B)$. Similarly, we consider a sufficient condition. The following property states that every binomial choice should

[6] See Luce (1956); Sen (1971); Jamison and Lau (1973).

consistently yield a single alternative whenever they are intervened by simple choice architectures (connected by a single edge).

Axiom 1.7 (Strong Relevance). *For any $u, v \in X$, $Z((u,v))$ and $Z((v,u))$ are singleton.*

Proposition 1.2. *If a choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, IIP*, and Strong Relevance then $Z^* : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$ satisfies WARP.*

Proof. See Appendix 1.A.4.2.

1.5. Discussion

1.5.1. More on Axioms

The axioms imposed on $z : \mathcal{D} \to X$ and $Z : \mathcal{D} \twoheadrightarrow X$ are of the UNCAF type proposed by Chambers et al. (2014). In addition, all of these except IIP* become *vacuous* when we focus on the choice from sets ($z^* : 2^X \setminus \{\emptyset\} \to X$ and $Z^* : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$), meaning that choice data from sets might not be sufficient to falsify properties of choice behavior. IIP* induces the following property in the restricted domain. Being akin to the IIA axiom, it states that eliminating some alternatives that are rejected from disjoint submenus does not alter the choice from the given menu.

Axiom 1.8 (IIP* on Sets). *For any $V \in 2^X \setminus \{\emptyset\}$, it holds that (i) $Z^*(V) = Z^*(\bigcup_{V_j \in \{V_i\}_i} Z^*(V_j))$ for any partition $\{V_i\}_i$ of V, and (ii) if $\bigcup_{V_j \in \{V_i\}_i} Z^*(V_j) = V$ for every non-trivial partition $\{V_i\}_i$, then $Z^*(V) = V$.*

In the characterization of $Z : \mathcal{D} \twoheadrightarrow X$, DAD acts as a normative imposition rather than a necessary condition. It is imposed since Theorem 1.2 has a strong normative implication for the rationalization of $Z^* : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$ (the induced choice correspondence from sets) when we consider the connection between Z and the choice from the sets. We present a characterization of Z without DAD. For every $D \in \mathcal{D}$, let $\mathcal{H}(D) \subseteq [V(D)]^2$ be the collection of 2-element sets of isolated vertices in D. Formally, $\mathcal{H}(D) := \{\{u,v\} \in [V(D)]^2 \mid TD[\{u,v\}] = \{u,v\}\}$.

PROPOSITION 1.3. *A choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC* and IIP* if and only if there exist a unique nonempty correspondence $\varDelta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ and a unique pair of mappings $\gamma_C : [X]^2 \to [X]^1 \cup \{\emptyset\}, \gamma_I : [X]^2 \to [X]^1 \cup \{\emptyset\}$ such that*

$$Z(D) = \left(\bigcup_{\varphi \in \varPhi(\mathcal{A} \circ TD)} Z_{\varDelta} \left(\varphi \circ \mathcal{A} \circ TD \right) \right) \cap \Gamma^*(D);$$

$$\Gamma^*(D) := V(D) \setminus \left(\left(\bigcup_{\{u,v\} \in \mathcal{Y}(D)} \gamma_C(\{u,v\}) \right) \cup \left(\bigcup_{\{u,v\} \in \mathcal{H}(D)} \gamma_I(\{u,v\}) \right) \right)$$

for any $D \in \mathcal{D}$. Moreover, given γ_C, γ_I and $\varDelta$ the expression of Z is unique.

PROOF. See Appendix 1.A.4.3.

1.5.2. Procedural Invariant and Sampling

Here, we study how our characterization of choice translates between the choice function (the non-repeated experiment that forces single choices) and the choice correspondence (sufficiently accumulated choice data). The following corollaries show that the selection $z_{\succsim,\delta} : \mathcal{S} \to X$ and the elimination $Z_{\succsim_\delta} : \mathcal{S} \twoheadrightarrow X$ are compatible, and every singleton sample of Z reveals the unconditional choice observed in Z.[7]

COROLLARY 1.2.3. *Let $\succsim := \succ \cup \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$ be the preference obtained in Theorem 1.1. Then a unique pair of partial orders $Q_{\mathbf{0}}, Q_{\mathbf{1}} \subset X \times X$ exists such that: (i) for every $u, v \in X$, $u \succsim v$ if and only if $uQ_{\mathbf{0}}v$ or $uQ_{\mathbf{1}}v$, and (ii) $Z_{Q_\delta}(S) = \{z_{\succsim,\delta}(S)\}$ for all $S \in \mathcal{S}$.*

PROOF. See Appendix 1.A.5.

COROLLARY 1.2.4. *Suppose that choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ and binary relation $\mathcal{R}$ follow the statement in Theorem 1.2. Let $\hat{z} : \mathcal{D} \to X$ be a sample of Z such that $\hat{z}(D) \in Z(D)$ for every $D \in \mathcal{D}$. Denote by $\hat{\succsim} := \hat{\succ} \cup \hat{\sim}_{\mathbf{0}} \cup \hat{\sim}_{\mathbf{1}}$ the binary relation defined by $\hat{z} : \mathcal{D} \to X$ following Remark 1.1. Then, (i) $(\succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}}) \subseteq \hat{\succ}$, and (ii) for any $u, v \in X$, if $u \succsim_{\mathbf{0}} v$ or $u \succsim_{\mathbf{1}} v$, then $u \hat{\succsim} v$.*

PROOF. See Appendix 1.A.5.

Regarding the translation of choice behavior, the followings are invariant between the choice function and correspondence: (i) the sorting of choice architectures and the

[7]Notice that $\hat{z} : \mathcal{D} \to X$ in Corollary 1.2.4 does not necessarily satisfy DAC and IIP. The corollary only requires those binary relations to be defined in the same way as in Theorem 1.1, according to the data given by $\hat{z}$.

selection/elimination procedure from each sorted list, and (ii) the unconditional choice ($\succ^* := \succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}}$) observed in data of sufficient size. Moreover, the unconditional choice cannot be underestimated theoretically in choice functions. Hence, when $\Gamma(D) = V(D)$ for all $D \in \mathcal{D}$, the choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ is a *twofold* extension of $z : \mathcal{D} \to X$. That is, all possible sorted lists (i.e., $\varphi \in \Phi(\mathcal{A} \circ TD)$) are considered in the cumulative observation, with the extended description of preference.

1.5.3. Consideration Set Formation and Source of Stochasticity

In recent papers, consideration sets have been explicitly derived as the results of models (Masatlioglu and Nakajima (2013); Caplin et al. (2019)).[8] Some choice-theoretical studies also give insight into the formation of consideration sets.[9] As with these papers, our model has implications for the formation of consideration sets. For every $D \in \mathcal{D}$, Theorem 1.1 implies that $z(D) \in \{z_{\succsim,\delta}(\varphi_D \circ \mathcal{A} \circ TD) \mid \varphi_D \in \Phi(\mathcal{A} \circ TD)\}$ and does not falsify any outcome from the selection of topological sorting $\varphi_D \in \Phi(\cdot)$. Given the preference $\succsim := \succ \cup \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$, the realization of φ_D uniquely determines the choice from D. Since the set $\Phi(\cdot)$ depends on D, $z_{\succsim,\delta}(\varphi_D(\cdot)) = z_{\succsim,\delta}(\varphi'_D(\cdot))$ might hold for some $\varphi_D, \varphi'_D \in \Phi(\cdot)$, or there might be a $\succsim$-maximal alternative $v \in V(D)$ such that $v \neq z_{\succsim,\delta}(\varphi_D(\cdot))$ for all $\varphi_D \in \Phi(\cdot)$. In addition, for some $D, \tilde{D} \in \mathcal{D}$ with $V(D) \subseteq V(\tilde{D})$, it is possible that $z(D) \neq z_{\succsim,\delta}(\varphi_{\tilde{D}} \circ \mathcal{A} \circ T\tilde{D})$ for all $\varphi_{\tilde{D}} \in \Phi(\mathcal{A} \circ T\tilde{D})$ and there is a $\varphi'_{\tilde{D}} \in \Phi(\mathcal{A} \circ T\tilde{D})$ such that $z_{\succsim,\delta}(\varphi'_{\tilde{D}} \circ \mathcal{A} \circ T\tilde{D}) \in V(D)$, as if $z(D)$ is ignored from $\tilde{D}$. These observations support the notion of consideration sets. Consequently, the set $K(D) := \{z_{\succsim,\delta}(\varphi_D \circ \mathcal{A} \circ TD) \mid \varphi_D \in \Phi(\mathcal{A} \circ TD)\}$ can be identified as the consideration set of a given $D \in \mathcal{D}$. Note that, (i) the set $K(D)$ is obtained as the result of preference maximization and summarizes different resolutions of indifference, whereas the consideration set studied in the literature serves as the subdomain of preference maximization; (ii) our model identifies the consideration sets by isolating its formation mechanism, which is often sidestepped in the literature.

Manzini and Mariotti (2014) proposed a model that links the preference maximization to the stochastic choice data, where the source of stochasticity is given by a probabilistic membership of consideration sets. That is, each alternative is considered with a fixed probability. In an analogy, our model signifies the realization frequency of

[8] Caplin et al. (2019) captured the formation of consideration sets by optimizing the Shannon model, whereas in Masatlioglu and Nakajima (2013), consideration sets are obtained by searching process. For other econometric and experimental approaches, see Mehta et al. (2003); Caplin et al. (2011).

[9] For instance, *shortlisting* in Manzini and Mariotti (2007) and *categorizing* in Manzini and Mariotti (2012a).

topological sorting as a source of stochasticity. Given a choice architecture D, Proposition 1.1 implies that each topological sorting $\varphi_D \in \Phi(\mathcal{A} \circ TD)$ uniquely determines a subset $\nu(\varphi_D) := Z_\Delta(\varphi_D(\cdot)) \subseteq V(D)$, where, for every $e \in E(T \circ \varphi_D \circ \mathcal{A} \circ TD[\nu(\varphi_D)])$, we have $\Delta(e) = \{\mathbf{0}, \mathbf{1}\}$. If we suppose the cumulative observation $Z : \mathcal{D} \twoheadrightarrow X$ is endowed with choice frequency data, then our model is related to the choice frequency in the following manner. Upon generating the data, each $\varphi_D \in \Phi(\cdot)$ might be realized with a probability $p(\varphi_D; D)$, and for every $(u, v) \in \{(u, v) \in \mathcal{E} \mid \Delta((u, v)) = \{\mathbf{0}, \mathbf{1}\}\}$, the image of $\Delta((u, v))$ might collapse to $\{\mathbf{0}\}, \{\mathbf{1}\}$ or $\{\mathbf{0}, \mathbf{1}\}$ with probability $q(\{\mathbf{0}\}; (u, v)), q(\{\mathbf{1}\}; (u, v))$ and $q(\{\mathbf{0}, \mathbf{1}\}; (u, v))$, respectively. Then, under suitable numerical extensions of our axioms, one might characterize the choice frequency by those probabilities, or represent all the frequencies as a function of alternatives (e.g., Luce (1959)). Notably, since $\nu(\varphi_D)$ only contains $\succ^*$-maximal alternatives for every $D \in \mathcal{D}$ and any $\varphi_D \in \Phi(\cdot)$, our model conjectures that the choice reveals to be stochastic because of the random tie-breaking. This implication is in analogy to Aguiar et al. (2016), where the choice is described by satisficing with fixed preference and random search orders.

1.6. Applications

In the abstract, our model can be interpreted as choice under exogenous irreflexive orders, which are identified as digraph architectures. The architecture dependency of choice can be exercised to a wide spectrum of choice-relevant studies due to the abundance of real-world objects, rules, and information that can induce orders over the alternatives in question. In what follows, we discuss two strands of applications in terms of demand and equilibrium.

1.6.1. Choice Leading and Manipulation

One major strand of application is that, with the architecture dependency being unveiled, interested parties can utilize the real-world objects or information (e.g., hyperlink connections, rules, user ratings) to lead the choice intentionally via shaping the choice architectures.[10] Here, we focus our argument on frequent choice situations, hence on the choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$. The following result shows the possibility of such choice leading.

[10] Manzini et al. (2011) studied the choice manipulation problem in two-stage threshold framework. In their paper, the manipulation is specified by varying one psychological variable with the other variables being fixed; hence, they are implicit on how the manipulation is exercised. For the two-stage threshold model, see also Manzini et al. (2013).

PROPOSITION 1.4. *Suppose $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, and IIP*. Then, for every nonempty $V \in 2^X$, there exist unique subsets $Core(V), Cl(V) \subseteq V$ such that (i) $Core(V) \subseteq Z(D) \subseteq Cl(V)$ for any $D(V,E) \in \mathcal{D}$ and (ii) for every $V^* \subseteq Cl(V)$, a connected DAG $D^*(V,E^*) \in \mathcal{D}$ exists such that $(Core(V) \cup V^*) \subseteq Z(D^*)$. Moreover, if $(Core(V) \cup V^*) \subset Cl(V)$, there is a $D_*(V,E_*) \in \mathcal{D}$, in which $\{e \in E_* \mid \iota(e) = v \vee \tau(e) = v\} \neq \emptyset$ for all $v \in V$, such that $(Core(V) \cup V^*) \subseteq Z(D_*) \subset Cl(V)$.*

PROOF. See Appendix 1.A.4.4.

In the proposition, for a given menu V, (i) clarifies the spectrum of manipulation in the sense that the alternatives in $V \backslash Cl(V)$ or in $Core(V)$ are rejected or selected by the DM solely according to her preference; hence, the variation of architecture cannot alter the choice involving those alternatives. Probing further to (ii), given any submenu in that spectrum, a connected DAG is sufficient to lead the choice to include the submenu, while the latter part of (ii) reveals that, by a certain class of architectures, one can also lead the DM *not to choose* particular alternatives. Incorporating the discussion on the source of stochasticity, architecture shaping might also be related to leading the choice frequency. We now demonstrate both the leading choice and, under some simplifying assumptions, the leading choice frequency.

EXAMPLE 1.4 (Leading Choice). Let a DM's choice behavior $Z : \mathcal{D} \twoheadrightarrow X$ satisfy DAC*, DAD, and IIP*. Consider the menu $V = \{u, v, x, \tilde{x}, y\} \subset X$, where it reveals $xR^I_{\mathbf{0}}\tilde{x}$, $vR^P_{\mathbf{0}}u$, $vR^P_{\mathbf{0}}x(\tilde{x})$, $vR^P_{\mathbf{1}}y$, $x(\tilde{x})R^P_{\mathbf{0}}u$, $y \sim^* x$ and $y \sim^* u$. By default, V is endowed with the architecture $D_0(V, E_0)$ given in Figure 1.5. Since D_0 has a unique topological sorting $\varphi_0(D_0) = \overrightarrow{xu\tilde{x}yv}$, we have $Z(D_0) = Z_\Delta(\varphi_0(D_0)) = \{x, v\}$. Suppose the interested party intends to ensure $u, \tilde{x}$ *would be chosen as well*. This could be managed by shaping the architecture to $D(V, E)$ in the figure. In fact, D has topological sorting $\varphi_1(D) = \overrightarrow{ux\tilde{x}yv}$, $\varphi_2(D) = \overrightarrow{u\tilde{x}xyv}$, and $\varphi_3(D) = \overrightarrow{u\tilde{x}yxv}$, which yield $Z_\Delta(\varphi_1(D)) = \{u, x, v\}$, $Z_\Delta(\varphi_2(D)) = Z_\Delta(\varphi_3(D)) = \{u, \tilde{x}, v\}$. As a result, $Z(D) = \bigcup_{i\in\{1,2,3\}} Z_\Delta(\varphi_i(D)) = \{u, v, x, \tilde{x}\}$.

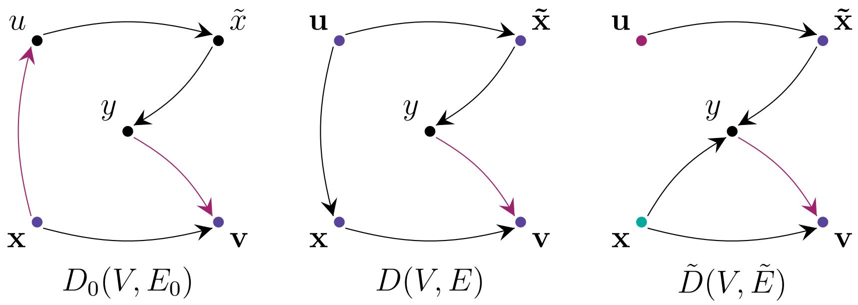

FIGURE 1.5 – Leading Choice (Frequency)

Example 1.5 (Leading Frequency). To simplify the argument, assume, in every i.i.d. choice instance and for every architecture $\hat{D}$, that: (i) the DM implements each topological sorting $\varphi \in \Phi(\mathcal{A} \circ T\hat{D})$ with equal probability and picks an alternative w uniformly from $Z_\Delta(\varphi(\cdot))$, and (ii) the realization of φ and the selection of w are i.i.d. Let $p(w, \hat{D})$ be the probability of w being chosen from $\hat{D}$. Then, it yields

$$p(w, \hat{D}) = \sum_{\varphi_i \in \left\{\varphi \mid w \in Z_\Delta(\varphi \circ \mathcal{A} \circ T\hat{D}))\right\}} \frac{1}{\left(\#\Phi(\mathcal{A} \circ T\hat{D})\right)\left(\#Z_\Delta(\varphi_i \circ \mathcal{A} \circ T\hat{D})\right)}.$$

Hence, for $D(V, E)$ in Figure 1.5, we have $p(x, D) = 1/9$ and $p(u, D) = 1/3$. In the figure, $\tilde{D}(V, \tilde{E})$ has topological sorting $\tilde{\varphi}_1(\tilde{D}) = \overrightarrow{xu\tilde{x}yv}$, $\tilde{\varphi}_2(\tilde{D}) = \overrightarrow{ux\tilde{x}yv}$, and $\tilde{\varphi}_3(\tilde{D}) = \overrightarrow{u\tilde{x}xyv}$. Notice that $Z_\Delta(\tilde{\varphi}_1(\tilde{D})) = \{x, v\}$, $Z_\Delta(\tilde{\varphi}_2(\tilde{D})) = \{u, x, v\}$, and $Z_\Delta(\tilde{\varphi}_3(\tilde{D})) = \{u, \tilde{x}, v\}$. Hence, we have $1/9 = p(x, D) < p(x, \tilde{D}) = 5/18$ and $1/3 = p(u, D) > p(u, \tilde{D}) = 2/9$. As a result, the interested party can *stimulate the frequency* of x and *reduce* that of u intentionally by shaping the architecture to $\tilde{D}(V, \tilde{E})$.

1.6.2. Choice Architectures as Games

Chambers et al. (2017) studied a sufficient condition under which the revealed preference formalization can reveal strategic group behavior, such as the Nash equilibrium. Similarly, in the following, we apply choice architectures as an alternative language of games with discrete strategies and present how our choice correspondence reveals the pure-strategy Nash equilibria and the competitive equilibrium.

Let A, B be two DMs and $V = \{u, v, x, y\}$ be the set of allocations. Suppose the DMs' preferences are given by the quasi-transitive binary relations $\succsim^A, \succsim^B \subset V \times V$. Given V, $\succsim^A$ and $\succsim^B$, define, for each DM i, a correspondence $\Delta^i : V \times V \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ and a mapping $\gamma^i : [V]^2 \to [V]^1 \cup \{\emptyset\}$ by

$$\begin{aligned} w \succ^i \tilde{w} &\iff \left[\begin{array}{c} \Delta^i((w, \tilde{w})) = \{\mathbf{0}, \mathbf{1}\} \wedge \Delta^i((\tilde{w}, w)) = \{\mathbf{1}\} \\ \wedge\, \gamma^i(\{w, \tilde{w}\}) = \{\tilde{w}\} \end{array}\right]; \\ w \sim^i \tilde{w} &\iff \left[\begin{array}{c} \Delta^i((w, \tilde{w})) = \{\mathbf{0}, \mathbf{1}\} \wedge \Delta^i((\tilde{w}, w)) = \{\mathbf{0}, \mathbf{1}\} \\ \wedge\, \gamma^i(\{w, \tilde{w}\}) = \{\emptyset\} \end{array}\right]. \end{aligned} \tag{$*$}$$

Let $D_A(V, E_A)$ and $D_B(V, E_B)$ be the digraphs given in Figure 1.6. Now, we consider the set of allocations $G(\{D_i\}_i, \{\succsim^i\}_i) := Z^A(D_A) \cap Z^B(D_B)$, where for $i = A, B$, $Z^i : \mathcal{D} \twoheadrightarrow V$ denotes the choice correspondence given in Proposition 1.1 with respect to (Δ^i, γ^i) defined in Eq $(*)$.

Example 1.6. (i) Suppose $x \succ^A u$, $u \succ^A v$, $v \succ^A y$, and $y \succ^B u$, $u \succ^B v$, $v \succ^B x$. Then, by Eq $(*)$, $Z^A(D_A) = \{x, v\}$, $Z^B(D_B) = \{y, v\}$. Hence, $G(\{D_A, D_B\}, \{\succ^A, \succ^B\}) = \{v\}$. (ii) Suppose $u \succ^A v, v \succ^A x, x \succsim^A y$ and $v \succ^B u, u \succ^B x, x \succsim^B y$ with

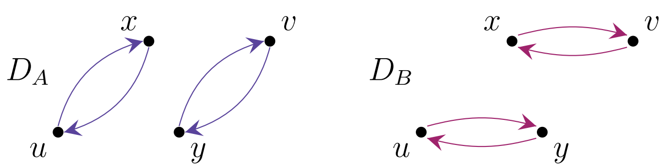

FIGURE 1.6 – 2×2-Games

$\succsim^A, \succsim^B$ being transitive. Then, we have $G(\cdot) = Z^A(D_A) = Z^B(D_B) = \{u, v\}$. (iii) Suppose $u \sim^A v, v \succ^A x, x \sim^A y$ and $x \sim^B y, y \succ^B u, u \sim^B v$ with $\succsim^A, \succsim^B$ being transitive. Then, $Z^A(D_A) = \{u, v\}$, $Z^B(D_B) = \{x, y\}$, and hence $G(\cdot) = \emptyset$.

Under D_A and D_B, the preference profiles $\succ^A$ and $\succ^B$ given in (i), (ii), and (iii) represent the *Prisoner's Dilemma*, *Battle of Genders*, and *Matching Pennies*, respectively. In each case, $G(\cdot)$ captures the pure-strategy Nash equilibria. Regarding this formalization, the followings should be noted.

REMARK 1.3. From each DM's perspective, the choice problem is not strategic per se, as it is degenerated to the choice from architectures. The strategic interaction, however, is implicitly captured by the following aspects.

(a) The choice objects are given as allocations of payoffs. Hence, for each i, the strict part of $\succsim^i$ might differ from i's strict preference over payoffs (e.g., $\succsim^i$ might incorporate envy).

(b) For each i, the intuition of (Δ^i, γ^i) in Eq $(*)$ is that, given $u \succ^i v$, if the game allows someone (including i) to alter the allocation from v to u, or it is free of alteration, then since i can always alter, u is the only possible allocation $(\Delta((v, u)) = \{\mathbf{1}\}, \gamma^i(\{u, v\}) = \{v\})$. If someone (including i) can exercise the alteration (u, v), then i must admit both u, v $(\Delta((u, v)) = \{\mathbf{0}, \mathbf{1}\})$, as other players might make the alteration even if i would not.

(c) In Figure 1.6, each $D_i(V, E_i)$ describes the possible alterations of i's that are allowed by the game. For example, $(x, y), (y, x) \notin E_A$ as these alterations require B to change her playing strategy.

Since the size of players is irrelevant to Remark 1.3-(a)(b)(c), the formalization and the property of $G(\cdot)$ can be naturally extended to any finite set of players N.

Let V be the finite set of allocations in question. The set of players is denoted by N, and each $i \in N$ is described by the quasi-transitive preference $\succsim^i \subset V \times V$. For each $i \in N$, define a correspondences $\Delta^i : V \times V \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ and a mapping $\gamma^i : [V]^2 \to [V]^1 \cup \{\emptyset\}$ following Eq $(*)$, and let D_i be the digraph on V with $E(D_i)$ containing all the edges on the allocations between which i can make the alteration. We consider the game described by $\mathcal{G} = (V, N, \{(\Delta^i, \gamma^i)\}_{i \in N}, \{D_i\}_{i \in N})$ and the set $G(\mathcal{G}) := \bigcap_{i \in N} Z^i(D_i)$, where each $Z^i : \mathcal{D} \twoheadrightarrow V$ is the correspondence given in Proposition 1.1 with respect to (Δ^i, γ^i).

PROPOSITION 1.5. *$G(\mathcal{G})$ is the set of all pure-strategy Nash equilibria of $\mathcal{G}$.*

PROOF. See Appendix 1.A.4.5.

Suppose D_i is the complete digraph for each $i \in N$, then by Eq $(*)$, each $Z^i(D_i)$ coincides with the maximization of $\succsim^i$. Hence, this formalization can also accommodate the individual choice with $G(\cdot) := \bigcap_{i \in N} Z^i(D_i)$ revealing the competitive equilibrium. Moreover, given a set of allocations V and the preference profile $\{\succsim^i\}_{i \in N}$, Proposition 1.4 implies that, if $\bigcap_{i \in N} Cl^i(V) \neq \emptyset$, then there exists a game $\{D_i\}_{i \in N}$ that has a pure-strategy Nash equilibrium. Similarly, if $\bigcap_{i \in N} Core^i(V) \neq \emptyset$, then both types of equilibrium always exist for any form of game.

Integrating the leading choice with the above discussion, given a discrete set of allocations and the preference profiles of N players, a social planner can lead a desirable Nash equilibrium to exist (e.g., mechanism design or implementation) by designing specific rules D_i as choice architectures for each $i \in N$. The related axiomatization then becomes equivalent to such an architecture design. A straightforward example is the rights structures studied by Koray and Yildiz (2018).

1.6.3. Price as A Signal

Price is also an example of the information that induces orders (not necessarily linear) over the alternatives as choice architectures.[11] Price is often incorporated into the budget constraint in the standard revealed preference theory. Theorem 1.1, however, can accommodate the preference maximization in a discrete choice with the *signaling effect* of price (Milgrom and Roberts (1986)).

In this case, $u \sim_{\mathbf{0}} v$ indicates that the DM is indifferent between u and v and thus prefers the one with the lower price, while $u \sim_{\mathbf{1}} v$ suggests she is unable to compare u, v and is convinced that the higher price reflects the higher quality such that she tends to compromise on price. Consequently, $\succsim$ divides X into two categories, where every $\sim_{\mathbf{0}}$-indifferent set does not overlap with any $\sim_{\mathbf{1}}$-indifferent set (Claim 1.3), meaning that the DM always prefers one alternative over the other between two categories. Given a

[11] For instance, given a menu V, let $(u, v) \in E(D)$ only if the hundreds digit of the price of v exceeds that of u. Then, the digraph D represents a nonlinear order on V. For the theoretical foundation of such a specification, see Matějka and McKay (2015).

choice architecture ($D \in \mathcal{D}$) and the induced price perception as being a linear order over the underlying menu (φ_D), the procedure $z_{\succsim,\delta} : \mathcal{S} \to X$ maximizes the preference $\succsim$, then resolves $\sim_{\mathbf{0}}$ and $\sim_{\mathbf{1}}$ by minimizing the price (expenditure) and maximizing the quality signaling, respectively.[12]

1.7. Related Literature

The study of choice with explicit information was pioneered by Rubinstein and Salant (2006, 2012); Salant and Rubinstein (2008). Their earlier paper considered choice with linear orders over alternatives (hence, from lists). Our model is a generalization in the sense that digraphs represent the orders that are not necessarily linear, or even cyclic. For the choice function, our selection procedure from sorted lists coincides with their characterization, which can also accommodate Simon (1955). Their later paper studied choice with choice-relevant yet alternative-irrelevant properties in terms of frames. The major difference between frames and architectures is that we specify architectures in a homogeneous way as digraphs to allow our model to be more translatable to various sources of architectures. Rubinstein and Salant (2012) proposed a general model in which a DM reveals different preferences under different frames. In our model, the description of preference is consistent throughout, and the realization of topological sorting explains the revealed inconsistency of choice.

Another strand of literature related to our model is the searching approach (Caplin et al. (2011); Masatlioglu and Nakajima (2013)). In Masatlioglu and Nakajima (2013), iterative searching serves as a formation mechanism of consideration sets. Neither their choice function nor ours is informative about the identification of a search path or the realization of topological sorting. In our model, however, for every given choice architecture, the potential topological sorting $\Phi(\mathcal{A} \circ TD)$ is predetermined by D, meaning that one can predict the candidates of choice $\bigcup_{\varphi \in \Phi(\cdot)} z_{\succsim,\delta}(\varphi(\cdot))$. If each $z_{\succsim,\delta}(\varphi(\cdot))$ varies sufficiently, one might infer the realized φ_D from $z(D) = z_{\succsim,\delta}(\varphi_D(\cdot))$. A limitation of our model is that we implicitly assume the DM would inspect the entire given architecture.[13] Hence, our model is in an analogy of the hybrid of the fixed-sample-size search and marginal search. los Santos et al. (2012), for example, provided an empirical study on these two types of searching.

[12] In marketing literature, a similar interpretation is reported, e.g., in Wathieu and Bertini (2007). For a more stylized analysis, see Erdem et al. (2008); Schmidbauer and Stock (2018).

[13] In some situations, this might be resolved practically. For instance, the server of an online shopping site (e.g., Amazon) records the browsing history of each DM. According to the data, one can construct an actual architecture by taking the induced subgraph over the reviewed pages and including all the one-step-ahead pages.

Probing further, Manzini and Mariotti (2007, 2012a,b) are seminal in incorporating choice procedures into the preference maximization. In Manzini and Mariotti (2007) (RSM), the choice is explained by the sequential maximization of asymmetric binary relations. Our choice function is related to the canonical RSM in the sense that the DM maximizes her preference $\succsim = \succ \cup \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$ in the first stage and resolves the second stage among $\sim_{\mathbf{0}}$-indifferent (resp., $\sim_{\mathbf{1}}$-indifferent) alternatives. By definition of $\sim_{\mathbf{0}}$ and $\sim_{\mathbf{1}}$, the second stage can also be considered as an order maximization with respect to the topological sorting.

1.A. Proofs

1.A.1. Preliminaries

Given a $D \in \mathcal{D}$, let $V_C(D)$ be the set of all vertices contained in cycles in D. Formally, $V_C(D) := \{v \in V(D) \mid \exists C \in \mathcal{C}(D), v \in V(C)\}$.

We say a set of vertices $A \subseteq V(D)$ satisfies property $\mathcal{L}$ in D, denoted by $A \vDash \mathcal{L}(D)$, if the following condition is satisfied by any $u, v \in A$:

$$\exists P \in \mathcal{P}(D),\ uPv \subset D \iff \nexists P' \in \mathcal{P}(D),\ vP'u \subset D.$$
$$((u,v) \in E(TD) \iff (v,u) \notin E(TD))$$

Claim 1.1. *For any $D \in \mathcal{D}$, if $V(D) \vDash \mathcal{L}(D)$, then D is a DAG and has a unique topological sorting $\varphi \in \Phi(D)$.*

Proof. Let D be an arbitrary digraph that satisfies $V(D) \vDash \mathcal{L}(D)$. Suppose $\mathcal{C}(D) \neq \emptyset$, and fix an arbitrary $C \in \mathcal{C}(D)$. Then, for any $u, v \in V(C)$, it holds that

$$\exists P \in \mathcal{P}(C), uPv \subset C \iff \exists P' \in \mathcal{P}(C), vP'u \subset C.$$

This contradicts to $V(D) \vDash \mathcal{L}(D)$. Hence, D is a DAG as $\mathcal{C}(D) = \emptyset$.

Since D is a DAG under $V(D) \vDash \mathcal{L}(D)$, it follows that $\Phi(D) \neq \emptyset$.[14] Suppose $\varphi, \tilde{\varphi} \in \Phi(D)$ and $\varphi \neq \tilde{\varphi}$. Then, there must be $u, v \in V(D)$ such that $(u,v) \in E(T\varphi(D))$ and $(v,u) \in E(T\tilde{\varphi}(D))$. Meanwhile, $V(D) \vDash \mathcal{L}(D)$ implies that $(u,v) \in E(TD)$ or $(v,u) \in E(TD)$ holds exclusively in TD. As a result, either $E(T\varphi(D)) = E(T\tilde{\varphi}(D))$, or one of φ and $\tilde{\varphi}$ is not a topological sorting of D. By contradictions, $\varphi, \tilde{\varphi} \in \Phi(D)$ implies $\varphi = \tilde{\varphi}$. Hence, D has a unique topological sorting. *Q.E.D.*

[14] By Szpilrajn's extension theorem, it is well-know that every DAG has a topological sorting.

1.A.2. Proof of Theorem 1.1

Since it is easy to verify that $z(D) = z_{\succsim,\delta}(\varphi \circ \mathcal{A} \circ TD)$ satisfies DAC and IIP, we show the sufficiency part. The proof is conducted by the following series of claims.

For every $u, v \in X$, define the following binary relations.

$$\begin{aligned}
\succ &:= \{(u,v) \mid z((u,v)) = z((v,u)) = u\}\,;\\
\sim_{\mathbf{0}} &:= \{(u,v) \mid z((u,v)) = u \wedge z((v,u)) = v\}\,;\\
\sim_{\mathbf{1}} &:= \{(u,v) \mid z((u,v)) = v \wedge z((v,u)) = u\}\,.
\end{aligned}$$

Define accordingly $\succsim := \succ \cup \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$, then $\succsim \subset X \times X$ is connex on X.

Claim 1.2. *$\succ$, $\sim_{\mathbf{0}}$, and $\sim_{\mathbf{1}}$ are transitive.*

Proof. Let $u, v, w \in X$ be arbitrary, and denote by $\overrightarrow{uvw}$ the string such that $V(\overrightarrow{uvw}) = \{u, v, w\}$ and $E(\overrightarrow{uvw}) = \{(u,v), (v,w)\}$.

(a) Suppose $u \succ v$ and $v \succ w$. Consider $S = \overrightarrow{uvw}$ and $S' = \overrightarrow{wvu}$. Then, we have $z((u,v)) = u$ and $z((v,w)) = v$. Since $\{(u,v), \{w\}\}$ and $\{(v,w), \{x\}\}$ define two induced partitions of S, by IIP, we have $z(TS[\{u,w\}]) = z(S) = z(TS[\{u,v\}])$. As a result, $z((u,w)) = z((u,v)) = u$. Similarly, it follows that $z(TS'[\{v,u\}]) = z(S') = z(TS'[\{w,u\}])$. Hence, $z((w,u)) = z((v,u)) = u$. That is, $u \succ w$.

(b) Suppose $u \sim_{\mathbf{0}} v$ and $v \sim_{\mathbf{0}} w$. By IIP, we have $z(TS[\{u,w\}]) = z(TS[\{v,u\}]) = Z(S)$. This implies $z((u,w)) = z((u,v)) = u$ by the definition of $\sim_{\mathbf{0}}$. Analogously, it holds that $z(TS'[\{w,u\}]) = z(S') = z(TS'[\{w,v\}])$, which implies $z((w,u)) = z((w,v)) = w$. Hence, $u \sim_{\mathbf{0}} w$. (c) The transitivity of $\sim_{\mathbf{1}}$ follows similarly. *Q.E.D.*

Claim 1.3. *For every $u, v, w \in X$, it is impossible to have $u \sim_{\mathbf{0}} v$ and $v \sim_{\mathbf{1}} w$.*

Proof. Assume that $u \sim_{\mathbf{0}} v$ and $v \sim_{\mathbf{1}} w$. Let $S = \overrightarrow{uwv}$ and $S' = \overrightarrow{vuw}$. For S, consider induced partitions $\{(u,v), \{w\}\}$ and $\{(v,w), \{u\}\}$. By IIP, it follows that $z(TS[\{u,w\}]) = z(S) = z(TS[\{u,v\}])$. Hence, $z((u,w)) = z((u,v)) = u$. Meanwhile, for S', consider induced partitions $\{(v,w), \{u\}\}$ and $\{(u,v), \{w\}\}$. Then, we have $z(TS'[\{u,w\}]) = z(S') = z(TS'[\{v,w\}])$. As a result, $z((u,w)) = z((v,w)) = w$. This contradicts to $z((u,w)) = u$ obtained in $S = \overrightarrow{uwv}$. *Q.E.D.*

Under Claim 1.2, $\sim_{\mathbf{0}}$ and $\sim_{\mathbf{1}}$ are symmetric and transitive. Since $\succsim \subset X \times X$ is connex on X, it follows Claim 1.3 that, for every $\succsim$-indifferent set A, if there exist $u, v \in A$ with $u \sim_{\mathbf{0}} v$, then $x \sim_{\mathbf{0}} y$ for all $x, y \in A$. Consequently, define a bivalent function $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ by as follows, then $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ is well-defined.

$$\delta(v) := \begin{cases} \mathbf{1}, & \exists u \in X, [u \neq v \wedge u \sim_{\mathbf{1}} v] \\ \mathbf{0}, & \text{otherwise} \end{cases}. \qquad \forall v \in X$$

For a given $\succsim$-indifferent set A, simply write $A \in X/\sim_{\mathbf{0}}$ (resp., $A \in X/\sim_{\mathbf{1}}$) if $\delta(v) = \mathbf{0}$ (resp., $\mathbf{1}$) for any $v \in A$.

Claim 1.4. *$z(D)$ is $\succsim$-maximal in $V(D)$ for any $D \in \mathcal{D}$.*

Proof. Let $x, y \in X$ be arbitrary and suppose $x \succ y$. By definition of $\succsim$, $z((x, y)) = z((y, x)) = x$. Note that the possible connected DAG on $\{x, y\}$ is (x, y) or (y, x). Hence by DAC, $z(D) \neq y$ for any D with $x, y \in V(D)$. That is, for any $D \in \mathcal{D}$ and any $v \in V(D)$, if there is a $u \in V(D)$ such that $u \succ v$, then $v \neq z(D)$. *Q.E.D.*

Given a $D \in \mathcal{D}$, let $M(D)$ be the set of all $\succsim$-maximal vertices in $V(D)$. Formally,

$$M(D) := \{v \in V(D) \mid \nexists u \in V(D),\, u \succ v\}\,.$$

Then, for every $D \in \mathcal{D}$, it follows that $\delta(v) = \mathbf{0}$ for any $v \in M(D)$ ($M(D) \in X/\sim_{\mathbf{0}}$), or $\delta(v) = \mathbf{1}$ for every $v \in M(D)$ ($M(D) \in X/\sim_{\mathbf{1}}$), exclusively.

Claim 1.5. *The followings hold for all $D \in \mathcal{D}$:*
(a) $M(D) \in X/\sim_{\mathbf{0}} \implies \{e \in E(\mathcal{A} \circ TD[M(D)]) \mid \tau(e) = z(D)\} = \emptyset$;
(b) $M(D) \in X/\sim_{\mathbf{1}} \implies \{e \in E(\mathcal{A} \circ TD[M(D)]) \mid \iota(e) = z(D)\} = \emptyset$.

Proof. Fix an arbitrary $D \in \mathcal{D}$ and let $z(D) = v$.

(a) Suppose a $u \in M(D)$ exists with satisfying $TD[\{u, v\}] = (u, v)$. Let $\{V_i\}_i$ be an arbitrary partition of $V(D)$ such that $V_j \cap M(D) \neq \emptyset$ for all $V_j \in \{V_i\}_i$, and a $V_k \in \{V_i\}_i$ exists with satisfying $V_k = \{u, v\}$. Fix this k, and for each i, let $D_i := TD[V_i]$. Then, $\{D_i\}_i$ forms an induced partition of D. By Claim 1.4, $z(D_i) \in M(D)$ for all i, and $z(D_k) = u$, since $u \sim_{\mathbf{0}} v$ with $TD[\{u, v\}] = (u, v)$. That is, $v \notin \{z(D_i)\}_i$. Hence, we have $z(D) = v \neq z(TD[\{z(D_i)\}_i])$ and $v \in V(D_k) \setminus \{z(D_k)\}$. By IIP-(ii), $(u, v) \in E(TD)$ if and only if $(v, u) \in E(TD)$, which contradicts to $TD[\{u, v\}] = (u, v)$. Thus, by Claim 1.4, $M(D) \in X/\sim_{\mathbf{0}}$ implies $TD[\{u, v\}] \neq (u, v)$ for any $u \in M(D)$, meaning that $\{e \in E(\mathcal{A} \circ TD[M(D)]) \mid \tau(e) = v\} = \emptyset$.

(b) Suppose there exists a $u \in M(D)$ such that $TD[\{u, v\}] = (v, u)$. Let $\{D_i\}_i$ be the induced partition of D defined in (a). By Claim 1.4, $\{z(D_i)\}_i \subset M(D)$. Since $u \sim_{\mathbf{1}} v$ and $TD[\{u, v\}] = (v, u)$, we have $z(D_k) = u$. Thus, it yields $z(D) = v \neq z(TD[\{z(D_i)\}_i])$ and $v \in V(D_k) \setminus \{z(D_k)\}$. In analogy to (a), a contradiction occurs against $TD[\{u, v\}] = (v, u)$ under IIP-(ii). Hence, $M(D) \in X/\sim_{\mathbf{1}}$ implies that $TD[\{u, v\}] \neq (v, u)$ for all $u \in M(D)$. The statement follows. *Q.E.D.*

Claim 1.6. *For any $D \in \mathcal{D}$, $M(D) \vDash \mathcal{L}(D)$ implies $z(D) = z(TD[M(D)])$.*

Proof. Fix an arbitrary $D \in \mathcal{D}$ that satisfies $M(D) \vDash \mathcal{L}(D)$.

Let $\{V_i\}_i$ be an arbitrary partition of $V(D)$ such that $\#(V_j \cap M(D)) = 1$ for all $V_j \in \{V_i\}_i$. Define a digraph $D_j := TD[V_j]$ for each $V_j \in \{V_i\}_i$, then $\{D_i\}_i$ forms an induced partition of D. By Claim 1.4 and $M(D) \vDash \mathcal{L}(D)$, we have $z(D) \in M(D) = \{z(D_i)\}_i$ and $\{z(D_i)\}_i \vDash \mathcal{L}(D)$. That is, IIP-(i)(ii) are falsified for $\{D_i\}_i$. Hence, by IIP, we have $z(D) = z\left(TD\left[\{z(D_i)\}_i\right]\right) = z\left(TD[M(D)]\right)$. *Q.E.D.*

Claim 1.7. *For any $D \in \mathcal{D}$, if $M(D) \vDash \mathcal{L}(D)$, then $z(D) = z_{\succsim,\delta}(\varphi \circ \mathcal{A} \circ TD)$ for all $\varphi \in \Phi(\mathcal{A} \circ TD)$.*

Proof. Let $D \in \mathcal{D}$ be an arbitrary digraph that satisfies $M(D) \vDash \mathcal{L}(D)$. Under Claim 1.6, it suffices to show $z(TD[M(D)]) = z_{\succsim,\delta}(\varphi \circ \mathcal{A} \circ TD)$ for all $\varphi \in \Phi(\mathcal{A} \circ TD)$.

Note that $V(S) \vDash \mathcal{L}(S)$ for any $S \in \mathcal{S}$. Hence, by Claim 1.6, if there is a $S \in \mathcal{S}$ such that $V(S) = M(D)$ and $TS = TD[M(D)]$, then we have $z(TD[M(D)]) = z(S)$. Since $M(D) \vDash \mathcal{L}(D)$ implies $M(D) \vDash \mathcal{L}(TD[M(D)])$, by Claim 1.1, $TD[M(D)]$ has a unique topological sorting. Denote by φ_M this topological sorting, then we have $T\varphi_M(TD[M(D)]) = TD[M(D)]$. Moreover, by Claim 1.1, we have $TD[M(D)] = \mathcal{A} \circ TD[M(D)]$. As a result, $z(TD[M(D)]) = z(\varphi_M(\mathcal{A} \circ TD[M(D)]))$.

Suppose that $M(D) \in X/{\sim_{\mathbf{0}}}$. Then, by Claim 1.5-(a), if $(u,v) \in E(T\varphi_M(\mathcal{A} \circ TD[M(D)]))$, then $z(\varphi_M(\mathcal{A} \circ TD[M(D)])) \neq v$. Since $M(D) \vDash \mathcal{L}(\varphi_M(\mathcal{A} \circ TD[M(D)])$, $z(\varphi_M(\mathcal{A} \circ TD[M(D)])) = u$ implies that $(u,v) \in E(T\varphi_M(\mathcal{A} \circ TD[M(D)]))$ for all $v \in M(D)$. Suppose now $M(D) \in X/{\sim_{\mathbf{1}}}$. Then, by Claim 1.5-(b), $z(\varphi_M(\mathcal{A} \circ TD[M(D)])) = u$ implies that $(v,u) \in E(T\varphi_M(\mathcal{A} \circ TD[M(D)]))$ for all $v \in M(D)$. Hence, $z(\varphi_M(\mathcal{A} \circ TD[M(D)])) = z_{\succsim,\delta}(\varphi_M(\mathcal{A} \circ TD[M(D)]))$.

Given $\mathcal{A} \circ TD$ being a DAG, let $\varphi \in \Phi(\mathcal{A} \circ TD)$ be arbitrary. Then, we have $M(D) \vDash \mathcal{L}(\varphi_M(\mathcal{A} \circ TD[M(D)]))$ and $V(D) \vDash \mathcal{L}(\varphi(\mathcal{A} \circ TD))$. Since φ_M is the unique topological sorting of $\mathcal{A} \circ TD[M(D)]$, it follows that $T\varphi(\mathcal{A} \circ TD)[M(D)] = \varphi_M(\mathcal{A} \circ TD[M(D)])$. Hence, by Claim 1.4, we have $z_{\succsim,\delta}(\varphi \circ \mathcal{A} \circ TD) = z_{\succsim,\delta}(\varphi_M(\mathcal{A} \circ TD[M(D)]))$ for all $\varphi \in \Phi(\mathcal{A} \circ TD)$. The claim follows. *Q.E.D.*

Completion of Sufficiency. By the definition of topological sorting, for any DAG $D \in \mathcal{D}$ and any $u \in V(D)$, it follows that

$$\begin{aligned} &[\forall w \in V(D),\, (u,w) \notin E(TD)] \implies \left[\begin{array}{r} \exists \varphi \in \Phi(D), \forall w \in V(D), \\ (w,u) \in E(T\varphi(D)) \end{array}\right]; \\ &[\forall w \in V(D),\, (w,u) \notin E(TD)] \implies \left[\begin{array}{r} \exists \varphi \in \Phi(D), \forall w \in V(D), \\ (u,w) \in E(T\varphi(D)) \end{array}\right]. \end{aligned} \tag{1.1}$$

Let $D \in \mathcal{D}$ be an arbitrary digraph, and let $z(D) = v$. By Claim 1.5, we have (i) if $M(D) \in X/{\sim_{\mathbf{0}}}$, then $(u,v) \notin E(\mathcal{A} \circ TD)$ for all $u \in M(D)$, and (ii) $(v,u) \notin E(\mathcal{A} \circ TD)$

for any $u \in M(D)$ when $M(D) \in X/\sim_{\mathbf{1}}$. Since $\mathcal{A} \circ TD$ is a DAG, by Eq (1.1),

$$[\forall u \in M(D), \delta(u) = \mathbf{0}] \implies \left[\begin{array}{c} \exists \varphi_D \in \Phi(\mathcal{A} \circ TD), \forall u \in M(D), \\ (z(D), u) \in E(T\varphi_D(\mathcal{A} \circ TD)) \end{array} \right];$$

$$[\forall u \in M(D), \delta(u) = \mathbf{1}] \implies \left[\begin{array}{c} \exists \varphi_D \in \Phi(\mathcal{A} \circ TD), \forall u \in M(D), \\ (u, z(D)) \in E(T\varphi_D(\mathcal{A} \circ TD)) \end{array} \right].$$

Thus, it yields $z(D) = z_{\succsim,\delta}(\varphi_D \circ \mathcal{A} \circ TD)$.

Now we show the uniqueness. Let $D, \tilde{D} \in \mathcal{D}$ be arbitrary digraphs such that $M(D) \cap M(\tilde{D}) \neq \emptyset$. Clearly, we have $\delta(u) = \delta(\tilde{u})$ for any $u \in M(D), \tilde{u} \in M(\tilde{D})$. Suppose that $M(D) \vDash \mathcal{L}(D), M(\tilde{D}) \vDash \mathcal{L}(\tilde{D})$, and without loss of generality, assume $\delta(u) = \mathbf{0}$ for all $u \in M(D)$. Then, it follows Claim 1.1 that every $\varphi_D \in \Phi(\mathcal{A} \circ TD)$ yields the same permutation on $M(D)$, and so does every $\varphi_{\tilde{D}} \in \Phi(\mathcal{A} \circ T\tilde{D})$ on $M(\tilde{D})$. Hence, by Claim 1.7, if a $v \in M(D) \cap M(\tilde{D})$ exists such that $(v, u) \in E(TD)$ and $(v, \tilde{u}) \in E(T\tilde{D})$ for all $u \in M(D)$ and any $\tilde{u} \in M(\tilde{D})$, then

$$z(D) = z_{\succsim,\delta}(\varphi_D \circ \mathcal{A} \circ TD) = v = z_{\succsim,\delta}(\varphi_{\tilde{D}} \circ \mathcal{A} \circ T\tilde{D}) = z(\tilde{D}),$$

for all $\varphi_D \in \Phi(\mathcal{A} \circ TD)$ and any $\varphi_{\tilde{D}} \in \Phi(\mathcal{A} \circ T\tilde{D})$. That is, for any $D, \tilde{D} \in \mathcal{D}$, $z(D) = z(\tilde{D})$ whenever every topological sorting of $\mathcal{A} \circ TD$ and $\mathcal{A} \circ T\tilde{D}$ yields the same first (or the same last) $\succsim$-maximal vertex. The uniqueness follows. *Q.E.D.*

1.A.3. Proof of Theorem 1.2

We show only the sufficiency part via the following series of claims.

Define the following binary relations on X.

$$\begin{aligned}
\succ^* &:= \{(x, y) \mid Z((x, y)) = Z((y, x)) = \{x\}\}; \\
\sim^* &:= \{(x, y) \mid Z((x, y)) = Z((y, x)) = \{x, y\}\}; \\
R_{\mathbf{0}}^P &:= \{(x, y) \mid Z((x, y)) = \{x\} \wedge Z((y, x)) = \{x, y\}\}; \\
R_{\mathbf{0}}^I &:= \{(x, y) \mid Z((x, y)) = \{x\} \wedge Z((y, x)) = \{y\}\}; \\
R_{\mathbf{1}}^P &:= \{(x, y) \mid Z((x, y)) = \{x, y\} \wedge Z((y, x)) = \{x\}\}; \\
R_{\mathbf{1}}^I &:= \{(x, y) \mid Z((x, y)) = \{y\} \wedge Z((y, x)) = \{x\}\}.
\end{aligned}$$

Define $\mathcal{R} := \succ^* \cup \sim^* \cup R_\delta^P \cup R_\delta^I$, where $\delta = \mathbf{0}, \mathbf{1}$, then $\mathcal{R}$ is connex on X.

Claim 1.8. *$\succ^*$, $\left(\succ^* \cup R_\delta^P\right)$, and R_δ^I are transitive for $\delta = \mathbf{0}, \mathbf{1}$.*

Proof. Let $u, v, w \in X$ be arbitrary, and without loss of generality, let $\delta = \mathbf{0}$. Denote by $\overrightarrow{uvw}$ the string S such that $V(S) = \{u, v, w\}$ and $E(S) = \{(u, v), (v, w)\}$.

Suppose $u \succ^* v$ and $v \succ^* w$ and consider $\overrightarrow{uvw}$. By IIP*, $Z\left(\overrightarrow{uvw}\right) = Z((u, v)) = \{u\}$. Also IIP* implies that $Z((u, w)) = Z\left(\overrightarrow{uvw}\right)$, hence $Z((u, w)) = \{u\}$. Similarly for $\overrightarrow{wvu}$, it follows that $Z((w, u)) = Z(\overrightarrow{wvu}) = Z((v, u)) = \{u\}$. Thus $u \succ^* w$.

Now suppose $u(\succ^* \cup R^P_{\mathbf{0}})v$ and $v(\succ^* \cup R^P_{\mathbf{0}})w$. Since $\succ^*$ is transitive, it suffices to show: (i) $uR^P_{\mathbf{0}}v$ and $vR^P_{\mathbf{0}}w$ implies $u(\succ^* \cup R^P_{\mathbf{0}})w$; (ii) $u \succ^* v$ and $vR^P_{\mathbf{0}}w$ implies $u(\succ^* \cup R^P_{\mathbf{0}})w$. (i) Assume $uR^P_{\mathbf{0}}v$, $vR^P_{\mathbf{0}}w$ and consider $\overrightarrow{uvw}, \overrightarrow{wuv}$. By IIP* and the definition of $R^P_{\mathbf{0}}$, $Z((u,v)) = Z(\overrightarrow{uvw}) = Z((u,w))$. Hence, we have $Z((u,w)) = Z((u,v)) = \{u\}$. Again, by IIP*, $Z((w,u)) = Z(\overrightarrow{wuv}) = Z(T\overrightarrow{wuv}[Z(w,u) \cup \{v\}])$ for $\overrightarrow{wuv}$. If $Z((w,u)) = \{w\}$, it follows that $Z(\overrightarrow{wuv}) = Z(T\overrightarrow{wuv}[Z(w,u) \cup \{v\}]) = Z((w,v)) = \{v,w\}$. Since this contradicts to $Z(\overrightarrow{wuv}) = Z((w,u)) = \{w\}$, it yields that $Z((w,u)) \neq \{w\}$. Thus, we have $u\left(\succ^* \cup R^P_{\mathbf{0}}\right)w$. (ii) Assume $u \succ^* v$, $vR^P_{\mathbf{0}}w$ and consider $\overrightarrow{uvw}, \overrightarrow{vwu}$. By IIP*, $Z((u,w)) = Z(\overrightarrow{uvw}) = Z((u,v)) = \{u\}$ and $Z((w,u)) = Z(\overrightarrow{vwu}) = Z((v,u)) = \{u\}$. Thus, $u \succ^* v$ and $vR^P_{\mathbf{0}}w$ imply $u \succ^* w$. Therefore, $(\succ^* \cup R^P_{\mathbf{0}}) \subset X \times X$ is transitive.

Suppose $uR^I_{\mathbf{0}}v$, $vR^I_{\mathbf{0}}w$ and consider $\overrightarrow{uvw}, \overrightarrow{wvu}$. Then by IIP* we have $Z((u,w)) = Z(\overrightarrow{uvw}) = Z((u,v)) = \{u\}$ holds for $\overrightarrow{uvw}$; while $Z((w,u)) = Z(\overrightarrow{wvu}) = Z((w,v)) = \{w\}$ for $\overrightarrow{wvu}$. Hence, it yields that $uR^I_{\mathbf{0}}w$. *Q.E.D.*

Claim 1.9. *For any $u,v,w \in X$,*

(a) $[uR^P_{\mathbf{0}}v \wedge vR^P_{\mathbf{1}}w] \implies u \succ^* w$; *(b)* $[uR^P_{\mathbf{1}}v \wedge vR^P_{\mathbf{0}}w] \implies u \succ^* w$;
(c) $[uR^P_{\mathbf{0}}v \wedge wR^P_{\mathbf{1}}v] \implies u \sim^* w$; *(d)* $[uR^P_{\mathbf{0}}v \wedge uR^P_{\mathbf{1}}w] \implies v \sim^* w$;
(e) $[uR^P_{\mathbf{0}}v \wedge vR^I_{\mathbf{0}}w] \implies uR^P_{\mathbf{0}}w$; $[uR^P_{\mathbf{1}}v \wedge vR^I_{\mathbf{1}}w] \implies uR^P_{\mathbf{1}}w$;
(f) $[uR^I_{\mathbf{0}}v \wedge vR^P_{\mathbf{0}}w] \implies uR^P_{\mathbf{0}}w$; $[uR^I_{\mathbf{1}}v \wedge vR^P_{\mathbf{1}}w] \implies uR^P_{\mathbf{1}}w$;
(g) $[u \succ^* v \wedge vR^I_{\delta}w] \implies u \succ^* w$, $\delta = \mathbf{0},\mathbf{1}$;
(h) $[uR^I_{\delta}v \wedge v \succ^* w] \implies u \succ^* w$, $\delta = \mathbf{0},\mathbf{1}$;
(i) $[u \sim^* v \wedge vR^I_{\delta}w] \implies u \sim^* w$, $\delta = \mathbf{0},\mathbf{1}$.

Proof. We show (a), (c), (e), (g) and (i) since the rest can be shown similarly.

(a) Suppose $uR^P_{\mathbf{0}}v$ and $vR^P_{\mathbf{1}}w$. For $\overrightarrow{uwv}$, IIP* implies that $Z((u,w)) = Z(\overrightarrow{uwv}) = Z((u,v)) = \{u\}$. Similarly for $\overrightarrow{wuv}$, it holds that $Z((w,u)) = Z(\overrightarrow{wuv}) = Z((u,v)) = \{u\}$. Hence, $u \succ^* w$.

(c) Suppose $uR^P_{\mathbf{0}}v$ and $wR^P_{\mathbf{1}}v$. For $S = \overrightarrow{uwv}$, it holds that $Z(S) = Z((u,w)) = Z(TS[Z((u,w)) \cup \{v\}])$. If $Z((u,w)) = \{w\}$ then it yields that $Z(S) = Z((u,w)) = \{v,w\}$, which suggests a contradiction. Hence, $u \in Z((u,w))$. For $S' = \overrightarrow{vuw}$, we have $Z(S') = Z((u,w)) = Z(TS'[Z((u,w)) \cup \{v\}])$, which implies $w \in Z((u,w))$. Consider $\tilde{S} = \overrightarrow{vwu}$ and $\tilde{S}' = \overrightarrow{wuv}$. By IIP*, $Z(\tilde{S}) = Z(T\tilde{S}[Z((w,u)) \cup \{v\}]) = Z((w,u)) = Z(T\tilde{S}'[Z((w,u)) \cup \{v\}]) = Z(\tilde{S}')$. If $Z((w,u)) = \{u\}$ then $Z(\tilde{S}) = Z(T\tilde{S}[\{u,v\}]) = \{u,v\} \neq Z((w,u))$. If $Z((w,u)) = \{w\}$, then we have $Z(\tilde{S}') = Z(T\tilde{S}'[\{w,v\}]) = \{w,v\} \neq Z((w,u))$. Thus, $Z((w,u)) = Z((u,w)) = \{u,w\}$.

(e) Suppose $uR^P_{\mathbf{0}}v, vR^I_{\mathbf{0}}w$. Let $S = \overrightarrow{uvw}, S' = \overrightarrow{vwu}$, and $S'' = \overrightarrow{wvu}$. Then, IIP* yields $Z(S) = Z((u,w)) = Z((u,v)) = \{u\}$, $Z(S') = Z(TS'[Z((w,u)) \cup \{v\}]) = Z((v,u)) = \{u,v\}$, and $Z(S'') = Z(TS''[Z((w,u)) \cup \{v\}]) = Z((w,u))$. If $u \notin Z((w,u))$ then $Z(TS'[Z((w,u)) \cup \{v\}]) = Z((w,v)) = \{w\}$, which contradicts to $Z(S') = \{u,v\}$. If $w \notin Z((w,u))$ then it yields $Z(TS''[Z((w,u)) \cup \{v\}]) = Z((v,u)) = \{u,v\}$. This contradicts to $Z(S'') = Z((w,u))$, as $v \notin Z((w,u))$. Hence, $Z((u,w)) = \{u\}$ and $Z((w,u)) = \{u,w\}$. The case $uR^P_{\mathbf{1}}v, vR^I_{\mathbf{1}}w$ follows analogously.

(g) Let $u \succ^* v, vR^I_{\mathbf{0}}w$ and $S = \overrightarrow{uvw}, S' = \overrightarrow{vwu}$. By IIP*, $Z(S) = Z((u,w)) = Z((u,v)) = \{u\}$ and $Z(S') = Z((w,u)) = Z((v,u)) = \{u\}$. Thus, $Z((u,w)) = Z((w,u)) = \{u\}$, that is, $u \succ^* w$. The case $\delta = \mathbf{1}$ can be shown similarly.

(i) Suppose $u \sim^* v, vR^I_{\mathbf{0}}w$, and consider $S_1 = \overrightarrow{uvw}, S_2 = \overrightarrow{uwv}, S_3 = \overrightarrow{wuv}, S_4 = \overrightarrow{vwu}$. By IIP*, it follows that $Z(TS_1[Z((u,w)) \cup \{v\}]) = Z(S_1) = Z(TS_1[Z((v,w)) \cup \{u\}]) = Z((u,v)) = \{u,v\}$, which implies $u \in Z((u,w))$. For S_2, IIP* implies that $Z(TS_2[Z((u,w)) \cup \{v\}]) = Z(S_2) = Z(TS_2[Z((w,v)) \cup \{u\}]) = Z((u,w))$. If $Z((u,w)) = \{u\}$, it yields the contradiction $\{u\} = Z((u,w)) = Z(S_2) = Z((u,v)) = \{u,v\}$. Hence, $Z((u,w)) = \{u,w\}$. For S_3, it holds that $Z(TS_3[Z((w,u)) \cup \{v\}]) = Z(S_3) = Z(TS_3[Z((w,v)) \cup \{u\}]) = Z((w,u))$. Similarly, $Z((w,u)) = \{u\}$ yields the contradiction $\{u,v\} = Z((u,v)) = Z(S_3) = Z((w,u)) = \{u\}$, meaning that $w \in Z((w,u))$. For S_4, we have $Z(TS_4[Z((w,u)) \cup \{v\}]) = Z(S_4) = Z(TS_4[Z((v,w)) \cup \{u\}]) = Z((v,u)) = \{u,v\}$, which implies $u \in Z((w,u))$. As a result, $Z((u,w)) = Z((w,u)) = \{u,w\}$. That is, $u \sim^* w$. The case $\delta = \mathbf{1}$ follows similarly. *Q.E.D.*

Claim 1.10. *For any $u,v,w \in X$, it is impossible to have*
(a) $uR^I_{\mathbf{0}}v \wedge vR^I_{\mathbf{1}}w$;
(b) $uR^I_{\mathbf{0}}v \wedge vR^P_{\mathbf{1}}w$ (resp., $uR^P_{\mathbf{1}}v \wedge vR^I_{\mathbf{0}}w$);
(c) $uR^I_{\mathbf{1}}v \wedge vR^P_{\mathbf{0}}w$ (resp., $uR^P_{\mathbf{0}}v \wedge vR^I_{\mathbf{1}}w$).

Proof. (a) Assume that there exist $u,v,w \in X$ such that $uR^I_{\mathbf{0}}v$ and $vR^I_{\mathbf{1}}w$. For $\overrightarrow{uwv}$, IIP* implies that $Z((u,w)) = Z(\overrightarrow{uwv}) = Z((u,v)) = \{u\}$, while it follows that $Z((u,w)) = Z(\overrightarrow{vuw}) = Z((v,w)) = \{w\}$ for $\overrightarrow{vuw}$. A contradiction occurs.

(b) Suppose there exist $u,v,w \in X$ that satisfy $uR^I_{\mathbf{0}}v$ and $vR^P_{\mathbf{1}}w$. Consider $\overrightarrow{uwv}$ and $\overrightarrow{vuw}$. By IIP* and definition of $R^I_{\mathbf{0}}, R^P_{\mathbf{1}}$, we have $Z((u,w)) = Z(\overrightarrow{uwv}) = Z((u,v)) = \{u\}$. Since $Z((u,w)) = \{u\}$, it again follows IIP* that $Z(\overrightarrow{vuw}) = Z((v,w)) = \{v,w\}$ and $Z(\overrightarrow{vuw}) = Z((v,u)) = \{v\}$, which contradict to each other. The case $uR^P_{\mathbf{1}}v$ and $vR^I_{\mathbf{0}}w$ follows similarly by considering the same pair of digraphs.

(c) Let $uR^I_{\mathbf{1}}v$ and $vR^P_{\mathbf{0}}w$. Then, by IIP*, $Z(T\overrightarrow{uwv}[Z((u,w)) \cup \{v\}]) = Z(\overrightarrow{uwv}) = Z((w,v)) = \{w,v\}$ for $\overrightarrow{uwv}$. Hence, $w \in Z((u,w))$. Meanwhile, for $\overrightarrow{vuw}$, we have $Z((u,w)) = Z(\overrightarrow{vuw}) = Z((v,u)) = \{u\}$, which contradicts to $w \in Z((u,w))$. *Q.E.D.*

Let $\succsim_\delta := \succ^* \cup R_\delta$ and $R_\delta := R^P_\delta \cup R^I_\delta$, where $\delta = \mathbf{0}, \mathbf{1}$. For a given $D \in \mathcal{D}$, let $M^*(D) \subseteq V(D)$ be the set of all $\succ^*$-maximal elements in $V(D)$. Formally,

$$M^*(D) := \{v \in V(D) \mid \nexists u \in V(D), u \succ^* v\}.$$

Denote by $C(\{u,v\})$ the 2-cycle on $\{u,v\}$. Let $\gamma : [X]^2 \to [X]^1 \cup \{\emptyset\}$ be the mapping given by

$$\gamma(\{u,v\}) := \begin{cases} \{u\}, & Z(C(\{u,v\})) = \{v\} \\ \{v\}, & Z(C(\{u,v\})) = \{u\} \\ \emptyset, & \text{otherwise} \end{cases}, \qquad \forall \{u,v\} \in [X]^2.$$

Accordingly, define a correspondence $\Gamma : \mathcal{D} \twoheadrightarrow X$ by

$$\Gamma(D) := V(D) \setminus \left(\bigcup_{\{u,v\} \in \mathcal{Y}(D)} \gamma(\{u,v\}) \right), \quad \forall D \in \mathcal{D}.$$

Claim 1.11. *Let $M^*_\Gamma(D) = M^*(D) \cap \Gamma(D)$, then $Z(D) \subseteq M^*_\Gamma(D)$ for all $D \in \mathcal{D}$.*

Proof. For any $D \in \mathcal{D}$, if there is a $C \in \mathcal{C}(D)$ with $\#V(C) = n$, then given any $V \in [V(C)]^k$ with $2 \leq k \leq n$, there is a $\tilde{C} \in \mathcal{C}(TD)$ such that $V = V(\tilde{C})$. In particular, $TD[\{u,v\}] = C(\{u,v\})$ holds for any $\{u,v\} \in \mathcal{Y}(D)$. The claim is trivial if $\mathcal{C}(D) = \emptyset$. Suppose $\mathcal{C}(D) \neq \emptyset$, and let $v \in \bigcup_{\{x,y\} \in \mathcal{Y}(D)} \gamma(\{x,y\})$ be arbitrary. Then, a $u \in V_C(D)$ exists such that $\{u,v\} \in \mathcal{Y}(D)$, and thus $TD[\{u,v\}] = C(\{u,v\})$. By definition of γ, $\gamma(\{u,v\}) = \{v\}$ implies $Z(C(\{u,v\})) = \{u\}$. Hence, $v \notin Z(C(\{u,v\})) = Z(TD[\{u,v\}])$, which yields $v \notin Z(D)$ by IIP*. As a result, $v \in \bigcup_{\{x,y\} \in \mathcal{Y}(D)} \gamma(\{x,y\})$ implies $v \notin Z(D)$ for any $v \in V(D)$, meaning that $Z(D) \subseteq \Gamma(D)$.

Fix an arbitrary $D \in \mathcal{D}$. Let $u, v \in V(D)$ and suppose that $u \succ^* v$. For the set $\{u,v\}$, the possible connected DAGs are (u,v) and (v,u). Also, $u \succ^* v$ implies $v \notin Z((u,v))$ and $v \notin Z((v,u))$. Hence, by DAC*, we have $v \notin Z(D)$, as $\{u,v\} \subseteq V(D)$. Since $u, v \in V(D)$ are arbitrary, it yields that, for any $D \in \mathcal{D}$ and any $v \in V(D)$, $v \notin Z(D)$ if there is a $u \in V(D)$ such that $u \succ^* v$. That is, $Z(D) \subseteq M^*(D)$. *Q.E.D.*

Claim 1.12. *For any $D \in \mathcal{D}$, any $u \in V(D)$, and any $v \in Z(D)$,*
(a) $uR_{\mathbf{0}}v \implies TD[\{u,v\}] \neq (u,v)$;
(b) $uR_{\mathbf{1}}v \implies TD[\{u,v\}] \neq (v,u)$.

Proof. Let $D \in \mathcal{D}$, $u \in V(D)$, and $v \in Z(D)$ be arbitrary. Set $V = \{u,v\}$ and $V' = V(D) \setminus V$, then $\{V, V'\}$ defines a partition of $V(D)$. Thus, by IIP*,

$$Z(D) = Z\left(TD\left[Z(TD[V']) \cup Z(TD[V])\right]\right). \tag{1.2}$$

(a) Suppose that $uR_{\mathbf{0}}v$ and $TD[V] = (u,v)$. Then, $Z(TD[V]) = \{u\}$. Since $v \notin Z(TD[V'])$, by Eq (1.2), $v \notin Z(D) = Z\left(TD\left[Z(TD[V']) \cup \{u\}\right]\right)$. This contradicts to $v \in Z(D)$. As a result, $uR_{\mathbf{0}}v$ implies $TD[V] \neq (u,v)$.

(b) Let $uR_{\mathbf{1}}v$ and $TD[V] = (v,u)$. Then, $Z(TD[V]) = \{u\}$. By Eq (1.2), we have $Z(D) = Z\left(TD\left[Z(TD[V']) \cup \{u\}\right]\right)$, which implies $v \notin Z(D)$ since $v \notin Z(TD[V'])$. Hence, by contradiction, $uR_{\mathbf{1}}v$ implies $TD[V] \neq (v,u)$. The claim follows. *Q.E.D.*

CLAIM 1.13. *For any $D \in \mathcal{D}$, if $V(D) \vDash \mathcal{L}(D)$ then $Z(D) = Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD)$ for the unique $\varphi \in \Phi(\mathcal{A} \circ TD)$.*

PROOF. Fix an arbitrary such $D \in \mathcal{D}$. Then, by Claim 1.1, D is a DAG and $\#\Phi(D) = 1$. Thus, $\Gamma(D) = V(D)$ and $\Phi(D) = \Phi(\mathcal{A} \circ TD)$. Let $\Phi(\mathcal{A} \circ TD) = \{\varphi\}$.

Suppose $v \in Z(D)$. By Claim 1.11, $Z(D) \subseteq M^*(D)$. Since D is a DAG, by Claim 1.12, for any $u \in V(D)$, $(u, v) \in E(TD)$ implies $\neg(uR_{\mathbf{0}}v)$, and $\neg(uR_{\mathbf{0}}v)$ if $(v, u) \in E(TD)$. Hence, $Z(D) \subseteq Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD)$.

Now suppose $v \notin Z(D)$. Then, by IIP*, a $V_1 \subset V(D)$ exists such that $v \in V_1$ and $v \notin Z(TD[V_1])$. Fix this V_1. Then, it follows IIP* that there exists a $V_2 \subset V_1$ that satisfies $v \in V_2$ and $v \notin Z(TD[V_2])$. Following the iterative manner, a $u \in V(D)$ exists such that $v \notin Z(TD[\{u, v\}])$. Since D is a DAG, it yields that

$$\begin{aligned} TD[\{u, v\}] = (u, v) &\implies [u \succ^* v \vee uR_{\mathbf{0}}v]\,; \\ TD[\{u, v\}] = (v, u) &\implies [u \succ^* v \vee uR_{\mathbf{1}}v]\,. \end{aligned}$$

Thus, we have $v \notin Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD)$, which implies $Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \subseteq Z(D)$. *Q.E.D.*

COMPLETION OF SUFFICIENCY. Fix an arbitrary $D \in \mathcal{D}$.

Suppose $v \in Z(D)$. By Claim 1.12, for all $u \in V(D)$, it holds that

$$\begin{aligned} uR_{\mathbf{0}}v &\implies \mathcal{A} \circ TD[\{u, v\}] \neq (u, v); \\ uR_{\mathbf{1}}v &\implies \mathcal{A} \circ TD[\{u, v\}] \neq (v, u). \end{aligned} \tag{1.3}$$

Hence, a $\varphi \in \Phi(\mathcal{A} \circ TD)$ exists such that (i) $T\varphi(\mathcal{A} \circ TD)[\{u, v\}] = (v, u)$ for all $u \in \{u \in V(D) \mid uR_{\mathbf{0}}v\}$, and (ii) $T\varphi(\mathcal{A} \circ TD)[\{u, v\}] = (u, v)$ for all $u \in \{u \in V(D) \mid uR_{\mathbf{1}}v\}$. Moreover, by Claim 1.11, $v \in M^*_\Gamma(D)$. As a result, it follows that $v \in Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD)$ for some $\varphi \in \Phi(\mathcal{A} \circ TD)$, and $v \in \Gamma(D)$. Consequently, we have $Z(D) \subseteq \bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} (Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \cap \Gamma(D))$.

Suppose there is a $v \in V(D)$ and a $\varphi \in \Phi(\mathcal{A} \circ TD)$ such that $v \in Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \cap \Gamma(D)$. Then, by definition of $Z_{\succsim_\delta} : \mathcal{S} \twoheadrightarrow X$, we have (i), (ii) and $\neg(u \succ^* v)$ held for all $u \in V(D)$. Hence, Eq (1.3) holds for all $u \in V(D)$. By DAD, $v \in Z(\{u, v\})$ as $v \in M^*(D)$ and $Z(\{u, v\}) = Z((u, v)) \cup Z((v, u))$. Moreover, if $v \in V_C(D)$, then a $u \in V_C(D)$ exists such that $\{u, v\} \in \mathcal{Y}(D)$ and $TD[\{u, v\}] = C(\{u, v\})$. Since $v \in \Gamma(D)$, Eq (1.3) implies that $v \in Z(TD[\{u, v\}])$ whenever $E(TD[\{u, v\}]) \neq \emptyset$. As a result, $v \in Z(TD[\{u, v\}])$ for all $u \in V(D)$. By IIP*, it follows that $v \in Z(TD[V])$ for any $V \in [V(D)]^3$ such that $v \in V$. Following the iterative manner, $v \in Z(TD[V_{n-1}])$ for all $V_{n-1} \in [V(D)]^{n-1}$ with $v \in V_{n-1}$, where $n = \#V(D)$. Hence, it yields $v \in Z(D)$ when there is a $\varphi \in \Phi(\mathcal{A} \circ TD)$ such that $v \in (Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \cap \Gamma(D))$. That is, $\bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} (Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \cap \Gamma(D)) \subseteq Z(D)$. Since $D \in \mathcal{D}$ is arbitrary,

$$Z(D) = \bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} \Big(Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \cap \Gamma(D)\Big), \quad \forall D \in \mathcal{D}.$$

Now we show that the binary relation $\mathcal{R} \subset X \times X$ given by $\mathcal{R} := \succsim_{\mathbf{0}} \cup \succsim_{\mathbf{1}} \cup \sim^*$ is quasi-transitive. Let $u, v, w \in X$ be arbitrary. Denote by $\mathcal{R}^P, \mathcal{R}^I$ the asymmetric and symmetric part of $\mathcal{R}$, respectively. Then, $u\mathcal{R}v$ if and only if $(u \succ^* v) \vee (uR^P_{\mathbf{0}}v) \vee (uR^P_{\mathbf{1}}v) \vee (uR^I_{\mathbf{0}}v) \vee (uR^I_{\mathbf{1}}v) \vee (u \sim^* v)$, and $v\mathcal{R}u$ if and only if $(v \succ^* u) \vee (vR^P_{\mathbf{0}}u) \vee (vR^P_{\mathbf{1}}u) \vee (uR^I_{\mathbf{0}}v) \vee (uR^I_{\mathbf{1}}v) \vee (u \sim^* v)$. Hence, we have $\mathcal{R}^P = (u\mathcal{R}v) \wedge \neg(v\mathcal{R}u) = \succ^* \cup R^P_{\mathbf{0}} \cup R^P_{\mathbf{1}}$, and $\mathcal{R}^I = (u\mathcal{R}v) \wedge (v\mathcal{R}u) = \sim^* \cup R^I_{\mathbf{0}} \cup R^I_{\mathbf{1}}$. Since $\mathcal{R}$ is quasi-transitive if and only if $\mathcal{R}^P$ is transitive, it suffices to show $\mathcal{R}^P$ is transitive. By Claim 1.8, $(u \succ^* v) \wedge (v \succ^* w) \Rightarrow (u \succ^* w)$, $(u \succ^* v) \wedge (vR^P_\delta w) \Rightarrow (u \succ^* w)$ for $\delta = \mathbf{0}, \mathbf{1}$, and $(uR^P_\delta v) \wedge (vR^P_\delta w) \Rightarrow (u \succ^* w) \vee (uR^P_\delta w)$ for $\delta = \mathbf{0}, \mathbf{1}$. Moreover, by Claim 1.9-(a)(b), we have $(uR^P_{\mathbf{0}}v) \wedge (vR^P_{\mathbf{1}}w) \Rightarrow (u \succ^* w)$ and $(uR^P_{\mathbf{1}}v) \wedge (vR^P_{\mathbf{0}}w) \Rightarrow (u \succ^* w)$. Therefore, $\mathcal{R}^P \subset X \times X$ is transitive, equivalently, $\mathcal{R} \subset X \times X$ is quasi-transitive.

The transitivity of $\mathcal{R}^P$ and Claim 1.9-(e)(f)(g)(h) imply the transitivity of $\succsim_{\mathbf{0}} = \succ^* \cup R^P_{\mathbf{0}} \cup R^I_{\mathbf{0}}$ and $\succsim_{\mathbf{1}} = \succ^* \cup R^P_{\mathbf{1}} \cup R^I_{\mathbf{1}}$. Since $\Phi(\mathcal{A} \circ TD)$ is singleton for those $D \in \mathcal{D}$ with $V(D) \vDash \mathcal{L}(D)$, thus by Claim 1.13, $Z(D) = Z_{\succsim_\delta}(\varphi \circ TD)$ is uniquely determined for all such $D \in \mathcal{D}$. Hence, given $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$, the choice procedure is unique. *Q.E.D.*

1.A.4. Propositions

1.A.4.1. Proof of Proposition 1.1

Proof. Define a nonempty correspondence $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ by

$$\Delta((u,v)) := \begin{cases} \{\mathbf{0}\}, & Z((u,v)) = \{u\} \\ \{\mathbf{1}\}, & Z((u,v)) = \{v\} \\ \{\mathbf{0},\mathbf{1}\}, & Z((u,v)) = \{u,v\} \end{cases}, \qquad \forall (u,v) \in \mathcal{E}.$$

Then, by the definitions of $\succ^*, R^P_\delta, R^I_\delta$, it follows that

$$\begin{aligned} \neg(u \succ^* v) &\iff [\Delta((u,v)) \neq \{\mathbf{0}\} \vee \Delta((v,u)) \neq \{\mathbf{1}\}]; \\ \neg(uR^I_{\mathbf{0}}v) &\iff [\Delta((u,v)) \neq \{\mathbf{0}\} \vee \Delta((v,u)) \neq \{\mathbf{0}\}]; \\ \neg(uR^I_{\mathbf{1}}v) &\iff [\Delta((u,v)) \neq \{\mathbf{1}\} \vee \Delta((v,u)) \neq \{\mathbf{1}\}]; \\ \neg(uR^P_{\mathbf{0}}v) &\iff [\Delta((u,v)) \neq \{\mathbf{0}\} \vee \Delta((v,u)) \neq \{\mathbf{0},\mathbf{1}\}]; \\ \neg(uR^P_{\mathbf{1}}v) &\iff [\Delta((u,v)) \neq \{\mathbf{0},\mathbf{1}\} \vee \Delta((v,u)) \neq \{\mathbf{1}\}]. \end{aligned}$$

Fix an arbitrary $S \in \mathcal{S}$. By Theorem 1.2, it suffices to show $Z_{\succsim_\delta}(S) = Z_\Delta(S)$. Suppose $v \in Z_{\succsim_\delta}(S)$. By the definition of $Z_{\succsim_\delta}$, for any $u \in V(S)$, we have

$$\begin{aligned} TS[\{u,v\}] = (u,v) &\Rightarrow \left[\neg(u \succ^* v) \wedge \neg(uR^P_{\mathbf{0}}v) \wedge \neg(uR^I_{\mathbf{0}}v)\right] \\ &\Leftrightarrow \Delta((u,v)) \neq \{\mathbf{0}\}; \\ TS[\{u,v\}] = (v,u) &\Rightarrow \left[\neg(u \succ^* v) \wedge \neg(uR^P_{\mathbf{1}}v) \wedge \neg(uR^I_{\mathbf{1}}v)\right] \\ &\Leftrightarrow \Delta((v,u)) \neq \{\mathbf{1}\}. \end{aligned} \tag{1.4}$$

Since $V(S) \vDash \mathcal{L}(S)$, $\#E(TS[V]) = 1$ for all $V \in [V(S)]^2$. As a result, $v \in Z_{\succsim_\delta}(S)$ implies $\Delta(e) \neq \{\mathbf{0}\}$ for all $e \in \{e \in E(TS) \mid \tau(e) = v\}$; and $\Delta(\tilde{e}) \neq \{\mathbf{1}\}$ for all $\tilde{e} \in \{e \in E(TS) \mid \iota(e) = v\}$. Hence, $Z_{\succsim_\delta}(S) \subseteq Z_\Delta(S)$. Suppose $v \in Z_\Delta(S)$. By the definition of Z_Δ and Eq (1.4), we have $\neg(u \succsim_{\mathbf{0}} v)$ when $TS[\{u,v\}] = (u,v)$; and $\neg(u \succsim_{\mathbf{1}} v)$ when $TS[\{v,u\}] = (v,u)$. That is, $v \in Z_{\succsim_\delta}(S)$. Therefore, $Z_\Delta(S) \subseteq Z_{\succsim_\delta}(S)$. *Q.E.D.*

1.A.4.2. *Proof of Proposition 1.2*

Since $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD and IIP*, Theorem 1.2 can be applied.

Under Strong Relevance, the binary relation $Q :=\succ^* \cup R^I_{\mathbf{0}} \cup R^I_{\mathbf{1}}$ is connex on X. By Claim 1.10, $uR^I_{\mathbf{0}}v$ and $vR^I_{\mathbf{1}}w$ cannot hold simultaneously for any $u, v, w \in X$. Define an indicator function $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ by

$$\delta(v) = \begin{cases} \mathbf{1} & \exists u \in X, uR^I_{\mathbf{1}}v \\ \mathbf{0} & \text{otherwise} \end{cases}.$$

Then, $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ is well-defined, and it holds that $\delta(u) = \delta(v)$ whenever $\neg(u \succ^* v)$ and $\neg(v \succ^* u)$. To simplify the statements, given a nonempty $A \subseteq X$, write $A \in X/R^I_{\mathbf{0}}$ (resp., $A \in X/R^I_{\mathbf{1}}$) if $uR^I_{\mathbf{0}}v$ (resp., $uR^I_{\mathbf{1}}v$) for any $u, v \in A$. Let $z_{Q,\delta} : \mathcal{S} \to X$ be the choice function defined in the same way in Theorem 1.1, with respect to the binary relation Q and the indicator function δ.

Claim 1.14. *$Z(D) = \bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} (\{z_{Q,\delta}(\varphi \circ \mathcal{A} \circ TD)\} \cap \Gamma(D))$ for any $D \in \mathcal{D}$.*

Proof. Fix an arbitrary $D \in \mathcal{D}$.

Without loss of generality, suppose $M^*(D) \in X/R^I_{\mathbf{0}}$.

Assume $v \in Z(D)$. Then, by Claim 1.11 and 1.12, (i) $v \in M^*_\Gamma(D)$, and (ii) $(u,v) \in E(TD)$ only if $(v,u) \in E(TD)$ for any $u \in M^*(D)$. That is, $v \in V_C(TD[M^*(D)])$, or $\{e \in E(TD[M^*(D)]) \mid \tau(e) = v\} = \emptyset$. As a result, it follows that $\{e \in E(\mathcal{A} \circ TD[M^*_\Gamma(D)]) \mid \tau(e) = v\} = \emptyset$. Hence, a $\varphi \in \Phi(\mathcal{A} \circ TD)$ exists such that, for any $u \in M^*_\Gamma(D)$, there exists a $P \in \mathcal{P}(\varphi \circ \mathcal{A} \circ TD)$ that satisfies $vPu \subset \varphi(\mathcal{A} \circ TD)$. Thus, it yields $Z(D) \subseteq \bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} (\{z_{Q,\delta}(\varphi \circ \mathcal{A} \circ TD)\} \cap \Gamma(D))$.

Now suppose $v \in \left(\bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} (\{z_{Q,\delta}(\varphi \circ \mathcal{A} \circ TD)\} \cap \Gamma(D))\right) \setminus Z(D)$. Then, we have $\{e \in E(T\varphi(\mathcal{A} \circ TD)[M^*_\Gamma(D)]) \mid \tau(e) = v\} = \emptyset$ for some $\varphi \in \Phi(\mathcal{A} \circ TD)$. By IIP*, a $V \subset M^*_\Gamma(D)$ exists such that $v \in V$ and $v \notin Z(TD[V])$. Following the iterative manner, a $u \in M^*_\Gamma(D)$ exists such that $v \notin Z(TD[\{u,v\}])$. Since $M^*_\Gamma(D) \subseteq M^*(D) \in X/R^I_{\mathbf{0}}$ and $\{e \in E(T\varphi(\mathcal{A} \circ TD)[M^*_\Gamma(D)]) \mid \tau(e) = v\} = \emptyset$ for some $\varphi \in \Phi(\mathcal{A} \circ TD)$, $v \notin Z(TD[\{u,v\}])$ implies that $TD[\{u,v\}] = \{u,v\}$, or $TD[\{u,v\}] = C(\{u,v\})$. By DAD, we have $Z(\{u,v\}) = \{u,v\}$, as $\{u,v\} = (u,v) \cap (v,u)$. Hence, $TD[\{u,v\}] =$

$C(\{u,v\})$ and $Z(C(\{u,v\})) = \{u\}$. This contradicts to $v \in M^*_\Gamma(D)$. Consequently, $v \in \bigcup_{\varphi \in \Phi(\mathcal{A}\circ TD)} (\{z_{Q,\delta}(\varphi \circ \mathcal{A} \circ TD)\} \cap \Gamma(D))$ implies $v \in Z(D)$, and the claim holds for all $D \in \mathcal{D}$ such that $M^*(D) \in X/R^I_{\mathbf{0}}$. *Q.E.D.*

Completion. Let $V \in 2^X \setminus \{\emptyset\}$ be arbitrary. Denote by $\omega(V)$ the string that links every consecutive elements following a given permutation of V. Let $\Omega(V)$ be the set of all such strings on V. Clearly, $\Phi(\mathcal{A} \circ T((V,\emptyset))) = \Omega(V)$ and $\Gamma((V,\emptyset)) = V$. Hence, by Claim 1.14, $Z^*(V) = Z((V,\emptyset)) = \{z_{Q,\delta}(\omega(V)) \mid \omega \in \Omega(V)\}$. Note that, given $\delta : X \to \{\mathbf{0},\mathbf{1}\}$, each $\omega \in \Omega(V)$ represents a particular linear order on V, and $Q \subset X \times X$ is connex and transitive. Therefore, by Rubinstein and Salant (2006, Proposition 3, .pp 10-11), $Z^* : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$ satisfies WARP. *Q.E.D.*

1.A.4.3. Proof of Proposition 1.3

Proof. Let $\succ^*, R_{\mathbf{0}}, R_{\mathbf{1}}, \sim^* \subset X \times X$ and $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0},\mathbf{1}\}$ follow the definitions given in Theorem 1.2 and Proposition 1.1. Since $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC* and IIP*, Claim 1.8-Claim 1.12 hold as primitives.

Define a mapping $\gamma_I : [X]^2 \to [X]^1 \cup \{\emptyset\}$ by

$$\gamma_I(\{u,v\}) := \begin{cases} \{u\}, & Z(\{u,v\}) = \{v\}; \\ \{v\}, & Z(\{v,u\}) = \{u\}; \\ \emptyset, & \text{otherwise} \end{cases}.$$

Moreover, for every $\{u,v\} \in [X]^2$, let $\gamma_C(\{u,v\}) = \gamma(\{u,v\})$. Then, by IIP*, for every $D \in \mathcal{D}$, the followings hold for every $u, v \in V(D)$.

$$[TD[\{u,v\}] = \{u,v\} \wedge \gamma_I(\{u,v\}) = \{v\}] \implies v \notin Z(D);$$
$$[TD[\{u,v\}] = C(\{u,v\}) \wedge \gamma_C(\{u,v\}) = \{v\}] \implies v \notin Z(D).$$

Hence, for every $D \in \mathcal{D}$, we have

$$Z(D) \subseteq \Gamma^*(D) := V(D) \setminus \left(\left(\bigcup_{\{u,v\} \in \mathcal{Y}(D)} \gamma_C(\{u,v\}) \right) \cup \left(\bigcup_{\{u,v\} \in \mathcal{H}(D)} \gamma_I(\{u,v\}) \right) \right).$$

Fix an arbitrary $D \in \mathcal{D}$.

Suppose there exist a $v \in V(D)$ and a $\varphi_D \in \Phi(\mathcal{A} \circ TD)$ such that $v \in Z_\Delta(\varphi_D \circ \mathcal{A} \circ TD) \cap \Gamma^*(D)$. By the definitions of Z_Δ and Γ^*, for every $u \in V(D)$, it holds that

$$TD[\{u,v\}] = (u,v) \implies \Delta((u,v)) \neq \{\mathbf{0}\} \iff v \in Z((u,v));$$
$$TD[\{u,v\}] = (v,u) \implies \Delta((v,u)) \neq \{\mathbf{1}\} \iff v \in Z((v,u));$$
$$TD[\{u,v\}] = \{u,v\} \implies \gamma_I(\{u,v\}) \neq \{v\} \iff v \in Z(\{u,v\});$$
$$TD[\{u,v\}] = C(\{u,v\}) \implies \gamma_C(\{u,v\}) \neq \{v\} \iff v \in Z(C(\{u,v\})).$$

Hence, by IIP*, $v \in Z(TD[V_3])$ for any $V_3 \in [V(D)]^3$ with $v \in V_3$. By iteration, we have $v \in Z(TD[V_{n-1}])$ for any $V_{n-1} \in [V(D)]^{n-1}$, where $\#V(D) = n$. As a result, $v \in Z(TD[V])$ for every $V \subset V(D)$ that contains v. If an induced partition $\{D_i\}_i$ of D exists such that $V = \bigcup_{D_j \in \{D_i\}_i} Z(D_j) \subset V(D)$, then by IIP*-(i), $v \in Z(TD[V]) = Z(D)$. Oppositely, if $\bigcup_{D_j \in \{D_i\}_i} Z(D_j) = V(D)$ for all induced partition $\{D_i\}_i$, then by IIP*-(ii), $v \in V(D) = Z(D)$. Thus, it follows that $\bigcup_{\varphi_D \in \Phi(\mathcal{A} \circ TD)} (Z_\Delta(\varphi_D \circ \mathcal{A} \circ TD) \cap \Gamma^*(D)) \subseteq Z(D)$.

Suppose $v \in Z(D)$. Then, by Claim 1.12, for any $u \in V(D)$, $\mathcal{A} \circ TD[\{u, v\}] \neq (u, v)$ if $uR_{\mathbf{0}}v$, and $\mathcal{A} \circ TD[\{u, v\}] \neq (v, u)$ when $uR_{\mathbf{1}}v$. In addition, by Claim 1.11, we have $\neg(u \succ^* v)$ for all $u \in V(D)$. Under the definition of $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$, for every $e \in E(\mathcal{A} \circ TD)$, $\Delta(e) \neq \{\mathbf{1}\}$ if $\iota(e) = v$, and $\Delta(e) \neq \{\mathbf{0}\}$ when $\tau(e) = v$. Hence, a $\varphi_D \in \Phi(\mathcal{A} \circ TD)$ exists such that $v \in Z_\Delta(\varphi_D \circ \mathcal{A} \circ TD)$. Since $v \in Z(D) \subseteq \Gamma^*(D)$, it yields $Z(D) \subseteq \left(\bigcup_{\varphi_D \in \Phi(\mathcal{A} \circ TD)} Z_\Delta(\varphi_D \circ \mathcal{A} \circ TD)\right) \cap \Gamma^*(D)$. The proposition follows. *Q.E.D.*

1.A.4.4. Proof of Proposition 1.4

Proof. Since $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, and IIP*, Theorem 1.2 and Proposition 1.1 hold. Fix an arbitrary nonempty $V \in 2^X$.

(i) For any $D, \tilde{D} \in \mathcal{D}$ with $V(D) = V(\tilde{D}) = V$, it follows that $M^*(D) = M^*(\tilde{D})$. By Claim 1.11, $Z(D) \subseteq M^*(D) \cap \Gamma(D)$ for all $D \in \mathcal{D}$. Hence, let

$$Cl(V) := \{v \in V \mid \forall u \in V, \neg(u \succ^* v)\},$$

then it holds that $Z(D) \subseteq Cl(V)$ for all $D(V, E) \in \mathcal{D}$. By Claim 1.12, if a $v \in Cl(V)$ exists such that $v \in Z(D)$ for all $D(V, E) \in \mathcal{D}$, then $v \succ^* u$ or $v \sim^* u$, and $\gamma(\{u, v\}) \neq \{v\}$, for all $u \in V$. Define $Core(V) \subseteq V$ by

$$Core(V) := \left\{ v \in V \,\middle|\, \forall u \in V, \begin{bmatrix} \neg(v \succ^* u) \Rightarrow v \left(R^P_{\mathbf{0}} \cup R^P_{\mathbf{1}} \cup \sim^*\right) u \\ \wedge \quad \gamma(\{u, v\}) \neq \{v\} \end{bmatrix} \right\}.$$

Then, it follows that $Core(V) \subseteq Z_{\succsim_\delta}(S)$ for all $S(V, E) \in \mathcal{S}$, and $Core(V) \subseteq \Gamma(D)$ for all $D(V, E) \in \mathcal{D}$. That is, for any $D(V, E) \in \mathcal{D}$, $Core(V)$ would be selected from every $\varphi \in \Phi(\mathcal{A} \circ TD)$. Hence, $Core(V) \subseteq Z(D) \subseteq Cl(V)$ for any $D(V, E) \in \mathcal{D}$.

(ii) Let $\hat{V} := V \setminus Cl(V)$. By definition, for any $u \in \hat{V}$, there is a $v \in Cl(V)$ such that $v \succ^* u$. Thus, by Claim 1.8 and Claim 1.9-(g)(h), we have $\neg(uR_{\mathbf{0}}v \vee uR_{\mathbf{1}}v)$ for any $u \in \hat{V}, v \in Cl(V)$.[15] Let $\hat{E} \subset Cl(V) \times \hat{V}$ be the set that satisfies $v \succ^* u$ for any $(v, u) \in \hat{E}$, and $\#\{e \in \hat{E} \mid \tau(e) = u\} = 1$ for all $u \in \hat{V}$.

Claim 1.8 implies that $R^P_{\mathbf{0}}, R^P_{\mathbf{1}}$ are transitive on $Cl(V)$, and by DAD, $Z(\{u, v\}) = \{u, v\}$ for every $u, v \in Cl(V)$. Let $V^1_{\mathbf{0}} \subseteq Cl(V)$ be the largest $R^I_{\mathbf{0}}$-indifferent subset of $Cl(V)$, and let $V^2_{\mathbf{0}}$ be the largest such subset of $Cl(V) \setminus V^1_{\mathbf{0}}$, and so on. Then, it yields a collection of $R^I_{\mathbf{0}}$-indifferent subsets $\{V^1_{\mathbf{0}}, \dots, V^k_{\mathbf{0}}\}$. Similarly, let $\{V^1_{\mathbf{1}}, \dots, V^l_{\mathbf{1}}\}$ be the

[15] It is shown in the proof of Claim 1.8 that for any $u, v, w \in X$, $u \succ^* v, vR^P_\delta w$ imply $u \succ^* w$.

collection of $R_{\mathbf{1}}^{I}$-indifferent subsets. Denote by V_0 the set of remaining elements of $Cl(V)$, and let $\{V^1, \ldots, V^m\}$ be the collection of all singleton subsets of V_0. Then, by Claim 1.10, $\mathcal{V}(V) := \{V^1, \ldots, V^m, V_{\mathbf{0}}^1, \ldots, V_{\mathbf{0}}^k, V_{\mathbf{1}}^1, \ldots, V_{\mathbf{1}}^l\}$ defines a partition of $Cl(V)$. By construction, $R_{\mathbf{0}}^P \cup R_{\mathbf{1}}^P \cup \sim^*$ is connex on V_0, and Claim 1.10, Claim 1.9-(a)(b) jointly imply that $v^0 \sim^* v^1$ for any $v^0 \in V_{\mathbf{0}}^i, v^1 \in V_{\mathbf{1}}^j$ with $1 \leq i \leq k, 1 \leq j \leq l$. Moreover, by Claim 1.9 and the transitivity of $R_{\mathbf{0}}^P, R_{\mathbf{1}}^P$, it follows that:

a1) $\forall u, v, w \in Cl(V),\ \neg\left(uR_{\mathbf{0}}^P v \wedge vR_{\mathbf{1}}^P w\right) \wedge \neg\left(uR_{\mathbf{1}}^P v \wedge vR_{\mathbf{0}}^P w)\right)$;

a2) $\forall V^i, V^j \in \mathcal{V}(V), \forall Q \in \{R_{\mathbf{0}}^P, R_{\mathbf{1}}^P, \sim^*\}$,

$$\exists u \in V^i, \exists v \in V^j, uQv \Longrightarrow \forall v^i, \forall v^j, v^i Q v^j;$$

a3) $\forall V^i, V^j, V^h \in \mathcal{V}(V)$,

$$\begin{aligned}
&\left[\begin{array}{r} \forall u \in V^i, \forall v \in V^j, \forall w \in V^h, \\ uR_{\mathbf{0}}^P w \ \wedge\ vR_{\mathbf{1}}^P w \end{array}\right] \Rightarrow \left[\begin{array}{r} \forall V^r \in \mathcal{V}(V), \forall \tilde{v} \in V^r, \\ \tilde{v}\left(R_{\mathbf{0}}^P \cup R_{\mathbf{1}}^P \cup \sim^*\right) w \end{array}\right]; \\
&\left[\begin{array}{r} \forall u \in V^i, \forall v \in V^j, \forall w \in V^h, \\ wR_{\mathbf{0}}^P u \ \wedge\ wR_{\mathbf{1}}^P v \end{array}\right] \Rightarrow \left[\begin{array}{r} \forall V^r \in \mathcal{V}(V), \forall \tilde{v} \in V^r, \\ w\left(R_{\mathbf{0}}^P \cup R_{\mathbf{1}}^P \cup \sim^*\right) \tilde{v} \end{array}\right].
\end{aligned} \tag{1.5}$$

Hence, under the transitivity of $R_{\mathbf{0}}^P, R_{\mathbf{1}}^P$, a permutation $(V_1, \ldots V_n)$ of $\mathcal{V}(V)$ exists such that, for any $V_i, V_j \in \{V_1, \ldots V_n\}$, $i < j$ implies

$$\forall v_i \in V_i, \forall v_j \in V_j,\ (v_j R_{\mathbf{0}}^P v_i) \vee (v_i R_{\mathbf{1}}^P v_j) \vee (v_j \sim^* v_i). \tag{1.6}$$

Fix this permutation. Since there is an $i \in \{1, \ldots, n\}$ such that $u \in V_i$ for every $v \in Cl(V)$, to simplify the statement, let v_i (resp., v_j, v_h, u_r, w_t) imply $v_i \in V_i$ (resp., V_j, V_h, V_r, V_t) in $(V_1, \ldots, V_n)$. Then, by Eq (1.5) and Eq (1.6), we have

$$\left[\begin{array}{l} \exists V_i, V_j, V_h \in \mathcal{V}(V), \\ v_i R_{\mathbf{0}}^P v_h \ \wedge\ v_i R_{\mathbf{1}}^P v_j \end{array}\right] \Rightarrow \left[\begin{array}{l} \forall V_r, V_t \in \mathcal{V}(V) : r \geq i \geq t, \\ \forall v_r \in V_r, \forall v_t \in V_t, \forall v_i \in V_i, \\ v_i\left(R_{\mathbf{1}}^P \cup \sim^*\right) v_r \ \wedge\ v_i\left(R_{\mathbf{0}}^P \cup \sim^*\right) v_t \end{array}\right]; \tag{1.7}$$

$$\left[\begin{array}{l} \exists V_i, V_j, V_h \in \mathcal{V}(V), \\ v_j R_{\mathbf{0}}^P v_i \ \wedge\ v_h R_{\mathbf{1}}^P v_i \end{array}\right] \Rightarrow \left[\begin{array}{l} \forall V_r, V_t \in \mathcal{V}(V) : r \geq i \geq t, \\ \forall v_r \in V_r, \forall v_t \in V_t, \forall v_i \in V_i, \\ v_r\left(R_{\mathbf{0}}^P \cup \sim^*\right) v_i \ \wedge\ v_t\left(R_{\mathbf{1}}^P \cup \sim^*\right) v_i \end{array}\right]. \tag{1.8}$$

Fix an arbitrary $V^* \subseteq Cl(V)$. Without loss of generality, suppose that $Core(V) \subseteq V^*$. Define the following subsets of $\tilde{V} := Cl(V) \setminus V^*$:

$$\begin{aligned}
\tilde{V}_\delta^I &:= \left\{u \in \tilde{V} \mid \exists v \in Cl(V),\ vR_\delta^I u\right\}, \quad \delta = \mathbf{0}, \mathbf{1}; \\
\tilde{V}_\delta^P &:= \left\{u \in \tilde{V} \mid \exists v \in Cl(V),\ vR_\delta^P u \ \wedge\ \nexists v' \in Cl(V),\ uR_\delta^I v'\right\}, \quad \delta = \mathbf{0}, \mathbf{1}; \\
\tilde{V}_+ &:= \left\{u \in \tilde{V} \mid \forall v \in Cl(V), u\left(R_{\mathbf{0}}^P \cup R_{\mathbf{1}}^P \cup \sim^*\right) v\right\}.
\end{aligned}$$

Then, by (a1), $\tilde{V} = \tilde{V}_{\mathbf{0}}^P \cup \tilde{V}_{\mathbf{1}}^P \cup \tilde{V}_{\mathbf{0}}^I \cup \tilde{V}_{\mathbf{1}}^I \cup \tilde{V}_+$, and every pair of these sets excepting $\tilde{V}_{\mathbf{0}}^P, \tilde{V}_{\mathbf{1}}^P$ are disjoint, where $\tilde{V}_{\mathbf{0}}^P \cap \tilde{V}_{\mathbf{1}}^P$ satisfies Eq (1.8). Moreover, if $\tilde{V}_+ \neq \emptyset$, then, for any $u \in \tilde{V}_+$, a $v \in Cl(V)$ exists such that $\gamma(\{u, v\}) = \{u\}$.

The idea is that, if any of $\tilde{V}_\delta^I$ and $\tilde{V}_\delta^P$ is nonempty, say $\tilde{V}_{\mathbf{0}}^I \neq \emptyset$, then construct a connected DAG $D = D^* = D_*$ that satisfies $Core(V) \cup V^* \subseteq Z(D) \subset Cl(V)$ and

$\tilde{V}_{\mathbf{0}}^I \setminus Z(D) \neq \emptyset$. Otherwise, define a $D_*(V, E_*)$ that satisfies $\tilde{V}_+ \cap Z(D_*) = \emptyset$, where $\{e \in E_* \mid \iota(e) = v \vee \tau(e) = v\} \neq \emptyset$ holds for all $v \in V$.

Let $\tilde{E} \subset V \times V$ and $J \subset \{1, \ldots, n\}$ be the sets given by the following criteria:

b1) for any $u_i \in \tilde{V}_{\mathbf{0}}^I$ and any $u_s \in \tilde{V}_{\mathbf{1}}^I$,

$$[V_i \cap V^* \neq \emptyset] \Rightarrow \left[\forall v \in V_i \cap V^*,\, (v, u_i) \in \tilde{E}\right];$$
$$\begin{bmatrix} V_i \cap V^* = \emptyset \quad \wedge \\ \exists k \geq i, v_k R_{\mathbf{0}}^P u_i \end{bmatrix} \Rightarrow \begin{bmatrix} i \in J \ \wedge\ \forall v_j \in V_j,\, (v_j, u_i) \in \tilde{E} \\ (j = \min_{k \geq i} k \ \text{ s.t. }\ v_k R_{\mathbf{0}}^P u_i) \end{bmatrix};$$
$$[V_s \cap V^* \neq \emptyset] \Rightarrow \left[\forall v \in V_s \cap V^*,\, (u_s, v) \in \tilde{E}\right];$$
$$\begin{bmatrix} V_s \cap V^* = \emptyset \quad \wedge \\ \exists l \leq s, v_l R_{\mathbf{1}}^P u_s \end{bmatrix} \Rightarrow \begin{bmatrix} s \in J \ \wedge\ \forall v_r \in V_r,\, (u_s, v_r) \in \tilde{E} \\ (r = \max_{l \leq s} l \ \text{ s.t. }\ v_l R_{\mathbf{1}}^P u_s) \end{bmatrix};$$

b2) for any $V_i \in \mathcal{V}(V)$ that satisfies $V_i \subseteq \tilde{V}_{\mathbf{0}}^I$ and $\neg(v_j R_{\mathbf{0}}^P u_i)$ for all $j \geq i$, we have $i \in J$, and a unique $v_i^0 \in V_i$ exists such that

$$\begin{aligned}
&\forall u \in V_i \setminus \{v_i^0\},\, (v_i^0, u) \in \tilde{E}; \\
&\begin{bmatrix} i \neq 1 \wedge \\ \exists k \leq i,\, V_k \cap V^* \neq \emptyset \end{bmatrix} \Rightarrow \begin{bmatrix} \forall v_h \in V_h \cap V^*,\, (v_h, v_i^0) \in \tilde{E} \\ (h = \max_{k \leq i} k \ \text{ s.t. }\ V_k \cap V^* \neq \emptyset) \end{bmatrix}; \\
&\begin{bmatrix} i \neq n \wedge \\ \exists k \geq i,\, V_k \cap V^* \neq \emptyset \end{bmatrix} \Rightarrow \begin{bmatrix} \forall v_j \in V_j \cap V^*, \forall u \in V_i \setminus \{v_i^0\}, \\ (u, v_j) \in \tilde{E} \\ (j = \min_{k \geq i} k \ \text{ s.t. }\ V_k \cap V^* \neq \emptyset) \end{bmatrix}.
\end{aligned} \tag{1.9}$$

b3) for any $V_s \in \mathcal{V}(V)$ which satisfies $V_s \subseteq \tilde{V}_{\mathbf{1}}^I$ and $\neg(v_r R_{\mathbf{1}}^P u_s)$ for all $r \leq s$, we have $s \in J$, and a unique $v_s^1 \in V_s$ exists such that

$$\begin{aligned}
&\forall u \in V_s \setminus \{v_s^1\},\, (u, v_s^1) \in \tilde{E}; \\
&\begin{bmatrix} s \neq n \wedge \\ \exists l \geq s,\, V_l \cap V^* \neq \emptyset \end{bmatrix} \Rightarrow \begin{bmatrix} \forall v_t \in V_t \cap V^*,\, (v_s^1, v_t) \in \tilde{E} \\ (t = \min_{l \geq s} l \ \text{ s.t. }\ V_l \cap V^* \neq \emptyset) \end{bmatrix}; \\
&\begin{bmatrix} s \neq 1 \wedge \\ \exists l \leq s,\, V_l \cap V^* \neq \emptyset \end{bmatrix} \Rightarrow \begin{bmatrix} \forall v_r \in V_r \cap V^*, \forall u \in V_s \setminus \{v_s^1\}, \\ (v_r, u) \in \tilde{E} \\ (r = \max_{l \leq s} l \ \text{ s.t. }\ V_l \cap V^* \neq \emptyset) \end{bmatrix}.
\end{aligned} \tag{1.10}$$

b4) for any $u_i \in \tilde{V}_{\mathbf{0}}^P$ and any $u_s \in \tilde{V}_{\mathbf{1}}^P \setminus \tilde{V}_{\mathbf{0}}^P$,

$$\begin{aligned}
&i \in J \ \wedge\ \forall v_j \in V_j : (j = \min_{k \geq i} k \ \text{ s.t. }\ v_k R_{\mathbf{0}}^P u_i),\, (v_j, u_i) \in \tilde{E}; \\
&s \in J \ \wedge\ \forall v_r \in V_r : (r = \max_{l \leq s} l \ \text{ s.t. }\ v_l R_{\mathbf{1}}^P u_s),\, (u_s, v_r) \in \tilde{E}.
\end{aligned}$$

Let $J^c := \{1, \ldots, n\} \setminus J$. Define a set $E_J \subset Cl(V) \times Cl(V)$ by

$$E_J := \left\{(v_i, v_j) \ \middle|\ v_i \in V_i, v_j \in V_j \wedge i \in J^c \wedge j = \min_{k > i,\, k \in J^c} k\right\}.$$

Define $D^*(V, E^*)$ by $E := \tilde{E} \cup E_J \cup \hat{E}$. Then, by construction, D^* is a connected DAG. Moreover, by (b1)–(b4) and Eq (1.6)Eq (1.7)Eq (1.8), for any $e \in E(TD^*)$,

$\varDelta(e) \neq \{\mathbf{1}\}$ when $\iota(e) \in V^*$, and $\varDelta(e) \neq \{\mathbf{0}\}$ when $\tau(e) \in V^*$. Hence, for any $v \in V^*$, a $\varphi \in \varPhi(D^*)$ exists such that $v \in Z_\varDelta(\varphi(D^*))$.[16] Consequently, $(Core(V) \cup V^*) \subseteq Z(D^*)$.

Suppose that $Core(V) \cup V^* \cup \tilde{V}_+ \subset Cl(V)$. Then, not all of (b1)–(b4) are vacuous. By (b4), for any $u \in \tilde{V}^P_{\mathbf{0}} \cup \tilde{V}^P_{\mathbf{1}}$, an $e \in E^*$ exists such that $u \notin Z(e)$. By (b1)–(b3), for every $u \in \tilde{V}^I_{\mathbf{0}} \cup \tilde{V}^I_{\mathbf{1}}$, if u is neither the vertex v^0_i in Eq (1.9) nor v^1_s in Eq (1.10), then an $e \in E^*$ exists such that $u \notin Z(e)$. Since D^* is a DAG, $TD^*[\{u, v\}]$ cannot be a cycle for any $u, v \in V$. Hence, by IIP*, $Core(V) \cup V^* \cup \tilde{V}_+ \subset Cl(V)$ implies $Z(D^*) \subset Cl(V)$.

Suppose that $Cl(V) \setminus (Core(V) \cup V^*) = \tilde{V}_+$. In this case, $\tilde{E} = \emptyset$. Let $\tilde{E}_\gamma \subset Cl(V) \times Cl(V)$ be the set that satisfies:

c1) $\forall u, v \in Cl(V), (u, v) \in \tilde{E}_\gamma \iff (v, u) \in \tilde{E}_\gamma$;

c2) $\forall u \in \tilde{V}_+, \exists! v \in Cl(V), \left[\gamma(\{u, v\}) = \{u\} \wedge (u, v), (v, u) \in \tilde{E}_\gamma\right]$.

Let $V_\gamma := \{v \in Cl(V) \mid \exists u \in Cl(V), (u, v) \in \tilde{E}_\gamma\}$, and let $\tilde{J} := \{i \in \{1, \dots, n\} \mid V_i \not\subseteq \tilde{V}_+\}$.[17] Define a set $E_{\tilde{J}} \subset V^* \times V^*$ accordingly by

$$E_{\tilde{J}} := \left\{ (v_i, v_j) \;\middle|\; v_i \in V_i \setminus V_\gamma, v_j \in V_j \setminus V_\gamma \,\wedge\, i \in \tilde{J} \,\wedge\, j = \min_{k > i,\, k \in \tilde{J}} k \right\}.$$

Define $D_*(V, E_*)$ by $E_* = \hat{E} \cup \tilde{E}_\gamma \cup E_{\tilde{J}}$. Then, for any $v \in V$, a $u \in V$ exists such that $(u, v) \in E_*$ or $(v, u) \in E_*$. By Eq (1.6)Eq (1.7)Eq (1.8) and Theorem 1.2, for any $v \in Core(V) \cup V^*$ and any $u \in V$, $v \in Z(TD_*[\{u, v\}])$. In particular, since a $v \in Cl(V)$ exists such that $TD[\{u, v\}] = C(\{u, v\})$ with $\gamma(\{u, v\}) = \{u\}$, thus $u \notin \Gamma(D_*)$ holds for any $u \in \tilde{V}_+$. Hence, by Theorem 1.2 and IIP*, $(Core(V) \cup V^*) = Z(D_*) \subset Cl(V)$. *Q.E.D.*

1.A.4.5. Proof of Proposition 1.5

Proof. Given a $\mathcal{G} = \left(V, N, \{(\varDelta^i, \gamma^i)\}_{i \in N}, \{D_i\}_{i \in N}\right)$, denote by $\mathcal{N}(\mathcal{G})$ the set of Nash equilibria of $\mathcal{G}$. Let $v \in V$ be an arbitrary allocation.

Suppose $v \in G(\mathcal{G})$. By definition, $v \in Z^i(D_i)$ for every $i \in N$, meaning that, for every $i \in N$, (a) $\{v\} \neq \gamma^i(\{u, v\})$ for any $u \in V$ with $TD_i[\{u, v\}] = C(\{u, v\})$, and (b) a $\varphi_i \in \varPhi(\mathcal{A} \circ TD_i)$ exists such that $v \in Z_{\varDelta^i}(\varphi_i \circ \mathcal{A} \circ TD_i)$, under Proposition 1.1. Hence, it follows that $\varDelta^i((u, v)) \neq \{\mathbf{0}\}$ for every $u \in V$ that satisfies $T\varphi_i(\cdot)[\{u, v\}] = (u, v)$, and $\varDelta^i((v, u)) \neq \{\mathbf{1}\}$ for all $u \in V$ with $T\varphi_i(\cdot)[\{u, v\}] = (v, u)$. By Eq $(*)$, since $\varDelta^i((w, \tilde{w})) \neq \{\mathbf{0}\}$ for any $w, \tilde{w} \in V$, we have $TD_i[\{u, v\}] \neq (v, u)$ and $TD_i[\{u, v\}] \neq C(\{u, v\})$ for all $u \in V$ with $\varDelta^i((v, u)) = \{\mathbf{1}\}$. As a result, for every $i \in N$ and any $u \in V$, $u \succ^i v$ implies that $TD_i[\{u, v\}] = (u, v)$ or $TD_i[\{u, v\}] = \{u, v\}$. Thus, $G(\mathcal{G}) \subseteq \mathcal{N}(\mathcal{G})$ as $v \in \mathcal{N}(\mathcal{G})$.

Now suppose $v \in \mathcal{N}(\mathcal{G})$. Then, for every $i \in N$, we have $v \succsim^i u$ for any $u \in V$ with $(v, u) \in E(TD_i[\{u, v\}])$. By Eq $(*)$, it yields that, for every $i \in N$, $\varDelta^i((v, u)) = \{\mathbf{0}, \mathbf{1}\}$

[16] See Eq (1.4) ff. in Appendix 1.A.4.1.

[17] Note that, for every $u_i \in \tilde{V}_+$, $V_i = \{u_i\}$ is singleton.

for any $u \in V$ with $TD_i[\{u,v\}] = (v,u)$, and $\gamma^i(\{u,v\}) \neq \{v\}$ for any $u \in V$ with $TD_i[\{u,v\}] = C(\{u,v\})$. Moreover, $\Delta^i((w,\tilde{w})) \neq \{\mathbf{0}\}$ for any $w, \tilde{w} \in V$. Hence, for every $i \in N$, a $\varphi_i \in \Phi(\cdot)$ exists such that $v \in Z_{\Delta^i}(\varphi_i(\cdot))$, meaning that $v \in Z^i(D_i)$ for all $i \in N$. Consequently, $\mathcal{N}(\mathcal{G}) \subseteq G(\mathcal{G})$ as $v \in G(\mathcal{G})$. *Q.E.D.*

1.A.5. Corollaries

PROOF OF COROLLARY 1.2.1. Let $\omega(V)$ be the string that links every consecutive elements following a given permutation of V. Denote by $\Omega(V)$ the set of all such strings on V. Then, for any $V \in 2^X \setminus \{\emptyset\}$, each $\omega \in \Omega(V)$ corresponds to a permutation of V. Clearly, $T((V,\emptyset)) = (V,\emptyset)$, $\mathcal{A}((V,\emptyset)) = (V,\emptyset)$, and $\Gamma((V,\emptyset)) = V$ for any $V \in 2^X \setminus \{\emptyset\}$. Thus, $\Phi(\mathcal{A} \circ TD) = \Omega(V)$. Consequently, by Proposition 1.1, for any $V \in 2^X \setminus \{\emptyset\}$,

$$Z((V,\emptyset)) = \bigcup_{\omega \in \Omega(V)} Z_{\Delta}(\omega(V)) = V \setminus \left(\bigcap_{\omega \in \Omega(V)} V \setminus Z_{\Delta}(\omega(V)) \right).$$

Let $Q(V) := \bigcap_{\omega \in \Omega(V)} V \setminus Z_{\Delta}(\omega(V))$. Fix a $V \in 2^X \setminus \{\emptyset\}$ and a $q \in Q(V)$ arbitrarily. Then, by the definition of Z_{Δ}, for any $\omega \in \Omega(V)$, a $v \in V$ exists such that $\Delta((v,q)) = \{\mathbf{0}\}$, or $\Delta((q,v)) = \{\mathbf{1}\}$. Let $\omega^+(V)$ denote the string in which $E(T\omega^+(V)[\{v,q\}]) = (v,q)$ for all v with $\Delta((q,v)) = \{\mathbf{1}\}$. Note that $q \in Q(V)$ implies $q \notin Z_{\Delta}(\omega^+(V))$. Hence, a $v \in V$ exists such that $\Delta((v,q)) = \{\mathbf{0}\}$ and $\Delta((q,v)) = \{\mathbf{1}\}$. Consequently, $Q(V) = \{q \in V \mid \exists v \in V, \{\Delta((v,q)) = \{\mathbf{0}\} \wedge \Delta((q,v)) = \{\mathbf{1}\}\}\}$. As a result, $Z^*(V) = Z((V,\emptyset)) = V \setminus Q(V) = \{v \in V \mid \nexists u \in V, u \succ^* v\}$ for all $V \in 2^X \setminus \{\emptyset\}$. By Claim 1.8, $\succ^* \subset X \times X$ is transitive. *Q.E.D.*

PROOF OF COROLLARY 1.2.2. Given Theorem 1.2 and Corollary 1.2.1, it suffices to show $\succ^*$ is a semiorder under Relevance. By Relevance, for any $u, v \in X$, it is impossible to have $Z((u,v)) = \{u,v\}$ and $Z((v,u)) = \{v,u\}$. That is, $\sim^* = \emptyset$. Hence, define a binary relation $\succsim^S := \succ^* \cup R^P_\delta \cup R^I_\delta$, then $\succsim^S \subset X \times X$ is connex, where $\delta = \{\mathbf{0},\mathbf{1}\}$. By Claim 1.9-(c)(d), for any $u,v,w \in X$, it is impossible to have $uR^P_{\mathbf{0}}v$ and $wR^P_{\mathbf{1}}v$. Let $\succ^S := \succ^*$ and $\sim^S := R^P_\delta \cup R^I_\delta$. Clearly, $\succ^S$ is asymmetric. Let $u,v,x,y \in X$ be arbitrary. Suppose $u \succ^* x$, $x \sim^* y$, and $y \succ^S v$. Then, by Claim 1.8 and Claim 1.9-(g)(h), $u \succ^S y$, and hence, $u \succ^S v$. Suppose $x \succ^S u$, $u \succ^S y$, and $u \sim^S v$. If $uR^I_\delta v$ for $\delta = \mathbf{0},\mathbf{1}$, then by Claim 1.9-(g)(h), $x \succ^* v$ and $v \succ^* y$. If $uR^P_\delta v$ for $\delta = \mathbf{0},\mathbf{1}$, then by Claim 1.8, $x \succ^S v$. If $vR^P_\delta u$ for $\delta = \mathbf{0},\mathbf{1}$, then by Claim 1.8, $v \succ^S y$. Thus, $x \sim^S v$ and $v \sim^S y$ cannot be true simultaneously. Hence, $\succ^S \subset X \times X$ is a semiorder. *Q.E.D.*

PROOF OF COROLLARY 1.2.3. (i) By Theorem 1.1, $\succsim \subset X \times X$ has a unique decomposition $\succsim = \succ \cup \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$. Let $Q_{\mathbf{0}} := \succ \cup \sim_{\mathbf{0}}$ and $Q_{\mathbf{1}} := \succ \cup \sim_{\mathbf{1}}$. Then, $(Q_{\mathbf{0}}, Q_{\mathbf{1}})$ is the pair that satisfies the statement (i).

(ii) Let $S \in \mathcal{S}$ be arbitrary. By the definition of Z_{Q_δ} and $(Q_{\mathbf{0}}, Q_{\mathbf{1}})$, we have $Z_{Q_\delta}(S) \subseteq M(S)$. Then, by Claim 1.3, $Z_{Q_\delta}(S) \in X/{\sim_{\mathbf{0}}}$ or $Z_{Q_\delta}(S) \in X/{\sim_{\mathbf{1}}}$. Suppose $Z_{Q_\delta}(S) \subseteq M(S) \in X/{\sim_{\mathbf{0}}}$. Then, for any $u \in M(S)$, if $\{e \in E(TS[M(S)]) \mid \tau(e) = u\} \neq \emptyset$, then $u \notin Z_{Q_\delta}(S)$. For S, a unique $v \in M(S)$ exists such that $\{e \in E(TS[M(S)]) \mid \tau(e) = v\} = \emptyset$, i.e., the first $\succsim$-maximal vertex. Suppose $Z_{Q_\delta}(S) \subseteq M(S) \in X/{\sim_{\mathbf{1}}}$, then for any $u \in M(S)$, $\{e \in E(TS[M(S)]) \mid \iota(e) = u\} \neq \emptyset$ implies $u \notin Z_{Q_\delta}(S)$. Similarly, there is a unique $v \in M(S)$ that satisfies $\{e \in E(TS[M(S)]) \mid \iota(e) = v\} = \emptyset$, the last $\succsim$-maximal vertex. Hence, $Z_{Q_\delta}(S)$ is singleton in both cases, and by the definition of $z_{\succsim,\delta}$, we have $Z_{Q_\delta}(S) = \{z_{\succsim,\delta}(S)\}$. *Q.E.D.*

Proof of Corollary 1.2.4. (i) By Theorem 1.2, $(u,v) \in (\succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}})$ if and only if $Z((u,v)) = Z((v,u)) = \{u\}$. Hence, for all $(u,v) \in (\succsim_{\mathbf{0}} \cup \succsim_{\mathbf{1}})$, we have $\hat{z}((u,v) = \hat{z}((v,u)) = u$, as $\hat{z}(D) \in Z(D)$ for all $D \in \mathcal{D}$. Thus, it yields that $u \hat{\succ} v$ for all $(u,v) \in (\succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}})$.

(ii) Let $u, v \in X$ be arbitrary and suppose that $u \succsim_{\mathbf{0}} v$ or $u \succsim_{\mathbf{1}} v$. Then, by the definition of $\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}}$, we have $v \notin Z((u,v))$ or $v \notin Z((v,u))$. Since $\hat{z}(D) \in Z(D)$ for all $D \in \mathcal{D}$, it follows that $v \neq \hat{z}((u,v))$ or $v \neq \hat{z}((v,u))$. Hence, we have $(u \hat{\succ} v) \vee (u \hat{\sim}_{\mathbf{0}} v) \vee (u \hat{\sim}_{\mathbf{1}} v)$. That is, $u \hat{\succsim} v$. *Q.E.D.*

CHAPTER 2

Uncertainty Attitude and Variable Information Structures

2.1. Introduction

Seminal models of decision making under uncertainty often consider information *implicit* and *fixed* (e.g., Anscombe and Aumann (1963); Schmeidler (1989); Gilboa and Schmeidler (1989); Klibanoff et al. (2005)). On this account, the information a decision maker (DM) perceived, as well as the imprecision or ambiguity inherent in the decision problem, are often reflected by *parts* (conditions) of the representations.[1] Meanwhile, attitude towards uncertainty is a primary feature to which the revealed choice behavior is often attributed. However, when we apply the seminal models to the environments of uncertainty that involve fixed contingencies with associated consequences yet allow exogenous information about the likelihoods of contingencies to vary, we might experience difficulties in isolating the influence of information on the uncertainty attitude which the DM's choice exhibits. Namely, seminal models often become uninformative about whether and how the revealed uncertainty attitude translates among the choices made under different information. Explicating the translatability of uncertainty attitude requires a model that accommodates information-dependency of choice under uncertainty, thereby necessitating a formalism that takes variable information as primitive. The objective of this chapter is to develop such a model by explicitly incorporating variable information into the benchmark subjective expected utility (SEU, Anscombe and Aumann (1963)) framework and connect the representation of preference to behavioral definitions of uncertainty attitude.

In literature, there are attempts to treat exogenous information as variables. Olszewski (2007); Ahn (2008); Gajdos et al. (2004, 2008); Hayashi (2012) modeled information by sets of probability distributions over either the state space or the outcome space. Since information is incorporated as a margin of choice objects, these models thus inevitably require "rich" observations on "large" domains of choice.[2] Among

[1] For example, the perceived information is reflected by the unique set of priors in Gilboa and Schmeidler (1989), by the first- and second-order belief in Klibanoff et al. (2005); Seo (2009).

[2] For instance, the preference studied in Gajdos et al. (2008) includes rankings over different acts paired with different sets of probabilities over the state space, that is, rankings of the form $(f, P) \succsim (g, Q)$, where f and g are acts while P and Q denote sets of probabilities.

the cited papers, Gajdos et al. (2008); Hayashi (2012) connected subjective beliefs to objective information and achieved a separation between the way the DM processes information and the machinery by which she assesses choice objects. Due to this, information being specified by sets of probabilities becomes indispensable for the representations obtained in these papers. Such a specification is arguably restrictive since, in some instances, objective information about likelihoods of contingencies might be too vague to be captured by sets of probabilities, yet nonetheless influences choice.[3] In this strand of literature, information is variable in the sense that it may deliver different underlying knowledge of likelihoods with various degrees of imprecision.

We, however, focus on a subtler aspect of variable information. In the real-world, a fixed knowledge of likelihoods of contingencies (e.g., a statistical observation) can be described in different frames, and one may interpret each of such descriptions as a piece of information about likelihoods. We refer to those descriptive frames as *information structures*. The DM's choice may depend on the information structure in which a fixed knowledge of likelihoods of contingencies is described to her, even when the contingencies and associated consequences are fixed and of which she is fully aware. More notably, the content of the knowledge becomes irrelevant for the machinery of information-dependency in this vein.

	R	B	Y	G
ℓ_1	\$100	\$100	\$0	\$0
ℓ_2	\$100	\$0	\$100	\$0
ℓ_3	\$0	\$100	\$0	\$100
ℓ_4	\$0	\$0	\$100	\$100

	Announced frequencies $\rho(\cdot)$
I_{rb}	$\rho(\{\text{R},\text{B}\}) \approx .5,\ \rho(\{\text{Y}\}) \approx .2,\ \rho(\{\text{G}\}) \approx .3$
I_{ry}	$\rho(\{\text{R},\text{Y}\}) \approx .5,\ \rho(\{\text{B}\}) \approx .2,\ \rho(\{\text{G}\}) \approx .3$
I_g	$\rho(\{\text{R},\text{B},\text{Y}\}) \approx .7,\ \rho(\{\text{G}\}) \approx .3$
I_N	——

TABLE 2.1 – Four-Color Urn with Variable Information

To clarify this idea, consider a four-color urn (Ellsberg (1961)) in the following manner. An urn contains balls in four colors: Red (R), Black (B), Yellow (Y) and Green (G). The total quantity and that of each color are unknown. However, from a series of trials in each of which one ball was drawn from and returned into the urn, the host observed that the frequency $\rho(\cdot)$ of each color being drawn is *approximately* as

[3]The choice problem illustrated in Table 2.1 is an example.

follows: $\rho(\{\mathrm{R}\}) \approx \rho(\{\mathrm{G}\}) \approx .3$ and $\rho(\{\mathrm{B}\}) \approx \rho(\{\mathrm{Y}\}) \approx .2$. Yet these frequencies are undisclosed to the DM. Now, a single ball will be drawn from the urn. As illustrated in Table 2.1, the DM is asked to rank bets $\{\ell_1, \ell_2, \ell_3, \ell_4\}$, after a piece of information in $\{I_{rb}, I_{ry}, I_g, I_N\}$ is announced. In these hypothetically parallel situations, the DM may reveal: $\ell_1 \sim \ell_4 \succ \ell_3 \succsim \ell_2$ when I_{rb} is given; $\ell_2 \sim \ell_3 \succ \ell_4 \succsim \ell_a$ for I_{ry}; $\ell_4 \sim \ell_3 \succ \ell_2 \sim \ell_1$ for I_g; and $\ell_1 \sim \ell_2 \sim \ell_3 \sim \ell_4$ for I_N. In this example, the available knowledge of likelihoods of contingencies, "$\rho(\{\mathrm{R}\}) \approx \rho(\{\mathrm{G}\}) \approx .3$ and $\rho(\{\mathrm{B}\}) \approx \rho(\{\mathrm{Y}\}) \approx .2$," is *fixed*, and information varies in the sense that this knowledge is described in different frames. Nevertheless, if we discriminate the rankings observed under different information, the example suggests that such variation of information may lead the DM to rank the bets differently. Moreover, changes of information do not always alter rankings over some bets (e.g., $\ell_3 \succ \ell_1$ for both I_{ry} and I_g, and $\ell_4 \succ \ell_2$ for both I_{rb} and I_g), meaning that choice behavior may possess certain consistency up to such variation of information, which in turn is related closely to the information-dependency of choice alluded to above.

We, therefore, consider an environment wherein a fixed knowledge of likelihoods of contingencies is described in variable information structures. Incorporated into the SEU framework, we specify information structures as partitions of the state space and study a family of preferences over Anscombe-Aumann (AA) acts which are indexed by partitions.[4] This formalism is of significant interest since it yields naturally the separation between the situation being "more ambiguous" and the preference revealing "more uncertainty-averse" with minimum technical requirements.

Equipped with this primitive, we axiomatize a representation which we call *bi-criterion expected utility* (BCEU). The representation admits a functional form of the following twofold expectation:

$$U(f, L) = \sum_{E \in \Omega(L)} \left(\int_E u(f)\, dV \right) \frac{P_L(E)}{V(E)};$$
$$\Omega(L) = \{E \in L \mid V(E) > 0\},$$

where f is an AA act, L is a partition of the state space, and the integral takes the form of Choquet integral. As for its elements, u is a von Neumann-Morgenstern (vNM) index, V is a capacity over the state space, and P_L is a probability measure defined on the σ-algebra induced by L. When information is given with structure L, the DM assesses AA act f as if she first computes the conditional expected utility of f on each learnable event $E \in L$ with respect to vNM index u and capacity V, then evaluates the overall expected payoff with respect to probability P_L. Capacity V can be viewed as the DM's pure subjective belief, while P_L captures how she interprets the fixed

[4] Dillenberger et al. (2014) studied a similar specification of information by a parsimonious model which they named partition-learning.

knowledge when it is given in information structure L. We also find that the suggested decision rule is equivalent to that given by Choquet expected utility (CEU, Schmeidler (1989)) with partition-dependent belief formation. In this way, our model also separates, from the machinery of act assessment, the functional relationship between information structures and willingness-to-bet of each event.[5] We then relate BCEU to the definition of comparative uncertainty aversion by Ghirardato and Marinacci (2002) and that proposed by Epstein (1999) to study several characteristic conditions regarding the translatability of uncertainty attitude.

The rest of the chapter is organized as follows: Section 2.2 introduces the preliminaries. Section 2.3 presents the representations under different impositions of axioms. Translatability of uncertainty attitude are studied in Section 2.4. Finally, Section 2.5 summarizes discussion. Proofs are included in Appendix 2.A.

2.2. Preliminaries

Let S be a finite set of states and $\mathscr{P}(S)$ be the power set of S. Each element of $\mathscr{P}(S)$ is called an event and is denoted typically by E, B or D. Let Σ be a σ-algebra of S. Then, we call a set function $\mu : \Sigma \to [0,1]$ a capacity on Σ if $\mu(\emptyset) = 0$, $\mu(S) = 1$, and for any $B, D \in \Sigma$, $B \subseteq D$ implies $\mu(B) \leq \mu(D)$. All the integrals stand for the Choquet integral throughout.

Denote by L a typical partition of S, and let $\Sigma(L)$ be the σ-algebra (of S) generated by L with respect to the set union and intersection. In particular, let $\underline{L} = \{S\}$ and $\overline{L} = \{\{s\} \mid s \in S\}$. Denote by $\mathcal{L}$ the set of all partitions of S (including $\underline{L}$). Each partition $L \in \mathcal{L}$ corresponds to an information structure. Define $\mathscr{L} = \{(L, \tilde{L}) \in \mathcal{L} \times \mathcal{L} \mid \Sigma(L) \subseteq \Sigma(\tilde{L})\}$. That is, if $(L, \tilde{L}) \in \mathscr{L}$, then $\tilde{L}$ carries more detailed (finer) information about the underlying knowledge of likelihoods than L does. Clearly, $\mathcal{L}$ defines a partial order under which $(\mathcal{L}, \mathscr{L})$ becomes a lattice. Let A be the set of consequences. Denote by $\Pi(A)$ the set of all simple lotteries over A. An act is defined by function $f : S \to \Pi(A)$. Let $\mathcal{H}$ denote the set of all acts, and denote by $\mathcal{H}_C$ the set of all constant acts in $\mathcal{H}$. The act space $\mathcal{H}$ is endowed with a mixture operation defined by

$$\Big(\alpha f + (1-\alpha) g\Big)(s) = \alpha f(s) + (1-\alpha) g(s), \quad \forall s \in S, \forall \alpha \in [0,1].$$

For any $f, g \in \mathcal{H}$ and $E \in \mathscr{P}(S)$, let $[fEg]$ be the binary act such that for all $s \in S$, $[fEg](s) = f(s)$ when $s \in E$, and $[fEg](s) = g(s)$ otherwise.

[5] Ghirardato and Marinacci (2001) studied a general class of preferences, which they name biseparable preferences, and introduced the term willingness-to-bet to interpret the monotone set function obtained in the representation. It is known that CEU is a form of biseparable preferences.

For every $L \in \mathcal{L}$, let $\succsim_L \subset \mathcal{H} \times \mathcal{H}$ be a binary relation, where $\succ_L$ and $\sim_L$ denote the asymmetric and symmetric part of $\succsim_L$, respectively. For every $L \in \mathcal{L}$, we extend $\succsim_L$ to $\Pi(A)$ by identifying lotteries with constant acts. The object being of our interest is the family of those binary relations $\{\succsim_L\}_{L \in \mathcal{L}}$.

For every $L \in \mathcal{L}$, let $\mathcal{H}_L$ denote the set of all $\Sigma(L)$-measurable acts. For every $L \in \mathcal{L}$ and any $f \in \mathcal{H}$, denote by $\underline{f}^L$ the constant act such that $f(s) \succsim_L \underline{f}^L(s)$ on S and $f(\hat{s}) = \underline{f}^L(\hat{s})$ for some $\hat{s} \in S$. Define, for any $L \in \mathcal{L}$, $D \in \mathscr{P}(S)$ and $c \in \mathcal{H}_L$,

$$\mathcal{C}_D^L[c] = \left\{ h \in \mathcal{H} \mid h = [hDc] \wedge c = \underline{h}^L \right\}; \quad \mathcal{C}_D^L = \bigcup_{c \in \mathcal{H}_C} \mathcal{C}_D^L[c].$$

Given $L \in \mathcal{L}$ and $E \in L$, it is worth noting the implication of acts in $\mathcal{C}_E^L$ (and those in $\mathcal{C}_E^L[c]$ for a fixed $c \in \mathcal{H}_C$) in terms of the demand for informativeness, since this type of acts play a central role in formalizing axioms, and the suggested implication is closely related to the normative appeal of our axioms. By definition, an $f \in \mathcal{C}_E^L$ can be viewed as a guaranteed minimal payoff $\underline{f}^L(\cdot)$ (according to $\succsim_L$) associated with a "gain-bet" over the sub-events of E. Thus, $\succsim_L$ over these acts, say $f, g \in \mathcal{C}_E^L$, arguably depend on the DM's assessment of the gain-bets induced by f and g, and on the trade-off between the anticipated utility increments delivered by those gain-bets and the utility given by the corresponding minimal payoffs. Notice that $\underline{f}^L(\cdot)$ and $\underline{g}^L(\cdot)$ do not involve any subjective uncertainty, and that the gain-bets are given over the sub-events of E. Hence, the uncertainty inherent in ranking f and g can be described by $(E, \mathscr{P}(E))$ which is beyond the descriptive power of L. In particular, when $f, g \in \mathcal{C}_E^L[c]$ for a fixed c, they deliver a "sure-thing" payoff since $c(\cdot) = \underline{f}^L(\cdot) = \underline{g}^L(\cdot)$. Thus, it is reasonable to ascribe the preference over f and g to the DM's assessment of the aforementioned gain-bets which solely depends on how she judges the *relative* likelihoods of event in $(E, \mathscr{P}(E))$. Consequently, the information carried by L is (should be) irrelevant for such a demand for informativeness.[6]

2.3. Axioms and Results

In this section, we first present the baseline axioms that characterize our general representation, then study two of its specific variations under stronger impositions in terms of consistency.

[6]For example, let $L = \{\{\mathrm{R}, \mathrm{B}, \mathrm{Y}\}, \{\mathrm{G}\}\}$ and $E = \{\mathrm{R}, \mathrm{B}, \mathrm{Y}\}$ in the four-color urn considered in Section 2.1. Then, L uniquely identifies information I_g, and $\ell_1, \ell_2 \in \mathcal{C}_E^L[\$0]$ provided that $\$100 \succsim_L \$0$. In this case, I_g is completely irrelevant for the ranking between ℓ_1 and ℓ_2 because the rankings over such acts (should) depend solely on the information about how the total quantity of "non-green" balls is distributed over R, B and Y, instead of the information about the quantity itself.

2.3.1. Bi-Criterion Expected Utility

The basic axioms that characterize our general representation are of two types. Axiom 2.1-Axiom 2.6 are the assumptions for rationality which are applied to each fixed information structure L. In particular, the first five are the standard impositions (Fishburn (1970); Schmeidler (1989)) and collectively denoted as the SEU-CEU axioms. The subsequent axioms (Axiom 2.7 and Axiom 2.8) address assumptions for consistency across variable information structures.

Axiom 2.1 (Order). *For any $L \in \mathcal{L}$, $\succsim_L$ is complete and transitive.*

Axiom 2.2 (Continuity). *For any $L \in \mathcal{L}$ and any $f, g, h \in \mathcal{H}$ with $f \succ_L g \succ_L h$, there exist $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ_L g \succ_L \beta f + (1-\beta)h$.*

Axiom 2.3 (Monotonicity). *For any $L \in \mathcal{L}$ and $f, g \in \mathcal{H}$, if $f(s) \succsim_L g(s)$ on S, then $f \succsim_L g$.*

Axiom 2.4 (Nondegeneracy). *For every $L \in \mathcal{L}$, not for all $f, g \in \mathcal{H}$, $f \sim_L g$.*

Axiom 2.5 (Relevance Independence, RI).

(I) For every $L \in \mathcal{L}$, any $f, g, h \in \mathcal{H}_L$, and any $\alpha \in (0,1)$, $f \succsim_L g$ if and only if $\alpha f + (1-\alpha)h \succsim_L \alpha g + (1-\alpha)h$.

(II) For every $L \in \mathcal{L}$, any $E \in L$, every pairwise comonotonic $f, g, h \in \mathcal{C}_E^L$, and any $\alpha \in (0,1)$, $f \succsim_L g$ if and only if $\alpha f + (1-\alpha)h \succsim_L \alpha g + (1-\alpha)h$.[7]

RI can be interpreted in line with the objective and subjective rationality introduced by Gilboa et al. (2010). They argued that a choice is rational in the objective sense if the DM can convince others of the rightness of her choices, while it is subjectively rational if others cannot convince the DM that she is making the wrong choice. As we translate into the AA framework, their paper considered the independence axiom an assumption for the objective rationality.[8] For a fixed partition L, the uncertainty inherent in $\Sigma(L)$-measurable acts is described by $(S, \Sigma(L))$, and the given information structure L can

[7]For a fixed $L \in \mathcal{L}$, acts $f, g \in \mathcal{H}$ are said to be *comonotonic* if, for any $s, t \in S$, $f(s) \succ_L f(t)$ implies $g(s) \succsim_L g(t)$.

[8]$\succsim$ on $\mathcal{H}$ satisfies *independence* if for all $f, g, h \in \mathcal{H}$ and $\alpha \in (0,1)$, $f \succsim_L g$ if and only if $\alpha f + (1-\alpha)h \succsim_L \alpha g + (1-\alpha)h$.

accommodate such a demand for informativeness. Since the choices revealed by $\succsim_L$ on $\mathcal{H}_L$ convey how the DM interprets the given information, the information carried by L thus provides certain statistical evidence for the choices. Hence, we consider $\succsim_L$ on $\mathcal{H}_L$ reflects the choices that are objectively rational, and thus impose the independence axiom upon the restriction of $\succsim_L$ on $\mathcal{H}_L$ (i.e., RI-(I)). Meanwhile, we maintain that the comonotonic independence is normative for the preference over certain acts when the given information is irrelevant for the assessment of those acts. Consider the implication of RI-(II) in conjunction with RI-(I). As argued in the previous section, an act in $\mathcal{C}^L_E$, for a learnable event $E \in L$, can be viewed as a minimal payoff associated with a gain-bet over the sub-events of E. Notice that $\mathcal{H}_L \cap \mathcal{C}^L_E \neq \emptyset$, and that each act in this intersection involves a degenerated gain-bet (constant) over the sub-events of E.[9] For such acts, RI-(I) implies that the trade-off between the degenerated gain-bets and the minimal payoffs is performed in a way obeying the independence axiom.[10] Thus, RI-(II) is essentially an assumption for the assessment of the gain-bets induced by acts in $\mathcal{C}^L_E$ which is purely subjective. As a result, we consider comonotonic independence normative for $\succsim_L$ on each $\mathcal{C}^L_E$.

Axiom 2.6 (Dominance). *For any $L \in \mathcal{L}$, any $h \in \mathcal{H}$ and $f \in \mathcal{H}_L$, if $\underline{f}^L \succsim_L \underline{h}^L$ and $[hE\underline{h}^L] \succsim_L [fE\underline{h}^L]$ (resp., $\precsim_L$) for all $E \in L$, then $h \succsim_L f$ (resp., $\precsim_L$).*

Let f be a $\Sigma(L)$-measurable act and h be a normal act. For each learnable event $E \in L$, $[fE\underline{h}^L]$ and $[hE\underline{h}^L]$ offer a sure-thing payoff $\underline{h}^L(\cdot)$ provided that $\underline{f}^L \succsim_L \underline{h}^L$. As argued, the preference over $[fE\underline{c}]$ and $[hE\underline{c}]$ conveys how the DM's evaluates the gain-bets over E that are respectively induced by f and h. Notice that $[fE\underline{h}^L]$ induces a degenerated gain-bet over learnable event E. Thus, Dominance states that, given an act h and its minimal payoff, if for every learnable event the DM prefers the subjective gain-bet induced by h more than (resp., less than) the gain yielded by an objective lottery, then she must prefer h over act f that gives the corresponding lottery on each learnable event (resp., prefers f over h).

Our first result observes that the DM's preference for each fixed information structure admits the CEU representation, where the influence of information structure is manifested in the formation of the non-additive belief (the capacity). The implication of the result will be discussed after the formal statement. For every $L \in \mathcal{L}$, let $\mathscr{S}(L)$ be the semi-ring (of S) given by $\mathscr{S}(L) = \bigcup_{E \in L} \mathscr{P}(E)$.

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[9]In the example given in Section 2.1, let $L = \{\{\mathrm{R},\mathrm{B}\},\{\mathrm{Y}\},\{\mathrm{G}\}\}$ and $E = \{\mathrm{R},\mathrm{B}\}$. Then, $\ell_1 \in \mathcal{H}_L \cap \mathcal{C}^L_E$ provided that \$100 $\succsim_L$ \$0, and it involves a degenerated gain-bet over $\{\mathrm{R},\mathrm{B}\}$ since both $\{\mathrm{R}\}$ and $\{\mathrm{B}\}$ yield \$100.

[10]This is also compatible with RI-(II) since acts in $\mathcal{H}_L \cap \mathcal{C}^L_E$ are pairwise comonotonic.

THEOREM 2.1. *Given an $L \in \mathcal{L}$, the following statements are equivalent.*

(I) $\succsim_L \subset \mathcal{H} \times \mathcal{H}$ *satisfies the SEU-CEU axioms and Dominance.*

(II) There exist an affine function $u_L : \Pi(A) \to \mathbb{R}$ *and a unique monotone set function* $V_L : \mathscr{S}(L) \to [0,1]$ *such that for all* $f, g \in \mathcal{H}$,

$$f \succsim_L g \iff \int_S u_L(f)\, d\mu_L \geq \int_S u_L(g)\, d\mu_L,$$

where $\mu_L : \mathscr{P}(S) \to [0,1]$ *is the unique capacity such that for all* $B \in \mathscr{P}(S)$,

$$\mu_L(B) = \sum_{E \in L} V_L(B \cap E).$$

Moreover, u_L *is unique up to positive linear transformation (p.l.t.).*

PROOF. See Appendix 2.A.2.

For a fixed information structure, the SEU-CEU axioms and Dominance jointly characterize the following utility representation. Presented with the information given in structure L, the DM places a weight on every sub-event of each learnable event in L. Such a weighting is captured by the monotone set function $V_L : \mathscr{S}(L) \to [0,1]$. Notably, V_L assigns a probability $V_L(E)$ to each learnable event $E \in L$ (notice that $\mu_L(S) = 1 = \sum_{E \in L} V_L(E)$) which reflects her judgment of the likelihoods of learnable events based on the given information. According to this weighting, the DM judges the likelihoods of other events by unpacking the event into the largest sub-events of the learnable ones and summing the weights of those sub-events given by V_L. This defines the non-additive belief μ_L over $\mathscr{P}(S)$. The influence of information structure is captured by such a process of belief formation, and the latter exhibits framing effect since the assessed likelihood of an event depends on how this event is unpacked, that is, depends on which events are learnable. The utility for each act is then computed by CEU with respect to belief μ_L and a vNM index u_L.

Now, we introduce the additional axioms which deal with consistency properties. In formalizing these axioms, we consider preferences over a specific subset of acts and focus on the demand for informativeness discussed in Section 2.2. The basic idea is that if two pieces of information carried by different structures are both irrelevant for assessing a set of acts, then the preference over this set of acts should be consistent under those information structures. The next axiom, Certainty, states that preferences over objective lotteries (hence, preferences over constant acts) are consistent across variable information structures.

AXIOM 2.7 (Certainty). *For any* $L, L' \in \mathcal{L}$ *and any* $c, c' \in \mathcal{H}_C$, $c \succsim_L c'$ *if and only if* $c \succsim_{L'} c'$.

Axiom 2.8 (Consistency of Irrelevance, CoI). *For any* $(L, \tilde{L}) \in \mathscr{L}$*, any* $E \in \tilde{L} \setminus L$*, and any* $c \in \mathcal{H}_C$*, if* $\mathcal{C}_E^{\tilde{L}}[c] \cap \mathcal{C}_E^{L}[c]$ *is not* $\succsim_{\tilde{L}}$*-indifferent, then for any* $f, g \in \mathcal{C}_E^{\tilde{L}}[c] \cap \mathcal{C}_E^{L}[c]$*,* $f \succsim_L g$ *if and only if* $f \succsim_{\tilde{L}} g$*.*[11]

CoI considers information refinements in conjunction with sets of acts which differ from some fixed constant act only on a event. It suggests that, if sub-events of an event E are not learnable from information structure L, nor from a finer information structure $\tilde{L}$, then acts that deliver improvements from some fixed constant act only on E are ranked the same under L and $\tilde{L}$. Let $L, \tilde{L}$ and E be as in the statement. As argued in Section 2.2, preference over acts $f, g \in \mathcal{C}_E^{\tilde{L}}[c] \cap \mathcal{C}_E^{L}[c]$ (should) solely depends on the DM's evaluation of the gain-bets over sub-events of E, which are induced by f and g. Since $E \in \tilde{L} \setminus L$, both $\tilde{L}$ and L are irrelevant for ranking f and g, which is thus purely subjective. Due to this, f and g should be ranked the same under L and $\tilde{L}$, even though $\tilde{L}$ is finer than L and E itself is learnable from $\tilde{L}$. On this account, we consider CoI a normative assumption for consistency.

Now, we are ready to characterize the entire family of preferences $\{\succsim_L\}_{L\in\mathcal{L}}$. The next theorem shows that the SEU-CEU axioms, Dominance, Certainty, and CoI characterize $\{\succsim_L\}_{L\in\mathcal{L}}$ by a general representation which we call bi-criterion expected utility. We first state the formal definition of this representation.

Definition 2.1 (Bi-Criterion Expected Utility). We say $\{\succsim_L\}_{L\in\mathcal{L}}$ admits a *bi-criterion expected utility* (BCEU) representation if there exist an affine function $u : \Pi(A) \to \mathbb{R}$, a unique capacity $V : \mathscr{P}(S) \to [0, 1]$, and for every $L \in \mathcal{L}$, there is a unique probability measure $P_L : \Sigma(L) \to [0, 1]$ such that for all $L \in \mathcal{L}$ and $f, g \in \mathcal{H}$, $f \succsim_L g$ if and only if $U(f, L) \geq U(g, L)$, where

$$U(f, L) = \sum_{E\in\Omega(L)} \left(\int_E u(f)\, dV\right) \frac{P_L(E)}{V(E)};$$
$$\Omega(L) = \{E \in L \mid V(E) > 0\}.$$

Moreover, u is unique up to p.l.t.

When such a tuple $(u, V, \{P_L\}_{L\in\mathcal{L}})$ exists, we call it a BCEU representation.

BCEU can be read as follows. Each AA act delivers information-independent utility $u(\cdot)$, and also being independent of information, the DM holds a subjective belief V about the state space. Given the information carried by structure L, the DM entertains a probabilistic interpretation of information P_L over the learnable events. To evaluate act f, she first coarsens the inherent uncertainty by computing the conditional expected

[11]Given an $L \in \mathcal{L}$, a nonempty set of acts $\mathcal{F} \subseteq \mathcal{H}$ is $\succsim_L$*-indifferent* if for any $f, g \in \mathcal{F}$, $f \sim_L g$.

utility on each learnable event in the sense of CEU applying her subjective belief V, so that it can be fully assessed under the descriptive power of information structure L. The overall utility for f is then evaluated by taking the expectation of those conditional expected utilities on learnable events against P_L. The twofold expectation suggests that the DM tends to evaluate AA acts in a way adapting to the descriptive frames given by information structures, where the coarsening at the first stage and non-additivity of subjective belief allow for framing effect. Although the framing effect exhibited in BCEU and the one observed in Theorem 2.1 have different manifestations, the next theorem also shows that the resulting decision rules are behaviorally equivalent.

THEOREM 2.2. *The following statements are equivalent.*

(I) $\{\succsim_L\}_{L\in\mathcal{L}}$ *satisfies the SEU-CEU axioms, Dominance, Certainty and CoI.*

(II) $\{\succsim_L\}_{L\in\mathcal{L}}$ *admits a BCEU representation.*

(III) There exist an affine real function u *on* $\Pi(A)$*, a unique capacity* V *on* $\mathscr{P}(S)$*, and for every* $L \in \mathcal{L}$*, there is a unique probability measure* P_L *on* $\Sigma(L)$ *such that for all* $L \in \mathcal{L}$ *and* $f, g \in \mathcal{H}$*,*

$$f \succsim_L g \iff \int_S u(f)\, d\mu_L \geq \int_S u(g)\, d\mu_L,$$

where for every $L \in \mathcal{L}$*,* μ_L *is the unique capacity such that for all* $B \in \mathscr{P}(S)$*,*

$$\mu_L(B) = \sum_{E\in\Omega(L)} \frac{V(E\cap B)P_L(E)}{V(E)}.$$

Moreover, u *is unique up to p.l.t.*

PROOF. See Appendix 2.A.3.

We refer to the formulation of each μ_L given in statement (III) as bi-criterion belief formation. As in BCEU, the DM holds a subjective belief V, and given the information carried by L, she also entertains a probabilistic interpretation P_L over the learnable events. Since L equips each $E \in L$ with stronger evidence, she may adjust her judgment of likelihoods based on the subjective interpretation P_L, and such an adjusted belief is captured by capacity μ_L. The intuition behind the formulation of μ_L is akin to support theory studied by Tversky and Koehler (1994); Rottenstreich and Tversky (1997). They took as primitive descriptions of events, called hypotheses, and considered probability judgment of hypotheses under variable evaluation frames, which consist of mutually exclusive hypotheses. Incorporated into our belief formation, information structure L coincides with a complete collection of mutually exclusive hypotheses. Presented with this evaluation frame, for event B, the DM rather considers how likely the realization of uncertainty would meet the focal hypothesis B given each hypothesis E in evaluation frame L is met. This process leads to the conditional likelihoods $V(B\cap E)/V(E)$, and the adjusted belief $\mu_L(B)$ is obtained by taking the expectation of those conditional likelihoods over L.[12] This is because the realization of uncertainty would meet one of the hypotheses in L, and each $E \in L$ is assigned to a probability $P_L(E)$ by interpreting the information carried by L.

Apart from characterizations, the theorem maintains that BCEU is behaviorally equivalent to the CEU with bi-criterion belief formation. This equivalence is of significant interest since it reduces the information-dependency of choice that hinged on information structures to that of beliefs within CEU. Namely, one can study the implications of the suggested information-dependency of choice by only focusing on the information-adjusted beliefs obtained in the theorem.

Remark 2.1. Recall that $\underline{L} = \{S\}$ and $\overline{L} = \{\{s\} \mid s \in S\}$. It is a straightforward consequence that $\succsim_{\underline{L}}$ admits a standard CEU representation with respect to capacity V, while $\succsim_{\overline{L}}$ has a standard SEU representation with respect to probability measure $P_{\overline{L}} : \mathscr{P}(S) \to [0, 1]$.

2.3.2. More on Consistency

The general representation previously characterized does not specify any connection neither between the subjective belief V and subjective interpretation P_L for each $L \in \mathcal{L}$, nor between subjective interpretations of the information carried by different information structures. Here, we characterize, under stronger axioms of consistency, two specific forms of BCEU which build such connections. The axioms presented in what follows (Axiom 2.9 and Axiom 2.10) are also formulated based on the idea which says preferences over a set of acts should be consistent whenever the corresponding information is irrelevant for assessing those acts. However, they involve more restrictive interpretations of "irrelevance" compared with Certainty and CoI, respectively.

Axiom 2.9 (Consistency of Separation, CoS). *For any $(L, \tilde{L}) \in \mathscr{L}$, $E \in (L \cap \tilde{L}) \cup \{\emptyset\}$ and any $f, g \in \mathcal{C}_E^{\tilde{L}} \cap \mathcal{C}_E^{L}$, $f \succsim_L g$ if and only if $f \succsim_{\tilde{L}} g$.*

CoS states that, if information structure $\tilde{L}$ is finer than L, and an event E is learnable from both L and $\tilde{L}$, then acts that yield improvements from some constant acts only on E (i.e. acts in $\mathcal{C}_E^{\tilde{L}} \cap \mathcal{C}_E^{L}$) are ranked the same under L and $\tilde{L}$. For acts in $\mathcal{C}_E^{\tilde{L}} \cap \mathcal{C}_E^{L}$, CoS considers such a variation of information structures irrelevant because, although $\tilde{L}$ carries the finer information about the likelihoods of sub-events of $S \setminus E$,

[12]The expression of conditional likelihood, $V(B \mid E) = V(B \cap E)/V(E)$, is characterized as an updating rule of ambiguous belief in Gilboa and Schmeidler (1993).

each act in $\mathcal{C}_E^{\tilde{L}} \cap \mathcal{C}_E^L$ is constant and delivers its minimal payoff on $S \setminus E$. In addition, it is a straightforward observation that CoS implies Certainty since it follows that $\mathcal{C}_E^{\tilde{L}} \cap \mathcal{C}_E^L = \mathcal{H}_C$ when $E = \emptyset$.

The following proposition characterizes a class of BCEU representations, as well as the equivalent CEU with bi-criterion belief formation, wherein the subjective interpretations of information P_L are compatible across variable information structures.

Proposition 2.1. *The following statements are equivalent.*

(I) $\{\succsim_L\}_{L\in\mathcal{L}}$ *satisfies the SEU-CEU axioms, Dominance, CoI and CoS.*

(II) $\{\succsim_L\}_{L\in\mathcal{L}}$ *admits a BCEU representation* $(u, V, \{P_L\}_{L\in\mathcal{L}})$*, where a unique probability measure* P *exists on* $\mathscr{P}(S)$ *such that for all* $L \in \mathcal{L}$ *and* $B \in \Sigma(L)$*,* $P_L(B) = P(B)$*.*

(III) There exist an affine real function u *on* $\Pi(A)$*, a unique capacity* V *on* $\mathscr{P}(S)$*, and a unique probability measure* P *on* $\mathscr{P}(S)$ *such that for all* $L \in \mathcal{L}$ *and* $f, g \in \mathcal{H}$*,* $f \succsim_L g$ *if and only if* $\int_S u(f)\, d\mu_L \geq \int_S u(g)\, d\mu_L$*, where for every* $L \in \mathcal{L}$*,* μ_L *is the unique capacity such that for all* $B \in \mathscr{P}(S)$*,*

$$\mu_L(B) = \sum_{E\in\Omega(L)} \frac{V(E\cap B)P(E)}{V(E)}.$$

Moreover, u *is unique up to p.l.t.*

Proof. See Appendix 2.A.4.

We denote the BCEU representation characterized in the proposition by (u, V, P). Recall that, in our formalism, information varies in the sense that a fixed knowledge of likelihoods of contingencies is described in variable information structures. Thus, representation (u, V, P) suggests that the DM's interpretation of information is determined by the underlying knowledge of likelihoods, thereby being independent of information structures in which this knowledge is transmitted. In other words, the DM's judgment of P rejects framing effects, and there is no specified connection between purely subjective belief V and P.

In what follows, we present another class of BCEU representations wherein each P_L and V are related such that the DM's subjective interpretations of information exhibit framing effects.

Axiom 2.10 (Strong Consistency of Irrelevance, SCoI). *For any* $(L, \tilde{L}) \in \mathscr{L}$*, any* $E, E' \in \tilde{L}$*, any* $c \in \mathcal{H}_C$*, and for any* $f \in \mathcal{C}_E^{\tilde{L}}[c] \cap \mathcal{C}_E^L[c]$ *and* $g \in \mathcal{C}_{E'}^{\tilde{L}}[c] \cap \mathcal{C}_{E'}^L[c]$*,* $f \succsim_L g$ *if and only if* $f \succsim_{\tilde{L}} g$*.*

PROPOSITION 2.2. *The following statements are equivalent.*

(I) $\{\succsim_L\}_{L\in\mathcal{L}}$ *satisfies the SEU-CEU axioms, Dominance, Certainty and SCoI.*

(II) $\{\succsim_L\}_{L\in\mathcal{L}}$ *admits a BCEU representation* $(u, V, \{P_L\}_{L\in\mathcal{L}})$, *where for any* $L \in \mathcal{L}$ *and* $E \in L$, $P_L(E) = V(E)/(\sum_{D\in L} V(D))$.

(III) There exist an affine real function u *on* $\Pi(A)$ *and a unique capacity* V *on* $\mathscr{P}(S)$ *such that for all* $L \in \mathcal{L}$ *and* $f, g \in \mathcal{H}$,

$$f \succsim_L g \iff \int_S u(f)\, d\mu_L \geq \int_S u(g)\, d\mu_L,$$

where for every $L \in \mathcal{L}$, μ_L *is the unique capacity such that for all* $B \in \mathscr{P}(S)$,

$$\mu_L(B) = \frac{\sum_{E\in L} V(B\cap E)}{\sum_{E\in L} V(E)}.$$

Moreover, u *is unique up to p.l.t.*

PROOF. See Appendix 2.A.5.

Denote by (u, V) a typical BCEU representation given in statement (II). For a fixed information structure L, the DM interprets the information as if she performs a non-extensional judgment of the likelihoods of learnable events based on her pure subjective belief V. Such a judgment is expressed by the same formula with the one characterized as part of the partition-dependent expected utility (PDEU) representation introduced by Ahn and Ergin (2010) and is also compatible with support theory developed by Tversky and Koehler (1994); Rottenstreich and Tversky (1997). However, in our formalism, the connection between P_L and V has an intrinsically disparate behavioral interpretation. It can be viewed as a manifestation of confirmation bias studied by Tversky and Kahneman (1974); Nickerson (1998), that is, the DM tends to interpret information in a way that supports her pure subjective belief V, where the non-additivity of V allows for framing effects. We address more details about such a difference in Section 2.5.1.

2.4. Uncertainty Attitude

In this section, we connect BCEU to the behavioral definitions of comparative uncertainty attitude and explore the implied characteristic conditions regarding the revealed uncertainty attitude and its translatability under variable information structures. For simplicity, given a BCEU preference, the subsequent analysis will assume that $\mathscr{P}(S)$ does not have any V-null event.

Ghirardato and Marinacci (2002) proposed a behavioral definition of comparative uncertainty aversion with respect to preferences over Savage's domain (Savage (1954)). We restate this definition in our formalism.

DEFINITION 2.2 (Ghirardato and Marinacci (2002)). Given two orderings $\succsim$ and $\succsim'$ on $\mathcal{H}$, we say $\succsim$ is *more uncertainty averse* than $\succsim'$ if for every $c \in \mathcal{H}_C$ and any $f \in \mathcal{H}$, $c \succsim' f$ implies $c \succsim f$, and $c \succ' f$ implies $c \succ f$.

Our first result considers two BCEU preferences with the same vNM index. Doing so allows us to fix the risk attitude of the two families of preferences and focus only on the uncertainty attitude revealed from choices of acts. Applying BCEU to the above definition, the following proposition characterizes a necessary and sufficient condition for a BCEU preference under a given information structure being more uncertainty averse than the other one observed under a different information structure. Besides, we also consider a special case that the two BCEU preferences are identical in the absence of exogenous information (i.e., under $\underline{L}$).

PROPOSITION 2.3. *Let $\{\succsim_L\}_{L\in\mathcal{L}}$ and $\{\succsim'_L\}_{L\in\mathcal{L}}$ be BCEU preferences that admit representation $(u, V, \{P_L\}_{L\in\mathcal{L}})$ and $(u, V', \{P'_L\}_{L\in\mathcal{L}})$, respectively.*

(I) For any $L, \tilde{L} \in \mathcal{L}$, $\succsim_L$ is more uncertainty averse than $\succsim'_{\tilde{L}}$ if and only if for all $B \in \mathscr{P}(S)$,

$$\sum_{E\in L} \frac{V(B\cap E)P_L(E)}{V(E)} \leq \sum_{\tilde{E}\in\tilde{L}} \frac{V'(B\cap\tilde{E})P'_{\tilde{L}}(\tilde{E})}{V'(\tilde{E})}.$$

(II) Suppose that $\succsim_{\underline{L}} = \succsim'_{\underline{L}}$ on $\mathcal{H}\times\mathcal{H}$. Then, for every $L \in \mathcal{L}$, the followings are equivalent: (a) $\succsim_L$ is more uncertainty averse than $\succsim'_L$. (b) $\succsim'_L$ is more uncertainty averse than $\succsim_L$. (c) $P_L = P'_L$ on $\Sigma(L)$. (d) $\succsim_L = \succsim'_L$ on $\mathcal{H}\times\mathcal{H}$.

PROOF. See Appendix 2.A.6.

By Theorem 2.2, a BCEU preference can also be represented by a CEU with bi-criterion belief formation. Thus, statement (I) of the proposition states that $\succsim_L$ is more uncertainty averse than $\succsim'_{\tilde{L}}$ if and only if the information-adjusted belief $\mu'_{\tilde{L}}$ that governs $\succsim'_{\tilde{L}}$ dominates μ_L event-wise. Statement (II) shows that, if $\{\succsim_L\}_{L\in\mathcal{L}}$ and $\{\succsim'_L\}_{L\in\mathcal{L}}$ reflect the same behavior in the absence of exogenous information (i.e., they are identical in purely subjective sense), then facing the same information structure L, the revealed uncertainty attitudes of $\succsim_L$ and $\succsim'_L$ are comparable according to Definition 2.2 if and only if they yield the identical behavior.

In the presence of exogenous information, it is of interest to relate the strength of informativeness to the degree of uncertainty attitude revealed from choices of acts, and in our formalism, $\mathscr{L}$ is indeed an indicator of informativeness. Although it might be intuitive to argue that choices under coarse information would exhibit a higher degree of uncertainty aversion than those under finer information do, this is not true in general for BCEU preferences. The result below presents necessary and sufficient conditions for this statement.

PROPOSITION 2.4. *Let* $\{\succsim_L\}_{L\in\mathcal{L}}$ *admit a BCEU representation* $(u, V, \{P_L\}_{L\in\mathcal{L}})$. *Then, the following statements are equivalent.*

(I) For every $(L, \tilde{L}) \in \mathscr{L}$, $\succsim_L$ *is more uncertainty averse than* $\succsim_{\tilde{L}}$.

(II) There exists a unique probability measure $P \in Core(V)$ *such that* $\{\succsim_L\}_{L\in\mathcal{L}}$ *admits BCEU representation* (u, V, P).[13] *Moreover, for any* $B \in \mathscr{P}(S)$ *and any nonempty disjoint* $D, D' \in \mathscr{P}(S)$,

$$\frac{V(B\cap(D\cup D'))P(D\cup D')}{V(D\cup D')} \leq \frac{V(B\cap D)P(D)}{V(D)} + \frac{V(B\cap D')P(D')}{V(D')}.$$

PROOF. See Appendix 2.A.7.

Epstein (1999) also proposed a definition of comparative uncertainty attitude based on an exogenously identified set of unambiguous acts. The paper defined unambiguous acts using measurability with respect to some fixed algebra, which is closely related to λ-system studied by Zhang (2002); Epstein and Zhang (2001). In our model, each information structure L also yields naturally a focal algebra $\Sigma(L)$, and RI discriminates the set of $\Sigma(L)$-measurable acts $\mathcal{H}_L$ to some extent. Although $\mathcal{H}_L$ does not necessarily, in primitive, corresponds to the set of unambiguous acts, BCEU suggests that $\succsim_L$ on $\mathcal{H}_L$ admits standard SEU representation, meaning that $\mathcal{H}_L$, in result, can be viewed as so under L.[14] Thus, in what follows, we apply BCEU to Epstein's definition. Since this definition is also given on Savage's domain, we start with restating the definition using our notations.

DEFINITION 2.3 (Epstein (1999)). Let Σ be a σ-algebra of S, and denote by $\mathcal{F}$ the set of all Σ-measurable acts in $\mathcal{H}$. Given Σ and two orderings $\succsim$ and $\succsim'$ on $\mathcal{H}$, we say $\succsim$ is *more Σ-uncertainty averse* than $\succsim'$ if for every $f \in \mathcal{F}$ and any $h \in \mathcal{H}$, $f \succsim' h$ implies $f \succsim h$, and $f \succ' h$ implies $f \succ h$.

Now, we consider two BCEU preferences, $\{\succsim_L\}_{L\in\mathcal{L}}$ and $\{\succsim'_L\}_{L\in\mathcal{L}}$, and study the characteristic conditions for $\succsim_L$ being more uncertainty averse than $\succsim'_L$ in the sense of Definition 2.3. However, unlike Proposition 2.3, we only consider comparisons under

[13] For a capacity μ, $Core(\mu)$ denotes the *core* of μ, that is, the set of all probability measures which dominates μ eventwise.

[14] In light of this, Epstein and Zhang proposed a definition of unambiguous events with respect to a given preference relation (Epstein and Zhang (2001, Definition, .p 273)), and it can be shown that events in $\Sigma(L)$ are unambiguous with respect to $\succsim_L$ when we apply this definition to $\mathcal{H}_L$.

the same information structure. This is due to that Definition 2.3 requires a commonly fixed set of unambiguous acts.

Proposition 2.5. *Let* $\{\succsim_L\}_{L\in\mathcal{L}}$ *and* $\{\succsim'_L\}_{L\in\mathcal{L}}$ *admit BCEU* $(u, V, \{P_L\}_{L\in\mathcal{L}})$ *and* $(u, V', \{P'_L\}_{L\in\mathcal{L}})$*, respectively. Then, for any* $L \in \mathcal{L}$*, the followings are equivalent.*

(I) $\succsim_L$ *is more* $\Sigma(L)$*-uncertainty averse than* $\succsim'_L$*.*

(II) $P_L = P'_L$ *on* $\Sigma(L)$*, and for any* $E \in L$ *and* $B \in \mathscr{P}(E)$*,* $V(B)/V(E) \leq V'(B)/V'(E)$*.*

Proof. See Appendix 2.A.8.

Epstein (1999) also introduced a definition of absolute (definitive) uncertainty aversion based on the suggested definition of comparative uncertainty aversion. Our last result shows that, for every BCEU preference and any information structure, if it reveals uncertainty aversion in the sense of Schmeidler (1989, .p 582), then it is uncertainty aversion according to Epstein (1999).

Proposition 2.6. *Let* $\{\succsim_L\}_{L\in\mathcal{L}}$ *admit a BCEU representation* $(u, V, \{P_L\}_{L\in\mathcal{L}})$*. Then, for any* $L \in \mathcal{L}$*, statement (I) implies (II).*

(I) For any $f, g \in \mathcal{H}$ *and any* $\alpha \in (0,1)$*,* $f \sim_L g$ *implies* $\alpha f + (1-\alpha)g \succsim_L f$*.*

(II) There exists a probability measure $P : \mathscr{P}(S) \to [0,1]$ *such that if denote by* $\succsim^P \subset \mathcal{H} \times \mathcal{H}$ *the preference that admits SEU representation* (u, P)*, then* $\succsim_L$ *is more* $\Sigma(L)$*-uncertainty averse than* $\succsim^P$*.*

Proof. See Appendix 2.A.9.

2.5. Discussion

2.5.1. Causes of Framing Effects

As to the formalism, the closet model of which we are aware is the PDEU introduced by Ahn and Ergin (2010). On a finite state space, the primitive of PDEU is also a family of partition-indexed preferences $\{\succeq_L\}_{L\in\mathcal{L}}$ over AA acts. Yet for each partition L, the corresponding preference is given over the set of $\Sigma(L)$-measurable acts instead of the full domain, that is, $\succeq_L \subset \mathcal{H}_L \times \mathcal{H}_L$. A PDEU representation (u, v) has the form

$$\forall L \in \mathcal{L}, \forall h \in \mathcal{H}_L, \quad U_L^{PD}(h) = \int_S u(h)\, d\rho_L,$$

where $\rho_L : \Sigma(L) \to [0,1]$ is the unique probability such that

$$\forall E \in L, \quad \rho_L(E) = \frac{v(E)}{\sum_{D\in L} v(D)}.$$

If we disregard interpretations, then for every partition L, the BCEU given in Proposition 2.2 (i.e., (u, V)) induces, on $\mathcal{H}_L$, the exact functional form of PDEU. In particular, framing effects are expressed by the same formula in PDEU and BCEU (u, V), respectively. However, this is nothing but a fortunate coincidence.

The fundamental difference between BCEU and PDEU lies on the disparate aspects to which framing effects are attributed, respectively. In their paper, each partition corresponds to a description of the state space, and under each description L, an act cannot be identified unless it is $\Sigma(L)$-measurable. Presented with a description L, the perceived uncertainty inherent in the choice problem is thus $(S, \Sigma(L))$, meaning that the variation of such descriptions leads the DM to entertain different coarse understandings of the state space. In this way, the framing effect exhibited in the judgment of each ρ_L is ascribed to unforeseen contingencies, thereby being an (un)awareness issue. Similar approaches that relate choice behavior to (un)awareness also appear in Nehring (2000); Ghirardato (2001); Dekel et al. (2001); Epstein et al. (2007); Karni and Vierø (2013).

Contrarily, in our model, the uncertainty inherent in choice problem is $(S, \mathscr{P}(S))$ across variable information structures. However, for each information structure L, the information carried by L may equip $(S, \Sigma(L))$ with stronger evidence about likelihoods than what the DM entertains previously. Consequently, the implied framing effects are due to the friction between awareness $(S, \mathscr{P}(S))$ and descriptive power of exogenous information $(S, \Sigma(L))$.

2.5.2. Distinguishing Perception and Attitude

Some studies discriminated perceived uncertainty and the DM's attitude towards it to isolate a robust characterization of uncertainty attitude. Nehring (1999, 2009) considered these notions based on incomplete orderings over events, to which he referred as comparative likelihood relations, that summarize information about likelihoods available to the DM. Ghirardato et al. (2004); Hayashi (2012) addressed this separation in the vein of the multiple-prior model (MP, Gilboa and Schmeidler (1989)), wherein the DM's perception of uncertainty is related to the set of priors, and her revealed degree of uncertainty attitude is attributed to other conditions of the corresponding representations. In Hayashi (2012)'s approach, perception of uncertainty is predetermined by exogenously given sets of probabilities that correspond to imprecise information available to the DM. Contrarily in Ghirardato et al. (2004), it is revealed from choices by characterizing, in their terminology, unambiguous preference, hence being a subjective matter. A primary implication of their results is that the characterization of uncertainty attitude revealed from choices is *independent* of the perceived uncertainty.

BCEU, however, has a different implication on this aspect.

Let $\{\succsim_L^A\}_{L\in\mathcal{L}}$ and $\{\succsim_L^B\}_{L\in\mathcal{L}}$ be BCEU preferences, and for $i \in \{A, B\}$, denote by $(u^i, V^i, \{P_L^i\}_{L\in\mathcal{L}})$ the representation. Let $\{\mu_L^i\}_{L\in\mathcal{L}}$ be the family of capacities derived from V^i and $\{P_L^i\}_{L\in\mathcal{L}}$ as in Theorem 2.2, and for every $L \in \mathcal{L}$, let $I^i(\cdot, L)$ denote the functional $f \mapsto \int_S u^i(f)\, d\mu_L^i$. Then, $I^i(\cdot, L)$ represents $\succsim_L^i$ on $\mathcal{H}$.

For $i \in \{A, B\}$, assume that

(S0) $\forall L \in \mathcal{L}, \forall f, g \in \mathcal{H}, \forall \alpha \in (0,1),\ f \sim_L^i g \Rightarrow \alpha f + (1-\alpha) g \succsim_L^i f$.

Then, by Theorem 2.2 and Schmeidler's proposition (Schmeidler (1989, .pp 582-583)), for any $L \in \mathcal{L}$, μ_L^i is convex, and CEU coincides with MP in the sense that

$$I^i(h, L) = \min_{\rho_L^i \in Core(\mu_L^i)} \int_S u^i(h)\, d\rho_L^i, \quad \forall h \in \mathcal{H}.$$

By definition, $\mu_L^i = P_L^i$ on $\Sigma(L)$, meaning that

(A0) $\forall \rho_L^i \in Core(\mu_L^i), \forall B \in \Sigma(L),\ \rho_L^i(B) = P_L^i(B)$.

Suppose further that $u^A = u^B = u$. Then, by Proposition 2.3, for any $L \in \mathcal{L}$,

(H0) $\succsim_L^A$ is more uncertainty averse than $\succsim_L^B$ iff $Core(\mu_L^A) \supseteq Core(\mu_L^B)$.

Connecting (A0) to (H0), we have $P_L^A = P_L^B$ on $\Sigma(L)$. Finally, consider acts in $\mathcal{H}_L$. For $i \in \{A, B\}$, due to the previous observations, it is easy to verify that for any $f, g \in \mathcal{H}_L$,

(G0) $\forall h \in \mathcal{H}, \forall \alpha \in (0,1),\ f \succsim_L^i g \Leftrightarrow \alpha f + (1-\alpha) h \succsim_L^i \alpha g + (1-\alpha) h$;

(G1) $f \succsim_L^i g$ iff $\forall \rho_L^i \in Core(\mu_L^i),\ \int_S u(f)\, d\rho_L^i \geq \int_S u(g)\, d\rho_L^i$;

(G2) $f \succsim_L^A g \Leftrightarrow f \succsim_L^B g$.

To summarize, if the preferences of two BCEU maximizers are (definitive) uncertainty averse in the spirit of Schmeidler (1989) (i.e., condition (S0)), then facing the same information structure, the one with a larger set of priors is revealed to be more uncertainty averse than the other in the sense of Ghirardato and Marinacci (2002). This observation (i.e., condition (H0)) also agrees with the characterization given by Hayashi (2012). Moreover, in this case, they have the same unambiguous preferences, which reveal the same degree of ambiguity in the sense of Ghirardato et al. (2004) (conditions (G0)-(G2)).[15]

The presented argument differs from the implication given by Ghirardato et al. (2004). In this case, $\succsim_L^A$ and $\succsim_L^B$ revealing the same degree of ambiguity becomes necessary for the exhibited attitudes towards uncertainty being comparable, while according to the cited paper whether $\succsim_L^A$ and $\succsim_L^B$ reveal the same degree of ambiguity or not does not, in principle, concern if one is more uncertainty averse than the other.

[15]See Ghirardato et al. (2004) cf. Definition 3, .p 143.

2.5.3. Prediction Rule

For every partition L, the bi-criterion belief formation can be viewed as the process in which the DM predicts the likelihood of each event based on the information carried by L. Then, one may think of $\mu_L(\cdot)$ as the indicator of such a prediction and the determination of μ_L as the corresponding prediction rule.

Recall that, in Proposition 2.1, the family of probability measures $\{P_L\}_{L\in\mathcal{L}}$ is determined by a unique P. Thus, given a BCEU representation (u, V, P), the bi-criterion prediction rule (belief formation) associated with the CEU-counterpart can be rewritten as follows. Let $m : \mathscr{P}(S) \times \mathcal{L} \to [0, 1]$ be the function such that for all $L \in \mathcal{L}$ and $B \in \mathscr{P}(S)$, $m(B, L) = \mu_L(B)$. In addition, given V and P, define a function $v : \mathscr{P}(S) \times \mathscr{P}(S) \to [0, 1]$ by

$$v(B, D) = \frac{V(B \cap D)P(D)}{V(D)}.$$

Then, by Proposition 2.1, we have, for any $B \in \mathscr{P}(S)$ and $L \in \mathcal{L}$,

$$m(B, L) = \sum_{E\in L} v(B, E).$$

The above expression of "global" bi-criterion belief formation takes a reduced form of the case-based prediction rule introduced by Gilboa and Schmeidler (2003), where $\mathscr{P}(S)$ corresponds to both the set of eventualities and that of cases, and each partition L can be interpreted as the predictor's (DM's) perceived knowledge.

2.A. Proofs

2.A.1. Preliminaries

In what follows, we present some auxiliary observations and notations which will be invoked in the subsequent proofs. The following lemma shows that the the SEU-CEU axioms, Dominance and Certainty characterize, for every $L \in \mathcal{L}$, a standard SEU representation for the preference over $\Sigma(L)$-measurable acts, wherein the lottery assessment does not depend on variable partitions.

Lemma 2.1. *The followings are equivalent.*

(I) $\{\succsim_L\}_{L\in\mathcal{L}}$ satisfies the SEU-CEU axioms, Dominance and Certainty.

(II) There exist an affine function $u : \Pi(A) \to \mathbb{R}$, and a unique family of probability measures $\{P_L : \Sigma(L) \to [0,1]\}_{L\in\mathcal{L}}$ such that for any $L \in \mathcal{L}$ and $f, g \in \mathcal{H}_L$,

$$f \succsim_L g \iff \int_S u(f)\, dP_L \geq \int_S u(f)\, dP_L.$$

Moreover, u_L is unique up to p.l.t.

Proof. Assume statement (I). Fix an arbitrary $L \in \mathcal{L}$.

Then, $\succsim_L \subset \mathcal{H}_L \times \mathcal{H}_L$ satisfies Order, Continuity, Monotonicity and Independence. Thus, by AA theorem (Anscombe and Aumann (1963)), there exist an affine function $u_L : \Pi(A) \to \mathbb{R}$ and a unique probability measure $P_L : \Sigma(L) \to [0,1]$ such that for every $f, g \in \mathcal{H}_L$, $f \succsim_L g$ if and only if $\sum_{E\in L} x^f_E P_L(E) \geq \sum_{E\in L} x^g_E P_L(E)$, where for any $f \in \mathcal{H}_L$ and $E \in L$, $x^f_E = u_L(f(s))$ for $s \in E$. Moreover, u_L is unique up to p.l.t. Note that, for any $f \in \mathcal{H}_L$, $f \mapsto \sum_{E\in L} x^f_E P_L(E)$ coincides with the Choquet integral when P_L is additive on $\Sigma(L)$. As a result, for any $f, g \in \mathcal{H}_L$, $f \succsim_L g$ if and only if $\int_S u_L(f)\, dP_L \geq \int_S u_L(g)\, dP_L$.

By Certainty, $(\mathcal{H}_C, \succsim_L)$ and $(\mathcal{H}_C, \succsim_{\hat{L}})$ are order isomorphic for any $L \in \mathcal{L}$. Notice that $\Pi(A)$ is a mixture space with respect to the lottery mixture. As a result, for any $L \in \mathcal{L}$, a bounded continuous and strictly increasing function $\varphi_L : \mathbb{R} \to \mathbb{R}$ exists such that $u_{\hat{L}} = \varphi_L \circ u_L$. Set $u = u_{\hat{L}}$. Since u and u_L are unique up to p.l.t. with satisfying mixture linearity, φ_L is thus linear for any $L \in \mathcal{L}$. Hence, for every $L \in \mathcal{L}$, the previous equivalence is satisfied by u.

The opposite direction is trivial. *Q.E.D.*

Lemma 2.2. *The following statement hold.*

(a) Let $\underline{L} = \{S\}$. $\succsim_{\underline{L}}$ on $\mathcal{H}$ satisfies the SEU-CEU axioms if and only if there exist an affine real function $\underline{u}$ on $\Pi(A)$ and a unique capacity V on $\mathscr{P}(S)$ such that for all $f, g \in \mathcal{H}$, $f \succsim_{\underline{L}} g$ if and only if $\int_S \underline{u}(f)\, dV \geq \int_S \underline{u}(g)\, dV$. Moreover, $\underline{u}$ is unique up to p.l.t.

(b) Let $\overline{L} = \{\{s\} \mid s \in S\}$. $\succsim_{\overline{L}}$ on $\mathcal{H}$ satisfies the SEU-CEU axioms if and only if there exist an affine real function $\overline{u}$ on $\Pi(A)$ and a unique probability measure P on $\mathscr{P}(S)$ such that for all $f, g \in \mathcal{H}$, $f \succsim_{\overline{L}} g$ if and only if $\int_S \overline{u}(f)\, dP \geq \int_S \overline{u}(g)\, dP$. Moreover, $\overline{u}$ is unique up to p.l.t.

Proof. (a) Clearly, $\mathcal{C}^{\overline{L}}_S = \mathcal{H}$ and $\mathcal{H}_{\underline{L}} = \mathcal{H}_C$. Thus, for $\underline{L}$, RI-(II) implies RI-(I), and coincides with Comonotonic Independence axiom. Therefore, the statement is a direct consequence of Schmeidler's theorem (Schmeidler (1989)).

(b) Let $\overline{L} = \{\{s\} \mid s \in S\}$. Then, it holds that $\mathcal{H}_{\overline{L}} = \mathcal{H}$ and $\Sigma(L) = \mathscr{P}(S)$. As a result, the statement is implied by Lemma 2.1. *Q.E.D.*

Consider an arbitrary $L \in \mathcal{L}$. By the previous lemma, there exists a $u_L : \Pi(A) \to \mathbb{R}$ that represents the induced $\succsim_L$ on $\Pi(A)$ (hence, on $\mathcal{H}_C$), and u_L is unique up to p.l.t.[16] Hence, without loss of generality, suppose that $u_L(f(s)) \in X$ for every $f \in \mathcal{H}$ and $s \in S$, where $X \subset \mathbb{R}$ is a closed convex set satisfying $[-1, 1] \subseteq X$.

For each $f \in \mathcal{H}$, define a function $\psi_f : S \to X$ by $\psi_f(s) = (u_L \circ f)(s)$, and typically write ψ whenever the choice of $f \in \mathcal{H}$ does not concern the argument. Let $\Psi = \{u_L \circ f \mid f \in \mathcal{H}\}$ be the set of all such functions. Let $\succcurlyeq_L \subset \Psi \times \Psi$ be the binary relation such that for every $\psi_f, \psi_g \in \Psi$,

$$\psi_f \succcurlyeq_L \psi_g \Longleftrightarrow f \succsim_L g.$$

Then, $\geq$ and $\succcurlyeq_L$ coincide on X. Denote by Ψ_L and Ψ_C the set of $\Sigma(L)$-measurable functions and that of constant functions, respectively. Let $\underline{\psi}^L$ be the Ψ-counterparts (with respect to $\succcurlyeq_L$) of the one defined on $\mathcal{H}$. Then, for every $\psi, \psi' \in \Psi$, it holds that $\underline{\psi}^L(s) = \min_{s \in S} \psi(s)$ on S. For every $D \in \mathscr{P}(S)$ and $\psi_x \in \Psi_C$, define

$$\mathcal{M}_D^L[\psi_x] = \left\{ \psi \in \Psi \mid \psi = [\psi D \psi_x] \wedge \min_{s \in S} [\psi D \psi_x](s) = x \right\},$$

where $x = \psi_x(s)$ on S. Accordingly, let $\mathcal{M}_D^L = \bigcup_{\psi_x \in \Psi_C} \mathcal{M}_D^L[\psi_x]$.

Claim 2.1. *For any $L \in \mathcal{L}$, $D \in \mathscr{P}(S)$ and any $\psi_z \in \Psi_C$, $\mathcal{M}_D^L$ and $\mathcal{M}_D^L[\psi_z]$ are closed under the mixture (henceforth, convex).*

Proof. Fix an arbitrary $D \in \mathscr{P}(S)$. Suppose that $\psi, \psi' \in \mathcal{M}_D^L$ with $\psi = [\psi D \psi_x]$ and $\psi' = [\psi' D \psi_y]$, where $\psi_x(s) = x$ and $\psi_y(s) = y$ on S. Then, for any $\alpha \in [0, 1]$, we have $\alpha\psi + (1-\alpha)\psi' = [(\alpha\psi + (1-\alpha)\psi')D(\alpha\psi_x + (1-\alpha)\psi_y)]$. Since $\psi(s) \geq x$ and $\psi'(s) \geq y$ for any $s \in D$, it follows that $(\alpha\psi + (1-\alpha)\psi')(s) \geq (\alpha x + (1-\alpha)y)$ for any $s \in D$, which implies $(\alpha\psi + (1-\alpha)\psi') \in \mathcal{M}_D^L$.

For any $\psi_z \in \Psi_C$, the convexity of $\mathcal{M}_D^L[\psi_z]$ can be shown by letting $\psi_x = \psi_y = \psi_z$ in the above proof. *Q.E.D.*

2.A.2. Proof of Theorem 2.1

The necessity part is trivial given statement (II). The sufficiency part will follow all the notations and claims given in Appendix 2.A.1. Assume statement (I).

Lemma 2.3. *For every $E \in L$, there exists a unique monotone set function $\mu_L^E : \mathscr{P}(E) \to [0, 1]$ such that for any $\psi, \psi' \in \mathcal{M}_E^L$, $\psi \succcurlyeq_L \psi'$ if and only if $J_E(\psi) \geq J_E(\psi')$, where $J_E(\psi) = \int_E \psi d\mu_L^E + \left(1 - \mu_L^E(E)\right) \min_{s \in S} \psi(s)$.*

[16] Notice that the existence of $u_L : \Pi(A) \to \mathbb{R}$ does not require Certainty.

Proof. Fix an arbitrary $E \in L$. By Claim 3.7, $\mathcal{M}_E^L$ is convex. The restriction of $\succsim_L$ on $\mathcal{M}_E^L$ satisfies Order, Continuity, Monotonicity, Comonotonic Independence, and Nondegeneracy. Thus, by Schmeidler (1989, the prood of Theorem, .p 579), a functional $J_E : \mathcal{M}_E^L \to \mathbb{R}$ exists such that for every $\psi, \psi' \in \mathcal{M}_E^L$, $\psi \succsim_L \psi'$ if and only if $J_E(\psi) \geq J_E(\psi')$, and J_E satisfies:

(a) $\forall \psi_x \in \Psi_C,\ J_E(\psi_x) = x$;

(b) $J_E(\psi) > J_E(\psi')$ implies $J_E(\alpha\psi + (1-\alpha)\psi'') > J_E(\alpha\psi' + (1-\alpha)\psi'')$ for any pairwise comonotonic $\psi, \psi', \psi'' \in \mathcal{M}_E^L$ and any $\alpha \in (0,1)$; and

(c) $\forall \psi, \psi' \in \Psi,\ [\forall s \in S,\ \psi(s) \geq \psi'(s)] \Longrightarrow J_E(\psi) \geq J_E(\psi')$.

Define a monotone set function $\nu^E : \mathscr{P}(E) \cup \{S\} \to [0,1]$ by $\nu^E(B) = J_E(\mathbf{1}_B)$ for every $B \in \mathscr{P}(E) \cup \{S\}$, where $\mathbf{1}_B$ is the indicator function of B. Then, by Schmeidler's corollary (Schmeidler (1986, .p 258)), ν^E is the unique monotone set function on $\mathscr{P}(E) \cup \{S\}$ such that for every $\psi \in \mathcal{M}_E^L$, $J_E(\psi) = \int_S \psi d\nu^E$.[17]

Fix an arbitrary $\psi \in \mathcal{M}_E^L$. Let $(x^{(i)}, \dots x^{(K)})$ be the permutation of $\{\psi(s) \mid s \in S\}$ such that for every $i \in \{1, \dots K-1\}$, $x^{(i)} < x^{(i+1)}$, and let $\nu_{(i)}^E = \nu^E(\{s \in S \mid \psi(s) \geq x^{(i)}\})$. Then, by (a) and (i), $\nu^E(E) \geq \nu_{(2)}^E$ and $\nu_{(1)}^E = 1$. Thus, (i) yields that

$$
\begin{aligned}
J_E(\psi) &= \sum_{i=1}^{K} x^{(i)} \left(\nu_{(i)}^E - \nu_{(i+1)}^E\right) \\
&= x^{(1)} \left(1 - \nu^E(E)\right) + x^{(1)} \left(\nu^E(E) - \nu_{(2)}^E\right) + \sum_{i=2}^{K} x^{(i)} \left(\nu_{(i)}^E - \nu_{(i+1)}^E\right) \\
&= \int_E \psi d\nu^E + \left(1 - \nu^E(E)\right) \min_{s \in S} \psi(s).
\end{aligned}
$$

Let $\mu_L^E : \mathscr{P}(E) \to [0,1]$ be the restriction of ν^E on $\mathscr{P}(E)$. Then, by the uniqueness of ν^E, μ_L^E is the unique monotone set function such that J_E represents $\succsim_L$ on $\mathcal{M}_E^L$, where for all $\psi \in \mathcal{M}_E^L$, $J_E(\psi) = \int_E \psi d\mu_L^E + (1 - \mu_L^E(E)) \min_{s \in S} \psi(s)$. *Q.E.D.*

Let $P_L : \Sigma(L) \to [0,1]$ be the capacity obtained in Lemma 2.1. Let I_L be the functional given by $I_L(\psi) = \int_S \psi dP_L$ for every $\psi \in \Psi_L$. Let $N(L)$ be the set of all P_L-null cells in L. Formally,

$$N(L) = \{E \in L \mid P_L(E) = 0\}.$$

Then, by the SEU-CEU axioms, $N(L) \subset L$.

Claim 2.2. *For every $E \in L$, $\mu_L^E(E) = P_L(E)$.*

Proof. Fix an arbitrary $E \in L$. Clearly, $\Psi_L \cap \mathcal{M}_E^L \neq \emptyset$, and by Claim 2.1, $\mathcal{M}_E^L \cap \Psi_L$ is convex. Since both $J_E : \mathcal{M}_E^L \to \mathbb{R}$ and $I_L : \Psi_L \to \mathbb{R}$ represent $\succsim_L$ on $\mathcal{M}_E^L \cap \Psi_L$, there

[17] Caution should be exercised on the domain of ν^E. Let $L_E = \{\{s\} \mid s \in E\} \cup \{S \setminus E\}$ and $\mathcal{B}_E = \{(S \setminus E) \cup B \mid B \subset E\}$. Then, any $\psi \in \mathcal{M}_E^L$ is $\Sigma(L_E)$-measurable, and $\{\mathcal{B}_E, \mathscr{P}(E) \cup \{S\}\}$ partitions $\Sigma(L_E)$. However, for any $B \in \mathcal{B}_E$, $\mathbf{1}_B \notin \mathcal{M}_E^L$. Thus, ν^E cannot be defined on $\mathcal{B}_E$.

is a continuous and strictly increasing function $\varphi_E : J_E(\mathcal{M}_E^L \cap \Psi_L) \to \mathbb{R}$ such that for every $\psi \in \mathcal{M}_E^L \cap \Psi_L$, $\varphi_E\left(J_E(\psi)\right) = I_L(\psi)$. Note that $\Psi_C \subset (\Psi_L \cap \mathcal{M}_E^L)$. Thus, we have $\varphi_E(x) = x$ for every $x \in X$. As a result, for every $[\psi_y E \psi_z] \in \Psi_L \cap \mathcal{M}_E^L$, it follows that

$$\begin{aligned} J_E([\psi_y E \psi_z]) &= \mu_L^E(E)y + (1 - \mu_L^E(E))z \\ &= P_L(E)y + (1 - V_L(E))z = I_L([\psi_y E \psi_z]), \end{aligned}$$

where $y = \psi_y(s)$ and $z = \psi_z(s)$ on S. Set $x = y - z$. Since ψ_z and ψ_y are arbitrary, the above equality implies that $\mu_L^E(E)x = P_L(E)x$ for every $x \in \mathbb{R}_+$. Therefore, for every $E \in L$, $\mu_L^E(E) = P_L(E)$ as E is assumed to be arbitrary. *Q.E.D.*

Claim 2.3. *Given a $\psi \in \Psi$ and a $\psi_L \in \Psi_L$, let ψ_z be the constant function such that $\psi_z(s) = z = \min\{\underline{\psi}^L(s), \underline{\psi_L}^L(s)\}$. Then, $[\psi E \psi_z] \sim_L [\psi_L E \psi_z]$ for every $E \in L$ if and only if for every $E \in L \setminus N(L)$, $\psi_L(s) = (\int_E \psi d\mu_L^E)/P_L(E)$ on E.*

Proof. Fix a $\psi \in \Psi$ and a $\psi_L \in \Psi_L$ arbitrarily. Let $x = \underline{\psi}^L(s) = \min_{s \in S} \psi(s)$ on S. By construction, it holds that $[\psi E \psi_z] \in \mathcal{M}_E^L$ and $[\psi_L E \psi_z] \in \mathcal{M}_E^L \cap \Psi_L$ for every $E \in L$. Hence, by Lemma 3.2, $[\psi E \psi_z] \sim_L [\psi_L E \psi_z]$ for every $E \in L$ if and only if $J_E([\psi E \psi_z]) = J_E([\psi_L E \psi_z])$ for every $E \in L$. Meanwhile, for any $E \in L$, $J_E([\psi E \psi_z]) = J_E([\psi_L E \psi_z])$ if and only if $\int_E \psi d\mu_L^E = \mu_L^E(E)\psi_L^E$, where $\psi_L^E = \psi_L(t)$ for any $E \in L$ and $t \in E$. Thus, under Claim 2.2, the followings are equivalent: (i) for every $E \in L$, $[\psi E \psi_z] \sim_L [\psi_L E \psi_z]$; (ii) for every $E \in L$, $\int_E \psi d\mu_L^E = P_L(E)\psi_L^E$.

By Claim 2.2, for all $E \in L \setminus N(L)$, $\mu_L^E(E) = P_L(E) > 0$. As a result, for all $E \in L \setminus N(L)$ and $s \in E$, we have $\psi_L(s) = (\int_E \psi d\mu_L^E)/P_L(E)$ on E. *Q.E.D.*

Claim 2.4. *Suppose that a mapping $Q_L : \Psi \to \Psi_L$ exists such that for all $\psi \in \Psi$, (i) $\min_{s \in S} Q_L(\psi)(s) \geq \min_{s \in S} \psi(s)$ on S; and (ii) $[\psi E \underline{\psi}^L] \sim_L [Q_L(\psi) E \underline{\psi}^L]$ holds for any $E \in L$. Then, for any $\psi, \psi' \in \Psi$, $\psi \succcurlyeq_L \psi'$ if and only if*

$$\int_{\bigcup (L \setminus N(L))} Q_L(\psi) P_L(E) \geq \int_{\bigcup (L \setminus N(L))} Q_L(\psi') P_L(E).$$

Proof. Let Q_L be such a mapping.

By (i) and Monotonicity, $\underline{Q_L(\psi)}^L \succcurlyeq_L \underline{\psi}^L$ holds for all $\psi \in \Psi$. Hence, by Dominance, (ii) implies $\psi \sim_L Q_L(\psi)$. Therefore, by transitivity, for any $\psi, \psi' \in \Psi$, $\psi \succcurlyeq_L \psi'$ if and only if $Q_L(\psi) \succcurlyeq_L Q_L(\psi')$. Note that $Q_L(\psi), Q_L(\psi') \in \Psi_L$. Thus, by Lemma 2.1, $\psi \succcurlyeq_L \psi'$ if and only if $I_L(Q_L(\psi)) \geq I_L(Q_L(\psi'))$, where for any $\psi_L \in \Psi_L$, $I_L(\psi) = \int_S \psi_L dP_L$.

For any $\psi_L \in \Psi_L$, it follows that

$$\int_S \psi_L dP_L = \sum_{E \in L} \psi_L^E P_L(E) = \sum_{E \in L \setminus N(L)} \psi_L^E P_L(E) = \int_{\bigcup (L \setminus N(L))} \psi_L dP_L,$$

where for every $E \in L$, $\psi_L^E = \psi_L(s)$ for $s \in E$, and the second equality holds since $E \in N(L)$ implies $P_L(E) = 0$. As a result, the claim follows. *Q.E.D.*

SUFFICIENCY. Let $\Delta(L) = L \setminus N(L)$.

Let $Q_L : \Psi \to \Psi_L$ be a mapping such that $\min_{s\in S} Q_L(\psi)(s) \geq \min_{s\in S} \psi(s)$ on S. Then, by Claim 2.3, Q_L satisfies condition (ii) in Claim 2.4 if and only if for any $\psi \in \Psi$, $E \in \Delta(L)$ and $s \in E$, $Q_L(\psi)(s) = \left(\int_E \psi d\mu_L^E\right)/P_L(E)$. Therefore, a mapping $Q_L : \Psi \to \Psi_L$ satisfies (i) and (ii) in Claim 2.4 if and only if it can be written by

$$Q_L(\psi) = \sum_{E\in\Delta(L)} \left(\int_E \psi d\mu_L^E\right) \frac{\mathbf{1}_E}{P_L(E)} + \sum_{\hat{E}\in N(L)} \left(x_{\hat{E}} + \min_{s\in S} \psi(s)\right) \mathbf{1}_{\hat{E}},$$

where $x_{\hat{E}} \in \mathbb{R}_+$ for each $\hat{E} \in N(L)$.

Therefore, by Claim 2.4, for any $\psi, \psi' \in \Psi$,

$$\begin{aligned} \psi \succsim_L \psi' \iff & \int_{\bigcup \Delta(L)} Q_L(\psi) dP_L \geq \int_{\bigcup \Delta(L)} Q_L(\psi') dP_L; \\ & Q_L(\psi) = \sum_{E\in\Delta(L)} \left(\int_E \psi d\mu_L^E\right) \frac{\mathbf{1}_E}{P_L(E)}. \end{aligned} \tag{2.1}$$

Recall that $\mathscr{S}(L) = \bigcup_{E\in L} \mathscr{P}(E)$. Define a set function $V_L : \mathscr{S}(L) \to [0,1]$ such that for all $B \in \mathscr{S}(L)$ and $E \in L$, $B \in \mathscr{P}(E)$ implies $V_L(B) = \mu_L^E(B)$. Then, V_L is well-defined and monotone. Therefore, for any $\psi \in \Psi$,

$$\begin{aligned} \int_{\bigcup \Delta(L)} Q_L(\psi) dP_L &= \int_{\bigcup \Delta(L)} \left\{ \sum_{E\in\Delta(L)} \left(\int_E \psi d\mu_L^E\right) \frac{\mathbf{1}_E}{P_L(E)} \right\} dP_L \\ &= \sum_{E\in\Delta(L)} \int_E \psi d\mu_L^E = \sum_{E\in L} \int_E \psi dV_L. \end{aligned}$$

In addition, the uniqueness of $\{\mu_L^E\}_{E\in L}$ implies that of V_L.

Let $\mathcal{Q}_L$ denote the functional $\psi \mapsto \sum_{E\in L} \int_E \psi dV_L$. Then, $\mathcal{Q}_L : \Psi \to \mathbb{R}$ satisfies: (i) for all $x \in X$, $\mathcal{Q}_L(x\mathbf{1}) = x$; (ii) for any $\psi, \psi' \in \Psi$, $\psi(s) \geq \psi'(s)$ on S implies $\mathcal{Q}_L(\psi) \geq \mathcal{Q}_L(\psi')$; and (iii) for any pairwise comonotonic $\psi, \psi', \tilde{\psi} \in \Psi$ and any $\alpha \in (0,1)$, $\mathcal{Q}_L(\psi) > \mathcal{Q}_L(\psi')$ implies $\mathcal{Q}_L(\alpha\psi + (1-\alpha)\tilde{\psi}) > \mathcal{Q}_L(\alpha\psi' + (1-\alpha)\tilde{\psi})$. Therefore, by Schmeidler's corollary, defining $\mu_L(B) = \mathcal{Q}_L(\mathbf{1}_B)$ for all $B \in \mathscr{P}(S)$, then for any $\psi \in \Psi$, we have

$$\mathcal{Q}_L(\psi) = \int_S \psi \, d\mu_L.$$

Finally, for all $B \in \mathscr{P}(S)$, it follows that

$$\mu_L(B) = \mathcal{Q}_L(\mathbf{1}_B) = \sum_{E\in L} V_L(B \cap E).$$

Q.E.D.

2.A.3. Proof of Theorem 2.2

Invoke all the notations and claims given in Appendix 2.A.1. The proof will be conducted by showing that both (I) and (III) are equivalent to (II). First, we show the equivalence between (I) and (II). Assume statement (I). Under Certainty, Lemma 2.1

implies that $\{u_L\}_{L\in\mathcal{L}}$ can be fixed by a $u : \Pi(A) \to \mathbb{R}$, meaning that $\psi \mapsto \underline{\psi}^L$ and $\mathcal{M}_E^L$ are no longer subject to the choice of $L \in \mathcal{L}$. Thus, in what follows, we simply write $\underline{\psi}$ and $\mathcal{M}_E$, respectively.

For every $L \in \mathcal{L}$, let P_L be the unique probability measure obtained in Lemma 2.1, and let $\Delta(L) = \{E \in L \mid P_L(E) > 0\}$. Recall that $\underline{L} = \{S\}$. Let $V : \mathscr{P}(S) \to [0, 1]$ be the unique capacity obtained in Lemma 2.2-(a).

Claim 2.5. *Under Certainty and CoI, the following statements hold.*

(a) For every $L \in \mathcal{L}$ and any $E \in L$, $E \in \Delta(L)$ implies $V(E) > 0$.

(b) For any $L \in \mathcal{L}$ and $E \in \Delta(L)$, there exists a $\vartheta_L^E \in \mathbb{R}_{++}$ such that for every $B \in \mathscr{P}(E)$, $\mu_L^E(B) = \vartheta_L^E V(B)$.

Proof. Fix an arbitrary $L \in \mathcal{L}$.

(a) Fix an $E \in L$ and a $\psi_x \in \Psi_C$, where $x = \psi_x(s)$ on S. Consider $\psi, \psi' \in \mathcal{M}_E[\psi_x]$. Then, by Lemma 3.2, $\psi \succcurlyeq_L \psi'$ if and only if $\int_E \psi d\mu_L^E \geq \int_E \psi' d\mu_L^E$. Notice that μ_L^E is monotone, and that $\int_E \psi d\mu_L^E + (1 - \mu_L^E(E))x \geq x$ holds for any $\psi \in \mathcal{M}_E[\psi_x]$. Thus, $\mathcal{M}_E[\psi_x]$ is not $\succcurlyeq_L$-indifferent if and only if $\mu_L^E(E) > 0$. Hence, by Claim 2.2, for any $\psi_x \in \Psi_C$, $\mathcal{M}_E[\psi_x]$ is not $\succcurlyeq_L$-indifferent if and only if $E \in \Delta(L)$.

Suppose $E \in \Delta(L)$. Then, for any $\psi_x \in \Psi_C$, there exist $\psi, \psi' \in \mathcal{M}_E[\psi_x]$ such that $\psi \succ_L \psi'$. Thus, by CoI, $\psi \succ_{\underline{L}} \psi'$ holds for every such $\psi, \psi' \in \mathcal{M}_E[\psi_x]$, meaning that for any $\psi_x \in \Psi_C$, $\mathcal{M}_E[\psi_x]$ is not $\succcurlyeq_{\underline{L}}$-indifferent. By Lemma 2.2-(a), the latter statement is equivalent to $V(E) > 0$.

(b) Fix an $E \in \Delta(L)$ and a $\psi_z \in \Psi_C$ arbitrarily. Then, by CoI, for any $\psi, \psi' \in \mathcal{M}_E[\psi_z]$, $\psi \succcurlyeq_L \psi'$ if and only if $\psi \succcurlyeq_{\underline{L}} \psi'$. Let $\psi_z(s) = z$ on S. Then, by Lemma 2.3 and Lemma 2.2-(a), $\int_E \psi d\mu_L^E + (1 - \mu_L^E(E))z \geq \int_E \psi' d\mu_L^E + (1 - \mu_L^E(E))x$ if and only if $\int_S \psi dV \geq \int_S \psi' dV$. Notice that $\int_S \psi dV = \int_E \psi dV + (1 - V(E))z$. Thus, for any $\psi, \psi' \in \mathcal{M}_E[\psi_z]$, $\int_E \psi d\mu_L^E \geq \int_E \psi' d\mu_L^E$ if and only if $\int_E \psi dV \geq \int_E \psi' dV$.

By (a), $E \in \Delta(L)$ implies $V(E) > 0$. Let $[\psi_y E \psi_z] \in \mathcal{M}_E[\psi_z] \cap \Psi_L$, and suppose $\psi_y(s) = y > z$ on S. Then, it holds that $\int_E [\psi_y E \psi_z] d\mu_L^E = \mu_L^E(E)y$ and $\int_E [\psi_y E \psi_z] dV = V(E)y$. Set $\vartheta_L^E = \mu_L^E(E)/V(E)$. Then, $V(E) > 0$ and $\mu_L^E(E) > 0$ imply $\vartheta_L^E > 0$. Therefore, by CoI, for any $\psi \in \mathcal{M}_E[\psi_z]$ that satisfies $\psi \sim_L [\psi_y E \psi_z]$, we have $\int_E \psi d\mu_L^E = \mu_L^E(E)y = \vartheta_L^E V(E)y = \vartheta_L^E(\int_E \psi dV)$. Since $\psi_z \in \Psi_C$ and $[\psi_y E \psi_z] \in \mathcal{M}_E[\psi_z]$ are arbitrary, thus $\int_E \psi d\mu_L^E = \vartheta_L^E \int_E \psi dV$ holds for any $\psi \in \mathcal{M}_E$. As a result, $\mu_L^E(B) = \vartheta_L^E V(B)$ for every $B \in \mathscr{P}(E)$. *Q.E.D.*

Proof of (I) $\implies$ (II). For every $L \in \mathcal{L}$, let $\Omega(L) = \{E \in L \mid V(E) > 0\}$.

By Claim 2.3 and Claim 2.4, the equivalence obtained in Eq (2.1) holds for every $L \in \mathcal{L}$. Given Q_L in Eq (2.1), define a mapping $Q : \Psi \times \mathcal{L} \to \Psi$ by $Q(\psi, L) = Q_L(\psi)$.

Then, for any $L \in \mathcal{L}$ and $\psi, \psi' \in \Psi$, $\psi \succcurlyeq_L \psi'$ if and only if $\int_{\bigcup \Delta(L)} Q(\psi, L) dP_L \geq \int_{\bigcup \Delta(L)} Q(\psi', L) dP_L$.

By Claim 2.5-(a), for any $L \in \mathcal{L}$, $\Delta(L) \subseteq E \in \Omega(L)$. By Claim 2.2 and Claim 2.5-(b), for every $L \in \mathcal{L}$ and $E \in \Delta(L)$, $P_L(E) = \vartheta_L^E V(E)$ with $\vartheta_L^E > 0$. Therefore, for any $L \in \mathcal{L}$ and $\psi \in \Psi$, it holds that

$$Q(\psi, L) = \sum_{E \in \Omega(L)} \vartheta_L^E \left(\int_E \psi dV \right) \frac{\mathbf{1}_E}{\vartheta_L^E V(E)} = \sum_{E \in \Omega(L)} \left(\int_E \psi dV \right) \frac{\mathbf{1}_E}{V(E)}. \tag{2.2}$$

As a result, for any $L \in \mathcal{L}$ and $\psi \in \Psi$,

$$\int_{\bigcup \Delta(L)} Q(\psi, L) dP_L = \int_{\bigcup \Omega(L)} Q(\psi, L) dP_L = \sum_{E \in \Omega(L)} \left(\int_E \psi dV \right) \frac{P_L(E)}{V(E)}.$$

The proof is complete. *Q.E.D.*

Proof of (II) $\Longrightarrow$ (I). Suppose that $\{\succsim_L\}_{L \in \mathcal{L}}$ admits a BCEU representation $(u, V, \{P_L\}_{L \in \mathcal{L}})$. The SEU-CEU axioms, Dominance and Certainty are trivially implied.

(CoI) Let $(L, \tilde{L}) \in \mathscr{L}$ be arbitrary. For any $\tilde{E} \in \tilde{L}$ and $c \in \mathcal{H}_C$, $\mathcal{C}_{\tilde{E}}^{\tilde{L}}[c]$ is not $\succsim_{\tilde{L}}$-indifferent if and only if $P_L(E) > 0$. Let $\Delta(L) = \{E \in L \mid P_L(E) > 0\}$.

Fix an $\tilde{E} \in \Delta(\tilde{L})$ and a $c \in \mathcal{H}_C$ arbitrarily. Given that $(L, \tilde{L}) \in \mathscr{L}$, let $E \in L$ be the unique cell such that $\tilde{E} \subseteq E$. By definition, $\Delta(L) \subseteq \Omega(L)$ and $\Delta(\tilde{L}) \subseteq \Omega(\tilde{L})$. Thus, by the monotonicity of V, $\tilde{E} \in \Delta(\tilde{L})$ implies $E \in \Delta(L)$. In turn, we have $V(E) > 0$ and $V(\tilde{E}) > 0$. Consequently, for any $h \in \mathcal{C}_{\tilde{E}}^{\tilde{L}}[c]$, it holds that

$$\begin{aligned}
U(h, \tilde{L}) &= \tilde{\beta} \left(\int_{\tilde{E}} u(h) dV \right) + (1 - P_{\tilde{L}}(\tilde{E})) u_c; \\
U(h, L) &= \beta \left(\int_{\tilde{E}} u(h) dV + (V(E) - V(\tilde{E})) u_c \right) + (1 - P_L(E)) u_c \\
&= \beta \left(\int_{\tilde{E}} u(h) dV \right) + \left(1 - \beta V(\tilde{E}) \right) u_c,
\end{aligned}$$

where $\tilde{\beta} = P_{\tilde{L}}(\tilde{E}) / V(\tilde{E}) > 0$, $\beta = P_L(E)/V(E) > 0$, and $u_c = u(c(s))$ on S.

Therefore, for any $f, g \in \mathcal{C}_{\tilde{E}}^{\tilde{L}}[c]$, $U(f, \tilde{L}) \geq U(g, \tilde{L})$ if and only if $\int_{\tilde{E}} u(f) dV \geq \int_{\tilde{E}} u(g) dV$, and $U(f, L) \geq U(g, L)$ if and only if $\int_{\tilde{E}} u(f) dV \geq \int_{\tilde{E}} u(g) dV$. As a result, $\succsim_L = \succsim_{\tilde{L}}$ on $\mathcal{C}_{\tilde{E}}^{\tilde{L}}[c] \times \mathcal{C}_{\tilde{E}}^{\tilde{L}}[c]$. *Q.E.D.*

Proof of (II) $\Longleftrightarrow$ (III). Fix an arbitrary $L \in \mathcal{L}$.

Let $V : \mathscr{P}(S) \to [0,1]$ be a capacity and $P_L : \Sigma(L) \to [0,1]$ be a probability measure. Define $\Delta(L) = \{E \in L \mid P_L(E) > 0\}$ and $\Omega(L) = \{E \in L \mid V(E) > 0\}$. Suppose $\Delta(L) \subseteq \Omega(L)$. Define a set function $\mu_L : \mathscr{P}(S) \to [0,1]$ by

$$\mu_L(B) = \sum_{E \in \Delta(L)} \frac{V(B \cap E) P_L(E)}{V(E)}.$$

Invoke the notations given in Appendix 2.A.1. Define a functional $\mathcal{Q}_L : \Psi \to \mathbb{R}$ by

$$\mathcal{Q}_L(\psi) = \sum_{E \in \Omega(L)} \left(\int_E \psi \, dV \right) \frac{P_L(E)}{V(E)}.$$

By Theorem 2.2, $\{\succsim_L\}_{L \in \mathcal{L}}$ admits the BCEU representation $(u, V, \{P_L\}_{L \in \mathcal{L}})$ if and only if for all $L \in \mathcal{L}$, $\mathcal{Q}_L$ represents $\succcurlyeq_L$ on Ψ. Thus, given that $L \in \mathcal{L}$ is fixed arbitrarily, it suffices to show that μ_L is a capacity, and for all $\psi \in \Psi$, $\mathcal{Q}_L(\psi) = \int_S \psi \, d\mu_L$.

Clearly, $\mu_L(S) = 1$ and $\mu_L(\emptyset) = 0$. Let $B, D \in \mathscr{P}(S)$ be such that $B \subset D$. Then, for all $E \in L$, we have $(B \cap E) \subseteq (D \cap E)$. Thus, it follows that

$$\mu_L(D) - \mu_L(B) = \sum_{E \in \Omega(L)} (V(D \cap E) - V(B \cap E)) \frac{P_L(E)}{V(E)} \geq 0.$$

As a result, μ_L is a capacity, and by definition, for all $B \in \mathscr{P}(S)$, $\mu_L(B) = \mathcal{Q}_L(\mathbf{1}_B)$.

Fix an arbitrary $\psi \in \Psi$. Let $\tau : S \to \{1, \ldots, n\}$ be a bijection such that for all $s, t \in S$, $\tau(s) \geq \tau(t)$ implies $\psi(s) \geq \psi(t)$. Fix this τ.

Given τ, for all $s \in S$, define $e^\tau(s) = \{t \in S \mid [\tau(t) > \tau(s)] \wedge [\exists E \in L,\, t, s \in E]\}$. Then, for any $E \in L$ and $s \in S$, $e^\tau(s) \cap E \neq \emptyset$ if and only if $s \in E$. Moreover, for every $s \in S$, let $b^\tau(s) = \{t \in S \mid \tau(t) \geq \tau(s)\}$.

Given τ, define a function $y^\tau : S \to \mathbb{R}$ by $y^\tau(s) = \psi(s)$ if $\tau(s) = 1$, and $y^\tau(s) = \psi(s) - \psi(\tau^{-1}(\tau(s) - 1))$ otherwise. In addition, given τ, define $p^\tau : S \to [0, 1]$ by

$$p^\tau(s) = (V(\{s\} \cup e^\tau(s)) - V(e^\tau(s))) \sum_{E \in \Omega(L)} \frac{P_L(E) \mathbf{1}_E}{V(E)}.$$

Then, for all $s \in S$, we have $\mathcal{Q}_L(\mathbf{1}_{b^\tau(s)}) = \sum_{t \in b^\tau(s)} p^\tau(t)$.

Therefore, it follows that

$$\begin{aligned} \mathcal{Q}_L(\psi) = \sum_{s \in S} \psi(s) p^\tau(s) &= \sum_{s \in S} y^\tau(s) \left(\sum_{t \in b^\tau(s)} p^\tau(t) \right) \\ &= \sum_{s \in S} y^\tau(s) \mathcal{Q}_L(\mathbf{1}_{b^\tau(s)}) = \int_S \psi \, d\mu_L. \end{aligned}$$

The proof is complete as ψ and τ are fixed arbitrarily. *Q.E.D.*

2.A.4. Proof of Proposition 2.1

The equivalence between (II) and (III) is implied by Theorem 2.2. The equivalence between (I) and (II) will be concluded.

Proof of (I) $\implies$ (II). Assume statement (I), and let $\overline{L} = \{\{s\} \mid s \in S\}$.

Since CoS implies Certainty, by Theorem 2.2, $\{\succsim_L\}_{L \in \mathcal{L}}$ admits a BCEU representation $(u, V, \{P_L\}_{L \in \mathcal{L}})$. Fix this tuple, and for every $L \in \mathcal{L}$, define $\Delta(L) = \{E \in L \mid P_L(E) > 0\}$. Then, by the definition of BCEU, $\Delta(L) \subseteq \Omega(L)$ holds for all $L \in \mathcal{L}$. It suffices to show that, for any $(L, \tilde{L}) \in \mathscr{L}$, $L \cap \tilde{L} \neq \emptyset$ implies $P_L(E) = P_{\tilde{L}}(E)$ whenever

$E \in L \cap \tilde{L}$. Then, given that $(\mathcal{L}, \mathscr{L})$ is a lattice, for any $L \in \mathcal{L}$, P_L is the restriction of $P_{\overline{L}}$ on $\Sigma(L)$, and $P = P_{\overline{L}}$ is the unique probability claimed in (II).

Invoke the notations given in Appendix 2.A.1.

Let $(L, \tilde{L}) \in \mathscr{L}$ be an arbitrary such that $L \cap \tilde{L} \neq \emptyset$. Fix an $E \in L \cap \tilde{L}$.

Suppose $E \notin \Delta(\tilde{L})$. Then, for any $\psi_z \in \Psi_C$, $\mathcal{M}_E[\psi_z]$ is $\succcurlyeq_{\tilde{L}}$-indifferent. Thus, by CoS, $\mathcal{M}_E[\psi_z]$ is also $\succcurlyeq_L$-indifferent, meaning that $E \notin \Delta(L)$. That is, $P_L(E) = P_{\tilde{L}}(E) = 0$. Suppose $E \in \Delta(\tilde{L})$. Then, $\Delta(\tilde{L}) \subseteq \Omega(\tilde{L})$ implies that $E \in \Omega(\tilde{L})$. In addition, for any $\psi_z \in \Psi_C$, $\mathcal{M}_E[\psi_z]$ is not $\succcurlyeq_{\tilde{L}}$-indifferent. Thus, by CoS, $\mathcal{M}_E[\psi_z]$ is not $\succcurlyeq_L$-indifferent. Therefore, we have $E \in \Delta(L)$, thereby $E \in \Omega(L)$. Fix $\psi_x, \psi_y \in \Psi_C$ arbitrarily, where $\psi_x(s) = x$ and $\psi_y(s) = y$ on S. Let $\psi \in \mathcal{M}_E[\psi_x]$ and $\psi' \in \mathcal{M}_E[\psi_y]$ be arbitrary such that $\psi \sim_L \psi'$. Then, by Theorem 2.2, $zP_L(E) + (1 - P_L(E))\, x = z'P_L(E) + (1 - P_L(E))\, y$, where $z = (\int_E \psi \, dV)/V(E)$ and $z' = (\int_E \psi' \, dV)/V(E)$. By CoS, we have $\psi \sim_{\tilde{L}} \psi'$, which implies $zP_{\tilde{L}}(E) + (1 - P_{\tilde{L}}(E))\, x = z'P_{\tilde{L}}(E) + (1 - P_{\tilde{L}}(E))\, y$. Therefore, we have $(z - z' + y - x)\,(P_L(E) - P_{\tilde{L}}(E)) = 0$. Since ψ, ψ', ψ_x and ψ_y are arbitrary, thus $z - z' + y - x$ is not always zero. As a result, $P_L(E) = P_{\tilde{L}}(E)$.

The proof is complete since $(L, \tilde{L})$ and E are fixed arbitrarily. *Q.E.D.*

Proof of (II) $\Longrightarrow$ (I). Assume statement (II).

The SEU-CEU axioms and Dominance are trivial, while CoI is implied by Theorem 2.2.

(CoS) Let $(L, \tilde{L}) \in \mathscr{L}$ be arbitrary.

When $L \cap \tilde{L} = \emptyset$, CoS yields Certainty for $\succsim_L$ and $\succsim_{\tilde{L}}$. Thus, consider $L \cap \tilde{L} \neq \emptyset$, and let $E \in L \cap \tilde{L}$. Fix $\psi_x, \psi_y \in \Psi_C$ arbitrarily, where $\psi_x(s) = x$ and $\psi_y(s) = y$ on S. Let $\psi \in \mathcal{M}_E[\psi_x]$ and $\psi' \in \mathcal{M}_E[\psi_y]$ be arbitrary.

Suppose $P(E) = 0$. Then, $\psi' \succ_{\tilde{L}} \psi$ if and only if $y > x$, and $\psi' \succ_L \psi$ if and only if $y > x$. CoS follows. Now, suppose $P(E) > 0$. Then, by the definition of BCEU, $V(E) > 0$. Thus, by statement (II), $\psi' \succ_{\tilde{L}} \psi$ if and only if $(z' - y)P(E) + y > (z - x)P(E) + x$, where $z = (\int_E \psi \, dV)/V(E)$ and $z' = (\int_E \psi \, dV)/V(E)$. Similarly, $\psi' \succ_L \psi$ if and only if $(z' - y)P(E) + y > (z - x)P(E) + x$. Hence, we have $\psi' \succ_{\tilde{L}} \psi$ if and only if $\psi' \succ_L \psi$. As a result, CoS holds. *Q.E.D.*

2.A.5. Proof of Proposition 2.2

The equivalence between (II) and (III) is a straightforward consequence of Theorem 2.2. Thus, we focus on the equivalence between (I) and (II).

Assume statement (I). Since SCoI implies CoI, by Theorem 2.2, $\{\succsim_L\}_{L \in \mathcal{L}}$ admits a BCEU representation $(u, V, \{P_L\}_{L \in \mathcal{L}})$. Fix this tuple, and for every $L \in \mathcal{L}$, define $\Delta(L) = \{E \in L \mid P_L(E) > 0\}$. Let $\underline{L} = \{S\}$.

Claim 2.6. *Under Certainty and SCoI, the following statements hold.*

(a) For any $L \in \mathcal{L}$ and $E \in L$, $E \in \Delta(L)$ if and only if $E \in \Omega(L)$.

(b) For any $L \in \mathcal{L}$, there exists a unique $\vartheta_L \in \mathbb{R}_{++}$ such that for any $E \in L$, $P_L(E) = \vartheta_L V(E)$.

Proof. Fix an arbitrary $L \in \mathcal{L} \setminus \{\underline{L}\}$. Then, $(\underline{L}, L) \in \mathscr{L}$.

Invoke the notations given in Appendix 2.A.1.

(a) By Claim 2.5-(a), for any $E \in L$, $E \in \Delta(L)$ implies $E \in \Omega(L)$. Hence, it suffices to show the opposite direction. Let $E \in L$, and suppose that $E \notin \Delta(L)$. Then, by Lemma 2.3, $\mathcal{M}_E[\psi_z]$ is $\succcurlyeq_L$-indifferent for any $\psi_z \in \Psi_C$. Thus, SCoI implies that $\mathcal{M}_E[\psi_z]$ is also $\succcurlyeq_{\underline{L}}$-indifferent for all $\psi_z \in \Psi_C$. By Nondegeneracy and Lemma 2.2-(a), it follows that $V(E) = 0$. As a result, $E \in \Omega(L)$ implies $E \in \Delta(L)$.

(b) Let $E, E' \in L$. Fix an arbitrary $\psi_x \in \Psi_C$, and suppose that $\psi \in \mathcal{M}_E[\psi_x]$ and $\psi' \in \mathcal{M}_{E'}[\psi_x]$ satisfy $\psi \sim_{\underline{L}} \psi'$. Then, SCoI implies $\psi \sim_L \psi'$.

By Lemma 2.2-(a), $\psi \sim_{\underline{L}} \psi'$ if and only if $\int_S \psi dV = \int_S \psi' dV$, which yields that

$$\int_E \psi dV - V(E)x = \int_{E'} \psi' dV - V(E')x,$$

where $x = \psi_x(s)$ on S. On the other hand, by Lemma 2.3, J_E and $J_{E'}$ represents $\succcurlyeq_L$ on $\mathcal{M}_E$ and $\mathcal{M}_{E'}$ respectively. By definition, $\Psi_C = \mathcal{M}_E \cap \mathcal{M}_{E'}$, and for every $\psi_z \in \Psi_C$, it holds that $J_E(\psi_z) = J_{E'}(\psi_z) = z$, where $z = \psi_z(s)$ on S. Thus, by transitivity, $\psi \sim_{\underline{L}} \psi'$ if and only if $J_E(\psi) = J_{E'}(\psi')$. Similarly, $J_E(\psi) = J_{E'}(\psi')$ yields

$$\int_E \psi \mu_L^E - \mu_L^E(E)x = \int_{E'} \psi' \mu_L^{E'} - \mu_L^{E'}(E')x.$$

Since (a) implies that $\Delta(L) = \Omega(L)$, thus Claim 3.11-(b) holds for any $E \in L$. As a result, under Claim 2.5-(b), the above equations jointly imply that

$$\vartheta_L^E \left(\int_E \psi dV - V(E)x \right) = \vartheta_L^{E'} \left(\int_E \psi dV - V(E)x \right).$$

Hence, $\vartheta_L^E = \vartheta_L^{E'}$ as E, E', ψ_x, ψ and ψ' are chosen arbitrarily. Let $\vartheta_L = \vartheta_L^E$ for an arbitrary $E \in L$. Then, $\vartheta_L \in \mathbb{R}_{++}$ is the unique number satisfies the claim. *Q.E.D.*

Proof of (I) $\Longrightarrow$ (II). Let $L \in \mathcal{L}$ be arbitrary. Then, by Claim 2.6-(b), for any $E \in L$, we have $P_L(E) = \vartheta_L V(E)$, meaning that $\sum_{E \in L} P_L(E) = \vartheta_L(\sum_{E \in L} V(E))$. Since P_L is a probability measure, it thus yields that $\vartheta_L = 1/(\sum_{E \in L} V(E))$.

As a result, for any $L \in \mathcal{L}$ and $E \in L$, $P_L(E) = V(E)/(\sum_{D \in L} V(D))$. *Q.E.D.*

Proof of (II) $\Longrightarrow$ (I). Assume statement (II).

The SEU-CEU axioms, Dominance and Certainty are trivial.

(SCoI) Fix an arbitrary $L \in \mathcal{L}$. Then, by (II), for any $h \in \mathcal{H}$,

$$U(h, L) = \sum_{E \in \Omega(L)} \left(\int_E u(h) dV \right) \frac{P_L(E)}{V(E)} = \frac{1}{\sum_{E \in L} V(E)} \sum_{E \in \Omega(L)} \int_E u(h) dV.$$

Note that for any $L \in \mathcal{L}$, $\sum_{E \in L} V(E)$ is a constant, and the above equality holds for any $L \in \mathcal{L}$. Thus, for any $L \in \mathcal{L}$ and $f, g \in \mathcal{H}$, $f \succsim_L g$ if and only if

$$\sum_{E \in \Omega(L)} \int_E u(f) dV \geq \sum_{E \in \Omega(L)} \int_E u(g) dV.$$

Let $I^V : \mathcal{H} \times \mathcal{L} \to \mathbb{R}$ denote the functional given by $I^V(h, L) = \sum_{E \in \Omega(L)} \int_E u(h) dV$.

Let $(L, \tilde{L}) \in \mathscr{L}$, $E, E' \in \tilde{L}$, and $c \in \mathcal{H}_C$. Then, fix an $f \in \mathcal{C}_E^{\tilde{L}}[c]$ and a $g \in \mathcal{C}_{E'}^{\tilde{L}}[c]$ arbitrarily. Let $D, D' \in L$ denote the cells that satisfy $E \subseteq D$ and $E' \subseteq D'$. Let $u_c = u(c(s))$ on S. Then, exactly one of the following cases holds.

Case 1) $E, E' \notin \Omega(\tilde{L})$. Then, we have $I^V(f, \tilde{L}) = I^V(g, \tilde{L}) = \sum_{\tilde{E} \in \Omega(\tilde{L})} u_c V(\tilde{E})$ and $I^V(f, L) = I^V(g, L) = \sum_{\tilde{D} \in \Omega(L)} u_c V(\tilde{D})$. Hence, $f \sim_L g$ and $f \sim_{\tilde{L}} g$.

Case 2) $E \in \Omega(\tilde{L})$ and $E' \notin \Omega(\tilde{L})$. Then, it holds that $I^V(g, \tilde{L}) = u_c \sum_{\tilde{E} \in \Omega(\tilde{L})} V(\tilde{E})$ and $I^V(g, L) = u_c \sum_{\tilde{D} \in \Omega(L)} V(\tilde{D})$. Moreover, by the monotonicity of V, $D \in \Omega(L)$. Thus, it yields that

$$\begin{aligned} I^V(f, \tilde{L}) &= \int_E u(f) dV + u_c \sum_{\tilde{E} \in (\Omega(\tilde{L}) \setminus \{E\})} V(\tilde{E}) \\ &= \int_E (u(f) - u_c \mathbf{1}_S) dV + u_c \sum_{\tilde{E} \in \Omega(\tilde{L})} V(\tilde{E}); \\ I^V(f, L) &= \int_E u(f) dV + (V(D) - V(E)) u_c + u_c \sum_{\tilde{D} \in (\Omega(L) \setminus \{D\})} V(\tilde{D}) \\ &= \int_E (u(f) - u_c \mathbf{1}_S) dV + u_c \sum_{\tilde{D} \in \Omega(L)} V(\tilde{D}). \end{aligned}$$

Note that $f \in \mathcal{C}_E^{\tilde{L}}[c]$ implies $\int_E u(f) dV \geq u_c V(E)$. Thus, it holds that $I^V(f, \tilde{L}) \geq u_c \sum_{\tilde{E} \in \Omega(\tilde{L})} V(\tilde{E})$ and $I^V(f, L) \geq u_c \sum_{\tilde{D} \in \Omega(L)} V(\tilde{D})$. As a result, $f \succsim_L g$ and $f \succsim_{\tilde{L}} g$.

Case 3) $E, E' \in \Omega(\tilde{L})$. Then, by the monotonicity of V, we have $D, D' \in \Omega(L)$. In this case, the equations in Case 2 also hold for g with respect to E'. Since $u_c \sum_{\tilde{E} \in \Omega(\tilde{L})} V(\tilde{E})$ and $u_c \sum_{\tilde{D} \in \Omega(L)} V(\tilde{D})$ are constants, it thus yields that

$$\begin{aligned} f \succsim_L g &\iff \int_E (u(f) - u_c \mathbf{1}_S) dV \geq \int_{E'} (u(g) - u_c \mathbf{1}_S) dV \\ &\iff f \succsim_{\tilde{L}} g. \end{aligned}$$

As a result, it holds in all cases that $f \succsim_L g$ if and only if $f \succsim_{\tilde{L}} g$. Consequently, SCoI is satisfied as $(L, \tilde{L})$, E, E', c, f and g are arbitrary. *Q.E.D.*

2.A.6. Proof of Proposition 2.3

Let $\{\succsim_L\}_{L \in \mathcal{L}}$ and $\{\succsim'_L\}_{L \in \mathcal{L}}$ be as assumed, and denote by U and U' the corresponding functionals. The following claim translates Definition 2.2 into BCEU.

CLAIM 2.7. *For every $L, \tilde{L} \in \mathcal{L}$, $\succsim_L$ is more uncertainty averse than $\succsim'_L$ if and only if for all $h \in \mathcal{H}$, $U(h, L) \leq U'(h, \tilde{L})$.*

PROOF. Let $L, \tilde{L} \in \mathcal{L}$ be arbitrary. Notice that $\{\succsim_L\}_{L\in\mathcal{L}}$ and $\{\succsim'_L\}_{L\in\mathcal{L}}$ admit the BCEU representations which comprise a same affine function u on $\Pi(A)$. Thus, for all $c \in \mathcal{H}_C$, $U(c, L) = U'(c, \tilde{L})$.

Suppose $\succsim_L$ is more uncertainty averse than $\succsim'_L$. Let $f \in \mathcal{H}$ be an act that satisfies $U(f, L) > U'(f, \tilde{L})$. Then, by Continuity, there exists a $c \in \mathcal{H}_C$ such that $U(f, L) > u_c > U'(f, \tilde{L})$, where $u_c = U(c, L) = U'(c, \tilde{L})$. Thus, we have $f \succ_L c$ and $c \succ'_{\tilde{L}}$, which yields a contradiction. The opposite direction is trivial. *Q.E.D.*

STATEMENT (I). Fix $L, \tilde{L} \in \mathcal{L}$ arbitrarily.

Given V, V', P_L and $P'_{\tilde{L}}$, define set functions $\mu_L : \mathscr{P}(S) \to [0, 1]$ and $\mu_{\tilde{L}} : \mathscr{P}(S) \to [0, 1]$ respectively by

$$\mu_L(B) = \sum_{E\in L} \frac{V(B\cap E)P_L(E)}{V(E)}; \quad \mu'_{\tilde{L}}(B) = \sum_{\tilde{E}\in\tilde{L}} \frac{V'(B\cap\tilde{E})P'_{\tilde{L}}(\tilde{E})}{V'(\tilde{E})}.$$

Then, by Theorem 2.2, for all $h \in \mathcal{H}$, $U'(h, \tilde{L}) \geq U(h, L)$ if and only if $\int_S u(h)\, d\mu'_{\tilde{L}} \geq \int_S u(h)\, d\mu_L$. Hence, the previous claim, $\succsim_L$ is more uncertainty averse than $\succsim'_{\tilde{L}}$ if and only if for all $h \in \mathcal{H}$, $\int_S u(h)\, d\mu'_{\tilde{L}} \geq \int_S u(h)\, d\mu_L$. Therefore, it suffices to show that the followings are equivalent:

(i) For all $h \in \mathcal{H}$, $\int_S u(h)\, d\mu'_{\tilde{L}} \geq \int_S u(h)\, d\mu_L$.

(ii) For all $B \in \mathscr{P}(S)$, $\mu_L(B) \leq \mu'_{\tilde{L}}(B)$.

Assume (i). Then, since u is unique up to p.l.t., for any $B \in \mathscr{P}(S)$, it holds that $\int_S \mathbf{1}_B\, d\mu_L \leq \int_S \mathbf{1}_B\, d\mu'_{\tilde{L}}$. This implies (ii) since for all $B \in \mathscr{P}(S)$, $\mu_L(B) = \int_S \mathbf{1}_B\, d\mu_L$ (resp., $\mu'_{\tilde{L}}(B)$). Now, assume (ii). For an arbitrary $h \in \mathcal{H}$, let $(s_1, \ldots, s_n)$ be a permutation of S such that for every $i \in \{1, \ldots, n-1\}$, $u(h(s_i)) \leq u(h(s_{i+1}))$. Fix this permutation. For every $i \in \{2, \ldots, n\}$, let $m_i = \mu_L(\{s_j \mid j \geq i\})$, $m'_i = \mu'_{\tilde{L}}(\{s_j \mid j \geq i\})$, and $d_i = u(h(s_i)) - u(h(s_{i-1}))$. Then, by construction and (ii), for every $i \in \{2, \ldots, n\}$, we have $d_i \geq 0$ and $m'_i \geq m_i$. Thus, it follows that

$$\begin{aligned}\int_S u(h)\, d\mu'_{\tilde{L}} - \int_S u(h)\, d\mu_L &= u(h(s_1)) + \sum_{i=2}^{n} d_i m'_i - \left(u(h(s_1)) + \sum_{i=2}^{n} d_i m_i\right) \\ &= \sum_{i=2}^{n} d_i\,(m'_i - m_i) \;\geq\; 0.\end{aligned}$$

Therefore, (ii) implies (i) since $h \in \mathcal{H}$ is given arbitrarily.

As a result, (i) and (ii) are equivalent. The proof is complete. *Q.E.D.*

STATEMENT (II). Fix an arbitrary $L \in \mathcal{L}$, and assume $\succsim_{\underline{L}} = \succsim'_{\underline{L}}$. Then, by Theorem 2.2, it follows immediately that $V = V'$ on $\mathscr{P}(S)$.

By (I), statement (a) (resp., (b)) holds if and only if for all $B \in \mathscr{P}(S)$,

$$\sum_{E \in L} \frac{V(B \cap E)}{V(E)} \left(P'_L(E) - P_L(E)\right) \geq 0 \quad (\textit{resp.}, \leq).$$

Notice that $E \cap E' = \emptyset$ holds for any $E, E' \in L$. Thus, for every $E \in L$, setting $E = B$ yields $P'_L(E) \geq P_L(E)$ (resp., $\leq$). Given that $\sum_{E \in L} P'_L(E) = \sum_{E \in L} P_L(E) = 1$, $P'_L(E) = P_L(E)$ holds for all $E \in L$. Hence, by additivity, $P_L = P'_L$ on $\Sigma(L)$. Therefore, both statement (a) and (b) are equivalent to (c).

In addition, under $V = V'$, the equivalence between (c) and (d) is a trivial consequence of Theorem 2.2. *Q.E.D.*

2.A.7. Proof of Proposition 2.4

Let $\{\succsim_L\}_{L \in \mathcal{L}}$ be as assumed, and let U be the corresponding functional.

Proof of (I) $\Longrightarrow$ (II). Assume statement (I).

Since $(L, \overline{L}) \in \mathscr{L}$ holds for all $L \in \mathcal{L}$, thus for every $L \in \mathcal{L}$, $\succsim_L$ is more uncertainty averse than $\succsim_{\overline{L}}$. By Proposition 2.3-(I), for any $L \in \mathcal{L}$ and $E \in L$, it holds that $P_L(E) \leq P_{\overline{L}}(E)$ as $\sum_{s \in E} P_{\overline{L}}(\{s\}) = P_{\overline{L}}(E)$. Given that $\sum_{E \in L} P_L(E) = \sum_{E \in L} P_{\overline{L}}(E) = 1$, this yields that $P_L = P_{\overline{L}}$ on $\Sigma(L)$. Set $P = P_{\overline{L}}$. Then, for all $L \in \mathcal{L}$, P_L is the restriction of P on $\Sigma(L)$. Therefore, $\{\succsim_L\}_{L \in \mathcal{L}}$ admits BCEU representation (u, V, P). Meanwhile, by assumption, $\succsim_{\underline{L}}$ is more uncertainty averse than $\succsim_{\overline{L}}$. Hence, by Proposition 2.3-(I), for any $B \in \mathscr{P}(S)$, it holds that

$$\frac{V(B \cap S)P(S)}{V(S)} = V(B) \leq \sum_{s \in S} \frac{V(B \cap \{s\})P(\{s\})}{V(\{s\})} = \sum_{s \in B} P(\{s\}) = P(B).$$

Therefore, $P \in \mathit{Core}(V)$. Fix a $B \in \mathscr{P}(S)$ and nonempty disjoint $D, D' \in \mathscr{P}(S)$ arbitrarily. Let $(L, \tilde{L}) \in \mathscr{L}$ be the unique pair of partitions such that $\tilde{L} \setminus L = \{D, D'\}$. Then, we have $(D \cup D') \in L$. By assumption, $\succsim_L$ is more uncertainty averse than $\succsim_{\tilde{L}}$. Thus, by Proposition 2.3-(I), it follows that

$$\begin{aligned} &\frac{V(B \cap (D \cup D'))P(D \cup D')}{V(D \cup D')} + \sum_{E \in L \cap \tilde{L}} \frac{V(B \cap E)P(E)}{V(E)} \\ &\quad \leq \quad \frac{V(B \cap D)P(D)}{V(D)} + \frac{V(B \cap D')P(D')}{V(D')} + \sum_{E \in L \cap \tilde{L}} \frac{V(B \cap E)P(E)}{V(E)}. \end{aligned}$$

As a result, statement (I) implies (II). *Q.E.D.*

Proof of (II) $\Longrightarrow$ (I). Assume statement (II), and fix $(L, \tilde{L}) \in \mathscr{L}$ arbitrarily.

By assumption, for any $\tilde{E} \in \tilde{L} \setminus L$ and $D \in \mathscr{P}(\tilde{E})$, it holds that Given V and P, let μ_L and $\mu_{\tilde{L}}$ be the capacities defined as in Theorem 2.2. For every $\tilde{E} \in \tilde{L} \setminus L$, let

$\eta(\tilde{E})$ be the unique cell in L such that $\tilde{E} \subset \eta(\tilde{E}) \in L$. Then, for any $B \in \mathscr{P}(S)$, it holds that

$$\begin{aligned}\mu_L(B) - \mu_{\tilde{L}}(B) &= \sum_{E \in L \setminus \tilde{L}} \frac{V(B \cap E)P(E)}{V(E)} - \sum_{\tilde{E} \in \tilde{L} \setminus L} \frac{V(B \cap \tilde{E})P(\tilde{E})}{V(\tilde{E})} \\ &= \sum_{E \in L \setminus \tilde{L}} \left(\frac{V(B \cap E)P(E)}{V(E)} - \sum_{\tilde{E} \in \tilde{L} \colon E = \eta(\tilde{E})} \frac{V(B \cap \tilde{E})P(\tilde{E})}{V(\tilde{E})} \right).\end{aligned}$$

Let $E \in L \setminus \tilde{L}$ be arbitrary. Let K and K' be arbitrary partitions of E that satisfy $1 < |K| < |K'|$, and for any $D' \in K'$, there is a unique $D \in K$ such that $D' \subseteq D$. Then, by the last condition in (II), we have

$$\frac{V(B \cap E)P(E)}{V(E)} \leq \sum_{D \in K} \frac{V(B \cap D)P(D)}{V(D)} \leq \sum_{D' \in K'} \frac{V(B \cap D')P(D')}{V(D')}.$$

Thus, for any $E \in L \setminus \tilde{L}$, it holds that

$$\frac{V(B \cap E)P(E)}{V(E)} - \sum_{\tilde{E} \in \tilde{L} \colon E = \eta(\tilde{E})} \frac{V(B \cap \tilde{E})P(\tilde{E})}{V(\tilde{E})} \leq 0,$$

which implies $\mu_L(B) \leq \mu_{\tilde{L}}(B)$. Therefore, by Proposition 2.3-(I), for any $(L, \tilde{L}) \in \mathscr{L}$, $\succsim_L$ is more uncertainty averse than $\succsim_{\tilde{L}}$. Statement (II) implies (I). *Q.E.D.*

2.A.8. Proof of Proposition 2.5

Let $\{\succsim_L\}_{L \in \mathcal{L}}$ and $\{\succsim'_L\}_{L \in \mathcal{L}}$ be as assumed. Fix an arbitrary $L \in \mathcal{L}$. By assumption, for any $c \in \mathcal{H}_C$, we have $U(c, L) = U'(c, L)$. Given the representation tuples, let μ_L and μ'_L be the capacities defined as in Theorem 2.2 respectively for $\succsim_L$ and $\succsim'_L$.

Proof of (II) $\Longrightarrow$ (I). Assume statement (II). Then, for all $B \in \mathscr{P}(S)$, we have $\mu_L(B) \leq \mu'_L(B)$. Thus, as shown in the proof of Proposition 2.3-(I), for any $h \in \mathcal{H}$, it holds that $U(h, L) \leq U'(h, L)$. Moreover, notice that $P_L = P'_L$ on $\Sigma(L)$. Hence, for any $f \in \mathcal{H}_L$, $U(f, L) = U'(f, L)$.

Let $f \in \mathcal{H}_L$ and $h \in \mathcal{H}$ be arbitrary. Suppose $f \succsim'_L h$ (resp., $\succ'_L$). Then, it follows that $U(f, L) = U'(f, L) \geq U'(h, L) \geq U(h, L)$ (resp., $U(f, L) = U'(f, L) > U'(h, L) \geq U(h, L)$). Therefore, we have $f \succsim_L h$ (resp., $\succ_L$), meaning that $\succsim_L$ is more $\Sigma(L)$-uncertainty averse than $\succsim'_L$. *Q.E.D.*

Proof of (I) $\Longrightarrow$ (II). Assume statement (I). Then, for any $c \in \mathcal{H}_C$ and $f \in \mathcal{H}_L$, $c \succsim'_L f$ (resp., $\succ'_L$) implies $c \succsim_L f$ (resp., $\succ_L$). Thus, by Claim 2.7, for any $f \in \mathcal{H}_L$, it holds that $U(f, L) \leq U'(f, L)$. In addition, by Theorem 2.2, for any $f \in \mathcal{H}_L$, we have $\sum_{E \in L} x^f_E P_L(E) \leq \sum_{E \in L} x^f_E P'_L(E)$, where for all $E \in L$, $x^f_E = u(f(s))$ for $s \in E$. Therefore, it follows that $P_L = P'_L$ on $\Sigma(L)$. As a result, for all $f \in \mathcal{H}_L$,

$U(f,L) = U'(f,L)$. Let $f \in \mathcal{H}_L$ and $h \in \mathcal{H}$ be arbitrary such that $f \succsim'_L h$ (resp., $\succ'_L$). Then, by assumption, we have $f \succsim_L h$ (resp., $\succ_L$). Recall that $U(f,L) = U'(f,L)$. Set $x^f = U(f,L) = U'(f,L)$. Then, $x^f \geq U'(h,L)$ (resp., $>$) implies $x^f \geq U(h,L)$ (resp., $>$), meaning that $U'(h,L) \geq U(h,L)$. Since f and h are arbitrary, thus for any $h \in \mathcal{H}$, we have $U'(h,L) \geq U(h,L)$. As shown in the proof of Proposition 2.3-(I), this is equivalent to that for all $B \in \mathscr{P}(S)$, $\mu'_L(B) \geq \mu_L(B)$.

Therefore, for any $E \in L$ and $D \in \mathscr{P}(E)$, it yields that $(V'(D)P'_L(E))/V'(E) = \mu'_L(D) \geq \mu_L(D) = (V(D)P_L(E))/V(E)$. Given that $P_L = P'_L$ on $\Sigma(L)$, we have $V'(D)/V'(E) \geq V(D)P_L(E)/V(E)$. *Q.E.D.*

2.A.9. Proof of Proposition 2.6

Proof. Let $\{\succsim_L\}_{L\in\mathcal{L}}$ be as assumed, and fix an $L \in \mathcal{L}$. Assume statement (I).

Given V and P_L, define a capacity $\mu_L : \Sigma(L) \to [0,1]$ as in Theorem 2.2. Then, by Theorem 2.2, for any $f, g \in \mathcal{H}$, $f \succsim_L g$ if and only if $\int_S u(f)\, d\mu_L \geq \int_S u(g)\, d\mu_L$. Thus, by Schmeidler's proposition (Schmeidler (1989, .pp 582-583)), $\succsim_L$ satisfies statement (I) if and only if μ_L is convex on $\mathscr{P}(S)$.[18] Given that μ_L is convex, $Core(\mu_L) \neq \emptyset$, and for any $h \in \mathcal{H}$, we have

$$U(h,L) = \int_S u(h)\, d\mu_L = \min_{P \in Core(\mu_L)} \int_S u(h)\, dP.$$

Meanwhile, by definition, for any $B \in \Sigma(L)$, it follows that

$$\mu_L(B) = \sum_{E^* \in L \colon E^* \subseteq B} \frac{V(E^*)P_L(E^*)}{V(E^*)} + \sum_{E \in L \colon E \cap B = \emptyset} \frac{V(\emptyset)P_L(E)}{V(E)} = P_L(B).$$

Therefore, for any $P \in Core(\mu_L)$, $P_L = P$ on $\Sigma(L)$.

Fix an arbitrary $P \in Core(\mu_L)$. Let $\succsim^P \subset \mathcal{H} \times \mathcal{H}$ be the preference such that for all $f, g \in \mathcal{H}$, $f \succsim^P g$ if and only if $\int_S u(f)\, dP \geq \int_S u(g)\, dP$. Let I^P denote this functional. Then, by the above arguments, for any $h \in \mathcal{H}$, $U(h,L) = \min_{P' \in Core(\mu_L)} \int_S u(h)\, dP' \leq I^P(h)$, and in particular, for any $f \in \mathcal{H}_L$, $U(f,L) = I^P(f)$. Thus, for any $f \in \mathcal{H}_L$ and $h \in \mathcal{H}$, $I^P(f) \geq I^P(h)$ (resp., $>$) implies $U(f,L) = I^P(f) \geq I^P(h) \geq U(h,L)$ (resp., $U(f,L) = I^P(f) > I^P(h) \geq U(h,L)$), thereby yielding $U(f,L) \geq U(h,L)$ (resp., $>$). Therefore, by Definition 2.3, $\succsim_L$ is more $\Sigma(L)$-uncertainty averse than $\succsim^P$.

As a result, statement (I) implies (II). The proof is complete. *Q.E.D.*

[18]A capacity $\mu : \mathscr{P}(S) \to [0,1]$ is said to be *convex* (or *supermodular*) if for any $B, D \in \mathscr{P}(S)$, $\mu(B \cup D) + \mu(B \cap D) \geq \mu(B) + \mu(D)$.

CHAPTER 3

Framing Information

3.1. Introduction

3.1.1. Objectives

Models of decision making under uncertainty have identified a range of significant behavioral consequences of (un)awareness or unforeseen contingencies. The general approach of a decision maker (DM) with a coarse understanding of the contingencies is adopted in Ghirardato (2001); Dekel et al. (2001); Epstein et al. (2007). In particular, Ahn and Ergin (2010) studied a framework wherein coarse descriptions of contingencies are given exogenously, while Karni and Vierø (2013, 2017) developed models of the evolution of DMs' belief in the wake of growing awareness. In this chapter, we focus on a different perspective of awareness issue and consider an environment wherein the DM has a complete understanding of contingencies yet perceives the *friction* between awareness and the *frame of information* given by exogenous information. In many problems of choice under uncertainty, some knowledge about the contingencies in question is often available to the DM, and her choice may exhibit particular dependence on objective information that describes that knowledge of contingencies. Studies featuring such dependency mainly focused on situations wherein the variation of information also includes that of the underlying knowledge of contingencies.[1] However, in real-life situations, one may present a fixed knowledge of contingencies to the DM in different descriptive frames, and each description corresponds to a piece of information about the contingencies. In such situations, choices often depend on the framing of information even when the contingencies in question are fixed and of which she is fully aware. Explicating this dependence is the objective of this chapter.

To clarify this idea, consider the intuition behind our approach in the context of the following extended four-color urn experiment (Ellsberg (1961)). An urn contains balls in four colors: Red (R), Black (B), Yellow (Y), and Green (G). The total quantity and

[1] For example, in Gajdos et al. (2004, 2008); Hayashi (2012), information is assumed to take the form of a set of probability distributions over the state space, and thus different pieces of information may suggest disparate underlying statistical knowledge of the state space.

that of each color are unknown. The host have run a series of trials in each of which one ball was drawn from and returned into the urn, and observes that the frequency $\rho(\cdot)$ of each color being drawn is *approximately* as follows: $\rho(\{\mathrm{R}\}) \approx \rho(\{\mathrm{G}\}) \approx .3$ and $\rho(\{\mathrm{B}\}) \approx \rho(\{\mathrm{Y}\}) \approx .2$. Yet those frequencies are undisclosed to the DM. Now, a ball

Fixed Available Knowledge $\rho(\cdot)$
$\rho(\{\mathrm{B}\}) \approx \rho(\{\mathrm{Y}\}) \approx .2,\ \rho(\{\mathrm{R}\}) \approx \rho(\{\mathrm{G}\}) \approx .3,$
$\rho(\{\mathrm{B,Y}\}) \approx .4,\ \rho(\{\mathrm{R,G}\}) \approx .6,$
$\rho(\{\mathrm{R,B}\}) \approx \rho(\{\mathrm{R,Y}\}) \approx \rho(\{\mathrm{B,G}\}) \approx \rho(\{\mathrm{Y,G}\}) \approx .5,$
$\rho(\{\mathrm{R,B,Y}\}) \approx \rho(\{\mathrm{B,Y,G}\}) \approx .7,\ \rho(\{\mathrm{R,B,G}\}) \approx \rho(\{\mathrm{R,Y,G}\}) \approx .8,$

	R	B	Y	G
ℓ_1	\$100	\$100	\$0	\$0
ℓ_2	\$100	\$0	\$100	\$0
ℓ_3	\$0	\$100	\$0	\$100
ℓ_4	\$0	\$0	\$100	\$100

	Announced Frequencies
I_{rb}	$\rho(\{\mathrm{R,B}\}) \approx .5,\ \rho(\{\mathrm{Y}\}) \approx .2,\ \rho(\{\mathrm{G}\}) \approx .3$
I_{ry}	$\rho(\{\mathrm{R,Y}\}) \approx .5,\ \rho(\{\mathrm{B}\}) \approx .2,\ \rho(\{\mathrm{G}\}) \approx .3$
I_g	$\rho(\{\mathrm{R,B,Y}\}) \approx .7,\ \rho(\{\mathrm{G}\}) \approx .3$
I_N	——

TABLE 3.1 – Framing of Information in Four-Color Urn

will be drawn from the urn, and the DM is asked to rank bets $\{\ell_1, \ell_2, \ell_3, \ell_4\}$, after a piece of information in $\{I_{rb}, I_{ry}, I_g, I_N\}$ is announced. The bets and information are given as in Table 3.1. In these hypothetically parallel situations, the DM may reveal: $\ell_1 \sim \ell_4 \succ \ell_3 \succsim \ell_2$ when I_{rb} is given; $\ell_2 \sim \ell_3 \succ \ell_4 \succsim \ell_1$ for I_{ry}; $\ell_4 \sim \ell_3 \succ \ell_2 \sim \ell_1$ for I_g; and $\ell_1 \sim \ell_2 \sim \ell_3 \sim \ell_4$ for I_N.[2]

In this example, as shown in the table, there is a fixed knowledge of the contingencies in question, and the suggested pieces of information describe this knowledge in different frames. More notably, in each of the parallel situations, the DM is fully aware of all the possible outcomes from the ball drawn. Nevertheless, different information frames lead the DM to rank the bets differently. One may argue that the suggested different

[2]The combination of rankings for I_{rb} and I_{rg} is in the spirit of Machina (2009, Reflection Example, .p 390). As argued by Machina (2009), whether the DM reveals such rankings (or, whether she even exhibits any strict ranking under I_{rb} or I_{ry}) is an empirical matter, however, if she has a strict ranking either over $\{\ell_1, \ell_2\}$ or over $\{\ell_3, \ell_4\}$ in whichever direction under I_{rb}, this ranking would be reversed under I_{ry}.

rankings are due to the numbers assigned to the frequencies rather than the variation of information frames. Yet this is not quite comprehensive. Although the exact form of rankings, e.g., which bet in $\{\ell_3, \ell_4\}$ is ranked higher under I_{ry}, indeed depends on those numbers, we maintain that information frames do have certain "structural" influence on the machinery that governs the DM's ranking, and that such an influence is independent of the content (in this case, the numbers) of the underlying knowledge of contingencies. To see this, one may consider the "dual" property, consistency. In the example, ℓ_1 and ℓ_2 (resp., ℓ_3 and ℓ_4) are ranked the same under I_N and I_g, and notice that this ranking would still be consistent no matter what number in $(0, 1)$ is assigned to $\rho(\{\mathrm{G}\})$.

Explicating the information-dependency of choice that hinged on information frames is of significant interest. First, this approach allows us to explore the connection between characteristic conditions of such dependence and those of revealed attitude towards uncertainty as in, for example, Hayashi (2012); Li (2018). Moreover, in the wake of Machina (2009, 2014), the suggested dependence explains the behavioral pattern known as ambiguity-aversion. That is, it may relate ambiguity aversion and frame-dependency in the same vein. Uncovering such dependency is of importance not only from the decision-theoretical perspective but also for economic applications. In the economic problems that involve both uncertainty and variable information, decision models are often considered models of beliefs. However, the dependency we alluded to suggests that the frame of information per se may also be an argument of the models, hence providing an additional dimension to the related analysis.

We, therefore, focus on an environment wherein the DM entertains a complete awareness of contingencies, and a fixed exogenous knowledge of uncertainty is described coarsely to her within variable information frames. The objective is to develop a model to accommodate the influence that the friction between awareness and variable information frames exerts on decision making under uncertainty.

3.1.2. Frame-Adaptivity of Choice

We invoke the standard Anscombe and Aumann (henceforth, AA) framework and take as primitive a family of information-dependent preferences over AA acts that are indexed by partitions of the state space.[3] Each partition corresponds to an information frame in which a fixed knowledge of contingencies is described to the DM.

[3]Caution should be exercised on the difference between the *standard* AA framework given by Fishburn (1970) and the original one introduced in Anscombe and Aumann (1963).

We impose two types of baseline axioms on the family of preferences, the *uniform* and *consistency* axioms. Those axioms characterize a general representation that takes the following partition-dependent expectation form:

$$\mathcal{U}(f,L)=\int_S U(f,L)dV_L; \quad U(f,L)=\sum_{E\in L}\left(\int_E u(f)dV\right)\frac{\mathbf{1}_E}{V(E)},$$

where f is an AA act, L is a partition of the state space S, $\mathbf{1}_E$ denotes the indicator function of event E, and the integrals are Choquet integrals. As for its elements, u is a von Neumann-Morgenstern (vNM) index, V is a capacity on S, and for each partition L, V_L is a null-additive capacity on the σ-algebra induced by L. The representation suggests that the DM makes choices as if she entertains a twofold assessment of acts featuring information-frame-dependency. When the fixed exogenous knowledge of contingencies is described in information frame L, she firstly computes the conditional expected payoff on each learnable event against her subjective belief V in a rank-dependent manner. This process coarsens the inherent uncertainty such that it can be assessed in the frame L. She then applies V_L, which captures how she interprets the given information, to evaluate the overall expected payoff based on the conditional expected payoff on each learnable event. Such an assessment manifests itself as if the DM evaluates AA acts in a way adapting the given frame of information L. Hence, we call the representation *frame-adaptive expected utility* (FAEU).

One interpretation of the frame-adaptivity is that exogenous information alters the salience of the frame in which that information is encoded (Taylor and Thompson (1982)). Presented with the information given by L, descriptive frame L is perceived to be more salient such that the DM may discriminate her awareness of contingencies (i.e., the state space) and the information frame, and tends to evaluate AA acts according to frame L (Kahneman (2011, .p 324)). Bordalo et al. (2012) developed a model of choice under risk featuring salience in which the true probabilities given by lotteries are distorted in favor of salient payoffs. In our model, however, salience is attributed to the frames which the DM entertains to resolve the inherent uncertainty.

Having the general representation, we focus on two specific forms of FAEU characterized under modified impositions of consistency axioms. These representations suggest different manifestations of frame-adaptivity and highlight attitude towards frames of information and the degree of salience effect, respectively. In each form of representation, the influence of the suggested behavioral feature is captured by a family of normalized parameters. Given the same mapping U as in the general FAEU, the first representation, which we call ω-FAEU, takes the form

$$\mathcal{U}^{\omega}(f,L)=\omega_L\int_S U(f,L)d\hat{V}+(1-\omega_L)\min_{s\in S}U(f,L),$$

where for every partition L, parameter $\omega_L \in (0,1]$ captures the DM's attitude towards the informativeness of frame L. The second representation, S-FAEU, has the form

$$\mathcal{U}^S(f,L) = \vartheta_L \int_S \left\{ U(f,L) - \left(\min_{s \in S} U(f,L) \right) \mathbf{1}_S \right\} dV + \min_{s \in S} U(f,L),$$

where for each L, positive parameter ϑ_L reflects the degree of salience.

As mentioned, our representations allow us to explore the translatability of uncertainty attitude as in Li (2018). We connect frame-adaptive models to the definition of (definitive) uncertainty attitude given by Schmeidler (1989) and to that of comparative uncertainty attitude proposed by Ghirardato and Marinacci (2002) and isolate particular properties regarding whether and how the uncertainty attitude exhibited under a given information frame is related to those revealed under other frames. Applying different forms of FAEU, we also conduct a comparative analysis to study the separating roles of the conditions of representations in determining the degree of uncertainty attitude revealed under each given information frame. Later in the chapter, we discuss the possibility of attributing ambiguity-aversion to the frame-adaptive reasoning and explore the applications of frame-adaptive models to economic problems wherein the interested party can manipulate individual behavior via releasing information in suitable frames to achieve desirable goals.

The rest of the chapter is organized as follows: Section 3.2 introduces the preliminaries. In Section 3.3, basic axioms and general representation are presented. Section 3.4 presents the suggested two specific forms of frame-adaptive expected utility under stronger impositions of axioms. Translatability of uncertainty attitude is studied in Section 3.5. Finally, Section 3.6 discusses ambiguity-aversion and applications to economic problems. Proofs are included in appendices.

3.2. Model

3.2.1. Partitions as Information Frames

Let S be a finite set of states and $\mathscr{P}(S)$ be the power set of S. Each element of $\mathscr{P}(S)$ is called an event and is denoted typically by E, B, or D. Denote by L a typical partition of S, and let $\Sigma(L)$ be the σ-algebra (of S) generated by L. Let $\mathcal{L}$ be the set of all partitions of S. In particular, let $\underline{L} = \{S\}$ and $\overline{L} = \{\{s\} \mid s \in S\}$. Each partition L is interpreted as an information frame. Define $\mathscr{L} = \{(L, \tilde{L}) \in \mathcal{L} \times \mathcal{L} \mid \Sigma(L) \subseteq \Sigma(\tilde{L})\}$. That is, $\tilde{L}$ is finer than L whenever $(L, \tilde{L}) \in \mathscr{L}$. Intuitively, if $(L, \tilde{L}) \in \mathscr{L}$, then $\tilde{L}$ carries more detailed information about the underlying knowledge of likelihoods than L does. Endowed with $\mathscr{L}$, partially ordered set $(\mathcal{L}, \mathscr{L})$ becomes a lattice.

REMARK 3.1. Regarding the transparency of information frames, one may consider the following formalism. Let $\mathcal{K}$ be a fixed knowledge of contingencies which is specified either by a mapping $\mathcal{K} : \mathscr{P}(S) \to Y$ with Y being a co-domain of discourse or by a binary predicate $\mathcal{K} \subset \mathscr{P}(S) \times \mathscr{P}(S)$. For every $L \in \mathcal{L}$, let $\mathcal{K}_L : L \to Y$, or $\mathcal{K}_L \subset L \times L$, denote the restriction of $\mathcal{K}$ on L. For each partition L, $\mathcal{K}_L$ corresponds to a description of knowledge $\mathcal{K}$ (hence, a piece of information). Each $L \in \mathcal{L}$ uniquely specifies the descriptive frame — that is, the restricted domain of $\mathcal{K}$ whose elements are mutually exclusive and collectively describe the state space — of information $\mathcal{K}_L$, thereby being referred to as an information frame.

For example, in the four-color urn example discussed in Section 3.1.1, the underlying knowledge of contingencies $\mathcal{K}$ has interpretation "the approximated frequencies of events observed in trials," and the information frame of I_{rb} corresponds to partition $\{\{\mathrm{R}, \mathrm{B}\}, \{\mathrm{Y}\}, \{\mathrm{G}\}\}$.

3.2.2. Preliminaries

For every $B \in \mathscr{P}(S)$, $\mathbf{1}_B : X \to \{0, 1\}$ denotes the indicator function of B. Let Σ be a σ-algebra of S. We call a set function $\mu : \Sigma \to [0, 1]$ a capacity on Σ if $\mu(\emptyset) = 0$, $\mu(S) = 1$, and for any $B, D \in \Sigma$, $B \subseteq D$ implies $\mu(B) \leq \mu(D)$. A capacity μ is said to be *null-additive* if $\mu(B) = 0$ implies $\mu(B \cup D) = \mu(D)$ for every disjoint $B, D \in \mathcal{X}$. All the integrals stand for the Choquet integral throughout.

Let A be the set of consequences assumed to be a separable metric space. Denote by $\Pi(A)$ the set of all simple lotteries over A. An act is defined by function $f : S \to \Pi(A)$, and denote by $\mathcal{H}$ the set of all acts. Let $\mathcal{H}_C \subset \mathcal{H}$ be the set of all constant acts. The act space $\mathcal{H}$ is endowed with a mixture operation defined by

$$\big(\alpha f + (1-\alpha) g\big)(s) = \alpha f(s) + (1-\alpha) g(s), \quad \forall s \in S, \forall \alpha \in [0, 1].$$

For any $f, g \in \mathcal{H}$ and $E \in \mathscr{P}(S)$, let $[fEg]$ be the binary act such that for all $s \in S$, $[fEg](s) = f(s)$ when $s \in E$, and $[fEg](s) = g(s)$ otherwise. For every $L \in \mathcal{L}$, let $\succsim_L \subset \mathcal{H} \times \mathcal{H}$ be a binary relation where $\succ_L$ and $\sim_L$ denote the asymmetric and symmetric part of $\succsim_L$, respectively. For every $L \in \mathcal{L}$, we extend $\succsim_L$ to $\Pi(A)$ by identifying lotteries with constant acts. The object being of our interest is the family of those binary relations $\{\succsim_L\}_{L \in \mathcal{L}}$.

For every $L \in \mathcal{L}$, act h is said to be $\Sigma(L)$-*measurable* if for any $\pi \in \Pi(A)$, $\{s \in S \,|\, h(s) = \pi\} \in \Sigma(L)$. Let $\mathcal{H}_L$ denote the set of all $\Sigma(L)$-measurable acts. For every $L \in \mathcal{L}$ and any $f \in \mathcal{H}$, denote by $\underline{f}^L$ the constant act such that $f(s) \succsim_L \underline{f}^L(s)$ on S, and $f(\hat{s}) = \underline{f}^L(\hat{s})$ for some $\hat{s} \in S$.

DEFINITION 3.1 (Relevant Acts). Given an $L \in \mathcal{L}$, a $D \in \mathscr{P}(S)$, an act f is called an (L, D)-*relevant act* (simply (L, D)-*act*) if $f = [f D \underline{f}^L]$. In addition, for a fixed $c \in \mathcal{H}_L$, we call an act f an (L, D, c)-*act* if $f = [f D c]$ and $c = \underline{f}^L$.

Note that the notion of (L, D)-acts does not require event D to be a cell of partition L. That is, the given partition L affects the specification of (L, D)-acts solely through the induced preference $\succsim_L$ over lotteries. Let $\mathcal{C}_D^L$ and $\mathcal{C}_D^L[c]$ denote the set of all (L, D)-acts and that of all (L, D, c)-acts, respectively.

Formulations of our key axioms, as well as their normative implications, will be based on the following properties of $\Sigma(L)$-measurable acts and (L, D)-acts in terms of demand for informativeness in the DM's assessment of AA acts.

REMARK 3.2. For an $L \in \mathcal{L}$, in order to rank the $\Sigma(L)$-measurable acts, the DM may need information about the likelihoods of events in L. Thus, if the underlying knowledge of contingencies is knowledge of likelihoods, then the information carried by L provides a reference for such demand of informativeness. In this way, the inherent uncertainty in evaluating $\Sigma(L)$-measurable acts can be accommodated by the descriptive power of information frame L.

REMARK 3.3. For a fixed $L \in \mathcal{L}$ and an $E \in L$, an (L, E)-act $f \in \mathcal{C}_E^L$ guarantees its payoff on $S \setminus E$ as the least preferred payoff (with respect to $\succsim_L$) regardless of the realized state, meaning that f can be viewed as a minimal payoff $\underline{f}^L(s)$ associated with a "gain-bet" over the sub-events of E. For example, let $L = \{\{\mathrm{R}, \mathrm{B}, \mathrm{Y}\}, \{\mathrm{G}\}\}$ and $E = \{\mathrm{R}, \mathrm{B}, \mathrm{Y}\}$ in the four-color urn discussed in Section 3.1.1, and suppose \$100 $\succ_L$ \$50 $\succ_L$ \$30 $\succ_L$ \$0. Then, the following table gives an illustration of (L, E)-acts with the induced minimal payoffs and gain-bets.

	R	B	Y	G
ℓ_1	\$100	\$100	\$0	\$0
ℓ_5	\$50	\$100	\$30	\$30

	ℓ_1	ℓ_5
Minimal Payoff	\$0	\$30
Gain-Bets	$\begin{pmatrix} \{\mathrm{R}, \mathrm{B}\} & \text{\$100 instead of \$0} \\ \{\mathrm{Y}\} & \text{\$0 instead of \$0} \end{pmatrix}$	$\begin{pmatrix} \{\mathrm{R}\} & \text{\$50 instead of \$30} \\ \{\mathrm{B}\} & \text{\$100 instead of \$30} \\ \{\mathrm{Y}\} & \text{\$30 instead of \$30} \end{pmatrix}$

TABLE 3.2 – (L, E)-acts, Minimal Payoff, and Induced Gain-Bets

Presumably, rankings over (L, E)-acts depend on (i) the DM's assessment of the gain-bets induced by those acts, and (ii) trade-off between the anticipated utility increments delivered by those gain-bets and utility given by the corresponding minimal payoffs. Since the uncertainty inherent in those gain-bets is described by $(E, \mathscr{P}(E))$, it is thus plausible to argue that both of the assessment of gain-bets and the aforementioned trade-off would depend on how she judges the likelihoods of sub-events of E. However, this is beyond the descriptive power of frame L, hence being purely-subjective. In particular, for a $c \in \mathcal{H}_C$, since (L, E, c)-acts share common "sure-thing" payoff c, rankings over these acts can be ascribed to (i), which solely depends on her judgment of the *relative* likelihoods of the sub-events of E. Hence, the information frame L is completely irrelevant for such demand for informativeness.

3.3. Basic Axioms and General Representation

First, we present the baseline axioms that characterize the general representation of $\{\succsim_L\}_{L\in\mathcal{L}}$. These axioms are akin to those considered by Li (2018) and categorized into two types: *uniform* and *consistency* axioms.

3.3.1. Uniform Axioms

The first five axioms are applied to each fixed partition L and collectively denoted as the uniform axioms.

Axiom 3.1 (Order). *For any $L \in \mathcal{L}$, $\succsim_L$ is complete and transitive.*

Axiom 3.2 (Continuity). *For any $L \in \mathcal{L}$ and any $f, g, h \in \mathcal{H}$ with $f \succ_L g \succ_L h$, there exist $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ_L g \succ_L \beta f + (1-\beta)h$.*

Axiom 3.3 (Monotonicity). *For any $L \in \mathcal{L}$, any $f, g \in \mathcal{H}$ and any $h \in \mathcal{H}_L$*

(I) $f(s) \succsim_L g(s)$ on S implies $f \succsim_L g$.

(II) if $\underline{h}^L \succsim_L \underline{f}^L$ and $[fE\underline{f}^L] \succsim_L [hE\underline{f}^L]$ (resp., $\precsim_L$) for all $E \in L$, then $f \succsim_L h$ (resp., $\precsim_L$).

Axiom 3.4 (Nondegeneracy). *For every $L \in \mathcal{L}$, not for all $f, g \in \mathcal{H}$, $f \sim_L g$.*

Axiom 3.5 (Comonotonic Relevance Independence, CRI).

(I) For every $L \in \mathcal{L}$, every pairwise comonotonic $f, g, h \in \mathcal{H}_L$, and any $\alpha \in (0,1)$, $f \succsim_L g$ if and only if $\alpha f + (1-\alpha)h \succsim_L \alpha g + (1-\alpha)h$.[4]

(II) For every $L \in \mathcal{L}$, $E \in L$, any pairwise comonotonic $f, g, h \in \mathcal{C}_E^L$, and any $\alpha \in (0,1)$, $f \succsim_L g$ if and only if $\alpha f + (1-\alpha)h \succsim_L \alpha g + (1-\alpha)h$.

Presented with the information given in frame L, the DM may discriminate the uncertainty accommodated by the descriptive power of L, $(S, \Sigma(L))$, and the uncertainty of which she is aware $(S, \mathscr{P}(S))$. It is because the given information may either carry the knowledge of likelihoods of learnable events or alter the salience of contingencies that are describable in frame L. Hence, there might be friction between her awareness and the exogenously given information frame.

In CRI, we maintain that such a difference in perception of uncertainty matters when we consider comonotonic independence axiom as in the Choquet expected utility model (CEU, Schmeidler (1989)).[5] Precisely, the comonotonic independence is not as normatively appealing for the acts which involve the trade-off between uncertainty with different perceptions as it is for those who do not. As argued in Remark 3.2, frame L accommodates the uncertainty inherent in $\Sigma(L)$-measurable acts, meaning that $\succsim_L$ over $\mathcal{H}_L$ does not involve such a trade-off. We thus find the comonotonic independence normative for $\succsim_L$ on $\mathcal{H}_L$.[6] The implication of CRI-(II) should be discussed in conjunction with that of CRI-(I). Given that $\mathcal{H}_L \cap \mathcal{C}_E^L \neq \emptyset$, consider first $f, g, h \in \mathcal{H}_L \cap \mathcal{C}_E^L$. Clearly, f, g, and h are pairwise comonotonic. Hence, CRI-(I) implies that the restriction of $\succsim_L$ on $\mathcal{H}_L \cap \mathcal{C}_E^L$ obeys the independence axiom. Meanwhile, following the argument in Remark 3.3, f, g and h can be respectively viewed as a minimal payoff associated with a gain-bet over E, yet they differ from acts in $\mathcal{C}_E^L \setminus \mathcal{H}_L$ in the sense that they induce degenerated gain-bets over E. Thus, CRI-(I) implies that the trade-off between the anticipated increment from the gain-bets over E and the minimal payoffs is exercised in a way obeying the independence axiom. Due to this, CRI-(II) is essentially reduced to an assumption for the evaluation of the gain-bets over E induced by acts in $\mathcal{C}_E^L$. As discussed in Remark 3.3, L is irrelevant for the assessment of those gain-bets, which is purely subjective, hence there being no trade-off between uncertainty with different perceptions in evaluating the anticipated increments of those gain-bets. Therefore, for each learnable event $E \in L$, we consider the comonotonic independence normative for the restriction of $\succsim_L$ on $\mathcal{C}_E^L$.

[4]For a fixed $L \in \mathcal{L}$, acts $f, g \in \mathcal{H}$ are said to be *comonotonic* if, for any $s, t \in S$, $f(s) \succ_L f(t)$ implies $g(s) \succsim_L g(t)$.

[5]The comonotonic independence can be restated in our notation: for any $L \in \mathcal{L}$, any pairwise comonotonic $f, g, h \in \mathcal{H}$, and any $\alpha \in (0,1)$, $f \succsim_L g$ iff $\alpha f + (1-\alpha)h \succsim_L \alpha g + (1-\alpha)h$.

[6]Note that $\mathcal{H}_L$ and $\mathcal{C}_E^L$ for each $E \in L$ are closed under the mixture operation. Otherwise, it becomes invalid to consider the mixture on these sub-domains.

3.3.2. Consistency Axioms

The general intuition behind our baseline consistency axioms is that if two pieces of information carried by different frames are both irrelevant for the assessment of a set of acts, then the preference over this set of acts should be consistent under those information frames.

Axiom 3.6 (Certainty). *For any $L, L' \in \mathcal{L}$ and any $c, c' \in \mathcal{H}_C$, $c \succsim_L c'$ if and only if $c \succsim_{L'} c'$.*

Axiom 3.7 (Consistency of Irrelevance, CoI). *For any $(L, \tilde{L}) \in \mathscr{L}$, any $E \in \tilde{L}$, and any $c \in \mathcal{H}_C$, if $\mathcal{C}_E^{\tilde{L}}[c]$ is not $\succsim_{\tilde{L}}$-indifferent, then for any $f, g \in \mathcal{C}_E^{\tilde{L}}[c] \cap \mathcal{C}_E^{L}[c]$, $f \succsim_L g$ if and only if $f \succsim_{\tilde{L}} g$.*[7]

Certainty states that the preference over constant acts should stay identical across all information frames. Such consistency separates the resolution of uncertainty and that of physical risk governed by objective probabilities. CoI suggests that for an information frame L, if sub-events of an event E are not learnable from L, nor a finer frame $\tilde{L}$, then acts that yield improvements from some fixed constant act only on E should be ranked the same under L and $\tilde{L}$, even when E itself is learnable from $\tilde{L}$.

For example, let $L = \{\{\text{R}, \text{B}, \text{Y}\}, \{\text{G}\}\}$, $\tilde{L} = \{\{\text{R}, \text{B}\}, \{\text{Y}\}, \{\text{G}\}\}$, and $E = \{\text{R}, \text{B}\}$ in the four-color urn given in Section 3.1.1. Consider bet ℓ_1 and ℓ_6 given in the following table respectively under information I_{rb} and I_g. Suppose \$120 $\succ_L$ \$100 $\succ_L$ \$80 $\succ_L$ \$0

	R	B	Y	G
ℓ_1	\$100	\$100	\$0	\$0
ℓ_6	\$120	\$80	\$0	\$0

	Information: Frequencies in The Trials
I_{rb}	$\rho(\{\text{R}, \text{B}\}) \approx .5$, $\rho(\{\text{Y}\}) \approx .2$, $\rho(\{\text{G}\}) \approx .3$
I_g	$\rho(\{\text{R}, \text{B}, \text{Y}\}) \approx .7$, $\rho(\{\text{G}\}) \approx .3$

Table 3.3 – CoI in Four-Color Urn

(resp., $\succ_{\tilde{L}}$). Then, we have $\ell_1, \ell_6 \in \mathcal{C}_E^{L}[\$0] \cap \mathcal{C}_E^{\tilde{L}}[\$0]$. Thus, CoI requires that ℓ_1 and ℓ_6 should be ranked the same under information I_{rb} and I_g.

[7] Given an L, we say a nonempty set of acts $\mathcal{F} \subseteq \mathcal{H}$ is $\succsim_L$-*indifferent* if $f \sim_L g$ holds for any $f, g \in \mathcal{F}$.

3.3.3. General Representation: Frame-Adaptive Expected Utility

We now have all the necessary elements to characterize our general representations. First, we consider $\succsim_L$ for each fixed information frame L. Our first result establishes an equivalence between the uniform axioms and a representation wherein information-frame-dependency is captured by a twofold partition-dependent integral.

THEOREM 3.1. *Given an $L \in \mathcal{L}$, the following statements are equivalent.*

(I) $\succsim_L \subset \mathcal{H} \times \mathcal{H}$ satisfies the uniform axioms.

(II) An affine function $u_L : \Pi(A) \to \mathbb{R}$ and a capacity $v_L : \mathscr{P}(S) \to [0,1]$ exist such that, for any $f, g \in \mathcal{H}$,

$$f \succsim_L g \iff \int_{\bigcup \Delta(L)} U_L(f) dv_L \geq \int_{\bigcup \Delta(L)} U_L(g) dv_L, \quad \text{where}$$
$$U_L(f) = \sum_{E \in \Delta(L)} \left(\int_E u_L(f) dv_L \right) \frac{\mathbf{1}_E}{v_L(E)};$$
$$\Delta(L) = \{E \in L \mid v_L(E) > 0\}.$$

Moreover, v_L is null-additive on $\Sigma(L)$, and is unique in the sense that: if there is another capacity v' being such, then $v'(B) = v_L(B)$ for every $B \in (\bigcup_{E \in L} \mathscr{P}(E) \cup \Sigma(L))$. Finally, u_L is unique up to positive linear transformations (p.l.t.), and $\Delta(L) \neq \emptyset$.

PROOF. See Appendix 3.B.1.

In the theorem, capacity v_L can be interpreted as the DM's willingness-to-bet or subjective belief. Presented with information frame L, act f is evaluated in the following way. First, U_L translates f into a $\Sigma(L)$-measurable utility act $U_L(f)$ which maps each learnable event $E \in L$ to a utility level which can be roughly interpreted by the conditional expected utility on the learnable event obtained with respect to v_L. Mapping U_L indicates that the DM coarsens the uncertainty inherent in f such that utility act $U_L(f)$ can be assessed within information frame L, thereby reflecting information-frame-dependency which is manifested as if the DM tends to evaluate f in a way adapting frame L. Then, the overall utility of f is evaluated by computing the rank-dependent expectation of utility act $U_L(f)$ against v_L. The suggested uniqueness of v_L is also significant. One possible explanation can be given related to rational (motivated) inattention. Since the DM entertains the aforementioned frame-adaptive assessment of AA acts, she only needs to judge the likelihoods of the event that are relevant to coarsening the uncertainty or to computing the overall expected utility. Thus, given information frame L, it is unnecessary to consider likelihoods of the events which are not included in $(\bigcup_{E \in L} \mathscr{P}(E)) \cup \Sigma(L)$, and utility of acts must not depend on the subjective likelihoods which v_L assigns to those events.

Recall that $\overline{L} = \{\{s\} \mid s \in S\}$ and $\underline{L} = \{S\}$. As direct consequences, it can be shown that $\succsim_{\overline{L}}$ and $\succsim_{\underline{L}}$ admit standard CEU representations on its full domain.

Corollary 3.1.1. *The following statements hold.*

(I) $\succsim_{\underline{L}} \subset \mathcal{H} \times \mathcal{H}$ *satisfies the uniform axioms if and only if there exist an affine function* $\underline{u} : \Pi(A) \to \mathbb{R}$ *and a unique capacity* $v : \mathscr{P}(S) \to [0,1]$ *such that* $f \succsim_{\underline{L}} g$ *if and only if* $\int_S \underline{u}(f)\, dv \geq \int_S \underline{u}(g)\, dv$ *for every* $f, g \in \mathcal{H}$. *Moreover,* $\underline{u}$ *is unique up to p.l.t.*

(II) $\succsim_{\overline{L}} \subset \mathcal{H} \times \mathcal{H}$ *satisfies the uniform axioms if and only if there exist an affine function* $\hat{u} : \Pi(A) \to \mathbb{R}$ *and a unique null-additive capacity* $\hat{V}$ *on* $\mathscr{P}(S)$ *such that, for any* $f, g \in \mathcal{H}$, $f \succsim_{\overline{L}} g$ *if and only if* $\int_S \hat{u}\,(f)\, d\hat{V} \geq \int_S \hat{u}\,(g)\, d\hat{V}$. *Moreover,* $\hat{u}$ *is unique up to p.l.t.*

Proof. (I) By Theorem 3.1, $\underline{L} = \{S\}$ implies $\Delta(\underline{L}) = \underline{L} = \{S\}$. Hence, $v_{\underline{L}}$ is unique since it holds that $\Sigma(S) = \mathscr{P}(S) = \mathscr{P}(\bigcup \Delta(\underline{L}))$. Also, $\Delta(\underline{L})$ being singleton implies that $U_{\underline{L}}(f) \in \mathcal{H}_C$ for every $f \in \mathcal{H}$. Therefore, for every $f, g \in \mathcal{H}$, $f \succsim_{\underline{L}} g$ if and only if $U_{\underline{L}}(f) \geq U_{\underline{L}}(g)$. Set $v(B) = v_{\underline{L}}(B)$ for every $B \in \mathscr{P}(S)$. Then, for every $f \in \mathcal{H}$, it follows that $U_{\underline{L}}(f) = \int_S \underline{u}(f)\, dv$.

(II) $\overline{L} = \{\{s\} \mid s \in S\}$ implies $\mathcal{H}_{\overline{L}} = \mathcal{H}$, since $\Sigma(\overline{L}) = \mathscr{P}(S)$. Thus, by Lemma 3.1 in Appendix 3.A.2, the equivalence follows. *Q.E.D.*

In what follows, we present the general representation theorem for $\{\succsim_L\}_{L \in \mathcal{L}}$. Clarify a notion. We say a functional $\mathcal{U} : \mathcal{H} \times \mathcal{L} \to \mathbb{R}$ *represents* $\{\succsim_L\}_{L \in \mathcal{L}}$ on $\mathcal{H}$ if for all $L \in \mathcal{L}$ and $f, g \in \mathcal{H}$, $f \succsim_L g$ if and only if $\mathcal{U}(f, L) \geq \mathcal{U}(g, L)$. We can now formally define the general utility representation.

Definition 3.2 (Frame-Adaptive Expected Utility). We say $\{\succsim_L\}_{L \in \mathcal{L}}$ admits a *Frame-Adaptive Expected Utility* (FAEU) representation if there exist an affine function $u : \Pi(A) \to \mathbb{R}$, a unique capacity $V : \mathscr{P}(S) \to [0,1]$, and for every $L \in \mathcal{L}$, there is a unique null-additive capacity $V_L : \Sigma(L) \to [0,1]$ such that $\mathcal{U}$ is well-defined and represents $\{\succsim_L\}_{L \in \mathcal{L}}$ on $\mathcal{H}$, where

$$\mathcal{U}(f, L) = \int_{\bigcup \Delta(L)} \left\{ \sum_{E \in \Delta(L)} \left(\int_E u(f) dV \right) \frac{\mathbf{1}_E}{V(E)} \right\} dV_L;$$
$$\Delta(L) = \{E \in L \mid V_L(E) > 0\}.$$

Moreover, u is unique up to p.l.t.

When such a tuple $(u, V, \{V_L\}_{L \in \mathcal{L}})$ exists, we call it an FAEU representation.

In comparison with Theorem 3.1, FAEU involves a globally unique capacity V on $\mathscr{P}(S)$ capturing the DM's purely subjective belief, which is used to coarsen the uncertainty inherent in each act so that it can be accommodated by the given information frame. For each partition L, capacity V_L reflects the DM's subjective interpretation of the information carried by L.

THEOREM 3.2. *$\{\succsim_L\}_{L\in\mathcal{L}}$ satisfies the uniform axioms, Certainty and CoI if and only if it admits an FAEU representation.*

PROOF. See Appendix 3.B.2.

3.4. Frame-Adaptivity: Attitude and Salience

Here, we study two specific forms of FAEU under stronger impositions of consistency axioms. In either of the representations, frame-adaptivity is performed in a parameterized manner with respect to a family of parameters that indicate the strength of the effect of the corresponding behavioral feature towards each information frame. The intuition behind the consistency axioms considered in this section is in line with the one discussed previously. That is, if two information frames are both irrelevant for the assessment of a particular set of acts, then these acts should be ranked the same under the two information frames. However, the axioms presented in this section are formulated based on more restrictive interpretations of irrelevance.

3.4.1. Attitude towards Informativeness

The general form of FAEU does not establish any connection between the DM's subjective interpretations of information carried by different frames. Here, we study a form of FAEU in which the DM's interpretation of information is invariant across variable information frames. As to the axioms, we focus on additional consistency properties when finer information frames yield redundancy of informativeness for a particular set of acts.

AXIOM 3.8 (Coarseness Aversion). *For any $(L,\tilde{L})\in\mathscr{L}$, any $f\in\mathcal{H}_L$ and $c\in\mathcal{H}_C$, $f\succsim_L(\succ_L)\,c$ implies $f\succsim_{\tilde{L}}(\succ_{\tilde{L}})\,c$.*

AXIOM 3.9 (Consistency of Redundancy, CoR). *For any $(L,\tilde{L})\in\mathscr{L}$, any $D,D'\in\Sigma(L)\setminus\{S,\emptyset\}$, and any $c\in\mathcal{H}_C$, if neither $\mathcal{C}^L_{S\setminus D}[c]$ nor $\mathcal{C}^L_{S\setminus D'}[c]$ is $\succsim_L$-indifferent, then for any $f\in\mathcal{C}^L_D[c]\cap\mathcal{H}_L$ and $g\in\mathcal{C}^L_{D'}[c]\cap\mathcal{H}_L$, $f\succsim_L g$ if and only if $f\succsim_{\tilde{L}} g$.*

Coarseness Aversion is, in the formulation, akin to the comparative uncertainty aversion defined by Ghirardato and Marinacci (2002). Let f and c be as in the statement. The axiom states that if the DM (weakly) prefers act f to objective lottery c under an information frame that fully accommodates the uncertainty inherent in that act, then redundancy of informativeness resulted by any finer information frame does not reverse this ranking. It can be shown that Coarseness Aversion implies Certainty.

Let L, $\tilde{L}$, D, D', f and g be as in the statement of CoR. Notice that, both D and D' are describable within information frame L as $D, D' \in \Sigma(L) \setminus \{S, \emptyset\}$. Thus, $f \in \mathcal{C}^L_D[c] \cap \mathcal{H}_L$ and $g \in \mathcal{C}^L_{D'}[c] \cap \mathcal{H}_L$ jointly imply that the uncertainty inherent in f and g can be accommodated by frame L, and that they involve a same minimal payoff. Hence, the finer frame $\tilde{L}$ yields redundancy of informativeness for the acts. As a result, CoR states that if act f and g have a common minimal payoff, then additional redundancy of informativeness does not alter the ranking over f and g, which is revealed under the frame that accommodates the uncertainty inherent in assessing the acts.

We now formally define the first special form of FAEU.

Definition 3.3 (ω-FAEU). We say $\{\succsim_L\}_{L\in\mathcal{L}}$ admits an ω-*FAEU* representation if the followings are satisfied:

(1) There exist an affine function u on $\Pi(A)$, a unique capacity V on $\mathscr{P}(S)$, a unique null-additive capacity $\hat{V}$ on $\mathscr{P}(S)$ and a unique normalized family of positive parameters $\boldsymbol{\omega} = \{\omega_L\}_{L\in\mathcal{L}}$ with $\omega_{\overline{L}} = 1$ such that $\mathcal{U}^\omega$ is well-defined and represents $\{\succsim_L\}_{L\in\mathcal{L}}$ on $\mathcal{H}$, where

$$\mathcal{U}^\omega(f, L) = \omega_L \int_{\bigcup \Omega(L)} U(f, L) d\hat{V} + (1 - \omega_L) \min_{s \in \bigcup \Omega(L)} U(f, L);$$

$$U(f, L) = \sum_{E\in\Omega(L)} \left(\int_E u(f) dV \right) \frac{\mathbf{1}_E}{V(E)};$$

$$\Omega(L) = \{E \in L \mid \hat{V}(E) > 0\}.$$

(2) For any $L, \tilde{L} \in \mathcal{L} \setminus \{\underline{L}\}$, $\omega_L \in (0, 1]$, and $(L, \tilde{L}) \in \mathscr{L}$ implies $\omega_L \le \omega_{\tilde{L}}$. The family of parameters $\boldsymbol{\omega}$ is unique in the sense that: $\omega_{\underline{L}}$ is arbitrary, while for every $L \in \mathcal{L} \setminus \{\underline{L}\}$, ω_L is unique. Moreover, u is unique up to p.l.t.

When such a tuple $(u, V, \hat{V}, \boldsymbol{\omega})$ exists, we call it an ω-FAEU representation.

The decision rule given by ω-FAEU differs from that the general FAEU suggests only on the machinery by which the frame-adaptivity in the second stage is manifested. First, the DM interprets information in a compatible way such that her interpretations of information given in variable frames are summarized by the unique capacity $\hat{V}$ on the finest algebra $\mathscr{P}(S)$. Presented with the information given in frame L, the

DM coarsens the uncertainty inherent in each act based on purely subjective belief V as in the general FAEU. Yet having adapted to frame L, she considers the worst-case learnable event such that the overall utility of each act is evaluated by a convex combination of (i) the rank-dependent expected utility of the corresponding utility act $U(\cdot, L)$ obtained applying her subjective interpretation $\hat{V}$, and (ii) the worst-case conditional expected utility given by the utility act $U(\cdot, L)$. Such a convex combination is taken with respect to a parameter ω_L which, in conjunction with the implication of Coarseness Aversion, reflects her attitude towards the informativeness of L. Notably, Definition 3.3-(2) suggests that the DM puts more weight on the worst-case conditional expected utility while facing the coarser frame.

Probing further, it is possible in general to have

$$\left\{ s \in S \;\middle|\; u(f(s)) = \min_{s \in \bigcup \Omega(L)} U(f, L)(s) \right\} = \emptyset;$$

$$\min_{s \in \bigcup \Omega(L)} U(f, L)(s) > \min_{s \in S} u(f)(s).$$

Since $U(\cdot, L)$ is a functional of u and V, the above observation thus implies that the worst-case which the DM takes into consideration is purely subjective. Moreover, in ω-FAEU, this worst-case conditional expected utility level plays a role related to a subjective reference point. If we adopt such an interpretation of $\min_{s \in \bigcup \Omega(L)} U(\cdot, L)$, then ω-FAEU indeed connects the frame-adaptivity and an endogenous reference-dependency which differs from the one studied by Kőszegi and Rabin (2006, 2007).

Theorem 3.3. *$\{\succsim_L\}_{L \in \mathcal{L}}$ admits an ω-FAEU representation if and only if it satisfies the uniform axioms, CoI, CoR and Coarseness Aversion.*

Proof. See Appendix 3.B.3.

In what follows, we present a special case of ω-FAEU in which the frame-adaptivity appears in the same vein as in the general FAEU, yet the DM holds a consistent interpretation of information across variable frames. The following axiom states that act f and g should be ranked the same under L and a finer frame $\tilde{L}$, whenever L can accommodate the demand for informativeness of both f and g.

Axiom 3.10 (Strong Consistency of Redundancy, SCoR). *For any $(L, \tilde{L}) \in \mathscr{L}$, and for any $f, g \in \mathcal{H}_L$, $f \succsim_L g$ if and only if $f \succsim_{\tilde{L}} g$.*

Corollary 3.3.1. *The following statements are equivalent.*

(I) $\{\succsim_L\}_{L \in \mathcal{L}}$ satisfies the uniform axioms, SCoR and CoI.

(II) $\{\succsim_L\}_{L\in\mathcal{L}}$ *admits an* ω*-FAEU representation* $(u, V, \hat{V}, \boldsymbol{\omega})$ *in which* $\omega_L = 1$ *for any* $L \in \mathcal{L} \setminus \{\underline{L}\}$. *That is,* $\mathcal{U}^{SCoR}$ *represents* $\{\succsim_L\}_{L\in\mathcal{L}}$ *on* $\mathcal{H}$, *where*

$$\mathcal{U}^{SCoR}(f,L) = \int_{\bigcup \Omega(L)} U(f,L)d\hat{V}; \quad U(f,L) = \sum_{E\in\Omega(L)} \left(\int_E u(f)dV\right)\frac{\mathbf{1}_E}{V(E)}.$$

Proof. See Appendix 3.B.3.1.

Let $(u, V, \hat{V})$ denote this representation.

3.4.2. Pure Salience Effects

Here, we explore another special form of FAEU in which the DM's subjective likelihoods of events are not affected by the variation of information frames. That is to say, variable information influences choices solely through altering the salience of frames in which information is described to the DM. Preferences admit such a form of FAEU might be observed when the underlying knowledge of contingencies does not contain information about the likelihoods of contingencies in question.

Axiom 3.11 (Strong Consistency of Irrelevance, SCoI). *For any* $(L, \tilde{L}) \in \mathscr{L}$, *any* $E, E' \in \tilde{L}$, *any* $c \in \mathcal{H}_C$, *and for any* $f \in \mathcal{C}_E^{\tilde{L}}[c] \cap \mathcal{C}_E^{L}[c]$ *and* $g \in \mathcal{C}_{E'}^{\tilde{L}}[c] \cap \mathcal{C}_{E'}^{L}[c]$, $f \succsim_L g$ *if and only if* $f \succsim_{\tilde{L}} g$.

SCoI is a strictly restrictive version of CoI. It states that for a frame L and a finer one $\tilde{L}$, if $\tilde{L}$ does not unpack the sub-events of event E and E', then acts that deliver improvements from some fixed objective lottery (constant act) only on E and E', respectively, should be ranked the same under L and $\tilde{L}$.

Definition 3.4 (Salience-FAEU). We say $\{\succsim_L\}_{L\in\mathcal{L}}$ admits a Salience-FAEU (S-FAEU) representation if the followings are satisfied.

(1) There exist an affine function u on $\Pi(A)$, a unique capacity V on $\mathscr{P}(S)$, and a unique normalized family of positive parameters $\boldsymbol{\vartheta} = \{\vartheta_L\}_{L\in\mathcal{L}}$ with $\vartheta_{\underline{L}} = 1$ such that $\mathcal{U}^S$ is well-defined and represents $\{\succsim_L\}_{L\in\mathcal{L}}$ on $\mathcal{H}$, where

$$\mathcal{U}^S(f,L) = \vartheta_L \int_{\bigcup \Omega(L)} \left\{U(f,L) - \left(\min_{s\in\bigcup\Omega(L)} U(f,L)\right)\mathbf{1}_S\right\} dV + \min_{s\in\bigcup\Omega(L)} U(f,L);$$

$$U(f,L) = \sum_{E\in\Omega(L)} \left(\int_E u(f)dV\right)\frac{\mathbf{1}_E}{V(E)};$$

$$\Omega(L) = \{E \in L \mid V(E) > 0\}.$$

(2) For all $L \in \mathcal{L}$, define a set function $V_L : \Sigma(L) \to [0,1]$ by: $V_L(B) = 1$ if $V(S \setminus B) = 0$, and $V_L(B) = \vartheta_L V(B)$ otherwise. Then, V_L is a null-additive capacity. Finally, u is unique up to p.l.t.

When such a tuple $(u, V, \boldsymbol{\vartheta})$ exists, we call it an S-FAEU representation.

Each information frame L carries a parameter ϑ_L which captures the degree of the salience of L. As in ω-FAEU, the DM takes the worst-case conditional expected utility level $\min_{s \in \bigcup \Omega(L)} U(\cdot, L)$ into account. However, S-FAEU suggests that the DM rather focuses on the Choquet expected utility-increment from $\min_{s \in \bigcup \Omega(L)} U(\cdot, L)$, where the salience may exert influence on her assessment. Thus, the overall utility of each act is computed by adding the modified expected utility-increment to the reference utility level given by $\min_{s \in \bigcup \Omega(L)} U(\cdot, L)$.

Theorem 3.4. *$\{\succsim_L\}_{L \in \mathcal{L}}$ admits an S-FAEU representation if and only if it satisfies the uniform axioms, Certainty, CoR and SCoI.*

Proof. See Appendix 3.B.4.

The following result establishes a connection between ω-FAEU and S-FAEU. It states that they are equivalent if and only if $\{\succsim_L\}_{L \in \mathcal{L}}$ does not exhibit attitude towards informativeness and has a fixed degree of salience across variable information frames.

Corollary 3.4.1. *The following statements are equivalent.*

(I) $\{\succsim_L\}_{L \in \mathcal{L}}$ satisfies the uniform axioms, SCoR and SCoI.

(II) $\{\succsim_L\}_{L \in \mathcal{L}}$ admits an S-FAEU representation $(u, V, \boldsymbol{\vartheta})$ wherein a $\vartheta \in \mathbb{R}_{++}$ exists such that for all $L \in \mathcal{L} \setminus \{\underline{L}\}$, $\vartheta_L = \vartheta$.

(III) $\{\succsim_L\}_{L \in \mathcal{L}}$ admits an ω-FAEU representation $(u, V, \hat{V})$ in which there is a $\vartheta \in \mathbb{R}_{++}$ such that for all $B \in \mathscr{P}(S)$, $V(S \setminus B) > 0$ implies $\hat{V}(B) = \vartheta V(B)$.

Proof. See Appendix 3.B.4.1.

Let (u, V, ϑ) denote this equivalent representation.

Throughout the rest of the chapter, we call $\{\succsim_L\}_{L \in \mathcal{L}}$ an FAEU (resp., ω- or S-FAEU) preference whenever it admits an FAEU (resp., ω- or S-FAEU) representation. For simplicity, given an FAEU preference, the subsequent analysis will always assume that for every $L \in \mathcal{L}$, measurable space $(S, \mathscr{P}(S))$ does not have any nonempty V_L-null (resp., $\hat{V}$-null) event. Then, by definitions, it follows immediately that $V(B) > 0$ holds for all nonempty $B \in \mathscr{P}(S)$. It can be shown that such a non-nullity is equivalent to imposing, instead of Nondegeneracy, the following assumption on $\{\succsim_L\}_{L \in \mathcal{L}}$.

AXIOM 3.12 (Richness). *For any* $L \in \mathcal{L}$ *and* $E \in L$, *not for all* $c \in \mathcal{H}_C$, $\mathcal{C}_E^L[c]$ *is* $\succsim_L$*-indifferent.*

3.5. Translatability of Uncertainty Attitude

In this section, we establish a connection between frame-adaptivity and the translatability of revealed attitude towards uncertainty. In what follows, since we focus our discussion on FAEU preferences, meaning that Certainty will always be considered, we thus use $\mathcal{C}_D$, instead of $\mathcal{C}_D^L$, to denote the set of (L, D)-acts.

3.5.1. Definitive Properties

Here, we focus on the implied properties of uncertainty attitude in a definitive manner. Schmeidler (1989) proposed a behavioral definition of uncertainty attitude and provided a characterization using the capacity obtained in the CEU model (see also Gilboa and Schmeidler (1989)). We restate this definition using our notations.

DEFINITION 3.5 (Uncertainty Attitude). Given an $L \in \mathcal{L}$ and a set of acts $\mathcal{F} \subseteq \mathcal{H}$ that is closed under the mixture operation, we say the preference $\succsim_L \subset \mathcal{H} \times \mathcal{H}$ reveals *uncertainty aversion* (resp., *loving*) on $\mathcal{F}$ if for any $f, g \in \mathcal{F}$ and any $\alpha \in (0, 1)$, $f \sim_L g$ implies $\alpha f + (1 - \alpha)g \succsim_L f$ (resp., $\alpha f + (1 - \alpha)g \precsim_L f$).

Under this definition, we consider a question that if $\succsim_L$ for a fixed $L \in \mathcal{L}$ reveals uncertainty aversion (resp., loving) on a convex sub-domain, say $\mathcal{F}$, then for which $\tilde{L} \in \mathcal{L}$ and $\tilde{\mathcal{F}} \subseteq \mathcal{H}$ we can predict that $\succsim_{\tilde{L}}$ reveals uncertainty aversion (resp., loving) on $\tilde{\mathcal{F}}$ without further observations. Explicating this question is intrinsically to isolate every inference kernel $(\succsim_L, \mathcal{F})$ of uncertainty attitude and the corresponding scope of inference. The following proposition observes a general property regarding tranlatability of uncertainty attitude which is satisfied by any FAEU preference. We start with the formal definition of this property. Define $\mathcal{R} \subset \mathcal{L} \times \mathscr{P}(S)$ by $\mathcal{R} = \{(L, D) \mid [\emptyset \neq D] \wedge [\exists E \in L, D \subseteq E]\}$.

DEFINITION 3.6 (Relevance Translation). We say $\{\succsim_L\}_{L \in \mathcal{L}}$ reveals *relevance translation of uncertainty attitude* (simply, *relevance translation*) if it satisfies the following condition: for every $(L, D) \in \mathcal{R}$, if $\succsim_L$ reveals uncertainty aversion (resp., loving) on $\mathcal{C}_D$, then for any $(\tilde{L}, \tilde{D}) \in \mathcal{R}$, $\succsim_{\tilde{L}}$ reveals uncertainty aversion (resp., loving) on $\mathcal{C}_{\tilde{D}}$ whenever $\tilde{D} \subseteq D$.

PROPOSITION 3.1. *Every FAEU preference reveals the relevance translation.*

PROOF. See Appendix 3.C.1.

Now, we state additional translation properties which are satisfied by ω- and S-FAEU, respectively. Define $\overline{\mathcal{R}} \subset \mathcal{L} \times \mathscr{P}(S)$ by $\overline{\mathcal{R}} = \{(L, B) \mid B \in \Sigma(L) \setminus \{S, \emptyset\}\}$.

PROPOSITION 3.2. *The following statements holds.*

(I) Let $\{\succsim_L\}_{L\in\mathcal{L}}$ be an ω-FAEU preference. For every $(L, B) \in \overline{\mathcal{R}}$, if $\succsim_L$ reveals uncertainty aversion (resp., loving) on $\mathcal{C}_B \cap \mathcal{H}_L$, then for all $(\tilde{L}, \tilde{B}) \in \overline{\mathcal{R}}$, $\succsim_{\tilde{L}}$ reveals uncertainty aversion (resp., loving) on $\mathcal{C}_{\tilde{B}} \cap \mathcal{H}_{\tilde{L}}$ whenever $\tilde{B} \subseteq B$ and there exists a $K \subseteq (L \cap \tilde{L})$ such that $\tilde{B} = \bigcup K$.

(II) Let $\{\succsim_L\}_{L\in\mathcal{L}}$ be an S-FAEU preference. For every $(L, D) \in \mathcal{R}$, if $\succsim_L$ exhibits uncertainty aversion (resp., loving) on $\mathcal{C}_D$, then for all $(\hat{L}, B) \in \overline{\mathcal{R}}$, $\succsim_{\hat{L}}$ exhibits uncertainty aversion (resp., loving) on $\mathcal{C}_B \cap \mathcal{H}_{\hat{L}}$ whenever $B \subseteq D$.

PROOF. See Appendix 3.C.2.

The following result shows that when $\{\succsim_L\}_{L\in\mathcal{L}}$ admits an ω-FAEU representation $(u, V, \hat{V})$, the translatability of uncertainty attitude can be related to the measurability given each information frame.

PROPOSITION 3.3. *Let $\Sigma_{\mathcal{F}}$ be a σ-algebra of S and $\mathcal{F}$ be the set of all $\Sigma_{\mathcal{F}}$-measurable acts. Suppose that $\{\succsim_L\}_{L\in\mathcal{L}}$ admits an ω-FAEU representation $(u, V, \hat{V})$. Let $(L, \tilde{L}) \in \mathscr{L}$ be such that $\succsim_L$ and $\succsim_{\tilde{L}}$ reveal uncertainty aversion (resp., loving) on $\mathcal{F}$.*

(I) If $\Sigma_{\mathcal{F}} \subseteq \Sigma(L)$, then for any $\hat{L} \in \mathcal{L}$ that satisfies $\Sigma_{\mathcal{F}} \subseteq \Sigma(\hat{L})$, $\succsim_{\hat{L}}$ reveals uncertainty aversion (resp., loving) on $\mathcal{F}$.

(II) If $\Sigma(L) \subseteq \Sigma_{\mathcal{F}} \subseteq \Sigma(\tilde{L})$, then for any $\hat{L} \in \mathcal{L}$, $\succsim_{\hat{L}}$ reveals uncertainty aversion (resp., loving) on $\mathcal{F}$ if $(L, \hat{L}) \in \mathscr{L}$.

(III) If $\Sigma(\tilde{L}) \subseteq \Sigma_{\mathcal{F}}$, then for any $\hat{L} \in \mathcal{L}$ satisfying $\Sigma(L) \subseteq \Sigma(\hat{L}) \subseteq \Sigma(\tilde{L})$, $\succsim_{\hat{L}}$ reveals uncertainty aversion (resp., loving) on $\mathcal{F}$.

PROOF. See Appendix 3.C.3.

3.5.2. Comparative Property: The Role of Framing of Information

In what follows, we study how information frames influence the revealed uncertainty attitude in a comparative manner. To this end, we follow the definition of comparative uncertainty aversion proposed by Ghirardato and Marinacci (2002). We restate the formal definition using our notations.

DEFINITION 3.7 (Comparative Uncertainty Attitude). Let $\mathcal{F} \subseteq \mathcal{H}$ be an arbitrary set of acts that satisfies $\mathcal{H}_C \subseteq \mathcal{F}$. Given two orderings on $\mathcal{H}$, $\succsim$ and $\succsim'$, we say $\succsim$ is *more uncertainty averse* than $\succsim'$ if for every $c \in \mathcal{H}_C$ and any $f \in \mathcal{F}$, $c \succsim' f$ implies $c \succsim f$, and $c \succ' f$ implies $c \succ f$.

In accordance with this definition, we say $\succsim$ is *as uncertainty averse as* $\succsim'$ if $\succsim$ is more uncertainty averse than $\succsim'$, and $\succsim'$ is more uncertainty averse than $\succsim$. The next result presents the necessary and sufficient conditions for an FAEU preference revealing the same degree of uncertainty aversion across nontrivial information frames.

PROPOSITION 3.4. *For any FAEU preference $\{\succsim_L\}_{L\in\mathcal{L}}$, the followings statements are equivalent.*

(I) *For every $L, L' \in \mathcal{L} \setminus \{\underline{L}\}$, $\succsim_L$ is as uncertainty averse as $\succsim_{L'}$.*

(II) *$\{\succsim_L\}_{L\in\mathcal{L}}$ admits an ω-FAEU representation (u, V, P), where P is a probability measure on $\mathscr{P}(S)$, and there exists a $\vartheta \in \mathbb{R}_{++}$ such that for all $B \in \mathscr{P}(S) \setminus \{S\}$, $P(B) = \vartheta V(B)$.*

(III) *$\{\succsim_L\}_{L\in\mathcal{L}}$ satisfies all the axioms given in Section 3.3 and Section 3.4. Moreover, for every $L \in \mathcal{L}$, any $f, g, h \in \mathcal{H}_L$, and any $\alpha \in (0, 1)$, $f \succsim_L g$ if and only if $\alpha f + (1-\alpha)h \succsim_L \alpha g + (1-\alpha)h$.*

PROOF. See Appendix 3.C.4.

The representation given in statement (II) of the proposition is more restrictive than representation (u, V, ϑ) characterized in Corollary 3.4.1. In other words, FAEU (resp., ω- or S-FAEU) in general does not ensure that the degree of uncertainty aversion is invariant across nontrivial information frames. It means that the framing of information, as well as the DM's reaction to it, do play a role in shaping the degree of uncertainty attitude revealed from choices. Besides, it can be shown that an FAEU preference reveals the same degree of uncertainty attitude across all information frames if and only if it admits a global subjective expected utility representation. Thus, the DM holds a probabilistically sophisticated belief in the sense of Machina and Schmeidler (1992) under variable information frames.

The last result presents a comparative analysis of uncertainty attitude. In order to isolate the role of the DM's reaction to information frames, we focus on two "ex-ante" identical FAEU preferences, $\{\succsim_L\}_{L\in\mathcal{L}}$ and $\{\succsim'_L\}_{L\in\mathcal{L}}$, with respect to the presence of nontrivial information frames, which means that they have the same vNM index and purely subjective belief such that $\succsim_{\underline{L}} = \succsim'_{\underline{L}}$ on $\mathcal{H} \times \mathcal{H}$.

PROPOSITION 3.5. *Let* $\{\succsim_L\}_{L\in\mathcal{L}}$ *and* $\{\succsim'_L\}_{L\in\mathcal{L}}$ *be FAEU preferences that satisfy* $u = u'$ *and* $V = V'$. *Then, for every* $L \in \mathcal{L} \setminus \{\underline{L}\}$, *the following statements hold.*

(I) $\succsim_L$ *is more uncertainty averse than* $\succsim'_L$ *if and only if for all* $B \in \Sigma(L)$, $V_L(B) \leq V'_L(B)$.

(II) Suppose that $\{\succsim_L\}_{L\in\mathcal{L}}$ *and* $\{\succsim'_L\}_{L\in\mathcal{L}}$ *are* ω*-FAEU preferences. Then,* $\succsim_L$ *is more uncertainty averse than* $\succsim'_L$ *if and only if for every* $B \in \Sigma(L) \setminus \{S\}$, $\omega_L \hat{V}(B) \leq \omega'_L \hat{V}'(B)$.

(III) Suppose that $\{\succsim_L\}_{L\in\mathcal{L}}$ *and* $\{\succsim'_L\}_{L\in\mathcal{L}}$ *are S-FAEU preferences. Then,* $\succsim_L$ *is more uncertainty averse than* $\succsim'_L$ *if and only if* $\vartheta_L \leq \vartheta'_L$.

PROOF. See Appendix 3.C.5.

3.6. Discussion

3.6.1. "Ambiguity-Aversion" and Machina's Concern

Machina (2009, 2014) maintained that models based on tail-separability (e.g., CEU) may have particular difficulties in accommodating ambiguity-averse behavior (see also Baillon et al. (2011)). In our model, tail-separability is also imposed as a part of basic axioms (CRI), yet in a restricted manner. In what follows, we apply our model to the thought-experimental examples proposed in these papers and subsequently discuss that the frame-adaptive reasoning might be a plausible approach to explaining Ellsberg-type behavior.

REFLECTION EXAMPLE. Suppose there is another four-color urn containing 50 balls in R or B, and 50 balls in Y or G. Consider the choice between $\{f_1, f_2\}$ and that between $\{f_3, f_4\}$ in Table 3.4 (in the table, $x > 0$). Clearly, CEU cannot accommodate the reversed strict rankings such as $f_1 \succ f_2$ and $f_3 \prec f_4$.[8]

	50 balls		50 balls	
	R	B	Y	G
f_1	\$ x	\$$2x$	\$ x	\$ 0
f_2	\$ x	\$ x	\$$2x$	\$ 0
f_3	\$ 0	\$$2x$	\$ x	\$ x
f_4	\$ 0	\$ x	\$$2x$	\$ x

TABLE 3.4 – Reflection Example

[8] L'Haridon and Placido (2010) reported that 90% of subjects revealed strict rankings in analogous problems, and that 70% revealed reversed rankings.

The example is a choice problem under information frame $L = \{\{R, B\}, \{Y, G\}\}$. Set $S = \{R, B, Y, G\}$, and let (u_L, v_L) denote the representation tuple obtained in Theorem 3.1. To sharpen the argument, we present FAEU accommodates reversed strict rankings for this example in an intuitive way: for all $E, E' \in L$, $v_L(E) = v_L(E')$, and for all $s, t \in S$, $v_L(\{s\}) = v_L(\{t\})$. Adding to which, if (u_L, v_L) satisfies:

(i) $u_L(\$0) < u_L(\$x) < u_L(\$2x)$;

(ii) $\forall E \in L, \forall s \in E,\ u_L(\$x) > \left(1 - \frac{v_L(\{s\})}{v_L(E)}\right) u_L(\$0) + \frac{v_L(\{s\})}{v_L(E)} u_L(\$2x)$; and

(iii) $\forall E \in L, \forall s \in E,\ 2v_L(\{s\}) \neq \frac{v_L(\{s\})}{v_L(E)}$,

then FAEU yields reversed strict rankings, where the direction of the strict ranking between f_1 and f_2 aligns with that of inequality (iii). (See Appendix 3.D.1 for details.)

Slightly-Bent Coin. Consider a coin that is bent slightly in an unknown direction and an urn containing an unknown quantity of balls in two colors: R and B. The mechanics of the coin-flip does not depend on the content of the urn. Also, the coin-flip and ball-drawing will be performed only once simultaneously. As discussed in Machina (2014), CEU cannot deliver a strict ranking in either direction between Bet I and II given in the following table if $\{HR\}$ and $\{TB\}$ are assigned an equal value of measure due to the symmetrical information (in the table, $x > 0$).

Bet I (h_1)				Bet II (h_2)		
	R	B			R	B
H	$+\$\,x$	$\$\,0$		H	$\$\,0$	$\$\,0$
T	$-\$\,x$	$\$\,0$	versus	T	$-\$\,x$	$+\$\,x$

Table 3.5 – Slightly-Bent Coin Problem

Due to the independence between coin-flip and ball-drawing, the description suggests that the probability of $\{HR, HB\}$ and $\{TR, TB\}$ would be very close to 1/2. Thus, the example is intrinsically the choice between two bets under information frame $L = \{\{HR, HB\}, \{TR, TB\}\}$. Let (u_L, v_L) be the representation tuple given in Theorem 3.1. For this example, if tuple (u_L, v_L) satisfies:

(i) $u_L(-\$\,x) < u_L(\$0) < u_L(+\$\,x)$;

(ii) $\forall E \in L, \forall s \in E,\ \left(1 - \frac{v_L(\{s\})}{v_L(E)}\right) u_L(-\$\,x) + \frac{v_L(\{s\})}{v_L(E)} u_L(+\$\,x) < u_L(\$0)$; and

(iii) $\forall E, E' \in L, \forall s \in E, \forall t \in E',\ \frac{v_L(\{s\})}{v_L(\{t\})} \neq \frac{(1-v_L(E))}{v_L(E')}$,

then FAEU yields a strict ranking between Bet I and II with the direction aligning with that of the inequality in (iii). (For details, see Appendix 3.D.1.)

In his analysis, Machina evinced a seminal viewpoint which says that, in situations of objective, subjective, and mixed objective/subjective uncertainty, the DM's trade-offs between subjective and objective (sub-)bets result in ambiguity-aversion. Our model, on the other hand, suggests that, in the presence of exogenous information, what we observe as ambiguity-aversion is a *manifestation* of DM evaluating the acts described in the frame of awareness (i.e., the frame yielded by identifying acts) in a way adapting to the *coarse* frames given by exogenous information.

To a certain extent, this explanation supports the insights offered by Machina (2014). Namely, that a piece of exogenous information leads a particular algebra of the state space to involve objective uncertainty is exactly a typical environment wherein the DM entertains coarse frames given by exogenous information and performs the frame-adaptive reasoning. Note that this is not the only type of situation in which the DM considers a coarse frame in decision making. Slightly-Bent Coin Problem can be a relevant example. Machina (2014, Section IV) suggested that choices of an ambiguity-averter may also exhibit different attitudes towards different degrees of ambiguity. In our formalism, however, information endowing a particular algebra with uncertainty in a disparate degree is another analogy of situations that lead the DM to exhibit ambiguity aversion via the frame-adaptive reasoning.

3.6.2. Example: Allocation with Framing of Information

In what follows, we firstly apply FAEU to an analogy of allocation problems with exogenous knowledge of uncertainty, then discuss its general application to related economic problems. The following example shows that a planner can distribute the market uncertainty solely through releasing the knowledge of uncertainty in a suitable frame, without creating asymmetry of information.

Distributing Uncertainty. Suppose $S = \{r, s, t\}$ describes the uncertainty which a market involves, and let $\mathcal{L}$ be the family of information frames that describe a fixed statistical knowledge of S. Set $A = \mathbb{R}_+$. The market comprises individual 1 and 2 whose behaviors can be described by ω-FAEU preferences $\succsim^1$ and $\succsim^2$, respectively. Denote by $(u^1, V^1, P, \boldsymbol{\omega}^1)$ and $(u^2, V^2, P, \boldsymbol{\omega}^2)$ the representations.

To simplify the argument, assume the followings.

(A1) Let $e = (e_r, e_s, e_t)$ be the state-contingent total endowments, where $e_r < e_s < e_t$. Thus, aggregate market uncertainty exists.

(A2) P is a probability measure on $\mathscr{P}(S)$. Namely, if $\overline{L} = \{\{r\}, \{s\}, \{t\}\}$ is released, the knowledge would be interpreted as a common probabilistic belief.

(A3) u^1 is strictly increasing strictly concave and differentiable, while u^2 is the identity function. That is, $\succsim^1$ is strictly risk averse, while $\succsim^2$ is risk neutral.

(A4) For all $L \in \mathcal{L} \setminus \{\overline{L}\}$, $\omega_L^1 > \omega_L^2$.

Now, let $x^i = (x_r^i, x_s^i, x_t^i)$ be the ex-ante allocation for individual i, and consider the allocation problem (x^1, x^2) in the spirit of Rigotti et al. (2008).

Suppose the statistical knowledge is released in information frame $\overline{L}$. Under assumptions (A1)-(A3), it is well-known that, an interior feasible allocation (x^1, x^2) is Pareto-optimal if and only if $x_r^1 = x_s^1 = x_t^1$. Thus, under $\overline{L}$, any interior Pareto-optimal allocation would lead individual 2 to burden the entire market uncertainty. However, under (A4), if the statistical knowledge is released in frame $L = \{\{s\}, \{r, t\}\}$, then any interior Pareto-optimal allocation would require individual 1 to share the market uncertainty to some extent. (See Appendix 3.D.2 for detail.)

In economic problems that involve uncertainty and information, decision models are often considered models of beliefs. In other words, the variation of information is reflected by that of (the set of) beliefs in the models. However, FAEU allows one to consider not only the variation of knowledge about uncertainty (e.g., Ellsberg's urns with different contents) but also that of frames in which each knowledge is revealed (e.g., for a fixed urn, announcing the quantities of different combinations of colors), hence enlarging the scope of analysis and observations for those economic problems.

As for a typical instance of such applications, one can consider, as in the above example, a market framework comprising an information sender and receivers, wherein the sender can design a suitable information frame in order to lead the receivers' decision to achieve some desirable goals, such as implementing an equilibrium with particular properties. Incorporating the previous ambiguity-related discussion, leading individuals' decisions by selecting a suitable information frame to release the knowledge of uncertainty can be viewed as doing so via a design of ambiguity. This approach is similar in nature to the one adopted by di Tillio et al. (2017).

3.A. Preliminary Observations

3.A.1. Axioms and Null-Additive Capacities

Some preliminary claims will be stated explicitly.

Claim 3.1. *The following statements hold.*

(a) *SCoR implies Certainty, CoR and Coarseness Aversion, and Coarseness Aversion implies Certainty.*
(b) *SCoI implies CoI.*

Proof. (a) First, assume SCoR. Let $\underline{L} = \{S\}$. Then, for every $L \in \mathcal{L}$, $(\underline{L}, L) \in \mathscr{L}$. By construction, $\mathcal{H}_{\underline{L}} = \mathcal{H}_C$. Thus, by SCoR, for every $L \in \mathcal{L}$ and any $c, c' \in \mathcal{H}_C$, $c \succsim_L c'$ if and only if $c \succsim_{\underline{L}} c'$. Certainty is implied. For all $(L, \tilde{L}) \in \mathscr{L}$, $c \in \mathcal{H}_C$ and $D \in \Sigma(L) \setminus \{S, \emptyset\}$, we have $\mathcal{C}^L_D[c] \subset \mathcal{H}_L$. Thus, CoR holds. Coarseness Aversion is in an analogy as $\mathcal{H}_C \subseteq \mathcal{H}_L$ for all $L \in \mathcal{L}$.

Now, assume Coarseness Aversion. Let $\overline{L} = \{\{s\} \mid s \in S\}$, and fix an $L \in \mathcal{L} \setminus \overline{L}$ arbitrarily. Then, $(L, \overline{L}) \in \mathscr{L}$. By Coarseness Aversion, for all $c, c' \in \mathcal{H}_C$, $c \succsim_L c'$ implies $c \succsim_{\overline{L}} c'$, and $c \succ_L c'$ implies $c \succ_{\overline{L}} c'$. Thus, $c \succsim_L c'$ if and only if $c \succsim_{\overline{L}} c'$.

(b) Set $E = E'$ in SCoI. Then, for any $(L, \tilde{L}) \in \mathscr{L}$, $E \in \tilde{L}$, $c \in \mathcal{H}_C$ and any $f, g \in \mathcal{C}^{\tilde{L}}_E[c]$, $f \succsim_L g$ if and only if $f \succsim_{\tilde{L}} g$. Thus, CoI is satisfied. *Q.E.D.*

The following claims state some properties of null-additive capacities and the corresponding Choquet integrals. These properties will be frequently applied to the subsequent proofs. Let X be a nonempty finite set.

Claim 3.2. *Let Σ be a σ-algebra of X and $v : \Sigma \to [0, 1]$ be a null-additive capacity. Suppose $\beta \in \mathbb{R}_{++}$ is a parameter that satisfies $1/\beta \geq \max_{B \in \Sigma:\, v(X \setminus B) > 0} v(B)$. Define a set function $m : \Sigma \to [0, 1]$ by: $m(B) = \beta v(B)$ if $v(X \setminus B) > 0$, and $m(B) = 1$ otherwise. Then, m is a null-additive capacity on Σ.*

Proof. Let v, β and m be as above. First, we show that m is a capacity on Σ.

Let $B, D \in \Sigma$ be such that $B \subset D$. Since v is a capacity, it holds that $v(B) \leq v(D)$, and $v(X \setminus D) > 0$ implies $v(X \setminus B) > 0$. Thus, one of the following cases holds. (a1) $v(X \setminus D) > 0$. Then, by definition, $m(D) = \beta v(D)$ and $m(B) = \beta v(B)$. Given that $\beta > 0$, we have $m(B) \leq m(D)$. (a2) $v(X \setminus D) = 0$ and $v(X \setminus B) > 0$. Then, it follows that $m(D) = 1$ and $m(B) = \beta v(B)$. By assumption, we have $v(B) \leq \max_{\tilde{B} \in \Sigma: v(X \setminus \tilde{B})} v(\tilde{B}) \leq 1/\beta$, which yields $m(B) \leq 1 = m(D)$. (a3) $v(X \setminus D) = 0$ and $v(X \setminus B) = 0$. Then, it holds that $m(B) = m(D) = 1$.

As a result, for all $B, D \in \Sigma$, $B \subset D$ implies $m(B) \leq m(D)$. Clearly, $m(\emptyset) = 0$ and $m(X) = 1$ as $m(\emptyset) = \beta v(\emptyset)$ and $v(X \setminus X) = 0$. Hence, m is a capacity on Σ.

Now, we show the null-additivity.

Let $D_0 \in \Sigma$ be such that $m(D_0) = 0$. Given that $\beta > 0$, by assumption, we have $v(D_0) = 0$. Let $B \in \Sigma$ be a subset such that $B \cap D_0 = \emptyset$, and let $D = B \cup D_0$. Since v is null-additive and $v(D_0) = 0$, it thus follows that $v(D) = v(B)$, and that $v(X \setminus B) > 0$ if and only if $v(X \setminus D) > 0$. (b1) $v(X \setminus B) > 0$. Then, by assumption, $m(D) = \beta v(D)$ and $m(B) = \beta v(B)$. Thus, we have $m(D) = m(B)$. (b2) $v(X \setminus B) = 0$. Then, it follows that $m(D) = m(B) = 1$.

Hence, for any disjoint $B, D_0 \in \Sigma$, $m(D_0) = 0$ implies $m(B \cup D_0) = m(B)$.

Therefore, m is a null-additive capacity on Σ. *Q.E.D.*

CLAIM 3.3. *Let L be a partition of X, and denote by Y the set of all bounded $\Sigma(L)$-measurable $\mathbb{R}$-valued functions on X. Let $v : \Sigma(L) \to [0,1]$ and $v' : \Sigma(L) \to [0,1]$ be capacities. Then, the following statements are equivalent: (i) For all $y \in Y$, $\int_X y\,dv \geq \int_X y\,dv'$; (ii) For all $B \in \Sigma(L)$, $v(B) \geq v'(B)$.*

PROOF. Let E denote a typical cell in L.

Define a set $\overline{Y} \subset Y$ by

$$\overline{Y} = \{\overline{y} \in Y \mid \forall E \in L, \forall x \in E, \forall x' \in X \setminus E,\, \overline{y}(x) \neq \overline{y}(x')\}.$$

Given an $\overline{y} \in \overline{Y}$, define a set $Y[\overline{y}] \subset Y$ by

$$Y[\overline{y}] = \{y \in Y \mid \forall x, x' \in X,\, [\overline{y}(x) \geq \overline{y}(x') \Rightarrow y(x) \geq y(x')]\}.$$

Then, it follows that $Y = \bigcup_{\overline{y} \in \overline{Y}} Y[\overline{y}]$.

Let $\tau : L \to \{1, \ldots, |L|\}$ be a bijection. Given v and τ, define a function $v_\tau : L \to [0,1]$ by $v_\tau(E) = v(\bigcup\{\hat{E} \in L \mid \tau(\hat{E}) \geq \tau(E)\})$. Similarly, given v' and τ, define $v'_\tau : L \to [0,1]$ by $v'_\tau(E) = v'(\bigcup\{\hat{E} \in L \mid \tau(\hat{E}) \geq \tau(E)\})$.

Given τ and $y \in Y$, let $d^y_\tau : L \to \mathbb{R}$ be the function such that: (i) $d^y_\tau(E) = y(x)$ for $x \in E$ if $\tau(E) = 1$; and (ii) for every $E \in L$ with $\tau(E) > 1$, $d^y_\tau(E) = y(x) - y(\hat{x})$ for $x \in E$ and $\hat{x} \in \hat{E}$, where $\tau(\hat{E}) = \tau(E) - 1$.

For every $\overline{y} \in \overline{Y}$, let $\tau_{\overline{y}}$ be the above mentioned bijection such that for any $E, \hat{E} \in L$, $x \in E$ and $\hat{x} \in \hat{E}$, $\tau_{\overline{y}}(E) \geq \tau_{\overline{y}}(\hat{E})$ implies $\overline{y}(x)) \geq \overline{y}(\hat{x})$. Clearly, for every $\overline{y} \in \overline{Y}$, such $\tau_{\overline{y}}$ uniquely exists, and yields a permutation of L. Given $\overline{y} \in \overline{Y}$ and the permutation $(E^{\tau_{\overline{y}}(E)})_{\tau_{\overline{y}}(E)=1}^{|L|}$ induced by $\tau_{\overline{y}}$, for every $y \in Y[\overline{y}]$, let $\mathbf{d}^y_{\tau_{\overline{y}}} \in \mathbb{R}_+^{|L|-1}$ be the vector given by $\mathbf{d}^y_{\tau_{\overline{y}}} = (d^y_{\tau_{\overline{y}}}(E^2) \cdots d^y_{\tau_{\overline{y}}}(E^{|L|}))$. Analogously, let $\mathbf{v}_{\tau_{\overline{y}}}, \mathbf{v}'_{\tau_{\overline{y}}} \in \mathbb{R}_+^{|L|-1}$ be those induced by $v_{\tau_{\overline{y}}}$ and $v'_{\tau_{\overline{y}}}$, respectively. Given $\overline{y} \in \overline{Y}$, let $C[\overline{y}] = \{\mathbf{d}^y_{\tau_{\overline{y}}} \mid y \in Y[\overline{y}]\}$, and denote by $C^*[\overline{y}]$ the dual cone of $C[\overline{y}]$.

Then, for any $\overline{y} \in \overline{Y}$ and $y \in Y[\overline{y}]$, it follows that

$$\begin{aligned}\int_X y\,dv - \int_X y\,dv' &= \sum_{E \in L : \tau_{\overline{y}}(E) > 1} d^y_{\tau_{\overline{y}}}(E)\left(v_{\tau_{\overline{y}}}(E) - v'_{\tau_{\overline{y}}}(E)\right)\\ &= \mathbf{d}^y_{\tau_{\overline{y}}} \cdot \left(\mathbf{v}_{\tau_{\overline{y}}} - \mathbf{v}'_{\tau_{\overline{y}}}\right).\end{aligned}$$

Therefore, statement (i) holds if and only if for any $\overline{y} \in \overline{Y}$, $\mathbf{v}_{\tau_{\overline{y}}} - \mathbf{v}'_{\tau_{\overline{y}}} \in C^*[\overline{y}]$. Denote the latter statement by (i0).

For a fixed $\overline{y} \in \overline{Y}$, (a) functions in $Y[\overline{y}]$ are pairwise comonotonic, and (b) for any $y \in Y$, if y and $\overline{y}$ are comonotonic, then $y \in Y[\overline{y}]$. In addition, by the properties of Choquet integral, for every positive affine transformation a, it holds that $\int_X a \circ y\,dv = a(\int_X y\,dv)$, where $a \circ y$ denotes the pointwise affine transformation. Thus, it yields $C[\overline{y}] = \mathbb{R}_+^{|L|-1}$, meaning that $C[\overline{y}]$ is self-dual. Note that, for every $E \in L$ and $\overline{y} \in \overline{Y}$,

$v'_{\tau_{\overline{y}}}(E) = v_{\tau_{\overline{y}}}(E) = 1$ whenever $\tau_{\overline{y}}(E) = 1$. Therefore, (i0) holds if and only if for any $E \in L$ and $\overline{y} \in \overline{Y}$, $v_{\tau_{\overline{y}}}(E) \geq v'_{\tau_{\overline{y}}}(E)$.

By construction, for every $B \in \Sigma(L)$, there exist an $\overline{y} \in \overline{Y}$ and an $E \in L$ such that $B = \bigcup\{\hat{E} \in L \mid \tau_{\overline{y}}(\hat{E}) \geq \tau_{\overline{y}}(E)\}$. As a result, (i0) holds if and only if for all $B \in \Sigma(L)$, $v(B) \geq v'(B)$. *Q.E.D.*

CLAIM 3.4. *Let L be a partition of X. Let $v : \Sigma(L) \to [0,1]$ and $\hat{v} : \Sigma(L) \to [0,1]$ be a null-additive capacity and a capacity on $\Sigma(L)$, respectively. Given v, define $\Delta(L) = \{B \in L \mid v(B) > 0\}$. Let $y : X \to \mathbb{R}$ be an arbitrary $\Sigma(L)$-measurable function. Then, the following statements hold.*

(a) $\int_X y\, dv = \int_{\bigcup \Delta(L)} y\, dv$.

(b) Suppose that for any $D \in \Sigma(L)$, $v(D) = 0$ if and only if $\hat{v}(D) = 0$, and that there is a $\beta \in \mathbb{R}_{++}$ such that for any $B \in \Sigma(L)$, $v(B) = 1$ when $\hat{v}(X \setminus B) = 0$ and $v(B) = \beta\hat{v}(B)$ otherwise. Then,

$$\int_{\bigcup \Delta(L)} y\, dv = \beta \int_{\bigcup \Delta(L)} \left\{ y - \left(\min_{x \in \bigcup \Delta(L)} y(x) \right) \mathbf{1}_X \right\} d\hat{v} + \min_{x \in \bigcup \Delta(L)} y(x).$$

(c) Let $v, \hat{v}$ and β be as in (b). If $\hat{v}$ is null-additive, then

$$\int_{\bigcup \Delta(L)} y\, dv = \beta \int_{\bigcup \Delta(L)} y\, d\hat{v} + (1 - \beta) \min_{x \in \bigcup \Delta(L)} y(x).$$

PROOF. Note that y is constant on every $E \in L$. Let $y_E = y(x)$ for $x \in E$.

Let $p(y) = (E^1, \ldots, E^{|L|})$ be a permutation of L such that for all $i \in \{1, \ldots, |L|-1\}$, $y_{E^{i+1}} \geq y_{E^i}$. Fix this $p(y)$.

(a) For any $i \in \{1, \ldots, |L|\}$, let $y^i = y_{E^i}$, and for any $i \in \{1, \ldots, |L|-1\}$, let $v^i = v(\bigcup_{j=i}^{|L|} E^j) - v(\bigcup_{j=i+1}^{|L|} E_j)$, and let $v^{|L|} = v(E^{|L|})$. Denote by $\mathbf{k}$ the set of indices such that $E^k \in \Delta(L)$ for any $k \in \mathbf{k}$, and let $\mathbf{j} = \{1, \ldots, |L|\} \setminus \mathbf{k}$. Then, by null-additivity, for any $j \in \mathbf{j}$, we have $v^j = 0$. Therefore, it yields that

$$\begin{aligned} \int_X y\, dv = \sum_{i=1}^{|L|} y^i v^i &= \sum_{k \in \mathbf{k}} y^k v^k + \sum_{j \in \mathbf{j}} y^j v^j \\ &= \sum_{k \in \mathbf{k}} y^k v^k = \int_{\bigcup \Delta(L)} y\, dv. \end{aligned}$$

(b) By the null-additivity of v, we have $v(\bigcup \Delta(L)) = 1$, and for any $\mathcal{D} \subset \Delta(L)$, it holds that $v(X \setminus (\bigcup \mathcal{D})) > 0$, which is by assumption equivalent to $\hat{v}(X \setminus (\bigcup \mathcal{D})) > 0$. Thus, by assumption, for any $\mathcal{D} \subset \Delta(L)$, $v(\bigcup \mathcal{D}) = \beta\hat{v}(\bigcup \mathcal{D})$.

Let $\hat{p}(y) = (E^1, \ldots, E^m)$ be the permutation of $\Delta(L)$ induced by $p(y)$. Let $y_1 = y_{E^1}$, and for any $i \in \{2, \ldots, m\}$, let $y_i = y_{E^i} - y_{E^{i-1}}$, $v_i = v(\bigcup_{j=i}^m E^j)$, and $\hat{v}_i = \hat{v}(\bigcup_{j=i}^m E^j)$.

Then, for any $i \in \{2, \dots, m\}$, $v^i = \beta \hat{v}_i$. Therefore, it follows that

$$\int_{\bigcup \Delta(L)} y \, dv = y_1 + \sum_{i=2}^{m} y_i v_i = y_1 + \beta \left(0 \cdot \hat{v}(\textstyle\bigcup \Delta(L)) + \sum_{i=2}^{m} y_i \hat{v}_i \right)$$
$$= y_1 + \beta \int_{\bigcup \Delta(L)} (y - y_1 \mathbf{1}_X) \, d\hat{v}.$$

Clearly, $y_1 = \min_{x \in \bigcup \Delta(L)} y(x)$. Hence, the statement follows.

(c) Let v, $\hat{v}$ and $\beta \in \mathbb{R}_{++}$ be as in (b), and suppose that $\hat{v}$ is null-additive.

Let $\underline{y} = \min_{x \in \bigcup \Delta(L)} y(x)$. Then, by (b), it holds that

$$\int_{\bigcup \Delta(L)} y \, dv = \beta \int_{\bigcup \Delta(L)} \left(y - \underline{y} \mathbf{1}_X \right) d\hat{v} + \underline{y}.$$

By assumption, for any $D \in \Sigma(L)$, $v(D) > 0$ if and only if $\hat{v}(D) > 0$. Thus, it follows that $\Delta(L) = \{B \in L \mid \hat{v}(B) > 0\}$. Consequently, by the null-additivity of $\hat{v}$, we have $\hat{v}(\bigcup \Delta L) = 1$. As a result, the above equation yields

$$\int_{\bigcup \Delta(L)} y \, dv = \beta \left(\int_{\bigcup \Delta(L)} \left(y - \underline{y} \mathbf{1}_X \right) d\hat{v} + \underline{y} \hat{v}(\textstyle\bigcup \Delta L) \right) + \left(1 - \beta \hat{v}(\textstyle\bigcup \Delta L) \right) \underline{y}$$
$$= \beta \int_{\bigcup \Delta(L)} y \, d\hat{v} + (1 - \beta) \underline{y}.$$

The proof is complete. *Q.E.D.*

3.A.2. An Auxiliary Lemma

The following auxiliary result states that we have the standard CEU representation with respect to *null-additive* capacities for the acts that are $\Sigma(L)$-measurable under each given information frame.

LEMMA 3.1. *The following statements are equivalent.*

(I) $\{\succsim_L\}_{L \in \mathcal{L}}$ *satisfies the uniform axioms and Certainty.*

(II) There exists an affine function $u : \Pi(A) \to \mathbb{R}$, *and for every* $L \in \mathcal{L}$, *there is a unique null-additive capacity* $V_L : \Sigma(L) \to [0, 1]$ *such that for all* $L \in \mathcal{L}$ *and* $f, g \in \mathcal{H}_L$,

$$f \succsim_L g \iff \int_S u(f) dV_L \geq \int_S u(g) dV_L.$$

Moreover, u is unique up to p.l.t.

3.A.2.1. Proof of Lemma 3.1

CLAIM 3.5. *For every* $L \in \mathcal{L}$, *the following statements hold.*

(a) Under statement (II), Monotonicity-(I) implies Monotonicity-(II) on $\mathcal{H}_L$.

(b) CRI-(I) implies CRI-(II) on $\mathcal{H}_L$.

Proof. Fix an $L \in \mathcal{L}$ and $f, g \in \mathcal{H}_L$ arbitrarily. Assume statement (II).

(a) Assume Monotocinity-(I), and suppose that $g \succ_L f$. Define $G, N \subseteq L$ by

$$G = \left\{\hat{E} \in L \mid \forall t \in \hat{E},\ g(t) \succ_L f(t)\right\};$$
$$N = \{E_0 \in L \mid V_L(E_0) = 0\}.$$

Then, Monotonicity-(I) implies $G \neq \emptyset$ as $f, g \in \mathcal{H}_L$ and $g \succ_L f$. Also, by the null-additivity of V_L, we have $N \subset L$. Suppose that $G \setminus N \neq \emptyset$, and fix an $\hat{E} \in G \setminus N$. Set $c = \underline{f}^L$ if $\underline{g}^L \succsim_L \underline{g}^L$, and $c = \underline{g}^L$ otherwise. Then, since $[f\hat{E}c], [g\hat{E}c] \in \mathcal{H}_L$, thus

$$\begin{aligned}\int_S u([f\hat{E}c])dV_L &= \left(1 - V_L(\hat{E})\right) u^c + V_L(\hat{E})\hat{u}^f \\ &< \left(1 - V_L(\hat{E})\right) u^c + V_L(\hat{E})\hat{u}^g = \int_S u([g\hat{E}c])dV_L,\end{aligned}$$

where $u^c = u(c(s))$ on S, and $\hat{u}^f = u(f(t))$, $\hat{u}^g = u(g(t))$ for $t \in \hat{E}$. By statement (II), this inequality is equivalent to $[f\hat{E}c] \prec_L [g\hat{E}c]$. As a result, $f \prec_L g$ implies the existence of an $\hat{E} \in L$ that satisfies $[f\hat{E}c] \prec_L [g\hat{E}c]$. Thus, under Claim 3.1, Monotonicity-(II) follows. Therefore, it now suffices to show $G \setminus N \neq \emptyset$.

Suppose $G \setminus N = \emptyset$. Then, $N \subset L$ yields $G \subset L$. For g, fix a permutation $p(g) = (E^{g,(1)}, \ldots, E^{g,(|L|)})$ of L such that $u(g(s)) \geq u(g(t))$ for each $s \in E^{g,(i+1)}$ and $t \in E^{g,(i)}$, and let $q(g) = (u^{g,(1)}, \ldots, u^{g,(|L|)})$ be the implied $\geq$-increasing sequence of $u(g)$. Accordingly, set $v^{g,(i)} = V_L(\bigcup_{j=i}^{|L|} E^{g,(j)}) - V_L(\bigcup_{j=i+1}^{|L|} E^{g,(j)})$ for $i \in \{1, \ldots, |L|-1\}$, and let $v^{g,(|L|)} = V_L(E^{g,(|L|)})$. Then, it holds that $\int_S u(g)dV_L = \sum_{i=1}^{|L|} u^{g,(i)}v^{g,(i)}$. Since V_L is null-additive, $E^{g,(i)} \in G$ implies $v^{g,(i)} = 0$ as $G \subseteq N$. Hence, given $p(g)$ and $q(g)$, if define a $\geq$-increasing sequence $\tilde{q}(g) = (\tilde{u}^{g,(1)}, \ldots, \tilde{u}^{g,(|N|)})$ by

$$\tilde{u}^{g,(i)} = \begin{cases} u^{g,(i)}, & E^{g,(i)} \notin G \\ u^{g,(\iota(i))}, & E^{g,(i)} \in G \end{cases}, \text{where}$$

$$\iota(i) = \begin{cases} \arg\max\limits_{j \leq i} E^{g,(j)} \notin G, & \exists j \leq i,\ E^{g,(j)} \notin G \\ 1, & \forall j \leq i,\ E^{g,(j)} \in G \end{cases},$$

then it follows that $\sum_{i=1}^{|L|} u^{g,(i)}v^{g,(i)} = \sum_{i=1}^{|L|} \tilde{u}^{g,(i)}v^{g,(i)}$. As a result, we have $\int_S u(g)dV_L = \int_S u(\tilde{g})dV_L$, where $\tilde{g} \in \mathcal{H}_L$ denotes the act that yields $\tilde{q}(g)$ under $p(g)$. Similarly, let $p(f)$, $q(f)$ and $(v^{f,(i)})_{i=1,\ldots,|L|}$ be the f-counterparts of $p(g), q(g)$ and $(v^{g,(i)})_{i=1,\ldots,|L|}$. Given $p(f)$ and $q(f)$, define a sequence $\tilde{q}(f) = (\tilde{u}^{f,(1)}, \ldots, \tilde{u}^{f,(|N|)})$ by

$$\tilde{u}^{f,(i)} = \begin{cases} u^{f,(i)}, & E^{f,(i)} \notin G \\ u^{f,(\tau(i))}, & E^{f,(i)} \in G \end{cases}, \text{where}$$

$$\tau(i) = \begin{cases} \arg\min\limits_{j \geq i} E^{f,(j)} \notin G, & \exists j \geq i,\ E^{f,(j)} \notin G \\ |L|, & \forall j \geq i,\ E^{f,(j)} \in G \end{cases},$$

and let $\tilde{f}$ be the act that yields $\tilde{q}(f)$ under $p(f)$. Then, it holds that $\int_S u(f)dV_L = \sum_{i=1}^{|L|} u^{f,(i)}v^{f,(i)} = \sum_{i=1}^{|L|} \tilde{u}^{f,(i)}v^{f,(i)} = \int_S u(\tilde{f})dV_L$. Moreover, since $u(f(s)) \geq u(g(s))$ on every $E \in L \setminus G$, the definitions of $\tilde{q}(f)$ and $\tilde{q}(g)$ imply that $u(\tilde{f}(s)) \geq u(\tilde{g}(s))$ on S (i.e., on every $E \in L$). Therefore, it yields $\int_S u(\tilde{f})dV_L \geq \int_S u(\tilde{g})dV_L$, meaning that $\int_S u(f)dV_L \geq \int_S u(g)dV_L$. By statement (II), the last inequality is equivalent to $f \succsim_L g$, which contradicts to $g \succ_L f$. As a result, $G \setminus N \neq \emptyset$.

(b) On $\mathcal{H}_L$, (b) is trivial. *Q.E.D.*

NECESSITY. Order, Monotonicity-(I), CRI-(I) and Certainty hold as direct consequences of Choquet integral. Hence, Claim 3.5 completes the necessity. *Q.E.D.*

Now, assume statement (I).

CLAIM 3.6. *Suppose that, for a fixed $L \in \mathcal{L}$, there exist a $u : \Pi(A) \to \mathbb{R}$ and a capacity $V : \Sigma(L) \to [0,1]$ such that $I : \mathcal{H}_L \to \mathbb{R}$ given by $I(f) = \int_S u(f)dV$ represents $\succsim_L$ on $\mathcal{H}_L$. Then, V is null-additive.*

PROOF. Fix an arbitrary $L \in \mathcal{L}$. Let $N = \{E \in L \mid V(E) = 0\}$.

By assumption, for any $E \in L$, $c \in \mathcal{H}_C$ and $h \in \mathcal{C}^L_E[c] \cap \mathcal{H}_L$, it holds that

$$I(h) = \int_S u(h)dV = V(E)u_E(h) + (1 - V(E))u^c,$$

where $u^h_E = u(h(s))$ on E and $u^c = u(c(s))$ on S.

First, we show that $N \neq L$. Suppose $N = L$. Then, for all $E \in L$ and $f, g \in \mathcal{H}_L$, we have $[fE\underline{c}] \sim_L [gE\underline{c}]$, where $\underline{c} = \underline{f}^L$ if $\underline{g}^L \succsim_L \underline{f}^L$, and $\underline{c} = \underline{g}^L$ otherwise. Thus, by Monotonicity-(II), for all $f, g \in \mathcal{H}_L$, $f \sim_L g$, meaning that $\mathcal{H}_C$, as well as $\Pi(A)$, is $\succsim_L$-indifferent. Under Monotonicity-(I), this in turn implies that $\mathcal{H}$ is $\succsim_L$-indifferent, which contradicts to Nondegeneracy. As a result, $N \subset L$.

Moreover, for any $\hat{E} \in N$, $c \in \mathcal{H}_C$ and $h \in \mathcal{C}^L_E[c] \cap \mathcal{H}_L$, we have $I(h) = (1 - V(\hat{E}))u^c$. Thus, for all $\hat{E} \in N$ and $c \in \mathcal{H}_C$, $\mathcal{C}^L_E[c] \cap \mathcal{H}_L$ is $\succsim_L$-indifferent

Fix an $\hat{E} \in N$. Let $\mathcal{H}_L(\hat{E})$ be the set given by

$$\mathcal{H}_L(\hat{E}) = \left\{ h \in \mathcal{H}_L \;\middle|\; \begin{array}{l} [\forall E \in L, \forall t \in E, \forall s \in S \setminus E, \neg(h(t) \sim_L h(s))] \\ \wedge\, [\exists s \in S \setminus \hat{E}, \forall t \in \hat{E},\ h(s) \succ_L h(t)] \end{array} \right\}.$$

For a fixed $h \in \mathcal{H}_L(\hat{E})$, define a set $D_{\hat{E}}(h) \subset \mathcal{H}_L$ by

$$D_{\hat{E}}(h) = \left\{ [c\hat{E}h] \;\middle|\; \begin{array}{l} c \in \mathcal{H}_C \ \wedge\ [\forall t \in \hat{E}, c(t) \succ_L h(t)] \wedge \\ \quad [\forall s \in S \setminus \hat{E}, \forall t \in \hat{E},\ h(s) \succsim_L c(t) \Leftrightarrow h(s) \succsim_L h(t)] \end{array} \right\}.$$

Then, we have $h \in D_{\hat{E}}(h)$. Moreover, for any $\hat{h} \in D_{\hat{E}}(h)$, it holds that: (d1) for every $E \in L$, $[\hat{h}E\underline{h}^L] \in \mathcal{C}^L_E[\underline{h}^L] \cap \mathcal{H}_L$; (d2) for every $E \in L \setminus \{\hat{E}\}$, $[\hat{h}E\underline{h}^L] = [hE\underline{h}^L]$; and (d3) $\hat{h}$ and h are comonotonic. Since $\mathcal{C}^L_{\hat{E}}[\underline{h}^L] \cap \mathcal{H}_L$ is $\succsim_L$-indifferent, thus for any $\hat{h} \in D_{\hat{E}}(h)$, (d1) and (d2) jointly imply that $[\hat{h}E\underline{h}^L] \sim_L [hE\underline{h}^L]$ holds for every $E \in L$. Hence, by Monotonicity-(II), $\hat{h} \sim_L h$, meaning that $D_{\hat{E}}(h)$ is also $\succsim_L$-indifferent.

As in the proof of Claim 3.5, define $p(h) = (E^{h,(1)}, \ldots, E^{h,(|L|)})$, $(v^{h,(i)})_i$ and $q(h) = \{u^{h,(1)}, \ldots, u^{h,(|L|)}\}$ with respect to u and V. By (d3), for every $\hat{h} \in D_{\hat{E}}(h)$, $p(\hat{h})$ uniquely exists. Hence, for any $\hat{h} \in D_{\hat{E}}(h)$, we have $p(\hat{h}) = p(h)$ and $(v^{\hat{h},(i)})_i = (v^{h,(i)})_i$. Let j be the index such that $\hat{E} = E^{h,(j)}$ in $p(h)$. Then, by the definition of $D_{\hat{E}}(h)$, for every $\hat{h} \in D_{\hat{E}}(h)$, $u^{\hat{h},(i)} \neq u^{h,(i)}$ if and only if $i = j$. Recall that $D_{\hat{E}}(h)$ is $\succsim_L$-indifferent. Thus, for any $\hat{h} \in D_{\hat{E}}(h)$, it follows that

$$\begin{aligned}
I(h) = I(\hat{h}) &\iff \sum_{i=1}^{|L|} u^{h,(i)} v^{h,(i)} = \sum_{i=1}^{|L|} u^{\hat{h},(i)} v^{\hat{h},(i)} \\
&\iff u^{h,(j)} v^{h,(j)} + \sum_{i \neq j} u^{h,(i)} v^{h,(i)} = u^{\hat{h},(j)} v^{\hat{h},(j)} + \sum_{i \neq j} u^{\hat{h},(i)} v^{\hat{h},(i)} \\
&\iff u^{h,(j)} v^{h,(j)} = u^{\hat{h},(j)} v^{h,(j)}.
\end{aligned}$$

Since $u^{h,(j)} \neq u^{\hat{h},(j)}$, the last equality yields $v^{h,(j)} = 0$. That is, $V(B \cup \hat{E}) = V(B)$, where $B = \bigcup_{i=j+1}^{|L|} E^{h,(i)}$.

As a result, for a given $h \in \mathcal{H}_L(\hat{E})$, $D_{\hat{E}}(h)$ leads to a unique $B_h \in \Sigma(L)$ that satisfies $B_h \cap \hat{E} = \emptyset$ and $V(B_h \cup \hat{E}) = V(B_h)$. Note that, for every $h \in \mathcal{H}_L(\hat{E})$, the definition of $D_{\hat{E}}(h)$ does not impose any restriction on h. Thus, there exists a mapping $\sigma : \mathcal{H}_L(\hat{E}) \to \Sigma(L)$ such that $V(\hat{E} \cup \sigma(h)) = V(\sigma(h))$ for any $h \in \mathcal{H}_L(\hat{E})$. Then, it holds that $\{B \in \Sigma(L) \mid B \cap \hat{E} = \emptyset\} = \{\sigma(h) \mid h \in \mathcal{H}_L(\hat{E})\}$. Therefore, the proof is complete as $\hat{E} \in N$ is fixed arbitrarily. *Q.E.D.*

Sufficiency. Fix an arbitrary $\hat{L} \in \mathcal{L}$. Then, $\succsim_{\hat{L}} \subset \mathcal{H}_{\hat{L}} \times \mathcal{H}_{\hat{L}}$ satisfies Order, Continuity, Monotonicity-(I) and Comonotonic Independence. Hence, by Schmeidler's theorem (Schmeidler (1989, .pp 578-579)), there exist an affine function $u_{\hat{L}} : \Pi(A) \to \mathbb{R}$ and a unique capacity $V_{\hat{L}} : \Sigma(\hat{L}) \to [0, 1]$ such that for any $f, g \in \mathcal{H}_{\hat{L}}$,

$$f \succsim_{\hat{L}} g \iff \int_S u_{\hat{L}}(f) dV_{\hat{L}} \geq \int_S u_{\hat{L}}(g) dV_{\hat{L}},$$

where $u_{\hat{L}}$ is unique up to p.l.t. As a result, the above equivalence holds for every $L \in \mathcal{L}$ since $\hat{L}$ is assumed to be arbitrary.

By Certainty, $(\mathcal{H}_C, \succsim_L)$ and $(\mathcal{H}_C, \succsim_{\hat{L}})$ are order isomorphic for any $L \in \mathcal{L}$. Notice that $\Pi(A)$ is a mixture space with respect to the lottery mixture. As a result, for any $L \in \mathcal{L}$, a bounded continuous and strictly increasing function $\varphi_L : \mathbb{R} \to \mathbb{R}$ exists such that $u_{\hat{L}} = \varphi_L \circ u_L$. Set $u = u_{\hat{L}}$. Since u and u_L are unique up to p.l.t. with satisfying mixture linearity, φ_L is thus linear for any $L \in \mathcal{L}$. Hence, for every $L \in \mathcal{L}$, the previous equivalence is satisfied by u.

Finally, Claim 3.6 implies that V_L is null-additive for every $L \in \mathcal{L}$. *Q.E.D.*

3.A.3. Utility Acts

Fix an arbitrary $L \in \mathcal{L}$. Then, by Lemma 3.1, there exists a $u_L : \Pi(A) \to \mathbb{R}$ that represents the induced $\succsim_L$ on $\Pi(A)$ (hence, on $\mathcal{H}_C$), and u_L is unique up to p.l.t.[9] Hence, wlog, suppose that $u_L(f(s)) \in X$ on S for every $f \in \mathcal{H}$, where $X \subset \mathbb{R}$ is a closed convex set satisfying $[-1, 1] \subseteq X$.

For each $f \in \mathcal{H}$, define a function $\psi_f : S \to X$ by $\psi_f(s) = (u_L \circ f)(s)$, and typically write ψ whenever the choice of $f \in \mathcal{H}$ does not concern the argument. Let $\Psi = \{u_L \circ f \mid f \in \mathcal{H}\}$ be the set of all such functions. Let $\succcurlyeq_L \subset \Psi \times \Psi$ be the binary relation such that for every $\psi_f, \psi_g \in \Psi$,

$$\psi_f \succcurlyeq_L \psi_g \Longleftrightarrow f \succsim_L g.$$

Then, $\geq$ and $\succcurlyeq_L$ coincide on X. Denote by Ψ_L and Ψ_C the set of $\Sigma(L)$-measurable functions and that of constant functions, respectively.

Let $\underline{\psi}^L$ be the Ψ-counterparts (with respect to $\succcurlyeq_L$) of the one defined on $\mathcal{H}$. Then, for every $\psi, \psi' \in \Psi$, it holds that $\underline{\psi}^L(s) = \min_{s \in S} \psi(s)$ on S. For every $D \in \mathscr{P}(S)$, let $\mathcal{M}_D^L[\psi_x]$ be the set of all functions such that $\psi = [\psi D \psi_x]$ with $[\psi D \psi_x](s) \geq x$ for all $s \in S$, where $\psi_x(s) = x$ on S. Formally, given a $\psi_x \in \Psi_C$,

$$\mathcal{M}_D^L[\psi_x] = \left\{ [\psi D \psi_x] \mid \min_{s \in S} [\psi D \psi_x](s) = x \right\}.$$

Accordingly, for every $D \in \mathscr{P}(S)$, let $\mathcal{M}_D^L = \bigcup_{\psi_x \in \Psi_C} \mathcal{M}_D^L[\psi_x]$.

Claim 3.7. *For any $L \in \mathcal{L}$, $D \in \mathscr{P}(S)$ and any $\psi_z \in \Psi_C$, $\mathcal{M}_D^L$ and $\mathcal{M}_D^L[\psi_z]$ are closed under the mixture (henceforth, convex).*

Proof. Fix an arbitrary $D \in \mathscr{P}(S)$. Suppose that $\psi, \psi' \in \mathcal{M}_D^L$ with $\psi = [\psi D \psi_x]$ and $\psi' = [\psi' D \psi_y]$, where $\psi_x(s) = x$ and $\psi_y(s) = y$ on S. Then, for any $\alpha \in [0, 1]$, we have $\alpha\psi + (1-\alpha)\psi' = [(\alpha\psi + (1-\alpha)\psi')D(\alpha\psi_x + (1-\alpha)\psi_y)]$. Since $\psi(s) \geq x$ and $\psi'(s) \geq y$ for any $s \in D$, it follows that $(\alpha\psi + (1-\alpha)\psi')(s) \geq (\alpha x + (1-\alpha)y)$ for any $s \in D$, which implies $(\alpha\psi + (1-\alpha)\psi') \in \mathcal{M}_D^L$. For any $\psi_z \in \Psi_C$, the convexity of $\mathcal{M}_D^L[\psi_z]$ can be shown by letting $\psi_x = \psi_y = \psi_z$ in the above proof. *Q.E.D.*

3.B. Proofs for Section 3.3 and Section 3.4

The proofs are based on all the preliminary observations given in Appendix 3.A.

[9]In the proof of Lemma 3.1, the existence of $u_L : \Pi(A) \to \mathbb{R}$ does not require Certainty.

3.B.1. Proof of Theorem 3.1

The sufficiency part will be concluded by the following series of claims.

LEMMA 3.2. *Given an $E \in L$, let $L_E = \{\{s\} \mid s \in E\} \cup \{S \setminus E\}$. Then, for every $E \in L$, there exists a capacity $\mu_L^E : \Sigma(L_E) \to [0,1]$ such that for any $\psi, \psi' \in \mathcal{M}_E^L$, $\psi \succsim_L \psi'$ if and only if $J_E(\psi) \geq J_E(\psi')$, where $J_E : \Psi \to \mathbb{R}$ is given by*

$$J_E(\psi) = \int_E \psi d\mu_L^E + \Big(1 - \mu_L^E(E)\Big) \min_{s\in S} \psi(s).$$

Moreover, for every $E \in L$, μ_L^E is unique in the sense that if there is another capacity $\mu : \Sigma(L_E) \to [0,1]$ being such, then $\mu_L^E(B) = \mu(B)$ for every $B \in \mathscr{P}(E)$.

PROOF. Fix an arbitrary $E \in L$. Notice that any $\psi \in \mathcal{M}_E^L$ is $\Sigma(L_E)$-measurable, and by Claim 3.7, $\mathcal{M}_E^L$ is convex. Then, the restriction of $\succsim_L$ on $\mathcal{M}_E^L$ satisfies Order, Continuity, Monotonicity-(I), Comonotonic Independence, and Nondegeneracy. Thus, by Schmeidler (1989, the prood of Theorem, .p 579), there exists a unique functional $J_E : \mathcal{M}_E^L \to \mathbb{R}$ such that for any $\psi, \psi' \in \mathcal{M}_E^L$, $\psi \succsim_L \psi'$ if and only if $J_E(\psi) \geq J_E(\psi')$, and satisfies:

(a) $\forall \psi_x \in \Psi_C,\ J_E(\psi_x) = x$;

(b) for any pairwise comonotonic $\psi, \psi', \psi'' \in \mathcal{M}_E^L$ and $\alpha \in (0,1)$, $J_E(\psi) > J_E(\psi')$ implies $J_E(\alpha\psi + (1-\alpha)\psi'') > J_E(\alpha\psi' + (1-\alpha)\psi'')$;

(c) $\forall \psi, \psi' \in \Psi,\ [\forall s \in S,\ \psi(s) \geq \psi'(s)] \Longrightarrow J_E(\psi) \geq J_E(\psi')$.

Moreover, for any $\psi \in \mathcal{M}_E^L$, we have $\psi(s) \in X$ on S with $[-1,1] \subseteq X$, and any $\psi \in \mathcal{M}_E^L$ satisfies the following conditions:

(i) $t \in S \setminus E \implies \psi(t) = \min_{s\in S} \psi(s)$;

(ii) $\forall x \in X,\ \{s \mid \psi(s) > x\} \in (\mathscr{P}(E) \cup \{S\}) \subset \Sigma(L_E)$.

Define a monotone set function $\nu^E : \mathscr{P}(E) \cup \{S\} \to [0,1]$ by $\nu^E(B) = J_E(\mathbf{1}_B)$ for every $B \in \mathscr{P}(E) \cup \{S\}$, where $\mathbf{1}_B$ is the indicator function of B. Then, by Schmeidler's corollary (Schmeidler (1986, .p 258)), ν^E is the unique monotone set function on $\mathscr{P}(E) \cup \{S\}$ such that, for every $\psi, \psi' \in \mathcal{M}_E^L$,

$$\psi \succsim_L \psi' \iff \int_S \psi \, d\nu^E \geq \int_S \psi' \, d\nu^E.$$ [10]

Fix an arbitrary $\psi \in \mathcal{M}_E^L$, and suppose that $\psi(t) = \min_{s\in S} \psi(s)$ for some $t \in E$, wlog. Let $(x^{(i)}, \dots x^{(K)})$ be the permutation of $\{\psi(s) \mid s \in S\}$ such that $x^{(i)} < x^{(i+1)}$ for $i \in \{1, \dots K-1\}$, and let $\nu_{(i)}^E$ denote the quantity of $\nu^E(\{s \in S \mid \psi(s) \geq x^{(i)}\})$.

[10] Under (a), (b) and (c), Schmeidler's corollary does not imply that ν_E is a capacity on $\Sigma(L_E)$. To see this, let $\mathcal{B}_E = \{(S \setminus E) \cup B \mid B \subset E\}$. Then, $\{\mathcal{B}_E, \{S\}, \mathscr{P}(E)\}$ forms a partition of $\Sigma(L_E)$. Note that $\mathbf{1}_B \notin \mathcal{M}_E^L$ for every $B \in \mathcal{B}_E$ as $\mathbf{1}_B$ violates (i) when $B \in \mathcal{B}_E$. Nonetheless, by condition (i) and $\Psi_C \subset \mathcal{M}_E^L$, Schmeidler's corollary does imply the suggested representation.

Then, by (a) and (i), $\nu^E(E) \geq \nu^E_{(2)}$ and $\nu^E_{(1)} = 1$. Thus,

$$\begin{aligned} J_E(\psi) &= \int_S \psi \, d\nu^E \\ &= x^{(1)} \left(1 - \nu^E(E)\right) + x^{(1)} \left(\nu^E(E) - \nu^E_{(2)}\right) + \sum_{i=2}^{K} x^{(i)} \left(\nu^E_{(i)} - \nu^E_{(i+1)}\right) \\ &= \int_E \psi d\nu^E + \left(1 - \nu^E(E)\right) \min_{s \in S} \psi(s). \end{aligned}$$

Since ν^E is the unique monotone set function that satisfies the above equation, thus if there is a capacity $\mu : \Sigma(L_E) \to [0,1]$ which satisfies this equation, then $\mu(B) = \nu^E(B)$ holds for every $B \in (\mathscr{P}(E) \cup \{S\})$. Hence, the claimed uniqueness follows.

Now, it suffices to show that there is an extension, $\mu^E_L : \Sigma(L_E) \to [0,1]$, of ν^E that forms a capacity on $\Sigma(L_E)$. This is shown by an example as follows. For every $B \in \Sigma(L_E)$, let $\mathcal{D}(B) = (\mathscr{P}(E) \cup \{S\}) \cap \mathscr{P}(B)$. Given ν^E, fix a real number $\nu \in [0, 1 - \nu^E(E)]$ and define a set function $\mu^E_L : \Sigma(L_E) \to [0,1]$ by

$$\mu^E_L(B) = \begin{cases} \nu & B = S \setminus E \\ \max_{D \in \mathcal{D}(B)} \nu^E(D) & B \neq S \setminus E \end{cases}.$$

Then, since ν^E is monotone with respect to the set inclusion, it follows that: (c1) $\mu^E_L(\emptyset) = 0$ and $\mu^E_L(S) = 1$, (c2) for any $B \in \Sigma(L_E)$ and $D \in \mathscr{P}(B)$ $\mu^E_L(B) \geq \mu^E_L(D)$, and (c3) for any $B \in \mathscr{P}(E)$, $\mu^E_L(B) = \nu^E(B)$. Hence, μ^E_L is a capacity on $\Sigma(L_E)$. The proof is complete as $E \in L$ is assumed to be arbitrary. *Q.E.D.*

Let $V_L : \Sigma(L) \to [0,1]$ be the capacity obtained in Lemma 3.1. Denote by $I(\psi)$ the functional derived in the proposition. That is, $I(\psi) = \int_S \psi dV_L$ for every $\psi \in \Psi_L$. Let $N(L)$ be the set of all V_L-null cells in L. Formally,

$$N(L) = \{E \in L \mid V_L(E) = 0\}.$$

Then, by null-additivity, $N(L) \neq L$.

CLAIM 3.8. *For every $E \in L$, $\mu^E_L(E) = V_L(E)$.*

PROOF. Fix an arbitrary $E \in L$. By construction, $\Psi_L \cap \mathcal{M}^L_E \neq \emptyset$. Clearly, $\mathcal{M}^L_E \cap \Psi_L$ is convex, hence being simply connected. Since $J_E : \mathcal{M}^L_E \to \mathbb{R}$ and $I : \Psi_L \to \mathbb{R}$ represents the same preference $\succcurlyeq_L$ on $\mathcal{M}^L_E \cap \Psi_L$, there is a continuous and strictly increasing function $\varphi_E : J_E(\mathcal{M}^L_E \cap \Psi_L) \to \mathbb{R}$ such that for every $\psi \in \mathcal{M}^L_E \cap \Psi_L$, $\varphi_E(J_E(\psi)) = I(\psi)$. Note that $\Psi_C \subset (\Psi_L \cap \mathcal{M}^L_E)$. Thus, we have $\varphi_E(x) = x$ for every $x \in X$. As a result, for every $[\psi_y E \psi_z] \in \Psi_L \cap \mathcal{M}^L_E$, it follows that

$$\begin{aligned} J_E([\psi_y E \psi_z]) &= \mu^E_L(E)y + (1 - \mu^E_L(E))z \\ &= V_L(E)y + (1 - V_L(E))z = I([\psi_y E \psi_z]), \end{aligned}$$

where $y = \psi_y(s)$ and $z = \psi_z(s)$ on S. Set $x = y - z$. Since ψ_z and ψ_y are arbitrary, the equality implies that $\mu_L^E(E)x = V_L(E)x$ for every $x \in \mathbb{R}_+$. Therefore, for every $E \in L$, $\mu_L^E(E) = V_L(E)$ as E is assumed to be arbitrary. *Q.E.D.*

Claim 3.9. *Given a $\psi \in \Psi$ and a $\psi_L \in \Psi_L$, let $\psi_z \in \Psi_C$ be the constant function such that $\psi_z(s) = z = \min\{\min_{s\in S}\psi(s), \min_{s\in S}\psi_L(s)\}$ on S.*

(a) Suppose $z = \min_{s\in S}\psi(s)$. Then, $[\psi E \psi_z] \sim_L [\psi_L E \psi_z]$ for every $E \in L$ if and only if for every $E \in L \setminus N(L)$, $\psi_L(s) = \left(\int_E \psi d\mu_L^E\right)/V_L(E)$ on E.

(b) Suppose $N(L) = \emptyset$. Then, $[\psi E \psi_z] \sim_L [\psi_L E \psi_z]$ for every $E \in L$ only if $z = \min_{s\in S}\psi(s)$.

Proof. Fix a $\psi \in \Psi$ and a $\psi_L \in \Psi_L$ arbitrarily. Let $x = \min_{s\in S}\psi(s)$ on S. By construction, it holds that $[\psi E\psi_z] \in \mathcal{M}_E^L$ and $[\psi_L E\psi_z] \in \mathcal{M}_E^L \cap \Psi_L$ for every $E \in L$. Hence, by Lemma 3.2, $[\psi E\psi_z] \sim_L [\psi_L E\psi_z]$ for every $E \in L$ if and only if $J_E([\psi E\psi_z]) = J_E([\psi_L E\psi_z])$ for every $E \in L$. Thus, for any $E \in L$, it holds that

$$\begin{aligned}
&J_E([\psi E\psi_z]) = J_E([\psi_L E\psi_z]) \\
\iff &\int_E [\psi E\psi_z] d\mu_L^E + \left(1 - \mu_L^E(E)\right) z = \left(1 - \mu_L^E(E)\right) z + \mu_L^E(E)\psi_L^E \\
\iff &\int_E \psi d\mu_L^E = \mu_L^E(E)\psi_L^E,
\end{aligned}$$

where $\psi_L^E = \psi_L(t)$ for $t \in E$. Under Claim 3.8, it yields that

$$\begin{bmatrix} \forall E \in L, \\ [\psi E\psi_z] \sim_L [\psi_L E\psi_z] \end{bmatrix} \iff \begin{bmatrix} \forall E \in L, \\ \int_E \psi d\mu_L^E = V_L(E)\psi_L^E \end{bmatrix}. \tag{3.1}$$

(a) Set $\psi_z = \underline{\psi}^L$. Then, for every $E \in L$, $[\psi_L E\psi_x] \in \mathcal{M}_E^L \cap \Psi_L$. Thus, the above equivalences also hold for x and $\underline{\psi}^L$. Fix an arbitrary $E \in L$. Then, $[\psi E\underline{\psi}^L] \sim_L [\psi_L E\underline{\psi}^L]$ if and only if $\int_E \psi d\mu_L^E = V_L(E)\psi_L^E$. By Claim 3.8, when $E \in N(L)$, this equality becomes trivial as $\mu_L^E(E) = V_L(E) = 0$. Yet, since $\mu_L^E(E) = V_L(E) > 0$ when $E \in L \setminus N(L)$, it thus follows that $\psi_L^E = \left(\int_E \psi d\mu_L^E\right)/V_L(E)$. As a result, for every $E \in L \setminus N(L)$ and any $t \in E$,

$$\psi_L(t) = \frac{1}{V_L(E)}\left(\int_E \psi d\mu_L^E\right).$$

(b) Suppose $N(L) = \emptyset$. It suffices to show $\min_{s\in S}\psi_L(s) = z$ implies $z = x$. Assume $\min_{s\in S}\psi_L(s) = z$. Then, since $\psi_L \in \Psi_L$, there is an $\hat{E} \in L$ such that $[\psi_L \hat{E}\psi_z] = \psi_z$. Fix this $\hat{E} \in L$. By Eq (3.1), it holds that $\int_{\hat{E}} \psi d\mu_L^{\hat{E}} = V_L(\hat{E})z$. Under Claim 3.8, $N(L) = \emptyset$ implies $\mu_L^{\hat{E}}(\hat{E}) = V_L(\hat{E}) > 0$. Thus, it follows that

$$\int_{\hat{E}} \psi d\mu_L^{\hat{E}} \geq \min_{s\in\hat{E}} \psi(s)V_L(\hat{E}) \geq V_L(\hat{E})x \geq V_L(\hat{E})z,$$

where the last inequality is implied by $z = \min\{\min_{s\in S}\psi(s), \min_{s\in S}\psi_L(s)\}$. As a result, $\int_{\hat{E}} \psi d\mu_L^{\hat{E}} = V_L(\hat{E})z$ yields that $\min_{s\in\hat{E}}\psi(s) = \min_{s\in S}\psi(s) = x = z$. *Q.E.D.*

CLAIM 3.10. *Suppose that a mapping $Q_L : \Psi \to \Psi_L$ exists such that, for every $\psi \in \Psi$, (i) $\min_{s\in S} Q_L(\psi)(s) \geq \min_{s\in S} \psi(s)$ on S; and (ii) $[\psi E \underline{\psi}^L] \sim_L [Q_L(\psi) E \underline{\psi}^L]$ holds for any $E \in L$. Then, for every $\psi, \psi' \in \Psi$, $\psi \succcurlyeq_L \psi'$ if and only if*

$$\int_{\bigcup (L\setminus N(L))} Q_L(\psi) dV_L \geq \int_{\bigcup (L\setminus N(L))} Q_L(\psi') dV_L.$$

PROOF. By (i), it holds that $\underline{Q_L(\psi)}^L \succcurlyeq_L \underline{\psi}^L$ for every $\psi \in \Psi$. Hence, under Monotonicity-(II), condition (ii) implies that $\psi \sim_L Q_L(\psi)$. Therefore, by transitivity, for every $\psi, \psi' \in \Psi$, $\psi \succcurlyeq_L \psi'$ if and only if $Q_L(\psi) \succcurlyeq_L Q_L(\psi')$. Note that $Q_L(\psi), Q_L(\psi') \in \Psi_L$. Thus, by Lemma 3.1, $\psi \succsim_L \psi'$ if and only if $I(Q_L(\psi)) \geq I(Q_L(\psi'))$, where for any $\psi_L \in \Psi_L$, $I(\psi) = \int_S \psi_L dV_L$.

Recall that V_L is null-additive, and that $L \setminus N(L) = \{E \in L \mid V_L(E) > 0\}$. As a result, by Claim 3.4-(a), for any $\psi_L \in \Psi_L$, $I(\psi_L) = \int_{\bigcup (L\setminus N(L))} \psi_L dV_L$. *Q.E.D.*

SUFFICIENCY. Let $\Delta(L) = L \setminus N(L)$.

Let $Q_L : \Psi \to \Psi_L$ be a mapping such that $\min_{s\in S} Q_L(\psi)(s) \geq \min_{s\in S} \psi(s)$ on S. Then, by Claim 3.9-(a), Q_L satisfies condition (ii) in Claim 3.10 if and only if for any $\psi \in \Psi$, $E \in \Delta(L)$ and $s \in E$, $Q_L(\psi)(s) = \left(\int_E \psi d\mu_L^E\right) / V_L(E)$. Therefore, a mapping $Q_L : \Psi \to \Psi_L$ satisfies (i) and (ii) in Claim 3.10 if and only if it can be written by

$$Q_L(\psi) = \sum_{E\in\Delta(L)} \left(\int_E \psi d\mu_L^E\right) \frac{\mathbf{1}_E}{V_L(E)} + \sum_{\hat{E}\in N(L)} \left(x_{\hat{E}} + \min_{s\in S} \psi(s)\right) \mathbf{1}_{\hat{E}},$$

where $x_{\hat{E}} \in \mathbb{R}_+$ for each $\hat{E} \in N(L)$.

By null-additivity, for any such Q_L and any $\psi \in \Psi$, it holds that

$$\int_{\bigcup \Delta(L)} Q_L(\psi) dV_L = \int_{\bigcup \Delta(L)} \left\{ \sum_{E\in\Delta(L)} \left(\int_E \psi d\mu_L^E\right) \frac{\mathbf{1}_E}{V_L(E)} \right\} dV_L.$$

Therefore, by Claim 3.10, for any $\psi, \psi' \in \Psi$,

$$\psi \succcurlyeq_L \psi' \iff \mathcal{Q}_L(\psi) \geq \mathcal{Q}_L(\psi'), \text{ where}$$
$$\mathcal{Q}_L(\psi) = \int_{\bigcup \Delta(L)} Q_L(\psi) dV_L; \quad Q_L(\psi) = \sum_{E\in\Delta(L)} \left(\int_E \psi d\mu_L^E\right) \frac{\mathbf{1}_E}{V_L(E)}. \tag{3.2}$$

Let $\mathcal{B}_L = (\bigcup_{E\in L} \mathscr{P}(E)) \cup \Sigma(L)$, and define a set function $v : \mathcal{B}_L \to [0,1]$ by

$$v(B) = \begin{cases} V_L(B), & B \in \Sigma(L); \\ \mu_L^E(B), & \exists E \in \Delta(L), B \subseteq E; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $(\bigcup_{E\in L} \mathscr{P}(E)) \cap \Sigma(L) = L$, and by Claim 3.8, $V_L(E) = \mu_L^E(E)$ for every $E \in L$. Thus, v is well-defined. Since V_L and each $\mu_L^E : \Sigma(L_E) \to [0,1]$ are monotone with respect to the set inclusion, Claim 3.8 implies that v is also monotone. By null-additivity, $V_L(\bigcup \Delta(L)) = V_L(S) = 1$, meaning that $v(\bigcup \Delta(L)) = 1$. Hence, v is the unique monotone set function on $\mathcal{B}_L$ such that $v(\bigcup \Delta(L)) = 1$ and

$$\mathcal{Q}_L(\psi) = \int_{\bigcup \Delta(L)} \left\{ \sum_{E \in \Delta(L)} \left(\int_E \psi dv \right) \frac{\mathbf{1}_E}{v(E)} \right\} dv, \quad \forall \psi \in \Psi.$$

Given that $v(\bigcup \Delta(L)) = 1$, any monotone extension $v_L : \mathscr{P}(S) \to [0,1]$ of v satisfies equivalence in Eq (3.2) with respect to the above equation, and any such v_L forms a capacity on $\mathscr{P}(S)$. Hence, the existence of capacity v_L on $\mathscr{P}(S)$ follows. The uniqueness of v proves the second statement in (II).

Finally, the null-additivity of V_L implies $\Delta(L) \neq \emptyset$. *Q.E.D.*

Necessity. Order, Continuity and Monotonicity-(I) are trivial.
CRI-(I) is implied by Lemma 3.1, while Lemma 3.2 concludes CRI-(II).
Monotonicity-(II) follows the equivalences in Claim 3.9 and Claim 3.10. *Q.E.D.*

3.B.2. Proof of Theorem 3.2

First, we show the sufficiency part. The proof will follow all the notations and claims given in Appendix 3.A and Appendix 3.B.1.

Under Certainty, Lemma 3.1 implies that $\{u_L\}_{L \in \mathcal{L}}$ can be fixed by a $u : \Pi(A) \to \mathbb{R}$, meaning that $\{\psi \mapsto \underline{\psi}^L\}_{L \in \mathcal{L}}$ is not subject to $L \in \mathcal{L}$. Accordingly, for every $L, \tilde{L} \in \mathcal{L}$, it holds that $\mathcal{M}_E^L = \mathcal{M}_E^{\tilde{L}}$ whenever $E \in L \cap \tilde{L}$. Thus, in what follows, we simply write $\underline{\psi}$ and $\mathcal{M}_E$, respectively.

Let $\underline{L} = \{S\}$ and $\overline{L} = \{\{s\} \mid s \in S\}$. Let $\hat{V} : \mathscr{P}(S) \to [0,1]$ denote the null-additive capacity obtained in Corollary 3.1.1-(II). Denote by $V : \mathscr{P}(S) \to [0,1]$ the unique capacity obtained in Corollary 3.1.1-(I), and let $N(S)$ be the set of V-null events. Formally, $N(S) = \{B \in \mathscr{P}(S) \mid V(B) = 0\}$. Given an $L \in \mathcal{L}$, let $V_L : \Sigma(L) \to [0,1]$ be the null-additive capacity obtained in Lemma 3.1. Then, $\hat{V} = V_{\overline{L}}$ in particular.

Claim 3.11. *Under Certainty and CoI, the following statements hold.*

(a) For every $L \in \mathcal{L}$ and any $E \in L$, $E \in \Delta(L)$ implies $E \notin N(S)$.

(b) For any $L \in \mathcal{L}$ and every $E \in \Delta(L)$, there exists a $\vartheta_L^E \in \mathbb{R}_{++}$ such that for every $B \in \mathscr{P}(E)$, $\mu_L^E(B) = \vartheta_L^E V(B)$.

Proof. Fix an arbitrary $L \in \mathcal{L}$.

(a) Fix an $E \in L$ and a $\psi_x \in \Psi_C$, where $x = \psi_x(s)$ on S. Consider $\psi, \psi' \in \mathcal{M}_E[\psi_x]$. Then, by Lemma 3.2,

$$\psi \succcurlyeq_L \psi' \iff J_E(\psi) \geq J_E(\psi') \iff \int_E \psi d\mu_L^E \geq \int_E \psi' d\mu_L^E.$$

Since μ_L^E is monotone and $J_L(\psi) \geq x$ holds for any $\psi \in \mathcal{M}_E[\psi_x]$, thus $\mathcal{M}_E[\psi_x]$ is not $\succcurlyeq_L$-indifferent if and only if $\mu_L^E(E) > 0$. Hence, by Claim 3.8, for any $\psi_x \in \Psi_C$, $\mathcal{M}_E[\psi_x]$ is not $\succcurlyeq_L$-indifferent if and only if $E \in \Delta(L)$.

Suppose that $E \in \Delta(L)$. Then, for any $\psi_x \in \Psi_C$, there exist $\psi, \psi' \in \mathcal{M}_E[\psi_x]$ such that $\psi \succ_L \psi'$. Thus, by CoI, $\psi \succ_{\underline{L}} \psi'$ holds for every such $\psi, \psi' \in \mathcal{M}_E[\psi_x]$, meaning that $\mathcal{M}_E[\psi_x]$ is not $\succcurlyeq_{\underline{L}}$-indifferent for any $\psi_x \in \Psi_C$. By Corollary 3.1.1-(I), this statement is equivalent to $V(E) > 0$. As a result, $E \in \Delta(L)$ implies $E \notin N(S)$.

(b) Fix an $E \in \Delta(L)$ and a $\psi_z \in \Psi_C$ arbitrarily. Then, by CoI, for any $\psi, \psi' \in \mathcal{M}_E[\psi_z] \setminus \{\psi_z\}$, $\psi \succcurlyeq_L \psi'$ if and only if $\psi \succcurlyeq_{\underline{L}} \psi'$. By Lemma 3.2 and Corollary 3.1.1-(I), it yields that $J_E(\psi) \geq J_E(\psi')$ if and only if $\int_S \psi dV \geq \int_S \psi' dV$. Let $\psi_z(s) = z$ on S. Then, $J_E(\psi) = \int_E \psi d\mu_L^E + (1 - \mu_L^E(E))z$ and $\int_S \psi dV = \int_E \psi dV + (1 - V(E))z$. Thus, for every $E \in \Delta(L)$ and any $\psi, \psi' \in \mathcal{M}_E[\psi_z] \setminus \{\psi_z\}$

$$\int_E \psi d\mu_L^E \geq \int_E \psi' d\mu_L^E \iff \int_E \psi dV \geq \int_E \psi' dV.$$

By (a), $E \in \Delta(L)$ implies $E \notin N(S)$, meaning that $V(E) > 0$ and $\mu_L^E(E) > 0$. Let $[\psi_y E \psi_z] \in \mathcal{M}_E[\psi_z] \cap \Psi_L$, and suppose $\psi_y(s) = y > z$ on S. Clearly, it holds that $\int_E [\psi_y E \psi_z] d\mu_L^E = \mu_L^E(E)y$ and $\int_E [\psi_y E \psi_z] dV = V(E)y$. Set $\vartheta_L^E = \mu_L^E(E)/V(E)$. Then, since $V(E) > 0$ and $\mu_L^E(E) > 0$, we have $\vartheta_L^E > 0$. Therefore, by CoI, for every $\psi \in \mathcal{M}_E[\psi_z] \setminus \{\psi_z\}$ that satisfies $\psi \sim_L [\psi_y E \psi_z]$, it follows that $\int_E \psi d\mu_L^E = \mu_L^E(E)y = \vartheta_L^E V(E)y = \vartheta_L^E(\int_E \psi dV)$. Since $\psi_z \in \Psi_C$ and $[\psi_y E \psi_z] \in \mathcal{M}_E[\psi_z] \setminus \{\psi_z\}$ are arbitrary,

$$\int_E \psi d\mu_L^E = \vartheta_L^E \int_E \psi dV$$

holds for any $\psi \in \mathcal{M}_E \setminus \Psi_C$. As a result, this equality yields that $\mu_L^E(B) = \vartheta_L^E V(B)$ for every $B \in \mathscr{P}(E)$. *Q.E.D.*

Sufficiency. By Claim 3.9 and Claim 3.10, the equivalence obtained in Eq (3.2) holds for every L. Given $\mathcal{Q}_L$ and Q_L in Eq (3.2), define a functional $\mathcal{Q} : \Psi \times \mathcal{L} \to \mathbb{R}$ and a mapping $Q : \Psi \times \mathcal{L} \to \Psi$ respectively by $\mathcal{Q}(\psi, L) = \mathcal{Q}_L(\psi)$ and $Q(\psi, L) = Q_L(\psi)$. Then, for all $L \in \mathcal{L}$ and $\psi, \psi' \in \Psi$, $\psi \succcurlyeq_L \psi'$ if and only if $\mathcal{Q}(\psi, L) \geq \mathcal{Q}(\psi', L)$.

Probing further, by Claim 3.11-(a), for any $L \in \mathcal{L}$, $\Delta(L) = \{E \in L \mid V_L(E) > 0\} \subseteq \{E \in L \mid V(E) > 0\}$. By Claim 3.8 and Claim 3.11-(b), for every $L \in \mathcal{L}$ and $E \in \Delta(L)$, $V_L(E) = \vartheta_L^E V(E)$ with $\vartheta_L^E > 0$. Therefore, for any $L \in \mathcal{L}$ and $\psi \in \Psi$, it holds that

$$Q(\psi, L) = \sum_{E \in \Delta(L)} \vartheta_L^E \left(\int_E \psi dV \right) \frac{\mathbf{1}_E}{\vartheta_L^E V(E)} = \sum_{E \in \Delta(L)} \left(\int_E \psi dV \right) \frac{\mathbf{1}_E}{V(E)}.$$

For every $L \in \mathcal{L}$, V_L is null-additive on $\Sigma(L)$, and $Q(\psi, L) \in \Psi_L$ for any $\psi \in \Psi$. Thus, by Claim 3.10 and Claim 3.4-(a), for any $L \in \mathcal{L}$ and $\psi \in \Psi$,

$$\mathcal{Q}(\psi, L) = \int_{\bigcup \Delta(L)} Q(\psi, L) dV_L; \quad Q(\psi, L) = \sum_{E \in \Delta(L)} \left(\int_E \psi dV \right) \frac{\mathbf{1}_E}{V(E)}. \tag{3.3}$$

Q.E.D.

Necessity. Suppose that $\{\succsim_L\}_{L \in \mathcal{L}}$ admits a representation $(u, V, \{V_L\}_{L \in \mathcal{L}})$.

(Uniform Axioms) Fix an arbitrary $L \in \mathcal{L}$. Since $\Delta(L) \subseteq \{E \in L \mid V(E) > 0\}$, we have $V(E) > 0$ whenever $V_L(E) > 0$. Let $\mathcal{B}_L = (\bigcup_{E \in L} \mathscr{P}(E)) \cup \Sigma(L)$ and define a monotone set function $v_L : \mathcal{B}_L \to [0, 1]$ by

$$v_L(B) = \begin{cases} V_L(B), & B \in \Sigma(L); \\ \frac{V(B)V_L(E)}{V(E)}, & B \in \mathscr{P}(E), E \in \Delta(L); \\ 0, & \text{otherwise}, \end{cases} \qquad \forall B \in \mathcal{B}_L.$$

Then, v_L is well-defined. Thus, for any $f \in \mathcal{H}$, it follows that

$$\mathcal{U}(f, L) = \int_{\bigcup \Delta(L)} \left\{ \sum_{E \in \Delta(L)} \left(\int_E u(f) dv_L \right) \frac{\mathbf{1}_E}{v_L(E)} \right\} dv_L.$$

By the uniqueness of V_L, v_L is the unique monotone set function on $\mathcal{B}_L$ with satisfying the above equation. Therefore, by Theorem 3.1, uniform axioms hold.

(CoI) Let $(L, \tilde{L}) \in \mathscr{L}$ be arbitrary.

For any $\tilde{E} \in \tilde{L}$ and any $c \in \mathcal{H}_C$, $\mathcal{C}^{\tilde{L}}_{\tilde{E}}[c]$ is not $\succsim_{\tilde{L}}$-indifferent if and only if $\tilde{E} \in \Delta(\tilde{L})$.

Fix an $\tilde{E} \in \Delta(\tilde{L})$ and a $c \in \mathcal{H}_C$ arbitrarily. Let $E \in L$ be the unique cell such that $\tilde{E} \subseteq E$. Recall that $\Delta(L) \subseteq \{E \in L \mid V(E) > 0\}$ and $\Delta(\tilde{L}) \subseteq \{E \in \tilde{L} \mid V(E) > 0\}$. Thus, by the monotonicity of V, $\tilde{E} \in \Delta(\tilde{L})$ implies $E \in \Delta(L)$. In turn, we have $V(E) > 0$ and $V(\tilde{E}) > 0$. Consequently, for any $h \in \mathcal{C}^{\tilde{L}}_{\tilde{E}}[c]$, it holds that

$$\begin{aligned} \mathcal{U}(h, \tilde{L}) &= \tilde{\vartheta} \left(\int_{\tilde{E}} u(h) dV \right) + (1 - V_{\tilde{L}}(\tilde{E})) u_c; \\ \mathcal{U}(h, L) &= \vartheta \left(\int_{\tilde{E}} u(h) dV + (V(E) - V(\tilde{E})) u_c \right) + (1 - V_L(E)) u_c \\ &= \vartheta \left(\int_{\tilde{E}} u(h) dV \right) + (1 - \beta) \, u_c, \end{aligned}$$

where $\tilde{\vartheta} = V_{\tilde{L}}(\tilde{E})/V(\tilde{E}) > 0$, $\vartheta = V_L(E)/V(E) > 0$, $\beta = (V(\tilde{E})V_L(E))/V(E)$ and $u_c = u(c(s))$ on S.

Therefore, for any $f, g \in \mathcal{C}^{\tilde{L}}_{\tilde{E}}[c]$, $\mathcal{U}(f, \tilde{L}) \geq \mathcal{U}(g, \tilde{L})$ if and only if $\int_{\tilde{E}} u(f) dV \geq \int_{\tilde{E}} u(g) dV$, and $\mathcal{U}(f, L) \geq \mathcal{U}(g, L)$ if and only if $\int_{\tilde{E}} u(f) dV \geq \int_{\tilde{E}} u(g) dV$. As a result, $\succsim_L = \succsim_{\tilde{L}}$ on $\mathcal{C}^{\tilde{L}}_{\tilde{E}}[c] \times \mathcal{C}^{\tilde{L}}_{\tilde{E}}[c]$. *Q.E.D.*

3.B.3. Proof of Theorem 3.3

First, we show the necessity part.

CLAIM 3.12. *Suppose that $\{\succsim_L\}_{L\in\mathcal{L}}$ admits an ω-FAEU representation $(u, V, \hat{V}, \boldsymbol{\omega})$. For every $L \in \mathcal{L}$, define a set function $V_L : \Sigma(L) \to [0,1]$ by: $V_L(B) = \omega_L \hat{V}(B)$ if $\hat{V}(S\setminus B) > 0$, and $V_L(B) = 1$ otherwise. Then, $\{\succsim_L\}_{L\in\mathcal{L}}$ admits the FAEU representation $(u, V, \{V_L\}_{L\in\mathcal{L}})$.*

PROOF. Let $\{\succsim_L\}_{L\in\mathcal{L}}$ and $\{V_L\}_{L\in\mathcal{L}}$ be as assumed.

By definition, for every $L \in \mathcal{L} \setminus L$, $1/\omega_L \geq 1$, and $\hat{V}$ is null-additive. These yield $1/\omega_L \geq 1 \geq \max_{B\in\Sigma(L):\hat{V}(S\setminus B)>0} \hat{V}(B)$, meaning that for all $L \in \mathcal{L} \setminus \{\underline{L}\}$, V_L and $\hat{V}$ satisfy the assumptions of Claim 3.2. Therefore, for every $L \in \mathcal{L}$, $V_L : \Sigma(L) \to [0,1]$ is a null-additive capacity.

Given the defined family $\{V_L\}_{L\in\mathcal{L}}$, for every $L \in \mathcal{L}$, let $\Delta(L) = \{E \in L \mid V_L(E) > 0\}$. Then, it follows trivially that $\Omega(L) = \Delta(L)$ holds for all $L \in \mathcal{L}$.

Hence, for all $L \in \mathcal{L}$, $\hat{V}$ and V_L satisfy the assumptions of Claim 3.4-(b). Therefore, by Claim 3.4-(b), for every $L \in \mathcal{L}$ and any $f \in \mathcal{H}$, it follows that

$$\mathcal{U}^\omega(f, L) = \int_{\bigcup \Delta(L)} \left\{ \sum_{E\in\Delta(L)} \left(\int_E u(f)dV \right) \frac{\mathbf{1}_E}{V(E)} \right\} dV_L. \tag{3.4}$$

Given that the tuple $(u, V, \hat{V}, \boldsymbol{\omega})$ is unique, for every $L \in \mathcal{L}$, $V_L : \Sigma(L) \to [0,1]$ defined above is the unique null-additive capacity that satisfies Eq (3.4).

As a result, $\{\succsim_L\}_{L\in\mathcal{L}}$ admits the FAEU $(u, V, \{V_L\}_{L\in\mathcal{L}})$. *Q.E.D.*

NECESSITY. Suppose that $\{\succsim_L\}_{L\in\mathcal{L}}$ admits a representation $(u, V, \hat{V}, \boldsymbol{\omega})$. Then, by Claim 3.12, $\{\succsim_L\}_{L\in\mathcal{L}}$ admits FAEU representation $(u, V, \{V_L\}_{L\in\mathcal{L}})$, where $\{V_L\}_{L\in\mathcal{L}}$ is induced by $\hat{V}$ and $\boldsymbol{\omega}$ as in the claim. Thus, Theorem 3.2 proves the uniform axioms, Certainty and CoI.

(Coarseness Aversion) Given Certainty, we only need to consider $\mathcal{L} \setminus \{\underline{L}\}$.

Let $(L, \tilde{L}) \in \mathscr{L}$ be such that $L \neq \underline{L}$ and $\tilde{L} \neq \underline{L}$.

By the definition in Claim 3.12, V_L and $V_{\tilde{L}}$ are null-additive. Thus, by Claim 3.4-(a), for all $h \in \mathcal{H}_L$, it holds that $\int_{\bigcup \Delta(L)} u(h)dV_L = \int_S u(f)dV_L$, and so does for $V_{\tilde{L}}$ and $\tilde{L}$. Moreover, under the definition in Claim 3.12, $\omega_{\tilde{L}} \geq \omega_L$ implies that for every $B \in \Sigma(L)$, $V_{\tilde{L}}(B) \geq V_L(B)$. Thus, by Claim 3.3, for all $h \in \mathcal{H}_L$, it holds that $\int_S u(h)dV_{\tilde{L}} \geq \int_S u(h)dV_L$. That is, for all $h \in \mathcal{H}_L$, $\mathcal{U}^\omega(h, \tilde{L}) \geq \mathcal{U}^\omega(h, L)$.

Since $\mathcal{U}^\omega(c, L) = \mathcal{U}^\omega(c, \tilde{L})$ holds for all $(L, \tilde{L}) \in \mathscr{L}$ and $c \in \mathcal{H}_C$, the above inequality is equivalent to Coarseness Aversion.

(CoR) Let $(L, \tilde{L}) \in \mathscr{L}$ be arbitrary, and let $\{V_L\}_{L\in\mathcal{L}}$ be the aforementioned family.

Recall that $V_L : \Sigma(L) \to [0,1]$ is null-additive, and that $\Omega(L) = \Delta(L) \subseteq \{E \in L \mid V(E) > 0\}$. Thus by Eq (3.4) and Claim 3.4-(a), for any $h \in \mathcal{H}_L$, $\mathcal{U}^\omega(h, L) = \int_S u(h) dV_L$ and $\mathcal{U}^\omega(h, \tilde{L}) = \int_S u(h) dV_{\tilde{L}}$. Note that $\omega_L, \omega_{\tilde{L}} \in \mathbb{R}_{++}$. Hence, by the definition given in Claim 3.12, for any $D \in \Sigma(L)$, $\hat{V}(S \setminus D) > 0$ implies $V_L(D) = \beta V_{\tilde{L}}(D)$, where $\beta = \omega_L / \omega_{\tilde{L}} > 0$.

Suppose that there exist $D, D' \in \Sigma(L) \setminus \{S, \emptyset\}$, a $c \in \mathcal{H}_C$, an $f \in \mathcal{C}^L_D[c] \cap \mathcal{H}_L$ and a $g \in \mathcal{C}^L_{D'}[c] \cap \mathcal{H}_L$ such that $f \succsim_L g$ and $f \prec_{\tilde{L}} g$. Then, it holds that

$$\int_D (u(f) - u_c \mathbf{1}_S) dV_L \geq \int_{D'} (u(g) - u_c \mathbf{1}_S) dV_L;$$
$$\int_D (u(f) - u_c \mathbf{1}_S) dV_{\tilde{L}} < \int_{D'} (u(g) - u_c \mathbf{1}_S) dV_{\tilde{L}},$$

where $u_c = u(c(s))$ on S. Note that both sides of the above inequalities are non-negative as $f \in \mathcal{C}^L_D[c] \cap \mathcal{H}_L$ and $g \in \mathcal{C}^L_{D'}[c] \cap \mathcal{H}_L$. Suppose that $V_L(B) = \beta V_{\tilde{L}}(B)$ holds for every $B \in ((\mathscr{P}(D) \cup \mathscr{P}(D')) \cap \Sigma(L))$. Then, it follows that $\int_D (u(f) - u_c \mathbf{1}_S) dV_L = \beta \int_D (u(f) - u_c \mathbf{1}_S) dV_{\tilde{L}}$ and $\int_{D'} (u(g) - u_c \mathbf{1}_S) dV_L = \beta \int_{D'} (u(g) - u_c \mathbf{1}_S) dV_{\tilde{L}}$. Given that $\beta > 0$, the above pair of inequalities is self-contradictory. Therefore, there exists a $B \in ((\mathscr{P}(D) \cup \mathscr{P}(D')) \cap \Sigma(L))$ such that $V_L(B) \neq \beta V_{\tilde{L}}(B)$. Fix this B. Then, by the definition given in Claim 3.12, we have $\hat{V}(S \setminus B) = 0$. Suppose that, wlog, $B \in (\mathscr{P}(D) \cap \Sigma(L))$. Then, the monotonicity of $\hat{V}$ implies $\hat{V}(S \setminus D) = 0$, meaning that $\bigcup \Omega(L) \subseteq D$. Hence, for any $h \in \mathcal{C}^L_{S \setminus D}[c]$, it yields that $\mathcal{U}^\omega(h, L) = u_c$ as $h(s) = c(s)$ on $\bigcup \Omega(L)$. That is, $\mathcal{C}^L_{S \setminus D}[c]$ is $\succsim_L$-indifferent.

As a result, for any $D, D' \in \Sigma(L) \setminus \{S, \emptyset\}$ and any $c \in \mathcal{H}_C$, if there exist $f \in \mathcal{C}^L_D[c] \cap \mathcal{H}_L$ and $g \in \mathcal{C}^L_{D'}[c] \cap \mathcal{H}_L$ such that $f \succsim_L g$ and $f \prec_{\tilde{L}} g$, then either $\mathcal{C}^L_{S \setminus D}[c]$ or $\mathcal{C}^L_{S \setminus D'}[c]$ is $\succsim_L$-indifferent. CoR holds. *Q.E.D.*

The sufficiency part will follow all the notations and claims given in Appendix 3.A.

Suppose $\{\succsim_L\}_{L \in \mathcal{L}}$ satisfies the given axioms. By Claim 3.1-(b), Certainty is implied. Thus, by Theorem 3.2, $\{\succsim_L\}_{L \in \mathcal{L}}$ admits an FAEU $(u, V, \{V_L\}_{L \in \mathcal{L}})$. Fix this tuple, and let $\hat{V}$ denote the null-additive capacity $V_{\overline{L}}$.

Claim 3.13. *Under Certainty and CoR, the following statements hold.*

(a) For any $L \in \mathcal{L}$ and any $B \in \Sigma(L)$, $V_L(B) > 0$ if and only if $\hat{V}(B) > 0$.

(b) For any $L \in \mathcal{L}$, an $\omega_L \in \mathbb{R}_{++}$ exists such that, for every $B \in \Sigma(L)$, $\hat{V}(S \setminus B) > 0$ implies $V_L(B) = \omega_L \hat{V}(B)$.

Proof. Fix an arbitrary $L \in \mathcal{L}$.

(a) Fix a $B \in \Sigma(L)$. For any $\psi_z \in \Psi_C$ and any $\psi \in \mathcal{M}_E[\psi_z] \cap \Psi_L$, we have

$$\int_S \psi dV_L = (1 - V_L(B)) z + \int_B \psi dV_L,$$

where $z = \psi_z(s)$ on S. Thus, by Lemma 3.1, $V_L(B) = 0$ if and only if $\mathcal{M}_E[\psi_z] \cap \Psi_L$ is $\succcurlyeq_L$-indifferent for any $\psi_z \in \Psi_C$.

Suppose that $V_L(B) = 0$. Then, $\mathcal{M}_E[\psi_z] \cap \Psi_L$ is $\succcurlyeq_L$-indifferent for any $\psi_z \in \Psi_C$, and by null-additivity, $V_L(S \setminus B) = 1$. Under Nondegeneracy, this in turn implies that $\mathcal{M}_{S\setminus B}[\psi_z]$ is not $\succcurlyeq_L$-indifferent. Thus, by CoR, $\succcurlyeq_L = \succcurlyeq_{\overline{L}}$ on $\mathcal{M}_B[\psi_z] \cap \Psi_L$, meaning that $\mathcal{M}_B[\psi_z] \cap \Psi_L$ is also $\succcurlyeq_{\overline{L}}$-indifferent. Since $\Sigma(L) \subseteq \Sigma(\overline{L})$ implies $\Psi_L \subseteq \Psi_{\overline{L}}$, by Lemma 3.1, $\mathcal{M}_B[\psi_z] \cap \Psi_L$ is $\succcurlyeq_{\overline{L}}$-indifferent if and only if $\hat{V}(B) = 0$. Hence, $V_L(B) = 0$ implies $\hat{V}(B) = 0$.

Suppose that $V_L(B) > 0$, and that there is a $\psi_z \in \Psi_C$ such that $\mathcal{M}_{S\setminus B}[\psi_z]$ is $\succcurlyeq_L$-indifferent. Similarly, we have $V_L(S \setminus B) = 0$, and hence $V_L(B) = 1$ by null-additivity. Under Nondegeneracy, the latter implies that $\mathcal{M}_B[\psi_z]$ is not $\succcurlyeq_L$-indifferent for every $\psi_z \in \Psi_C$. Thus, by CoR, for any $\psi_z \in \Psi_C$, $\succcurlyeq_L = \succcurlyeq_{\overline{L}}$ on $\mathcal{M}_{S\setminus B}[\psi_z] \cap \Psi_L$, meaning that $\hat{V}(S \setminus B) = 0$. By null-additivity, it yields that $\hat{V}(B) = 1$. Now, assume that $V_L(B) > 0$, and that $\mathcal{M}_{S\setminus B}[\psi_z]$ is not $\succcurlyeq_L$-indifferent for any $\psi_z \in \Psi_C$. Then, by CoR, $\succcurlyeq_L = \succcurlyeq_{\overline{L}}$ on $\mathcal{M}_B[\psi_z] \cap \Psi_L$, meaning that $\mathcal{M}_B[\psi_z] \cap \Psi_L$ is not $\succcurlyeq_{\overline{L}}$-indifferent. Thus, by Lemma 3.1, $\hat{V}(B) > 0$. Therefore, $V_L(B) > 0$ implies $\hat{V}(B) > 0$.

As a result, for any $B \in \Sigma(L)$, $V_L(B) > 0$ if and only if $\hat{V}(B) > 0$.

(b) Let $\mathcal{B} = \{B \in \Sigma(L) \setminus \{S, \emptyset\} \mid \hat{V}(S \setminus B) > 0\}$. Fix an arbitrary $B \in \mathcal{B}$. By (a), we have $V_L(S \setminus B) > 0$, which implies that $\mathcal{M}_{S\setminus B}[\psi_z]$ is not $\succcurlyeq_L$-indifferent for any $\psi_z \in \Psi_C$. Thus, by CoR, for any $\psi_z \in \Psi_C$, $\succcurlyeq_L = \succcurlyeq_{\overline{L}}$ on $\mathcal{M}_B[\psi_z] \cap \Psi_L$. Note that $(\mathcal{M}_B[\psi_z] \cap \Psi_L) \subseteq \Psi_{\overline{L}} = \Psi$, and that, by Claim 3.7, $\mathcal{M}_B[\psi_z] \cap \Psi_L$ is convex. Thus, by Lemma 3.1, $I_L(\psi) = \int_S \psi dV_L$ and $\hat{I}(\psi) = \int_S \psi d\hat{V}$ represent the same preference on $\mathcal{M}_B[\psi_z] \cap \Psi_L$. Therefore, for any $\psi_z \in \Psi_C$, there exists a continuous and strictly increasing function $\varphi_{B,z} : \hat{I}(\mathcal{M}_B[\psi_z] \cap \Psi_L) \to \mathbb{R}$ such that, for any $\psi \in \mathcal{M}_B[\psi_z] \cap \Psi_L$,

$$\int_S \psi dV_L = \varphi_{B,z}\left(\int_S \psi d\hat{V}\right), \tag{3.5}$$

where $\hat{I}(\mathcal{M}_B[\psi_z] \cap \Psi_L)$ denotes the range of $\hat{I}$ on $\mathcal{M}_B[\psi_z] \cap \Psi_L$.

Recall that, for any $\psi \in \psi$ and every $s \in S$, $\psi(s) \in X \subset \mathbb{R}$, and that $[-1, 1] \subseteq X$. Let $\mathbf{0}$ denote the constant function that assigns 0 to every $s \in S$. Then, for every $D \in \Sigma(L) \cap \mathscr{P}(B)$ and any $x \in \hat{I}(\mathcal{M}_B[\mathbf{0}] \cap \Psi_L)$, we have $x\mathbf{1}_D \in \mathcal{M}_B[\mathbf{0}] \cap \Psi_L$. Thus, by Eq (3.5) and comonotonic additivity, $\varphi_{B,0} : \hat{I}(\mathcal{M}_B[\mathbf{0}] \cap \Psi_L) \to \mathbb{R}_+$ satisfies: (d1) $\varphi_{B,0}(0) = 0$; (d2) for any $\alpha \in (0, 1)$ and any $x, y \in \hat{I}(\mathcal{M}_B[\mathbf{0}] \cap \Psi_L)$, $\alpha\varphi_{B,0}(x) + (1 - \alpha)\varphi_{B,0}(y) = \varphi_{B,0}(\alpha x + (1 - \alpha)y)$; and (d3) for every $D \in \Sigma(L) \cap \mathscr{P}(B)$, $V_L(D) = \varphi_{B,0}(\hat{V}(D))$.

To see that (d2) is satisfied, let $\psi, \psi' \in \mathcal{M}_B[\mathbf{0}] \cap \Psi_L$ be arbitrary comonotonic functions. Due to Claim 3.7, for any $\alpha \in (0, 1)$, we have $(\alpha\psi + (1-\alpha)\psi') \in \mathcal{M}_B[\mathbf{0}] \cap \Psi_L$. In addition, by comonotonic additivity, $I_L(\alpha\psi + (1 - \alpha)\psi') = \alpha I_L(\psi) + (1 - \alpha)I_L(\psi')$ and $\hat{I}(\alpha\psi + (1 - \alpha)\psi') = \alpha\hat{I}(\psi) + (1 - \alpha)\hat{I}(\psi')$. Thus, under Eq (3.5), it follows that $\alpha\varphi_{B,0}(\hat{I}(\psi)) + (1-\alpha)\varphi_{B,0}(\hat{I}(\psi')) = \varphi_{B,0}(\hat{I}(\alpha\psi + (1-\alpha)\psi')) = \varphi_{B,0}(\alpha\hat{I}(\psi) + (1-\alpha)\hat{I}(\psi'))$. Since ψ and ψ' are arbitrary, so are $\hat{I}(\psi)$ and $\hat{I}(\psi')$ in $\hat{I}(\mathcal{M}_B[\mathbf{0}] \cap \Psi_L)$. Thus, (d2) holds.

By (d1) and (d2), $\varphi_{B,0}$ is linear on $\hat{I}(\mathcal{M}_B[\mathbf{0}] \cap \Psi_L)$ and there exists a $\omega_L^B \in \mathbb{R}_{++}$ such that $\varphi_{B,0}(x) = \omega_L^B x$, where the strict positiveness is implied by (a).

Let $B, B' \in \mathcal{B}$ be arbitrary, and suppose that $\hat{I}(\mathcal{M}_B[\mathbf{0}] \cap \Psi_L) \subseteq \hat{I}(\mathcal{M}_{B'}[\mathbf{0}] \cap \Psi_L)$. By CoR, for any $\psi \in \mathcal{M}_B[\mathbf{0}] \cap \Psi_L$ and $\psi' \in \mathcal{M}_{B'}[\mathbf{0}] \cap \Psi_L$, $\psi \sim_L \psi'$ if and only if $\psi \sim_{\overline{L}} \psi'$. Thus, by Eq (3.5), for any $x \in \hat{I}(\mathcal{M}_B[\mathbf{0}] \cap \Psi_L)$, it holds that $\omega_L^B x = \varphi_{B,0}(x) = \varphi_{B',0}(x) = \omega_L^{B'} x$, which yields $\omega_L^B = \omega_L^{B'}$. As a result, by (d3), a unique $\omega_L \in \mathbb{R}_{++}$ exists such that, for any $B \in \mathcal{B}$, $V_L(B) = \omega_L \hat{V}(B)$. *Q.E.D.*

Claim 3.14. *Under Coarseness Aversion, (a) for any $(L, \tilde{L}) \in \mathscr{L}$ such that $L \neq \underline{L}$ and $\tilde{L} \neq \underline{L}$, $\omega_L \leq \omega_{\tilde{L}}$; and (b) for all $L \in \mathcal{L} \setminus \{\underline{L}\}$, $\omega_L \in (0, 1]$.*

Proof. (a) Let $(L, \tilde{L}) \in \mathscr{L}$ be arbitrary, and suppose $L \neq \underline{L}$ and $\tilde{L} \neq \underline{L}$.

Let I_L and $I_{\tilde{L}}$ denote the functionals on Ψ_L given by $I_L(\psi) = \int_S \psi dV_L$ and $I_{\tilde{L}}(\psi) = \int_S \psi dV_{\tilde{L}}$, respectively. By Lemma 3.1, I_L represents $\succcurlyeq_L$ on Ψ_L, and $I_{\tilde{L}}$ represents $\succcurlyeq_{\tilde{L}}$ on Ψ_L. Thus, by Coarseness Aversion, for all $\psi \in \Psi_L$, we have $I_L(\psi) \leq I_{\tilde{L}}(\psi)$.

Therefore, by Claim 3.3, for all $B \in \Sigma(L)$, $V_L(B) \leq V_{\tilde{L}}(B)$. Recall that V_L, $V_{\tilde{L}}$ and $\hat{V}$ are null-additive. Thus, by Claim 3.13-(b) implies that for all $B \in \Sigma(L)$, $V_L(B) = \omega_L \hat{V}(B)$ and $V_{\tilde{L}} = \omega_{\tilde{L}} \hat{V}(B)$ whenever $\hat{V}(S \setminus B) > 0$, where $\omega_L, \omega_{\tilde{L}} \in \mathbb{R}_{++}$. Under null-additivity, Claim 3.13-(b) also implies that $V_L(B) = V_{\tilde{L}}(B) = \hat{V}(B) = 1$ whenever $\hat{V}(S \setminus B) = 0$. As a result, $V_L(B) \leq V_{\tilde{L}}(B)$ holds for all $B \in \Sigma(L)$ if and only if $\omega_L \leq \omega_{\tilde{L}}$.

(b) Given $V_{\overline{L}} = \hat{V}$, Claim 3.13-(b) implies that $\omega_{\overline{L}} = 1$. Thus, (b) is a direct consequence of (a) since for all $L \in \mathcal{L} \setminus \{\underline{L}\}$, $(L, \overline{L}) \in \mathscr{L}$. *Q.E.D.*

Sufficiency. Let $\mathcal{Q}$ and Q be the functional and mapping given in Eq (3.3).

Under the uniform axioms, Certainty and CoI, $\mathcal{Q}$ represents $\{\succcurlyeq_L\}_{L \in \mathcal{L}}$ on Ψ with respect to V and $\{V_L\}_{L \in \mathcal{L}}$ fixed above.

For every $L \in \mathcal{L}$, let $\Omega(L) = \{E \in L \mid \hat{V}(E) > 0\}$. Then, Claim 3.13-(a) implies that $\Omega(L) = \Delta(L) = \{E \in L \mid V_L(E) > 0\}$. Thus, by null-additivity and Claim 3.11-(a), $\Omega(L) \neq \emptyset$ and $\Omega(L) \subseteq \{E \in L \mid V(E) > 0\}$.

Under Claim 3.13-(a)(b), for any $L \in \mathcal{L}$, V_L and $\hat{V}$ satisfy the assumptions in Claim 3.4-(c). In particular, the assumptions are vacuous for $\underline{L}$. As a result, by Claim 3.4-(c), for any $L \in \mathcal{L}$ and $\psi \in \Psi$,

$$\begin{aligned}
\mathcal{Q}(\psi, L) =& \omega_L \int_{\bigcup \Omega(L)} Q(\psi, L) d\hat{V} + (1 - \omega_L) \min_{s \in \bigcup \Omega(L)} Q(\psi, L), \\
Q(\psi, L) =& \sum_{E \in \Omega(L)} \left(\int_E \psi dV \right) \frac{\mathbf{1}_E}{V(E)}.
\end{aligned} \tag{3.6}$$

Claim 3.14 concludes the rest of the proof. *Q.E.D.*

3.B.3.1. Proof of Corollary 3.3.1

Sufficiency. By Claim 3.1, SCoR implies Coarseness Aversion and CoR. Thus, by Theorem 3.3, $\{\succsim_L\}_{L\in\mathcal{L}}$ admits an ω-FAEU representation $(u, V, \hat{V}, \boldsymbol{\omega})$.

Hence, it suffices to show $\omega_L = 1$ for every $L \in \mathcal{L} \setminus \{\underline{L}\}$.

Fix an arbitrary $L \in \mathcal{L} \setminus \{\underline{L}\}$. Then, we have $\mathcal{H}_C \subseteq \mathcal{H}_L \subset \mathcal{H}_{\overline{L}} = \mathcal{H}$, and by SCoR, $\succsim_L = \succsim_{\overline{L}}$ on $\mathcal{H}_L \times \mathcal{H}_L$. Thus, for any $h \in \mathcal{H}_L$ and any $c \in \mathcal{H}_C$, $\mathcal{U}(h, L) = \mathcal{U}(c, L)$ if and only if $\mathcal{U}(h, \overline{L}) = \mathcal{U}(c, \overline{L})$. Let $u_c = u(c(s))$ on S. Then, we have $\mathcal{U}(c, L) = \mathcal{U}(c, \overline{L}) = u_c$, which implies $\mathcal{U}(h, L) = \mathcal{U}(h, \overline{L})$.

Note that, (i) for any $h \in \mathcal{H}_L$, $U(h, L)(s) = u(h(s))\mathbf{1}_{\bigcup \Omega(L)}$ and $U(h, \overline{L})(s) = u(h(s))\mathbf{1}_{\bigcup \Omega(\overline{L})}$ on S; (ii) $\Omega(L) = \{E \in L \mid \hat{V}(E) > 0\}$ and $\Omega(\overline{L}) = \{\{s\} \mid \hat{V}(\{s\}) > 0\}$; (iii) $\hat{V} : \mathscr{P}(S) \to [0, 1]$ is null-additive; and (iv) $\omega_{\overline{L}} = 1$. By (i), (ii) and (iii), for any $h \in \mathcal{H}_L$, it holds that $\int_{\bigcup \Omega(L)} U(h, L) d\hat{V} = \int_{\bigcup \Omega(\overline{L})} U(h, \overline{L}) d\hat{V} = \int_{\bigcup \Omega(L)} u(h) d\hat{V} = \int_S u(h) d\hat{V}$. Thus, under (iv), for any $h \in \mathcal{H}_L$, we have

$$\mathcal{U}(h, L) = \omega_L \int_{\bigcup \Omega(L)} u(h) d\hat{V} + (1 - \omega_L) \min_{s \in \bigcup \Omega(L)} u(h);$$
$$\mathcal{U}(h, \overline{L}) = \int_{\bigcup \Omega(L)} u(h) d\hat{V}.$$

As a result, $\mathcal{U}(h, L) = \mathcal{U}(h, \overline{L})$ yields that

$$(1 - \omega_L) \int_{\bigcup \Omega(L)} u(h) d\hat{V} = (1 - \omega_L) \min_{s \in \bigcup \Omega(L)} u(h(s)).$$

Clearly, $\int_{\bigcup \Omega(L)} u(h) d\hat{V} \geq \min_{s \in \bigcup \Omega(L)} u(h(s))$. Since the above equality holds for any $h \in \mathcal{H}_L$, it thus follows that $\omega_L = 1$.

The proof is complete as $L \in \mathcal{L} \setminus \{\underline{L}\}$ is assumed to be arbitrary. *Q.E.D.*

Necessity. Assume statement (II).

By Theorem 3.3, the uniform axioms and CoI are satisfied.

(SCoR) Since $\hat{V} : \mathscr{P}(S) \to [0, 1]$ is null-additive, by Claim 3.4-(a), for any $L \in \mathcal{L}$ and any $h \in \mathcal{H}_L$, we have $\int_{\bigcup \Omega(L)} u(h) d\hat{V} = \int_S u(h) d\hat{V}$. This implies that $\mathcal{U}^{SCoR}(h, L) = \mathcal{U}^{SCoR}(h, \overline{L})$. Fix a $(L, \tilde{L}) \in \mathscr{L}$. Then, we have $\mathcal{H}_L \subset \mathcal{H}_{\tilde{L}}$. Thus, it follows that $\mathcal{U}^{SCoR}(h, L) = \mathcal{U}^{SCoR}(h, \tilde{L}) = \mathcal{U}^{SCoR}(h, \overline{L})$. As a result, for any $(L, \tilde{L}) \in \mathscr{L}$ and any $f, g \in \mathcal{H}_L$, $f \succsim_L g$ if and only if $f \succsim_{\tilde{L}} g$. *Q.E.D.*

3.B.4. Proof of Theorem 3.4

Suppose $\{\succsim_L\}_{L\in\mathcal{L}}$ satisfies the axioms. By Claim 3.1-(b), CoI is also satisfied. By Theorem 3.2, $\{\succsim_L\}_{L\in\mathcal{L}}$ admits an FAEU $(u, V, \{V_L\}_{L\in\mathcal{L}})$. Fix this tuple. Let $\hat{V}$ denote the null-additive capacity $V_{\overline{L}}$. For every $L \in \mathcal{L}$ and $E \in L$, let $\vartheta_L^E, \omega_L \in \mathbb{R}_{++}$ be the numbers respectively obtained in Claim 3.11-(b) and Claim 3.13-(b).

Claim 3.15. *Under Certainty, CoR and SCoI, the following statements hold.*

(a) For any $L \in \mathcal{L}$ and $E \in L$, $E \in \Delta(L)$ if and only if $V(E) > 0$.

(b) For any $L \in \mathcal{L}$, there exists a unique $\vartheta_L \in \mathbb{R}_{++}$ such that for any $E \in L$, $V_L(E) = \vartheta_L V(E)$.

(c) For any $B \in \mathscr{P}(S)$, $\hat{V}(B) = \vartheta_{\overline{L}} V(B)$ whenever $V(S \setminus B) > 0$.

Proof. Fix an arbitrary $L \in \mathcal{L} \setminus \{\underline{L}\}$. Then, $(\underline{L}, L) \in \mathscr{L}$.

(a) By Claim 3.11-(a), for any $E \in L$, $E \in \Delta(L)$ implies $V(E) > 0$. Hence, it suffices to show the opposite direction. Let $E \in L$, and suppose that $E \notin \Delta(L)$. Then, by Lemma 3.2, $\mathcal{M}_E[\psi_z]$ is $\succcurlyeq_L$-indifferent for any $\psi_z \in \Psi_C$. Thus, SCoI implies that $\mathcal{M}_E[\psi_z]$ is also $\succcurlyeq_{\underline{L}}$-indifferent for all $\psi_z \in \Psi_C$. By Nondegeneracy and Corollary 3.1.1-(I), it follows that $V(E) = 0$. As a result, $V(E) > 0$ implies $E \in \Delta(L)$.

(b) Let $E, E' \in L$. Fix an arbitrary $\psi_x \in \Psi_C$, and suppose that $\psi \in \mathcal{M}_E[\psi_x] \setminus \{\psi_x\}$ and $\psi' \in \mathcal{M}_{E'}[\psi_x] \setminus \{\psi_x\}$ satisfy $\psi \sim_{\underline{L}} \psi'$. Then, SCoI implies $\psi \sim_L \psi'$.

By Corollary 3.1.1-(I), $\psi \sim_{\underline{L}} \psi'$ if and only if $\int_S \psi dV = \int_S \psi' dV$, which yields

$$\begin{aligned} \int_S \psi dV = \int_S \psi' dV \iff & \int_E \psi dV + (1 - V(E))x = \int_{E'} \psi' dV + (1 - V(E'))x \\ \iff & \int_E \psi dV - V(E)x = \int_{E'} \psi' dV - V(E')x, \end{aligned}$$

where $x = \psi_x(s)$ on S. On the other hand, by Lemma 3.2, J_E and $J_{E'}$ represents $\succcurlyeq_L$ on $\mathcal{M}_E$ and $\mathcal{M}_{E'}$ respectively. By definition, $\Psi_C = \mathcal{M}_E \cap \mathcal{M}_{E'}$, and $J_E(\psi_z) = J_{E'}(\psi_z) = z$ for every $\psi_z \in \Psi_C$, where $z = \psi_z(s)$ on S. Thus, by transitivity, $\psi \sim_{\underline{L}} \psi'$ if and only if $J_E(\psi) = J_{E'}(\psi')$. Similarly, $J_E(\psi) = J_{E'}(\psi')$ yields

$$\int_E \psi \mu_L^E - \mu_L^E(E)x = \int_{E'} \psi' \mu_L^{E'} - \mu_L^{E'}(E')x.$$

Since (a) implies that there is no $E \in L$ satisfying $V(E) > 0$ and $E \notin \Delta(L)$, thus Claim 3.11-(b) holds for any $E \in L$. As a result, under Claim 3.11-(b), the above equations jointly imply that

$$\vartheta_L^E \left(\int_E \psi dV - V(E)x \right) = \vartheta_L^{E'} \left(\int_E \psi dV - V(E)x \right).$$

Hence, $\vartheta_L^E = \vartheta_L^{E'}$ as E, E', ψ_x, ψ and ψ' are chosen arbitrarily.

Let $\vartheta_L = \vartheta_L^E$ for an $E \in L$. Then, ϑ_L is the unique number satisfies the claim.

(c) By (a) and Claim 3.13-(a), it holds that for any $B \in \mathscr{P}(S)$, $\hat{V}(B) > 0$ if and only if $V(B) > 0$. By (b), there is a unique $\vartheta_L \in \mathbb{R}_{++}$ such that $V_L(E) = \vartheta_L V(E)$ holds for any $E \in L$. Thus, by Claim 3.13-(b), for any $E \in L$, if $V(S \setminus E) > 0$, then $V_L(E) = \omega_L \hat{V}(E) = \vartheta_L V(E)$. Fix such an $E \in L$, and let $\tilde{L} \in \mathcal{L}$ be a partition such that $E \in (L \cap \tilde{L})$. Similarly, for every such $\tilde{L}$, we have $V_{\tilde{L}}(E) = \omega_{\tilde{L}} \hat{V}(E) = \vartheta_{\tilde{L}} V(E)$. Given that $\omega_L, \omega_{\tilde{L}}, \vartheta_L, \vartheta_{\tilde{L}} \in \mathbb{R}_{++}$, we have $\hat{V}(E) = (\vartheta_L/\omega_L)V(E) = (\vartheta_{\tilde{L}}/\omega_{\tilde{L}})V(E)$, which implies $(\vartheta_L/\omega_L) = (\vartheta_{\tilde{L}}/\omega_{\tilde{L}})$. Thus, for any $L, \tilde{L} \in \mathcal{L}$, $L \cap \tilde{L} \neq \emptyset$ implies $(\vartheta_L/\omega_L) = (\vartheta_{\tilde{L}}/\omega_{\tilde{L}})$.

As a result, for any $L \in \mathcal{L} \setminus \{\underline{L}\}$, $(\vartheta_L/\omega_L) = (\vartheta_{\overline{L}}/\omega_{\overline{L}})$.

To see this is satisfied, let $(L_1, \ldots, L_n)$ be an n-sequence of partitions such that: (i) for any $i \in \{1, \ldots, n-1\}$, $(L_i, L_{i+1}) \in \mathscr{L}$ with $|L_{i+1}| - |L_i| = 1$; and (ii) $L_1 = \underline{L}$ and $L_n = \overline{L}$. Then, for any $i \in \{2, \ldots, n-1\}$, $L_i \cap L_{i+1} \neq \emptyset$. Thus, it holds that $(\vartheta_{L_i}/\omega_{L_i}) = (\vartheta_{L_{i+1}}/\omega_{L_{i+1}})$. As a result, for any $i \in \{2, \ldots, n-1\}$, $(\vartheta_{L_i}/\omega_{L_i}) = (\vartheta_{\overline{L}}/\omega_{\overline{L}})$. Clearly, $\mathscr{L}$ is a partial order over $\mathcal{L}$, and the partially ordered set $(\mathcal{L}, \mathscr{L})$ forms a lattice. Thus, $\mathcal{L}$ is the union of all such n-sequences, due to which, the aforementioned equality holds for every $L \in \mathcal{L} \setminus \{\underline{L}\}$.

By Claim 3.13-(b), $\omega_{\overline{L}} = 1$. Then, for any $L \in \mathcal{L} \setminus \{\underline{L}\}$, $(\vartheta_L/\omega_L) = \vartheta_{\overline{L}}$. As a result, for any $B \in \mathscr{P}(S)$, $\hat{V}(B) = \vartheta_{\overline{L}} V(B)$ whenever $V(S \setminus B) > 0$. *Q.E.D.*

Sufficiency. Let $\mathcal{Q}$ and Q be as in Eq (3.3). By Theorem 3.2, $\mathcal{Q}$ represents $\{\succcurlyeq_L\}_{L \in \mathcal{L}}$ on Ψ with respect to V and $\{V_L\}_{L \in \mathcal{L}}$ fixed above.

By Claim 3.15-(c), for any $B \in \mathscr{P}(S)$, $\hat{V}(B) > 0$ if and only if $V(B) > 0$. For every $L \in \mathcal{L}$, let $\Omega(L) = \{E \in L \mid \hat{V}(E) > 0\}$. Then, by Claim 3.13-(a) and Claim 3.15-(a), it follows that $\Omega(L) = \Delta(L) = \{E \in L \mid V(E) > 0\}$.

For all $L \in \mathcal{L} \setminus \{\underline{L}\}$, set $\vartheta_L = \omega_L \vartheta_{\overline{L}}$. Given the above observation, by Claim 3.13-(b) and Claim 3.15-(c), the following statement is satisfied:

$$(P): \quad \forall L \in \mathcal{L}, \forall B \in \Sigma(L),\ V(S \setminus B) > 0 \Longrightarrow V_L(B) = \vartheta_L V(B).$$

Thus, for every $L \in \mathcal{L} \setminus \{\underline{L}\}$, the defined ϑ_L coincides with the one obtained in Claim 3.15-(b). In particular, by Claim 3.11-(b) and Claim 3.15-(b), $\vartheta_{\underline{L}} = 1$ and $V_{\underline{L}}(B) = \vartheta_{\underline{L}} V(B)$ holds for all $B \in \Sigma(\underline{L})$. Hence, for any $L \in \mathcal{L}$, null-additive capacity V_L and V satisfy the assumptions of Claim 3.4-(b) with respect to ϑ_L.

Therefore, by Claim 3.4-(b), for all $L \in \mathcal{L}$ and $\psi \in \Psi$,

$$\mathcal{Q}(\psi, L) = \vartheta_L \int_{\bigcup \Omega(L)} \{Q(\psi, L) - q_L(\psi)\mathbf{1}_S\}\, dV + q_L(\psi),$$

where $q_L(\psi) = \min_{s \in \bigcup \Omega(L)} Q(\psi, L)(s)$.

Finally, property (P) proves statement (2) in Definition 3.4. *Q.E.D.*

Necessity. Suppose $\{\succsim_L\}_{L \in \mathcal{L}}$ admits an S-FAEU representation $(u, V, \boldsymbol{\vartheta})$. Let $\mathcal{U}^S$ be the functional.

For all $L \in \mathcal{L}$, let V_L be the null-additive capacity defined as in Definition 3.4-(2). Then, for every $L \in \mathcal{L}$, $\Omega(L) = \{E \in L \mid V_L(E) > 0\}$. Thus, by Claim 3.4, for all $L \in \mathcal{L}$ and $h \in \mathcal{H}$, it holds that $\mathcal{U}^S(h, L) = \int_{\bigcup \Omega(L)} U(h, L) dV_L$. Therefore, $\{\succsim_L\}_{L \in \mathcal{L}}$ admits the FAEU representation $(u, V, \{V_L\}_{L \in \mathcal{L}})$. As a result, Theorem 3.2 proves he uniform axioms and Certainty.

(CoR) Given Certainty, we only need to consider partitions in $\mathcal{L} \setminus \{\underline{L}\}$.

Let $(L, \tilde{L}) \in \mathscr{L}$ be arbitrary such that $L \neq \underline{L}$ and $\tilde{L} \neq \underline{L}$, and let V_L and $V_{\tilde{L}}$ be the null-additive capacities defined above. Then, by Claim 3.4-(a), for all $f \in \mathcal{H}_L$, $\mathcal{U}^S(h, L) = \int_S u(h) dV_L$ and $\mathcal{U}^S(h, \tilde{L}) = \int_S u(h) dV_{\tilde{L}}$. Set $\beta = \vartheta_L / \vartheta_{\tilde{L}}$. Then, by Definition 3.4-(2), for all $B \in \Sigma(L)$, $V_{\tilde{L}}(S \setminus D) > 0$ implies $V_L(D) = \beta V_{\tilde{L}}(D)$.

The rest part is exactly same as that in the necessity of CoR for Theorem 3.3.

(SCoI) Let $(L, \tilde{L}) \in \mathscr{L}$, $E, E' \in \tilde{L}$, and $c \in \mathcal{H}_C$ be arbitrary. Given these, fix an $f \in \mathcal{C}_E^{\tilde{L}}[c]$ and a $g \in \mathcal{C}_{E'}^{\tilde{L}}[c]$ arbitrarily. Denote by D and D' the cells in L that satisfy $E \subseteq D$ and $E' \subseteq D'$. Let $u_c = u(c(s))$ on S.

Suppose that $E, E' \notin \Omega(\tilde{L})$. Then, since $\Omega(\tilde{L}) = \{E \in \tilde{L} \mid V(E) > 0\}$, it thus holds that $U(f, \tilde{L}) = U(g, \tilde{L}) = [(u \circ c)(\bigcup \Omega(\tilde{L}))\mathbf{0}]$ and $U(f, L) = U(g, L) = [(u \circ c)(\bigcup \Omega(L))\mathbf{0}]$. As a result, we have $\mathcal{U}^S(f, \tilde{L}) = \mathcal{U}^S(g, \tilde{L}) = \mathcal{U}^S(f, L) = \mathcal{U}^S(g, L) = u_c$, meaning that $f \succsim_L g$ if and only if $f \succsim_{\tilde{L}} g$.

Suppose that, wlog, $E \in \Omega(\tilde{L})$. Let $\tilde{m}(f) = \min_{s \in \bigcup \Omega(\tilde{L})} U(f, \tilde{L})$ and $m(f) = \min_{s \in \bigcup \Omega(L)} U(f, L)$. Given that $E \in \Omega(\tilde{L})$, exactly one of the following cases holds.

Case 1) $\Omega(\tilde{L}) = \{E\}$. Then, by the monotonicity of V, $\Omega(L) = \{D\}$. Then, it follows immediately that $\mathcal{U}^S(h, \tilde{L}) = \tilde{m}(f)$ and $\mathcal{U}^S(h, L) = m(f)$, where $\tilde{m}(f) = (1/V(E)) \int_E u(f) dV$ and $m(f) = (1/V(D))(\int_E u(f) dV + (V(D) - V(E))u_c)$. Since $f \in \mathcal{C}_E^{\tilde{L}}[c]$, it thus holds that $\int_E u(f) dV \geq V(E) u_c$. Hence, we have $\mathcal{U}^S(f, \tilde{L}) \geq u_c$ and $\mathcal{U}^S(f, L) \geq u_c$. Moreover, $E' \notin \Omega(\tilde{L})$ implies that $\mathcal{U}^S(g, \tilde{L}) = \mathcal{U}^S(g, L) = u_c$. As a result, we have $\mathcal{U}^S(f, \tilde{L}) \geq \mathcal{U}^S(g, \tilde{L})$ and $\mathcal{U}^S(f, L) \geq \mathcal{U}^S(g, L)$.

Case 2) $E' \notin \Omega(\tilde{L})$, $\Omega(L) = \{D\}$ and $\Omega(\tilde{L})$ is not singleton. Then, it still holds that $\mathcal{U}^S(g, \tilde{L}) = \mathcal{U}^S(g, L) = u_c$. As in Case 1, $\Omega(L) = \{D\}$ implies that $\mathcal{U}^S(f, L) = m(f) = (1/V(D))(\int_E u(f) dV + (V(D) - V(E))u_c)$. Similarly, we have $\mathcal{U}^S(f, L) \geq \mathcal{U}^S(g, L)$. Since $\Omega(\tilde{L})$ is not singleton and $f \in \mathcal{C}_E^{\tilde{L}}[c]$, thus $\tilde{m}(f) = u_c$, meaning that

$$\mathcal{U}^S(f, \tilde{L}) = \vartheta_{\tilde{L}} \int_E \{u(f) - u_c \mathbf{1}_S\} \, dV + u_c \ \geq \ u_c.$$

Therefore, it follows that $\mathcal{U}^S(f, \tilde{L}) \geq \mathcal{U}^S(g, \tilde{L})$ and $\mathcal{U}^S(f, L) \geq \mathcal{U}^S(g, L)$.

Case 3) $E' \notin \Omega(L)$ and neither $\Omega(L)$ nor $\Omega(\tilde{L})$ is singleton. Then, as in Case 2, $\mathcal{U}^S(g, \tilde{L}) = \mathcal{U}^S(g, L) = u_c$. Moreover, that $\Omega(L)$ and $\Omega(\tilde{L})$ are not singleton implies $\tilde{m}(f) = m(f) = u_c$. Thus, it yields that

$$\mathcal{U}^S(f, \tilde{L}) = \vartheta_{\tilde{L}} \int_E \{u(f) - u_c \mathbf{1}_S\} \, dV + u_c \ \geq \ u_c;$$
$$\mathcal{U}^S(f, L) = \vartheta_L \int_E \{u(f) - u_c \mathbf{1}_S\} \, dV + u_c \ \geq \ u_c.$$

As a result, we have $f \succsim_L g$ and $f \succsim_{\tilde{L}} g$.

Case 4) $E, E' \in \Omega(\tilde{L})$. Then, the above equations hold for both f and g. Thus, it follows that

$$\begin{aligned} f \succsim_L g &\iff \int_E \{u(f) - u_c \mathbf{1}_S\} \, dV \geq \int_{E'} \{u(g) - u_c \mathbf{1}_S\} \, dV \\ &\iff f \succsim_{\tilde{L}} g. \end{aligned}$$

As a result, it holds in all cases that $f \succsim_L g$ if and only if $f \succsim_{\tilde{L}} g$. Since $(L, \tilde{L}), E, E', c, f$ and g are arbitrary, SCoI is thus satisfied. *Q.E.D.*

3.B.4.1. Proof of Corollary 3.4.1

The proof will be conducted by showing both (I) and (II) are equivalent to (III). First, the following claim shows the equivalence between (II) and (III).

Claim 3.16. *Let $V : \mathscr{P}(S) \to [0,1]$ be a capacity and $\vartheta \in \mathbb{R}_{++}$ be a number such that defining a set function $\hat{V} : \mathscr{P}(S) \to [0,1]$ by $\hat{V}(B) = 1$ if $V(S \setminus B) = 0$, and $\hat{V}(B) = \vartheta V(B)$ otherwise, yields a null-additive capacity. Then, for any affine function u on $\Pi(A)$, (u, V, ϑ) is an S-FAEU representation if and only if $(u, V, \hat{V})$ is an ω-FAEU representation.*

Proof. Fix an arbitrary u, and let V, ϑ and $\hat{V}$ be as assumed.

Then, V, ϑ and $\hat{V}$ satisfies the assumptions of Claim 3.4-(b). Therefore, for any $L \in \mathcal{L}$ and $h \in \mathcal{H}$, it follows that

$$\int_{\bigcup \Omega(L)} U(h, L)\, \hat{V} = \vartheta \int_{\bigcup \Omega(L)} \left\{U(h, L) - \underline{u}_L^h \mathbf{1}_S\right\} dV + \underline{u}_L^h,$$

where $\underline{u}_L^h = \min_{s \in \bigcup \Omega(L)} U(h, L)$ and U is the mapping given in Definition 3.2. *Q.E.D.*

Now, we show the equivalence between (I) and (III). Assume statement (I). Then, by Claim 3.1, $\{\succsim_L\}_{L \in \mathcal{L}}$ satisfies all the axioms previously postulated. Due to this, the sufficiency parts will be based on all the notations and claims made previously.

By Theorem 3.2, $\{\succsim_L\}_{L \in \mathcal{L}}$ admits an FAEU representation $(u, V, \{V_L\}_{L \in \mathcal{L}})$. Fix this tuple, and let $\hat{V} = V_{\overline{L}}$. By Claim 3.15-(c), for any $B \in \mathscr{P}(S)$, $\hat{V}(B) > 0$ if and only if $V(B)$. Let $\Omega(L) = \{E \in L \mid \hat{V}(E) > 0\}$ and $\Delta(L) = \{E \in L \mid V_L(E) > 0\}$. Then, Claim 3.13-(a) and Claim 3.15-(a) jointly imply that $\Omega(L) = \Delta(L) = \{E \in L \mid V(E) > 0\}$. For every $L \in \mathcal{L}$, let $\omega_L, \vartheta_L \in \mathbb{R}_{++}$ be as in Claim 3.11-(b) and Claim 3.15-(b). Then, under Corollary 3.3.1, the following claim concludes the sufficiency of (III).

Claim 3.17. *Under SCoR and SCoI, there exists a $\vartheta \in \mathbb{R}_{++}$ such that for all $L \in \mathcal{L}$ and any $B \in \Sigma(L) \setminus \{S\}$, $V_L(B) = \vartheta V(B)$.*

Proof. As shown in Corollary 3.3.1, SCoR implies that $\omega_L = 1$ holds for all $L \in \mathcal{L} \setminus \{\underline{L}\}$. Hence, for any $L \in \mathcal{L}$, $V_L : \Sigma(L) \to [0,1]$ is the restriction of $\hat{V}$ on $\Sigma(L)$.

Let $L, \tilde{L} \in \mathcal{L}$ be such that $L \cap \tilde{L} \neq \emptyset$. Then, by Claim 3.15-(b) and the above statement, for any $E \in L \cap \tilde{L}$, it holds that $\hat{V}(E) = \vartheta_L V(E) = \vartheta_{\tilde{L}} V(E)$. Since $\vartheta_L, \vartheta_{\tilde{L}} \in \mathbb{R}_{++}$, we have $\vartheta_L = \vartheta_{\tilde{L}}$. Thus, for any $L, \tilde{L} \in \mathcal{L}$, $\vartheta_L = \vartheta_{\tilde{L}}$ whenever $L \cap \tilde{L} \neq \emptyset$.

As shown in the proof of Claim 3.15-(c), this implies that $\vartheta_L = \vartheta_{\overline{L}}$ holds for any $L \in \mathcal{L} \setminus \{\underline{L}\}$.

Under Claim 3.15-(c), it now suffices to show that $\hat{V}(B) = \vartheta_{\overline{L}} V(B)$ also holds for $B \in \mathscr{P}(S) \setminus \{S\}$ that satisfies $V(S \setminus B) = 0$. Let $B \in \mathscr{P}(S) \setminus \{S\}$ be such that $V(S \setminus B) = 0$. Then, there exists an $\hat{L} \in \mathcal{L} \setminus \{\underline{L}\}$ such that $B \in \hat{L}$, and by Claim 3.15-(b), we have $\hat{V}(B) = \vartheta_{\hat{L}} V(B)$. Thus, $\vartheta_{\hat{L}} = \vartheta_{\overline{L}}$ implies $\hat{V}(B) = \vartheta_{\overline{L}} V(B)$.

Therefore, in conjunction with Claim 3.15-(c), it holds for every $B \in \mathscr{P}(S) \setminus \{S\}$ that $\hat{V}(B) = \vartheta_{\overline{L}} V(B)$. Note that B has to be a proper subset of S because no $L \in \mathcal{L} \setminus \{\underline{L}\}$ includes S as a cell. Let $\vartheta = \vartheta_{\overline{L}}$. Then, for all $L \in \mathcal{L} \setminus \{L\}$ and $B \in \Sigma(L) \setminus S$, $V_L(B) = \vartheta V(B)$. *Q.E.D.*

Necessity of (III). Assume statement (III). The uniform axioms and SCoR are implied by Corollary 3.3.1, while under Claim 3.16, SCoI follows. *Q.E.D.*

3.C. Proofs for Section 3.5

3.C.1. Proof of Proposition 3.1

Proof. Suppose $\{\succsim_L\}_{L\in\mathcal{L}}$ admits an FAEU representation $(u, V, \{V_L\}_{L\in\mathcal{L}})$.

Fix an arbitrary $(L, D) \in \mathcal{R}$, and let $E \in L$ be the cell such that $D \subsetneq E$. Then, for every $h \in \mathcal{C}_D$, it holds that

$$\begin{aligned}
\mathcal{U}(h, L) &= \left(\frac{\int_E u(h)dV}{V(E)}\right) V_L(E) + (1 - V_L(E)) \min_{s\in S} u(h) \\
&= \left(\frac{\int_D u(h)dV}{V(E)} + \left(1 - \frac{V(D)}{V(E)}\right) \min_{s\in S} u(h)\right) V_L(E) + (1 - V_L(E)) \min_{s\in S} u(h) \\
&= \beta \left(\frac{\int_D u(h)dV}{V(D)}\right) + (1 - \beta) \min_{s\in S} u(h),
\end{aligned}$$

where $\beta = (V(D)V_L(E))/V(E)$. By the monotonicity of V and V_L, we have $\beta \in (0, 1)$. Define $V_D : \mathscr{P}(D) \to [0, 1]$ by $V_D(B) = V(B)/V(D)$ for every $B \in \mathscr{P}(D)$. Then, V_D forms a capacity on $\mathscr{P}(D)$ as V is a capacity on $\mathscr{P}(S)$. Thus, for any $f \in \mathcal{C}_D$, it follows that

$$\mathcal{U}(h, L) = \beta \left(\int_D u(h)dV_D\right) + (1 - \beta) \min_{s\in S} u(h).$$

For every $h \in \mathcal{H}$, let I_D be the functional given by $I_D(h) = \int_D u(h)dV_D$.

Let $f, g \in \mathcal{C}_D$ be such that $f \sim_L g$, and given an $\alpha \in (0, 1)$, let $l^\alpha = \alpha f + (1 - \alpha)g$. Simply let $\underline{u}^f = \min_{s\in S} u(f)$ (resp., $\underline{u}^g$ and $\underline{u}^{l^\alpha}$). By Claim 3.7, $\mathcal{C}_D$ is convex, meaning that $l^\alpha \in \mathcal{C}_D$ for all $\alpha \in (0,1)$. Thus, for all $\alpha \in (0,1)$, it yields that

$$\begin{aligned}\mathcal{U}(l^\alpha, L) &= \beta I_D(l^\alpha) + (1-\beta)\underline{u}^{l^\alpha} \\ &= \beta I_D(l^\alpha) + (1-\beta)\left(\alpha \underline{u}^f + (1-\alpha)\underline{u}^g\right).\end{aligned}$$

The second line is implied by the definition of $\mathcal{C}_D$. That is, for any $h \in \mathcal{C}_D$ and $s \in S \setminus D$, $\underline{u}^h = u(h(s))$. Moreover, $\mathcal{U}(f,L) = \mathcal{U}(g,L)$ implies that $(1-\beta)\left(\underline{u}^f - \underline{u}^g\right) = \beta\left(I_D(g) - I_D(f)\right)$. Hence, we have

$$\mathcal{U}(l^\alpha, L) - \mathcal{U}(f, L) = \beta\Big(I_D(l^\alpha) - \alpha I_D(f) - (1-\alpha) I_D(g)\Big).$$

Since $\beta > 0$, by definition, $\succsim_L$ reveals uncertainty aversion (resp., loving) on $\mathcal{C}_D$ if and only if for any $f, g \in \mathcal{C}_D$ and $\alpha \in (0,1)$, $f \sim_L g$ implies $I_D(l^\alpha) \geq \alpha I_D(f) + (1-\alpha) I_D(g)$ (resp., $\leq$). Therefore, by Schmeidler's proposition (Schmeidler (1989, .pp 582-583)), $\succsim_L \subset \mathcal{C}_D \times \mathcal{C}_D$ reveals uncertainty aversion (resp., loving) if and only if V_D is convex (resp., concave) on $\mathscr{P}(D)$.

Since V and V_D differ only in positive multiplication on $\mathscr{P}(D)$ with respect to a constant multiplier $1/V(D)$, thus the convexity (resp., concavity) of V and V_D are equivalent on $\mathscr{P}(D)$. As a result, $\succsim_L \subset \mathcal{C}_D \times \mathcal{C}_D$ reveals uncertainty aversion (resp., loving) if and only if V is convex (resp., concave) on $\mathscr{P}(D)$.

Suppose that V is convex (resp., concave) on $\mathscr{P}(D)$. Then, for every $\tilde{D} \subseteq D$, V is convex (resp., concave) on $\mathscr{P}(\tilde{D})$. Let $(\tilde{L}, \tilde{D}) \in \mathcal{R}$ be arbitrary such that $\tilde{D} \subseteq D$. Let $\tilde{E} \in \tilde{L}$ be the cell that satisfies $\tilde{D} \subseteq \tilde{E}$. Then, for every $f \in \mathcal{C}_{\tilde{D}}$, it follows that

$$\mathcal{U}(f, \tilde{L}) = \tilde{\beta}\left(\int_{\tilde{D}} u(f) dV_{\tilde{D}}\right) + (1-\tilde{\beta}) \min_{s \in S} u(f),$$

where $\tilde{\beta} = (V(\tilde{D}) V_{\tilde{L}}(\tilde{E}))/V(\tilde{E}) > 0$, and $V_{\tilde{D}}(B) = V(B)/V(\tilde{E})$ for every $B \in \mathscr{P}(\tilde{D})$.

By assumption, $V_{\tilde{D}}$ is also convex (resp., concave) on $\mathscr{P}(\tilde{D})$. Thus, as shown in the proof for $\succsim_L$ on $\mathcal{C}_D$, $\succsim_{\tilde{L}} \subset \mathcal{C}_{\tilde{D}} \times \mathcal{C}_{\tilde{D}}$ reveals uncertainty aversion (resp., loving).

The proof is complete as $(L, D), (\tilde{L}, \tilde{D}) \in \mathcal{R}$ are assumed to be arbitrary. *Q.E.D.*

3.C.2. Proof of Proposition 3.2

Statement (I). Suppose $\{\succsim_L\}_{L \in \mathcal{L}}$ admits an ω-FAEU representation $(u, V, \hat{V}, \boldsymbol{\omega})$. For every $L \in \mathcal{L}$, let V_L be the capacity such that for all $D \in \Sigma(L) \setminus \{S\}$, $V_L(D) = \omega_L \hat{V}(D)$. Let $(L, B) \in \overline{\mathcal{R}}$ be arbitrary.

By Claim 3.12, for all $h \in \mathcal{C}_B \cap \mathcal{H}_L$, $\mathcal{U}^\omega(h, L) = \int_B u(h) dV_L + (1 - V_L(B)) u(\underline{h})$, where $\underline{h}$ is the constant act such that for all $t \in S \setminus B$, $\underline{h}(s) = h(t)$. Let I_B denote the functional $I_B(\cdot) = \int_B u(\cdot) dV_L$. Let $f, g \in \mathcal{C}_B \cap \mathcal{H}_L$ be arbitrary such that $f \sim_L g$. As in the previous proof, given an $\alpha \in (0,1)$, let $l^\alpha = \alpha f + (1-\alpha) g$. By construction, for all $\alpha \in (0,1)$, $l^\alpha \in \mathcal{C}_B \cap \mathcal{H}_L$. Thus, it follows that $\mathcal{U}^\omega(l^\alpha, L) - \mathcal{U}^\omega(f, L) = I_B(l^\alpha) - I_B(f) + (1 - V_L(B))\left(\underline{u}^{l^\alpha} - \underline{u}^f\right)$, where $\underline{u}^f = u(\underline{f})$. Notice that $\underline{u}^{l^\alpha} = \alpha \underline{u}^f + (1-\alpha)\underline{u}^g$.

Thus, $\underline{u}^{l^\alpha} - \underline{u}^f = (1-\alpha)(\underline{u}^g - \underline{u}^f)$. Moreover, $f \sim_L g$ implies $(1 - V_L(B))(\underline{u}^g - \underline{u}^f) = I_B(f) - I_B(f)$. Hence, it yields that

$$\mathcal{U}^\omega(l^\alpha, L) - \mathcal{U}^\omega(f, L) = I_B(l^\alpha) - \Big(\alpha I_B(f) + (1-\alpha) I_B(g)\Big).$$

Given this, by definition, $\succsim_L$ exhibits uncertainty aversion (resp., loving) on $\mathcal{C}_B \cap \mathcal{H}_L$ if and only if for all $f, g \in \mathcal{C}_B \cap \mathcal{H}_L$ and $\alpha \in (0,1)$, $I_B(l^\alpha) \geq \alpha I_B(f) + (1-\alpha) I_B(g)$ (resp., $\leq$). Therefore, the first statement is equivalent to that V_L is convex (resp., concave) on $\mathscr{P}(B) \cap \Sigma(L)$. Recall that $V_L(D) = \omega_L \hat{V}(D)$ holds for all $D \in \Sigma(L) \setminus \{S\}$. As a result, $\succsim_L$ exhibits uncertainty aversion (resp., loving) on $\mathcal{C}_B \cap \mathcal{H}_L$ if and only if $\hat{V}$ is convex (resp., concave) on $\mathscr{P}(B) \cap \Sigma(L)$.

Suppose $\succsim_L$ exhibits uncertainty aversion (resp., loving) on $\mathcal{C}_B \cap \mathcal{H}_L$. Then, as shown above, $\hat{V}$ is convex (resp., concave) on $\mathscr{P}(B) \cap \Sigma(L)$. Let $(\tilde{L}, \tilde{B}) \in \overline{\mathcal{R}}$ be arbitrary such that $\tilde{B} \subseteq B$ and there is a $K \subseteq (L \cap \tilde{L})$ satisfying $\tilde{B} = \bigcup K$. Then, it holds that $\mathscr{P}(\tilde{B}) \cap \Sigma(\tilde{L}) = \mathscr{P}(\tilde{B}) \cap \Sigma(L) \subseteq (\mathscr{P}(B) \cap \Sigma(L))$. Thus, $\hat{V}$ is convex (resp., concave) on $\mathscr{P}(\tilde{B}) \cap \Sigma(\tilde{L})$, and so is $V_{\tilde{L}}$. Therefore, as shown above, $\succsim_{\tilde{L}}$ exhibits uncertainty aversion (resp., loving) on $\mathcal{C}_{\tilde{B}} \cap \mathcal{H}_{\tilde{L}}$. *Q.E.D.*

Statement (II). Suppose $\{\succsim_L\}_{L \in \mathcal{L}}$ admits an S-FAEU representation $(u, V, \boldsymbol{\vartheta})$.

Let $(L, D) \in \mathcal{R}$ be arbitrary, and suppose that $\succsim_L$ reveals uncertainty aversion (resp., loving) on $\mathcal{C}_D$. Then, as shown in the proof of Proposition 3.1, V is convex (resp., concave) on $\mathscr{P}(D)$. Let $(\hat{L}, B) \in \overline{\mathcal{R}}$ be arbitrary such that $B \subseteq D$. Then, V is convex (resp., concave) on $\mathscr{P}(B)$. Given V and $\vartheta_{\hat{L}}$, define a capacity $V_{\hat{L}} : \Sigma(\hat{L}) \to [0,1]$ as in Definition 3.4-(2). Then, since V is convex (resp., concave) on $\mathscr{P}(B)$, $V_{\hat{L}}$ is convex (resp., concave) on $\mathscr{P}(B) \cap \Sigma(\hat{L})$. In addition, by Claim 3.4-(b), for all $h \in \mathcal{C}_B \cap \mathcal{H}_{\hat{L}}$, it holds that $\mathcal{U}^S(h, \hat{L}) = \int_S u(h) dV_{\hat{L}}$. Therefore, as shown in the proof of (I), $\succsim_{\hat{L}}$ reveals uncertainty aversion (resp., loving) on $\mathcal{C}_B \cap \mathcal{H}_{\hat{L}}$. *Q.E.D.*

3.C.3. Proof of Proposition 3.3

Proof. Suppose that $\{\succsim_L\}_{L \in \mathcal{L}}$ admits an ω-FAEU representation $(u, V, \hat{V})$, and that $\succsim_L$ and $\succsim_{\tilde{L}}$ reveal uncertainty aversion (resp., loving) on $\mathcal{F}$.

(I) When $\Sigma_{\mathcal{F}} \subseteq \Sigma(L)$, the statement is a consequence of SCoR.

(II) Suppose that $\Sigma(L) \subseteq \Sigma_{\mathcal{F}} \subseteq \Sigma(\tilde{L})$. Then, by (I), the statement holds for any $\hat{L} \in \mathcal{L}$ that satisfies $\Sigma_{\mathcal{F}} \subseteq \Sigma(\hat{L})$. The remaining part is a special case of (III) in which $\Sigma_{\mathcal{F}}$ and $\Sigma(\tilde{L})$ coincide.

(III) Suppose that $\Sigma(\tilde{L}) \subseteq \Sigma_{\mathcal{F}}$. Then, $\mathcal{H}_{\tilde{L}} \subseteq \mathcal{F}$. Since functional $I_{\tilde{L}}(h) = \int_S u(h) d\hat{V}$ represents $\succsim_{\tilde{L}}$ on $\mathcal{H}_{\tilde{L}}$, by Schmeidler's proposition, $\succsim_{\tilde{L}} \subset \mathcal{F} \times \mathcal{F}$ revealing uncertainty aversion (resp., loving) implies that $\hat{V}$ is convex (resp., concave) on $\Sigma(\tilde{L})$. Moreover,

$\Sigma(L) \subseteq \Sigma_{\mathcal{F}}$ implies that $\mathcal{C}_E \subset \mathcal{F}$ holds for every $E \in L$. By assumption, $\succsim_L$ reveals uncertainty aversion (resp., loving) on each $\mathcal{C}_E$. As shown in the proof of Proposition 3.1, $\succsim_L \subset \mathcal{F} \times \mathcal{F}$ revealing uncertainty aversion (resp., loving) implies that V is convex (resp., concave) on $\mathscr{S}(L)$, where $\mathscr{S}(L)$ is the semi-ring of S given by $\mathscr{S}(L) = \bigcup_{E \in L} \mathscr{P}(E)$.

Fix an arbitrary $\hat{L} \in \mathcal{L}$ such that $(L, \hat{L}), (\hat{L}, \tilde{L}) \in \mathscr{L}$. Then, it holds that $\mathscr{S}(\hat{L}) \subseteq \mathscr{S}(L)$. Thus, V and $\hat{V}$ are convex (resp., concave) on $\mathscr{S}(\hat{L})$ and $\Sigma(\hat{L})$, respectively.

Let $\mathcal{U}$ be the functional given in Corollary 3.3.1. Then, $\mathcal{U}(h, \hat{L}) = \int_S U(h, \hat{L}) d\hat{V}$ represents $\succcurlyeq_{\hat{L}}$ on $\mathcal{F}$. Let $f, g \in \mathcal{F}$ be arbitrary such that $f \sim_{\hat{L}} g$. Recall that V is convex (resp., concave) on $\mathscr{S}(\hat{L}) = \bigcup_{E \in \hat{L}} \mathscr{P}(E)$, and $\hat{V}$ is convex (resp., concave) on $\Sigma(\hat{L})$. Thus, for any $\alpha \in (0,1)$, it follows that

$$\begin{aligned}
\int_S U(\alpha f + (1-\alpha) g, \hat{L}) d\hat{V} &\geq \int_S \left(\alpha U(f, \hat{L}) + (1-\alpha) U(g, \hat{L})\right) d\hat{V} \quad \text{(resp., } \leq) \\
&\geq \alpha \int_S U(f, \hat{L}) d\hat{V} + (1-\alpha) \int_S U(g, \hat{L}) d\hat{V} \quad \text{(resp., } \leq).
\end{aligned}$$

As a result, for any $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, $f \sim_{\hat{L}} g$ implies $\alpha f + (1-\alpha) g \succsim_{\hat{L}} f$ (resp., $\precsim_{\hat{L}}$). Hence, for any $\hat{L} \in \mathcal{L}$ that satisfies $(L, \hat{L}), (\hat{L}, \tilde{L}) \in \mathscr{L}$, $\succsim_{\hat{L}}$ reveals uncertainty aversion (resp., loving) on $\mathcal{F}$. *Q.E.D.*

3.C.4. Proof of Proposition 3.4

3.C.4.1. Preliminaries for The Proof

We introduce some notations and claims which are essential for the proof.

Suppose that $\{\succsim_L\}_{L \in \mathcal{L}}$ admits an FAEU representation $(u, V, \{V_L\}_{L \in \mathcal{L}})$.

Let $L \in \mathcal{L}$ be arbitrary, and let $|L| = m$.

Fix an $f \in \mathcal{H}$. Given $\succsim_L$, let $p_L(f) = (E^i)_{i=1}^m$ be a permutation of L such that:

(A1) for every $i \in \{2, \ldots, m\}$, $(\int_{E^i} u(f) dV)/V(E^i) \geq (\int_{E^{i-1}} u(f) dV)/V(E^{i-1})$.

Fix this $p_L(f)$, and equip $\overline{L}$ with a lexicographic index $(((i,r))_{r=1}^{|E^i|})_{i=1}^m$ such that:

(A2) for every $i \in \{1, \ldots, m\}$, $\bigcup_{r=1}^{|E^i|} \{s\}^{(i,r)} = E^i$; and

(A3) for any $i \in \{1, \ldots, m\}$ and $r \in \{2, \ldots, |E^i|\}$, $u(f(t)) \leq u(f(t'))$ for $t \in \{s\}^{(i,r-1)}$ and $t' \in \{s\}^{(i,r)}$.

Let $o_L(f)$ denote a typical indexed partition that satisfies (A1)-(A3), i.e.,

$$o_L(f) = \left\{\{s\}^{(1,1)}, \ldots, \{s\}^{(k,|E^k|)}, \{s\}^{(k+1,1)}, \ldots, \{s\}^{(m,|E^m|)}\right\}.$$

Denote by $O_L(f)$ the set of all indexed partitions over $\overline{L}$ induced by f and $\succsim_L$ in the way satisfying (A1)-(A3).

Given $f \in \mathcal{H}$ and $o_L(f) \in O_L(f)$, let $v_L(f) = (v_L^1, \ldots, v_L^m)$ be the m-sequence such that for each $k \in \{1, \ldots, m-1\}$, $v_L^k = V_L(\bigcup_{i=k}^m E^i) - V_L(\bigcup_{i=k+1}^m E^i)$, and $v_L^m = V_L(E^m)$. For each $i \in \{1, \ldots, m\}$, let $v^{L,i}(f) = (v^{L,(i,1)}, \ldots, v^{L,(i,|E^i|)})$ be the $|E^i|$-sequence given by $v^{L,(i,r)} = (V(\bigcup_{j=r}^{|E^i|}\{s\}^{(i,j)}) - V(\bigcup_{j=r+1}^{|E^i|}\{s\}^{(i,j)}))/V(E^i)$ for each $r \in \{1,\ldots,|E^i|-1\}$, and $v^{L,(i,|E^i|)} = V(\{s\}^{(i,|E^i|)})/V(E^i)$. Define a function $\mathbf{v}_{o_L(f)} : \overline{L} \to [0,1]$ by

$$\mathbf{v}_{o_L(f)}(\{s\}^{(i,r)}) = v^{L,(i,r)}v_L^i.$$

Finally, given an $f \in \mathcal{H}$, define a function $\mathbf{u}^f : \overline{L} \to \mathbb{R}$ by $\mathbf{u}^f(\{s\}) = u(f(s))$.

Slightly abuse the notation and consider $\mathbf{v}_{o_L(f)}$ and $\mathbf{u}^f$ also to be vectors in n-dimensional Euclidean space. That is, suppose also that $\mathbf{u}^f, \mathbf{v}_{o_L(f)} \in \mathbb{R}^n$.

Let $\overline{\mathcal{H}}$ be the set of acts given by

$$\overline{\mathcal{H}} = \{h \in \mathcal{H} \mid \forall s,t \in S,\ [s \neq t \Longrightarrow u(h(s)) \neq u(h(t))]\}.$$

Then, for any $h \in \overline{\mathcal{H}}$, the followings hold: (i) for every $L \in \mathcal{L}$, $h \notin \mathcal{H}_L$ unless $L = \overline{L}$; and (ii) For $\overline{L}$, $o_{\overline{L}}(h)$ uniquely exists. Thus, whenever $\overline{L}$ and $h \in \overline{\mathcal{H}}$ are considered, we simply write $\mathbf{v}_h$ instead of $\mathbf{v}_{o_{\overline{L}}(h)}$.

Given $L \in \mathcal{L}$, $h \in \overline{\mathcal{H}}$ and $o_L(h) \in O_L(h)$, define $\mathcal{F}[h \,|\, o_L(h)]$ and $C[h \,|\, o_L(h)]$ by

$$\mathcal{F}[h \,|\, o_L(h)] = \left\{ f \in \mathcal{H} \;\middle|\; \begin{array}{c} [\forall s,t \in S,\ u(h(s)) \geq u(h(t)) \Rightarrow u(f(s)) \geq u(f(t))] \\ \wedge\ [o_L(h) \in O_L(f)] \end{array} \right\};$$

$$C[h \,|\, o_L(h)] = \left\{ \mathbf{u}^f \;\mid\; f \in \mathcal{F}[h \,|\, o_L(h)] \right\}.$$

Claim 3.18. *Suppose $\{\succsim_L\}_{L\in\mathcal{L}}$ admit an FAEU representation $(u, V, \{V_L\}_{L\in\mathcal{L}})$. Then, for every $L \in \mathcal{L}$, the following statements hold.*

(a) *For any $h \in \overline{\mathcal{H}}$ and $o_L(h) \in O_L(h)$, $C[h \,|\, o_L(h)]$ is a convex cone.*

(b) *$\succsim_L$ is as uncertainty averse as $\succsim_{\overline{L}}$ if and only if for any $h \in \overline{\mathcal{H}}$ and $o_L(h) \in O_L(h)$, $\mathbf{v}_h - \mathbf{v}_{o_L(h)} = \mathbf{0}$.*

Proof. Let $L \in \mathcal{L}$ be arbitrary.

(a) Fix $h \in \overline{\mathcal{H}}$ and $o_L(h) \in O_L(h)$ arbitrarily.

Clearly, $(\mathcal{H}_C \cup \{h\}) \subset \mathcal{F}[h \,|\, o_L(h)]$. Hence, $\{\mathbf{0}\} \subset C[h \,|\, o_L(h)]$. Let $f \in \mathcal{F}[h \,|\, o_L(h)]$ be such that $h \neq f$. Let g be an act such that for every $s \in S$, $u(g(s)) = \alpha u(h(s)) + \beta u(h(f))$, where $\alpha \geq 0$ and $\beta \geq 0$.

Given that $o_{\overline{L}}$ is the unique for h, it thus yields the unique index system that satisfies (A1)-(A3) for h and f simultaneously. Hence, by the definition of $\mathcal{F}[h \,|\, o_L(h)]$, for all $s,t \in S$, $u(h(s)) \geq u(h(t))$ implies $u(g(s)) \geq u(g(t))$. This implies that f, g and h are pairwise comonotonic on each $E^i \in L$. Hence, by (A1) and comonotonicity, for every $i \in \{1,\ldots,m-1\}$, it follows further that

$$\begin{aligned}
\frac{\int_{E^i} u(g)dV}{V(E^i)} &= \frac{\int_{E^i} \alpha u(f) + \beta u(h)dV}{V(E^i)} \\
&= \frac{\alpha \int_{E^i} u(f)dV}{V(E^i)} + \frac{\beta \int_{E^i} u(h)dV}{V(E^i)} \\
&\leq \frac{\alpha \int_{E^{i+1}} u(f)dV}{V(E^{i+1})} + \frac{\beta \int_{E^{i+1}} u(h)dV}{V(E^{i+1})} = \frac{\int_{E^{i+1}} u(g)dV}{V(E^{i+1})}.
\end{aligned}$$

This implies $o_L(h) \in O_L(g)$.

Therefore, we have $g \in \mathcal{F}[h \,|\, o_L(h)]$, meaning that $\mathbf{u}^g \in C[h \,|\, o_L(h)]$. As a result, $C[h \,|\, o_L(h)]$ is a convex cone in $\mathbb{R}^n$.

(b) Clearly, for any $c \in \mathcal{H}_C$, $\mathcal{U}(c, L) = \mathcal{U}(c, \overline{L})$. Thus, $\succsim_L$ is as uncertainty averse as $\succsim_{\overline{L}}$ if and only if for any $h \in \overline{\mathcal{H}}$ and $o_L(h) \in O_L(h)$, and for any $f \in \mathcal{F}[h \,|\, o_L(h)]$, $\mathcal{U}(f, \overline{L}) = \mathcal{U}(f, L)$.

Fix $h \in \overline{\mathcal{H}}$ and $o_L(h) \in O_L(h)$ arbitrarily. Then, for any $f \in \mathcal{F}[h \,|\, o_L(h)]$, we have

$$\mathcal{U}(f, L) = \sum_{\{s\} \in \overline{L}} \mathbf{u}^f(\{s\}) \mathbf{v}_{o_L(f)}(\{s\}) = \sum_{\{s\} \in \overline{L}} \mathbf{u}^f(\{s\}) \mathbf{v}_{o_L(h)}(\{s\}).$$

The second equality follows the definition of $\mathcal{F}[h \,|\, o_L(f)]$. That is, function $\mathbf{v}_{o_L(\cdot)}$ solely depend on the indices given by $o_L(\cdot)$, rather than the specific act by which the indexed partition $o_L(\cdot)$ is induced. Thus, given that $o_L(h) \in O_L(f)$ for all $f \in \mathcal{F}[h \,|\, o_L(h)]$, the second equality holds.

Therefore, for any $f \in \mathcal{F}[h \,|\, o_L(h)]$, it follows that

$$\mathcal{U}(f, \overline{L}) - \mathcal{U}(f, L) = \sum_{\{s\} \in \overline{L}} \mathbf{u}^f(\{s\}) \left(\mathbf{v}_h(\{s\}) - \mathbf{v}_{o_L(f)}(\{s\}) \right) = \mathbf{u}^f \cdot \left(\mathbf{v}_h - \mathbf{v}_{o_L(h)} \right).$$

Consequently, the followings are equivalent:

(i) $\succsim_L$ is as uncertainty averse as $\succsim'_{\tilde{L}}$.

(ii) $\forall h \in \overline{\mathcal{H}}, \forall o_L(h) \in O_L(h), \forall \mathbf{u} \in C[h \,|\, o_L(h)], \quad \mathbf{u} \cdot (\mathbf{v}_h - \mathbf{v}_{o_L(h)}) = 0.$

By (a), for any $h \in \overline{\mathcal{H}}$ and $o_L(h) \in O_L(h)$, $C[h \,|\, o_L(h)]$ is a convex cone such that $\{\mathbf{0}\} \subset C[h \,|\, o_L(h)]$. This implies that $\mathbf{v}_h - \mathbf{v}_{o_L(h)}$ lies in both the dual and polar cone of $C[h \,|\, o_L(h)]$. Therefore, statement (ii) holds if and only if for all $h \in \overline{\mathcal{H}}$ and $o_L(h) \in O_L(h)$, $\mathbf{v}_h - \mathbf{v}_{o_L(h)} = \mathbf{0}$. *Q.E.D.*

3.C.4.2. Proof of The Proposition

Let $\{\succsim_L\}_{L \in \mathcal{L}}$ be an arbitrary FAEU preference, and let $(u, V, \{V_L\}_{L \in \mathcal{L}})$ denote the representation. Fix this tuple. The proof will be concluded by showing both (I) and (III) are equivalent to (II).

(II) $\Leftrightarrow$ (III). Let u and V be as fixed, and let P and ϑ be as in (II).

By Claim 3.16, $\{\succsim_L\}_{L \in \mathcal{L}}$ admits the ω-FAEU representation (u, V, P) if and only if it admits the S-FAEU representation (u, V, ϑ).[11]

[11]For the fixed tuple $(u, V, \{V_L\}_{L \in \mathcal{L}})$, recall that we assumed, for all $L \in \mathcal{L}$, $(S, \Sigma(L))$ does not have any nonempty V_L-null event. By the definition of FAEU, this implies that $(S, \mathscr{P}(S))$ does not have any nonempty V-null event. Hence, given that $\{\succsim_L\}_{L \in \mathcal{L}}$ admits the representation $(u, V, \{V_L\}_{L \in \mathcal{L}})$, if $\{\succsim_L\}_{L \in \mathcal{L}}$ precisely admits the representation (u, V, P) as in (II), then $(S, \mathscr{P}(S))$ does not have any nonempty P-null or V-null event. Therefore, the following statements are equivalent: (i) for all $B \in \mathscr{P}(S) \setminus \{S\}$, $P(B) = \vartheta V(B)$; and (ii) for all $B \in \mathscr{P}(S)$, $P(B) = 1$ if $V(S \setminus B) = 0$, and

Therefore, by Theorem 3.4 and Corollary 3.4.1, $\{\succsim_L\}$ satisfies all the axioms given previously, in which Richness is trivial. Notice that P is a probability measure on $\mathscr{P}(S)$. Thus, for all $L \in \mathcal{L}$, $\succsim_L$ satisfies independence on $\mathcal{H}_L$. *Q.E.D.*

(I) $\Leftrightarrow$ (II). Let $\{\succsim_L\}_{L\in\mathcal{L}}$ and $(u, V, \{V_L\}_{L\in\mathcal{L}})$ be as fixed, and let $\hat{V}$ denote $V_{\overline{L}}$. Invoke the notations given in Appendix 3.C.4.1.

Define the following properties of $\{\succsim_L\}_{L\in\mathcal{L}}$ and $(u, V, \{V_L\}_{L\in\mathcal{L}})$:

(P1) *For every $L \in \mathcal{L} \setminus \{\underline{L}\}$, $\succsim_L$ is as uncertainty averse as $\succsim_{\overline{L}}$.*

(P2) *For any $L \in \mathcal{L} \setminus \{\underline{L}\}$, $h \in \overline{\mathcal{H}}$ and $o_L(h) \in O_L(h)$, $\mathbf{v}_h - \mathbf{v}_{o_L(h)} = \mathbf{0}$.*

Then, by definition, $\{\succsim_L\}_{L\in\mathcal{L}}$ satisfies statement (I) if and only if it satisfies (P1). In addition, by Claim 3.18-(b), $\{\succsim_L\}_{L\in\mathcal{L}}$ satisfies (P1) if and only if it satisfies (P2). As a result, statement (I) is equivalent to (P2).

Now, assume (P2).

For a fixed $L \in \mathcal{L} \setminus \{\underline{L}\}$, let $f \in \overline{\mathcal{H}}$ be an act that satisfies the following condition:

$$\exists o_L(f) \in O_L(f), \exists s^0 \in S,\ \{s \in S \mid u(f(s)) \geq u(f(s^0))\} \in \Sigma(L).$$

Then, for this f and $o_L(f)$, $\mathbf{v}_h - \mathbf{v}_{o_L(f)} = \mathbf{0}$ yields that $V_L(D_f) = \hat{V}(D_f)$, where $D_f = \{s \in S \mid u(f(s)) \geq u(f(s^0))\}$. Note that $L \in \mathcal{L} \setminus \{\underline{L}\}$ and such $f \in \overline{\mathcal{H}}$ are arbitrary. Thus, for all $L \in \mathcal{L} \setminus \{\underline{L}\}$ and $D \in \Sigma(L)$, $V_L(D) = \hat{V}(D)$. As a result, $\{\succsim_L\}_{L\in\mathcal{L}}$ admits the ω-FAEU representation $(u, V, \hat{V})$, where $\hat{V} = V_{\overline{L}}$.

Therefore, by (P2), for any nonempty $E \subset S$, any $s^* \in E$, $D \subset E$, and any $B, \overline{B} \subset S$, if $s^* \notin D$, $s^* \notin \overline{B}$, $E \cap B = \emptyset$ and $D \subseteq \overline{B}$, then it holds that

$$\frac{V(\{s^*\} \cup D) - V(D)}{V(E)} \left(\hat{V}(E \cup B) - \hat{V}(B)\right) = \hat{V}(\{s^*\} \cup \overline{B}) - \hat{V}(\overline{B}). \tag{3.7}$$

Note that $E \neq S$ since $L \neq \underline{L}$ in (P2), and S is not a cell for any $L \in \mathcal{L} \setminus \{\underline{L}\}$.

Set $D = \overline{B} = B = \emptyset$ in Eq (3.7). Then, for any $E \subset S$ and $s \in E$, it yields that

$$\frac{\hat{V}(\{s\})}{V(\{s\})} = \frac{\hat{V}(E)}{V(E)}.$$

For a given $s \in S$, let ϑ_s denote this ratio. Then, $\vartheta_s V(E) = \hat{V}(E)$ holds for any $E \subset S$ such that $s \in E$. Thus, for any $s, s' \in S$, it follows that $\vartheta_s = \vartheta_{s'}$. As a result, there is a unique $\vartheta \in \mathbb{R}_{++}$ such that for every $B \in \mathscr{P}(S) \setminus \{S\}$, $\vartheta V(B) = \hat{V}(B)$.

Set $D = B = \emptyset$. Then, for any $s \in S$ and any $E, D \subset S$ that satisfy $s \in E$ and $s \notin D$, Eq (3.7) yields that

$$\frac{\hat{V}(\{s\})}{\hat{V}(E)} = \frac{V(\{s\})}{V(E)} = \frac{\hat{V}(\{s\} \cup D) - \hat{V}(D)}{\hat{V}(E)},$$

where the first equality follows the previous equation. Hence, for any $s \in S$ and $D \subseteq S$, we have $\hat{V}(\{s\}) = \hat{V}(\{s\} \cup D) - \hat{V}(D)$ whenever $s \notin D$. By iteration, for any disjoint

$P(B) = \vartheta V(B)$ otherwise. This is because, under the non-nullity, $B = S$ is the unique event such that $V(S \setminus B) = 0$. As a result, P, V and ϑ satisfy the assumption of Claim 3.16.

$B, D \subset S$, it follows that $\hat{V}(B) = \hat{V}(B \cup D) - \hat{V}(D)$. Thus, $\hat{V} : \mathscr{P}(S) \to [0,1]$ is additive. That is, $\hat{V}$ is the probability measure P in (II).

As a result, (P2) implies statement (II).

Conversely, assume (II).

For every $L \in \mathcal{L}$, let P_L be the restriction of P on $\Sigma(L)$. Then, V and $\{P_L\}_{L\in\mathcal{L}}$ satisfy: (i) for every $L \in \mathcal{L} \setminus \{\underline{L}\}$, P_L is a probability measure; (ii) for every $B \in \mathscr{P}(S) \setminus \{S\}$, $P(B) = \vartheta V(B)$; and (iii) for any disjoint $B, D \subset S$, $B \cup D \neq S$ implies $V(B \cup D) - V(D) = V(B)$. Note that (iii) is implied by (i) and (ii).

Let $\mathcal{U}$ be the functional that represents $\{\succsim_L\}_{L\in\mathcal{L}}$, and let $L \in \mathcal{L} \setminus \{\underline{L}\}$ be arbitrary. Then, for any $f \in \mathcal{H}$ and $o_L(f) \in O_L(f)$, it holds that

$$\mathcal{U}(f, L) = \sum_{\{s\}^{(i,r)} \in \overline{L}} u(f(s^{(i,r)}))\mathbf{v}_{o_L(f)}(\{s\}^{(i,r)}),$$

where for every (i, r), $s^{(i,r)}$ denotes the unique state in $\{s\}^{(i,r)}$.

Moreover, by the definition of $\mathbf{v}_{o_L(f)}$, for any $\{s\}^{(k,j)} \in \overline{L}$, it follows that

$$\begin{aligned}
\mathbf{v}_{o_L(f)}(\{s\}^{(k,j)}) &= \frac{V(\{s\}^{(k,j)} \cup D) - V(D)}{V(E^k)} \Big(P_L(E^k \cup B) - P_L(B)\Big) \\
&= \frac{V(\{s\}^{(k,j)})P(E^k)}{V(E^k)} \\
&= P(\{s\}^{(k,j)}),
\end{aligned}$$

where $D = \bigcup_{r=j+1}^{|E^k|} \{s\}^{(k,r)}$ and $B = \bigcup_{i=k+1}^{n} E^i$. The second line is implied by (i) and (iii), while the last line follows (ii).

Therefore, under (i), for any $L \in \mathcal{L} \setminus \{\underline{L}\}$ and $f \in \mathcal{H}$,

$$\mathcal{U}(f, L) = \sum_{s\in S} u(f(s))P(\{s\}) = \int_S u(f)dP = \mathcal{U}(f, \overline{L}).$$

As a result, statement (II) implies (I). The proof is complete. *Q.E.D.*

3.C.5. Proof of Proposition 3.5

Fix an arbitrary $L \in \mathcal{L} \setminus \{\underline{L}\}$.

Statement (I). Let $\{\succsim_L\}_{L\in\mathcal{L}}$ and $\{\succsim'_L\}_{L\in\mathcal{L}}$ be FAEU preferences.

Denote by $(u, V, \{V_L\}_{L\in\mathcal{L}})$ and $(u', V', \{V'_L\}_{L\in\mathcal{L}})$ the representations, where $u = u'$ and $V = V'$. Let $\mathcal{U}$, U, $\mathcal{U}'$ and U' be the associated functionals and mappings.

Then, $u = u'$ and $V = V'$ jointly imply that $U(\cdot, L) = U'(\cdot, L)$ on $\mathcal{H}$. Thus, for any $f \in \mathcal{H} \setminus \mathcal{H}_C$, there exists an $h \in \mathcal{H}_L \setminus \mathcal{H}_C$ such that $f \sim_L h$ and $f \sim'_L h$. In addition, $u = u'$ implies that $\mathcal{U}(c, L) = \mathcal{U}'(c, L)$ holds for all $c \in \mathcal{H}_C$. Therefore, by definition, $\succsim_L$ is more uncertainty averse than $\succsim'_L$ if and only if for all $h \in \mathcal{H}_L$, $\mathcal{U}(h, L) \leq \mathcal{U}'(h, L)$. As a result, Claim 3.3 concludes the proof. *Q.E.D.*

STATEMENT (II). Let $\{\succsim_L\}_{L\in\mathcal{L}}$ and $\{\succsim'_L\}_{L\in\mathcal{L}}$ be as assumed in (II).

Let $\mathcal{U}^\omega$ and $\mathcal{U}^{\omega\prime}$ be the corresponding functionals. Then, as shown in (I), $\succsim_L$ is more uncertainty averse than $\succsim'_L$ if and only if for all $h \in \mathcal{H}_L$, $\mathcal{U}^{\omega\prime}(h, L) \geq \mathcal{U}^\omega(h, L)$.

Given $\hat{V}$ and ω_L, define a set function $V_L : \Sigma(L) \to [0, 1]$ as in Claim 3.12. Similarly, define V'_L using $\hat{V}'$ and ω'_L. Then, by the claim, for all $h \in \mathcal{H}_L$, $\mathcal{U}^\omega = \int_S u(h)\, dV_L$ and $\mathcal{U}^{\omega\prime}(h, L) = \int_S u(h)\, dV'_L$. Therefore, by Claim 3.3, $\mathcal{U}^{\omega\prime}(h, L) \geq \mathcal{U}^\omega(h, L)$ holds for all $h \in \mathcal{H}_L$ if and only if for all $B \in \Sigma(L)$, $V'_L(B) \geq V_L(B)$.

By the definition given in Claim 3.12, the latter statement is equivalent to that for all $B \in \Sigma(L) \setminus \{S\}$, $\omega'_L\hat{V}'(B) \geq \omega_L\hat{V}(B)$. Consequently, (II) holds.

Now, assume that $\hat{V} = \hat{V}'$. Then, for all $B \in \Sigma(L) \setminus \{S\}$, $\omega'_L\hat{V}'(B) \geq \omega_L\hat{V}(B)$ if and only if $\omega'_L \geq \omega_L$. *Q.E.D.*

STATEMENT (III). Suppose $\{\succsim_L\}_{L\in\mathcal{L}}$ and $\{\succsim'_L\}_{L\in\mathcal{L}}$ are S-FAEU preferences.

Let $\mathcal{U}^S$ and $\mathcal{U}^{S\prime}$ be the corresponding functionals. Given $u = u'$ and $V = V'$, as shown in (I), $\succsim_L$ is more uncertainty averse than $\succsim'_L$ if and only if for all $h \in \mathcal{H}_L$, $\mathcal{U}^S(h, L) \leq \mathcal{U}^{S\prime}(h, L)$.

For all $h \in \mathcal{H}_L$, it holds that

$$\mathcal{U}^{S\prime}(h, L) - \mathcal{U}^S(h, L) = (\vartheta'_L - \vartheta_L) \int_S \left\{ u(h) - \left(\min_{s\in S} u(h(s)) \right) \mathbf{1}_S \right\} dV,$$

where the integral part is non-negative. Therefore, $\mathcal{U}^S(h, L) \leq \mathcal{U}^{S\prime}(h, L)$ holds for all $h \in \mathcal{H}_L$ if and only if $\vartheta_L \leq \vartheta'_L$. *Q.E.D.*

3.D. DETAILS FOR SECTION 3.6

3.D.1. Details of Section 3.6.1

<u>REFLECTION EXAMPLE</u>. Let U_L be as in Theorem 3.1, and denote by $\mathcal{U}_L$ the functional. Given v_L, assume $v_L(\{B\}) = v_L(\{Y\})$ and $v_L(\{R, B\}) = v_L(\{Y, G\})$. Given the tuple (u_L, v_L), let $u^0 = u_L(\$0)$, $u^1 = u_L(\$x)$, $u^2 = u_L(\$2x)$, $v = v_L(\{Y, G\})$ and $p = v_L(\{Y\})/v$. Then, assumption (i), (ii) and (iii) imply the followings:

(A1) $u^0 < u^1 < u^2$;

(A2) $u^1 > (1 - p)u^0 + pu^2$; and

(A3) $2vp \neq p$.

By the definitions of $\mathcal{U}_L$ and U_L, $v_L(\{R, B\}) = v_L(\{Y, G\})$ and $v_L(\{B\}) = v_L(\{Y\})$ jointly imply that $\mathcal{U}_L(f_1) \geq \mathcal{U}_L(f_2)$ if and only if $\mathcal{U}_L(f_3) \leq \mathcal{U}_L(f_4)$.

Thus, consider only f_1 and f_2. Then, it holds that $U_L(f_1) = ((1-p)u^1+pu^2)\mathbf{1}_{\{R,B\}}+ ((1-p)u^0+pu^1)\mathbf{1}_{\{Y,G\}}$, and that $U_L(f_2) = u^1\mathbf{1}_{\{R,B\}}+((1-p)u^0+pu^2)\mathbf{1}_{\{Y,G\}}$. By (A1), we have $(1-p)u^1 + pu^2 > (1-p)u^0 + pu^1$. Thus, under (A2), it follows that

$$\mathcal{U}_L(f_1) = (1-v)\left((1-p)u^0 + pu^1\right) + v\left((1-p)u^1 + pu^2\right);$$
$$\mathcal{U}_L(f_2) = (1-v)\left((1-p)u^0 + pu^2\right) + vu^1.$$

Therefore, it yields that $\mathcal{U}_L(f_1) - \mathcal{U}_L(f_2) = (2vp - p)(u^2 - u^1)$. As a result, by (A1) and (A3), $\mathcal{U}_L(f_1) \neq \mathcal{U}_L(f_2)$, and $2vp > p$ if and only if $\mathcal{U}_L(f_1) > \mathcal{U}_L(f_2)$.

<u>Slightly-Bent Coin</u>. Let U_L be as in Theorem 3.1, and denote by $\mathcal{U}_L$ the functional. Given u_L, let $u^- = u_L(-\$\,x)$, $u^0 = u_L(\$\,0)$, and $u^+ = u_L(+\$\,x)$. In addition, given v_L, define $v^H = v_L(\{HR, HB\})$, $v^T = v_L(\{TR, TB\})$, $p^{hr} = v_L(\{HR\})/v^H$, and $p^{tb} = v_L(\{TB\})/v^T$. Then, assumption (i), (ii) and (iii) imply the followings:

(A1) $u^- < u^0 < u^+$;
(A2) $(1-p^{tb})u^- + p^{tb}u^+ < u^0$; and
(A3) $v^H p^{hr} \neq (1-v^H)p^{tb}$, where $p^{tb} > 0$ and $p^{hr} > 0$.

By the definition of U_L, it holds that

$$U_L(h_1) = ((1-p^{hr})u^0 + p^{hr}u^+)\mathbf{1}_{\{HR,HB\}} + ((1-p^{tb})u^- + p^{tb}u^0)\mathbf{1}_{\{TR,TB\}};$$
$$U_L(h_2) = u^0\mathbf{1}_{\{HR,HB\}} + ((1-p^{tb})u^- + p^{tb}u^+)\mathbf{1}_{\{TR,TB\}}.$$

Given $p^{tb} > 0$, (A1) implies that $(1-p^{hr})u^0 + p^{hr}u^+ > (1-p^{tb})u^- + p^{tb}u^0$. On the other hand, for h_2, (A2) yields $u^0 > (1-p^{tb})u^- + p^{tb}u^+$. Thus, it follows that

$$\mathcal{U}_L(h_1) = v^H\left((1-p^{hr})u^0 + p^{hr}u^+\right) + (1-v^H)\left((1-p^{tb})u^- + p^{tb}u^0\right);$$
$$\mathcal{U}_L(h_2) = v^H u^0 + (1-v^H)\left((1-p^{tb})u^- + p^{tb}u^+\right).$$

As a result, we have $\mathcal{U}_L(h_1) - \mathcal{U}_L(h_2) = (v^H p^{hr} - (1-v^H)p^{tb})(u^+ - u^0)$. By (A1), $u^+ - u^0 > 0$. Therefore, (A3) implies that $\mathcal{U}_L(h_1) \neq \mathcal{U}_L(h_2)$. In particular, if $v^H p^{hr} > (1-v^H)p^{tb}$ (resp., $<$), then $\mathcal{U}_L(h_1) > \mathcal{U}_L(h_2)$ (resp., $<$).

3.D.2. Details of Section 3.6.2

Suppose fixed statistical knowledge is released in the form of $L = \{\{r\}, \{s,t\}\}$.

Let $\boldsymbol{x} = (x^1, x^2)$ be an interior feasible allocation such that $x^1_r = x^1_s = x^1_t = x$. We show that $\boldsymbol{x}$ cannot be Pareto-optimal under L (i.e., for $\succsim^1_L$ and $\succsim^2_L$).

Given the representation tuples, let $p = P(\{s,t\})$, and for each $i \in \{1,2\}$, let $v^i_t = V^i(\{t\})/V^i(\{s,t\})$. By construction, $x^2 = (e_r - x, e_s - x, e_t - x)$. By (A1), we have $x^2_r < x^2_s < x^2_t$. Thus, for $\boldsymbol{x}$, it follows that $\mathcal{U}^{\omega\,1}(x^1) = u^1(x)$, and that

$$\mathcal{U}^{\omega\,2}(x^2) = (1-\omega^2_L)x^2_r + \omega^2_L\left((1-p)x^2_r + (1-v^2_t)px^2_s + v^2_t px^2_t\right)$$
$$= (1-\omega^2_L p)x^2_r + \omega^2_L\Big((1-v^2_t)px^2_s + v^2_t px^2_t\Big).$$

Set $\beta = \omega_L^2 p/(1-\omega_L^2 p)$. By construction, we have $\beta > 0$. Let $\delta > 0$ be an arbitrary real number such that $\delta < \min\{(1+\beta)(e_s - e_r), x/\beta\}$. Let $\hat{\boldsymbol{x}} = (\hat{x}^1, \hat{x}^2)$ denote the allocation given by $\hat{x}^1 = (x - \beta\delta, x + \delta, x + \delta)$ and $\hat{x}^2 = (x_r^2 + \beta\delta, x_s^2 - \delta, x_t^2 - \delta)$.

For $\hat{x}^2$, $\delta < (1+\beta)(e_s - e_r)$ implies $\hat{x}_r^2 < \hat{x}_s^2 < \hat{x}_t^2$. Thus, it follows immediately that $\mathcal{U}^{\omega\,2}(\hat{x}^2) = \mathcal{U}^{\omega\,2}(x^2)$. Moreover, for $\hat{x}^1$, we have

$$\begin{aligned}\mathcal{U}^{\omega\,1}(\hat{x}^1) &= (1-\omega_L^1)u^1(x-\beta\delta) + \omega_L^1\Big((1-p)u^1(x) + pu^1(x+\delta)\Big)\\ &= (1-\omega_L^1 p)u^1(x-\beta\delta) + \omega_L^1 p u^1(x+\delta).\end{aligned}$$

Set $\alpha = \omega_L^1 p$. Then, it yields that

$$\mathcal{U}^{\omega\,1}(\hat{x}^1) - \mathcal{U}^{\omega\,1}(x^1) = (1-\alpha)\Big(u^1(x-\beta\delta) - u^1(x)\Big) + \alpha\Big(u^1(x+\delta) - u^1(x)\Big).$$

By (A3), we have $u^1(x-\beta\delta) - u^1(x) > -\beta\delta u^{1\prime}(x)$ and $u^1(x+\delta) - u^1(x) > \delta u^{1\prime}(x)$. Thus, $\mathcal{U}^{\omega\,1}(\hat{x}^1) - \mathcal{U}^{\omega\,1}(x^1) > (\alpha - \beta + \alpha\beta)\delta u^{1\prime}(x)$. By construction, $\alpha - \beta + \alpha\beta = (\omega_L^1 - \omega_L^2)p/(1-\omega_L^2 p)$. Hence, (A4) implies that $\alpha - \beta + \alpha\beta > 0$. Therefore, under (A1)-(A4), $\mathcal{U}^{\omega\,1}(\hat{x}^1) - \mathcal{U}^{\omega\,1}(x^1) > (\alpha - \beta + \alpha\beta)\delta u^{1\prime}(x) > 0$.

As a result, $\boldsymbol{x}$ is not Pareto-optimal.

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