Donaldson–Thomas Invariants of Threefold Flops

by

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Abstract

This thesis is about a class of complex algebraic threefolds known as *flops*, which are an important part of the Minimal Model Program in birational geometry. Threefold flops are commonly studied via their enumerative invariants, and here we focus on one such type of invariant: *refined Donaldson–Thomas invariants.* We develop theoretical aspects of refined Donaldson–Thomas theory for threefold flops, which allow us to understand their *stability conditions* and *cyclic* A_∞*-deformation theory.* With these new methods, we are able to sidestep common computational barriers in the field and fully determine the Donaldson–Thomas invariants for an infinite family of flops, which includes many new examples. Our results show that a refined version of the strong-rationality conjecture of Pandharipande–Thomas holds in this setting, and also that refined Donaldson–Thomas invariants are not sufficiently fine to determine flops. Where possible we work motivically, computing invariants in the Grothendieck ring of varieties, but we also produce Hodge theoretic realisations of the invariants.

Contents

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Finally, I want to thank my parents for their unending love & support, for encouraging me during all my studies, and for inspiring me to be both creative and analytical.

Author’s declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

Although the field of algebraic geometry has undergone many changes over the years, its main aim is still to understand *algebraic varieties*: spaces that are (locally) defined by polynomial equations. One of the most effective ways to unravel the geometry of an algebraic variety is to calculate an *invariant*: a quantity or structure that represents some intrinsic aspect of the geometry of the space. Invariants provide a very concrete picture of the geometry of a variety, and can be used to study how this geometry behaves under deformations, birational transformations, and other such operations.

In this thesis we study *Donaldson–Thomas invariants*, which are *virtual counts* of objects in the *derived category* of a complex variety of dimension 3. Classically, these counts come in the form of a number, and can be calculated as a weighted Euler characteristic on the *moduli space* of certain classes of objects. In recent years there has been a great development in the *refinement* of these quantities: instead of a numerical count, one can obtain a more interesting object that encodes new information about the geometry of the moduli space. However, the road towards refinement is also full of new hurdles, and one of the aims of this thesis is to overcome these.

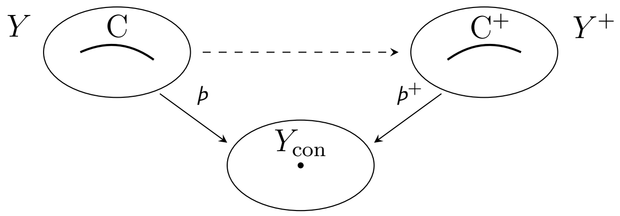
We will focus our attention on *flopping curves*, a type of curve in a threefold that has an associated birational transformation–a flop–which is crucial for the classification of minimal models in the *Minimal Model Program* as developed in the '80s and '90s. In the current millennium, flops are predominantly studied via *noncommutative* and *categorical* methods, and it has been discovered that the flopping behaviour can be nicely captured as a symmetry of the derived category. We will show how this categorical point of view is compatible with the DT theory of flops, and how the derived symmetries can be used to greatly simplify the calculation of invariants. In the process we develop several novel techniques, which we will explain below. These techniques allow us to completely determine the DT theory of a large class of flops, most of which are new and will be constructed explicitly. Our results show the depth which the refinement brings to DT theory and provide important evidence towards a conjecture on the relation between the different invariants found in this context.

§ 1.1 | Background and motivation

§ 1.1.1 | Threefold flops

Perhaps one of the greatest achievements of 20th century algebraic geometry is the development of the *minimal model program* (MMP), first for surfaces by Castelnuovo and Enriques [Enr49], and later for threefolds by Mori, Kollar, Reid [Rei83; Mor88; KoM98], among many others. The aim of this program is to classify birational equivalence classes of varieties by finding a suitable representative in each class. Such a representative is given by a *minimal model*, a variety Y for which the canonical bundle has nonnegative degree $K_Y \cdot \mathrm{C} \geq 0$ on all embedded rational curves $\mathrm{C} \subset Y$. Minimal models are known to exist for smooth surfaces, and also for threefolds if one allows them to have mild singularities.

A big difference between dimensions 2 and dimension 3 (and higher) is that the minimal models of threefolds are not unique. Instead, minimal models are connected by *flops*: if $\mathrm{C} \subset Y$ is a suitable curve with $K_Y \cdot \mathrm{C} = 0$, then there exists a second minimal model Y^+ and a diagram of birational maps as illustrated in the following cartoon:



The flop $Y \dashrightarrow Y^+$ is the composition of the *flopping contraction* $\mathfrak{p}$, which contracts C to a singular point $p \in Y_{\mathrm{con}}$, followed by a resolution to a second curve $\mathrm{C}^+ \subset Y^+$. One can also flop back: the curve C^+ also satisfies $K_{Y^+} \cdot \mathrm{C}^+ = 0$, and the flop $Y^+ \dashrightarrow Y$ along C^+ is inverse to $Y \dashrightarrow Y^+$. Hence, "flopping" is a symmetric operation, and this symmetry makes flops very rich objects of study.

In this thesis we restrict ourselves to studying flops of *smooth* varieties Y, and in particular focus on the class of *simple* flops, in which the flopping curve $\mathrm{C} \subset Y$ is an irreducible rational curve $\mathrm{C} \simeq \mathbb{P}^1$. Because flops are a local phenomenon, we will moreover focus on local (and sometimes even formal) neighbourhoods of flopping curves and of the singularities to which they contract. Because the flopping behaviour is intrinsic to the geometry of Y (for a fixed C), we will henceforth also refer to the space Y itself as "the flop", and hope that the reader will forgive us this abuse of terminology.

§ 1.1.2 | Invariants of flops

Various invariants have been constructed to categorise (simple) flops, including the normal bundle invariants [Lau81], the ADE-type [Kaw94], the length invariant [Kol89], the width invariant [Rei83], and Gopakumar-Vafa invariants [Kat06].

The scheme-theoretic fibre of a flopping contraction $p\colon Y \to Y_{\mathrm{con}}$ is in general not reduced, and its defining ideal sheaf has a *length* ℓ: there is a chain of inclusions

$$\mathrm{C} \subset 2\mathrm{C} \subset \ldots \subset \ell\mathrm{C},$$

of thickened curves $i\mathrm{C}$ ranging between $\mathrm{C} \simeq \mathbb{P}^1$ and the fibre $\ell\mathrm{C}$. Hence, the length is a rough measure of the complexity of the exceptional locus: for $\ell = 1$ the fibre is simply a smooth curve $\ell\mathrm{C} = \mathrm{C} = \mathbb{P}^1$, while the cases $\ell \geq 2$ have more complicated geometry.

The flops of length $\ell = 1$ were completely classified by Reid [Rei83] and form a family in a single integer parameter known as the *width*. The width has an intrinsic interpretation as a multiplicity associated to the class of the curve C in the appropriate Hilbert scheme of Y. For higher length flops one can associate such multiplicities $n_i \in \mathbb{Z}$ to each thickened curve $i\mathrm{C} \subset Y$. These invariants are known as (genus 0) *Gopakumar-Vafa* (GV) invariants and derive from Gromov-Witten curve counts [BKC99]. Motivated by the nice classification in the length $\ell = 1$ case, one might hope that the ℓ–tuples

$$(n_1, n_2, \ldots, n_\ell) \in \mathbb{Z}^\ell$$

completely classify the higher length flops, but a recent counter-example of Brown–Wemyss [BW17] shows that this is *false*. Hence, GV invariants are insufficient to completely capture the geometry of threefold flops, and a more intricate theory is needed to fully understand them.

§ 1.1.3 | Donaldson–Thomas invariants

Another way of looking at Gopakumar-Vafa invariants is as numerical *Donaldson–Thomas* invariants: Katz showed [Kat06] they correspond to virtual counts of sheaves with K-theory class $[i\mathrm{C}] := [\mathcal{O}_{i\mathrm{C}}]$. If one wants to understand higher length flops, the logical next step is look for new invariants in the framework of Donaldson–Thomas theory, developed by Joyce–Song [JS08], Kontsevich–Soibelman [KS08], and others.

The rough setup for this theory is as follows. We consider the category $\mathrm{D}^b_{\mathrm{C}}(Y)$ of objects in the derived category of Y with (set-theoretic) support on the curve $\mathrm{C} \subset Y$. The K-theory class of such an object can be expressed as a pair (r, χ) of a rank and Euler-characteristic, and there is a moduli space parametrising objects with that rank/Euler characteristic. By integrating a certain measure on this moduli space space, one can define a "virtual count" $\mathrm{DT}^{\mathrm{num}}_{(r,\chi)} \in \mathbb{Q}$ of its points, yielding a *partition function*

$$\Phi^{\mathrm{num}}(t) := \sum_{(r,\chi)} \mathrm{DT}^{\mathrm{num}}_{(r,\chi)} \cdot t^{(r,\chi)}.$$

The K-theory classes $[i\mathrm{C}]$ correspond to the rank/Euler pairs $(i, 1)$, and the GV invariants can be recovered from the coefficients $\mathrm{DT}_{(i,1)}$. However, the partition function can express more information, and one can potentially determine new invariants by analysing this power series.

Not only are there *more* invariants to be found in Donaldson–Thomas theory, the invariants can also be *refined*: whereas the GV invariants are integers, one can practice DT theory in a richer ring of invariants, which allows one to better express the geometry of the moduli spaces and of the flop itself. At the cutting edge of modern DT theory are the *motivic* refinement, and refinement in *monodromic mixed Hodge structures*, each of which is captured by their respective refined partition function

$$\Phi(t) := \sum_{(r,\chi)} \mathrm{DT}_{(r,\chi)} \cdot t^{(r,\chi)}, \quad \Phi^{\mathrm{mmhs}}(t) := \sum_{(r,\chi)} \mathrm{DT}^{\mathrm{mmhs}}_{(r,\chi)} \cdot t^{(r,\chi)}.$$

These partition function have only been calculated for a limited number of flops: the conifold/Atiyah flop [MMNS12] and the Pagoda flops [DM17], which together form the class of length $\ell = 1$ flops. The purpose of this thesis is to develop the Donaldson–Thomas theory of higher length flops, and to determine the degree to which this theory improves on the GV invariants. This motivates us to investigate two questions.

The first question is about the complexity of the partition functions: although $\Phi(t)$ is a power series with infinitely many coefficients, one expects these coefficients to be related by multiple-cover formulas and wall-crossing relations.

Question 1. On how much data do the (refined) partition functions depend?

In the numerical case, an expected answer is provided by the *strong rationality conjecture* of Pandariphande–Thomas [PT09], which is known to hold in certain cases by work of Toda [Tod14]. Roughly, this conjecture proposes that the partition function $\Phi^{\mathrm{num}}(t)$ can be expressed as a multiple-cover formula involving *BPS invariants* $\mathrm{BPS}^{\mathrm{num}}_{(r,\chi)}$ which are independent of χ. This conjecture would therefore imply that the DT theory can be reconstructed from invariants

$$\mathrm{BPS}^{\mathrm{num}}_{[\mathrm{pt}]}, \quad \mathrm{BPS}^{\mathrm{num}}_{[\mathrm{C}]}, \quad \mathrm{BPS}^{\mathrm{num}}_{[2\mathrm{C}]}, \quad \dots, \quad \mathrm{BPS}^{\mathrm{num}}_{[\ell\mathrm{C}]},$$

associated to the class [pt] with rank/Euler pair $(0, 1)$, and to the classes $[i\mathrm{C}]$ with rank/Euler pairs $(i, 1)$, which yield the GV invariants. In other words, the numerical DT theory of the flop is expected to be controlled by the GV invariants and a count of points on the curve. We will investigate this conjecture for the numerical invariants as well as in the refined setting, where an analogue of the strong rationality conjecture is expected to hold.

Because the strong rationality conjecture predicts that the DT theory is controlled by the GV invariants, we will also investigate if the refinements of these invariants give a more precise description of the geometry. In particular we ask:

Question 2. Do the refined BPS invariants distinguish simple flops?

The answer is naturally "yes" for $\ell = 1$ flops, as the *unrefined* GV invariants are a complete invariant for this family, but very little is known about the general case. We will therefore go to the next level and try to answer these questions for length 2 flops.

§ 1.2 | Results on length $\ell = 2$ flops

To start our analysis of $\ell = 2$ flops we require a representative collection of examples. Several candidates exist in the literature: the first examples were constructed by Laufer [Lau81]; a few more examples have been constructed by Pinkham [Pin83]; and finally there is the recent example of Brown–Wemyss [BW17]. In chapter 2 we construct an infinite family $\{Y_f\}$ of length $\ell = 2$ flopping contractions

$$\flat_f : Y_f \to \operatorname{Spec} R_f$$

indexed by a parameter $f \in \mathbb{C}[y]$. For specific choices of f this family recovers all the examples mentioned above, but also contains many new[1] examples.

By studying the stability conditions and deformation theory for *noncommutative models* of length $\ell = 2$ flops, as explained below, we deduce that the partition function of any flop can be presented as the following generating function:

$$\begin{aligned}\Phi(t) = \operatorname{Sym}\Bigg(&\sum_{k\geq 0} \frac{\mathrm{BPS}_{k[\mathrm{pt}]}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot t^{(0,k)} \\ &+ \sum_{k,n\geq 0} \frac{\mathrm{BPS}_{k[\mathrm{C}]}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot \left(t^{(k,kn)} + t^{(-k,k(n+1))}\right) \\ &+ \sum_{k,n\geq 0} \frac{\mathrm{BPS}_{k[2\mathrm{C}]}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot \left(t^{(2k,k(n+1))} + t^{(-2k,kn)}\right)\Bigg),\end{aligned} \tag{1.1}$$

where the exponents on t are rank/Euler pairs (r, χ), the factor $\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}$ is the virtual motive of the algebraic torus, and $\mathrm{BPS}_{k[\mathrm{pt}]}$, $\mathrm{BPS}_{k[\mathrm{C}]}$, and $\mathrm{BPS}_{k[2\mathrm{C}]}$ are the *motivic BPS invariants* that express the deformation theory of, respectively, the point sheaves $\mathcal{O}_p$ for $p \in \mathrm{C}$, and the structure sheaves $\mathcal{O}_{\mathrm{C}}, \mathcal{O}_{2\mathrm{C}}$ of the curve C and thickened curve 2C.

We calculate these BPS invariants for the entire family $\{Y_f\}$, and find that they depend on two parameters $a \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{\infty\}$ derived from the polynomial f.

Theorem A (Theorem 6.2)**.** *For a flop in the family $\{Y_f\}$ with parameters a, b derived from f, the BPS invariants appearing in the expansion 1.1 are as follows.*

[1] Recently, this family was also constructed and studied by Kawamata in independent work [Kaw20] which appeared while writing this thesis.

The BPS invariants associated to the point sheaves are

$$\mathrm{BPS}_{k[\mathrm{pt}]} = \mathbb{L}^{-\frac{3}{2}}[\mathbb{P}^1] \quad \textit{for } k \geq 1.$$

The BPS invariants associated to the curve class 2C *are*

$$\mathrm{BPS}_{k[2\mathrm{C}]} = \begin{cases} \mathbb{L}^{-\frac{1}{2}}(1-[\mu_a]) & k=1 \\ 0 & \textit{for } k>1 \end{cases}$$

The first BPS invariant associated to the curve class C *is*

$$\mathrm{BPS}_{[\mathrm{C}]} = \begin{cases} \mathbb{L}^{-1}(1-[D_{4a}])+2 & a \leq b, \\ \mathbb{L}^{-1}(1-[D_{2b+1}])+3 & a > b. \end{cases}$$

where D_{4a} and D_{2b+1} are curves of genus a resp. b with a monodromy action of μ_{4a} and μ_{2b+1} respectively. For $k \geq 2$ the BPS invariants have the realisation

$$\mathrm{BPS}^{\mathrm{mmhs}}_{k[\mathrm{C}]} = \begin{cases} \chi_{\mathrm{mmhs}}\left(\mathbb{L}^{-\frac{1}{2}}(1-[\mu_a])\right) & k=2 \\ 0 & \textit{for } k>2 \end{cases}$$

in the Grothendieck ring of monodromic mixed Hodge structures.

Here $\mathbb{L} = [\mathbb{A}^1]$ denotes the Lefschetz motive, μ_n is the group scheme of nth roots of unity, and χ_{mmhs} is a realisation map into monodromic mixed Hodge structures.

Theorem A confirms the strong rationality conjecture at the MMHS–level of refinement: if we present the partition function as a generating function

$$\Phi^{\mathrm{mmhs}}(t) = \mathrm{Sym}\left(\sum_{(r,\chi)} \frac{\mathrm{BPS}^{\mathrm{mmhs}}_{(r,\chi)}}{\chi^{\mathrm{mmhs}}(\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}})} \cdot t^{(r,\chi)}\right)$$

indexed by rank/Euler characteristic pairs (r,χ) then the results imply the following.

Corollary 1.1. *The* MMHS*-refined BPS invariants* $\mathrm{BPS}^{\mathrm{mmhs}}_{(r,\chi)}$ *do not depend on the Euler characteristic χ, and are given by*

$$\mathrm{BPS}^{\mathrm{mmhs}}_{(r,\chi)} = \begin{cases} \mathrm{BPS}^{\mathrm{mmhs}}_{[\mathrm{pt}]} & r=0 \\ \mathrm{BPS}^{\mathrm{mmhs}}_{[\mathrm{C}]} & r=\pm 1 \\ \mathrm{BPS}^{\mathrm{mmhs}}_{2[\mathrm{C}]} & r=\pm 2 \\ 0 & \textit{otherwise} \end{cases}$$

Hence, the strong rationality conjecture holds at this level of refinement.

Because the point count $\mathrm{BPS}_{[\mathrm{pt}]}$ is the same for all flops in the family $\{Y_f\}$, the DT theory is essentially controlled by the invariants $\mathrm{BPS}^{\mathrm{mmhs}}_{[\mathrm{C}]}$, $\mathrm{BPS}^{\mathrm{mmhs}}_{[2\mathrm{C}]}$, which are refinements of the GV invariants. We expect a similar result to hold in the motivic refinement, but were unable to compare the motivic invariants for the classes $k \cdot [2\mathrm{C}]$ corresponding to $(r, \chi) = (2k, k)$, with the invariants for the classes $2k[\mathrm{C}]$ corresponding to $(r, \chi) = (2k, 2k)$.

Our results also show the extent of the refinement: for the first time we obtain invariants which carry interesting monodromy *and* Hodge structure, whereas the earlier $\ell = 1$ examples yielded invariants of Lefschetz type [MMNS12] or invariants with monodromy but only a very elementary Hodge structure [DM17]. However, this refinement is no better at distinguishing flops than the GV invariants: for every $a > 1$ there are flops in the family $\{Y_f\}$ with parameters

$$(a, b) = (a, a), \dots, (a, 2a - 1), (a, \infty)$$

which are non-isomorphic, but for which the MMHS–refined BPS invariants are equal.

Corollary 1.2. *The* MMHS*–realisations of the DT invariants* ***do not*** *determine flops.*

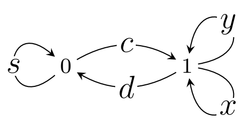
This corollary strengthens the result of [BW17] and puts it in a wider context, as the two examples they use form a subset of our family and their GV invariants can be deduced from the refined invariants we find. As in [BW17], we also compare with the noncommutative contraction algebra invariant of [DW16], which *does* distinguish these flops. Corollary 1.2 suggests that, even when using a refinement, some essential aspect of the noncommutative deformation theory is lost in the calculation of DT invariants.

§ 1.3 | Noncommutative methods

To obtain the above results we will work with a *noncommutative model*: for each threefold Y in our family we find a quiver with potential (Q, W) for which the Jacobi algebra $\mathrm{Jac}(Q, W)$ is derived equivalent to Y via a pair of R–linear functors

$$\mathrm{D}^b(\mathrm{coh}\, Y) \underset{\Psi^{-1}}{\overset{\Psi}{\rightleftarrows}} \mathrm{D}^b(\mathrm{mod}\, \mathrm{Jac}(Q, W))$$

Across this derived equivalence one can then perform *noncommutative* DT theory, as developed in the work of Szendrői [Sze08] and Kontsevich–Soibelman [KS08]. Instead of counting moduli of objects in $\mathrm{D}^b(Y)$, the DT invariants we calculate will count $\mathrm{Jac}(Q, W)$–modules for the family of quivers with potential shown in figure 1.1.

$Q\colon$ 

$$W = x^2 y - f(y) + y^2 cd - sdc + 2 f_{\mathrm{even}}\left(s^{\frac{1}{2}}\right)$$

Figure 1.1: Our family of quivers with potential.

As we are only interested in the objects $\mathrm{D}^b_{\mathrm{C}}(Y)$ supported on $\mathrm{C} \subset Y$, we will moreover restrict to the *nilpotent* modules $\operatorname{nilp}\operatorname{Jac}(Q,W)$. This nilpotency corresponds to a restriction to a formal neighbourhood of C, as the equivalence Ψ restricts to an equivalence

$$\mathrm{D}^b_{\mathrm{C}}(\operatorname{coh} Y) \underset{\Psi^{-1}}{\overset{\Psi}{\rightleftarrows}} \mathrm{D}^b(\operatorname{nilp}\operatorname{Jac}(Q,W))$$

The derived equivalence also induces an isomorphism between the K-theory groups $\mathrm{K}_0(\operatorname{nilp}\operatorname{Jac}(Q,W)) = \mathbb{Z}[S_0] \oplus \mathbb{Z}[S_1]$ and $\mathrm{K}_0(\mathrm{D}_{\mathrm{C}}(Y))$, which maps the classes of the nilpotent simples S_0, S_1 to the classes of the corresponding objects in $\mathrm{D}^b_{\mathrm{C}}(Y)$:

$$[S_0] \mapsto [\mathcal{O}_{2\mathrm{C}}(-1)[1]], \quad [S_1] \mapsto [\mathcal{O}_{\mathrm{C}}(-1)].$$

Hence, we can calculate our DT partition function $\Phi(t) = \sum_\delta \mathrm{DT}_\delta \cdot t^\delta$ in terms of virtual counts DT_δ of δ-dimensional nilpotent $\operatorname{Jac}(Q,W)$–modules for each dimension vector $\delta = \delta_0[S_0] + \delta_1[S_1]$, and then interpret the invariants DT_δ as the virtual counts $\mathrm{DT}_{(r,\chi)}$ of the objects in $\mathrm{D}^b_{\mathrm{C}}(Y)$ for the corresponding rank/Euler pair

$$(r,\chi) = \delta_0(-2,1) + \delta_1(1,0).$$

Although the reader may find this setup somewhat convoluted, the noncommutative framework will have some strong advantages for the calculation of DT invariants.

Firstly, the moduli spaces of a quiver with potential have the natural structure of a *critical locus*, which allows one to more easily define and analyse the refined DT invariants. Secondly, the moduli spaces are obtained via a *GIT construction*, which allows one to break them down by using *stability conditions*: a stability condition induces a decomposition

$$\Phi(t) = \prod^{\curvearrowright}_{\theta\in(0,1]} \Phi^\theta(t),$$

where the product ranges over the *phases* θ of *semistable objects*, and each $\Phi^\theta(t)$ is a virtual count of semistable objects with phase θ. Because the moduli spaces of semistable objects are usually easier to deal with, this can break down the calculation of the partition function into manageable chunks. Each power series $\Phi^\theta(t)$ can be presented as a generating function

$$\Phi^\theta(t) =: \operatorname{Sym}\left(\sum_{\Theta(\delta)=\theta} \frac{\mathrm{BPS}_\delta}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot t^\delta\right).$$

in terms of the refined BPS invariants for the dimension vectors with phase θ. Our goal is therefore to identify stability conditions for the quivers with potential associated to flops; to find the phases and dimension vectors of semistable objects; and to compute the BPS invariants for these dimension vectors.

The classification of stability conditions and semistable objects for a general quiver with potential can be a hard problem, as the representation theory is often too complicated to work out by hand. One of the main innovations in this thesis is that we develop a way to overcome this problem for threefold flops, by showing a compatibility between the DT theory of the quiver with potential and the *tilting theory* of its *completed* Jacobi algebra.

Briefly, we show that there is a *duality* between the tilting complexes of the completion $\Lambda := \widehat{\mathrm{Jac}}(Q,W)$ of the Jacobi algebra $\mathrm{Jac}(Q,W)$, and the semistable objects in the category of finite dimensional modules

$$\mathrm{fdmod}\,\Lambda \simeq \mathrm{nilp}\,\mathrm{Jac}(Q,W).$$

This means that the dimension vectors of semistable modules can be deduced from the K–theory classes of the tilting complexes, which are known by recent work of Hirano–Wemyss and Donovan–Wemyss [HW19; DW19b], Moreover, we show that the tilts induce isomorphisms between the moduli spaces that respect the critical locus structure, which implies that BPS invariants associated to these spaces are the same. As a result, we deduce that the DT theory of flops can be reconstructed from the deformation theory of only three types of objects.

§ 1.3.1 | Stability & tilting theory

The tilting complexes of simple flops have recently been determined [HW19], and are most conveniently represented in the form of the *g-vector fan* pictured (for $\ell = 2$) in figure 1.2a. This is a wall-and-chamber structure in the vector space $\mathrm{K}_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}}$, wherein every chamber corresponds to a tilting object $T = T_i \oplus T_{i+1}$ and the g-vectors

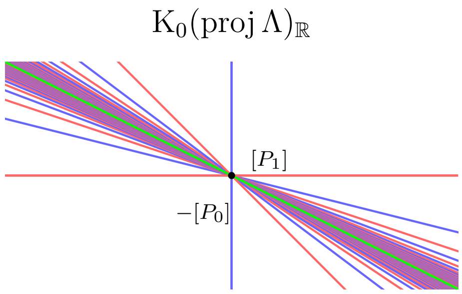
$$[T_i], [T_{i+1}] \in \mathrm{K}_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}},$$

of the summands span the walls that bound the chamber. Tilting objects that are separated by a finite number of *wall-crossings* are related by a sequence of *mutations*, a fact which is used in [HW19] to generate all tilting modules.

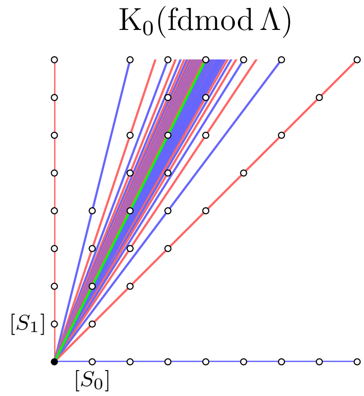
By construction, each tilting complex $T \in \mathcal{K}^b(\mathrm{proj}\,\Lambda)$ defines a derived equivalence

$$- \overset{\mathbf{L}}{\otimes} T\colon\; \mathrm{D}^b(\mathrm{mod}\,\mathrm{End}(T)) \xrightarrow{\sim} \mathrm{D}^b(\mathrm{mod}\,\Lambda),$$

and hence any module for the endomorphism algebra induces an object in the derived category of the flop. We show that these derived equivalences can be used to find the stable objects in the subcategory $\mathrm{fdmod}\,\Lambda$ of finite dimensional modules with respect to an appropriate stability condition. Concretely, if a summand T_i spans a wall in the south-east quadrant of the hyperplane arrangement, we show that it can be completed



(a) g-vectors of tilting complexes



(b) dimension vectors of semi-stables

Figure 1.2: For a generic stability condition, the dimension vectors of (semi-)stable objects are on the rays 1.2b perpendicular to the tilting hyperplane arrangement 1.2a. Each ray is spanned by the dimension vector of a twist/shift of $\mathcal{O}_{\mathrm{C}}$ (red), $\mathcal{O}_{2\mathrm{C}}$ (blue) or $\mathcal{O}_p$ for $p \in \mathrm{C}$ (green).

to a tilting complex T such that there exists a stable module

$$S \overset{\mathbf{L}}{\otimes}_{\mathrm{End}(T)} T \in \mathrm{D}^b(\mathrm{fdmod}\,\Lambda)$$

where S denotes the simple $\mathrm{End}(T)$–module which satisfies $\mathrm{Hom}_{\mathrm{End}(T)}(T_i, S) = 0$. By construction, the class of $[S \overset{\mathbf{L}}{\otimes} T]$ in $\mathrm{K}_0(\mathrm{fdmod}\,\Lambda)$ is then perpendicular to the wall spanned by $[T_i]$ with respect to the Euler pairing

$$\langle -, - \rangle \colon\ \mathrm{K}_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}} \otimes \mathrm{K}_0(\mathrm{fdmod}\,\Lambda) \to \mathbb{R},$$

giving the dual wall-and-chamber structure in figure 1.2b. The simples of the tilted algebras were identified in [DW19b] and have a nice geometric interpretation: across the equivalence $\mathrm{D}^b(\mathrm{mod}\,\Lambda) \simeq \mathrm{D}^b(\mathrm{coh}\,Y)$, they correspond to shifts/twists of the structure sheaves $\mathcal{O}_{\mathrm{C}}$, $\mathcal{O}_{2\mathrm{C}}$ of the curve and the scheme-theoretic fibre. Another class of stable objects is given by the images of point sheaves $\mathcal{O}_p$ for $p \in \mathrm{C}$, which have a dimension vector that is perpendicular to the accumulation ray in 1.2a. Together, these two classes form the complete set of stable modules.

Theorem B (Theorem 4.1)**.** *There exists a Bridgeland stability condition on the triangulated category* $\mathrm{D}^b(\mathrm{fdmod}\,\Lambda)$ *with heart* $\mathrm{fdmod}\,\Lambda$*, for which the stable objects correspond to the the following complexes of sheaves*

$$\begin{array}{rrl} & \mathcal{O}_p & \textit{for } p \in \mathrm{C}, \\ \mathcal{O}_{2\mathrm{C}}(n), & \mathcal{O}_{\mathrm{C}}(n-1) & \textit{for } n \geq 0, \\ \mathcal{O}_{2\mathrm{C}}(n)[1], & \mathcal{O}_{\mathrm{C}}(n-1)[1] & \textit{for } n < 0, \end{array}$$

via the equivalence $\mathrm{D}^b(\mathrm{fdmod}\,\Lambda) \simeq \mathrm{D}^b_C(Y)$.

To prove that the the above set of stable modules is indeed complete, we move to the context of finite dimensional algebras. We show that the g-vectors and stable modules of Λ coincide with those of a finite-dimensional fibre $\Lambda/I\Lambda$ over a fat point $\operatorname{Spec} R/I$. With this reduction step, the theorem follows from a similar statement discovered by finite-dimensional algebraists [BST19; Asa21].

We should remark that, although we give the proof of Theorem B for $\ell = 2$ flops, the techniques used to prove Theorem B do not explicitly depend on the length assumption. In the general case, the stable modules correspond to point sheaves and the objects in the "simples helix" of [DW19b].

§ 1.3.2 | Tilting preserves potentials

With the classification of stable objects in Theorem B, the DT partition function of the flop can be put into a generating function involving the BPS invariants for the classes of the stable objects. Indexed by rank/Euler pairs this yields the following:

$$\begin{aligned}\Phi(t) = \operatorname{Sym}\Bigg(& \sum_{k\geq 0} \frac{\mathrm{BPS}_{k[\mathrm{pt}]}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot t^{(0,k)} \\ & + \sum_{k,n\geq 0} \left(\frac{\mathrm{BPS}_{k[\mathcal{O}_{\mathrm{C}}(n-1)]}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot t^{(k,kn)} + \frac{\mathrm{BPS}_{k[\mathcal{O}_{\mathrm{C}}(-2-n)[1]]}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot t^{(-k,k(n+1))} \right) \\ & + \sum_{k,n\geq 0} \left(\frac{\mathrm{BPS}_{k[\mathcal{O}_{2\mathrm{C}}(n)]}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot t^{(2k,k(n+1))} + \frac{\mathrm{BPS}_{k[\mathcal{O}_{2\mathrm{C}}(-1-n)[1]]}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot t^{(-2k,kn)} \right) \Bigg).\end{aligned}$$

The terms $\mathrm{BPS}_{k[\mathrm{pt}]}$ yield a count of 0-dimensional sheaves, which are extensions between point sheaves, while the other invariants are virtual counts of the self-extensions of the stable modules corresponding to

$$\mathcal{F} = \mathcal{O}_{\mathrm{C}}(n)[m] \quad \text{and} \quad \mathcal{F} = \mathcal{O}_{2\mathrm{C}}(n)[m].$$

For these stable modules, the contribution to the BPS invariants is determined by their cyclic A_∞–deformation theory, which is captured by a noncommutative *potential*. In the setting of cluster algebras it is known that these potentials are preserved under mutation [KY11]. The quivers we consider are not of cluster type, as they consist of loops and 2-cycles, but we show an analogous result for the 'mutation' induced by the tilting complexes.

If the flop $Y \to Y_{\mathrm{con}} = \operatorname{Spec} R$ is tilting equivalent to an R–algebra A, we show that potentials are preserved by *R–linear standard equivalences* that satisfy a homological condition: if an R–linear functor $F\colon \mathrm{D}^b(\operatorname{mod} A) \to \mathrm{D}^b(\operatorname{mod} A)$ lifts to a differentially graded enhancement of the derived category there is an induced R–linear action

$$\mathrm{HH}_3(F)\colon \mathrm{HH}_3(A) \to \mathrm{HH}_3(A),$$

on Hochschild homology. We show that F preserves potentials if this action is a scalar.

Theorem C (Theorem 5.1)**.** *Let* $F\colon \mathrm{D}^b(\operatorname{mod} A) \to \mathrm{D}^b(\operatorname{mod} A)$ *be an* R*–linear standard equivalence such that* $\mathrm{HH}_3(F) = \lambda \in \mathbb{C}^\times$*. Then for every pair of nilpotent modules* M, N *with simple endomorphism algebras, which are related by* $F(M) \simeq N$*, the potentials* W_M *and* $\lambda \cdot W_N$ *are equivalent.*

Any functor induced by a tilting complex is automatically an R–linear standard equivalence. Moreover, we show that the homological condition is always satisfied in our setting and hence preserve the potentials of objects supported on the curve. This is what allows us to show that the BPS invariants for the classes $k[\mathcal{O}_{\mathrm{C}}(n)[m]]$ and $k[\mathcal{O}_{2\mathrm{C}}(n)[m]]$ do not depend on the twist by n or the shift by m, and therefore equal

$$\mathrm{BPS}_{k[\mathrm{C}]} = \mathrm{BPS}_{k[\mathcal{O}_{\mathrm{C}}]}, \quad \mathrm{BPS}_{k[2\mathrm{C}]} = \mathrm{BPS}_{k[\mathcal{O}_{2\mathrm{C}}]},$$

respectively. This greatly reduces the amount of classes for which we need to compute DT invariants.

To prove Theorem C we follow the approach of Kontsevich–Soibelman [KS08; KS09] by working with a cyclic A_∞–enhancement $(\mathcal{H}, \sigma)$ of the derived category $\mathrm{D}^b_{\mathrm{nilp}}(A)$ of nilpotent modules. The cyclic structure σ encodes the Calabi–Yau property of the category $\mathrm{D}^b_{\mathrm{nilp}}(A) \simeq \mathrm{D}^b_{\mathrm{C}}(Y)$ and endows each object with its potential. The cyclic structure is determined up to homotopy by its Hochschild cohomology class $[\sigma] \in \mathrm{HH}^3(\mathcal{H}, \mathcal{H}^*)$ and any auto-equivalence of $\mathcal{H}$ which preserves this class also preserves the potentials. The categories $\mathrm{D}^b(\operatorname{mod} A)$ and $\mathrm{D}^b_{\mathrm{nilp}}(A)$ are related by duality, and at the level of Hochschild (co-)homology this duality induces a map

$$\Upsilon\colon \mathrm{HH}_3(A) \to \mathrm{HH}^3(\mathcal{H}, \mathcal{H}^*),$$

for which we follow the construction of Brav–Dyckerhoff [BD19]. We show that an R–linear standard equivalence $F\colon \mathrm{D}^b(\operatorname{mod} A) \to \mathrm{D}^b(\operatorname{mod} A)$ induces an autoequivalence of $\mathcal{H}$ whose action on $\mathrm{HH}^3(\mathcal{H}, \mathcal{H}^*)$ is completely determined by the R–linear action of F on $\mathrm{HH}_3(A)$: there is a unique R–linear map making the diagram

$$\begin{array}{ccc} \mathrm{HH}_3(A) & \xrightarrow{\Upsilon} & \mathrm{HH}^3(\mathcal{H}, \mathcal{H}^*) \\ {\scriptstyle \mathrm{HH}_3(F)}\downarrow & & \uparrow \\ \mathrm{HH}_3(A) & \xrightarrow{\Upsilon} & \mathrm{HH}^3(\mathcal{H}, \mathcal{H}^*) \end{array}$$

commute. In this way the homological condition in Theorem C translates to a preservation of the cyclic structure up to homotopy, and from there to a preservation of potentials.

If Y is Calabi–Yau, the proof can be interpreted in terms of Calabi–Yau structures: a holomorphic volume form on Y defines a *left* CY structure in $\mathrm{HH}_3(Y) \simeq \mathrm{HH}_3(A)$, while the cyclic structure σ defines a *right* CY structure. A functor satisfying the condition

$\mathrm{HH}_3(F) = \lambda$ scales the Calabi–Yau volume linearly, and via the (weak) duality Υ (which maps left CY structures to right CY structures) it induces the inverse scaling on σ. However, Theorem C relies on a *relative* condition $\mathrm{HH}_3(F) = \lambda$, which does not require the existence of a volume form, and therefore holds even if Y is not globally Calabi–Yau.

§ 1.4 | Outline of the thesis

Chapter 2 starts with a short review of the geometry of threefold flops, and then delves into our construction of a family of length $\ell = 2$ flops in Theorem 2.7.

Chapter 3 is a short introduction to (motivic) Donaldson–Thomas theory, which sets up the methodology for calculating invariants for quivers with potential. In particular, we stress the essential role played by stability conditions.

Chapter 4 explains the close relation between stability condition and tilting theory for length $\ell = 2$ flops. The result is Theorem 4.1, which completely classifies the moduli of stable objects with support on the exceptional curve into three families.

Chapter 5 uses the framework of cyclic A_∞-categories to develop a condition for derived equivalences to interact nicely with the DT theory of a quiver with potential. The main result is Theorem 5.1, which works in a very general setting, but we also provide sufficient corollaries to apply the theorem to flops.

All paths converge in chapter 6, where we finally calculate the invariants of our family of length $\ell = 2$ flops using the techniques we developed in chapter 4 and chapter 5.

§ 1.5 | Notation and conventions

In this thesis we work over the field of complex numbers. All schemes will be separated, locally of finite type over $\mathbb{C}$, and we will use the term *variety* to mean a reduced separated scheme of finite type over $\mathbb{C}$. When X is moreover affine we will write $\mathbb{C}[X]$ for its coordinate ring. We write $\operatorname{coh} X$, $\operatorname{QCoh} X$ for the abelian categories of coherent and quasicoherent sheaves on a scheme X and abbreviate

$$\mathrm{D}(X) := \mathrm{D}(\operatorname{QCoh} X), \quad \mathrm{D}^b(X) := \mathrm{D}^b(\operatorname{coh} X), \quad \mathrm{D}^{\mathrm{perf}}(X),$$

for the unbounded derived category, the bounded derived category, and the category of perfect complexes. We will also use the notation $\mathrm{D}^b_{\mathrm{C}}(X)$ for the subcategory of complexes with cohomology supported on a fixed curve $\mathrm{C} \subset X$.

At several points we will also use *stacks*, by which we mean Artin stacks which are locally of finite type over $\mathbb{C}$. These will almost exclusively be quotient stacks X/G of a scheme X by some linear algebraic group G. Contrary to the more popular notation we do not write square brackets around stack quotients, as this notation is reserved for the corresponding motive. To make the distinction between stacks and schemes clear we will write all stacks in "Euler script" font, e.g. $\mathcal{X}, \mathcal{Y}, \mathcal{M}, \mathcal{N}, \ldots$

Unless specified otherwise, the term *algebra* will refer to a finitely generated unital algebra over $\mathbb{C}$ which is not assumed to be commutative. If A is an algebra we write $\operatorname{Mod} A$, $\operatorname{mod} A$, $\operatorname{proj} A$ for its category of right modules, finitely generated right modules, and projective right modules respectively; when necessary, left modules will be interpreted as right modules over A^{op}. We abbreviate

$$\mathrm{D}(A) := \mathrm{D}(\operatorname{Mod} A), \quad \mathrm{D}^b(A) := \mathrm{D}^b(\operatorname{mod} A), \quad \mathrm{D}^{\mathrm{perf}}(A) := \mathcal{K}^b(\operatorname{proj} A),$$

as in the case of schemes. We also write $\operatorname{fdmod} A$ for the abelian category of finite dimensional modules, and denote its derived category as $\mathrm{D}^b_{\mathrm{fd}}(A) = \mathrm{D}^b(\operatorname{fdmod} A)$.

By convention, a quiver Q will consist of a finite set of vertices Q_0 and a finite set of arrows Q_1. If $v_0, \ldots, v_n \in Q_1$ is a sequence of vertices and $a_i \colon v_{i-1} \to v_i$ are arrows between them, we will write $a_n \cdots a_1$ for the associated path from v_0 to v_n. If R is a commutative algebra, we write RQ for the path algebra of Q over R and $\widehat{RQ}$ for its completion with respect to path length. We write $\mathbb{C}Q_{\mathrm{cyc}}$ for the vector space of paths in $\mathbb{C}Q$ up to cyclic permutation and $\widehat{\mathbb{C}Q}_{\mathrm{cyc}}$ for its completion. The term *potential* will be used both for an element of $\mathbb{C}Q_{\mathrm{cyc}}$ (respectively $\widehat{\mathbb{C}Q}_{\mathrm{cyc}}$), and for a lift of it to a linear combination of cycles in $\mathbb{C}Q$ (respectively $\widehat{\mathbb{C}Q}$); we assure the reader that confusing the two does not lead to any problems.

Given a quiver Q, we let $\Delta = \mathbb{N}Q_0$ denote the monoid of dimension vectors. By a representation of dimension $\delta \in \Delta$ with values in a commutative algebra R, we will mean an RQ_0-linear map $\rho \colon RQ \to \operatorname{End}_R(\bigoplus_{v \in Q_0} R^{\oplus \delta_v})^{\mathrm{op}}$. Every representation corresponds to a right RQ-module in the natural way, and we make a habit of identifying the two. We denote the representation variety of Q by

$$\operatorname{Rep}_\delta(Q) := \prod_{a \colon v \to w \in Q_1} \operatorname{Mat}_{\delta_w \times \delta_v}(\mathbb{C}),$$

and identify its R-points with the set of R-valued representation of dimension δ in the natural way. The representation variety is acted on by the linear algebraic group $\mathrm{GL}_\delta := \prod_{v \in Q_0} \mathrm{GL}_{\delta_v}$ via conjugation, and we write

$$\mathcal{M}_{Q,\delta} := \operatorname{Rep}_\delta(Q) /\!\!/ \mathrm{GL}_\delta, \quad \mathcal{M}^\theta_{Q,\delta} := \operatorname{Rep}_\delta(Q) /\!\!/^\theta \mathrm{GL}_\delta, \quad \mathfrak{M}_{Q,\delta} := \operatorname{Rep}_\delta(Q)/\mathrm{GL}_\delta,$$

for the scheme-theoretic quotient, the GIT quotient with respect to a linear character $\theta \colon \Delta \to \mathbb{R}$ as in [Kin94], and the stack-theoretic quotient. We will drop the quiver Q from the notation if it is clear from context, and will drop the dimension vector δ from the notation to indicate the disjoint union over all dimension vectors.

All other notation will be explained in due course.

Chapter 2

Constructing Flops

§ 2.1 | Threefold flops

The goal of the minimal model program is to reduce any suitable variety to a minimal representative in its birational equivalence class: there should be a *minimal model* Y, a normal variety with at most mild singularities such that the canonical divisor has non-negative intersection $K_Y \cdot \mathrm{C} \geq 0$ with every rational curve $\mathrm{C} \subset Y$. This is achieved via an algorithm consisting of a series of birational modifications based on *contractions* of curve classes, as described by Kollar–Mori [KoM98]. Because we will not run the MMP explicitly, we will not keep track of the curve classes that are contracted and use the following simplified definition of a contraction morphism.

Definition 2.1. Let Y be a normal variety. Then a *contraction* onto a normal variety Y_{con} is a morphism of varieties $\flat\colon Y \to Y_{\mathrm{con}}$ which is projective, surjective, and satisfies the condition $\mathbf{R}\flat_* \mathcal{O}_Y = \mathcal{O}_{Y_{\mathrm{con}}}$.

There are several types of contractions used in the MMP, such as fibre-type contractions, divisorial contractions, and small contractions, which include flips and flops.

Definition 2.2. We say a contraction $\flat\colon Y \to Y_{\mathrm{con}}$ is *small* if it is birational, and the components of the exceptional locus $\mathrm{Ex}(\flat) \subset Y$ are of codimension ≥ 2 in Y.

In the setting of threefolds, a small contraction $\flat\colon Y \to Y_{\mathrm{con}}$ contracts curves to isolated points. This is because the condition $\flat_* \mathcal{O}_Y \simeq \mathcal{O}_{Y_{\mathrm{con}}}$ guarantees that for every point $p \in Y_{\mathrm{con}}$ the fibre $f^{-1}(p)$ is connected. Hence, each fibre $f^{-1}(p)$ is either a single point (in which case $\flat$ is an isomorphism over p), or $f^{-1}(p)$ is a connected curve contained in $\mathrm{Ex}(\flat)$, which can only happen for finitely many points.

Definition 2.3. Let $\flat\colon Y \to Y_{\mathrm{con}}$ be a small contraction of threefolds and assume the canonical divisor K_Y is $\mathbb{Q}$-Cartier,[1] then $\flat$ is a *flopping contraction* if every exceptional curve $\mathrm{C} \subset \mathrm{Ex}(\flat)$ has intersection $K_Y \cdot \mathrm{C} = 0$.

[1] This guarantees that the intersection number is well-defined. It is automatic if Y is smooth.

Every threefold flopping contraction from a variety with mild (terminal) singularities gives rise to a unique flop [KoM98, Theorem 6.14]. Namely, there exists a normal variety Y^+ with mild singularities and a (nontrivial) birational map $Y \dashrightarrow Y^+$ called a *flop* which factors as

$$\begin{array}{ccc} Y & \dashrightarrow & Y^+ \\ & {\scriptstyle \mathpzc{p}} \searrow \quad \swarrow {\scriptstyle \mathpzc{p}^+} & \\ & Y_{\mathrm{con}} & \end{array}$$

where $\mathpzc{p}^+ \colon Y^+ \to Y_{\mathrm{con}}$ is a second flopping contraction. Because of the condition $K_Y \cdot \mathrm{C} = 0$, flopping a minimal model produces a new minimal model, and in fact all birational minimal models are connected by a finite sequence of flops [Kaw08].

In this thesis we focus on simple flopping contractions $\mathpzc{p} \colon Y \to Y_{\mathrm{con}}$, where Y is smooth and $\mathrm{C} \subset Y$ is a rational curve $\mathrm{C} \simeq \mathbb{P}^1$, contracted to a point $p \in Y_{\mathrm{con}}$. Simple flops are categorised by their *length invariant*: the scheme-theoretic fibre $Y \times_{Y_{\mathrm{con}}} \{p\}$ is in general not reduced, and the length ℓ is defined by Kollar [Kol89] as the length of its structure sheaf over the generic point of C. Moreover, there exists (see [BKC99]) a chain of subschemes

$$\mathrm{C} \subset 2\mathrm{C} \subset \ldots \subset \ell\mathrm{C}$$

which interpolates between C and the scheme-theoretic fibre ℓC.

§ 2.1.1 | The universal flop

It follows from a theorem of Reid [Rei83, Theorem 1.1] that the base of a smooth flopping contraction is a *compound Du Val* (cDV) singularity, which means that a general hyperplane section through the singularity is a Du Val surface singularity. For (smooth) simple flops, Katz–Morrison [KaM92] show that the ADE-type of the Du Val singularity is determined by the length invariant. Moreover, in their proof they show that the base of every simple threefold flop appears as a *deformation* of the associated Du Val singularity and therefore appears in a semi-universal family of such deformations. Curto–Morrison [CM13] explicitly construct this family for flops of length 2, which they call the *universal flop of length* $\ell = 2$. It is a resolution $\pi \colon \mathcal{Y} \to \mathcal{Y}_{\mathrm{con}}$ of the affine hypersurface $\mathcal{Y}_{\mathrm{con}} := \operatorname{Spec} \mathcal{R}$ with equation

$$u^2 - rw^2 + 2zvw - sv^2 + (rs - z^2)t^2 \in \mathbb{C}[r, s, t, u, v, w, z].$$

Similar universal flops for lengths $\ell \geq 3$ were also constructed by Karmazyn [Kar19].

Besides a commutative resolution, the base of the universal flop also has a noncommutative resolution: Curto–Morrison [CM13] use a matrix factorisation to construct a reflexive module $N \in \operatorname{mod} \mathcal{R}$ such that the algebra

$$\Lambda := \operatorname{End}_{\mathcal{R}}(\mathcal{R} \oplus N)$$

is a *noncommutative crepant resolution* (NCCR) as defined in the work of Van den Bergh [VdB04a]. This means that Λ is an $\mathcal{R}$-algebra with global dimension equal to $\dim \mathcal{R}$, which is Cohen-Macaulay as an $\mathcal{R}$-module. The algebra map $\mathcal{R} \to \Lambda$ can be thought of as the analogue of the resolution $\pi\colon \mathcal{Y} \to \mathcal{Y}_{\mathrm{con}}$, and has similar homological properties.

One can make the analogy between the commutative resolutions and noncommutative resolutions more precise. Van den Bergh [VdB04b] shows that if $f\colon X \to X_{\mathrm{con}}$ is a contraction with fibres of dimension ≤ 1, then there exists a *tilting bundle* $\mathcal{O}_X \oplus \mathcal{F}$ on X such that $\mathcal{F}^*$ is generated by global sections, and this tilting bundle induces a derived equivalence

$$\mathrm{D}^b(X) \underset{-\overset{\mathbf{L}}{\otimes}_X(\mathcal{O}_X\oplus\mathcal{F}).}{\overset{\mathbf{R}\mathrm{Hom}_X(\mathcal{O}_X\oplus\mathcal{F},-)}{\rightleftarrows}} \mathrm{D}^b(\mathrm{End}_X(\mathcal{O}_X \oplus \mathcal{F})),$$

For the universal flop $\mathcal{Y}$ one can choose the tilting bundle to be a lift of $\mathcal{R} \oplus N$: there exists a vector bundle $\mathcal{N} \in \operatorname{coh} \mathcal{Y}$ such that $\mathcal{T} := \mathcal{O}_{\mathcal{Y}} \oplus \mathcal{N}$ is a tilting bundle, $\mathcal{N}^*$ is generated by global sections, and which lifts N in the sense that $\pi_* \mathcal{N} = N$. One checks, as in [VdB04b], that there is an isomorphism $\mathrm{End}_{\mathcal{Y}}(\mathcal{T}) \simeq \Lambda$, so that $\mathcal{T}$ provides a bridge between the commutative and noncommutative realms.

Noncommutative resolutions are the lifeblood of the homological approach to the MMP developed in [DW16; Wem18], which aims to recover the birational operations of the MMP via mutations of NCCRs and derived functors. A particularly important ingredient in this approach is the *contraction algebra*.

Definition 2.4 ([DW16])**.** Let $f\colon X \to \operatorname{Spec} R$ be a birational contraction with fibres of dimension ≤ 1, and let A denote the endomorphism algebra of the Van den Bergh tilting bundle $\mathcal{O}_X \oplus \mathcal{F}$ described above. Then the associated *contraction algebra* is

$$A_{\mathrm{con}} := A/[\mathcal{O}_X],$$

where $[\mathcal{O}_X]$ is the two-sided ideal of maps factoring through $\operatorname{add} \mathcal{O}_X$.

By [DW19a, Theorem 1.1], the support of the contraction algebra is precisely the image of the exceptional locus on the base, and can therefore be used to detect the dimension of the contracted locus. In particular, the universal flop of length 2 has a contraction algebra

$$\Lambda_{\mathrm{con}} := \mathrm{End}_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}} \oplus \mathcal{N})/[\mathcal{O}_Y] \simeq \mathrm{End}_{\mathcal{R}}(\mathcal{R} \oplus N)/[\mathcal{R}],$$

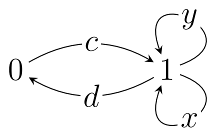
which is a finite $\mathcal{R}$-algebra with $\operatorname{Supp}_{\mathcal{Y}_{\mathrm{con}}} \Lambda_{\mathrm{con}} = \operatorname{Sing} \mathcal{Y}_{\mathrm{con}}$.

Our aim for this chapter is to construct explicit families of threefold flops, in both their commutative and noncommutative form by taking a threefold slice of the universal flop over a ring map $\mathcal{R} \to R$. To choose such a slice appropriately we require an explicit expression for Λ and $\mathcal{Y}$ as (noncommutative) spaces over $\mathcal{R}$.

§ 2.1.2 | The universal flop as a quiver with relations

An explicit form for Λ was constructed by Karmazyn [Kar19, Example 5.1], which we recall here in our choice of notation. Let $\mathbb{H} := \mathbb{C}[r, s, t, T]$, then Λ is isomorphic to the quiver algebra $\mathbb{H}\mathcal{Q}/I$ for the following quiver and relations:

$\mathcal{Q}$:



$$\left.\begin{array}{c} dc = t \cdot 1_0, \quad x^2 = r \cdot 1_1, \quad y^2 = s \cdot 1_1 \\ (cd + x + y - \frac{1}{2} t \cdot 1_1)^2 = T \cdot 1_1 \end{array}\right\} I$$

where we write $1 = 1_0 + 1_1$ for the idempotent splitting of the unit. Under this isomorphism, the base ring $\mathcal{R}$ acts via a map $\varphi\colon \mathcal{R} \to \mathbb{H}\mathcal{Q}/I$ defined on generators as

$$\begin{gathered} \varphi(r) = r, \quad \varphi(s) = s, \quad \varphi(t) = t, \\ \varphi(u) = \tfrac{1}{2} d[x, y]c + \tfrac{1}{2}([x, y]cd + cd[x, y] - t \cdot [x, y]) \\ \varphi(v) = dxc + (xcd + cdx - t \cdot x), \\ \varphi(w) = dyc + (ycd + cdy - t \cdot y), \\ \varphi(z) = \tfrac{1}{2}(\varphi(v) + \varphi(w) + r + s - T + \tfrac{1}{4} t^2) \cdot 1_0 - \tfrac{1}{2}(xy + yx), \end{gathered}$$

where $[x, y] = xy - yx$ denotes the commutator. From now on we will identify $\Lambda = \mathbb{H}\mathcal{Q}/I$ as $\mathcal{R}$-algebras via the above structure.

In [Kar19, Example 5.1] it is shown that the universal flop $\mathcal{Y}$ is isomorphic to a moduli scheme of stable representations of $\mathcal{Q}$. Consider the dimension vector $\delta = (1, 2)$, then the space $\operatorname{Rep}_\delta(\mathcal{Q})$ parametrises representations of $\mathcal{Q}$ via the tautological family τ:

$$\tau(x) = \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix}, \quad \tau(y) = \begin{pmatrix} y_{00} & y_{01} \\ y_{10} & y_{11} \end{pmatrix}, \quad \tau(c) = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}, \quad \tau(d) = \begin{pmatrix} d_0 & d_1 \end{pmatrix},$$

of which the coefficients generate the coordinate ring $\mathbb{C}[\operatorname{Rep}_\delta(\mathcal{Q})]$. Let Δ denote the lattice of dimension vectors for $\mathcal{Q}$, and consider $\theta, \theta^+\colon \Delta \to \mathbb{R}$

$$\theta\colon (a, b) \mapsto 2a - b, \quad \theta^+\colon (a, b) \mapsto b - 2a.$$

as King-stability conditions on $\operatorname{Rep}_\delta(\mathcal{Q})$. This gives rise to two moduli spaces

$$\begin{aligned} \mathcal{M}^\theta_\delta(\Lambda) &:= (\operatorname{Spec} \mathbb{H} \times \mathcal{M}^\theta_\delta(\mathcal{Q})) \cap \{\tau(I) = 0\}, \\ \mathcal{M}^{\theta^+}_\delta(\Lambda) &:= (\operatorname{Spec} \mathbb{H} \times \mathcal{M}^{\theta^+}_\delta(\mathcal{Q})) \cap \{\tau(I) = 0\}, \end{aligned}$$

which come equipped with maps to $\operatorname{Spec} \mathcal{R}$ by restricting a representation to the subalgebra $1_0 \Lambda 1_0 \simeq \mathcal{R}$. This moduli construction recovers the universal flop.

Proposition 2.5 ([Kar19, Proposition 3.9])**.** *The following diagrams are equivalent:*

$$\begin{array}{ccc} \mathcal{Y} & & \mathcal{Y}^+ \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi^+} & \\ & \mathcal{Y}_{\mathrm{con}} & \end{array} \qquad \simeq \qquad \begin{array}{ccc} \mathcal{M}^{\theta}_{\delta}(\Lambda) & & \mathcal{M}^{\theta^+}_{\delta}(\Lambda) \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec}\mathcal{R} & \end{array}$$

In the following paragraph we will describe the structure of $\mathcal{M}^{\theta}_{\delta}(\Lambda)$ explicitly by gauge fixing. We will not give any proofs, but refer the reader to [Kar19, Example 5.1] for a full treatment. Given a field extension $\mathbb{K} \supset \mathbb{C}$, a $\mathbb{K}$-valued representation ρ is stable for θ exactly if the images of $\rho(c), \rho(yc), \rho(xc) \colon \mathbb{K} \to \mathbb{K}^2$ span $\mathbb{K}^2$. Hence, the stable locus $\operatorname{Rep}^{\theta}_{\delta}(\mathcal{Q})$ is the union of the open subspaces

$$V_x := \{\det(\rho(c) \mid \rho(xc)) \neq 0\}\,, \quad V_y := \{\det(\rho(c) \mid \rho(yc)) \neq 0\}\,.$$

These subspaces are trivial GL_2-torsors $V_x \simeq \mathrm{GL}_2 \times U'_x$, $V_y \simeq \mathrm{GL}_2 \times U'_y$ over affine spaces $U'_x, U'_y \simeq \mathbb{A}^8$ with respect to the action of $\mathrm{GL}_2 \simeq \{\mathrm{Id}\} \times \mathrm{GL}_2 \subset \mathrm{GL}_{\delta}$. Here $U'_x, U'_y \subset \operatorname{Rep}_{\delta}(\mathcal{Q})$ are described by the family ρ_x, ρ_y of representations

$$\rho_x(c) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \rho_x(d) = \begin{pmatrix} d_0 & d_1 \end{pmatrix}, \ \rho_x(x) = \begin{pmatrix} 0 & x_{01} \\ 1 & x_{11} \end{pmatrix}, \ \rho_x(y) = \begin{pmatrix} y_{00} & y_{01} \\ y_{10} & y_{11} \end{pmatrix},$$
$$\rho_y(c) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \rho_y(d) = \begin{pmatrix} d_0 & d_1 \end{pmatrix}, \ \rho_y(x) = \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix}, \ \rho_y(y) = \begin{pmatrix} 0 & y_{01} \\ 1 & y_{11} \end{pmatrix}.$$

It follows that $\mathcal{M}^{\theta}_{\delta}(\mathcal{Q}) \simeq U'_x \cup U'_y$ is a gluing of two affine charts. As a result $\mathcal{M}^{\theta}_{\delta}(\Lambda)$ can be obtained by extending each chart to $\mathbb{H}$ and imposing the relations generating the ideal I. On U'_x these relations reduce to the following system of equations:

$$x_{11} = 0, \quad y_{00} + y_{11} = 0, \ t = d_0, \quad r = x_{01},$$
$$s = y_{00}^2 + y_{01}y_{10}, \quad T = (y_{00} - \tfrac{1}{2}d_0)^2 + (1 + y_{10})(d_1 + x_{01} + y_{01}),$$

which eliminate the generators of $\mathbb{H}$ and impose the condition $\operatorname{tr}(\tau(x)) = \operatorname{tr}(\tau(y)) = 0$, yielding a subspace $U_x \subset U'_x$ isomorphic to $\mathbb{A}^6$. Similarly the system of equations

$$x_{00} + x_{11} = 0, \quad x_{11} = 0, \ t = d_0, \quad r = x_{00}^2 + x_{01}x_{10}$$
$$s = y_{01}, \quad T = (x_{00} - \tfrac{1}{2}d_0)^2 + (1 + x_{10})(d_1 + y_{01} + x_{01}),$$

cuts out a subspace $U_y \simeq \mathbb{A}^6$ in U'_y, and it follows that $\mathcal{M}^{\theta}_{\delta}(\Lambda)$ is a gluing of two copies of $\mathbb{A}^6$. The map $U_x \to \operatorname{Spec}\mathcal{R}$ is described by the ring map

$$\begin{aligned} r \mapsto x_{01}, \quad s \mapsto y_{00}^2 + y_{01}y_{10}, \quad t \mapsto d_0, \quad u \mapsto d_1 y_{00} + \tfrac{1}{2}d_0(x_{01}y_{10} - y_{01}), \\ v \mapsto d_1, \quad w \mapsto d_0 y_{00} + d_1 y_{10}, \quad z \mapsto x_{01}y_{10} + y_{01}, \end{aligned} \tag{2.1}$$

and similarly, the map $U_y \to \operatorname{Spec} \mathcal{R}$ is described by the ring map

$$\begin{gathered} r \mapsto x_{00}^2 + x_{01}x_{10}, \quad s \mapsto y_{01}, \quad t \mapsto d_0, \quad u \mapsto d_1 x_{00} + \tfrac{1}{2} d_0 (x_{01} - y_{01}x_{10}), \\ v \mapsto d_0 x_{00} + d_1 x_{10}, \quad w \mapsto d_1, \quad z \mapsto x_{01} + y_{01}x_{10}. \end{gathered} \tag{2.2}$$

The above is a sufficiently explicit description for the construction of our flops below.

§ 2.2 | Slicing the universal flop

We will now construct a family of length 2 flops based on a choice of *quiver with potential*, which is to serve as a model for the contraction algebras of these flops. Because the contraction algebra is supported over the image of the exceptional locus, it is finite dimensional if and only if that image is a set of points. Hence, the finiteness of the contraction algebra can be used as a condition for a contraction to be small.

Let $f = f(y) \in \mathbb{C}[y]$ be a polynomial divisible by y^3 and define the potential

$$W_f := x^2 y - f(y),$$

on the two loop quiver Q_2. The associated Jacobi algebra has the form

$$\operatorname{Jac}(Q_2, W_f) = \frac{\mathbb{C}\langle x, y\rangle}{(xy + yx,\ x^2 - f'(y))},$$

where f' denotes the first derivative of f. Let $f_{\text{even}}, f_{\text{odd}} \in \mathbb{C}[y]$ denote the polynomials of even and odd parity respectively such that $f = f_{\text{even}} + f_{\text{odd}}$.

Lemma 2.6. *Suppose* $f_{\text{even}} \neq 0$*, then* $\operatorname{Jac}(Q_2, W_f)$ *is finite dimensional.*

Proof. Let $f \in \mathbb{C}[y]$ be a polynomial divisible by y^3 and $\operatorname{Jac}(Q_2, W_f)$ the Jacobi algebra defined by the relations

$$xy + yx = 0 \quad \text{and} \quad x^2 = f'(y).$$

We will use a proof analogous to [BW17, Lemma 4.6] to show that the Jacobi algebra is a finite dimensional algebra. Firstly, observe that the two relations allow us to rewrite any element of $\operatorname{Jac}(Q_2, W_f)$ as an element of the span

$$\operatorname{span}\{y^i, y^j x \mid i, j \geq 0\}.$$

Let $f'_{\text{even}}(y)$, $f'_{\text{odd}}(y)$ denote the derivatives of the even part and odd parts of f. Then $f'_{\text{even}}(y)$ anti-commutes with x because it has odd parity, and f'_{odd} commutes with x. Hence, multiplying the second relation by x on the left and on the right yields:

$$f'_{\text{even}}(y)x + f'_{\text{odd}}(y)x = (f'(y))x = x^3 = x(f'(y)) = -f'_{\text{even}}(y)x + f'_{\text{odd}}(y)x,$$

which implies $f'_{\text{even}}(y)x = 0$ in $\operatorname{Jac}(Q_2, W_f)$. With the assumption $f_{\text{even}} \neq 0$, this gives a linear dependence relation in $\operatorname{Jac}(Q_2, W_f)$ for the leading term of $f'_{\text{even}}(y)x$, and it follows that one can eliminate $y^j x$ from the spanning set for $j \geq \deg f'_{\text{even}}(y)$. Multiplying the relation just found by x yields

$$0 = f'_{\text{even}}(y)x^2 = f'_{\text{even}}(y)f'(y),$$

which also allows us to write the leading term of $f'_{\text{even}}(y)f'(y)$ as a linear combination of lower degree terms. This shows that y^i can be eliminated from the spanning set for $i \geq \deg f'_{\text{even}}(y)f'(y)$. It follows that any element of $\operatorname{Jac}(Q_2, W_f)$ is contained in

$$\operatorname{span}\{y^i, y^i x \mid 0 \leq i \leq \deg f'_{\text{even}}(y) + \deg f'(y),\ 0 \leq j < \deg f'_{\text{even}}(y)\}.$$

Because this is finite, it follows that the Jacobi algebra is finite dimensional. □

In the remainder of this section we show that $\operatorname{Jac}(Q_2, W_f)$ is in fact the contraction algebra of a certain *slice* of the universal flop. Indeed, given a parameter $f \in \mathbb{C}[y]$ we define a slice $\mathfrak{p}_f \colon Y_f \to \operatorname{Spec} R_f$, where R_f is an $\mathcal{R}$-algebra of the form[2]

$$R_f = \frac{\mathbb{C}[r, s, u, v]}{(u^2 + r(r - f'_{\text{odd}}(s^{\frac{1}{2}}))^2 - sv^2 + r(f'_{\text{even}}(s^{\frac{1}{2}})^2).}$$

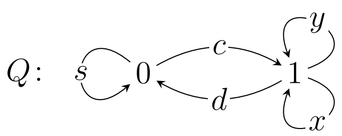
and claim that it has an NCCR $\Lambda := \Lambda \otimes_{\mathcal{R}} R_f$ with contraction algebra isomorpic to $\operatorname{Jac}(Q_2, W_f)$. Moreover, we will show that this NCCR can itself be expressed as a quiver with potential. After checking all the necessary technical details in the following three subsections, we then obtain a family of length 2 flops, yielding the following theorem.

Theorem 2.7. *Let $f \in \mathbb{C}[y]$ be divisible by y^3 with $f_{\text{even}} \neq 0$. Then*

- *the map $\mathfrak{p}_f \colon Y_f \to \operatorname{Spec} R_f$ is a flopping contraction of curves in a smooth threefold Y_f,*
- *the length over the origin is $\ell = 2$,*
- *$\mathcal{T} \otimes_R R_f \in \operatorname{coh} Y_f$ is a tilting bundle with endomorphism algebra $\Lambda \simeq \Lambda \otimes_R R_f$,*
- *there is an isomorphism $\Lambda \simeq \operatorname{Jac}(Q, W_f)$.*

§ 2.2.1 | Slicing the NCCR

Fix a polynomial $f(y) = f_{\text{even}}(y) + f_{\text{odd}}(y) \in \mathbb{C}[y]$ divisible by y^3, and consider the following quiver with potential:



$$W_f = x^2 y - f(y) + ycd + cdy - sdc + 2f_{\text{even}}\left(s^{\frac{1}{2}}\right)$$

[2] Although we write a fractional exponent $s^{\frac{1}{2}}$ in this definition, the reader may verify that the relations are in fact polynomial due to the parity of f_{even} and f_{odd}.

The Jacobi algebra $\mathrm{Jac}(Q, W_f)$ is the quotient of the path algebra $\mathbb{C}Q$ by the ideal generated by the cyclic derivatives of W_f, which are as follows:

$$\partial_s W_f = f'_{\text{even}}(s^{\frac{1}{2}})s^{-\frac{1}{2}}, \quad \partial_c W_f = dy^2 - ds, \quad \partial_d W_f = y^2c - sc,$$
$$\partial_x W_f = xy + yx, \quad \partial_y W_f = x^2 - f'(y) + ycd + cdy.$$

In this subsection we show that this Jacobi algebra can be obtained as a base change of the NCCR Λ of the universal flop over a ring $R_f := \mathcal{R}/J$, for $J = (g_1, g_2, g_3)$ generated by the following sequence

$$g_1 = z, \ g_2 = r - w - f'_{\text{odd}}(s^{\frac{1}{2}}), \ g_3 = t - f'_{\text{even}}(s^{\frac{1}{2}})s^{-\frac{1}{2}}. \tag{2.3}$$

The following lemma verifies that $\Lambda := \Lambda \otimes_{\mathcal{R}} R_f$ does indeed recover the Jacobi algebra.

Lemma 2.8. *There is a $\mathbb{C}\mathcal{Q}$-algebra isomorphism*

$$\mathrm{Jac}(Q, W_f) \simeq \Lambda/J\Lambda \simeq \Lambda$$

Proof. We will give a direct proof by constructing two homomorphisms

$$\varphi\colon \mathbb{C}Q \to \Lambda/J\Lambda, \quad \psi\colon \Lambda \to \mathrm{Jac}(Q, W_f),$$

showing they descend along the quotients $\mathbb{C}Q \to \mathrm{Jac}(Q, W_f)$ and $\Lambda \to \Lambda/J\Lambda \simeq \Lambda$, and find that this yields a pair of mutually inverse maps between $\mathrm{Jac}(Q, W_f)$ and Λ.

The map ψ is the $\mathbb{C}\mathcal{Q}$-algebra homomorphism defined on the generators of $\mathbb{H}$ as

$$\psi(r) = x^2 + dyc + f'_{\text{odd}}(s^{\frac{1}{2}}), \quad \psi(s) = s + y^2, \quad \psi(t) = f'_{\text{even}}(\psi(s))\psi(s)^{-1},$$
$$\psi(T) = (cd + x + y - \tfrac{1}{2}\psi(t))^2 + dxc + s + f'_{\text{odd}}(s^{\frac{1}{2}}) + \tfrac{1}{4}\psi(t)^2 \cdot 1_0.$$

To verify that this is a homomorphism, we should check (1) that r, s, t, T are mapped into the centre of $\mathrm{Jac}(Q, W_f)$, (2) the defining relations of the universal flop hold. We check both conditions, starting with (1). The centrality of $\psi(s)$ follows easily from the relations $xy + yx = 0$, $y^2c = cs$, and $dy^2 = sd$:

$$[\psi(s), s] = [s, s] = 0, \quad [\psi(s), x] = y^2x - xy^2 = 0, \quad [\psi(s), y] = y^3 - y^3 = 0,$$
$$[\psi(s), c] = y^2c - cs = 0, \quad [\psi(s), d] = sd - dy^2 = 0.$$

Because $\psi(s)$ is central, so is any polynomial expressed in terms of $\psi(s)$, and in particular $\psi(t)$ is central. Next, the element $\psi(r)$ commutes with x,y, and s by the identity

$$[\psi(r), x] = x^3 - x^3 = 0, \quad [\psi(r), y] = x^2y - yx^2 = 0,$$
$$[\psi(r), s] = dycs - sdyc + [f'_{\text{odd}}(s^{\frac{1}{2}}), s] = dycs - dy^3c = 0.$$

From the relation $dc = \psi(t)\cdot 1_0$, it follows that

$$\begin{aligned}[\psi(r), c] &= x^2c - c(dyc + f'_{\text{odd}}(s^{\frac{1}{2}})) \\ &= f'(y)c - ycdc - cdyc - cdyc + cf'_{\text{odd}}(s^{\frac{1}{2}}) = [\psi(t), yc] = 0\end{aligned}$$

and similarly for d. Finally, we check that $\psi(T)$ is central. The commutator with s is

$$\begin{aligned}[\psi(T), s] &= [dxc, s] + [s + f'_{\text{odd}}(s^{\frac{1}{2}}) + \tfrac{1}{4}\psi(t)^2\cdot 1_0, s] \\ &= dxy^2c - sdxc = dy^2xc - sdxc = 0.\end{aligned}$$

Next, the relation $x^2 - f'(y) + ycd + cdy$ implies

$$\begin{aligned}[\psi(T), x] &= cdyx + ycdx - xcdy - xycd - \psi(t)[x, y] \\ &= (-ycdx - x^3 + f'(y)x) + ycdx + (xycd + x^3 - xf'(y)) - xycd \\ &\quad - xf'_{\text{even}}(y) + f'_{\text{even}}(y)x \\ &= f'_{\text{odd}}(y)x - xf'_{\text{odd}}(y) = 0.\end{aligned}$$

and $[\psi(T), y]$ vanishes by a similar computation. Lastly, there is the identity

$$\begin{aligned}\psi(T)c &= cdxc + xcdc + cdyc + ycdc + x^2c + y^2c - \psi(t)(x+y)c + \tfrac{1}{4}\psi(t)^2c \\ &= cdxc + cdyc + (xc + yc)(dc - f'_{\text{even}}(s^{\frac{1}{2}})s^{-\frac{1}{2}}) + (f'(y) - ycd + cdy)c \\ &\quad + cs + \tfrac{1}{4}c\psi(t)^2 \\ &= c(dxc + s + f'_{\text{odd}}(s^{\frac{1}{2}}) + \tfrac{1}{4}\psi(t)^2) + yc(f'_{\text{even}}(s^{\frac{1}{2}})s^{-\frac{1}{2}} - dc) = c\psi(T),\end{aligned}$$

which shows that $\psi(T)$ commutes with c, and a similar computation for d shows that $\psi(T)$ is indeed central.

For (2) the relations $s\cdot 1_1 = y^2$, $r\cdot 1_1 = x^2$ and $T\cdot 1_1 = (cd + x + y - \frac{1}{2}t)^2$ hold in the image by construction, while $cd = f_{\text{even}}(s^{\frac{1}{2}})s^{-\frac{1}{2}} = \psi(t\cdot 1_0)$ is one of the defining relations of $\text{Jac}(Q, W_f)$. It follows that $\psi\colon \Lambda \to \text{Jac}(Q, W_f)$ is a well-defined homomorphism, which is moreover surjective because the generators x, y, c, d, s are in the image. By construction, its kernel contains the ideal $J\Lambda$ because the image of the generators of J vanish:

$$\begin{aligned}\psi(z) = (xy + yx) + (dxc + dyc + \psi(r\cdot 1_0) + \psi(s\cdot 1_0) - \psi(T) + \tfrac{1}{4}\psi(t)^2\cdot 1_0) = 0 \\ \psi(r - dyc + ycd + cdy - ty - f'_{\text{odd}}(s^{\frac{1}{2}})) = (x^2 - f'(y) + ycd + cdy) = 0 \\ \psi(t - f_{\text{even}}(s^{\frac{1}{2}})s^{-\frac{1}{2}}) = 0.\end{aligned}$$

It follows that ψ descends to a surjective homomorphism $\psi\colon \Lambda/J\Lambda \to \text{Jac}(Q, W_f)$. Conversely, the map $\varphi\colon \mathbb{C}Q \to \Lambda/J\Lambda$ is simply the $\mathbb{C}\mathcal{Q}$-algebra homomorphism that sends the additional loop s to $\varphi(s) = s\cdot e_0$, which is clearly well-defined. We claim that the kernel contains the relations of $\mathrm{Jac}(Q, W_f)$:

$$\begin{aligned}
\varphi(f'_{\text{even}}(s^{\frac{1}{2}})s^{-\frac{1}{2}} - dc) &= f'_{\text{even}}(s^{\frac{1}{2}})s^{-\frac{1}{2}} \cdot 1_0 - dc = t \cdot 1_0 - dc = 0 \\
\varphi(dy^2 - sd) &= ds - sd = 0 \\
\varphi(y^2c - cs) &= sc - sc = 0 \\
\varphi(xy + yx) &= z \cdot 1_1 = 0 \\
\varphi(x^2 - f'(y) + cdy + ycd) &= (r - f'_{\text{odd}}(s^{\frac{1}{2}}) + cdy + ycd - f'_{\text{even}}(s^{\frac{1}{2}})) \cdot 1_1 = 0.
\end{aligned}$$

Hence φ descends to a homomorphism $\varphi\colon \mathrm{Jac}(Q, W_f) \to \Lambda/J\Lambda$. One checks explicitly that it is an inverse for ψ, so that $\mathrm{Jac}(Q, W_f) \simeq \Lambda/J\Lambda$ as $\mathbb{C}\mathcal{Q}$-algebras. □

Because the contraction algebra of Λ is obtained by deleting the vertex 0 in the quiver, we now also find that the algebra $\Lambda_{\text{con}} := \Lambda_{\text{con}} \otimes_{\mathcal{R}} R_f$ has a superpotential description

$$\Lambda_{\text{con}} \simeq \mathrm{Jac}(Q_2, W_f),$$

where W_f is restricted to the subquiver Q_2 generated by x and y. By Lemma 2.6, this algebra is finite dimensional if the parameter f satisfies $f_{\text{even}} \neq 0$.

We wish to compare the R_f-algebra Λ to the base change of the commutative resolution $\pi\colon \mathcal{Y} \to \mathcal{Y}_{\text{con}}$. Let $Y_{\text{con}} = \operatorname{Spec} R_f$ then we obtain a map $\mathpzc{p} = \mathpzc{p}_f\colon Y \to Y_{\text{con}}$ from the fibre product $Y = \mathcal{Y} \times_{\mathcal{Y}_{\text{con}}} Y_{\text{con}}$. A priori it is not clear if the map $\mathpzc{p}\colon Y \to Y_{\text{con}}$ is a flop, or even a contraction morphism, nor that the tilting bundle $\mathcal{T}$ induces a derived equivalence between $\mathrm{D}^b(\Lambda)$ and $\mathrm{D}^b(Y)$.

§ 2.2.2 | Sufficiently nice slices

In this subsection we formulate a criterion that guarantees that a map $\mathpzc{p}\colon Y \to Y_{\text{con}}$ cut out from the universal flop $\pi\colon \mathcal{Y} \to \mathcal{Y}_{\text{con}}$ is a contraction morphism, and that the tilting bundle $\mathcal{T}$ restricts to a tilting bundle on Y. To do this we break the base change up into a sequence of codimension 1 slices.

Setup 2.9. Let $f\colon \mathcal{X} \to \mathcal{X}_{\text{con}}$ be a projective, surjective map between integral varieties. Then by a *slice* we will mean a diagram

$$\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \\
\big\downarrow {\scriptstyle f|_{X_{\text{con}}}} & & \big\downarrow {\scriptstyle f} \\
X_{\text{con}} & \longrightarrow & \mathcal{X}_{\text{con}}
\end{array}$$

where $X_{\text{con}} \hookrightarrow \mathcal{X}_{\text{con}}$ is an integral closed subvariety cut out by a nontrivial element $g \in \mathrm{H}^0(\mathcal{X}_{\text{con}}, \mathcal{O}_{\mathcal{X}_{\text{con}}})$, such that $X := \mathcal{X} \times_{\mathcal{X}_{\text{con}}} X_{\text{con}}$ is also integral.

Lemma 2.10. *Let $f\colon \mathcal{X} \to \mathcal{X}_{\text{con}}$ be a map between integral varieties which is projective, surjective, and satisfies $\mathbf{R}f_*\mathcal{O}_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}_{\text{con}}}$. Then for every slice as in Setup 2.9 the map*

$f|_{X_{\mathrm{con}}}\colon X \to X_{\mathrm{con}}$ *also satisfies these three conditions.*

Proof. Surjectivity and projectivity are preserved under arbitrary base change (see e.g. [Stacks, Section 02WE]), so every slice $f|_{X_{\mathrm{con}}}\colon X \to X_{\mathrm{con}}$ is again a surjective, projective map between integral varieties. Hence it suffices to show that it satisfies the condition $\mathbf{R}(f|_{X_{\mathrm{con}}})_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{\mathrm{con}}}$. Let $g \in \mathrm{H}^0(\mathcal{X}_{\mathrm{con}}, \mathcal{O}_{\mathcal{X}_{\mathrm{con}}})$ be a regular section which cuts out $X_{\mathrm{con}} \hookrightarrow \mathcal{X}_{\mathrm{con}}$, then g can be viewed as a nontrivial section on $\mathcal{X}$ via

$$\mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \mathrm{Hom}_{\mathcal{X}}(f^*\mathcal{O}_{\mathcal{X}_{\mathrm{con}}}, \mathcal{O}_{\mathcal{X}}) \simeq \mathrm{Hom}_{\mathcal{X}_{\mathrm{con}}}(\mathcal{O}_{\mathcal{X}_{\mathrm{con}}}, f_*\mathcal{O}_{\mathcal{X}}) \simeq \mathrm{H}^0(\mathcal{X}_{\mathrm{con}}, \mathcal{O}_{\mathcal{X}_{\mathrm{con}}}).$$

Because $\mathcal{X}$ is integral, for each open $U \subset \mathcal{X}$ the ring $\mathcal{O}_{\mathcal{X}}(U)$ is a domain and the map $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) \to \mathcal{O}_{\mathcal{X}}(U)$ is injective. In particular, for each U the restriction $g|_U$ is a nonzerodivisor and induces an injective map $g|_U\cdot\colon \mathcal{O}_{\mathcal{X}}(U) \to \mathcal{O}_{\mathcal{X}}(U)$. Hence, there is a short exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \xrightarrow{g\cdot} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_X \to 0, \tag{2.4}$$

which defines the structure sheaf of X. Applying $\mathbf{R}f_*$ yields the long exact sequence

$$\ldots \to \mathbf{R}^n f_*\mathcal{O}_{\mathcal{X}} \to \mathbf{R}^n f_*\mathcal{O}_{\mathcal{X}} \to \mathbf{R}^n f_*\mathcal{O}_X \to \mathbf{R}^{n+1} f_*\mathcal{O}_{\mathcal{X}} \to \ldots.$$

Because $\mathbf{R}f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{\mathrm{con}}}$, the entries $\mathbf{R}^n f_*\mathcal{O}_{\mathcal{X}}$ vanish for $n \geq 1$, and it follows that

$$\mathbf{R}f_*\mathcal{O}_X \simeq \mathcal{O}_{\mathcal{X}_{\mathrm{con}}}/g\mathcal{O}_{\mathcal{X}_{\mathrm{con}}} \simeq \mathcal{O}_{X_{\mathrm{con}}}.$$ □

Hence, the criterion in Setup 2.9 is almost sufficient to show that a contraction slices out to a contraction: one only needs to verify that the varieties X and X_{con} are normal. It is also sufficient to guarantee that tilting bundles restrict to tilting bundles.

Lemma 2.11. *Let* $f\colon \mathcal{X} \to \mathcal{X}_{\mathrm{con}}$ *be a map as in Lemma 2.10, and suppose there exists a tilting bundle* $\mathcal{F} \in \mathrm{coh}\,\mathcal{X}$*. Then for every slice as in Setup 2.9 the bundle* $\mathcal{F} \otimes_{\mathcal{X}} \mathcal{O}_X$ *is again tilting on* X*, and*

$$\mathrm{End}_X(\mathcal{F} \otimes_{\mathcal{X}} \mathcal{O}_X) \simeq \mathrm{End}_{\mathcal{X}}(\mathcal{F})/g\,\mathrm{End}_{\mathcal{X}}(\mathcal{F}),$$

where $g \in \mathrm{H}^0(\mathcal{X}_{\mathrm{con}}, \mathcal{O}_{\mathcal{X}_{\mathrm{con}}})$ *denotes the section defining* X_{con}*.*

Proof. Let $g \in \mathrm{H}^0(\mathcal{X}_{\mathrm{con}}, \mathcal{O}_{\mathcal{X}_{\mathrm{con}}})$ be a regular section defining $X_{\mathrm{con}} \hookrightarrow \mathcal{X}_{\mathrm{con}}$. Because $\mathcal{F}$ is a tilting bundle, it locally free and the functor $\mathcal{F} \otimes_{\mathcal{X}} -$ preserves the short exact sequence in (2.4), yielding

$$0 \to \mathcal{F} \xrightarrow{g\cdot} \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{X}} \mathcal{O}_X \to 0.$$

The tilting property moreover implies that $\mathrm{Ext}^i_{\mathcal{X}}(\mathcal{F}, \mathcal{F}) = 0$, so that the corresponding entries in the long exact sequence

$$\ldots \to \operatorname{Ext}^i_{\mathcal{X}}(\mathcal{F},\mathcal{F}) \to \operatorname{Ext}^i_{\mathcal{X}}(\mathcal{F},\mathcal{F}) \to \operatorname{Ext}^i_{\mathcal{X}}(\mathcal{F},\mathcal{F}\otimes \mathcal{O}_X) \to \operatorname{Ext}^{i+1}_{\mathcal{X}}(\mathcal{F},\mathcal{F}) \to \ldots,$$

of $\mathbf{R}\mathrm{Hom}_{\mathcal{X}}(\mathcal{F},-)$ vanish. It follows by adjunction that

$$\operatorname{Ext}^i_{\mathcal{X}}(\mathcal{F},\mathcal{F}\otimes_{\mathcal{X}} \mathcal{O}_X) \simeq \operatorname{Ext}^i_{X}(\mathcal{F}\otimes_{\mathcal{X}} \mathcal{O}_X,\mathcal{F}\otimes_{\mathcal{X}} \mathcal{O}_X) = 0$$

for all $i > 0$, and $\operatorname{End}_X(\mathcal{F}\otimes_{\mathcal{X}} \mathcal{O}_X) \simeq \operatorname{End}_{\mathcal{X}}(\mathcal{F})/g\operatorname{End}_{\mathcal{X}}(\mathcal{F})$. To show that $\mathcal{F}\otimes_{\mathcal{X}} \mathcal{O}_X$ is tilting it therefore suffices to show that it is a generator in $\mathrm{D}(X) = \mathrm{D}(\mathrm{QCoh}\,X)$, in the sense that

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{F}\otimes_{\mathcal{X}} \mathcal{O}_X,\mathcal{G}) = 0 \quad\implies\quad \mathcal{G}=0,$$

for all $\mathcal{G} \in \mathrm{D}(X)$. This follows by adjunction: let $i\colon X \hookrightarrow \mathcal{X}$ denote the closed immersion of X as a subscheme of $\mathcal{X}$, then because $\mathcal{F}$ is a generator in $\mathrm{D}(\mathcal{X})$

$$0 = \mathbf{R}\mathrm{Hom}_X(\mathcal{F}\otimes_{\mathcal{X}} \mathcal{O}_X,\mathcal{G}) \simeq \mathbf{R}\mathrm{Hom}_X(\mathbf{L}i^*\mathcal{F},\mathcal{G}) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{X}}(\mathcal{F},\mathbf{R}i_*\mathcal{G}) \implies \mathbf{R}i_*\mathcal{G}=0,$$

and hence $\mathcal{G}=0$ because i is affine. The result follows. □

§ 2.2.3 | Slicing out flops

For the rest of this section we fix a parameter $f \in \mathbb{C}[y]$ which is divisible by y^3, and consider the associated map þ: $Y \to Y_{\mathrm{con}} = \operatorname{Spec} R_f$. By construction, Y_{con} is generated by a sequence $g_1, g_2, g_3 \in \mathcal{R}$, so that þ fits into a diagram

$$\begin{array}{ccccccc} Y = Y_3 & \longrightarrow & Y_2 & \longrightarrow & Y_1 & \longrightarrow & \mathcal{Y} \\ \downarrow \text{þ} & & \downarrow & & \downarrow & & \downarrow \pi \\ Y_{\mathrm{con}} = \operatorname{Spec} R_3 & \longrightarrow & \operatorname{Spec} R_2 & \longrightarrow & \operatorname{Spec} R_1 & \longrightarrow & \operatorname{Spec} \mathcal{R} \end{array} \tag{2.5}$$

where $R_k = \mathcal{R}/(g_1,\ldots,g_k)$ and $Y_k = \mathcal{Y}\times_{\operatorname{Spec}\mathcal{R}} \operatorname{Spec} R_k$. We first verify that each square in this diagram is a slice as in Setup 2.9. It suffices to show that all the spaces in the diagram are integral.

Lemma 2.12. *The rings R_1, R_2, R_3 are integral domains.*

Proof. By inspection, each generator contains a different linear term. Therefore each generator eliminates a different generator of $\mathcal{R}$, yielding the quotients:

$$\begin{aligned} R_1 &\simeq \frac{\mathbb{C}[r,s,t,u,v,w]}{(u^2 - rw^2 - sv^2 + rst^2)} \\ R_2 &\simeq \frac{\mathbb{C}[r,s,t,u,v]}{(u^2 - r(r - f'_{\mathrm{odd}}(s^{\frac{1}{2}})^2 - sv^2 + rst^2)} \\ R_3 &\simeq \frac{\mathbb{C}[r,s,u,v]}{(u^2 - r(r - f'_{\mathrm{odd}}(s^{\frac{1}{2}})^2 - sv^2 + r(f'_{\mathrm{even}}(s^{\frac{1}{2}}))^2)}. \end{aligned}$$

Each of these quotients is defined by a single equation of the form $u^2 - h$ for some polynomial h which has degree 0 in u. Such a hypersurface is reducible if and only if h is a square, but this is clearly not the case as the r-degree of h is odd. It follows that the rings R_1, R_2, and R_3 are domains. □

Lemma 2.13. *The space Y_1, Y_2, Y_3 are smooth and connected, hence integral.*

Proof. As recalled in §2.1.2, the space $\mathcal{Y}$ is covered by two charts U_x and U_y which are both isomorphic to $\mathbb{A}^6$. Each space Y_k is therefore covered by the base change of these two charts over $\mathcal{R} \to R_k$. Because $\operatorname{Spec} R_k$ contains the origin for each k, it follows that Y_k contains the curve $\mathrm{C} \subset \mathcal{Y}$ supported over it. Hence, the base changes of U_x and U_y still have a nonzero intersection in Y_k, which implies Y_k is connected. It remains to check that U_x and U_y remain smooth after base change.

The ring map $\mathcal{R} \to \mathbb{C}[U_x]$ in (2.1) maps the generators $g_1, g_2, g_3 \in J$ to

$$\begin{aligned} g_1 &\mapsto u_1 := y_{01} + x_{01}y_{10}, \\ g_2 &\mapsto u_2 := d_0 - f'_{\text{even}}((y_{00}^2 + y_{10}y_{01})^{\frac{1}{2}})(y_{00}^2 + y_{10}y_{01})^{-\frac{1}{2}}, \\ g_3 &\mapsto u_3 := x_{01} - (d_0y_{00} + d_1y_{10}) - f'_{\text{odd}}((y_{00}^2 + y_{01}y_{10})^{\frac{1}{2}}). \end{aligned}$$

The first two equations have independent linear terms and cut the chart down to the rings $\mathbb{C}[U_x]/(u_1) \simeq \mathbb{C}[d_0, d_1, x_{01}, y_{00}, y_{10}]$ and $\mathbb{C}[U_x]/(u_1, u_2) \simeq \mathbb{C}[d_1, x_{01}, y_{00}, y_{10}]$, which are regular. Eliminating these variables from u_3 yields

$$u_3 = x_{01} - f'_{\text{even}}((y_{00}^2 - x_{01}y_{10}^2)^{\frac{1}{2}})(y_{00}^2 - x_{01}y_{10}^2)^{-\frac{1}{2}}y_{00} + d_1y_{10} - f'_{\text{odd}}((y_{00}^2 - x_{01}y_{10}^2)^{\frac{1}{2}}).$$

A short argument shows that the critical locus of u_3 in $\operatorname{Spec} \mathbb{C}[d_1, x_{01}, y_{00}, y_{10}]$ is trivial: setting $\partial_{d_1} u_3 = y_{10}$ to 0 yields $(\partial_{x_{01}} u_3)|_{y_{10}=0} = 1$. It follows that $\mathbb{C}[U_x]/(u_1, u_2, u_3)$ is also regular, so that $U_x \cap Y_k$ is smooth for $k = 1, 2, 3$.

The smoothness on the other chart is verified in a similar way. The ring map $\mathcal{R} \to \mathbb{C}[U_y]$ in (2.2) maps the relations g_1, g_2, g_3 to

$$\begin{aligned} g_1 &\mapsto v_1 := x_{01} + y_{01}x_{10}, \\ g_2 &\mapsto v_2 := d_0 - f'_{\text{even}}(y_{01}^{\frac{1}{2}})y_{01}^{-\frac{1}{2}}, \\ g_3 &\mapsto v_3 := x_{00}^2 + x_{01}x_{10} - d_1 - f'_{\text{odd}}(y_{01}^{\frac{1}{2}}). \end{aligned}$$

In this case, the polynomials v_1, v_2, and v_3 each contain an independent linear term, so that $\mathbb{C}[U_y]/(u_1) \simeq \mathbb{C}[d_0, d_1, x_{00}, x_{10}, y_{01}]$, $\mathbb{C}[U_y]/(u_1, u_2) \simeq \mathbb{C}[d_1, x_{00}, x_{10}, y_{01}]$, and $\mathbb{C}[U_y]/(u_1, u_2, u_3) \simeq \mathbb{C}[x_{00}, x_{10}, y_{01}]$ are all regular. The result follows. □

Corollary 2.14. *The squares in* (2.5) *are slices as in Setup 2.9.*

Because $\pi \colon \mathcal{Y} \to \mathcal{Y}_{\text{con}}$ is a contraction, Lemma 2.10 now guarantees that $p\colon Y \to$

Spec R is a projective surjective map with $\mathbf{R}\rho_*\mathcal{O}_Y = \mathcal{O}_{Y_{\mathrm{con}}}$. To show that it is a small contraction we have to show that it is also a birational map between normal varieties with $\mathrm{Ex}(\rho)$ of codimension 2. These additional requirements are satisfied if Y_{con} has a sufficiently small intersection with $\mathrm{Sing}\,\mathcal{Y}_{\mathrm{con}}$.

Lemma 2.15. *Suppose $Y_{\mathrm{con}} \subset \mathcal{Y}_{\mathrm{con}}$ intersects $\mathrm{Sing}\,\mathcal{Y}_{\mathrm{con}}$ in a finite set of points, then $\rho\colon Y \to Y_{\mathrm{con}}$ is a small contraction.*

Proof. Because ρ satisfies $\mathbf{R}\rho_*\mathcal{O}_Y = \mathcal{O}_{Y_{\mathrm{con}}}$, it follows that $R = \Gamma(Y_{\mathrm{con}}, \rho_*\mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)$ is a normal integral domain, as Y is an integral normal scheme. Hence Y_{con} is again normal and it follows that $\rho\colon Y \to Y_{\mathrm{con}}$ is a contraction. We check that it is small.

Let $U = \mathcal{Y}_{\mathrm{con}} \setminus \mathrm{Sing}\,\mathcal{Y}_{\mathrm{con}}$, then $\pi\colon \mathcal{Y} \to \mathcal{Y}_{\mathrm{con}}$ restricts to an isomorphism $\pi|_U\colon \pi^{-1}(U) \xrightarrow{\sim} U$. Hence the restriction

$$\rho|_{Y_{\mathrm{con}}\cap U} = \pi|_{Y_{\mathrm{con}}\cap U}\colon \rho^{-1}(Y_{\mathrm{con}} \cap U) \to Y_{\mathrm{con}} \cap U$$

is also an isomorphism. Because the complement of $Y_{\mathrm{con}} \cap U$ is precisely the intersection $Y_{\mathrm{con}} \cap \mathrm{Sing}\,\mathcal{Y}_{\mathrm{con}}$, which is a finite number of points by assumption, the exceptional locus consists of the 1-dimensional fibres over these points. It follows that ρ is small. □

The condition of Lemma 2.15 can be checked easily by using the base change of the contraction algebra.

Lemma 2.16. *Suppose the parameter $f \in \mathbb{C}[y]$ satisfies $f_{\mathrm{even}} \neq 0$, then $Y_{\mathrm{con}} \cap \mathrm{Sing}\,\mathcal{Y}_{\mathrm{con}}$ is a finite set of points which contains the origin.*

Proof. By [DW19a, Theorem 1.1] the support of the contraction algebra Λ_{con} is given by $\mathrm{Supp}_{\mathcal{Y}_{\mathrm{con}}} \Lambda_{\mathrm{con}} = \mathrm{Sing}\,\mathcal{Y}_{\mathrm{con}}$. Hence, the base change $\Lambda_{\mathrm{con}} \otimes_{\mathcal{R}} R_f$ has support

$$\mathrm{Supp}_{Y_{\mathrm{con}}}(\Lambda_{\mathrm{con}} \otimes_{\mathcal{R}} R_f) = Y_{\mathrm{con}} \cap \mathrm{Supp}_{\mathcal{Y}_{\mathrm{con}}}\Lambda_{\mathrm{con}} = Y_{\mathrm{con}} \cap \mathrm{Sing}\,\mathcal{Y}_{\mathrm{con}}.$$

It follows from Lemma 2.8 that $\Lambda_{\mathrm{con}} \otimes_{\mathcal{R}} R_f \simeq \mathrm{Jac}(Q_2, W_f)$, which is finite dimensional by Lemma 2.6. It follows that $\mathrm{Supp}_{Y_{\mathrm{con}}} \Lambda_{\mathrm{con}} \otimes_{\mathcal{R}} R_f$ is a finite set of points. Moreover, the singularity at the origin $\mathfrak{o} \in \mathcal{Y}_{\mathrm{con}}$ is clearly contained in this because the generators in (2.3) are contained in this maximal ideal. □

Using the above lemmas, the proof of the theorem is now straightforward.

Proof of Theorem 2.7. By Corollary 2.14 the map $\rho\colon Y \to Y_{\mathrm{con}}$ is constructed from a sequence of slices, hence Lemma 2.10 implies that ρ is projective, surjective, and satisfies $\mathbf{R}\rho_*\mathcal{O}_Y = \mathcal{O}_{Y_{\mathrm{con}}}$. Because the parameter $f \in \mathbb{C}[y]$ satisfies the assumptions of Lemma 2.16, it follows that $Y_{\mathrm{con}} \cap \operatorname{Sing} \mathcal{Y}_{\mathrm{con}}$ is a finite number of points. Hence, Lemma 2.15 implies that $\not{p}$ is a small contraction.

Because Y_{con} is a hypersurface, it has Gorenstein singularities and it follows that the canonical sheaf $\omega_{Y_{\mathrm{con}}}$ is a line bundle. The spaces Y and Y_{con} are normal, so the line bundles ω_Y and $\omega_{Y_{\mathrm{con}}}$ are determined by their restriction to the complement of the codimension ≥ 2 subsets $\mathrm{Ex}(\not{p})$ and $\not{p}(\mathrm{Ex}(\not{p}))$. Because $\not{p}$ is an isomorphism on these complements, it follows that $\omega_Y = \not{p}^* \omega_{Y_{\mathrm{con}}}$. For every exceptional curve $\mathrm{C} \subset \mathrm{Ex}(\not{p})$ the line bundle $\omega_{Y_{\mathrm{con}}}$ is trivial on some neighbourhood $U \ni \not{p}(\mathrm{C})$. Hence

$$\omega_Y|_{\not{p}^{-1}(U)} \simeq \not{p}^*(\omega_{Y_{\mathrm{con}}}|_U) \simeq \not{p}^*(\mathcal{O}_{Y_{\mathrm{con}}}|_U) \simeq \mathcal{O}_Y|_{\not{p}^{-1}(U)}.$$

Because $\not{p}^{-1}(U)$ is an open neighbourhood of C, it follows that $K_Y \cdot \mathrm{C} = 0$. This shows that $\not{p}$ is a flopping contraction.

The fibre of $\not{p}$ over the origin coincides with the fibre of the universal flop over the maximal ideal $\mathfrak{o} = (r, s, t, u, v, w, z) \subset \mathcal{R}$. The image of $\mathfrak{o}$ along the map $\mathcal{R} \to \mathbb{C}[U_x]$ is

$$\begin{aligned}(x_{01}, y_{00}^2 + y_{01}y_{10}, d_0, d_1 y_{00} + \tfrac{1}{2}d_0(x_{01}y_{10} - y_{01}), d_1, d_0 y_{01} - d_1 y_{00}, y_{01} + y_{10}x_{01}) \\ = (x_{01}, d_0, d_1, y_{01}, y_{00}^2),\end{aligned}$$

which cuts out a line of multiplicity 2 in the chart $U_x \subset \mathcal{Y}$. Hence the scheme theoretic fibre has length 2 at the generic point of $\mathrm{C} \subset \mathcal{Y}$, and $\not{p}$ has length $\ell = 2$ over this point.

Finally, it follows from Lemma 2.11 that $\mathcal{T} \otimes_{\mathcal{R}} R_f$ is a a tilting bundle on Y with endomorphism algebra Λ, which is isomorphic to $\mathrm{Jac}(Q, W_f)$ by Lemma 2.8. □

We remark that the proof Lemma 2.13 implies that the family we have constructed has the following curious property, which will be helpful for the calculation of DT invariants in a later chapter.

Lemma 2.17. *The singularity Y_{con} has absolute units $R_f^\times = \mathbb{C}^\times$.*

Proof. The map $\not{p}\colon Y \to Y_{\mathrm{con}}$ is surjective and therefore in particular a dominant map. Because Y and Y_{con} are moreover integral, there is an injective ring homomorphism

$$R_f = \mathcal{O}_{Y_{\mathrm{con}}}(Y_{\mathrm{con}}) \to \mathcal{O}_Y(U_y) \simeq \mathbb{C}[x_{00}, x_{10}, y_{01}]$$

Because any ring homomorphism maps units to units, there is an induced injective map $R_f^\times \hookrightarrow \mathbb{C}[x_{00}, x_{10}, y_{01}]^\times = \mathbb{C}^\times$. Since $R_f^\times$ contains $\mathbb{C}^\times$, the result follows. □

§ 2.2.4 | New examples of length 2 flops

Having constructed our family of flops, we now wish to verify that the family actually contains (analytically) *new* examples: we wish to find parameters $f \in \mathbb{C}[y]$ for which the formal neighbourhood of the origin in Spec R_f is distinct from earlier known examples. Consider the following choices of parameters:

$$f_n(y) = \tfrac{1}{2n} y^{2n}, \quad f_{n,m}(y) = \tfrac{1}{2n} y^{2n} + \tfrac{1}{2m+1} y^{2m+1},$$

for $n \geq 2$, $1 \leq m < 2n-1$. Then these parameters have nontrivial even part, and hence give rise to a flopping contraction $\flat_f \colon Y_f \to \operatorname{Spec} R_f$, with R_f defined by the hypersurface equations

$$U_n = u^2 + r^3 - sv^2 + rs^{2n-1}, \quad U_{n,m} = u^2 + r(r - s^m)^2 - sv^2 + rs^{2n-1}.$$

For $n = 2$ this recovers existing examples from the literature: the flop U_2 is Laufer's flop [Lau81] which formed the first example of a length 2 flop, $U_{2,1}$ is Pinkham's deformation [Pin83], and $U_{2,2}$ is the flop constructed by Brown–Wemyss [BW17]. The (classical) enumerative invariants of these two deformations behave differently: the former is distinguished from Laufer's example by its GV invariants, while [BW17] shows that the latter is not. The families $\{U_n\}$ and $\{U_{n,m}\}$ extends this triumvirate: for each n there is a quasihomogeneous singularity U_n, and two regimes of non-quasihomogeneous singularities $U_{n,m}$ for $m < n$ and $m \geq n$ respectively. In this section, we will show that each of these is an analytically distinct hypersurface singularity by comparing their Tjurina numbers. The classical and refined invariants will be analysed in chapter 6.

Definition 2.18. Let $g \in \mathbb{C}[[x_1, \ldots, x_n]]$, then its *Tjurina number* is defined as

$$T(g) := \dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1, \ldots, x_n]]}{(\mathrm{d}g, g)},$$

where $(\mathrm{d}g, g) = (\partial_i g \mid i = 1, \ldots, n) + (g)$ is the ideal of the formal critical locus.

To compute these invariants we construct a *standard basis*, following the example of Hironaka [Hir64]. Standard bases are the local/analytic analogue of *Gröbner bases* used in polynomial rings, and similar algorithms and tricks apply to them; for these we use the papers of Becker [Bec90; Bec93], which also offer a good introduction.

Let $\prec$ denote the degree-lexographical order on the set of monomials in $\mathbb{C}[[x_1, \ldots, x_n]]$: this is the unique relation such that $1 \prec x_1 \prec \ldots \prec x_n$ and such that $x^a \prec x^b$ implies $m \cdot x^a \prec m \cdot x^b$ for all monomials m. For a polynomial $g = \sum_a c_a x^a$ we denote its *least monomial* $\mathrm{lm}(g) = x^a$, which is the the $\prec$-smallest monomial for which $c_a \neq 0$. We also write $\mathrm{lt}(g) = c_b x^b = c_b \,\mathrm{lm}(g)$ for the least term, which includes the nonzero coefficient.

Definition 2.19. Let $I \subset \mathbb{C}[[x_1, \ldots, x_n]]$ be an ideal, and $G \subset I$ a generating set. Then G is a *standard basis* if for all $h \in I$ there exists $g \in G$ such that $\mathrm{lm}(g) \mid \mathrm{lm}(h)$.

From the definition, it is clear that a standard basis $G = \{g_1, \dots, g_m\}$ of I satisfies

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1, \dots, x_n]]}{I} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1, \dots, x_n]]}{(\mathrm{lm}(g_1), \dots, \mathrm{lm}(g_m))}.$$

Because the algebra on right hand side is a quotient by monomial relations, this makes the dimension much easier to compute that the left hand side. In particular, this is useful for computing Tjurina numbers. To find standard bases, we use the S*-series* of the generators:

$$S_{ij} := g_i \tfrac{\mathrm{lcm}(\mathrm{lm}(g_i),\mathrm{lm}(g_j))}{\mathrm{lt}(g_i)} - g_j \tfrac{\mathrm{lcm}(\mathrm{lm}(g_i),\mathrm{lm}(g_j))}{\mathrm{lt}(g_j)},$$

for all pairs i, j with $1 \leq i < j \leq m$. By [Bec90, Corollary 2.3] the set G is a standard basis if and only if each S-series has a *standard representation* with respect to G: that is there exist coefficients $q_k \in \mathbb{C}[[x_1, \dots, x_n]]$ such that

$$S_{ij} = \sum_{i=1}^{m} q_k g_k \quad \text{with} \quad \mathrm{lm}(S_{ij}) \preceq \mathrm{lm}(q_k g_k) \quad \text{for all } k \text{ with } q_k \neq 0.$$

The following two criteria guarantee that S_{ij} has a standard representation: firstly $S_{ij} = 0$ if g_i and g_j are unit multiples of a monomial, and secondly there is the product criterion proved in [Bec93, Theorem 3.1], which states that S_{ij} has a standard representation with respect to $\{g_i, g_j\}$ if $\gcd(\mathrm{lm}(g_i), \mathrm{lm}(g_j)) = 1$.

Returning to our setting, we want to construct a standard basis for the ideals $(\mathrm{d}U_n, U_n)$, $(\mathrm{d}U_{n,m}, U_{n,m})$ associated to the hypersurfaces in $\mathbb{C}[[u, v, r, s]]$ considered above.

Lemma 2.20. *The ideal* $(\mathrm{d}U_n, U_n)$ *has standard basis*

$$G = \{g_1, \dots, g_6\} := \{u,\ sv,\ 3r^2 + s^{2n-1},\ v^2 - (2n-1)rs^{2n-2},\ rs^{2n-1},\ s^{4n-2}\}.$$

Proof. The equations g_1, g_2, g_3, g_4 are (up to scaling) the derivatives of U_n, and therefore generate $(\mathrm{d}U_n)$, and because U_n is quasihomogeneous they also generate $(\mathrm{d}U_n, U_n) = (\mathrm{d}U_n)$. The S-series

$$S_{24} = v \cdot g_2 - s \cdot g_4 = (2n-1)rs^{2n-1} = (2n-1) \cdot g_5$$

produces the element g_5, and likewise, the element g_6 is obtained from the S-series

$$S_{35} = \tfrac{1}{3}s^{2n-1} \cdot g_3 - r \cdot g_2 = -\tfrac{1}{3}s^{4n-2} = -\tfrac{1}{3} \cdot g_6.$$

The set $G = \{g_1, \dots, g_6\}$ now forms a standard basis: G clearly generates $(\mathrm{d}U_n, U_n)$; S_{24}, S_{35} have a standard representation by construction; and all other S-series are guaranteed to have a standard representation by monomial and product criterion. □

With this standard basis, the Tjurina number can be calculated as the dimension of

$$\frac{\mathbb{C}[s, r, v, u]}{(u, vs, r^2, v^2, rs^{2n-1}, s^{4n-2})}.$$

This algebra has a monomial basis $\{1, v, r, vr, rs, \ldots, rs^{2n-2}, s, \ldots, s^{4n-3}\}$, so that

$$T(U_n) = \dim_{\mathbb{C}} \frac{\mathbb{C}[s, r, v, u]}{(u, vs, r^2, v^2, rs^{2n+1}, s^{4n+2})} = 6n - 1.$$

The standard bases for the family $U_{n,m}$ are split depending on the regime.

Lemma 2.21. *Let $n \geq 2$ and $1 \leq m < 2n-1$. Then $(\mathrm{d}U_{n,m}, U_{n,m})$ has a* standard basis $G = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ *where*

$$g_1 = u, \quad g_2 = sv, \quad g_3 = 3r^2 - 4rs^m + s^{2m} + s^{2n-1},$$
$$g_4 = v^2 + 2mr^2s^{m-1} - 2mrs^{2m-1} - (2n-1)rs^{2n-2},$$
$$g_5 = \begin{cases} rs^{2n-1} & n \leq m, \\ 2mrs^{2m} - 3(2n-1)rs^{2n-1} - 2ms^{3m} & m < n \end{cases},$$
$$g_6 = s^{2n+m-1}.$$

Proof. We will only consider the regime $n \leq m$ for simplicity, the other regime is similar. The polynomials g_1, g_2, g_3, g_4 are again (up to scaling) the derivatives of $U_{n,m}$ and hence generate $(\mathrm{d}U_{n,m})$. Because the polynomial $U_{n,m}$ is not quasihomogeneous, the generalised Euler operator with weights $\frac{1}{2}$, $\frac{3m-1}{6m}$, $\frac{1}{3}$, $\frac{1}{3m}$ gives a nontrivial generator:

$$(\tfrac{1}{2}u\partial_u + \tfrac{3m-1}{6m}v\partial_v + \tfrac{1}{3}r\partial_r + \tfrac{1}{3m}s\partial_s)U_{n,m} = U_{n,m} + \tfrac{2n-2m-1}{3m}rs^{2n-1}.$$

Hence, the set $\{g_1, g_2, g_3, g_4, g_5 = rs^{2n-1}\}$ generates $(\mathrm{d}U_{n,m}, U_{n,m})$. The element $g_6 = s^{2n+m-1}$ can be obtained from the S-series S_{24}:

$$\begin{aligned} S_{24} = sv^2 - s \cdot g_4 &= -2mr^2s^m + 2mrs^{2m} + (2n-1)rs^{2n-1} \\ &= \tfrac{2m}{3} \cdot g_3 + (2n - 1 + \tfrac{2m}{3}s^{2m-2n+1}) \cdot g_5 + (1 - \tfrac{2m}{3}s^{2m-2n+1}) \cdot g_6, \end{aligned}$$

and hence $g_6 \in (\mathrm{d}U_{n,m}, U_{n,m})$ because $1 - \frac{2m}{3}s^{2m-2n+1}$ is a unit. Moreover, this is a standard representation of S_{24} with respect to G as $\mathrm{lm}(S_{24}) = \mathrm{lm}(g_3) \prec \mathrm{lm}(g_5) \prec \mathrm{lm}(g_6)$. By inspection, the only other S-series not eliminated by the criteria is S_{35}, which has standard basis

$$\begin{aligned} S_{35} = \tfrac{1}{3}s^{2n-1} \cdot g_3 - r^2s^{2n-1} &= -\tfrac{4}{3}rs^{2n+m-1} + \tfrac{1}{3}s^{2n+2m-1} + \tfrac{1}{3}s^{4n-2} \\ &= -\tfrac{4}{3}s^m \cdot g_5 + (\tfrac{1}{3}s^m + \tfrac{1}{3}s^{2n-1}) \cdot g_6, \end{aligned}$$

as $\mathrm{lm}(S_{35}) = \mathrm{lm}(-\frac{4}{3}s^m g_5) \prec \mathrm{lm}((\frac{1}{3}s^m + \frac{1}{3}s^{2n-1})g_6)$. It follows that G is indeed a standard basis for $(\mathrm{d}U_{n,m}, U_{n,m})$. ☐

The Tjurina numbers of the hypersurfaces $U_{n,m}$ can now be calculated as the dimension of the algebras

$$\frac{\mathbb{C}[[s, r, v, u]]}{(u, sv, r^2, v^2, rs^{2n-1}, s^{2n+m-1})}, \quad \frac{\mathbb{C}[[s, r, v, u]]}{(u, sv, r^2, v^2, rs^{2m}, s^{2n+m-1})},$$

for the respective regimes $n \leq m$ and $m < n$, which have monomial bases of cardinalities $4n + m$ and $2n + 3m$ respectively. Hence,

$$T(U_{n,m}) = \begin{cases} 2n + 3m & m < n \\ 4n + m & n \leq m \end{cases}$$

Fixing n, this yields $2n - 1$ hypersurfaces $U_{n,m}$, U_n with different Tjurina numbers:

regime	$U_{n,m}$ $(1 \leq m < n)$	$U_{n,m}$ $(n \leq m < 2n - 1)$	U_n
Tjurina numbers	$2n + 3, \quad \dots, \quad 5n - 3$	$5n, \quad \dots, \quad 6n - 2$	$6n - 1$

Chapter 3

The DT Toolbox

In this chapter we set up the machinery of DT theory for quivers with potential.

Let Q be a finite quiver with monoid of dimension vectors $\Delta = \mathbb{N}Q_0$. As before, we write $\mathrm{Rep}_\delta(Q)$ for the affine space of representations, which is acted on by the linear algebraic group $\mathrm{GL}_\delta = \prod_{v\in Q_0} \mathrm{GL}_{\delta_v}$. It is well-known that the orbits of the GL_δ-action correspond to isomorphism classes of representations, and therefore the quotient stack

$$\mathcal{M}_\delta := \mathrm{Rep}_\delta(Q)/\,\mathrm{GL}_\delta,$$

parametrises isomorphism classes of representations, or equivalently of $\mathbb{C}Q$-modules, of dimension vector δ. We drop the subscript to denote the disjoint union $\mathcal{M} := \coprod_{\delta\in\Delta} \mathcal{M}_\delta$, which is the moduli stack of $\mathrm{fdmod}\,\mathbb{C}Q$. For each δ, there is a map $\mathcal{M}_\delta \to \mathcal{M}_\delta$ onto the coarse moduli *scheme* $\mathcal{M}_\delta$, which parametrises semisimple modules. For $v \in Q_0$ let $S_v \in \mathrm{mod}\,\mathbb{C}Q$ denote the unique nilpotent simple at the vertex v, then $\mathcal{N}_\delta \subset \mathcal{M}_\delta$ denotes the fibre of $\mathcal{M}_\delta \to \mathcal{M}_\delta$ over the semisimple module $\bigoplus_{v\in Q_0} S_i^{\delta_v}$. Again, we drop the subscript to denote the disjoint union $\mathcal{N} = \coprod_{\delta\in\Delta} \mathcal{N}_\delta$.

If $W \in \mathbb{C}Q_{\mathrm{cyc}}$ is a potential, the trace of W defines a GL_δ-equivariant function $\mathrm{tr}(W)$ on $\mathrm{Rep}_\delta(Q)$ for each $\delta \in \Delta$, and hence a regular function $\mathrm{tr}(W)$ on $\mathcal{M}$. This function has a well-defined stacky critical locus $\mathcal{M}_{Q,W}$, whose intersection with $\mathcal{N}$ we denote by

$$\mathcal{C} = \mathcal{C}_{Q,W} := \mathcal{M}_{Q,W} \cap \mathcal{N}.$$

The closed points $\mathcal{C}(\mathbb{C})$ correspond to the isomorphism classes of the nilpotent $\mathbb{C}Q$-modules that satisfy the relations in the Jacobi algebra, i.e. $\mathrm{Jac}(Q,W)$-modules. The goal of motivic Donaldson–Thomas theory is to assign a motivic invariant which "counts" these classes. To do this, one constructs a *motivic vanishing cycle* $\phi_{\mathrm{tr}(W)}$ in some ring of motivic measures. Integrating the vanishing cycle over the components $\mathcal{C}_\delta$ defines a generating function

$$\Phi(t) = \Phi_{Q,W}(t) := \sum_{\delta\in\Delta} \int_{\mathcal{C}_\delta} \phi_{\mathrm{tr}(W)} \cdot t^\delta,$$

with motivic coefficients. This generating function is the *DT partition function* and its coefficients the *DT invariants*, which are a motivic refinement of the enumerative DT invariants of Joyce–Song [JS08]. The partition function can be greatly simplified using stability conditions and a multiple-cover formula.

§ 3.1 | Refined invariants

The intention of motivic Donaldson–Thomas theory is to refine enumerative invariants. Instead of the ring of integers, the desired invariant ring for a motivic theory is a version of the Grothendieck ring of varieties $\mathrm{K}(\mathrm{Var}/\mathbb{C})$: the ring generated by isomorphism classes of reduced separated schemes of finite type over $\mathbb{C}$ subject to the cut-and-paste relations

$$[X] = [Z] + [X \setminus Z] \text{ for } Z \subset X \text{ a closed subvariety},$$

with multiplication $[X] \cdot [Y] = [X \times Y]$ and unit $[\mathrm{pt}] = [\operatorname{Spec} \mathbb{C}]$. This is however not quite the right target ring for motivic DT theory: besides some technical modifications, it is crucial to keep track of *monodromy*. The invariants are therefore defined in some equivariant version $\mathrm{Mot}^{\widehat{\upmu}}(\mathbb{C})$ of the ring of varieties. The invariants will be defined via motivic integration, and we therefore also work in a relative setting: we require a ring of *motivic measures* $\mathrm{Mot}^{\widehat{\upmu}}(\mathcal{M})$ over a suitable stack $\mathcal{M}$, and a method of integrating these with respect to relative classes $\mathrm{K}(\mathrm{St}/\mathcal{M})$. We briefly recall this generalisation here, and point the reader to [DM15b] for a more complete treatment.

Let St denote the category of Artin stacks, locally of finite type over $\mathbb{C}$, having affine stabilisers. A *stack with monodromy*, is an $\mathcal{X} \in \mathrm{St}$ equipped with a good action of the group-scheme $\upmu_n$ of nth roots of unity, i.e. an action such that the orbit of any closed point is contained in an affine neighbourhood. It is convenient to interpret the monodromy as an action of the limit $\widehat{\upmu}$ of the inverse system $\{z^a \colon \upmu_{an} \to \upmu_n\}_{a,n\in\mathbb{N}}$ formed by these groups. Given a stack $\mathcal{M} \in \mathrm{St}$, interpreted as a stack with trivial monodromy, we consider finite type morphisms $\mathcal{X} \to \mathcal{M}$ from stacks with monodromy $\mathcal{X}$ to $\mathcal{M}$. We will call two such morphisms $f \colon \mathcal{X} \to \mathcal{M}$ and $g \colon \mathcal{Y} \to \mathcal{M}$ *equivalent* if there is $\widehat{\upmu}$-equivariant isomorphism $h \colon \mathcal{X} \to \mathcal{Y}$ such that $f = g \circ h$. If $\mathcal{M} \in \mathrm{St}$ is of finite type we let $\mathrm{K}^{\widehat{\upmu}}(\mathrm{St}/\mathcal{M})$ denote the abelian group generated by the equivalences classes $[\mathcal{X} \to \mathcal{M}]$ of such maps, subject to the relations

$$[\mathcal{X} \xrightarrow{f} \mathcal{M}] = [\mathcal{Z} \xrightarrow{f|_{\mathcal{Z}}} \mathcal{M}] + [\mathcal{X} \setminus \mathcal{Z} \xrightarrow{f|_{\mathcal{X}\setminus\mathcal{Z}}} \mathcal{M}],$$
$$0 = [\mathcal{Y} \xrightarrow{f\circ g} \mathcal{M}] - [\mathbb{A}^r \times \mathcal{X} \xrightarrow{f\circ \mathrm{pr}_{\mathcal{X}}} \mathcal{M}]$$

for closed substacks $\mathcal{Z} \subset \mathcal{X}$, and $\widehat{\upmu}$-equivariant vector bundles $g \colon \mathcal{Y} \to \mathcal{X}$ of rank r. For $\mathcal{M} \in \mathrm{St}$ not of finite type, the above defines a group $\mathrm{K}^{\widehat{\upmu}}_{pre}(\mathrm{St}/\mathcal{M})$ and we define

$$\mathrm{K}^{\widehat{\upmu}}(\mathrm{St}/\mathcal{M}) := \mathrm{K}^{\widehat{\upmu}}_{pre}(\mathrm{St}/\mathcal{M}) / \cap_{\mathcal{U}\subset\mathcal{M}} \mathrm{K}^{\widehat{\upmu}}_{pre}(\mathrm{St}/(\mathcal{M} \setminus \mathcal{U})),$$

where the intersection ranges over the open substacks $\mathcal{U} \subset \mathcal{M}$ which are of finite type. We also let $\mathrm{K}(\mathrm{St}/\mathcal{M}) \subset \mathrm{K}^{\hat{\mu}}(\mathrm{St}/\mathcal{M})$ denote the subgroup generated by classes $[\mathcal{X} \to \mathcal{M}]$ for which $\mathcal{X}$ carries the trivial $\hat{\mu}$ action. Any finite type map $j \colon \mathcal{M} \to \mathcal{N}$ induces a push-forward $j_* \colon \mathrm{K}^{\hat{\mu}}(\mathrm{St}/\mathcal{M}) \to \mathrm{K}^{\hat{\mu}}(\mathrm{St}/\mathcal{N})$ and a pull-back $j^* \colon \mathrm{K}^{\hat{\mu}}(\mathrm{St}/\mathcal{N}) \to \mathrm{K}^{\hat{\mu}}(\mathrm{St}/\mathcal{N})$ via

$$j_*[f \colon \mathcal{X} \to \mathcal{M}] = [j \circ f \colon \mathcal{X} \to \mathcal{N}], \quad j^*[f \colon \mathcal{X} \to \mathcal{N}] = [j^* f \colon \mathcal{X} \times_{\mathcal{N}} \mathcal{M} \to \mathcal{M}].$$

For $\mathcal{Z} \subset \mathcal{M}$ a substack we write $|_{\mathcal{Z}}$ for the pullback along the inclusion.

Any variety X can be interpreted as a finite type stack, and the classes $[X \to \mathcal{M}]$ generate a subgroup $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathcal{M}) \subset \mathrm{K}^{\hat{\mu}}(\mathrm{St}/\mathcal{M})$. In particular, for $\mathcal{M} = \operatorname{Spec} \mathbb{C}$ one obtains the absolute motives $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$, which have a ring structure with an exotic product (see [Loo02], where this product is called the "join"), which restricts to the usual product on $\mathrm{K}(\mathrm{Var}/\mathbb{C}) \subset \mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$. We write absolute motives simply as $[\mathcal{X}]$, ignoring the structure morphism, and use the notation

$$\mathbb{L} := [\mathbb{A}^1] \in \mathrm{K}(\mathrm{Var}/\mathbb{C}) \subset \mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$$

for the Lefschetz motive. With the exotic product on $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$ the Lefschetz motive moreover has a square root which is of the form (see e.g. [DM15b, Example 4.3])

$$\mathbb{L}^{\frac{1}{2}} = 1 - [\mu_2] \in \mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C}).$$

The ring $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$ acts on $\mathrm{K}^{\hat{\mu}}(\mathrm{St}/\mathcal{M})$ and $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathcal{M})$ for any $\mathcal{M} \in \mathrm{St}$. For a class $[X]$ with trivial monodromy, this action is simply

$$[X] \cdot [\mathcal{Y} \to \mathcal{M}] = [X \times \mathcal{Y} \to \mathcal{M}].$$

In particular it makes sense to define a localisation

$$\mathrm{Mot}^{\hat{\mu}}(\mathcal{M}) := \mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathcal{M}) \left[[\mathrm{GL}_n]^{-1} \mid n \in \mathbb{N} \right],$$

and write again $\mathrm{Mot}^{\hat{\mu}}(\mathbb{C}) = \mathrm{Mot}^{\hat{\mu}}(\operatorname{Spec} \mathbb{C})$. This localisation already recovers the stacky version: by [DM15b, Proposition 2.8] the map $\mathrm{Mot}^{\hat{\mu}}(\mathcal{M}) \to \mathrm{K}^{\hat{\mu}}(\mathrm{St}/\mathcal{M})$ which sends

$$[\mathrm{GL}_n]^{-1} \cdot [X \to \mathcal{M}] \mapsto [\mathrm{pt}/\mathrm{GL}_n \times X \to \mathcal{M}].$$

is an isomorphism of $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$-modules. We will refer to elements of $\mathrm{Mot}^{\hat{\mu}}(\mathcal{M})$ as motivic measures, as they have well-defined integrals: for $i \colon \mathcal{X} \to \mathcal{M}$ a stack over $\mathcal{M}$ with $a \colon \mathcal{X} \to \operatorname{Spec} \mathbb{C}$ of finite type over $\mathbb{C}$, the integral of $m \in \mathrm{Mot}^{\hat{\mu}}(\mathcal{M})$ is

$$\int_{\mathcal{X}} m := a_* i^* m \in \mathrm{Mot}^{\hat{\mu}}(\mathbb{C}),$$

and one can show that this integral only depends on the class $[i \colon \mathcal{X} \to \mathcal{M}]$ in $\mathrm{K}(\mathrm{St}/\mathcal{M})$.

We will collect our motivic invariants in generating series, expressed as elements of a ring of multi-variate motivic power series: if $S = \mathbb{N}S_0$ is a free monoid on a finite set S_0 we let

$$\mathrm{Mot}^{\widehat{\mu}}(\mathbb{C})[[S]] = \mathrm{Mot}^{\widehat{\mu}}(\mathbb{C})[[t^s \mid s \in S_0]].$$

Such rings have an additional pre-λ-ring structure [DM15b, §3], defined by a map

$$\mathrm{Sym}\colon \mathrm{Mot}^{\widehat{\mu}}(\mathbb{C})[[S]] \to 1 + \mathrm{Mot}^{\widehat{\mu}}(\mathbb{C})[[S]],$$

called the *plethystic exponential*, which satisfies the exponential identities

$$\mathrm{Sym}(0) = 1, \quad \mathrm{Sym}(a+b) = \mathrm{Sym}(a)\,\mathrm{Sym}(b),$$
$$\mathrm{Sym}(a \cdot t^s) = 1 + a \cdot t^s + \ldots \text{higher order terms} \ldots$$

The plethystic exponential allows one to systematically derive multiple-cover formulas for motivic invariants: starting with an ansatz $\mathrm{Sym}(\sum_{s\in S} a_s t^s)$, one can re-write any power series with constant term 1 as a plethystic exponential by computing the a_s term-wise.

§ 3.1.1 | Motivic vanishing cycles

The motivic vanishing cycle is a rule which assigns to a regular function $f\colon \mathcal{M} \to \mathbb{A}^1$ on a smooth stack $\mathcal{M}$ a motivic measure $\phi_f \in \mathrm{Mot}^{\widehat{\mu}}(\mathcal{M})$, and provides a measure of the critical locus of f. Its construction proceeds in successive levels of generality.

1) For a regular function $f\colon M \to \mathbb{A}^1$ on a smooth scheme M Denef–Loeser [DL99] construct the vanishing cycle via a certain rational function, defined by the (homogeneous) lifts $f_n\colon \mathcal{L}(M) \to \mathbb{A}^1$ of f to the arc-space of M.
2) For a regular function $f\colon \mathcal{M} \to \mathbb{A}^1$ on a quotient stack $\mathcal{M} = M/G$ of a smooth scheme M by a linear algebraic group G one can define as in [DM15b]

$$\phi_f = \mathbb{L}^{\dim G/2} \cdot [\mathrm{B}G] \cdot q_* \phi_{f\circ q} \in \mathrm{Mot}^{\widehat{\mu}}(\mathcal{M}),$$

where $q\colon M \to \mathcal{M}$ is the quotient, and $\phi_{f\circ q} \in \mathrm{Mot}^{\widehat{\mu}}(M)$ is defined as above.
3) For a general $\mathcal{M} \in \mathrm{St}$, the vanishing cycle ϕ_f is recovered from an open cover of $\mathcal{M}$ by suitable quotient stacks, via the cut-and-paste relations.

To compute the motivic measure on a smooth scheme M explicitly, Denef–Loeser [DL99] provide the following algorithm.

Let $f : M \to \mathbb{A}^1$ be a nonconstant regular function on a smooth scheme of pure dimension d, and write $M_0 := f^{-1}(0)$ for the associated divisor. Let $p : \widetilde{M} \to M$ be an *embedded resolution* of M_0, i.e. p is an isomorphism away from M_0 and the pull-back $E := p^*M_0 = m_1E_1 + \ldots + m_nE_n$ has normal crossings[1] in a neighbourhood of $p^{-1}(M_0)$.

[1] This is weaker than the *simple* normal crossing (snc) condition, as we allow multiplicities.

For any non-empty $I \subset \operatorname{Irr}(E)$ of the set $\operatorname{Irr}(E) = \{E_1, \ldots, E_n\}$ let

$$E_I := \bigcap_{E_i \in I} E_i, \quad E_I^\circ := E_I \setminus \bigcup_{E_i \in \operatorname{Irr}(E) \setminus I} E_i.$$

The spaces E_I° form a stratification[2] of $p^{-1}(X_0)$, and for each stratum there exists a cover $D_I \to E_I$, étale over E_I°, with Galois group μ_{m_I} for $m_I := \gcd\{m_i\}_{E_i \in I}$. By the construction in [Loo02], the action of μ_{m_I} on the cover D_I is canonical. The vanishing cycle is then computed by the following formula [DL99; Loo02]:

$$\phi_f = \mathbb{L}^{-\frac{\dim M}{2}} \Big([M_0 \hookrightarrow M_0] - \sum_{\varnothing \neq I \subset \operatorname{Irr}(E)} (1 - \mathbb{L})^{|I|-1} [D_I^\circ \to M_0 \hookrightarrow M] \Big), \tag{3.1}$$

where D_I° is understood to carry the monodromy defined by the μ_{m_I}-action. We will use this identity explicitly in one of our computations.

The motivic integration formula can be quite difficult to use in practice, as it is often hard to find embedded resolutions explicitly. There are several results which can reduce this complexity, the first of which concerns the case where the function is homogenous.

Theorem 3.1 ([DM15b, Theorem 5.9])**.** *Let M be a smooth variety of dimension* $\dim M = d$ *with a sufficiently nice $\mathbb{G}_m$-action, and suppose $f\colon M \to \mathbb{A}^1$ is homogenous of order n. Then the motivic measure is equal to*

$$\phi_f = \mathbb{L}^{-d/2} \left([f^{-1}(0) \to M] - [f^{-1}(1) \to M] \right),$$

where $f^{-1}(1)$ carries the residual μ_n-action and the monodromy on $f^{-1}(0)$ is trivial.

Next we recall the *motivic Thom-Sebastiani identity* which will be indispensable in what follows.

Theorem 3.2 ([GLM06])**.** *Let $f\colon \mathcal{M} \to \mathbb{A}^1$ and $g\colon \mathcal{M}' \to \mathbb{A}^1$ be functions on smooth stacks, and $\mathcal{X} \subset \mathcal{M}$, $\mathcal{X}' \subset \mathcal{M}$ closed substacks, then*

$$\int_{\mathcal{X} \times \mathcal{Y}} \phi_{f+g} = \int_{\mathcal{X}} \phi_f \cdot \int_{\mathcal{Y}} \phi_g.$$

Consider the homogeneous function $z^2\colon \mathbb{A}^1 \to \mathbb{A}^1$, which has absolute vanishing cycle

$$\int_{\mathbb{A}^1} \phi_{z^2} = \mathbb{L}^{-\frac{1}{2}}(1 - [\mu_2]) = 1.$$

By repeated use of the the Thom-Sebastiani theorem, one can use the above identity to derive the following statement about quadratic forms.

Corollary 3.3. *Let $q\colon \mathbb{A}^n \to \mathbb{A}^1$ be a nondegenerate quadratic form, then $\int_{\mathbb{A}^n} \phi_q = 1$.*

[2]Here and in the rest of the paper, by a stratification of a space we simply mean a decomposition into locally closed subspaces.

§ 3.2 | The motivic Hall algebra

Let Q be a quiver with moduli stack $\mathcal{M}$ as before. Given a potential $W \in \mathbb{C}Q_{\mathrm{cyc}} := \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$, the critical locus $\mathcal{M}_{Q,W}$ of $\mathrm{tr}(W)\colon \mathcal{M} \to \mathbb{A}^1$ parametrises those $\mathbb{C}Q$-modules that satisfy the relations in the Jacobi algebra

$$\mathrm{Jac}(Q, W) := \mathbb{C}Q/(\partial W/\partial a \mid a \in Q_1),$$

where $\partial W/\partial a \in \mathbb{C}Q$ denotes the cyclic derivative of W with respect to an arrow a. Because $\mathrm{mod}\,\mathrm{Jac}(Q, W)$ is an abelian category, the points of $\mathcal{M}_{Q,W}$ are related by short-exact sequences, which endow $\mathrm{K}(\mathrm{St}/\mathcal{M}_{Q,W})$ with an algebra structure, the *motivic Hall algebra* [Joy07]. A helpful introduction to motivic Hall algebras can be found in [Bri12].

Let $\Delta = \mathbb{N}Q_0$ denote the monoid of dimension vectors of Q. Given dimension vectors $\delta_1, \delta_2 \in \Delta$, there is a moduli stack $\mathcal{E}\mathrm{xt}_{\delta_1,\delta_2}$ whose S-points for an $\mathbb{C}$-algebra S, are the isomorphism classes of short-exact sequences

$$0 \to M_1 \to N \to M_2 \to 0$$

for $[M_i] \in \mathcal{M}_{Q,W,\delta_i}(S)$ and $[N] \in \mathcal{M}_{Q,W,\delta_1+\delta_2}(S)$. There are three projections, of the form

$$p_i\colon \mathcal{E}\mathrm{xt}_{\delta_1,\delta_2} \to \mathcal{M}_{Q,W,\delta_i}, \quad q\colon \mathcal{E}\mathrm{xt}_{\delta_1,\delta_2} \to \mathcal{M}_{Q,W,\delta_1+\delta_2},$$

which map a s.e.s. to the respective modules M_i, N. Given a pair of finite-type maps $f_i\colon \mathcal{X}_i \to \mathcal{M}_{Q,W,\delta_i}$, there is a pullback diagram

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{g} & \mathcal{E}\mathrm{xt}_{\delta_1,\delta_2} & \xrightarrow{q} & \mathcal{M}_{Q,W,\delta_1+\delta_2} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p_1\times p_2} & & \\ \mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{f_1\times f_2} & \mathcal{M}_{Q,W,\delta_1} \times \mathcal{M}_{Q,W,\delta_2} & & \end{array} \tag{3.2}$$

The *convolution product* $\star\colon \mathrm{K}(\mathrm{St}/\mathcal{M}_{Q,W,\delta_1}) \times \mathrm{K}(\mathrm{St}/\mathcal{M}_{Q,W,\delta_2}) \to \mathrm{K}(\mathrm{St}/\mathcal{M}_{Q,W,\delta_1+\delta_2})$ of the classes $[f_i\colon \mathcal{X} \to \mathcal{M}_{Q,W,\delta_i}]$ is defined by the top row in the diagram:

$$[\mathcal{X}_1 \xrightarrow{f_1} \mathcal{M}_{Q,W,\delta_1}] \star [\mathcal{X}_2 \xrightarrow{f_2} \mathcal{M}_{Q,W,\delta_2}] = [\mathcal{Y} \xrightarrow{q\circ g} \mathcal{M}_{Q,W,\delta_1+\delta_2}].$$

This endows $\mathrm{K}(\mathrm{St}/\mathcal{M}_{Q,W})$ with the structure of an algebra over $\mathrm{K}(\mathrm{St}/\mathbb{C})$. Restricting to the nilpotent locus $\mathcal{C} = \mathcal{N} \cap \mathcal{M}_{Q,W}$, we obtain a subgroup $\mathrm{K}(\mathrm{St}/\mathcal{C}) \subset \mathrm{K}(\mathrm{St}/\mathcal{M}_{Q,W})$ which is closed under the convolution product. The *motivic Hall algebra* of $\mathcal{C}$ is the pair

$$\mathcal{H}(Q, W) := (\mathrm{K}(\mathrm{St}/\mathcal{C}), \star)$$

Let $\mathrm{Mot}^{\widehat{\mu}}(\mathbb{C})[[\Delta]]$ denote the ring of motivic power series over the monoid Δ of dimension vectors. Any class $[\mathcal{X} \to \mathcal{C}] \in \mathrm{K}(\mathrm{St}/\mathcal{C})$ has a decomposition $[\mathcal{X} \to \mathcal{C}] = \sum_{\delta\in\Delta}[\mathcal{X}_\delta \to \mathcal{C}_\delta]$

over the components of $\mathcal{C}$. Hence, there is a well-defined integration

$$[\mathcal{X} \to \mathcal{C}] \to \int_{[\mathcal{X}\to\mathcal{C}]} \phi_{\mathrm{tr}(W)}|_{\mathcal{C}} := \sum_{\delta\in\Delta} \int_{\mathcal{X}_\delta} \phi_{\mathrm{tr}(W)} \cdot t^{\delta}.$$

Remarkably, the integration map $\int_\bullet \phi_{\mathrm{tr}(W)}|_{\mathcal{C}} \colon \mathcal{H}(Q,W) \to \mathrm{Mot}^{\widehat{\mu}}(\mathbb{C})[[\Delta]]$ obtained this way is a $\mathrm{K}(\mathrm{St}/\mathbb{C})$-*algebra homomorphism*: this follows from [DM15a, Proposition 6.19] with the proof [Thu15] of the integral identity in [KS08]. The *Donaldson–Thomas partition function* of (Q,W) is the integral over the canonical element $[\mathrm{Id}\colon \mathcal{C} \to \mathcal{C}]$:

$$\Phi(t) = \int_{[\mathrm{Id}\colon\, \mathcal{C}\to\mathcal{C}]} \phi_{\mathrm{tr}(W)}|_{\mathcal{C}}$$

With this setup, the partition function is controlled by the algebraic identities in the Hall algebra: and any decomposition of the canonical element gives a decomposition of the partition function.

§ 3.2.1 | Decomposition through stability

Definition 3.4. Let $\mathcal{A}$ be an abelian category of finite rank: $\mathrm{K}_0(\mathcal{A}) \simeq \mathbb{Z}^{\oplus n}$. Then a stability condition on $\mathcal{A}$ is a group homomorphism $Z\colon \mathrm{K}_0(\mathcal{A}) \to \mathbb{C}$ such that any nonzero object of $M \in \mathcal{A}$ is mapped to a nonzero vector $Z([M])$ with phase

$$\Theta([M]) := \mathrm{Arg}(Z([M])) \in (0,\pi].$$

A nonzero object $M \in \mathcal{A}$ is *semistable* if for every subobject $N \hookrightarrow M$ there is an inequality

$$\Theta([N]) \leq \Theta([M]) \leq \Theta([M/N]).$$

The object M is *stable* if this inequality is strict for $N \notin \{0, M\}$. The semistable objects of a phase $\theta \in (0,\pi]$ together with the zero-object, form an abelian subcategory $\mathcal{A}^{\theta} \subset \mathcal{A}$.

For the abelian category $\mathcal{A} = \mathrm{nilp}\,\mathrm{Jac}(Q,W)$ of nilpotent modules, the Grothendieck group $\mathrm{K}_0(\mathrm{nilp}\,\mathrm{Jac}(Q,W))$ is the Grothendieck construction on the monoid Δ of dimension vectors. Moreover, every object has a finite composition series, i.e. $\mathrm{nilp}\,\mathrm{Jac}(Q,W)$ is a *finite length category*. The finite length property implies the existence of *Harder-Narasimhan* filtrations: if Θ is a phase function for a stability condition, then for any $M \in \mathrm{nilp}\,\mathrm{Jac}(Q,W)$ there exists a *unique* filtration

$$0 = M_0 \subset M_1 \subset \ldots M_n = M$$

where the subquotients M_i/M_{i-1} are semistable and the phases satisfy an inequality

$$\Theta(M_1/M_0) > \Theta(M_2/M_1) > \ldots \Theta(M_n/M_{n-1}).$$

The HN-filtration is unique, and hence defines a constructible function on $\mathcal{C}$ which associates to a module the tuple $(\theta_1, \ldots, \theta_n)$ of phases of its semistable subquotients, which defines a stratification of this stack [Rei02]. For $\theta \in (0, \pi]$ let $\mathcal{C}^\theta \subset \mathcal{C}$ denote the (open) substack of semistable modules of phase θ, then the stratification gives the following identity in the motivic Hall algebra:

$$[\mathcal{C} \to \mathcal{C}] = [\mathcal{C}_0 \hookrightarrow \mathcal{C}] + \sum_{n \in \mathbb{N}} \sum_{\theta_1 > \ldots > \theta_n} [(\mathcal{C}^{\theta_1} \setminus \mathcal{C}_0) \hookrightarrow \mathcal{C}] \star \cdots \star [(\mathcal{C}^{\theta_n} \setminus \mathcal{C}_0) \hookrightarrow \mathcal{C}]. \tag{3.3}$$

For each phase θ, the integration map sends the element $[\mathcal{C}^\theta \hookrightarrow \mathcal{C}]$ to a power series

$$\Phi^\theta(t) := \int_{[\mathcal{C}^\theta \hookrightarrow \mathcal{C}]} \phi_{\mathrm{tr}(W)}|_{\mathcal{C}} = \sum_{\delta \in \Delta} \int_{\mathcal{C}^\theta_\delta} \phi_{\mathrm{tr}(W)} \cdot t^\delta,$$

and the identity (3.3) translates to the following result of Kontsevich–Soibelman [KS08].

Lemma 3.5. *The following equality holds in the ring of motivic power series:*

$$\Phi(t) = \prod_{\theta \in (0,\pi]}^{\curvearrowright} \Phi^\theta(t), \tag{3.4}$$

where the product is taken clock-wise over all phases.

The identity (3.3) depends only the Harder-Narasimhan filtrations induced by the stability condition and not on the specific homomorphism $Z\colon \mathrm{K}_0(\mathrm{nilp}\,\mathrm{Jac}(Q, W)) \to \mathbb{C}$ chosen. We therefore fix the following notion of equivalence, which preserves the decomposition (3.4).

Definition 3.6. Two stability conditions $Z, Z'\colon \mathrm{K}_0(\mathrm{nilp}\,\mathrm{Jac}(Q, W)) \to \mathbb{C}$ are *equivalent* if they induce the same Harder-Narasimhan filtration on every nonzero representation.

Not every choice of stability condition will give a good decomposition of the partition function. For instance, the stability condition $Z\colon \mathrm{K}_0(\mathrm{nilp}\,\mathrm{Jac}(Q, W)) \to \mathbb{C}$ that maps all modules onto a single ray with phase θ gives the trivial relation $\Phi(t) = \Phi^\theta(t)$. We make the following genericity assumption, which guarantees that the decomposition is optimal.

Definition 3.7. Let $Z\colon \mathrm{K}_0(\mathrm{nilp}\,\mathrm{Jac}(Q, W)) \to \mathbb{C}$ be a stability condition with Θ its phase function, then Z is *generic* if for every pair of Z-semistable representations N, M

$$\Theta(N) = \Theta(M) \iff [N] = q \cdot [M] \in \mathrm{K}_0(\mathrm{nilp}\,\mathrm{Jac}(Q, W)) \quad \text{for some } q \in \mathbb{Q}.$$

Let Z be a generic stability condition, and θ a phase for which a semistable module exists. Then the genericity implies that the dimension vectors of semistable modules M with $\Theta(M) = \theta$ are multiples of a common, indivisible dimension vector $\delta \in \Delta$. Using the plethystic exponential, one may therefore expand the partition function of phase θ as

$$\Phi^\theta(t) =: \operatorname{Sym}\left(\sum_{n\in\mathbb{N}} \frac{\mathrm{BPS}_{n\delta}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot t^{n\delta}\right),$$

where the quantities $\mathrm{BPS}_{n\delta}$ are the *motivic BPS invariants* for the phase θ.

§ 3.3 | Formal noncommutative functions on a point

In chapter 4 we identify a stability condition and a set of phases for the quiver with potential of length 2 flops. With one exception, there exists a unique stable module M for each of these phases θ. In this setting the semistable locus $\mathcal{C}^\theta$ parametrises the extensions of M, and the DT/BPS invariants are determined by the deformation theory of M: one has

$$\Phi^\theta(t) = \Phi_{\mathcal{Q}_M,\mathcal{W}_M}(t^{[M]}).$$

for some potential $\mathcal{W}_M$ on a "noncommutative neighbourhood" of M described by an N-loop quiver $\mathcal{Q}_M$. The potential $\mathcal{W}_M$ is defined, up to a *formal* coordinate change, by a cyclic minimal A_∞-structure on $\operatorname{Ext}^\bullet(M,M)$. We will prove a few results that allow us to work with formal coordinate changes, saving the A_∞-deformation theory for chapter 5.

Lemma 3.8. *Let $f, g\colon Y \to \mathbb{A}^1$ be nonconstant regular functions on a smooth scheme, and $Z \subset Y$ a closed subscheme with $X \supset Z$ a formal neighbourhood in Y. Suppose there exists an automorphism $t\colon X \to X$ that identifies the germs $f|_X \circ t = g|_X$, then*

$$\int_Z \phi_f = \int_Z \phi_g.$$

Proof. By the definition of the vanishing cycle in [DL99], the integral of $\int_Z \phi_f$ is the (well-defined) value at $T = \infty$ (see [DM15b, §5]) of a generating series

$$\sum_{n\geq 1} \int_{\mathcal{L}_n(Y)|_Z} \phi_{f_n} \cdot T^n = \sum_{n\geq 1} \mathbb{L}^{-(n+1)\dim Y/2}\left([(f_n|_Z)^{-1}(0)] - [(f_n|_Z)^{-1}(1)]\right) \cdot T^n$$

where $\mathcal{L}_n(Y)|_Z$ is the space of order n arcs in Y with support on Z, and

$$f_n|_Z\colon \mathcal{L}_n(Y)|_Z \xrightarrow{\mathcal{L}_n(f)|_Z} \mathcal{L}_n(\mathbb{A}^1) \simeq \mathbb{A}^n \xrightarrow{(z_1,\ldots,z_n)\mapsto z_n} \mathbb{A}^1,$$

is the nth homogeneous component of the lift of f to the arc spaces. Every arc in with support on Z can be identified with an arc in a thickening of Z in Y. The automorphism $t\colon X \to X$ restricts to an automorphism on every finite thickening of Z and hence induces an automorphism $t_n\colon \mathcal{L}_n(Y)|_Z \to \mathcal{L}_n(Y)|_Z$ which satisfies $f_n|_Z \circ t_n = g_n|_Z$. In particular,

$$[(f_n|_Z)^{-1}(\lambda)] = [t_n^{-1}((g_n|_Z)^{-1}(\lambda))] = [(g_n|_Z)^{-1}(\lambda)] \in \operatorname{Mot}^{\widehat{\mu}}(\mathbb{C}),$$

for $\lambda = 0, 1$. It follows that the generating series for f_n and g_n coincide, and hence their values $\int_Z \phi_f$ and $\int_Z \phi_g$ at $T = \infty$ are equal. □

Let $(\mathcal{Q}, \mathcal{W})$ be a quiver with potential and $I = (a \mid a \in \mathcal{Q}_1)$ the two-sided ideal generated by its arrows. Then the path algebra has an I-adic completion $\widehat{\mathbb{C}\mathcal{Q}} = \lim_n \mathbb{C}\mathcal{Q}/I^n$ and the potential has a well-defined noncommutative cyclic germ $\widehat{\mathcal{W}} \in \widehat{\mathbb{C}\mathcal{Q}}_{\mathrm{cyc}} := \lim_n (\mathbb{C}\mathcal{Q}/I^n)_{\mathrm{cyc}}$. Given two potentials $\mathcal{W}$,$\mathcal{W}'$ it therefore makes sense to ask if the germs $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{W}'}$ are related by an I-adic endomorphism $\psi \in \mathrm{End}(\widehat{\mathbb{C}\mathcal{Q}})$. We have the following.

Lemma 3.9. *Let $\mathcal{Q}$ be a quiver with potentials $\mathcal{W}, \mathcal{W}' \in (\mathbb{C}\mathcal{Q})_{\mathrm{cyc}}$, Suppose there exists an I-adic automorphism $\psi \colon \widehat{\mathbb{C}\mathcal{Q}} \to \widehat{\mathbb{C}\mathcal{Q}}$ such that $\psi(\widehat{\mathcal{W}}) = \widehat{\mathcal{W}'}$ then*

$$\Phi_{\mathcal{Q},\mathcal{W}}(t) = \Phi_{\mathcal{Q},\mathcal{W}'}(t).$$

Proof. Fix a dimension vector δ, and let $\{X^{(n)} \to X^{(m)}\}_{m \geq n}$ denote the directed system of subschemes $X^{(n)} \subset \mathrm{Rep}_\delta(\mathcal{Q})$ defined by all powers I^m of I. Any cyclic path $a \in (\mathbb{C}\mathcal{Q}/I^n)_{\mathrm{cyc}}$ has a well-defined trace $\mathrm{tr}(a) \colon X^{(n)} \to \mathbb{A}^1$, which satisfies

$$\mathrm{tr}(\mathcal{W}_n) = \mathrm{tr}(\mathcal{W})|_{X^{(n)}},$$

for $\mathcal{W}_n \in (\mathbb{C}\mathcal{Q}/I^n)_{\mathrm{cyc}}$ the value of $\mathcal{W}$ in the quotient. An endomorphism $\psi_n \in \mathrm{End}(\mathbb{C}\mathcal{Q}/I^n)$ induces a map $t_n \colon X^{(n)} \to X^{(n)}$ such that $\mathrm{tr}(a) \circ t_n = \mathrm{tr}(\psi_n(a))$. In particular

$$\mathrm{tr}(\mathcal{W})|_{X^{(n)}} \circ t_n = \mathrm{tr}(\mathcal{W}_n) \circ t_n = \mathrm{tr}(\psi_n(\mathcal{W}_n)).$$

The I-adic isomorphism $\psi \in \mathrm{End}(\widehat{\mathbb{C}\mathcal{Q}})$ consists of a compatible sequence $(\psi_n)_{n \geq 1}$ of isomorphisms of $\mathbb{C}\mathcal{Q}/I^n$ for each n such that $\psi_n(\mathcal{W}_n) = \mathcal{W}'_n$. Let X be the colimit of the $X^{(n)}$, and let $t \colon X \to X$ be the isomorphism associated to the sequence $t_n \colon X^{(n)} \to X^{(n)}$ of isomorphisms induced by the ψ_n. Then for each n

$$\mathrm{tr}(\mathcal{W})|_{X^{(n)}} \circ t_n = \mathrm{tr}(\psi_n(\mathcal{W}_n)) = \mathrm{tr}(\mathcal{W}'_n) = \mathrm{tr}(\mathcal{W}')|_{X^{(n)}},$$

which shows that $\mathrm{tr}(\mathcal{W})|_X \circ t = \mathrm{tr}(\mathcal{W}')|_X$. Let $C_\delta \subset \mathrm{Rep}_\delta(Q)$ be the nilpotent part of the critical locus, i.e. $\mathcal{C}_\delta = C_\delta / \mathrm{GL}_\delta$. Then X is a formal neighbourhood of C_δ, and it follows from Lemma 3.8 that

$$\int_{\mathcal{C}_\delta} \phi_{\mathrm{tr}(\mathcal{W})} = \frac{\mathbb{L}^{\dim \mathrm{GL}_\delta/2} \int_{C_\delta} \phi_{\mathrm{tr}(\mathcal{W})}}{[\mathrm{GL}_\delta]} = \frac{\mathbb{L}^{\dim \mathrm{GL}_\delta/2} \int_{C_\delta} \phi_{\mathrm{tr}(\mathcal{W}')}}{[\mathrm{GL}_\delta]} = \int_{\mathcal{C}_\delta} \phi_{\mathrm{tr}(\mathcal{W}')}$$

The equality $\Phi_{\mathcal{Q},\mathcal{W}}(t) = \Phi_{\mathcal{Q},\mathcal{W}'}(t)$ follows by comparing coefficients for each δ. □

Using formal coordinate changes, the potential on an N-loop quiver can be brought into a simplified standard form, which consists of a minimal and quadratic part: let $\mathcal{Q}$ be an N-loop quiver with loops $x_1, \ldots, x_n, y_1, \ldots, y_{N-n}$ then we consider potentials of the form $\mathcal{W}_{\min} + q$, where such that $\mathcal{W}_{\min}$ is a polynomial which consists of degree ≥ 3

terms in the x_i and q is a nondegenerate quadratic form in the y_i. This quadratic form does not contribute to the invariants.[3]

Lemma 3.10. *Let $\mathcal{Q}$ be an N-loop quiver with a potential $\mathcal{W} = \mathcal{W}_{\min} + q$ as above, then*

$$\Phi_{\mathcal{Q},\mathcal{W}}(t) = \Phi_{\mathcal{Q}_{\min},\mathcal{W}_{\min}}(t),$$

where $\mathcal{W}_{\min}$ is interpreted as a potential on the quiver $\mathcal{Q}_{\min}$ with loops $x_1, \ldots, x_n$.

Proof. For each $k \in \mathbb{N}$ the variety $\mathrm{Rep}_k(\mathcal{Q})$ decomposes as a product $\mathrm{Rep}_k(\mathcal{Q}_{\min}) \times \mathbb{A}^{N-n}$ and $\mathrm{tr}(\mathcal{W})$ is the two terms in $\mathrm{tr}(\mathcal{W}_{\min}) + \mathrm{tr}(q)$ restrict to the respective factors. The trace of a noncommutative nondegenerate quadratic form is a nondegenerate quadratic form in the usual sense, hence by Thom-Sebastiani 3.2 and Corollary 3.3

$$\int_{\mathfrak{C}_{\mathcal{Q},k}} \phi_{\mathrm{tr}(\mathcal{W}_{\min}+q)} = \int_{\mathfrak{C}_{\mathcal{Q}_{\min},k}} \phi_{\mathrm{tr}(\mathcal{W}_{\min})} \cdot \int_{\mathbb{A}^{N-n}} \phi_{\mathrm{tr}(q)} = \int_{\mathfrak{C}_{\mathcal{Q}_{\min},k}} \phi_{\mathrm{tr}(\mathcal{W}_{\min})}.$$

and the equality $\Phi_{\mathcal{Q},\mathcal{W}}(t) = \Phi_{\mathcal{Q}_{\min},\mathcal{W}_{\min}}(t)$ follows by comparing coefficients. □

§ 3.4 | Intermediate refinements

The motivic theory we described so far is a motivic refinement of the Donaldson–Thomas theory of Joyce–Song [JS08]. In their framework the partition function is an ordinary powerseries with rational coefficients that can be written as function in numerical BPS invariants, which are *integers*. In the motivic setup something similar happens: the motivic BPS invariants lie in the "integral" subring $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C}) \subset \mathrm{Mot}^{\hat{\mu}}(\mathbb{C})$ (see [DM15a, Conjecture 6.5, Corollary 6.25]).

There are various intermediate refinements between $\mathbb{Z}$ and $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$, which are more closely related to vanishing cycle cohomology. Following [Dav19], we will consider the following hierarchy of invariant rings

$$\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C}) \xrightarrow{\chi_{\mathrm{mmhs}}} \mathrm{K}_0(\mathrm{MMHS}) \xrightarrow{\chi_{\mathrm{hsp}}^{\mathrm{mmhs}}} \mathbb{Z}[u^{\pm\frac{1}{n}}, v^{\pm\frac{1}{n}} \mid n \in \mathbb{N}] \xrightarrow{\chi_{\mathrm{wt}}^{\mathrm{hsp}}} \mathbb{Z}[q^{\pm\frac{1}{2}}] \xrightarrow{\chi^{\mathrm{wt}}} \mathbb{Z}.$$

Here $\mathrm{K}_0(\mathrm{MMHS})$ is the Grothendieck ring of the category of *monodromic mixed Hodge structures*, and the map χ_{mmhs} assigns to $[X] \in \mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$ the class

$$\chi_{\mathrm{mmhs}}([X]) = [\mathrm{H}_c(X, \mathbb{Q})],$$

of the mixed Hodge structure on the compactly supported cohomology, with a monodromy induced by the action on X. The map $\chi_{\mathrm{hsp}}^{\mathrm{mmhs}}$ assigns to each monodromic mixed Hodge structure its *Hodge spectrum*: if H is a pure Hodge structure of dimension d

[3]In general such a quadratic form encodes *orientation data* on the A_∞-vector bundle associated to a family of modules. Here our family is a single module, i.e. a point, and the orientation data is immaterial.

with an action of μ_n, then the Hodge spectrum of its class $[H]$ is defined as

$$\chi_{\text{hsp}}^{\text{mmhs}}([H]) = \sum_{p+q=d} (-1)^d \dim_{\mathbb{C}} H_{\mathbb{C}}^{p,q,0} u^p v^q + \sum_{a\neq 0} \sum_{p+q=d} (-1)^d \dim_{\mathbb{C}} H_{\mathbb{C}}^{p,q,a} u^{p+\frac{a}{n}} v^{q+\frac{n-a}{n}}$$

where $\bigoplus_{p+q=d} H_{\mathbb{C}}^{p,q} \simeq H_{\mathbb{C}}$ is the Hodge decomposition and $H_{\mathbb{C}}^{p,q,a} \subset H^{p,q}$ is the subspace on which μ_n acts with weight a. Extending linearly yields a map defined on all of $\mathrm{K}_0(\mathrm{MMHS})$. The map $\chi_{\text{wt}}^{\text{hsp}}$ assigns the *weight-polynomial*

$$\chi_{\text{wt}}^{\text{hsp}}(h(u,v)) = h(q^{\frac{1}{2}}, q^{\frac{1}{2}}).$$

The map χ^{wt} is the evaluation at $q^{\frac{1}{2}} = 1$, and the composition $\chi\colon\ \mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathbb{C}) \to \mathbb{Z}$ calculates the classical Euler characteristic of a variety. We will find all these intermediate invariants for length 2 flops in chapter 6.

Chapter 4

Classifying Stable Objects

Let $p\colon Y \to Y_{\mathrm{con}} = \operatorname{Spec} R$ be a simple length 2 flopping contraction over a complete local ring $(R, \mathfrak{o})$, let Λ be its NCCR and write

$$\Psi\colon \ \mathrm{D}^b(Y) \to \mathrm{D}^b(\Lambda)$$

for the associated derived equivalence induced by Van den Bergh's tilting bundle [VdB04b] as in Chapter 2. This derived equivalence restricts to an equivalence $\mathrm{D}^b_{\mathrm{C}}(Y) \simeq \mathrm{D}^b_{\mathrm{fd}}(\Lambda)$ between the complexes of sheaves with support on the exceptional curve C, and complexes of finite dimensional modules.[1] The category $\operatorname{fdmod} \Lambda$ therefore forms a noncommutative model for the objects supported on C.

The goal of this chapter is to classify the stable modules in $\operatorname{fdmod} \Lambda$ for a generic stability condition. Our approach relies on a close connection between stability and tilting theory: as shown in [HW19] the summands of tilting bundles on Y generate an affine hyperplane arrangement in the real Grothendieck group $\mathrm{K}_0(\operatorname{proj} \Lambda)_{\mathbb{R}}$, determining a wall-and-chamber structure. We show that each hyperplane determines a unique stable module, and find the complexes of sheaves on Y that maps to them.

Theorem 4.1. *Suppose that $Z\colon \mathrm{K}_0(\operatorname{flmod} \Lambda) \to \mathbb{C}$ is a generic stability condition, such that the phase function Θ satisfies $\Theta(S_0) > \Theta(S_1)$. Then the Z-stable modules are the images of the following objects under the equivalence $\Psi\colon \mathrm{D}^b(Y) \to \mathrm{D}^b(\Lambda)$:*

$$\begin{array}{rrl} & \mathcal{O}_p & \text{for } p \in \mathrm{C}, \\ \mathcal{O}_{2\mathrm{C}}(n), & \mathcal{O}_{\mathrm{C}}(n-1) & \text{for } n \geq 0, \\ \mathcal{O}_{2\mathrm{C}}(n)[1], & \mathcal{O}_{\mathrm{C}}(n-1)[1] & \text{for } n < 0. \end{array}$$

The proof uses the main result of [BST19], which shows a connection between stability and wall-and-chamber structures generated by silting complexes for finite dimensional algebras. Their theorem applies in particular to the fibres $\Lambda/I\Lambda \simeq \Lambda \otimes_R R/I$ of the NCCR over thickenings of the closed point of $\operatorname{Spec} R$ defined by an ideal $I \subset \mathfrak{o}$. We

[1]These are exactly the subcategories of *homologically finite* objects in their respective categories.

show that for one such thickening, the wall-and-chamber structure induced by the silting complexes of $\Lambda/I\Lambda$ coincides with the wall-and-chamber structure induced by the tilting bundles on Y.

§ 4.1 | King stability

Let A be an algebra. Then $\mathrm{Hom}_A(-,-)$ defines a K-theory pairing between the K-theory of $\mathrm{proj}\, A$ with real coefficients and of the category $\mathrm{fdmod}\, A$:

$$\langle -,-\rangle \colon \mathrm{K}_0(\mathrm{proj}\, A)_{\mathbb{R}} \otimes_{\mathbb{Z}} \mathrm{K}_0(\mathrm{fdmod}\, A) \to \mathbb{R}, \quad \langle [P], [M_j]\rangle = \dim_{\mathbb{C}} \mathrm{Hom}_A(P_i, M_i).$$

This pairing is known to be nondegenerate if A is either finite dimensional or if A is finite over a complete local commutative ring. Any element $v \in \mathrm{K}_0(\mathrm{proj}\, A)_{\mathbb{R}}$ induces a group homomorphism $Z_v \colon \mathrm{K}_0(\mathrm{fdmod}\, A) \to \mathbb{C}$ which maps a class $M \in \mathrm{fdmod}\, A$ to

$$Z_v([M]) = \dim_{\mathbb{C}} M \cdot i + \langle v, [M]\rangle.$$

Because the image of any M lies in the upper half-plane, this is a well-defined central charge on the abelian category $\mathrm{fdmod}\, A$, and hence defines a stability condition. These stability conditions are closely related to King's stability conditions [Kin94]: a nonzero module M is *King-(semi)stable* for $v \in \mathrm{K}_0(\mathrm{proj}\, A)_{\mathbb{R}}$ if and only if it is Z_v-(semi)stable and $\langle v, [M]\rangle = 0$. We will therefore refer to the elements of $\mathrm{K}_0(\mathrm{proj}\, A)$ as *King-stability parameters*. Every King stability parameter defines a thick abelian subcategory

$$\mathcal{S}_v(A) := \{M \mid M \text{ is } Z_v\text{-semistable}, \langle v, [M]\rangle = 0\} \cup \{0\},$$

and by the finite length property, each $\mathcal{S}_v(A)$ is the thick subcategory of $\mathrm{fdmod}\, A$ generated by the King-stable modules for the parameter v.

Returning to our setting, we consider the NCCR Λ of a simple $\ell = 2$ flop over $(R, \mathfrak{o})$. The Grothendieck group $\mathrm{K}_0(\mathrm{fdmod}\, \Lambda)$ is generated by the two simple modules S_0, S_1, and $\mathrm{K}_0(\mathrm{proj}\, \Lambda)$ is generated by the projective covers $P_i \twoheadrightarrow S_i$. As shown below, there are (up to equivalence) only two generic stability conditions on $\mathrm{fdmod}\, \Lambda$, which are described by King stability parameters $v = v_0[P_0] + v_1[P_1]$ with $v_0 < v_1$ and $v_0 > v_1$.

Lemma 4.2. *A King-stability parameter $v = v_0[P_0] + v_1[P_1]$ defines a generic stability condition Z_v in the sense of 3.7 if and only if $v_0 \neq v_1$.*

Proof. Suppose $v_0 = v_1$, then $Z_v([S_0]) = Z_v([S_1])$. Since $[S_0]$ is not a rational multiple of $[S_1]$ in $\mathrm{K}_0(\mathrm{fdmod}\, \Lambda)$, Z_v is not generic. Conversely, suppose $v_0 \neq v_1$ and let $a = a_0[S_0] + a_1[S_1]$, $b = b_0[S_0] + b_1[S_1]$ be classes such that $Z_v(a) = r \cdot Z_v(b)$ for some $r \in \mathbb{R}_{>0}$. Then

$$(a_0 + a_1) \cdot i + (a_0 + a_1)v_0 + a_1(v_0 - v_1) = r(b_0 + b_1) \cdot i + r(b_0 + b_1)v_0 + rb_1(v_0 - v_1).$$

which implies $a_0 + a_1 = r(b_0 + b_1)$ and $a_1(v_0 - v_1) = rb_1(v_0 - v_1)$. Because $v_0 - v_1 \neq 0$ by assumption, it follows that $a = r \cdot b$, hence Z_v is generic. □

Lemma 4.3. *Let $Z\colon \mathrm{K}_0(\mathrm{fdmod}\, A) \to \mathbb{C}$ be a generic stability condition and M a Z-stable module. Then there exists a King stability parameter $v \in \mathrm{K}_0(\mathrm{proj}\,\Lambda)$ such that Z is equivalent to Z_v and $M \in \mathcal{S}_v(\Lambda)$.*

Proof. Let $v = v_0[P_0] + v_1[P_1]$ be a nonzero vector that is perpendicular to $[M]$. If Z is generic, then $Z([S_0])$, $Z([S_1])$ form an $\mathbb{R}$-linear basis for $\mathbb{C}$ and there is an $\mathbb{R}$-linear transformation mapping the basis vectors to $i - v_0$, $i - v_1$. Let $\Theta\colon \mathrm{K}_0(\mathrm{fdmod}\, A) \to [0, \pi)$ be the phase function of Z, then $\Theta(a) \leq \Theta(b)$ for $a, b \in \mathrm{K}_0(\mathrm{fdmod}\, A)$ if and only if the signed area of the parallelogram spanned by $Z(a), Z(b)$ is positive. Any orientation preserving $\mathbb{R}$-linear transformation of $\mathbb{C}$ preserves the sign of the area, hence Z is equivalent to Z_v if the ordered basis $i - v_0, i - v_1$ has the same orientation as the ordered basis $Z([S_0]), Z([S_1])$, and is equivalent to Z_{-v} otherwise. In particular, $M \in \mathcal{S}_v(\Lambda)$ or $M \in \mathcal{S}_{-v}(\Lambda)$ depending on this orientation. □

Consider an ideal $I \subset \mathfrak{o}$ such that R/I is artinian, i.e. $\mathrm{Spec}\, R/I$ is a fat point. Then the *fibre of* Λ *over* $\mathrm{Spec}\, R/I$ is the finite dimensional algebra

$$\Lambda/I\Lambda \simeq \Lambda \otimes_R R/I.$$

The categories $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Lambda/I\Lambda$ are related by the pair of adjoint functors

$$- \otimes_\Lambda \Lambda/I\Lambda\colon\ \mathrm{mod}\,\Lambda \rightleftarrows \mathrm{mod}\,\Lambda/I\Lambda : (-)_\Lambda,$$

of extension and restriction of scalars. Because I is contained in the radical $\mathfrak{o} \subset R$, the functor $- \otimes_\Lambda$ preserves/reflects projectives and $(-)_\Lambda$ preserves/reflects simples. Hence there are induced isomorphisms between the K-theory spaces:

$$\zeta\colon\ \mathrm{K}_0(\mathrm{proj}\,\Lambda)_\mathbb{R} \to \mathrm{K}_0(\mathrm{proj}\,\Lambda/I\Lambda)_\mathbb{R}, \quad [(-)_\Lambda]\colon \mathrm{K}_0(\mathrm{fdmod}\,\Lambda/I\Lambda) \to \mathrm{K}_0(\mathrm{fdmod}\,\Lambda),$$

which are adjoint with respect to the K-theory pairing $\langle -, - \rangle$. The first isomorphism identifies King stability parameters on Λ and $\Lambda/I\Lambda$, and as the following lemma shows, the second identifies the dimension vectors of stable modules.[2]

Proposition 4.4. *Let $v \in \mathrm{K}_0(\mathrm{proj}\,\Lambda)_\mathbb{R}$, then $(-)_\Lambda$ identifies $\zeta(v)$-stable $\Lambda/I\Lambda$-modules with v-stable Λ-modules. In particular*

$$\mathcal{S}_v(\Lambda) = \mathrm{thick}(\mathcal{S}_{\zeta(v)}(\Lambda/I\Lambda)_\Lambda)$$

Proof. Let Θ_v and $\Theta_{\zeta(v)}$ denote the phase functions of Z_v and $Z_{\zeta(v)}$. The exact functor $(-)_\Lambda$ embeds $\mathrm{fdmod}\,\Lambda/I\Lambda$ into $\mathrm{fdmod}\,\Lambda$ as a Serre subcategory in $\mathrm{mod}\,\Lambda$. Hence, for

[2] This same result was observed in [DM17] and used to compute stable modules for length 1 flops.

any module $N \in \operatorname{fdmod} \Lambda/I\Lambda$ the submodules of its image N_Λ are precisely the images of its of submodules. It moreover follows from the adjunction that $Z_v((-)_\Lambda) = Z_{\zeta(v)}(-)$:

$$Z_v([N_\Lambda]) = \dim_{\mathbb{C}} N_\Lambda \cdot i + \langle v, [N_\Lambda]\rangle = \dim_{\mathbb{C}} N \cdot i + \langle \zeta(v), [N]\rangle = Z_{\zeta(v)}([N]).$$

for all $N \in \operatorname{fdmod} \Lambda/I\Lambda$. Hence N_Λ is King (semi)stable for v if and only if N is King (semi)stable for $\zeta(v)$, and the functor $(-)_\Lambda$ restricts to an exact embedding

$$(-)_\Lambda \colon \mathcal{S}_{\zeta(v)}(\Lambda/I\Lambda) \to \mathcal{S}_{\zeta(v)}(\Lambda/I\Lambda)_\Lambda \subset \mathcal{S}_v(\Lambda),$$

By the finite length property, $\mathcal{S}_v(\Lambda)$ is generated via extension by its stable modules, so it suffices to show that any stable module in $\mathcal{S}_v(\Lambda)$ is in the image of $\mathcal{S}_{\zeta(v)}(\Lambda/I\Lambda)$. Suppose $M \in \operatorname{fdmod} \Lambda$ is Z_v-stable and let $c \in I$. Because c is central in Λ it induces an endomorphism $f \colon M \to M$. The submodule $\operatorname{im} f \subset M$ satisfies $\Theta_v([\operatorname{im} f]) \leq \Theta_v([M])$ by semistability, and because $\operatorname{im} f$ is also a quotient

$$0 \to \ker f \to M \to \operatorname{im} f \to 0,$$

it follows that $\Theta_v([\operatorname{im} f]) = \Theta_v([M])$. Hence $\operatorname{im} f = M$ or $\operatorname{im} f = 0$. Because I is contained in the radical $\mathfrak{o} \subset R$ and $c \in I$, it follows from Nakayama's lemma that $\operatorname{im} f = cM \neq M$, which implies that f acts trivially on M. It follows that $M \simeq (M/IM)_\Lambda$ lies in the image of $(-)_\Lambda$, which finishes the proof. □

§ 4.2 | Tilting theory of the length 2 flop

In what follows A denotes an algebra for which the homotopy category of projectives, which we will write as $\mathcal{K}^b(\operatorname{proj} A)$, is Krull-Schmidt.

Definition 4.5. A complex $T \in \mathcal{K}^b(\operatorname{proj} A)$ is

- *basic* if its Krull-Schmidt decomposition has no repeated summands,
- a *2-term complex* if T is concentrated in degrees -1 and 0,
- *partial tilting* if $\operatorname{Ext}^i_A(T,T) = 0$ for all $i \neq 0$,
- *tilting* if it is partial tilting and T generates $\mathcal{K}^b(\operatorname{proj} A)$ as a triangulated category.

The set of basic 2-term tilting complexes is denoted $\operatorname{tilt} A$.

A famous result of Rickard [Ric89] shows that any tilting complex T determines a derived equivalence between A and the endomorphism algebra of T, via the functors

$$\mathbf{R}\mathrm{Hom}_A(T,-)\colon \mathrm{D}^b(A) \leftrightarrows \mathrm{D}^b(\mathrm{End}_{\mathrm{D}(A)}(T)) : - \overset{\mathbf{L}}{\otimes}_{\mathrm{End}_{\mathrm{D}(A)}(T)} T.$$

If a module $M \in \operatorname{mod} A$ has a projective resolution that is a 2-term tilting complex, then we say that M is a (classical) tilting module.

The tilting theory of NCCRs for Gorenstein threefold singularities is now well understood [IR08; IW10; IW11; Wem18]. Let $\operatorname{ref} R$ denote the set of reflexive modules: recall that a module $M \in \operatorname{mod} R$ is *reflexive* if the natural map $M \to M^{**}$ is an isomorphism, where $(-)^* := \operatorname{Hom}_R(-, R)$ denotes the R-linear dual. By [IW11, Theorem 1.4] any NCCR of R is isomorphic to $\Gamma = \operatorname{End}_R(M)$ for some reflexive $M \in \operatorname{ref} R$, and the NCCRs are connected by tilting modules, since $\operatorname{Hom}_R(M, -)\colon \operatorname{ref} R \to \operatorname{mod} \Gamma$ defines a bijection

$$\{L \in \operatorname{ref} R \mid \operatorname{End}_R(L) \text{ an NCCR}\} \xrightarrow{\sim} \{\text{tilting modules in } \operatorname{ref} \Gamma\} \tag{4.1}$$

where $\operatorname{ref} \Gamma \subset \operatorname{mod} \Gamma$ denotes the set of modules that are reflexive over R.

Let $p\colon Y \to \operatorname{Spec} R$ be a length 2 flop, and let $\mathcal{O}_Y \oplus \mathcal{N}$ be its Van den Bergh tilting bundle. Hirano–Wemyss [HW19] show that there are NCCRs $\Lambda_i := \operatorname{End}_R(M_i)$ for all $i \in \mathbb{Z}$, corresponding to the reflexive modules

$$M_{2k} := p_*\mathcal{O}_Y(k) \oplus p_*\mathcal{N}(k), \quad M_{2k-1} := p_*\mathcal{O}_Y(k-1) \oplus p_*\mathcal{N}(k).$$

The bijection in (4.1) relates the NCCRs Λ_i to our preferred NCCR $\Lambda_0 \simeq \Lambda$, via the tilting modules $\operatorname{Hom}_R(M_0, M_i)$. For $i \in \mathbb{Z}$, let $T_i \in \mathcal{K}^b(\operatorname{proj} \Lambda)$ be the minimal projective resolutions

$$T_{2k} \twoheadrightarrow \operatorname{Hom}_R(M_0, p_*\mathcal{O}_Y(k)), \quad T_{2k-1} \twoheadrightarrow \operatorname{Hom}_R(M_0, p_*\mathcal{N}(k)),$$

of the summands. Likewise, the modules $\operatorname{Hom}_R(M_i, M_0)$ are tilting in $\mathcal{K}^b(\operatorname{proj} \Lambda^{\mathrm{op}})$ with endomorphism algebra Λ_i^{op}. Let $F_i \in \mathcal{K}^b(\operatorname{proj} \Lambda^{\mathrm{op}})$ denote the minimal projective resolutions

$$F_{2k} \twoheadrightarrow \operatorname{Hom}_R(p_*\mathcal{O}_Y(k), M_0), \quad F_{2k-1} \twoheadrightarrow \operatorname{Hom}_R(p_*\mathcal{N}(k), M_0),$$

and write $E_i \in \mathcal{K}^b(\operatorname{proj} \Lambda)$ for the shifted dual $E_i = (F_i)^*[1]$.

Lemma 4.6. *The complexes $T_{i-1} \oplus T_i$ and $E_{i-1} \oplus E_i$ are in $\operatorname{tilt} \Lambda$ for all $i \in \mathbb{Z}$.*

Proof. Because the tilting module $\operatorname{Hom}_R(M_i, M_0)$ is reflexive, it follows from the generalised Auslander-Buchsbaum formula [IW10, Lemma 2.16] that $\operatorname{Hom}_R(M_i, M_0)$ has projective dimension ≤ 1. Hence its minimal resolution $T_{i-1} \oplus T_i$ is a 2-term tilting complex, which is furthermore basic since

$$\operatorname{End}_{\mathrm{D}(\Lambda)}(T_{i-1} \oplus T_i) \simeq \Lambda_i = \operatorname{End}_R(M_i),$$

is a basic algebra. It follows from [IR08] that for each i the dual M_i^* of M_i defines an NCCR

$$\operatorname{End}_R(M_i^*) \simeq \operatorname{End}_R(M_i)^{\mathrm{op}} = \Lambda_i^{\mathrm{op}},$$

and $\operatorname{Hom}_R(M_0^*, M_i^*) \simeq \operatorname{Hom}_R(M_i, M_0)$ is a tilting Λ^{op}-module. It follows from a similar argument that $F_{i-1} \oplus F_i$ is a basic 2-term tilting complex in $\mathcal{K}^b(\operatorname{proj}\Lambda^{\mathrm{op}})$. By [IR08, Corollary 3.4], the R-linear dual $(-)^*$ defines an exact duality

$$(-)^* \colon \mathcal{K}^b(\operatorname{proj}\Lambda^{\mathrm{op}}) \leftrightarrows \mathcal{K}^b(\operatorname{proj}\Lambda) : (-)^*,$$

which implies $E_{i-1} \oplus E_i = (F_{i-1} \oplus F_i)^*[1]$ is also a basic 2-term tilting complex. □

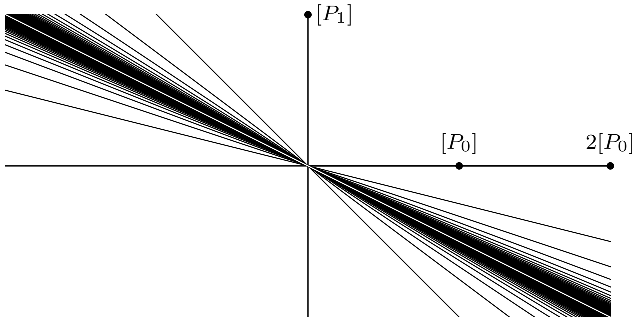


Figure 4.1: Wall-and-chamber structure of the $\ell = 2$ flop.

For a basic complex $U \in \mathcal{K}^b(\operatorname{proj}\Lambda)$ with decomposition $U = U_1 \oplus \ldots \oplus U_n$, the indecomposable summands define *g-vectors* $[U_j] \in \mathrm{K}_0(\operatorname{proj}\Lambda)_{\mathbb{R}}$, which span the cone

$$\operatorname{cone}(U) := \{\textstyle\sum_i \lambda_i \cdot [U_i] \mid \lambda_i \geq 0\} \subset \mathrm{K}_0(\operatorname{proj}\Lambda)_{\mathbb{R}}.$$

If $U \in \operatorname{tilt}\Lambda$, then by [AI12, Theorem 2.8] the g-vectors of U form a basis of $\mathrm{K}_0(\operatorname{proj}\Lambda)_{\mathbb{R}}$.

In particular, U has exactly $\operatorname{rk}\mathrm{K}_0(\operatorname{proj}\Lambda) = 2$ indecomposable summands and the interior $\operatorname{cone}^\circ(U)$ is a non-empty open subspace of $\mathrm{K}_0(\operatorname{proj}\Lambda)_{\mathbb{R}}$. In this way, one obtains a wall-and-chamber structure in $\mathrm{K}_0(\operatorname{proj}\Lambda)_{\mathbb{R}}$ whose walls correspond to the partial tilting complexes E_i and T_i and whose chambers correspond to the interior cones of tilting complexes. Using the results of [HW19], we find that this wall-and-chamber structure is the hyperplane arrangement of figure 4.1. Explicitly, the g-vectors are as follows.

Lemma 4.7. *The g-vectors of the complexes T_i are*

$$[T_i] = \begin{cases} [P_0] + n \cdot (2[P_0] - [P_1]) & \text{if } i = 2n \\ [P_1] + 2n \cdot (2[P_0] - [P_1]) & \text{if } i = 2n-1. \end{cases}$$

and $[E_i] = -[T_i]$.

Proof. As shown in [HW19], there is an isomorphism $\epsilon\colon \Lambda_{2n} \xrightarrow{\sim} \Lambda$ for all $n \in \mathbb{Z}$. Moreover, the isomorphism defined by the tilting module $\operatorname{Hom}_R(M_0, M_{2n})$

$$\mathrm{K}_0(\operatorname{proj}\Lambda) \xrightarrow{[\mathbf{R}\mathrm{Hom}_\Lambda(\mathrm{Hom}_R(M_0,M_{2n}),-)]} \mathrm{K}_0(\operatorname{proj}\Lambda_{2n}) \xrightarrow{\epsilon} \mathrm{K}_0(\operatorname{proj}\Lambda),$$

maps the class $[T_{2n}]$ to $[P_0]$ and the class $[T_{2n-1}]$ to $[P_1]$. By [HW19, Theorem 7.4, Lemma 7.6] this isomorphism can be presented in the basis $[P_0], [P_1]$ as the matrix

$$\begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}^n = \begin{pmatrix} 1-2n & -4n \\ n & 1+2n \end{pmatrix}. \tag{4.2}$$

The g-vectors of T_{2n} and T_{2n-1} can then be computed from the inverse:

$$[T_{2n}] = (1+2n)[P_0] - n[P_1], \quad [T_{2n-1}] = 4n[P_0] + (1-2n)[P_1].$$

Likewise, each tilting module $\mathrm{Hom}_R(M_{2n}, M_0)$ defines an isomorphism

$$\mathrm{K}_0(\mathrm{proj}\,\Lambda) \xrightarrow{\epsilon^{-1}} \mathrm{K}_0(\mathrm{proj}\,\Lambda_{2n}) \xrightarrow{[\mathbf{R}\mathrm{Hom}_{\Lambda_{2n}}(\mathrm{Hom}_R(M_{2n},M_0),-)]} \mathrm{K}_0(\mathrm{proj}\,\Lambda),$$

which maps $[P_0]$ to $[F^*_{2n}]$ and $[P_1]$ to $[F^*_{2n-1}]$. This isomorphism can also be presented as the inverse of the matrix (4.2) by [HW19, Rem. 7.5], hence

$$[E_i] = -[F^*_i] = -[T_i]. \qquad \square$$

§ 4.3 | From tilting to silting on the fibre

In [BST19] and [Asa21] it is shown how to recover the subcategories $\mathcal{S}_v(A)$ of semistable modules over a finite dimensional algebra A using *silting theory*.

Definition 4.8. Let A be an algebra such that $\mathcal{K}^b(\mathrm{proj}\,A)$ is Krull-Schmidt. Then a complex $U \in \mathcal{K}^b(\mathrm{proj}\,A)$ is called

- *pre-silting* if $\mathrm{Hom}_{\mathcal{K}^b(\mathrm{proj}\,A)}(U, U[i]) = 0$ for $i > 0$,
- *silting* if it is pre-silting and generates $\mathcal{K}^b(\mathrm{proj}\,A)$ as a triangulated category.

The set of isomorphism classes of basic 2-term silting complex is denoted $\mathrm{silt}\,A$.

Clearly, the set $\mathrm{tilt}\,\Lambda$ of tilting complexes is contained in $\mathrm{silt}\,\Lambda$, so that silting is suitable generalisation. The set $\mathrm{silt}\,\Lambda$ is moreover partially ordered: one considers $U \leq V$ if and only if $\mathrm{Hom}_{\mathcal{K}^b(\mathrm{proj}\,A)}(U, V[i]) = 0$ for all $i > 0$.

To apply the results of [BST19] and [Asa21] to our geometric setting, we will relate the silting theory of Λ with that of a finite dimensional fibre $\Lambda/I\Lambda$.

Proposition 4.9. *There exists an ideal $I \subset \mathfrak{o}$ for which $\Lambda/I\Lambda$ is finite dimensional, such that the functor $-\otimes_R R/I\colon \mathcal{K}^b(\mathrm{proj}\,\Lambda) \to \mathcal{K}^b(\mathrm{proj}\,\Lambda/I\Lambda)$ induces a map of posets*

$$\mathrm{silt}\,\Lambda \to \mathrm{silt}\,\Lambda/I\Lambda$$

Proof. Because R is a Gorenstein local of dimension 3, the maximal ideal $\mathfrak{o}$ contains an ideal $I \subset \mathfrak{o}$ generated by a regular sequence $g_1, g_2, g_3 \in I$. Hence, R/I is artinian and $\Lambda/I\Lambda \simeq \Lambda \otimes_R R/I$ is finite dimensional. Because Λ is an NCCR, it is moreover a maximal Cohen-Macaulay over R, and so g_1, g_2, g_3 is also a regular sequence for any projective Λ-module. As a result, if $U = U^1 \to U^0$ is a basic 2-term silting complex of projective Λ-modules, there are induces short exact sequences in chain complexes:

$$0 \longrightarrow U/I_{k-1}U \xrightarrow{g_k} U/I_{k-1}U \longrightarrow U/I_kU \longrightarrow 0 \tag{4.3}$$

where the successive quotients by $I_k = (g_1, \ldots, g_k)$ slice down to yield a 2-term complex of projectives over the finite dimensional algebra $\Lambda/I\Lambda$. Applying $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(U, -)$ yields the following long exact sequence in cohomology:

$$\cdots \longrightarrow \mathrm{Hom}_{\mathrm{D}(\Lambda)}(U, U/I_{k-1}U[i]) \longrightarrow \mathrm{Hom}_{\mathrm{D}(\Lambda)}(U, U/I_kU[i]) \longrightarrow \mathrm{Hom}_{\mathrm{D}(\Lambda)}(U, U/I_{k-1}U[i+1]) \xrightarrow{g_k} \mathrm{Hom}_{\mathrm{D}(\Lambda)}(U, U/I_{k-1}U[i+1]) \longrightarrow \cdots$$

Because U is silting, $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(U, U[i]) = 0$ for $i > 0$ and it also follows by (finite) induction that $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(U, U/I_kU[i]) = 0$ for $i > 0$. In particular, there is an adjunction

$$\mathrm{Hom}_{\mathrm{D}(\Lambda/I\Lambda)}(U/IU, U/IU[i]) \simeq \mathrm{Hom}_{\mathrm{D}(\Lambda)}(U, U/IU[i]) = 0 \quad \forall i > 0,$$

which implies that U/IU is a 2-term pre-silting complex in $\mathcal{K}^b(\mathrm{proj}\,\Lambda/I\Lambda)$, and that the map $- \otimes_R R/I \colon \mathrm{End}_\Lambda(U) \to \mathrm{End}_{\Lambda/I\Lambda}(U \otimes_R R/I)$ induces an algebra isomorphism

$$\mathrm{End}_{\mathrm{D}(\Lambda)}(U)/I\,\mathrm{End}_{\mathrm{D}(\Lambda)}(U) \xrightarrow{\sim} \mathrm{End}_{\mathrm{D}(\Lambda/I\Lambda)}(U/IU) \tag{4.4}$$

Because $\mathrm{End}_{\mathrm{D}(\Lambda)}(U)$ is a complete algebra and I is contained in the radical, it follows that idempotents lift over the quotient $R \to R/I$. Hence, any indecomposable summand of U remains indecomposable in the quotient U/IU. Because U is a basic 2-term silting complex, it has exactly $\mathrm{rk}\,\mathrm{K}_0(\Lambda) = 2$ indecomposable summands, and therefore U/IU is a basic presilting complex with 2 indecomposable summands. By [AIR14, Proposition 3.3] a basic presilting complex for a finite dimensional algebra is silting if and only if it has the maximal number of indecomposable summands. Hence U/IU is in fact *silting*, because $\Lambda/I\Lambda$ is finite dimensional.

The above shows that $- \otimes_R R/I$ restricts to a map $\mathrm{silt}\,\Lambda \to \mathrm{silt}\,\Lambda/I\Lambda$, which we claim to be a morphism of posets. To see this, consider $U, V \in \mathrm{silt}\,\Lambda$ with $V \geq U$, then by applying $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(V, -)$ to the short exact sequence (4.3) one sees that

$$\mathrm{Hom}_{\mathrm{D}(\Lambda/I\Lambda)}(V/I, U/IU[i]) \simeq \mathrm{Hom}_{D(\Lambda)}(V, U/IU[i]) = 0 \quad \forall i > 0$$

which shows that $V/IV \geq U/IU$ in $\mathrm{silt}\,\Lambda/I\Lambda$ as claimed. □

Remark 4.10. In independent work by Kimura [Kim20], which appeared while writing this thesis, it is shown that the above map is a bijection in a much more general setting.

Using the map $\mathrm{silt}\,\Lambda \to \mathrm{silt}\,\Lambda/I\Lambda$, the results of [BST19] and [Asa21] now yield the following proposition.

Proposition 4.11. *Let* $U = U_1 \oplus U_2 \in \operatorname{silt}\Lambda$*, then for any stability parameter* v*,*

- *if* $v \in \operatorname{cone}^\circ(U)$ *then the subcategory* $\mathcal{S}_v(\Lambda)$ *is trivial,*
- *if* $v \in \operatorname{cone}^\circ(U_i)$ *then the subcategory* $\mathcal{S}_v(\Lambda)$ *contains a unique stable module.*

Proof. It follows from Proposition 4.9 that $U/IU \in \operatorname{silt}\Lambda/I\Lambda$ with g-vectors

$$[U_i/IU_i] = \zeta([U_i]) \in \mathrm{K}_0(\operatorname{proj}\Lambda/I\Lambda).$$

If v lies in $\operatorname{cone}^\circ(U_i)$ then $\zeta(v)$ lies in $\operatorname{cone}^\circ(U_i/IU_i)$, so it follows from [BST19, Theorem 1.1] that $\mathcal{S}_{\zeta(v)}(\Lambda/I\Lambda)$ contains a unique stable module N. By Proposition 4.4

$$\mathcal{S}_v(\Lambda) = \operatorname{thick}(N_\Lambda),$$

so that N_Λ is the unique stable module in $\mathcal{S}_v$. Likewise, if $v \in \operatorname{cone}^\circ(U)$, then [BST19, Theorem 1.1] implies $\mathcal{S}_{\zeta(v)}(\Lambda) = 0$ and hence $\mathcal{S}_v(\Lambda) = 0$ is trivial. □

Suppose $U, V \in \operatorname{silt}\Lambda$ share a summand $U_1 = V_1$ and $U > V$, then the larger silting complex U is called the Bongartz completion of U_1, as in [AIR14].

Proposition 4.12. *Suppose* $U \in \operatorname{silt}\Lambda$ *is the Bongartz completion of a summand* U_1*, then* $\operatorname{Hom}_{\mathrm{D}(\Lambda)}(U, -)$ *restricts to an abelian equivalence*

$$\mathcal{S}_{[U_1]}(\Lambda) \xrightarrow{\sim} \operatorname{fdmod}\operatorname{End}_{\mathrm{D}(\Lambda)}(U)/(e),$$

where (e) *denotes the two-sided ideal of the idempotent* $e\colon U \to U_1 \to U$*.*

Proof. Let $M \in \mathcal{S}_{[U_1]}(\Lambda)$ be the unique stable module, then $M = N_\Lambda$ for some stable module $N \in \mathcal{S}_{[U_1/IU_1]}(\Lambda/I\Lambda)$ by Proposition 4.4. By Proposition 4.9 the complex U/IU is in $\operatorname{silt}\Lambda/I\Lambda$ and is the Bongartz completion of U_1/IU_1. Because $\Lambda/I\Lambda$ is finite dimensional, the silting version [Asa21, Proposition 4.1] of [BST19, Theorem 1.1] then implies that

$$\operatorname{Hom}_{\mathrm{D}(\Lambda/I\Lambda)}(U/IU, N[i]) = \begin{cases} S & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

where S is the simple $\overline{\Gamma} := \operatorname{End}_{\mathrm{D}(\Lambda)}(U/IU)$-module that is killed by the idempotent $\overline{e}\colon U/IU \to U_1/IU_1 \to U/IU$. By (4.4) the algebra $\overline{\Gamma}$ is a quotient of $\Gamma := \operatorname{End}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(U)$ by an ideal contained in the radical, hence $(-)_\Gamma\colon \operatorname{fdmod}\overline{\Gamma} \to \operatorname{fdmod}\Gamma$ maps S to a simple module S_Γ. Because Γ is I-adically complete, the idempotent $\overline{e} \in \overline{\Gamma}$ lifts to the idempotent $e\colon U \to U_1 \to U$. By adjunction,

$$\operatorname{Hom}_{\mathrm{D}(\Lambda)}(U, M[i]) = \operatorname{Hom}_{\mathrm{D}(\Lambda/I\Lambda)}(U/IU, N[i])_\Gamma = \begin{cases} S_\Gamma & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Because $\mathcal{S}_{[U_1]}(\Lambda)$ is generated by its stable modules and $\operatorname{fdmod}\Gamma/(e) \subset \operatorname{fdmod}\Gamma$ is generated by S_Γ, it follows that U defines an additive functor

$$\operatorname{Hom}_{\mathrm{D}(\Lambda)}(U,-)\colon \mathcal{S}_{[U_1]}(\Lambda) \to \operatorname{fdmod}\Gamma/(e),$$

which is exact by the vanishing of $\operatorname{Hom}_{\mathrm{D}(\Lambda)}(U, M[i])$ for $i \neq 0$. □

§ 4.4 | Identifying the stable modules

The results of the previous section imply that the hyperplane arrangement of figure 4.1 controls the stability of Λ: if $v \in \mathrm{K}_0(\operatorname{proj}\Lambda)_{\mathbb{R}}$ is a King stability parameter such that $\mathcal{S}_v(\Lambda)$ is nontrivial, then either

$$v = [T_i] \quad \text{or} \quad v = [E_i]$$

for some $i \in \mathbb{Z}$, or v lies on the accumulation hyperplane spanned by $2[P_0] - [P_1]$. In the former case $\mathcal{S}_v(\Lambda)$ contains a unique stable module M and $\mathcal{S}_v(\Lambda) = \operatorname{thick}\{M\}$. The objects in $\mathrm{D}^b(Y)$ corresponding to these stable modules have been identified in [DW19b], and hence we may conclude the following.

Lemma 4.13. *Let v_i denote the g-vector $v_i = [T_i]$, then for all $n \geq 0$,*

$$\mathcal{S}_{v_{2n}}(\Lambda) = \operatorname{thick}\Psi(\mathcal{O}_{\mathrm{C}}(n-1)), \quad \mathcal{S}_{v_{2n+1}}(\Lambda) = \operatorname{thick}\Psi(\mathcal{O}_{2\mathrm{C}}(n)).$$

Likewise, let w_i denote the g-vector $w_i = [E_i]$, then for all $n < 0$.

$$\mathcal{S}_{w_{2n}}(\Lambda) = \operatorname{thick}\Psi(\mathcal{O}_{\mathrm{C}}(n-1)[1]), \quad \mathcal{S}_{w_{2n+1}}(\Lambda) = \operatorname{thick}\Psi(\mathcal{O}_{2\mathrm{C}}(n)[1]).$$

Proof. Let $i \geq 0$. By the construction in [HW19], the tilting complexes $T_{i-1} \oplus T_i$ and $T_i \oplus T_{i+1}$ are obtained via finite sequence of mutations:

$$P_0 \oplus P_1 \dashrightarrow \ldots \dashrightarrow T_{i-1} \oplus T_i \dashrightarrow T_i \oplus T_{i+1},$$

from the largest element[3] $P_0 \oplus P_1$ in the silting order to $T_i \oplus T_{i+1}$. The silting order is known to be monotonic with respect to mutation, see e.g. [IW20, Theorem 9.34], which shows that $T_{i-1} \oplus T_i > T_i \oplus T_{i+1}$. Therefore $T = T_{i-1} \oplus T_i$ is the Bongartz-completion of T_i, and Proposition 4.12 implies that

$$\mathcal{S}_{v_i}(\Lambda) = \operatorname{thick} S \overset{\mathbf{L}}{\otimes}_{\Lambda_i} T$$

for $S \in \operatorname{mod}\Lambda_i$ the simple that is annihilated by the idempotent $T \to T_i \to T$. The

[3] Note that $\operatorname{Ext}^1(P_0 \oplus P_1, -) = 0$ because P_i are the projectives

images of these simples were calculated in [DW19b, Theorem 4.13]; explicitly:

$$S \overset{\mathbf{L}}{\otimes}_{\Lambda_i} T \simeq \begin{cases} \Psi(\mathcal{O}_{\mathrm{C}}(n-1)) & \text{if } i = 2n \\ \Psi(\mathcal{O}_{2\mathrm{C}}(n)) & \text{if } i = 2n+1 \end{cases}$$

We proceed similarly for the case $i < 0$ using the complexes $E_i = F_i^*[1]$. The tilting complexes F_i are again related by a sequence of mutations in $\mathcal{K}^b(\operatorname{proj}\Lambda^{\mathrm{op}})$

$$F_{i-1} \oplus F_i \dashrightarrow F_i \oplus F_{i+1}. \dashrightarrow \ldots \dashrightarrow P_0^{\mathrm{op}} \oplus P_1^{\mathrm{op}},$$

so that $F_i \oplus F_{i+1} > F_{i-1} \oplus F_i$ with respect to the silting order. Because $(-)^*$ is an exact duality between $\mathcal{K}^b(\operatorname{proj}\Lambda^{\mathrm{op}})$ and $\mathcal{K}^b(\operatorname{proj}\Lambda)$, it follows that

$$\operatorname{Ext}^1_\Lambda(E_{i-1} \oplus E_i, E_i \oplus E_{i+1}) \simeq (\operatorname{Ext}^1(F_i \oplus F_{i+1}, F_{i-1} \oplus F_{i+1}))^* = 0,$$

which shows that $E = E_{i-1} \oplus E_i$ is the Bongartz-completion of E_i in $\operatorname{tilt}\Lambda$. Hence, it follows from Proposition 4.12 that

$$\mathcal{S}_{w_i}(\Lambda) = \operatorname{thick} S \overset{\mathbf{L}}{\otimes}_{\Lambda_i} E$$

for $S \in \operatorname{mod}\Lambda_i$ the simple module that is annihilated by the idempotent $E \to E_i \to E$. Because Λ is 3-CY [IR08, Theorem 3.8] shows that there are natural isomorphisms

$$\begin{aligned} (-) \overset{\mathbf{L}}{\otimes}_{\Lambda_i} E &\simeq \mathbf{R}\mathrm{Hom}_{\Lambda_i}(\mathbf{R}\mathrm{Hom}_{\Lambda_i^{\mathrm{op}}}(E, \Lambda_i^{\mathrm{op}}), -) \\ &\simeq \mathbf{R}\mathrm{Hom}_{\Lambda_i}(E^*, -) \\ &= \mathbf{R}\mathrm{Hom}_{\Lambda_i}(F_{i-1} \oplus F_i, -)[1]. \end{aligned}$$

For $i < 0$ the image of S under the functor $\mathbf{R}\mathrm{Hom}_{\Lambda_i}(F_{i-1} \oplus F_i, -)$ was also calculated in [DW19b, Proposition 4.13]. Shifting their results by [1] then yields

$$S \overset{\mathbf{L}}{\otimes}_{\Lambda_i} E = \begin{cases} \Psi(\mathcal{O}_{\mathrm{C}}(n-1)[1]) & \text{if } i = 2n \\ \Psi(\mathcal{O}_{2\mathrm{C}}(n)[1]) & \text{if } i = 2n+1. \end{cases}$$ □

For v on the ray spanned by $2[P_0] - [P_1]$, the vector v is perpendicular to the class of the module $\Psi(\mathcal{O}_p)$ where $\mathcal{O}_p$ is structure sheaf of a point $p \in \mathrm{C}$. These modules are stable, and we can show that these are the only stable modules in $\mathcal{S}_v(\Lambda)$ in a way similar to the proof of Nakamura's conjecture in [BKR01, §8].

Lemma 4.14. *For $p \in \mathrm{C}$ let $\mathcal{O}_p$ denote the skyscraper sheaf on p. If $v \in \mathrm{K}_0(\operatorname{proj}\Lambda)_{\mathbb{R}}$ is a positive real multiple of $2[P_0] - [P_1]$, then*

$$\mathcal{S}_v(\Lambda) = \operatorname{thick}\{\Psi(\mathcal{O}_p) \mid p \in \mathrm{C}\}.$$

Proof. Because the projectives P_0, P_1 are the images of the bundles $\mathcal{O}_Y$ and $\mathcal{N}$ respectively, for each $p \in \mathrm{C}$ the skyscraper sheaf $\mathcal{O}_p$ is mapped to an object which satisfies

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_\Lambda(P_0, \Psi(\mathcal{O}_p)) &\simeq \mathbf{R}\mathrm{Hom}_Y(\mathcal{O}_Y, \mathcal{O}_p) \simeq \mathbb{C},\\ \mathbf{R}\mathrm{Hom}_\Lambda(P_1, \Psi(\mathcal{O}_p)) &\simeq \mathbf{R}\mathrm{Hom}_Y(\mathcal{N}, \mathcal{O}_p) \simeq \mathbb{C}^{\mathrm{rk}\,\mathcal{N}} = \mathbb{C}^2.\end{aligned}$$

Hence $\Psi(\mathcal{O}_p)$ is a module of dimension vector $(1, 2)$. A module of this dimension vector is King stable for a multiple of $2[P_0] - [P_1]$ if $\langle v, [N]\rangle < 0$ for any proper submodule N, or equivalently that the dimension vectors of its proper submodules are a multiple of $(0, 1)$. The module $\Psi(\mathcal{O}_p)$ cannot contain a submodule of dimension vector $(1, 1)$, because any such submodule would induce a nontrivial quotient map $\Psi(\mathcal{O}_p) \twoheadrightarrow S_1$, but

$$\mathrm{Hom}_\Lambda(\Psi(\mathcal{O}_p), S_1) \simeq \mathrm{Hom}_Y(\mathcal{O}_p, \mathcal{O}_{\mathrm{C}}(-1)) = 0$$

shows that this is not possible. Likewise, $\Psi(\mathcal{O}_p)$ cannot contain S_0 as a submodule:

$$\mathrm{Hom}_\Lambda(S_0, \Psi(\mathcal{O}_p)) \simeq \mathrm{Hom}_Y(\mathcal{O}_{2\mathrm{C}}(-1)[1], \mathcal{O}_p) \simeq \mathrm{Ext}^{-1}(\mathcal{O}_{2\mathrm{C}}(-1), \mathcal{O}_p) = 0.$$

It follows that $\Psi(\mathcal{O}_p)$ is indeed a stable module in $\mathcal{S}_v(\Lambda)$ for every $p \in \mathrm{C}$.

Now suppose there exists a module $M \in \mathcal{S}_v(\Lambda)$ which is not isomorphic to $\Psi(\mathcal{O}_p)$ for any $p \in \mathrm{C}$. We claim that $\mathrm{Hom}_\Lambda(M, \Psi(\mathcal{O}_p)) = 0$ for all $p \in \mathrm{C}$. If $f : M \to \Psi(\mathcal{O}_p)$ is a homomorphism, then $\mathrm{im}\, f$ is a submodule of $\Psi(\mathcal{O}_p)$, which implies $\langle v, \mathrm{im}\, f\rangle \leq 0$, but $\mathrm{im}\, f$ is also a quotient module of M, which implies $\langle v, \mathrm{im}\, f\rangle \geq 0$. Hence $\mathrm{im}\, f = \Psi(\mathcal{O}_p)$ and f is an isomorphism, which contradicts the assumption, or $f = 0$.

By [Bri02] the complex $\Psi^{-1}(M)$ is a *perverse sheaf of perversity* 0 and is thus quasi-isomorphic to a complex of sheaves supported in degrees $-1, 0$. However, the vanishing of $\mathrm{Hom}_\Lambda(M, \Psi(\mathcal{O}_p))$ for all p implies that the cohomology sheaf $\mathcal{H}^0(\Psi^{-1}(M)) \in \mathrm{coh}\, Y$ satisfies

$$\mathcal{H}^0(\Psi^{-1}(M))_p \simeq \mathrm{Hom}_Y(\Psi^{-1}(M), \mathcal{O}_p) \simeq \mathrm{Hom}_\Lambda(M, \Psi(\mathcal{O}_p)) = 0,$$

over every point $p \in \mathrm{C}$. Hence, $\mathcal{H}^0(\Psi^{-1}(M))$ has empty support and it follows that $\Psi^{-1}(M)$ is quasi-isomorphic to $\mathcal{F}[1]$ for some sheaf $\mathcal{F} \in \mathrm{coh}\, Y$.

Because Y is quasiprojective, there is an embedding $j : Y \hookrightarrow \bar{Y}$ into a projective variety, and the sheaf $j_*\mathcal{F} \otimes_{\overline{Y}} \mathcal{L}$ has Euler characteristic $\chi(\mathcal{F} \otimes_{\overline{Y}} \mathcal{L}) \geq 0$ for some sufficiently ample line bundle $\mathcal{L}$ on $\bar{Y}$. The King stability condition $\langle v, [M]\rangle = 0$ implies that $[\mathcal{F}] = -n[\mathcal{O}_p]$ for some $n \geq 0$, so that by the positivity

$$0 \leq \chi(\mathcal{F} \otimes_{\overline{Y}} \mathcal{L}) = -n\chi(\mathcal{O}_p \otimes_{\overline{Y}} \mathcal{L}) = -n\chi(\mathcal{O}_p) = -n,$$

which implies $n = 0$. It follows $[M] = -[\mathcal{F}] = 0$, so that M is a module with dimension vector $(0, 0)$, and is therefore not stable by definition. It follows that all stable modules in $\mathcal{S}_v(\Lambda)$ are isomorphic to $\Psi(\mathcal{O}_p)$ for some $p \in \mathrm{C}$, which yields the equality. □

The proof of the main theorem is now immediate.

Proof of Theorem 4.1. Let $Z\colon \mathrm{K}_0(\operatorname{fdmod}\Lambda) \to \mathbb{C}$ be a stability condition for which the phase function Θ satisfies $\Theta([S_0]) > \Theta([S_1])$ as in the assumption of the theorem. Then for every Z-stable module $M \in \operatorname{fdmod}\Lambda$ it follows from Lemma 4.3 that

$$M \in \mathcal{S}_v(\Lambda) \quad \text{for some } v = v_0[P_0] + v_1[P_1],\ v_0 > v_1.$$

If v is a multiple of $2[P_0] - [P_1]$, it follows from Lemma 4.14 that $M \simeq \Psi(\mathcal{O}_p)$ for some point $p \in \mathrm{C}$. Otherwise, it follows from Proposition 4.11 that v lies on one of the hyperplanes in the lower-right quadrant of figure 4.1. But then Lemma 4.7 implies that v is a multiple of v_i for $i \geq 0$ or w_i for $i < 0$, and it follows that

$$\mathcal{S}_v(\Lambda) = \mathcal{S}_{v_i}(\Lambda) \quad \text{or} \quad \mathcal{S}_v(\Lambda) = \mathcal{S}_{w_i}(\Lambda).$$

Hence it follows by Lemma 4.13 that M is isomorpic to one of the objects $\Psi(\mathcal{O}_{\mathrm{C}}(n-1))$, $\Psi(\mathcal{O}_{2\mathrm{C}}(n))$ for $n \geq 0$, or one of the objects $\Psi(\mathcal{O}_{\mathrm{C}}(n-1)[1])$, $\Psi(\mathcal{O}_{2\mathrm{C}}(n)[1])$ for $n < 0$. $\square$

Chapter 5

Calabi-Yau Structures

Let $p\colon Y \to \operatorname{Spec} R$ be a flopping contraction as before, which is (at least in a neighbourhood of the flopping curve C) described by a Jacobi algebra $A = \operatorname{Jac}(Q, W)$. In this chapter we relate the DT theory of objects in $D^b_C(Y) \simeq D^b_{\text{nilp}}(A)$ to their deformation theory, which is captured by a Calabi–Yau enhancement. The Calabi–Yau structure endows every module $M \in \operatorname{nilp} A \subset D^b_{\text{nilp}}(A)$ with a quiver with minimal potential $(\mathcal{Q}_M, \mathcal{W}_M)$, which determines its contribution to the DT theory of (Q, W).

In this model it becomes possible to compare potentials of different objects M and N related by $N \simeq F(M)$ via a *standard* derived equivalence F, i.e. a derived equivalence that lifts to an enhancement of $D^b(A)$. This includes in particular all tilting functors defined in chapter 4. Any standard equivalence has an action on Hochschild homology

$$\operatorname{HH}_\bullet(F)\colon \operatorname{HH}_\bullet(A) \to \operatorname{HH}_\bullet(A).$$

We formulate the following sufficient condition for the potentials to be preserved by F in terms of the action $\operatorname{HH}_3(F)$ on Hochschild homology.

Theorem 5.1. *Suppose $F\colon D^b(A) \to D^b(A)$ is an R-linear standard equivalence which acts on Hochschild homology as $\operatorname{HH}_3(F) = \lambda \in \mathbb{C}^\times$. Then for every pair of nilpotent modules $M, N \in \operatorname{nilp} A$ such that $\operatorname{End}_A(M) \simeq \operatorname{End}_A(N) \simeq \mathbb{C}$ and $F(M) \simeq N$, the associated minimal potentials $\mathcal{W}_M$ and $\lambda \cdot \mathcal{W}_N$ are equivalent.*

In chapter 6 we will consider the DT theory contributions of certain stable modules $M \in \operatorname{nilp} A$, which will be obtained as a motivic integral over the stack $\mathcal{P}_M = \coprod_{k \geq 0} \mathcal{P}_{M,k}$ of extensions of M. The theorem has the following consequence for these contributions.

Corollary 5.2. *Let $M, N \in \operatorname{nilp} A$ be modules with $\operatorname{End}_A(M) \simeq \operatorname{End}_A(N) \simeq \mathbb{C}$, and suppose $F\colon D^b(A) \to D^b(A)$ is a functor that satisfies the condition of Theorem 5.1 and maps M to $F(M) \simeq N$. Then for all $k \geq 0$ there is an equality*

$$\int_{\mathcal{P}_{M,k}} \phi_{\operatorname{tr}(W)} = \int_{\mathcal{P}_{N,k}} \phi_{\operatorname{tr}(W)}.$$

In Chapter 6 we will use this corollary to compute the partition functions $\Phi^\theta(t)$ when the stack $\mathcal{C}^\theta = \mathcal{P}_M$ parameterises extensions of a stable module M.

For the family of length 2 flops we work with, the units $R^\times$ of the base are absolute, and the homological condition is automatically satisfied.

Proposition 5.3. *Suppose R has absolute units $R^\times \simeq \mathbb{C}^\times$, then the assumptions of Theorem 5.1 are satisfied by every R-linear standard derived auto-equivalence of* $\mathrm{D}^b(A)$.

To prove the theorem requires us to relate the Hochschild homology of a smooth DG-enhancement $\mathcal{A}$ of $\mathrm{D}^b(A)$ with (a version of) the Hochschild cohomology of a proper DG-enhancement $\mathcal{N}$ of $\mathrm{D}^b_{\mathrm{nilp}}(A)$, and to show that this relation is compatible with derived equivalences. This relation comes from a pairing on Hochschild homology reviewed in §5.1 and is related to Koszul duality as we show in §5.2. The potentials are defined on the minimal model of $\mathcal{N}$, which as we explain in §5.3.1 is given by a *cyclic* A_∞-category of twisted complexes. The cyclic inner product on this category expresses the Calabi–Yau property, and is the crucial additional structure which allows one to define the potentials as we recall in §5.3.

§ 5.1 | Hochschild homology

We recall the notion of Hochschild (co-)homology and Calabi–Yau structures on DG-categories and A_∞-categories. Detailed introductions to the theories of DG and A_∞ categories can be found in [Kel06] and [Lef03] respectively. In what follows we work over the base-field $\mathbb{C}$, all DG-/A_∞-categories are assumed to be small and all A_∞-categories are assumed to have strict units. If $\mathcal{C}$ is a DG-/A_∞-category we write $\mathcal{P}\mathrm{erf}\mathcal{C}$ for its DG-category of perfect complexes.

Given a DG-/A_∞-category $\mathcal{C}$, the Hochschild complex is (see e.g. [Kel06, §5.3])

$$\mathbf{C}(\mathcal{C}) := \left(\bigoplus_{k\geq 0} \bigoplus_{c_i \in \mathrm{Ob}\,\mathcal{C}} \mathcal{C}(c_1, c_0) \otimes (\mathcal{C}(c_2, c_1) \otimes \ldots \otimes \mathcal{C}(c_0, c_k)), b \right)$$

where the differential b is given by application of the composition $\circ$ and differential d if $\mathcal{C}$ is a DG-category, and involves also the higher multiplications in case $\mathcal{C}$ is an A_∞-category (see e.g. [Gan13] or the appendix to [She16]). Its homology is the *Hochschild homology* of $\mathcal{C}$:

$$\mathrm{HH}_\bullet(\mathcal{C}) := \mathrm{H}^{-\bullet}\mathbf{C}(\mathcal{C}).$$

The complex $\mathbf{C}(\mathcal{C})$ is an explicit model for the derived tensor product $\mathcal{C} \overset{\mathbf{L}}{\otimes}_{\mathcal{C}^e} \mathcal{C}$ of $\mathcal{C}$ as a bimodule over itself. If $\mathcal{C}$ is a *smooth* DG-category, i.e. a DG-category that is perfect as a $\mathcal{C}^e$-module then the elements of $\mathrm{HH}_k(\mathcal{C})$ can be interpreted as morphisms:

$$\mathrm{HH}_k(\mathcal{C}) \simeq \mathrm{H}^0(\mathcal{C} \overset{\mathbf{L}}{\otimes}_{\mathcal{C}^e} \mathcal{C}[-k]) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C}^e)}(\mathcal{C}^!, \mathcal{C}[-k]),$$

where $\mathcal{C}^! := \mathbf{R}\mathrm{Hom}_{\mathcal{C}^e}(\mathcal{C}, \mathcal{C}^e)$ denotes the derived bimodule dual.

Write $(-)^* = \mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$ for the *linear* dual, then the *Hochschild cohomology* with coefficients in $\mathcal{C}^*$ is the cohomology of the dual complex:

$$\mathrm{HH}^\bullet(\mathcal{C}, \mathcal{C}^*) := \mathrm{H}^\bullet(\mathbf{C}(\mathcal{C})^*)$$

Recall that $\mathcal{C}$ is *proper* if the cohomology $\mathrm{H}^\bullet\mathcal{C}(c, c')$ of the underlying complex is finite dimensional for all $c, c' \in \mathrm{Ob}\,\mathcal{C}$. For proper DG-/$A_\infty$-categories one can identify the cohomology classes in $\mathrm{HH}^\bullet(\mathcal{C}, \mathcal{C}^*)$ with morphisms through the adjunction:

$$\mathrm{HH}^k(\mathcal{C}, \mathcal{C}^*) \simeq \mathrm{H}^k\,\mathrm{Hom}_{\mathbb{C}}(\mathcal{C} \overset{\mathbf{L}}{\otimes}_{\mathcal{C}^e} \mathcal{C}, \mathbb{C}) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C}^e)}(\mathcal{C}, \mathcal{C}^*[k]).$$

The Hochschild (co-)homology can be used to define the two (dual) versions of the Calabi-Yau property.

Definition 5.4. A (weak) *left k-Calabi–Yau structure* on a smooth DG-category $\mathcal{C}$ is a cycle $\nu \in \mathbf{C}_k(\mathcal{C})$ such that $[\nu] \in \mathrm{Hom}_{\mathrm{D}(\mathcal{C}^e)}(\mathcal{C}^!, \mathcal{C}[-k])$ is an isomorphism.

Definition 5.5. A (weak) *right k-Calabi–Yau structure* on a proper DG-/A_∞-category $\mathcal{C}$ is a cocycle $\xi \in (\mathbf{C}_k(\mathcal{C}))^*$ such that $[\xi] \in \mathrm{Hom}_{\mathrm{D}(\mathcal{C}^e)}(\mathcal{C}, \mathcal{C}^*[-k])$ is an isomorphism.

Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a DG-/A_∞-functor, then application of F defines a chain map $\mathbf{C}(F)\colon \mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{D})$. For a DG-functor the map $\mathbf{C}(F)$ is defined pointwise:

$$\mathcal{C}(c_1, c_0) \otimes \ldots \otimes \mathcal{C}(c_0, c_k) \xrightarrow{F\otimes\ldots\otimes F} \mathcal{D}(F(c_1), F(c_0)) \otimes \ldots \otimes \mathcal{D}(F(c_0), F(c_k)),$$

and for an A_∞-functor $F = (F_k)_{k\geq 1}$ it also involves the higher maps (see [Gan13, §2.9]). We denote its dual as $\mathbf{C}(F)^*\colon \mathbf{C}(\mathcal{D}) \to \mathbf{C}(\mathcal{C})$, and write

$$\mathrm{HH}_\bullet(F)\colon\ \mathrm{HH}_\bullet(\mathcal{C}) \to \mathrm{HH}_\bullet(\mathcal{D}), \quad \mathrm{HH}^\bullet(F)\colon\ \mathrm{HH}^\bullet(\mathcal{D}) \to \mathrm{HH}^\bullet(\mathcal{C}).$$

for the induced maps on (co-)homology. If a DG category $\mathcal{C}$ is smooth and proper, it admits a perfect pairing $\mathrm{HH}_\bullet(\mathcal{C}) \simeq \mathrm{HH}^\bullet(\mathcal{C}, \mathcal{C}^*)$ which is compatible with DG-functors (see [Shk13]), and identifies left and right Calabi–Yau structures. This is the DG-categorical analogue of the Mukai pairing for smooth projective schemes [Cal03].

In the noncompact Calabi-Yau setting we work in, all DG-categories of interest (e.g. enhancements of $\mathrm{D}^b(Y)$) are smooth but *not* proper. There nonetheless exists a pairing when restricting to a subcategory $\mathcal{N} \subset \mathcal{C}$ of *compactly supported* objects, as shown in [BD19]. Recall that an object $p \in \mathcal{C}$ is compactly supported if $\mathcal{C}(c, p) \in \mathcal{P}\mathrm{erf}\mathbb{C}$ for all $c \in \mathcal{C}$. If $\mathcal{N} \subset \mathcal{C}$ is the full DG-subcategory on a set of compactly supported objects, then the diagonal bimodule $\mathcal{C}$ defines a DG-functor

$$\mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{N} \to \mathcal{P}\mathrm{erf}\mathbb{C},$$

into $\mathcal{P}\mathrm{erf}\mathbb{C}$, and the Hochschild complex construction yields a morphism $\mathbf{C}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{N}) \xrightarrow{\mathbf{C}(\mathcal{C}(-,-))} \mathbf{C}(\mathcal{P}\mathrm{erf}\mathbb{C})$. Recall (see e.g. [Lod97, §4.2.1]) that there is a shuffle product $\nabla\colon \mathbf{C}(\mathcal{C}^{\mathrm{op}}) \otimes \mathbf{C}(\mathcal{N}) \to \mathbf{C}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{N})$, which maps a pair of classes $\mathbf{f} = f_0[f_1 \mid \ldots \mid f_n]$, $\mathbf{g} = g_0[g_1 \mid \ldots \mid g_m]$ (written in bar notation) to the class

$$\nabla(\mathbf{f}, \mathbf{g}) = \sum_{\sigma} \pm (f_0 \otimes g_0)[\sigma_1 | \ldots | \sigma_{m+n}]$$

where the sum is over the (n,m)-shuffles of the terms $f_1 \otimes 1, \ldots f_n \otimes 1, 1 \otimes g_1, \ldots, 1 \otimes g_m$. Using the shuffle product, one defines a pairing between the Hochschild complexes

$$\mathbf{C}(\mathcal{C}^{\mathrm{op}}) \otimes \mathbf{C}(\mathcal{N}) \xrightarrow{\nabla} \mathbf{C}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{N}) \xrightarrow{\mathbf{C}(\mathcal{C}(-,-))} \mathbf{C}(\mathcal{P}\mathrm{erf}\mathbb{C}), \tag{5.1}$$

and this yields a pairing on cohomology:

$$\langle -, - \rangle_{\mathcal{N}}\colon \mathrm{HH}_\bullet(\mathcal{C}^{\mathrm{op}}) \otimes \mathrm{HH}_\bullet(\mathcal{N}) \to \mathrm{HH}_\bullet(\mathcal{P}\mathrm{erf}\mathbb{C}) \simeq \mathrm{HH}_\bullet(\mathbb{C}) \simeq \mathbb{C}.$$

If $\nu \in \mathrm{HH}_d(\mathcal{C}) \simeq \mathrm{HH}_d(\mathcal{C}^{\mathrm{op}})$ is the Hochschild class of a (weak) left Calabi-Yau structure on $\mathcal{C}$, then its dual $\langle \nu, - \rangle \in (\mathrm{HH}_\bullet(\mathcal{N}))^* \simeq \mathrm{HH}^\bullet(\mathcal{N}, \mathcal{N}^*)$ is the class of a (weak) right Calabi-Yau structure on $\mathcal{N}$ (see [BD19, Theorem 3.1]); although not every right Calabi–Yau structure arises in this way. Just as the Mukai-pairing of a smooth projective variety is preserved under Fourier-Mukai transforms (see [Cal03]), so is the above pairing preserved under suitable DG-functors.

Lemma 5.6. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a quasi-fully-faithful DG-functor that maps a compactly supported subcategory $\mathcal{N}$ to $\mathcal{N}'$, then $\langle \mathrm{HH}_\bullet(F^{\mathrm{op}})(-), \mathrm{HH}_\bullet(F)(-) \rangle_{\mathcal{N}'} = \langle -, - \rangle_{\mathcal{N}}$.*

Proof. First, note that the map $\mathbf{C}(F^{\mathrm{op}} \otimes F)$ is given by term-wise application, and therefore commutes with the shuffle product:

$$\nabla \circ (\mathbf{C}(F^{\mathrm{op}}) \otimes \mathbf{C}(F)) = \mathbf{C}(F^{\mathrm{op}} \otimes F) \circ \nabla.$$

Because F is quasi-fully-faithful, for all $M \in \mathcal{C}$, $N \in \mathcal{N}$ there are quasi-isomorphisms

$$F_{M,N}\colon \mathcal{C}(M,N) \to \mathcal{D}(F(M), F(N)),$$

which are natural in M, N. This data defines a DG-natural transformation between the functors $\mathcal{C}(-,-)$ and $\mathcal{D}(-,-) \circ (F^{\mathrm{op}} \otimes F)$, which is a homotopy equivalence. It follows that the DG-functors $\mathbf{C}(\mathcal{C}(-,-)) \circ \nabla$ and $\mathbf{C}(\mathcal{D}(-,-)) \circ \nabla \circ \mathbf{C}(F^{\mathrm{op}} \otimes F)$ are homotopic, from which the claim follows by [Kel99, Lemma 3.4]:

$$\begin{aligned}
\langle \mathrm{HH}_\bullet(F^{\mathrm{op}})(-), \mathrm{HH}_\bullet(F)(-) \rangle_{\mathcal{N}'} &= \mathrm{H}^0(\mathbf{C}(\mathcal{D}(-,-)) \circ \nabla \circ (\mathbf{C}(F^{\mathrm{op}}) \otimes \mathbf{C}(F))) \\
&\simeq \mathrm{H}^0(\mathbf{C}(\mathcal{D}(-,-)) \circ \mathbf{C}(F^{\mathrm{op}} \otimes F) \circ \nabla) \\
&\simeq \mathrm{H}^0(\mathbf{C}(\mathcal{C}(-,-)) \circ \nabla) = \langle -, - \rangle_{\mathcal{N}}.
\end{aligned}$$

□

Some of the DG-categories we consider are defined over a commutative $\mathbb{C}$-algebra R. To avoid technical issues, we will however only work with DG-categories and Hochschild homology over $\mathbb{C}$ and view them as being equipped with a compatible R-linear structure. This R-action still induces an R-module structure on the Hochschild homology: given $f_0[f_1 \mid \dots \mid f_n] \in \mathbf{C}(\mathcal{C})$ (in bar notation) the action of $r \in R$ is simply

$$f_0[f_1 \mid \dots \mid f_n] \mapsto r f_0[f_1 \mid \dots \mid f_n].$$

The action is compatible with the Hochschild differential, so that $\mathrm{HH}_\bullet(\mathcal{C})$ is a graded R-module. If a DG-functor $F\colon \mathcal{C} \to \mathcal{D}$ is R-linear, the induced chain map $\mathbf{C}(F)\colon \mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{D})$ is also R-linear, and similarly for the maps $\mathrm{HH}_\bullet(F)$ and $\mathrm{HH}^\bullet(F)$. The R-linear structure is compatible with the pairing in Lemma 5.6 in the following sense.

Lemma 5.7. *If $\mathcal{C}$ is an R-linear DG-category $\mathcal{N} \subset \mathcal{A}$ a subcategory of compactly supported objects, then the pairing is R-linear: $\langle r \cdot -, - \rangle_{\mathcal{N}} = \langle -, r \cdot - \rangle_{\mathcal{N}}$ for all $r \in R$.*

Proof. For clarity, we write $G\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{N} \to \mathcal{P}\mathrm{erf}\mathcal{C}$ for the functor that maps a pair of morphisms $(f\colon c' \to c, g\colon p \to p')$ in $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{N}$ to the map

$$G(f,g)\colon \mathcal{C}(c,p) \to \mathcal{C}(c',p'), \quad h \mapsto g \circ h \circ f.$$

By inspection this satisfies $G(r \cdot f, g) = G(f, r \cdot g)$ because the composition commutes with the R-action. Applying the shuffle product now yields

$$\begin{aligned}(\mathbf{C}(G) \circ \nabla)(r \cdot \mathbf{f}, \mathbf{g}) &= \sum \pm G(r f_0, g_0)[G(\sigma_1) \mid \dots \mid G(\sigma_{n+m})] \\ &= \sum \pm G(f_0, r g_0)[G(\sigma_1) \mid \dots \mid G(\sigma_{n+m})] \\ &= (\mathbf{C}(G) \circ \nabla)(\mathbf{f}, r \cdot \mathbf{g}).\end{aligned}$$

The same identity then holds in cohomology, making $\langle -, - \rangle_{\mathcal{N}}$ an R-linear pairing. □

§ 5.2 | Koszul duality

Let A be a (module-)finite algebra over a commutative Noetherian $\mathbb{C}$-algebra R, and assume it is homologically smooth over $\mathbb{C}$. Then the DG-category of perfect complexes $\mathcal{A} := \mathcal{P}\mathrm{erf} A$ is a smooth DG-category which is moreover R-linear. Given a maximal ideal $\mathfrak{m} \subset R$ there is a full DG-subcategory $\mathcal{N} \subset \mathcal{A}$ of objects with cohomology supported on $\mathfrak{m} \in \operatorname{Spec} R$, i.e. $\mathrm{H}^0\mathcal{N} = \mathrm{D}^{\mathrm{perf}}_{\mathfrak{m}}(A) \subset \mathrm{D}^{\mathrm{perf}}(A)$. These are compactly supported objects and hence induce a pairing $\langle -, - \rangle_{\mathcal{N}}$ as in (5.1).

The objects in $\mathrm{D}^{\mathrm{perf}}_{\mathfrak{m}}(A)$ have finite length: they are obtained as a finite extension of shifts of the simple A-modules supported over $\mathfrak{m}$. Hence $\mathrm{D}^{\mathrm{perf}}_{\mathfrak{m}}(A)$ is generated by some finite sum $S = \bigoplus_i S_i$ of simple modules. Let $\mathbf{p}S \in \mathcal{N}$ be the associated perfect complex, so that the DG-algebra

$$E := \mathcal{A}(\mathbf{p}S, \mathbf{p}S),$$

computes $\mathbf{R}\mathrm{End}_A(S)$. Because S generates, the embedding $E \to \mathcal{N}$ is a Morita equivalence, hence defines a quasi-isomorphism $\mathbf{C}(E) \to \mathbf{C}(\mathcal{N})$ between the Hochschild complexes. Likewise, $\mathcal{A}^{\mathrm{op}}$ is Morita equivalent to $\mathcal{A}^{\mathrm{op}}(A, A) \simeq A$, giving a quasi-isomorphism $\mathbf{C}(A) \to \mathbf{C}(\mathcal{A}^{\mathrm{op}})$. The pairing therefore restricts to a pairing between Hochschild homologies of (DG-)algebras

$$\langle -, - \rangle_{\mathcal{N}} \colon \mathrm{HH}_\bullet(A) \otimes \mathrm{HH}_\bullet(E) \to \mathbb{C},$$

and by adjunction this gives a morphism of R-modules

$$\Upsilon \colon \mathrm{HH}_\bullet(A) \to \mathrm{HH}_\bullet(E)^* = \mathrm{HH}^\bullet(E, E^*)$$

In general this map fails to be an isomorphism (certainly for flops) but this is to be expected: we may as well have replaced A by a suitable localisation. In fact, one can replace A by its $\mathfrak{m}$-adic completion, in which case the analogous map is an isomorphism due to Koszul duality [VdB10, Corollary D.2].

Proposition 5.8. *The map Υ factors through the completion of $\mathrm{HH}_\bullet(A)$ as*

$$\Upsilon \colon \mathrm{HH}_\bullet(A) \to \mathrm{HH}_\bullet(A) \otimes_R \widehat{R} \simeq \mathrm{HH}^\bullet(E, E^*).$$

Proof. As remarked before, the Hochschild homology and its dual compute derived bimodule morphisms: there are R-linear isomorphisms

$$\mathrm{HH}_\bullet(A) \simeq \mathbf{R}\mathrm{Hom}_{A^e}(A^!, A), \quad \mathrm{HH}^\bullet(E, E^*) \simeq \mathbf{R}\mathrm{Hom}_{E^e}(E, E^*).$$

It follows from the proof of [BD19, Theorem 3.1], the composition of these isomorphisms with the map $\Upsilon \colon \mathrm{HH}_\bullet(A) \to \mathrm{HH}^\bullet(E, E^*)$ is induced by the following derived functor

$$\mathbf{R}\mathrm{Hom}_A(S, \mathbf{R}\mathrm{Hom}_A(-, S)) \colon \mathrm{D}^{\mathrm{perf}}(A^e) \to \mathrm{D}^{\mathrm{perf}}(E^e)^{\mathrm{op}},$$

which maps A to E and $A^!$ to E^*. Let $\widehat{R}$ be the completion of R at $\mathfrak{m}$, then because R is Noetherian we may identify the completion $\widehat{M}$ of any finitely generated R-module M with $M \otimes_R \widehat{R}$. In particular, the completion of A is the base-change $\Lambda \simeq A \otimes_R \widehat{R}$. This completion is a pseudocompact algebra, which Van den Bergh shows [VdB10] is Koszul dual to E. Let $\mathrm{D}^{\mathrm{perf}}_{\mathrm{pc}}(\Lambda^e)$ denote the category of perfect complexes of pseudocompact Λ-bimodules (see e.g. the appendix of [KY11]). By Koszul duality, the functor

$$\mathbf{R}\mathrm{Hom}_\Lambda(S, \mathbf{R}\mathrm{Hom}_\Lambda(-, S)) \colon \mathrm{D}^{\mathrm{perf}}_{\mathrm{pc}}(\Lambda^e) \to \mathrm{D}^{\mathrm{perf}}(E^e)^{\mathrm{op}}, \tag{5.2}$$

is an equivalence of triangulated categories. In particular, it defines an isomorphism

$\mathbf{R}\mathrm{Hom}_{\Lambda^e}(\Lambda^!, \Lambda) \to \mathbf{R}\mathrm{Hom}_{E^e}(E, E^*)$, making the following diagram of R-modules commute:

$$\begin{array}{ccc} \mathbf{R}\mathrm{Hom}_{A^e}(A^!, A) & \xrightarrow{\mathbf{R}\mathrm{Hom}_A(S, \mathbf{R}\mathrm{Hom}_A(-,S))} & \\ \downarrow{\scriptstyle -\otimes_R \widehat{R}} & \searrow & \\ \mathbf{R}\mathrm{Hom}_{\Lambda^e}(\Lambda^!, \Lambda) & \xrightarrow{\mathbf{R}\mathrm{Hom}_\Lambda(S, \mathbf{R}\mathrm{Hom}_\Lambda(-,S))} & \mathbf{R}\mathrm{Hom}_{E^e}(E, E^*) \end{array}$$

where $- \otimes_R \widehat{R}$ is the map induced by the completion functor (which is exact). The R-module $\mathbf{R}\mathrm{Hom}_{A^e}(\Lambda^!, \Lambda)$ is obtained by base-change from the Hochschild homology:

$$\mathbf{R}\mathrm{Hom}_{\Lambda^e}(\Lambda^!, \Lambda) \simeq \mathbf{R}\mathrm{Hom}_{A^e}(A^!, A) \otimes_R \widehat{R} \simeq \mathrm{HH}_\bullet(A) \otimes_R \widehat{R}.$$

Let K denote the composition of this isomorphism with (5.2), then Υ is the composition

$$\mathrm{HH}_\bullet(A) \xrightarrow{-\otimes_R \widehat{R}} \mathrm{HH}_\bullet(A) \otimes_R \widehat{R} \xrightarrow{K} \mathrm{HH}^\bullet(E, E^*). \qquad \square$$

Suppose $F\colon \mathcal{A} \to \mathcal{A}$ is an R-linear quasi-equivalence preserving $\mathcal{N}$, then it induces R-linear endomorphisms $\mathrm{HH}_\bullet(F)$ on $\mathrm{HH}_\bullet(A) \simeq \mathrm{HH}_\bullet(\mathcal{A}^{\mathrm{op}})$ and $\mathrm{HH}^\bullet(F)$ on $\mathrm{HH}^\bullet(E, E^*) \simeq \mathrm{HH}^\bullet(\mathcal{N}, \mathcal{N}^*)$. By the previous proposition, the actions are related as follows:

Proposition 5.9. *Let $F\colon \mathcal{A} \to \mathcal{A}$ be an R-linear quasi-equivalence preserving $\mathcal{N}$, then*

$$\mathrm{HH}^\bullet(F) = K \circ (\mathrm{HH}_\bullet(F)^{-1} \otimes_R \widehat{R}) \circ K^{-1}$$

for $K\colon \mathrm{HH}_\bullet(A) \otimes_R \widehat{R} \to \mathrm{HH}_\bullet(E, E^*)$ the isomorphism from the previous proposition.

Proof. By Lemma 5.6 the pairing $\langle -, - \rangle_{\mathcal{N}}$ is invariant under the simultaneous action of $\mathrm{HH}_\bullet(F)$ on both arguments. Hence, by adjunction the map Υ satisfies

$$\mathrm{HH}^\bullet(F) \circ \Upsilon \circ \mathrm{HH}_\bullet(F) = \Upsilon,$$

for any quasi-fully faithful functor F. If F is a quasi-equivalence, then $\mathrm{HH}_\bullet(F)$ is moreover invertible, so that

$$\mathrm{HH}^\bullet(F) \circ \Upsilon = \Upsilon \circ \mathrm{HH}_\bullet(F)^{-1}. \tag{5.3}$$

Let $c\colon \mathrm{HH}_\bullet(A) \to \mathrm{HH}_\bullet(A) \otimes_R \widehat{R}$ denote the completion map. Then by Proposition 5.8 above, there is a factorisation $\Upsilon = K \circ c$, and we can consider the following diagram of R-modules

$$\begin{array}{ccccc} \mathrm{HH}_\bullet(A) & \xrightarrow{c} & \mathrm{HH}_\bullet(A) \otimes_R \widehat{R} & \xrightarrow{K} & \mathrm{HH}_\bullet(E, E^*) \\ \downarrow{\scriptstyle \mathrm{HH}_\bullet(F)^{-1}} & & \downarrow{\scriptstyle \mathrm{HH}_\bullet(F)^{-1} \otimes_R \widehat{R}} & & \downarrow{\scriptstyle \mathrm{HH}^\bullet(F)} \\ \mathrm{HH}_\bullet(A) & \xrightarrow{c} & \mathrm{HH}_\bullet(A) \otimes_R \widehat{R} & \xrightarrow{K} & \mathrm{HH}_\bullet(E, E^*) \end{array}$$

The outer compositions agree by (5.3), and by the universal property of the completion $\mathrm{HH}_\bullet(F)^{-1} \otimes_R \widehat{R}$ is the unique map which makes the left inner square commute. Hence the right-inner square also commutes and the result follows. □

Corollary 5.10. *Suppose $F\colon \mathcal{A} \to \mathcal{A}$ is an R-linear quasi-equivalence and which acts on $\mathrm{HH}_d(\mathcal{A})$ as multiplication $\mathrm{HH}_d(F) = r\cdot$ by a unit $r \in R^\times$. Then $\mathrm{HH}^{-d}(F) = r^{-1}\cdot$.*

Remark 5.11. In the context of CY structures, Proposition 5.8 shows that any right CY structure for the objects $\mathcal{N}$ supported on $\mathfrak{m}$ is determined by a left CY structure defined in a formal neighbourhood of $\mathfrak{m}$, and that a 'global' left CY structure restricts to this formal neighbourhood. Although not every right CY structure for $\mathcal{N}$ is the image of a global left CY structure, Proposition 5.9 shows that the action of a global equivalence on the right CY structures on $\mathcal{N}$ is nonetheless determined by its action on the global left CY structures.

§ 5.3 | Cyclic A_∞-categories

We would like to endow the properly supported objects in our 3-CY categories with a *potential* that expresses their deformation theory, and compare the potentials of objects related by a derived auto-equivalence. To this end we use A_∞-categories equipped with a *cyclic structure* which are a strict version of a right Calabi–Yau structure. Because we can, we assume that all A_∞-categories/functors/modules are strictly unital.

Given an A_∞-category $\mathcal{C}$, we write $\mathcal{C}\overset{\infty}{\mathrm{mod}}\,\mathcal{C}$ for its DG-category of A_∞-bimodules. The Hom-complex between bimodules $M, N \in \mathcal{C}\overset{\infty}{\mathrm{mod}}\,\mathcal{C}$ is of the form

$$\mathcal{C}\overset{\infty}{\mathrm{mod}}\,\mathcal{C}(M,N) := \left(\bigoplus_{i,j\geq 0} \mathrm{Hom}_{\mathbb{C}}(\mathcal{C}^{\otimes i} \otimes M \otimes \mathcal{C}^{\otimes j}, N), \mathrm{d}\right),$$

and so any degree k bimodule map $\alpha\colon M \to N[k]$ is given by its components $\alpha_{i,j}$. Any A_∞ category $\mathcal{C}$ is a bimodule over itself, and so is its linear dual $\mathcal{C}^*$ by pre-composition. Given an A_∞-functor $F\colon \mathcal{C} \to \mathcal{D}$ there is a pullback $F^*\colon \mathcal{D}\overset{\infty}{\mathrm{mod}}\,\mathcal{D} \to \mathcal{C}\overset{\infty}{\mathrm{mod}}\,\mathcal{C}$, which identifies $F^*\mathcal{M}(c,c') = \mathcal{M}(F(c), F(c'))$. The functor also gives a morphism $F\colon \mathcal{C} \to F^*\mathcal{D}$ in a natural way, so that we may complete any bimodule morphism $\alpha\colon \mathcal{D} \to \mathcal{D}^*$ to a bimodule morphism $\mathcal{C} \to \mathcal{C}^*$ via the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;F\;} & F^*\mathcal{D} \\ \downarrow & & \downarrow \alpha \\ \mathcal{C}^* & \xleftarrow{\;F^*\;} & F^*\mathcal{D}^* \end{array}$$

in $\mathcal{C}\overset{\infty}{\mathrm{mod}}\,\mathcal{C}$. By slight abuse of notation we denote the dashed vertical arrow as $F^*\alpha$. In this bimodule formalism, a cyclic structure is defined as follows.

Definition 5.12. Let $\mathcal{C}$ be a finite dimensional A_∞-category, by which we mean that $\mathcal{C}(c,c')$ is a finite dimensional vectorspace for all $c, c' \in \mathrm{Ob}\,\mathcal{C}$. A cyclic structure on $\mathcal{C}$ is an A_∞-bimodule homomorphism $\sigma = (\sigma_{i,j})\colon \mathcal{C} \to \mathcal{C}^*[-3]$ such that:

1) the higher maps $\sigma_{i,j}$ for $(i,j) \neq (0,0)$ vanish,
2) for all $a, b \in \operatorname{Ob}\mathcal{C}$ the map $\sigma_{0,0}(a,b)\colon \mathcal{C}(a,b) \to \mathcal{C}(b,a)^*$ is an isomorphism,
3) the dual $\sigma^*\colon \mathcal{C}^{**}[3] \to \mathcal{C}^*$ is identified with σ via the isomorphism $\mathcal{C} \simeq \mathcal{C}^{**}$ and obvious shift

Under these conditions the pair $(\mathcal{C}, \sigma)$ is a *cyclic A_∞-category.* A *cyclic A_∞-functor* $F\colon (\mathcal{C}, \sigma) \to (\mathcal{D}, \sigma')$ is given by the those A_∞-functors $F\colon \mathcal{C} \to \mathcal{D}$ such that $F^*\sigma' = \sigma$.

Objects in a cyclic A_∞-category are endowed with a potential. Let $(\mathcal{C}, \sigma)$ be a cyclic A_∞-category and $T \in \operatorname{Ob}\mathcal{C}$ an object with endomorphism A_∞-algebra $\mathcal{C}_T := \mathcal{C}(T,T)$, which has a cyclic structure $\sigma|_T\colon \mathcal{C}_T \to \mathcal{C}_T^*$ given by the restriction of σ. Then the potential of T is the noncommutative formal function

$$\mathcal{W} = \mathcal{W}_T \in \left(\bigoplus_{k\geq 1} (\mathcal{C}_T^1)^{\otimes k}\right)^*$$

which maps the $k+1$ tuple $f_0 \otimes \ldots \otimes f_k$ of degree 1 elements to

$$\mathcal{W}(f_0, \ldots, f_k) := \sigma(f_0)(m_k(f_1, \ldots, f_k)). \tag{5.4}$$

Let $N = \dim_{\mathbb{C}} \mathcal{C}_T^1$ and define $\mathcal{Q}_T$ to be the N-loop quiver, then $\mathcal{W}$ may be regarded as a formal potential $\mathcal{W} \in \widehat{\mathbb{C}\mathcal{Q}}_{\mathrm{cyc}}$. If $F\colon (\mathcal{C}, \sigma) \to (\mathcal{D}, \sigma')$ is a cyclic A_∞-functor then Kajiura [Kaj07, Proposition 4.16] shows that there is an induced formal homomorphism $\widehat{\mathbb{C}\mathcal{Q}}_{F(T)} \to \widehat{\mathbb{C}\mathcal{Q}}_T$ of the quiver algebras which maps the potential $\mathcal{W}_{F(T)}$ to $\mathcal{W}_T$.

For a general A_∞-functor F the pullback $F^*\sigma$ of a cyclic structure fails to be cyclic, but can be made cyclic via a perturbation, as Kontsevich–Soibelman [KS09] have shown. Given a cyclic A_∞-category $(\mathcal{C}, \sigma)$, the map $\sigma = \sigma_{0,0}$ defines an cochain in the dual Hochschild complex via the isomorphism[1]

$$\bigoplus_{c,c' \in \operatorname{Ob}\mathcal{C}} \operatorname{Hom}_{\mathbb{C}}(\mathcal{C}(c,c'), \mathcal{C}^*(c,c')) \simeq \bigoplus_{c,c' \in \operatorname{Ob}\mathcal{C}} \operatorname{Hom}_{\mathbb{C}}(\mathcal{C}(c,c') \otimes \mathcal{C}(c',c), \mathbb{C}) \subset \mathbf{C}(\mathcal{C})^*,$$

and its homotopy class coincides with a class $[\sigma] \in \mathrm{HH}^{-3}(\mathcal{C}, \mathcal{C}^*)$. If $F\colon \mathcal{C} \to \mathcal{D}$ is an A_∞-functor onto a second cyclic A_∞-category $(\mathcal{D}, \sigma')$, then $\mathrm{HH}^\bullet(F)[\sigma']$ corresponds to the homotopy class of the bimodule morphism $F^*\sigma'$. One can therefore ask that the condition $F^*\sigma' = \sigma$ holds up to homotopy:

$$\mathrm{HH}^\bullet(F)[\sigma'] = [\sigma].$$

If this condition holds, there exists an automorphism of $\mathcal{C}$ that perturbs $F^*\sigma$ to σ. These automorphisms are described in [CL10] in the setting of A_∞-algebras.

Lemma 5.13. *Let (C, σ) and (D, σ') be minimal cyclic A_∞-algebras with an A_∞-homomorphism $f\colon C \to D$. Suppose $\mathrm{HH}^\bullet(f)([\sigma']) = [\sigma]$, then there exists an A_∞-*

[1] N.B. one checks that this isomorphism is compatible with the Hochschild and bimodule differential. It extends to a quasi-isomorphism $\mathbf{C}(\mathcal{C})^* \to \mathcal{C} \operatorname{mod}^\infty \mathcal{C}(\mathcal{C}, \mathcal{C}^*)$, see e.g. [Gan13].

automorphism $g\colon C \to C$ *such that the composition* $f \circ g$ *is a* cyclic A_∞*-homomorphism.*

Proof. See the proof of [CL10, Proposition 7.4]. □

This result applies to the endomorphism A_∞-algebras of objects in a cyclic A_∞-category.

Lemma 5.14. *Let* $(\mathcal{C}, \sigma)$ *and* $(\mathcal{D}, \sigma')$ *be minimal cyclic* A_∞*-categories and* $F\colon \mathcal{C} \to \mathcal{D}$ *a quasi-fully-faithful* A_∞*-functor which satisfies* $\mathrm{HH}^\bullet(F)[\sigma'] = [\sigma]$*. Then for every* $M \in \mathcal{C}$ *there exists a cyclic* A_∞*-algebra isomorphism* $(\mathcal{C}_M, \sigma|_M) \to (\mathcal{D}_{F(M)}, \sigma'|_{F(M)})$*.*

Proof. If an A_∞-functor between minimal A_∞-categories is quasi-fully-faithful, then the restrictions $F|_M\colon \mathcal{C}_M \to \mathcal{D}_{F(M)}$ are A_∞-isomorphisms. By the perturbation Lemma 5.13 it suffices to shows that this preserves the Hochschild cohomology classes of the cyclic structures. Let $i_{F(M)}$ and i_M denote the inclusion functors of $\mathcal{D}_{F(M)}$ and $\mathcal{C}_M$, then

$$\mathrm{HH}^\bullet(F|_M)[\sigma'|_{F(M)}] = \mathrm{HH}^\bullet(i_{F(M)} \circ F|_M)[\sigma'] = \mathrm{HH}^\bullet(i_M)(\mathrm{HH}^\bullet(F)[\sigma']) = [\sigma|_M]. \quad \square$$

Although the lemma allows one to compare cyclic structures of an object with its image, we are usually interested in objects up to quasi-isomorphism. The following lemma tells us that the cyclic structure (hence the potential) is preserved under quasi-isomorphism.

Lemma 5.15. *Let* $(\mathcal{C}, \sigma)$ *be a minimal cyclic* A_∞*-category and* $M, N \in \mathrm{Ob}\,\mathcal{C}$*. If* M *and* N *are isomorphic in* $\mathrm{H}^0\mathcal{C}$*, then* $(\mathcal{C}_M, \sigma|_M) \simeq (\mathcal{C}_N, \sigma|_N)$ *as cyclic* A_∞*-algebras.*

Proof. Consider the DG-envelope $\mathcal{D}$ of $\mathcal{C}$, i.e. a DG-category with the same set of objects with $\mathcal{C}$ as its minimal model. Let $u \in \mathcal{C}(M, N)$ and $u^{-1} \in \mathcal{C}(N, M)$ be the lifts of the isomorphism in $\mathrm{H}^0\mathcal{D} = \mathrm{H}^0\mathcal{C}$ and its homotopy inverse. There is an induced map

$$u \circ - \circ u^{-1}\colon \mathcal{D}_M \to \mathcal{D}_N,$$

which gives rise to DG-bimodule morphisms $\overline{u}\colon \mathcal{D}_M \to \mathcal{D}_N$ and $\overline{u}^*\colon \mathcal{D}_N^* \to \mathcal{D}_M^*$. If $\alpha\colon \mathcal{D} \to \mathcal{D}^*[k]$ is a DG-bimodule morphism, with restrictions $\alpha|_N$ and α_M, then

$$\begin{aligned}(\overline{u}^* \circ \alpha|_N \circ \overline{u})(f)(g) &= \alpha(u \circ f \circ u^{-1}, u \circ g \circ u^{-1}) \\ &= \alpha(f \circ u^{-1} \circ u, g \circ u^{-1} \circ u).\end{aligned}$$

Because $u^{-1} \circ u$ is homotopic to the identity, it follows that for any such $\alpha\colon \mathcal{D} \to \mathcal{D}^*[k]$

$$[\alpha|_M] = [\overline{u}^* \circ \alpha|_N \circ \overline{u}] = \mathrm{HH}^\bullet(\overline{u})[\alpha|_N].$$

The same holds for the induced map $\mathcal{C}_M \to \mathcal{C}_N$ on the minimal model, so the result follows from the perturbation Lemma 5.13. □

Remark 5.16. Note that the existence of a quasi-isomorphism $M \simeq N$ in $\mathcal{C}$ is much stronger than the existence of a A_∞-isomorphism $\mathcal{C}_M \simeq \mathcal{C}_N$, as the latter is not guaranteed to satisfy the homotopy-cyclic condition.

§ 5.3.1 | Cyclic minimal models

There is a standard cyclic A_∞-category associated to a choice of quiver with potential.

Definition 5.17. Let (Q, W) be a quiver with potential and for vertices $v, w \in Q_0$ denote by $Q(v, w)$ the set of arrows from v to w. The A_∞-category $\mathcal{D} = \mathcal{D}_{Q,W}$ has objects $\operatorname{Ob}\mathcal{D} = Q_0$ and morphism spaces

$$\mathcal{D}(v,w) = \begin{cases} \mathbb{C}1_v \oplus \mathbb{C}Q(w,v)^*[1] \oplus \mathbb{C}Q(v,w)[2] \oplus \mathbb{C}1_v^*[3] & v = w \\ \mathbb{C}Q(w,v)^*[1] \oplus \mathbb{C}Q(v,w)[2] & \text{otherwise} \end{cases}$$

The higher products are required to have 1_v as strict units, for each $a \in Q(v, w)$

$$m_2(a^*, a) = 1_v^*, \quad m_2(a, a^*) = 1_w^*,$$

and for any chain of arrows $a_1, \dots, a_k$ in Q where $a_1 \in Q(v, w')$ and $a_k \in Q(v', w)$,

$$m_k(a_k^*, \dots, a_1^*) = \sum_{a \in Q(w,v)} c^a_{a_1 \cdots a_k} \cdot a,$$

where $c^a_{a_k \cdots, a_1}$ is the coefficient of $a_1 \cdots a_k$ in the cyclic derivative $\partial W / \partial a \in \mathbb{C}Q$ of the potential. All other compositions are zero, and in particular $\mathcal{D}$ is minimal.

As Kontsevich and Soibelman show [KS09], there is a cyclic structure on $\mathcal{D}$ defined by choice of trace: given a linear map $\operatorname{tr}_Q \colon \bigoplus_{v \in Q_0} \mathcal{D}^3(v, v) \to \mathbb{C}$ the pairing

$$\sigma(f)(g) = \operatorname{tr}_Q(m_2(f, g)),$$

is a cyclic structure $\sigma \colon \mathcal{D} \to \mathcal{D}^*$. In particular, we may fix the trace which sends the generators $1_v^* \mapsto 1 \in \mathbb{C}$, so that $\sigma(a^*)(a) = \sigma(a)(a^*) = a^*(a) = 1$ for all arrows. With this choice of cyclic structure, the potential of the cyclic A_∞-category $(\mathcal{D}, \sigma)$ agrees with W as an element of the completed path algebra $\widehat{\mathbb{C}Q}$ and hence encodes the same data as the quiver with potential.

We now wish to extend $\mathcal{D}$ to a cyclic A_∞-model for nilpotent modules, so that we may describe the DT invariants in terms of a local potential. This model is provided by the A_∞-category of twisted complexes $\operatorname{tw}\mathcal{D}_{Q,W}$, a definition of which can be found in [Lef03, §7]. It is quasi-equivalent to the DG-category $\mathcal{P}\mathrm{erf}\mathcal{D}_{Q,W}$ of perfect complexes, but in contrast to $\mathcal{P}\mathrm{erf}\mathcal{D}_{Q,W}$ the A_∞-category $\operatorname{tw}\mathcal{D}_{Q,W}$ is finite dimensional and admits a cyclic structure. The cyclic structure extends the cyclic structure on $\mathcal{D}_{Q,W}$, and we will therefore again denote it by σ. If W is a finite potential then the potential $\mathcal{W}_T$ is also finite for any $T \in \operatorname{tw}\mathcal{D}_{Q,W}$.

Theorem 5.18 (see [Dav11, Theorem 7.1.3]). *Let $M \in \operatorname{nilp}\operatorname{Jac}(Q,W)$ be a module with $\operatorname{End}_{\operatorname{Jac}(Q,W)}(M) \simeq \mathbb{C}$ and let $\mathcal{P} \subset \mathfrak{M}$ denote locus of repeated self-extensions of M. Then there exists a twisted complex T such that*

$$\int_{[\mathcal{P}\to\mathfrak{C}]} \phi_{\operatorname{tr}(W)}|_{\mathfrak{C}} = \Phi_{\mathcal{Q}_T,\mathcal{W}_T}(t^{[M]}), \tag{5.5}$$

where $\mathcal{Q}_T$ is the N-loop quiver of T with potential $\mathcal{W}_T$ as defined in (5.4).

The potential $\mathcal{W}_T$ of a twisted complex is too coarse of an invariant to track under derived quasi-equivalences and we will instead consider the *minimal* potential.

Let T be a twisted complex in $\mathcal{C} := \operatorname{tw}\mathcal{D}_{Q,W}$ corresponding to a nilpotent module M with $\operatorname{End}_A(M) \simeq \mathbb{C}$. The cyclic decomposition theorem [Kaj07, Theorem 5.15] gives a splitting of the cyclic endomorphism A_∞-algebra $\mathcal{C}_T$ of T: there is a cyclic A_∞-isomomorphism

$$(\mathcal{C}_T, \sigma|_T) \xrightarrow{\sim} (\mathrm{H}^\bullet\mathcal{C}_T, \sigma_{\min}) \oplus (L_T, \sigma'), \tag{5.6}$$

where $(\mathrm{H}^\bullet\mathcal{C}_T, \sigma_{\min})$ is *the cyclic minimal model*, a cyclic minimal A_∞-algebra structure on the cohomology of $\mathcal{C}_T$, and (L_T, σ') is a linearly contractible A_∞, i.e. a cyclic A_∞-algebra with $m_k = 0$ for $k \geq 2$. There is an associated splitting of the set of loops

$$(\mathcal{Q}_T)_1 = \{x_1, \dots, x_n\} \sqcup \{y_1, \dots, y_{N-n}\} \tag{5.7}$$

of the N-loop quiver $\mathcal{Q}_T$, so that x_i form a basis for the cohomology $\mathrm{H}^1\mathcal{C}_T$. Let $\mathcal{Q}_{\min,T}$ be the subquiver of $\mathcal{Q}_T$ generated by the n-loops $\{x_1, \dots, x_n\}$, then the *minimal potential*

$$\mathcal{W}_{\min,T} = \mathcal{W}_{\min,T}(x_1, \dots, x_n),$$

on $\mathcal{Q}_{\min,T}$ is the noncommutative formal function of $(\mathrm{H}^\bullet\mathcal{C}_T, \sigma_{\min})$ as in (5.4). Likewise, the linearly contractible summand (L, σ') has a potential $q = q(y_1, \dots, y_{N-n})$, which is a nondegenerate quadratic form. The splitting (5.6) induces formal isomorphism $\psi_T \colon \widehat{\mathbb{C}\mathcal{Q}}_T \to \widehat{\mathbb{C}\mathcal{Q}}_T$ such that $\psi_T(\mathcal{W}_T) = \mathcal{W}_{\min,T} + q$.

If $\mathcal{W}_{\min,T}$ is again a *finite* potential, then the partition function $\Phi_{\mathcal{Q}_{\min,T},\mathcal{W}_{\min,T}}(t)$ is well-defined, and Lemma 3.10 implies that it is equal to the partition function $\Phi_{\mathcal{Q}_T,\mathcal{W}_T}(t)$. Even if the minimal potential is a formal powerseries, it can still be used to compare the partition functions associated to two twisted complexes.

Lemma 5.19. *Let $T_1, T_2 \in \operatorname{Ob}\mathcal{C}$ be twisted complexes corresponding to nilpotent modules $M, N \in \operatorname{nilp}A$ with simple endomorphism algebras as above. If there exists a formal isomorphism $\psi \colon \widehat{\mathbb{C}\mathcal{Q}}_{\min,T_1} \to \widehat{\mathbb{C}\mathcal{Q}}_{\min,T_2}$ between their complete path algebras such that*

$$\psi(\mathcal{W}_{\min,T_1}) = \lambda \cdot \mathcal{W}_{\min,T_2},$$

for some scalar $\lambda \in \mathbb{C}^\times$*, then the partition functions of* T_1*,* T_2 *are equal:*

$$\Phi_{\mathcal{Q}_{T_1},\mathcal{W}_{T_1}}(t) = \Phi_{\mathcal{Q}_{T_2},\mathcal{W}_{T_2}}(t). \tag{5.8}$$

Proof. Without loss of generality, we can identify the first n loops in the N_1-loop quiver $\mathcal{Q}_{T_1}$ with the first n loops in the N_2-loop quiver $\mathcal{Q}_{T_2}$, and write the splitting in (5.7) as

$$(\mathcal{Q}_{T_1})_1 = \{x_1, \dots, x_n\} \sqcup \{y_1, \dots, y_{N_1-n}\}, \quad (\mathcal{Q}_{T_2})_2 = \{x_1, \dots, x_n\} \sqcup \{z_1, \dots, z_{N_2-n}\},$$

so that ψ is a formal automorphism of the quiver generated by the variables x_i. The potentials $\mathcal{W}_{\min,T_1}$, $\mathcal{W}_{\min,T_2}$ are functions in the variables x_i, and the quadratic terms q_1, q_2 of the linearly contractible summands for T_1 and T_2 are functions in the variables y_i and z_i respectively. Let $\mathcal{Q}$ be the $N_1 + N_2 - n$-loop quiver with loops

$$\mathcal{Q}_1 = \{x_1, \dots, x_n\} \sqcup \{y_1, \dots, y_{N_1-n}\} \sqcup \{z_1, \dots, z_{N_2-n}\}.$$

Then the formal isomophisms ψ_{T_1}, ψ_{T_2}, ψ lift to formal automorphisms of the completed path algebra $\widehat{\mathbb{C}\mathcal{Q}}$ in the obvious way, and satisfy:

$$\begin{aligned} \psi_{T_1}(\mathcal{W}_{T_1} + \lambda \cdot q_2) &= \mathcal{W}_{\min,T_1} + q_1 + \lambda \cdot q_2, \\ \psi(\mathcal{W}_{\min,T_1} + q_1 + \lambda \cdot q_2) &= \lambda \cdot \mathcal{W}_{\min,T_2} + q_1 + \lambda \cdot q_2, \\ \psi_{T_2}(\lambda \cdot \mathcal{W}_{T_2} + q_1) &= \lambda \cdot \mathcal{W}_{\min,T_2} + q_1 + \lambda \cdot q_2. \end{aligned}$$

By inspection, the composition $\psi_{T_2}^{-1} \circ \psi \circ \psi_{T_1}$ maps $\mathcal{W}_{T_1} + \lambda \cdot q_2$ to $\lambda \cdot \mathcal{W}_{T_2} + q_1$. Hence, Lemma 3.9 and Lemma 3.10 imply that

$$\Phi_{\mathcal{Q}_{T_1},\mathcal{W}_{T_1}}(t) \overset{3.10}{=} \Phi_{\mathcal{Q},\mathcal{W}_{\min,T_1}+\lambda\cdot q_2}(t) \overset{3.9}{=} \Phi_{\mathcal{Q},\lambda\cdot\mathcal{W}_{\min,T_2}+q_1}(t) \overset{3.10}{=} \Phi_{\mathcal{Q}_{T_2},\lambda\cdot\mathcal{W}_{T_2}}(t).$$

The partition function of $(\mathcal{Q}_{T_2}, \lambda \cdot \mathcal{W}_{T_2})$ is independent of λ as the vanishing cycle of $\mathrm{tr}(\lambda \cdot \mathcal{W}_{T_2}) = \lambda \cdot \mathrm{tr}(\mathcal{W}_{T_2})$ depends only on the zero locus of the function. □

In view of the above, it suffices to work with the cyclic minimal model $\mathrm{H}^\bullet \mathrm{tw}\, \mathcal{D}_{Q,W}$ of the cyclic A_∞-category $\mathrm{tw}\, \mathcal{D}_{Q,W}$.

§ 5.3.2 | Cyclic minimal models associated to finite R-algebras

We return to the setting of §5.2 where A is an algebra over a commutative Noetherian $\mathbb{C}$-algebra R, which is smooth over $\mathbb{C}$. We let $\Lambda = A \otimes_R \widehat{R}$ denote the completion of A at a choice of maximal ideal $\mathfrak{m} \subset R$, and let E denote the Koszul dual of A in $\mathfrak{m}$.

If the completion is isomorphic to a completed Jacobi algebra of a quiver with potential (Q, W), then the following theorem of Van den Bergh relates the Koszul dual to the A_∞-category of (Q, W).

Theorem 5.20 (See [VdB10, Theorem 12.1])**.** *Suppose the completion* Λ *is isomorphic to* $\widehat{\mathrm{Jac}}(Q, W)$ *for some quiver with potential* (Q, W)*. Then* $\mathcal{D}_{Q,W}$ *is* A_∞*-quasi-*

isomorphic to the DG algebra E.

If A satisfies the conditions of the theorem we then obtain the following chain of quasi-equivalences

$$U\colon \mathcal{H} \xrightarrow{\sim \text{q.e}} \operatorname{tw}\mathcal{D}_{Q,W} \xrightarrow{\sim \text{q.e}} \mathcal{P}\mathrm{erf}\mathcal{D}_{Q,W} \xrightarrow{\sim \text{q.e}} \mathcal{P}\mathrm{erf}\, E \xrightarrow{\sim \text{q.e}} \mathcal{N}.$$

where $\mathcal{H} := \mathrm{H}^\bullet \operatorname{tw}\mathcal{D}_{Q,W}$ is the cyclic minimal model of $\operatorname{tw}\mathcal{D}_{Q,W}$ and $\mathcal{N} \subset \mathcal{A} = \mathcal{P}\mathrm{erf}\, A$ is the DG-subcategory of objects supported on the maximal ideal $\mathfrak{m} \subset R$ as in §5.2. Via the equivalence U we can relate the Hochschild actions of autoequivalences on $\mathcal{H}$ and $\mathcal{N}$, yielding the following.

Proof of Theorem 5.1. Let A be an algebra with a completion isomorphic to $\widehat{\mathrm{Jac}}(Q,W)$, and write $\mathcal{A} = \mathcal{P}\mathrm{erf}\, A$ as before. If $F\colon \mathcal{A} \to \mathcal{A}$ is an R-linear quasi-equivalence, such that $\mathrm{HH}_3(F) = \lambda \in \mathbb{C}^\times$, then by Corollary 5.10 it acts on $\mathrm{HH}^{-3}(\mathcal{N},\mathcal{N}^*)$ as

$$\mathrm{HH}^{-3}(F) = \lambda^{-1} \cdot .$$

By [Lef03, Theorem 9.2.0.4], the A_∞-functor $U\colon \mathcal{H} \to \mathcal{N}$ has a quasi-inverse $U^{-1}\colon \mathcal{N} \to \mathcal{H}$. Then the composition $F' := U^{-1} \circ F \circ U$ is a quasi-auto-equivalence on $\mathcal{H}$ and acts as

$$\mathrm{HH}^{-3}(F') = \mathrm{HH}^{-3}(U^{-1}) \circ \mathrm{HH}^{-3}(F) \circ \mathrm{HH}^{-3}(U) = \lambda^{-1},$$

on $\mathrm{HH}^{-3}(\mathcal{H},\mathcal{H}^*)$. This shows that the functor F' satisfies the homotopy-cyclic condition

$$\mathrm{HH}^{-3}(F')([\lambda \cdot \sigma]) = [\sigma],$$

with respect to the cyclic structures σ and $\lambda \cdot \sigma$ on $\mathcal{H}$. Let $T \in \operatorname{Ob}\mathcal{H}$ be a twisted complex, then Lemma 5.14 shows that there exists a cyclic A_∞-algebra isomorphism

$$(\mathcal{H}_T, \sigma|_T) \to (\mathcal{H}_{F(T)}, \lambda \cdot \sigma|_{F'(T)}). \tag{5.9}$$

Now suppose $M, N \in \mathrm{nilp}A$ are modules with $\mathrm{End}_A(M) \simeq \mathrm{End}_A(N) \simeq \mathbb{C}$ such that there exists a quasi-isomorphism $F(M) \simeq N$ in the derived category $\mathrm{D}_\mathfrak{m}(A) \simeq \mathrm{H}^0\mathcal{N}$. Then they can be represented by the twisted complexes $T_1, T_2 \in \operatorname{Ob}\mathcal{H}$ such that $U(T_1) \simeq M$ and $U(T_2) \simeq N$. Because $F(M) \simeq N$, it then it also follows that $F'(T_1) \simeq T_2$ in $\mathrm{H}^0\mathcal{H}$. Combining the map (5.9) with Lemma 5.15, we obtain a cyclic A_∞-isomorphism

$$(\mathcal{H}_{T_2}, \lambda \cdot \sigma|_{T_2}) \to (\mathcal{H}_{F(T_1)}, \lambda \cdot \sigma|_{F'(T_1)}) \to (\mathcal{H}_{T_1}, \sigma|_{T_1}).$$

In particular, there is an isomorphism $\psi\colon \widehat{\mathbb{C}Q}_{\min,T_1} \to \widehat{\mathbb{C}Q}_{\min,T_2}$ of the completed path algebras which maps $\mathcal{W}_M = \mathcal{W}_{\min,T_1}$ to the potential defined by the noncommutative formal function associated to the minimal cyclic A_∞-algebra $(\mathcal{H}_{T_2}, \lambda \cdot \sigma|_{T_2})$:

$$\begin{aligned}\psi(\mathcal{W}_M)(f_0, \dots, f_i) &= \sum_{i=2}^{\infty} (\lambda \cdot \sigma|_{T_2})(f_0)(m_i(f_1, \dots, f_i)) \\ &= \lambda \cdot \sum_{i=2}^{\infty} \sigma|_{T_2}(f_0)(m_i(f_1, \dots, f_i)) \\ &= \lambda \cdot \mathcal{W}_{\min,T_2}(f_0, \dots, f_i).\end{aligned}$$

Hence $\psi(\mathcal{W}_M) = \lambda \cdot \mathcal{W}_{\min,T_2} = \lambda \cdot \mathcal{W}_N$ as claimed. □

With Theorem 5.1 established, the proof of the corollary now follows almost directly from Theorem 5.18 and Lemma 5.19.

Proof of Corollary 5.2. By assumption $M \in \operatorname{nilp} A$ and $F(M) \in \operatorname{nilp} A$ are modules with $\operatorname{End}_A(M) \simeq \operatorname{End}_A(F(M)) \simeq \mathbb{C}$, so Theorem 5.18 implies that

$$\sum_{k \geq 0} \int_{\mathcal{P}_{M,k}} \phi_{\operatorname{tr}(W)} \cdot t^{[M]} = \Phi_{\mathcal{Q}_{T_1}, \mathcal{W}_{T_1}}(t^{[M]}),$$
$$\sum_{\kappa \geq 0} \int_{\mathcal{P}_{F(M),k}} \phi_{\operatorname{tr}(W)} \cdot t^{[F(M)]} = \Phi_{\mathcal{Q}_{T_2}, \mathcal{W}_{T_2}}(t^{[F(M)]}),$$

for some twisted complexes $T_1, T_2 \in \operatorname{tw} \mathcal{D}_{Q,W}$ corresponding to M and $F(M)$ respectively. Theorem 5.1 shows that there exists a formal isomorphism between the completed path algebras of $\mathcal{Q}_{\min,T_1}$ and $\mathcal{Q}_{\min,T_2}$ which maps $\mathcal{W}_{\min,T_2}$ to $\lambda \cdot \mathcal{W}_{\min,T_1}$ for some scalar $\lambda \in \mathbb{C}^\times$. Hence, Lemma 5.19 show that

$$\sum_{k \geq 0} \int_{\mathcal{P}_{M,k}} \phi_{\operatorname{tr}(W)} \cdot t^{[M]} = \Phi_{\mathcal{Q}_{T_1}, \mathcal{W}_{T_1}}(t^{[M]}) = \Phi_{\mathcal{Q}_{T_2}, \mathcal{W}_{T_2}}(t^{[M]}) = \sum_{k \geq 0} \int_{\mathcal{P}_{F(M),k}} \phi_{\operatorname{tr}(W)} \cdot t^{[M]},$$

and the result follows after comparing coefficients. □

§ 5.4 | The case of flops

Let Y be a threefold and suppose $p \colon Y \to \operatorname{Spec} R$ a small contraction. Then the bounded complexes of locally free sheaves form an R-linear DG-category $\mathcal{P}\mathrm{erf}\, Y$, which forms an enhancement of $\mathrm{D}^{\mathrm{perf}}(Y)$.

Lemma 5.21. *Let Y be a smooth quasi-projective threefold Y, then $\mathcal{P}\mathrm{erf}\, Y$ is a smooth DG-category with Hochschild homology $\mathrm{HH}_3(\mathcal{P}\mathrm{erf}\, Y) \simeq \mathrm{H}^0(Y, \omega_Y)$.*

Proof. The smoothness of $\mathcal{P}\mathrm{erf}\, Y$ follows from [Orl16; Lun10]. As shown in [Kel98] the Hochschild homology $\mathrm{HH}_\bullet(\mathcal{P}\mathrm{erf}\, Y)$ coincides with the geometric Hochschild homology $\mathrm{HH}_\bullet(Y)$. Because Y is smooth, the Hochschild-Kostant-Rosenberg theorem (see e.g.

[Lod97, Theorem 3.4.4]) induces a decomposition

$$\mathrm{HH}_d(Y) \simeq \bigoplus_{j-i=d} \mathrm{H}^i(Y, \Omega_Y^j)$$

where Ω_Y^j denotes the sheaf of differential j-forms on Y. Because Y is a threefold, it then follows that $\mathrm{HH}_3(Y) \simeq \mathrm{H}^0(Y, \Omega_Y^3) = \mathrm{H}^0(Y, \omega_Y)$ □

Lemma 5.22. *Suppose $p \colon Y \to \operatorname{Spec} R$ is a contraction with R Gorenstein, then*

$$\operatorname{Aut}_R(\mathrm{HH}_3(\mathcal{P}\mathrm{erf} Y)) \simeq R^\times.$$

Proof. Because R is Gorenstein, it has a canonical line bundle ω_R with $\omega_Y = p^! \omega_R$. Moreover, $\mathbf{R}p_* \mathcal{O}_Y \simeq R$ because p is a contraction, which implies $\mathrm{HH}_3(\mathcal{P}\mathrm{erf} Y)$ is

$$\mathrm{H}^0(Y, \omega_Y) = \mathrm{H}^0 \mathbf{R}\mathrm{Hom}_Y(\mathcal{O}_Y, p^! \omega_R) \simeq \mathrm{H}^0 \mathbf{R}\mathrm{Hom}_R(\mathbf{R}p_* \mathcal{O}_Y, \omega_R) \simeq \mathrm{H}^0(\operatorname{Spec} R, \omega_R).$$

Because ω_R is a line bundle, the R-linear endomorphism group is

$$\operatorname{End}_R(\mathrm{H}^0(\operatorname{Spec} R, \omega_R)) = \operatorname{End}_{\operatorname{Spec} R}(\omega_R) \simeq R,$$

and the automorphisms are the invertible elements $R^\times \subset R$. □

Now let $p \colon Y \to \operatorname{Spec} R$ be a flopping contraction of a curve $\mathrm{C} \subset Y$ in a smooth threefold onto a maximal ideal $\mathfrak{m} \subset R$ with R Gorenstein. Suppose Y admits a tilting bundle $\mathcal{P}$ with $A = \operatorname{End}_Y(\mathcal{P})$, and write $\mathcal{A} = \mathcal{P}\mathrm{erf} A$ and $\mathcal{N} \subset \mathcal{A}$ as before. Then there are R-linear quasi-inverse quasi-equivalences

$$- \otimes_{\mathcal{A}} \mathcal{P} \colon \mathcal{A} \to \mathcal{P}\mathrm{erf} Y, \quad (\mathcal{P}\mathrm{erf} Y)(\mathcal{P}, -) \colon \mathcal{P}\mathrm{erf} Y \to \mathcal{A},$$

which identifies $\mathrm{H}^0\mathcal{N}$ with $\mathrm{D}^b_{\mathrm{C}}(Y)$. If moreover, $\widehat{A} \simeq \widehat{\mathrm{Jac}}(Q, W)$ for some quiver with potential (Q, W), then the (minimal) potentials of objects in $\mathcal{H}$ compute DT-invariants for objects in $\mathrm{D}^b_{\mathrm{C}}(Y)$.

Proof of Proposition 5.3. If the base ring R satisfies $R^\times = \mathbb{C}^\times$, then it follows from Lemma 5.22 that the Hochschild homology of $\mathcal{A}$ has R-linear automorphisms

$$\operatorname{Aut}_R(\mathrm{HH}_3(\mathcal{A})) \simeq \operatorname{Aut}_R(\mathrm{HH}_3(\mathcal{P}\mathrm{erf} Y)) \simeq R^\times \simeq \mathbb{C}^\times.$$

Suppose $F \colon \mathcal{A} \to \mathcal{A}$ is an R-linear quasi-equivalence, then $\mathrm{HH}_3(F)$ is an R-linear autoequivalence of $\mathrm{HH}_3(\mathcal{A})$ and is therefore given by a nonzero scalar $\lambda \in \mathbb{C}^\times$. □

Chapter 6

Donaldson-Thomas Invariants

In this chapter we will finally compute the Donaldson–Thomas partition function of the length 2 flops we constructed in chapter 2. Our computation relies on the methods we introduced in chapter 4 & chapter 5, which reduce the computation to a finite number of cases. Because these methods work in a wider context than the family constructed in 2 we will first work in a more general setup.

Let $p\colon Y \to Y_{\mathrm{con}} = \operatorname{Spec} R$ be a flopping contraction of a length 2 curve $\mathrm{C} \subset Y$ in a smooth threefold onto $\mathfrak{o} \in \operatorname{Spec} R$, and suppose that there is an R-linear tilting equivalence

$$\Psi\colon\ \mathrm{D}^b(Y) \to \mathrm{D}^b(\mathrm{Jac}(Q,W)),$$

where (Q,W) is a quiver with potental such that $\mathrm{Jac}(Q,W)$ is an NCCR over R. We moreover assume that Q has vertices $Q_0 = \{0,1\}$, and that the vertex simples S_0, S_1 are the images of the objects $\mathcal{O}_{2\mathrm{C}}(-1)[1]$ and $\mathcal{O}_{\mathrm{C}}(-1)$ respectively. Then the category $\operatorname{nilp}\mathrm{Jac}(Q,W)$ of nilpotent modules is naturally identified with the category $\operatorname{fdmod}\Lambda$ of finite dimensional modules over the completion $\Lambda := \widehat{\mathrm{Jac}(Q,W)}$ at $\mathfrak{o}$. Hence the Donaldson–Thomas partition function

$$\Phi(t) = \Phi_{Q,W}(t) \in \mathrm{Mot}^{\widehat{\mu}}(\mathbb{C})[[t_0,t_1]],$$

counts objects in $\operatorname{fdmod}\Lambda$ for each dimension vector in $\Delta = \mathbb{N}Q_0$. In what follows we identify Δ with the positive cone $\mathbb{N}[S_0] \oplus \mathbb{N}[S_1] \subset \mathrm{K}_0(\operatorname{fdmod}\Lambda)$, and identify the K-theories $\mathrm{K}_0(\operatorname{fdmod}\Lambda) \simeq \mathrm{K}_0(\mathrm{D}^b_{\mathrm{C}}(Y))$ via the equivalence Ψ.

Let $P_0, P_1 \in \operatorname{mod}\Lambda$ be the projective covers of S_0 and S_1 as before and fix the stability parameter $v = 2[P_0] - [P_1]$. Then Theorem 4.1 implies that the Z_v-stable modules are

$$\Psi(\mathcal{O}_p), \qquad \left.\begin{matrix}\Psi(\mathcal{O}_{\mathrm{C}}(n-1))\\ \Psi(\mathcal{O}_{2\mathrm{C}}(n))\end{matrix}\right\} \text{ for } n \geq 0, \qquad \left.\begin{matrix}\Psi(\mathcal{O}_{\mathrm{C}}(n-1)[1])\\ \Psi(\mathcal{O}_{2\mathrm{C}}(n)[1])\end{matrix}\right\} \text{ for } n < 0,$$

The central charge $Z_v\colon\ \mathrm{K}_0(\operatorname{fdmod}\Lambda) \to \mathbb{C}$ maps the classes of these stable modules into the upper half-plane, as depicted in figure 6.1. Let $\Theta\colon\ \mathrm{K}_0(\operatorname{fdmod}\Lambda) \to (0,1]$ denote

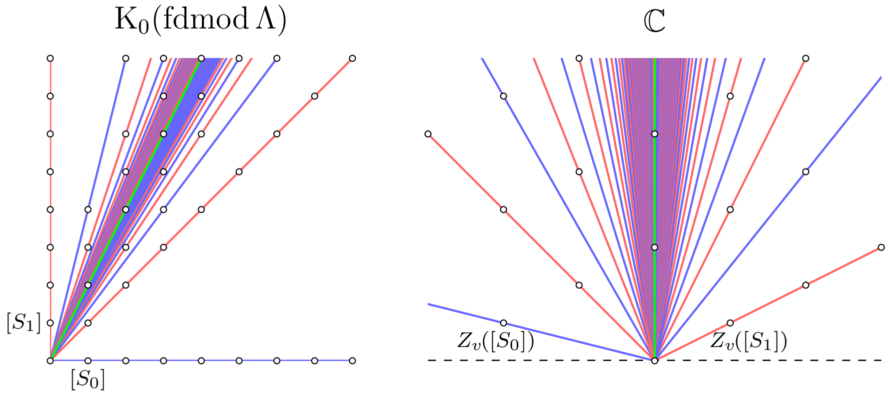


Figure 6.1: The dimension vectors of semistable modules (left) are mapped into the upper half-plane by the central charge Z_v of the parameter $v = 2[P_0] - [P_1]$. Semistable objects on the same coloured ray filter by a stable module of type $\mathcal{O}_C$ (red), $\mathcal{O}_{2C}$ (blue) or $\mathcal{O}_p$ (green).

the phase function of Z_v, and write the phases of the stable objects as $\theta_{\mathrm{pt}} = \Theta([\mathrm{pt}])$,

$$\theta_{C,n} = \begin{cases} \Theta([\mathcal{O}_C(n-1)]) & n \geq 0 \\ \Theta([\mathcal{O}_C(n-1)[1]]) & n < 0 \end{cases}, \quad \theta_{2C,n} = \begin{cases} \Theta([\mathcal{O}_{2C}(n)]) & n \geq 0 \\ \Theta([\mathcal{O}_{2C}(n-1)[1]]) & n < 0 \end{cases},$$

Then Lemma 3.5 yields a decomposition of the partition function into an ordered product

$$\Phi(t) := \prod_{n=-\infty}^{-1} \left(\Phi^{\theta_{2C,n}}(t) \cdot \Phi^{\theta_{C,n}}(t)\right) \cdot \Phi^{\theta_{\mathrm{pt}}}(t) \cdot \prod_{n=0}^{\infty} \left(\Phi^{\theta_{2C,n}}(t) \cdot \Phi^{\theta_{C,n}}(t)\right)$$

where $\Phi^{\theta}(t) := \int_{[\mathfrak{C}^{\theta} \hookrightarrow \mathfrak{C}]} \phi_{\mathrm{tr}(W)}|_{\mathfrak{C}}$ denotes the partition function counting semistable nilpotent modules of phase θ as defined in §3.2.1. Because Z_v is generic, each of these partition functions expands as a function of the BPS invariants

$$\Phi^{\theta}(t) = \mathrm{Sym}\left(\sum_{k \geq 1} \frac{\mathrm{BPS}_{k\delta}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} t^{k\delta}\right),$$

where δ denotes the unique K-theory class (or dimension vector) of the stable modules with phase θ. Hence, the DT partition function is completely determined by the BPS invariants associated to the sheaves $\mathcal{O}_C(n)[m]$, $\mathcal{O}_{2C}(n)[m]$ and $\mathcal{O}_p$ for $p \in C$.

If the base $\operatorname{Spec} R$ has absolute units $R^{\times} = \mathbb{C}^{\times}$ (as is the case for our family by Lemma 2.17) then the main theorem of chapter 5 implies that the BPS invariants do not depend on the twist/shift by n and m.

Proposition 6.1. *Suppose $R^{\times} = \mathbb{C}^{\times}$, then for all $k \in \mathbb{N}$ and $n, m \in \mathbb{Z}$ such that the object $\Psi(\mathcal{O}_C(n)[m])$ lies in* fdmod Λ, *there is an equality of motivic BPS invariants*

$$\mathrm{BPS}_{k[\mathcal{O}_C(n)[m]]} = \mathrm{BPS}_{k[C]},$$

where $[\mathrm{C}] := [\mathcal{O}_\mathrm{C}]$ *denotes the class of the curve. Likewise, for all* $k \in \mathbb{N}$ *and* $n, m \in \mathbb{Z}$ *such that the object* $\Psi(\mathcal{O}_{2\mathrm{C}}(n)[m])$ *lies in* $\operatorname{fdmod}\Lambda$*, there is an equality*

$$\mathrm{BPS}_{k[\mathcal{O}_{2\mathrm{C}}(n)[m]]} = \mathrm{BPS}_{k[2\mathrm{C}]},$$

where $[2\mathrm{C}] := [\mathcal{O}_{2\mathrm{C}}]$ *denote the class of the scheme fibre.*

Proof. Fix $n, m \in \mathbb{Z}$ such that $M = \Phi(\mathcal{O}_\mathrm{C}(n-1)[m])$ is stable. By Theorem 4.1 every semistable objects of phase $\theta_{\mathrm{C},n}$ is a repeated self-extensions of this stable module, which implies that $\mathcal{C}^{\theta_{\mathrm{C},n}}$ is equivalent to the stack $\mathcal{P}_M$ parametrising the self-extensions of M as in Theorem 5.18, and it follows that

$$\operatorname{Sym}\left(\sum_{k\geq 1}\frac{\mathrm{BPS}_{k[M]}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}t^{k[M]}\right) = \Phi^{\theta_{\mathrm{C},n}}(t) = \sum_{\kappa\geq 0}\int_{\mathcal{P}_{M,k}}\phi_{\operatorname{tr}(W)}\cdot t^{k[M]}.$$

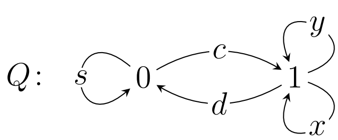
The derived functor $F = \Phi\circ(-\overset{\mathbf{L}}{\otimes}\mathcal{O}_Y(1-n)[-m])\circ\Phi^{-1}$ maps M to the stable module $F(M)\simeq\Phi(\mathcal{O}_\mathrm{C})$, and lifts to a DG enhancement of $\mathrm{D}^b(\Lambda)$ because it is a composition of tilting functors. By Proposition 5.3 the action $\mathrm{HH}_3(F)$ on Hoschild homology is a scalar multiplication, so that Corollary 5.2 implies that

$$\begin{aligned}\operatorname{Sym}\left(\sum_{k\geq 1}\frac{\mathrm{BPS}_{k[M]}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}t^{k[M]}\right) &= \sum_{k\geq 0}\int_{\mathcal{P}_{M,k}}\phi_{\operatorname{tr}(W)}\cdot t^{k[M]}\\ &= \sum_{k\geq 0}\int_{\mathcal{P}_{F(M),k}}\phi_{\operatorname{tr}(W)}\cdot t^{k[M]}\\ &= \operatorname{Sym}\left(\sum_{k\geq 1}\frac{\mathrm{BPS}_{k[\mathrm{C}]}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}t^{k[M]}\right).\end{aligned}$$

Hence the equalities $\mathrm{BPS}_{k[M]} = \mathrm{BPS}_{k[\mathrm{C}]}$ follow after comparing coefficients. The proof for the curve class $[2\mathrm{C}]$ is analogous. □

Hence, in this general setup the motivic DT partition function is controlled by the invariants $\mathrm{BPS}_{k[\mathrm{pt}]}$, $\mathrm{BPS}_{k[\mathrm{C}]}$, and $\mathrm{BPS}_{k[2\mathrm{C}]}$ associated to the point class $[\mathrm{pt}] = [\mathcal{O}_p]$ and the curve classes $[\mathrm{C}]$ and $[2\mathrm{C}]$.

Now we return to the family constructed in Theorem 2.7. Fix a parameter $f \in \mathbb{C}[y]$ divisible by y^3 such that $f_{\text{even}} \neq 0$, let $p\colon Y \to Y_{\text{con}} = \operatorname{Spec} R_f$ be the associated flop, and (Q, W_f) its quiver with potential, as pictured in figure 6.2. Let $a, b \in \mathbb{N}\cup\{\infty\}$



$$W_f = x^2y - f(y) + ycd + cdy - sdc + 2f_{\text{even}}\left(s^{\frac{1}{2}}\right)$$

Figure 6.2: The family (Q, W_f), repeated for the reader's convenience.

denote the maximal integers such that y^{2a} divides $f_{\mathrm{even}}(y)$ and y^{2b} divides $f_{\mathrm{odd}}(y)$, where we set $b = \infty$ if $f_{\mathrm{odd}} = 0$. Then we compute BPS invariants $\mathrm{BPS}_{k[\mathrm{pt}]}$, $\mathrm{BPS}_{k[\mathrm{C}]}$ and $\mathrm{BPS}_{k[2\mathrm{C}]}$ at the following levels of refinement.

Theorem 6.2. *The BPS invariants associated to the point sheaves on* C *are*

$$\mathrm{BPS}_{k[\mathrm{pt}]} = \mathbb{L}^{-\frac{3}{2}}[\mathbb{P}^1] \quad \textit{for } k \geq 1. \tag{6.1}$$

The BPS invariants associated to the curve class 2C *are*

$$\mathrm{BPS}_{[2\mathrm{C}]} = \mathbb{L}^{-\frac{1}{2}}(1 - [\mu_a]), \quad \mathrm{BPS}_{k[2\mathrm{C}]} = 0 \quad \textit{for } k > 1. \tag{6.2}$$

The first BPS invariant associated to the curve class C *is*

$$\mathrm{BPS}_{[\mathrm{C}]} = \begin{cases} \mathbb{L}^{-1}(1 - [D_{4a}]) + 2 & a \leq b, \\ \mathbb{L}^{-1}(1 - [D_{2b+1}]) + 3 & a > b. \end{cases} \tag{6.3}$$

where D_{4a} and D_{2b+1} are curves of genus a resp. b which carry a monodromy representation of μ_{4a} and μ_{2b+1} respectively. The other BPS invariants have the realisation

$$\mathrm{BPS}^{\mathrm{mmhs}}_{2[\mathrm{C}]} = \chi_{\mathrm{mmhs}}\left(\mathbb{L}^{-\frac{1}{2}}(1 - [\mu_a])\right), \tag{6.4}$$

$$\mathrm{BPS}^{\mathrm{mmhs}}_{k[\mathrm{C}]} = 0 \quad \textit{for } k > 2, \tag{6.5}$$

where $\mathrm{BPS}^{\mathrm{mmhs}}_{\delta} := \chi_{\mathrm{mmhs}}(\mathrm{BPS}_{\delta})$ denotes the realisation in the ring $\mathrm{K}_0(\mathrm{MMHS})$ of monodromic mixed Hodge structures.

Proof. The proof is split over the rest of the chapter: identity (6.1) is shown in Proposition 6.13, identity (6.2) is shown in Lemma 6.8, identity (6.3) is shown in Proposition 6.22, identity (6.4) is shown in Proposition 6.19, and identity (6.5) in Lemma 6.15. □

Corollary 6.3. $\mathrm{K}_0(\mathrm{MMHS})$*-refined DT-invariants do not determine flops.*

Proof. The Tjurina-number calculation in §2.2.4 shows that for each $a \geq 2$, the flops defined by the parameters

$$f(y) = y^{2a}, \quad f(y) = y^{2a} + y^{2a+1}, \quad \ldots \quad f(y) = y^{2a} + y^{4a-1},$$

are not all pairwise non-isomorphic (even analytically). By inspection, Theorem 6.2 shows that $\mathrm{K}_0(\mathrm{MMHS})$-refined BPS invariants only depend on a and is not influenced by the perturbations $y^{2a+1}, \ldots, y^{4a-1}$ in the potentials. Because the BPS invariants determine the DT partition function, the result follows. □

As remarked in the introduction, the derived equivalence Ψ induces an isomorphism $[\Psi]\colon \mathrm{K}_0(\mathrm{D}^b_{\mathrm{C}}(Y)) \xrightarrow{\sim} \mathrm{K}_0(\mathrm{D}^b(\mathrm{fdmod}\,\Lambda))$ in K-theory, which identifies the dimension vectors $\delta \in \Delta \subset \mathrm{K}_0(\mathrm{fdmod}\,\Lambda)$ with K-theory classes $[\Psi]^{-1}(\delta) \in \mathrm{K}_0(\mathrm{D}^b_{\mathrm{C}}(Y))$. From a geometric point of view, a natural way to present $\mathrm{K}_0(\mathrm{D}^b_{\mathrm{C}}(Y))$ is as pairs of rank and Euler-characteristic: there is an isomorphism $\mathrm{K}_0(\mathrm{D}^b_{\mathrm{C}}(Y)) \xrightarrow{\sim} \mathbb{Z}^2$ which maps

$$[\mathcal{F}] \mapsto (r,\chi) = (\operatorname{rk}\mathcal{F}, \chi(\mathcal{F})),$$

for every sheaf $\mathcal{F}$ supported on C. The rank/Euler characteristic of the sheaves $\mathcal{O}_{\mathrm{C}}$, $\mathcal{O}_{2\mathrm{C}}$ were computed in [Kat06], and are $(r,\chi) = (1,1)$, $(r,\chi) = (2,1)$ respectively. Moreover, one can check that the twist $\mathcal{O}_Y(1)\otimes -$ acts on these pairs as $(r,\chi) \mapsto (r, \chi + r)$. Hence, the classes of S_0 and S_1 are mapped to the following (r,χ) pairs:

$$[\Psi^{-1}(S_0)] = [\mathcal{O}_{2\mathrm{C}}(-1)[1]] \mapsto (r,\chi) = (-2,1),$$
$$[\Psi^{-1}(S_1)] = [\mathcal{O}_{\mathrm{C}}(-1)] \mapsto (r,\chi) = (1,0).$$

In general, any dimension vector δ can be represented as a pair $(r,\chi) = (\delta_1 - 2\delta_0, \delta_0)$. Indexing the BPS invariants BPS_δ as $\mathrm{BPS}_{(r,\chi)}$ for (r,χ) corresponding to δ, we arrive at the *strong-rationality conjecture* (see [Tod15] and [Dav19] for the refined version) which proposes that $\mathrm{BPS}_{(r,\chi)}$ is independent of χ. Theorem 6.2 allows us to verify this conjecture for our family at the MMHS level of refinement.

Corollary 6.4. *Let $Y = Y_f$ be a length 2 flop as in Theorem 6.2, and consider a pair (r,χ) corresponding to some dimension vector $\delta \in \Delta$. Then*

$$\mathrm{BPS}^{\mathrm{mmhs}}_{(r,\chi)} = \begin{cases} \mathrm{BPS}^{\mathrm{mmhs}}_{[\mathrm{pt}]} & r = 0 \\ \mathrm{BPS}^{\mathrm{mmhs}}_{[\mathrm{C}]} & r = \pm 1 \\ \mathrm{BPS}^{\mathrm{mmhs}}_{2[\mathrm{C}]} & r = \pm 2 \\ 0 & \textit{otherwise} \end{cases}$$

In particular, $\mathrm{BPS}^{\mathrm{mmhs}}_{(r,\chi}$ is independent of χ, and the strong rationality conjecture holds in the MMHS*–refinement.*

Remark 6.5. We expect that the identities (6.4) and (6.5) lift to the motivic refinement, which would imply Corollary 6.4 also holds at this level. However, to produce the identities (6.4) and (6.5) we require the theory of monodromic mixed Hodge modules, which categorify $\mathrm{K}_0(\mathrm{MMHS})$. One might hope to lift the proof to the motivic setting via a similar categorification, but such a thing is currently not present in the literature.

Other refined invariants can be deduced by first determining the Hodge structure and monodromy on the curves D_{4a} and D_{2b+1}. The monodromy is concentrated on the middle cohomology and, as we show in §6.3.2, has the following form.

Proposition 6.6. *The Hodge decomposition of $\mathrm{H}^1(D_{4a},\mathbb{Q})$ consists of representations*

$$\mathrm{H}^{1,0}(D_{4a}) = \mathrm{H}^1(D_{4a}, \mathcal{O}_{D_{4a}}) \simeq \textstyle\bigoplus_{j=1}^a \xi^{2j-1+2a},$$
$$\mathrm{H}^{0,1}(D_{4a}) = \mathrm{H}^0(D_{4a}, \Omega^1_{D_{4a}}) \simeq \textstyle\bigoplus_{j=1}^a \xi^{2j-1}.$$

where ξ denotes the primitive μ_{4k}–representation. Likewise, the Hodge decomposition of $H^1(D_{2b+1}, \mathbb{Q})$ is the following direct sum of μ_{2b+1}-representations

$$\mathrm{H}^{1,0}(D_{4a}) = \mathrm{H}^1(D_{2b+1}, \mathcal{O}_{D_{2b+1}}) \simeq \bigoplus_{j=1}^b \xi^{b+j},$$
$$\mathrm{H}^{0,1}(D_{4a}) = \mathrm{H}^0(D_{2b+1}, \Omega^1_{D_{2b+1}}) \simeq \bigoplus_{j=1}^b \xi^{j}.$$

From the above, one can calculate the Hodge spectrum $\mathrm{hsp}_k := \chi_{\mathrm{hsp}}([\mathrm{BPS}_{k[\mathrm{C}]}])$.

Corollary 6.7. *The nontrivial Hodge spectrum realisations are*

$$\mathrm{hsp}_1(z_1, z_2) = \begin{cases} 1 + \sum_{j=1}^a \left(z_1^{\frac{2j-1}{4a}} z_2^{-\frac{2j-1}{4a}} + z_1^{-\frac{2j-1}{4a}} z_2^{\frac{2j-1}{4a}} \right) & a \leq b \\ 2 + \sum_{j=1}^b \left(z_1^{\frac{j}{2b+1}} z_2^{-\frac{j}{2b+1}} + z_1^{-\frac{j}{2b+1}} z_2^{\frac{j}{2b+1}} \right) & a > b \end{cases}$$

and

$$\mathrm{hsp}_2(z_1, z_2) = \sum_{j=1}^a z_1^{\frac{j}{a}} z_2^{\frac{a-j}{a}} - 1$$

By inspection, the weight-polynomial $\mathrm{wt}_k(q) = \mathrm{hsp}_k(q^{\frac{1}{2}}, q^{\frac{1}{2}})$ is constant in each case, and coincides with the numerical BPS invariants

$$\mathrm{wt}_1(q) = \mathrm{hsp}_1(q^{\frac{1}{2}}, q^{\frac{1}{2}}) = \min\{2a+1, 2b+2\} = \chi([\mathrm{BPS}_{[\mathrm{C}]}]),$$
$$\mathrm{wt}_2(q) = a - 1 = \chi([\mathrm{BPS}_{2[\mathrm{C}]}]).$$

The numerical BPS invariants determine the *Gopakumar-Vafa* numbers of the flop: for each $i = 1, \ldots, \ell$ this curve counting invariant is given by (see [Tod15])

$$n_i = \chi([\mathrm{BPS}_{[i\mathrm{C}]}]).$$

By the above calculation the Gopakumar-Vafa numbers of Y are therefore

$$(n_1, n_2) = \begin{cases} (2a+1, a-1) & a \leq b \\ (2b+2, a-1) & a > b \end{cases}$$

As shown by Toda [Tod15], the GV invariants also determine the dimension of the contraction algebra and of its abelianisation, which we can now easily compute:

$$\dim_{\mathbb{C}} \Lambda_{\mathrm{con}} = n_1 + 4n_2 = \begin{cases} 6a - 3 & a \leq b \\ 4a + 2b - 2 & a > b \end{cases},$$
$$\dim_{\mathbb{C}} \Lambda_{\mathrm{con}}^{\mathrm{ab}} = n_1 = \min\{2a, 2b+1\} + 1.$$

These same dimensions were also found independently by Kawamata [Kaw20] via a direct calculation.

We proceed with the calculation of the invariants.

§ 6.1 | Motivic invariants for the class [2C]

By Proposition 6.1, the BPS invariants $\mathrm{BPS}_{k[2C]}$ are equal to the BPS invariants associated to the class $[S_0] = [\mathcal{O}_{2C}(-1)[1]]$. Fix the phase $\theta = \theta_{2C,-1} = \Theta([S_0])$, then the BPS invariants are obtained via the motivic powerseries

$$\mathrm{Sym}\left(\sum_{k\geq 0} \frac{\mathrm{BPS}_{k[2C]}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} t_0^k\right) = \Phi^\theta(t) := \sum_{k\geq 0} \int_{\mathcal{C}^\theta_{(k,0)}} \phi_{\mathrm{tr}(W_f)} \cdot t_0^k,$$

where $\mathcal{C}^\theta = \mathcal{M}^\theta_{Q,W} \cap \mathcal{N}$ denotes the closed substack of $\mathcal{M}^\theta$ which parametrises nilpotent semistable $\mathrm{Jac}(Q,W)$-modules of phase θ, as defined in chapter 3.

Fix a dimension vector $\delta = (k, 0)$. Then every representation $\rho \in \mathrm{Rep}_\delta(Q)$ is trivial on all paths in Q that factors through the vertex 1, and is therefore completely determined by its value $\rho(s) \in \mathrm{Mat}_{k\times k}$ on the loop s. Moreover, the module corresponding to such a representation is automatically semistable, so that

$$\mathcal{M}^\theta_\delta = \mathcal{M}_\delta \simeq \mathrm{Mat}_{k\times k} \,/\, \mathrm{GL}_k \simeq \mathcal{M}_{\mathcal{Q},k},$$

where $\mathcal{M}_{\mathcal{Q},k}$ is the moduli stack of the one-loop quiver $\mathcal{Q}$ with loop s. This isomorphism identifies $\phi_{\mathrm{tr}(W_f)}$ with the vanishing cycle $\phi_{\mathrm{tr}(\mathcal{W})}$ of the potential $\mathcal{W} = 2f_{\mathrm{even}}(s^{\frac{1}{2}})$, and maps $\mathcal{C}^\theta_\delta$ to the closed substack $\mathcal{C}_{\mathcal{Q},k} \subset \mathcal{M}_{\mathcal{Q},k}$ which parametrises nilpotent $\mathrm{Jac}(\mathcal{Q},\mathcal{W})$-modules. It follows that

$$\Phi^\theta(t) = \sum_{k\geq 0} \int_{\mathcal{C}_{\mathcal{Q},k}} \phi_{\mathrm{tr}(\mathcal{W})} \cdot t_0^k = \Phi_{\mathcal{Q},\mathcal{W}}(t_0).$$

The DT partition function $\Phi_{\mathcal{Q},\mathcal{W}}(t)$ of a one-loop quiver with potential was found by Davison-Meinhardt [DM15b], and the BPS invariants for the class [2C] follow directly from their computation.

Lemma 6.8. *Let $a \in \mathbb{N}$ be the maximal integer such that s^a divides $2f_{\mathrm{even}}(s^{\frac{1}{2}})$, then*

$$\Phi^\theta(t) = \mathrm{Sym}\left(\frac{\mathbb{L}^{-\frac{1}{2}}(1 - [\mu_a])}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot t_0\right).$$

In particular $\mathrm{BPS}_{[2C]} = \mathbb{L}^{-\frac{1}{2}}(1 - [\mu_a])$ *and* $\mathrm{BPS}_{k[2C]} = 0$ *for* $k > 1$.

Proof. By [DM15b, Theorem 6.4] the right hand side is exactly $\Phi_{\mathcal{Q},\mathcal{W}}(t_0)$. □

§ 6.2 | Motivic point count

Next we calculate the BPS invariants for the phase $\theta = \theta_{\mathrm{pt}} = \Theta([\mathrm{pt}])$. As in chapter 3, we let $\mathcal{M}^\theta$ denote the moduli stack of semistable modules of phase θ, let $\mathcal{C} = \mathcal{M}_{Q,W_f} \cap \mathcal{N}$ be the stack of nilpotent representations satisfying the relations of the Jacobi algebra $\mathrm{Jac}(Q,W_f)$, and write $\mathcal{C}^\theta = \mathcal{C} \cap \mathcal{M}^\theta$. The BPS invariants for the phase θ are determined by the partition function

$$\Phi^\theta(t) = \int_{[\mathcal{C}^\theta \to \mathcal{C}]} \phi_{\mathrm{tr}(W)}|_{\mathcal{C}} = \sum_{k \geq 0} \int_{\mathcal{C}^\theta_{(k,2k)}} \phi_{\mathrm{tr}(W)} \cdot t_0^k t_1^{2k},$$

associated to the class $[\mathcal{C}^\theta \to \mathcal{C}]$ in the Hall algebra on $\mathcal{C}$. By Lemma 4.14 the only *stable* modules of phase θ are the images $\Psi(\mathcal{O}_p)$ of point sheaves supported on $p \in \mathrm{C}$ under the derived equivalence $\Psi\colon \mathrm{D}^b_{\mathrm{C}}(Y) \to \mathrm{D}^b_{\mathrm{fd}}(\Lambda)$. All other semistable modules are extensions of stable modules, and therefore correspond to the images of finite length sheaves supported on C.

We will show how the partition function decomposes along the support of these finite length sheaves. Fix a point $p \in \mathrm{C}$, and define $\mathcal{C}^p$ to be the closed substack of $\mathcal{C}^\theta$ parametrising semistable modules M such that $\Psi^{-1}(M)$ is a skyscraper sheaf on p. Likewise, let $\mathcal{C}^\circ$ be the open substack of $\mathcal{C}^\theta$ parametrising finite length sheaves with support in the complement $\mathrm{C} \setminus \{p\}$. Consider the partition functions

$$\Phi^\circ(t) = \int_{[\mathcal{C}^\circ \to \mathcal{C}]} \phi_{\mathrm{tr}(W_f)}|_{\mathcal{C}}, \quad \Phi^p(t) = \int_{[\mathcal{C}^p \to \mathcal{C}]} \phi_{\mathrm{tr}(W_f)}|_{\mathcal{C}},$$

obtained by integrating the vanishing cycle over the two substacks. Then we have the following.

Lemma 6.9. *There is a decomposition* $\Phi^\theta(t) = \Phi^\circ(t) \cdot \Phi^p(t)$.

Proof. Because the integration map is a homomorphism, it suffices to show the identity

$$[\mathcal{C}^\theta \to \mathcal{C}] = [\mathcal{C}^\circ \to \mathcal{C}] \star [\mathcal{C}^p \to \mathcal{C}] \tag{6.6}$$

in the motivic Hall algebra. Consider the substack $\mathcal{Y} \subset \mathcal{E}\mathrm{xt}$, parametrising those short exact sequences

$$0 \to M^\circ \to M \to M^p \to 0,$$

with M° in $\mathcal{C}^\circ$ and M^p in $\mathcal{C}^p_{\delta_2}$. The right hand side of (6.6) is then the class $[\mathcal{Y} \to \mathcal{C}]$ of the map $\mathcal{Y} \hookrightarrow \mathcal{E}\mathrm{xt} \to \mathcal{C}$ which maps a short-exact sequence to its middle term. Because an extension of semistable modules of phase θ is again semistable of phase θ, this map factors as $\mathcal{Y} \to \mathcal{C}^\theta \hookrightarrow \mathcal{C}$, and we claim that this factorisation identifies the classes $[\mathcal{Y} \to \mathcal{C}]$ and $[\mathcal{C}^\theta \hookrightarrow \mathcal{C}]$ in $\mathrm{K}(\mathrm{St}/\mathcal{C})$. By [Bri12, Lemma 3.2] it is sufficient to check that functor $\mathcal{Y}(\mathbb{C}) \to \mathcal{C}^\theta(\mathbb{C})$ on $\mathbb{C}$-points is an equivalences of categories. The support property guarantees that this is true: a semistable module M of phase θ is the image $M = \Psi(\mathcal{F})$ of a finite length sheaf on C, hence it is the unique extension $M \simeq M^\circ \oplus M^p$ of the modules

$$M^\circ := \Psi(\mathcal{F}|_{\mathbb{A}^1}), \quad M^p = \Psi(\mathcal{F}|_p),$$

in $\mathcal{C}^\circ(\mathbb{C})$ and $\mathcal{C}^p(\mathbb{C})$. Hence any object of $\mathcal{C}(\mathbb{C})$ lifts uniquely to $\mathcal{Y}(\mathbb{C})$. □

The explicit construction for the substacks $\mathcal{C}^p$ and $\mathcal{C}^\circ$ is analogous to the construction in §2.1.2. The moduli space $\mathrm{Rep}^\theta_{(1,2)}(Q)$ of semistable representations is covered by two open charts $\mathrm{Rep}^\theta_{(1,2)}(Q) = U_y \cup U_x$, where

$$U_y = \{\rho \mid \det(\rho(c) \mid \rho(yc)) \neq 0\}, \quad U_x := \{\rho \mid \det(\rho(c) \mid \rho(xc)) \neq 0\}.$$

The curve C parametrises the nilpotent representations of $\mathrm{Rep}^\theta_{(1,2)}(Q)$, so we can stratify accordingly: we fix the point p to correspond to the unique nilpotent representation in the complement of U_y, which is the semistable nilpotent representation ρ_p such that $\rho_p(yc) = 0$. With this choice of p, the stack $\mathcal{C}^\circ$ parametrises semistable nilpotent $\mathrm{Jac}(Q, W)$-representations ρ such that the matrix $\det(\rho(c) \mid \rho(yc))$ is invertible. Hence, $\mathcal{C}^\circ$ is precisely the intersection of the nilpotent locus $\mathcal{N}$ with the critial locus of $\mathrm{tr}(W)$ restricted to the open substack

$$\mathcal{M}^\circ := \coprod_{k\geq 0} \Big\{\rho \in \mathrm{Rep}_{(k,2k)}(Q) \mid (\rho(c) \mid \rho(yc)) \text{ is invertible}\Big\}/\,\mathrm{GL}_{(k,2k)}\,.$$

To compute the partition function Φ°, it is convenient to rewrite $\mathcal{M}^\circ$ as the moduli stack of a different quiver. Consider the quiver $\mathcal{Q}$ with a unique vertex and 9 loops

$$\mathcal{Q}_1 = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3\}.$$

Let $\mathrm{loc}_y\colon \mathbb{C}Q_{\mathrm{cyc}} \to \mathbb{C}\mathcal{Q}_{\mathrm{cyc}}$ be the composition of the trace map $\mathrm{tr}_{\mathcal{Q}}\colon \mathrm{Mat}_{3\times 3}(\mathbb{C}\mathcal{Q}) \to \mathbb{C}\mathcal{Q}$ with the homomorphism $\mathbb{C}Q \to \mathrm{Mat}_{3\times 3}(\mathbb{C}\mathcal{Q})$ defined on generators as

$$\begin{gathered} s \mapsto \begin{pmatrix} \gamma_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad d \mapsto \begin{pmatrix} 0 & \beta_1 & \beta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ x \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_2 & \beta_3 - \gamma_1\gamma_3 \\ 0 & \gamma_1 & \alpha_3 - \gamma_2 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_1 + \gamma_3 \\ 0 & 1 & \alpha_2 \end{pmatrix}. \end{gathered} \tag{6.7}$$

Then $\mathcal{W} = \mathrm{loc}_y(W_f) \in \mathbb{C}\mathcal{Q}_{\mathrm{cyc}}$ is a potential on $\mathcal{Q}$, and we have the following.

Lemma 6.10. *There is an isomorphism* $\mathcal{M}^\circ \xrightarrow{\sim} \mathcal{M}_{\mathcal{Q}}$ *that pulls back* $\mathrm{tr}(W_f)$ *to* $\mathrm{tr}(\mathcal{W})$*.*

Proof. Fix $\delta = (k, 2k)$ and consider the tautological representation τ on $\mathrm{Rep}_\delta(Q)$: the $\mathbb{C}[\mathrm{Rep}_\delta(Q)]$-valued representation corresponding to the identity across the isomorphism

$$\mathrm{Rep}_\delta(Q)(\mathbb{C}[\mathrm{Rep}_\delta(Q)]) \simeq \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Rep}_\delta(Q), \mathrm{Rep}_\delta(Q)).$$

Let $A = \Big(\tau(c)\,\Big|\,\tau(yc)\Big)$ denote the $\mathbb{C}[\mathrm{Rep}_\delta(Q)]$-valued $2k \times 2k$-matrix obtained by adjoining the block matrices $\tau(c)$ and $\tau(yc)$. Then $\mathcal{M}^\circ$ is the stack quotient $U/\,\mathrm{GL}_\delta$ of the invariant subspace

$$U = \operatorname{Spec} \mathbb{C}[\operatorname{Rep}_\delta(Q)][(\det A)^{-1}] \subset \operatorname{Rep}_\delta(Q).$$

There is a closed subspace $V \subset U$ defined by the vanishing of the $(2k)^2$ entries in the matrix $A - \mathrm{Id}_{2k\times 2k}$. We claim that U is a GL_{2k}-torsor over V with respect to the action of subgroup $\mathrm{GL}_{2k} \simeq \{\mathrm{Id}\} \times \mathrm{GL}_{2k} \subset \mathrm{GL}_\delta$. To show this, consider the invertible $\mathbb{C}[U]$-valued matrix

$$g = \left(\begin{array}{c|c} \mathrm{Id}_{k\times k} & 0 \\ \hline 0 & A^{-1} \end{array}\right) \in \mathrm{GL}_\delta(\mathbb{C}[U]).$$

then the family $g \cdot \tau$ of representations satisfies $\big(g\cdot\tau(c) \mid g\cdot\tau(yc)\big) = \mathrm{Id}_{2k\times 2k}$ and hence defines a map $U \to V$. The GL_{2k}-action restricts to a free & transitive action on the fibres of this map, which shows that U is indeed a GL_{2k}-torsor over V. It follows that

$$\mathcal{M}^\circ \simeq U/\,\mathrm{GL}_\delta \simeq V/\,\mathrm{GL}_k\,.$$

Because V is affine, any k-dimensional representation of $\mathcal{Q}$ with values in $\mathbb{C}[V]$ determines a map $V \to \operatorname{Rep}_k(\mathcal{Q})$ via $\operatorname{Rep}_k(\mathcal{Q})(\mathbb{C}[V]) \simeq \operatorname{Hom}_{\mathrm{Sch}}(V, \operatorname{Rep}_k(\mathcal{Q}))$. We can construct one such $\mathbb{C}[V]$-valued representation as follows. The tautological representation τ restricted to the space V is of the form

$$\tau(s), \quad \tau(c) = \left(\begin{array}{c} \mathrm{Id}_{k\times k} \\ \hline 0 \end{array}\right), \quad \tau(d) = \left(\begin{array}{c|c} d_0 & d_1 \end{array}\right),$$

$$\tau(x) = \left(\begin{array}{c|c} x_{00} & x_{01} \\ \hline x_{10} & x_{11} \end{array}\right), \quad \tau(y) = \left(\begin{array}{c|c} 0 & y_{01} \\ \hline \mathrm{Id}_{k\times k} & y_{11} \end{array}\right),$$

where $\tau(s), d_0, d_1, x_{00}, x_{01}, x_{10}, x_{11}, y_{01}, y_{11}$ are $\mathbb{C}[V]$-valued $k \times k$ matrices. Hence there is a representation $\sigma \in \operatorname{Rep}_k(\mathcal{Q})(\mathbb{C}[V])$ which maps the loops in $\mathcal{Q}$ to

$$\sigma(\alpha_1) = y_{01} - \tau(s), \quad \sigma(\alpha_2) = y_{11}, \quad \sigma(\alpha_3) = x_{11} + x_{00},$$
$$\sigma(\beta_1) = d_0, \quad \sigma(\beta_2) = d_1, \quad \sigma(\beta_3) = x_{01} + x_{10}\tau(s),$$
$$\sigma(\gamma_1) = x_{10}, \quad \sigma(\gamma_2) = x_{00}, \quad \sigma(\gamma_3) = \tau(s).$$

One checks that this map is a GL_k-equivariant isomorphism, and therefore yields an isomorphism $\mathcal{M}^\circ \simeq V/\,\mathrm{GL}_k \simeq \mathcal{M}_{\mathcal{Q},k}$ of moduli stacks. Moreover, by comparing the above with (6.7) one sees that the isomorphism identifies the functions $\operatorname{tr}(W_f)$ and $\operatorname{tr}(\mathcal{W})$ on the two spaces. Repeating this process for all k and taking the disjoint union gives the required isomorphism. □

The contribution of the stratum $\mathcal{C}^\circ$ can now be calculated as follows.

Lemma 6.11. *The contribution of the stratum* $\mathcal{C}^\circ$ *is*

$$\Phi^\circ(t) = \mathrm{Sym}\left(\sum_{k\geq 1}\frac{\mathbb{L}^{-\frac{3}{2}}[\mathbb{A}^1]}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\cdot t_0^k t_1^{2k}\right)$$

Proof. The potential $\mathcal{W} = \mathrm{loc}_y(W_f) \in \mathbb{C}\mathcal{Q}_{\mathrm{cyc}}$ has the following form:

$$\begin{aligned}\mathcal{W} &= \mathrm{loc}_y(x^2y - f(y) + cdy^2 - sdc + 2f_{\mathrm{even}}(s^{\frac{1}{2}}))\\ &= \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + [\gamma_1,\gamma_2]\gamma_3 + \alpha_1([\gamma_1,\gamma_2] + \alpha_3\gamma_1) - \alpha_3\gamma_1\gamma_3\\ &\quad + \alpha_2((\alpha_3-\gamma_2)^2 + \gamma_1\beta_3 - \gamma_1^2\gamma_3) - \mathrm{tr}_{\mathcal{Q}}\left(f\left(\begin{smallmatrix}0 & \alpha_1+\gamma_3\\ 1 & \alpha_2\end{smallmatrix}\right)\right) + 2f_{\mathrm{even}}(\gamma_3^{\frac{1}{2}}).\end{aligned}$$

We will construct an automorphism $\psi\colon \mathbb{C}\mathcal{Q}\to\mathbb{C}\mathcal{Q}$ which maps $\mathcal{W}$ to the simplified form $\sum_{i=1,2,3}\alpha_i\beta_i + \mathcal{W}_{\min}$, for some minimal potential $\mathcal{W}_{\min} = \mathcal{W}_{\min}(\gamma_1,\gamma_2,\gamma_3)$. By inspection, the potential can be written as

$$\begin{aligned}\mathcal{W} = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_1\cdot u_1 + \alpha_2\cdot u_2 + \alpha_3\cdot u_3\\ + [\gamma_1,\gamma_2]\gamma_3 - \mathrm{tr}_{\mathcal{Q}}\left(f\left(\begin{smallmatrix}0 & \gamma_3\\ 1 & 0\end{smallmatrix}\right)\right) + 2f_{\mathrm{even}}(\gamma_3^{\frac{1}{2}}).\end{aligned}$$

for nc-polynomials u_i of order ≥ 2 such that u_i only depends on the γ-variables and on α_j, β_j for $j > i$. Consider the automorphisms $\psi_1,\psi_2,\psi_3\colon \mathbb{C}\mathcal{Q}\to\mathbb{C}\mathcal{Q}$ which map

$$\psi_i(\beta_i) = \beta_i - u_i,$$

and send the other generators to themselves. Then $\psi_i(u_j) = u_j$ for $j > i$, and one sees that the composition $\psi := \psi_3\circ\psi_2\circ\psi_1$ maps $\mathcal{W}$ to $\psi(\mathcal{W}) = \sum_{i=1,2,3}\alpha_i\beta_i + \mathcal{W}_{\min}$ where

$$\mathcal{W}_{\min} = [\gamma_1,\gamma_2]\gamma_3 - \mathrm{tr}_{\mathcal{Q}}\left(f\left(\begin{smallmatrix}0 & \gamma_3\\ 1 & 0\end{smallmatrix}\right)\right) + 2f_{\mathrm{even}}(\gamma_3^{\frac{1}{2}}).$$

is the minimal potential on the loops γ. By inspection the last two terms cancel:

$$\mathrm{tr}_{\mathcal{Q}}\left(f\left(\begin{smallmatrix}0 & \gamma_3\\ 1 & 0\end{smallmatrix}\right)\right) = \mathrm{tr}_{\mathcal{Q}}\begin{pmatrix}f_{\mathrm{even}}(\gamma_3^{\frac{1}{2}}) & 0\\ 0 & f_{\mathrm{even}}(\gamma_3^{\frac{1}{2}})\end{pmatrix} + \mathrm{tr}_{\mathcal{Q}}\begin{pmatrix}0 & f_{\mathrm{odd}}(\gamma_3^{\frac{1}{2}})\gamma_3^{\frac{1}{2}}\\ f_{\mathrm{odd}}(\gamma_3^{\frac{1}{2}})\gamma_3^{-\frac{1}{2}} & 0\end{pmatrix} = 2f_{\mathrm{even}}(\gamma_3^{\frac{1}{2}})$$

so that $\mathcal{W}_{\min}$ is simply given by the cubic term $[\gamma_1,\gamma_2]\gamma_3$.

Let $\mathcal{J}\subset\mathcal{M}_{\mathcal{Q}}$ be a finite type substack of $\mathcal{M}_{\mathcal{Q}}$, then by motivic Thom-Sebastiani

$$\int_{\mathcal{J}}\phi_{\mathrm{tr}(W_f)} = \int_{\mathcal{J}}\phi_{\mathrm{tr}(\psi(\mathcal{W}))} = \int_{\mathcal{J}\cap\{\alpha_i=\beta_i=0\}}\phi_{\mathrm{tr}([\gamma_1,\gamma_2]\gamma_3)}.$$

In particular, this applies to the images $\mathcal{J}_k$ of the strata $\mathcal{C}^\circ_{(k,2k)}\subset\mathcal{M}^\circ$ under the isomorphism $\mathcal{M}^\circ\simeq\mathcal{M}_{\mathcal{Q}}$ in Lemma 6.10. It then follows from the main result of [BBS13] that the partition function $\Phi^\circ(t)$ is a motivic count of the points on $\mathbb{A}^1\subset\mathbb{A}^3$, which is

$$\Phi^\circ(t) = \sum_{k\geq 0}\int_{\mathcal{J}_k\cap\{\alpha_i=\beta_i=0\}}\phi_{\mathrm{tr}([\gamma_1,\gamma_2]\gamma_3)}\cdot t_0^k t_1^{2k} = \mathrm{Sym}\left(\sum_{n\geq 1}\frac{\mathbb{L}^{-\frac{3}{2}}[\mathbb{A}^1]}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\cdot t_0^k t_1^{2k}\right).\qquad\square$$

For the second partition function we proceed in a similar fashion. As in Lemma 6.10, for each $\delta = (k, 2k)$ define the matrix $A = \big(\tau(c) \big| \tau(xc)\big)$ where τ still denotes the tautological representation. Then there is an open neighbourhood $\mathcal{U}$ of $\mathcal{C}^p_\delta$ in $\mathcal{M}^\theta_\delta$ of the form $\mathcal{U} \simeq U/\operatorname{GL}_\delta$, where

$$U = \operatorname{Spec} \mathbb{C}[\operatorname{Rep}_\delta(Q)][\det A^{-1}].$$

As before, U is a GL_{2k}-torsor over the closed subspace $V \subset U$ defined by the entries of the matrix $A - \operatorname{Id}_{2k\times 2k}$, so that $\mathcal{U} \simeq V/\operatorname{GL}_k$. The restriction of τ to V is of the form

$$\tau(s), \quad \tau(c) = \left(\begin{array}{c} \operatorname{Id}_{k\times k} \\ \hline 0 \end{array}\right), \quad \tau(d) = \left(\begin{array}{c|c} d_0 & d_1 \end{array}\right),$$
$$\tau(x) = \left(\begin{array}{c|c} 0 & x_{01} \\ \hline \operatorname{Id}_{k\times k} & x_{11} \end{array}\right), \quad \tau(y) = \left(\begin{array}{c|c} y_{00} & y_{01} \\ \hline y_{10} & y_{11} \end{array}\right),$$

for $\mathbb{C}[V]$-valued $k \times k$-matrices $\tau(s), d_0, d_1, x_{01}, x_{11}, y_{00}, y_{01}, y_{10}, y_{11}$. and there is a GL_k-equivariant isomorphism $V \to \operatorname{Rep}_k(\mathcal{Q})$ determined by the family of representations $\sigma \in \operatorname{Rep}_k(\mathcal{Q})(\mathbb{C}[V])$ which takes the following values on generators:

$$\begin{aligned} \sigma(\alpha_1) &= -d_0, & \sigma(\alpha_2) &= x_{01}, & \sigma(\alpha_3) &= x_{11}, \\ \sigma(\beta_1) &= \tau(s) - y_{00}^2 - y_{01}y_{10}, & \sigma(\beta_2) &= y_{00} + y_{11}, & \sigma(\beta_3) &= y_{01}, \\ \sigma(\gamma_1) &= y_{10}, & \sigma(\gamma_2) &= y_{00}, & \sigma(\gamma_3) &= d_1, \end{aligned}$$

Again, the isomorphism of stacks $\mathcal{V}_k \simeq \mathcal{M}_{\mathcal{Q},k}$ identifies $\operatorname{tr}(W_f)$ with the trace of the potential $\mathcal{W} = \operatorname{loc}_x(W_f)$ on $\mathcal{Q}$, where $\operatorname{loc}_x \colon \mathbb{C}Q_{\text{cyc}} \to \mathbb{C}\mathcal{Q}_{\text{cyc}}$ is the composition of the trace $\operatorname{tr}_{\mathcal{Q}} \colon \operatorname{Mat}_{3\times 3}(\mathbb{C}\mathcal{Q}) \to \mathbb{C}\mathcal{Q}$ with the homomorphism $\mathbb{C}Q \to \operatorname{Mat}_{3\times 3}(\mathbb{C}\mathcal{Q})$ which maps the generators of $\mathcal{Q}$ to

$$s \mapsto \begin{pmatrix} \beta_1 + \gamma_2^2 + \beta_3\gamma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_2 \\ 0 & 1 & \alpha_3 \end{pmatrix},$$
$$d \mapsto \begin{pmatrix} 0 & -\alpha_1 & \gamma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_2 & \beta_3 \\ 0 & \gamma_1 & \beta_2 - \gamma_2 \end{pmatrix}.$$

One can check that the isomorphism identifies $\mathcal{C}^p$ with the critical locus $\mathcal{C}_{\mathcal{Q},\mathcal{W}}$.

Lemma 6.12. *There is an equality*

$$\Phi^p(t) = \operatorname{Sym}\left(\sum_{n\geq 1} \frac{\mathbb{L}^{-\frac{3}{2}}[\mathrm{pt}]}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot t_0^n t_1^{2n}\right).$$

Proof. As in Lemma 6.11, we calculate the motivic contribution via the potential

$$\begin{aligned}\mathcal{W} &= \mathrm{loc}_x(x^2y - f(y) + cdy^2 - sdc + 2f_{\mathrm{even}}(s^{\frac{1}{2}}))\\ &= \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \gamma_3[\gamma_1,\gamma_2] + \alpha_2\alpha_3\gamma_1 + \alpha_3^2(\beta_2-\gamma_2) + \gamma_3\beta_2\gamma_1\\ &\quad + 2f_{\mathrm{even}}((\beta_1+\gamma_2^2+\beta_3\gamma_1)^{\frac{1}{2}}) - \mathrm{loc}_x(f(y)).\end{aligned}$$

on the completed quiver $\widehat{\mathbb{C}\mathcal{Q}}$. Below, we will construct a formal automorphism of $\widehat{\mathbb{C}\mathcal{Q}}$ that maps $\mathcal{W}$ to a potential of the form $\sum_i \alpha_i\beta_i + \gamma_3[\gamma_1,\gamma_2]$. Note that this is sufficient to yield the required identity: the isomorphism $\mathcal{U} \simeq \mathcal{M}_Q$ identifies $\mathcal{C}^p$ with the nilpotent locus $\mathcal{N} \subset \mathcal{M}_Q$, and the theorem of [BBS13] again yields

$$\begin{aligned}\Phi^p(t) &= \sum_{k=1}^{\infty}\int_{\mathcal{C}^p_{(k,2k)}} \phi_{\mathrm{tr}(W_f)} t_0^k t_1^{2k} = \sum_{k=1}^{\infty}\int_{\mathcal{N}_k} \phi_{\mathrm{tr}(\sum_i \alpha_i\beta_i+\gamma_3[\gamma_1,\gamma_2])} t_0^k t_1^{2k}\\ &= \mathrm{Sym}\left(\sum_{k\geq 1}\frac{\mathbb{L}^{-3/2}[\mathrm{pt}]}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\cdot t_0^k t_1^{2k}\right).\end{aligned}$$

To construct the automorphism, we first write $\mathcal{W}$ as

$$\begin{aligned}\mathcal{W} &= \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \gamma_3[\gamma_1,\gamma_2] + u\cdot\beta_1 + \alpha_2(\alpha_3\gamma_1) + \alpha_3(-\alpha_3\gamma_2)\\ &\quad + w\cdot\beta_3 + (\alpha_3^2+\gamma_1\gamma_2+v)\cdot\beta_2 + 2f_{\mathrm{even}}(\gamma_2) - \mathrm{tr}_{\mathcal{Q}}\left(\begin{smallmatrix} f(\gamma_2) & 0\\ \cdots & f(-\gamma_2)\end{smallmatrix}\right),\end{aligned} \tag{6.8}$$

for $u = u(\beta_1,\beta_3,\gamma_1,\gamma_2)$ of order ≥ 1, and $v = v(\beta_2,\beta_3,\gamma_1,\gamma_2)$, $w = w(\beta_3,\gamma_1,\gamma_2)$ of order ≥ 3. Note moreover, that that the last two terms in (6.8) cancel by a parity argument:

$$\mathrm{tr}_{\mathcal{Q}}\left(\begin{smallmatrix} f(\gamma_2) & 0\\ \cdots & f(-\gamma_2)\end{smallmatrix}\right) = f_{\mathrm{even}}(\gamma_2) + f_{\mathrm{even}}(-\gamma_2) + f_{\mathrm{odd}}(\gamma_2) + f_{\mathrm{odd}}(-\gamma_2) = 2f_{\mathrm{even}}(\gamma_2).$$

Let $\psi_1,\psi_2,\psi_3 \colon \mathbb{C}\mathcal{Q} \to \mathbb{C}\mathcal{Q}$ be the automorphisms which map

$$\psi_1(\alpha_1) = \alpha_1 - u, \quad \psi_2(\beta_2) = \beta_2 - \alpha_3\gamma_1, \quad \psi_3(\alpha_2) = \alpha_2 - \alpha_3^2 - \gamma_1\gamma_2 - \psi_2(v),$$

and act as the identity on the other generators. By construction, these map $\mathcal{W}$ to

$$\begin{aligned}\psi_3(\psi_2(\psi_1(\mathcal{W}))) &= \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \gamma_3[\gamma_1,\gamma_2]\\ &\quad + w\cdot\beta_3 + \alpha_3\cdot(-\alpha_3\gamma_2 - \gamma_1\alpha_3^2 - \gamma_1^2\gamma_2 - \gamma_1\psi_2(v)).\end{aligned}$$

Because the terms on the bottom line are of order ≥ 3, one can use a recursive algorithm analogous to [DWZ08, §3] to further reduce the cross-terms order by order: one defines automorphisms $\psi_n \colon \mathbb{C}\mathcal{Q} \to \mathbb{C}\mathcal{Q}$, trivial up to order $n-1$, a sequence of nc-polynomials $w^{(n)}(\beta_1,\beta_3,\gamma_1,\gamma_2)$, $v^{(n)} = v^{(n)}(\beta_1,\gamma_1,\gamma_2)$, and $\mathcal{W}^{(n)}_{\min}(\gamma_1,\gamma_2)$ of orders $\geq n-1$, $\geq n-1$, and $\geq n$ respectively, such that $\psi_n(\psi_{n-1}(\cdots\psi_1(\mathcal{W})))$ is of the form

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \mathcal{W}^{(3)}_{\min} + \ldots + \mathcal{W}^{(n)}_{\min} + w^{(n)}\beta_3 + \alpha_3 v^{(n)}.$$

The existence of the above data can be shown by induction on the base case

$$\mathcal{W}_{\min}^{(3)} = \gamma_3[\gamma_1, \gamma_2], \quad w^{(3)} = w, \quad v^{(3)} = -\alpha_3\gamma_2 - \gamma_2\alpha_3^2 - \gamma_1^2\gamma_2 - \gamma_1\psi(v).$$

Suppose the data ψ_n, $\mathcal{W}_{\min}^{(n)}$, $v^{(n)}$, $w^{(n)}$ as above are given for $n \leq N$, then we construct the automorphism ψ_{N+1} by letting

$$\psi_{N+1}(\alpha_2) = \alpha_2 + w^{(N)}, \quad \psi_{N+1}(\beta_3) = \beta_3 + v^{(N)},$$

and sending all other generators to themselves. By assumption, this satisfies

$$\begin{aligned}\psi_{N+1}(\cdots(\psi_1(\mathcal{W}))) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \mathcal{W}_{\min}^{(3)} + \ldots + \mathcal{W}_{\min}^{(N)} \\ + w^{(N)}v^{(N)} - \psi_{N+1}(w^{(N)})v^{(N)} - w^{(N)}\psi_{N+1}(v^{(N)}). \\ + (\psi_{N+1}(w^{(N)}) - w^{(N)})\beta_3 + \alpha_3(\psi_{N+1}(v^{(N)}) - v^{(N)}).\end{aligned}$$

The bottom two lines contain only terms of order $\geq (N-1)^2 \geq N+1$. Hence, these lines can be written (up to cyclic permutation) as $\mathcal{W}_{\min}^{(n)} + w^{(n)}\beta_3 + \alpha_2 v^{(n)}$ for nc-polynomials of the claimed form. By induction the required data then exists for all $n \geq 4$.

The limit $\psi = \lim_{n\to\infty} \psi_n \circ \cdots \circ \psi_1$ is a well-defined formal automorphism $\widehat{\mathbb{C}\mathcal{Q}} \to \widehat{\mathbb{C}\mathcal{Q}}$, and maps $\mathcal{W}$ to the nc-powerseries

$$\psi(\mathcal{W}) = \sum_{i=1}^{3} \alpha_i\beta_i + \mathcal{W}_{\min} = \sum_{i=1}^{3} \alpha_i\beta_i + \gamma_3[\gamma_1, \gamma_2] + \mathcal{W}'_{\min},$$

where $\mathcal{W}'_{\min}$ a nc-powerseries in the variables γ_1, γ_2. Because ψ is an automorphism on $\widehat{\mathbb{C}\mathcal{Q}}$, it induces an isomorphism $\mathrm{Jac}(\mathcal{Q}, \psi(\mathcal{W})) \simeq \mathrm{Jac}(\mathcal{Q}, \mathcal{W}) \simeq \mathbb{C}[[\mathcal{O}_{Y,p}]]$ onto the ring of formal functions at p. In particular, the Jacobi algebra is commutative, so that the cyclic derivatives of $\mathcal{W}_{\min}$ are contained in the (completed) commutator ideal. In particular, because $\mathcal{W}'_{\min}$ is only a function of γ_1, γ_2 we find:

$$\partial_{\gamma_1}\mathcal{W}'_{\min} \equiv 0 \mod ([\gamma_1, \gamma_2])_{\mathrm{top}}, \quad \partial_{\gamma_2}\mathcal{W}'_{\min} \equiv 0 \mod ([\gamma_1, \gamma_2])_{\mathrm{top}}, \quad \partial_{\gamma_3}\mathcal{W}'_{\min} = 0.$$

A moment of reflection shows[1] that $\mathcal{W}_{\min} = \gamma_3[\gamma_1, \gamma_2] + q \cdot [\gamma_1, \gamma_2]$ for some nc-polynomial q of order ≥ 2. One final automorphism $\gamma_3 \mapsto \gamma_3 - q$ then maps $\mathcal{W}_{\min}$ to $\gamma_3[\gamma_1, \gamma_2]$, as required. The result follows. □

Adding up the contributions of $\mathcal{C}^\circ$ and $\mathcal{C}^p$ now yields the desired DT invariants.

Proposition 6.13. *The BPS invariants are* $\mathrm{BPS}_{k[\mathrm{pt}]} = \mathbb{L}^{-\frac{3}{2}}[\mathbb{P}^1]$ *for all* $k \geq 1$.

Proof. By Lemma 6.9 the partition function decomposes as $\Phi^\theta(t) = \Phi^\circ(t) \cdot \Phi^p(t)$, so it follows from Lemma 6.11, Lemma 6.12 and the properties of the plethystic exponential

[1] One can for example, apply the Euler identity $\sum_i \gamma_i \partial_{\gamma_i} H = n \cdot H$ to the homogeneous parts.

that

$$\Phi^\theta(t) = \mathrm{Sym}\left(\sum_{k\geq 0} \frac{\mathbb{L}^{-\frac{3}{2}}\left([\mathbb{A}^1]+[\mathrm{pt}]\right)}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}} t_0^k t_1^{2k}\right). \qquad \square$$

Remark 6.14. In the framework of [BBS13] the BPS invariants are interpreted as a virtual count of points, and are *defined* as the restriction of the *virtual motive* of Y:

$$[Y]_{\mathrm{virt}}\,|_{\mathrm{C}} = \mathbb{L}^{-3/2}[\mathbb{P}^1].$$

The proposition shows that the invariants $\mathrm{BPS}_{k[\mathrm{pt}]}$, which we compute in the framework of [KS08], are in fact given by this virtual motive.

§ 6.3 | Invariants for the class [C]

Finally we calculate the BPS invariants $\mathrm{BPS}_{[\mathrm{C}]}$. By Proposition 6.1 these invariants are equal to $\mathrm{BPS}_{k[S_1]}$ (as $[S_1] = [\mathcal{O}_{\mathrm{C}}(-1)]$) and can therefore be calculated via:

$$\mathrm{Sym}\left(\sum_{k\geq 1} \frac{\mathrm{BPS}_{k[\mathrm{C}]}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\cdot t_1^k\right) = \Phi^\theta(t) = \sum_{k\geq 0}\int_{\mathfrak{C}^\theta_{(0,k)}} \phi_{\mathrm{tr}(W_f)},$$

where $\theta = \theta_{\mathrm{C},0} = \Theta([S_1])$. As in §6.1, we see that the stability condition is trivial on the space $\mathrm{Rep}_{(0,k)}(Q)$, and the moduli space $\mathcal{M}^\theta$ is isomorphic to the moduli space $\mathcal{M}_{\mathcal{Q}}$ of a quiver $\mathcal{Q}$ with a unique vertex and loops $\mathcal{Q}_1 = \{x, y\}$. The potential restricts to $\mathcal{W} = x^2y - f(y) \in \mathbb{C}\mathcal{Q}_{\mathrm{cyc}}$ and the BPS invariants are determined by the partition function of $(\mathcal{Q}, \mathcal{W})$ via

$$\Phi^\theta(t) = \sum_{k\geq 0}\int_{\mathfrak{C}_{\mathcal{Q},k}} \phi_{\mathrm{tr}(\mathcal{W})}\cdot t_1^k = \Phi_{\mathcal{Q},\mathcal{W}}(t).$$

To calculate this partition function explicitly, one would have to use the integration formula of Denef–Loeser, which requires one to find an embedded resolution of the zero locus $\{\mathrm{tr}(\mathcal{W}) = 0\}$ in $\mathrm{Rep}_k(\mathcal{Q})$. We are able to find such an embedded resolution for $k = 1$, but for $k > 2$ the dimension of $\mathrm{Rep}_k(\mathcal{Q})$ is at least 8 and finding a suitable embedded resolution is rather complicated.

Instead, we will determine the realisations $\mathrm{BPS}^{\mathrm{mmhs}}_{k[\mathrm{C}]} := \chi_{\mathrm{mmhs}}(\mathrm{BPS}_{k[\mathrm{C}]})$ in the Grothendieck ring of monodromic mixed Hodge structures. As shown in [DM20], this realisation coincides with the class $[\mathcal{BPS}_k]$ of a monodromic mixed Hodge structure

$$\mathcal{BPS}_k := \mathrm{H}_c\left(\mathcal{M}_{\mathcal{Q},k}, \left(\phi^{\mathrm{mmhs}}_{\mathrm{tr}(\mathcal{W})}\,\mathrm{IC}_{\mathcal{M}_{\mathcal{Q},k}}\right)^{\mathrm{nilp}}\right),$$

where $\mathcal{M}_k \to \mathcal{M}_{\mathcal{Q},k} := \mathrm{Rep}_k(\mathcal{Q})/\!/\mathrm{GL}_k(\mathbb{C})$ is the associated *coarse moduli scheme* of $\mathcal{M}_{\mathcal{Q},k}$, and the cohomology with compact support takes values in the image of the intersection complex $\mathrm{IC}_{\mathcal{M}_{\mathcal{Q},k}}$ under the vanishing cycle functor $\phi^{\mathrm{mmhs}}_{\mathrm{tr}(\mathcal{W})}$, restricted to the nilpotent locus. The following follows from the main result of [Dav19].

Lemma 6.15. *$\mathcal{BPS}_k$ vanishes for $k > 2$.*

Proof. By [Dav19, Theorem B] the monodromic mixed Hodge structures $\mathcal{BPS}_k$ are all concentrated in degree 0, and by [Dav19, Proposition 5.2] the dimension $\dim_{\mathbb{C}}(\mathcal{BPS}_k)$ of their degree 0 part is given by the Gopakumar-Vafa invariant n_k of the flop. Because Y is a length $\ell = 2$ flop, the GV invariant n_k vanishes if $k > \ell = 2$ and $\mathcal{BPS}_k$ is therefore trivial for $k > 2$. □

Corollary 6.16. $\mathrm{BPS}^{\mathrm{mmhs}}_{k[\mathrm{C}]} = 0$ *for* $k > 2$.

§ 6.3.1 | The realisation for $k = 2$

The coarse moduli space $\mathcal{M}_{\mathcal{Q},2}$ is a smooth scheme.

Lemma 6.17. $\mathcal{M}_{\mathcal{Q},2} \simeq \mathbb{A}^5$

Proof. As shown by Procesi [Pro84], the ring of GL_2-invariant functions on the space of representations $\mathrm{Rep}_2(\mathbb{C}\langle x_1, \dots, x_n\rangle)$ of the free algebra $\mathbb{C}\langle x_1, \dots, x_n\rangle$ is the ring of trace functions $\mathrm{tr}(p)\colon \rho \mapsto \mathrm{tr}(\rho(p))$ of noncommutative polynomials $p \in \mathbb{C}\langle x_1, \dots, x_n\rangle$, subject to the relations

$$\begin{aligned} \mathrm{tr}(p_1p_2p_3) + \mathrm{tr}(p_1p_3p_2) = {} & \mathrm{tr}(p_1p_2)\mathrm{tr}(p_3) + \mathrm{tr}(p_1p_3)\mathrm{tr}(p_2) \\ & + \mathrm{tr}(p_1)\mathrm{tr}(p_2p_3) - \mathrm{tr}(p_1)\mathrm{tr}(p_2)\mathrm{tr}(p_3). \end{aligned} \tag{6.9}$$

for any triple of noncommutative polynomials p_1, p_2, p_3. In the case of two generators the trace ring is the polynomial ring (see [LP87, Proposition II.3.1]), hence

$$\mathcal{M}_2(\mathcal{Q}) = \mathrm{Rep}_2(\mathcal{Q}) /\!\!/ \mathrm{GL}_2 \simeq \mathrm{Spec}\, \mathbb{C}[\mathrm{tr}(x), \mathrm{tr}(y), \mathrm{tr}(x^2), \mathrm{tr}(y^2), \mathrm{tr}(xy)].$$ □

Because $\mathcal{M}_{\mathcal{Q},2}$ is smooth, its intersection complex $\mathrm{IC}_{\mathcal{M}_2(\mathcal{Q})}$ is trivial, and we can calculate the BPS invariant of the function $\mathrm{tr}(\mathcal{W})$ on the coarse scheme.

Lemma 6.18. *Let $\{0\} \subset \mathcal{M}_{\mathcal{Q},2}$ denote the origin, then there is an equality*

$$\mathrm{BPS}^{\mathrm{mmhs}}_{2[\mathrm{C}]} = \chi_{\mathrm{mmhs}}\left(\int_{\{0\}} \phi_{\mathrm{tr}(\mathcal{W})}\right),$$

where on the right-hand side $\mathrm{tr}(\mathcal{W})$ is regarded as a function on $\mathcal{M}_{\mathcal{Q},2}$.

Proof. Because $\mathcal{M}_{\mathcal{Q},2} \simeq \mathbb{A}^5$ is smooth of dimension 5, its intersection complex is simply

$$\mathrm{IC}_{\mathcal{M}_{\mathcal{Q},2}} = \underline{\mathbb{Q}}[\dim \mathcal{M}_{\mathcal{Q},2}] = \underline{\mathbb{Q}}[5],$$

where $\underline{\mathbb{Q}}$ denotes the constant sheaf with value $\mathbb{Q}$ on $\mathcal{M}_{\mathcal{Q},2}$. It then follows from the monodromic version of [DL98, Theorem 4.2.1] (see [Dav19, §2.7]), that

$$\chi_{\mathrm{mmhs}}\left(\int_{\{0\}}\phi_{\mathrm{tr}(\mathcal{W})}\right)=\left[\mathrm{H}_c\left(\mathcal{M}_{\mathcal{Q},2},\left(\phi^{\mathrm{mmhs}}_{\mathrm{tr}(\mathcal{W})}\underline{\mathbb{Q}}[5]\right)^{\mathrm{nilp}}\right)\right].$$

and the right hand side of the equation is precisely $[\mathcal{BPS}_2]=\mathrm{BPS}^{\mathrm{mmhs}}_{2[\mathrm{C}]}$. □

Proposition 6.19. *Let a be the maximal integer such that y^{2a} divides $f_{\mathrm{even}}(y)$, then*

$$\mathrm{BPS}^{\mathrm{mmhs}}_{2[\mathrm{C}]}=\chi_{\mathrm{mmhs}}\left(\mathbb{L}^{-\frac{1}{2}}(1-[\mu_a])\right).$$

Proof. Substituting $p_1=p_2=y$ and $p_3=y^n$ into (6.9), there is a relation

$$2\cdot\mathrm{tr}(y^{n+2})=\mathrm{tr}(y^2)\mathrm{tr}(y^n)+2\cdot\mathrm{tr}(y^{n+1})\mathrm{tr}(y)-\mathrm{tr}(y)^2\mathrm{tr}(y^n),$$

in the coordinate ring of $\mathcal{M}_{\mathcal{Q},2}$ for every $n>0$. Therefore, there exists a polynomial $v(\mathrm{tr}(y),\mathrm{tr}(y^2))$ so that $\mathrm{tr}(f(y))$ is of the form

$$\mathrm{tr}(f(y))=\mathrm{tr}(y)\cdot v(\mathrm{tr}(y),\mathrm{tr}(y^2))+f_{\mathrm{even}}(\mathrm{tr}(y^2)).$$

Likewise, substituting $p_1=p_2=x$, $p_3=y$ into equation (6.9) gives an expression

$$2\mathrm{tr}(x^2y)=\mathrm{tr}(x^2)\mathrm{tr}(y)+2\mathrm{tr}(xy)\mathrm{tr}(x)-\mathrm{tr}(x)^2\mathrm{tr}(y),$$

which gives an expression for $\mathrm{tr}(x^2y)$ as a polynomial in the traces of degree 2 terms. Choosing the coordinates

$$a_1=\tfrac{1}{2}\mathrm{tr}(x^2)-\tfrac{1}{2}\mathrm{tr}(x)^2-v(\mathrm{tr}(y),\mathrm{tr}(y^2)),\quad b_1=\mathrm{tr}(y),$$
$$a_2=\mathrm{tr}(xy),\quad b_2=\mathrm{tr}(x),\quad z=\mathrm{tr}(y),$$

one finds that $\mathrm{tr}(\mathcal{W})$ can be written as the polynomial

$$\mathrm{tr}(x^2y-f(y))=a_1b_1+a_2b_2-f_{\mathrm{even}}(z^{\frac{1}{2}}).$$

By assumption $f_{\mathrm{even}}(\gamma^{\frac{1}{2}})=u(\gamma)\cdot\gamma^a$ for $u(\gamma)$ invertible on some neighbourhood of the nilpotent locus. Then it follows from the Thom-Sebastiani identity that

$$\int_{\{0\}}\phi_{\mathrm{tr}(\mathcal{W})}=\int_{\{0\}}\phi_{a_1b_1+a_1b_2+u(z)\cdot z^a}=\int_{\mathbb{A}^1_z}\phi_{z^a}=\mathbb{L}^{-\frac{1}{2}}(1-[\mu_a]).$$ □

§ 6.3.2 | The BPS invariant for $k=1$

The first BPS invariant $\mathrm{BPS}_{[\mathrm{C}]}$ is the linear term in the partition function $\Phi_{\mathcal{Q},\mathcal{W}}(T)$: the plethystic exponential has the first order expansion $\mathrm{Sym}(\sum_{k>1}a_k\cdot T^k)=1+a_1\cdot T+\dots$ so the partition function is of the form

$$\Phi_{\mathcal{Q},\mathcal{W}}(T)=1+\frac{\mathrm{BPS}_{[\mathrm{C}]}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\cdot T+(\dots\text{higher order terms in }T\dots).$$

Hence $\mathrm{BPS}_{[\mathrm{C}]}$ can be calculated as the motivic integral of $\phi_{\mathrm{tr}(\mathcal{W})}$ on the origin in $\mathrm{Rep}_1(\mathcal{Q}) \simeq \mathbb{A}^2$. The function $\mathrm{tr}(\mathcal{W}) = x^2y - f(y) \in \mathbb{C}[x,y]$ has an isolated singularity at the origin, so we can fix an open neighbourhood $U \hookrightarrow \mathrm{Rep}_1(\mathcal{Q})$ of the origin $\{0\} \subset \mathbb{A}^2 \simeq \mathrm{Rep}_1(\mathcal{Q})$ which does not contain any other singularities. Then

$$\mathrm{BPS}_{[\mathrm{C}]} = (\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}) \cdot \int_{\mathcal{C}_{\mathcal{Q},1}} \phi_{\mathrm{tr}(\mathcal{W})} = \int_{\{0\}} \phi_{\mathrm{tr}(\mathcal{W})} = \int_U \phi_{\mathrm{tr}(\mathcal{W})}|_U.$$

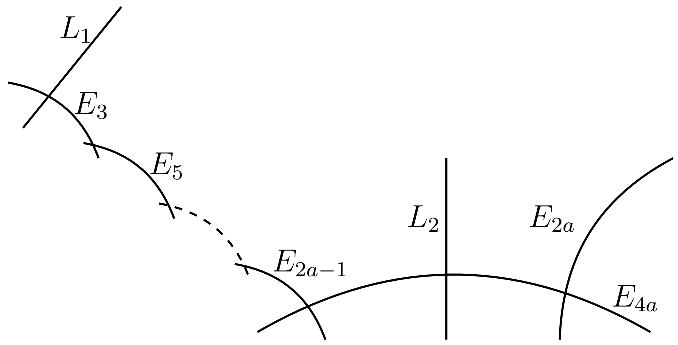
To calculate the right-hand side we construct an embedded resolution $h\colon X \to U$ of the divisor $Z := \{\mathrm{tr}(\mathcal{W}) = 0\}$ such that h^*Z has normal crossings: i.e. every prime component of h^*Z is a smooth codimension 1 subvariety of X and the intersection of any set of components is defined by a regular sequence.

As before let $a, b \in \mathbb{N}$ be the maximal integers such that y^{2a} divides $f_{\mathrm{even}}(y)$ and such that y^{2b} divides $f_{\mathrm{odd}}(y)$. Then the embedded resolution depends on a and b as follows.

Proposition 6.20. *If $a \leq b$ there exists an embedded resolution $h\colon X \to U$ such that*

$$h^*Z = L_1 + L_2 + \sum_{k=2}^{a} (2k-1) \cdot E_{2k-1} + 2a \cdot E_{2a} + 4a \cdot E_{4a},$$

where L_1 and L_2 are the components of the strict transform of Z and the E_i are exceptional curves. These components intersect each other as follows:

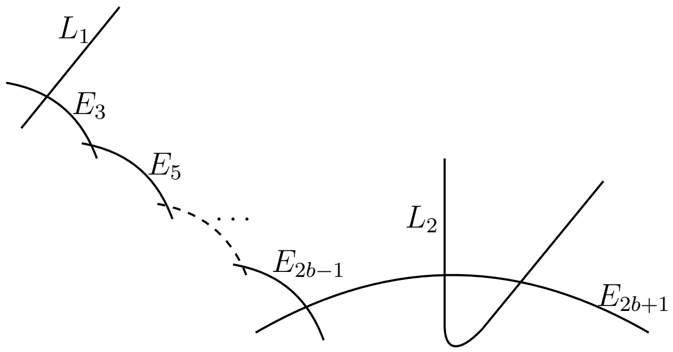


Proposition 6.21. *If $a > b$ there exists an embedded resolution $h\colon X \to U$ such that*

$$h^*Z = L_1 + L_2 + \sum_{k=2}^{b} (2k-1) \cdot E_{2k-1} + (2b+1) \cdot E_{2b+1},$$

where L_1 and L_2 are the components of the strict transform of Z and the E_i are excep-

tional curves. These components intersect each other as follows:



The resolutions can be found via a sequence of blowups of points, a straightforward but somewhat long computation which we include in appendix A.

To compute the motives we use the formula of Denef–Loeser recalled in §3.1.1. Write h^*Z as a sum $\sum_{i\in I} m_i E_i$ of prime divisors E_i with multiplicity $m_i > 0$ ranging over an index set I, and let E_J and E_J° be the strata for subsets $J \subset I$. Looijenga [Loo02] defines the following degree $m_I = \gcd\{m_j \mid j \in J\}$ cover $D_J \to E_J$ of E_J: let $g\colon \widetilde{X} \to \mathbb{A}^1$ be the normalisation of the base-change

$$\begin{array}{ccc} \mathbb{A}^1 \times_{\mathbb{A}^1} X & \longrightarrow & X \\ \downarrow & & \downarrow {\scriptstyle \mathrm{tr}(\mathcal{W})\circ h} \\ \mathbb{A}^1 & \xrightarrow{z \mapsto z^{m_I}} & \mathbb{A}^1 \end{array}$$

$\widetilde{X} \to X$ is a μ_I-fold cover of X, and $D_J \to E_J$ is the restriction to $E_J \subset X$. This cover has a canonical μ_{m_I}-action via its action on $\mathbb{A}^1$. We will also denote by $D_J^\circ \to E_J^\circ$ the restriction to the open subspace E_J°, which is a regular cover with Galois group μ_{m_J}. To ease notation, we write D_j, etc. instead of $D_{\{j\}}$, etc. if $J = \{j\}$ is a one-element set.

Proposition 6.22. *The BPS invariant is*

$$\mathrm{BPS}_{[C]} = \begin{cases} \mathbb{L}^{-1}(1 - [D_{4a}]) + 2 & a \le b \\ \mathbb{L}^{-1}(1 - [D_{2b+1}]) + 3 & a > b \end{cases}.$$

where D_{4a} has genus a with an μ_{4a}-action and D_{2b+1} is genus b with an μ_{2b+1} action.

Proof. Given a resolution as above, the Denef-Loeser formula for the motivic integral is

$$\mathbb{L}^{\dim U/2} \cdot \int_U \phi_{\mathrm{tr}(\mathcal{W})} = [Z] - \sum_{\varnothing \neq J \subset I} (1 - \mathbb{L})^{|J|-1} [D_J^\circ],$$

where D_J° carries the $\hat{\mu}$ action induced from the μ_{m_J}-action. For the case $a \le b$, the explicit expression can then be read off from the diagram in 6.20: write $E_1 = L_1$ and

$E_2 = L_2$ and let $I = \{1, 2, 3, 5, \ldots, 2a-1, 2a, 4a\}$ then the formula expands to

$$\begin{aligned}\mathbb{L} \cdot \int_U \phi_{\mathrm{tr}(\mathcal{W})} = [Z] &- [D_1^\circ] - [D_2^\circ] \\ &- (1-\mathbb{L})[D^\circ_{\{1,3\}}] - (1-\mathbb{L})[D^\circ_{\{2,4a\}}] \\ &- \sum_{i=2}^{a}[D^\circ_{2i-1}] - (1-\mathbb{L})\sum_{i=2}^{a-1}[D^\circ_{\{2i-1,2i+1\}}] \\ &- [D^\circ_{2a}] - [D^\circ_{4a}] - (1-\mathbb{L})[D^\circ_{\{4a,2a\}}] - (1-\mathbb{L})[D_{\{2a-1,4a\}}].\end{aligned}$$

We will reduce this expression line by line. The divisor L_1 appears with multiplicity $m_1 = 1$, so that $D_1 = L_1$ is a trivial cover and $D_1^\circ \subset L_1$ is the complement of the intersection point, which lies above the singularity of $\mathrm{tr}(\mathcal{W})$; similarly for L_2. Because $L_1 \sqcup L_2$ is the strict transform of Z, it is isomorphic to Z outside the singular locus, so that

$$[Z] - [D_1^\circ] - [D_2^\circ] = ([Z] - 1) - ([L_1] + [L_2] - 2) + 1 = 1.$$

Likewise, the intersection points of $L_1 \cap E_3$ and $L_2 \cap E_{4a}$ have a trivial cover, so that

$$-(1-\mathbb{L})[D^\circ_{\{1,3\}}] - (1-\mathbb{L})[D^\circ_{\{2,4a\}}] = 2\mathbb{L} - 2.$$

For $i = 2, \ldots, a-1$, the exceptional $E_{2i-1} \simeq \mathbb{P}^1$ has multiplicity $m_{2i-1} = 2i-1$ and intersects E_{2i+1} in a point with multiplicity $\gcd(2i-1, 2i+1) = 1$. It follows that each cover $D_{2i-1} \to E_{2i-1}$ is connected, and therefore restricts to a regular covering

$$D^\circ_{2i-1} \to E^\circ_{2i-1} \simeq \mathbb{G}_m,$$

for each $i = 2, \ldots, a$. The only connected cover is $D^\circ_{2i-1} \simeq \mathbb{G}_m$, which means that the map $D^\circ_{2i-1} \to E^\circ_{2i-1}$ is an equivariant isomorphism. Hence in $\mathrm{Mot}^{\hat{\mu}}(\mathbb{C})$ there is an equality

$$[D^\circ_{2i-1}] = [E^\circ_{2i-1}] = \mathbb{L} - 1.$$

It follows that these curves and their intersections contribute

$$-\sum_{i=2}^{a}[D^\circ_{2i-1}] - (1-\mathbb{L})\sum_{i=2}^{a-1}[D^\circ_{\{2i-1,2i+1\}}] = (a-1)(1-\mathbb{L}) - (a-2)(1-\mathbb{L}) = 1 - \mathbb{L}$$

Likewise, D_{2a-1} intersects D_{4a} in a point with multiplicity $\gcd(2a-1, 4a) = 1$ and contributes

$$-(1-\mathbb{L})[D_{\{2a,4a\}}] = \mathbb{L} - 1.$$

The curve E_{2a} only intersects E_{4a} in a single point, so that $E^\circ_{2a} \simeq \mathbb{A}^1$, which has only the trivial μ_{2a}-cover $D^\circ_{2a} = (\mathbb{A}^1)^{\sqcup 2a} \to \mathbb{A}^1$ for which μ_{2a} permutes the sheets. Hence there is an equivariant isomorphism $D^\circ_{2a} \simeq \mathbb{A}^1 \times \mu_{2a}$, and it follows that $[D_{2a}] = \mathbb{L}[\mu_{2a}]$. Likewise, the intersection $E_{2a} \cap E_{4a}$ is a point which is covered by $E_{\{2a,4a\}} = \mu_{2a}$ because the multiplicity is $\gcd(2a, 4a) = 2a$. Adding these two contributions gives:

$$-[D^\circ_{2a}] - (1-\mathbb{L})[D^\circ_{\{2a,4a\}}] = -\mathbb{L}[\mu_{2a}] - (1-\mathbb{L})[\mu_{2a}] = -[\mu_{2a}].$$

The curve E_{4a} intersects L_2 and E_{2a-1} in a point of multiplicity 1 and E_{2a} in a point of multiplicity $2a$, so $D_{4a} \to E_{4a}$ is a connected cover with Euler characteristic

$$\chi(D_{4a}) = 4a\chi(E^\circ_{4a}) + (2+2a) = 4a\chi(\mathbb{P}^1 - 3\mathrm{pt}) - (2+2a) = 2 - 2a.$$

Hence, D_{4a} is a smooth projective curve of genus a with equivariant motive

$$[D_{4a}] = [D^\circ_{4a}] + 2 + [\mu_{2a}].$$

Collection the terms found above, it follows that the motivic integral is

$$\begin{aligned}\int_U \phi_{\mathrm{tr}(\mathcal{W})} &= \mathbb{L}^{-1}\left(1 + 2\mathbb{L} - 2 + (1-\mathbb{L}) + (\mathbb{L}-1) - [\mu_{2a}] - [D_{4a}] + 2 + [\mu_{2a}]\right)\\ &= \mathbb{L}^{-1}(1 - [D_{4a}]) + 2.\end{aligned}$$

The case $a > b$ proceeds in much the same way, and yields the motivic integral

$$\begin{aligned}\mathbb{L}\int_U \phi_{\mathrm{tr}(\mathcal{W})} &= 1 + (b-1)(1-\mathbb{L}) + (b+2)(\mathbb{L}-1) - [D_{2b+1}] + 3\\ &= (1 - [D_{2b+1}]) + 3\mathbb{L},\end{aligned}$$

where D_{2b+1} is a genus b curve with an μ_{2b+1} action. □

To complete the calculation, we will make the Hodge structure and monodromy on the curves D_{4a} and D_{2b+1} explicit. We recall some generalities.

Suppose C_g is a smooth projective curve of genus g over $\mathbb{C}$ with $\rho\colon \mu_i \hookrightarrow \mathrm{Aut}(C_g)$ an action of μ_i. The components of its integral (co-)homology

$$\mathrm{H}^\bullet(C_g, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}^{2g}[1] \oplus \mathbb{Z}[2] \simeq \mathrm{H}_\bullet(C_g, \mathbb{Z}),$$

carry an induced action $\mathrm{H}^i(\rho, \mathbb{Z})$ of μ_i. Because the action preserves effective classes, it is trivial on $\mathrm{H}^0(C_g, \mathbb{Z})$ and $\mathrm{H}^2(C_g, \mathbb{Z})$, so we may concentrate on the middle cohomology. The middle cohomology of a smooth projective curve has a pure Hodge structure

$$\mathrm{H}^n(C_g, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{H}^n(C_g, \mathbb{C}) \simeq \textstyle\bigoplus_{p+q=n} \mathrm{H}^{p,q}(C_g),$$

with $\mathrm{H}^{p,q}(C_g) \simeq \overline{\mathrm{H}^{q,p}(C_g)}$. The summands $\mathrm{H}^{p,q}(C_g)$ are isomorphic to $\mathrm{H}^q(C_g, \Omega^p_{C_g})$ by the degeneration of the Hodge-to-de Rham spectral sequence, and the action of μ_i restricts to each summand in the Hodge decomposition

$$\mathrm{H}^1(C_g, \mathbb{C}) \simeq \mathrm{H}^1(C_g, \mathcal{O}_{C_g}) \oplus \mathrm{H}^0(C_g, \Omega^1_{C_g}),$$

Each summand is a g-dimensional complex representation of μ_i, which decomposes as a direct sum of irreducible representations labeled by their characters ξ^j. Moreover, $\mathrm{H}^1(C_g, \mathcal{O}_{C_g}) \simeq \overline{\mathrm{H}^0(C_g, \Omega^1_{C_g})}$ as dual representations.

Proof of Proposition 6.6. The curve D_{4a} is a ramified cover $q\colon D_{4a} \to \mathbb{P}^1$ of degree $4a$. By Birkhoff-Grothendieck, the push-forward $q_*\mathcal{O}_{D_{4a}}$ splits as a direct sum $\bigoplus_{i=0}^{4k} L_i$ of line bundles L_i on $\mathbb{P}^1$. It follows from [Ste77, Lemma 3.14] that this decomposition can be chosen to be invariant with respect to the monodromy action, with μ_{4a} acting with weight i on L_i. Furthermore, the degrees of these linebundles are determined by the multiplicities of the components that intersect E_{4a} in the diagram of proposition 6.20. The components E_{4a} intersects the components L_2, E_{2a}, E_{2a-1} of multiplicities $1, 2a, 2a-1$ each in a single point, so Steenbrink's formula yields

$$L_i := \mathcal{O}_{\mathbb{P}^1}\left(-i + \left\lfloor \frac{i}{4a} \right\rfloor + \left\lfloor \frac{2a \cdot i}{4a} \right\rfloor + \left\lfloor \frac{(2a-1)\cdot i}{4a} \right\rfloor\right),$$

where $\lfloor - \rfloor : \mathbb{Q} \to \mathbb{Z}$ is the floor function. Some pleasant modular arithmetic shows that

$$L_i \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) & i = 2j, \\ \mathcal{O}_{\mathbb{P}^1}(-1) & i = 2j-1, j \leq a, \\ \mathcal{O}_{\mathbb{P}^1}(-2) & i = 2j-1, j > a. \end{cases}$$

Because the morphism $q\colon D_{4a} \to \mathbb{P}^1$ is affine, $\mathrm{H}^1(D_{4a}, \mathcal{O}_{D_{4a}}) \simeq \mathrm{H}^1(\mathbb{P}^1, q_*\mathcal{O}_{D_{4a}})$ and the Hodge decomposition contains exactly a summand ξ^i for each i such that $L_i \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$:

$$\mathrm{H}^1(D_{4a}, \mathcal{O}_{D_{4a}}) = \xi^{2a+1} \oplus \xi^{2a+3} \oplus \cdots \oplus \xi^{4a-3} \oplus \xi^{4a-1}.$$

The second summand $\mathrm{H}^0(D_{4a}, \Omega^1_{D_{4a}})$ is obtained by duality. Likewise, the curve D_{2b+1} is a ramified cover $q\colon D_{2b+1} \to \mathbb{P}^1$ of degree $2b+1$ and the decomposition $q_*\mathcal{O}_{D_{2b+1}} = \bigoplus_{i=0}^{2b+1} L_i$ can be chosen invariantly, with μ_{2k+1} acting on L_i by weight i. Because the curve E_{2b+1} intersects E_{2b-1} of multiplicity $2b-1$ and has a double intersection with the curve L_2, which has multiplicity 1, these line bundles are

$$L_i := \mathcal{O}_{\mathbb{P}^1}\left(-i + 2\left\lfloor \frac{i}{2b+1} \right\rfloor + \left\lfloor \frac{(2b-1)\cdot i}{2b+1} \right\rfloor\right) \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) & i \leq b, \\ \mathcal{O}_{\mathbb{P}^1}(-2) & i > b. \end{cases}$$

Taking the first cohomology once more, one finds

$$\mathrm{H}^1(D_{2b+1}, \mathcal{O}_{D_{2b+1}}) = \xi^{b+1} \oplus \xi^{b+2} \oplus \cdots \oplus \xi^{2b-1} \oplus \xi^{2b},$$

with $\mathrm{H}^0(D_{2b+1}, \Omega^1_{D_{2b+1}})$ being the dual representation. □

Appendix A

Blowup calculations

Here we prove propositions 6.20 and 6.21 by constructing an embedded resolution over $U \subset \mathbb{A}^2$ of the divisor $Z \subset U$ defined by

$$Z := \{0 = \mathcal{W} = x^2 y - f(y)\}.$$

In what follows we decompose the parameter f as $f(y) = y^{k+1} \cdot u(y)$ for $k \geq 2$ such that the factor $u(y)$ is invertible on the neighbourhood U.

We construct an embedded resolution via a sequence of blowups. Consider the blowup $\pi\colon \operatorname{Bl}\mathbb{A}^2 \to \mathbb{A}^2$ of the origin, which is a gluing $\operatorname{Bl}\mathbb{A}^2 = \mathbb{A}^2 \cup \mathbb{A}^2$ of two affine charts, and write

$$\pi_x\colon \mathbb{A}^2 \to \mathbb{A}^2, \quad \pi_x(x,y) = (xy, y), \quad \pi_y\colon \mathbb{A}^2 \to \mathbb{A}^2, \quad \pi_y(x,y) = (x, xy),$$

for the restriction of π to these charts. Let $N = \left\lfloor \frac{k}{2} \right\rfloor$, then blowing up N times gives a resolution with $N+1$ charts, on which the resolution restricts to the maps

$$\pi_y, \quad \pi_x \circ \pi_y, \quad \pi_x \circ \pi_x \circ \pi_y, \quad \ldots, \quad \pi_x^{N-1} \circ \pi_y, \quad \pi_x^N.$$

The pullback of Z through the resolution is locally given by

$$(\pi_x^j \circ \pi_y)^* Z = \left\{ y^{2j+1} x^{2j+3} (1 - x^{k-2-2j} y^{k-2j} \cdot u(xy)) = 0 \right\}.$$

for $j < N$ on the first N charts and on the remaining chart by the equation

$$(\pi_x^N)^* Z = \left\{ y^{2N+1} (x^2 - y^{k-2N} \cdot u(y)) = 0 \right\}.$$

Then the pullback is normal-crossing on the former N charts.

Lemma A.1. *The divisor $(\pi_x^j \circ \pi_y)^* Z$ has normal-crossing singularities when restricted to the pre-image of $U \subset \mathbb{A}^2$.*

Proof. The pullback of Z is the sum of the following prime divisors with multiplicity

$$(\pi_x^j \circ \pi_y)^* Z = (2j+1)\cdot\{y=0\} + (2j+3)\cdot\{x=0\} + \{1 - x^{k-2-2j}y^{k-2j}\cdot u(xy) = 0\}.$$

Each of the prime divisors appearing in this sum is smooth on $(\pi_x^j \circ \pi_y)^{-1}(U)$, so it suffices to check that their intersections are generated by a regular system of parameters. The only intersection to consider is the intersection of the axes $\{y = 0\}$ and $\{x = 0\}$ in the origin. This is clearly normal-crossing because x, y is a regular system of parameters for the equation $xy = 0$. □

Lemma A.2. *If $k = 2N$ then $(\pi_x^N)^* Z$ is normal-crossing on $(\pi_x^N)^{-1}(U)$.*

Proof. For $k = 2N$, the pullback of Z is following sum of divisors with multiplicity

$$(\pi_x^N)^* Z = 2N \cdot \{y = 0\} + \{x^2 - u(y) = 0\}.$$

Note that $x^2 - u(y)$ is not necessarily irreducible, but nonetheless defines a smooth reduced curve in $(\pi_X^N)^{-1}(U)$. It therefore suffices to show that the intersections of this curve with the x-axis are generated by a regular system of parameters. Let c be one of the square roots of $u(0) \neq 0$, then the curve intersects the x-axis at the points $(c, 0)$ and $(-c, 0)$. The defining equation of the curve can be put into the form

$$x^2 - u(y) = x_- x_+ - (u(y) - c^2).$$

where $x_\pm := x \pm c$. Then x_+ is invertible at the point $(0, c)$ and

$$y, \quad x_- x_+ - (u(y) - c^2)$$

is a regular system of parameters for the equation $y(x_- x_+ - u(y) - c^2)$ in $\mathcal{O}_{(0,c)}$. It follows that $(\pi_x^N)^* Z$ is normal crossing at $(0, c)$, and similarly it is normal crossing at $(0, -c)$. □

The proof of proposition 6.21 now follows easily from the previous two lemmas.

Proof of proposition 6.21. The condition $a > b$ implies that y^{2b+1} is the lowest term in $f(y)$, so that the divisor Z is defined by the equation

$$y(x^2 - y^{2b} \cdot u(y)),$$

where $u(y)$ is invertible with a leading term that is odd. Hence, we set $N = b$, and define $h\colon X = \bigcup_{j=0}^N X_j \to U$ as the gluing of the $N + 1$ charts

$$X_0 = \pi_y^{-1}(U), \quad \ldots, \quad X_{N-1} = (\pi_x^{N-1} \circ \pi_y)^{-1}(U), \quad X_N = (\pi_X^N)(U),$$

as schemes over U via the maps $\pi_x^j \circ \pi_y$ and π_x^N. Then the previous two lemmas show that h^*Z is a normal-crossing divisor, and it remains to show that h^*Z is the sum of the prime divisors $L_1, \quad E_3, \quad \dots, \quad E_{2N+1}, \quad L_2$ with the stated multiplicities and intersections. On the chart X_0 the divisor h^*Z restricts to π_y^*Z, which is a sum of three prime divisors

$$L_1 = \{y = 0\}, \quad E_3|_{X_0} = \{x = 0\}, \quad L_2|_{X_0} = \{1 - x^{k-2}y^k u(xy) = 0\}$$

with multiplicities 1, 3 and 1 respectively. The lines L_1 and $E_3|_{X_0}$ meet in the origin and do no intersect $L_2|_{X_0}$. On the charts X_j for $j = 1, \dots, N-1$ the divisor h^*Z restricts to $(\pi_x^j \circ \pi_y^* Z)$, which is a sum of prime divisors

$$E_{2j+1}|_{X_j} = \{y = 0\}, \quad E_{2j+3}|_{X_j} = \{x = 0\}, \quad L_2|_{X_j} = \{1 - x^{k-2-2j}y^{k-2j}u(xy) = 0\}$$

with multiplicities $2j+1$, $2j+3$ and 1 respectively, with the former two intersecting in the origin. On the chart X_N the divisor h^*Z restricts to $(\pi_x^N)^*Z$, which is a sum of two prime divisors

$$E_{2N+1}|_{X_N} = \{y = 0\}, \quad L_2|_{X_N} = \{x^2 = u(y)\},$$

with multiplicities $2N+1$ and 1 respectively. By inspection, $E_3, \dots, E_{2N+1}$ form a chain of intersecting rational curves meeting eachother in a single point. Likewise L_1 meets E_3 in a single point, while L_2 meets E_{2N+1} in two points, which are the distinct solutions of $x^2 = u(0)$. □

For the defining equation in 6.20 the parameter $k = 2N+1 = 2a-1$ is odd, and $(\pi_x^N)^*Z$ is not normal crossing. One needs to blowup twice more.

Lemma A.3. *The following divisors are normal-crossing on the pre-images of U:*

$$\begin{aligned} (\pi_x^N \circ \pi_y)^*Z &= \{\ y^{2N+2}(x^2y - u(y)) = 0\ \} \\ (\pi_x^{N+1} \circ \pi_y)^*Z &= \{\ y^{2N+1}x^{4N+4}(1 - y \cdot u(xy)) = 0\ \} \\ (\pi_x^{N+2})^*Z &= \{\ y^{4N+4}x^{2N+2}(x - u(xy)) = 0\ \} \end{aligned}$$

Proof. In all three cases the axes $\{y = 0\}$ and $\{x = 0\}$ are smooth and intersect only in the origin. By assumption the polynomial u has a constant term, which implies the curves $x^2y = u(y)$, $1 = yu(xy)$, and $x = u(xy)$ are smooth. The $(\pi_x^N \circ \pi_y)^*Z$ is therefore normal-crossing, because the intersection

$$\{y = 0\} \cap \{x^2y - u(y) = 0\} = \varnothing.$$

The radical of the defining equation for the second divisor is $xy(1 - y \cdot u(xy))$. The curve $\{1 = y \cdot u(xy)\}$ does not intersect the axis $\{y = 0\}$ and intersects $\{x = 0\}$ in the point $p = (0, 1/u(0))$. The variable y is invertible in the local ring $\mathcal{O}_p$, so

$$x, \quad y(y - 1/u(0)),$$

is a regular system of parameters defining $xy(1 - y \cdot u(xy)$ in $\mathcal{O}_p$. It follows that the second divisor is normal-crossing. The radical of the third defining equation is $yx(x - u(xy))$. The curve $\{x = u(xy)\}$ does not intersect the axis $\{x = 0\}$ and intersects $\{y = 0\}$ in the point $p = (u(0), 0)$. The intersection is again normal crossing, as $yx(x - u(xy))$ has the regular system of parameters

$$y, \quad x(x - u(xy)),$$

because $u(0) \neq 0$ implies x is invertible in $\mathcal{O}_p$. □

The proof of proposition 6.20 now follows analogously to the proof of proposition 6.21.

Proof of proposition 6.20. The divisor Z is defined by the equation

$$y(x^2 - y^{2a-1} \cdot u(y)),$$

for $u(y)$ invertible on U. Set $N = a - 1$ and define $h\colon X = \bigcup_{j=0}^{N+2} X_j \to U$ as the gluing of the $N + 3$ charts

$$X_0 = \pi_y^{-1}(U), \quad \ldots, \quad X_{N+1} = (\pi_x^{N-1} \circ \pi_y)^{-1}(U), \quad X_{N+2} = (\pi_X^{N+2})(U),$$

as schemes over U via the maps $\pi_x^j \circ \pi_y$ and π_x^{N+2}. As in the proof of 6.21 we obtain a curve L_1 of multiplicity 1 in X_0 and a chain of exceptional $\mathbb{P}^1$'s $E_3, \ldots, E_{2N+1}$ of multiplicities $3, \ldots, 2N + 1$ glued from the lines in the charts $X_0, \ldots, X_N$. The remaining terms are E_{4N+4}, which is glued from

$$E_{4N+4}|_{X_{N+1}} = \{\ x^{4N+4} = 0\ \}, \quad E_{4N+4}|_{X_{N+2}} = \{\ y^{4N+4} = 0\ \},$$

and has multiplicity $4N + 4 = 4a$, the divisor E_{2N+2}, which is glued from

$$E_{2N+2}|_{X_N} = \{\ y^{4N+2} = 0\ \}, \quad E_{2N+2}|_{X_{N+2}} = \{\ x^{2N+2} = 0\ \},$$

and has multiplicity $2N + 2 = 2a$, and the curve L_2 which is given by the equation $x = u(xy)$ on the chart X_{N+2}. By inspection, E_{4N+4} meets L_2 and E_{2N+2} in separate points on the chart X_{N+2} and meets E_{2N+1} on the chart X_{N+1}. The components L_2 and E_{2N+2} do not intersect any other divisor. □

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