KAON PION SCATTERING

by

Mazharul Huq

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CONTENTS

CHAPTER I

INTRODUCTION

In processes involving pions such as pion-nucleon scattering, the pion-pion interaction plays a very important role.[1] Similarly, the kaon-pion interaction should take an important part in processes involving kaons. In particular, the low energy kaon-nucleon scattering is mainly determined by the two pion exchange and the hyperon (Λ, Σ) exchange.[2] The two pion exchange is described by a mechanism suggested by Barshay.[3] In conventional field theoretic language the interaction is said to take place through the Hamiltonian:

$$H_{K\pi} = 4\pi\lambda\,(K^{+}K)\,\phi_{\pi}^{2} \tag{1.1}$$

Attempts are made in this thesis to determine the low energy kaon-pion scattering amplitude. Since the interaction involved is strong in nature, the field theoretic approach runs into divergence difficulties. The programme based on unitarity, crossing and analyticity to determine the scattering amplitude is known as the S-matrix theory. Here, one does not encounter the divergence difficulties of the field theory and is concerned with only physically measurable quantities. The S-matrix theory approach is used in this thesis. A brief historical outline of the

S-matrix theory and its various aspects are described in this chapter. Since we are mainly concerned with partial wave dispersion relations various methods available for solving them are also discussed.

1.1. UNITARITY, CROSSING AND ANALYTICITY:

We consider a scattering process involving the four particles A, B, C and D with four-momenta p_1, p_2, p_3 and p_4 respectively as schematically represented by Fig. 1.1. It is a suitable convention

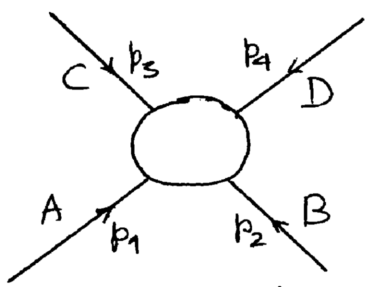


Fig 1.1.

to take incoming momenta as positive. By pairing the four particles - two incoming and two outgoing - the following three channels are defined:

$$\mathrm{I} \quad A(p_1) + B(p_2) \to C(-p_3) + D(-p_4)$$
$$\mathrm{II} \quad A(p_1) + \bar{C}(p_3) \to \bar{B}(-p_2) + D(-p_4)$$
$$\mathrm{III} \quad A(p_1) + \bar{D}(p_4) \to \bar{B}(-p_2) + C(-p_3)$$

Besides describing the process shown above, each channel also describes the PCT equivalent anti-particle process; as for example, channel I also describes the process $\bar{C}(p_3) + \bar{D}(p_4) \to \bar{A}(-p_1) + \bar{B}(-p_2)$ which is obtained from the particle process by changing the signs of all the four momenta. Whenever a particle is switched from incoming to outgoing it is to be replaced by the corresponding antiparticle and vice versa.

The four-momentum p_i is connected to the mass of the corresponding particle by:

$$p_i^2 = m_i^2 \tag{1.2}$$

Three scaler invariants can be defined from these four-momenta:

$$s = (p_1+p_2)^2 = (p_3+p_4)^2 \tag{1.3a}$$

$$u = (p_1+p_3)^2 = (p_2+p_4)^2 \tag{1.3b}$$

$$t = (p_1+p_4)^2 = (p_2+p_3)^2 \tag{1.3c}$$

It is easy to verify that s is positive time-like in channel I and is the square of the centre of mass energy there; and so are u and t in channels II and III respectively. While in each channel the remaining two variables are negative of the momentum transfer squared between the initial and the final states in the centre of mass system. So each variable plays the double role of energy in one channel and momentum transfer in the remaining channels. Conservation of four momenta imposes the constraint:

$$s+u+t = \sum_{i=1}^{4} m_i^2 \tag{1.4}$$

on the three scalar invariants, so only two out of the three are independent. This is as expected, because a two body scattering amplitude depends on two variables - the energy and the scattering angle. The latter is a function of the momentum transfer. In each channel, the suitable variables will be the centre of mass energy squared and the cosine of the scattering angle in the same system.

Taking channel I as an example, the physical region is defined by $s \geqslant s_{th}, -1 \leqslant \cos\theta \leqslant +1$; where s_{th} is the lowest physically possible value of the centre of mass energy squared, which is, of course, equal to the larger one of the two quantities $(m_1+m_2)^2$ and $(m_3+m_4)^2$. θ is the scattering angle in the centre of mass system. Similarly, the physical regions for channels II and III can be defined. In the space of the variables s, u and t these three physical regions are separated from each other by entirely unphysical regions.

<u>S-matrix</u>: A scattering process is described by the S-matrix which is an unitary operator mapping a set of incoming states on to a set of outgoing states. The $\underset{\sim}{T}$-matrix is defined by:

$$\underset{\sim}{S} = \underset{\sim}{1} + i\,\underset{\sim}{T} \tag{1.5}$$

Taking the matrix element between the final state f and the initial state i of both sides of the above equation:

$$\begin{aligned} S_{fi} \equiv \langle f | \underset{\sim}{S} | i \rangle &= \langle f | i \rangle + i \langle f | \underset{\sim}{T} | i \rangle \\ &= \delta_{fi} + i\,(2\pi)^4 \delta^{(4)}(p_f - p_i)\,T_{fi} \end{aligned} \tag{1.6}$$

where a factor $(2\pi)^4\,\delta^{(4)}(p_f - p_i)$ has been taken out in the definition of the matrix element T_{fi}. This takes into account the conservation of four-momenta. T_{fi}, in general, depends on the spins and isotopic spins of the particles taking part in the scattering process. It is always possible by taking out suitable kinematical, spin and isotopic-spin factors to express T_{fi} in terms of one or more scalar amplitudes. The number of amplitudes depends on the spin and isotopic-spin complications. In this chapter all such complications are ignored and so we are concerned with only one scalar amplitude. This is written as A. The differential cross section is connected to A by:

$$\frac{d\sigma}{d\Omega} = k \left| A \right|^2 \qquad (1.7)$$

where k is a kinematical factor.

<u>Lorentz invariance</u>: Any reasonable theory of elementary particles is expected to be invariant under Lorentz-transformations. In our case it is guaranteed by taking A to be a function of only scalar invariants. Since we are considering two body scattering process, A is a function of two variables. Any two of the set $\{s, u, t\}$ of scalar invariants may be chosen. Which pair, will depend on the channel under consideration. But when referred to in general, A will be written as $A(s,u,t)$.

Unitarity: Conservation of total probability requires that the S-matrix is unitary, that is;

$$\underset{\sim}{S}^{+}\underset{\sim}{S} = \underset{\sim}{S}\,\underset{\sim}{S}^{+} = \underset{\sim}{1} \qquad (1\cdot 8)$$

In terms of the $\underset{\sim}{T}$ operator this becomes:

$$\underset{\sim}{T} - \underset{\sim}{T}^{+} = i\,\underset{\sim}{T}^{+}\underset{\sim}{T} \qquad (1\cdot 9)$$

Then taking the matrix-element between f and i of both sides and after introducing a complete set of states $\sum_n |n\rangle\langle n| = 1$ between $\underset{\sim}{T}^{+}$ and $\underset{\sim}{T}$ on the right hand side one finally obtains;

$$\mathrm{Im}\, T_{fi} = \frac{(2\pi)^4}{2} \sum_n T_{fn}^{*} T_{ni}\, \delta^{(4)}(p_i - p_n) \qquad (1\cdot 10)$$

The left hand side is obtained by using: $T_{fi} - T_{fi}^{*} = 2i\,\mathrm{Im}\,T_{fi}$

p_i and p_n are the four-momenta of the initial and the intermediate states respectively. The summation over n includes all possible physical intermediate states allowed by conservation laws. Equation (1.10) may be rewritten as:

$$\mathrm{Im}\, T_{fi} = \frac{(2\pi)^4}{2} \sum_p \int \prod_n \left\{ d^4 p_n\, \theta(p_{0n})\, \delta(p_n^2 - m_n^2) \right\} \times \delta^{(4)}\Big(\sum_n p_n - p_i\Big)\, T_{fn}^{*}\, T_{ni} \qquad (1\cdot 11)$$

where the summation $\sum_p$ is over the number and types of particles in the intermediate state. The product $\prod_n$ runs over all the particles in the set p. The integrations over the four-momenta give the phase-space factor.

The unitarity condition is a general feature of every quantum mechanical theory.

<u>Crossing</u>: It has already been seen that Fig. 1.1. describes three different channels; each channel in turn corresponds to two reactions: the particle and the PCT equivalent anti-particle reaction. The postulate of crossing states that the same invariant amplitude - in our case $A(s,u,t)$ - taken as a function of the scalar kinematical invariants and continued to the appropriate values of these variables represents the actual scattering amplitude for all the three channels.

In perturbation theory diagrams of all orders are found to satisfy this postulate. There it is known as "Substitution law,"[4] and is stated as follows: no matter how the external lines of a given Feynman diagram are oriented the contribution of the diagram is the same. Crossing conditions are also found to be satisfied in the axiomatic field theory under suitable conditions. But there does not seem to be a way of proving it on the basis of the S-matrix theory. It is taken to be a postulate.

Since the physical regions of the three channels are mutually exclusive crossing conditions will be of no use unless the scattering amplitude has enough analytical properties to allow analytic continuation from one physical region to another.

<u>Analyticity</u>: The analytical properties of the scattering

amplitude in the kinematical variables have been subjected to extensive investigations since the early fifties. The forward scattering amplitude, which is a function of only the energy variable was the first to be investigated. Gell-Mann, Goldberger and Thirring[5] showed the amplitude for the scattering of photons in the forward direction to be analytic in the upper-half energy plane. Then Goldberger[6] was able to extend this proof of the analytical properties to the scattering of particles with mass. The next step was to derive similar results for non-forward directions. Heuristic derivations were given independently by various groups.[7]

Such analytical properties in the energy variable are due to the imposing of the restriction of the "Principle of microscopic causality" on the scattering amplitudes. This states that no signal can propagate with a speed greater than that of light in vacuo. In field theoretic language this means that the commutator (anticommutator) of boson (fermion) fields at two points separated by space like distance vanishes. Use of the above principle through the Jost-Lehmann-Dyson representation[8] for the vacuum expectation values of the commutators (anticommutators) forms the basis for the proofs of the analytical properties.

Rigorous proofs of analytical properties of the scattering amplitudes in the energy variable were obtained for the forward scattering by Symanzik[9] and for the non-forward scattering by Bogoliubov, Medvedev and Polivanov,[10] Bremermann, Oehme and Taylor[11] and Lehmann[12]. These proofs for the non-forward scattering amplitudes were valid provided the momentum transfer was less than a certain maximum and the masses of the particles satisfied certain inequalities.

To illustrate, how the analytical properties can be used to extract physical knowledge about the scattering process, the case of the forward scattering is considered. Forward direction corresponds to $t = 0$, when s is the centre of mass energy squared. The scattering amplitude $A(s,0)$, where the zero stands for t is analytic in the entire s-plane except for branch points and branch cuts on the real axis. If there exists any single particle state having the same quantum numbers as the initial state, there will be a pole at $s = m^2$, where m is the mass of the particle. Two cuts, both of them along the real axis, one taken from s_1 to $+\infty$ and the other from s_2 to $-\infty$ will account for all the branch points and cuts. s_1 is greater than s_2 unless the scattering process involves massless particles. So there is a gap on the real axis between the two cuts. A closed contour is drawn by going round the above two cuts and a circle of radius R, when $R \to \infty$. Application of the Cauchy's theorem on complex variables yields a representation for the scattering amplitude at any point inside the closed contour. If $A(s,0) \to 0$ as $s \to \infty$ the contribution from the large circle of radius R vanishes. In the case when $A(s,0) \to s^N$ as $s \to \infty$ by considering $A(s,0)/s^{N'}$ where $N' \gtrsim N$ instead of $A(s,0)$ the contributions from the large circle can be made to vanish. To avoid complications, it is assumed that there is no single particle state and $A(s,0) \to 0$ as $s \to \infty$. Then the representation is given by:

$$A(s,0) = \frac{1}{\pi}\int_{s_1}^{\infty} ds' \frac{\Im m\, A(s',0)}{s'-s} + \frac{1}{\pi}\int_{-\infty}^{s_2} ds' \frac{\Im m\, A(s',0)}{s'-s} \qquad (1.12)$$

where the property of real analyticity:

$$A(s,0) = A^*(s^*,0) \qquad (1.13)$$

has been used to express the discontinuities across the cuts in terms of the imaginary parts of the amplitude. The property of real analyticity is due to the fact that the amplitude $A(s,0)$ is real on the real axis between the two cuts.

Crossing relations allow us to write $\Im m\, A(s,0)$ in the second integral in Equation (1.12) in terms of $\Im m\, A(s,0)$ for $s \geqslant s_1$ that is, in terms of physical values. The unitarity condition Eqn (1.10) in the case of the forward scattering gives the optical theorem, which states that $\Im m\, A(s,0)$ is proportional to the total cross-section. By allowing s to approach the right hand cut $(s_1 \leqslant s \leqslant +\infty)$, that is, by putting $s \equiv s + i\epsilon$ when $\epsilon \to 0$ the real part of $A(s,0)$ on the right hand cut is obtained as a principal value integral over $\Im m\, A(s,0)$. Here use has been made of:

$$\frac{1}{s'-s\pm i\epsilon} = P\,\frac{1}{s'-s} \mp i\pi\delta(s'-s) \quad (1.14)$$

In a scattering experiment, the total cross sections are usually measured precisely. Thus, the analytical properties of the scattering amplitude through the representation Equation (1.12) determines the real part of the amplitude in terms of the total cross sections. Anderson, Davidon and Kruse[13] used Eqn (1.12) in the case of pion-nucleon scattering in the forward direction to test the validity of the analytic properties against experimental results. The agreement was fairly good.

A representation of the type of Eqn (1.12) is commonly known as dispersion relation. The origin of this name is in the investigation of the analytic properties of the scattering amplitude in the classical dispersion theory of light by Kramers[14] and Kronig[15] in 1926-27.

Using dispersion relations for $A(s,0)$ and for the derivatives of $A(s,t)$ with respect to t in the forward direction in the case of pion-nucleon scattering Chew, Goldberger, Low and Nambu[16] obtained approximate expressions for the individual partial waves. On the right hand side, only the dominant contributions to $\Im m\, A(s,0)$ such as, the contribution from the 3/2, 3/2 resonance in the pion-nucleon system, were kept. Thus the phase shifts for the lower partial waves in the low energy region were approximately determined. Among various applications of the single dispersion relations in the energy variable to the problems of pion-nucleon scattering[17], photo-production,[18] decay processes[19], processes involving strange particles[20], etc. worth mentioning are the determination of the pion-nucleon coupling constant[21] and the removal of the Fermi-Yang ambiguity for pion-nucleon phase shifts.[22]

Lehmann[12] in his proof of the single dispersion relations was also able to show that the scattering amplitude is analytic in a limited region in the Cos θ - plane around the physical values of Cos θ with the energy kept fixed. This analyticity is far from being enough to enable one to write useful dispersion relations in the momentum transfer for fixed energy.

In 1958, Mandelstam[23] wrote down a representation for the two body scattering amplitude by considering it to be analytic in the two variables: the energy and the momentum transfer, except for cuts along certain hyperplanes. In a subsequent paper[24] he proved that the fourth order diagrams in the perturbation theory do satisfy such a representation.

This led to the postulate of the principle of maximal analyticity[25], which states that the scattering amplitude

$A(s,u,t)$ is analytic in the variables s, u and t except for singularities demanded by unitarity. Since there are three channels, three different sets of singularities will appear in the scattering amplitude. Each set comes from the unitarity condition in the channel concerned.

Single particle states, stable or unstable, give rise to poles in the S-matrix theory. These poles are to be inserted into the S-matrix at the start of the process of determining the singularities demanded by unitarity. All one particle states having the same quantum numbers as the initial state of a certain scattering amplitude give rise to poles in that amplitude. In the S-matrix theory particles are classified as unstable or stable by the existence or non-existence of a decay threshold below the mass of the particle. When a pole is initially inserted into the S-matrix its parameters are arbitrary. If it is possible to determine these parameters through the requirements of self-consistency then the particle represented by the pole is said to be a bound state. When the parameters are truly arbitrary the particle is called "Elementary".

We now examine the unitarity condition, Eqn (1.11) to find out how it gives rise to singularities. When the energy on the real positive axis has been increased to an extent to make a new intermediate state, consisting of two or more particles, physically possible an additional contribution, which was identically zero before, is added to the right hand side. This gives a branch point on the positive real axis at this energy. As the energy increases, more and more massive intermediate states become physically possible giving rise to more and more branch points. These are known as normal threshold singularities.

The normal thresholds may be introduced in T_{fn}^* and T_{ni} on the right hand side of Eqn (1.11) to produce more singularities. These additional singularities can be fed back into the integrand to produce more. Continuing this iteration procedure all possible singularities of the scattering amplitude will be obtained, and it is postulated that there are no further singularities of the scattering amplitude.

Polkinghorne[26] has shown that the set of singularities obtained by iterating the unitarity condition is the same as the set obtained by considering all orders of perturbation theory. This allows us to use the Landau - Cutkosky[27] rules, first obtained for singularities in perturbation theory, to locate the singularities demanded by unitarity. The singularities of the simple diagrams drawn in Fig. 1.2 are as follows. Diagram (a) gives a pole in the scattering amplitude at $s = m^2$; diagram (b) produces a branch point on the positive real axis at $s = (m_1+m_2)^2$; while diagram (c) gives rise to singularities on the positive

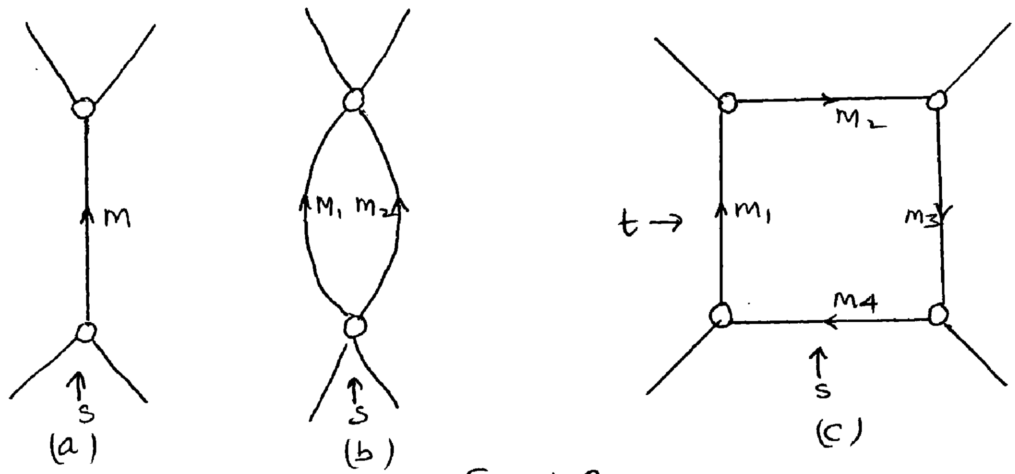


Fig 1.2

real axes in both s and t variables. The singularities lie on a curve in the s-t plane. This curve is asymptotic to the normal thresholds obtained by contracting out the lines m_1 and m_3 or m_2 and m_4, and gives the boundary of the contribution to the double spectral function of the Mandelstam representation from this diagram. In the case of diagram (c) the asymptotes are $t = (m_2+m_4)^2$ and $s = (m_1+m_3)^2$.

The locations of the singularities obtained from unitarity depend only on the masses of the particles in the physical intermediate state except in the case of weakly bound systems. Then a type of singularities appear which do not correspond to the mass of any physical intermediate state. These are called anomalous threshold singularities.[28] Since these may be obtained from the normal case by continuing in the masses of the particles, they too, arise out of the unitarity condition.

In the case of the two body scattering amplitude the singularities obtained from unitarity allow us to write down a representation for the scattering amplitude known after Mandelstam. Such a representation breaks down when anomalous thrwsholds appear. Many attempts[29] were made to prove Mandelstam representation for all orders of perturbation theory. Proofs exist only for diagrams up to the sixth order. We discuss the Mandelstam representation in the next section.

The unitarity condition couples the two body scattering amplitude with many particle systems. So, unless many - particle systems are known even the two body process cannot be determined completely. At the moment there is no general method of treating the many particle systems. At a particular energy only limited number of intermediate states are physically possible by energy conservation. The multiparticle states come into the picture away from the low energy region. It may be expected by making the energy appreciably low that the two body scattering process is isolated enough from the rest of the world for any calculation considering only the lowest intermediate states to make any sense.

1.2. MANDELSTAM REPRESENTATION:

A two body scattering amplitude having no anomalous threshold singularities allow this representation. Single particle states give rise to poles in the respective variables. Diagrams like Fig. 1.2(b) give singularities in one variable, that is, the single spectral terms. While the fourth (Fig. 1.2(c)) and higher order diagrams give the double spectral terms. The boundaries of the double spectral functions are determined by the fourth order diagrams. The representation then takes the form:

$$A(s,u,t) = \text{poles} + \frac{1}{\pi}\int ds' \frac{\rho_1(s')}{s'-s} + \frac{1}{\pi}\int du' \frac{\rho_2(u')}{u'-u} + \frac{1}{\pi}\int dt' \frac{\rho_3(t')}{t'-t}$$
$$+ \frac{1}{\pi^2}\iint ds'dt' \frac{A_{13}(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\iint ds'du' \frac{A_{12}(s',u')}{(s'-s)(u'-u)}$$
$$+ \frac{1}{\pi^2}\iint du'dt' \frac{A_{23}(u',t')}{(u'-u)(t'-t)} \qquad (1.15)$$

All possible subtractions have been ignored. Taking the discontinuities in s, u and t:

$$A_1(s,u,t) = \rho_1(s) + \frac{1}{\pi}\int dt' \frac{A_{13}(s,t')}{t'-t} + \frac{1}{\pi}\int du' \frac{A_{12}(s,u')}{u'-u} \qquad (1.16)$$
$$A_2(s,u,t) = \rho_2(u) + \frac{1}{\pi}\int ds' \frac{A_{12}(s',u)}{s'-s} + \frac{1}{\pi}\int dt' \frac{A_{23}(u,t')}{t'-t} \qquad (1.17)$$
$$A_3(s,u,t) = \rho_3(t) + \frac{1}{\pi}\int ds' \frac{A_{13}(s',t)}{s'-s} + \frac{1}{\pi}\int du' \frac{A_{23}(u',t)}{u'-u} \qquad (1.18)$$

where A_1, A_2 and A_3 are the discontinuities in s, u and t respectively. Using these discontinuities the Mandelstam representation may be rewritten in any one of the following three forms:-

Fixed s:

$$A(s,u,t) = \text{poles} + \frac{1}{\pi}\int ds' \frac{\rho_1(s')}{s'-s} + \frac{1}{\pi}\int du' \frac{A_2(s,u',\Sigma-s-u')}{u'-u}$$
$$+ \frac{1}{\pi}\int dt' \frac{A_3(s,\Sigma-s-t',t')}{t'-t} \qquad (1.19)$$

Fixed u:

$$A(s,u,t) = \text{poles} + \frac{1}{\pi}\int du' \frac{\rho_2(u')}{u'-u} + \frac{1}{\pi}\int ds' \frac{A_1(s',u,\Sigma-s'-u)}{s'-s} + \frac{1}{\pi}\int dt' \frac{A_3(\Sigma-u-t',u,t')}{t'-t} \tag{1.20}$$

Fixed t:

$$A(s,u,t) = \text{poles} + \frac{1}{\pi}\int dt' \frac{\rho_3(t')}{t'-t} + \frac{1}{\pi}\int ds' \frac{A_1(s',\Sigma-s'-t,t)}{s'-s} + \frac{1}{\pi}\int du' \frac{A_2(\Sigma-u'-t,u',t)}{u'-u} \tag{1.21}$$

where we have used: $s+u+t = \sum_{i=1}^{4} m_i^2 \equiv \Sigma$

The single dispersion relations studied before Mandelstam representation were of the type fixed t. In the forward direction when t = 0 the discontinuities A_1 and A_2 are entirely in the physical regions of channels I and II respectively. Later on, in our application of dispersion relations to the problem of kaon-pion scattering this type of single dispersion relations will be found to be very useful.

The Mandelstam representation gives the locations of the singularities. If one can devise a procedure to calculate the discontinuties across all the cuts, then the scattering amplitude will be completely known. The pole parameters corresponding to a single particle may possibly be determined by self-consistency requirements or may remain as arbitrary parameters. Similar is the case with subtraction constants. The masses of various particles may well have to be taken as arbitrary parameters. It has been hoped in the S-matrix theory that one will be able to determine everything in terms of only one parameter, possibly the mass of one particle to fix the scale. But such a hope has been far from realization. All calculations so far done have been done in a number of limited regions of the S-matrix with own set of parameters in each case.

The scattering amplitude $A(s,u,t)$ may be expanded in terms of individual partial waves in any one of the three channels using Legendre polynomials. The case of channel I is considered in the following discussion. The expansion may be written down as:

$$A(s,\cos\theta) = \sum_{l=0}^{\infty} (2l+1)\, P_l(\cos\theta)\, A_l(s) \qquad (1\cdot 22)$$

where $A_l(s)$ is the lth partial wave amplitude which is a function of only one variable s. Using the inverted form of Eqn (1.22):

$$A_l(s) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta\, P_l(\cos\theta)\, A(s,\cos\theta) \qquad (1\cdot 23)$$

individual partial wave amplitudes may be projected out of the Mandelstam representation. Then the analytic structures of these partial waves may be obtained. In the next chapter, it will be shown in detail how this is done for the kaon-pion system. For the present need, it is just sufficient to mention that in the general case the following singularities appear: a cut extending from the lowest threshold to infinity along the positive real axis known as the physical or the right hand cut, and one or more cuts which may or may not lie entirely on the real axis and extending to infinity along the negative direction. The later set of cuts is known as the unphysical or the left hand cut. The unitarity condition when expressed in terms of partial wave amplitudes allows us to write $A_l(s)$ as:

$$A_l(s) = k\, e^{i\delta_l} \sin\delta_l \tag{1.24}$$

where k is a kinematical factor which goes to a constant as $s \to +\infty$ and δ_l is the lth partial wave phase shift. This shows that $A_l(s)$ tends to a constant as s approaches infinity, then the following partial wave dispersion relation may be written down:

$$A_l(s) = a_l + \frac{s-s_0}{\pi}\int_R \frac{\operatorname{Im} A_l(s')}{(s'-s)(s'-s_0)}\,ds' + \frac{s-s_0}{\pi}\int_L \frac{\Delta A_l(s')}{(s'-s)(s'-s_0)}\,ds' \tag{1.25}$$

where one subtraction has been made at $s = s_0$ and $a_l = A_l(s_0)$ The discontinuities $\Delta A_l(s)$ across the left hand cut may be obtained in terms of A_2 and A_3 by using the fixed s dispersion relation Eqn (1.19). Both A_2 and A_3 are known if the double spectral functions are known. Alternatively A_2 and A_3 may be expressed in terms of channel II and III partial wave amplitudes by continuation using the Legendre polynomial expansion. Such a continuation is valid only in the nearby regions of the left hand cut.

The double spectral functions need not be known in this case. In the next section we shall discuss the various methods of solving the partial wave dispersion relations.

Among the various contributions to the double spectral functions there will be some from diagrams having elastic intermediate states in one of the three channels. These contributions are referred to as the elastic double spectral functions. The regions of such contributions appear as fringes along the boundaries of the double spectral functions. Iterative procedure may be designed to calculate these contributions. This is known as the strip approximation.[30] The numerical calculations are very much involved. This method was applied to the problem of pion-pion scattering by Bransden and Moffat[31] and Bransden, Burke, Moorhouse and Morgan[32]. Application has also been made to the problem of pion-nucleon scattering.[33]

Froissart[34] studied the asymptotic behaviour of a two body scattering amplitude involving scalar particles and satisfying the Mandelstam representation. He found the scattering amplitude to be bounded by

$$\text{Const.}\, s \ln^2 s$$

at the forward and backward angles, and by

$$\text{Const.}\, s^{3/4} \ln^{3/2} s$$

at any other fixed angle. This imposes a very serious restriction on the number of arbitrary subtractions in the

Mandelstam representation. Subtraction terms in the variable t are, in general, of the form $P_l(\cos\theta_t) f_l(t)$. When s is very large, $P_l(\cos\theta_t)$ behaves as s^l. So, subtraction terms with $l > 1$ violates the Froissart bounds. Froissart also showed that cancellations do not occur between the different terms of this type of subtraction constants or between these terms and the double spectral functions. Applying this argument to all three variables, the number of independent subtractions allowed in each case is only two.

A stable particle is represented by a pole on the physical sheet in the form:

$$g^2 \frac{P_l(\cos\theta_s)}{m^2 - s}$$

when $l > 1$, this term violates the Froissart bound. Similarly, resonant partial waves with $l > 1$ will also be in trouble.

In potential scattering, it has been shown by Regge[35] that the scattering amplitude is meromorphic in the complex angular momentum plane when Rel is within a certain range of values. The poles are not fixed. They start on the real axis and move to the right with increasing energy. At a certain value of the energy (at the threshold of the process) a particular pole moves into the upperhalf plane still continuing its rightward motion. Then ultimately when the energy has been increased to a certain value it turns back and starts moving left. Such poles are known as Regge poles. Whenever a Regge pole crosses, during its rightward motion, an integral value of Rel it gives rise to a bound state or a resonance depending on whether the pole is on the real axis or in the upper half plane.

The idea of the Regge poles has been extended to relativistic S-matrix theory by various people.[36] There, of course, like the other postulates of the theory no rigorous proof exists for the occurrence of such poles. In field theory Regge poles appear in the sum over an infinite set of ladder diagrams.[37]

By putting Regge poles into the S-matrix theory the divergence troubles with stable or unstable particles for $\ell > 1$ can be avoided. At first, Regge poles seemed to be satisfactory in explaining the high energy behaviour of scattering cross sections. Later experiments on πp and pp scattering cast some doubt on such an explanation. Moving Regge cuts may have to be brought into the picture. Then, certain theoretical investigations[38] suggesting the existence of such moving cuts in the complex l-plane have introduced serious complications. In short it may be stated that the Regge poles cannot explain the high energy scattering properly.

We leave the general discussion on the S-matrix theory at this point after making one remark. Although the partial wave dispersion relations follow straightforwardly from the

Mandelstam representation, their existence may be proved independently on the basis of the perturbation theory[39]. This allows us to have a bit more faith in them.

1.3. METHODS OF SOLVING THE PARTIAL WAVE DISPERSION RELATIONS.

In this section, various methods of solving the partial wave dispersion relations are discussed. The drawbacks of each method are also mentioned.

(i) N/D METHOD:

Here, the partial wave amplitude is written down as:

$$A_l(s) = N_l(s)/D_l(s) \tag{1.26}$$

Where $N_l(s)$ has got only the left hand cut and $D_l(s)$ has got only the right hand cut. The discontinuities of $N_l(s)$ and $D_l(s)$ across the cuts are given by:

$$\operatorname{Im} D_l(s) = \operatorname{Im}\left(\frac{1}{A_l(s)}\right) N_l(s) \quad \text{for } s \text{ on the right hand cut.} \tag{1.27}$$

$$\operatorname{Im} N_l(s) = \Delta A_l(s)\, D_l(s) \quad \text{for } s \text{ on the left hand cut.} \tag{1.28}$$

Normalizing $D_l(s) = 1$ at $s = s_0$, the following dispersion relations for $N_l(s)$ and $D_l(s)$ may be written down:

$$N_l(s) = a_l + \frac{s-s_0}{\pi}\int_L ds' \frac{\Delta A_l(s')\, D_l(s')}{(s'-s)(s'-s_0)} \tag{1.29}$$

$$D_l(s) = 1 + \frac{s-s_0}{\pi}\int_R ds' \frac{\operatorname{Im}\left(\frac{1}{A_l(s')}\right) N_l(s')}{(s'-s)(s'-s_0)} \tag{1.30}$$

The imaginary part of $1/A_l(s)$ on the physical cut is given by the unitarity condition for the partial wave amplitudes, Eqn (1.24). Substituting Eqn (1.29) in Eqn (1.30) one obtains a Fredholm equation for $D_l(s)$. The existence of solutions[40] for such an equation depends critically on the behaviour of $\Delta A_l(s)$ as $s \to -\infty$. $\Delta A_l(s)$ is normally calculated by using the crossing conditions to express it in terms of channels II and III physical amplitudes. As has already been pointed out that such a procedure involves a continuation using the Legendre polynomial expansion which is valid only in a limited region of the left hand cut. When a particle with spin equal to or greater than one is exchanged in channel I, $\Delta A_l(s)$ either tends to a large constant or blows up as $s \to -\infty$. Under this circumstance any solution to the partial wave dispersion relations using the above method of continuation either will not exist or will make no sense. A cut off may be introduced to avoid the troublesome region. In the case of N/D method with a cut off, the solutions are strongly dependent on the position of the cut off. Chew and Mandelstam[41] obtained such solutions for the pion-pion scattering. The cut off introduced an extra parameter into the pion-pion system.

Various simple approximations of the N/D equations can be made by replacing the left hand cut by a set of poles and calculating their parameters by the requirements of self-consistency[42]. All these approximations are very crude in nature and have limited applications. Another class of approximations,[43] commonly known as "Bootstrap calculations," may be devised by neglecting everything but the contributions from certain one-particle exchange terms on the left hand cut. Each exchange term has two parameters, the coupling constant and the mass of the particle exchanged. Most of these parameters may be determined by the requirements of self-consistency. As for example, if the particle exchanged can also appear in the direct channel, then equating $\mathrm{Re}\, D_\ell(s) = 0$ at the position of the mass of the particle and $\frac{d}{ds} \mathrm{Re}\, D_\ell(s)$ to the coupling constant the parameters may be determined self-consistently. In general, the numerical results of such a crude calculation disagrees badly with experimental results. The iteration is done only to the first order, when $D_\ell(s) = 1$ on the left hand cut. Any attempt to go beyond this makes the solution blow up. Recently, Gervais[44] extended this method to include the two particle exchange terms in certain approximations. The numerical results show slight improvements.

The solutions obtained for the N/D equation are not unique. Introducing zeros in $D_\ell(s)$ arbitrarily, poles may be generated in $A_\ell(s)$. These are known as CDD poles[45]. The parameters of these poles are entirely arbitrary. Such poles are taken to represent the elementary particles of the theory. The Froissart bounds restrict these poles to the total angular momentum states 0, $\frac{1}{2}$ and 1.

(ii) VARIATIONAL METHOD OF HAMILTON AND DONNACHIE[46]:

Let us consider the quantity;

$$g_l(s) = A_l(s)/(s-s_0)^{1/2} \qquad (1.31)$$

where s_o is the lower limit of the right hand integral in Eqn (1.25). Then ignoring possible subtractions the following dispersion relation may be written down;

$$\frac{\mathrm{Im}\, A_l(s)}{\sqrt{s-s_0}} = -\frac{1}{\pi} P \int_{s_0}^{\infty} ds' \frac{\mathrm{Re}\, A_l(s')}{(s'-s)\sqrt{s'-s_0}} - \frac{1}{\pi}\int_L ds' \frac{\Delta A_l(s')}{(s'-s)\sqrt{s_0-s'}} \qquad (1.32)$$

This type of inverted dispersion relation was first obtained by Gilbert.[47] Assuming suitable parametric form for $A_l(s)$ on the physical cut Hamilton and Donnachie developed a variational method using Eqns (1.25) and (1.32) together. Various parameters are calculated by varying them to make the solutions of the above dispersion relations satisfy unitarity as closely as possible. Good agreements are obtained in the applications to the problem of pion-pion scattering in the T = 1 J = 1 state by Oades[48] and to the problem of pion-nucleon scattering in the $\frac{3}{2}$, $\frac{3}{2}$ state by Hamilton and Donnachie.[46]

A serious drawback of this method is that the approximate nature of $A_l(s)$ on the physical cut has to be known in advance. Only resonant states can be dealt with fairly simply.

(iii) INVERSE AMPLITUDE METHOD:

In this method, instead of dealing with the amplitude $A_l(s)$ its inverse $1/A_l(s)$ is considered. To make the illustration showing how the method works very simple the following assumptions are made. It is not necessary for all of them to be true in a practical application. The partial wave amplitude $A_l(s)$ is assumed to have the physical cut $s_0 \leq s \leq +\infty$ and a simple left hand cut $-\infty \leq s \leq s_1$ along the real axis. It is assumed that there is no complex zero of $A_l(s)$ and that it goes to a constant at s_1 and s_o. All possible subtractions are ignored. Then the following dispersion relation can be written down

$$A_l^{-1}(s) = \frac{1}{\pi}\int_{s_0}^{\infty} ds' \frac{\mathrm{Im}\, A_l^{-1}(s')}{s'-s} + \frac{1}{\pi}\int_{-\infty}^{s_1} ds' \frac{\mathrm{Im}\, A_l^{-1}(s')}{s'-s} \qquad (1.33)$$

The imaginary part $\mathrm{Im}\, A_l^{-1}(s)$ on the right hand cut is given by ~~the~~ unitarity. It is purely a kinematical factor for elastic scattering. Then the contribution from the right hand cut is known analytically. On the left hand cut $\mathrm{Im}\, A_l^{-1}(s)$ is expressed in terms of $\mathrm{Im}\, A_l(s)$ as follows:

$$\mathrm{Im}\, A_l^{-1}(s) = -\frac{\mathrm{Im}\, A_l(s)}{[\mathrm{Re}\, A_l(s)]^2 + [\mathrm{Im}\, A_l(s)]^2} \qquad (1.34)$$

where $\mathrm{Re}\, A_l(s)$ is obtained from the dispersion relation Eqn (1.33) using

$$\mathrm{Re}\, A_l(s) = \frac{\mathrm{Re}\, A^{-1}(s)}{[\mathrm{Re}\, A_l^{-1}(s)]^2 + [\mathrm{Im}\, A_l^{-1}(s)]^2} \qquad (1.35)$$

$\mathrm{Im}\, A_l(s)$ in Eqn (1.34) can be obtained by using crossing in terms of physical quantities in channels II and III. It is evident from Eqn (1.34) that if $\mathrm{Im}\, A_l(s)$ blows up as $s \rightarrow -\infty$, then $\mathrm{Im}\, A_l^{-1}(s) \rightarrow 0$ as $s \rightarrow -\infty$. This effectively produces a cutoff in the dispersion relation.

Moffat[49] first wrote down the inverse amplitude dispersion relations for the pion-pion scattering. Then Bransden and Moffat[50] obtained numerical solutions to the problem by an iterative process based on the closed set formed by equations (1.33) - (1.35) and the crossing relations for the pion-pion system. The S-wave amplitudes have one subtraction each. The P-wave amplitude, which has just one total isotopic spin state, I = 1 has two parameters. The S-wave constants are related to the coupling constant of the pion-pion system, which is defined to be the value of the scattering amplitude at the symmetry point s = u = t = 4/3. The two P-wave parameters are obtained in terms of the S-waves using the derivative conditions at the symmetry point. Thus the iterative procedure gave solutions of the coupled S and P waves in the pion-pion scattering dependent on only one arbitrary parameter, the pion-pion coupling constant. The solutions have the P-wave resonance, called the rho meson. The position and the width of the resonance depended very much on the S-wave amplitudes. The low energy solutions were insensitive to the distant regions of the left hand cut as expected. Crossing was satisfied in the nearby portion of the left hand cut.

Later on, we shall use the inverse amplitude method to determine the S-waves in the kaon-pion scattering. It will be found to be much more complicated than the pion-pion case.

CHAPTER II

THE ANALYTICAL PROPERTIES OF THE KAON-PION SCATTERING AMPLITUDE

In this chapter we discuss the analytical properties of the kaon-pion scattering amplitude. The kinematics and various other details of the scattering amplitudes for all the three channels are given in section I. The Mandelstam representation is written down in the next section. Section III is devoted to the derivation of the analytical properties of the partial wave amplitudes for kaon-pion scattering. The discontinuities across the cuts are also obtained in this section.

2.1. KINEMATICS:

Fig. 2.1. represents schematically the kaon-pion scattering and the crossed processes. The three scalar invariants, which

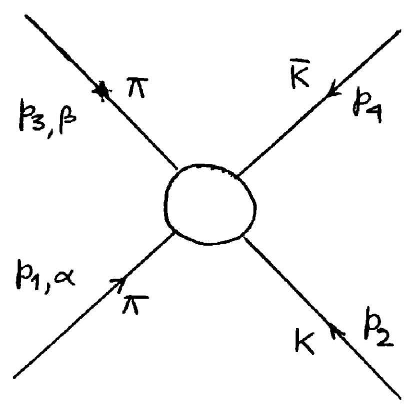


Fig 2.1.

may be formed from the four-momenta p_1, p_2, p_3 and p_4 are as follows*:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \qquad (2\cdot1a)$$

$$u = (p_1 + p_4)^2 = (p_2 + p_3)^2 \qquad (2\cdot1b)$$

$$t = (p_1 + p_3)^2 = (p_2 + p_4)^2 \qquad (2\cdot1c)$$

s, u and t are the energy variables in the following three channels

$$\mathrm{I} \quad \pi(p_1,\alpha) + K(p_2) \to \pi(-p_3,\beta) + K(-p_4)$$

$$\mathrm{II} \quad \pi(p_3,\beta) + K(p_2) \to \pi(-p_1,\alpha) + K(-p_4)$$

$$\mathrm{III} \quad \pi(p_1,\alpha) + \pi(p_3,\beta) \to \bar{K}(-p_2) + K(-p_4)$$

respectively. Both channels I and II describe the kaon-pion scattering and can be obtained from each other by interchanging the two pions. While channel III describes the annihilation-creation process $\pi\pi \to K\bar{K}$. Conservation of four-momenta requires that

$$s + u + t = 2m^2 + 2\mu^2 \equiv \Sigma \qquad (2\cdot2)$$

This reduces the number of independent variables from three to two. In each channel we shall use the square of the centre-of-mass energy and the cosine of the scattering angle in the same system as the two independent variables. The relations between these variables and s, u and t are as follows

---

* We have chosen the units so that $\hbar = c = \mu = 1$, where μ is the mass of the pion. The metric chosen is such that the scalar product is defined by $A\,B = A_0B_0 - \underset{\sim}{A}\cdot\underset{\sim}{B}$. The mass of a kaon is denoted by m.

Channel I:

Let

$$\underset{\sim}{p}_1^2 = \underset{\sim}{p}_2^2 = \underset{\sim}{p}_3^2 = \underset{\sim}{p}_4^2 = k^2$$

when k^2 is the square of the momentum in the centre of mass system. Then

$$s = m^2+\mu^2+2k^2 \pm 2\sqrt{(k^2+m^2)(k^2+\mu^2)} \qquad (2.3a)$$

$$t = -2k^2(1-\cos\theta) \qquad (2.3b)$$

$$u = \Sigma - s + 2k^2(1-\cos\theta) \qquad (2.3c)$$

where the cosine of the scattering angle, $\cos\theta$ is defined by

$$\cos\theta = \frac{\underset{\sim}{p}_1 \cdot \underset{\sim}{p}_3}{k^2} \qquad (2.4)$$

From Eqn. (2.3a) it is clear that s is a double valued function of k^2 and so the k^2- plane is two-sheeted. The (+)ve sign in the above mentioned equation corresponds to the physical sheet, because then $k^2 \geq 0$ for $s \geq (m+\mu)^2$ By s, this value on the physical sheet will be meant unless otherwise stated. k^2 may be expressed in terms of s as follows:

$$k^2 = \frac{[s-(m+\mu)^2][s-(m-\mu)^2]}{4s} \qquad (2.5)$$

The physical region for channel I is defined by:

$$s \geq (m+\mu)^2 \text{ and } -1 \leq \cos\theta \leq +1 .$$

Channel II:

The variables are all similar to those of channel I and can be obtained from the latter by interchanging s and u. A bar is placed on k^2 and $\cos\theta$ like $\bar{k}^2$ and $\cos\bar{\theta}$ to denote channel II quantities. Then

$$u = m^2+\mu^2+2\bar{k}^2 \pm 2\sqrt{(\bar{k}^2+m^2)(\bar{k}^2+\mu^2)} \qquad (2.6a)$$

$$t = -2\bar{k}^2(1-\cos\bar{\theta}) \qquad (2.6b)$$

$$s = \Sigma - u + 2\bar{k}^2(1-\cos\bar{\theta}) \qquad (2.6c)$$

$$\bar{k}^2 = \frac{[u-(m+\mu)^2][u-(m-\mu)^2]}{4u} \qquad (2.6d)$$

The physical region is now defined by

$$u \geqslant (m+\mu)^2 \quad \text{and} \quad -1 \leqslant \cos\bar{\theta} \leqslant +1$$

<u>Channel III</u>:

Let

$$\underset{\sim}{p}_1^2 = \underset{\sim}{p}_3^2 = q^2 \quad ; \quad \underset{\sim}{p}_2^2 = \underset{\sim}{p}_4^2 = p^2.$$

where p and q are the kaon and the pion momenta respectively in the centre of mass system. Then

$$s = -p^2-q^2+2pq\cos\varphi \qquad (2.7a)$$

$$u = -p^2-q^2-2pq\cos\varphi \qquad (2.7b)$$

$$t = 4(p^2+m^2) = 4(q^2+\mu^2) \qquad (2.7c)$$

$\cos\varphi$ is defined by

$$\cos\varphi = \frac{\underset{\sim}{p}\cdot\underset{\sim}{q}}{pq} \qquad (2.8)$$

The physical region is given by

$$t \geq 4m^2 \quad \text{and} \quad -1 \leq \cos\phi \leq +1$$

We can write down the S-matrix for the process described by channel I as follows:

$$S_{\beta\alpha} = \delta_{\beta\alpha} + i\,(2\pi)^4\,\delta^{(4)}(p_\alpha - p_\beta)\,T_{\beta\alpha} \qquad (2\cdot 9)$$

where the isotopic spin indices α and β for the pions are used to denote the initial and the final states respectively. $T_{\beta\alpha}$ is connected to the invariant scattering amplitude $A_{\beta\alpha}$ by

$$T_{\beta\alpha} = \frac{8\pi}{(16\,p_{01}\,p_{02}\,p_{03}\,p_{04})^{1/2}}\,A_{\beta\alpha} \qquad (2\cdot 10)$$

$A_{\beta\alpha}$ will be taken to be a function of s, u and t to guarantee relativistic invariance and it is related to the differential cross section for kaon-pion scattering by.

$$\frac{d\sigma}{d\Omega} = \frac{1}{s}\,|\,A_{\beta\alpha}(s,u,t)\,|^2 \qquad (2\cdot 11)$$

Using the unitarity condition, Eqn (1.10), one gets the optical theorem:

$$\sigma_{tot} = \frac{4\pi}{k\sqrt{s}}\,\mathrm{Im}\,A_{\alpha\alpha} \qquad (2\cdot 12)$$

where $A_{\alpha\alpha}$ denotes the forward scattering amplitude in channel I.

Crossing requires that the same scattering amplitude

$A(s,u,t)$ continued to appropriate values of the variables describe all the three channels. We denote

$$
\begin{aligned}
A(s,u,t) &\equiv A(s,\cos\theta) \text{ in channel I} \\
A(s,u,t) &\equiv A(u,\cos\bar{\theta}) \text{ in channel II} \\
A(s,u,t) &\equiv B(t,\cos\phi) \text{ in channel III}
\end{aligned}
\tag{2.13}
$$

In the kaon isotopic-spin space the scattering amplitude for channel I, $A_{\beta\alpha}$ may be written as:

$$A_{\beta\alpha} = \delta_{\beta\alpha} A^{(+)} + \tfrac{1}{2}[\tau_\beta, \tau_\alpha] A^{(-)} \tag{2.14}$$

where $A^{(+)}$ and $A^{(-)}$ are the symmetric and the anti-symmetric parts respectively. There can be two eigen-states of the total isotopic spin, $I = 1/2$ and $I = 3/2$ for the kaon-pion system. The scattering amplitudes for these eigen-states are related to $A^{(+)}$ and $A^{(-)}$ by (see appendix I):

$$A^{1/2} = A^{(+)} + 2A^{(-)} \tag{2.15}$$

$$A^{3/2} = A^{(+)} - A^{(-)} \tag{2.16}$$

A special case of crossing arises when the two pions are interchanged, that is when channels I and II are switched. Both these channels describe kaon-pion scattering. From Eqn (2.14) one finds that $A^{(+)}$ is symmetric under the exchange of the pions, while $A^{(-)}$ is anti-symmetric. Then we have

$$A^{(\pm)}(s,u,t) = \pm A^{(\pm)}(u,s,t) \tag{2.17}$$

where s and u are interchanged, because the interchange of the pions means that s $\leftrightarrow$ u. Eqn (2.17) is known as the "crossing symmetry" and is a very severe restriction on the scattering amplitudes. Using Eqn (2.15) and (2.16) the "crossing symmetry" condition can be written as:

$$A^{I}(s,\cos\theta) = \sum_{I'} \alpha_{II'} A^{I'}(u,\cos\bar{\theta}) \qquad (2.18)$$

where the crossing matrix $\alpha_{II'}$ is given by

$$\alpha_{II'} = \begin{pmatrix} -1/3 & 4/3 \\ 2/3 & 1/3 \end{pmatrix} \qquad (2.19)$$

In channel III the eigen-states of total isotopic-spin states are I = 0 and I = 1. Since two pions in the state of total isotopic-spin state I = 0 is symmetric under the interchange of the two pions, the scattering amplitude in channel III with I = 0 is proportional to $A^{(+)}$. Similarly the state of two pions with I = 1 is antisymmetric under the exchange of the pions so the scattering amplitude in this state is proportional to $A^{(-)}$. The constants of proportionality are determined in appendix I. Then

$$B^{0} = \sqrt{6}\, A^{(+)} \qquad (2.20)$$

$$B^{1} = 2\, A^{(-)} \qquad (2.21)$$

Expressing $A^{(+)}$ and $A^{(-)}$ in terms of amplitudes in the eigen-states of total isotopic-spin in channel I we have

$$B^{I}(t,\cos\phi) = \sum_{I'} \beta_{II'} A^{I'}(s,\cos\theta) \qquad (2.22)$$

The crossing matrix $\beta_{II'}$ is given by

$$\beta_{II'} = \begin{pmatrix} \sqrt{6}/3 & 2\sqrt{6}/3 \\ 2/3 & -2/3 \end{pmatrix} \qquad (2.23)$$

Eqn (2.22) may be inverted to give

$$A^I(s, \cos\theta) = \sum_{I'} \gamma_{II'} B^{I'}(t, \cos\varphi) \qquad (2.24)$$

where

$$\gamma_{II'} = \begin{pmatrix} 1/\sqrt{6} & 1 \\ 1/\sqrt{6} & -1/2 \end{pmatrix} \qquad (2.25)$$

Eqns (2.18), (2.22) and (2.24) will be very useful later on.

In channel I, the scattering amplitude $A^I(s, \cos\theta)$ may be expanded in a series of Legendre polynomials

$$A^I(s, \cos\theta) = \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) A_l^I(s) \qquad (2.26)$$

Reversing Eqn (2.26) the partial wave amplitude $A_l^I(s)$ may be expressed in terms of $A^I(s, \cos\theta)$ as follows:

$$A_l^I(s) = \frac{1}{2} \int_{-1}^{+1} d\cos\theta \, P_l(\cos\theta) \, A^I(s, \cos\theta) \qquad (2.27)$$

The unitarity condition, Eqn (1.11) when expressed in terms of the partial wave amplitudes takes a very simple diagonalized form (appendix II):

$$\mathrm{Im}\, A_l^I(s) = \frac{k}{\sqrt{s}} \, |A_l^I(s)|^2 \, R_l^I(s) \qquad (2.28)$$

where $R_l^I(s)$ is the coefficient of inelasticity defined by

$$R_l^I(s) = \sigma_{l\,tot}^I(s) / \sigma_{l\,el}^I(s) \qquad (2.29)$$

when the scattering is completely elastic $R_l^I(s) = 1$. In the case of the kaon-pion scattering, the amplitude is purely elastic in the region $(m+\mu)^2 \leqslant s \leqslant (m+3\mu)^2$.

From the unitarity condition, Eqn (2.28) it follows that:

$$A_l^I(s) = \frac{\sqrt{s}}{k} e^{i\delta_l^I} \sin\delta_l^I \qquad (2.30)$$

where δ_l^I is the phase shift, which is real in the elastic region and becomes complex when the scattering is inelastic. The unitarity condition, Eqn (2.28) can be rewritten in terms of the inverse amplitude $A_l^{I^{-1}}(s)$ as follows:

$$\operatorname{Im} A_l^{I^{-1}}(s) = -\frac{k}{\sqrt{s}} R_l^I(s) \qquad (2.31)$$

This is a very useful form of the unitarity condition, because when the scattering is fully elastic $\operatorname{Im} A_l^{I^{-1}}(s)$ is just a kinematical factor. Eqn (2.31) will be used in chapter IV in the inverse amplitude formulation for solving the partial wave dispersion relations.

In channel III, we make the following partial wave expansion:

$$B^I(t, \cos\phi) = \sum_{l=0}^{\infty} (2l+1)(pq)^l P_l(\cos\phi) B_l^I(t) \qquad (2.32)$$

The reverse of this is.

$$B_l^{I}(t) = \frac{1}{2(pq)^l}\int_{-1}^{+1} d\cos\varphi \, P_l(\cos\varphi) \, B^{I}(t,\cos\varphi) \quad (2.33)$$

The interchange s $\leftrightarrow$ u means that $\cos\varphi \leftrightarrow -\cos\varphi$. Then the "crossing symmetry" condition, Eqn (2.17) when applied to channel III gives

$$B^{(\pm)}(t,\cos\varphi) = \pm B^{(\pm)}(t,-\cos\varphi) \quad (2.34)$$

Making partial wave expansions of both sides, we have

$$\sum_{l=0}^{\infty}(2l+1)(pq)^l\left[1 \mp (-1)^l\right] P_l(\cos\varphi) B_l^{(\pm)}(t) = 0 \quad (2.35)$$

where, the relation $P_l(-\cos\varphi) = (-1)^l P_l(\cos\varphi)$ has been used. Since the summation over l forms a complete set, each term of the sum can be equated to zero separately. Then it follows that.

$$B_l^{0}(t) = 0 \quad \text{for } l \text{ odd}$$
$$B_l^{1}(t) = 0 \quad \text{for } l \text{ even}$$

because B_l^0 and B_l^1 are proportional to $B_l^{(+)}$ and $B_l^{(-)}$ respectively. Thus "crossing symmetry" reduces the number of amplitudes in a particular eigen-state of angular momentum in channel III from two to one. Since the two pion are in the initial state, "crossing symmetry" is just the Pauli exclusion principle.

Retaining only the two-pion intermediate state in the unitarity condition, Eqn (1.11) we have

$$\text{Im}\, B_l^I(t) = \frac{q}{\sqrt{t}} B_l^{I*}(t) A_l^{I\pi\pi}(t) \qquad (2.36)$$

where $A_l^{I\pi\pi}(t)$ is the pion-pion scattering amplitude and this can be written down as:

$$A_l^{I\pi\pi}(t) = \frac{\sqrt{t}}{q} e^{i\delta_l^{I\pi\pi}} \sin \delta_l^{I\pi\pi} \qquad (2.37)$$

Since $\text{Im}\, B_l^I(t)$ is real, $B_l^I(t)$ should have the same phase as $A_l^{I\pi\pi}(t)$ in the two pion approximation for the unitarity condition. Then

$$B_l^I(t) = G_l^I(t)\, e^{i\delta_l^{I\pi\pi}} \qquad (2.38)$$

where $G_l^I(t)$ is a real quantity. This is the "Final state theorem"[51]. Eqn (2.38) is exactly true only in the region $4\mu^2 \leq t \leq 16\mu^2$. Now the physical region in channel III starts at $t = 4m^2$. But Mandelstam[52] has shown that the unitarity condition can be extended to the region $4\mu^2 \leq t \leq 4m^2$.

We can find a very suitable point at which

$$s = u = s_0 \;;\; \cos\theta = \cos\bar{\theta} = \cos\phi = 0$$

$$\text{and} \quad t = -2k_0^2$$

where S_o is given by

$$S_0 = \frac{1}{3}\left[m^2 + \mu^2 + 2\sqrt{m^4 + \mu^4 - m^2\mu^2}\,\right] \qquad (2.39)$$

From the condition of "crossing symmetry", Eqn (2.17) it immediately follows that

$$A^{(-)}(s_0, s_0, t_0) = 0 \qquad (2.40)$$

So, at this point, referred from now on as the symmetry point the two isotopic spin states in channels I and II are equal to each other

$$A^{1/2}(s_0, 0) = A^{3/2}(s_0, 0) \qquad (2.41)$$

Since $B^1(t, \cos\varphi)$ is proportional to $A^{(-)}$ we have also.

$$B^1(t_0, 0) = 0 \qquad (2.42)$$

Differentiating both sides of Eqn (2.18) with respect to s and $\cos\theta$ at the symmetry point various derivative conditions for $A_l^I(s)$ may be obtained. In appendix III, the first derivative conditions are deduced. They are

$$a_1^{1/2} \approx 2.64 \left.\frac{dA_0^{1/2}(s)}{ds}\right|_{s=s_0} + 5.96 \left.\frac{dA_0^{3/2}(s)}{ds}\right|_{s=s_0} \qquad (2.43)$$

$$a_1^{3/2} \approx 2.98 \left.\frac{dA_0^{1/2}(s)}{ds}\right|_{s=s_0} + 5.17 \left.\frac{dA_0^{3/2}(s)}{ds}\right|_{s=s_0} \qquad (2.44)$$

where $a_1^I = A_1^I(s_0)$ are the values of the P-wave scattering amplitudes at the symmetry point.

Similarly from Eqn (2.24) we obtain the following first derivative conditions (see appendix III):

$$\left.\frac{dA_0^{1/2}(s)}{ds}\right|_{s=s_0} \approx -.031 \left.\frac{dB_0^0(t)}{dt}\right|_{t=t_0} + .1914\, B_1^1(t_0) \qquad (2.45)$$

$$\left.\frac{dA_0^{3/2}(s)}{ds}\right|_{s=s_0} \approx -.031 \left.\frac{dB_0^0(t)}{dt}\right|_{t=t_0} - .0957\, B_1^1(t_0) \qquad (2.46)$$

Many such conditions can be written down by evaluating the higher derivatives. But they will depend on higher partial waves to a greater extent.

2.2. <u>THE MANDELSTAM REPRESENTATION</u>:

The singularities of $A^{(\pm)}(s,u,t)$ in the variables s, u and t demanded by the unitarity conditions for the three channels may be obtained by studying the allowed diagrams for the scattering process. Conservation of G-parity will forbid even number of pions in the intermediate states for the kaon-pion scattering and odd number of pions in the intermediate states for the process $\pi\pi \rightarrow K\bar{K}$. With the Hamiltonian given by Eqn (1.1), the possible lower order diagrams are drawn in Fig. 2.2 and Fig. 2.3.

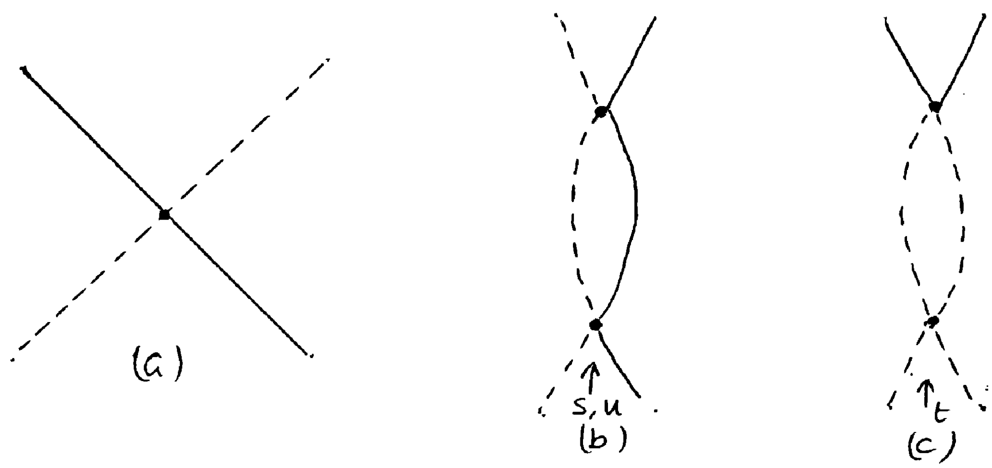


Fig 2.2. Chain diagrams.

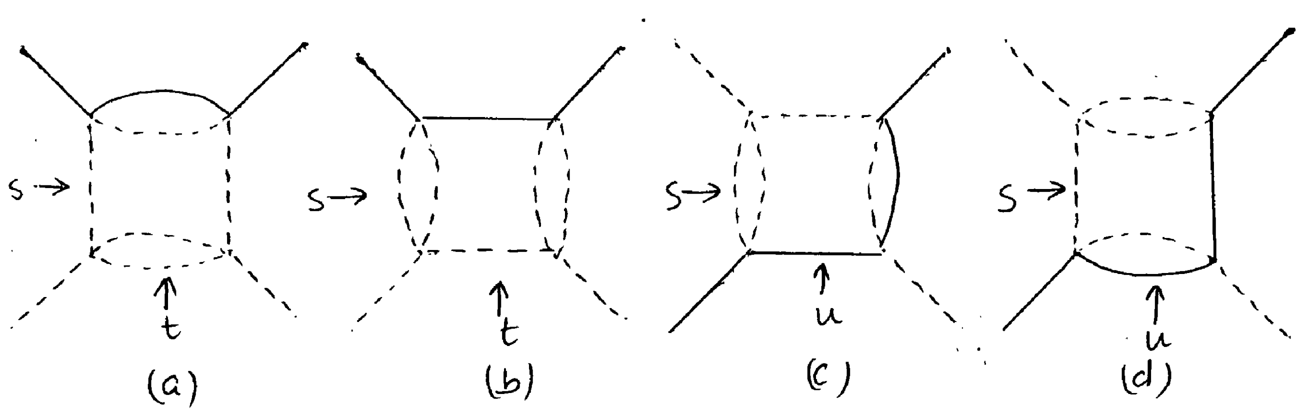


Fig 2.3. Box diagrams

Solid lines represent kaons and dotted lines represent pions. Diagram (a) of Fig. 2.2 corresponds to the $\lambda(K^+K)\phi_\pi^2$ term. In S-matrix theory this corresponds to an arbitrary constant, which is defined to be the coupling constant. This may be taken to be the value of the scattering amplitude at a particular point, which may conveniently be taken to be the symmetry point in our case. Diagram (a) together with unitarity leads to the "chain diagrams" as drawn in diagrams (b) and (c) of Fig. 2.2. The chain diagrams have singularities in only one variable and so give rise to the single spectral terms.

The diagrams of Fig. 2.3, commonly known as box diagrams give rise to the double spectral terms. Diagrams (a) - (d) give the outer most boundaries of the double spectral functions. These boundaries will be obtained in appendix IV.

The Mandelstam representation will mainly be used to determine the analytical properties of the partial wave amplitudes. Subtraction constants and single spectral terms ~~to~~ do not alter in any way these analytical properties. So the representation written without subtractions and single spectral terms will serve our purpose. This is as follows:

$$A^{(\pm)}(s,u,t) = \frac{1}{\pi^2}\iint ds'dt' \frac{A_{13}^{(\pm)}(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\iint ds'du' \frac{A_{12}^{(\pm)}(s',u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\iint du'dt' \frac{A_{23}^{(\pm)}(u',t')}{(u'-u)(t'-t)} \qquad (2\cdot47)$$

The boundaries of the double spectral functions as obtained in appendix IV are.

(a) For $A_{13}^{(\pm)}(s,t)$

I $\quad [t-4\mu^2][s-(m+3\mu)^2] - 32\mu^3(m+\mu) = 0 \qquad (2.48a)$

II $\quad [t-16\mu^2][s-(m+\mu)^2][s-(m-\mu)^2] - 64\mu^4 s = 0 \qquad (2.48b)$

(b) For $A_{12}^{(\pm)}(s,u)$

I $\quad [s-(m+\mu)^2][u-(m+3\mu)^2] - 16\mu^2(m+\mu)^2 = 0 \qquad (2.49a)$

II $\quad [u-(m+\mu)^2][s-(m+3\mu)^2] - 16\mu^2(m+\mu)^2 = 0 \qquad (2.49b)$

(c) For $A_{23}^{(\pm)}(u,t)$

The boundaries are the same as $A_{13}^{(\pm)}(s,t)$ and can be obtained from them by replacing s by u.

From the representation given by Eqn (2.47) the discontinuities in the variables s, u and t may be obtained. They are as follows:

$$A_1^{(\pm)}(s,u,t) = \frac{1}{\pi}\int dt' \frac{A_{13}^{(\pm)}(s,t')}{t'-t} + \frac{1}{\pi}\int du' \frac{A_{12}^{(\pm)}(s,u')}{u'-u} \qquad (2.50)$$

$$A_2^{(\pm)}(s,u,t) = \frac{1}{\pi}\int ds' \frac{A_{12}^{(\pm)}(s',u)}{s'-s} + \frac{1}{\pi}\int dt' \frac{A_{23}^{(\pm)}(u,t')}{t'-t} \qquad (2.51)$$

$$A_3^{(\pm)}(s,u,t) = \frac{1}{\pi}\int ds' \frac{A_{13}^{(\pm)}(s',t)}{s'-s} + \frac{1}{\pi}\int du' \frac{A_{23}^{(\pm)}(u',t)}{u'-u} \qquad (2.52)$$

With these discontinuities we can write down the single variable dispersion relations:

Fixed s:

$$A^{(\pm)}(s,u,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} du' \frac{A_2^{(\pm)}(s,u',\Sigma-s-u')}{u'-u} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{A_3^{(\pm)}(s,\Sigma-s-t',t')}{t'-t} \qquad (2.53)$$

Fixed u:

$$A^{(\pm)}(s,u,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \frac{A_1^{(\pm)}(s',u,\Sigma-s'-u)}{s'-s} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{A_3^{(\pm)}(\Sigma-u-t',u,t')}{t'-t} \qquad (2.54)$$

Fixed t:

$$A^{(\pm)}(s,u,t) = \frac{1}{\pi}\int_{(m+r)^2}^{\infty} ds' \frac{A_1^{(\pm)}(s',\Sigma-s'-t,t)}{s'-s} + \frac{1}{\pi}\int_{(m+r)^2}^{\infty} du' \frac{A_2^{(\pm)}(\Sigma-u'-t,u',t)}{u'-u} \tag{2.55}$$

These single dispersion relations, strictly speaking are not valid without any subtractions. The fixed t type will be used later on to obtain a sum rule for kaon-pion scattering and to determine certain set of parameters for $\pi\pi \rightarrow K\bar{K}$ amplitudes. Necessary subtractions will be made there. For the present need the single dispersion relations written in the above form will serve the purpose.

The application of crossing symmetry, Eqn (2.17) to the representation, Eqn (2.47) gives:

$$A_{13}^{(\pm)}(x,y) = \pm A_{23}^{(\pm)}(x,y) \tag{2.56}$$

$$A_{12}^{(\pm)}(x,y) = \pm A_{12}^{(\pm)}(y,x) \tag{2.57}$$

Putting this conditions in Eqns (2.50) - (2.52) we get

$$A_1^{(\pm)}(s,u,t) = \pm A_2^{(\pm)}(u,s,t) \tag{2.58}$$

$$A_3^{(\pm)}(s,u,t) = \pm A_3^{(\pm)}(u,s,t) \tag{2.59}$$

In the next section we shall derive the analytic properties of the kaon-pion partial wave scattering amplitude and obtain the discontinuities across the various cuts. The case of $\pi\pi \rightarrow K\bar{K}$ partial wave amplitudes will be taken up in chapter III.

2.3. ANALYTIC PROPERTIES OF THE PARTIAL WAVE AMPLITUDES FOR CHANNEL I:

The fixed s dispersion relation is used to determine the analytic properties of the partial wave amplitudes for kaon-pion scattering. The lth partial wave is projected out from Eqn (2.53) by using Eqn (2.27) as follows:

$$
\begin{aligned}
A_l^{(\pm)}(s) &= \frac{1}{2}\int_{-1}^{+1} dx\, P_l(x)\, A^{(\pm)}(s,x) \\
&= \frac{1}{2}\int_{-1}^{+1} dx\, P_l(x) \left\{ \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} du' \, \frac{A_2^{(\pm)}(s,u',\Sigma-s-u')}{u'-\Sigma+s-2k^2(1-x)} \right. \\
&\qquad \left. + \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \, \frac{A_3^{(\pm)}(s,\Sigma-s-t',t')}{t'+2k^2(1-x)} \right\} \qquad (2.60)
\end{aligned}
$$

where we have written x for $\cos\theta$ and used Eqn (2.3b) and (2.3c) to express u and t in terms of s and x. The singularities of $A_l^{(\pm)}(s)$ arise from two different sources. Firstly, any singularity of $A_2^{(\pm)}$ and $A_3^{(\pm)}$ will also appear in $A_l^{(\pm)}(s)$. Secondly, the vanishing of the denominators in Eqn (2.60) will give rise to singularities in $A_l^{(\pm)}(s)$.

The singularities of $A_2^{(\pm)}$ and $A_3^{(\pm)}$ are obtained by examining Eqns (2.51) and (2.52). The first denominators, in both of these equations give rise to a series of branch points on the real s axis. The first branch point appears at $s = (m+\mu)^2$ and corresponds to the physical threshold for kaon-pion scattering. The next branch point is at $s = (m+3\mu)^2$, the first inelastic threshold and the next one at $s = (m+5\mu)^2$ and so on. There ~~is~~ are no branch points at $s = (m+2\mu)^2$, $(m+4\mu)^2$, etc., because the conservation of G-parity forbids the even number of pions in channel I. A branch cut taken along the real axis in the range $(m+\mu)^2 \leq s \leq +\infty$ will account for all these branch points. This cut is known as the physical cut or the right hand cut, because it involves only the physical partial wave amplitudes for kaon-pion scattering. The second denominators in both of the two equations (2.51) and (2.52) give rise to singularities which cancel each other.

The singularities arising from the vanishing of the denominators in Eqn (2.60) can be obtained as follows. Both of the denominators are of the form a + bx. There are two situations which may give rise to singularities: (i) a = 0 and b = 0, and (ii) a ≠ 0 and b ≠ 0. For the first denominator $(u'-\Sigma+s-2k^2(1-x))$ condition (i) may be satisfied only at one point given by $u'=(m+\mu)^2$ and $s=(m-\mu)^2$. This gives a branch point at $s=(m-\mu)^2$. The second denominator $(t'+2k^2(1-x))$ cannot satisfy condition (i) since $t'\geqslant 4\mu^2$. The simplest way of determining the singularities arising from condition (ii) is to interchange the order of integrations in Eqn (2.60) and to perform the x-integration. Then we have

$$A_l^{(\pm)}(s) = -\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} du'\, A_2^{(\pm)}(s,u',\Sigma-s-u')\frac{1}{2k^2}Q_l\left(1-\frac{u'-\Sigma+s}{2k^2}\right)$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\, A_3^{(\pm)}(s,\Sigma-s-t',t')\frac{1}{2k^2}Q_l\left(1+\frac{t'}{2k^2}\right) \qquad (2.61)$$

where the Legendre polynomial of the second kind is defined by.

$$Q_l(a) = \frac{1}{2}\int_{-1}^{+1} dx \, \frac{P_l(x)}{a-x} \qquad (2.62)$$

It is well known that $Q_l(a)$ is analytic in the a-plane except for logarithmic branch points at a = $\pm$ 1. The cut may suitably be taken along the real axis from -1 to +1. Then the Q-function in the first term of Eqn (2.61) gives rise to the following singularities. The first branch point at $a = +1$ corresponds to $s = \Sigma - u'$. The physical threshold , $u' = (m+\mu)^2$ gives a branch point in s plane at $s = (m-\mu)^2$ The next branch point in u' at $u' = (m+3\mu)^2$ will give a branch point at $s = (m+\mu)(m-7\mu)$ and so on. When $u' \to +\infty$ s approaches $-\infty$. The second branch point of $Q_l(a)$ at $a = -1$ corresponds to $s = \frac{(m^2-\mu^2)^2}{u'}$ The physical threshold, $u' = (m+\mu)^2$ gives the branch point $s = (m-\mu)^2$ and the first inelastic threshold , $u' = (m+3\mu)^2$ gives a branch point at $s = (m^2-\mu^2)^2/(m+3\mu)^2$ and so on. As we approach the upper limit of u'-integration s tends to zero. So a branch cut on the real axis from $s = (m-\mu)^2$ to $s = -\infty$ will take into account all the singularities coming from the first term of Eqn (2.61).

For the second term in Eqn (2.61) the branch point $a = +1$ corresponds to $\frac{t'}{2k^2} = 0$. But since $t' \geq 4\mu^2$ it can only be satisfied when $k^2 = \pm\infty$. This happens when $s = 0$ and $s = -\infty$. So the second term in Eqn (2.61) gives rise to branch points at $s = 0$ and $s = -\infty$ for all values of t' corresponding to $a = +1$. The branch point $a = -1$ corresponds to $t' = -4k^2$. Solving for s we have

$$s = m^2 + \mu^2 - t/2 \pm \tfrac{1}{2}\sqrt{(t'-4m^2)(t'-4\mu^2)} \quad (2.63)$$

when $4\mu^2 < t < 4m^2$, s is complex and is real for $t' = 4\mu^2$ and $t' \geqslant 4m^2$. We first consider the later case. The branch point $t' = 4m^2$ gives a branch point in s at $s = -(m^2 - \mu^2)$ Taking the lower sign in the above equation it is found that as $t' \to +\infty$, s goes to $-\infty$. The upper sign corresponds to $s \to 0_-$ as $t' \to +\infty$. Then the higher branch points in t' give rise to branch points in s at the values of s tending to $-\infty$ with $t' \to +\infty$ for the lower sign and at the values of s tending to 0 from the negative side with $t' \to +\infty$. The point $t' = 4\mu^2$ gives a branch point at $s = m^2 - \mu^2$. We write $s = x + iy$ when $4\mu^2 < t' < 4m^2$. Where $x = m^2 + \mu^2 - t/2$ and $y = \frac{1}{2}\sqrt{(4m^2 - t')(t' - 4\mu^2)}$ Then evaluating $|s|^2 = x^2 + y^2$ we find that

$$|s|^2 = (m^2 - \mu^2)^2$$

This is the equation of a circle of radius $(m^2 - \mu^2)$ with centre at the origin. Then all the branch points in t' for $4\mu^2 < t' < 4m^2$ give rise to branch points in s lying on a circle of radius $(m^2 - \mu^2)$ corresponding to the branch point a = - 1 of $Q_l(a)$. As for example, the branch point $t' = 16\mu^2$ gives branch points in the s-plane at

$s = (m^2 - 7\mu^2) \pm i\sqrt{48\mu^2(m^2-4\mu^2)}$ So finally, the branch cuts taken along the circle $|s| = m^2-\mu^2$ and along $-\infty \leq s \leq 0$ on the real axis in the s-plane will account for all the singularities coming from the second term in Eqn (2.61).

Collecting all the singularities together we get the following analytical structure of $A_l^{(\pm)}(s)$ in the s-plane:

(i) The right hand cut: $(m+\mu)^2 \leq s \leq +\infty$

(ii) The left hand cut: $-\infty \leq s \leq (m-\mu)^2$

(iii) The circle cut: $|s| = m^2-\mu^2$

The unitarity condition, Eqn (2.30) shows that $A_l^I(s) \to$ constant as $s \to \infty$ So, a once subtracted dispersion relation may be written down for $A_l^I(s)$

$$A_l^I(s) = a_l^I + \frac{s-s_0}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \frac{\operatorname{Im} A_l^I(s')}{(s'-s)(s'-s_0)} + \frac{s-s_0}{\pi}\int_{-\infty}^{(m-\mu)^2} ds' \frac{\operatorname{Im} A_l^I(s')}{(s'-s)(s'-s_0)} + \frac{s-s_0}{\pi}\int_{\text{Circle}} ds' \frac{\Delta A_l^I(s')}{(s'-s)(s'-s_0)} \qquad (2.64)$$

where a subtraction is made at $s = s_0$ and a_l^I is the lth partial wave subtraction constant with isotopic spin I. The discontinuities across the cuts on the real axis equal the imaginary part as a consequence of the real analyticity property

$$A_l^I(s) = A_l^{I*}(s^*) \qquad (2.65)$$

That is

$$\lim_{\epsilon\to 0} \frac{1}{2i}\left[A_l^I(s+i\epsilon) - A_l^I(s-i\epsilon)\right] = \text{Im}\, A_l^I(s) \quad (2\cdot 66)$$

when s is on the real axis. $\Delta A_l^I(s)$ is the discontinuity across the circle cut given by:

$$\Delta A_l^I(s) = \frac{1}{2i}\left[A_l^I(s_{out}) - A_l^I(s_{in})\right] \quad (2\cdot 67)$$

where s_{out} and s_{in} are the values of s just outside and inside of the circle cut respectively.

The discontinuity across the right hand cut is given by the unitarity condition, Eqn (2.28). The discontinuities across the other cuts may be obtained by examining Eqn (2.61).

The discontinuity across the cut $-1 \leqslant a \leqslant +1$ of $Q_l(a)$ is given by:

$$Q_l(a+i\epsilon) - Q_l(a-i\epsilon) = -i\pi P_l(a) \quad (2\cdot 68)$$

We first determine the discontinuity across the left hand cut for contributions from channel II, that is for the first term in Eqn (2.61). Since the discontinuity is defined by going from $s+i\epsilon$ to $s-i\epsilon$ on the real axis, the argument of the Q_l- function is examined.

We have

$$a_l(s\pm i\epsilon) = a \mp i\epsilon \frac{\left[1 + \frac{(\Sigma - u' - s)(s^2 - (m^2-\mu^2)^2)}{4s^2k^2}\right]}{2k^2} \quad (2\cdot 69)$$

where $a(s+i\epsilon)$ and $a(s-i\epsilon)$ are the arguments of Q_l at $s+i\epsilon$ and $s-i\epsilon$ respectively. Then by examining the coefficient of $i\epsilon$ in the above equation for the range of u' - integration which gives $-1 \leqslant a \leqslant +1$ for a fixed s it is found that:

$$a(s \pm i\epsilon) = a \mp i\eta \qquad \text{when } 0 \leqslant s \leqslant (m-\mu)^2$$
$$= a \pm i\eta \qquad \text{when } s < 0$$

where η is positive and goes to zero as $\epsilon \to 0$. Now using Eqn (2.68) one can easily find out the discontinuity across the left hand cut for the first term of Eqn (2.61). It is as follows:

$$\operatorname{Im} \bar{A}_l^{(\pm)}(s) = -\frac{1}{4K^2}\int_{\Sigma-s}^{\frac{(m^2-\mu^2)^2}{s}} du'\, P_l\left(1+\frac{\Sigma-u'-s}{2K^2}\right) A_2^{(\pm)}(s,u',\Sigma-s-u') \qquad \text{when } 0 \leqslant s \leqslant (m-\mu)^2 \tag{2.70}$$

$$= \frac{1}{4K^2}\int_{(m+\mu)^2}^{\Sigma-s} du'\, P_l\left(1+\frac{\Sigma-u'-s}{2K^2}\right) A_2^{(\pm)}(s,u',\Sigma-s-u') \tag{2.71}$$

where a bar is placed on $\operatorname{Im} A_l^{(\pm)}(s)$ to indicate that it is the contribution from channel II.

When $0 \leqslant s \leqslant (m-\mu)^2$, $A_2^{(\pm)}(s,u',\Sigma-s-u')$ is entirely in the physical region of channel II, that is: $u' \geqslant (m+\mu)^2$ and $\cos\bar{\theta} = 1+\frac{\Sigma-u'-s}{2K^2}$ is in the range $-1 \leqslant \cos\bar{\theta} \leqslant +1$. Then the use of crossing symmetry gives:

$$A_2^{(\pm)}(s,u',\Sigma-s-u') = \pm \operatorname{Im} A^{(\pm)}(u',\cos\bar{\theta}) \tag{2.72}$$

For $s < 0$, $\cos\bar{\theta} \geqslant 1$ the equality applying only when the value of u' is at the upper limit of Eqn (2.71). But we shall still define $A_2^{(\pm)}(s,u',\Sigma-s-u')$ through Eqn (2.72). $\operatorname{Im} A^{(\pm)}(u',\cos\bar{\theta})$ can then be expressed in terms of physical partial wave amplitudes $A_l^{(\pm)}(u')$ by analytic continuation using the Legendre polynomial expansion in $\cos\bar{\theta}$. The boundaries of the double spectral functions determine the region in which such an expansion is convergent. In appendix IV it has been shown that the Legendre polynomial expansion for Eqn (2.71) is convergent only up to $s \gtrsim -27$ Combining Eqn (2.70) and (2.71) and using Eqn (2.18) we have

$$\operatorname{Im} \bar{A}_l^I(s) = -\frac{1}{4k^2}\int_{\Sigma - s}^{c(s)} du' P_l\left(1 + \frac{\Sigma - u' - s}{2k^2}\right) \sum_{I'} \alpha_{II'} \times \sum_{l'=0}^{\infty} (2l'+1) P_{l'}\left(1 + \frac{\Sigma - u' - s}{2k^2}\right) \operatorname{Im} A_{l'}^{I'}(u') \quad (2\cdot73)$$

where

$$c(s) = \frac{(m^2-\mu^2)^2}{s} \quad \text{when} \quad 0 \leq s \leq (m-\mu)^2$$
$$= (m+\mu)^2 \quad \text{when} \quad s < 0$$

Similarly examining the second term in Eqn (2.61) it is found that when s is on the left hand cut the argument of in the second term becomes:

$$a(s \pm i\epsilon) = a \mp i\epsilon \frac{s^2 - (m^2-\mu^2)^2}{8 s^2 k^4} t'$$

that is

$$a(s \pm i\epsilon) = a \pm i\eta \quad \text{when} \quad -(m^2-\mu^2) \leq s \leq 0$$
$$= a \mp i\eta \quad \text{when} \quad -\infty \leq s \leq -(m^2-\mu^2)$$

Then the contribution of channel III to the discontinuity on the left hand cut is given by:

$$\operatorname{Im} A_l^{I\pi\pi}(s) = -\epsilon(s + m^2 + \mu^2)\frac{1}{4k^2}\int_{4\mu^2}^{-4k^2} dt' P_l\left(1 + \frac{t'}{2k^2}\right) \sum_{I'} \gamma_{II'} \times \sum_{l'=0}^{\infty} (2l'+1) P_{l'}\left(\frac{s + p^2 + q^2}{2pq}\right) (pq)^{l'} \operatorname{Im} B_{l'}^{I'}(t') \quad \text{when } s < 0 \quad (2\cdot74)$$

where we have used:

$$A_3^{(\pm)}(s, \Sigma - s - t', t') = \mathrm{Im}\, B^{(\pm)}(t', \cos\varphi) \qquad (2.75)$$

$\mathrm{Im}\, B^{(\pm)}(t, \cos\varphi)$ is then expressed in terms of isotopic spin states by using Eqn (2.24) and expanded in Legendre polynomial series in $\cos\varphi$. Appendix IV shows that this expansion is convergent only up to $s \simeq -27$

On the circle cut, one can write $s = r e^{i\phi}$ where $r = m^2 - \mu^2$ is the radius of the circle. S can also be written in the form:

$$s_\pm(\lambda) = 2\lambda + m^2 + \mu^2 \pm 2i\sqrt{(\lambda + m^2)(-\lambda - \mu^2)} \qquad (2.76)$$

where $\lambda \equiv k^2$ and the (+)ve and the (-)ve sign corresponds to the upper and the lower half of the circle respectively. It is easy to see that s_{out} and s_{in} corresponds to $\lambda + \frac{i\epsilon}{2}\sin\phi$ and $\lambda - \frac{i\epsilon}{2}\sin\phi$ respectively. Then

$$a\begin{pmatrix} s_{out} \\ s_{in} \end{pmatrix} = a \mp i\epsilon \sin\phi \frac{t'}{4\lambda^2} \qquad (2.77)$$

That is

$$a\begin{pmatrix} s_{out} \\ s_{in} \end{pmatrix} = a \mp i\eta \text{ on the upper half of the circle.}$$
$$= a \pm i\eta \text{ on the lower half of the circle.}$$

Then the discontinuity across the circle cut becomes:

$$\Delta A_l^I(s) = \pm \frac{1}{4\lambda} \int_{4\mu^2}^{-4\lambda} dt'\, P_l\left(1 + \frac{t'}{2\lambda}\right) \sum_{I'} \gamma_{II'} \sum_{l'=0}^{\infty} (2l'+1)$$
$$\times (pq)^{l'} P_{l'}\left(\frac{s_\pm(\lambda) + p^2 + q^2}{2pq}\right) \mathrm{Im}\, B_{l'}^{I'}(t') \qquad (2.78)$$

Where the (+)ve sign corresponds to the upper half and the (-)ve sign to the lower half of the circle respectively. As shown in appendix IV, the Legendre polynomial expansion of $\mathrm{Im}\, B^{I'}(t', \cos\phi)$ is convergent everywhere on the circle cut.

The discontinuities Eqns (2.73),(2.74) and (2.78) may also be derived by examining Eqns (2.18) and (2.24).

In the next chapter we shall discuss the partial wave amplitudes for channel III.

CHAPTER III

$\pi\pi \rightarrow K\bar{K}$ PARTIAL WAVE AMPLITUDES

It has been seen in the last chapter that a knowledge of the partial wave scattering amplitudes for $\pi\pi \rightarrow K\bar{K}$ is necessary in the determination of the kaon-pion scattering amplitudes. This chapter is devoted to the formulation of an approximate method of solving the S and P waves of $\pi\pi \rightarrow K\bar{K}$ process in the low energy regions. Section I gives the analytical properties of the partial wave amplitudes. The discontinuities across the cuts are also obtained. In the next section the Omnes[53] method has been used to obtain approximate solutions for the S and P waves.

3.1. ANALYTICAL PROPERTIES OF THE PARTIAL WAVE AMPLITUDES FOR CHANNEL III

The fixed t dispersion relation is used to determine the analytical properties of the partial wave amplitudes, $B_l^{(\pm)}(t)$ Utilizing Eqn (2.33) to project out the lth partial wave from Eqn (2.55) we have

$$B_l^{(\pm)}(t) = \frac{1}{2(pq)^l}\int_{-1}^{+1} dx\, P_l(x)\, A^{(\pm)}(s,u,t)$$

$$= \frac{1}{2(pq)^l}\int_{-1}^{+1} dx\, P_l(x)\left\{\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\, \frac{A_1^{(\pm)}(s',\Sigma-s'-t,t)}{s'+p^2+q^2-2pqx}\right.$$

$$\left. + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} du'\, \frac{A_2^{(\pm)}(\Sigma-u'-t,u',t)}{u'+p^2+q^2+2pqx}\right\} \qquad (3.1)$$

where $\cos\phi$ is written as x and s and u are expressed in terms of t and x with the help of Eqns (2.7a) and (2.7b). From crossing symmetry it follows that

$$A_2^{(\pm)}(\Sigma-u'-t, u', t) = \pm A_1^{(\pm)}(u', \Sigma-u'-t, t) \qquad (3.2)$$

The replacing the integration variable u' in the second integral of Eqn (3.1) by s' one gets

$$B_l^{(\pm)}(t) = \frac{1}{2}\frac{1}{(pq)^l}\int_{-1}^{+1} dx\, P_l(x)\,\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\, A_1^{(\pm)}(s', \Sigma-s'-t, t)$$
$$\times\left\{\frac{1}{s'+p^2+q^2-2pqx} \pm \frac{1}{s'+p^2+q^2+2pqx}\right\} \qquad (3.3)$$

As in the case of kaon-pion scattering amplitudes the singularities arise from two different sources. Firstly, the absorptive parts $A_1^{(\pm)}$ and $A_2^{(\pm)}$ in Eqn (3.1) have singularities which may be obtained by examining Eqns (2.50) and (2.51). The denominator $(t'-t)$ in both of these equations gives rise to branch points in $B_l^{(\pm)}(t)$ at $t=4\mu^2$, $t=16\mu^2$ and so on. A branch cut taken along the real axis in the range $4\mu^2 \leq t \leq +\infty$ will cover all these branch points. The other denominators in Eqns (2.50) and (2.51) give rise to singularities which cancel out each other. Secondly, the vanishing of the denominators in Eqn (3.1) give rise to singularities of $B_l^{(\pm)}(t)$. These can be obtained by examining Eqn (3.3). The singularities are of end point type, that is they appear when $x = \pm 1$.

The first denominator can vanish at $x = +1$ for s' inside the range of the second integration of Eqn (3.3), that is for $s' \geq (m+\mu)^2$. This occurs at $t = -\frac{[s'-(m+\mu)^2][s'-(m-\mu)^2]}{s'}$ At the end point $x = -1$ the first denominator cannot vanish as shown below. This denominator at $x = -1$ is:

$$s' + p^2 + q^2 + 2pq = s' + t/2 - m^2 - \mu^2 + \tfrac{1}{2}\sqrt{(4m^2-t)(4\mu^2-t)}$$

For s' inside the integration limits, that is for $s' \geq (m+\mu)^2$ the combination $t/2 + \frac{1}{2}\sqrt{(4m^2-t)(4\mu^2-t)}$ is always greater than zero, whatever be the value of t. So, the denominator cannot vanish at $x = -1$ for $s' \geq (m+\mu)^2$. For the second denominator the points $x = +1$ and $x = -1$ interchange roles and give rise to the same set of singularities as the first denominator. The physical threshold, $s' = (m+\mu)^2$ thus gives rise to a branch point at $t = 0$. The next inelastic threshold gives a branch point at $t = -\frac{32\mu^2(m+\mu)(m+2\mu)}{(m+3\mu)^2} \simeq -18.9$ and so on. Higher and higher thresholds in s' give rise to branch points in the t plane on the negative real axis further and further from the origin. A branch cut taken along the real axis from $t = 0$ to $t = -\infty$ will cover all these branch points. The denominator $(pq)^l$ does not give rise to any singularities. This can be shown as follows. Since $B_l^{(+)}(t)$ is non zero only for even l values and $B_l^{(-)}(t)$ is non zero only for odd l values, the $(\pm)$ sign between the two denominators in Eqn (3.3) can be replaced by $(-1)^l$. Then the x-integration is performed after interchanging the order of the integrations. This gives

$$B_\ell^{(\pm)}(t) = \frac{1}{(pq)^{\ell+1}} \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} ds' \, A_1^{(\pm)}(s', \Sigma - s' - t, t) \, Q_\ell\left(\frac{s'+p^2+q^2}{2pq}\right) \tag{3.4}$$

To obtain the above equation we have used $Q_\ell(-a) = (-1)^\ell Q_\ell(a)$ for the second denominator. Now the term $(pq)^\ell$ vanishes at $p = 0$ and $q = 0$. So whenever we approach one of these two points.

$$a = \frac{s'+p^2+q^2}{2pq} \rightarrow +\infty$$

This allows the expansion of $Q_\ell(a)$ in a power series in a as follows:

$$Q_\ell(a) = \frac{\sqrt{\pi}\,\ell!}{2^{\ell+1}\,\Gamma(\ell+3/2)} \, a^{-\ell-1} \left\{ 1 + \frac{(\ell+1)(\ell+2)}{2(2\ell+3)} a^{-2} + \cdots \right\} \tag{3.5}$$

The leading term behaves like $(pq)^{\ell+1}$ and this cancels out the denominator in Eqn (3.4). The other terms approach zero faster than the denominator. Thus the denominator $(pq)^\ell$ in Eqn (3.3) gives rise to no singularity. Then $B_\ell^{(\pm)}(t)$ has the following singularities in the t-plane:

(i) Physical or the right hand cut: $4\mu^2 \leq t \leq +\infty$

(ii) The left hand cut: $-\infty \leq t \leq 0$

Ignoring all possible subtractions, the following dispersion relation can be written down for $B_\ell^{(\pm)}(t)$:

$$B_\ell^{(\pm)}(t) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \, \frac{\mathrm{Im}\, B_\ell^{(\pm)}(t')}{t'-t} + \frac{1}{\pi} \int_{-\infty}^{0} dt' \, \frac{\mathrm{Im}\, B_\ell^{(\pm)}(t')}{t'-t} \tag{3.6}$$

where the real analyticity property:

$$B_l^{(\pm)}(t) = B_l^{(\pm)*}(t^*) \qquad (3.7)$$

has been used to express the discontinuities across the cuts in terms of the imaginary parts of the amplitudes. That is

$$\lim_{\epsilon \to 0} \frac{1}{2i}\left[B_l^{(\pm)}(t+i\epsilon) - B_l^{(\pm)}(t-i\epsilon)\right] = \operatorname{Im} B_l^{(\pm)}(t) \qquad (3.8)$$

The discontinuity across the physical cut is given by the unitarity condition, which in the two pion approximation is:

$$\operatorname{Im} B_l^{I}(t) = \frac{q}{\sqrt{t}} B_l^{I*}(t) A_l^{I\pi\pi}(t) \qquad (3.9)$$

where $A_l^{I\pi\pi}(t)$ is the pion-pion scattering amplitude in the lth partial wave with total isotopic spin I. $B_l^{I}(t)$ is connected to $B_l^{(\pm)}$ by $B_l^{0} = \sqrt{6}\, B_l^{(+)}$ and $B_l^{1} = 2 B_l^{(-)}$.

The discontinuity across the left hand cut may be obtained by examining Eqn (3.4). It comes from the discontinuity of $Q_l(a)$ across the branch cut taken along $-1 \leq a \leq +1$ where $a = \dfrac{s' + p^2 + q^2}{2pq}$. This is given by

$$Q_l(a+i\epsilon) - Q_l(a-i\epsilon) = -i\pi P_l(a) \qquad (3.10)$$

The sign of the discontinuity on the left hand cut is decided by the imaginary part a(t) picks up when we put

$t = t \pm i\epsilon$. It is given by

$$a(t \pm i\epsilon) = a \pm i\epsilon \; \frac{4p_-^2 q_-^2 + (p_-^2 + q_-^2)(s' - p_-^2 - q_-^2)}{16 p_-^3 q_-^3}$$

where

$$p_- = \sqrt{m^2 - t/4} \quad \text{and} \quad q_- = \sqrt{\mu^2 - t/4}$$

one can easily find that the coefficient of $i\epsilon$ is positive for all values of s' in the range $(m+\mu)^2 \leq s' \leq (p_- + q_-)^2$ which gives $-1 < a \leq +1$ for a fixed t. Then $a(t \pm i\epsilon) = a \pm i\eta$ using Eqn (3.10) in Eqn (3.4) the discontinuity across the left hand cut is found to be given by:

$$\operatorname{Im} B_l^{(\pm)}(t) = -\int_{(m+\mu)^2}^{(p_- + q_-)^2} ds' \, \frac{1}{2(p_- q_-)^{l+1}} P_l\left(\frac{s' - p_-^2 - q_-^2}{2 p_- q_-}\right) A_1^{(\pm)}(s', \Sigma - s' - t, t) \tag{3.11}$$

On the right hand side of this equation we put:

$$A_1^{(\pm)}(s', \Sigma - s' - t, t) = \operatorname{Im} A^{(\pm)}(s', \cos\theta) \tag{3.12}$$

where

$$\cos\theta = 1 + \frac{t}{2k^2} \tag{3.13}$$

$\operatorname{Im} A^{(\pm)}(s', \cos\theta)$ is not exactly the imaginary part of the scattering amplitude in channel I, because $\cos\theta$ is not in the physical region. Examining Eqn (3.11) it can be seen that for values of s' in the range of the integration, $\cos\theta \leq -1$, the equality sign applies only when s' is at the upper limit of the integration. Then

$\mathrm{Im}\, A^{(\pm)}(s', \cos\theta)$ has to be analytically continued to express it
$\mathrm{Im}\, A^{(\pm)}(s', \cos\theta)$ has to be analytically continued to express it
in terms of physical amplitudes by expanding in a Legendre polynomial series in $\cos\theta$. This expansion is convergent only in a limited region of the l-plane. In appendix IV it has been shown that this region extends only up to $t \approx -32$ on the left hand cut. Lastly, expressing all the amplitudes in terms of respective eigenstates of total isotopic spin one gets:

$$\mathrm{Im}\, B_l^I(t) = -\int_{(m+\mu)^2}^{(p_-+q_-)^2} ds' \, \frac{1}{2(p_- q_-)^{l+1}} P_l\left(\frac{s'-p_-^2-q_-^2}{2p_-q_-}\right) \sum_{I'} \beta_{II'}$$
$$\times \sum_{l'=0}^{\infty} (2l'+1) P_{l'}\left(1+\frac{t}{2k^2}\right) \mathrm{Im}\, A_{l'}^{I'}(s') \qquad (3.14)$$

where the crossing matrix $\beta_{II'}$ is given by

$$\beta_{II'} = \begin{pmatrix} \sqrt{6}/3 & 2\sqrt{6}/3 \\ 2/3 & -2/3 \end{pmatrix} \qquad (3.15)$$

It is to be noted that Eqn (3.14) can also be obtained from

$$B^I(t, \cos\varphi) = \sum_{I'} \beta_{II'} A^{I'}(s, \cos\theta) \qquad (3.16)$$

by projecting out the lth partial wave and calculating the discontinuity across the left hand cut in the t-plane.

When t approaches the limiting point of the left hand cut at $t=0$ the integration over s' in Eqn (3.14) collapses. Because, when $t \to 0$ we have

$$(p_-+q_-)^2 \approx (m+\mu)^2 \left[1 - \frac{t}{4m\mu}\right]$$

So the upper limit approaches the lower limit as t goes to zero. Now very near the threshold in channel I:

$$\operatorname{Im} A_{l'}^{I'}(s') \simeq k^{4l'+1} \cdot \text{Constant}$$
$$\approx [s' - (m+\mu)^2]^{2l+1/2} \cdot \text{Constant}.$$

Examining the Legendre polynomial series expansion on the right hand side of Eqn (3.14) it is found that the s-wave term behaves like $[s' - (m+\mu)^2]^{1/2}$. For other terms $\operatorname{Im} A_{l'}^{I'}(s')$ goes to zero faster than the above quantity. Now if we express $P_{l'}(\cos\theta)$ in terms of $\cos\theta$, a term $(1+\frac{t}{2k^2})^{l'}$ appears which when combined with the behaviour of $\operatorname{Im} A_{l'}^{I'}(s')$ produces a term behaving like $[s' - (m+\mu)^2]^{1/2}$, but there will also be a factor $(t)^{l'}$ with it. So only the s-wave in the Legendre polynomial is leading when t approaches zero. The argument of $P_{l'}$, $\frac{s'+p^2+q^2}{2pq}$ approaches a constant in the above limit of t. Then we have

$$\operatorname{Im} B_l^I(s) \approx \text{Const.} \int_{(m+\mu)^2}^{(m+\mu)^2[1-\frac{t}{4m\mu}]} ds' \, [s' - (m+\mu)^2]^{1/2}$$
$$\approx \text{Const.} \, (-t)^{3/2} \tag{3.17}$$

Thus $\operatorname{Im} B_l^I(t)$ goes to zero as $(-t)^{3/2}$ when t approaches the origin along the left hand cut. This behaviour will have some important consequences in the next section.

3.2. APPROXIMATE SOLUTIONS FOR S AND P WAVE AMPLITUDES:

The unitarity condition in the two pion approximation, Eqn (3.9) shows that $B_l^I(t)$ has the same phase as the pion-pion scattering amplitude for $4\mu^2 \leq t \leq 16\mu^2$. One can define the quantity $B_l^I(t)D_l^I(t)$ where $D_l^I(t)$ has the phase $e^{-i\delta_l^{I\pi\pi}(t)}$ on the right hand cut. Then $B_l^I(t)D_l^I(t)$ will have the singularities in the t-plane:

(i) A cut on the positive t-axis: $16\mu^2 \leq t \leq +\infty$

(ii) The left hand cut of $B_l^I(t)$: $-\infty \leq t \leq 0$

By making the assumption that four-pion and other higher mass intermediate states have small contributions unless t is very large, the lower limit of cut (i) may be raised quite a bit from $t = 16\mu^2$. On the left hand cut the Legendre polynomial expansion in Eqn (3.14) is convergent only up to $t \simeq -32\mu^2$. So it is not justified to use this equation to calculate the discontinuity further beyond $t = -32\mu^2$. Since we are mainly interested in the low energy region on the right hand cut, say in the range $4\mu^2 \leq t \leq 50\mu^2$ it will not be a very bad approximation to cut off the left hand cut at $t = -32\mu^2$ and replace the neglected portion from $t = -32\mu^2$ to $t = -\infty$ by two poles at fixed positions on the real axis. The prescription giving these pole positions will be discussed shortly. The residues may be determined by comparison with kaon-pion scattering amplitude in the forward direction. We then write down the dispersion relation:

$$B_l^I(t) = \frac{1}{D_l^I(t)}\left\{\frac{1}{\pi}\int_{-32\mu^2}^{0} dt' \, \frac{\operatorname{Im} B_l^I(t')D_l^I(t')}{t'-t} + \frac{\alpha_l^I}{t+t_1} + \frac{\beta_l^I}{t+t_2}\right\} \qquad (3.19)$$

for the S and P waves. The right hand cut is neglected, because the contribution of this may be expected to be ~~expected to be~~ fairly small in the low energy region. The effect of neglecting this cut, as well as the effects of ignoring necessary subtractions may be expected to be absorbed in the parameters α_l^I and β_l^I .

The D_l^I's for pion-pion scattering are determined as follows:

(i) $l = 0$ I = 0 AMPLITUDE:

Experimental results indicate that there is a peak in the cross sections for this amplitude in the low energy region. The phase shift is expected to rise up to about 30^o near t = 5. This is the well known ABC anomaly. There seems to be some evidence of the existence of a resonance called the σ- meson[54] at a slightly higher energy. The mechanism producing such an s-wave resonance very near the threshold is not very clearly understood. Since the ABC anomaly seems to be well confirmed we shall consider only this. Hamilton et al[46] have already given an approximate solution for this amplitude by replacing the entire left hand cut by a pole, which reproduces the experimental result quite well. We shall use this method. It is outlined here for the sake of completeness. The s-wave amplitude we are considering may be written as:

$$A_0^0(t) = N_0^0(t) / D_0^0(t) \qquad (3.19)$$

where $N_0^0(t)$ has the left hand cut $-\infty \leq t \leq 0$ and $D_0^0(t)$

has the right hand cut $4\mu^2 \leq t \leq +\infty$ for the pion-pion scattering amplitude. The following dispersions can be written down immediately:

$$N_0^0(t) = \frac{1}{\pi}\int_{-\infty}^{0} dt' \frac{\operatorname{Im} A_0^0(t') D_0^0(t')}{t'-t} \qquad (3.20)$$

$$D_0^0(t) = 1 - \frac{t-t_0}{\pi}\int_{4\mu^2}^{\infty} dt' \sqrt{\frac{t'-4}{t'}} \frac{N_0^0(t')}{(t'-t)(t'-t_0)} \qquad (3.21)$$

where one subtraction is made in $D_0^0(t)$ at $t = t_0$ and $D_0^0(t)$ is normalized by taking $D_0^0(t_0) = 1$. The elastic unitarity condition for $\operatorname{Im} A_0^{0^{-1}}(t)$:

$$\operatorname{Im} A_0^{0^{-1}}(t) = -\sqrt{\frac{t-4}{t}} \qquad (3.22)$$

is used on the right hand cut.

Hamilton replaces the discontinuity on the left hand cut by a delta function contribution, $\operatorname{Im} A_0^0(t) = -\pi\,\Gamma\,\delta(t+t_s)$ Then after choosing $t_0 = -t_s$ we have

$$N_0^0(t) = \frac{\Gamma}{t+t_s} \qquad (3.23)$$

Substituting Eqn (3.23) in (3.21) the final expression for $D_0^0(t)$ is obtained.

$$D_0^0(t) = 1 - \frac{t+t_s}{\pi}\int_{4\mu^2}^{\infty} dt' \sqrt{\frac{t'-4}{t'}} \frac{\Gamma}{(t'-t)(t'+t_s)^2} \qquad (3.24)$$

The integration can be performed analytically. The derivation is given in appendix V. Then we have:

For $4\mu^2 \leq t \leq +\infty$

$$\mathrm{Re}\, D_0^0(t) = 1 + \frac{2\Gamma}{\pi}\left\{ \frac{1}{t+t_s}\left[\sqrt{\frac{t-4}{t}} \ln \frac{\sqrt{t}+\sqrt{t-4}}{2} - \sqrt{\frac{t_s+4}{t_s}} \ln \frac{\sqrt{t_s}+\sqrt{t_s+4}}{2} \right] + \frac{1}{2t_s} - \frac{2}{t_s(t_s+4)}\sqrt{\frac{t_s+4}{t_s}} \ln \frac{\sqrt{t_s}+\sqrt{t_s+4}}{2} \right\} \tag{3.25}$$

$$\mathrm{Im}\, D_0^0(t) = -\theta(t-4)\sqrt{\frac{t-4}{t}}\;\frac{\Gamma}{t+t_s} \tag{3.26}$$

For $-\infty \leq t \leq 0$

$$D_0^0(t) = 1 + \frac{2\Gamma}{\pi}\left\{ \frac{1}{t_s+t}\left[\sqrt{\frac{t-4}{t}} \ln \frac{\sqrt{-t}+\sqrt{4-t}}{2} - \sqrt{\frac{t_s+4}{t_s}} \ln \frac{\sqrt{t_s}+\sqrt{t_s+4}}{2} \right] + \frac{1}{2t_s} - \frac{2}{t_s(t_s+4)}\sqrt{\frac{t_s+4}{t_s}} \ln \frac{\sqrt{t_s}+\sqrt{t_s+4}}{2} \right\} \tag{3.27}$$

The phase shift may be obtained by using:

$$\delta_0^0(t) = \tan^{-1}\left[\sqrt{\frac{t-4}{t}}\;\frac{N_0^0(t)}{\mathrm{Re}\, D_0^0(t)} \right] \tag{3.28}$$

If the following choice:

$$t_s = 116 \quad \text{and} \quad \Gamma = 60$$

is made, then the phase shift is found to reach the maximum of about 30° at $t \approx 7.0$. After that it falls off slowly.

(ii) <u>l = 1 I = 1 AMPLITUDE:</u>

This amplitude has got the well known resonance called the $\rho-$ meson at the energy of about 750 MeV. The half width is roughly about 50 MeV. We choose the following form for $D_1'(t)$

$$D_1'(t) = \frac{t_R - t}{t_R} - i\theta(t-4)\frac{\Gamma_\rho}{t_R}\sqrt{\frac{(t-4)^3}{t}} \qquad (3.29)$$

where t_R and Γ_ρ are the position and the reduced width of the ρ-meson respectively.

The pole positions t_1 and t_2 are determined by a process first used by Balaz[55]. The neglected portion of the left hand cut is

$$\Delta_l^I(t) = \frac{1}{\pi}\int_{-\infty}^{-32\mu^2} dt' \frac{\operatorname{Im} B_l^I(t')\, D_l^I(t')}{t'-t} \qquad (3.30)$$

Substituting $x = -\mu^2/t'$ one gets

$$\Delta_l^I(t) = -\frac{1}{\pi}\int_0^{.03125} \frac{dx}{x}\, \frac{\operatorname{Im} B_l^I(-\mu^2/x)\, D_l^I(-\mu^2/x)}{1 + xt} \qquad (3.31)$$

Then approximating:

$$\frac{1}{1+xt} \approx \sum_{i=1}^{n} \frac{G_i(x)}{1 + x_i t} \qquad (3.32)$$

we have

$$\Delta_l^I(t) = \sum_{i=1}^{n} \frac{F_{li}^I}{t_i + t} \qquad (3.33)$$

where

$$F_{li}^I = -\frac{1}{\pi}\int_0^{.03125} \frac{dx}{x}\, \frac{\operatorname{Im} B_l^I(-\mu^2/x)\, D_l^I(-\mu^2/x)}{x_i}\, G_i(x) \qquad (3.34)$$

and $t_i = \mu^2/x_i$

The case with n = 2 is chosen for the present need. Then $G_1(x)$ and $G_2(x)$ are equations to straight lines:

$$G_{1,2}(x) = \frac{x - x_2}{x_1 - x_2}, \quad \frac{x - x_1}{x_2 - x_1} \qquad (3.35)$$

The pole positions t_1 and t_2 are determined by obtaining the best fit between the right hand side of Eqn (3.32) and $y = \frac{1}{1+xt}$ for the values of t in the range $4\mu^2 \leq t \leq 50\mu^2$. A numerical calculation shows that $t_1 \approx 40$ and $t_2 \approx 200$ gives a reasonably good fit for t in the range mentioned above. Eqn (3.33) is rewritten as:

$$\Delta_l^I(t) = \frac{\alpha_l^I}{t + t_1} + \frac{\beta_l^I}{t + t_2} \qquad (3.36)$$

To determine the residues α_l^I and β_l^I the fixed t dispersion relation is used. For the $A^{(+)}$ amplitude one subtraction is necessary. A very suitable point of subtraction is $s = (m+\mu)^2$ and $t = 0$ because at this point,

$$A^{(+)}(s, \cos\theta) = A^{(+)}((m+\mu)^2, 1) = a_0^{(+)} \qquad (3.37)$$

where $a_0^{(+)} = \frac{1}{3}(a_0^{1/2} + 2a_0^{3/2})$ is a combination of the s-wave scattering lengths defined by $a_0^I = A_0^I(s = (m+\mu)^2)$ Eqn (3.37) is a consequence of the threshold behaviour k^{2l} for $A_l^I(s)$.

Then we have

$$A^{(+)}(s,u,t) = a_0^{(+)} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \, \mathrm{Im}\, A^{(+)}(s', \Sigma - s' - t, t)$$

$$\times \left\{ \frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-(m+\mu)^2} - \frac{1}{s'-(m-\mu)^2} \right\} \qquad (3.38)$$

where the third and the fourth denominators are the subtraction terms for the first and the second integral in Eqn (2.52) respectively. Crossing symmetry has been used to express $A_2^{(\pm)}(s,u,t)$ in terms of $A_1^{(\pm)}(s,u,t)$ and then $A_1^{(+)}$ is equated to the imaginary part of the scattering amplitude for kaon-pion scattering. When $t=0$ this is fully justified because $A_1^{(+)}$ is then exactly the imaginary part of the forward scattering amplitude. Now, the s-wave amplitude $B_0^{(+)}(t)$ may be projected out from Eqn (3.38) as follows:

$$B_0^{(+)}(t) = a_0^{(+)} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \,\mathrm{Im}\, A^{(+)}(s',t)\left\{\frac{1}{2pq}\ln\left(1+\frac{4pq}{s'+p^2+q^2-2pq}\right) - \frac{1}{s'-(m+\mu)^2} - \frac{1}{s'-(m-\mu)^2}\right\} \qquad (3.39)$$

We evaluate Eqn (3.39) and its first derivative with respect to t at $t=0$. Retaining only the S and P waves of $\mathrm{Im}\, A^{(+)}(s',t)$ one obtains:

$$B_0^0(0) = \sqrt{6}\, a_0^{(+)} + \frac{\sqrt{6}}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \left[\mathrm{Im}\, A_0^{(+)}(s') + 3\,\mathrm{Im}\, A_1^{(+)}(s')\right] \times\left\{\frac{1}{2m\mu}\ln\left(1+\frac{4m\mu}{s'-(m+\mu)^2}\right) - \frac{1}{s'-(m+\mu)^2} - \frac{1}{s'-(m-\mu)^2}\right\} \qquad (3.40)$$

$$\left.\frac{d}{dt}B_0^0(t)\right|_{t=0} = \frac{\sqrt{6}}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \Bigg[\mathrm{Im}\, A_0^{(+)}(s')\left\{\frac{m^2+\mu^2}{16m^3\mu^3}\ln\left(1+\frac{4m\mu}{s'-(m+\mu)^2}\right) - \frac{s'(m^2+\mu^2)-(m^2-\mu^2)^2}{4m^2\mu^2[s'-(m+\mu)^2][s'-(m-\mu)^2]}\right\}$$
$$+ 3\,\mathrm{Im}\, A_1^{(+)}(s')\left\{\frac{1}{4m\mu}\left[\left(\frac{1}{k^2}+\frac{m^2+\mu^2}{4m^2\mu^2}\right)\ln\left(1+\frac{4m\mu}{s'-(m+\mu)^2}\right)+\frac{1}{m\mu}\right] - \frac{s'-m^2-\mu^2}{k^2}\left[\frac{1}{[s'-(m+\mu)^2][s'-(m-\mu)^2]}+\frac{1}{16m^2\mu^2}\right]\right\}\Bigg] \qquad (3.41)$$

where we have used $B_0^0(t) = \sqrt{6}\, B_0^{(+)}(t)$

For the $A^{(-)}$ amplitude there is no need for a subtraction in the fixed t dispersion relation. Because $A^{(-)} = A^{1/2} - A^{3/2}$, and Pomeranchuk's theorem[56] states that $A^{1/2} \approx A^{3/2}$ at high energies. So $A^{(-)} \to 0$ as $s \to +\infty$ Then one can write

$$A^{(-)}(s,u,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \,\mathrm{Im}\, A^{(-)}(s', \Sigma - s' - t, t)\left\{\frac{1}{s'-s} - \frac{1}{s'-u}\right\} \quad (3.42)$$

where crossing symmetry is used and $A_1^{(-)}$ has been equated to $\mathrm{Im}\, A^{(-)}$ for channel I. Projecting out the P-wave amplitude $B_1^{(-)}(t)$ from the above equation:

$$B_1^{(-)}(t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \,\mathrm{Im}\, A^{(-)}(s', t)$$

$$\times \frac{1}{2p^2q^2}\left\{ \frac{s'+p^2+q^2}{2pq} \ln\left(1 + \frac{4pq}{s'+p^2+q^2-2pq}\right) - 2\right\} \quad (3.43)$$

Then evaluating Eqn (3.43) and its first derivative with respect to t at $t = 0$ we have in the same approximation as in the case of Eqns (3.40 and (3.41):

$$B_1^1(0) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \left\{\mathrm{Im}\, A_0^{(-)}(s') + 3\,\mathrm{Im}\, A_1^{(-)}(s')\right\}$$

$$\times \frac{1}{m^2\mu^2}\left\{ \frac{s'-m^2-\mu^2}{2m\mu} \ln\left(1 + \frac{4m\mu}{s'-(m+\mu)^2}\right) - 2\right\} \quad (3.44)$$

$$\frac{d}{dt} B_1^1(t)\Big|_{t=0} = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \Big[\mathrm{Im}\, A_0^{(-)}(s') \Big\{ \frac{1}{m\mu}\Big(1 + \frac{3(m^2+\mu^2)}{4m^2\mu^2}(s'-m^2-\mu^2)\Big)$$

$$\times \ln\Big(1 + \frac{4m\mu}{s'-(m+\mu)^2}\Big) - \frac{1}{k^2} - \frac{3(m^2+\mu^2)}{m^2\mu^2}\Big\} \Big/ 4m^2\mu^2$$

$$+ 3\,\mathrm{Im}\, A_1^{(-)}(s') \Big\{ \frac{1}{m\mu}\Big[1 + (s'-m^2-\mu^2)\Big\{\frac{1}{k^2} + \frac{3(m^2+\mu^2)}{4m^2\mu^2}\Big\}\Big]$$

$$\times \ln\Big(1 + \frac{4m\mu}{s'-(m+\mu)^2}\Big) - \Big(\frac{5}{k^2} + \frac{3(m^2+\mu^2)}{m^2\mu^2}\Big)\Big\} \Big/ 4m^2\mu^2 \Big]$$

(3.45)

Eqns (3.40), (3.41), (3.44) and (3.45) may be used to determine α_l^I and β_l^I for the S and P waves in terms of the forward scattering amplitudes for channel I. Eqn (3.18) and its first derivative at $t = 0$ are evaluated as follows:

$$B_l^I(0)\, D_l^I(0) = \frac{1}{\pi}\int_{-32\mu^2}^{0} dt' \frac{\mathrm{Im}\, B_l^I(t')\, D_l^I(t')}{t'} + \frac{\alpha_l^I}{t_1} + \frac{\beta_l^I}{t_2} \quad (3.46)$$

$$\frac{d}{dt}\Big\{ B_l^I(t)\, D_l^I(t) \Big\}\Big|_{t=0} = \frac{1}{\pi}\int_{-32\mu^2}^{0} dt' \frac{\mathrm{Im}\, B_l^I(t')\, D_l^I(t')}{t'^2} - \frac{\alpha_l^I}{t_1^2} - \frac{\beta_l^I}{t_2^2} \quad (3.47)$$

where in both the integrals the upper limit $t' = 0$ does not give rise to any trouble because of the behaviour of $\mathrm{Im}\, B_l^I(t')$ as $t' \to 0$ given by Eqn (3.17). Then defining

$$X_l^I = B_l^I(0)\, D_l^I(0) - \frac{1}{\pi}\int_{-32\mu^2}^{0} dt' \frac{\mathrm{Im}\, B_l^I(t')\, D_l^I(t')}{t'} \quad (3.48)$$

$$Y_l^I = \frac{d}{dt}\left\{B_l^I(t)\,D_l^I(t)\right\}\Big|_{t=0} - \frac{1}{\pi}\int_{-32\mu^2}^{0} dt' \frac{\operatorname{Im} B_l^I(t')\,D_l^I(t')}{t'^2} \tag{3.49}$$

We get after solving Eqns (3.46) and (3.47) for α_l^I and β_l^I

$$\alpha_l^I = \frac{t_1^2}{t_1 - t_2}\left(X_l^I + t_2 Y_l^I\right) \tag{3.50}$$

$$\beta_l^I = -\frac{t_2^2}{t_1 - t_2}\left(X_l^I + t_1 Y_l^I\right) \tag{3.51}$$

Thus we have in our hands an approximate method of obtaining the solutions to S and P waves for $\pi\pi \rightarrow K\bar{K}$ in the low energy region. The solutions depend on the kaon-pion scattering to a great extent. Numerical results are discussed in chapter V .

CHAPTER IV

INVERSE AMPLITUDE DISPERSION RELATIONS FOR KAON PION PARTIAL WAVE AMPLITUDES

In this chapter we attempt to solve the partial wave dispersion relations for the kaon-pion scattering using the inverse amplitude method of Bransden and Moffat.[50] In section I the behaviours of $A_l^I(s)$ at the physical threshold, $s = (m+\mu)^2$ and at the crossed threshold, $s = (m-\mu)^2$ are discussed. These have very significant consequences in the inverse amplitude method. The dispersion relation for $A_l^I(s)$ is written down in section II. An unsuccessful attempt to solve the coupled S and P wave dispersion relations is also discussed very briefly in this section. The third and the last section gives a formulation for the S-wave amplitudes in which the high-energy contributions on the unphysical cuts from channels II and III are suppressed.

4.1. THE BEHAVIOUR OF $A_l^I(s)$ AT THE PHYSICAL THRESHOLD, $s = (m+\mu)^2$ AND AT THE CROSSED THRESHOLD, $s = (m-\mu)^2$:

The Mandelstam representation may be used to obtain the behaviour of $A_l^I(s)$ at the physical threshold, $s = (m+\mu)^2$ Projecting at the lth partial wave amplitude for channel I from the fixed S dispersion relation, Eqn (2.53) one gets:

$$A_l^{(\pm)}(s) = -\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} du'\, A_2^{(\pm)}(s, u', \Sigma-s-u')\, \frac{1}{2k^2}\, Q_l\left(1+\frac{\Sigma-u'-s}{2k^2}\right)$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\, A_3^{(\pm)}(s, \Sigma-s-t', t')\, \frac{1}{2k^2}\, Q_l\left(1+\frac{t'}{2k^2}\right) \qquad (4.1)$$

when s approaches $(m+\mu)^2$, k^2 goes to zero. So the arguments of both the Q_l - functions blow up in this limit. This allows the following power series expansion of Q_l for s very near the threshold:

$$Q_l(a) = \frac{l!\sqrt{\pi}}{2^{l+1}\Gamma(l+3/2)}\, a^{-l-1}\left\{1 + \frac{(l+1)(l+2)}{2(2l+3)}\, a^{-2} + \cdots\right\} \qquad (4.2)$$

An examination of the arguments of the Q_l - functions in Eqn (4.1) shows that both terms give a behaviour k^{2l} for $A_l^I(s)$ as s approaches the physical threshold. The above behaviour is for the real part of the amplitudes. By putting this in the unitarity condition:

$$\mathrm{Im}\, A_l^I(s) = \frac{k}{\sqrt{s}}\left|A_l^I(s)\right|^2 R_l^I \qquad (4.3)$$

we get the behaviour of k^{4l+1} for the imaginary part near the physical threshold.

The Mandelstam representation cannot be used to obtain the behaviour of $A_l^I(s)$ at the crossed threshold, $s=(m-\mu)^2$ Because at that point the first denominator of the fixed s-dispersion relation, Eqn (2.53) vanishes at $u'=(m+\mu)^2$ The crossing relation

$$A^I(s,\cos\theta) = \sum_{I'} \alpha_{II'}\, A^{I'}(u, \cos\bar{\theta}) \qquad (4.4)$$

can be used to investigate this. The lth partial wave amplitude, $A_l^I(s)$ may be projected out from the above equation:

$$A_l^I(s) = \frac{1}{2}\int_{-1}^{+1} dx\; P_l(x) \sum_{I'} \alpha_{II'} A^{I'}(u, \cos\bar{\theta}) \qquad (4.5)$$

where $\cos\theta$ has been written as x and

$$u = \Sigma - s + 2k^2(1-x) \qquad (4.6)$$

$$\cos\bar{\theta} = 1 + \frac{t}{2\bar{k}^2} = 1 - \frac{k^2(1-x)}{\bar{k}^2} \qquad (4.7)$$

k^2 and $\bar{k}^2$ are defined in terms of s and u respectively by Eqns (2.5) and (2.6d). When $0 \leqslant s \leqslant (m-\mu)^2$, $A^{I'}(u, \cos\bar{\theta})$ is entirely in the physical region, that is $u \geqslant (m+\mu)^2$ and $-1 \leqslant \cos\bar{\theta} \leqslant +1$ Then Eqn (4.5) becomes:

$$A_l^I(s) = \frac{1}{2}\int_{-1}^{+1} dx\, P_l(x) \sum_{I'} \alpha_{II'} \sum_{l'=0}^{\infty} (2l'+1)\, P_{l'}(\cos\bar{\theta})\, A_{l'}^{I'}(u) \qquad (4.8)$$

When we allow s to approach $(m-\mu)^2$ from the left, u is found to approach $(m+\mu)^2$ for all values of x. The threshold behaviour of $A_{l'}^{I'}(u)$ allows the following power series expansion in $\bar{k}^2$:

$$\begin{aligned} A_{l'}^{I'}(u) \simeq &\left\{ \bar{k}^{2l'} a_{l'}^{I'} + \bar{k}^{2l'+2} b_{l'}^{I'} + \cdots \right\} \\ &+ i \left\{ \bar{k}^{4l'+1} c_{l'}^{I'} + \bar{k}^{4l'+3} d_{l'}^{I'} + \cdots \right\} \end{aligned} \qquad (4.9)$$

Now $\bar{k}^2$ may be expanded in a power series in x for u very near $(m+\mu)^2$ by using first Eqn (2.6d) to express it in terms of u and then Eqn (4.6). The expansion is:

$$\bar{k}^2 \simeq \frac{\Sigma^2-(m^2-\mu^2)^2-s^2}{2(\Sigma-s)^2}k^2 + \frac{1}{2}\left\{\frac{(m^2-\mu^2)^2}{(\Sigma-s)^2}-1\right\}k^2x + \frac{(m^2-\mu^2)^2(1-x)^2}{(\Sigma-s)^3}k^4+\cdots \qquad (4.10)$$

Using Eqn (4.9) in Eqn (4.8) we have

$$\begin{aligned} A_l^I(s) = \frac{1}{2}\int_{-1}^{+1} dx\, P_l(x)\Big[&\left\{[a_0^{I'}+\bar{k}^2b_0^{I'}+\cdots]+i[\bar{k}c_0^{I'}+\bar{k}^3d_0^{I'}+\cdots]\right\} \\ &+3\left(1-\frac{k^2(1-x)}{\bar{k}^2}\right)\left\{[\bar{k}^2a_1^{I'}+\bar{k}^4b_1^{I'}+\cdots]+i[\bar{k}^5c_1^{I'}+\bar{k}^7d_1^{I'}+\cdots]\right\} \\ &+\frac{5}{2}\left\{3\left(1-\frac{k^2(1-x)}{\bar{k}^2}\right)^2-1\right\}\left\{[\bar{k}^4a_2^{I'}+\bar{k}^6b_2^{I'}+\cdots]+i[\bar{k}^9c_2^{I'}+\bar{k}^{11}d_2^{I'}+\cdots]\right\} \\ &+\frac{7}{2}\left\{5\left(1-\frac{k^2(1-x)}{\bar{k}^2}\right)^3-3\left(1-\frac{k^2(1-x)}{\bar{k}^2}\right)\right\}\left\{[\bar{k}^6a_3^{I'}+\bar{k}^8b_3^{I'}+\cdots] \right. \\ &\left. +i[\bar{k}^{13}c_3^{I'}+\bar{k}^{15}d_3^{I'}+\cdots]\right\}\Big] \end{aligned} \qquad (4.11)$$

The behaviours of the real and the imaginary parts of $A_l^I(s)$ are treated separately. For the real part the following property of the Legendre polynomials:

$$\int_{-1}^{+1} dx\, P_l(x)x^n = 0 \qquad \text{If } n<l \qquad (4.12)$$

implies that all contributions to $\mathrm{Re}\,A_l^I(s)$ in Eqn (4.11) come from only the terms on the right hand side having a power of x greater or equal to 1. A close examination reveals that all terms involving the power x^n have also a factor k^{2n}. This shows that the leading term in $\mathrm{Re}\,A_l^I(s)$ near the crossed threshold behaves as k^{2l} .

The case of the imaginary parts is very interesting. The leading term comes from the s-wave on the right hand side.

Retaining only the first order term in k^2 in Eqn (4.10) we have

$$\bar{k} = k\left[\frac{\Sigma^2-(m^2-\mu^2)^2-s^2}{2(\Sigma-s)^2}+\frac{1}{2}\left\{\frac{(m^2-\mu^2)^2}{(\Sigma-s)^2}-1\right\}x\right]^{1/2} \quad (4.13)$$

Then it follows that:

$$\operatorname{Im} A_l^I(s) = \frac{1}{2}\int_{-1}^{+1} dx\, P_l(x) \sum_{I'} \alpha_{II'}\, k\left[\frac{\Sigma^2-(m^2-\mu^2)^2-s^2}{2(\Sigma-s)^2}+\frac{1}{2}\left\{\frac{(m^2-\mu^2)^2}{(\Sigma-s)^2}-1\right\}x\right]^{1/2} C_0^{I'} \quad (4.14)$$

This gives the behaviour of $\operatorname{Im} A_l^I(s)$ at the crossed threshold as follows:

$$\operatorname{Im} A_l^I(s) \approx k \sum_{I'} \alpha_{II'} C_0^{I'} \quad (4.15)$$

where $C_0^{I'}$ is related to the s-wave scattering length $a_0^{I'}$. It is quite surprising that the real part has the same l-fold zero at the crossed threshold as at the physical threshold. While the imaginary part has a behaviour independent of the value of l.

From Eqn (4.11) a very interesting and important result may be written down for the s-wave amplitudes

$$b_0^I = \sum_{I'} \alpha_{II'}\, a_0^{I'} \quad (4.16)$$

where b_0^I is written for the amplitude, $A_0^I(s)$ at the crossed threshold, $s = (m-\mu)^2$. The above result will be very much useful in section III in the formulation of the s-wave dispersion relations.

4.2. INVERSE AMPLITUDE DISPERSION RELATIONS FOR $A_l^I(s)$:

Let us define:

$$G_l^I(s) = 1/A_l^I(s) \qquad (4.17)$$

From the unitarity condition for the partial wave amplitudes it follows that $A_l^I(s)$ goes to a constant as $s \to \infty$. Assuming this constant is non zero, $G_l^I(s)$ also goes to a constant at infinity. The inverse amplitude, $G_l^I(s)$ shares with $A_l^I(s)$ all the singularities the latter has got. In addition, any complex zero of $A_l^I(s)$ will give rise to more singularities of $G_l^I(s)$. It is assumed that no such complex zeros exist at least in the nearby region of the complex s-plane. In the next chapter it is shown that such an assumption may well be wrong. Then, with the above assumptions $G_l^I(s)$ has the following singularities:

(i) The physical cut: $(m+\mu)^2 \leqslant s \leqslant +\infty$

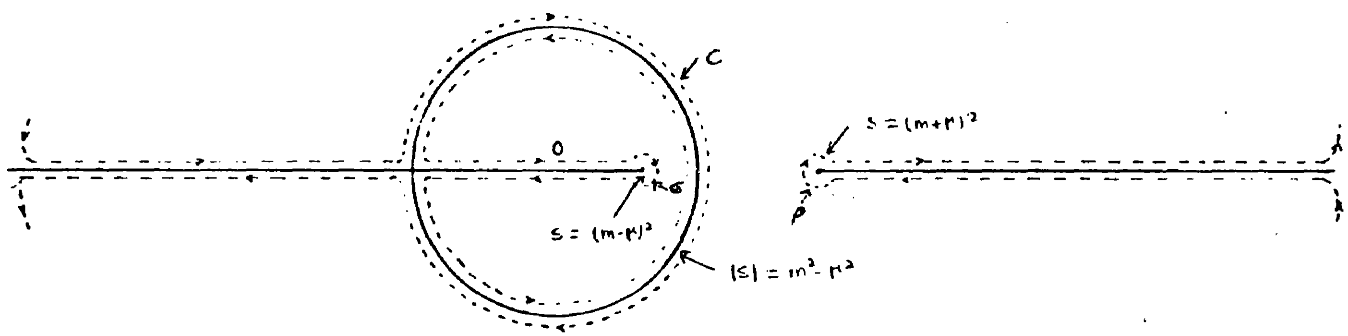
(ii) The left hand cut: $-\infty \leqslant s \leqslant (m-\mu)^2$

(iii) The circle cut: $|s| = m^2-\mu^2$

Since $G_l^I(s)$ goes to a constant at infinity, a once subtracted dispersion relation can be written down for it by applying Cauchy's theorem to the contour drawn in Fig. 4.1. Then we have

$$G_l^I(s) = \gamma_l^I + \lim_{\epsilon\to 0} \frac{s-s_0}{2\pi i}\int_{(m+\mu)^2}^{\infty} ds' \, \frac{G_l^I(s'+i\epsilon) - G_l^I(s'-i\epsilon)}{(s'-s)(s'-s_0)}$$

$$+ \lim_{\epsilon\to 0} \frac{s-s_0}{2\pi i}\int_{-\infty}^{(m-\mu)^2} ds' \, \frac{G_l^I(s'+i\epsilon) - G_l^I(s'-i\epsilon)}{(s'-s)(s'-s_0)}$$



$(m+\mu)^2 \simeq 21$
$(m-\mu)^2 \simeq 6{\cdot}7$
$(m^2-\mu^2) \simeq 13{\cdot}87$

FIG 4.1 THE CONTOUR IN S PLANE

$$+\frac{s-s_0}{2\pi i}\int_C ds'\,\frac{G_l^I(s'_{out})-G_l^I(s'_{in})}{(s'-s)(s'-s_0)}+\lim_{\rho\to 0}\frac{s-s_0}{2\pi i}\int_\rho ds'\,\frac{G_l^I(s')}{(s'-s)(s'-s_0)}$$

$$+\lim_{\sigma\to 0}\frac{s-s_0}{2\pi i}\int_\sigma ds'\,\frac{G_l^I(s')}{(s'-s)(s'-s_0)} \qquad (4.18)$$

Where the integral with a suffix C is over the circle cut. The integration variable on this cut is $s'=|s|e^{i\phi}$ where $|s|=m^2-\mu^2$. s'_{out} and s'_{in} are defined by $s'_{\substack{out\\in}}=(|s|\pm\epsilon)e^{i\phi}$ with $\epsilon\to 0$. The last two terms come from the small circles of radius ρ around the point $s=(m+\mu)^2$ and of radius σ around the point $s=(m-\mu)^2$ It is shown in appendix VI that:

$$\lim_{\rho\to 0}\frac{s-s_0}{2\pi i}\int_\rho ds'\,\frac{G_l^I(s')}{(s'-s)(s'-s_0)}=-(s-s_0)\sum_{n=0}^{l-1}\frac{[s-(m+\mu)^2]^{n-1}}{n!}\,g_l^{I(n)}((m+\mu)^2) \qquad (4.19)$$

and

$$\lim_{\sigma\to 0}\frac{s-s_0}{2\pi i}\int_\sigma ds'\,\frac{G_l^I(s')}{(s'-s)(s'-s_0)}=\lim_{\sigma\to 0}\frac{\text{Const.}}{\sigma^{1/2}} \qquad (4.20)$$

Cancelled by similar contribution from L.H.C.

These are the consequences of the behaviour of $A_l^I(s)$ at the physical and the crossed thresholds.

We now define

$$\lim_{\epsilon\to 0}\frac{1}{2i}\left[G_l^I(s-i\epsilon)-G_l^I(s+i\epsilon)\right]=F_l^I(s)\ ,\ (m+\mu)^2\leq s\leq+\infty \qquad (4.21)$$

$$\lim_{\epsilon\to 0}\frac{1}{2i}\left[G_l^I(s-i\epsilon)-G_l^I(s+i\epsilon)\right]=K_l^I(s)\ ,\ -\infty\leq s\leq(m-\mu)^2 \qquad (4.22)$$

$$\lim_{\epsilon\to 0}\frac{1}{2i}\left[G_l^I([|s|-\epsilon]e^{i\phi})-G_l^I([|s|+\epsilon]e^{i\phi})\right]=M_l^I(s),\ |s|=m^2-\mu^2 \qquad (4.23)$$

Then the inverse amplitude dispersion relation becomes

$$G_l^I(s) = \gamma_l^I + L_l^I(s,s_0) + N_l^I(s,s_0) + R_l^I(s,s_0) - (s-s_0)\sum_{n=0}^{l-1} \frac{[s-(m+\mu)^2]^{n-l}}{n!} g_l^{I(n)}((m+\mu)^2) - iT_l^I(s) \tag{4.24}$$

where

$$L_l^I(s,s_0) = -\frac{s-s_0}{\pi} P \int_{(m+\mu)^2}^{\infty} ds' \frac{F_l^I(s')}{(s'-s)(s'-s_0)} \tag{4.25}$$

$$N_l^I(s,s_0) = -\frac{s-s_0}{\pi} P \int_{-\infty}^{(m-\mu)^2} ds' \frac{K_l^I(s')}{(s'-s)(s'-s_0)} \tag{4.26}$$

$$R_l^I(s,s_0) = -\frac{s-s_0}{\pi} P \int_{C} ds' \frac{M_l^I(s')}{(s'-s)(s'-s_0)} \tag{4.27}$$

where the letter P in front of all the integrals means that we should take the principal value when the denominator may vanish, otherwise it should be ignored.

$$\begin{aligned} T_l^I(s) &= F_l^I(s) \quad \text{for } (m+\mu)^2 \leq s \leq +\infty \\ &= K_l^I(s) \quad \text{for } -\infty \leq s \leq (m-\mu)^2 \\ &= M_l^I(s) \quad \text{for } |s| = m^2-\mu^2 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{4.28}$$

The subtraction constants γ_l^I in Eqn (4.24) are the values of $G_l^I(s)$ at the point $s = s_0$. This may be chosen to be the symmetry point.

On the right hand cut the discontinuity is given by the unitarity condition:

$$F_l^I(s) = \frac{k}{\sqrt{s}} R_l^I(s) \tag{4.29}$$

where $R_l^I(s)$ is the coefficient of inelasticity. When the scattering is elastic $R_l^I(s) = 1$ and $L_l^I(s, s_0)$ can be determined analytically.

On the left hand cut the discontinuity may be written down as follows:

$$K_l^I(s) = \frac{\operatorname{Im} A_l^I(s)}{[\operatorname{Re} A_l^I(s)]^2 + [\operatorname{Im} A_l^I(s)]^2} \qquad (4.30)$$

$\operatorname{Im} A_l^I(s)$ can be calculated from crossing by using Eqns (2.73) and (2.74) in terms of physical quantities in channels II and III. The real part, $\operatorname{Re} A_l^I(s)$ is calculated from the dispersion relation, Eqn (4.24) through

$$\operatorname{Re} A_l^I(s) = \frac{\operatorname{Re} G_l^I(s)}{[\operatorname{Re} G_l^I(s)]^2 + [K_l^I(s)]^2} \qquad (4.31)$$

On the circle cut we may write

$$A_l^I(s_\pm) = X_l^I(s) \pm i \Delta A_l^I(s) \qquad (4.32)$$

$$G_l^I(s_\pm) = Y_l^I(s) \mp i M_l^I(s) \qquad (4.33)$$

where $s = |s| e^{i\phi}$ and $s_\pm = (|s| \pm \epsilon) e^{i\phi}$. The (+)ve sign corresponding to the outside and the (-)ve sign corresponding to the inside of the circle. Every quantity occurring in the above two equations is complex. From Eqn (4.32) and the definition of $G_l^I(s)$ we have

$$M_l^I(s) = \lim_{\epsilon\to 0} \frac{1}{2i}\left[G_l^I(s_-) - G_l^I(s_+)\right]$$
$$= \lim_{\epsilon\to 0} \frac{1}{2i}\left[\frac{1}{A_l^I(s_-)} - \frac{1}{A_l^I(s_+)}\right]$$
$$= \frac{\Delta A_l^I(s)}{[X_l^I(s)]^2 + [\Delta A_l^I(s)]^2} \qquad (4.34)$$

$\Delta A_l^I(s)$ can be calculated by using Eqn (2.78) in terms of physical quantities in channel III. $X_l^I(s)$ may be obtained from the dispersion relation, Eqn (4.24) by using Eqn (4.33):

$$X_l^I(s) = \frac{1}{2}\left[A_l^I(s_+) + A_l^I(s_-)\right]$$
$$= \frac{1}{2}\left[\frac{1}{G_l^I(s_+)} + \frac{1}{G_l^I(s_-)}\right]$$
$$= \frac{Y_l^I(s)}{[Y_l^I(s)]^2 + [M_l^I(s)]^2} \qquad (4.35)$$

where

$$Y_l^I(s) = \gamma_l^I + L_l^I(s,s_0) + N_l^I(s,s_0) + R_l^I(s,s_0)$$
$$-(s-s_0)\sum_{n=0}^{l-1} \frac{[s-(m+\mu)^2]^{n-l}}{n!}\, g_l^{I(n)}((m+\mu)^2) \qquad (4.36)$$

Examining Eqn (2.78) we can find the following property of $\Delta A_l^I(s)$:

$$\Delta A_l^I(s) = -\Delta A_l^{I*}(s^*) \qquad (4.37)$$

Using this and the property of real analyticity for $A_l^I(s)$ [In fact the above equation follows from the real analyticity of $A_l^I(s)$] one gets

$$M_l^I(s) = -M_l^{I*}(s^*) \qquad (4.38)$$

This also follows, straightforwardly, from the real analyticity property of $G_l^I(s)$. Then defining the variable $\lambda \equiv k^2$

on the circle cut by

$$S_{\pm}(\lambda) = 2\lambda + m^2 + \mu^2 \pm 2i\sqrt{(\lambda+m^2)(-\lambda-\mu^2)} \tag{4.39}$$

where the (+)ve and the (-)ve sign refer to the upper and the lower half of the circle respectively, we can write

$$R_l^I(s,s_0) = E_l^I(s,s_0) + E_l^{I*}(s^*,s_0) \tag{4.40}$$

where

$$E_l^I(s,s_0) = -\frac{s-s_0}{\pi}\int_{-m^2}^{-\mu^2} d\lambda \, \frac{S_+(\lambda)}{i\sqrt{(\lambda+m^2)(-\lambda-\mu^2)}} \, \frac{M_l^I(S_+(\lambda))}{(S_+(\lambda)-s)(S_+(\lambda)-s_0)} \tag{4.41}$$

We now describe very briefly the programme used unsuccessfully for solving the coupled S and P waves. The iterative procedure is practically the same as that for the S-wave case discussed in detail in the next chapter. So nothing is said here about it.

The S-wave amplitudes involve one parameter, the subtraction constant for each isotopic spin state. The behaviour at the physical threshold does not give rise to any parameter in this case. While, for the P-wave there are two parameters for each isotopic spin state, the subtraction constant and one constant for the threshold behaviour. Then we have

$$\begin{pmatrix} G_0^I(s) \\ G_1^I(s) \end{pmatrix} = \gamma_{0,1}^I + L_{0,1}^I(s,s_0) + N_{0,1}^I(s,s_0) + R_{0,1}^I(s,s_0) - \begin{pmatrix} 0 \\ \frac{s-s_0}{s-(m+\mu)^2} C_1^I \end{pmatrix} - i\,T_{0,1}^I(s) \tag{4.42}$$

where

$$C_1^{I} = g_1^{I(0)}\left((m+\mu)^2\right)$$

The symmetry point is chosen to be the subtraction point. Then, assuming the effect of D and higher waves to be small we have

$$\gamma_0^{1/2} = \gamma_0^{3/2} \approx -1/\lambda \qquad (4.43)$$

where $\lambda = -A^{(+)}(s_0, u_0, t_0)$ is defined to be the coupling constant of the theory and is taken to be an arbitrary parameter in the solutions.

The P-wave parameters may be determined as follows. Since $\cos\theta = 0$ at the symmetry point, we have

$$\frac{1}{\gamma_1^{I}} = A_1^{I}(s_0) \approx \frac{1}{3}\frac{\partial}{\partial\cos\theta} A^{I}(s,\cos\theta)\bigg|_{\substack{s=s_0\\ \cos\theta=0}} \qquad (4.44)$$

The threshhold behaviour of $A_1^{I}(s)$ gives

$$-\frac{(m+\mu)^2}{m\mu\left[(m+\mu)^2-s_0\right]}\frac{1}{C_1^{I}} = \frac{1}{3}\frac{\partial}{\partial\cos\theta}\left[\frac{A^{I}(s,\cos\theta)}{k^2}\right]\Bigg|_{s=(m+\mu)^2} \qquad (4.45)$$

The derivatives with respect to $\cos\theta$ are calculated from fixed t dispersion relations written in a particular form given in appendix VII.

Thus, an iterative method may be set up to solve the coupled S and P wave dispersion relations in terms of only one arbitrary parameter. This iterative scheme is a very complicated one involving the $\pi\pi \rightarrow K\bar{K}$ amplitudes which are determined in terms of $K\pi$ amplitudes. It was

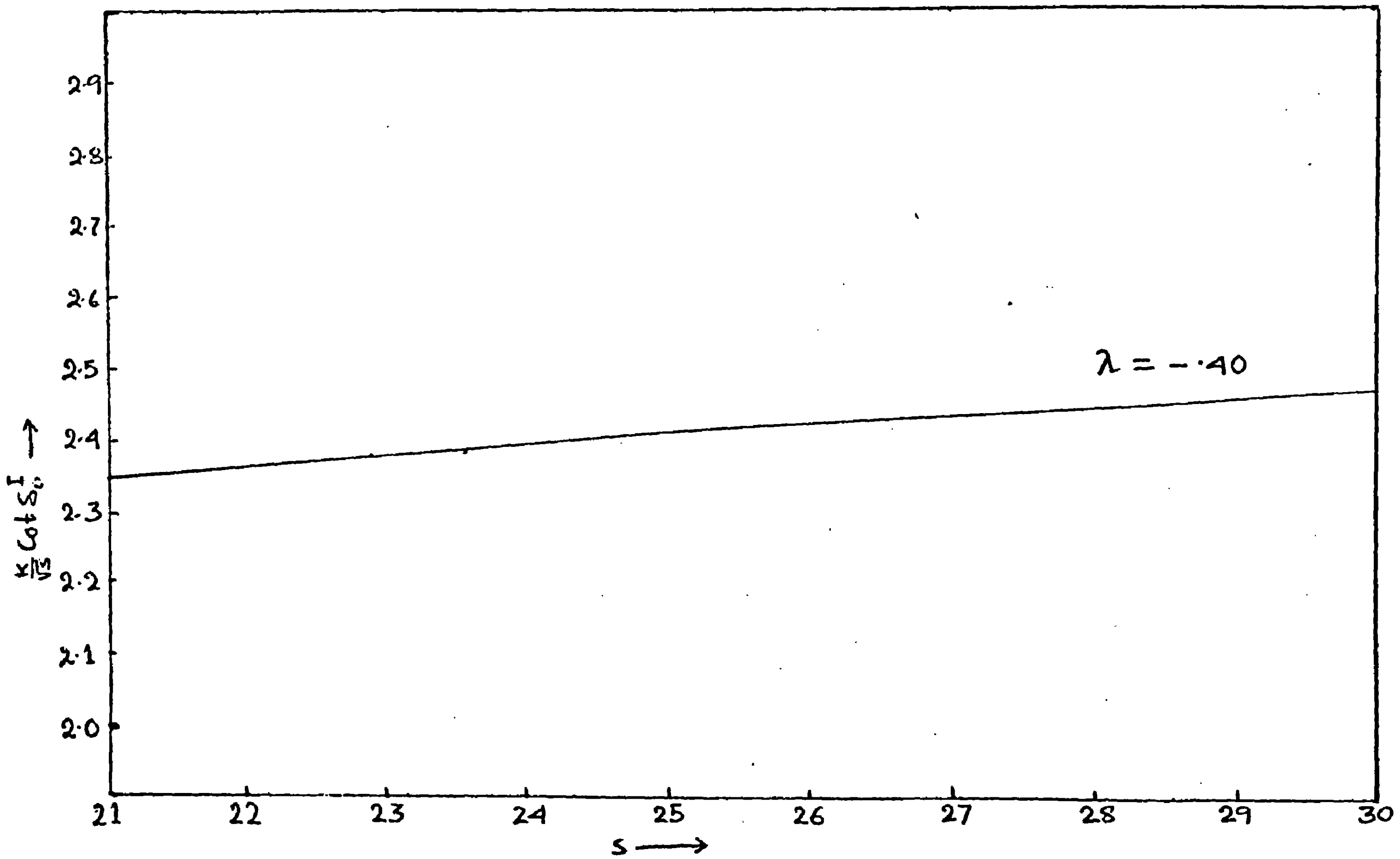


FIG 4·2 S-DOMINANT SOLUTION
NO ISOTOPIC SPIN SPLITTING

programmed on the digital computer IBM7090. Only stable solution that could be obtained was the familiar S dominant solution first obtained by Chew and Mandelstam[57] in the case of the pion-pion scattering. Otherwise, the iterative scheme was very unstable giving no solution at all. Fig. 4.2 show that S-dominant solution for $\lambda = -.40$ There is no isotopic spin splitting of the solutions.

4.3. THE S-WAVE INVERSE AMPLITUDE DISPERSION RELATIONS:

Let us examine Eqns (2.74) and (2.75) giving the contributions of channel II and III respectively to the imaginary parts on the left hand cut. For the crossed kaon-pion contribution the upper limit of integration over u' is $(M^2-\mu^2)^2/s$ in Eqn (2.74). This goes to infinity as s approaches zero from the right. Thus very high energy kaon-pion scattering contributes to ~~the~~ $\mathrm{Im}\, A_\ell^I(s)$ when s is quite near the origin. Similarly, in Eqn (2.75) the higher limit is $-4k^2$ in the t'- integration. Now $k^2 \to -\infty$ as $s \to 0-$. This brings in very high energy $\pi\pi \to K\bar{K}$ amplitudes. The following table shows the values of these upper limits and k^2 for various values of s around the origin:

S	Upper limit of crossed kπ u'	Upper limit of channel III t'	K^2
+ 2.0	70.5	-	11.2
+ 1.0	140.99	-	28.56
+ .5	281.98	-	63.68
+ .1	1409.9	-	345.56
- .1	-	1437.73	- 359.4
- 1	-	169.74	- 42.43
- 5	-	60.9	- 15.24
- 10	-	51.84	- 12.96

(In terms of u', t' 1 Bev $\approx$ 52.5)

Eqn (4.30) shows that when $\mathrm{Im}\, A_l^I(s)$ is large the discontinuity of the inverse amplitude, $K_l^I(s)$ is small. The high energy regions of channel II, which is, of course, kaon-pion scattering and of channel III are not known at all. If the low-energy solutions for these are extended to the high energy region there will be very large errors in the contributions to $\mathrm{Im}\, A_l^I(s)$ on the left hand cut near the origin through Eqn (2.74) and (2.75). Calculated this way $\mathrm{Im}\, A_l^I(s)$ around the origin is found to be quite large. This makes $K_l^I(s)$ quite small in this region. With this in mind the s-wave dispersion relations in the formulation given in the last section were solved. On the left hand cut the contributions of P-wave kaon-pion amplitudes were kept in the K^* approximation. The region around the origin was found not to be fully suppressed. Such a solution cannot be relied on. The above table also shows the behaviour of K^2 near the origin. Then, instead of considering $G_0^I(s)$, if we consider $G_0^I(s)/K^2$ the region around the origin may be expected to be fully suppressed. The application of Cauchy's theorem to the contour drawn in Fig. 4.1 gives the following dispersion relation:

$$G_0^I(s) = \frac{s-(m-\mu)^2}{4s}\gamma_0^I + \frac{s-(m+\mu)^2}{4s}\delta_0^I + L_0^I(s) + N_0^I(s) + R_0^I(s) - i\,T_0^I(s) \qquad (4.46)$$

where

$$L_0^I(s) = -\frac{K^2}{\pi} P\int_{(m+\mu)^2}^{\infty} ds' \frac{F_0^I(s')}{K^2(s')(s'-s)} \qquad (4.47)$$

$$N_0^I(s) = -\frac{K^2}{\pi} P\int_{-\infty}^{(m-\mu)^2} ds' \frac{K_0^I(s')}{K^2(s')(s'-s)} \qquad (4.48)$$

$$R_0^I(s) = -\frac{K^2}{\pi} P\int_{C} ds' \frac{M_0^I(s')}{K^2(s')(s'-s)} \qquad (4.49)$$

The first and the second term in Eqn (4.46) come from the integration round the circles of radius ρ around $s=(m+\mu)^2$ and of radius σ around $s=(m-\mu)^2$ respectively. Since $G_0^I(s)$ goes to a constant at both the physical and the crossed thresholds and K^2 goes to zero, these contributions are non zero. As shown in appendix VI these contributions are:

$$\lim_{\rho\to 0} \frac{k^2}{2\pi i}\int_\rho ds' \frac{G_0^I(s')}{k^2(s')(s'-s)} = k^2 \frac{\gamma_0^I}{s-(m+\mu)^2} = \frac{s-(m-\mu)^2}{4s}\gamma_0^I \quad (4.50)$$

$$\lim_{\sigma\to 0} \frac{k^2}{2\pi i}\int_\sigma ds' \frac{G_0^I(s')}{k^2(s')(s'-s)} = k^2 \frac{\delta_0^I}{s-(m-\mu)^2} = \frac{s-(m+\mu)^2}{4s}\delta_0^I \quad (4.51)$$

The letter P in front of the integrals in Eqns (4.47) - (4.49) has the same significance as before. In Eqn (4.47) $k^2(s')$ goes to zero at $s'=(m+\mu)^2$ and $F_0^I(s')\approx k(s')$ for $s'\to(m+\mu)^2$ Similarly for Eqn (4.48) at the point $s'=(m-\mu)^2$, $k^2(s')$ goes to zero and $K_0^I(s')$ behaves as $k(s')$. It is shown in appendix VI that these behaviours do not give rise to any difficulty.

Using elastic unitarity the integration in Eqn (4.47) may be performed. The derivations for S on various cuts are given in appendix V. Using the variable $\omega = s - m^2 - \mu^2$ on the real axis we have.

On the right hand cut for which $2m\mu \leq \omega \leq +\infty$

$$L_0^I(\omega) = -\frac{1}{2\pi}\frac{\sqrt{\omega^2-4m^2\mu^2}}{\omega+m^2+\mu^2}\ln\frac{\omega-\sqrt{\omega^2-4m^2\mu^2}}{2m\mu} \quad (4.52)$$

On the left hand cut for which $-\infty \leq \omega \leq -2m\mu$

$$L_0^I(\omega) = -\frac{1}{2\pi}\frac{\sqrt{\omega^2-4m^2\mu^2}}{\omega+m^2+\mu^2}\ln\frac{-\omega+\sqrt{\omega^2-4m^2\mu^2}}{2m\mu} \quad (4.53)$$

On the circle cut L_0^I is complex. Using the variable λ we have

$$\mathrm{Re}\, L_0^I(\lambda) = -\frac{\lambda}{(m^2-\mu^2)\pi}\left\{\sqrt{\frac{\lambda+m^2}{-\lambda}}\tan^{-1}\sqrt{\frac{\lambda+m^2}{-\lambda}} + \sqrt{\frac{\lambda+\mu^2}{\lambda}}\ln\frac{\sqrt{-\lambda}+\sqrt{-\lambda-\mu^2}}{\mu}\right\}$$

(4.54)

$$\mathrm{Im}\, L_0^I(\lambda) = -\frac{\lambda}{(m^2-\mu^2)\pi}\left\{\sqrt{\frac{\lambda+m^2}{-\lambda}}\ln\frac{\sqrt{-\lambda}+\sqrt{-\lambda-\mu^2}}{\mu} - \sqrt{\frac{\lambda+\mu^2}{\lambda}}\tan^{-1}\sqrt{\frac{\lambda+\mu^2}{\lambda}}\right\} \quad (4.55)$$

The subtraction constants γ_0^I and δ_0^I are related to the scattering length a_0^I and the value of $A_0^I(s)$ at $s=(m-\mu)^2$ written as b_0^I respectively, as follows:

$$\gamma_0^I = \frac{(m+\mu)^2}{m\mu}\frac{1}{a_0^I} \quad (4.56)$$

$$\delta_0^I = -\frac{(m-\mu)^2}{m\mu}\frac{1}{b_0^I} \quad (4.57)$$

Now b_0^I may be expressed in terms of a_0^I by using Eqn (4.16). The scattering lengths $a_0^{1/2}$ and $a_0^{3/2}$ can be written as follows

$$a_0^{1/2} = a^{(+)} + 2a^{(-)} \quad (4.58)$$

$$a_0^{3/2} = a^{(+)} - a^{(-)} \quad (4.59)$$

The fixed t dispersion relation for $A^{(-)}(s,u,t)$ when written down without any subtraction is;

$$A^{(-)}(s,u,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\,\mathrm{Im}\,A^{(-)}(s',\Sigma-s'-t,t)\left\{\frac{1}{s'-s} - \frac{1}{s'-u}\right\} \quad (4.60)$$

Evaluating this equation at the point $A^{(-)}(s,u,t)$ we get an expression for $a^{(-)}$

$$a^{(-)} = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\,\mathrm{Im}\,A^{(-)}(s',1)\,\frac{m\mu}{k^2(s')\,s'} \quad (4.61)$$

where the index 1 stands for $\cos\theta$ which is equal to unity for $t=0$. The above sum rule is a very convergent one and allows us to express $a_0^{1/2}$ and $a_0^{3/2}$ in terms of one parameter $a_0^{(+)}$. Thus we have the S-wave dispersion relations dependent on only one parameter. The iteration procedure and the numerical solutions are discussed in the next chapter.

CHAPTER V

NUMERICAL SOLUTIONS FOR THE S-WAVE DISPERSION RELATIONS

In this chapter we discuss the numerical solutions obtained for the S-wave inverse amplitude dispersion relations. The iteration procedure is discussed in section I. All the equations required here have already been given in the last three chapters, but for convenience most of them are rewritten in suitable forms as required. The results of the numerical calculations done for various values of $a_o^{(+)}$ are given in section II. Section III gives a discussion of the results and on the problem in general.

5.1. ITERATION SCHEME:

It is useful to write down in the beginning all the variables used in the numerical calculations. In channel I, on the real axis we define $\omega = s - m^2 - \mu^2$. Then the left and the right hand cuts become $-\infty \leq \omega \leq -2m\mu$ and $2m\mu \leq \omega \leq +\infty$ respectively. Since the cuts extend up to infinity the following transformations are made

On the right hand cut: $z = 2m\mu/\omega$ so that when $2m\mu \leq \omega \leq +\infty$ one has $1 \geq z \geq 0$

On the left hand cut: $x = -2m\mu/\omega$ so that when $-\infty \leq \omega \leq -2m\mu$ one has $0 \leq x \leq 1$

In the numerical calculations 49 equidistant points are taken in both x and z variables in the interval 0 to + 1. But in writing down equations s and w corresponding to these mesh points are used as required for convenience. On the circle cut we use the variable ϕ defined by $s = (m^2-\mu^2)\, e^{i\phi}$ 33 mesh points are used here. In terms of ϕ one has

$$\lambda = \frac{m^2-\mu^2}{2}\cos\phi - \frac{m^2+\mu^2}{2} \quad \text{and} \quad \sqrt{(\lambda+m^2)(-\lambda-\mu^2)} = \frac{m^2+\mu^2}{2}\sin\phi$$

In $\pi\pi \to K\bar{K}$ channel 33 mesh points are taken in t on the left hand cut for $-32\mu^2 \leq t \leq 0$. On the right hand cut for the P wave the variable t is used, while for the S wave the variable $y = 4\mu^2/t$ is used. 49 equidistant points in y and t are taken for $4\mu^2 \leq t \leq 50\mu^2$. The choice of y as the variable in the S wave has the effect of concentrating more points in the low energy region.

The contributions from ρ and ABC to the D-functions can be calculated by using Eqns (3.25) - (3.27) and (3.29). The kaon-pion amplitudes contribute to the discontinuities across the left hand cuts of channels I and II and to the fixed t dispersion relations. We retain only the S wave and the K^* contributions. The S waves arise during iteration. The K^* contributions are to be input as data and are given below. Using a Breit Wigner formula for

$$A_1^{1/2}(s) = \frac{k^2\,\Gamma_{K^*}}{(s_r-s) - i\theta[s-(m+\mu)^2]\,\Gamma_{K^*}\frac{k^3}{\sqrt{s}}} \qquad (5.1)$$

we have

$$\mathrm{Im}\, A_1^{1/2}(s) = \frac{\frac{k^5}{\sqrt{s}}\,\Gamma_{K^*}}{(s_r-s)^2 + \Gamma_{K^*}^2\,\frac{k^6}{s}} \qquad (5.2)$$

where Γ_{K^*} is the reduced width defined by $\Gamma_{K^*} = \frac{S_r}{k_r^3}\Delta E$,
ΔE being the full width of half maximum in energy.
is the value of the momentum variable k at the resonance position S_r . The experimental results $E_r = 880$ Mev, $\Delta E = 50$ Mev give $S_r = 41$, $k_r^2 = 4.2$ and $\Gamma_{K^*} = 1.7$ In the sharp resonance approximation, Eqn (5.2) becomes

$$\mathrm{Im}\, A_1^{1/2}(s) = \pi \Gamma_{K^*} k_r^2 \delta(s - S_r) \qquad (5.3)$$

The various contributions can be written down as follows.

On the left hand cut of channel III (See Eqn (3.14)):

$$\mathrm{Im}\, B^0_{0K^*}(t) = -\frac{\sqrt{6}\pi\Gamma_{K^*} k_r^2}{2p_-q_-}\left(1 + \frac{t}{2k_r^2}\right) \text{ for } t \leq -4k_r^2$$
$$= 0 \quad \text{otherwise} \qquad (5.4)$$

$$\mathrm{Im}\, B^1_{1K^*}(t) = -\pi\Gamma_{K^*} k_r^2 \frac{2S_r - \Sigma + t}{4(p_-q_-)^3}\left(1 + \frac{t}{2k_r^2}\right) \text{ for } t \leq -4k_r^2$$
$$= 0 \quad \text{otherwise} \qquad (5.5)$$

The the fixed t dispersion relations determining the residues of the Balaz poles in channel III (See Eqns (3.40), (3.41), (3.44) and (3.45)):

$$B^0_{0K^*}(0) = \sqrt{6}\,\Gamma_{K^*} k_r^2 \left\{ \frac{1}{2m\mu} \ln\left(1 + \frac{4m\mu}{W_r - 2m\mu}\right) - \frac{2W_r}{W_r^2 - 4m^2\mu^2} \right\} \qquad (5.6)$$

$$\frac{d}{dt} B^0_{0K^*}(t)\Big|_{t=0} = \sqrt{6}\,\Gamma_{K^*} k_r^2 \left\{ \frac{1}{4m\mu}\left[\left(\frac{1}{k_r^2} + \frac{m^2+\mu^2}{4m^2\mu^2}\right)\ln\left(1 + \frac{4m\mu}{W_r - 2m\mu}\right) + \frac{1}{m\mu}\right] - \frac{W_r}{k_r^2}\left[\frac{1}{W_r^2 - 4m^2\mu^2} + \frac{1}{16m^2\mu^2}\right]\right\} \qquad (5.7)$$

$$B^1_{1K^*}(0) = \frac{\Gamma_{K^*} k_r^2}{m^2\mu^2}\left\{ \frac{W_r}{2m\mu}\ln\left(1 + \frac{4m\mu}{W_r - 2m\mu}\right) - 2\right\} \qquad (5.8)$$

$$\frac{d}{dt} B^1_{1K^*}(t)\Big|_{t=0} = \frac{\Gamma_{K^*} k_r^2}{4m^2\mu^2}\left\{ \frac{1}{m\mu}\left[1 + W_r\left(\frac{1}{k_r^2} + \frac{3(m^2+\mu^2)}{4m^2\mu^2}\right)\right]\ln\left(1 + \frac{4m\mu}{W_r - 2m\mu}\right) - \left(\frac{5}{k_r^2} + \frac{3(m^2+\mu^2)}{m^2\mu^2}\right)\right\} \qquad (5.9)$$

To the sum rule, Eqn (4.61)

$$a^{(-)}_{K*} = \Gamma_{K*} \frac{m\mu}{s_\gamma} \qquad (5.10)$$

And finally on the left hand cut of channel I:

$$\operatorname{Im} A^{I}_{0K*}(s) = -\frac{1}{4k^2}\int_{\Sigma - s}^{c(s)} ds' \begin{pmatrix} -1 \\ 2 \end{pmatrix} \left(1 + \frac{\Sigma - s' - s}{2k^2(s')}\right) \operatorname{Im} A^{1/2}_{1}(s') \qquad (5.11)$$

where

$$c(s) = \frac{(m^2-\mu^2)^2}{s} \quad \text{for} \quad 0 \le s \le (m-\mu)^2$$
$$= (m+\mu)^2 \quad \text{for} \quad s < 0$$

$\operatorname{Im} A^{1/2}_{1}(s)$ is given by Eqn (5.2). Here the delta function approximation is not used, because it is better to have $\operatorname{Im} A^{I}_{0K*}(s)$ as a smooth function for the inverse amplitude formulation.

We now describe a typical iteration cycle where all the quantities required are either given by a previous cycle or input as data. How the iteration is started initially is discussed later on.

An iteration cycle is commenced with the calculations of the $\pi\pi \to K\bar{K}$ amplitudes. Using $\operatorname{Im} A^{I}_{0}(s)$ obtained in the previous cycle or input as data one can calculate the following contributions.

On the left hand cut (Eqn (3.14)):

$$\operatorname{Im} B^{0}_{0s}(t) = -\int_{c(t)}^{1} \frac{dz}{z^2} \frac{2m\mu}{\sqrt{6}\, p_- q_-} \left\{\operatorname{Im} A^{1/2}_{0}(z) + 2 \operatorname{Im} A^{3/2}_{0}(z)\right\} \qquad (5.12)$$

$$\operatorname{Im} B^{1}_{1s}(t) = -\int_{c(t)}^{1} \frac{dz}{z^2} \frac{2m\mu\left(\frac{4m\mu}{z}+t\right)}{12(p_- q_-)^3} \left\{\operatorname{Im} A^{1/2}_{0}(z) - \operatorname{Im} A^{3/2}_{0}(z)\right\} \qquad (5.13)$$

where

$$c(t) = \frac{4m\mu}{4p_- q_- - t}$$

The contributions to the fixed t dispersion relations (Eqns (3.40), (3.41), (3.44) and (3.45)) are:

$$B^0_{0s}(0) = \sqrt{6}\,a_0^{(+)} + \frac{\sqrt{6}}{3\pi}\int_0^1 \frac{dz}{z^2}\left\{\mathrm{Im}\,A_0^{1/2}(z) + 2\,\mathrm{Im}\,A_0^{3/2}(z)\right\}\left\{\ln\left(1+\frac{2z}{1-z}\right) - \frac{2z}{1-z^2}\right\} \qquad (5.14)$$

$$\left.\frac{d}{dt}B^0_{0s}(t)\right|_{t=0} = \frac{1}{\sqrt{6}\,m\mu}\,\frac{1}{\pi}\int_0^1 \frac{dz}{z^2}\left\{\mathrm{Im}\,A_0^{1/2}(z) + 2\,\mathrm{Im}\,A_0^{3/2}(z)\right\}\left\{\frac{m^2+\mu^2}{4m\mu}\ln\left(1+\frac{2z}{1-z}\right) - \frac{z(a+z)}{1-z^2}\right\} \qquad (5.15)$$

$$B^1_{1s}(0) = \frac{2}{3m\mu}\,\frac{1}{\pi}\int_0^1 \frac{dz}{z^2}\left\{\mathrm{Im}\,A_0^{1/2}(z) - \mathrm{Im}\,A_0^{3/2}(z)\right\}\left\{\frac{1}{z}\ln\left(1+\frac{2z}{1-z}\right) - 2\right\} \qquad (5.16)$$

$$\left.\frac{d}{dt}B^1_{1s}(t)\right|_{t=0} = \frac{1}{6m^2\mu^2}\,\frac{1}{\pi}\int_0^1 \frac{dz}{z^2}\left\{\mathrm{Im}\,A_0^{1/2}(z) - \mathrm{Im}\,A_0^{3/2}(z)\right\}\left\{\left(1+\frac{3(m^2+\mu^2)}{2m\mu}\,\frac{1}{z}\right)\ln\left(1+\frac{2z}{1-z}\right) - \frac{2(1+az)}{1-z^2} - \frac{3(m^2+\mu^2)}{m\mu}\right\} \qquad (5.17)$$

where $a = \dfrac{m^2+\mu^2}{2m\mu}$ (this is used when necessary)

The total contributions are obtained by adding the S wave and the K^* contributions. The residues of the Balaz poles can then be calculated by using Eqns (3.48) - (3.51). Finally from Eqn (3.18) we evaluate $\mathrm{Im}\,B_0^0(t)$ and $\mathrm{Im}\,B_1^1(t)$ on the right hand cut

$$\mathrm{Im}\,B_l^I(t) = \mathrm{Im}\left\{\frac{1}{D_l^I(t)}\right\}\left[\frac{1}{\pi}\int_{-32\mu^2}^{0} dt'\,\frac{\mathrm{Im}\,B_l^I(t')\,D_l^I(t')}{t'-t} + \frac{\alpha_l^I}{t+t_1} + \frac{\beta_l^I}{t+t_2}\right] \qquad (5.18)$$

This completes the calculations of the $\pi\pi \to K\bar{K}$ amplitudes. Now the contributions of $\mathrm{Im}\,B_l^I(t)$ to kaon-pion scattering amplitudes may be calculated by using Eqns (2.74) and (2.78). We rewrite them for the S waves in the following forms.

On the circle cut

$$\mathrm{Re}\,\Delta A_l^I(\lambda) = \frac{1}{\sqrt{6}\,\lambda}\int_{-1/\lambda}^{1} dy\,\frac{1}{y^2}\,\mathrm{Im}\,B_0^0(y)$$

$$+\frac{1}{4\lambda}\int_{4\mu^2}^{-4\lambda} dt\begin{pmatrix}1\\-1/2\end{pmatrix}3(\lambda+t/4)\,\mathrm{Im}\,B_1^1(t) \qquad (5.19)$$

$$\mathrm{Im}\,\Delta A_l^I(\lambda) = \frac{3\sqrt{(\lambda+m^2)(-\lambda-\mu^2)}}{4\lambda}\int_{4\mu^2}^{-4\lambda} dt\begin{pmatrix}3/2\\-3/4\end{pmatrix}\mathrm{Im}\,B_1^1(t) \qquad (5.20)$$

On the left hand cut

$$A_l^{I\pi\pi}(\omega) = -\epsilon(\omega+2m^2)\frac{1}{\sqrt{6}\,k^2}\int_{-1/k^2}^{1} dy\,\frac{1}{y^2}\,\mathrm{Im}\,B_0^0(y)$$

$$-\epsilon(\omega+2m^2)\frac{1}{4k^2}\int_{4\mu^2}^{-4k^2} dt\begin{pmatrix}3/2\\-3/4\end{pmatrix}(\omega+t/2)\,\mathrm{Im}\,B_1^1(t) \qquad (5.21)$$

$$\text{for}\quad \omega < -(m^2+\mu^2)$$

In the column vectors the top and the bottom factors correspond to $I = 1/2$ and $I = 3/2$ states respectively. This convention is used everywhere. In Eqn (5.21) the lower limit of the first term and the upper limit of the second term extend beyond t = 50, but we introduce a cut off at this point. Since the kaon-pion dispersion relations are written in a form to suppress the high energy contributions on the left hand cut, this cut off has practically no effect.

Now the contributions of the S wave kaon-pion amplitudes on the left hand cut are given by

$$\mathrm{Im}\,A_{0S}^I(x) = -\frac{m\mu}{6}\frac{1}{k^2}\int_{c(x)}^{x}\frac{dz}{z^2}\left\{\begin{pmatrix}-1\\2\end{pmatrix}\mathrm{Im}\,A_0^{1/2}(z)+\begin{pmatrix}4\\1\end{pmatrix}\mathrm{Im}\,A_0^{3/2}(z)\right\} \qquad (5.22)$$

where

$$c(x) = \frac{ax-1}{a-x} \quad \text{when } 1/a \leq x \leq 1$$

$$= 1 \quad \text{when } 0 \leq x \leq 1/a$$

The total discontinuity across the left hand cut is obtained by adding the contributions of Eqns (5.11), (5.21) and (5.22). The discontinuities of the inverse amplitudes, $K_0^I(x)$'s are obtained by iterating the following two equations (for future reference we call it loop 1)

$$K_0^I(x) = \frac{\mathrm{Im}\, A_0^I(x)}{[\mathrm{Re}\, A_0^I(x)]^2 + [\mathrm{Im}\, A_0^I(x)]^2} \tag{5.23}$$

$$\mathrm{Re}\, A_0^I(x) = \frac{\mathrm{Re}\, A_0^{I^{-1}}(x)}{[\mathrm{Re}\, A_0^{I^{-1}}(x)]^2 + [K_0^I(x)]^2} \tag{5.24}$$

where

$$\mathrm{Re}\, A_0^{I^{-1}}(x) = -\frac{1-x}{4(ax-1)}\gamma_0^I - \frac{1+x}{4(ax-1)}\delta_0^I + L_0^I(x) + N_0^I(x) + R_0^I(x) \tag{5.25}$$

The constants γ_0^I and δ_0^I are taken from the previous cycle and $L_0^I(x)$ can be calculated from Eqn (4.53). While $N_0^I(x)$ and $R_0^I(x)$ are calculated from $K_0^I(x)$ and $M_0^I(\phi)$ obtained from either the previous cycle or input data as follows

$$R_0^I(x) = -\frac{k^2}{\pi}\int_0^\pi d\phi \frac{1}{\lambda}\left\{\sqrt{(\lambda+m^2)(-\lambda-\mu^2)}\, \frac{\mathrm{Re}\, M_0^I(\phi)}{\lambda-k^2} + \mathrm{Im}\, M_0^I(\phi)\, \frac{\lambda - \frac{m\mu(a-x)}{ax-1}}{\lambda-k^2}\right\} \tag{5.26}$$

$$N_0^I(x) = -\frac{xk^2(x)}{\pi} P\int_0^1 dx' \frac{\frac{K_0^I(x')}{x'k^2(x')} - \frac{K_0^I(x)}{xk^2(x)}}{x'-x} - \frac{K_0^I(x)}{\pi}\ln\frac{1-x}{x} \tag{5.27}$$

where the singularity in the principal value integral is taken out by using the following technique

$$P\int_a^b dx \frac{f(x)}{x-z} = P\int_a^b dx \frac{f(x)-f(z)}{x-z} + f(z)\ln\left|\frac{b-z}{a-z}\right| \tag{5.28}$$

Once $K_0^I(x)$'s are calculated by loop 1 they may be put in Eqn (5.27) to obtain new values of $N_0^I(x)$'s . Then using these in Eqn (5.25) keeping everything else unchanged loop 1 may be repeated to give new values of $K_0^I(x)$'s. This is repeated a specified number of times (loop 2).

Now we can go to the circle cut and calculate $M_0^I(\phi)$ by iterating the following equations (loop 3)

$$M_0^I(\phi) = \frac{\Delta A_0^I(\phi)}{[X_0^I(\phi)]^2 + [\Delta A_0^I(\phi)]^2} \tag{5.29}$$

$$X_0^I(\phi) = \frac{G_0^I(\phi)}{[G_0^I(\phi)]^2 + [M_0^I(\phi)]^2} \tag{5.30}$$

where $\Delta A_0^I(\phi)$'s are given by Eqns (5.19) and (5.20) and

$$\begin{aligned} \mathrm{Re}\, G_0^I(\phi) = {} & \tfrac{1}{4}\left(1 - \tfrac{m-\mu}{m+\mu}\cos\phi\right)\gamma_0^I + \tfrac{1}{4}\left(1 - \tfrac{m+\mu}{m-\mu}\cos\phi\right)\delta_0^I \\ & + \mathrm{Re}\, L_0^I(\phi) + \mathrm{Re}\, N_0^I(\phi) + \mathrm{Re}\, R_0^I(\phi) \end{aligned} \tag{5.31}$$

$$\begin{aligned} \mathrm{Im}\, G_0^I(\phi) = {} & \left(\tfrac{1}{4}\tfrac{m-\mu}{m+\mu}\gamma_0^I + \tfrac{1}{4}\tfrac{m+\mu}{m-\mu}\delta_0^I\right)\sin\phi + \mathrm{Im}\, L_0^I(\phi) + \mathrm{Im}\, N_0^I(\phi) \\ & + \mathrm{Im}\, R_0^I(\phi) \end{aligned} \tag{5.32}$$

the method of calculating the real and imaginary parts of Eqns (5.29) and (5.30) is illustrated as follows. If A,B and C are all complex and

$$A = \frac{C}{B^2 + C^2}$$

Then

$$\mathrm{Re}\, A = \frac{\mathrm{Re}\, C\, \mathrm{Re}\, D + \mathrm{Im}\, C\, \mathrm{Im}\, D}{[\mathrm{Re}\, D]^2 + [\mathrm{Im}\, D]^2}, \quad \mathrm{Im}\, A = \frac{\mathrm{Im}\, C\, \mathrm{Re}\, D - \mathrm{Re}\, C\, \mathrm{Im}\, D}{[\mathrm{Re}\, D]^2 + [\mathrm{Im}\, D]^2}$$

where

$$\mathrm{Re}\, D = [\mathrm{Re}\, B]^2 + [\mathrm{Re}\, C]^2 - [\mathrm{Im}\, C]^2 - [\mathrm{Im}\, B]^2, \quad \mathrm{Im}\, D = 2\,(\mathrm{Re}\, B\, \mathrm{Im}\, B + \mathrm{Re}\, C\, \mathrm{Im}\, C)$$

The constants γ_0^I and δ_0^I are taken from the previous iteration as before. $L_0^I(\phi)$ is given by Eqns (4.54) and (4.55). Using $K_0^I(x)$ obtained by loop 1 and loop 2 we can calculate $N_0^I(\phi)$ on the circle cut:

$$\mathrm{Re}\, N_0^I(\phi) = -\frac{\lambda}{\pi}\int_0^1 dx\; K_0^I(x)\,\frac{\frac{m\mu}{x}+\lambda}{m\mu(1-x^2)(\lambda-k^2)} \tag{5.33}$$

$$\mathrm{Im}\, N_0^I(\phi) = \frac{\lambda\sqrt{(\lambda+m^2)(-\lambda-\mu^2)}}{\pi}\int_0^1 dx\; \frac{K_0^I(x)}{m\mu(1-x^2)(\lambda-k^2)} \tag{5.34}$$

$R_0^I(\phi)$ can be obtained from $M_0^I(\phi)$ given by the previous iteration or input data.

$$\mathrm{Re}\, R_0^I(\phi) = -\frac{\lambda}{\pi}P\int_0^\pi d\phi'\left\{\frac{\mathrm{Re}\,M_0^I(\phi')\frac{\sin\phi'}{\lambda'} - \mathrm{Re}\,M_0^I(\phi)\frac{\sin\phi}{\lambda}}{\cos\phi' - \cos\phi} + \frac{\mathrm{Im}\,M_0^I(\phi')}{\lambda'}\right\} \tag{5.35}$$

$$\mathrm{Im}\, R_0^I(\phi) = -\frac{\lambda\sin\phi}{\pi}P\int_0^\pi d\phi'\;\frac{\frac{\mathrm{Im}\,M_0^I(\phi')}{\lambda'} - \frac{\mathrm{Im}\,M_0^I(\phi)}{\lambda}}{\cos\phi' - \cos\phi} \tag{5.36}$$

The same technique is used as before to calculate the principal value integrals. But now

$$\int_0^\pi d\phi'\,\frac{1}{\cos\phi' - \cos\phi} = 0$$

so there is no logarithmic term. Eqns (5.35) and (5.36) may again be evaluated after $M_0^I(\phi)$ has been recalculated by loop 3. The results are inserted in Eqns (5.31) and (5.32) keeping everything else unchanged to start loop 3 again. This is done a specified number of times (loop 4). When this is completed we can go back to Eqn (5.23) and repeat loops 1 to 4 (loop 5).

Now with the values of $K_0^I(x)$ and $M_0^I(\phi)$ obtained by iterating through loops 1 to 5 may be used to calculate quantities on the right hand cut as follows

$$N_0^I(z) = \frac{2k^2(z)}{\pi}\int_0^1 dx\, \frac{K_0^I(x)}{x k^2(x)(x+z)} \qquad (5.37)$$

$$R_0^I(z) = -\frac{k^2}{\pi}\int_0^{\pi} d\phi\, \frac{1}{\lambda}\left\{ \operatorname{Re} M_0^I(\phi)\, \frac{\sqrt{(\lambda+m^2)(-\lambda-\mu^2)}}{\lambda-k^2} + \operatorname{Im} M_0^I(\phi)\, \frac{\lambda + \frac{m\mu(a+z)}{az+1}}{\lambda-k^2} \right\} \qquad (5.38)$$

$$\frac{k}{\sqrt{s}} \cot\alpha_0^I \equiv \operatorname{Re} A_0^{I^{-1}}(z) = \frac{1+z}{4(az+1)}\gamma_0^I + \frac{1-z}{4(az+1)}\delta_0^I + L_0^I(z) + N_0^I(z) + R_0^I(z) \qquad (5.39)$$

$\alpha_0^I \rightarrow$ phase shifts

$$\operatorname{Im} A_0^I(z) = \frac{\frac{k}{\sqrt{s}}}{\left[\frac{k}{\sqrt{s}}\cot\alpha_0^I\right]^2 + \frac{k^2}{s}} \qquad (5.40)$$

Putting $\operatorname{Im} A_0^I(z)$ in the sum rule, Eqn (4.61) one gets

$$a_0^{(-)} = \Gamma_{K^*}\frac{m\mu}{S_r} + \frac{2}{3\pi}\int_0^1 dz\, \frac{\operatorname{Im} A_0^{1/2}(z) - \operatorname{Im} A_0^{3/2}(z)}{1-z^2} \qquad (5.41)$$

The first term is the K^* contribution. Then using Eqns (4.16) and (4.56) - (4.59) the constants γ_0^I and δ_0^I may be recalculated. At this point we go back to Eqn (5.22) and recalculate the contributions of the S-wave amplitudes on the left hand cut. After that loops 1 to 5 are repeated. When this is done a number of times (loop 6) one goes back to Eqn (5.12) to recalculate the $\pi\pi \rightarrow K\bar{K}$ amplitudes with the new values of $\operatorname{Im} A_0^I(z)$. This outermost loop (loop 7) is repeated until convergence is achieved.

When the iteration is started for the very first time the S-waves are approximated by

$$\operatorname{Im} A_0^I(z) = \frac{k}{\sqrt{s}}\left[a_0^I\right]^2 \qquad (5.42)$$

For a particular value of $a_0^{(+)}$ from Eqn (5.41) we can obtain $a_0^{1/2}$ and $a_0^{3/2}$ by using Eqns (5.42), (4.58) and (4.59). The $\pi\pi \rightarrow K\bar{K}$ amplitudes are calculated using Eqn (5.42) in Eqns (5.12) - (5.17). Then Eqn (5.22) can be calculated. The constants γ_0^I and δ_0^I are evaluated using Eqns (4.16) and (4.56) - (4.59). We put $N_0^I = R_0^I = 0$ and calculate $K_0^I(x)$ and $M_0^I(\phi)$ from loops 1 and 3. Once $K_0^I(x)$ and $M_0^I(\phi)$ are known the usual iteration cycle can be started.

Eqn (4.16) when written in full is

$$\begin{pmatrix} b_0^{1/2} \\ \\ b_0^{3/2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}a_0^{1/2} + \frac{4}{3}a_0^{3/2} \\ \\ \frac{2}{3}a_0^{1/2} + \frac{1}{3}a_0^{3/2} \end{pmatrix} \qquad (5\text{-}43)$$

Examining the above equation it is found that if $a_0^{1/2}$ and $a_0^{3/2}$ are both of the same sign and $|a_0^{1/2}| > 4|a_0^{3/2}|$ then $a_0^{1/2}$ and $b_0^{1/2}$ are of opposite signs. This means that $A_0^{1/2}(s)$ has a zero on the real axis between $(m-r)^2 \leq s \leq (m+r)^2$. Again when $a_0^{1/2}$ and $a_0^{3/2}$ are of opposite signs zeros may appear under suitable conditions in either $A_0^{1/2}(s)$ or $A_0^{3/2}(s)$ or in both. Since the scattering amplitude is real on the real axis between $(m-r)^2 \leq s \leq (m+r)^2$ these zeros will appear as poles in the inverse amplitudes. In our numerical calculations we avoid such zeros. A possible way of dealing with them is discussed in section III. The results of the numerical calculations are given in the next section.

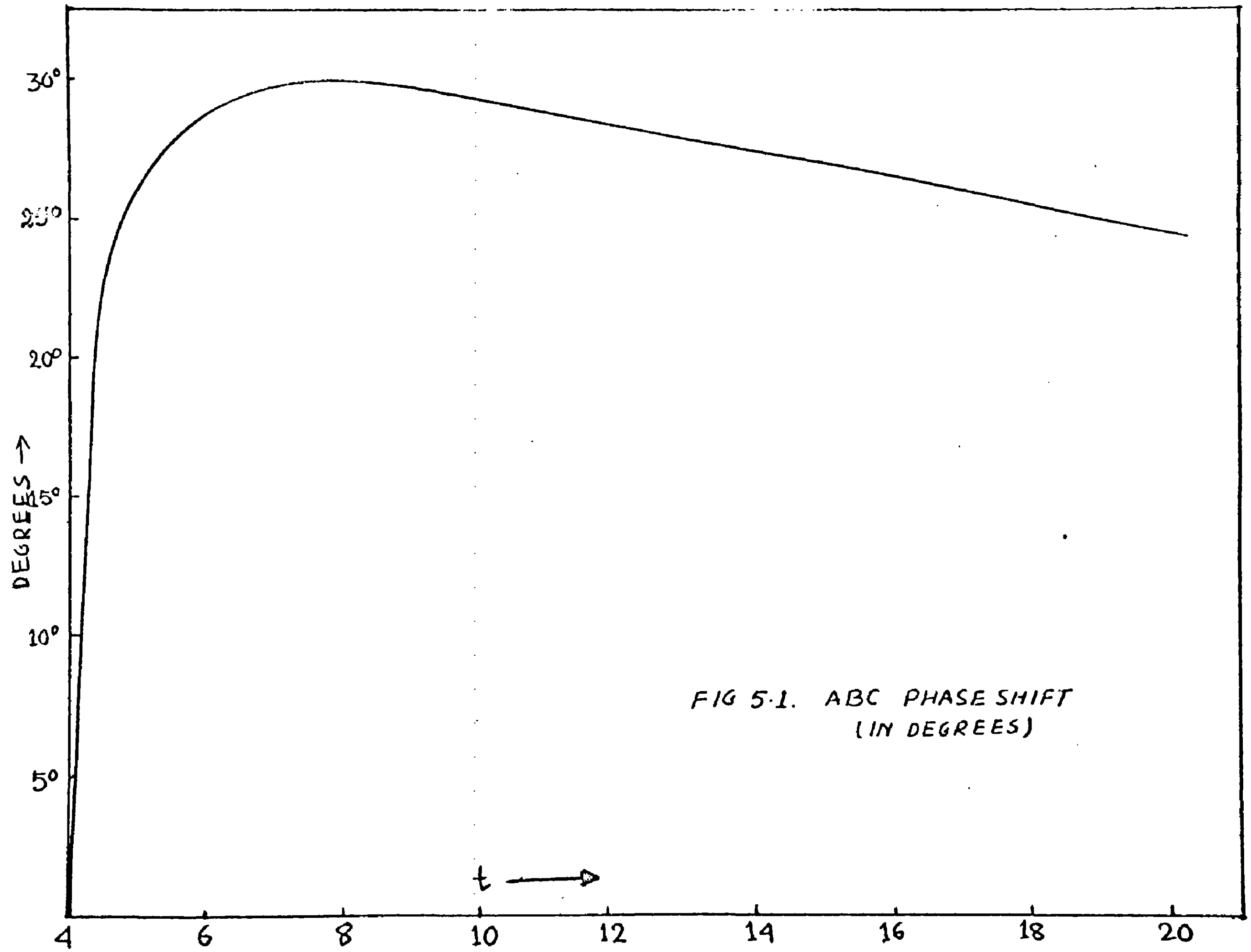


FIG 5·1. ABC PHASE SHIFT
(IN DEGREES)

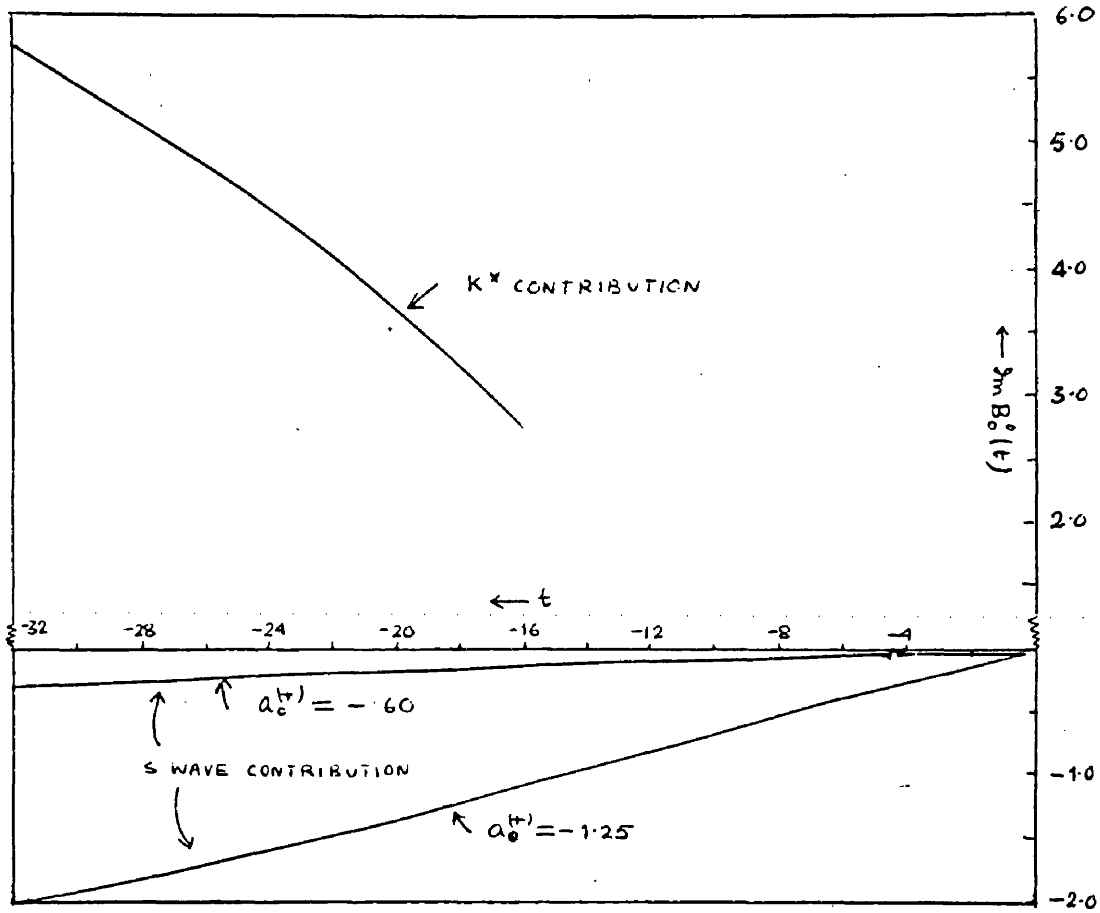


FIG 5·2 $\mathrm{Im}\, B_0^0(t)$ ON LHC

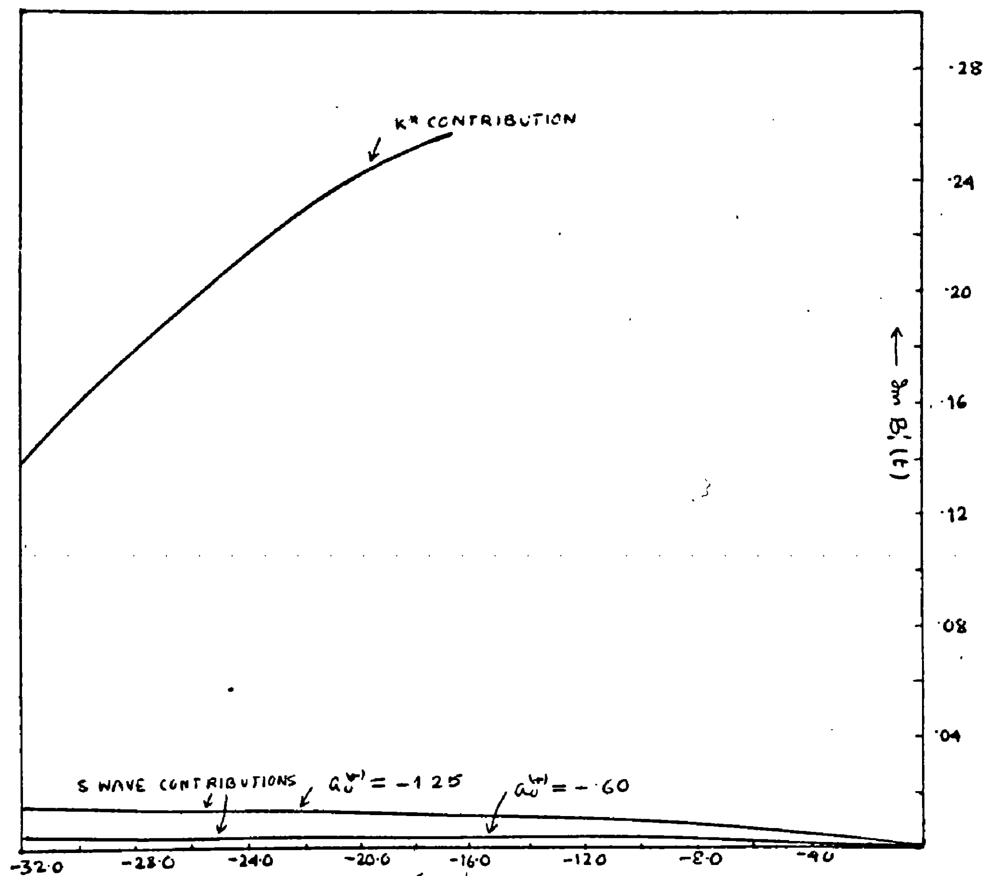


FIG 5·3 $\mathrm{Im}\, B_1^1(t)$ ON LHC

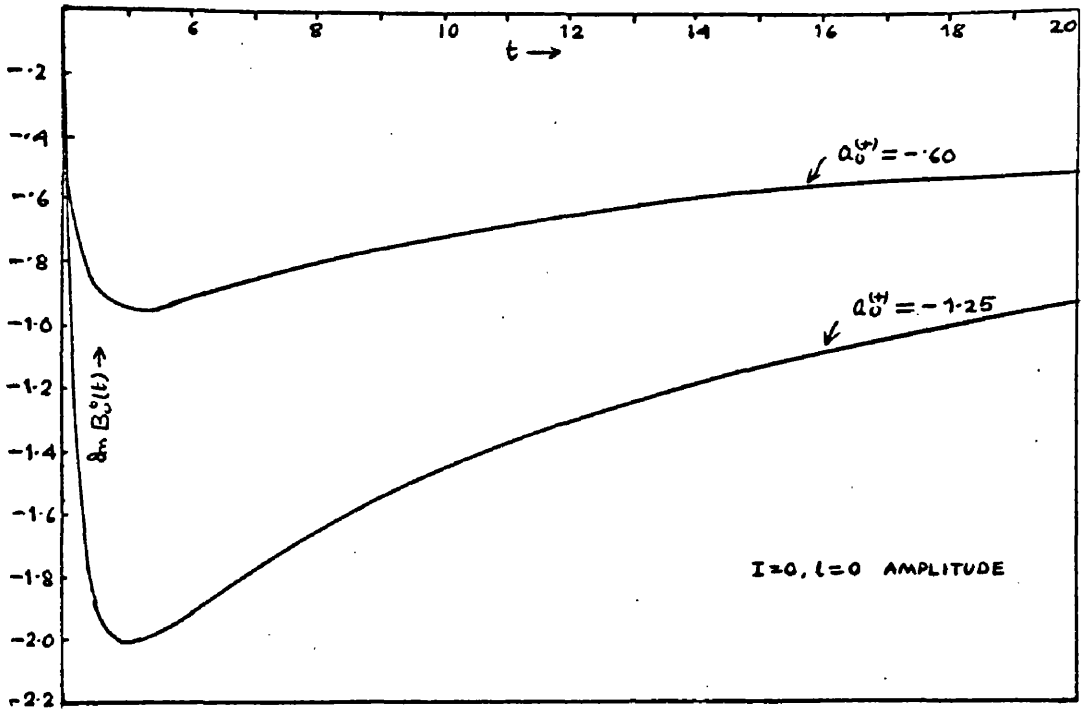


FIG 5.4 Im $B_0^0(t)$ on RHC

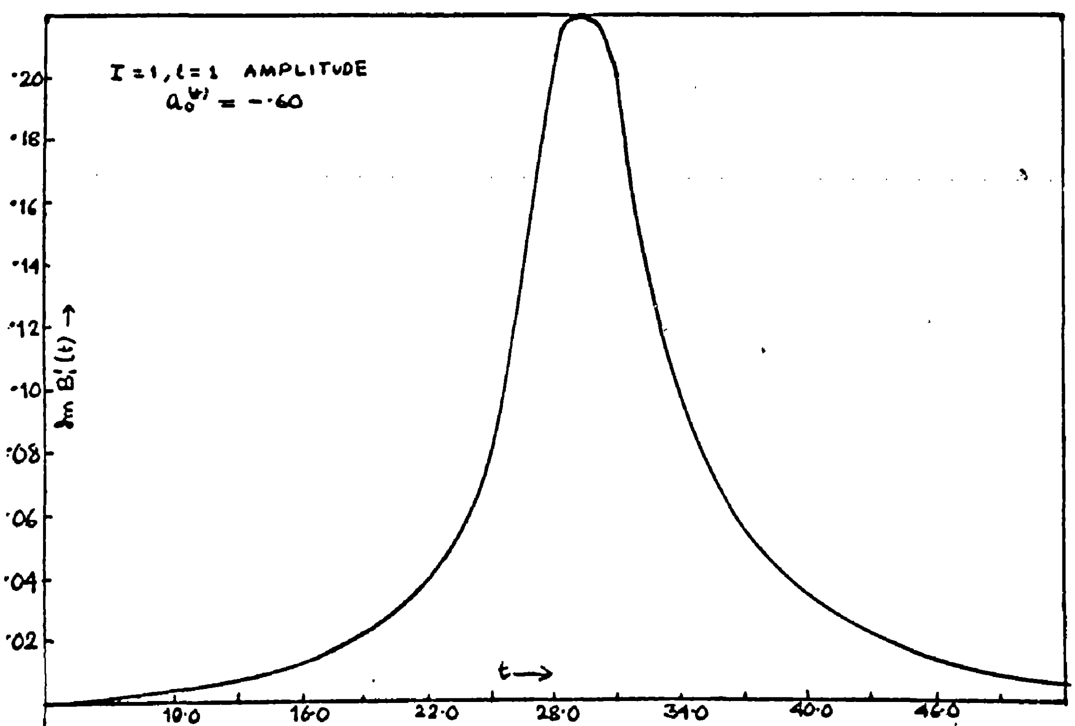


FIG 5.5 Im $B_1^1(t)$ ON RHC

5.2. RESULTS OF THE NUMERICAL CALCULATIONS:

The numerical calculations were performed on the Elliott 803 computer at the computing laboratory, the University of Durham. If one wants to avoid the above mentioned zeros of the scattering amplitudes, the values of $a_0^{(+)}$ close to zero cannot be investigated. Because in this case $a_0^{1/2}$ and $a_0^{3/2}$ are of opposite signs due to the rather large splitting introduced by K^* through Eqn (5.41) (the contribution of K^* to $a_0^{(-)}$ is about +.15)

For negative values of $a_0^{(+)}$ (negative enough to give the same sign for $a_0^{1/2}$ and $a_0^{3/2}$) rapidly convergent solutions are found. When the solution for a particular value of $a_0^{(+)}$ is obtained it is used as input for the next run. With such an input the numbers of circuits made around the various loops to achieve convergence in a typical case are as follows: loops 1 - 5, 2 circuits each, loop 6 - 3 circuits and finally loop 7 - 4 circuits. The time required on the machine for this is about six hours. The description of the results starts with the $\pi\pi \to K\bar{K}$ amplitudes.

(i) $I=0, l=0$ AMPLITUDE:

Fig. 5.1 shows the phase shifts we used to calculate $D_0^0(t)$. The K^* contributions on the left hand cut provide a strong repulsion, while the S-wave $K\pi$ contributions are attractive. Fig. 5.2 gives the discontinuity on the left hand cut for $-32\mu^2 \leq t \leq 0$. Since K^* contribution is obtained in a delta function approximation it starts at

$t \simeq -16.8$. The magnitudes of the K^* contributions are many times larger than those of the S-waves. The following table shows the values of the residues of the Balaz poles α_0^0 and β_0^0 for various values of $a_0^{(+)}$.

$a_0^{(+)}$	α_0^0	β_0^0	$\alpha_0^{0\prime}$	$\beta_0^{0\prime}$
-.35	77.29	-377.66	65.76	-298.18
-.50	61.91	-352.76	51.08	-274.49
-.60	51.54	-338.6	41.29	-259.30
-1.25	-21.89	-325.16	-22.35	-158.21
-1.5	-51-75	-329.59	-46.83	-119.33

where $\alpha_0^{0\prime}$ and $\beta_0^{0\prime}$ are the values of the residues when only the K^* contributions and $B_{0S}^0(0)=\sqrt{6}a_0^{(+)}$ are retained. The imaginary parts of $B_0^0(t)$ on the right hand cut is plotted in Fig. 5.4 for $a_0^{(+)} \approx -.60$ and -1.25. The values are negative and have a peak at t $\approx$ 5. The proportion of the S-wave contributions increases with increasing values of $|a_0^{(+)}|^2$. At $a_0^{(+)} \simeq -.60$ only about 10% of the total contribution at t $\simeq$ 5 comes from the S-wave $K\pi$ amplitudes.

(ii) <u>$I=1, l=1$ AMPLITUDE</u>:

In this case the effects of the S-wave amplitudes for πK scattering are practically zero. Fig. 5.3 gives $\mathrm{Im}\, B_1^1(t)$ on the left hand cut for $-32\mu^2 \leq t \leq 0$. In this nearby portion both the S-waves and the K^* contributions are repulsive. An examination of Eqn (5.5) shows that changes sign at $t = -2S_Y + \Sigma \approx -54$. So there are short range attractions coming from K^*. Now the residues α_1' and β_1' absorb the effects of neglecting the cut $16\mu^2 \leq t \leq +\infty$, the portion $-\infty \leq t \leq -32\mu^2$ of the left hand cut and any possible subtractions. These residues are given in the following table:

$a_0^{(+)}$	α_1'	β_1'	α_1''	β_1''
-.35	3.83	2.13	3.89	1.47
-.50	3.78	2.28	3.89	1.47
-.60	3.75	2.39	3.89	1.47
-1.25	3.70	3.29	3.89	1.47
-1.50	3.67	3.48	3.89	1.47

where α_1'' and β_1'' are obtained by retaining only the K^* contributions. Since α_1' and β_1' are both positive they give positive contributions to $\mathrm{Im}\, B_1'(t)$ given by Eqn (5.18). The integral gives negative contributions. Fig. 5.5 shows $\mathrm{Im}\, B_1'(t)$ on the right hand cut. There is a very narrow peak at the position of ρ and $\mathrm{Im}\, B_1'(t)$ is positive indicating that the contributions of the pole terms in Eqn (5.18) are more important than the integral.

Now we discuss the results obtained for the S-wave kaon-pion amplitudes. The values of the substraction constants are listed in the following table.

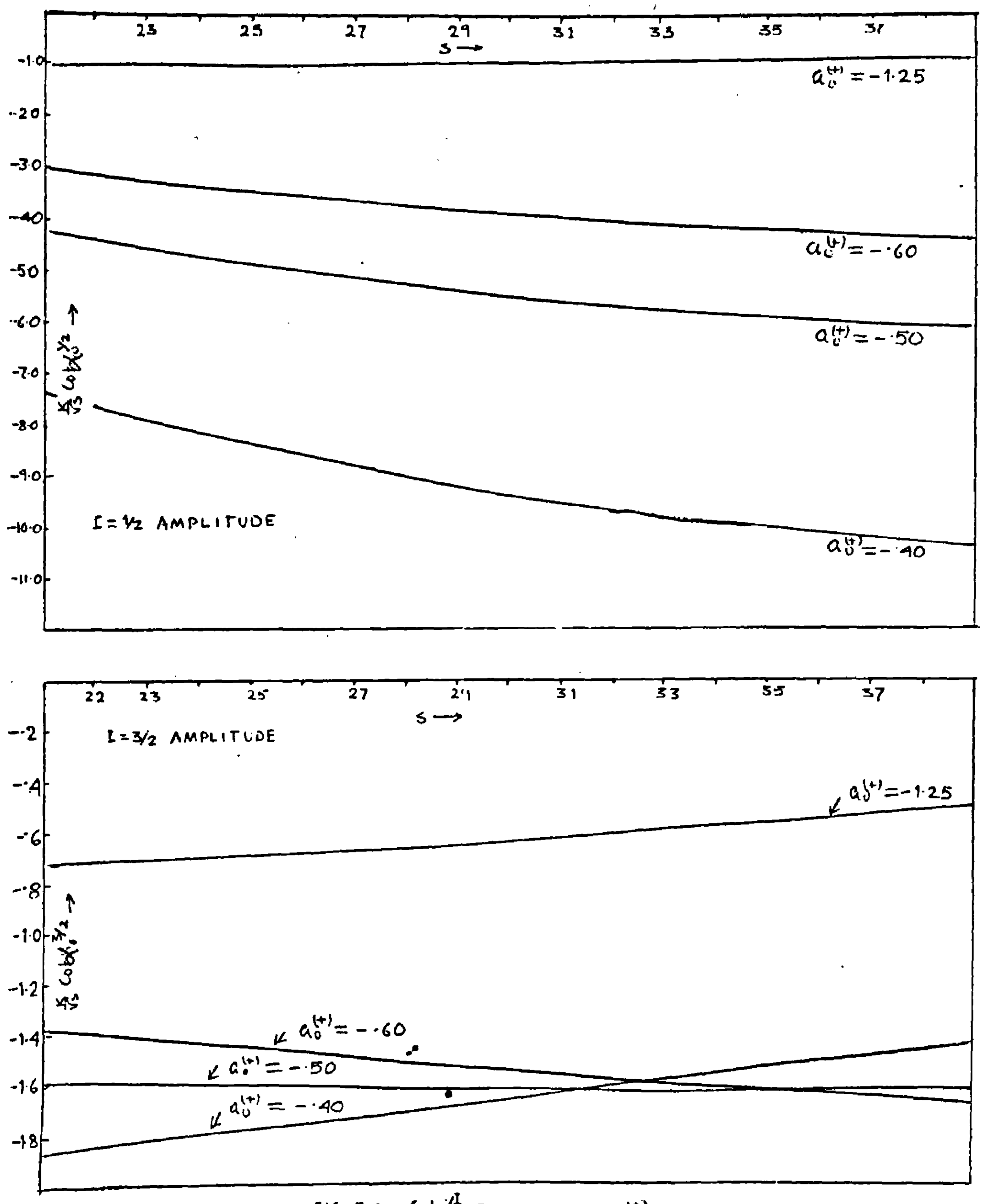


FIG 5·6 $\cot \alpha_0^I$ FOR VARIOUS $a_0^{(+)}$

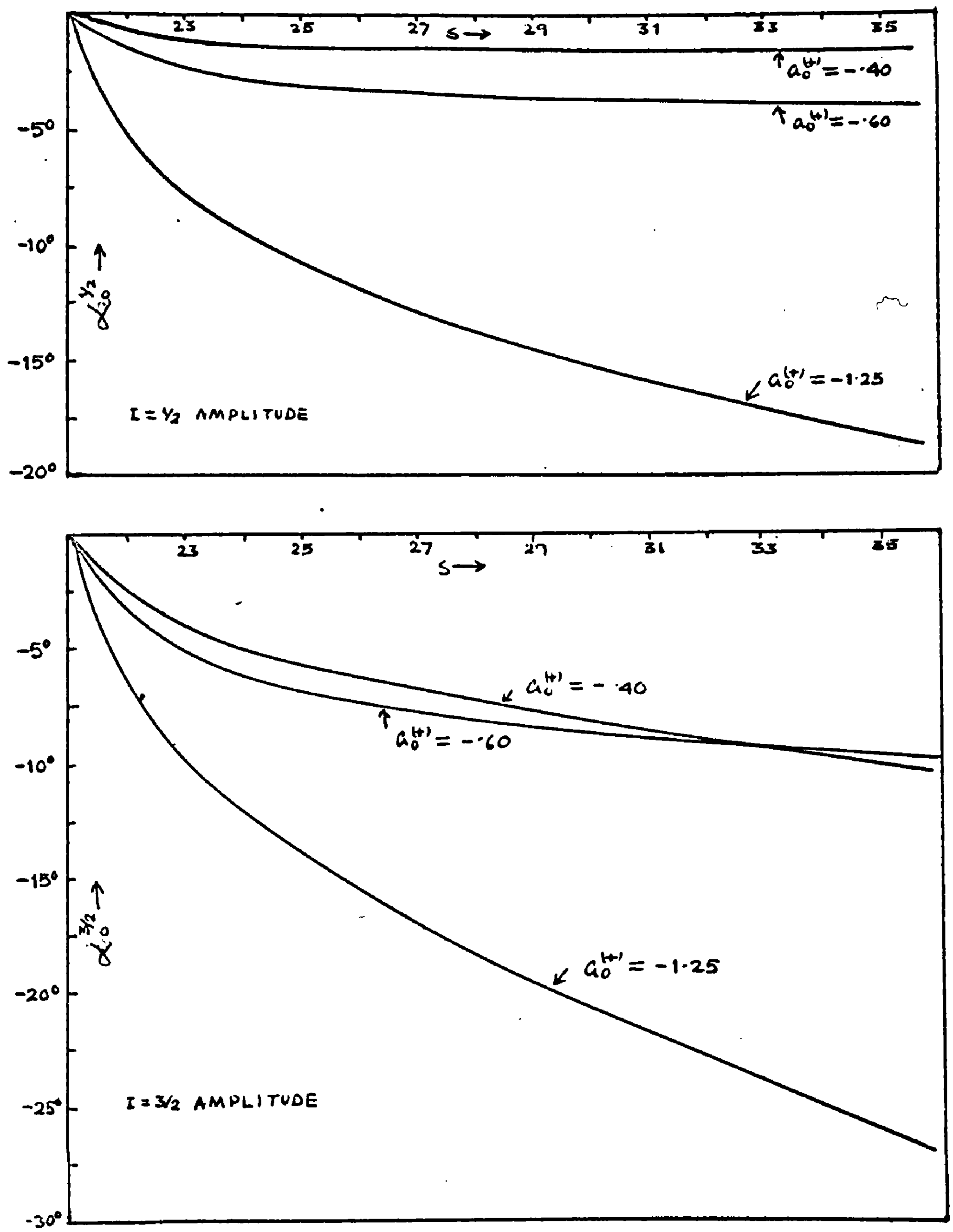


FIG 5·7 PHASE SHIFTS

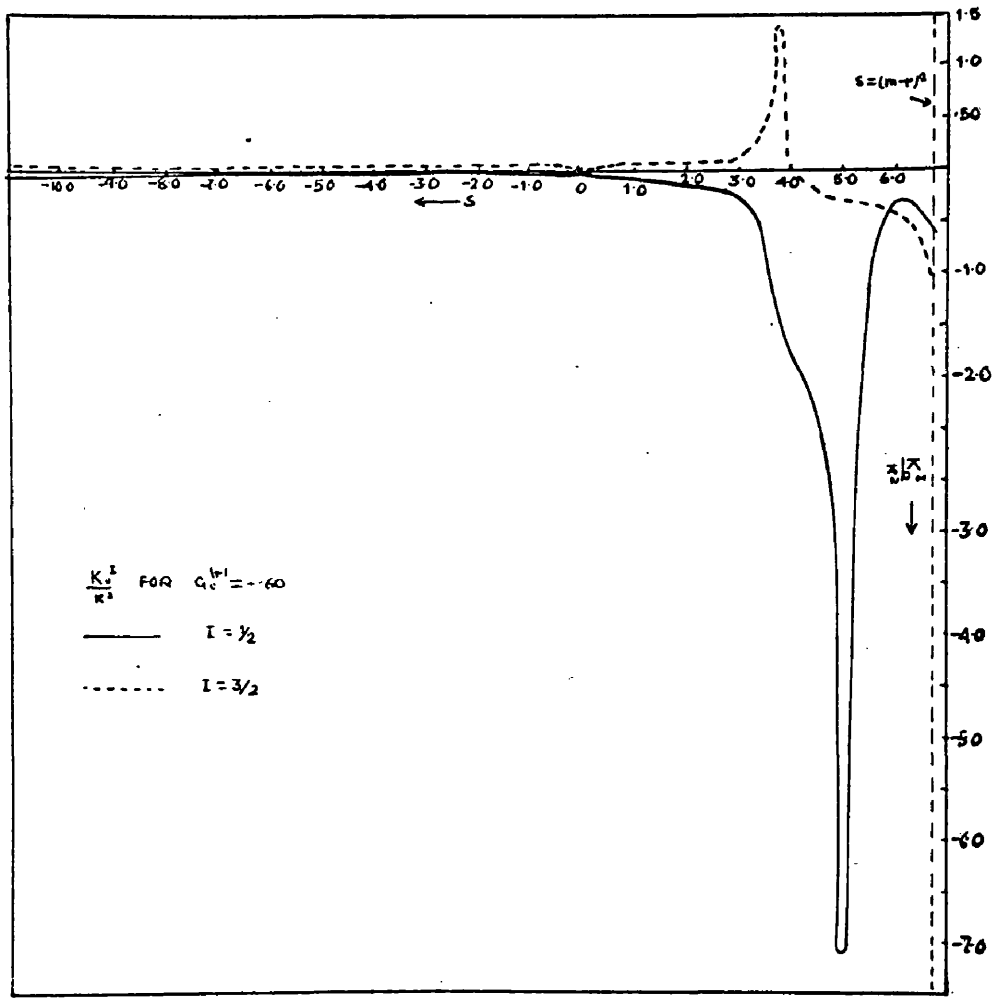


FIG 5·8 K_0^I/K^2

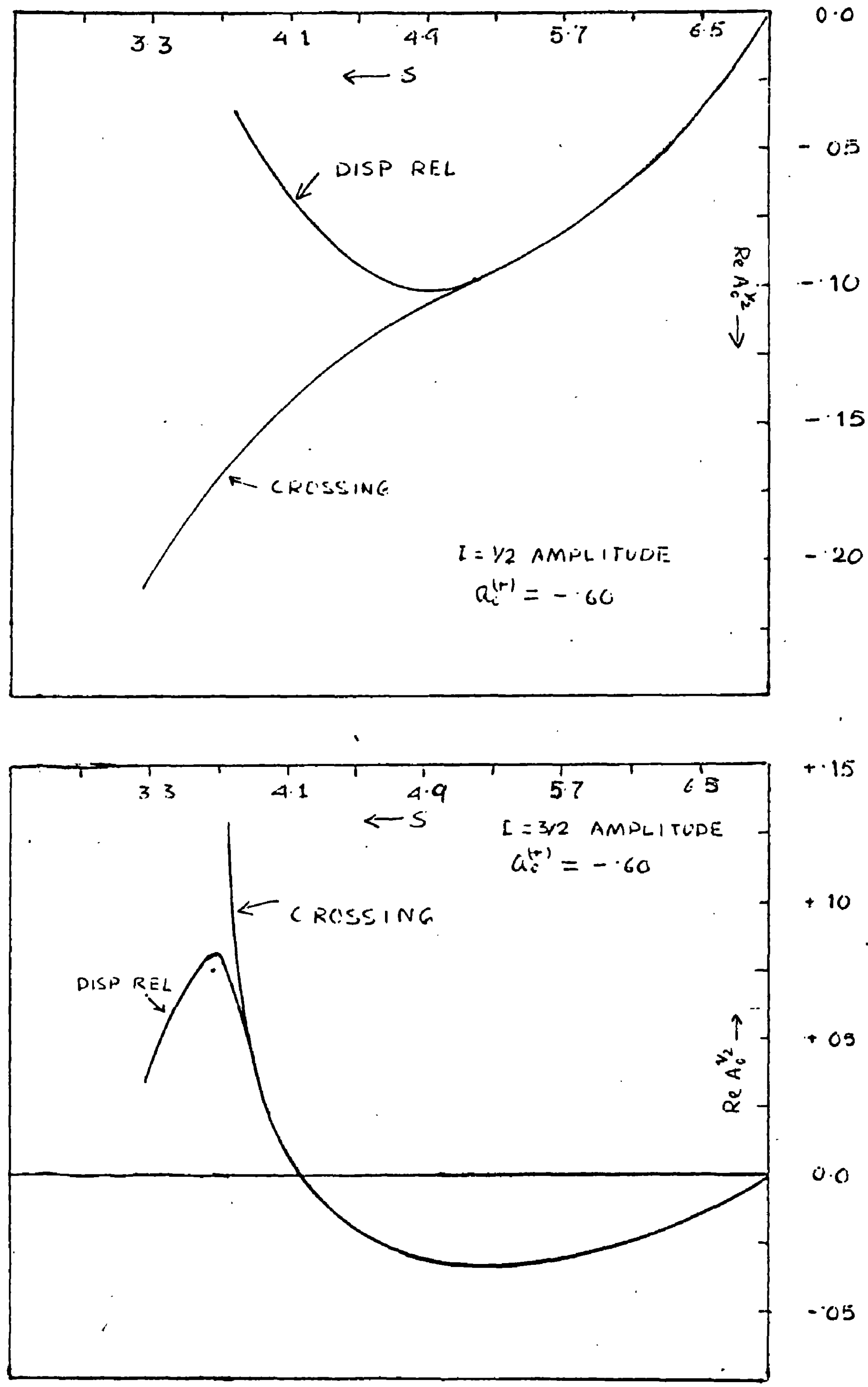


FIG 5·9 TEST ON CROSSING FOR IMAGINARY PARTS

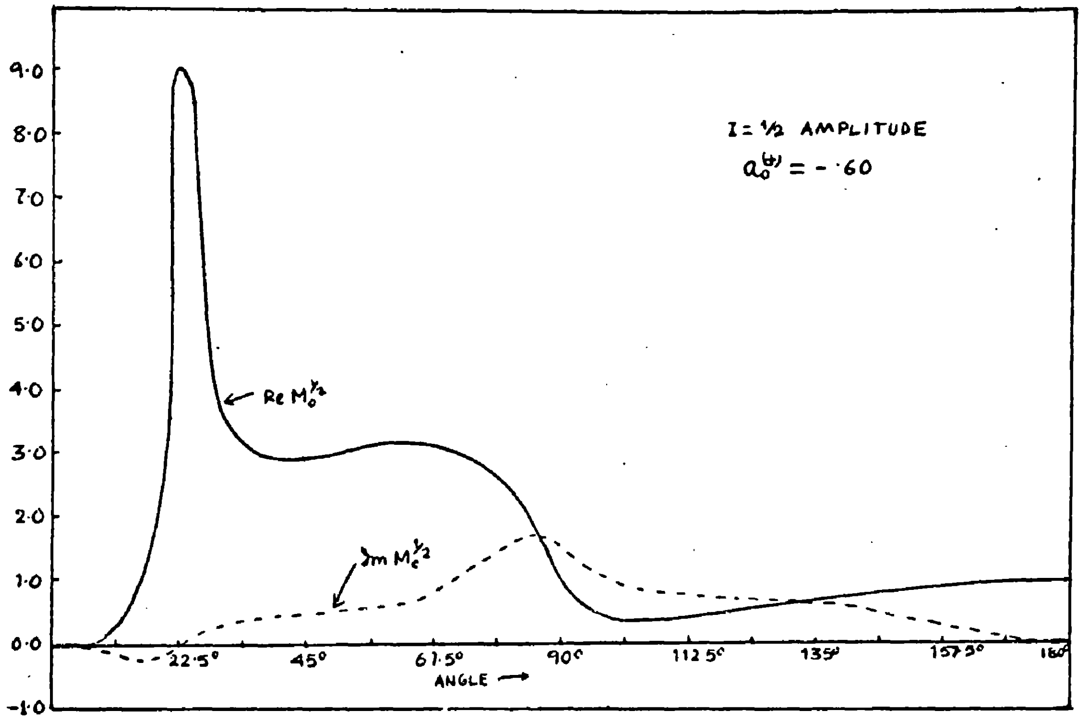


FIG 5.10

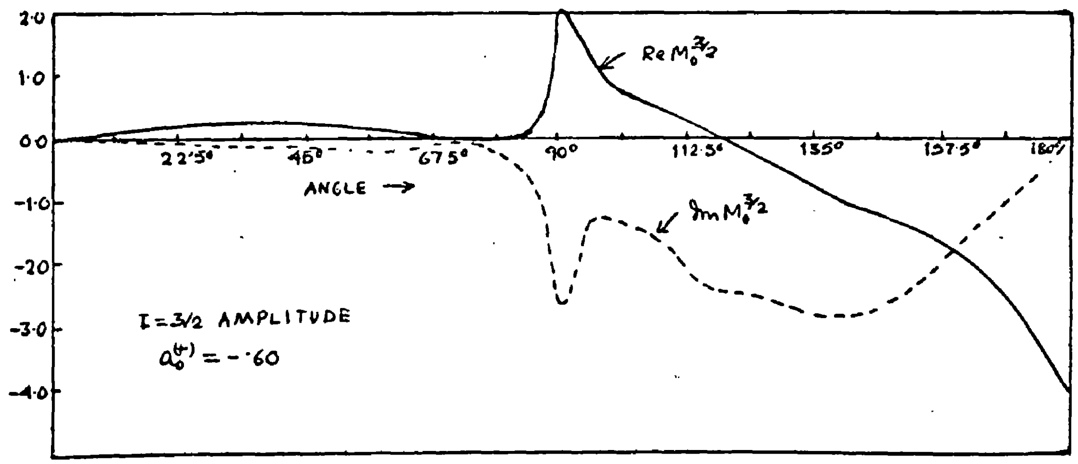


FIG 5.11

DISCONTINUITIES ACROSS THE CIRCLE CUT

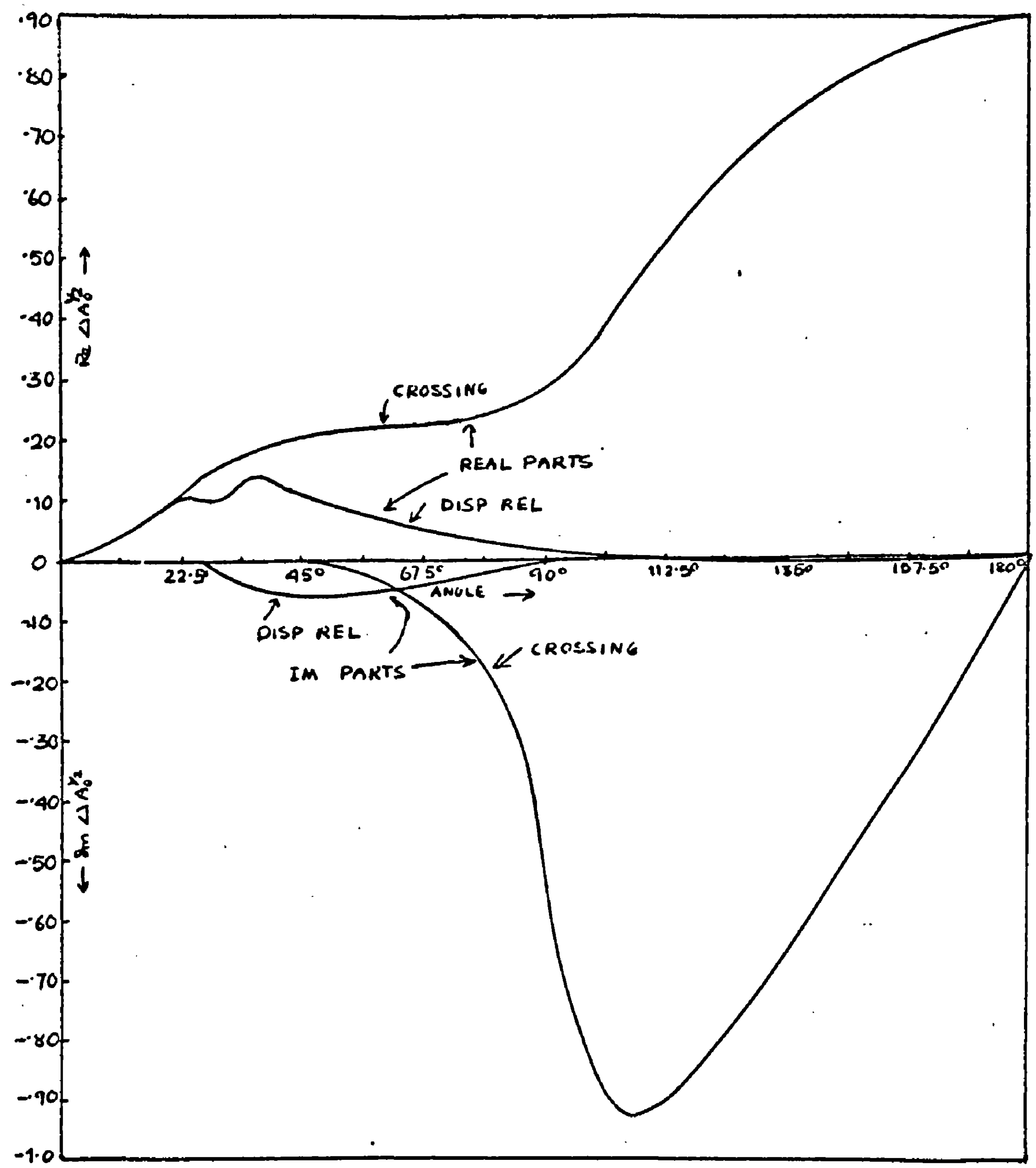


FIG 5·12 TEST ON CROSSING FOR $A_0^{1/2}$
ON CIRCLE CUT

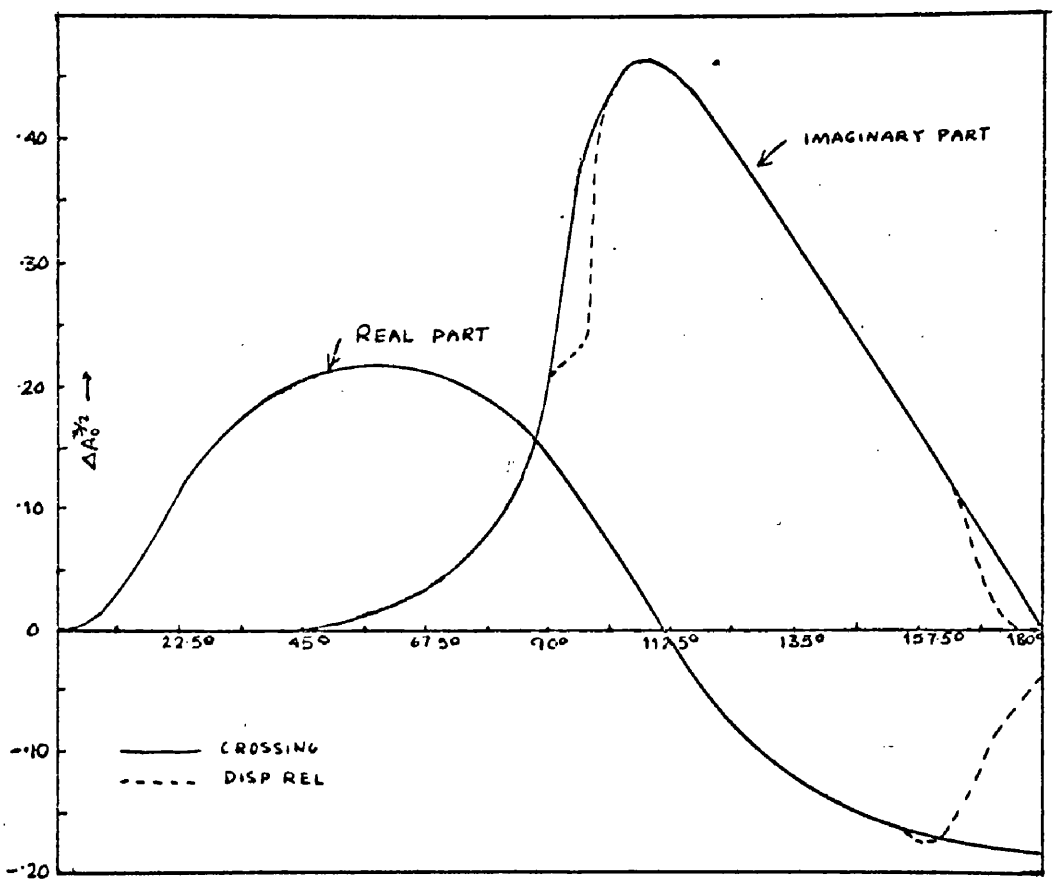


FIG 5·13 TEST ON CROSSING FOR $A_0^{3/2}$
ON CIRCLE CUT

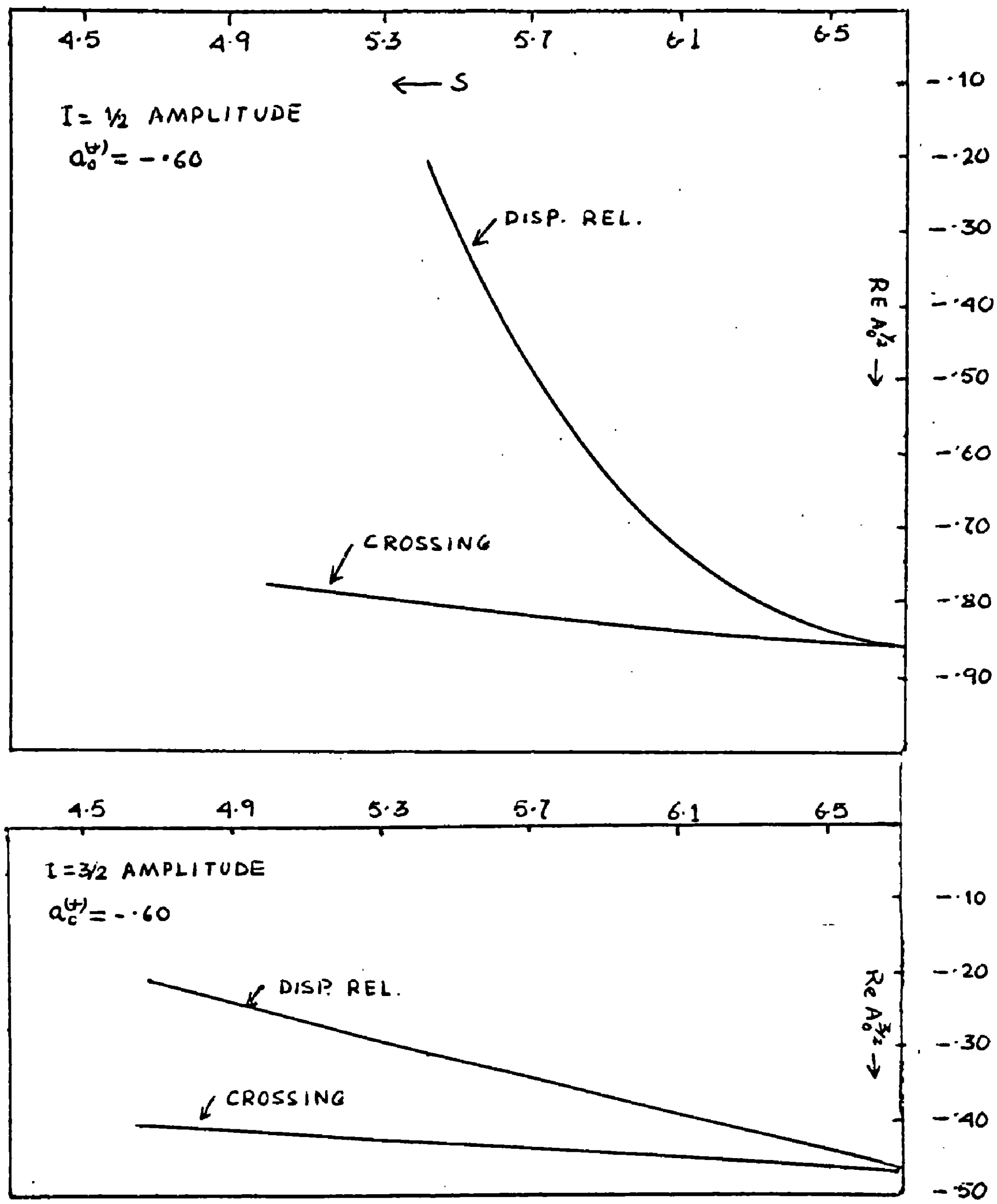


FIG. 5·14 TEST ON CROSSING FOR REAL PARTS

$a_0^{(+)}$	$\gamma_0^{1/2}$	$\delta_0^{1/2}$	$\gamma_0^{3/2}$	$\delta_0^{3/2}$
-.35	-68.92	3.04	-12.16	8.58
-.50	-24.73	2.45	-9.29	5.06
-.60	-.7.25	2.17	-8.04	3.97
-1.25	-5.56	1.29	-4.35	1.62
-1.5	-4.46	1.11	-3.68	1.33

The real parts of the inverse amplitude, $Re\,A_0^{I^{-1}}(s) = \frac{k}{\sqrt{s}} \cot \alpha_0^I$ are plotted in Fig. 5.6 and the phase shifts α_0^I are drawn in Fig. 5.7. The behaviours of $\frac{k}{\sqrt{s}} \cot \alpha_0^I$ indicate that $A_0^I(s)$ can be very well approximated by effective range formulae. The phase shifts are negative for both $I = 1/2$ and $I = 3/2$ states. the state $I = 3/2$ is much more repulsive than the $I = 1/2$ state. A close examination of the equations giving the discontinuities across the left hand cut and the circle cut (Eqns (5.11) (5.19) - (5.21) and (5.22)) reveal the following behaviours of various contributions.

On the L.H.C.

Contribution of	to $I = 1/2$	$I = 3/2$
S wave $K\pi$, $I = 1/2$	repulsive	attractive
S wave $K\pi$, $I = 3/2$	attractive	attractive
K^*	Long range attraction and short range repulsion	Long range repulsion and short range attraction
ρ and ABC lumped* together	Long range repulsion and short range attraction	Long range attraction and short range repulsion

* These contributions start at S = O. By long range we mean the contributions from the nearer regions and by short range

from further away regions.

On the circle cut

ABC	repulsion	repulsion
ρ	repulsion	attraction

It is to be noticed that the ABC contributions will be attractive for large positive values of $a_0^{(+)}$. The longest range force comes from ABC contributions because the front of the circle cut is nearest to the physical region. Since the high energy regions are suppressed in our formulation the contributions of the short range forces are absorbed in the subtraction constants. The long range part of the left hand cut, $0 \leqslant s \leqslant (m-\mu)^2$ and the front of the circle (ABC contribution) account for the variations in α_0^I in the low energy region. We have plotted $K_0^I(s)/k^2$ in Fig. 5.8. In both the isotopic spin states there is a sharp peak corresponding to the K^* contributions on the left hand cut. The region around the origin is very well suppressed Cutting off the left hand cut anywhere beyond the origin has practically no effect in the low energy solutions. It is also possible be using

$$\mathrm{Im}\, A_0^{I\,D.R.}(x) = \frac{K_0^I(x)}{[\mathrm{Re}\, A_0^{I'}(x)]^2 + [K_0^I(x)]^2} \qquad (5.44)$$

to calculate the imaginary part of the scattering amplitude for $0 \leqslant s \leqslant (m-\mu)^2$. For consistency this should agree with $\mathrm{Im}\, A_0^I(x)$ calculated from crossing (Eqns (5.11) and (5.22)). The agreement is reasonably good for both I = 1/2 and I = 3/2 states in the nearby regions for all values of

$a_0^{(+)}$ and becomes slightly better with the increase of $|a_0^{(+)}|$ Fig. 5.9 shows the case of $a_0^{(+)} = -.60$. We see that the agreement is quite good up to $s \approx 4.5$.

The discontinuities of the inverse amplitudes across the circle cut $M_0^I(\phi)$ are shown in Figs (5.10) and (5.11) for $a_0^{(+)} = -.60$. The real part has a very large peak near the front part of the circle in the I = 1/2 state. The peak in the I = 3/2 state occurs a bit further away. The imaginary parts in both I = 1/2 and I = 3/2 states have peaks beyond $\phi \approx 85^0$. Similar to the case of the left hand cut, we can obtain the discontinuities of the scattering amplitudes across the circle cut from the dispersion relations by using

$$\Delta A_0^{I\,D.R.}(\phi) = \frac{M_0^I(\phi)}{[G_0^I(\phi)]^2 + [M_0^I(\phi)]^2} \qquad (5.45)$$

For consistency this should agree with $\Delta A_0^I(\phi)$ obtained from Eqns (5.19) and (5.20). For the I = 1/2 state the agreement is good only for the front of the circle. Whereas, for the I = 3/2 state very good agreement is found for the entire circle cut. The case of I = 1/2 states improves a bit with the increase of $|a_0^{(+)}|$. Figs 5.12 and 5.13 give the results when $a_0^{(+)} = -.60$.

The region $0 \leq s \leq (M-\mu)^2$ is entirely in the crossed physical region, so it may be expected that the real part of the amplitude, $A_0^I(s)$ given by the dispersion relation in this region should agree with that obtained from the crossing relation

$$\mathrm{Re}\, A_0^I(s) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta \sum_{I'} \alpha_{II'}\, \mathrm{Re}\, A^{I'}(u, \cos\bar{\theta}) \quad (5.46)$$

when S is very near the crossed threshold $s = (m-M)^2$ only the S-waves are important on the right hand side of the above equation. Eqn (5.46) has been calculated for S near the crossed threshold by retaining only the S-wave on the right hand side Fig. 5.14 shows the results for $a_0^{(+)} = -.60$ The agreement is reasonably good.

Lastly, of the symmetry point one has $A^{1/2}(s,\cos\theta) = A^{3/2}(s,\cos\theta)$ Assuming D and higher waves to be small it follows that $A_0^{1/2}(s_0) \approx A_0^{3/2}(s_0)$. This is found to be approximately satisfied only for $a_0^{(+)} \approx -.60$.

For positive values of $a_0^{(+)}$ the front of the circle cut is attractive in both I = 1/2 and I = 3/2 states. In the state I = 1/2 the net effect of this and the attractive long range force coming from the exchange of K^* enhances $A_0^{1/2}(s)$ in the low energy region. As a result at a certain stage of the iteration process $a_0^{1/2}$ becomes greater than $4a_0^{3/2}$ and thus a zero appears in the scattering amplitude $A_0^{1/2}(s)$. The iteration process starts oscillating violently and it is not possible to obtain a stable solution.

5.3. DISCUSSION OF THE RESULTS AND CONCLUSIONS:

In this thesis we have managed to obtain the low energy solutions for the S-wave kaon-pion scattering amplitudes depending on only one parameter. This is chosen to be the combination, $a_0^{(+)} = \frac{1}{3}(a_0^{1/2} + 2a_0^{3/2})$ of the scattering lengths. The solutions for the values of $a_0^{(+)}$ in the range -.35 to -1.5 were obtained. They satisfy crossing resonably well. The region $-.35 < a_0^{(+)} < 0$ cannot be investigated, because of the zeros appearing in the scattering amplitudes. The region $a_0^{(+)} < -1.5$ was not investigated due to lack of machine time.

The numerical solutions give no clear evidence that a particular value of $a_0^{(+)}$ is preferable to the rest, except that $A_0^{1/2}(s_0) \simeq A_0^{3/2}(s_0)$ when $a_0^{(+)} \simeq -.60$. The agreement between the discontinuities of the scattering amplitudes obtained from the inverse amplitude dispersion relations and from the crossing relations improve slightly as $a_0^{(+)}$ is made more negative. When the magnitude of $a_0^{(+)}$ is quite large the contributions of the S-wave amplitudes on the left hand cut become comparable with other contributions. The above improvement in agreement may be due to this. So finally, it may be concluded from our numerical results that solutions for the S-wave scattering amplitudes exist for negative values of $a_0^{(+)}$ with a very slight preference for $a_0^{(+)} \simeq -.60$

If the physical solutions lie clear of the regions where zeros of the scattering amplitude may develop, we need not bother at all about these zeros. If this is not true the zeros should be taken into account. One possible way of dealing with these zeros on the real axis is suggested here. The position of a particular zero on the real axis in the scattering amplitude gives the position of the corresponding pole in the inverse amplitude. The scattering amplitudes are real on the real axis between the left and the right hand cuts. So the slope of the amplitude at the position of the zero gives the residue of the pole. Assuming that the scattering amplitude involving such a zero for $0 \leq s \leq (m-r)^2$ is well behaved in this region, the pole parameters for the inverse amplitude may be calculated in a particular cycle from the position of the zero and the slope at the position of the zero for the scattering amplitude given by the previous cycle. Such a calculation would require a big machine and it is not possible to predict beforehand whether it would give convergent solutions.

Although we could not find stable solutions for positive values of $a_0^{(+)}$ it is hoped that a more involved iteration scheme may yield convergent solutions. One needs a bigger and a faster machine to perform such calculations. In the intermediate stages of our unstable iteration process there is evidence of a peak in the amplitude, $A_0^{1/2}(s)$ in the low energy region. Whether this would remain in the final convergent solution, if obtainable, is impossible to tell beforehand. It is worth mentioning at this point that there are some experimental evidences[58] of the existence of a resonance at 730 MeV, commonly called the K' or the κ meson. The isotopic spin state of this is I = 1/2. Initially, it was thought to be a P wave resonance. Recent investigations[59] suggest that it is probably an S wave resonance.

The width is only of the order of 10 MeV.

Here, we did not make any attempt to calculate the P-wave amplitudes in the low energy region. The I = 1/2 state contains the K^* resonance at 880 MeV. If this amplitude is considered, the exchange of K^* on the left hand cut produces a repulsive force. The exchange of the ρ meson may produce attractive force strong enough to generate the resonance if the product of the coupling constants for the $\rho K\bar{K}$ and the $\rho\pi\bar{\pi}$ vertices has a negative sign and has got a magnitude[60], which is many times larger than that obtained from other considerations.[61] Then the alternative is to consider the multichannel problem. Bootstrap calculations have been performed by Diu et al[62] and Capps[63] by coupling the $K\pi$ and the $K\eta$ channels. The K^* could be bootstraped quite easily in this case, but the results are not very much satisfactory. The inclusion of contributions from other intermediate states including the two particle continum seems to be necessary. The other P wave amplitude with I = 3/2 may be expected to be small in the low energy region.

The first calculations of the S wave amplitudes for the scattering were performed by Lee[64] and Lee and Cho[65]. They did not put in any contributions for the exchange of the K^* resonance and so the solutions are similar to our S dominant solutions.

In K^+p scattering the nearest singularities on the left hand cut come from the process $\pi\pi \to K\bar{K}$ in the $I=0, \ell=0$

state. This later process, in turn, depends on both $K\pi$ and $\pi\pi$ scatterings. The pion-pion scattering in the state $I=0, \ell=0$ give the phases of the process $\pi\pi \to K\bar{K}$ mentioned above. Martin and Spearman[66] tried to fit the experimental results of K^+p scattering by parametrization of the dispersion relations. They expressed the $\pi\pi \to K\bar{K}$ amplitude in terms of one parameter, the combination $a_0^{(+)} = \frac{1}{3}(a_0^{1/2} + 2a_0^{3/2})$ of the $K\pi$ S wave scattering lengths. To obtain this the contributions coming from the exchange of K^* and the $K\pi$ system in the S wave were retained on the left hand cut of the $\pi\pi \to K\bar{K}$ amplitude. For the S-wave amplitude of $K\pi$ scattering they used

$$\operatorname{Im} A_0^I(s) = \left[a_0^I(s)\right]^2 \frac{k}{\sqrt{s}}$$

The pion-pion scattering amplitude in the state $I=0, \ell=0$ was put in the one pole approximation given by Hamilton et al, which is used in our calculations. For the best fit to the K^+p experimental results it was found by them that the S wave amplitudes for the kaon-pion scattering should be repulsive in the low energy region. The value of $a_0^{(+)}$ suggested by Martin and Spearman is

$$a_0^{(+)} = -.60 \pm .14$$

This is remarkably close to the value slightly preferred by our calculations.

Martin and Vick[67] attempted to calculate the kaon-pion scattering amplitudes by using parametric forms for the dispersion relations. Only the nearby portions of the unphysical cuts were retained, the rest was replaced by one or two poles. No attempts were made to generate the K^* resonance. The experimental results for it were used to reduce the number of free parameters. They determined the $\pi\pi \rightarrow K\bar{K}$ amplitudes using the same method as we have done. Sum rules were used to cut down the number of free parameters. The remaining free parameters were determined by minimization. They found that the short range poles gave appreciably large contributions, comparable to those coming from the nearby portions of the various cuts. The solutions for the S wave amplitudes indicate a negative and small value (practically zero) for $a_0^{(+)}$ the P wave, in the state I = 3/2 was found to very small.

Finally, we may conclude that our calculations have shown that the low energy kaon-pion scattering problem can be solved in terms of one parameter, which was chosen in our case to be $a_0^{(+)}$. If this parameter can be obtained accurately from some other source, the low energy region of the kaon-pion system would be known.

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APPENDIX I

EIGEN-STATES OF TOTAL ISOTOPIC SPIN

The scattering amplitude $A_{\beta\alpha}$ may be written in terms of a symmetric part, $A^{(+)}$ and an antisymmetric part $A^{(-)}$ in the kaon isotopic spin space as follows

$$A_{\beta\alpha} = \delta_{\beta\alpha} A^{(+)} + \tfrac{1}{2}[\tau_\beta, \tau_\alpha] A^{(-)} \qquad (I.1)$$

The projection operators for the eigenstates of total isotopic spin, $I = 1/2, 3/2$ may be obtained in the following way. The operator $(\underset{\sim}{\tau}/2 + \underset{\sim}{I})^2 = \underset{\sim}{I}^2$ has the eigen values $I^2 = \frac{3}{2}(\frac{3}{2}+1) = \frac{15}{4}$ and $I^2 = \frac{1}{2}(\frac{1}{2}+1) = \frac{3}{4}$ from which it follows that $(\underset{\sim}{\tau}\cdot\underset{\sim}{I})_{I=3/2} = 1$ and $(\underset{\sim}{\tau}\cdot\underset{\sim}{I})_{I=1/2} = -2$. Then the projection operators are

$$P_{3/2} = \frac{2 + \underset{\sim}{\tau}\cdot\underset{\sim}{I}}{3}, \quad P_{1/2} = \frac{1 - \underset{\sim}{\tau}\cdot\underset{\sim}{I}}{3} \qquad (I.2)$$

We have

$$\langle\beta|\,\underset{\sim}{\tau}\cdot\underset{\sim}{I}\,|\alpha\rangle = i\epsilon_{\beta j\alpha}\tau_j = \delta_{\beta\alpha} - \tau_\beta\tau_\alpha$$

So

$$\langle\beta|P_{1/2}|\alpha\rangle = \tau_\beta\tau_\alpha/3 \qquad (I.3)$$

$$\langle\beta|P_{3/2}|\alpha\rangle = \delta_{\beta\alpha} - \tau_\beta\tau_\alpha/3 \qquad (I.4)$$

According to our definition of the projection operators

$$A_{\beta\alpha} = \sum_I A^I \langle\beta|P_I|\alpha\rangle$$

$$= A^{1/2}\frac{\tau_\beta\tau_\alpha}{3} + A^{3/2}\left(\delta_{\beta\alpha} - \frac{\tau_\beta\tau_\alpha}{3}\right) \qquad (I.5)$$

The above equation may be rewritten as

$$A_{\beta\alpha} = A^{1/2}\,\frac{\delta_{\beta\alpha} + \frac{1}{2}[T_\beta, T_\alpha]}{3} + A^{3/2}\,\frac{2\delta_{\beta\alpha} - \frac{1}{2}[T_\beta, T_\alpha]}{3} \tag{I.6}$$

Comparing this with Eqn (1.1) we find

$$A^{(+)} = \frac{1}{3}\left(A^{1/2} + 2A^{3/2}\right) \tag{I.7}$$

$$A^{(-)} = \frac{1}{3}\left(A^{1/2} - A^{3/2}\right) \tag{I.8}$$

Expressing $A^{1/2}$, $A^{3/2}$ in terms of $A^{(\pm)}$ one gets

$$A^{1/2} = A^{(+)} + 2A^{(-)} \tag{I.9}$$

$$A^{3/2} = A^{(+)} - A^{(-)} \tag{I.10}$$

For channel III we may write

$$\langle jk|A|\beta\alpha\rangle = \sum_{I=0,1} A^I \langle jk|P^I|\beta\alpha\rangle \tag{I.11}$$

where j and k denotes the states of K and $\bar{K}$ respectively. P^I is the projection operator for the eigenstate of total isotopic spin state I and is defined by

$$P^I = \sum_{I_z} |I, I_z(K)\rangle\langle(\pi)\, I, I_z| \tag{I.12}$$

where (K) and (π) denote kaon and pion states respectively.

From the matrix element

$$\langle jk|P^I|\beta\alpha\rangle = \chi_j^\dagger P^I_{\beta\alpha}\chi_k \tag{I.13}$$

by using Eqn (1.12) one gets

$$\mathrm{Tr}\, P^{I\dagger}_{\alpha'\beta'} P^I_{\alpha\beta} = \mathcal{P}^I_{\alpha'\beta',\alpha\beta} \tag{I.14}$$

when

$$\rho^{I}_{\alpha'\beta',\alpha\beta} = \sum_{I_z} \langle \alpha'\beta' | I, I_z(\pi) \rangle \langle (\pi) I, I_z | \alpha\beta \rangle \qquad (I.15)$$

where ρ^{I} is the pion-pion isotopic spin projection operator and is given by

$$\rho^{0}_{\alpha'\beta',\alpha\beta} = \frac{1}{3}\delta_{\alpha'\beta'}\delta_{\alpha\beta} \quad , \quad \rho^{1}_{\alpha'\beta',\alpha\beta} = \frac{1}{2}\left(\delta_{\alpha\alpha'}\delta_{\beta\beta'} - \delta_{\alpha\beta'}\delta_{\alpha'\beta}\right)$$

It easily follows that

$$P^{0}_{\beta\alpha} = \frac{1}{\sqrt{6}}\delta_{\beta\alpha} \quad , \quad P^{1}_{\beta\alpha} = \frac{1}{4}[\tau_\beta, \tau_\alpha] \qquad (I.16)$$

Then

$$\langle jk | A | \beta\alpha \rangle = A^{0}\frac{1}{\sqrt{6}}\delta_{\beta\alpha} + A^{1}\frac{1}{4}[\tau_\beta, \tau_\alpha] \qquad (I.17)$$

Comparing with Eqn (I.1) it follows after writing B for the scattering amplitude in channel III that

$$B^{0} = \sqrt{6}\,A^{(+)} \quad , \quad B^{1} = 2A^{(-)} \qquad (I.18)$$

APPENDIX II

UNITARITY CONDITION FOR PARTIAL WAVE AMPLITUDES

We consider the case of channel I. Retaining only the two particle intermediate state consisting of a pion of four momentum p_5 and a kaon of four momentum p_6 in the unitarity condition, Eqn (1.11) one gets

$$\mathrm{Im}\, A_{fi} = \frac{1}{\pi}\int d^4p_5 d^4p_6\, \delta(p_5^2-\mu^2)\delta(p_6^2-\mu^2)\,\theta(p_{05})\,\theta(p_{06}) \times \delta^{(4)}(p_5+p_6-p_1-p_2) A_{nf}^* A_{ni} \qquad (\mathrm{II}\cdot 1)$$

Both A_{nf}^* and A_{ni} describe kaon-pion scattering. For the process $i \to n$ the scattering angle θ_1 is defined by $\cos\theta_1 = \underset{\sim}{p_1}\cdot\underset{\sim}{p_5}/|p_1 p_5|$ and for the process $f \to n$ the scattering angle θ_2 is defined by $\cos\theta_2 = \underset{\sim}{p_3}\cdot\underset{\sim}{p_5}/|p_3 p_5|$

For conveienence the vector $\underset{\sim}{p_1}$ is taken along the positive z axis and the vector $\underset{\sim}{p_3}$ in the yz plane. The angle between the plane containing the vectors $\underset{\sim}{p_1}$ and $\underset{\sim}{p_5}$ and the plane-yz is defined to be Φ. Then the various vectors are

$$\underset{\sim}{p_1} = k(0,0,1)$$
$$\underset{\sim}{p_3} = k(0,\sin\theta,\cos\theta)$$
$$\underset{\sim}{p_5} = k(\sin\theta_1\sin\Phi,\sin\theta_1\cos\Phi,\cos\theta_1)$$

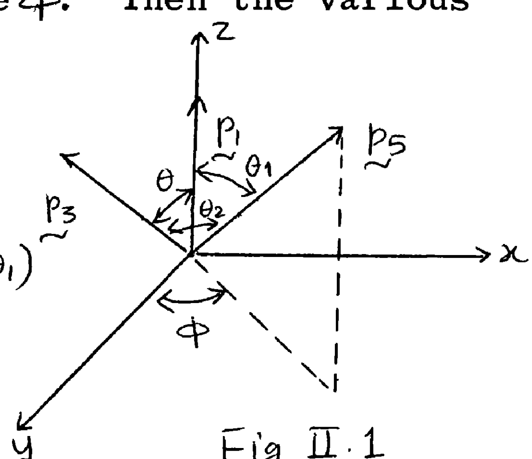


Fig II.1

Then

$$\cos\theta_2 = \frac{\underset{\sim}{p_3}\cdot\underset{\sim}{p_5}}{|p_3 p_5|} = \cos\theta_1\cos\theta + \sin\theta_1\sin\theta\cos\Phi \qquad (\mathrm{II}\cdot 2)$$

The d^4p_6 integration in Eqn (II.1) can be performed by using the delta function $\delta^{(4)}(p_5+p_6-p_1-p_2)$. Then the dp_{05} integration may be performed with the help of the delta function, $\delta(p_5^2-\mu^2)$ This leads to

$$\mathrm{Im}\, A_{fi} = \frac{1}{\pi}\int k^2 dk \sin\theta_1 d\theta_1 d\phi \frac{1}{4p_{05}p_{06}} \delta\left[p_{06}-\sqrt{k^2+m^2}\right] A_{nf}^* A_{ni} \qquad (\mathrm{II}.3)$$

$$\text{where } p_{05}+p_{06} = \sqrt{s}\,, \quad p_{05} = \sqrt{k^2+\mu^2}$$

and we have used $d^3\underset{\sim}{p_5} = k^2 dk \sin\theta_1 d\theta_1\, d\phi$. Changing the variable of integration from k to $\sqrt{k^2+m^2}+\sqrt{k^2+\mu^2}$ and using

$$dk = \frac{\sqrt{k^2+m^2}\sqrt{k^2+\mu^2}}{k\left[\sqrt{k^2+m^2}+\sqrt{k^2+\mu^2}\right]}\, d\left(\sqrt{k^2+m^2}+\sqrt{k^2+\mu^2}\right)$$

the integration over the variable k may be performed with the help of the delta function. Because the argument of the delta function can be written as follows: $p_{06}-\sqrt{k^2+m^2} = \sqrt{s}-\left(\sqrt{k^2+m^2}+\sqrt{k^2+\mu^2}\right)$ Then expressing A_{fi} in terms of isotopic spin states:

$$\mathrm{Im}\, A^I(s,\cos\theta) = \frac{1}{4\pi}\frac{k}{\sqrt{s}}\int \sin\theta_1 d\theta_1 d\phi\, A^{I*}(s,\cos\theta_2) A^I(s,\cos\theta_1) \qquad (\mathrm{II}.4)$$

Now we project out the lth partial wave amplitude and expand A^{I*} and A^I on the right hand side in terms of partial wave amplitudes

$$\mathrm{Im}\, A_l^I(s) = \frac{1}{4\pi}\frac{k}{\sqrt{s}}\frac{1}{2}\int_{-1}^{+1} d\cos\theta \int_{-1}^{+1} d\cos\theta_1 \int_0^{2\pi} d\phi \sum_{l'}\sum_{l''}(2l'+1)(2l''+1)$$

$$\times P_{l'}(\cos\theta_1)\, P_{l''}(\cos\theta_2)\, A_{l''}^{I*}(s)\, A_{l'}^I(s) \qquad (\mathrm{II}.5)$$

where the integration over θ_1 has been replaced by one over $\cos\theta_1$. $\cos\theta_2$ may be expressed in terms of $\cos\theta, \cos\theta_1$ and $\cos\phi$ by using Eqn (II.2). Then we have

$$P_{l''}(\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos\phi) = P_{l''}(\cos\theta)P_{l''}(\cos\theta_1)$$
$$+ 2\sum_{m=1}^{l''} \frac{(l''-m)!}{(l''+m)!}(-1)^m P_{l''}^m(\cos\theta)P_{l''}^m(\cos\theta_1)\cos m\phi \tag{II.6}$$

The second term does not contribute, because

$$\int_0^{2\pi} d\phi \cos m\phi = 0 \tag{II.7}$$

The orthonormality property of the Legendre polynomial gives the final expression

$$\mathrm{Im}\, A_l^I(s) = \frac{k}{\sqrt{s}}\left|A_l^I(s)\right|^2 \tag{II.8}$$

Inclusion of other intermediate states gives positive contributions, so we can write

$$\mathrm{Im}\, A_l^I(s) \geq \frac{k}{\sqrt{s}}\left|A_l^I(s)\right|^2$$

in the general case. This gives the form

$$\mathrm{Im}\, A_l^I(s) = \frac{k}{\sqrt{s}}\left|A_l^I(s)\right|^2 R_l^I(s) \tag{II.9}$$

The unitarity condition in the two pion approximation, Eqn (2.36) may be obtained exactly in the same way.

APPENDIX III

THE DERIVATIVE CONDITIONS

We obtain the first derivative conditions for Eqn (2.18) by differentiating with respect to s and $\cos\theta$ at the symmetry point. It is done as follows:

$$\left.\frac{\partial A^{I}(s,\cos\theta)}{\partial s}\right|_{\substack{s=s_0\\ \cos\theta=0}} = \sum_{I'} \alpha_{II'} \left\{ \frac{\partial u}{\partial s} \left.\frac{\partial A^{I'}(u,\cos\bar\theta)}{\partial u}\right|_{\substack{u=s_0\\ \cos\bar\theta=0}} + \frac{\partial \cos\bar\theta}{\partial s} \left.\frac{\partial A^{I'}(u,\cos\bar\theta)}{\partial \cos\bar\theta}\right|_{\substack{u=s_0\\ \cos\bar\theta=0}} \right\} \quad (\text{III}\cdot 1)$$

$$\left.\frac{\partial A^{I}(s,\cos\theta)}{\partial \cos\theta}\right|_{\substack{s=s_0\\ \cos\theta=0}} = \sum_{I'} \alpha_{II'} \left\{ \frac{\partial u}{\partial \cos\theta} \left.\frac{\partial A^{I'}(u,\cos\bar\theta)}{\partial u}\right|_{\substack{u=s_0\\ \cos\bar\theta=0}} + \frac{\partial \cos\bar\theta}{\partial \cos\theta} \left.\frac{\partial A^{I'}(u,\cos\bar\theta)}{\partial \cos\bar\theta}\right|_{\substack{u=s_0\\ \cos\bar\theta=0}} \right\} \quad (\text{III}\cdot 2)$$

We have

$$\left.\frac{\partial u}{\partial s}\right|_{\substack{s=s_0\\ \cos\theta=0}} = \frac{\partial}{\partial s}\left[\Sigma - s + 2k^2(1-\cos\theta)\right]\Big|_{\substack{s=s_0\\ \cos\theta=0}} = -1 + 2\left.\frac{dk^2}{ds}\right|_{s=s_0} = -\frac{s_0^2+(m^2-\mu^2)^2}{2s_0^2} \equiv A \ (\text{say})$$

$$\left.\frac{\partial \cos\bar\theta}{\partial s}\right|_{\substack{s=s_0\\ \cos\bar\theta=0}} = \frac{\partial}{\partial s}\left[1 - \frac{k^2(1-\cos\theta)}{\bar k^2}\right]\Big|_{\substack{s=s_0\\ \cos\theta=0}} = -\frac{1}{k_0^2}\left.\frac{dk^2}{ds}\right|_{s=s_0} + \frac{1}{k_0^2}\frac{dk^2}{ds}\left.\frac{\partial u}{\partial s}\right|_{\substack{s=s_0\\ \cos\theta=0}}$$

$$= -\frac{1}{k_0^2}\left.\frac{dk^2}{ds}\right|_{s=s_0}\left\{1-A\right\} = -\frac{1}{k_0^2}\frac{(s_0^2-(m^2-\mu^2)^2)(3s_0^2+(m^2-\mu^2)^2)}{8s_0^4} \equiv B \ (\text{say})$$

$$\left.\frac{\partial u}{\partial \cos\theta}\right|_{\substack{s=s_0\\ \cos\theta=0}} = \frac{\partial}{\partial \cos\theta}\left[\Sigma - s + 2k^2(1-\cos\theta)\right]\Big|_{\substack{s=s_0\\ \cos\theta=0}} = -2k_0^2 \equiv C \ (\text{say})$$

$$\left.\frac{\partial \cos\bar\theta}{\partial \cos\theta}\right|_{\substack{s=s_0\\ \cos\theta=0}} = \frac{\partial}{\partial \cos\theta}\left[1 - \frac{k^2(1-\cos\theta)}{\bar k^2}\right]\Big|_{\substack{s=s_0\\ \cos\theta=0}} = 1 - 2\left.\frac{dk^2}{ds}\right|_{s=s_0} = -A$$

Approximating

$$\frac{\partial}{\partial s}A^{I}(s,\cos\theta) \approx \frac{dA_0^{I}(s)}{ds}, \quad \frac{\partial}{\partial \cos\theta}A^{I}(s,\cos\theta) \approx 3A_1^{I}(s_0) = 3a_1^{I}$$

we get from Eqn (III.1)

$$a_1^{1/2} \approx -\frac{1+3A}{9B}\frac{dA_0^{1/2}(s)}{ds} + \frac{4}{9B}\frac{dA_0^{3/2}(s)}{ds} \qquad (\text{III}\cdot 3)$$

$$a_1^{3/2} \approx \frac{2}{9B}\frac{dA_0^{1/2}(s)}{ds} + \frac{1-3A}{9B}\frac{dA_0^{3/2}(s)}{ds} \qquad (\text{III}\cdot 4)$$

The same equations may be obtained from Eqn (III.2).

Putting numerical values we get

$$a_1^{1/2} \approx 2.64\,\frac{dA_0^{1/2}(s)}{ds} + 5.96\,\frac{dA_0^{3/2}(s)}{ds} \qquad (\text{III}\cdot 5)$$

$$a_1^{3/2} \approx 2.98\,\frac{dA_0^{1/2}(s)}{ds} + 5.17\,\frac{dA_0^{3/2}(s)}{ds} \qquad (\text{III}\cdot 6)$$

Similarly differentiating Eqn (2.24) with respect to S we have

$$\left.\frac{\partial A^I(s,\cos\theta)}{\partial s}\right|_{\substack{s=s_0\\ \cos\theta=0}} = \sum_{I'} \gamma_{II'}\left\{ \frac{\partial t}{\partial s}\left.\frac{\partial B^{I'}(t,\cos\phi)}{\partial t}\right|_{\substack{t=t_0\\ \cos\phi=0}} + \frac{\partial\cos\phi}{\partial s}\left.\frac{\partial B^{I'}(t,\cos\phi)}{\partial\cos\phi}\right|_{\substack{t=t_0\\ \cos\phi=0}}\right\} \qquad (\text{III}\cdot 7)$$

where

$$\left.\frac{\partial t}{\partial s}\right|_{\substack{s=s_0\\ \cos\theta=0}} = \frac{\partial}{\partial s}\left[-2k^2(1-\cos\theta)\right] = -2\left.\frac{dk^2}{ds}\right|_{s=s_0} = -\frac{s_0^2-(m^2-\mu^2)^2}{2s_0^2} \equiv E \text{ (say)}$$

$$\left.\frac{\partial\cos\phi}{\partial s}\right|_{\substack{s=s_0\\ \cos\theta=0}} = \frac{1+\frac{1}{2}\frac{\partial t}{\partial s}}{2p_0q_0} - \frac{(s_0+p_0^2+q_0^2)(p_0^2+q_0^2)}{16p_0^3q_0^3}\frac{\partial t}{\partial s} \equiv F \text{ (say)}$$

Then approximating

$$\frac{\partial}{\partial s}A^I(s,\cos\theta) \approx \frac{dA_0^I(s)}{ds} \;;\quad \frac{\partial}{\partial t}B^I(t,\cos\phi) \approx \frac{dB_0^0(t)}{dt}$$

$$\frac{\partial}{\partial\cos\phi}B^I(t,\cos\phi) \approx 3\,B_1^1(t_0)$$

We get

$$\frac{dA_0^I(s)}{ds} \simeq \frac{E}{\sqrt{6}}\frac{dB_0^0(t)}{dt} + 3\begin{pmatrix}1\\ -1/2\end{pmatrix} F\, B_1^1(t_0) \qquad (\text{III}\cdot 8)$$

Putting numerical values one gets

$$\frac{dA_0^I(s)}{ds} \approx -.031\frac{dB_0^0(t)}{dt} + \begin{pmatrix} .1914 \\ -.0957 \end{pmatrix} B_1^1(t_0) \qquad (\mathrm{III}.9)$$

APPENDIX IV

BOUNDARIES OF THE DOUBLE SPECTRAL FUNCTIONS AND THE REGIONS OF CONVERGENCE OF THE LEGENDRE POLYNOMIAL EXPANSIONS ON THE LEFT HAND CUTS.

The box diagrams determining the boundaries of the double spectral functions for kaon-pion scattering are drawn below

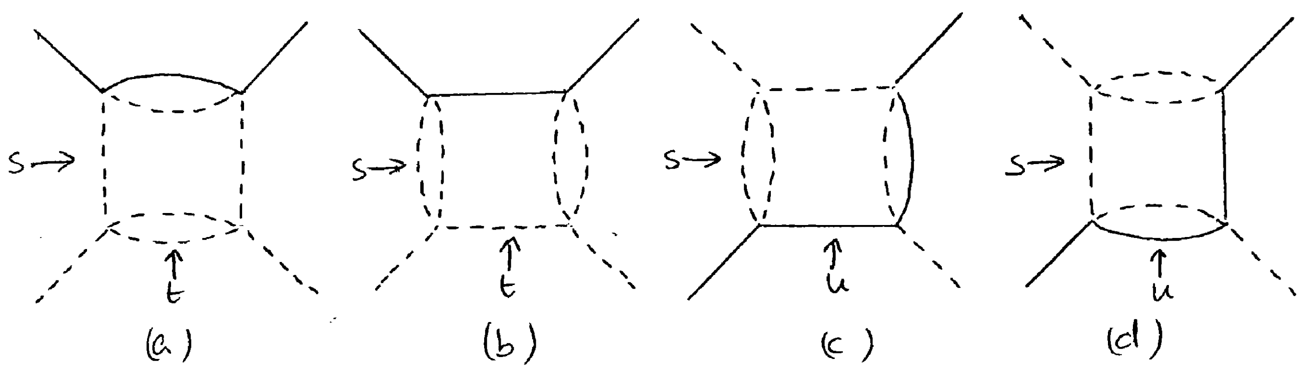


Fig IV.1

The boundary determined by a particular diagram is decided by the masses of the intermediate states. Let us consider the general box diagram

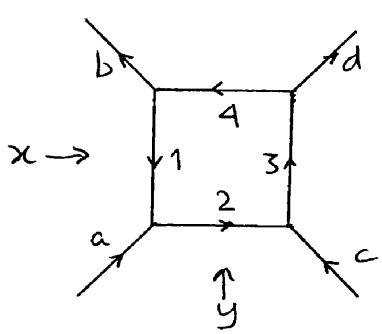


Fig IV.2

x and y are the independent variables. The masses of the particles a, b, c, d, 1, 2, 3 and 4 are denoted by $m_a, m_b, m_c, m_d, m_1, m_2, m_3,$ and m_4

respectively. We define

$$y_{ij} = \frac{k_i k_j}{m_i m_j} \qquad (i,j = 1,2,3,4) \tag{IV.1}$$

The four-momentum k_i is defined by

$$k_i^2 = m_i^2 \tag{IV.2}$$

y_{12}, y_{23}, y_{34} and y_{14} are functions of only the internal and external masses, while y_{13} and y_{24} depend on the variables x and y. Application of energy-momentum conservation to each corner separately leads to

$$y_{12} = \frac{m_1^2 + m_2^2 - m_a^2}{2m_1 m_2} \tag{IV.3}$$

$$y_{23} = \frac{m_2^2 + m_3^2 - m_c^2}{2m_2 m_3} \tag{IV.4}$$

$$y_{34} = \frac{m_3^2 + m_4^2 - m_d^2}{2m_3 m_4} \tag{IV.5}$$

$$y_{14} = \frac{m_1^2 + m_4^2 - m_b^2}{2m_1 m_4} \tag{IV.6}$$

$$y_{13} = \frac{m_1^2 + m_3^2 - y}{2m_1 m_3} \tag{IV.7}$$

$$y_{24} = \frac{m_2^2 + m_4^2 - x}{2m_2 m_4} \tag{IV.8}$$

The Landau-Cutkosky rules give the following condition for the boundary of the double spectral function determined by the above box diagram

$$\begin{vmatrix} 1 & y_{12} & y_{13} & y_{14} \\ y_{12} & 1 & y_{23} & y_{24} \\ y_{13} & y_{23} & 1 & y_{34} \\ y_{14} & y_{24} & y_{34} & 1 \end{vmatrix} = 0 \tag{IV.9}$$

Diagram (a) of Fig. IV.1 contributes to the double spectral functions $A_{13}^{(\pm)}$ we have

$$y_{12} = y_{23} = y_{34} = y_{14} = 1.$$

$$y_{13} = 1 - t/2\mu^2$$

$$y_{24} = \frac{4\mu^2 + (m+\mu)^2 - s}{4\mu(m+\mu)}$$

Then the application of Eqn (IV.9) gives the boundary

$$[t-4\mu^2][s-(m+3\mu)^2] - 32\mu^3(m+\mu) = 0 \qquad (IV.10)$$

Diagram (b), too contributes to $A_{13}^{(\pm)}$ and gives the boundary

$$[t-16\mu^2][s-(m+\mu)^2][s-(m-\mu)^2] - 64\mu^4 s = 0 \qquad (IV.11)$$

Diagrams (c) and (d) contribute to the spectral function $A_{12}^{(\pm)}$ and Eqn (IV.9) gives the boundary

$$[s-(m+\mu)^2][u-(m+3\mu)^2] - 16\mu^2(m+\mu)^2 = 0 \qquad (IV.12)$$

for diagram (c), while the boundary for diagram (d) is obtained by interchanging s and u in the above equation.

The boundaries of $A_{23}^{(\pm)}$ are the same as $A_{13}^{(\pm)}$ and are obtained from Eqn (IV.10) and Eqn (IV.11) by replacing s by u.

Now we proceed to determine the regions of convergence for the Legendre polynomial expansions on the left hand cuts. The Neumann's expansion theorem which states: If $f(z)$ is an analytic function, regular within and on an ellipse C with foci at the points of affix $\pm$ 1, it can be expanded as a series of Legendre polynomials

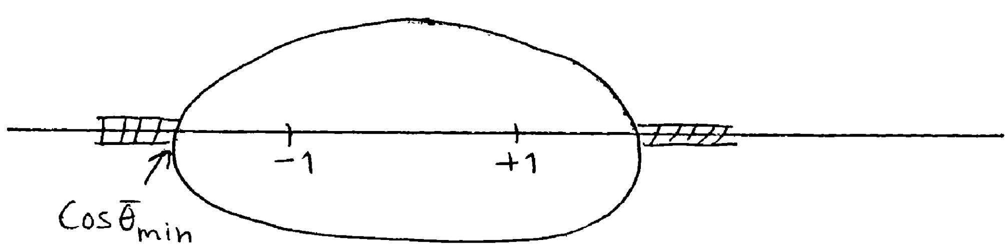
$$f(z) = \sum_{0}^{\infty} a_n P_n(z)$$

which converges uniformly when z lies within or on a smaller ellipse C_1, confocal with C; determines these regions of convergence.

Let us first consider the expansion on the right hand side of Eqn (2.73). Im A(u, Cos $\bar{\Theta}$) is analytic and regular in $\cos\bar{\theta}$ unless one of the denominators in

$$A_2(s,u,t) = \frac{1}{\pi}\int ds' \frac{A_{12}(s',u,t)}{s'-s} + \frac{1}{\pi}\int du' \frac{A_{23}(s,u',t)}{u'-u} \qquad (\text{IV}\cdot 13)$$

vanishes. The nearest singularity in $\cos\bar{\theta}$ coming from the vanishing of any one of the two denominators for a particular value of u is determined by the boundary of the double spectral term concerned. How the boundary determines the size of the ellipse inside which the expansion is convergent is illustrated below for the case of the first term.



where $\cos\bar{\theta}_{min}$ is determined by

$$\cos\bar{\theta}_{min} = 1 + \frac{\Sigma - u - C_{1,2}(u)}{2K^2} \qquad (\text{IV}\cdot 14)$$

with

$$C_1(u) = \frac{16\mu^2(m+\mu)^2}{u-(m+3\mu)^2} + (m+\mu)^2$$

$$C_2(u) = \frac{16\mu^2(m+\mu)^2}{u-(m+\mu)^2} + (m+3\mu)^2$$

which are the boundaries of $A_{12}^{(\pm)}$. Similarly for the second term

$$\cos\bar{\theta}_{min} = 1 + \frac{c_{3,4}(u)}{2\bar{k}^2} \qquad (IV\text{-}15)$$

where

$$c_3(u) = \frac{32\mu^3(m+\mu)}{u-(m+3\mu)^2} + 4\mu^2$$

$$c_4(u) = \frac{16\mu^4}{\bar{k}^2} + 16\mu^2$$

are the boundaries of $A_{23}^{(\pm)}$. Now $\cos\bar{\theta}$ in $\operatorname{Im} A(u,\cos\bar{\theta})$ is given by

$$\cos\bar{\theta} = 1 + \frac{\Sigma - s - u}{2\bar{k}^2} \qquad (IV\text{-}16)$$

For $0 \leqslant s \leqslant (m-\mu)^2$ we have $-1 \leqslant \cos\bar{\theta} \leqslant +1$ and so there is no trouble in the expansion. For $s < 0$, so long is inside the smallest of the ellipses determined by Eqns (IV.14) and (IV.15) for $(m+\mu)^2 \leqslant u \leqslant \Sigma - s$ the expansion is convergent. Numerical calculations show that this is so for values of s only up to ≈ -27. The smallest ellipse is given by C_4.

Similarly, in Eqn (2.74) $\operatorname{Im} B(t,\cos\varphi)$ is analytic and regular in $\cos\varphi$ unless any one of the denominators in

$$A_3(s,u,t) = \frac{1}{\pi}\int ds' \frac{A_{13}(s',u,t)}{s'-s} + \frac{1}{\pi}\int du' \frac{A_{23}(s,u',t)}{u'-u} \qquad (IV\text{-}17)$$

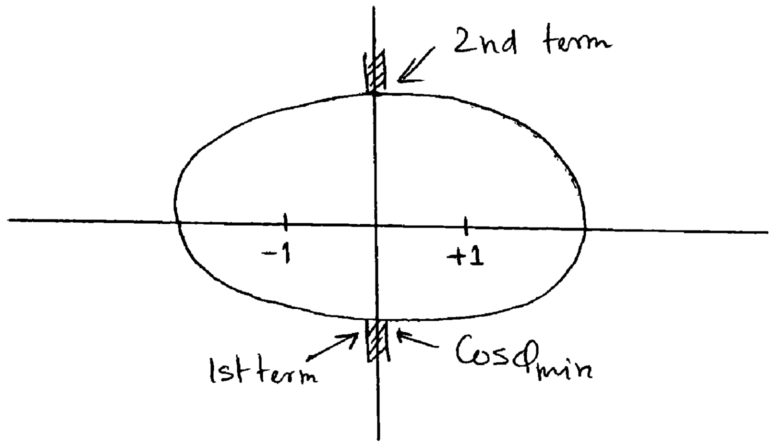
vanishes. We have to consider only the first term, since the second term gives the point on the opposite side of the ellipse.

There are two cases:

(i) $4\mu^2 < t < 4m^2$

$$i\cos\varphi_{min} = -\frac{c_{5,6}(t)+t/2-m^2-\mu^2}{2\sqrt{(t/4-\mu^2)(m^2-t/4)}} \qquad (\mathrm{IV}.18a)$$

that is



(ii) $t > 4m^2$

$$\cos\varphi_{min} = \frac{c_{5,6}(t)+t/2-m^2-\mu^2}{2\sqrt{(t/4-\mu^2)(t/4-m^2)}} \qquad (\mathrm{IV}.18b)$$

where

$$c_5(t) = \frac{32\mu^3(m+\mu)}{t-4\mu^2} + (m+3\mu)^2$$

$$c_6(t) = m^2+\mu^2+\frac{32\mu^4}{t-16\mu^2} + \sqrt{\left(m^2+\mu^2-\frac{32\mu^4}{t-16\mu^2}\right)^2-(m^2-\mu^2)^2}$$

On the real axis in the s-plane for $s<0$, $\cos\varphi$ in $\mathrm{Im}\,B(t,\cos\varphi)$ is real and is given by

$$\cos\varphi = \frac{s+t/2-m^2-\mu^2}{2\sqrt{(t/4-\mu^2)(t/4-m^2)}} \qquad (\mathrm{IV}.19)$$

Numerical calculations show that $\cos\varphi$ is inside the smallest ellipse determined by Eqns (IV.18a,b) for $4\mu^2 \lesssim t \lesssim -4k^2$ so long s is greater than $s \approx -27$.

On the circle cut (Eqn (2.78)) $\cos\varphi$ is complex and is given by

$$\cos\phi = \frac{2\lambda + t/2 \pm 2i\sqrt{(\lambda+m^2)(-\lambda-\mu^2)}}{2i\sqrt{(t/4-\mu^2)(m^2-t/4)}} \qquad (IV.20)$$

In this case it is found that $\cos\phi$ is within the smallest ellipse determined by Eqn (IV.18a) everywhere on the circle. So the expansion in Legendre polynomial is convergent for all points on the circle cut.

For channel III the expansion involved is one in $\cos\theta$ for $\mathrm{Im}\, A(s,\cos\theta)$ on the left hand cut in the t-plane. The nearest singularity in $\cos\theta$ comes from the vanishing of any one of the denominators in

$$A_1(s,u,t) = \frac{1}{\pi}\int dt' \frac{A_{13}(s,u,t')}{t'-t} + \frac{1}{\pi}\int du' \frac{A_{12}(s,u',t)}{u'-u} \qquad (IV.21)$$

The smallest ellipses for various boundaries are

$$\cos\theta_{min} = 1 + \frac{C_{7,8}(s)}{2k^2} \qquad (IV.22)$$

with

$$C_7(s) = \frac{32\mu^3(m+\mu)}{s-(m+3\mu)^2} + 4\mu^2$$

$$C_8(s) = \frac{16\mu^4}{k^2} + 16\mu^2$$

and

$$\cos\theta_{min} = 1 + \frac{\Sigma - s - C_{9,10}(s)}{2k^2} \qquad (IV.23)$$

with

$$C_9(s) = \frac{16\mu^2(m+\mu)^2}{s-(m+\mu)^2} + (m+3\mu)^2$$

$$C_{10}(s) = \frac{16\mu^2(m+\mu)}{s-(m+3\mu)^2} + (m+\mu)^2$$

It is found by numerical calculation that $\cos\theta$ in the expansion for $\mathrm{Im}\, A(s, \cos\theta)$ given by

$$\cos\theta = 1 + \frac{t}{2k^2} \qquad (\mathrm{IV}.24)$$

is inside the smallest of the ellipses determined by Eqns (IV.22) and (IV.23) for $(m+r)^2 \leq s \leq (p_- + q_-)^2$ only up to $t \approx -32r^2$

The smallest ellipse is given by C_8 .

APPENDIX V

CERTAIN IMPORTANT INTEGRALS

(i) $D_0^0(t)$ in one pole approximation.

We have

$$D_0^0(t) = 1 - \frac{t+t_s}{\pi}\int_{4\mu^2}^{\infty} dt' \sqrt{\frac{t'-4}{t'}}\ \frac{\Gamma}{(t'-t)(t'+t_s)^2}$$

$$= 1 - \frac{\Gamma}{t+t_s}\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \sqrt{\frac{t'-4}{t'}}\left\{\frac{1}{t'-t} - \frac{1}{t'+t_s} - \frac{t+t_s}{(t'+t_s)^2}\right\}$$

$$= 1 - \frac{2\Gamma}{t+t_s}\frac{1}{\pi}\int_0^1 du\left\{\frac{t-4}{t}\ \frac{1}{u^2-\frac{t-4}{t}} + \frac{t_s+4}{t_s}\ \frac{1}{\frac{t_s+4}{t_s}-u^2} - \frac{4(t+t_s)}{t_s^2}\ \frac{u^2}{\left(\frac{t_s+4}{t_s}-u^2\right)^2}\right\} \qquad (V.1)$$

where

$$u^2 = \frac{t'-4}{t'}$$

The integrations for the three terms may be performed very easily giving

(a) $4\mu^2 \leq t \leq +\infty$ or $-\infty \leq t \leq 0$

$$D_0^0(t) = 1 + \frac{2\Gamma}{\pi}\left\{\frac{1}{t+t_s}\left[\sqrt{\frac{t-4}{t}}\ \ln\frac{\sqrt{|t|}+\sqrt{|t-4|}}{2} - C_1\right] + C_2\right\} \qquad (V.2)$$

(b) $0 \leq t \leq 4\mu^2$

$$D_0^0(t) = 1 + \frac{2\Gamma}{\pi}\left\{\frac{1}{t+t_s}\left[\sqrt{\frac{4-t}{t}}\ \tan^{-1}\sqrt{\frac{4-t}{t}} - C_1\right] + C_2\right\} \qquad (V.3)$$

where

$$C_1 = \sqrt{\frac{t_s+4}{t_s}}\ \ln\frac{\sqrt{t_s}+\sqrt{t_s+4}}{2}$$

$$C_2 = \frac{1}{2t_s} - \frac{2C_1}{t_s(t_s+4)}$$

In particular for t = o it follows from (V.2) or (V.3) that

$$D_0^0(t) = 1 + \frac{2\Gamma}{\pi}\frac{1}{t_s}\left\{\frac{3}{2} - \frac{t_s+6}{t_s+4}C_1\right\} \qquad (V.4)$$

Differentiating Eqn (V.1) we get

$$\left.\frac{dD_0^0(t)}{dt}\right|_{t=0} = \frac{2\Gamma}{\pi}\frac{1}{t_s^2}\left\{-1 - \frac{t_s}{12} + C_1\right\} \qquad (V.5)$$

$$\left.\frac{dD_0^0(t)}{dt}\right|_{t=t_0} = \frac{2\Gamma}{\pi}\frac{1}{(t_0+t_s)^2}\left\{\frac{t_0+t_s}{2t_0} + \frac{t_0^2-6t_0-2t_s}{t_0^2}\sqrt{\frac{t_0}{4-t_0}}\tan^{-1}\sqrt{\frac{t_0}{4-t_0}} + C_1\right\} \qquad (V.6)$$

(ii) $L_0^I(s)$ on various cuts.

In elastic approximation we have

$$L_0^I(\omega) = -\frac{K^2}{\pi}\int_{2m\mu}^{\infty} d\omega' \frac{\sqrt{\omega'^2-4m^2\mu^2}}{2(\omega'+m^2+\mu^2)} \frac{1}{K^2(\omega')(\omega'-\omega)} \qquad (V.7)$$

where $\omega = s-m^2-\mu^2$. Substituting $\omega' = 2m\mu \operatorname{Sec}\theta$ one gets

$$L_0^I(\omega) = -\frac{2K^2}{\pi}\int_0^{\pi/2} d\theta \frac{1}{2m\mu - \cos\theta} \qquad (V.8)$$

For ω on the real axis the integration is easily performed giving

(a) $\omega \geq 2m\mu$

$$L_0^I(\omega) = -\frac{1}{2\pi}\frac{\sqrt{\omega^2-4m^2\mu^2}}{\omega+m^2+\mu^2} \ln \frac{\omega-\sqrt{\omega^2-4m^2\mu^2}}{2m\mu} \qquad (V.9)$$

(b) $\omega \leq -2m\mu$

$$L_0^I(\omega) = -\frac{1}{2\pi}\frac{\sqrt{\omega^2-4m^2\mu^2}}{\omega+m^2+\mu^2} \ln \frac{\sqrt{4m^2\mu^2-\omega^2}-\omega}{2m\mu} \qquad (V.10)$$

On the circle cut $L_0^I(\omega)$ is complex:

$$\operatorname{Re} L_0^I(\omega) = -\frac{2\lambda}{\pi}\int_0^{\pi/2} d\theta \frac{2m\mu - \operatorname{Re}\omega\cos\theta}{(2m\mu-\operatorname{Re}\omega\cos\theta)^2+(\operatorname{Im}\omega\cos\theta)^2} \qquad (V.11)$$

$$\operatorname{Im} L_0^I(\omega) = -\frac{2\lambda}{\pi}\int_0^{\pi/2} d\theta \frac{\operatorname{Im}\omega\cos\theta}{(2m\mu-\operatorname{Re}\omega\cos\theta)^2+(\operatorname{Im}\omega\cos\theta)^2} \qquad (V.12)$$

where $\operatorname{Re}\omega = 2\lambda$, $\operatorname{Im}\omega = 2\sqrt{(\lambda+m^2)(-\lambda-\mu^2)}$

After changing the variable by using

$$\cos\theta = -\frac{m-\mu\operatorname{Sec}\xi}{\mu - m\operatorname{Sec}\xi}$$

one gets

$$\operatorname{Re} L_0^I(\omega) = -\frac{\lambda}{\sqrt{m^2-\mu^2}\,\pi}\int_0^{\cos^{-1}\frac{\mu}{m}} d\xi \frac{(\lambda+m^2)\mu + m(-\lambda-\mu^2)\cos\xi}{(\lambda+m^2)\mu^2+m^2(-\lambda-\mu^2)\cos^2\xi} \qquad (V.13)$$

$$\mathrm{Im}\, L_0^{I}(\omega) = \frac{2\sqrt{(\lambda+m^2)(-\lambda-\mu^2)}}{(m^2-\mu^2)^{3/2}}\,\frac{\lambda}{\pi}\int_0^{\cos^{-1}\frac{\mu}{m}} d\xi\, \frac{m\cos\xi-\mu}{(\lambda+m^2)\mu^2+m^2(-\lambda-\mu^2)\cos^2\xi} \tag{V.14}$$

The integrations can be easily performed giving

$$\mathrm{Re}\, L_0^{I}(\omega) = -\frac{\lambda}{m^2-\mu^2}\,\frac{1}{\pi}\left\{\sqrt{\frac{\lambda+m^2}{-\lambda}}\tan^{-1}\sqrt{\frac{\lambda+m^2}{-\lambda}}+\sqrt{\frac{\lambda+\mu^2}{\lambda}}\ln\frac{\sqrt{-\lambda}+\sqrt{-\lambda-\mu^2}}{\mu}\right\} \tag{V.15}$$

$$\mathrm{Im}\, L_0^{I}(\omega) = -\frac{\lambda}{m^2-\mu^2}\,\frac{1}{\pi}\left\{\sqrt{\frac{\lambda+m^2}{-\lambda}}\ln\frac{\sqrt{-\lambda}+\sqrt{-\lambda-\mu^2}}{\mu}-\sqrt{\frac{\lambda+\mu^2}{\lambda}}\tan^{-1}\sqrt{\frac{\lambda+m^2}{-\lambda}}\right\} \tag{V.16}$$

APPENDIX VI

THE THRESHOLD BEHAVIOURS

The threshold behaviours of the partial wave amplitudes (both at the physical and the crossed thresholds) have very important consequences in the inverse amplitude dispersion relations. Firstly, in Eqn (4.18) the integral taken along a small circle of radius ρ around the physical threshold $s=(m+\mu)^2$ when $\rho \to 0$ is

$$\underset{\rho\to 0}{Lt}\ \frac{s-s_0}{2\pi i}\int_\rho ds' \frac{G_l^I(s')}{(s'-s)(s'-s_0)} = \underset{\rho\to 0}{Lt}\ \frac{s-s_0}{2\pi i}\int_\rho ds' \frac{g_l^I(s')}{(s'-s)(s'-a)^l} \quad (\text{VI}\cdot 1)$$

The threshold behaviour $A_l \approx k^{2l}$ is used to write

$$\frac{G_l^I(s')}{(s'-s)(s'-s_0)} = \frac{g_l^I(s')}{(s'-s)(s'-a)^l} \qquad \text{where } a=(m+\mu)^2$$

using the expansions

$$g_l^I(s') = \sum_{n=0}^{\infty} g_l^{I(n)}(a)\frac{(s'-a)^n}{n!} \quad \text{with } g_l^{I(n)} = \left.\frac{d^n}{ds^n} g_l^I(s)\right|_{s=a}$$

and

$$\frac{1}{s'-s} = -\sum_{n=0}^{\infty}\frac{(s'-a)^n}{(s-a)^{n+1}}$$

we get after changing the variable by $s'=a+\rho e^{i\phi}$

$$-\underset{\rho\to 0}{Lt}\ \frac{s-s_0}{2\pi}\int_0^{2\pi} d\phi\, \rho e^{i\phi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_l^{I(n)}(a)\ \frac{\rho^n e^{in\phi}}{n!}\ \frac{\rho^m e^{im\phi}}{(s-a)^{m+1}}\ \frac{e^{-il\phi}}{\rho^l}$$

$$= -(s-s_0)\sum_{n=0}^{l-1}\frac{[s-(m+\mu)^2]^{n-l}}{n!}\, g_l^{I(n)}(a) \quad (\text{VI}\,2)$$

where

$$\int_0^{2\pi} d\phi\, e^{\pm in\phi} = 0 \quad \text{if } n \text{ is integer and } n\neq 0$$
$$= 2\pi \quad \text{if } n=0$$

has been used. The minus sign is due to the clockwise direction of the contour around the circle with radius ρ (see Fig. 4.1).

At the crossed threshold, the real part of $A_l^I(s)$ behaves like k^{2l} and the imaginary part behaves like k. So for the inverse amplitude

$$G_l^I(s) = \frac{1}{\mathrm{Re}A_l^I(s) + i\,\mathrm{Im}A_l^I(s)} = \frac{\mathrm{Re}A_l^I(s) - i\,\mathrm{Im}A_l^I(s)}{[\mathrm{Re}A_l^I(s)]^2 + [\mathrm{Im}A_l^I(s)]^2}$$

the real part behaves like k^{2l-2} and the imaginary part behaves like $\frac{1}{k}$ for $l>0$, while for $l=0$ the real part is constant at the crossed threshold and the imaginary part behaves as k. So it is clear that for $l=0$ the contribution of the integration round the small circle σ in Eqn (4.18) is zero. This is also the case with the real part for $l>0$. For the imaginary part we put

$$\frac{\mathrm{Im}\,G_l^I(s)}{s-s_0} = -\frac{K_l^I(s)}{s-s_0} = -\frac{C_l^I}{(b-s)^{1/2}} \quad \text{with } b=(M-\mu)^2$$

Then the integration round the small circle σ becomes

$$\underset{\sigma\to 0}{Lt}\ \frac{s-s_0}{2\pi i}\int_\sigma ds' \frac{G_l^I(s')}{(s'-s)(s'-s_0)} = -\underset{\sigma\to 0}{Lt}\ \frac{s-s_0}{2\pi}\int_\sigma ds' \frac{C_l^I}{(b-s')^{1/2}(s'-s)}$$

$$= 0 \qquad \text{if } s \neq b$$

$$= \underset{\sigma\to 0}{Lt}\ \frac{s-s_0}{2\pi}\int_\sigma ds' \frac{C_l^I}{(b-s')^{3/2}} \quad \text{if } s=b$$

$$= \underset{\sigma\to 0}{Lt}\ \frac{s-s_0}{\pi}\ \frac{2C_l^I}{\sigma^{1/2}} \qquad (\mathrm{VI}.3)$$

Now the integration the real axis

$$\underset{\sigma\to 0}{Lt}\ -\frac{s-s_0}{\pi}\int_{-\infty}^{+b-\sigma} ds' \frac{K_l^I(s')}{(s'-s)(s'-s_0)} = -\underset{\sigma\to 0}{Lt}\ \frac{s-s_0}{\pi}\int_{-\infty}^{b-\sigma} ds' \frac{C_l^I}{(b-s')^{1/2}(s'-s)}$$

$$= 0 \quad \text{if } s\neq b$$

$$= -\underset{\sigma\to 0}{Lt}\ \frac{s-s_0}{\pi}\ \frac{2C_l^I}{\sigma^{1/2}} \qquad (\mathrm{VI}.4)$$

where we have retained only the relevant contribution to $K_l^I(s')$ which blows up at $s' = b$. It is clear that (VI.3) and (VI.4) cancel out each other. This justifies the writing of Eqn (4.20).

In the case of the S-wave dispersion relation for $G_0^I(s)/k^2$ it follows from arguments similar to those applied to obtained Eqn (VI.2) that the contributions of the small circles of radius ρ and σ around the physical and the crossed thresholds respectively to $G_0^I(s)/k^2$ are

$$\frac{\gamma_0^I}{s-(m+\mu)^2} \quad \text{and} \quad \frac{\delta_0^I}{s-(m-\mu)^2}$$

Multiplying by k^2 the first two terms of Eqn (4.46) are obtained.

Since both $F_0^I(s)$ and $K_0^I(s)$ behaves as $k(s)$ near the respective threshold similar situations as Eqn (VI.4) arise for the integrations on the real axis. The integrations around the small circle do not cancel out these infinite terms. But now the integrals are multiplied by k^2 which vanishes at the thresholds and so there is no trouble.

APPENDIX VII

FIXED t DISPERSION RELATIONS

The fixed t dispersion relation for $A^{(+)}(s,u,t)$ is

$$A^{(+)}(s,u,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \, \mathrm{Im}\, A^{(+)}(s',t)\left\{\frac{1}{s'-s} + \frac{1}{s'-u}\right\} \qquad (\mathrm{VII}.1)$$

We project out the $\ell = 0$ amplitude for channel III from Eqn (VII.1.)

$$A^{(+)}(s,u,t) = B_0^{(+)}(t) + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \, \mathrm{Im}\, A^{(+)}(s',t)\left\{\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{2pq}\ln\left(1 + \frac{4pq}{s'+(p-q)^2}\right)\right\} \qquad (\mathrm{VII}.2)$$

Now a partial wave dispersion relation for $B_0^{(+)}(t)$ may be written down with a subtraction at the symmetry point

$$B_0^{(+)}(t) = b_0^{(+)} + \frac{t-t_0}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\mathrm{Im}\, B_0^{(+)}(t')}{(t'-t)(t'-t_0)} + \frac{t-t_0}{\pi}\int_{-\infty}^{0} dt' \frac{\mathrm{Im}\, B_0^{(+)}(t')}{(t'-t)(t'-t_0)} \qquad (\mathrm{VII}.3)$$

Using Eqn (3.14) the integral over the left hand cut becomes

$$\frac{t-t_0}{\pi}\int_{-\infty}^{0} dt' \frac{\mathrm{Im}\, B_0^{(+)}(t')}{(t'-t)(t'-t_0)} = -\frac{1}{\pi}\int_{-\infty}^{0} dt' \int_{(m+\mu)^2}^{(p_-+q_-)^2} ds' \frac{1}{2p_-q_-}\mathrm{Im}\, A^{(+)}(s',t) \times\left\{\frac{1}{t'-t} - \frac{1}{t'-t_0}\right\} \qquad (\mathrm{VII}.4)$$

where

$$p_- = \sqrt{m^2 - t'/4} \quad \text{and} \quad q_- = \sqrt{\mu^2 - t'/4}$$

Only the S and P waves are retained in $\mathrm{Im}\, A^{(+)}(s',t)$ everywhere. Then the t' integration in Eqn (VII.4) can be performed after interchanging the order of the integrations.

This gives

$$-\frac{1}{\pi}\int_{-\infty}^{0} dt' \int_{(m+\mu)^2}^{\infty} ds' \frac{1}{2p_-q_-} \operatorname{Im} A^{(+)}(s',t') \left\{ \frac{1}{t'-t} - \frac{1}{t'-t_0} \right\}$$

$$= -\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \int_{-\infty}^{-4k^2} dt' \left\{ \operatorname{Im} A_0^{(+)}(s') + 3\left(1+\frac{t'}{2k^2}\right) \operatorname{Im} A_1^{(+)}(s') \right\}$$

$$\times \frac{1}{2p_-q_-} \left\{ \frac{1}{t'-t} - \frac{1}{t'-t_0} \right\}$$

$$= \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \Big[\operatorname{Im} A^{(+)}(s',t) \frac{1}{2pq} \ln\left(1+\frac{4pq}{s'+(p-q)^2}\right)$$
$$- \operatorname{Im} A^{(+)}(s',t_0) \frac{1}{2p_0q_0} \ln\left(1+\frac{4p_0q_0}{s'+(p_0-q_0)^2}\right)\Big] \quad (\text{VII}\cdot 5)$$

where $p_0^2 = t_0/4 - m^2$ and $q_0^2 = t_0/4 - \mu^2$ and we have used

$$\int_{-\infty}^{-4k^2} dt' \frac{1}{p_-q_-(t'-t)} = \frac{1}{pq} \ln \frac{s'(q^2-p^2)+(m^2-\mu^2)(p-q)^2}{s'(q^2-p^2)+(m^2-\mu^2)(p+q)^2}$$
$$= -\frac{1}{pq} \ln\left(1+\frac{4pq}{s'+(p-q)^2}\right) \quad [\text{since } m^2-\mu^2 = q^2-p^2]$$

Substituting Eqns (VII.3.) and (VII.5) in Eqn (VII.2.) we get

$$A^{(+)}(s,u,t) = b_0^{(+)} + \frac{t-t_0}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\operatorname{Im} B_0^{(+)}(t')}{(t'-t)(t'-t_0)}$$
$$+ \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \Big[\operatorname{Im} A^{(+)}(s',t) \left\{ \frac{1}{s'-s} + \frac{1}{s'-u} \right\}$$
$$- \operatorname{Im} A^{(+)}(s',t_0) \frac{1}{2p_0q_0} \ln\left(1+\frac{4p_0q_0}{s'+(p_0-q_0)^2}\right)\Big] \quad (\text{VII}\cdot 6)$$

This can be rewritten as

$$A^{(+)}(s,u,t) = -\lambda + \frac{t-t_0}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\operatorname{Im} B_0^{(+)}(t')}{(t'-t)(t'-t_0)}$$
$$+ \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \left[\operatorname{Im} A^{(+)}(s',t)\left\{\frac{1}{s'-s}+\frac{1}{s'-u}\right\} - \operatorname{Im} A^{(+)}(s',t_0)\left\{\frac{1}{s'-s_0}+\frac{1}{s'-u_0}\right\}\right] \quad (\text{VII}\cdot 7)$$

where we used Eqn (VII.6) evaluated at the symmetry point to express $b_0^{(+)}$ in terms of λ

$$-\lambda = b_0^{(+)} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \Big[\mathrm{Im}\, A^{(+)}(s',t_0)\Big\{\frac{1}{s'-s_0} + \frac{1}{s'-u_0}\Big\} - \mathrm{Im}\, A^{(+)}(s',t_0)\frac{1}{2p_0q_0}\ln\Big(1+\frac{4p_0q_0}{s'+(p_0-q_0)^2}\Big)\Big] \qquad (\mathrm{VII}.8)$$

For the amplitude $A^{(-)}(s,u,t)$ the fixed t dispersion relation is

$$A^{(-)}(s,u,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\, \mathrm{Im}\, A^{(-)}(s',t)\Big\{\frac{1}{s'-s} - \frac{1}{s'-u}\Big\} \qquad (\mathrm{VII}.9)$$

Projecting out $B_1^{(-)}(t)$ from this one gets

$$A^{(-)}(s,u,t) = 3pq\cos\phi\, B_1^{(-)}(t) + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\, \mathrm{Im}\, A^{(-)}(s',t) \times\Big\{\frac{1}{s'-s} - \frac{1}{s'-u} - \frac{3pq\cos\phi}{2p^2q^2}\Big[\frac{s'+p^2+q^2}{2pq}\ln\Big(1+\frac{4pq}{s'+(p-q)^2}\Big)-2\Big]\Big\} \qquad (\mathrm{VII}.10)$$

The following partial wave dispersion relation is written down for $B_1^{(-)}(t)$:

$$B_1^{(-)}(t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\mathrm{Im}\, B_1^{(-)}(t')}{t'-t} + \frac{1}{\pi}\int_{-\infty}^{0} dt' \frac{\mathrm{Im}\, B_1^{(-)}(t')}{t'-t} \qquad (\mathrm{VII}.11)$$

Similar considerations as in the case of $A^{(+)}(s,u,t)$ leads to:

$$A^{(-)}(s,u,t) = \frac{3pq\cos\phi}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\mathrm{Im}\, B_1^{(-)}(t')}{t'-t} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\, \mathrm{Im}\, A^{(-)}(s',t)\Big\{\frac{1}{s'-s} - \frac{1}{s'-u}\Big\} \qquad (\mathrm{VII}.12)$$

Taking proper combinations of Eqns (VII.7) and (VII.12)

$$
\begin{aligned}
A^{I}(s,u,t) = & -\lambda + \sqrt{6}\,\frac{t-t_0}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\operatorname{Im} B_0^0(t')}{(t'-t)(t'-t_0)} \\
& + \frac{3(s-u)}{4}\begin{pmatrix} 1 \\ -1/2 \end{pmatrix}\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\operatorname{Im} B_1^1(t')}{t'-t} \\
& + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \left[\frac{\operatorname{Im} A^{I}(s',t)}{s'-s} - \frac{\operatorname{Im} A^{I}(s',t_0)}{s'-s_0} \right. \\
& \left. + \sum_{I'} \alpha_{II'} \left\{ \frac{\operatorname{Im} A^{I'}(s',t)}{s'-u} - \frac{\operatorname{Im} A^{I'}(s',t_0)}{s'-u_0} \right\} \right] \qquad (\underline{VII}.13)
\end{aligned}
$$

where we have used

$$
0 = \frac{1}{s'-s_0} - \frac{1}{s'-u_0} \qquad , \text{ because } s_0 = u_0
$$

Eqn (VII.13) is used to calculate the derivatives with respect to Cos Θ in Eqns (4.44) and (4.45).

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