STABILITY OF LAMINAR FLOWS OF

A PERFECT MAGNETOFLUID

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C O N T E N T S

INTRODUCTION

This thesis reviews work done on stability of laminar flows of non-conducting fluids and makes extensions where possible to perfectly conducting fluids in the presence of a magnetic field parallel to the flow.

Dissipative effects are not studied, so that the fluids have no viscosity or electrical resistance. The fluids are incompressible.

The object is to classify the velocity and magnetic profiles of the laminar flows into stable and unstable groups. In spite of the simplicity of the systems studied, simple rules are difficult to find. Those available are summarised in the appendix.

The remaining results have the disadvantage that they involve the frequencies and amplitudes of the possible disturbances.

CHAPTER 1

REVIEW

1. The stability problem

The system to be studied is a laminar flow of an incompressible inviscid fluid with density ρ pressure p, and velocity field $\underset{\sim}{v}$ given by

$$\underset{\sim}{v} = U(y)\,\hat{\underline{x}}$$

between flat plates located at $y = y_1$ and $y = y_2$. (See figure 1)

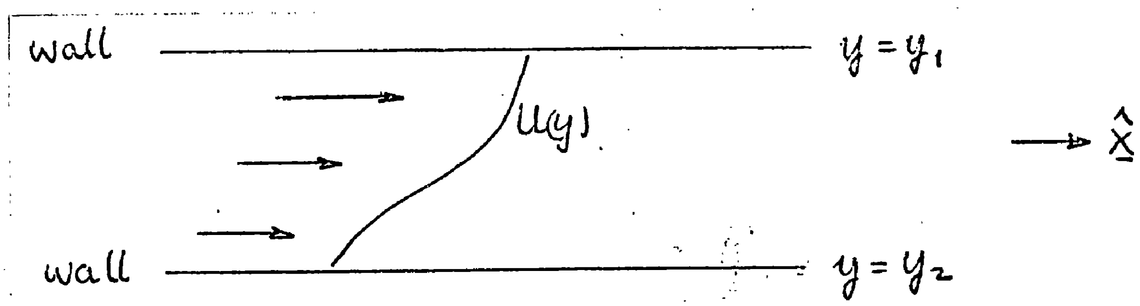

Figure 1

The flow is completely described by its velocity profile U(y), and is always a proper equilibrium because the fluid equations

$$\rho \frac{d\underset{\sim}{v}}{dt} = -\nabla p$$

$$\operatorname{div} \underset{\sim}{v} = 0$$

are satisfied for arbitrary U(y), provided there is no equilibrium pressure gradient. An equilibrium pressure gradient could occur only in association with a temperature gradient, but this effect will be ignored. Any instabilities detected are solely due to the velocity gradient.

Small perturbations $\underset{\sim}{v}^{(1)}(x,y,z,t)$, $p^{(1)}(x,y,z,t)$ are superposed on the equilibrium, and their time dependence deduced from the fluid equations. Since the perturbations are small they are linearly related, and any one perturbed quantity gives full information. Eliminating all quantities except $v_y^{(1)}(x,y,z,t)$ in the fluid equations,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v_y^{(1)} - U''\frac{\partial}{\partial x}v_y^{(1)} = 0. \qquad \text{(i)}^{*}$$

Since the coefficients are independent of x and z it is useful to Fourier transform.

$$\bar{v}_y^{(1)}(y,t) = \iint v_y^{(1)}(x,y,z,t)\, e^{-i(kx+mz)}\, dx\, dz$$

$$\therefore \left(U + \frac{1}{ik}\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial y^2} - (k^2+m^2)\right)\bar{v}_y^{(1)} - U''\bar{v}_y^{(1)} = 0 \qquad \text{(ii)}$$

Assuming exponential time dependence,

$$\bar{v}_y^{(1)}(y,t) = v(y)\, e^{-ikct}$$

$$\therefore \quad v'' - \left(k^2 + m^2 + \frac{U''}{U-c}\right)v = 0 \qquad \text{(iii)}$$

The stability problem can now be formulated as an eigenvalue problem. The physical boundary conditions are that $v_y^{(1)}(x,y,z,t)$ vanishes when $y = y_1$ and $y = y_2$. It follows that the correct boundary conditions on $v(y)$ are

$$v(y_1) = v(y_2) = 0$$

If $c \neq U(y_1)$ equation (iii) is regular at $y = y_1$ and there exists a solution $v(y,c)$ such that $v(y_1) = 0$ thus satisfying

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* Equations in the Review Chapter only are numbered by small Roman numerals (i), (ii), (iii)....

one boundary condition. If there exists a value of $c = c(k,m)$ such that $v(y_2) = 0$ and $v(y)$ is regular in the real interval (y_1, y_2) then c is an eigenvalue with corresponding eigenfunction $v(y)$.

If c is complex ($c = c_r + ic_i$) where $kc_i > 0$ then the solution $V(y)\exp(-ikct)$ increases unboundedly with time and the equilibrium is unstable. If no such c exists then the flow is stable, at least to modes of oscillation with exponential time dependence. (Non exponential time dependence is considered on page 91)

No difference is made to the stability problem if $k^2 + m^2$ is replaced by k^2 and only two dimensional perturbations like $v(y)\exp(ik\,(x-ct))$ considered. It is not necessary in general to consider the sign of c_i, because the existence of an eigenfunction $v(y,c)$ implies the existence of the eigenfunction $v^*(y,c^*)$.

The stability problem, then, is to find eigenvalues c ($c_i \neq 0$) of the equation

$$v'' - \left(k^2 + \frac{U''}{U-c}\right)v = 0 \qquad \text{(iv)}$$

subject to the boundary conditions $v(y_1) = v(y_2) = 0$. This problem was examined by Rayleigh (1880–).

2. Rayleigh's Inflexion Condition

It is clear that the most useful aim is to find conditions on $U(y)$ necessary or sufficient for stability.

Conditions involving v, k or c are less illuminating. Rayleigh (1880) discovered the necessary condition for instability that U(y) must have a point of inflexion. That is, for some y_0 in the real interval (y_1, y_2), $U''(y_0) = 0$.

Rayleigh's condition is easily proved from (iv)

$$L(v) = 0 \qquad \text{(iv)}$$

$$\therefore \; v^* L(v) - v\, L(v^*) = 0$$

$$\therefore \; \int_{y_1}^{y_2} \left(v^* L(v) - v\, L(v^*)\right) = 0$$

$$\therefore \; -2ic_i \int_{y_1}^{y_2} \frac{U''|v|^2}{|U-c|^2} = 0$$

By inspection of the integrand U" must change sign if the integral is to vanish. Assuming U(y) is a regular function of y, it follows that U" must have a zero, which confirms that Rayleigh's condition is necessary for instability. It is difficult to find any further conditions to match Rayleigh's condition in simplicity and power. No complete division of profiles U(y) into stable and unstable classes exists.

Rayleigh's condition can be derived in such a way as to make its physical meaning clearer.* The fluid equation

$$\rho \frac{d\underset{\sim}{v}}{dt} = -\nabla p$$

can be rewritten to display momentum flux $\frac{\partial \underset{\sim}{J}}{\partial t}$ at a point in space,

$$\frac{\partial \underset{\sim}{J}}{\partial t} + \operatorname{div} \underline{\underline{P}} = 0$$

where the pressure tensor $\underline{\underline{P}}$ is given by

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* This derivation is repeated in detail on page 33 in the case of a magnetofluid in the presence of a magnetic field.

$$P_{ij} = p\delta_{ij} + \rho v_i v_j$$

In the present case

$$\underset{\sim}{v} = U(y) + v^{(1)}(x,y,z,t)$$

where $v^{(1)}$ is periodic in x, z and t, so that the mean x-momentum density growth in a plane of fixed y is

$$\frac{\partial \bar{J}_x}{\partial t} = \rho \frac{\partial}{\partial y} \overline{V_x^{(1)} V_y^{(1)}}$$

where the mean value is with respect to x and z

$$\therefore \frac{\partial \bar{\bar{J}}_x}{\partial t} = -\frac{\rho e^{2kc_i t}}{4ik(m^2+k^2)}\left[k^2 W'[v] + m^2 2i\, \mathcal{I}m\left\{\frac{U'}{U-c}|v|^2\right\}'\right]$$

where $W[v] = v^* v' - v^{*\prime} v$.

$$\therefore \frac{\partial \bar{\bar{J}}_x}{\partial t} = \frac{\rho c_i e^{2kc_i t}}{2k(m^2+k^2)}\left[k^2 \frac{U''|v|^2}{|U-c|^2} + m^2\left(\frac{U''|v|^2}{|U-c|^2}\right)'\right]$$

By momentum conservation, the total x-momentum arriving between the walls is zero

$$\therefore 0 = \int_{y_1}^{y_2} \frac{\partial \bar{J}_x}{\partial t}\, dy$$

$$= \frac{\rho k e^{2kc_i t}}{2(m^2+k^2)} \int_{y_1}^{y_2} \frac{U''|v|^2}{|U-c|^2}\, dy$$

As before, U" must have a zero if the integral is to vanish.

3. Viscous Flows

The present work is entirely on inviscid fluids. In this section viscous flows are considered only to show the fundamental changes caused by the introduction of viscosity. These are such as to forbid tentative use of inviscid results in the presence of even vanishingly small viscosity.

A paradox in the relationship between viscous and inviscid laminar flows arose with Rayleigh's inflexion condition. Inviscid laminar flows with parabolic velocity profile (Plane Poiseuille flow) are stable because they have no point of inflexion. The seemingly plausible physical argument that viscous forces will tend to stabilise by damping out disturbances indicates that viscous plane Poiseuille flows should be stable also. But Reynolds had shown by theory and experiment that they are not. Although this paradox was explained by Prandtl (1922) who showed that the viscous forces near the walls caused instability, much more work had to be done subsequently on the behaviour of viscous flows in the limit of vanishing viscosity.

The difficulty arises from the change in order of the governing perturbation equations when viscosity vanishes. The viscous perturbation equation for v(y) is the fourth order Orr-Sommerfeld equation

$$\nabla^2\nabla^2 v = ikR\{(U-c)\nabla^2 v - U'' v\} \qquad (v)$$

where the Reynolds number R tends to infinity as the coefficient of viscosity tends to zero. As $R \to \infty$, Rayleigh's second order equation (iv) is recovered. The asymptotic solutions of (v) for large R do not necessarily tend to solutions of (iv). Nor are all possible solutions of (iv) expressible as the limit of a solution of (v) as $R \to \infty$.

Lin (1955) reviewed work done on asymptotic solutions of the Orr - Sommerfeld equation. In general, an asymptotic solution expanded about a singularity in the complex plane of y does not have the same form in all sectors of the plane. (Stoke's phenomenon). When $c_i > 0$, the asymptotic solutions of (v) do tend to solutions of (iv) in the whole real line (y_1, y_2), but when $c_i < 0$ they do not. For a uniform mathematical theory, therefore, solutions of (iv) with $c_i < 0$, and valid on (y_1, y_2) are rejected. It then ceases to be true that solutions of (iv) occur in complex conjugate pairs, but results obtained for real c in the limit $c_i \rightarrow 0$ become independent of the direction of the limit ($c_i > 0$ or $c_i < 0$), because the branch of the solutions is properly defined.

This reinterpretation of the inviscid solutions never leads to a changed decision on stability or instability of the inviscid flows, but adequately clarifies the relationship with viscous flows. In the present work finite viscosity (and conductivity) are never considered so that the ideal equations can be used naively, without fear of contradiction arising.

4. Tollmien's sufficient conditions for instability

Tollmien (1935) described Rayleigh's work, and commented on the absence of sufficient conditions for instability. He showed that symmetric velocity profiles with a point of inflexion are unstable. (Symmetric here means symmetric about the centre of (y_1, y_2)) His method was to search for complex eigenvalues near known real eigenvalues, and he in fact found two sets of complex eigenvalues near two different real eigenvalues.

One of the real eigenvalues was $c_o = U(y_o)$ $(U''(y_o) = 0)$. From equation (iv) for v(y),

$$v'' - \left(k^2 + \frac{U''}{U-c}\right)v = 0 \qquad \text{(iv)}$$

the choice $c = U(y_o)$ ensures that no singularities occur. Tollmien showed that for some $k^2 = k_o^2$ a symmetric real eigenfunction v_o exists, and then developed a perturbation theory to produce an expansion for $c = c(k^2)$ near k_o^2, which showed that complex eigenvalues exist near c_o. The expansion is

$$c = c_o + \Delta c_r + i\,\Delta c_i \qquad \text{(vi)}$$

where $$\Delta c_r\left\{L + \frac{U_o'''^2}{U_o'^2}\frac{\pi^2}{L^2}\right\} = -\frac{\Delta k^2}{2L}\int_{y_1}^{y_2} v_o^2\,dy$$

$$\Delta c_i = -\frac{U_o'''}{U_o'^2}\frac{\pi}{L}\Delta c_r$$

and where
$$L = P\int_{y_1}^{y_2}\frac{U''|v|^2}{|U-c|^2}\,dy$$

This type of expansion is not useful for a magnetofluid in the presence of a parallel magnetic field because the real eigenvalue c_o exists only when very restrictive conditions are applied to the velocity and magnetic profiles.

Tollmien also considered the well known real solution $v = U(y)$ $(k^2 = 0)$ which is an eigenfunction when $c = U(y_1)$ (=0 say), and showed that it is the only real eigenfunction occuring for all symmetric profiles (with or without inflexion) for all values of k^2. His perturbation theory in this case showed that nearby complex eigenvalues exist whenever $U''(y_1) > 0$.

Since U(y) is symmetric this condition implies that U" has a zero. But inflexions can exist even when $U''(y_1) < 0$, so a smaller class of profiles is being shown to possess unstable modes near this real mode. The expansion for $c = c(k^2)$ is

$$c_r = \frac{k^2}{2U_1'}\int_{y_1}^{y_2} U^2 dy$$

$$c_i = \frac{\pi U_0''}{U_0'^2} c_r^2 \qquad \text{(vii)}$$

The real solution $v = U(y)$ $(k^2 = 0)$ exists for magnetofluid flows. A perturbation theory similar to Tollmien's is developed later (page 67) and a sufficient condition for instability obtained in terms of the velocity and magnetic profiles.

5. Lin's work on laminar flows

In a series of three papers Lin (1945) studied laminar flows of viscous and inviscid fluids, paying special attention to the relationship between the two. To obtain a unified viscous - inviscid theory , he accepted inviscid solutions only when they were valid asymptotic forms of the Orr - Sommerfeld equation. As already indicated, no attempt will be made to imitate Lin's procedure, as this work deals solely with inviscid fluids.

Lin established a new class of unstable flows by perturbation about a neutral solution v_0 with eigenvalue $c_0 = U(y_0)$ $(U''(y_0) = 0)$ in the case where $U'(y) > 0$ and $\frac{U''}{U - c_0} < 0$.

Given these conditions, and a sufficient negative upper bound on $\frac{U''}{U-c_c}$ the existence of the real eigenfunction is assured. When the real eigenfunction exists, a nearby complex solution always exists for $k^2 < k_o^2$, and its eigenvalue c is given by

$$\frac{dc}{dk^2} = (A + iB)^{-1} \qquad \text{(viii)}$$

where

$$A = P\int_{y_1}^{y_2} \frac{U''}{(U-c_o)^2}\,dy \quad ; \quad B = \frac{\pi U_c'''}{U_o'} V_o^2 \, .$$

Once more there is little opportunity to extend relations like (viii) to magnetofluid flows because the existence of real eigenfunctions imposes undue restrictions on the velocity and magnetic profiles.

Lin also gave a physical mechanism for instability based on vorticity conservation and displaying the role of the point of inflexion. Since vorticity is not conserved in the presence of a magnetic field, this interpretation cannot be extended.

6. Recent results for inviscid laminar flows

Rosenbluth and Simon (1964) solved the eigenvalue problem when $k^2 = 0$, and $U'(y) > 0$. They made use of the equation

$$\{(U - c)^2 f'\}' - k^2 (U - c)^2 f = 0 \qquad \text{(ix)}$$

obtained from (iv) by the transformation $v = (U-c)f$.

When $k^2 = 0$,

$$f(y) = \int_{y_1}^{y} \frac{dy}{(U-c)^2} \quad ,$$

and c is an eigenvalue if and only if

$$G(c) \equiv f(y_2) = \int_{y_1}^{y_2} \frac{dy}{(U-c)^2}$$

has complex zeros. Using the standard method of the Nyquist Diagram for searching for zeros of complex functions, they showed that G(c) has a complex zero whenever

$$-\left[\frac{1}{U'(U-U_0)}\right]_{y_1}^{y_2} - \int_{y_1}^{y_2} \frac{U''\,dy}{U'^2(U-U_0)} > 0 \qquad (x)$$

where $U''_0 = 0$. That (x) should hold is a necessary and sufficient condition for the existence of complex eigenvalues with $k^2 = 0$, and therefore is a sufficient condition for instability. In the particular case where U"(y) has just one zero Lin's result that instability occurs for $k^2 < k_0^2$ ($k_0^2 \geqslant 0$) showed that (x) is necessary and sufficient for instability.

Rosenbluth and Simon's method is imitated (page 58) for the case of a magnetofluid with $U'(y) > 0$ in the presence of a constant magnetic field parallel to the flow, to produce a generalisation of the sufficient condition (x). There is no special result corresponding to the necessary and sufficient condition.

In an interesting pedagogic paper Case (1960) complained that the literature on fluid stability tended to ignore the possibility that perturbations could have time dependences other than exponential. He examined the asymptotic time dependence of non-exponential modes for laminar flows, and showed them to be stable. Case pointed out that this was known to Lord Rayleigh, but was worth expressing in modern terms. Instead of assuming exponential time dependence of the Fourier transform $v^{(1)}(y,t)$, governed by equation (ii), the Laplace transform

$$V_p(y) = \int_0^\infty v^{(1)}(y,t)\, e^{-pt}\, dt$$

is examined by means of the new equation (replacing (iv))

$$V_p''(y) - \left(k^2 + \frac{U''}{U + \frac{p}{ik}}\right) V_p = \frac{\left(\frac{\partial^2}{\partial y^2} - k^2\right) v^{(1)}(y,0)}{ik\left(U + \frac{p}{ik}\right)} \qquad \text{(xi)}$$

where $v^{(1)}(y,o)$ is the initial perturbation fixed arbitrarily at $t = 0$. A new definition of instability is required. The flow is unstable whenever the asymptotic time dependence of $v^{(1)}(y,t)$ for large t is unbounded, for any given choice of initial disturbance. Case solved equation (xi) for v_p in terms of the Green's funtion, inverted the Laplace transform and deduced the asymptotic time dependence of $v^{(1)}(y,t)$. Ignoring exponential growth or decay (which also emerges in this procedure) the result was at worst a sinusoidal vibration with constant amplitude. Laminar flows are therefore stable to non exponential modes.

Case's procedure is carried out in detail for magnetofluid flows, in Chapter 5.

7. Laminar magnetofluid flows

The linearised magnetofluid equations for laminar flow of a finitely conducting and viscous incompressible magnetofluid in the presence of a magnetic field parallel to the flow were written down by Michael (1953). For equilibrium velocity profiles U(y) and magnetic profiles H(y) the perturbation equations take the form

$$\left.\begin{aligned} \rho(U+\tfrac{\omega}{k})\nabla^2 v - \rho U'' v &= \mu_0 H \nabla^2 h + \frac{\eta}{ik}\nabla^2\nabla^2 v \\ \rho(U+\tfrac{\omega}{k}) h &= \mu_0 H v + \frac{1}{\sigma i k}\nabla^2 h \end{aligned}\right\} \quad (xii)$$

where the complex amplitudes h(y) and v(y) represent the y-components of the magnetic and velocity perturbations (full form h(y)exp(i(kx + mz) + iwt), v(y)exp(i(kx + mz) + iwt)), η is the coefficient of viscosity and σ is the electrical conductivity. It is, of course, more usual to express equations (xii) in dimensionless form.

Michael did not use his equations except to show that purely magnetic perturbations ($v \equiv 0$) are always stable.

It is interesting that the equilibrium profiles, which may be chosen arbitrarily when $\eta = \sigma = 0$, are now subject to the severe restrictions

$$U'''(y) = 0$$

$$H''(y) = 0.$$

which simplify the equations (xii). This is an additional factor complicating the comparison of perfect (non-dissipative) magnetofluids with viscous, resistive ones. When Lin* discusses this point for non conducting fluids he justifies arbitrary choice of U(y) because , of course, the magnetofluid equations plainly allow it for inviscid fluids and also because

* Book (1958) p115

nearly parallel flows of viscous fluids can have almost arbitrary velocity profiles. This has to be justified for each case on its merits.

Some work has been done recently with equations similar to (xii) in both cartesian and cylindrical geometries. For example Drazin (1960) and Hunt (1965) have done stability calculations for extreme values of the parameters η and σ. A parallel magnetic field tends to stabilize laminar flows in most cases but both workers report exceptional circumstances in which the magnetic field is a destabilising influence.

No work seems to have been published specifically on laminar flows of perfect magnetofluids, but some relevant work has been done on general flows of perfect magnetofluids, and is described in the next section.

8. General formalism for stability of Lagrangian Systems

Recent literature on stability of magnetofluid flows has placed the problem on a general basis applicable to any Lagrangian system, that is any system governed by equations second order in time derivatives and derivable from a Lagrangian.

Frieman and Rotenberg (1960) made a study of the stability of a compressible non-dissipative magnetofluid. A system with flow field $\underset{\sim}{v}$ and magnetic field $\underset{\sim}{h}$ is an equilibrium (stationary) flow provided only that it satisfies the magnetofluid equations with $\frac{\partial}{\partial t} \equiv 0$. (The magnetofluid equations and their linearisation is discussed in detail for laminar flows on page 22). The equilibrium is subjected to small perturbations which are governed by the linearised magnetofluid equations in their full form ($\frac{\partial}{\partial t} \neq 0$ now). The equation for the displacement field $\underset{\sim}{\xi}(\underset{\sim}{x},t)$ from the equilibrium position $\underset{\sim}{x}$ at time t takes the form

$$N \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} + 2iP \frac{\partial \underset{\sim}{\xi}}{\partial t} + Q = 0 \qquad \text{(xiii)}$$

where N, iP and Q are hermitian operators (N positive definite) containing space differentiations and the equilibrium fields. All information about the perturbed quantities is contained in $\underset{\sim}{\xi}$. Introducing the time dependence exp iwt, (xiii) becomes

$$H(w) X = 0 \qquad \text{(xiv)}$$

where $\underset{\sim}{\xi} = X(\underset{\sim}{x})\exp iwt$ and $H = -w^2N + 2iwP + Q$. The operator H(w) is non hermitian when w is complex.

When the flow field $\underset{\sim}{v} \equiv 0$, P = 0 and the static magnetofluid problem then has an equation of the form

$$F X = -\omega^2 \rho X \qquad \text{(xv)}$$

where F is hermitian. Thus, w^2 is real and w, if complex, must be pure imaginary. The introduction of the flow is seen to cause the possibility of overstability (strictly complex ω) associated with the non hermitian operator. The rapid progress made in static stability problems using the self adjoint equation (xv) is not possible with flow problems using (xiv). In particular (xv) can be expressed as a variational problem leading to the "Energy principle" formalism of Bernstein et al (1958)

Frieman and Rotenberg established a general sufficient condition for stability. From (xiv), taking the scalar product $\langle \chi | H | \chi \rangle$ and solving the resulting quadratic equation for w,

$$w = -\frac{\langle \chi | iP | \chi \rangle - \left\{ \langle \chi | iP | \chi \rangle^2 + \langle \chi | N | \chi \rangle \langle \chi | Q | \chi \rangle \right\}^{\frac{1}{2}}}{\langle \chi | N | \chi \rangle} \qquad \text{(xvi)}$$

Since the operators N, iP and Q are hermitian the scalar products are real. Therefore w is real and the system is stable if

$$\Delta = \langle \chi | iP | \chi \rangle^2 + \langle \chi | N | \chi \rangle \langle \chi | Q | \chi \rangle \geqslant 0$$

Since N is positive definite this condition can be replaced by the apparently less powerful but simpler condition

$$\langle \chi | Q | \chi \rangle \geqslant 0 \qquad \text{(xvii)}$$

In Frieman and Rotenberg's paper, Q is given by

$$Q\underset{\sim}{\chi} = \underset{\sim}{\nabla}(\gamma p \underset{\sim}{\nabla}.\underset{\sim}{\chi} + \underset{\sim}{\chi}.\underset{\sim}{\nabla}p - \underset{\sim}{H}.\underset{\sim}{\nabla}\times(\underset{\sim}{\chi}\times\underset{\sim}{H})) + \underset{\sim}{H}.\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\times(\underset{\sim}{\chi}\times\underset{\sim}{H}))$$
$$+ \underset{\sim}{\nabla}\times(\underset{\sim}{\chi}\times\underset{\sim}{H}).\underset{\sim}{\nabla}\underset{\sim}{H} + \underset{\sim}{\nabla}(\rho\underset{\sim}{\chi}\,\underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{v} - \rho\underset{\sim}{v}\,\underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{\chi})$$

where γ is the ratio of the specific heats of the (compressible) magnetofluid and p is the magnetofluid pressure and ρ its density.

Frieman and Rotenberg also developed a perturbation theory for small flow velocities to show that if a static equilibrium is stable then any flow equilibrium formed by superposing a small flow field is also stable. Their method was to expand in powers of a small parameter ϵ representing v/v_s or v/v_A where v_s and v_A are respectively the equilibrium sound and alfven velocities. Solutions near a static solution χ_0, ω_0 are written

$$\chi = \chi_0 + \epsilon\chi_1 + \epsilon^2\chi_2 + \cdots$$

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$$

By the hermitian nature of the static problem w_0^2 is real. If $w_0^2 \neq 0$ w has the same nature as w_0 and stability (or instability) persists. When $w_0 = 0$ the persistence of stability is not assured since

$$\omega_1^2 = -\frac{\langle\chi_0|Q_2|\chi_0\rangle}{\langle\chi_0|N|\chi_0\rangle} \qquad \text{(xviii)}$$

Stability persists if $\langle\chi_0|Q_2|\chi_0\rangle < 0$, instability occurs if $\langle\chi_0|Q_2|\chi_0\rangle > 0$ and the next order must be examined if $\langle\chi_0|Q_2|\chi_0\rangle = 0$.

The procedure of establishing a real mode of oscillation of a stable equilibrium to see whether stability persists was placed on a general basis by Low (1961). Given a system whose equations of motion are second order in time and derivable from a Lagrangian, stability does persist except for a circumstance depending only on the original equilibrium, and its stable mode of oscillation. The most general Lagrangian giving rise to the required equation of motion is

$$L = \tfrac{1}{2}\langle\dot{\xi}|N|\dot{\xi}\rangle + \langle\dot{\xi}|iP|\xi\rangle - \tfrac{1}{2}\langle\xi|Q|\xi\rangle \qquad \text{(xix)}$$

where the operators N, iP and Q are hermitian and dot means partial time derivative.

The equation of motion is

$$N\ddot{\underset{\sim}{\xi}} + 2iP\dot{\underset{\sim}{\xi}} + Q\underset{\sim}{\xi} = 0$$

and if the time dependence is like exp(iwt) the expression (xvi) is reproduced for w. If $\Delta > 0$ it follows that a small change in the equilibrium (and, therefore, in N, iP and Q) and a small change in the stable mode χ_o leave $\Delta > 0$ and stability persists. However if $\Delta = 0$ the small changes could well make $\Delta < 0$ whence complex values of w and instability. A real neutral mode with real eigenvalue w_0 and with the extra property that $\Delta = 0$ is called a marginally stable mode. A necessary and sufficient condition that a mode be marginal is

$$w_o = - \frac{\langle \chi_o | iP | \chi_o \rangle}{\langle \chi_o | N | \chi_o \rangle} .$$

The usefulness of Low's result is that no stable mode need be examined for a nearby instability unless $\Delta = 0$. "Nearby" has two senses : -

(1) A nearby perturbation about the same equilibrium : e.g., represented by a small change in wave number k.

(2) A nearby perturbation about a slightly different equilibrium : e.g., represented by the introduction of a small parameter provided the Lagrangian nature of the system is not destroyed. Thus the introduction of small viscosity is not permissible.

Completely new modes may arise for the new equilibrium and no information is available about them. It is also useful that Low's result is so general, applying equally for non-dissipative magnetofluids and to a collision free gas of charged particles interacting through their own (self consistent) average electric field and governed by the Collisionless Boltzmann equation.

Low interpreted his result in terms of a conserved 'energy' emerging naturally from the Lagrangian formulation. The quantity

$$E = \tfrac{1}{2}\langle \dot{\xi}|N|\dot{\xi}\rangle + \tfrac{1}{2}\langle \xi|Q|\xi\rangle$$

is a constant of the motion. Its time average is

$$E = \tfrac{1}{2}\omega^2\langle \chi|N|\chi\rangle + \tfrac{1}{4}\langle \chi|Q|\chi\rangle$$

where $\underset{\sim}{\xi} = \chi(\underset{\sim}{x})\exp i w t$. From expression (xvi) for w

$$w^2 = \frac{\langle \chi|P|\chi\rangle^2 + \Delta^2 \pm 2\Delta\langle \chi|P|\chi\rangle}{\langle \chi|N\chi\rangle^2} .$$

It follows that

$$E = \tfrac{1}{2}w\Delta .$$

It is therefore usually true that a stable mode can only have a possible nearby unstable mode when it has zero energy associated with it. This is not necessarily true when $w_o=0$, unless the marginally stable solution χ_o is unique (non-degenerate).

Low was unable to prove that the marginally stable modes actually possess unstable modes nearby, but he conjectured that this might be the case.

Laval et al (1964) took the matter further and succeeded in developing a perturbation theory which quantitatively evaluated the squared frequency change $(\Delta w)^2$ due to a small change $\Delta\alpha$ in an arbitrary parameter α of the equilibrium or its marginally stable mode χ_o. The result showed that $(\Delta w)^2/\Delta\alpha$, when non zero, has a sign independent of the sign of $\Delta\alpha$, so that one sign or other of $\Delta\alpha$ predicts $(\Delta w)^2 < 0$ and the introduction of unstable modes.

The result was achieved by expanding about the marginally stable solution $(\chi_0;\ w_0, \alpha_0)$ in terms of a small parameter ϵ. Thus

$$\chi_0 = \chi_0 + \epsilon\chi_1 + \epsilon^2\chi_2 + \cdots$$

For consistency $|\Delta\alpha| = \epsilon^2$. From the linearised magnetofluid equation (xiv)

$$H(\omega)\chi = 0 \qquad (xiv)$$

it follows that

$$H_0\chi_1 = -\omega_1\left(\frac{\partial H}{\partial \omega}\right)_0 \chi_0$$
$$H_0\chi_2 = -\left(\frac{\partial H}{\partial \epsilon^2}\right)_0 \chi_0 - \omega_1\left(\frac{\partial H}{\partial \omega}\right)_0 \chi_1 - \omega_2\left(\frac{\partial H}{\partial \omega}\right)_0 \chi_0 + \omega_1^2 N\chi_0 .$$

Excluding the possibility of degeneracy of χ_0, solutions exist if and only if χ_0 is orthogonal to the right sides. Hence,

$$(\Delta\omega)^2 = \frac{\langle \chi_0 | \left(\frac{\partial H}{\partial \alpha}\right)_0 | \chi_0 \rangle}{\langle \chi_0 | N | \chi_0 \rangle + \omega_1^2 \langle \chi_1 | H_0 | \chi_1 \rangle}\,\Delta\alpha \qquad (xx)$$

and $(\Delta w)^2 < 0$ for one choice of the sign of $\Delta\alpha$.

Exceptional circumstances invalidating (xx) arise when χ_0 is degenerate, when the operators possess singularities, when the numerator or denominator in (xx) is zero, and when the sign of $\Delta\alpha$ is restricted (e.g., if $\alpha = k^2$ and $\alpha_0 = 0$ then it is unphysical that $\Delta\alpha < 0$)

Laval et al examined the case of two fold degeneracy and showed that $(\Delta w)^2$ is positive or negative for all values of $\Delta\alpha$ according as the expression

$$\langle\chi_0^{(1)}|Q_1|\chi_0^{(1)}\rangle\langle\chi_0^{(2)}|Q_1|\chi_0^{(2)}\rangle - \langle\chi_0^{(1)}|Q_1|\chi_0^{(2)}\rangle^2$$

is positive or negative. Degeneracy is usual in laminar flows because of the degree of symmetry.

The occurrence of singularities in the operators is not discussed in the general analysis but they are invariably present in the case of laminar flows discussed in the succeeding chapters.

CHAPTER 2

LAMINAR FLOWS OF A PERFECT MAGNETOFLUID

Magnetofluid equations - linearised perturbation equations - necessary conditions for instability - physical interpretation - real eigenvalues - almost real eigenvalues - large k^2.

I. Magnetofluid equations

The results reviewed in Chapter 1 will now be adapted, where possible, to the case of a perfectly conducting magnetofluid in the presence of a parallel magnetic field. The equilibria studied are laminar flows

$$\underset{\sim}{v} = U(y)\,\hat{\underline{x}}$$

of a perfectly conducting, inviscid, incompressible magnetofluid between flat plates located at $y = y_1$ and $y = y_2$, in the presence of a magnetic field

$$\underset{\sim}{h} = H(y)\,\hat{\underline{x}}$$

The magnetic field may be expressed in terms of the associated Alfven velocity $\underset{\sim}{a} = \sqrt{\frac{\mu_o}{\rho}}\,\underset{\sim}{h}$ so that,

$$\underset{\sim}{a} = A(y)\,\hat{\underline{x}}$$

where μ_o = permeability of free space

ρ = magnetofluid density.

All the quantities $\underset{\sim}{v}$, U(y), $\underset{\sim}{a}$ and A(y) now have dimensions of velocity. In particular, the magnetic field is expressed as a velocity field.

These equilibria are fully specified by the flow and magnetic profiles U(y) and A(y). The profiles may be chosen arbitrarily because they always satisfy the magneto-fluid equati

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= -\frac{\nabla p}{\rho} + \operatorname{Curl}\mathbf{a}\times\mathbf{a} \\
\frac{\partial\mathbf{a}}{\partial t} &= \operatorname{Curl}(\mathbf{v}\times\mathbf{a}) \\
\operatorname{div}\mathbf{v} &= 0 \\
\operatorname{div}\mathbf{h} &= 0
\end{aligned}
\qquad (1)
$$

where p(y) = magnetofluid static pressure. It is noteworthy that the presence of the magnetic field naturally leads to an equilibrium pressure gra~~h~~dient. The magnetofluid equations can be modified to eliminate p and display a useful symmetry between $\mathbf{v}$ and $\mathbf{a}$.

$$
\begin{aligned}
\operatorname{Curl}\left\{\left(\mathbf{v}\cdot\nabla+\frac{\partial}{\partial t}\right)\mathbf{v}\right\} &= \operatorname{Curl}\left\{(\mathbf{a}\cdot\nabla)\,\mathbf{a}\right\} \\
\left(\mathbf{v}\cdot\nabla+\frac{\partial}{\partial t}\right)\mathbf{a} &= \mathbf{a}\cdot\nabla\,\mathbf{v} \\
\operatorname{div}\mathbf{v} &= 0 \\
\operatorname{div}\mathbf{a} &= 0
\end{aligned}
\qquad (2)
$$

The form of (2) is not altered by both of the interchanges

$$
\begin{aligned}
\mathbf{v}\cdot\nabla + \frac{\partial}{\partial t} &\longleftrightarrow \mathbf{a}\cdot\nabla \\
\mathbf{v} &\longleftrightarrow \mathbf{a}
\end{aligned}
\qquad (3)
$$

Physically, the real velocity field has been interchanged with the Alfven velocity field, and total time rates of change along the flow ($\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$) interchanged with convective rates of change along the Alfven "flow" ($\mathbf{a}\cdot\nabla$). The symmetry ceases to be valid when finite conductivity or finite viscosity are introduced.

2. Linearised perturbation equations

The mathematical stability problem is formulated by superposing on the equilibrium values of the quantities $\underset{\sim}{U}$, $\underset{\sim}{a}$, p small perturbations $\underset{\sim}{v}^{(1)}$, $\underset{\sim}{a}^{(1)}$, $p^{(1)}$ and examining their subsequent time growth or decay by means of the magnetofluid equations. If the perturbations are very small they are linearly related, so that all but one quantity can be eliminated from the magnetofluid equations, leaving one linear partial differential equation in the variables, x, y, z and t.

For this system it is convenient to eliminate all but two quantities $v_y^{(1)}(x,y,z,t)$ and $a_y^{(1)}(x,y,z,t)$, where the subscript y denotes the y-component of the appropriate vector, to obtain the coupled pair of linear partial differential equations

$$\begin{aligned} \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v_y^{(1)} - U''\frac{\partial}{\partial x}v_y^{(1)} - A\frac{\partial}{\partial x}\nabla^2 a_y^{(1)} + A''\frac{\partial}{\partial x}a_y^{(1)} &= 0 \\ \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)a_y^{(1)} - A\frac{\partial}{\partial x}v_y^{(1)} &= 0 \end{aligned} \qquad (4)$$

Solutions obtained from (4) are acceptable only if they satisfy the magnetofluid equations (1) and are finite for all values of x, y, z, t. All the derivatives involved in generating the other perturbed quantities must also be finite. It is not necessary that all derivatives involved in elimination of the other perturbed quantities be finite. For example the terms neglected in the first of equations (4) are

$$\frac{\partial}{\partial y}\,\mathrm{div}(\underset{\sim}{v}^{(1)}\cdot\nabla\underset{\sim}{v}^{(1)}) - \nabla^2(\underset{\sim}{v}^{(1)}\cdot\nabla v_y^{(1)})$$

which involve third order space derivatives. But even if these are infinite the solution $v_y^{(1)}$ will be acceptable provided all the other (six) quantities occurring in the magneto-fluid equations can be generated without infinities, and satisfy the magnetofluid equations.

Since the coefficients in (4) are independent of x and z it is useful to Fourier transform with respect to x and z...

$$v_y^{(1)}(x,y,z,t) = \iint v_y^{(1)}(k,m;y,t)\, e^{i(kx+mz)}\, dk\, dm$$
$$a_y^{(1)}(x,y,z,t) = \iint a_y^{(1)}(k,m;y,t)\, e^{i(kx+mz)}\, dk\, dm .$$

Rewriting equations (4) in terms of the transforms with independent variables y and t...

$$\begin{aligned} \left(U + \frac{1}{ik}\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial y^2} - k^2 - m^2\right) v_y^{(1)} - U'' v_y^{(1)} &= A\left(\frac{\partial^2}{\partial y^2} - k^2 - m^2\right) a_y^{(1)} - A'' a_y^{(1)} \\ \left(U + \frac{1}{ik}\frac{\partial}{\partial t}\right) a_y^{(1)} &= A\, v_y^{(1)} \end{aligned} \qquad (5)$$

At this stage the assumption is made that $a_y^{(1)}$ and $v_y^{(1)}$ are exponentially varying in time. That is, for some complex number $c = c_r + ic_i$

$$a_y^{(1)}(y,t) = a(y)\, e^{-ikct}$$
$$v_y^{(1)}(y,t) = v(y)\, e^{-ikct}$$

(Non exponential time variation is considered in Chapter 5.)

Substituting in (5) and eliminating $a(y)$, a single linear second order differential equation is obtained for v(y)...

$$\left\{\left[1 - \frac{A^2}{(U-c)^2}\right] v'\right\}' - \left[(m^2+k^2)\left(1 - \frac{A^2}{(U-c)^2}\right) + \frac{\left(U'\left(1 - \frac{A^2}{(U-c)^2}\right)\right)'}{U-c}\right] v = 0 \qquad (6)$$

As the magnetic field vanishes ($A \to 0$) Rayleigh's equation (iii), page 2, is recovered...

$$V'' - (k^2 + m^2 + \frac{U''}{U-c})V = 0 \qquad (iii)$$

The interchanges now take the form

$$\begin{aligned} U-c &\longleftrightarrow A \\ V &\longleftrightarrow a \end{aligned} \qquad (7)$$

As an example of the power of these interchanges, the equation for $a(y)$ can be generated by inspecting (6)...

$$\left\{\left[1 - \frac{(U-c)^2}{A^2}\right]a'\right\}' - \left[(m^2+k^2)\left(1 - \frac{(U-c)^2}{A^2}\right) + \frac{\left(A'\left(1 - \frac{(U-c)^2}{A^2}\right)\right)'}{A}\right]a = 0$$

From physical considerations it is clear that $v_y^{(1)}(x,y,z,t)$ must vanish at the solid walls, i.e. when $y = y_1$ and $y = y_2$. This occurs if and only if $v(y) = 0$ at $y = y_1$ and $y = y_2$. It follows, by the way, that the y-component of the magnetic field also vanishes at the walls, independently of the electrical properties of the walls.

A statement of the mathematical problem is now possible. Let m^2 and k^2 be fixed non negative numbers. Provided $U(y_1) - c \neq A(y_1)$, equation (6) for $v(y)$ is regular at y_1 and a solution $\bar{v}(y, c)$ vanishing at $y = y_1$ exists. If for some value of $c = c(k, m)$, $\bar{v}(y, c)$ vanishes also at $y = y_2$ and $\bar{v}(y, c)$ is regular in the real interval (y_1, y_2) then c is an eigenvalue with corresponding eigenfunction $\bar{v}(y, c)$ representing a valid physical disturbance. The time dependence of this disturbance is $\exp(-ikct)$ so that instability occurs when $kc_i > 0$. If no such eigenvalue exists for any values of k and m, the equilibrium under consideration is stable at least to exponential modes. In practice it will not be necessary to show $kc_i > 0$ explicitly because, from (6), $\bar{v}(c^*) = (\bar{v}(c))^*$ is also an eigenfunction, so that any complex eigenvalue ($c_i \neq 0$) will establish instability.

No difference is made to the outcome of the stability problem if $k^2 + m^2$ is replaced by k^2 in equation (6) and only two dimensional perturbations independent of z considered. That is, the perturbations are proportional to exp $ik(x-ct)$ and propagate in the x-direction with speed c_r, wavelength $2\pi/k$ and growth rate kc_i.

It will be convenient to have available two equivalent forms of equation (6). Transforming to the function $f(y) = v(y)/(U(y) - c)$ equation (6) becomes

$$\{[(U-c)^2 - A^2]\, f'\}' - k^2[(U-c)^2 - A^2]\, f = 0 \qquad (8)$$

subject to boundary conditions $(U(y_1) - c)\, f(y_1) = 0$ and $(U(y_2) - c)\, f(y_2) = 0$. The function $f(y)$ usually behaves like $\xi(y)$, the amplitude of the displacement $\xi_y^{(1)}(x,y,z,t)$ of a fluid element from equilibrium in the direction of y. More precisely,

$$\xi(y) = \begin{cases} \dfrac{v(y)}{ik(U(y)-c)} & (k \neq 0,\ U(y) \neq c) \\[2ex] \dfrac{v(y)}{v'(y)} & (k = 0 \text{ or } U(y) = c) \end{cases}$$

Thus, the function $f(y) \equiv v(y)/(U(y) - c)$ may have a first order pole when $U(y) = c$ but still represent a valid physical disturbance.

Transforming to the function $F(y) = X^{\frac{1}{2}} f(y)$ equation (6) becomes

$$F'' - \left[k^2 + \frac{2XX'' - X'^2}{4X^2}\right]F = 0 \qquad (9)$$

where $X = (U - c)^2 - A^2$,

subject to the boundary conditions $X^{-\frac{1}{2}}(y_1)(U(y_1) - c)F(y_1) = 0$ and $X^{-\frac{1}{2}}(y_2)(U(y_2) - c)F(y_2) = 0$.

3. Necessary conditions for instability

A number of necessary conditions for instability (whence corresponding sufficient conditions for stability) can be established by assuming c is a complex eigenvalue with $c_i \neq 0$ and examining the consequences.

One powerful sufficient condition for stability is

$$A^2(y) > U^2(y) \qquad (f)$$

for all values of y in the interval (y_1, y_2). This condition is derived at the end of this section. The physical relevance is probably that the Alfven speed is large enough to propagate disturbances ahead of the moving fluid. The result (f) is well known as a special case of the sufficient condition published by Frieman and Rotenberg (1960), and illustrates the general principle that large enough magnetic fields tend to stabilise magnetofluid flows. For lower values of the magnetic field instabilities do exist, and sometimes the magnetic field exhibits a destabilising effect.

Three other simple results can be obtained directly from (8),

$$\{[(U-c)^2-A^2]f'\}' - k^2[(U-c)^2-A^2]f = 0 \qquad (8).$$

Assuming $c_i \neq 0$ f is regular in the interval y_1, y_2 so that (8) may be multiplied by f^* and integrated from y_1 to y_2.

$$\int_{y_1}^{y_2}\{[(U-c)^2-A^2]f'\}'f^* - k^2[(U-c)^2-A^2]|f|^2 = 0$$

Therefore, integrating by parts

$$\int_{y_1}^{y_2}[(U-c)^2-A^2](|f'|^2+k^2|f|^2)\,dy = 0$$

since $f(y_1) = f(y_2) = 0$.

Taking real and imaginary parts,

$$\int_{y_1}^{y_2}(U-c_r)(|f'|^2+k^2|f|^2)\,dy = 0$$

and $\int_{y_1}^{y_2}[(U-c_r)^2-c_i^2-A^2](|f'|^2+k^2|f|^2) = 0$

If these integrals are to vanish, $U - c_r$ and $(U - c_r)^2 - A^2 - c_i^2$ must both change sign in (y_1, y_2). Thus

$$c_r = U(y) \text{ somewhere} \quad \ldots \quad (a)$$

placing upper and lower bounds on c_r, and revealing that unstable disturbances must travel at the flow speed somewhere.

Also,

$$c_i^2 = (U(y) - c_r)^2 - A(y)^2 \text{ somewhere} \quad \ldots.(b)$$

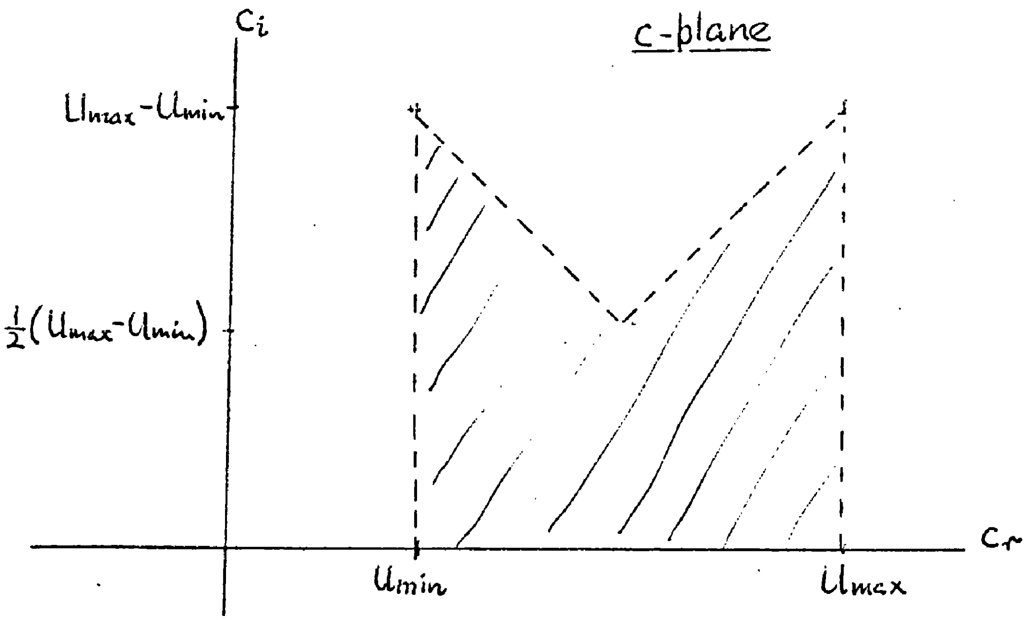
placing an upper bound on c_i^2. (See figure 2.)

If (b) is to be attained then

$$A^2(y) < (U(y) - c_r)^2$$

for some values of y.

$$\therefore \; A^2_{min} < (U_{max} - U_{min})^2 \quad \ldots(c)$$



Complex eigenvalues may occur only in the shaded region or its reflection in the real axis, due to conditions (a) and (b) with $A \equiv 0$.

Figure 2.

Two more necessary conditions can be obtained directly from equation (9),

$$F'' - \left[k^2 + \frac{2XX''-X'^2}{4X^2}\right]F = 0 . \qquad (9)$$

Assuming $c_i \neq 0$, it follows that $X \neq 0$ in the real interval (y_1, y_2), equation (9) has no singularities and F is regular in that interval.

$$\therefore \int_{y_1}^{y_2}\left\{F''F^* - \left[k^2 + \frac{2XX''-X'^2}{4X^2}\right]|F|^2\right\}dy = 0$$

$$\therefore \int_{y_1}^{y_2}\left\{|F'|^2 + \left[k^2 + \frac{2XX''-X'^2}{4X^2}\right]|F|^2\right\}dy = 0$$

since $F(y_1) = F(y_2) = 0$.

Taking real and imaginary parts,

$$\int_{y_1}^{y_2}\left\{|F'|^2 + \left[k^2 + Re\left\{\frac{2XX''-X'^2}{4X^2}\right\}\right]|F|^2\right\}dy = 0$$

and

$$\int_{y_1}^{y_2} Im\left\{\frac{2XX''-X'^2}{4X^2}\right\}|F|^2 = 0 .$$

By inspection of the integrands two more necessary conditions for instability are

$$\mathcal{I}m\left\{\frac{2XX''-X'^2}{4X^2}\right\} = 0 \qquad \text{somewhere} \qquad \text{(d)}$$

and $$k^2 + \mathcal{R}e\left\{\frac{2XX''-X'^2}{4X^2}\right\} < 0 \qquad \text{for some values of } y. \qquad \text{(e)}$$

In the limit of vanishing magnetic field ($A \to 0$) the result (d) coincides with Rayleigh's necessary condition for instability (page 3), that $U''(y) = 0$ somewhere. To display this fact,

$$\mathcal{I}m\left\{\frac{2XX''-X'^2}{4X^2}\right\} = \frac{1}{2}\frac{\mathcal{I}m\left\{|X|\frac{X'}{X}\right\}'}{|X|}$$

$$\therefore\; G(y) \equiv \tfrac{1}{2}\,\mathcal{I}m\left\{|X|\,\frac{X'}{X}\right\}'$$

$$= c_i\,\frac{U'((U-c_r)^2+c_i^2+A^2) - 2AA'(U-c_r)}{[(U-c_r-A)^2+c_i^2]^{\frac{1}{2}}\,[(U-c_r+A)^2+c_i^2]^{\frac{1}{2}}}$$

Condition (d) states that $G(y)$ has a turning point for some value of y in the interval (y_1, y_2). As $A \to 0$,

$$G(y) \to c_i U' \quad \therefore\; G'(y) \to c_i U''$$

$$\therefore\; U''(y) = 0 \quad \text{somewhere.}$$

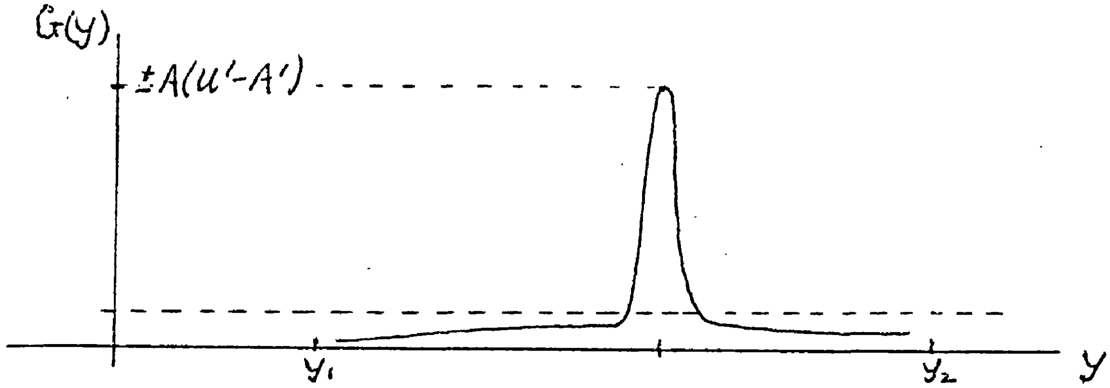
Condition (d) places no restriction on $A(y)$ and $U(y)$ when c_i is very small. For when $(U - c_r \pm A) \gg c_i$

$$G(y) = O(c_i)$$

but at points where $U - c_r = \pm A$, at least one of which exists in (y_1, y_2) from condition (c),

$$G(y) = \pm A(U'-A')$$

so that $G(y)$ rises and falls rapidly (see figure 3), $G'(y)$ must have a turning point and (d) cannot fail to be satisfied.



G(y) must have a turning point when c_i is small. Hence condition (d) must always be satisfied when c_i is small.

Figure 3.

For larger values of c_i, (d) is a restrictive condition but does not seem to yield simple general results restricting the profiles A(y) and U(y) independently of the value of c.

Frieman and Rotenberg's general sufficient condition for stability ((xvii), page 16) takes a simple form for laminar flows. Expressing equation (8) for f(y) in the form

$$H(c)\, f = 0$$

and taking the scalar product $\int_{y_1}^{y_2} f^* H f \, dy$ the resulting quadratic equation for c has discriminant

$$\left[\int_{y_1}^{y_2} U\left(|f'|^2+k^2|f|^2\right)dy\right]^2 - \int_{y_1}^{y_2}\left(|f'|^2+k^2|f|^2\right)dy \, . \int_{y_1}^{y_2}(U^2-A^2)\left(|f'|^2+k^2|f|^2\right)dy$$

The reality of c (and therefore stability) is assured by Frieman and Rotenberg's condition

$$\int_{y_1}^{y_2}(A^2-U^2)\left(|f'|^2+k^2|f|^2\right)dy > 0 \qquad \text{... special case of (xvii).}$$

and hence also by the condition

$$A^2(y) > U^2(y) \quad \text{for all } y \text{ in } (y_1,\, y_2) \qquad \text{(f).}$$

Condition (f) applies no matter what the frame of reference in which U(y) is measured. If A(y) has a zero(f) will not be satisfied unless U(y) is defined to be zero in the same place. If A(y) has more than one zero it is not possible in general to satisfy (f). Condition (f) is more powerful than condition (a), but not necessarily more powerful than the frequency dependent results from which (a) was derived.

4. Physical Interpretation

In Chapter 1 a physical interpretation of Rayleigh's necessary condition was given, in terms of momentum conservation. Condition (d) has a somewhat similar interpretation, as follows.

Returning to the magnetofluid equation of motion for general velocity and magnetic fields, $\underset{\sim}{v}$, $\underset{\sim}{h}$

$$\rho \frac{d\underset{\sim}{v}}{dt} = -\nabla p + \mu_0 \operatorname{Curl} \underset{\sim}{h} \times \underset{\sim}{h}$$

and rewriting to display momentum flux $\frac{\partial \underset{\sim}{J}}{\partial t}$ at a point in space,

$$\frac{\partial \underset{\sim}{J}}{\partial t} + \operatorname{div} \underset{=}{P} = 0$$

where the pressure tensor $\underset{=}{P}$ is given by

$$P_{ij} = (p + \tfrac{1}{2}\mu_0 h_i^2)\delta_{ij} + \rho v_i v_j - \mu_0 h_i h_j$$

The growth of momentum at a point in space is due to the rate of inflow of fluid momentum $\rho v_i v_j$ the rate of momentum creation by the electromagnetic forces, $\rho h_i h_j$ and the pressure forces $p + \frac{1}{2}\mu_0 h_i^2$ which are partly hydrostatic and partly magnetic.

The momentum flux in the x-direction is

$$\frac{\partial J_x}{\partial t} = \frac{\partial}{\partial x}\left(p + \tfrac{1}{2}h_x^2\right) + \frac{\partial}{\partial y}\left(\rho v_x v_y - \mu_0 h_x h_y\right) + \frac{\partial}{\partial z}\left(\rho v_x v_z - \mu_0 h_x h_z\right)$$

For the particular fields under study

$$\underset{\sim}{v} = U(y)\,\hat{\underline{x}} + \underset{\sim}{v}^{(1)}$$

$$\underset{\sim}{h} = H(y)\,\hat{\underline{x}} + \underset{\sim}{h}^{(1)}$$

where $\underset{\sim}{v}^{(1)}(x,y,z,t)$ $\underset{\sim}{h}^{(1)}(x,y,z,t)$ are periodic in x, z and t, the mean x-momentum growth in a plane of fixed y is

$$\frac{\partial \bar{J}_x}{\partial t} = \rho\frac{\partial}{\partial y}\left(\overline{v_x^{(1)}v_y^{(1)} - a_x^{(1)}a_y^{(1)}}\right)$$

where the mean value is with respect to z, and $\underset{\sim}{a} = \sqrt{\frac{\mu_0}{\rho}}\,\underset{\sim}{h}$.

The total momentum growth between the walls is $\int_{y_1}^{y_2}\frac{\partial \bar{J}_x}{\partial t}dy$ and is trivially zero from the fact that the integrand is a complete differential and from the boundary conditions. However, the integrand can be written in terms of the profiles U(y) and A(y) to derive the restricting condition (d) on the profiles.

Although it has been shown that stability decisions based on equation (6) are not changed by putting m = 0, it is not true that $\frac{\partial \bar{J}_x}{\partial t}$ is unaltered by putting m = 0. For general m, $v_x^{(1)}(x,y,z,t)$ can be expressed in terms of $v_y^{(1)}(x,y,z,t)$, from the magnetofluid equations.

$$v_x^{(1)} = \left[-\frac{k^2}{m^2+k^2}\frac{V'}{ik} - \frac{m^2}{m^2+k^2}\frac{U'}{U-c}\frac{V}{ik}\right]e^{ik(x-ct)+imz}$$

$$v_y^{(1)} = V e^{ik(x-ct)+imz}.$$

$$\therefore \frac{\partial}{\partial y}\overline{V_x^{(1)}V_y^{(1)}}$$

$$= -\frac{e^{2kc_i t}}{2}\frac{\partial}{\partial y}\,\mathcal{R}e\left\{\frac{k^2}{m^2+k^2}\frac{V'V^*}{ik}+\frac{m^2}{m^2+k^2}\frac{U'}{U-c}\frac{|V|^2}{ik}\right\}$$

$$= -\frac{e^{2kc_i t}}{4ik(m^2+k^2)}\left[k^2 W'[v] + m^2 2i\,\mathcal{I}m\left\{\frac{U'}{U-c}|V|^2\right\}'\right]$$

where $W[v] = V^*V' - V^{*\prime}V$.

Similarly, using the interchanges (7)

$$\frac{\partial}{\partial y}\overline{a_x^{(1)}a_y^{(1)}}$$

$$= -\frac{e^{2kc_i t}}{4ik(m^2+k^2)}\left[k^2 W'[a] + m^2 2i\,\mathcal{I}m\left\{\frac{A'}{A}|a|^2\right\}'\right]$$

It is still trivial that the momentum integral vanishes, but $W[v]$ can be expressed in terms of the profiles, from (6)

$$V'' - \frac{\left(\frac{A^2}{(U-c)^2}\right)'}{1-\frac{A^2}{(U-c)^2}}V' - \left[k^2+m^2+\frac{\left(U'\left(1-\frac{A^2}{(U-c)^2}\right)\right)'}{(U-c)\left(1-\frac{A^2}{(U-c)^2}\right)}\right]V = 0 \qquad (6)$$

$$\therefore 2i\,\mathcal{I}m\{V''V^*\} - 2i\,\mathcal{I}m\left\{\frac{\left(\frac{A^2}{(U-c)^2}\right)'}{1-\frac{A^2}{(U-c)^2}}V'V^*\right\} - 2i\,\mathcal{I}m\left\{\frac{\left(U'\left(1-\frac{A^2}{(U-c)^2}\right)\right)'}{(U-c)\left(1-\frac{A^2}{(U-c)^2}\right)}\right\}|V|^2 = 0$$

$$\therefore W'[v] - \mathcal{R}e\left\{\frac{\left(\frac{A^2}{(U-c)^2}\right)'}{1-\frac{A^2}{(U-c)^2}}\right\}W[v] = 2i\,\mathcal{I}m\left\{\frac{\left(U'\left(1-\frac{A^2}{(U-c)^2}\right)\right)'}{(U-c)\left(1-\frac{A^2}{(U-c)^2}\right)}\right\}|V|^2 + i\,\mathcal{I}m\left\{\frac{\left(\frac{A^2}{(U-c)^2}\right)'}{1-\frac{A^2}{(U-c)^2}}\right\}|V|^{2\prime}$$

This linear first order differential equation for $W[v]$ has integrating factor $\left|1-\frac{A^2}{(U-c)^2}\right|$ and has the solution

$$W[v] = i\,\mathcal{I}m\left\{\frac{\left(\frac{A^2}{(U-c)^2}\right)'}{1-\frac{A^2}{(U-c)^2}}\right\} + \frac{i|U-c|^2}{|(U-c)^2-A^2|}\int_{y_1}^{y_2}\mathcal{I}m\left\{\frac{((U-c)^2-A^2)'}{(U-c)^2-A^2}|(U-c)^2-A^2|\right\}'\frac{|V|^2}{|U-c|^2}\,dy$$

Using the interchanges (7) the corresponding solution for $W[a]$ is immediate...

$$W[a] = i\,\mathcal{I}m\left\{\frac{\left(\frac{(U-c)^2}{A^2}\right)'}{1-\frac{(U-c)^2}{A^2}}\right\} + \frac{iA^2}{|(U-c)^2-A^2|}\int_{y_1}^{y_2}\mathcal{I}m\left\{\frac{((U-c)^2-A^2)'}{(U-c)^2-A^2}|(U-c)^2-A^2|\right\}'\frac{|a|^2}{A^2}\,dy$$

$$\therefore \int_{y_1}^{y_2} \frac{\partial \bar{\bar{J}}_x}{\partial t}\, dy$$

$$= \frac{i(|U_2 - c|^2 - A_2^2)}{|X_2|^2} \int_{y_1}^{y_2} \mathcal{I}m\left\{|X| \frac{X'}{X}\right\} |f|^2 dy \,.$$

where, as before $X \equiv (U - c)^2 - A^2$ and $f \equiv v/(U - c) \equiv a/A$.

It is now clear that x-momentum conservation demands

$$\mathcal{I}m\left\{|X| \frac{X'}{X}\right\} = 0 \qquad \text{somewhere} \qquad \text{(d)}$$

The possibility that $|U(y_2) - c|^2 = A^2(y_2)$ is unimportant as this may only occur for an isolated value of c. Since the perturbation equations are regular on the real axis when $c_i \neq 0$ any complex eigenvalue c ($c_i \neq 0$) must have neighbouring eigenvalues for which $|U(y_2) - c|^2 \neq A^2(y_2)$ and for which (d) holds. Thus by continuity of $\mathcal{I}m\left\{|X| \frac{X'}{X}\right\}$ with respect to c, (d) is not excused from holding when it happens that $|U(y_2) - c|^2 = A^2(y_2)$.

5. Real eigenvalues

In the study of non-conducting laminar flows, progress was made by establishing the existence of a real eigenvalue and searching for nearby complex eigenvalues. However, in the presence of a parallel magnetic field there are not usually any real eigenvalues. To prove this, and for other purposes, it is important to know how the perturbation amplitudes behave near singularities of the appropriate differential equation.

Working with $f(y)$, from (8),

$$(Xf')' - k^2 Xf = 0 \qquad (8)$$

where $X(y) = (U(y) - c)^2 - A^2(y)$

$$\therefore \mathcal{L}[f] \equiv f'' + \frac{X'}{X} f' - k^2 f = 0 \qquad (10)$$

Singularities occur at values of y (in general complex) for which $X(y) = 0$. Since X has been defined for real y, an analytic continuation to the complex y-plane is implied. If $X(y_o) = 0$, then the series expansion for f in terms of $\eta = y - y_o$, given by

$$f(\rho, \eta) = \sum_{s=0}^{\infty} c_s(\rho) \eta^{\rho+s}$$

is a solution of (8) for some value of ρ, where the c_s and ρ depend on the order (n) of the zero of X. To obtain this solution a series expansion of $\frac{X'}{X}$ is required.

$$\frac{X'}{X} = \frac{n}{\eta} + \sum_{s=0}^{\infty} \alpha_s \eta^s$$

The coefficients α_s are known in terms of the profile values $A(y_o)$, $U(y_o)$ and their derivatives, and also depend on n. In particular

$$\alpha_o = \frac{1}{n+1} \frac{X_o^{(n+1)}}{X_o^{(n)}}$$

where the subscripts on the right side refer to values at $y = y_o$ but the subscript in α_o means $S = 0$. If it happens that $X(y)$ has a single zero at y_o ($X'(y_o) \neq 0$ and $n = 1$) then

$$\alpha_o = \frac{1}{2} \frac{X_o''}{X_o'} = \frac{1}{2} \frac{(U_o - c)U_o'' - A_o A_o'' + U_o'^2 - A_o'^2}{(U_o - c)U_o' - A_o A_o'}$$

Substituting the series expansion for $\frac{X'}{X}$ into (10)

$$\mathcal{L}[f(\rho,\eta)] = \sum_{s=0}^{\infty}(\rho+s)(\rho+s+n-1)c_s(\rho)\eta^{\rho+s-2} + \sum_{s=0}^{\infty} d_s(\rho)\eta^{\rho+s-1} - k^2\sum_{s=0}^{\infty}c_s(\rho)\eta^{\rho+s}$$

where

$$d_s(\rho) = \sum_{s'+s''=s}\alpha_{s''}(\rho+s')c_{s'}\,.$$

$$\therefore \mathcal{L}[f(\rho,\eta)] = \rho(\rho+n-1)c_0(\rho)\eta^{\rho-2} + [(\rho+1)(\rho+n)c_1(\rho)+\rho\alpha_0 c_0(\rho)]\eta^{\rho-1} + \sum_{s=0}^{\infty}[(\rho+s+2)(\rho+s+n+1)c_{s+2}(\rho)+d_{s+1}(\rho)-k^2c_s(\rho)]\eta^{\rho+s}.$$

Defining

$$c_{s+2}(\rho) = \frac{k^2c_s(\rho)-d_{s+1}(\rho)}{(\rho+s+2)(\rho+s+n+1)} \qquad (s\geq 0)$$

it follows that

$$\mathcal{L}[f(\rho,\eta)] = \rho(\rho+n-1)c_0(\rho)\eta^{\rho-2} + [(\rho+1)(\rho+n)c_1(\rho)+\rho\alpha_0 c_0(\rho)]\eta^{\rho-1}$$

so that one solution to the equation $\mathcal{L}[f] = 0$ is given by

$$\rho=0\,,\ c_0=1\,,\ c_1=0$$

$$f_1(\eta) = \sum_{s=0}^{\infty}c_s(0)\eta^s \qquad (11a)$$

To obtain a second solution, define

$$c_0(\rho)=0\,;\ c_1(\rho) = \frac{\rho+n}{\rho+1}$$

$$\therefore \mathcal{L}\left[\frac{\partial f}{\partial\rho}\right] = \frac{\partial}{\partial\rho}\{(\rho+n)^2\eta^{\rho+1}\}$$

$$\therefore \mathcal{L}\left[\frac{\partial f}{\partial\rho}\Big|_{\rho=-n}\right] = 0$$

Thus, a second solution is given by

$$\rho=-n\,,\ c_0(\rho)=0,\ c_1(\rho) = \frac{\rho+n}{\rho+1}$$

$$f_2(\eta) = \sum_{s=0}^{\infty}c_s'(-n)\eta^{s-n} + \operatorname{Log}\eta\sum_{s=0}^{\infty}c_s(-n)\eta^{s-n}. \qquad (11b)$$

The solution f_2 is always infinite when $\eta = 0 (y=y_0)$. This observation is now checked for all $n \neq 2$. The special case n = 2 is considered later.

When n = 1

$$f_1 = 1 + \tfrac{1}{4}k^2\eta^2 + \cdots$$

$$f_2 = -\alpha_0\eta + \operatorname{Log}\eta\,[1 + \tfrac{1}{4}k^2\eta^2 + \cdots]$$

so that f_2 has a logarithmic singularity.

When n > 2

$$f_1 = 1 + \frac{1}{2(n+1)}k^2\eta^2 + \cdots$$

$$f_2 = -\frac{1}{n-1}\frac{1}{\eta^{n-1}}\cdots + \operatorname{Log}\eta\,[c_n(-n) + c_{n+1}(-n)\eta + \cdots]$$

so that f_2 has a pole of order n - 1.

These singularities, if they occur on the real interval (y_1, y_2) make f_2 inadmissible. Any eigenfunction must then behave like f_1 near $y = y_0$.

For complex values of c ($c_i \neq 0$) the zeros of $X \equiv (U - c)^2 - A^2$ occur in the complex plane of y, so that $f(y)$ is regular in the real interval (y_1, y_2). Whenever $f(y_1) = f(y_2) = 0$, $f(y)$ is bound to be an eigenfunction representing a valid physical disturbance (i.e., to be finite with sufficient finite derivatives in (y_1, y_2)).

But for real values of c, all quantities in equation (8) are real and zeros of X occur only on the real axis. If c is to be a real eigenvalue (with $k^2 \neq 0$) then at least one zero of X must occur in the interval (y_1, y_2). This follows by comparing (8) with the general equation for a function $f(y)$

$$(p(y)f')' + q(y)f = 0$$

which is well known to be non oscillating, and therefore to possess at most one zero, in regions where $p(y)$ and $q(y)$ are non zero with opposite sign. In equation (8) $p(y) = X(y)$ and $q(y) = -k^2X(y)$. If $X(y) \neq 0$ and $k^2 \neq 0$ in (y_1, y_2) then $f(y)$ has at most one zero in (y_1, y_2) and cannot be an eigenfunction.

Thus, when $k^2 \neq 0$, the existence of a real eigenvalue c implies the existence of a point y_0 in the interval (y_1, y_2) such that $X(y_0) = 0$. The series solutions (11) then apply at $y = y_0$.

Assuming $X(y_0) = 0$ is either a first order zero, or greater than second order zero, it has been shown any eigenfunction must behave like $f_1(\eta)$ near y_0, because $f_2(\eta)$ is infinite. It follows that no eigenfunction exists. To prove this, consider the behaviour of any solution $f(y)$ of (8) in the region (y_0, y_2), assuming without loss of generality that y_0 is the first singularity to the left of y_2. Near $y = y_0$

$$f(y) \sim f_1(\eta) = 1 + \frac{1}{2(n+1)}k^2\eta^2 \qquad (\eta = y - y_0)$$

$$\therefore \quad f(y_0) = 1 \; ; \; f'(y_0) = 0 .$$

Either $f(y) > 0$ for all $y \geq y_0$ or there exists a unique point y_p such that $f(y_p) = 0$ and $f(y) > 0$ for all y with $y_0 \leq y < y_p$. Suppose y_p exists and $y_0 < y_p \leq y_2$. From (8)

$$Xf' = \int_{y_0}^{y} k^2 X f \qquad (y \geq y_0)$$

Since, in the interval $y_0 \leq y < y_p$, $X(y) \neq 0$ and $f(y) > 0$, it follows Xf' has the sign of Xf in that interval.

$\therefore$ f' has the sign of f (positive) for $y_0 < y \leq y_p$

$\therefore$ $0 = f(y_p) > f(y_0) = 1$, which is false.

$\therefore$ y_p, if it exists, is greater than y_2

$\therefore$ $f(y) > 0$ in the interval (y_0, y_2)

$\therefore$ $f(y_2) \neq 0$

$\therefore$ f is not an eigenfunction (see figure 4).

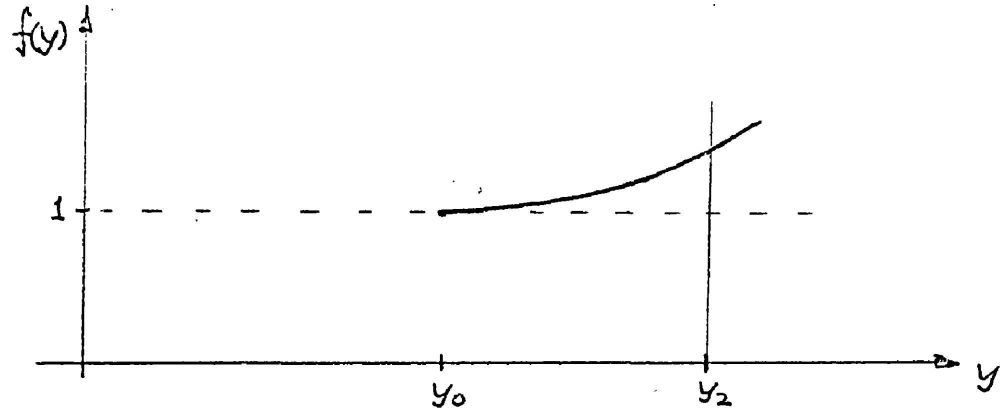


Figure 4

The possibility that $y_0 = y_2$ has not been examined, but clearly $f_1(\eta) \neq 0$ when $\eta = 0$ and no eigenfunction exists in this special case.

It has now been established that no real eigenvalue c can exist unless there exists a point y_0 in the interval (y_1, y_2) such that

$$X(y_0) = X'(y_1) = 0 \; ; \; X''(y_0) \neq 0 \qquad (n = 2)$$

Even assuming that such a point y_0 exists for some real value of c, it is exceptional that c should be an eigenvalue. To prove this, the series expansions for n = 2 are necessary.

When n = 2, the series solutions can be written

$$f_1(\eta) = 1 + \frac{1}{6}k^2\eta^2 \ldots$$

$$f_2(\eta) = \frac{1}{\eta} - 2\alpha_0 \ldots + \alpha_0 \operatorname{Log} \eta \left[1 + \frac{1}{6}k^2\eta^2 + \cdots\right]$$

where $\alpha_0 = \dfrac{X_0'''}{3X_0'}$.

By arguments similar to the case of $n \neq 2$ no real eigenvalues exist due to the infinity in $f_2(\eta)$, provided $\alpha_0 \neq 0$.

But when $\alpha_0 = 0$, i.e., $X_0''' = 0$.

$$f_1(\eta) = 1 + \frac{1}{6}k^2\eta^2 \ldots$$

$$f_2(\eta) = \frac{1}{\eta} + \frac{1}{2}(k^2 + \alpha_1)\eta \ldots$$

and there is a circumstance in which these solutions could represent a valid physical disturbance. By definition $f(y) \equiv v(y)/(U(y) - c)$ may have a first order pole when $U(y) = c$, so that $f_2(\eta)$ is physical if $c = U(y_0)$.

Summing up, for $k^2 \neq 0$, no real number c is an eigenvalue unless there is a point y_0 in (y_1, y_2) such that

$$X_0 = X_0' = X_0''' = 0,$$

and $c = U_0$.

That is, unless

$$\left.\begin{aligned} A_0 &= 0 \\ U_0''U_0' - A_0''A_0' &= 0 \\ \text{and } c &= U_0 \end{aligned}\right\} \qquad (12)$$

The conditions (12) are very restrictive on the profiles $A(y)$ and $U(y)$, and even if satisfied do not ensure the existence of a real eigenvalue. Conditions (12) actually ensure that equation (6) for $v(y)$ is not singular at $y = y_0$. That is, the functions appearing in

$$\left\{\left[1-\frac{A^2}{(U-c)^2}\right]v'\right\}' - \left[k^2\left(1-\frac{A^2}{(U-c)^2}\right) + \frac{\left(U'\left(1-\frac{A^2}{(U-c)^2}\right)\right)'}{U-c}\right]v = 0 \qquad (6)$$

are all regular at $y = y_0$. Similarly, equation (9) for $F(y)$ has no singularity at $y = y_0$.

In the limit of vanishing magnetic field ($A(y) \to 0$) conditions (12) reduce to Rayleigh's necessary condition for the existence of a real eigenvalue c

$$\left.\begin{aligned} U''(y_0) &= 0 \\ \text{and} \quad c &= U(y_0) \end{aligned}\right\} . \qquad (12)'$$

Also, equation (6) reduces to Rayleigh's equation

$$v'' - \left(k^2 + \frac{U''}{U-c}\right)v = 0$$

which clearly has no singularity at y_0. It has been shown in the literature (see Chapter 1) that when (12') is satisfied there are both real and complex eigenvalues for a class of profiles $U(y)$.

When $A(y) \not\equiv 0$ the corresponding system subject to conditions (12) does possess real and complex eigenvalues for some profiles $U(y)$ and $A(y)$*. But conditions (12) are

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* For example if $A(y) = \alpha U(y)$, α = const., and $U''/U < -B^2$ where the constant $B^2 > \pi^2/(y_2-y_1)^2$, then the system possesses a real eigenvalue $c = 0$ for some $k^2 = k_0^2$ and complex eigenvalues nearby.

so restrictive that a detailed discussion is probably not justified. If a real eigenvalue c_0 and its solution f_0 are known then the prescription of Laval et al (page 19) allows an immediate decision on the existence of neighbouring complex eigenvalues. The complex neighbours exist if and only if

$$c_0 = \frac{\int_{y_1}^{y_2}(U(|f'|^2+k^2|f|^2))dy}{\int_{y_1}^{y_2}(|f'|^2+k^2|f|^2)dy} .$$

The case of $k^2 = 0$ has not been considered. When $k^2 = 0$ equation (6) has solutions

$$v_1(y) = U(y) - c$$

$$v_2(y) = (U(y)-c)\int^y \frac{dy}{(U-c)^2 - A^2} .$$

For real $c, v_2(y)$ is not admissible. The solution $v_1(y)$ is regular and is an eigenfunction if and only if $U(y_1) = U(y_2)$. The eigenvalue in this case is $c = U(y_1) = U(y_2)$. Such eigenvalues, representing physical disturbances with long wavelength and speed of propagation equal to the flow velocity at the walls always exist for symmetric velocity profiles, independently of the magnetic profile $A(y)$. In Chapter 3 it is shown that for suitable $A(y)$ unstable eigenvalues always exist nearby, even when no instability exists for $A(y) \equiv 0$.

6. Almost Real Eigenvalues

Although real eigenvalues do not usually exist, complex eigenvalues with unboundedly small imaginary part may. It is shown in this section that the existence of any complex eigenvalue for a given value of k^2 implies the existence of "almost real" eigenvalues for larger values of k^2. As these eigenvalues approach the real axis, the corresponding eigenfunctions become badly behaved, and when the real axis is reached they cease to be physical because of the singular point in the real interval (y_1, y_2).

The existence of almost real eigenvalues can be exploited to show that for unstable systems $X(y)$ usually has two zeros in (y_1, y_2) for some value of c_r. The exceptions for which one zero of $X(y)$ suffices include the systems satisfying the restrictions (12) which were shown necessary for the existence of a real eigenvalue.

Suppose now that a complex eigenvalue $c_1 = c(k_1)(c_i \neq 0)$ exists for a fixed value of k_1 of k. Since the perturbation equations are regular in the real interval (y_1, y_2) when $c_i \neq 0$, their solutions are regular in that interval. For example, equation (8) for $f(y,c,k)$

$$\{[(U-c)^2 - A^2]f'\}' - k^2[(U-c)^2 - A^2]f = 0 \qquad (8)$$

is regular on the real axis when $c_i \neq 0$ so that its solutions $f(y,c,k)$ are regular functions of y, c and k on (y_1, y_2). Thus there exists a regular solution $\bar{f}(y,c,k)$ vanishing at y_1, and c is an eigenvalue if and only if

$$G(c, k^2) \equiv \bar{f}(y_2, c, k) = 0,$$

where G is a regular function of c and k. By the implicit function theorem it follows that c is a regular function of k. As k increases from the value k_1, c moves continuously from the value c_1. The necessary conditions (a) and (b) show that c must remain in the region $U_{min} < c_r < U_{max}$; $0 < c_i < (U_{max} - U_{min})^2 - A_{min}^2$. The present purpose is to show that c must come arbitrarily near to the real axis.

As k^2 increases c either continues to exist for indefinitely large k^2, or ceases to exist for values of k^2 greater than some value k_0^2. In the next section, the case of large k^2 is studied and an upper bound of the order $k^{-\frac{1}{2}}$ is placed on c_i, so that c is almost real. The only other possibility is that there is a k_0^2 such that $c(k^2)$ exists for $k_1^2 \leq k^2 < k_0^2$, but does not exist in a right hand neighbourhood of k_0^2. Suppose $c_0 = \lim_{k^2 \to k_0^2} c(k^2)$. This limit must exist since c is a regular function of k^2 for $k^2 < k_0^2$. But c_0 need not be an eigenvalue. The only possible location of c_0 is the real axis, because a complex value of c_0 would imply that the function $c = c(k^2)$ was regular at k_0^2, so that values of c would exist for $k^2 \geqslant k_0^2$. It follows that c can be brought arbitrarily close to the real axis by choosing k^2 sufficiently close to k_0^2.

The existence of two zeros of X(y) for some value of c_r can now be derived as a necessary condition for instability. Suppose the flow is unstable. The complex eigenvalues c exist and, as shown above, almost real values of c exist.

When c_i is small, there is a point y_o in the complex plane of y, and very close to the real line (y_1, y_2) such that $X(y_o) = 0$. This can be shown from the definition of $X \equiv (U - c_r)^2 - c_i^2 - A^2 + 2(U - c_r)ic_i$ and condition (b), page 29, which states that $Re\{X\}$ vanishes in (y_1, y_2) if c is to be an eigenvalue. Where (b) holds (at y_c, say), $Re\{X_c\} = 0$ and $\mathcal{I}m\{X_c\} = 2(U - c_r)c_i$ (small) so that X nearly has a zero. It follows that the analytic continuation of X to complex values of y has a zero nearby, in fact distant approximately $[m!\, 2\beta i c_i / X_c^{(m)}]^{\frac{1}{m}}$ from y_c, where $\beta = [C_i^2 + A_c^2]^{\frac{1}{2}}$ and m is the order of the first non zero derivative of X(y) at $y = y_c$.

From equation (9) for F it has been shown that

$$I(c) \equiv \int_{y_1}^{y_2} \mathcal{I}m\left\{\frac{2XX'' - X'^2}{4X^2}\right\} |F|^2 = 0$$

where, as before, $X = (U - c)^2 - A^2$.

Since almost real eigenvalues exist it follows that

$$\lim_{c_i \to 0} \{I(c)\} = 0 .$$

To evaluate the limit, the prescription in the footnote is employed. Since the integrand contains a factor c_i it is

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ootnote :

$$\lim_{a \to 0} \int_{\xi_1}^{\xi_2} \frac{a f(\xi) d\xi}{(\xi - x)^2 + a^2} = \pm \pi f(x) .$$

The plus sign is applicable when $a > 0$ and $\xi_2 > \xi_1$ and a factor (-1) occurs for each condition altered.

small, except near zeros of X. Near y_0 ($X(y_0) = 0$) expansions can be made in terms of $\eta = y - y_0$.

$$\frac{2XX''-X'^2}{4X^2} = \frac{n}{4\eta^2}\,\frac{(n-2) + \frac{2(n-1)}{n+1}\frac{X_0^{(n+1)}}{X_0^{(n)}}\eta}{1 + \frac{2}{n+1}\frac{X_0^{(n+1)}}{X_0^{(n)}}\eta} + O(1)$$

where n is the order of the zero of X.

The behaviour of $F(\eta) \equiv X^{\frac{1}{2}}(\eta)\,f(\eta)$ is already known (page 38).

The expression $\lim\limits_{c_i \to 0}\{I(c)\}$ is examined separately for the three cases n = 1, n = 2, n > 2.

When n = 1,

$$F(\eta) \sim \eta^{\frac{1}{2}} \operatorname{Log} \eta$$

$$\therefore \lim_{c_i \to 0} I(c) = \lim_{\eta_i \to 0} \int_{y_0-}^{y_0+} \mathcal{I}m\left\{-\frac{1}{4\eta^2} - \frac{X_0''}{X_0'}\frac{1}{\eta}\right\} |\eta|\,|\operatorname{Log}\eta|^2 d\eta_r + O(\eta_i)$$

$$= \lim_{\eta_i \to 0} \int_{y_0-}^{y_0+} \left(\frac{\eta_i\eta_r}{2(\eta_r^2+\eta_i^2)^2} - \frac{X_0''}{4X_0'}\frac{\eta_i}{\eta_r^2+\eta_i^2}\right) |\eta|\,|\operatorname{Log}\eta|^2 d\eta_r + O(\eta_i)$$

$$= \lim_{\eta_i \to 0} \int_{y_0-}^{y_0+} \frac{\eta_i\eta_r}{2(\eta_r^2+\eta_i^2)^{3/2}} |\operatorname{Log}\eta|^2 d\eta_r$$

which does not exist. If $\lim\limits_{c_i \to 0} I(c)$ is to be zero, at least one other zero of X must exist to allow cancellation of the infinities.

When n = 2

$$F(\eta) \sim \eta^{-\frac{1}{2}}$$

$$\therefore \lim_{c_i \to 0} I(c) = -\frac{1}{3}\frac{X_0'''}{X_0''}\eta_i \int_{y_0-}^{y_0+} \frac{d\eta_r}{(\eta_r^2+\eta_i^2)^{3/2}}$$

which does not exist unless $X_0''' \doteq 0$, in which case the limit does exist and is zero. Thus, if $X_0''' \neq 0$ there are two zeros of X, but if $X_0''' = 0$ one zero of X may suffice for the existence of nearly real eigenvalues. The necessary conditions (12) for the existence of a real eigenvalue include the condition $X_0''' = 0$.

When $n > 2$

$$F(\eta) \sim \eta^{-n+\frac{1}{2}}$$

$$\therefore \lim_{c_i \to 0} I(c) = \lim_{\eta_i \to 0} \int_{y_c-}^{y_c+} \mathrm{Im}\left\{ \frac{n(n-2)}{4\eta^2} \right\} \frac{1}{|\eta|^{2n-3}} \, d\eta_r$$

which does not exist.

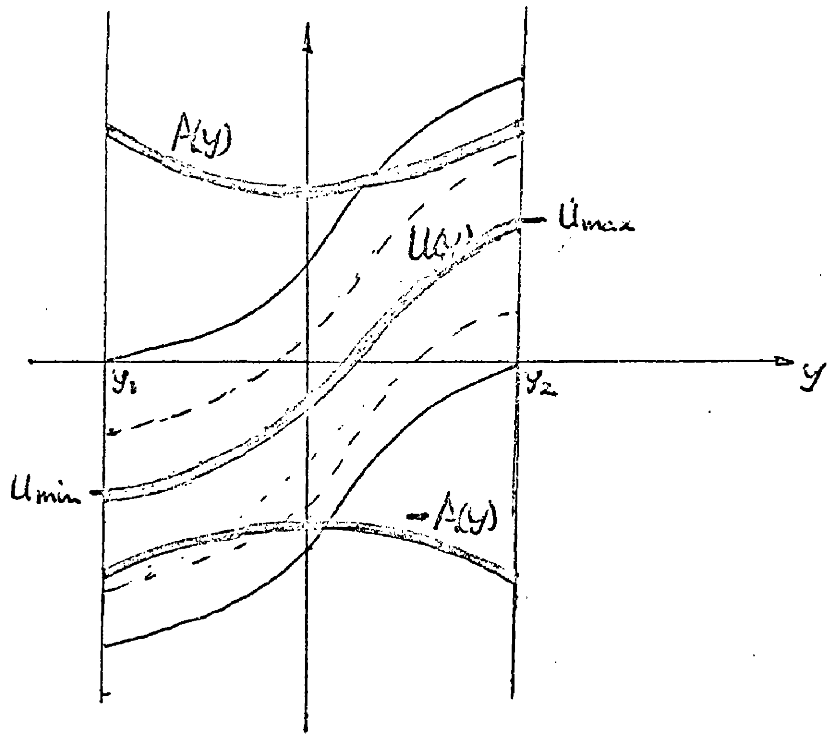
The conclusion is that instabilities cannot occur unless there are two zeros of $X(y)$ in (y_1, y_2), for some value of c_r, except in the unusual circumstance in which X possesses a second order zero coinciding with a zero of X'''.

The existence of two zeros of $X(y)$ for some value of c_r (when $c_i = 0$) has consequences on the flow profiles. There exist two points y_s, y_t in (y_1, y_2) such that $X_s = X_t = 0$ for some value of c_r.

$$\therefore U_s - c_r = \pm A_s$$

$$U_t - c_r = \pm A_t .$$

A rapid decision on the stability of a given pair of profiles is possible if graphs of $A(y)$ and $-A(y)$ fail to intersect a graph of $U(y) - c_r$ where c_r varies such that $U_{min} < c_r < U_{max}$. (See figure 5.)



Example showing profiles A(y), U(y) such that X(y) cannot have two zeros for any value of c_r between Umin and Umax.

Figure 5

Since $X_s = X_t = 0$ it follows that $X' = 0$ for some y-value between y_s and y_t.

$$\therefore (U - c_r)U' - AA' = 0$$

between y_s and y_t. It may be possible to construct a graph of $\frac{AA'}{U'}$ to examine whether it intersects the graph of $U(y) - c_r$ for any value of the parameter c_r. If not, X(y) cannot have two zeros.

The non existence of real eigenvalues, demonstrated in the last section frustrated any attempts to find real eigenvalues and search for nearby complex ones, as has been done in the literature for non conducting fluids. Assuming that complex eigenvalues do exist for some magnetofluid flows, this section demonstrates the existence of real values of c which allow a singular solution of the boundary problem. These are not eigenfunctions but do have complex (regular) eigenfunctions nearby.

It might be possible to find such real singular solutions and develop a perturbation theory to search for nearby complex eigenfunctions.

No such theory is developed in this work, but real singular solutions are discussed briefly below. For the special case of symmetric flows, instabilities are found by searching near an unphysical real solution (page 67).

If F(y) is a real solution of (9) satisfying the boundary conditions at y_1 and y_2, but not regular in the interval (y_1, y_2) it is easy to show that two singularities occur in (y_1, y_2).

From (9) the Wronskian function $F^{(1)\prime}F^{(2)} - F^{(1)}F^{(2)\prime}$ for any two linearly independent solutions of (9) is constant, except at singularities of (9) where discontinuities may occur. Singularities of (9) occur only at zeros of X(y), and the behaviour of F is known (page 38) in terms of η and n.

When n = 1

$F^{(1)} \sim \eta^{\frac{1}{2}}$; $F^{(2)} \sim \eta^{\frac{1}{2}} \operatorname{Log} \eta$

$\therefore W \sim \frac{1}{2} \operatorname{Log} \eta$.

Hence, W has a discontinuity $\pm\frac{1}{2}i\pi$. *

But W = 0 at each end point, therefore two singularities must occur to allow cancellation of the discontinuities.

When n = 2

$$F^{(1)} \sim \eta^{\frac{1}{2}} ; \quad F^{(2)} \sim \eta^{-\frac{1}{2}} + \alpha_0 \eta^{\frac{1}{2}} \operatorname{Log} \eta \qquad (\alpha_0 = X_0'''/3X_0')$$

$$\therefore \; W \sim \tfrac{1}{2}\alpha_0 \operatorname{Log} \eta + \text{poles}.$$

Hence, W has a discontinuity $\pm\frac{1}{2}\alpha_0 i\pi$, and two singularities are necessary for cancellation unless $\alpha_0 = 0$.

When n > 2 W has a discontinuity $\frac{1}{2}c_n'(-n)\, i\pi$ where $c_n'(-n)$ occurs in the series solution for $F(\eta)$, and has a complicated dependence on the profiles A(y), U(y).

The necessary conditions for the occurrence of two zeros of X(y) now become necessary conditions for the existence of singular real solutions, except when there exists a point y_0 such that $X(y_0) = X'(y_0) = X'''(y_0) = 0$.

---

* In the absence of some treatment like Lin's (page 6) for non conducting fluids the sign is ambiguous and depends on whether the real solution is treated as the limit $c_i \to 0+$ or $0-$. (Alternatively, on the choice of the branch of Log η).

7. Large k^2. (Short wavelength)

When $k^2 = \infty$ the system is stable for all profiles $U(y)$ and $A(y)$. This can be proved from (8),

$$(Xf')' - k^2 Xf = 0. \qquad (8)$$

When $k^2 = \infty$, $Xf \equiv 0$ and non zero f occurs only when $X \equiv 0$, i.e., $(U - c)^2 - A^2 \equiv 0$. This cannot occur unless $U(y) \pm A(y)$ is constant, when Alfven waves may propagate with real velocity $c = U \pm A$. In any event, the system is stable.

The object of this section is to prove that the system is stable for sufficiently large (finite) values of k^2, dependent only on the profiles $A(y)$ and $U(y)$. The necessary condition for instability (e), page 31, viz.,

$$E < 0 \text{ for some } y \text{ in } (y_1, y_2) \ldots \qquad (e)$$

where $E = k^2 + \mathcal{R}e\left\{\frac{2XX'' - X'^2}{4X^2}\right\}$, is difficult to satisfy when k^2 is large. But $\mathcal{R}e\left\{\frac{2XX'' - X'^2}{4X^2}\right\}$ is unboundedly large near zeros of $X(y)$, and as $c_i \to 0$ these zeros approach the real axis of y. Investigation will show that (e) always does hold for sufficiently small values of c_i, no matter how large k^2. Although this allows no decision on stability or instability an upper limit on c_i is obtained, and further investigation shows complete stability for large k^2.

First, expression E is examined remote from zeros of $X(y, c)$ to see for what values of c it may be negative, even when k^2 is large. It is already known (page 22) that complex eigenvalues c must lie in the region $U_{min} < c_r < U_{max}$, $0 < c_i^2 < ((U_{max} - c_r)^2 - A_{min}^2)$. Let B^2 be a fixed upper bound for $|2XX'' - X'^2|$ as y, c_r and c_i vary within (known) finite real domains. (Such a bound does not exist when X(y) is discontinuous, i.e., when U(y) or A(y) is discontinuous; hence discontinuous profiles examined in Chapter 4 may be unstable for large k^2.)

Let ϵ be a fixed small positive number. When $c_i \geq \epsilon > 0$,

$$
\begin{aligned}
|X|^2 &= ((U-c_r)^2 - c_i^2 - A^2)^2 + 4(U-c_r)^2 c_i^2 \\
&= ((U-c_r-A)^2 + c_i^2)((U-c_r+A)^2 + c_i^2) \\
&> c_i^4 \\
&\geq \epsilon^4 \quad \text{for all } c_r, y.
\end{aligned}
$$

so that E can be made positive for all c_r in (U_{min}, U_{max}) and all y in (y_1, y_2) by choosing $k > \frac{B}{2\epsilon^2}$, provided $c_i \geq \epsilon > 0$. Inverting this argument, a large value of k ensures that (e) cannot be satisfied unless $c_i < \epsilon = \left(\frac{B}{2k}\right)^{\frac{1}{2}}$.

When $\epsilon \geq c_i > 0$ the singularities of E (zeros of X) lie close to the real axis. In the neighbourhood of the singularities,

$$
E = k^2 + \mathcal{R}e\left\{ \frac{n}{4\eta^2} \frac{(n-2) + \frac{2(n-1)}{n+1}\frac{X_0^{(n+1)}}{X_0^{(n)}}\eta}{1 + \frac{2}{n+1}\frac{X_0^{(n+1)}}{X_0^{(n)}}\eta} \right\} + O(1)
$$

where X has an nth order zero at y_0 and $\eta = y - y_0$. Variation only of η need be considered because η_r varies like y and η_i like c_i. The value of c_r fixes the position y_0

of the singularity.

In the three separate cases n = 1, n = 2 and n > 2 it will be shown that E is negative for a range of values of η_r, so that condition (e) is always satisfied and no decision on stability is possible. But in each case the stronger necessary condition for instability

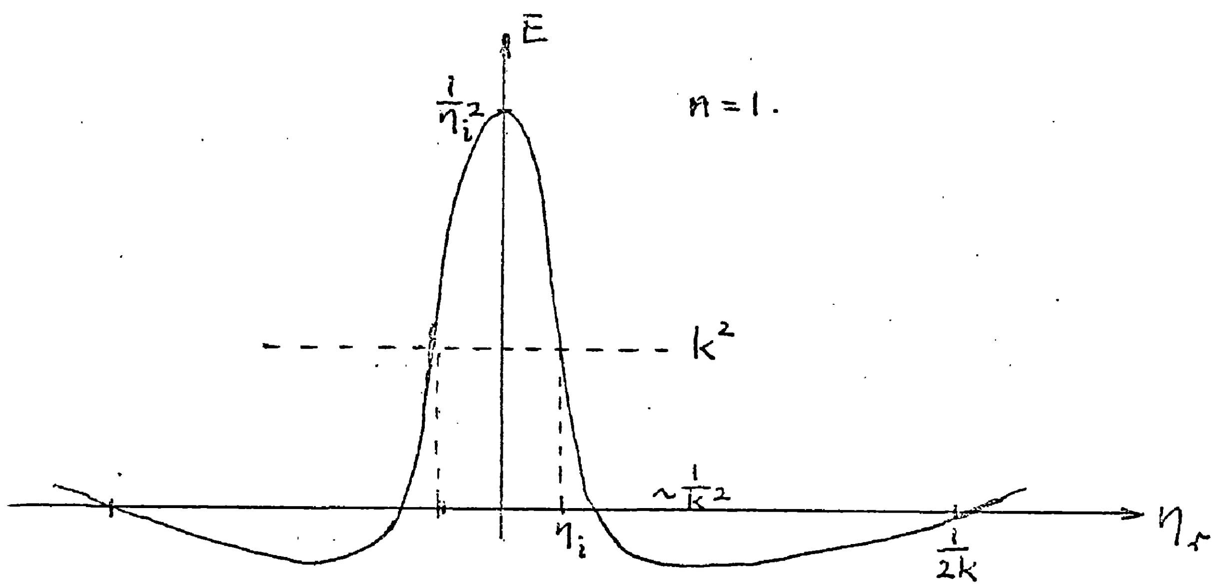
$$\int_{y_1}^{y_2}(|F'|^2 + E|F|^2)\,dy = 0 \qquad .(15)$$

from which (e) was derived (page 31) cannot be satisfied, whence stability for large k^2 is assured.

When n = 1

$$E = k^2 + \frac{\eta_i^2 - \eta_r^2}{4(\eta_i^2+\eta_r^2)^2} + O\left(\frac{1}{|\eta|}\right)$$

When $|\eta_r| > \frac{1}{2k}$, k^2 still dominates and $\eta_i \ll \eta_r \; (\therefore \ll \frac{1}{2k})$
when $-\frac{1}{2k} < \eta_r < \frac{1}{2k}$ it is negative for $\eta_i \ll \eta_r \; (\therefore \ll \frac{1}{2k})$ and positive for $\eta_r \lesssim \eta_i$ (See figure 6.)



E always negative for sufficiently small c_i (η_i) when n = 1.

Figure 6

But the integral (15) can be examined, for $-\frac{1}{2k} < \eta_r < \frac{1}{2k}$ and $\eta_i \ll \eta_r$, with a knowledge of $F(\eta)$. Since n = 1

$$F(\eta) \sim \eta^{\frac{1}{2}} \operatorname{Log} \eta$$

$$\therefore \int_{y_1}^{y_2} (|F'|^2 + E|F|^2)\, dy$$

$$> \int_{y_1}^{y_2} E|F|^2 dy$$

$$> \sum 2 \int_0^{1/2k} \frac{\eta_i^2 - \eta_r^2}{4(\eta_i^2+\eta_r^2)^{3/2}} (\operatorname{Log}|\eta|)^2 d\eta_r \qquad \left(\begin{matrix}\text{sum over}\\ \text{singularities}\end{matrix}\right)$$

$$> 2\sum \left\{ \int_0^{\eta_i} \frac{\eta_i^2-\eta_r^2}{4(\eta_i^2+\eta_r^2)^{3/2}} (\operatorname{Log}|\eta|)^2 d\eta_r - \int_{\eta_i}^{1/2k} \frac{\eta_r^2-\eta_i^2}{4(\eta_i^2+\eta_r^2)^{3/2}} (\operatorname{Log}|\eta|)^2 d\eta_r \right\}$$

$$> 2\sum \left\{ \frac{(\log \sqrt{2}\,\eta_i)^2}{\eta_i} \int_0^{\eta_i} \frac{\eta_i^2-\eta_r^2}{4(\eta_i^2+\eta_r^2)^{3/2}} - \left(\frac{1}{2k} - \eta_i\right) Max \left\{ \frac{(\eta_r^2-\eta_i^2)(\operatorname{Log}|\eta|)^2}{4(\eta_r^2+\eta_i^2)^{3/2}} \right\} \right\}$$

$$> 2\sum \left\{ \frac{(\log \eta_i)^2}{\eta_i} - k(\log k)^2 \right\}$$

$$> 0 .$$

When n = 2

$$E = k^2 + \frac{1}{3} \frac{X_0'''}{X_0''} \frac{\eta_r}{|\eta|^2} + O(1)$$

The expression E is positive when $\eta_r \gtrsim \frac{1}{k}$. Depending on the sign of X_0'''/X_0'' E is negative when $|\eta_r| \ll 1/k$ for one sign of η_r and positive for the other.

The integral (15) is plainly positive because

$$F \sim \eta^{-\frac{1}{2}}$$

$$\therefore \int_{y_1}^{y_2} \left(|F'|^2 + \left(k^2 + \frac{1}{3}\frac{X_0'''}{X_0''}\frac{\eta_r}{|\eta|^2}\right)|F|^2 \right) dy$$

$$> \int_{y_1}^{y_2} \left(\frac{1}{|\eta|^3} + k^2 + \frac{1}{3}\frac{X_0'''}{X_0'}\frac{\eta_r}{|\eta|^3} \right) dy$$

$$> 0 \quad \text{since } \eta_r << 1 .$$

When n > 2

$$E = k^2 + \frac{n(n-2)(\eta_r^2 - \eta_i^2)}{4|\eta|^4}$$

which is positive for all $\eta_r > \eta_i$ and negative only when $\frac{1}{k} > \eta_i >> \eta_r$.

But $F \sim \eta^{\frac{1}{2}-n}$ and

$$\int_{y_1}^{y_2} (|F'|^2 + E\,|F|^2)\, dy$$

$$> \sum \int_{-1/2k}^{1/2k} \left(\frac{(\frac{1}{2}-n)^2}{|\eta|^{2n+1}} + \frac{n(n-2)(\eta_r^2 - \eta_i^2)}{4|\eta|^{2n+3}} \right) d\eta_r$$

$$= \sum \int_{-1/2k}^{1/2k} \frac{(5n^2 - 4n + 1)\eta_r^2 + (3n^2+1)\eta_i^2}{4|\eta|^{2n+3}}\, d\eta_r$$

$$> 0 \quad \text{since } n > 2 .$$

The conclusion is that the necessary condition for instability

$$\int_{y_1}^{y_2} \left(|F'|^2 + \left(k^2 + \mathcal{R}e\left\{\frac{2XX'' - X'^2}{4X^2}\right\}\right)|F|^2 \right) dy = 0$$

can never be satisfied for values of k^2 chosen sufficiently large.

CHAPTER 3

LONG WAVE INSTABILITIES

Sufficient condition for instability - magnetic field stabilises or destabilises? - instability of symmetric flows

1. Sufficient condition for instability

Rosenbluth and Simon (1964) solved the stability problem for laminar flow of an inviscid incompressible non-conducting fluid when $k^2 = 0$, and hence wrote down a sufficient condition for instability (chapter 1, page 10). Their method can be extended to perfectly conducting fluids in the presence of a magnetic field. To obtain a simple sufficient condition (namely (16)) it will be supposed that

$$U'(y) > 0$$

$$A(y) = \text{constant}.$$

In these circumstances, Rayleigh's condition that $U(y)$ has a point of inflexion will emerge as a necessary condition for instability when $k^2 = 0$. This ceases to be true when $k^2 \neq 0$ or when $A(y) \neq$ constant, and recourse must be made to the more complicated (frequency dependent) necessary condition for instability (d), page 31. The condition

$$A < \tfrac{1}{2}(U_{max} - U_{min}) \qquad (c)'$$

also emerges as a necessary condition for instability when $k^2 = 0$. This too ceases to be true for other values of k^2 or for arbitrary $A(y)$, but the weaker condition (c) which states $A < U_{max} - U_{min}$ does hold.

Violation of (c) ensures stability, probably because the alfven speed is then sufficiently large to allow propagation of a disturbance beyond (faster than) the flow. This interpretation survives the strengthening of (c) to (c)', for the special $k^2 = 0$ modes of this section, because violation of (c)' allows propagation beyond the flow in at least one direction.

The derivation of the sufficient condition (16), below, now begins. From equation (8) for $f(y, c)$, when $k^2 = 0$,

$$\{[(U-c)^2 - A^2]\, f'\}' = 0$$

$$\therefore f(y,c) = \int_{y_1}^{y} \frac{dy}{(U-c)^2 - A^2} \qquad \text{since } f(y_1, c) = 0.$$

The system is unstable to long wave modes ($k^2 = 0$) if and only if $G(c) \equiv f(y_2, c)$ has complex zeros with $c_i \neq 0$. The relation

$$G(c) = \frac{1}{2A}\int_{y_1}^{y_2} \frac{dy}{U-c-A} - \frac{1}{2A}\int_{y_1}^{y_2} \frac{dy}{U-c+A}$$

can be used to plot the variation of $G(c)$ as c is varied around the contour of figure 7. $G(c)$ has zeros inside this contour if and only if the plot of G encloses the origin.

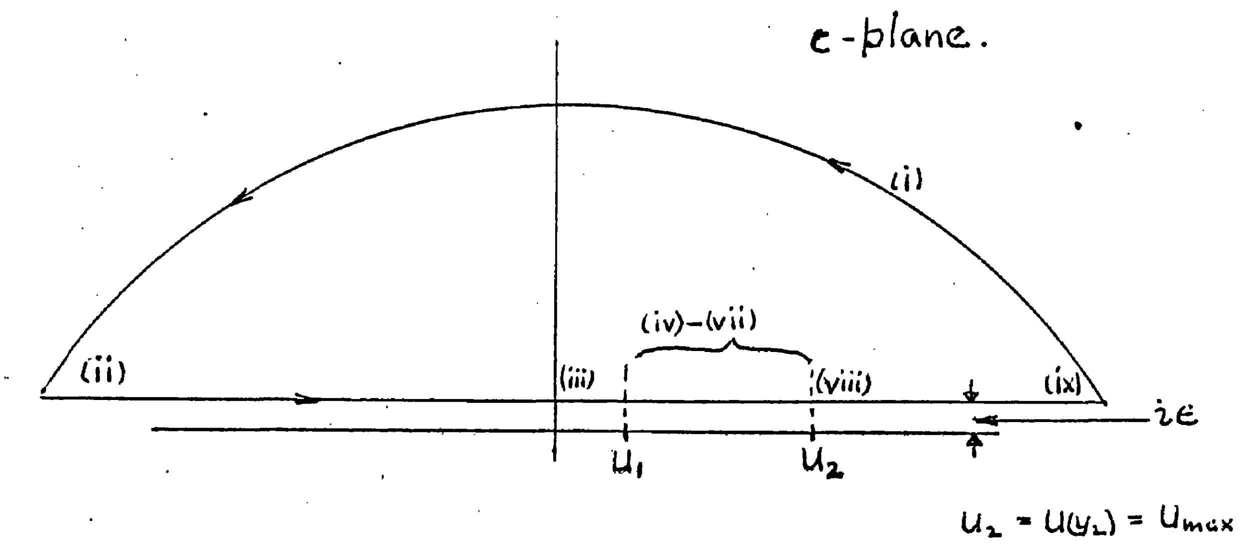


Figure 7

The variation of G can be sketched (figure 8) from the following information, remembering that U'(y) > 0 and A(y) is constant.

(i) $c = Re^{i\theta}$ (R large)...

$G \sim e^{-2i\theta}/R^2$.

(ii) $-\infty < c_r - A < c_r + A < U_1$; $c_i = \epsilon$ ...

$Re\{G\} > 0$, $Im\{G\} \sim +\epsilon$.

(iii) $c_r - A < c_r + A \cong U(y_1)$ ...

$Re\{G\} \sim +\infty$, $Im\{G\} = \phi$ where $0 \leq \phi \leq \pi$

(iv) $c_r - A < U_1 < c_r + A < U_2$

$Re\{G\} = ?$, $Im\{G\} > 0$

<u>EITHER</u> $A < \frac{1}{2}(U_{max} - U_{min})$

(v)
$c_r - A \sim U_1 < c_r + A < U_2$
$Re\{G\} \sim -\infty$, $Im\{G\} = ?$

(vi)
$U_1 < c_r - A < c_r + A < U_2$
$Re\{G\} = ?$, $Im\{G\} = ?$

(vii) $c_r - A \sim U_2$
$Re\{G\} = -\infty$, $Im\{G\} = ?$

(viii) $U_2 < c_r - A$
$Re\{G\} = ?$, $Im\{G\} < 0$

<u>OR</u> $A \geqslant \frac{1}{2}(U_{max} - U_{min})$.

(v)
$c_r - A \leqslant U_1 < c_r + A \sim U_2$
$Re\{G\} \sim -\infty$, $Im\{G\} = ?$

(vi)
$c_r - A < U_2 < c_r + A$
$Re\{G\} < 0$, $Im\{G\} = ?$

(vii) $c_r - A \sim U_2$
$Re\{G\} \sim -\infty$, $Im\{G\} = ?$

(viii) $U_2 < c_r - A$
$Re\{G\} = ?$, $Im\{G\} < 0$.

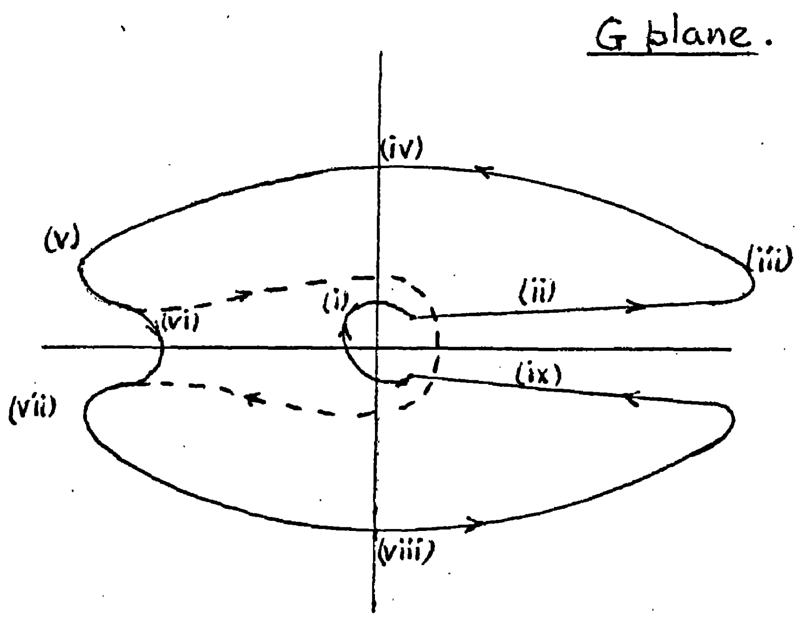


Figure 8

In one of the regions (v), (vi), (vii) $\mathcal{I}m\{G\}$ vanishes and there is a zero of G(c) (the origin is encircled) if and only if $\mathcal{R}e\{G\} > 0$ there. When $A > \frac{1}{2}(U_{max} - U_{min})$, $\mathcal{R}e\{G\} < 0$ in all three regions, implying no zero, hence stability. Thus the condition $A < \frac{1}{2}(U_{max} - U_{min})$ has emerged as a necessary condition for instability.

When $A < \frac{1}{2}(U_{max} - U_{min})$, $\mathcal{R}e\{G\}$ may have either sign in the region (vi), so that instability is possible if $\mathcal{I}m\{G\}$ does vanish in region (vi) and $\mathcal{R}e\{G\} > 0$ there. If $\mathcal{I}m\{G\}$ vanishes at (v) or (vii), then $\mathcal{R}e\{G\} \sim -\infty$ and no instability occurs. Thus, for instability, it is necessary that $\mathcal{I}m\{G\} = 0$ in region (vi), and this implies Rayleigh's condition, because ...

$$\mathcal{I}m\{G\} = 0 \qquad \text{in region (vi)}$$

$$\Rightarrow \lim_{c_i \to 0} \left\{ \frac{1}{2A}\int_{y_1}^{y_2} \frac{c_i\,dy}{(U-c_r-A)^2+c_i^2} - \frac{1}{2A}\int_{y_1}^{y_2} \frac{c_i\,dy}{(U-c_r+A)^2+c_i^2} \right\} = 0,$$

$$\Rightarrow \frac{\pi}{2A\,U'(y_s)} - \frac{\pi}{2A\,U'(y_t)} = 0,$$

where $U(y_s) = c_r - A$

$U(y_t) = c_r + A,$

$\Rightarrow$ there are two distinct points y_s, y_t in the interval y_1, y_2 such that

$U'(y_s) = U'(y_t)$

$\Rightarrow$ $U''(y_o) = 0$ for some y_o in (y_s, y_t), q.e.d.

The sufficient condition for instability can now be stated.

If there exist two points y_s, y_t such that

$U_s - U_t = 2A$

$U_s' - U_t' = 0$

and c is defined by

$c_r = \frac{1}{2}(U_s + U_t)$

$c_i = 0$

and if

$$M \equiv \int_{y_1}^{y_2} \frac{dy}{(U-c_r)^2 - A^2} > 0 \qquad (16)$$

then instability occurs.

The instability occurs when $k^2 = 0$. If it is understood that $k^2 = 0$, the condition is necessary and sufficient for instability. The constraints on c_r, y_s, y_t ensure that the integral is real.

As the magnetic field vanishes ($A \rightarrow 0$) the condition reduces to Rosenbluth and Simon's

$$R \equiv -\frac{1}{U'(U-c_r)} - \int_{y_1}^{y_2} \frac{U''\,dy}{U'^2(U-c_r)} > 0$$

where $c_r = U(y_o)$

and $U''(y_o) = 0$.

This limit is better displayed when (16) is rewritten in the form

$$M \equiv -\left[\frac{1}{U'}\ \frac{\mathrm{Log}\left\{\frac{U-c_r+A}{U-c_r-A}\right\}}{2A}\right]_{y_1}^{y_2} - \int_{y_1}^{y_2}\frac{U''}{U'^2}\ \frac{\mathrm{Log}\left\{\frac{U-c_r+A}{U-c_r-A}\right\}}{2A} > 0$$

2. Magnetic field stabilises or destabilises?

It has been shown that any flow $U(y)$ is always stabilised by a magnetic field $A(y)$ such that $|A(y)| > |U(y)|$ for all y. It is tempting to conjecture that an increasing magnetic field is always a stabilising influence. This will be shown untrue, at least for long wave modes.

When $U'(y) > 0$ and A is constant, $M(A) > 0$ is a necessary and sufficient condition for instability of $k^2 = 0$ modes. The sign of $\frac{dM}{dA}$ will reveal whether or not increasing A has a stabilising effect.* In the case of marginal stability ($M = 0$) for a given value of

---

* Strictly speaking, knowledge of the sign of dM/dA is useless unless $M = 0$. The sign of M merely indicates whether or not the plot of $G(c)$ (figure 8) encircles the origin. Thus if $dM/dA > 0$ for a given value of A, where $M < 0$, no difference is made to stability unless $dM/dA > 0$ over a finite range of values of A sufficient to cause M to change sign. When $dM/dA > 0$ it will be said that the magnetic field tends to stabilise.

A, the sign of $\frac{dM}{dA}$ is crucial to the stability problem, because an infinitesimal change in A causes stability or causes instability.

$$M(A) = \frac{1}{2A}\int_{y_1}^{y_2}\frac{dy}{U-U_t} - \frac{1}{2A}\int_{y_1}^{y_2}\frac{dy}{U-U_s}$$

where $U_t - U_s = 2A$

$U_t + U_s = 2c_r$

$U_t' - U_s' = 0$

It follows that $\dot{c}_r(A) = \frac{U_s'' + U_t''}{U_s'' - U_t''}$, where differentiation with respect to A is denoted by dot, and differentiation with respect to y by dash.

Keeping $A \neq 0$ at first, M(A) can be obtained by formal differentiation ...

$$\dot{M}(A) = -\frac{M}{A} + \frac{\dot{U}_t}{2A}\int_{y_1}^{y_2}\frac{dy}{(U-U_t)^2} - \frac{\dot{U}_s}{2A}\int_{y_1}^{y_2}\frac{dy}{(U-U_s)^2}.$$

The integrals on the right do not exist, but formally represent the proper expression

$$\dot{M}(A) = -\frac{M}{A} - \frac{1}{A}\left[\frac{A\dot{c}_r + U - c_r}{U'(U-U_t)(U-U_s)}\right]_{y_1}^{y_2} - \frac{1}{A}\,\mathcal{P}\int_{y_1}^{y_2}\frac{(A\dot{c}_r + U - c_r)\,U''\,dy}{U'^2(U-U_t)(U-U_s)}$$

Provided $\dot{M}(A) \neq 0$ a change ΔA in A causes a change $\dot{M}.\Delta A$ in M. The factor $\Delta A/A$ appears, and ensures that the change in M does not depend on the direction of the magnetic field, but only on the change in magnitude. This had to be the case because the perturbation equations do not depend on the sign of A. Increasing the magnitude of A is a destabilising influence only if $\dot{M}.\Delta A > 0$. In particular if M(A) = 0 the flow is marginally stable and an increase ΔA in the magnitude of A causes instability if and only if

$$-\left[\frac{A\dot{c}_r + U - c_r}{U'(U-U_t)(U-U_s)}\right]_{y_1}^{y_2} - P\int_{y_1}^{y_2}\frac{(A\dot{c}_r + U - c_r)U''}{U'^2(U-U_t)(U-U_s)}\,dy > 0$$

where $\dot{c}_r = (U_s'' + U_t'')/(U_s'' - U_t'')$.

Now examining the case A = 0, it happens that $\dot{M}(0) = 0$. The points y_t and y_s coincide at y_o where $U''(y_o) = 0$ and $\dot{c}_r(0) = 0$. It is necessary to calculate second derivatives and these are given formally by

$$\ddot{M}(0) = 2\ddot{c}_r\int_{y_1}^{y_2}\frac{dy}{(U-U_o)^3} + 2\int_{y_1}^{y_2}\frac{dy}{(U-U_o)^4}$$

where $\ddot{c}_r(0) = -U_o^{(iv)}/3U_o'U_o'''$. In terms of integrals which exist, $\ddot{M}(0)$ is given by

$$\frac{3}{2}\ddot{M}(0) = -\left[\frac{1}{U'(U-U_o)^3}\right]_{y_1}^{y_2} - P\int_{y_1}^{y_2}\frac{\frac{U_o^{(iv)}}{3U_o'U_o'''} + \frac{U''}{U'^2}}{(U-U_o)^3}\,dy\,.$$

The introduction of a small magnetic field ΔA to a field-free flow causes a change $\frac{1}{2}\ddot{M}(0)(\Delta A)^2$ in M and hence tends to destabilise only if $\ddot{M}(0) > 0$. In the special case when M(0) = 0 (i.e., R = 0) the introduction of ΔA actually causes instability if and only if

$$-\left[\frac{1}{U'(U-U_o)^3}\right]_{y_1}^{y_2} - P\int_{y_1}^{y_2}\frac{\frac{U_o^{(iv)}}{3U_o'U_o''} + \frac{U''}{U'^2}}{(U-U_o)^3} > 0 \qquad (17)$$

where $U_o'' = 0$. The sign of the boundary term is strictly negative, and always makes a stability contribution.

3. Instability of symmetric flows

Specialising to symmetric flows, a sufficient condition for instability can be established. The condition is interesting because it shows that any such flow that is stable in the absence of a magnetic field can be made unstable by the superposition of a suitable magnetic field parallel to the flow. For convenience $y_1 = 0$ and $y_2 = 2b$. The following are the conditions on the profiles.

(a) The velocity profile, $U(y)$, is symmetric,
$U(y) = U(2b - y)$.

(b) The magnetic field profile is symmetric,
$A(y) = A(2b - y)$.

(c) The magnetic field vanishes at the walls,
$A_o = 0$.

(d) The fluid velocity vanishes at the walls,
$U_o = 0$.

(e) The local flow energy exceeds the local magnetic energy, $U^2(y) > A^2(y)$.

Given these conditions, the system will be shown unstable whenever

$$U_o'' > \frac{A_o'}{U_o'} A_o'' \tag{18}$$

The derivation of (18) crucially depends on (a), (b) and (c). No generality is lost by imposing (d), and (e) could have been relaxed for some but not all values of y. If (e) were relaxed for all values of y, that is if $A^2(y) > U^2(y)$ for all y, then condition (18) for instability cannot be established, which is not surprising because in those circumstances the flow must be stable, according to the sufficient condition for stability (f), page 32.

The sufficient condition for instability (18) bears an interesting relation to known results in ordinary (non-conducting) fluid dynamics.* As the magnetic field vanishes (18) becomes $U_0'' > 0$, which result was obtained by Tollmien as a sufficient condition for instability of non-conducting fluids. If U_0'' is positive it follows by symmetry of U(y) that $U''(y_c) = 0$ for some critical y_c, so that U(y) satisfies Rayleigh's necessary condition for instability of any laminar flow (not necessarily symmetric) of inviscid incompressible non-conducting fluids. Thus all of Tollmien's unstable flows are seen to satisfy Rayleigh's necessary condition (see figure 9)

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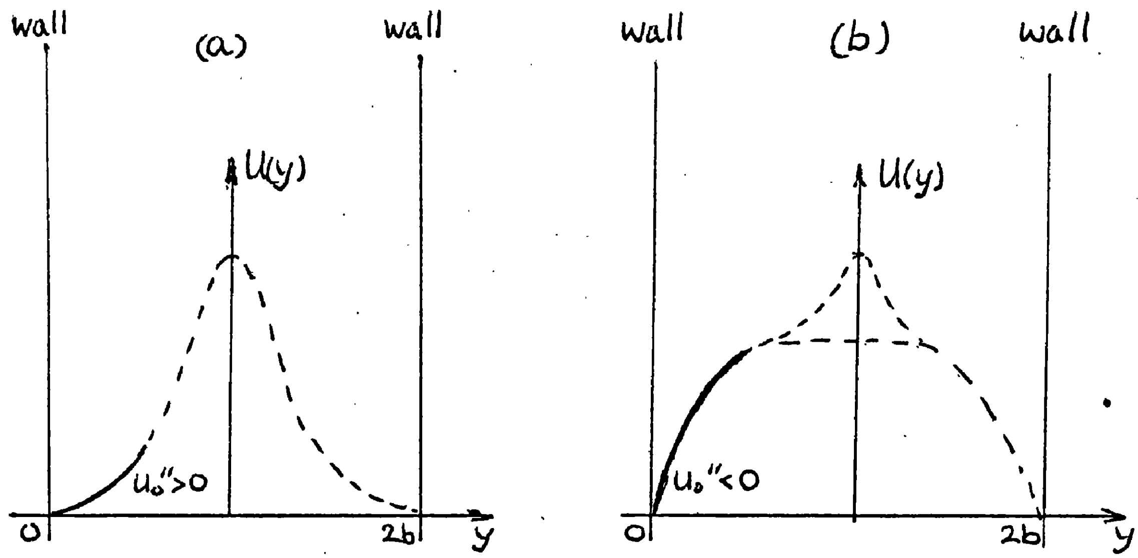
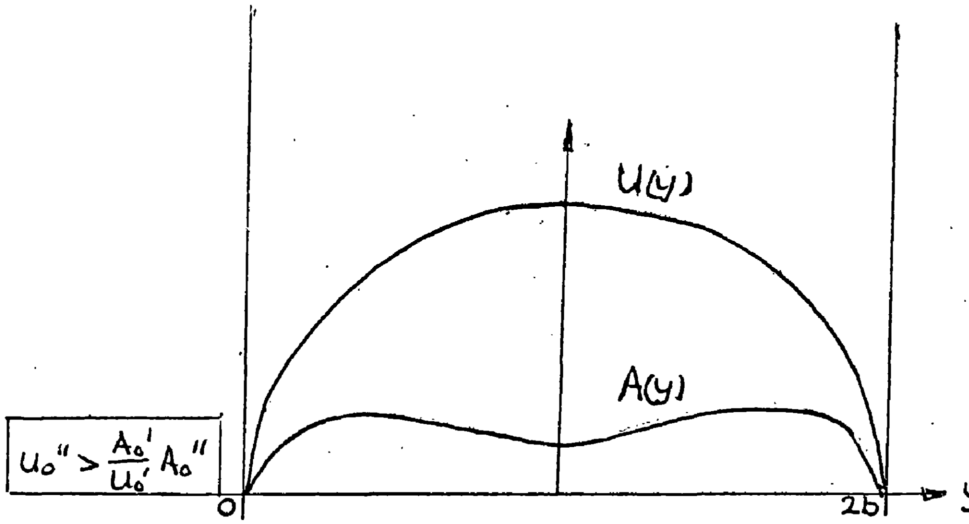
* Discussed in chapter 1.



(a) $U_0''>0$: point of inflexion must occur for symmetry.
(b) $U_0''<0$: point of inflexion may or may not occur.

Figure 9

But Rayleigh's condition may be violated by some flows for which $U_0'' < 0$. Such flows are stable in the absence of a magnetic field, but can always be made unstable by superposing a magnetic field such that (18) is satisfied. For example, inviscid incompressible laminar flow with parabolic profile, well known to be stable in the absence of a magnetic field, can be made unstable in this way (see figure 10).



Magnetic field profile suitable for making parabolic velocity profile unstable.

Figure 10

Working with equation (8) for f(y)

$$\{[(U-c)^2 - A^2] f'\}' - k^2 [(U-c)^2 - A^2] f = 0 \qquad (8)$$

subject to the boundary conditions $(U - c)f = 0$ when $y = 0$ and $y = 2b$, it is easy to spot the real eigen-solution $(f; k, c) = (1; 0, 0)$. This solution is marginally stable (definition, page 18) but is not covered by the general theory of Laval et al. The solution (1; 0, 0) is barely physical, since it implies an unbounded fluid velocity in the x direction. This is demonstrated by noting that $v = U$ and $\operatorname{div} v = 0$, so that the x velocity is $-xU'$ which is unbounded as x increases. But no difficulty will arise provided the nearby unstable solutions are properly physical.

Instability will be demonstrated by finding a complex solution (f; k, c) near the marginally stable solution (1; 0, 0). The complex solution f will be near 1 for most of the range $0 < y < 2b$. But near $y = 0$, f must rapidly fall to zero, to satisfy the boundary condition there. This will be facilitated by the presence of two singularities of the perturbation equation (8) in the complex y plane at those values of y for which $U(y) = c \pm A(y)$. For the marginally stable solution, $c = 0$ and $U_0 = A_0 = 0$ so that these singularities coincide at $y = 0$. If k^2 is a fixed small positive number and c is fixed arbitrarily as a complex number with small modulus and non-zero imaginary part, the perturbation equation (8) is regular at $y = 0$. A solution $f(y)$ vanishing at $y = 0$ must exist and so already satisfies the boundary condition at $y = 0$. By symmetry the same solution can be started from the second wall and the two parts joined in the middle $(y = b)$. Of course, the derivative at the join is unlikely to be continuous, but by demanding $f'(b-) = f'(b+) = 0$ at the join, f becomes an eigenfunction and c is fixed.

It is not possible to solve the perturbation equation (8) for completely arbitrary c, so that some initial information about c is necessary. Rather than seek this information in a laboriously logical way, the present purpose seems best served by assuming the result c in advance, solving for f, then solving for c again to see if the initial "assumption" was valid. This scheme will be shown to be self-consistent whenever condition (18) holds. Thus (18) is a sufficient condition for instability.

So, the following properties are tentatively ascribed to c :-

(i) $c_r \propto k^2$, (ii) $c_i \propto k^4$,

(iii) $c_r > 0$, (iv) $c_i > 0$.

Integration of the perturbation equation (8) is further divided into two regions. First in the region $0 < y < \epsilon$ near the wall, which is influenced by the singularities, and then in the region $\epsilon < y < b$ remote from the singularities where f(y) is likely to be close to the marginally stable solution f = 1. It will prove convenient to choose $\epsilon = k^{3/2}$.

From (8), for all y,

$$f'(y) = \frac{\text{const.}}{(U-c)^2 - A^2} + \frac{k^2 \int^y [(U-c)^2 - A^2] f}{(U-c)^2 - A^2} \tag{19}$$

When $0 < y < \epsilon$, the second term on the right is small compared with the first, which will be justified in retrospect. Therefore,

$$f'(y) = \frac{\text{const.}}{(U-c-A)(U-c+A)}$$

$$= \frac{\text{const.}}{y(U_0'-A_0') + \frac{1}{2}y^2(U_0''-A_0'') - c}$$

$$\cdot \frac{1}{y(U_0'+A_0') + \frac{1}{2}y^2(U_0''+A_0'') - c} \cdot \left[1 + O(y^3)\right],$$

by Taylor expansion in the range $0 < y < \epsilon$ where $\epsilon^3 \ll c_i$ Integrating this expression exactly, and demanding $f(0) = 0$, the following are the important terms of the solution f :-

$$f(y) = 1 - \frac{\log\left[\left(y - \frac{c}{U_0'-A_0'}\right)/\left(y - \frac{c}{U_0'+A_0'}\right)\right]}{\log\left[(U_0'+A_0')/U_0'-A_0'\right]}$$

$$+ \frac{2i\pi c_r A_0'(U_0''U_0' - A_0''A_0')}{(U_0'^2 - A_0'^2)^2 \log\left[(U_0'+A_0')/(U_0'-A_0')\right]}$$

$$+ \frac{1/2cA_0'}{(A_0'U_0'' - A_0''U_0')\log\left[(U_0'+A_0')/(U_0'-A_0')\right]}$$

$$\cdot \left\{\left(\frac{U_0''-A_0''}{U_0'-A_0'}\right)^2 \log\left[\frac{y(U_0'-A_0')}{c} - 1\right]\right.$$

$$\left. - \left(\frac{U_0''+A_0''}{U_0'+A_0'}\right)^2 \log\left[\frac{y(U_0'+A_0')}{c} - 1\right]\right\}. \qquad (20)$$

As c tends to zero f tends non-uniformly to 1, and the marginally stable solution is recovered. Remembering $c_i \ll c_r$, and $c_i > 0$, $c_r > 0$ so that the branch of the logarithmic functions is well defined, let $y \to \epsilon$. Then

$$f(y) = 1 + \frac{2i\pi c_r A_0'(U_0''U_0' - A_0''A_0')}{(U_0'^2 - A_0'^2)^2 \log[(U_0'+A_0')/(U_0'-A_0')]}$$

$$+ O(k^2 \log k) + i\,O(k^4 \log k)$$

provided that $\epsilon \gg c_r$. The choice $\epsilon = k^{3/2}$ makes ϵ small enough for the Taylor series expansion to hold ($\epsilon^3 \ll k^4$) and large enough to escape influence of the singularities ($\epsilon \gg k^2$).

It is already apparent that the neglect of the integral in (19) is justified. In the region $0 < y < \epsilon$, $\mathcal{Re}\{f\}$ is bounded above by a constant multiple of $-\log k^2$ and $\mathcal{Im}\{f\}$ is bounded above by a constant multiple of 1. Returning to (19) with this knowledge, and with the value of the constant of integration

$$f'(y) = \frac{-2A_0'c}{\log[(U_0'+A_0')/(U_0'-A_0')]} \cdot \frac{1}{(U-c)^2 - A^2}$$

$$+ \frac{O[k^2\epsilon(\epsilon^2 + 2i\epsilon c_i)(-\log k^2 + i)]}{(U-c)^2 - A^2}$$

$$= \frac{O(c)}{(U-c)^2 - A^2} + \frac{O(k^6) + i\,O(k^6)}{(U-c)^2 - A^2},$$

so that the second term is indeed negligible.

In the region $\epsilon < y < b$ all terms on the right side of (19) can be ignored and the perturbation equation (8) has the simpler solution

$$f(y) = \text{const} + O(k^2) + iO(k^4),$$

provided the constant is chosen as $O(1) + iO(k^2)$.

By joining the solutions at $y = \epsilon$, and neglecting real terms of order k^2 and imaginary terms of order k^4

a solution valid for $0 < y < b$ is obtained.

$$f(y) = 1 + \frac{2i\pi c_r A_0'(U_0''U_0' - A_0''A_0')}{(U_0'^2 - A_0'^2)^2 \log[(U_0'+A_0')/(U_0'-A_0')]} .$$

The function f derived in the region $0 < y < \epsilon$ and given by (20) is therefore valid for all y in the range $0 < y < b$, to a sufficient approximation. If, further, $f'(b) = 0$ exactly and $f(y) = f(2b - y)$ in the range $b < y < 2b$, f becomes an eigenfunction of the perturbation equation which can now be solved for c.

From (8),

$$\left[((U-c)^2 - A^2)f'\right]_0^b = k^2\int_0^b ((U-c)^2 - A^2) f\,dy$$

$$\therefore \quad -c^2 f'(0) = k^2\left(\int_0^\epsilon + \int_\epsilon^b\right)((U-c)^2 - A^2) f\,dy .$$

Since $\mathcal{Re}\{f\}$ is bounded above by a constant multiple of $-\log k^2$ and $\mathcal{Im}\{f\}$ by a constant multiple of 1 in the region $0 < y < \epsilon$, the first integral on the right behaves at worst like

$$k^2\epsilon(\epsilon^2 + 2i\epsilon c)(-\log k^2 + i) = O(k^6) + i\,O(k^6),$$

while the second integral is of order $k^2 + ik^4$. Thus the integral from 0 to ϵ may be neglected, and the following solution obtained for c :-

$$c_r = \frac{\log[(U_0'+A_0')/(U_0'-A_0')]}{2A_0'} k^2\int_0^b (U^2 - A^2)\,dy + O(k^4)$$

$$c_i = \frac{2\pi A_0'(U_0''U_0' - A_0''A_0')}{(U_0'^2 - A_0'^2)^2 \log[(U_0'+A_0')/(U_0'-A_0')]} c_r^2 + O(k^6) .$$

It remains to check that this solution for c satisfies the initial assumptions (i) - (iv), page 72.

Assumptions (i) - (iii) are obviously satisfied. Also $c_i > 0$ if $U_0''U_0' > A_0''A_0'$, that is if condition (18) holds. Thus the foregoing scheme is self-consistent whenever

$$U_0'' > (A_0'/U_0')A_0'' \tag{18}$$

so that (18) is a sufficient condition for instability.

If it had initially been assumed that $c_i < 0$, then the above expression for c_i would appear with opposite sign, so that (18) would remain unaltered.

CHAPTER 4

VACUUM BOUNDARIES

Boundary conditions - conditions for stability - an example - arbitrary density profile.

1. Boundary conditions

Still studying laminar flows between flat walls, it is possible for the magnetofluid to be kept off the walls by a magnetic field - an effect not at all permissible for non-conducting flows.

A possible equilibrium (figure 13) is a magnetofluid flow (still inviscid, incompressible and perfectly conducting) given by

$$\underset{\sim}{v} = U(y)\hat{\underline{x}} \qquad (-b < y < b)$$

parallel to a magnetic field given by

$$\underset{\sim}{h} = H(y)\hat{\underline{x}} \qquad (-b < y < b)$$

and contained by a vacuum magnetic field

$$\underset{\sim}{h}_v = H_v(y)\hat{\underline{x}} \qquad (B-b < |y| < B)$$

Equilibrium demands that H_V is harmonic and has the same value at both interfaces. The magnetofluid could flow nearer to one wall than the other.

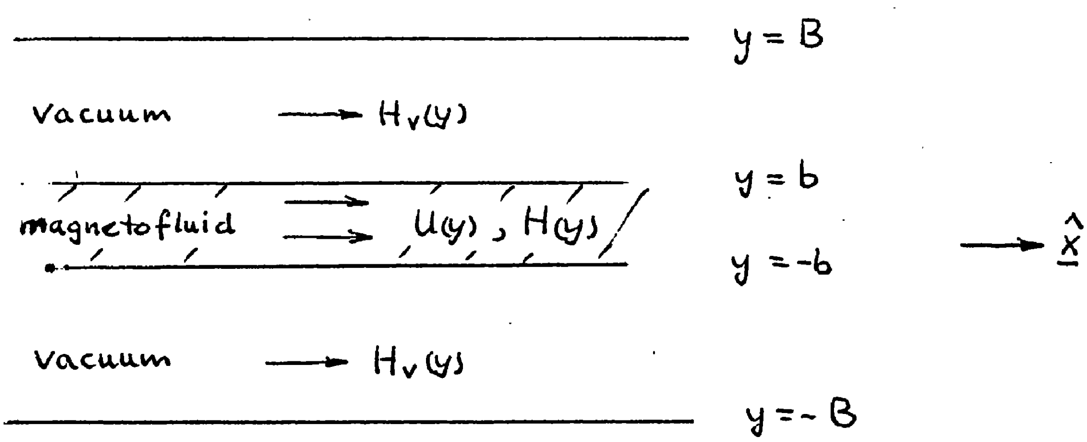


Figure 13

In the magnetofluid region, the perturbed equilibrium is described by the same equations as before. Only the boundary conditions change. The new boundary conditions are derived in this section.

When the equilibrium is perturbed, the vacuum field is easily calculated. It becomes $\underset{\sim}{h}_v = H_v(y)\hat{\underset{\sim}{x}} + h_v^{(1)}$ where

$$\operatorname{curl} \underset{\sim}{h}_v = \operatorname{div} \underset{\sim}{h}_v = 0 .$$

Each component of $\underset{\sim}{h}_v^{(1)}$ is harmonic, so that the y-component $h_{vy}^{(1)}$ can be written

$$h_{vy}^{(1)} = \alpha \sinh k(B-y)\, e^{ik(x-ct)}$$

which satisfies the boundary condition $h_{vy}^{(1)} = 0$ at the wall y = B. Physically, this occurs when the walls are perfectly conducting. Previously, when the magnetofluid touched the walls, it was not necessary to make any assumptions about the electrical properties of the walls. As before, only two dimensional perturbations are studied (m = 0). Allowing m > 0 does not change the boundary conditions, and only trivially alters the perturbation equations (replacing k^2 by $k^2 + m^2$).

At the fluid-vacuum interfaces pressure balance must be maintained. Thus for the equilibrium or the perturbed flow

$$p + \tfrac{1}{2}\mu_0 H^2 = \tfrac{1}{2}\mu_0 H_v^2 \qquad (25)$$

at the interfaces. In particular $H_v > H$ for equilibrium.

Linearising,

$$p^{(1)} + \mu_0 H h_x^{(1)} = \mu_0 H_v h_{vx}^{(1)}$$

using the notation of page 24 for the perturbed quantities.

In terms of complex amplitudes (i.e., coefficients of exp(ik(x - ct)) the pressure balance condition becomes

$$ikp - \mu_0 H h' = -\mu_0 H_v h_v'$$

A second boundary condition is required to eliminate the constant α. In both regions the magnetic field is parallel to the perturbed boundary

$$\therefore \quad \frac{h_{vy}^{(1)}}{H_v} = \frac{h_y^{(1)}}{H}$$

$$\therefore \quad \frac{h_v}{H_v} = \frac{h}{H}.$$

The new magnetofluid boundary condition is therefore

$$ikp - \mu_0 H h' = -\frac{\mu_0 H_v^2 h_v'}{H h_v} \cdot h$$

at each interface, and can be rewritten in terms of any one magnetofluid amplitude according to the perturbation equation being studied.

For example, equation (8) for f(y)

$$\left(\left((U-c)^2-A^2\right)f'\right)' - k^2\left((U-c)^2-A^2\right)f = 0 \qquad (8)$$

must now be solved subject to the boundary conditions

$$\left.\begin{aligned} \left((U-c)^2-A^2\right)f' - \Omega(k)f &= 0 \qquad (y=b) \\ \left((U-c)^2-A^2\right)f' + \Omega(k)f &= 0 \qquad (y=-b) \end{aligned}\right\} \qquad (26)$$

where

$$\Omega(k) = \frac{\mu_0 H_v^2}{\rho} \frac{k}{\tanh k(B-b)}$$

2. Conditions for stability

The necessary conditions for instability (a) - (c), page 29, still hold. Having assumed that a complex eigenvalue c exists ($c_i \neq 0$), any consequence is a necessary condition for instability. Proceeding as for rigid boundaries, when $c_i \neq 0$ f is regular in the interval (-b, b). From equation (8) for f, using the new boundary conditions (26),

$$\int_{-b}^{b}((U-c)^2-A^2)(|f'|^2+k^2|f|^2)\,dy$$
$$=\left[((U-c)^2-A^2)f'f^*\right]_{-b}^{b}$$
$$=\Omega\left(|f(b)|^2+|f(-b)|^2\right) \qquad (27)$$

where
$$\Omega=\frac{\mu_0 H_y^2}{\rho}\frac{k}{\tanh k(B-b)}\ .$$

Taking real and imaginary parts,
$$\int_{-b}^{b}(U-c_r)(|f'|^2+k^2|f|^2)\,dy=0$$
and
$$\int_{-b}^{b}((U-c_r)^2-A^2-c_i^2)(|f'|^2+k^2|f|^2)\,dy>0\ .$$

Since the first integral vanishes, $U - c_r$ must change sign in $(-b, b)$. Thus,

$$c_r = U(y) \text{ somewhere} \qquad (a)$$

placing upper and lower limits on c_r and revealing that unstable disturbances must travel at the flow speed somewhere. These remarks are identical to those made for rigid boundaries on page 29.

Since the second integral is positive $(U - c_r)^2 - c_i^2 - A^2$ must be positive for some values of y. It is also negative for other values of y (since $U - c_r$ has a zero) and hence is zero somewhere. Thus

$$c_i^2 = (U - c_r)^2 - A^2 \text{ somewhere} \qquad (b)$$

placing an upper bound on c_i^2.

In particular if (b) is to be attained

$$A^2(y) < (U_{max} - U_{min})^2 \qquad (c)$$

for some values of y.

Conditions (d) and (e) do not hold under the new boundary conditions. The physical interpretation of (d) in terms of momentum conservation is lost because the magnetofluid now interacts with the vacuum field.

Condition (f) derived on page 32, from Frieman and Rotenberg's general sufficient condition for stability. is strengthened by the introduction of free boundaries. Equation (27) is formally a quadratic equation in c with discriminant

$$\left[\int_{-b}^{b} U(|f'|^2+k^2|f|^2)\,dy\right]^2 - \int_{-b}^{b}(|f'|^2+k^2|f|^2)\,dy \cdot \left[\int_{-b}^{b}(U^2-A^2)(|f'|^2+k^2|f|^2)\,dy - \Omega(|f_b|^2+|f_{-b}|^2)\right].$$

c is real if and only if the discriminant is positive. Real c, and therefore stability, is assured if

$$\int_{-b}^{b}(A^2-U^2)(|f'|^2+k^2|f|^2)\,dy > -\Omega(|f_b|^2+|f_{-b}|^2)$$

In the case of rigid boundaries, the right hand side vanishes. In either case, stability always occurs if

$$A^2(y) > U^2(y) \qquad \text{(f)}$$

for all y.

The stability criterion for long waves ($k^2 = 0$) is slightly altered by the new boundary conditions, which make the system more stable. From equation (8) for $f(y)$, when $k^2 = 0$

$$f(y) = \int_{-b}^{y} \frac{X_{-b} f'_{-b}}{X(y)} dy + f_{-b}$$

$$\therefore \quad f(b) = X_{-b} f'_{-b} \cdot \left(\int_{-b}^{b} \frac{dy}{X} + \frac{f_{-b}}{X_{-b} f'_{-b}} \right).$$

From the boundary conditions (26),

$$X_b f_b' = \Omega f_b$$

where $\Omega = \mu_0 H_v^2/\rho \cdot 1/(B-b)$.

The long wave modes are unstable if and only if $G(c)$ $\left(\equiv f_b - \frac{X_b f_b'}{\Omega}\right)$ has a complex zero ($c_i \neq 0$). $G(c)$ is given by

$$G(c) = f_b - \frac{X_b f_b'}{\Omega}$$

$$= X_{-b} f'_{-b} \left(\int_{-b}^{b} \frac{dy}{X} - \frac{2}{\Omega} \right)$$

The case of rigid boundaries is recovered by letting $\Omega \to \infty$ that is by letting the vacuum field become infinitely large. When Ω is finite, $U'(y) > 0$ and A is constant, zeros of $G(c)$ inside the countour of figure 7, (page 60) occur less readily because the plot of $G(c)$, figure 8, is displaced to the left by the amount $\frac{2}{\Omega}$ and is less likely to encircle the origin.

The sufficient condition for stability given for rigid boundaries (page 63) can now be restated for free boundaries :- Profiles for which $U'(y) > 0$ and A = constant are unstable whenever

$$\int_{-b}^{b} \frac{dy}{(U-c)^2 - A^2} - \frac{2}{\Omega} > 0 \qquad (28)$$

with same definitions and restrictions as before.

3. <u>An example</u>

The case of trivial profiles (A, U, H_V all constant) was studied by Haas and Taylor* (1963) and shown to be stable. It is against intuition that a real flow should be stable for all velocities. The magnetofluid stream resembles a jet and instability might be conjectured for sufficiently high velocity. But the system here is so idealised geometrically that no physical frame of reference exists, and in fact all velocities U are indistinguishable in the stability problem. Haas and Taylor showed that the flow is made unstable, for velocities greater than a certain critical velocity, by a small dent in the rigid walls. The dent has the effect of introducing a frame of reference.

In this section the ideal (stable) problem is repeated and it is shown that a frame of reference is equally well provided by the introduction of a low

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* Their work is unpublished, but this section contains all the relevant details.

density magnetofluid into one of the vacuum regions. Also, it is already apparent from the last section that a slight inhomogeneity in the velocity U(y) also causes instability provided condition (28) is satisfied.

For the ideal problem with two vacuum regions equation (8) for f(y) becomes

$$f'' - k^2 f = 0 \qquad (\text{or } U - c = \pm A)$$

which has general solution $\alpha \sinh ky + \beta \cosh ky$. The boundary conditions (26) then yield four real eigenvalues given by

$$\begin{aligned} X &= \Omega_1 \\ X &= \Omega_2 \end{aligned} \qquad (29)$$

where

$$\Omega_1 = \frac{\mu_0 H_V^2}{\rho} \frac{\tanh kb}{\tanh k(\beta - b)}$$

$$\Omega_2 = \frac{\mu_0 H_V^2}{\rho} \coth k(\beta - b) \coth kb .$$

Since all six eigenvalues are real, the system is stable, and this is Haas and Taylor's result.

It is not to be expected that small changes in the system might cause instability because none of the four modes represented by $X = \Omega_1$, $X = \Omega_2$ is marginally stable (definition, page 18). This is shown from equation (27)

$$\int_{-b}^{b} X(|f'|^2 + k^2|f|^2)\,dy = \Omega(|f_b|^2 + |f_{-b}|^2)$$

which is formally a quadratic equation for c with discriminant

$$\Delta = \left[\int_{-b}^{b} U(|f'|^2+k^2|f|^2)dy\right]^2 - \int_{-b}^{b}(|f'|^2+k^2|f|^2)dy \cdot \left[\int_{-b}^{b}(U^2-A^2)(|f'|^2+k^2|f|^2)dy - \Omega(|f_{-b}|^2+|f_b|^2)\right].$$

For example, when $f = \alpha \sinh ky$, which corresponds to the real mode $X = \Omega_1$

$$\Delta \propto A^2+\Omega_1 \neq 0.$$

Thus the modes are not marginally stable, and no complex normal modes exist nearby. It will be shown, however, that linear instability may be caused by a slight change in the equilibrium.

If a magnetofluid with density ρ_1 and velocity U_1 is introduced into one of the vacuum regions equation (8) for f(y) has to be solved in the two regions $-b < y < b$ and $b < y < B$.

In the outer magnetofluid region $b < y < B$,

$$f'' - k^2 f = 0 \qquad (\text{or } U_1 - c = \pm A_1)$$

and from the boundary condition $(U_1 - c) f(B) = 0$ the solution is $f = \gamma \sinh k(y - B)$, where γ is a constant.

In the inner magnetofluid region $-b < y < b$

$$f'' - k^2 f = 0 \qquad (\text{or } U - c = \pm A)$$

$$f = \alpha \sinh ky + \beta \cosh ky$$

where α and β are constants. The three constants α, β and γ can be eliminated by means of three boundary conditions. At the magnetofluid interface where $y = b$ the pressure $(\rho X f')$ is continuous and the displacement (f) is continuous. At the vacuum interface where $y = -b$

there is pressure balance as described by (26). Thus,

$$\left.\begin{aligned} X\alpha\cosh kb + X\beta\sinh kb - X_1\gamma\frac{\rho_1}{\rho_2}\cosh k(B-b) &= 0 \\ \alpha\sinh kb + \beta\cosh kb + \gamma\sinh k(B-b) &= 0 \\ (X-\Omega_1)\alpha - \tanh kb(X-\Omega_2)\beta &= 0 \end{aligned}\right\} \quad (30)$$

where $X_1 = (U_1 - c)^2 - A_1{}^2$.

The eliminant of these equations is

$$\begin{vmatrix} X\cosh kb & X\sinh kb & -\frac{\rho_1}{\rho}X_1 \\ \sinh kb & \cosh kb & \tanh k(B-b) \\ X-\Omega_1 & -\tanh kb(X-\Omega_2) & 0 \end{vmatrix} = 0$$

When $\rho_1 = 0$ the problem with two vacuum regions is recovered and the eliminant becomes

$$2\tanh k(B-b)\sinh kb.(X-\Omega_1)(X-\Omega_2) = 0$$

which yields the four real eigenvalues of the ideal problem.

The effect of increasing ρ_1 from zero on the $X = \Omega_1$ mode may be deduced by differentiating the full eliminant ($\rho_1 \neq 0$) with respect to ρ_1. Assuming that the magnetic field H_V does not change with ρ_1, that is that the magnetofluid is introduced with zero pressure, the result is,

$$\left.\frac{\partial X}{\partial \rho_1}\right|_{\rho_1=0} = -\frac{\Omega_1(U_1-c)^2}{2\mu_0 H_v^2} \leq 0 .$$

Changes in X, and therefore in c, are real unless $U_1 = c$ for the stable mode. That is,

$$(U - U_1)^2 = A^2 + \Omega_1$$

in which case all derivatives of X with respect to ρ_1 vanish and the mode persists unchanged for all values of ρ_1.

But completely new modes arise and the system will be shown to possess a linear instability. It is necessary to return to equation (30) where X and X_1 are now operators ($X = (U + \frac{1}{ik}\frac{\partial}{\partial t})^2 - A^2$) and are time dependent.* Solving equation (30) for $\alpha(t)$...

$$\left[2(X-\Omega_1)(X-\Omega_2) + \frac{\rho_1}{\rho A^2}\left(\frac{\partial}{\partial t}+ikU_1\right)^2\left((X-\Omega_1)\Omega_2 + (X-\Omega_2)\Omega_1\right)\right]\alpha(t) = 0 .$$

Linear instabilities occur when the fourth order equation in $\frac{\partial}{\partial t}$ has a double root. Rather than study this problem, extra symmetry is caused by introducing a magnetofluid of density ρ_1 in both vacuum regions. Then equations (30) apply with $\beta = 0$ and the equation for $\alpha(t)$ becomes

$$\left[\frac{\partial^2}{\partial t^2} + \frac{2ik(\rho_1\Omega_1 U_1 + \rho A^2 U)}{\rho_1\Omega_1 + \rho A^2}\frac{\partial}{\partial t} + \frac{k^2(\rho(A^2+\Omega) - \rho_1\Omega_1 U_1^2 - \rho A^2 U^2)}{\rho_1\Omega_1 + \rho A^2}\right]\alpha(t) = 0$$

which predicts a linear instability when

$$(U-U_1)^2 = \left(1 + \frac{\rho}{\rho_1}\frac{A^2}{\Omega_1}\right)(A^2+\Omega_1)$$

Thus, when ρ_1 is small, high relative velocities

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* α, β, γ have lost their exponential time dependence. This situation is dealt with fully in Chapter 5.

$U - U_1$ lead to linear instability for some value of $\Omega_1(k)$. Similar modes arise from equations (30) with $\alpha = 0$.

4. Arbitrary density profile

The effect of the magnetic field in keeping the magnetofluid clear of the rigid walls is a special case of the more general effect that magnetic fields can support an equilibrium pressure gradient. The pressure gradient may well be associated with a density gradient, especially in the case of a conducting gas.

Assuming that $\rho(y)$ is non constant, but that the fluid is incompressible ($\frac{d\rho}{dt} = 0$) the equation (8) for $f(y)$ can be rewritten in the form

$$(\rho X f')' - k^2 \rho X f = 0 \qquad (33)$$

Clearly, replacing ρX by X equations (8) and (33) are formally identical. Assuming $\rho(y) > 0$, the magnetofluid makes contact with the rigid walls, and the old boundary conditions (page 27) apply.

The necessary conditions (a), (b) and (c), page 29, for instability still hold without alteration since

$$\int_{y_1}^{y_2} \rho\left((U-c)^2 - A^2\right)\left(|f'|^2 + k^2|f|^2\right) dy = 0 .$$

The necessary conditions (d) and (e) for instability hold if X is understood to mean ρX. Thus, from (d)

$$\mathrm{Im}\left\{\frac{2\rho X(\rho X)'' - (\rho X)'^2}{4(\rho X)^2}\right\} \quad \text{has a zero}$$

and from (e)

$$k^2 + \mathcal{R}e\left\{\frac{2\rho X(\rho X)'' - (\rho X)'^2}{4(\rho X)^2}\right\} < 0$$

for some values of y.

Condition (f) also holds unchanged because the discriminant Δ is ensured positive whenever

$$\int_{y_1}^{y_2} \rho\,(A^2 - U^2)(|f'|^2 + k^2|f|^2)\,dy > 0$$

and hence whenever $A^2(y) > U^2(y)$ for all y.

The conditions (12) on the existence of real eigenvalues become

$$\rho_0 X_0 = (\rho_0 X_0)' = (\rho_0 X_0)''' = 0 \quad ; \; c = U_0$$

that is,

$$U_0 - c = A_0 = 0$$

$$\rho_0'(U_0'^2 - A_0'^2) + U_0'' U_0' - A_0'' A_0' = 0$$

As the magnetic field vanishes ($A \to 0$) the second condition reduces to $\rho_0' U_0' + U_0'' = 0$ in which the variable density has caused a modification to Rayleigh's result $U_0'' = 0$. The modification is trivial since $\rho(y)$ would normally tend to a constant value as $A \to 0$, and therefore $\rho'(y) \equiv 0$.

The conclusion is that arbitrary continuous density profiles are simple to deal with and make no great modification to the stability problem. In the remaining sections only discontinuous density profiles with vacuum boundaries are considered.

CHAPTER 5

NON EXPONENTIAL TIME VARIATION

Inhomogeneous perturbation equations - rigid magnetofluid boundaries - free magnetofluid boundaries

1. Inhomogeneous perturbation equations

After derivation of the linearised equations (5), page 25, the assumption of exponential time dependence was made. It is easy to avoid this assupmption and study at least the asymptotic time dependence of non exponential modes. A slightly more general definition of stability is required. An initial perturbation is specified at $t = 0$. Instability is defined to occur whenever the subsequent perturbation is unbounded as $t \to \infty$.

When the initial values are specified the equations (5) can be laplace transformed.

$$\begin{aligned} \left(U + \frac{p}{ik}\right)\nabla^2 v_p - U'' v_p &= A\nabla^2 a_p - A'' a_p + \frac{1}{ik}\nabla^2 v_y\Big|_{t=0} \\ \left(U + \frac{p}{ik}\right) a_p &= A v_p + \frac{1}{ik}\, a_y\Big|_{t=0} \end{aligned} \qquad (35)$$

where

$$v_p(y,k,m) = \int_0^\infty \bar{v}_y(y,t;k,m)\, e^{-pt}\, dt$$

$$a_p(y,k,m) = \int_0^\infty \bar{a}_y(y,t;k,m)\, e^{-pt}\, dt \ .$$

and

$$\nabla^2 = \frac{\partial^2}{\partial y^2} - (k^2 + m^2)$$

The existence of the integrals is assured when $Re\{p\} > 0$

Eliminating either v_p or a_p, an inhomogeneous second order linear differential equation is obtained. For convenience both v_p and a_p are replaced by f_p, the laplace transform of the function $f_y(y, t; k, m)$ defined by $\frac{df_y}{dt} = v_y$. It follows that

$$(U + p/ik) f_p - \frac{1}{ik} f_y \Big|_{t=0} = v_p \ .$$

When $U + p/ik \neq 0$, f_p can safely be used instead of v_p, but when $U + p/ik = 0$, it will be necessary to return to v_p before making any stability decisions.

The equation for f_p is

$$(X f_p')' - k^2 X f_p = \frac{1}{ik}\left[(2U + p/ik + \frac{1}{ik}\frac{\partial}{\partial t}) \nabla^2 f_y + 2U' f_y'\right]_{t=0} \qquad (36)$$

where $X = (U + p/ik)^2 - A^2$.

If p/ik is replaced by $-c$, the homogeneous equation corresponding to (36) is just equation (8) for the complex amplitude f, obtained on the assumption of exponential time dependence. The laplace transform technique is therefore complete in that exponential time modes arise from the homogeneous equation, and all other modes from the inhomogeneous equation. The inhomogeneous equation can be solved, for given boundary conditions, in terms of the relevant Green's Function, which exists whenever the homogeneous equation has no eigensolution. (Of course, if the homogeneous equation does have an eigensolution the corresponding exponential mode can be examined, but to solve the non-homogeneous equation then requires more general treatment.)

If equation (36) is written in the form

$$L(f_p) = \beta$$

where β is a function of p and y, then solutions $f_p(y)$ can be expressed in terms of the Green's Function $G(y,\eta)$. This arises from the Green's Identity for any two solutions $u(\eta)$, $v(\eta)$ of the self adjoint equation $L(f) = 0$,

$$\int_{y_1}^{y_2} (u\,L(v) - v\,L(u))\,d\eta = \Big[X[uv' - vu']\Big]_{y_1}^{y_2}$$

Replacing $v(\eta)$ by $G(y,\eta)$ and $u(\eta)$ by $f_p(\eta)$ and choosing G to satisfy the conditions

$$L(G) = \delta(y-\eta)$$

$G(\eta)$ continuous when $\eta = y$

$G'(\eta)$ discontinuity $1/X(y)$ when $\eta = y$,

it follows that f_p is given by

$$f_p(y) = \int_{y_1}^{y_2} G(y,\eta)\,\beta(\eta)\,d\eta + \Big[X\Big(f_p(\eta)\frac{\partial G}{\partial \eta} - G\,\frac{\partial f_p}{\partial \eta}\Big)\Big]_{y_1}^{y_2}$$

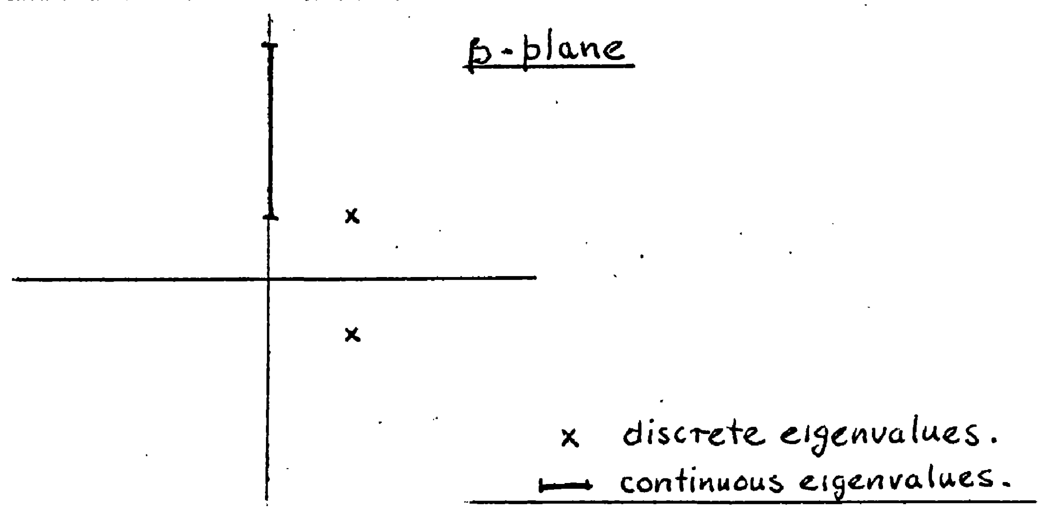
The boundary conditions for G are chosen such as to make $f_p(y)$ satisfy its boundary conditions. If a boundary condition of f_p is of the general form

$$\alpha f_p + \beta f_p' = \gamma$$

then the correct condition on G is

$$\alpha G + \beta G' = 0$$

For fixed p, the singularities of the integrand as η varies can be deduced from the differential equation. These cause similar singularities of f_p, as p varies with y fixed. Knowledge of the singularities of f_p in the complex plane of p (figure 14) allows immediate inversion of the laplace transform to find the asymptotic time dependence of $v_y(y, t, k, m)$.



Singularities of f_p occur where $p = ik(U \pm A)$, hence generating a line as y varies. Discrete singularities, not y-dependent, occur at values of p for which the homogeneous equation has a solution.

Figure 14

Although v_y is the fourier transform of the physical perturbation $v_y^{(1)}(x, y, z, t)$, the stability problem ends at this point, since $Re\{v_y e^{ikx}\}$ is itself a possible physical disturbance. The flow is unstable whenever

$v_y(t)$ is unbounded as $t \to \infty$ for any single initially prescribed perturbation.

It is interesting to note that the non-exponential modes include localised disturbances, since the laplace transform is inverted for a fixed value of y. The exponential time modes had the same time variation in all layers of the fluid, and so exhibited growth (or decay) of the initial perturbation in all parts of the flow. A further distinction is that the exponential modes are discrete while the non exponential modes are continuous. The continuous spectrum of eigenvalues is absent for viscous fluids (Lin (1961)).

In the remaining sections of this chapter, asymptotic time dependences are obtained for various systems.

2. Rigid magnetofluid boundaries

This section shows, in the case of laminar magnetofluid flows in contact with the walls, that the non exponential modes are usually stable. The physical boundary condition is that the transverse velocity perturbation v_y (x, y, z, t) must vanish at the walls. Since $f_y^{(1)}$ is defined by the equation $\frac{df_y^{(1)}}{dt} = V_y^{(1)}$ it follows that $f_p(y, k, m)$ must satisfy the condition

$$(U + p/ik) f_p = \frac{1}{ik} f_y |_{t=0} = 0$$

when $y = y_1$ and $y = y_2$. Equation (36) for f_p has to be solved subject to these boundary conditions.

The first step is to construct the Green's function for the problem. Suppose $\psi_1(y)$, $\psi_2(y)$ are linearly independent solutions of the homogeneous form of (36) such that ψ_1 satisfies the boundary condition at $y = y_1$ and ψ_2 satisfies the condition at $y = y_2$. Such functions exist, and are linearly independent whenever no eigensolution of the homogeneous problem exists. The defining eqations of the Green's function are satisfied by

$$G(y, \eta) = \psi_1(y_<)\, \psi_2(y_>)$$

where $y_<$ means the lesser of y, η, and $y_>$ the greater. As shown on page 93,

$$f_p(y) = \int_{y_1}^{y_2} G(y, \eta)\, \beta(\eta)\, d\eta + \left[\left(f_p(\eta) \frac{\partial G}{\partial \eta} - G \frac{\partial f_p}{\partial \eta}\right)\right]_{y_1}^{y_2}$$

$$= \int_{y_1}^{y_2} \psi_1(y_<)\, \psi_2(y_>)\, \beta(\eta)\, d\eta \, .$$

In this case the boundary term vanishes. All the functions in the integrand depend on p although the notation does not display this. Information about the initial value of the perturbation is contained in β. (β appears on the right hand side of equation (36), page 92, but its precise form is unimportant).

At first, to carry out the integration, let p be fixed with the value p_0. Singularities in the ψ's occur only at singularities of the differential equation, that is at points η_0 where $X(\eta_0) = 0$ and the zero is of order $n \geqslant 1$. As before, X is defined by

$$X = (U + p/ik)^2 - A^2.$$

Keeping y fixed, the worst possible behaviour of the integrand is arranged by choosing $\beta(\eta) = \delta(y-\eta)$ so that

$$f_p(y) = \psi_1(y_<)\,\psi_2(y_>)$$

where $y_<$ now means the lesser of y and η_0, and $y_>$ is the greater. The delta function initial perturbation does not imply that any physical quantities are infinite or even localised at the point $y = \eta_0$. A smooth function with a finite discontinuity in f_y' would suffice.

The behaviour of the solutions ψ near η_0 has already been found. In the case $n = 1$ where $X(\eta_0) = 0$ but $X'(\eta_0) \neq 0$, the solutions for f_1, f_2 on page 39 show that the highest possible singularity of the ψ's is $\log(\eta - \eta_0)$. Choosing $y = \eta_0$ to obtain the highest order singularity in f_p it follows that

$$f_p \sim [\log(p - p_0)]^2$$

for values of p near p_0.

Inverting the laplace transform

$$f_y(t) \sim \log t/t$$

and this is the asymptotic time dependence. The system is therefore stable.

In the case where $n \geqslant 2$ the behaviour of the ψ's is at worst $\sim 1/(\eta-\eta_0)^{n-1}$ so that

$$f_p \sim 1/(p-p_0)^{2n-2}$$

$$\therefore\ V_p \sim (U(\eta_0)+p/ik)/(p-p_0)^{2n-2}.$$

The nature of the singularity depends on the cause of the zero of $X'(\eta_0)$.

For, $$(U(\eta_0)+p/ik)^2 - A^2 = 0$$

$$\therefore\ U(\eta_0)+p/ik = \pm A$$

$$\therefore\ X'(\eta_0) = 2(U(\eta_0)+p/ik)(U'(\eta_0) \pm A'(\eta_0)).$$

The zero in X' can be due to a zero of $U + p/ik$ (and therefore to a zero in A — a null point in the magnetic field) or to a zero of $U' \pm A'$. In the former case

$$V_p \sim 1/(p-p_0)^{2n-3}$$

$$\therefore\ V_y(t) \sim \begin{cases} e^{p_0 t} & (n=2) \quad \ldots\ p_0 = -ik\,U(\eta_0) \\ t^{2(n-2)} & (n>2) \end{cases}$$

implying stability when $n = 2$ but instability when $n > 2$, while in the latter case

$$V_p \sim A(\eta_0)/(p-p_0)^{2n-2}$$

$$\therefore\ V_y(t) \sim t^{2n-3}$$

implying instability for all $n \geqslant 2$.

The conclusion for the present system is that the non exponential modes are usually stable. But flow profiles may be unstable if they satisfy certain restrictions, the simplest of which is

$$U'(y) \pm A'(y) = 0$$

for any value of y. The instability grows fastest near the point where the restriction holds.

3. Free magnetofluid boundaries

When the magnetofluid is kept off the walls by a vacuum magnetic field new non-exponential modes arise but normally no difference is made to the stability decision. The non-exponential modes are usually stable.

The boundary conditions were derived for the case of exponential time dependence and appear on page 80 Replacing $-c$ by $\frac{1}{ik}\frac{\partial}{\partial t}$,

$$\left(\left(U + \frac{1}{ik}\frac{\partial}{\partial t}\right)^2 - A^2\right) f_y' \pm \Omega(k) f_y = 0$$

where the minus sign holds at the magnetofluid-vacuum interface y = b and the plus sign at the interface y = -b. As before, the walls are situated at $y = \pm B$.

By laplace transform the boundary conditions on f_p are

$$X f_p' \pm \Omega(k) f_p = \frac{1}{ik}\left(2U + p/ik + \frac{1}{ik}\frac{\partial}{\partial t}\right) f_y \Big|_{t=0} \qquad (37)$$

where $X = (U + p/ik)^2 - A^2$; $\Omega(k) = \frac{\mu_0 H_v^2}{\rho} \frac{k}{\tanh k(B-b)}$.

The Green's function $G(y, \eta)$ is constructed as in the previous section, but the functions ψ_1, ψ_2 satisfy the new boundary conditions (in homogeneous form).

The solution for f_p is

$$f_p(y) = \int_{-b}^{b} \psi_1(y_<)\psi_2(y_>)\beta(\eta)d\eta + \frac{\psi_2(b)}{ik\Omega(k)}\left(2U(b)+p/ik+\frac{1}{ik}\frac{\partial}{\partial t}\right)f_y(b)\Big|_{t=0}\cdot\psi_1(y)$$

$$+\frac{\psi_1(-b)}{ik\Omega(k)}\left(2U(-b)+p/ik+\frac{1}{ik}\frac{\partial}{\partial t}\right)f_y(-b)\Big|_{t=0}\cdot\psi_2(y).$$

Since the behaviour of the integral has been discussed (last section) it remains only to examine the boundary terms. Singularities arise only in the functions ψ_1, ψ_2 and therefore only at singularities of the differential equation. The resulting time dependence is similar to that arising from the integral, but there is no need to introduce delta function perturbations. For example, choosing y = b, X(b) = 0, X'(b) ≠ 0 (n = 1)

$$f_p \sim [\log(p-p_0)]^2$$

$$f_y(t) \sim \log t/t$$

and the system is stable.

When n > 1 with A(b) = 0

$$V_y(t) \sim \begin{cases} e^{p_0 t} & (n=2) \\ t^{2(n-2)} & (n>2) \end{cases} \qquad \ldots\ p_0 = -ikU(b)$$

implying stability when n = 2 and instability when n > 2, but with U'(b) ± A'(b) = 0

$$V_y(t) \sim t^{2n-3}$$

implying instability for all $n \geqslant 2$.

The growth of instability is normally independent of the initial perturbation but (in the simplest case) depends on $U' \pm A'$ having a zero at the magnetofluid-vacuum interface. The instability is localised in the region of the interface.

It appears that trivial flow profiles ($U(y)$, $H(y)$, $H_V(y)$ all constant) may be unstable, since $U' \pm A' = 0$ everywhere. But further examination is necessary because $X'' \equiv 0$, and shows complete stability. The inhomogeneous perturbation equation reduces to

$$X(f_p'' - k^2 f_p) = \frac{1}{ik}\left(2U + p/ik + \frac{1}{ik}\frac{\partial}{\partial t}\right)\nabla^2 f_y\Big|_{t=0}$$

subject to boundary conditions

$$X f_p' \pm \Omega(k) f_p = \frac{1}{ik}\left(2U + p/ik + \frac{1}{ik}\frac{\partial}{\partial t}\right) f_y\Big|_{t=0}$$

when $y = \mp b$.

The Green's function is $G(y, \eta) = \psi_1(y_<)\psi_2(y_>)$ where

$$\psi_1 = \alpha_1 \sinh ky + \beta_1 \cosh ky$$

$$\psi_2 = \alpha_2 \sinh ky + \beta_2 \cosh ky$$

and the α's and β's are chosen to satisfy the boundary conditions. The solution f_p is given by

$$f_p(y) = \frac{1}{X(p)}\int_{-b}^{b} \psi_1(y_<)\psi_2(y_>)\beta(\eta)\,d\eta$$

$$+ \frac{2U + p/ik + \frac{1}{ik}\frac{\partial}{\partial t}}{ik\,\Omega(k)}\left[f_y(b)\psi_2(b)\psi_1(y) + f_y(-b)\psi_1(-b)\psi_2(y)\right].$$

No singularities arise from the ψ's (because the differential equation now has no singularities) so that singularities of f_p arise only from X(p)

$$\therefore f_p \sim 1/(p-p_0) \quad \text{where} \quad U + p_0/ik = \pm A \neq 0.$$

for values of p near p_0. Thus, $V_y(t) \sim e^{p_0 t}$, indicating stability whenever $A \neq 0$.

If $A \equiv 0$,

$$f_p \sim 1/(p-p_0)^2 \quad \text{where} \quad U + p/ik = 0$$

$$\therefore \; V_p \sim 1/(p-p_0)$$

$$\therefore \; V_y(t) \sim e^{p_0 t}$$

again indicating stability.

APPENDIX

CLASSIFICATION OF PROFILES A(y), U(y)

STABLE	UNSTABLE
$A^2(y) > U^2(y)$ p.32 $A(y) > U_{max} - U_{min}$ p.29	$U'(y) > 0$ $A = \text{const.}$ and $M(A) > 0$ p.63
$A \equiv 0$ $U''(y) \neq 0$ p.3	U(y) and A(y) symmetric, $U''U' - A''A' > 0$ at walls p.67
UNSTABLE $A \equiv 0,\ U'(y) > 0$ $R > 0$ p.11	$U'(y) \pm A'(y)$ has a zero p.99

The above table summarises the available results classifying laminar flows according to their velocity and magnetic profiles. Classification is far from complete.

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