On Pricing of Futures Contracts and Derivatives in The WTI Crude Oil Market

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Abstract

Ever since a stochastic process for valuing futures contracts was first introduced by Black in 1976, a large number of people have been drawn to this developing domain of quantitative finance. To be more specific, at the end of the last century, Schwartz built a factor model system, step by step, with different co-workers. Following that, crude oil has been experiencing an unprecedented boom since the beginning of this century. This commodity and its related financial products play an unprecedentedly important role in the financial markets and in our day-to-day life. In this thesis, the standards for WTI futures contracts and their options will be introduced after the introduction. Then, the original Schwartz (1997) model system, including the One-Factor model, the Two-Factor model and the Three-Factor model, is discussed in the following section. The thesis focuses next on augmenting the original Two-Factor model. For example, the Two-Factor model will be run based on different estimation methods and will be combined with an options pricing model. Lastly, the stochastic process of the volatility of the spot price of WTI crude oil will be inserted into the original Two-Factor model and the Three-Factor model, which means that new Three-Factor and Four-Factor models will be proposed in this thesis.

Contents

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Author's Declaration

I declare that, except where explicit reference is made to the contribution of others, that this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Printed Name: Zhe Zong

1 Introduction

To begin this thesis, it is essential to briefly consider the development of the commodity and derivative markets throughout the past several decades. There has been a dramatic change since the year 1973 in which the first options exchange opened in Chicago, and Black and Scholes published their famous option pricing model (Fan, 2008). Since then, the boom in financial derivative instruments, such as futures and options, has totally changed the way in which commodities are priced and valued (Geman, 2003).

As for the energy market, undoubtably, crude oil is the most indispensable physical commodity because it provides energy for every aspect of human's activity today. To be more precise, the primary distillates of crude oil (heating oil, aviation fuel, gasoline and fuel oil) are important for industrial production and daily life. Traditionally, oil was traded by long-term contracts but then the oil crisis broke out and the business changed (Alizadeh, Lin and Nomikos, 2008).

Around the world, the main crude oil exchanges include the New York Mercantile Exchange (NYMEX), the International Petroleum Exchange (IPE), the Dubai Mercantile Exchange (DME), the Singapore Exchange (SGX) and the Tokyo Commodity Exchange (TOCOM). In this thesis, the empirical analysis is based on WTI crude oil which is traded on the New York Mercantile Exchange, hence the other exchanges are only roughly introduced here, but the New York Mercantile Exchange is more clearly introduced in this thesis. The International Petroleum Exchange was renamed as ICE Futures in 2005 and is one of the most important centers for trading oil derivatives. As for crude oil, the IPE provides one of two benchmark prices based on the Brent crude around the world. In addition to this, the Brent crude is the base price that is used to trade crude oil in Europe. As for the Dubai Mercantile Exchange, it is the most important energy exchange in the Middle East, and its Oman Crude oil futures contract is the third largest crude oil price benchmark. Lastly, the Singapore Exchange and the Tokyo Commodity Exchange provide the price benchmarks for the Asian crude oil trade. They are beginning to play a more important role in the world's crude oil trade than ever before.

The New York Mercantile Exchange is owned and operated by the CME group (Chicago Mercantile Exchange) and is located in Manhattan (New York City). NYMEX handles billions of dollars' worth of energy products, metals and other commodities being traded on the trading floor and the overnight electronic trading computer systems. In the domain of energy products, the New York Mercantile Exchange (NYMEX) introduced heating oil and crude oil in 1978 and 1983 respectively.

After the introduction of WTI futures [1], it became a popular and successful tool to manage the risk on the price of the crude oil step by step. Exchange-price quotes began to take precedence over the previous price system in the middle of the 1980s with the overproduction of crude oil by OPEC (Organization of Petroleum Exporting Countries). At that time, the price of crude oil dropped from about 32 dollars per barrel to about 9.75 dollars per barrel in just five months due to this overproduction. Then, a large number of investors started to realize that crude oil had morphed into a highly risky investment, and they needed a tool–WTI crude oil futures contracts–to be provided in the market to hedge against the potential risk. Approximately five years later, Iraq invaded Kuwait in August of 1990. Due to the war, one quarter of world's oil supply was threatened with being cut off. Soaring prices led buyers to seek a tool to sustain an appropriate level of risk management at a time of extreme uncertainty. In a situation of such uncertainty, the markets flocked to futures contracts and a steady increase in trading volume has reinforced the function of WTI crude oil futures as a global benchmark for the price of crude oil. With the continuation of these huge uncertainties of the market, WTI futures must play an even more important role in financial markets in the future.

In addition, since the 1980s, oil has been traded as the biggest commodity in the world commodity markets. Simultaneously, trading oil has no longer only been seen to be a primary physical activity but also a more complicated, worldwide financial activity (Geman, 2003). Meanwhile, the prices of crude oil and its related commodities have

[1] Information about WTI Futures Contracts and Options on Futures Contracts traded in the New York Mercantile Exchange will be introduced in detail in the appendix

experienced an unprecedented boom. The oil price was relatively stable within a narrow range from 10 to 20 dollars (in 1997 dollars) per barrel between 1874 and 1974 (R$\ddot{u}$hl, 2008), but this stability ended during the oil crisis in 1973. The price of oil experienced a tremendous fluctuation from 12 dollars per barrel in 1973 to 95 dollars per barrel in January 1981, bottoming out at 21 dollars per barrel in July 1986. After a period of stability, the price of oil fluctuated strongly again. To be specific, after sliding to a low of 12 dollars per barrel in December 1998, the price of oil increased from 12 dollars to 145 dollars per barrel, which is the peak for the whole history of the oil business, in July 2008, and immediately dropped to under 40 dollars per barrel again before the end of 2008 (Smith, 2009).

In this thesis, three test periods are involved and the whole test period is from 2010 to 2014. To be more specific, they are: 2^{nd} Nov. 2010 to 31^{st} Oct. 2011; the entire year of 2012 and 1^{st} Mar. 2013 to 28^{st} Feb. 2014. During the period, the price of WTI crude oil fluctuates between 75 dollars per barrel and 115 dollars per barrel. Overall, the fluctuation of WTI crude oil is considerable during the test periods. To be more specific, the highest price of crude oil is about 115 dollars per barrel and the lowest trading price is about 75 dollars per barrel, and the both appear in the first testing period (2^{nd} Nov. 2010 to 31^{st} Oct. 2011). This might imply that the crude oil price is more volatile, since the crude oil had just created the historic record of the highest price in 2008. Even if the top and valley price are based on the futures price, they can represent the crude oil spot, because they are highly and causally correlated (The existence of linear and nonlinear causal relationship between the daily WTI spot and futures prices for short maturity has been proved (Bekiros and Diks, 2008)). This relationship can be seen in the following sections of this thesis.

On the other hand, the volatility of energy prices is actually greater than that of other financial assets, which means that the prices of energy commodities are more volatile than other commodities (Read, Goldberg and Fox-Peter, 2010). As for crude oil, the dynamics of crude oil prices have recently been characterized by highly volatile and an upward drift, which could imply that the markets for crude oil have been out of equilibrium (Askari and Krichene, 2004). To be more specific, the price of crude oil has experienced a tremendous boom since 1998. At the end of 1998, the price of crude oil bottomed out at 12 dollars per barrel, but after 10 years, the price peaked at about 145 dollars per barrel in 2008, and then the price of crude oil kept fluctuating around a relatively high level. The boom in crude oil has had, and continues to have, different impacts on both the macro-economy and micro-economy; viewed from the perspective of macro-economy, shocks in oil prices are changing many of the mechanisms (e.g. supply and demand routes, interest rate routes and so forth) (Jones, Leiby and Paik, 2003); on the other hand, viewed from the perspective of the micro-economy, the boom in crude oil obviously provides an enormous opportunity for investment in oil futures, whether it be short or long crude oil or its related assets. Hence, an increasing number of investors and scientists have paid more and more attention to properly valuing the futures contracts of crude oil and oil-related commodities.

In fact, as has been explained, crude oil and oil-related commodities have not only played increasingly crucial roles in our day-to-day lives, but also in the financial markets. Even though an increasing number of scientists have paid more and more attention to them, there are still lots of problems in the derivative markets. For example, one of the most important issues demanding to be solved is that there is still no practical and unified point of view for pricing a futures contract (not only for crude oil, but also for precious metals, agricultural products and so forth). In most financial institutions, investors still price a futures contract based on a simple relationship between the prices of futures contracts and the interest rate because there are too many factors which must be considered when people try to price a futures contract. Moreover, some of the most momentous factors cannot even be observed in the markets.

As previously stated, a well-known difficulty in the pricing of a futures contract is that there are a large number of factors that need to be considered, each of them impacting considerably on the value of a futures contract (e.g. the spot price of the underlying asset, the instantaneous convenience yield of the underlying commodity, the instantaneous interest rate of the cash market, and so forth). Every one of them can influence investors' valuations of futures contracts, but none of them can be directly observed in the markets by either investors or producers. Traditionally, what we can directly observe are only the traded prices of the futures contracts, which means that there is little observable information available for use in pricing them. In this context, the spot price of the underlying asset needs to be carefully reflected upon since it is not always clear what constitutes a spot price or what it represents. In the case of crude oil, investors cannot actually trade spot contracts of WTI at the CME, which leads many authors to make the assumption that the spot price is unobservable. However, elsewhere in the world a single portion of crude oil (even if it is a small part) is actually traded in spot oil markets, which means that somehow the spot price of WTI is observable. Even if the spot price for some kinds of commodities can somehow be observed directly in the markets (in the case of WTI, the US Energy Information Administration collects and publishes the spot price of WTI) and the instantaneous interest rate can be assumed to be replaced by the observed interest rate, the convenience yield of the underlying asset is absolutely unobservable in the markets. In addition, even though the prices of contracts can be easily obtained, they are traded with different maturities, which implies that we have to get rid of as much of the noise in the data as possible. Schwartz constructed a series of commodity futures pricing factor models with the Kalman filter in 1997 which were used as tools for getting rid of the noise, and then he and his co-worker proposed a rough idea for estimating parameters using the basic least squares method (Cortazar and Schwartz, 2002). Later, Cortazar and Naranjo (2006) presented an N-factor Gaussian model in which the Kalman filter was used as the estimation method. Recently, the least squares method and this dynamic system for pricing oil futures contracts have been combined in order to add the fourth factor - the exchange rate - to the Schwartz factor model system (Yan and Li, 2008).

This thesis aims to comprehensively discuss the main models that are currently being used to price futures contracts; to review classic models and advanced research in the fields of mathematical finance and quantitative finance; and to develop the current models used to price the futures contracts and options on futures contracts in the domain of WTI crude oil.

As for the motivation of this thesis, the main contributions can be summarized as follows: first, the usefulness of the Two-Factor model has been tested when the observed spot price of crude oil is used. To be more specific, the real observed spot price of WTI crude oil will be used in the Two-Factor model to replace the estimated spot price. Second, the parameters in the Two-Factor model will be considered as a part of state variables in order to estimate the stochastic parameters. Third, in this thesis the original Two-Factor and the original Three-Factor Schwartz (1997) models are developed into a new Three-Factor and a new Four-Factor model with consideration of a stochastic volatility of the spot price of WTI crude oil. The new models are considered to be very useful, because the stochastic volatility of the underlying asset has become a very important factor when people try to analyze the features of the spot price. Last, in this thesis the Schwartz Two-Factor is combined with the Hilliard and Reis (1998) model so that the European calls on WTI crude oil can be directly priced based on the observed prices by using the extended Kalman filter. This is a very useful development of the Schwartz (1997) model system, and the results are surprisingly good.

The models introduced in this thesis are suitable for all kinds of people who are working with the underlying commodity of a futures contract (in the case of this thesis, it is WTI crude oil, but the models can be easily used in pricing the futures contracts of other commodities), e.g. producers, consumers and speculators. To be more specific, producers can increase their output and sell more when the price observed in markets is higher than the estimated price based on the models; consumer can decrease their demand, when the price observed in markets is higher than the estimated price based on the models; last, as for the speculators, they can design multiple strategies based on the models introduced in this thesis. For example, first, the speculators can consider long WTI futures contracts if their estimated prices are higher than the observed trading prices. Second, the risk-seeking speculators can even long calls on WTI futures contracts if the estimated prices of futures contracts are higher than their observed trading prices. Third, the speculators can build a portfolio of WTI futures contracts with different maturities when the price of the underlying assets suddenly moves upward or downward, since the temporary trend of the backwardation and the contango is more easily found when WTI crude oil rapidly price-inverses, while all futures prices seem to be close, when the price of WTI crude oil has a clear trend of increase or decrease.

At the end of this section, there are several points which need to be clearly explained before introducing the models. First, in this thesis, each test period is short. To be more specific, only about 252 testing days (one year) were used to test the model in every part. Using short-term data in this field is rare to see, but the data is set like this with a special intention: there are two kinds of data that people can get with Bloomberg. One is the rollover data which means that the term to maturity is set to be certain. For example, when people want to search the price of a commodity with one month to maturity, the system shows the price from the historic contracts. This is the rollover price. On the other hand, people can search a particular contract with a certain maturity. The advantage of the second way is that people can trace a particular contract and price the chosen contract with a particular maturity. In addition, using this method, the term to maturity vector will be expanded as a term to maturity matrix. With the setting of the dynamics of the term to maturity, better fitting results will be expected. However, the disadvantage of this way is also obvious: since every contract has a limited trading period and a contract will expire after its maturity, the data sample is limited. In most studies, people used the rollover price with the Schwartz (1997) model system, and no one tried to use the other way to test the results in Schwartz (1997) model system. In this thesis, several chosen contracts will show the Schwartz (1997) model system is also useful in pricing a certain contract.

Second, based on the calculated standard errors from the original One-Factor and the original Two-Factor model, a small part of the standard errors of the estimated parameters are not very low. This might be caused by the short-term type of data [2], and it also might be caused by the data itself during that period (In section 6, the original Two-Factor model will be tested with new data to make a comparison with the new Three-Factor model. The standard errors of the estimated parameters are much better with the new data). On the other hand, the worse standard errors of some parameters

[2] See Appendix 8.6

can be seen as another motivation to make changes in the original Two-Factor model. To be more specific, instead of a single number of the estimated parameters, the original Two-Factor model with the stochastic parameters provides a way to improve the model with stochastic parameters (in section 3). On the one hand, if the standard error is seen as the only criteria, the estimated parameters are more reliable in the new Three-Factor model. On the other hand, in order to avoid using the standard error as the only criteria to evaluate the effect of the estimated parameters, the fitting results and the forward curves are also shown in each model. Moreover, using the forward curves to analyze the results is a popular way in this domain, because people can see both observed and model estimated prices of futures contracts with different maturities and test the effect of each model in a single figure.

Third, the Kalman filter and the extended Kalman filter are the major estimating methods used in this thesis. As one of the most popular estimating methods for the state space model, its drawback has been well known: the estimated parameters depend highly on the initial values of the unknown parameters. In addition, with more components added into the Schwartz (1997) model system, the number of unknown parameters increases quickly. The increasing number of unknown parameters and the uncertainty of their initial values increase the instability of the model system. To be more specific, practically, people need to try more sets of initial values of the unknown parameters and each attempt is more time consuming with more factors added into the model system, because sometimes a small change in an unknown parameter could hugely influence the estimated results.

The structure of this thesis can be summarized as follows:

•In section 2, the original Schwartz (1997) model system, including the original One-Factor model, Two-Factor model and Three-Factor model, are clearly introduced. In addition, each model is run with the historical data, and the results of each original model are shown and explained in each sub-section.

•In section 3, the original Two-Factor model, which is the most popular model in the Schwartz (1997) model system, is run with variations. Firstly, the observed spot price is inserted into the original Two-Factor model to see how the results are influenced. Secondly, the Two-Factor model is run in a different way, in order to observe the implied stochastic parameters. Similarly, each method is run with the historical data, and the results of each model are shown and explained in each sub-section.

•In section 4, the original Two-Factor and Three-Factor models in the Schwartz (1997) model system are developed into newly minted Three-Factor and Four-Factor models respectively. This development allows the volatility of the spot price of WTI crude oil to be tracked so that the models contain more information for the user. Similarly, each model is run with the historical data, and the results of each model are shown and explained in each sub-section.

•in section 5, the Two-Factor model is combined with an option valuation model, so that it can price not only futures contracts, but also the call options on futures contracts. Similarly, the model is run with the historical data, and the results of each model are shown and clearly explained in this section.

•In section 6, the traditional Schwartz factor model system, the transformations of the traditional Schwartz factor model system and its derivative models are compared. To be more specific, there are several subsections: Comparison of the Two-Factor Model With Different Estimation Algorithms; Comparison of the Two-Factor Model with Observed versus Unobserved Spot Price of the Underlying Asset; Comparison of The Original Three-Factor Model with The Stochastic σ_1 and The New Four-Factor Model; Comparison of The New Three-Factor Model and The New Four-Factor Model; Compare the Results in The Original Two-Factor Model and The New Three-Factor Model; Comparison of The Two-Factor Model with Stochastic Parameters and The Original Two-Factor Model and Comparison of The Two-Factor Model Based on European calls on Futures and The Original Two-Factor Model.

•In section 7, the thesis conclusions are presented based on the results from each model, and their comparisons.

•In section 8, the estimation methods that are used in the entire thesis are roughly introduced, including the Kalman filter algorithm, the extended Kalman filter algorithm and the Trust Region algorithm; the mathematical Black–Scholes Formulas and the history and the conditions of the futures contracts for WTI crude oil and their options, which are traded over the New York Mercantile Exchange, including the rationale for designing such a product, the trading volume, the unit of the contract, are introduced in detail.

2 Original Schwartz One-, Two- and Three-Factor model

The futures contracts pricing model was constructed in the 1990s based on stochastic processes and Itô's Lemma. To the author's knowledge, the Schwartz's model system is definitely the most famous model system for pricing a futures contract in financial markets. In this section, the One-Factor, Two-Factor and Three-Factor models and their empirical results will be introduced in detail. Before introducing Schwartz's Factor models, the stochastic process and Itô's Lemma will be roughly introduced at the beginning.

2.1 Stochastic Process and Itô's Lemma

2.1.1 Stochastic Process

Under the efficient market hypothesis, there are two basic presumptions. Firstly, the present price, which is not concerned with further information, is completely reflected in the past data. Secondly, any random arrival of new information will immediately affect the market.

An increment stochastic process consists of two parts. The first part is μdt, which can be predictable and can reflect the risk-free interest rate in the market. To be more precise, μ is an average level of growth of price, which is referred to as the drift. The second part is σdX, which means the random fluctuation in the price and whose mean is 0. More specifically, σ is the price volatility, and dX, which is referred to as a Wiener process, is a random variable from a normal distribution with mean 0 and variance dt. Then the stochastic process is constructed as:

$$dS/S = \sigma dX + \mu dt$$

(Wilmott, Howison and Dewynne, 1995)

2.1.2 Itô's Lemma

The Itô's Lemma is based on a result (see Hull, 1995), which is $dX^2 \to dt$ as $dt \to 0$. Suppose $f(S)$ is a smooth function, where S is the spot price and S is a stochastic process. From the Taylor series expansion, there is:

$$df = \frac{df}{dS}dS + \frac{1}{2}\frac{d^2f}{dS^2}dS^2 + \cdots$$

Recalling $dS/S = \sigma dX + \mu dt$, then $dS^2 = (\sigma SdX + \mu Sdt)^2$, which can go further as

$$dS^2 = \sigma^2 S^2 dX^2 + 2\sigma\mu S^2 dtdX + \mu^2 S^2 dt^2$$

is easily to obtain. Because of $dt \to 0$, the dS^2 is rewritten as

$$dS^2 = \sigma^2 S^2 dX^2 + \cdots$$

then, because of $dX^2 \to dt$, thus:

$$dS^2 \to \sigma^2 S^2 dt.$$

Up to now, $dS/S = \sigma dX + \mu dt$ can be rewritten as:

$$df = \frac{df}{dS}(\sigma SdX + \mu Sdt) + \frac{1}{2}\sigma^2 S^2 \frac{d^2f}{dS^2}dt$$

then

$$df = \sigma S\frac{df}{dS}dX + (\mu S\frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2f}{dS^2})dt$$

Consider a situation in which $f(\cdot)$ is not only concerned about S, but time t. Then, based on the Taylor series expansion, $f(S + dS, t + dt)$ can be expanded as:

$$df = \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + \cdots$$

Similar to the above process, the final equation is:

$$df = \sigma S\frac{\partial f}{\partial S}dX + (\mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t})dt$$

(Wilmott, Howison and Dewynne, 1995)

2.2 The Schwartz One-Factor Model and Its Empirical Results

2.2.1 The Schwartz One-Factor Model

At the beginning of the model system, only the spot price was considered as the state variable, which is unobservable:

$$dS = \kappa(\mu - lnS)Sdt + \sigma Sdz$$

where S is the unobservable spot price. Use Itô's Lemma and let $X = lnS$ which is characterized by the following stochastic process:

$$dX = \kappa(\alpha - X)dt + \sigma dz$$

with $\alpha = \mu - \frac{\sigma^2}{2\kappa}$. Here, κ is set as the measurement of the degree of mean reversion, which is greater than zero, α is the logarithmic price of the long run mean, σ is the volatility of the spot price of the commodity, and dz is an increment of a standard Brownian motion.

According to Ross (1995), assume there is no-arbitrage and insert a new variable λ with $\lambda = \alpha - \alpha^*$, where λ is defined as the market price of risk. Then, the above stochastic process is changed to:

$$dX = \kappa(\alpha^* - X)dt + \sigma dz^*$$

where $dz^*$is an increment to a Brownian motion under the equivalent martingale measure. Under the equivalent martingale measure, the conditional distribution X for the futures contract with the maturity T is normal. Then, the mean and the variance are

$$E_0[X(T)] = e^{-\kappa T}X(0) + (1 - e^{-\kappa T})\alpha^*$$

and

$$Var_0[X(T)] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa T})$$

respectively.

Solving the following PDE (partial differential equation) with the boundary condition $F(S, 0) = S$, means the futures price should be the spot price, when the time to maturity T=0:

$$\frac{1}{2}S^2\sigma^2 F_{SS} + \kappa(\mu - \lambda - lnS)SF_{SS} - F_T = 0$$

we can then easily obtain the solution:

$$F(S,T) = exp[e^{-T\kappa}lnS + ((1 - e^{-T\kappa})\alpha^*) + \frac{\sigma^2}{4\kappa}(1 - e^{-2T\kappa})]$$

2.2.2 Data and Assumptions

In the One-Factor model, in order to simplify the complex state space model, the weekends and other non-trading days have been ignored, which means that the trading days are considered to be continuous. Based on the efficient market hypothesis, powerful information from the non-trading days (e.g. weekends and Christmas holiday) can be immediately reflected in the price after the non-trading day. Hence, this is a reasonable and popular assumption in the financial world. The next assumption is that each futures contract is immediately executed on the first day when they are mature. This is an assumption of length of the maturity. Based on this assumption, the length of the maturity can be measured more accurately. In addition to those, in this thesis, instead of a single average T, the dynamics of the T with the time passing is added to the model, which is expected to get a better fit with the data.

As has been explained in the above sections, when the spot price of crude oil is set as an unobservable state variable, the observable futures prices are the only collected data when the model is implemented. In this thesis, the data of futures prices of WTI crude oil is collected from Bloomberg. Specifically, in this thesis, five futures contracts will be used in the One-Factor model. Their maturities are one month, two months, four months, seven months and ten months respectively, and the test period is the year from 2^{nd} Nov. 2010 to 31^{st} Oct. 2011 (252 trading days). To be more specific, the five futures contracts would mature in December 2011, January 2012, March 2012, June 2012 and September 2012 respectively. [3]

[3] Since in this thesis, the futures contracts are assumed to be immediately executed on the first day after they mature, in the section in which the same data is used and in section six, the term to maturity is set to

2.2.3 The Empirical Results of The Schwartz One-Factor Model

Figure 1 exhibits the estimated unobservable state variable, which is the spot price of WTI crude oil in the One-Factor model, using the Kalman filter. In Figure 1, it is not hard to observe that, in the test period, the estimated spot price of crude oil dropped to under 90 dollars per barrel from over 90 dollars at the beginning of the test period, and then, it rebounded and kept moving up to about 115 dollars per barrel, which is the peak of the entire test period. After the peak, the estimated spot price of crude oil experienced a long and continued decrease until it is about 75 dollars per barrel, which is the valley. At the end of the test period, the estimated spot price rebounded again. The downward jump in the second half of the test period can be explained by the fact that the International Energy Agent announced the release of 60 million barrels of their strategic petroleum reserve at the end of June 2011. As for the estimated parameters, overall the estimated parameters are not comparable with other models, since there is only one factor considered. To be more specific, based on the results in the One-Factor model, the degree of mean reversion is low for the spot price of WTI crude oil and the risk of investing in WTI crude oil is quite low.

Next, the five chosen futures contracts are shown together in Figures 2 to 3 in order to see whether there are significant differences between the observed prices of WTI crude oil futures contracts and the estimated prices of WTI crude oil futures contracts in the One-Factor model. To be specific, the real observed futures prices of the five chosen futures contracts are shown in Figure 2. Then the estimated model futures prices of the five chosen futures contracts, which are estimated from the One-Factor model, are shown in Figure 3. In order to see the trend clearly, in Figure 3 there are only 26 points chosen in the figure, which were picked the first trading day of each ten trading days. On the one hand, based on Figure 2, WTI crude oil did not follow a strict backwardation or a strict contango during the test period; on the other hand, it is not hard to see that the trend

reflect the trading length, while in the other parts the term to maturity reflects the month of the maturity. The different sets of the T aim to reflect that there is no significant error caused by the above assumption. Actually, the author of this thesis has tested the different sets of the term to maturity based on the same data and the results are almost the same, which means that there is no significant error caused by the above assumption. Due to the limitations of the scale of this thesis, the comparison of the different sets of the term to maturity based on the same data will not be shown in this thesis.

of the backwardation and the contango are easier to find when WTI crude oil fluctuates stably, while all futures prices seem to be close when the price of WTI crude oil rapidly increases or rapidly decreases. However, viewed from the perspective of Figure 3, the estimated prices of futures illustrate that WTI crude oil futures follow a strict contango, which means that a longer term to maturity makes a futures contract more valuable.

In order to observe the effectiveness of the One-Factor model, the comparisons with observed prices of each WTI crude oil futures in the One-Factor model are shown in Figures 7 to 11. The rank of the title of each picture corresponds to the rank of its maturity. For example, the title 'The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The One-Factor Model ' and the title 'The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The One-Factor Model ' correspond to the futures contracts which matured in December 2011 and January 2012 respectively, and so forth. It is not hard to see that the One-Factor model is useful in pricing a futures contract with a particular maturity, because all five pictures are showing that the estimated model price of each futures contract is really close to the real observed price of the futures contract.

Last, the forward curves from the One-Factor model are shown in Figures 4 and 6. To be more specific, first, on the 50^{th} trading day, the model estimated prices of the One-Factor model were slightly higher than the observed futures prices. Second, on the 100^{th} trading day, the model estimated prices of the One-Factor model crosses the observed prices: with an increase in the term to maturity, the estimated prices were firstly lower, then slightly higher, than the observed prices. Last, on the 200^{th} trading day, the model estimated prices of the One-Factor model were also crossed with the observed prices, but compared with Figure 5, the estimated prices were firstly higher, then slightly lower, than the observed prices. In other words, as expected, the estimated model prices from the One-Factor model are close to the observed prices.

Paremeters	Estimated result	Standard Error
κ	0.0566	0.0408
α	4.8544	0.1273
λ	0.4598	0.2454
σ	0.0029	0.0760

Table 2.2.3: Estimated Parameters from The One-Factor Model

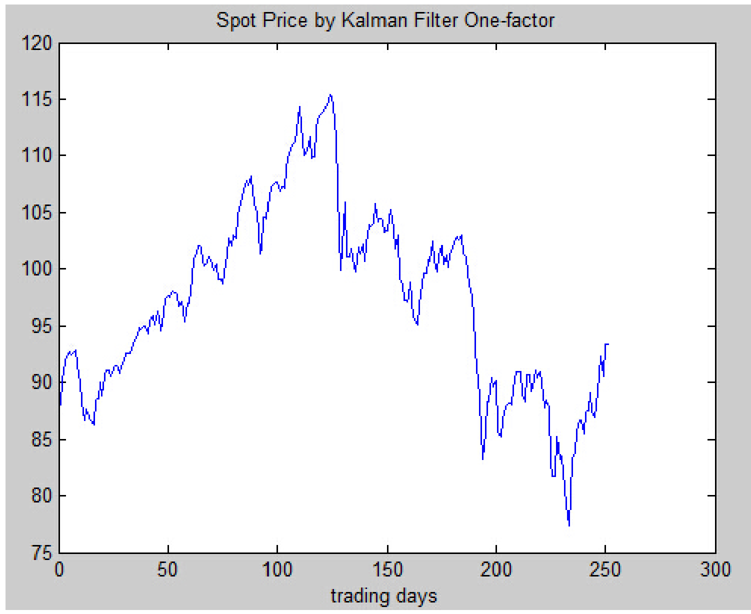


Figure 1: Estimated Spot Price from The One-Factor Model by The Kalman Filter

2.3 The Schwartz Two-Factor Model and Its Empirical Results

2.3.1 The Schwartz Two-Factor Model

With the development of research on the futures, an important notion – the convenience yield – was added to the model system. In order to introduce the convenience yield, the relationship between the future price of a commodity and its spot price is described as a basic equation $F^T(t) = S(t)e^{(r-\delta)(T-t)}$,where δ is defined as the convenience yield (here, it is a constant), which represents the yield if an investor holds the commodity instead of the futures contract of the commodity. As shown in the basic pricing for-

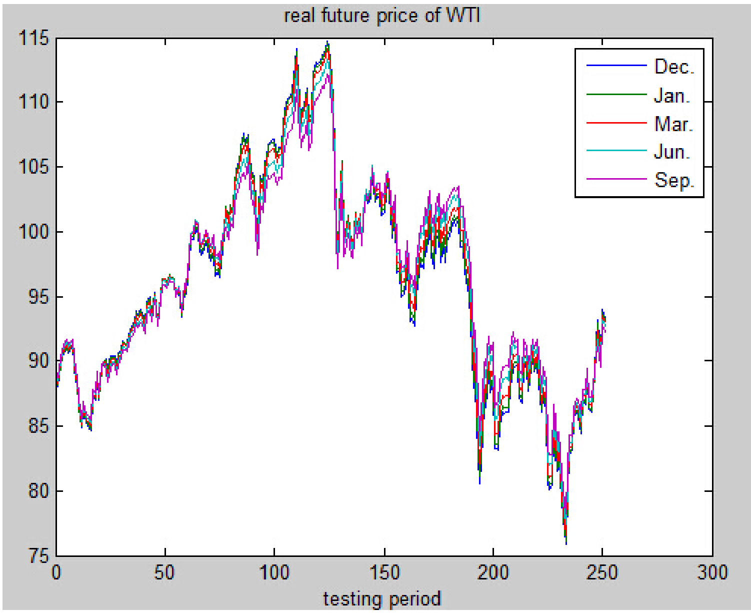


Figure 2: Observed Prices of WTI Futures Contracts with Different Maturities for The One-Factor Model

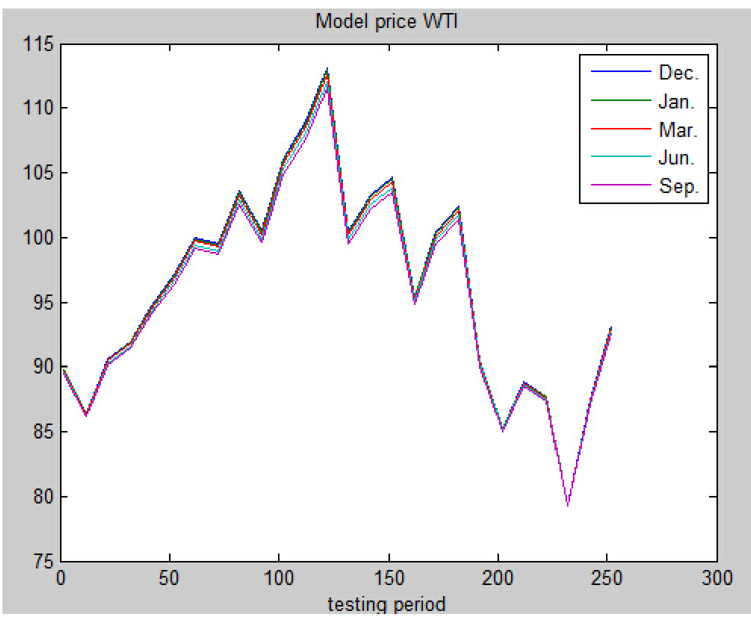


Figure 3: Estimated Prices of WTI Futures Contracts with Different Maturities for The One-Factor Model

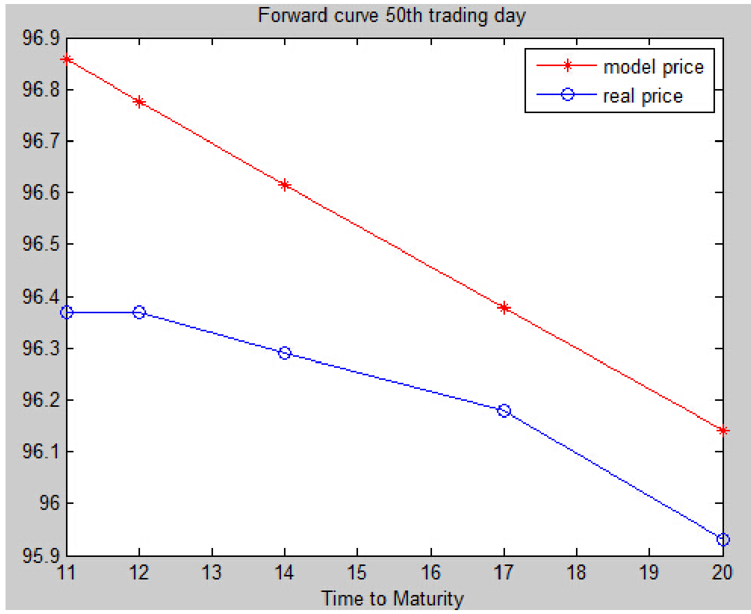


Figure 4: Forward Curves from The One-Factor Model on The 50th Testing day

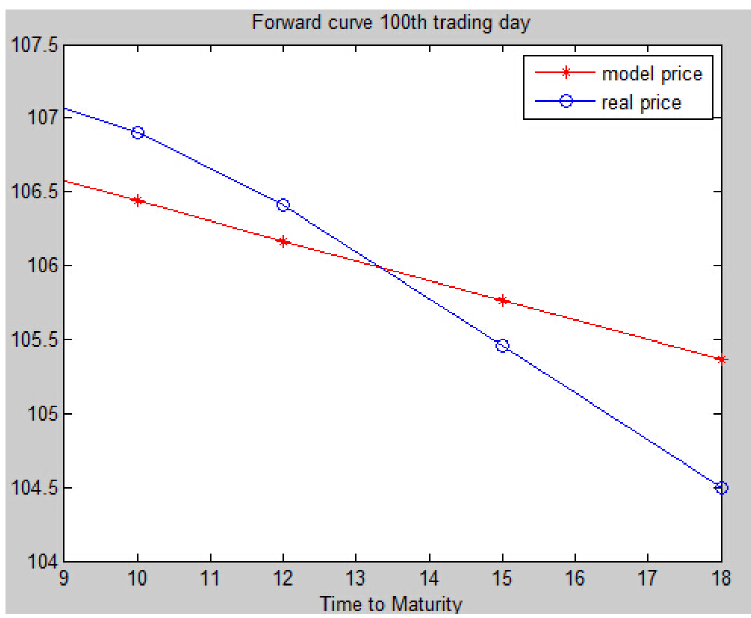


Figure 5: Forward Curves from The One-Factor Model on The 100th Testing day

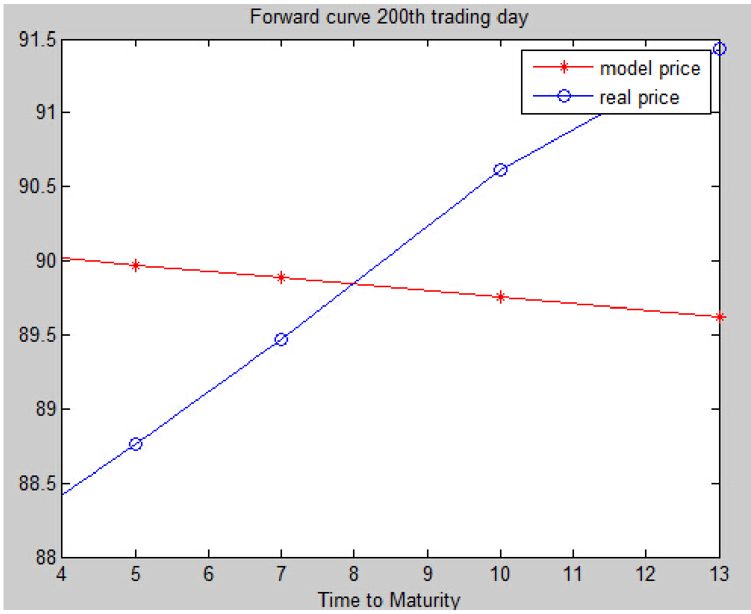


Figure 6: Forward Curves from The One-Factor Model on The 200th Testing day

mula, the part of time value is not measured by the single risk-free interest rate r, but a difference between the risk-free interest rate r and the convenience yield of the underlying commodity δ. As mentioned, the convenience yield of the underlying commodity is defined as a yield from holding the underlying commodity. With the development of pricing a particular futures contract, the convenience yield δ plays an extremely important role and has drawn more and more attention because, without any controversy, the convenience yield is absolutely not a directly observable variable, which means that it cannot be easily obtained, even though the alternative cost of carrying crude oil commodities or futures contracts must be considered for each investor or producer.

Because of this, there are various arguments about the convenience yield for discussion. Schwartz (1997) pointed out that convenience yield δ is one of the necessary variables for collecting the dynamics of futures prices. Casassus and Collin-Dufresne (2005) suggested that the convenience yield of an underlying asset might be obtained by the inverse mathematical function of a futures pricing model, which means that the convenience yield could be expressed as $\delta(t) = f^{-1}(F^T(t), S(t), r, T, t)$. In addition,

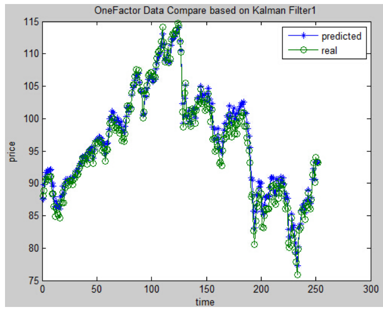


Figure 7: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The One-Factor Model

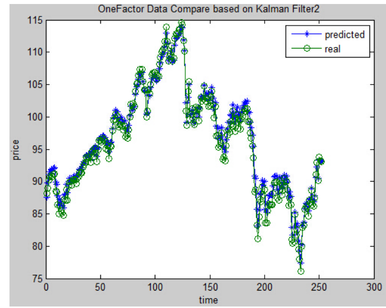


Figure 8: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The One-Factor Model

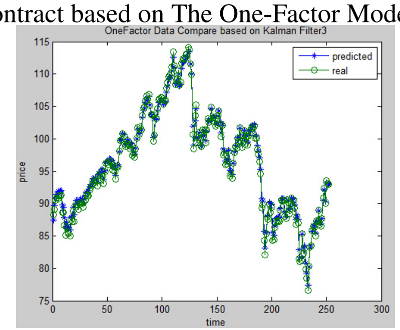


Figure 9: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The One-Factor Model

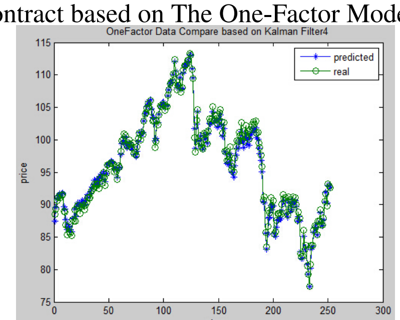


Figure 10: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The One-Factor Model

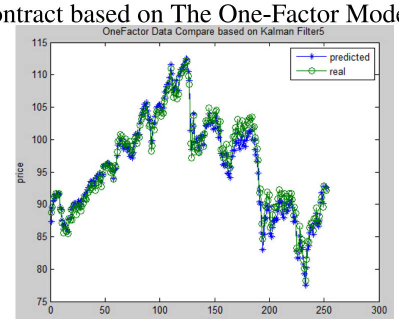


Figure 11: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The One-Factor Model

Lewis (2005) suggested that the convenience yield is an indication of an uncertain market and positively correlated with volatility across various commodity markets. Moreover, the convenience yield was considered as an exogenous random variable by Gibson and Schwartz (1990). On the other hand, a number of other studies considered the changes in the convenience yield to be an endogenous result of the interaction between supply, demand, and storage decisions (Brennan (1958), Deaton and Laroque (1992), Pindyck (2001), Routledge, Seppi and Spatt (2000), Working (1949)). Furthermore, Hong (2001) combined the views on exogenous and endogenous variables, and pointed to a strongly seasonal variation of the convenience yield due to the imbalance of supply and demand by studying the price spread of international oil.

In order to calculate the unobservable convenience yield, many studies have tried different approaches. For example, apart from Schwartz (1997) estimating the convenience yield form, his famous system of One-, Two- and Three-Factor models, which is a part of this thesis, there is also Carmona and Ludkovski (2004) who suggested a variant of the Schwartz (1997) model with time-dependent parameters to calculate the hidden state variables. Geman (2003) pointed out that the convenience yield is essentially the difference between the benefit from holding the physical commodity and the cost of storage, and the convenience yield should be strongly related to the level of the inventory. She also proposed that there is a negative relationship between the volatility of a commodity and the world inventory level, and hence the price of any particular commodity and its volatility are positively correlated. Cassassus and Collin-Dufresne (2005) successfully obtained a stochastic convenience yield which is implied from assumed observable prices of commodity futures and interest rates based on a similar system to the Schwartz (1997) model.

In the domain of quantitative finance, Schwartz (1997) set the convenience yield as the second stochastic process, which can be collected from the dynamics of futures prices. To be more specific, denoting δ to be the instantaneous convenience yield, then, the two processes can be described as:

$$dS = (\mu - \delta)Sdt + \sigma_1 Sdz_1$$

and

$$d\delta = \kappa(\alpha - \delta)dt + \sigma_2 dz_2$$

where μ and α represented the long-term return from investing in oil and the long-term convenience yield, respectively; κ is the coefficient of reverting; σ_1 and σ_2 are the volatilities of spot price and convenience yield; dz_1 and dz_2 are increments of Brownian motions and following normal distribution $N(0, dt^2)$ with $dz_1 dz_2 = \rho dt$, where ρ is the correlation coefficient between the two stochastic processes.

Under the risk neutral assumption, μ can be replaced by the interest rate r, which can be observed in the Two-Factor model, but later it is set to be unobservable in the Three-Factor model because in the Three-Factor model, the interest rate r is also considered to be unobservable. Similar to the One-Factor Model, the processes can be described as:

$$dS = (r - \delta)Sdt + \sigma_1 S dz_1^*$$

$$d\delta = [\kappa(\alpha - \delta) - \lambda]dt + \sigma_2 dz_2^*$$

Define $X = ln(S)$ and apply Itô's Lemma, the process for the log price can be written as:

$$dX = (\mu - \delta - \tfrac{1}{2}\sigma_1^2)dt + \sigma_1 S dz_1^*$$

In the Two-Factor model, the λ represents the convenience yield risk, which is called the market price of the convenience yield, and moreover, $dz_1^* z_2^* = \rho dt$. With the boundary condition $F(S, \delta, T = 0) = S$, the solution of the PDE:

$$\tfrac{1}{2}\sigma_1^2 S^2 F_{SS} + \sigma_1\sigma_2\rho S F_{S\delta} + \tfrac{1}{2}\sigma_2^2 F_{\delta\delta} + (r - \delta)SF_S + [\kappa(\alpha - \delta) - \lambda]F_\delta - F_T = 0$$

is shown as:

$$F(S, \delta, T) =$$
$$Sexp[-\delta\frac{(1-e^{-\kappa T})}{\kappa} + (r - \widehat{\alpha} + \frac{1}{2}\frac{\sigma_2^2}{\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa})T + \frac{1}{4}\sigma_2^2\frac{1-e^{-2\kappa T}}{\kappa^3} + (\widehat{\alpha}\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa})\frac{1-e^{-\kappa T}}{\kappa^2}]$$
with $\widehat{\alpha} = \alpha - \frac{\lambda}{\kappa}$.

2.3.2 Data and Assumptions

In the Two-Factor model, in order to simplify the complex state space model, the weekends and other non-trading days can be ignored, which means that the trading days are considered to be continuous. Based on the efficient market hypothesis, powerful information from the non-trading days (e.g. weekends and Christmas holiday) can be immediately reflected in price after the non-trading day. Hence, the continuous trading period is a reasonable and popular assumption in the financial world. The next assumption is that each futures contract is immediately executed on the first day they mature. This is an assumption about the length of the maturity. Based on this assumption, the length of the maturity can be measured more accurately. The third assumption is a measurement of the level of the cost of financing. In this section, the interest rate is assumed to be equal to 2% per year. Furthermore, to simplify the calculation, based on the non-arbitrage assumption the drift μ in the aforementioned model is replaced by the interest rate r, which means that the drift μ is set as a constant in the implementation. In addition, in this section, instead of a single average T, the dynamics of the T with the time passing is added to the model, which is expected to provide a better fit for the data.

As has been explained in the previous sections, when the spot price of crude oil is set as an unobservable state variable, the observable futures prices are the only data collected when the model is implemented. In this section, the data for futures prices of WTI crude oil is collected from Bloomberg. Specifically, in this section, five futures contracts will be used. Their maturities are one month, two months, four months, seven months and ten months respectively, and the test period is the year from 2^{nd} Nov. 2010 to 31^{st} Oct. 2011. To be more specific, the five futures contracts would mature at December 2011, January 2012, March 2012, June 2012 and September 2012 respectively.

2.3.3 The Empirical Results of The Schwartz Two-Factor Model

Figure 12 and Figure 13 exhibit the estimated unobservable state variables using the Kalman filter. On the one hand, the estimated spot price of crude oil is described in Figure 12. In Figure 12, it is not hard to observe that, in the test period, the estimated spot price of crude oil dropped to under 85 dollars per barrel from about 90 dollars at the beginning of the test period. Then it rebounds and keeps moving up to about 115 dollars per barrel, which is the peak of the entire test period. After the peak, the estimated spot price of crude oil experiences a long and continued decrease until it is about 75 dollars per barrel, which is the valley. At the end of the test period, the estimated spot price rebounds again. The downward slide in the second half of the test period can be explained by the International Energy Agents announcement of the release of 60 million barrels from the strategic petroleum reserve at the end of June 2011. On the other hand, Figure 13 shows the convenience yield of crude oil estimated by the Kalman filter in the same test period. At the beginning of the test period, the estimated convenience yield reached a relatively high value, then, after a short drop, it rapidly increased to the peak of the entire test period at about 3%. After the peak, it immediately slumps to the lowest point (about -8$\%$) of the entire period. At the end, the estimated convenience yield moves up to a relatively high value, about zero. As for the estimated parameters, first, the degree of mean reversion of the convenience yield of WTI crude oil is low; second, the long-term return investment on the convenience yield of WTI crude oil is positive during the test period; last, the risk on the spot price of WTI crude oil is higher than the risk on the convenience yield of WTI crude oil.

In most situations, rational people often consider that there is a strictly positive correlation between the spot price and the convenience yield of crude oil. This means that the convenience yield should move up when the spot price of crude oil enters a period of growth, whereas the convenience yield should decrease when the spot price is moving down. Indeed, for most of the trading days of our case, there is an obviously positive relationship between the estimated spot price and the estimated convenience yield of crude oil in the test period. However, from the perspective of Figure 12 and Figure 13, we can also easily see that around the 50^{th} trading day, the estimated spot price moved up, but the estimated convenience yield of crude oil suddenly slid down.

Then, the five chosen futures contracts are shown together in Figures 14 and 15 in order to see whether there are significant differences between the observed prices of WTI crude oil futures contracts and the estimated prices of WTI crude oil futures contracts in the Two-Factor model. To be more specific, the real observed futures prices of the five chosen futures contracts are shown in Figure 14. Then, the estimated model futures prices of the five chosen futures contracts, which are estimated from the Two-Factor model, are shown in Figure 15. In order to see the trend clearly, in Figure 15 only 26 points are chosen in the figure, which were picked on the first trading day of each ten trading days. In other words, there are no significant differences between the two figures. On the one hand, WTI crude oil did not follow a strict backwardation or a strict contango during the test period; on the other hand, it is not hard to see that the temporary trend of the backwardation and the contango is more easily found when WTI crude oil rapidly price-inverses, while all futures prices seem to be close, when the price of WTI crude oil has a clear trend of increase or decrease.

In order to observe the effectiveness of the Two-Factor model, the comparisons with observed prices of each WTI crude oil futures in the Two-Factor model are shown in Figures 19 to 23. The rank of the title of each picture corresponds to the rank of its maturity. For example, the title 'The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model' and the title 'The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model' correspond to the futures contracts which matured in December 2011 and January 2012 respectively, and so forth. It is not hard to see that the Two-Factor model is useful in pricing a futures contract with a particular maturity because all five pictures show that the estimated model price of each futures contract is really close to the real observed price of the futures contract.

Last, the forward curves from the Two-Factor model are shown within Figures 16-18. First, on the 50^{th} trading day, the model estimated prices of the Two-Factor model cross the observed prices. To be more specific, with the increase in the term to maturity, the estimated prices are firstly lower than the observed prices, then they are higher than the observed prices. Second, similar to the 50^{th} trading day, on the 100^{th} trading day, there was a crossover between the estimated model prices and the observed prices. Specifi-

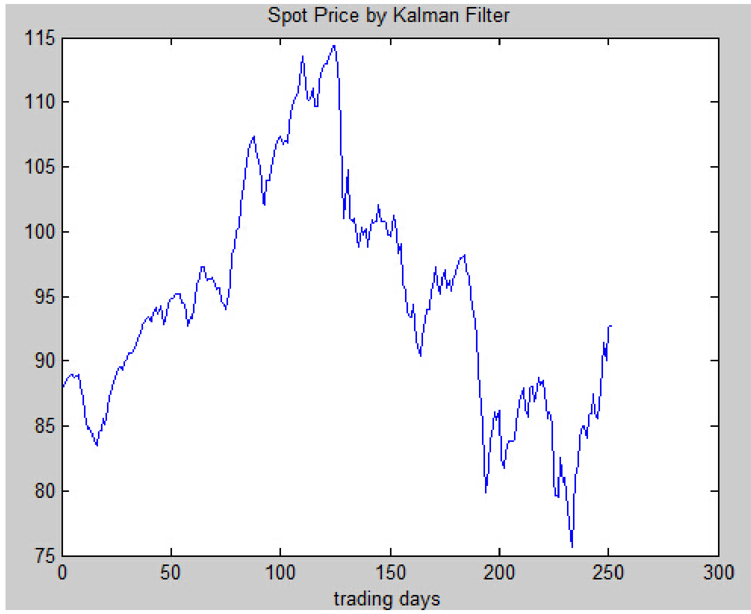


Figure 12: Estimated Spot Price from The Two-Factor Model by The Kalman Filter

cally, the situation is the same as the description for the 50^{th} trading day, but the difference between those two prices is smaller. Last, on the 200^{th} trading day, the model estimated prices of the Two-Factor model and the observed prices are almost the same. To be more specific, in Figure 18, the estimated prices are lower than the observed prices in the beginning and then they are higher than the observed prices, but the differences are really small. In other words, as expected, the estimated model prices from the Two-Factor model are close to the observed prices.

Paremeters	Estimated result	Standard Error
κ	0.2088	0.0054
α	0.0105	0.0752
λ	0.0305	0.0785
σ_1	0.6465	0.2778
σ_2	0.2998	0.1339
ρ	0.5904	0.2537

Table 2.3.3: Estimated Parameters from The Two-Factor Model

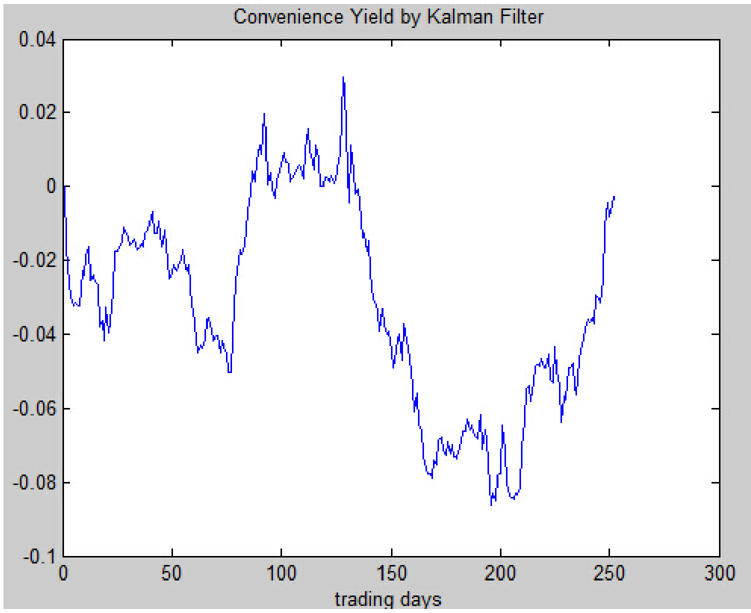


Figure 13: Estimated Convenience Yield from The Two-Factor Model by The Kalman Filter

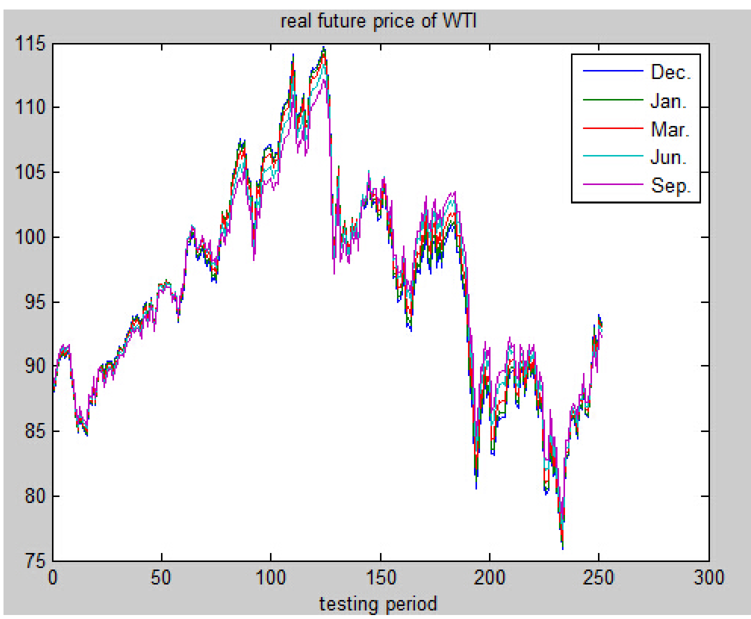


Figure 14: Observed Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model

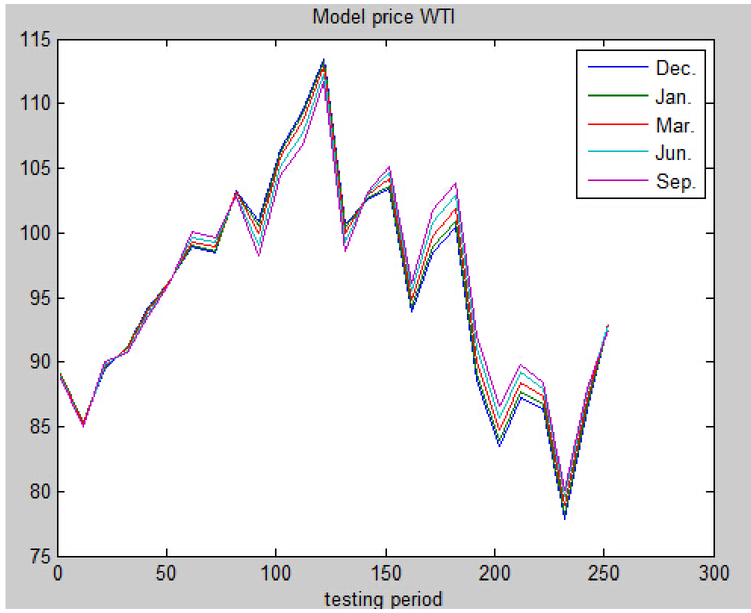


Figure 15: Estimated Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model

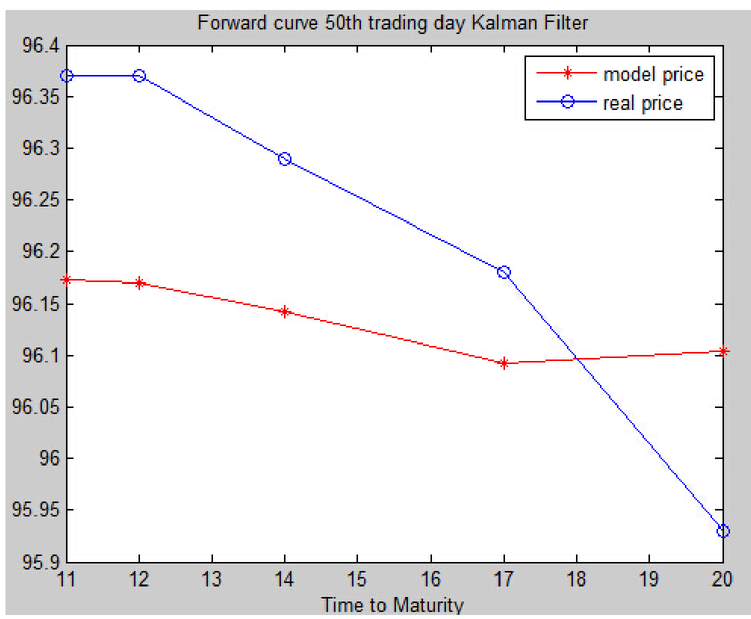


Figure 16: Forward Curves from The Two-Factor Model on The 50th day

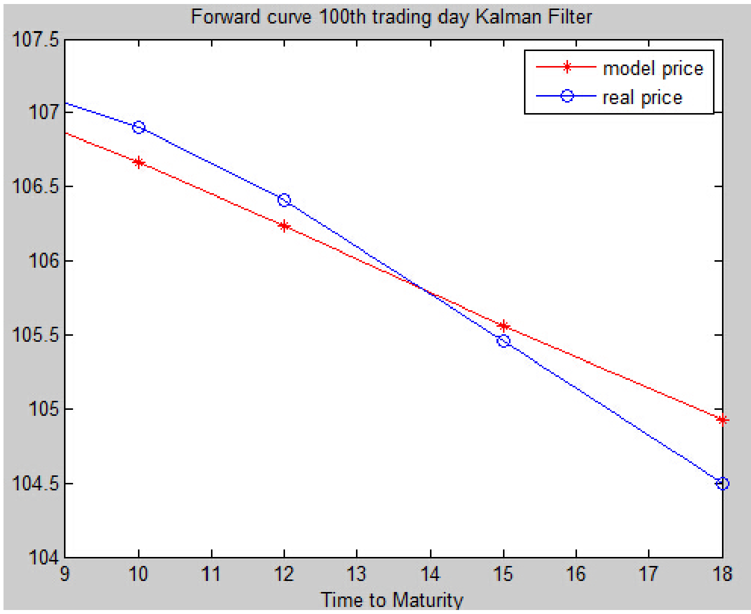


Figure 17: Forward Curves from The Two-Factor Model on The 100th day

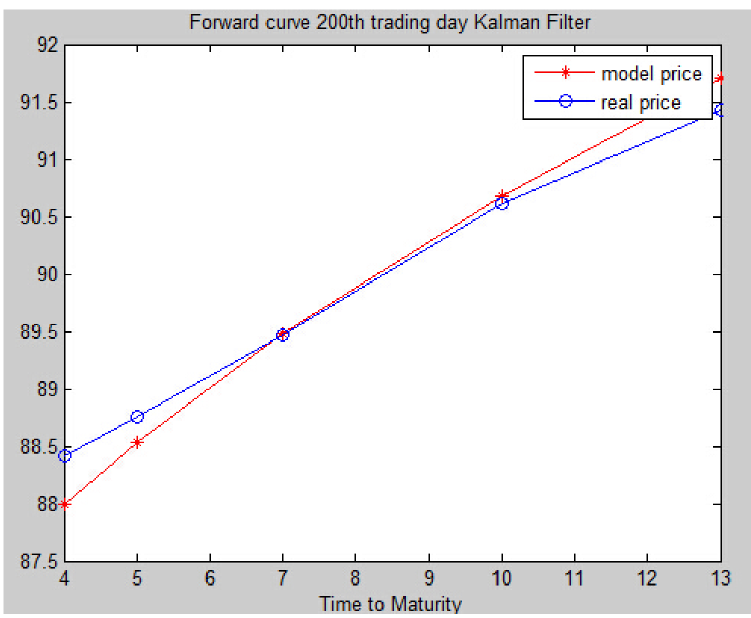


Figure 18: Forward Curves from The Two-Factor Model on The 200th day

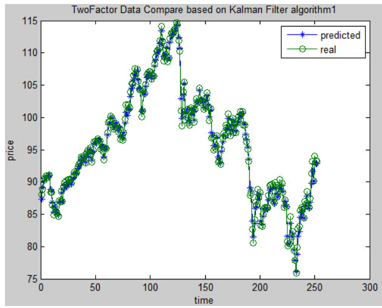


Figure 19: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model

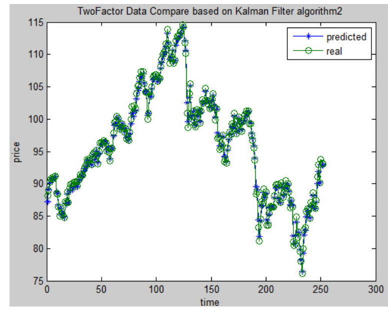


Figure 20: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model

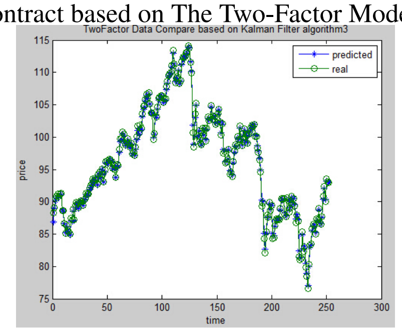


Figure 21: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The Two-Factor Model

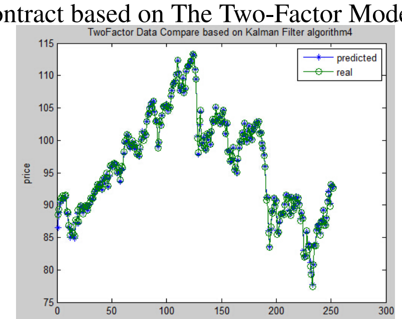


Figure 22: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The Two-Factor Model

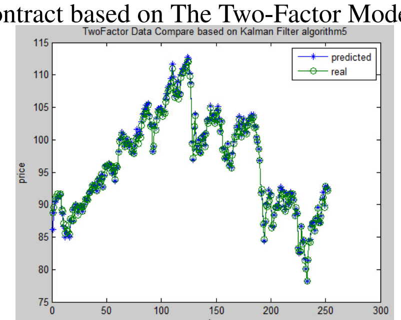


Figure 23: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The Two-Factor Model

2.4 The Schwartz Three-Factor Model and Its Empirical Results

2.4.1 The Schwartz Three-Factor Model

In 1997, Schwartz developed a new Three-Factor model in which the stochastic interest rate was considered to be the third unobservable factor. Since then, the instantaneous interest rate was introduced in the model system.

Similar to the Two-Factor model, the three stochastic processes can be described as follows:

$$dS = (\mu - \delta)Sdt + \sigma_1 Sdz_1^*$$

$$d\delta = \kappa(\widehat{\alpha} - \delta)dt + \sigma_2 dz_2^*$$

$$dr = \alpha(m^* - r)dt + \sigma_3 dz_3^*$$

with

$$dz_1^* dz_2^* = \rho_{12} dt$$

$$dz_1^* dz_3^* = \rho_{13} dt$$

$$dz_2^* dz_3^* = \rho_{23} dt$$

where μ, $\widehat{\alpha}$ and m^* representing the long-term return on investing in oil, the long-term convenience yield and the interest rate, respectively; κ and α are the coefficient of reverting in the stochastic processes, respectively; similarly, σ_1, σ_2 and σ_3 are the volatilities of spot price, convenience yield and the interest rate, respectively; dz_1^*, dz_2^* and dz_3^* are increments of Brownian motions and following normal distribution $N(0, dt^{1/2})$ with $dz_i^* dz_j^* = \rho_{ij} dt$, where ρ_{ij} is the correlation coefficient between the two stochastic processes i and j.

With the consideration of the three stochastic processes, the futures price formula must satisfy the following partial differential equation:

$$\frac{1}{2}\sigma_1^2 S^2 F_{SS} + \frac{1}{2}\sigma_2^2 F_{\delta\delta} + \frac{1}{2}\sigma_3^2 F_{rr} + \sigma_1\sigma_2\rho_{12} S F_{S\delta} + \sigma_1\sigma_3\rho_{13} S F_{Sr} + \sigma_2\sigma_3\rho_{23} F_{\delta r} + (r-\delta) S F_S + \kappa(\widehat{\alpha}-\delta)F_\delta + \alpha(m^* - r)F_r - F_T = 0$$

Schwartz proposed the solution of the above PDE in 1997, which is as follows:

$$F(S,\delta,r,T) = Sexp((-\delta\frac{1-e^{-\kappa T}}{\kappa}) + r\frac{1-e^{-\alpha T}}{\alpha} + C(T))$$

where

$$C(T) = \frac{(\kappa\widehat{\alpha}+\sigma_1\sigma_2\rho_{12})((1-e^{-\kappa T})-\kappa T)}{\kappa^2} - \frac{\sigma_2^2(4(1-e^{-\kappa T})-(1-e^{-2\kappa T})-2\kappa T}{4\kappa^3} - \frac{(\alpha m^*+\sigma_1\sigma_3\rho_{13})((1-e^{-\alpha T})-\alpha T)}{\alpha^2} - \frac{\sigma_3^2(4(1-e^{-\alpha T})-(1-e^{-2\alpha T})-2\alpha T)}{4\alpha^3} + \sigma_2\sigma_3\rho_{23}(\frac{(1-e^{-\kappa T})-(1-e^{-(\kappa+\alpha)T})+(1-e^{-\alpha T})}{\kappa\alpha(\kappa+\alpha)} + \frac{\kappa^2(1-e^{-\alpha T})+\alpha^2(1-e^{-\kappa T})-\kappa\alpha^2 T-\alpha\kappa^2 T}{\kappa^2\alpha^2(\kappa+\alpha)})$$

2.4.2 Data and Assumptions

In this section, in order to simplify the complex state space model, the weekends and other non-trading days can be ignored, which means that the trading days are considered to be continuous. Based on the efficient market hypothesis, powerful information from the non-trading days (e.g. weekends and Christmas holiday) can be immediately reflected in the price after the non-trading day. Hence, this is a reasonable and popular assumption in the financial world. The next assumption is that each futures contract is immediately executed on the first day they mature. This is an assumption about the length of the maturity. Based on this assumption, the length of the maturity can be measured more accurately. The third assumption is a measurement of the level of the drift of crude oil. Since the interest rate r is considered to be instantaneous, and under the risk neutral measure, the drift of crude oil equals the instantaneous interest rate, but is not a constant any more. In addition to these, in this section, instead of a single average T, the dynamics of the T over time are added to the model, which means the T keeps shortening with time passing in the running of this model, which is expected to provide a better fit for the data. Last, in order to compare the original Three-Factor model with the new Four-Factor model, which will be introduced in the fourth section, the Three-Factor model will be run with a change in estimating the parameter σ_1. The variation of the estimating method with the extended Kalman filter will be introduced in the section 3.2– The Original Two-Factor Model with Stochastic Parameters

As has been explained in the previous sections, since the spot price of crude oil is set as an unobservable state variable, the observable futures prices are the only collected data when the model is implemented. In this section, the data for futures prices of WTI crude oil is collected from Bloomberg. Specifically, in this section, twelve futures contracts will be used, their maturities are from February 2013 to January 2014, and the test period is the entire year of 2012.

2.4.3 The Empirical Results of The Schwartz Three-Factor Model

The estimated state variables of the Three-Factor model are shown within Figures 24-27. To be specific, Figure 24 exhibits the estimated spot price of WTI crude oil, which fluctuates between about 115 and about 80 dollars per barrel in the test period. In other words, the estimated spot price shows a downward trend in the test period. It fluctuates over 100 dollars per barrel and reaches the peak of about 113 dollars per barrel at the beginning of the test period, but then directly plummets to about 80 dollars per barrel, which is the valley in the entire test period. In the second half of the test period, it rebounds a little and then decreases again. Figure 25 shows the estimated convenience yield of the Three-Factor model. The estimated convenience yield also shows a downward trend in the entire test period. Specifically, the estimated convenience yield of WTI crude oil rapidly decreases from 0.02 to -0.04, and, after a rebound to about 0.02, the estimated convenience yield keeps moving down again until it is about -0.1 at the end of the test year. Figure 26 shows the estimated interest rate from the Three-Factor model, which fluctuates around zero during the entire test period. Specifically, it fluctuates in an interval between 0.04 and -0.07. Notably, for a great majority of the trading days, the estimated interest rate is negative. Furthermore, Figure 27 shows the implied stochastic volatility of the spot price in the Three-Factor model. After a stable fluctuation around 0.3, the implied σ_1 starts to move upward until 0.5, then it fluctuates between 0.3 and 0.5 in the second half of the test period. As for the estimated parameters, first, the degree of mean reversion of the convenience yield of WTI crude oil is not very high; second, the long-term return investment on the convenience yield of WTI crude oil is positive during the test period, while the long-term return investment on the interest rate is negative;

last, the risk to the spot price of WTI crude oil is higher than the risk to the convenience yield of WTI crude oil and the interest rate.

The model was run with 12 futures contracts. However, drawing 12 coloured lines in a picture would make the picture untidy. Hence, only the first six futures contracts are shown in the following figures. The six chosen futures contracts are shown together in Figure 28 and Figure 29 in order to see whether there are significant differences between the observed prices of WTI crude oil futures contracts and the estimated prices of the WTI crude oil futures contracts in the Three-Factor model. To be more specific, the real observed futures prices of the six chosen futures contracts are shown in Figure 28. Then, the estimated model futures prices of the six chosen futures contracts, which are estimated from the Three-Factor model, are shown in Figure 29. In order to see the trend clearly, in Figure 29, only 26 points are chosen in the figure, which were picked the first trading day of each ten trading days. In other words, there are no significant differences between the two figures. On the one hand, WTI crude oil did not follow a strict backwardation or a strict contango during the test period; on the other hand, it is not hard to see that the trend of the backwardation and/or the contango is more easily found when WTI crude oil fluctuates stably, while all futures prices seem to be close, when the price of WTI crude oil rapidly increases or rapidly decreases.

In order to observe the effectiveness of the Three-Factor model, the comparison with the observed futures prices of each WTI crude oil futures in the Three-Factor model are shown within Figure 33 to Figure 38. The rank of the title of each picture corresponds to the rank of its maturity. For example, the title 'The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Three-Factor Model' and the title 'The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Three-Factor Model' correspond to the futures contracts which matured in February 2013 and March 2013, respectively, and so forth. It is not hard to see that the Three-Factor model is useful for pricing a futures contract with a particular maturity because all six pictures show that the estimated model price of each futures contract is really close to the real observed price of the futures contract.

Last, the forward curves from the Three-Factor model are shown within Figure 30 to Figure 32. Firstly, on the 50^{th} trading day, the model estimated prices of the Three-Factor model are higher than the observed prices. To be more specific, with a longer term to maturity, the difference between the model price and the observed price seems to be smaller. Secondly, similar to the 50^{th} trading day, on the 100^{th} trading day, the model estimated prices of the factor model are also higher than the observed prices, and with a longer term to maturity, the difference between the model price and the observed price seems to be smaller. Lastly, on the 200^{th} trading day, the model estimated prices of the Three-Factor model are crossed with the observed prices. To be more specific, with a short term to maturity, the model estimated prices of the Three-Factor model are higher than the observed prices. In contrast, the model estimated prices of the Three-Factor model are lower than the observed prices with a long term to maturity.

Paremeters	Estimated result	Standard Error
κ	0.7660	0.0853
α	0.0511	0.0011
$\widehat{\alpha}$	0.0336	0.0175
m^*	-0.0074	SE1
σ_2	0.1220	SE2
σ_3	0.1322	SE3
ρ_{12}	-0.0106	SE4
ρ_{13}	-0.2751	0.1225
ρ_{23}	0.0184	SE5

Note:$SE1 = \sqrt{-2.81693e - 03 + ComputationalError1}$
$SE2 = \sqrt{-7.565022e - 05 + ComputationalError2}$
$SE3 = \sqrt{-4.085445e - 04 + ComputationalError3}$
$SE4 = \sqrt{-3.608047e - 02 + ComputationalError4}$
$SE5 = \sqrt{-3.762750e - 01 + ComputationalError5}$

Table 2.4.3: Estimated Parameters from The Three-Factor Model

[4]

[4]Some estimated SE equal square root of a small negative value, we consider that caused by the computational errors.

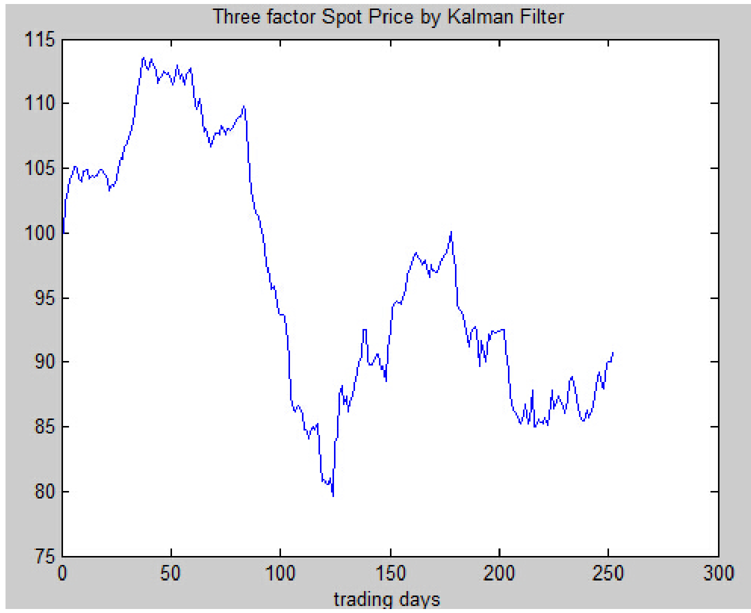


Figure 24: Estimated Spot Price from The Three-Factor Model by The Extended Kalman Filter

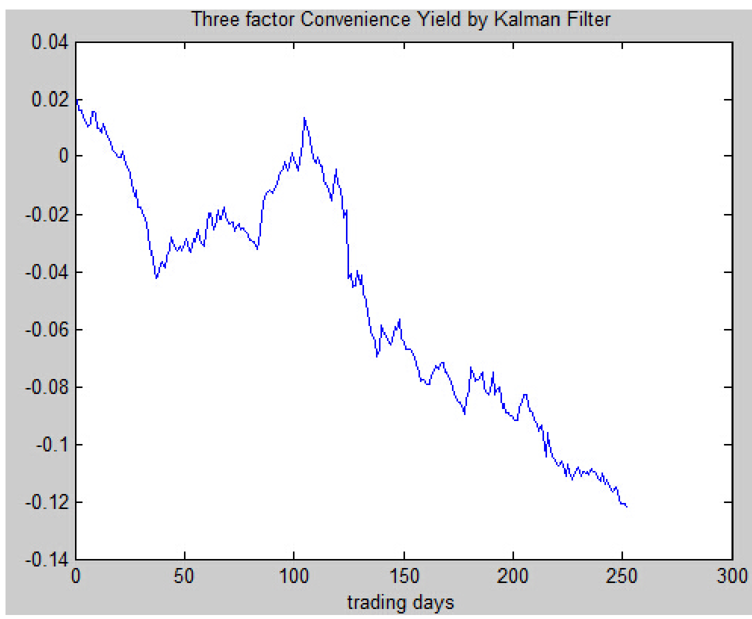


Figure 25: Estimated Convenience Yield from The Three-Factor Model by The Extended Kalman Filter

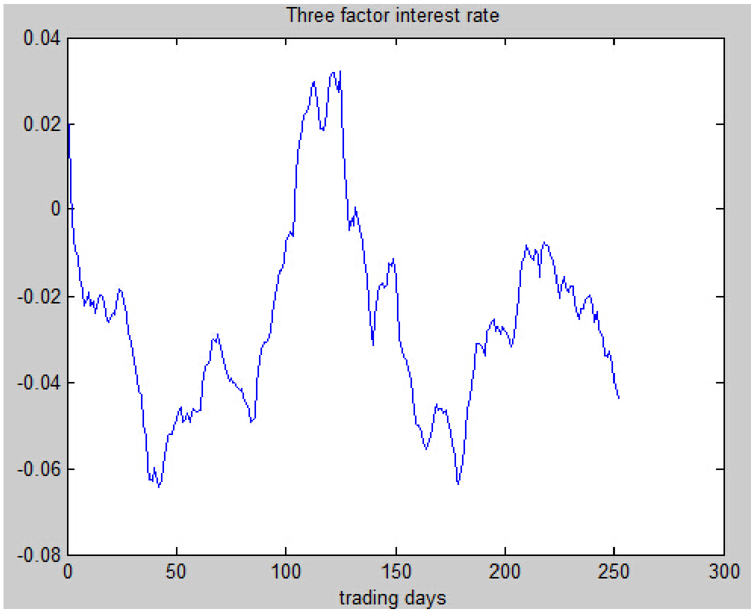


Figure 26: Estimated Interest Rate from The Three-Factor Model by The Extended Kalman Filter

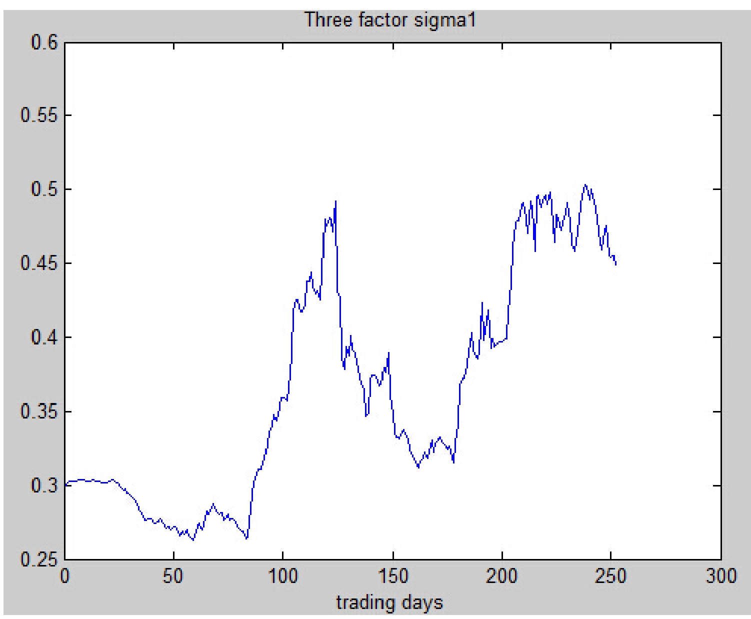


Figure 27: Estimated σ_1 from The Three-Factor Model by The Extended Kalman Filter

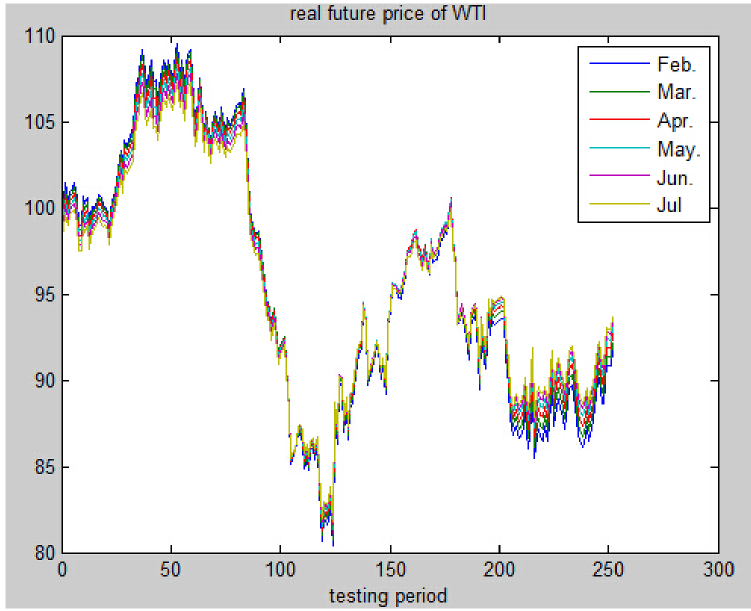


Figure 28: Observed Prices of WTI Futures Contracts with Different Maturities for The Three-Factor Model

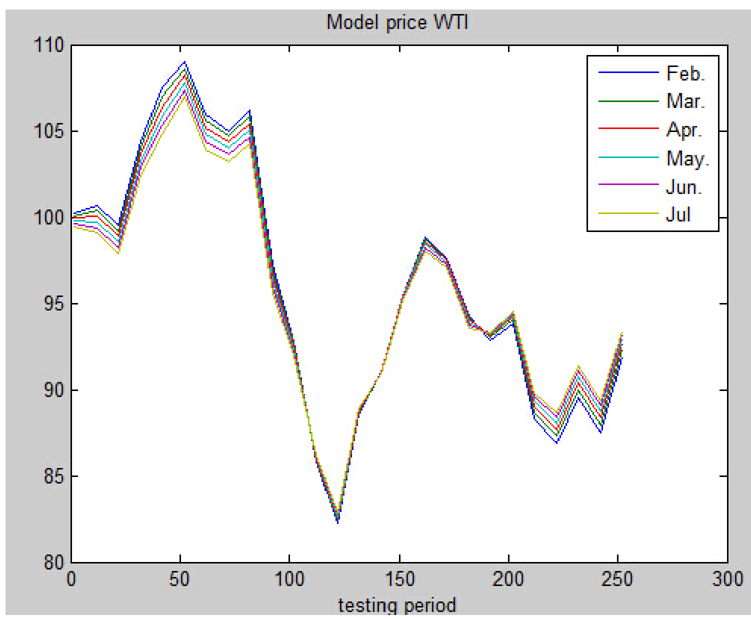


Figure 29: Estimated Prices of WTI Futures Contracts with Different Maturities for The Three-Factor Model

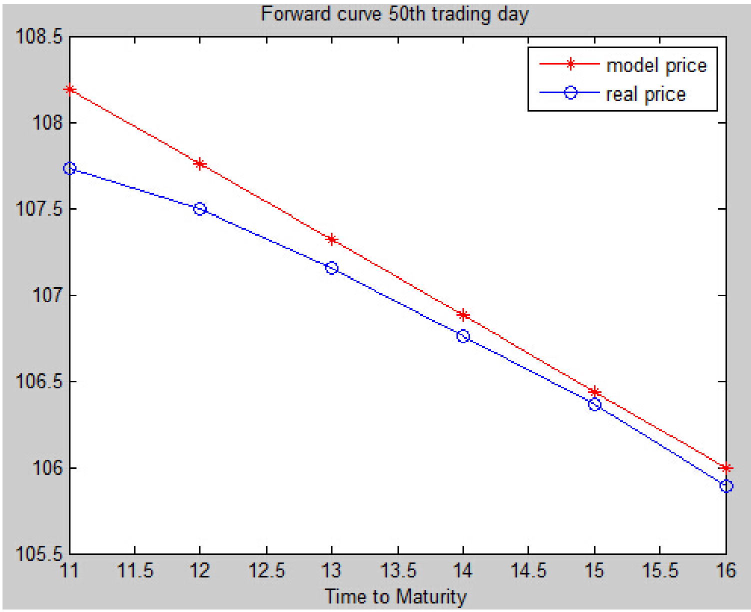


Figure 30: Forward Curves from The Three-Factor Model on The 50th day

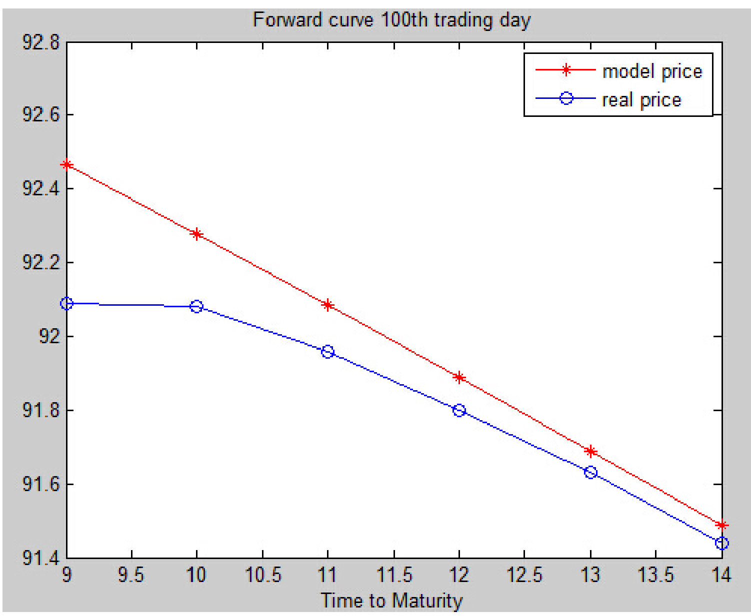


Figure 31: Forward Curves from The Three-Factor Model on The 100th day

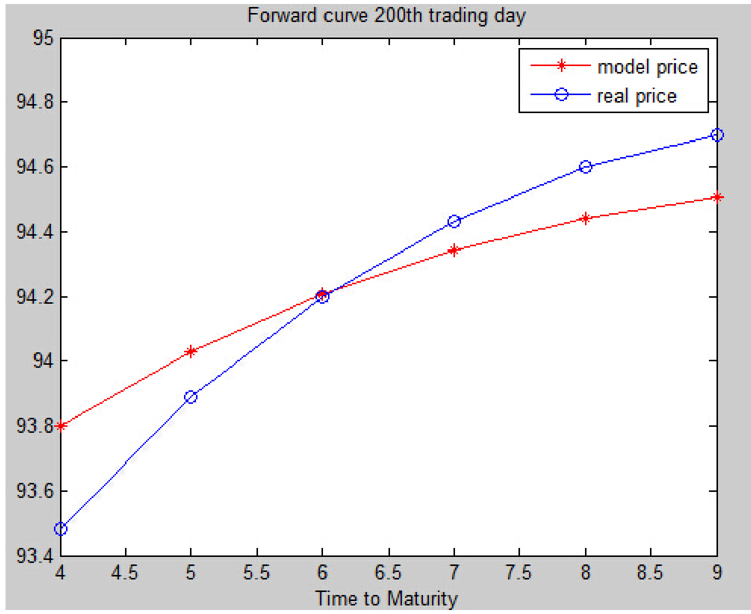


Figure 32: Forward Curves from The Three-Factor Model on The 200th day

3 The Original Two-Factor Model with Observed Spot Price and The Original Two-Factor Model with Stochastic Parameters

Since the One-Factor model does not take account of the convenience yield of the underlying asset, the One-Factor is considered to be an outdated model, because the convenience yield of the underlying asset is seen as the most important factor when people price a futures contract. Because of this, the Two-Factor model, which contains the stochastic spot price and the convenience yield of an asset, is chosen by a large number of researchers and scientists. However, assuming the spot price of crude oil being unobservable might not be the truth, a small proportion of crude oil is traded in spot oil markets. On the other hand, as has been shown in the last section, the standard errors of unknown parameters in the original Two-Factor model seem to state that some estimated parameters based on the maximum likelihood method are not very reliable, even if the unreliable results of parameters are caused by the data and not by the method. (In section

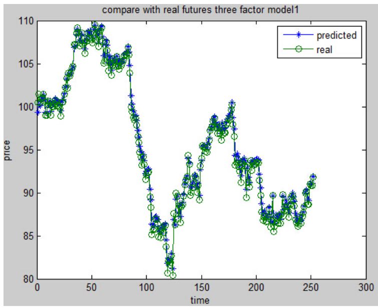


Figure 33: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Three-Factor Model

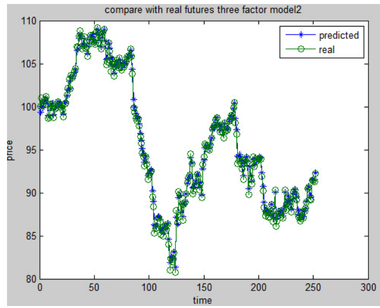


Figure 34: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Three-Factor Model

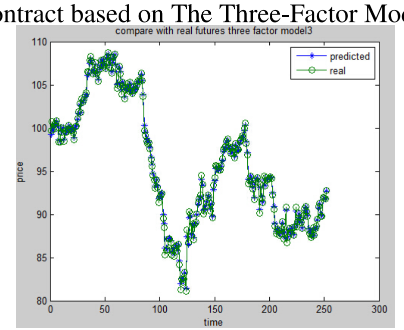


Figure 35: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The Three-Factor Model

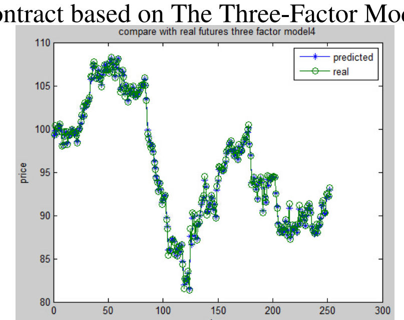


Figure 36: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The Three-Factor Model

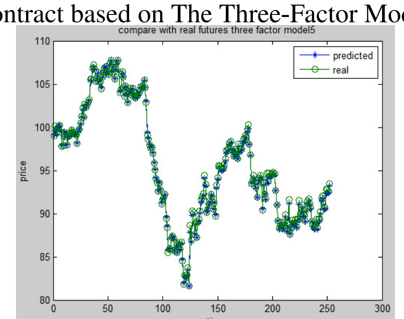


Figure 37: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The Three-Factor Model

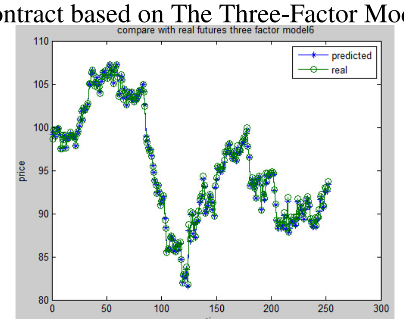


Figure 38: The Predicted and Observed Futures Prices for The sixth Testing Futures Contract based on The Three-Factor Model

6, all estimated parameters are far more reliable, when new data is used.) Hence, using more available information and tracing the path of unknown parameters might be the ways to improve the Two-Factor model. In this section, the original Two-Factor model will be run with two variations. First, the real observed spot price of WTI crude oil will be used in the Two-Factor model to replace the estimated spot price. Second, the parameters will be considered as a part of the state variables in order to estimate the stochastic parameters. Since the model is the same as that which was introduced in section 2.3, in this section, the mathematic model is not introduced.

3.1 The Original Two-Factor Model with Observed Spot Price

In the world, a proportion of crude oil (even if it is a small part) is actually traded in spot oil markets, which means sometimes the spot price of WTI is observable. In other words, the spot price of some kinds of commodities sometimes can be observed directly in the markets. In the case of WTI crude oil, the US Energy Information Administration collects and publishes the spot price of WTI crude oil, which is traded in the spot oil market in the US.

3.1.1 Data and Assumptions

In this model, in order to simplify the complex state space model, weekends and other non-trading days can be ignored, which means that the trading days are considered to be continuous. Based on the efficient market hypothesis, powerful information from the non-trading days (e.g. weekends and Christmas holiday) can be immediately reflected in the price after the non-trading day. Hence, this is a reasonable and popular assumption in the financial world. The next assumption is that each futures contract is immediately executed on the first day they mature. This is an assumption about the length of the maturity. Based on this assumption, the length of the maturity can be measured more accurately. The third assumption is a measurement of the level of the cost of financing. In this section, the interest rate is assumed to be equal to 2% per year. Furthermore, to simplify the calculation, based on the non-arbitrage assumption, the drift μ in the aforementioned model is replaced by the interest rate r, which means that the drift μ is set as a constant in the implementation. Additionally in this section, instead of a single average T, the dynamics of the T with the time passing is added to the model, which is expected to provide a better fit for the data.

As has been explained in the previous sections, when the spot price of crude oil is set as an unobservable state variable, the observable futures prices are the only data collected when the model is implemented. In this section, the data of futures prices of WTI crude oil is collected from Bloomberg. Specifically, in this section, five futures contracts will be used in the Two-Factor model with the observed spot price of WTI crude oil. Their maturities are one month, two months, four months, seven months and ten months, respectively, and the test period is the year from 2^{nd} Nov. 2010 to 31^{st} Oct. 2011. To be more specific, the five futures contracts would mature at December 2011, January 2012, March 2012, June 2012 and September 2012, respectively. When the spot price of WTI crude oil is considered to be observable, there is another time series which needs to be collected-the spot price of crude oil. In this section, the spot price of crude oil in WTI market is collected from the US Energy Information Administration.

3.1.2 The Empirical Results of The Original Two-Factor Model with Observed Spot Price

Figure 39 and Figure 40 show respectively, the observed spot price of WTI crude oil and the estimated unobservable state variable – the convenience yield of WTI crude oil – by using the Kalman filter. On the one hand, the observed spot price of crude oil is described in Figure 39. In Figure 39, it is not hard to see that in the test period the estimated spot price of crude oil fluctuates under 90 dollars per barrel at the beginning of the test period, and then rebounds and keeps moving up to about 115 dollars per barrel, which is the peak of the entire test period. After the peak, the observed spot price of crude oil experiences a long and continued decrease until it is about 75 dollars per barrel, which is the valley. At the end of the test period, the observed spot price rebounds again to about 95 dollars per barrel. The downward jump in the second half of the test period can be explained as the International Energy Agent's announcement of their release of 60 million barrels from their strategic petroleum reserve at the end of June 2011. On the other hand, Figure 40 shows the convenience yield of WTI crude oil estimated by the Kalman filter in the same test period. The situation can be expressed as follows: at the beginning of the period, the estimated convenience yield increased to a relatively high value during the entire testing days, then it suddenly plummeted to about -10%. Specifically, the convenience yield estimated by the Kalman filter dropped to about -12%. Then the estimated convenience yield of WTI rebounded hugely to about the early stage of the high value, which was close to 6%. In the second half of the period, the fluctuation of the convenience yield is smoother. To be more specific, it fluctuates around zero. As for the estimated parameters, first, the degree of mean reversion of the convenience yield of WTI crude oil is not very high; second, the long-term return investment on the convenience yield WTI crude oil is close to zero during the test period; last, the risk to the spot price of WTI crude oil is higher than the risk to the convenience yield of WTI crude oil.

Recalling the previously mentioned positive relationship between the estimated spot price and the estimated convenience yield of crude oil in section 2.3.3, the positive relationship is maintained between the observable spot price and the estimated convenience yield of WTI crude oil. However, the positive relationship is weakened because of the sudden plummet in the estimated convenience yield. Specifically, the estimated convenience yield plummets from nearly 0.06 to about -0.12, while the observed spot price of crude oil fluctuates around 85 dollars per barrel. Then the estimated convenience yield corrects its over-reflection. After that, the positive relationship re-emerges at the end of the test period.

Next, the five chosen futures contracts are shown together in Figure 41 and Figure 42, in order to see whether there are significant differences between the observed prices of WTI crude oil futures contracts and the estimated prices of crude oil futures contracts in the Two-Factor model with the observed spot price of WTI crude oil. To be more specific, the real observed futures prices of the five chosen futures contracts are shown in Figure 41. Then, the estimated model futures prices of the five chosen futures contracts, which are estimated from the Two-Factor model with the observed spot price of WTI crude oil, are shown in Figure 42. In order to see the trend clearly, in Figure 42 only 26 points are chosen in the figure, which were picked the first trading day of each ten trading days. On the one hand, as seen in Figure 41, WTI crude oil did not follow a strict backwardation or a strict contango during the test period; but on the other hand, based on the 26 points in Figure 42, the futures of WTI crude oil seem to follow a strict contango during the test period, which means that the futures are more valuable with a longer term to maturity.

In order to observe the effectiveness of the Two-Factor model, the fitting of each WTI crude oil futures in the Two-Factor model are shown in Figures 46-50. The rank of the title of each picture corresponds to the rank of its maturity. For example, the title 'The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model with The Observed Spot Price of WTI Crude Oil' and the title 'The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model with The Observed Spot Price of WTI Crude Oil' correspond to the futures contracts which matured in December 2011 and January 2012 respectively, and so forth. It is not hard to see that the Two-Factor model is useful in pricing a futures contract with a particular maturity because all five pictures are showing that the estimated model price of each futures contract is really close to the real observed price of the futures contract. To be more specific, on the one hand, all five futures have excellent fitting after the peak. However, the futures which would mature in June 2012 had the best result before the peak. As for the other four futures, before the peak the predicted prices seem to be lower than the observed prices with a shorter period to maturity, but higher with a longer period to maturity.

Last, the forward curves from the Two-Factor model are shown within Figures 43-45. First, on the 50^{th} trading day, the model estimated prices of the Two-Factor model cross with the observed prices. To be more specific, the estimated prices are lower than the observed prices in the beginning and then they are higher than the observed prices. Second, similar to the 50^{th} trading day, on the 100^{th} trading day, the model estimated prices of the Two-Factor model also cross with the observed prices. Specifically, the estimated prices are lower than the observed prices in the beginning and then they are higher than the observed prices. Last, on the 200^{th} trading day, the model estimated prices in the Two-Factor model with the observed spot price of WTI crude oil are slightly lower than the observed prices.

Paremeters	Estimated result	Standard Error
κ	0.5508	SE1
α	0.0024	0.1125
λ	0.0339	0.0646
σ_2	1.7989e-05	3.076819
ρ	-0.0076	0.2537

Note: the σ_1 is calculated as a standard deviation, since the spot price is observed and $SE1 = \sqrt{-9.381283e-08 + ComputationalError}$

Table 3.1.1: Estimated Parameters from The Two-Factor Model with Observed Spot Price

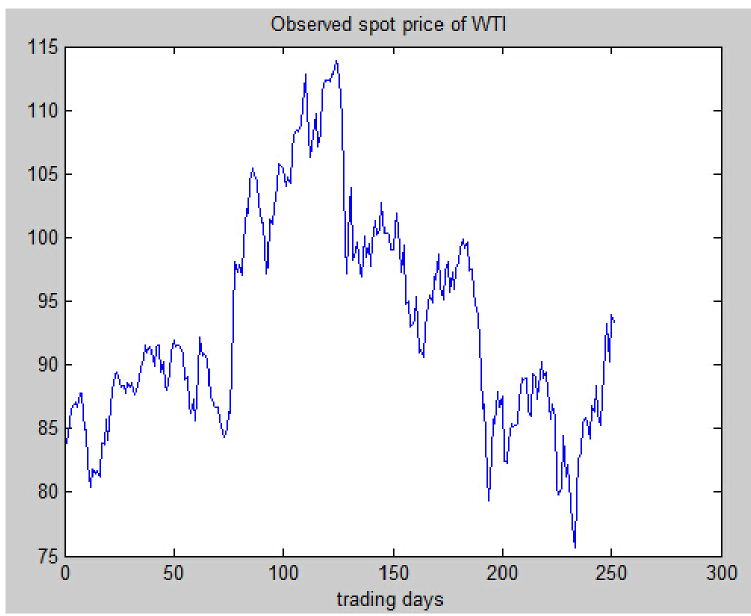


Figure 39: Observed WTI Spot Price During The Test Period

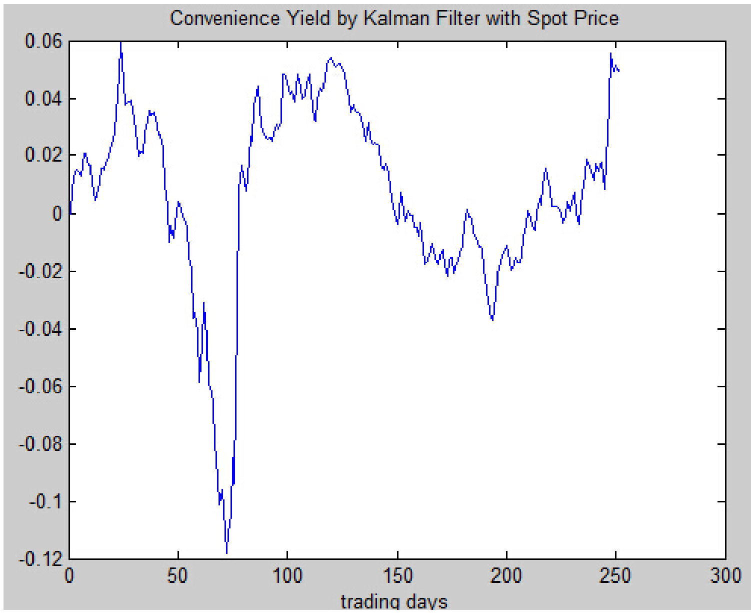


Figure 40: Estimated Convenience Yield from The Two-Factor Model With Observed WTI Spot Price

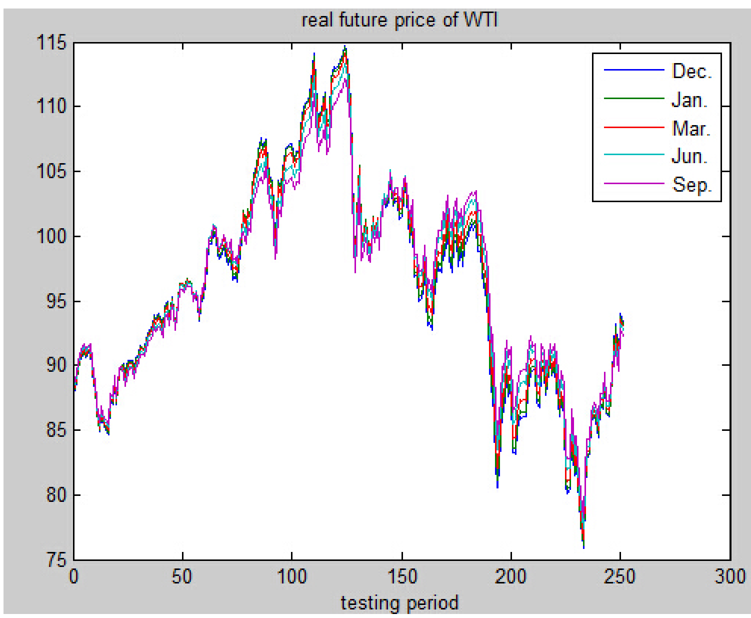


Figure 41: Observed Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model With Observed WTI Spot Price

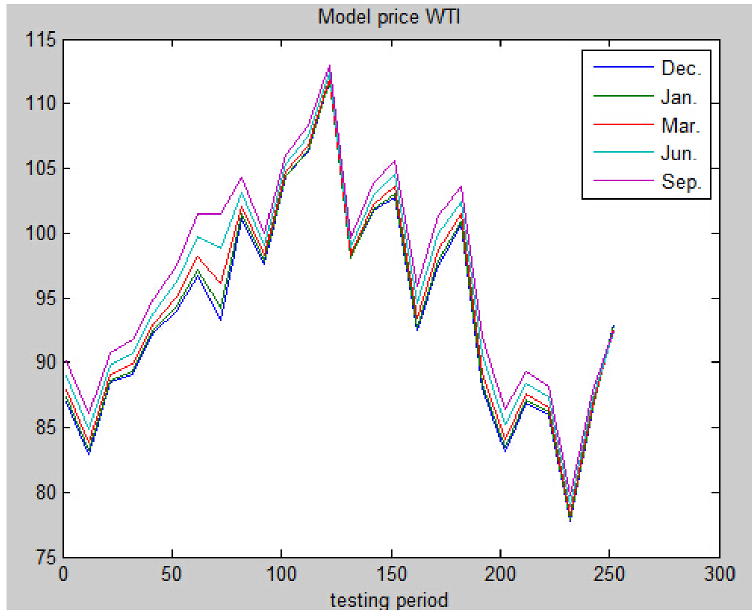


Figure 42: Estimated Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model With Observed WTI Spot Price

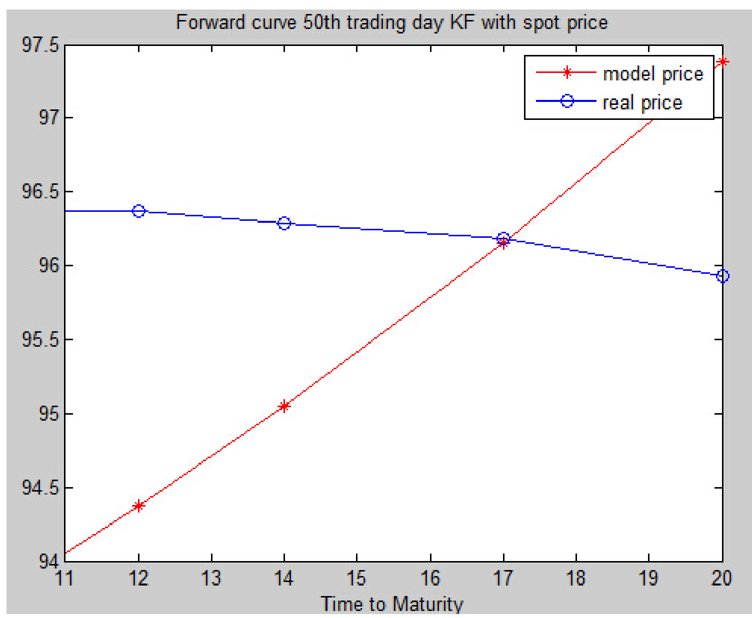


Figure 43: Forward Curves from The Two-Factor Model with Observed Spot Price on The 50th day

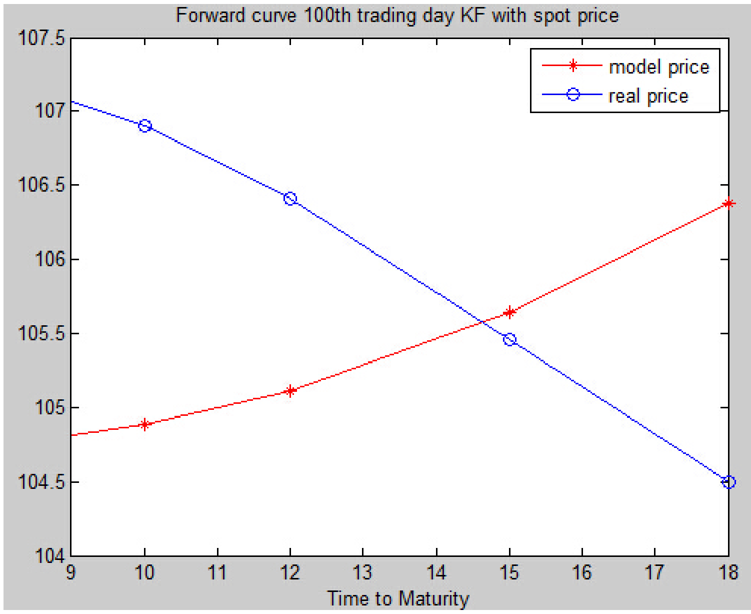


Figure 44: Forward Curves from The Two-Factor Model with Observed Spot Price on The 100th day

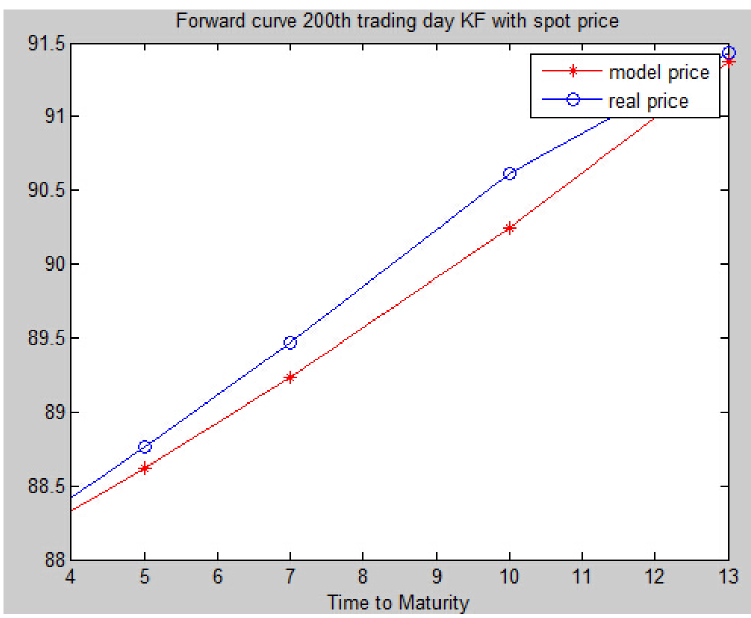


Figure 45: Forward Curves from The Two-Factor Model with Observed Spot Price on The 200th day

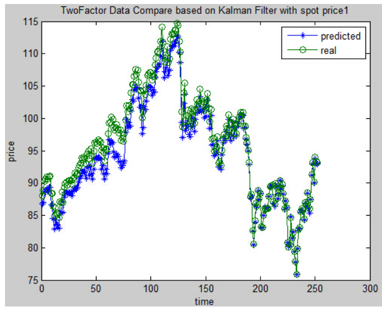


Figure 46: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model with The Observed Spot Price of WTI Crude Oil

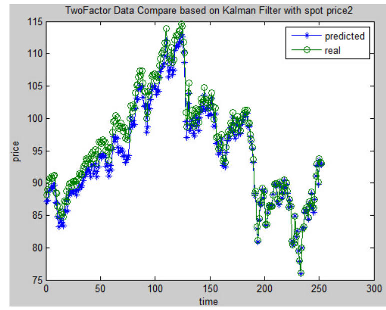


Figure 47: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model with The Observed Spot Price of WTI Crude Oil

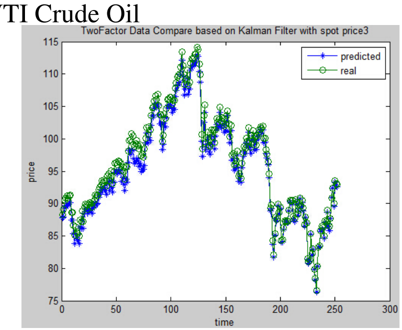


Figure 48: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The Two-Factor Model with The Observed Spot Price of WTI Crude Oil

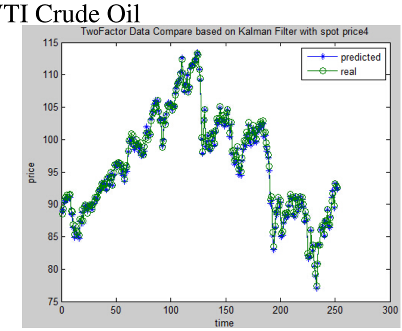


Figure 49: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The Two-Factor Model with The Observed Spot Price of WTI Crude Oil

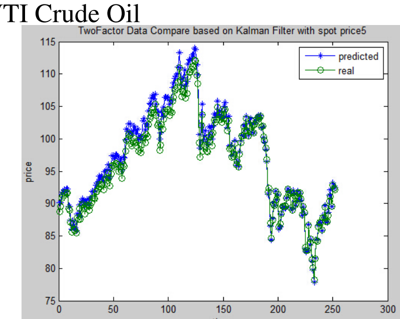


Figure 50: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The Two-Factor Model with The Observed Spot Price of WTI Crude Oil

3.2 The Original Two-Factor Model with Stochastic Parameters

As has been explained in the introduction, the calculated standard errors for some parameters are not very good, which might be caused by the short data. Actually, there is no certain evidence to show that the standard errors must be improved with longer data or that short data is the decisive reason for the worse standard errors in this model system. (In section 6, the original Two-Factor model will be tested with new data to make a comparison with the new Three-Factor model. The standard errors of the estimated parameters are much better with the new data.) However, the requirement of overcoming the potential problem is not negligible. Because of this, a way to get the stochastic parameters is figured out to show the implied parameters at each time point. In the Two-Factor model, in order to observe the dynamics of the parameters, the parameters are also seen as a part of the state variables, and they are iterated during the running of the filter. This implies that the model is not linear anymore. In order to simplify the model, the parameters are assumed to follow random walks, which means that the values of the parameters at step k are only dependent on themselves at step k-1 and a noise. In this way, the implied parameters can be shown in each step.

To be more specific, in the traditional Two-Factor model, the state equation and the measurement equation can be described as follows[5]:

$$\begin{cases} x_k = F_k x_{k-1} + v_{k-1} \\ z_k = H_k x_k + n_k \end{cases}$$

where

$$x = \begin{bmatrix} S \\ \delta \end{bmatrix}$$

[5] see appendix 8.1

and S is the spot price of the underlying asset and the δ is the convenience yield of the underlying asset.

Let a be the vector of the unknown parameters, then, a can be written as:

$$a = \begin{bmatrix} \kappa \\ \alpha \\ \lambda \\ \sigma_1 \\ \sigma_2 \\ \rho \end{bmatrix}$$

Since the unknown parameters are set to follow random walks, parameters at kth step depend only on themselves at $k-1$th step and a noise. Under this assumption, the a at kth step can be described as: $a_k = a_{k-1} + w_{k-1}$, where w is white noise. Use x_k and a_k, the new state and measurement equations can be constructed.

To be more specific, the state equations can be described as:

$$\begin{bmatrix} S \\ \delta \\ \kappa \\ \alpha \\ \lambda \\ \sigma_1 \\ \sigma_2 \\ \rho \end{bmatrix}_k = \begin{bmatrix} 1 & -\Delta t & 0 & 0 & 0 & \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & \Delta_3 & \Delta_4 & -\Delta t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_k \begin{bmatrix} S \\ \delta \\ \kappa \\ \alpha \\ \lambda \\ \sigma_1 \\ \sigma_2 \\ \rho \end{bmatrix}_{k-1} + v_{k-1}$$

in the case of the Two-Factor model $\Delta_1 = \Delta t \sigma_1$, $\Delta_2 = 1 - \Delta t \kappa$, $\Delta_3 = \Delta t(\lambda - \delta)$ and $\Delta_4 = \Delta t \kappa$.

In addition, the measurement equations can be described as:

$$\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial S} & \frac{\partial F_1}{\partial \delta} & \frac{\partial F_1}{\partial \kappa} & \frac{\partial F_1}{\partial \alpha} & \frac{\partial F_1}{\partial \lambda} & \frac{\partial F_1}{\partial \sigma_1} & \frac{\partial F_1}{\partial \sigma_2} & \frac{\partial F_1}{\partial \rho} \\ \frac{\partial F_2}{\partial S} & \frac{\partial F_2}{\partial \delta} & \frac{\partial F_2}{\partial \kappa} & \frac{\partial F_2}{\partial \alpha} & \frac{\partial F_2}{\partial \lambda} & \frac{\partial F_2}{\partial \sigma_1} & \frac{\partial F_2}{\partial \sigma_2} & \frac{\partial F_2}{\partial \rho} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_N}{\partial S} & \frac{\partial F_N}{\partial \delta} & \frac{\partial F_N}{\partial \kappa} & \frac{\partial F_N}{\partial \alpha} & \frac{\partial F_N}{\partial \lambda} & \frac{\partial F_N}{\partial \sigma_1} & \frac{\partial F_N}{\partial \sigma_2} & \frac{\partial F_N}{\partial \rho} \end{bmatrix} \begin{bmatrix} S \\ \delta \\ \kappa \\ \alpha \\ \lambda \\ \sigma_1 \\ \sigma_2 \\ \rho \end{bmatrix} + n$$

where

$$\frac{\partial F}{\partial S} = 1;$$

$$\frac{\partial F}{\partial \delta} = \frac{-1+e^{-\kappa T}}{\kappa};$$

$$\frac{\partial F}{\partial \kappa} = \frac{1}{4\kappa^4}(4e^{-\kappa T}T\kappa^2\rho\sigma_1\sigma_2 + 2\sigma_2^2 Te^{-2\kappa T}\kappa + 4e^{-\kappa T}T\alpha\kappa^3 - 4\delta Te^{-\kappa T}\kappa^3 + 4T\kappa^2\rho\sigma_1\sigma_2 - 4e^{-\kappa T}T\kappa^2\lambda - 4e^{-\kappa T}T\kappa\sigma_2^2 + 8e^{-\kappa T}\kappa\rho\sigma_1\sigma_2 + 3e^{-2\kappa T}\sigma_2^2 + 4e^{-\kappa T}\alpha\kappa^2 - 4\delta\kappa^2 e^{-\kappa T} - 4T\kappa^2\lambda - 4T\kappa\sigma_2^2 - 8\sigma_1\sigma_2\rho\kappa - 8e^{-\kappa T}\kappa\lambda - 12e^{-\kappa T}\sigma_2^2 - 4\alpha\kappa^2 + 4\delta\kappa^2 + 8\kappa\lambda + 9\sigma_2^2);$$

$$\frac{\partial F}{\partial \alpha} = \frac{-(\kappa T+e^{-\kappa T}-1)}{\kappa};$$

$$\frac{\partial F}{\partial \lambda} = \frac{\kappa T+e^{-\kappa T}-1}{\kappa^2};$$

$$\frac{\partial F}{\partial \sigma_1} = \frac{-\sigma_2\rho(\kappa T+e^{-\kappa T}-1)}{\kappa^2};$$

$$\frac{\partial F}{\partial \sigma_2} = -\frac{2T\kappa^2\rho\sigma_1+2e^{-\kappa T}\kappa\rho\sigma_1+e^{-2\kappa T}\sigma_2-2\kappa T\sigma_2-2\sigma_1\rho\kappa-4e^{-\kappa T}\sigma_2+3\sigma_2}{2\kappa^3};$$

and

$$\frac{\partial F}{\partial \rho} = \frac{-\sigma_1\sigma_2(\kappa T+e^{-\kappa T}-1)}{\kappa^2}$$

After iteration, not only the state variables(S and δ), but also unknown parameters (κ, α, λ, σ_1, σ_2 and ρ) can be tracked.

3.2.1 Data and Assumptions

In this model, in order to simplify the complex state space model, the weekends and other non-trading days can be ignored, which means that the trading days are considered to be continuous. Based on the efficient market hypothesis, powerful information from the non-trading days (e.g. weekends and Christmas holiday) can be immediately reflected in price after the non-trading day. Hence, this is a reasonable and popular assumption in the financial world. The next assumption is that each futures contract is immediately executed on the first day they mature. This is an assumption about the length of the maturity. Based on this assumption, the length of the maturity can be measured more accurately. The third assumption is a measurement of the level of the cost of financing. In this section, the interest rate is assumed to be equal to 2% per year. Furthermore, to simplify the calculation, based on the non-arbitrage assumption, the drift μ in the aforementioned model is replaced by the interest rate r, which means that the drift μ is set as a constant in the implementation. Additionally in this section, instead of a single average T, the dynamics of the T with the time passing is added to the model, which is expected to provide a better fit for the data.

As has been explained in the previous sections, since the spot price of crude oil is set as an unobservable state variable, the observable futures prices are the only collected data when the model is implemented. In this section, the data of futures prices of WTI crude oil is collected from Bloomberg. Specifically, since the extended Kalman filter is used first in this domain, in this section, ten futures contracts will be used in the Two-Factor model with stochastic parameters. Their maturities are between one month to ten months, respectively, and the test period is the year from 2^{nd} November 2010 to 31^{st} October 2011. To be more specific, the ten futures contracts matured between December 2011 and September 2012.

3.2.2 The Empirical Results of The Original Two-Factor Model with Stochastic Parameters

Figure 51 and Figure 52 exhibit the estimated unobservable spot price of WTI crude oil and the estimated convenience yield of WTI crude oil by the extended Kalman filter, respectively. On the one hand, the estimated spot price of crude oil is described in Figure 51. In Figure 51, it is not hard to see that, in the test period, the estimated spot price of crude oil fluctuates around 90 dollars per barrel at the beginning of the test period, and then, it rebounds and keeps moving up to about 115 dollars per barrel, which is the peak of the entire test period. After the peak, the estimated spot price of crude oil experiences a long and continued decrease until it reaches about 75 dollars per barrel, which is the valley. At the end of the test period, the estimated spot price rebounds again to about 95 dollars per barrel. The downward jump in the second half of the test period can be explained by the International Energy Agent's announcement of their release of 60 million barrels of their strategic petroleum reserve at the end of June 2011. On the other hand, Figure 52 shows the estimated convenience yield of WTI crude oil estimated by the extended Kalman filter in the same test period. The situation can be expressed as follows: at the beginning of the period, the estimated convenience yield increased from 0.02 to 0.03, and then kept decreasing to about -0.03.

In addition, the six estimated stochastic parameters are exhibited within Figures 53-58. To be more specific, the estimated dynamic parameters are roughly described as follows: first, κ increases stably from about 0.6 to about 0.7 in the entire test period. Second, α fluctuated above zero in the first half of the test period, then the estimated α drops and fluctuates under zero. Third, the estimated λ fluctuates around zero, and its interval of fluctuation is between -0.04 and 0.04. In addition, the estimated σ_1 moves upwards in the first half of the entire testing period, and then keeps moving down in the rest of the testing period. On the other hand, the estimated σ_2 experiences an opposite situation, it fluctuates around 0.2 in the first half of the testing period, then increases until at the end of the testing period it is about 0.3. Last, the movement of the estimated ρ looks like the estimated κ: it keeps moving up from 0.5 to about 0.65. Overall, first, the degree of mean reversion of the convenience yield of WTI crude oil is not very high; second, the long-term return investment on the convenience yield is positive for most testing days during the test period; last, the risk on the spot price of WTI crude oil is higher than the risk on the convenience yield of WTI crude oil.

The model was run with 10 futures contracts. However, drawing 10 coloured lines in a picture would make the picture untidy. Hence, in order to compare this method of estimation with the traditional method of estimation, only five futures were chosen, and their maturities are the same as the five futures chosen in the section 2.3. To be more specific, their maturities are one month, two months, four months, seven months and ten months respectively, and the test period is the year from 2^{nd} November 2010 to 31^{st} October 2011. To be more specific, the five futures contracts would mature at December 2011, January 2012, March 2012, June 2012 and September 2012 respectively. Next, the five chosen futures contracts are shown together in Figure 59 and Figure 60 in order to see whether there are significant differences between the observed prices of WTI crude oil futures contracts and the estimated prices of WTI crude oil futures contracts in the Two-Factor model with stochastic parameters. To be more specific, the real observed futures prices of the five chosen futures contracts are shown in Figure 59. The estimated model futures prices of the five chosen futures contracts, which are estimated from the Two-Factor model with stochastic parameters, are shown in Figure 60. In order to see the trend clearly, in Figure 60 only 26 points are chosen in the figure, which were picked the first trading day of each ten trading days. Overall, there are no significant differences between the two figures. On the one hand, WTI crude oil did not follow a strict backwardation or a strict contango during the test period; on the other hand, it is not hard to see that the temporary trend of the backwardation and the contango is more easily found when WTI crude oil rapidly price-inverses, while all futures prices seem to be close, when the price of WTI crude oil has a clear trend of increase or decrease.

In order to observe the effectiveness of all ten futures contracts, the comparison of each WTI crude oil futures in the Two factor model with stochastic parameters are shown within Figure 64-73. The rank of the title of each picture corresponds to the rank of its maturity. For example, the title 'The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters' and the title 'The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters' correspond to the futures contracts which matured in December 2011 and January 2012 respectively, and so forth. It is not hard to see that the Two-Factor model with stochastic parameters is useful in pricing a futures contract with a particular maturity because all ten pictures are showing that the estimated model price of each futures contract is really close to the real observed price of the futures contract.

Last, the forward curves from the Two-Factor model with stochastic parameters are shown within Figures 61-63. First, on the 50^{th} trading day, the model estimated prices of the Two-Factor model with stochastic parameters cross with the observed prices. To be more specific, with the increase in the term to maturity, the estimated prices are first of all lower than the observed prices, and then they are higher than the observed price. Second, similar to the 50^{th} trading day, on the 100^{th} trading day, there are two crossovers between the estimated model prices and the observed prices. Specifically, the situation is the same as the description for the 50^{th} trading day, but the difference between those two prices is smaller, when the term to maturity is short. Last, on the 200^{th} trading day, the model estimated prices of the Two-Factor model with stochastic parameters are lower than the observed prices. To be more specific, in Figure 63, the difference between the estimated prices and the observed prices is quite small, when the term to maturity is short, while the difference between the estimated prices and the observed prices is larger, when the term to maturity is longer. Overall, as expected, the estimated model prices from the Two-Factor model with stochastic parameters are close with the observed prices.

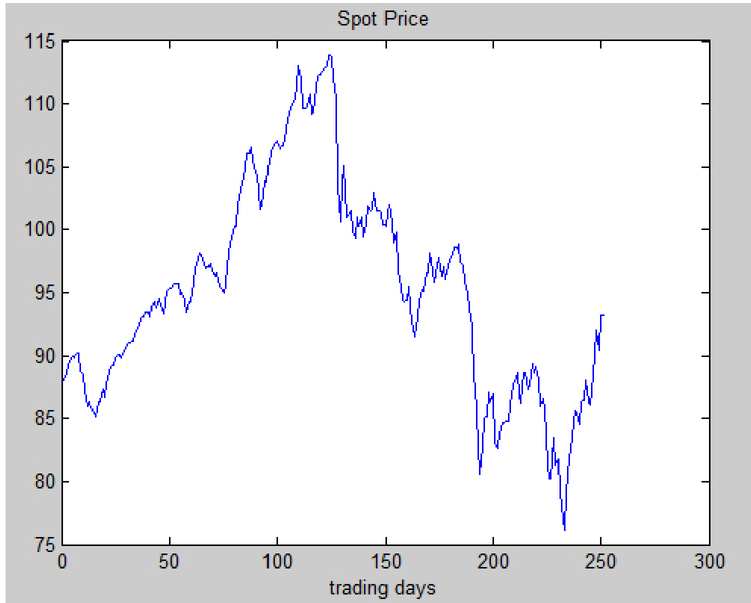


Figure 51: Estimated Spot Price from The Two-Factor Model With Stochastic Parameters

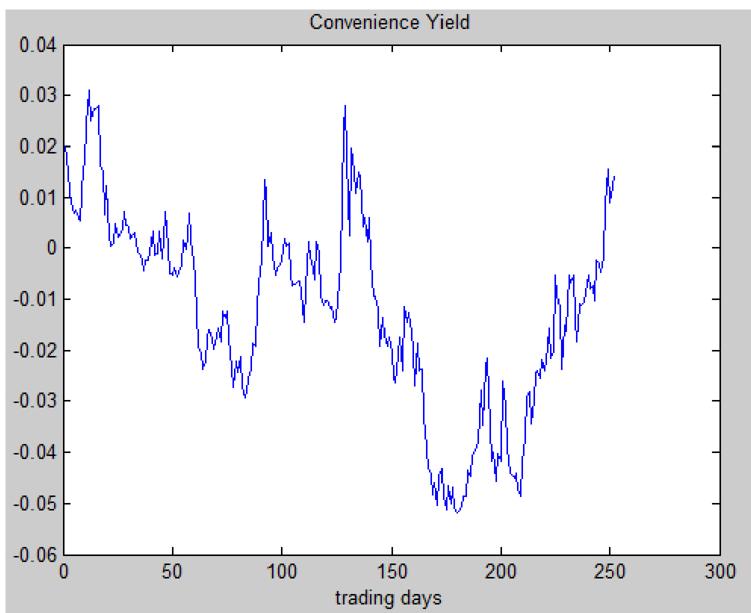
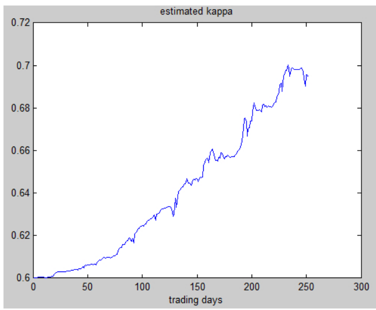


Figure 52: Estimated Convenience Yield from The Two-Factor Model With Stochastic Parameters



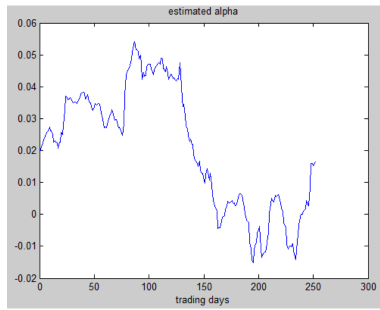
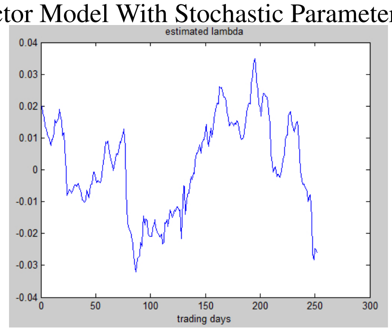


Figure 53: Estimated κ from The Two-Factor Model With Stochastic Parameters

Figure 54: Estimated α from The Two-Factor Model With Stochastic Parameters



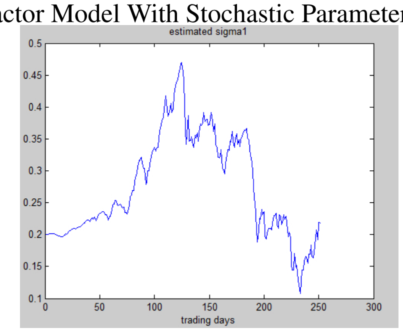
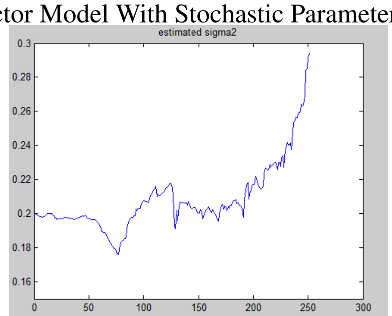


Figure 55: Estimated λ from The Two-Factor Model With Stochastic Parameters

Figure 56: Estimated σ_1 from The Two-Factor Model With Stochastic Parameters



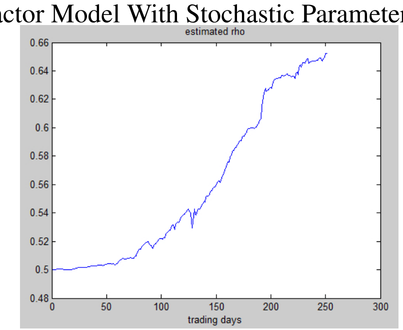


Figure 57: Estimated σ_2 from The Two-Factor Model With Stochastic Parameters

Figure 58: Estimated ρ from The Two-Factor Model With Stochastic Parameters

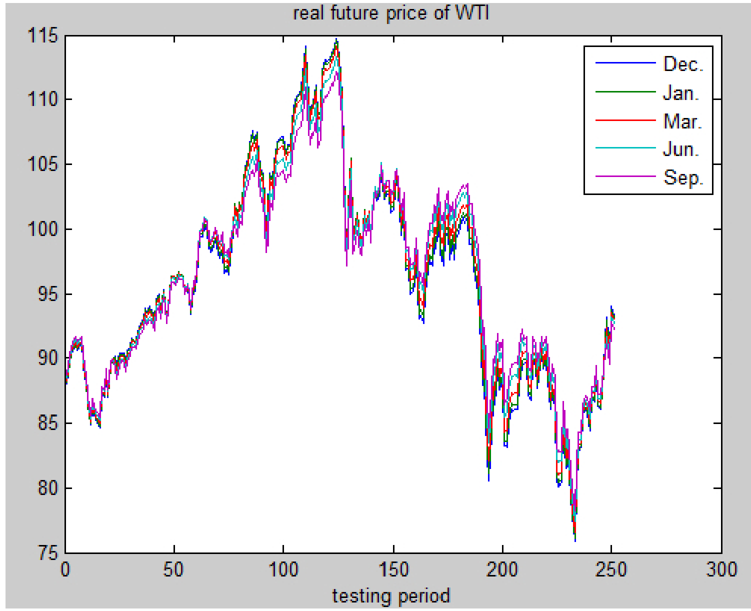


Figure 59: Observed Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model With Stochastic Parameters

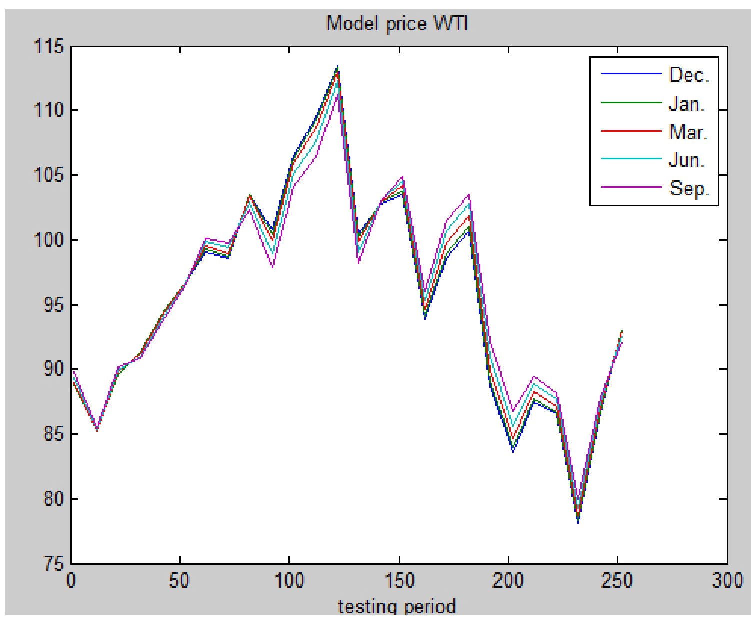


Figure 60: Estimated Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model With Stochastic Parameters

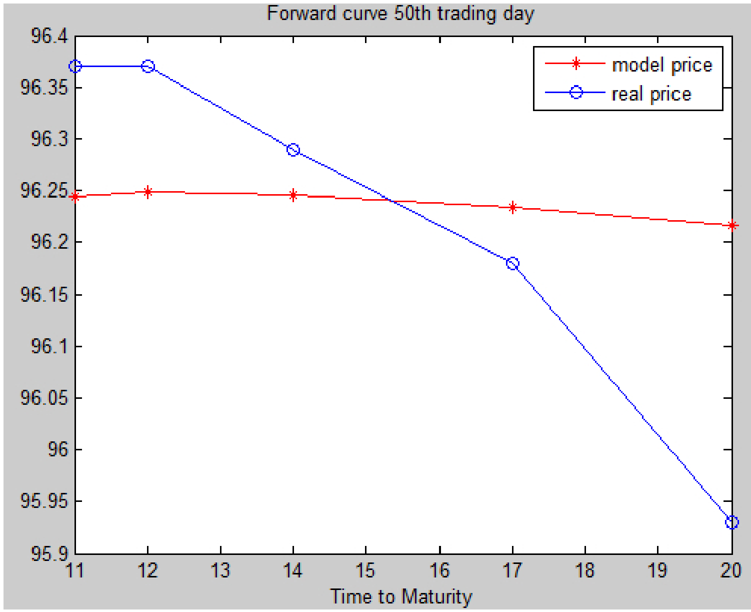


Figure 61: Forward Curves from The Two-Factor Model With Stochastic Parameters on The 50th day

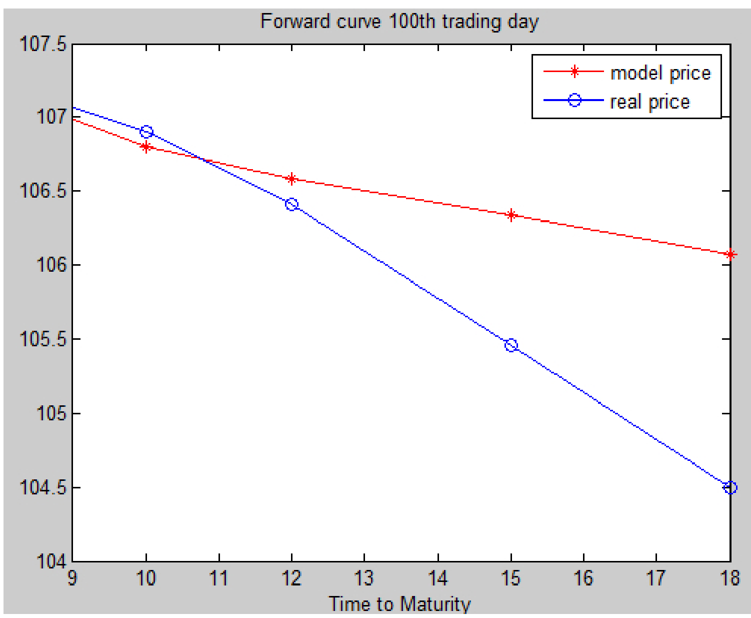


Figure 62: Forward Curves from The Two-Factor Model With Stochastic Parameters on The 100th day

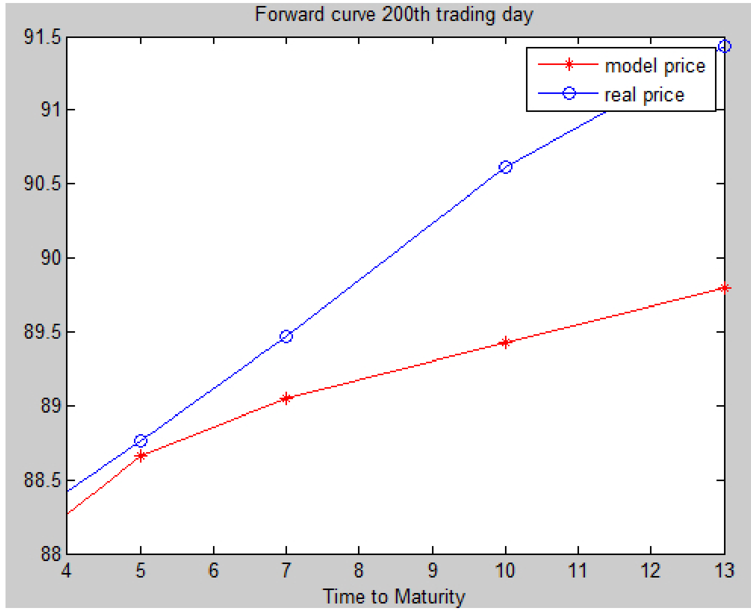


Figure 63: Forward Curves from The Two-Factor Model With Stochastic Parameters on The 200th day

4 A New Three-Factor Model and a New Four-Factor model based on the Schwartz (1997) model system

Nowadays, with the huge fluctuation in crude oil markets, measuring risk can be seen as the most important thing when people invest in crude oil, which makes calculating the volatility of the spot price of crude oil more crucial than ever. The volatility of the underlying asset of the futures contracts can be calculated by using different ways with the historical data, e.g. the exponentially weighted moving average model, GARCH model and other more complicated models and so forth, which means that the level of the volatility of the spot price is normally considered as a time varying variable. On the other hand, estimating stochastic volatility of the underlying asset plays one of the most important roles in the domain of mathematical finance. However, the original Two-Factor model estimates the volatility of the spot price of the underlying asset as a

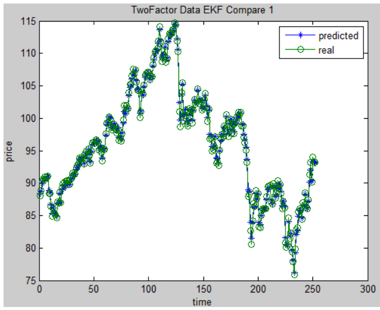


Figure 64: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters

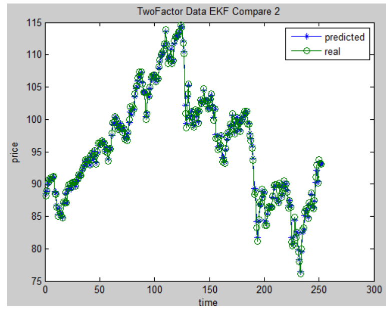


Figure 65: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters

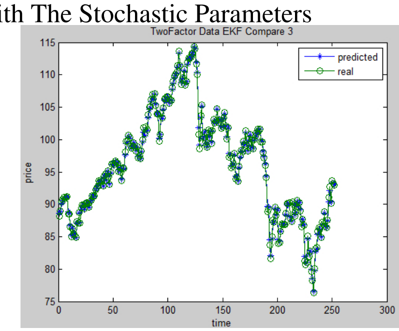


Figure 66: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters

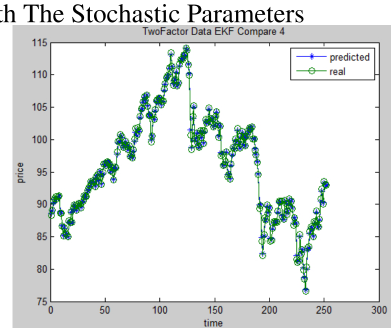


Figure 67: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters

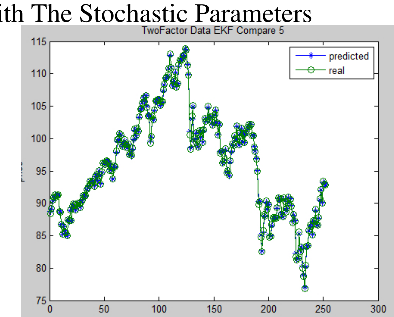


Figure 68: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters

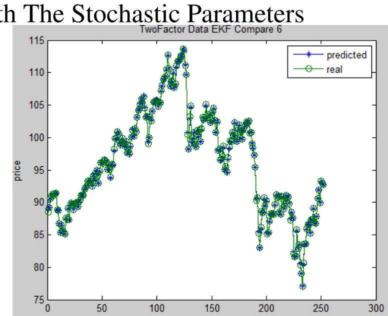
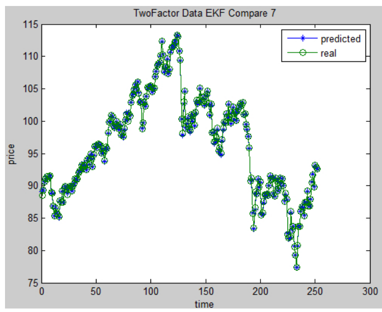


Figure 69: The Predicted and Observed Futures Prices for The sixth Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters



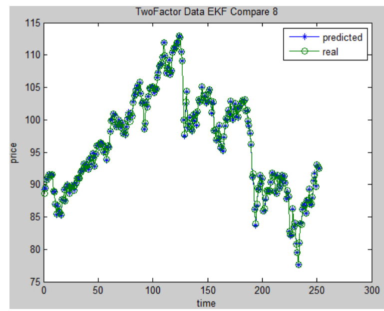
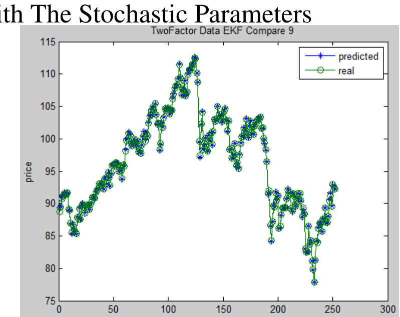


Figure 70: The Predicted and Observed Futures Prices for The Seventh Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters

Figure 71: The Predicted and Observed Futures Prices for The Eighth Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters



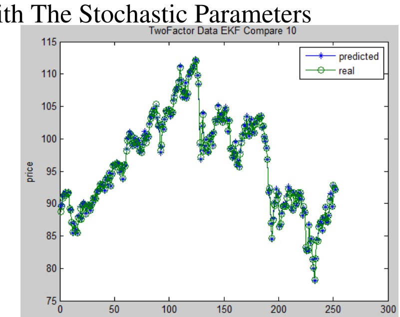


Figure 72: The Predicted and Observed Futures Prices for The Ninth Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters

Figure 73: The Predicted and Observed Futures Prices for The Tenth Testing Futures Contract based on The Two-Factor Model with The Stochastic Parameters

parameter. Hence, in this thesis the original Two-Factor and the original Three-Factor Schwartz (1997) models are developed into a new Three-Factor model and a new Four-Factor model respectively with consideration of a stochastic volatility of the spot price of WTI crude oil. In this section, the two new models for pricing a futures contract will be introduced in detail.

Adding the volatility of the spot price of the underlying asset to the original Schwartz (1997) model system is a crucial contribution in this thesis, since the stochastic volatility of spot price of WTI crude oil is a crucial index for participants in crude oil markets, because most participants make their decisions regarding investment or/and production based on the volatility of the spot price of crude oil. To be more specific, the volatility of the spot price of crude oil is assumed to be the mean reversion in this thesis, which means that it goes up, then must come down eventually. Because of this, risk-averters can temporarily leave crude oil markets when the volatility of the spot price of crude oil stays at a relatively high level, while the risk-seekers are attracted by the potentially higher return.

4.1 A New Three-Factor Model and Its Empirical Results

4.1.1 A New Three-Factor Model

Up to now, the Two-Factor model is still the most popular model. However, since the volatility of the spot price has been considered and accepted to be stochastic but not a simple constant, the traditional and original Two-Factor model might need to be developed in order to show a stochastic volatility of the spot price. Based on the original Two-Factor model, the new Three-Factor model can be described as follows The three stochastic processes are:

$$dS = (r - \delta)Sdt + \sigma_1 SdW_1$$

$$d\delta = \kappa_1(\alpha - \delta)dt + \sigma_2 dW_2$$

$$d\sigma_1 = \kappa_2(\theta - \sigma_1)dt + \sigma_3 dW_3$$

with

$$dW_1 dW_2 = \rho_{12} dt$$

$$dW_1 dW_3 = \rho_{13} dt$$

$$dW_2 dW_3 = \rho_{23} dt$$

where r, α, and θ represented the long-term return investment on oil, the long-term convenience yield, and the volatility of the spot price, respectively; κ_1, and κ_2 are the coefficient of reverting in the stochastic processes, respectively; similarly, σ_1, σ_2, and σ_3 are the volatilities of spot price, convenience yield, and the volatility of the spot price, respectively; dW_1, dW_2, and dW_3 are increments of Brownian motions and following normal distribution $N(0, dt^{1/2})$ with $dW_i dW_j = \rho_{ij} dt$, where ρ_{ij} is the correlation coefficient between the two stochastic processes i and j.

With consideration of the three stochastic processes, the futures price formula must satisfy the following partial differential equation:

$$\tfrac{1}{2}\sigma_1^2 S^2 F_{SS} + \tfrac{1}{2}\sigma_2^2 F_{\delta\delta} + \tfrac{1}{2}\sigma_3^2 F_{\sigma_1\sigma_1} + S\sigma_1\sigma_2\rho_{12}F_{S\delta} + S\sigma_1\sigma_3\rho_{13}F_{S\sigma_1} +$$
$$\sigma_2\sigma_3\rho_{23}F_{\delta\sigma_1} + (r-\delta)SF_S + \kappa_1(\alpha-\delta)F_\delta + \kappa_2(\theta-\sigma_1)F_{\sigma_1} - F_\tau = 0$$

Similar to previous studies[6], guess the solution of the PDE as:

$$F = Sexp(A + B\delta + C\sigma_1)$$

then, the PDE can be reduced as three ODEs as follows:

$$-1 - \kappa_1 B = \tfrac{\partial B}{\partial \tau}$$

$$\sigma_2\rho_{12}B + \sigma_3\rho_{13}C - \kappa_2 C = \tfrac{\partial C}{\partial \tau}$$

$$\tfrac{1}{2}\sigma_2^2 B^2 + \tfrac{1}{2}\sigma_3^2 C^2 + \sigma_2\sigma_3\rho_{23}BC + r + \kappa_1\alpha B + \kappa_2\theta C = \tfrac{\partial A}{\partial \tau}$$

B, C and A then can be easily solved. Last, the solution of the PDE can be obtained.

$$B = \tfrac{e^{-\kappa_1\tau}-1}{\kappa_1}$$

[6]Heston(1993) guessed a solution for the PDE for the Two-Factor model

$$C = (\rho_{12}\sigma_1(-exp((-\rho_{13}\sigma_2+\kappa_2)\tau)/(-\rho_{13}\sigma_2+\kappa_2)+exp(-\tau(\rho_{13}\sigma_2+\kappa_1-\kappa_2))/(-\rho_{13}\sigma_2-\kappa_1+\kappa_2))/\kappa_1-\rho_{12}\sigma_1(-1/(-\rho_{13}\sigma_2+\kappa_2)+1/(-\rho_{13}\sigma_2-\kappa_1+\kappa_2))/\kappa_1)exp(-(-\rho_{13}\sigma_2+\kappa_2)\tau)$$

$$A = (1/2)((2((-\rho_{13}\rho_{23}+\rho_{12})\sigma_2-\rho_{23}(\kappa_1-\kappa_2)))\sigma_2\rho_{12}\kappa_1(\rho_{13}\sigma_2-\kappa_2)\sigma_1^2 exp(\tau(\rho_{13}\sigma_2-\kappa_1-\kappa_2))/(\rho_{13}\sigma_2-\kappa_1-\kappa_2)+\kappa_1^2\rho_{12}^2\sigma_1^2\sigma_2^2 exp(-(2(-\rho_{13}\sigma_2+\kappa_2))\tau)/(2\rho_{13}\sigma_2-2\kappa_2)-(2(\rho_{13}\sigma_2+\kappa_1-\kappa_2))\rho_{12}\kappa_1(\sigma_1(-\rho_{13}\rho_{23}+\rho_{12})\sigma_2^2+\kappa_2(\kappa_1\rho_{13}\theta+\rho_{23}\sigma_1)\sigma_2-\theta\kappa_1\kappa_2^2)\sigma_1 exp((\rho_{13}\sigma_2-\kappa_2)\tau)/(\rho_{13}\sigma_2-\kappa_2)-(2(\rho_{13}\sigma_2+\kappa_1-\kappa_2))(((2\rho_{12}\rho_{13}\rho_{23}-\rho_{12}^2-\rho_{13}^2)\sigma_1^2+\alpha\rho_{13}^2\kappa_1^2)\sigma_2^2+((\kappa_1-2\kappa_2)(\rho_{12}\rho_{23}-\rho_{13})\sigma_1^2-\rho_{12}\rho_{13}\theta\kappa_1\kappa_2\sigma_1+\alpha\rho_{13}\kappa_1^2(\kappa_1-2\kappa_2))\sigma_2-\kappa_2((-\kappa_1+\kappa_2)\sigma_1^2-\kappa_2\theta\rho_{12}\sigma_1\kappa_1+\alpha\kappa_1^2(\kappa_1-\kappa_2)))(\rho_{13}\sigma_2-\kappa_2)exp(-\kappa_1\tau)/\kappa_1-(1/2)((-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_2^2-(2(\kappa_1-\kappa_2))(\rho_{12}\rho_{23}-\rho_{13})\sigma_2+(\kappa_1-\kappa_2)^2)(\rho_{13}\sigma_2-\kappa_2)^2\sigma_1^2 exp(-2\kappa_1\tau)/\kappa_1-2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2(((-(1/2)\rho_{12}^2-(1/2)\rho_{13}^2+\rho_{12}\rho_{13}\rho_{23})\sigma_1^2+\alpha\rho_{13}^2\kappa_1^2)\sigma_2^2-(2(((1/2)\rho_{12}\rho_{23}-(1/2)\rho_{13})\sigma_1^2+(1/2)\rho_{12}\rho_{13}\theta\kappa_1\sigma_1+\alpha\rho_{13}\kappa_1^2))\kappa_2\sigma_2+(\theta\kappa_1\rho_{12}\sigma_1+\kappa_1^2\alpha-(1/2)\sigma_1^2)\kappa_2^2)\tau)/(\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2)+r\tau-(1/2)((2((-\rho_{13}\rho_{23}+\rho_{12})\sigma_2-\rho_{23}(\kappa_1-\kappa_2)))\sigma_2\rho_{12}\kappa_1(\rho_{13}\sigma_2-\kappa_2)\sigma_1^2/(\rho_{13}\sigma_2-\kappa_1-\kappa_2)+\kappa_1^2\rho_{12}^2\sigma_1^2\sigma_2^2/(2\rho_{13}\sigma_2-2\kappa_2)-(2(\rho_{13}\sigma_2+\kappa_1-\kappa_2))\rho_{12}\kappa_1(\sigma_1(-\rho_{13}\rho_{23}+\rho_{12})\sigma_2^2+\kappa_2(\kappa_1\rho_{13}\theta+\rho_{23}\sigma_1)\sigma_2-\theta\kappa_1\kappa_2^2)\sigma_1/(\rho_{13}\sigma_2-\kappa_2)-(2(\rho_{13}\sigma_2+\kappa_1-\kappa_2))(((2\rho_{12}\rho_{13}\rho_{23}-\rho_{12}^2-\rho_{13}^2)\sigma_1^2+\alpha\rho_{13}^2\kappa_1^2)\sigma_2^2+((\kappa_1-2\kappa_2)(\rho_{12}\rho_{23}-\rho_{13})\sigma_1^2-\rho_{12}\rho_{13}\theta\kappa_1\kappa_2\sigma_1+\alpha\rho_{13}\kappa_1^2(\kappa_1-2\kappa_2))\sigma_2-\kappa_2((-\kappa_1+\kappa_2)\sigma_1^2-\kappa_2\theta\rho_{12}\sigma_1\kappa_1+\alpha\kappa_1^2(\kappa_1-\kappa_2)))(\rho_{13}\sigma_2-\kappa_2)/\kappa_1-(1/2)((-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_2^2-(2(\kappa_1-\kappa_2))(\rho_{12}\rho_{23}-\rho_{13})\sigma_2+(\kappa_1-\kappa_2)^2)(\rho_{13}\sigma_2-\kappa_2)^2\sigma_1^2/\kappa_1)/(\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2)(2((-\rho_{13}\rho_{23}+\rho_{12})\sigma_2-\rho_{23}(\kappa_1-\kappa_2))\sigma_2\rho_{12}\kappa_1(\rho_{13}\sigma_2-\kappa_2)\sigma_1^2/(\rho_{13}\sigma_2-\kappa_1-\kappa_2)+\kappa_1^2\rho_{12}^2\sigma_1^2\sigma_2^2/(2\rho_{13}\sigma_2-2\kappa_2)-(2(\rho_{13}\sigma_2+\kappa_1-\kappa_2))\rho_{12}\kappa_1(\sigma_1(-\rho_{13}\rho_{23}+\rho_{12})\sigma_2^2+\kappa_2(\kappa_1\rho_{13}\theta+\rho_{23}\sigma_1)\sigma_2-\theta\kappa_1\kappa_2^2)\sigma_1/(\rho_{13}\sigma_2-\kappa_2)-(2(\rho_{13}\sigma_2\kappa_1-\kappa_2))(((2\rho_{12}\rho_{13}\rho_{23}-\rho_{12}^2-\rho_{13}^2)\sigma_1^2+\alpha\rho_{13}^2\kappa_1^2)\sigma_2^2+((\kappa_1-2\kappa_2)(\rho_{12}\rho_{23}-\rho_{13})\sigma_1^2-\rho_{12}\rho_{13}\theta\kappa_1\kappa_2\sigma_1+\alpha\rho_{13}\kappa_1^2(\kappa_1-2\kappa_2))\sigma_2-\kappa_2((-\kappa_1+\kappa_2)\sigma_1^2-\kappa_2\theta\rho_{12}\sigma_1\kappa_1+\alpha\kappa_1^2(\kappa_1-\kappa_2)))(\rho_{13}\sigma_2-\kappa_2)/\kappa_1-(1/2)((-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_2^2-(2(\kappa_1-\kappa_2))(\rho_{12}\rho_{23}-\rho_{13})\sigma_2+(\kappa_1-\kappa_2)^2)(\rho_{13}\sigma_2-\kappa_2)^2\sigma_1^2/\kappa_1)$$

Then, C and A can be simplified as:

$C =$

$$(\frac{\rho_{12}\sigma_1(-\frac{e^{(-\rho_{13}\sigma_2+\kappa_2)\tau}}{\kappa_2-\rho_{13}\sigma_2}-\frac{e^{-\tau(\rho_{13}\sigma_2+\kappa_1-\kappa_2)}}{\kappa_2-\rho_{13}\sigma_2-\kappa_1})-\rho_{12}\sigma_1(-\frac{1}{\kappa_2-\rho_{13}\sigma_2}+\frac{1}{\kappa_2-\rho_{13}\sigma_2-\kappa_1})}{\kappa_1})e^{-(\kappa_2-\rho_{13}\sigma_2)\tau}$$

and $A =$

$$\frac{1}{2}\frac{(2((-\rho_{13}\rho_{23}+\rho_{12})\sigma_2-\rho_{23}(\kappa_1-\kappa_2))\sigma_2\rho_{12}\kappa_1(\rho_{13}\sigma_2-\kappa_2)\sigma_1^2\frac{e^{\tau(\rho_{13}\sigma_2-\kappa_1-\kappa_2)}}{\rho_{13}\sigma_2-\kappa_1-\kappa_2}}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}+$$

$$\frac{\kappa_1^2\rho_{12}^2\sigma_1^2\sigma_2^2\frac{e^{-2\tau(\kappa_2-\rho_{13}\sigma_2)}}{2\rho_{13}\sigma_2-2\kappa_2}}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}-$$

$$\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)\rho_{12}\kappa_1(\sigma_1(-\rho_{13}\rho_{23}+\rho_{12})\sigma_2^2+\kappa_2(\kappa_1\rho_{13}\theta+\rho_{23}\sigma_1)\sigma_2-\theta\kappa_1\kappa_2^2)\sigma_1\frac{e^{(\rho_{13}\sigma_2-\kappa_2)\tau}}{\rho_{13}\sigma_2-\kappa_2}}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}-$$

$$\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)(((2\rho_{12}\rho_{13}\rho_{23}-\rho_{12}^2-\rho_{13}^2)\sigma_1^2+\alpha\rho_{13}^2\kappa_1^2)\sigma_2^2)}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}+$$

$$\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)(((\kappa_1-2\kappa_2)(\rho_{12}\rho_{23}-\rho_{13})\sigma_1^2-\rho_{12}\rho_{13}\theta\kappa_1\kappa_2\sigma_1+\alpha\rho_{13}\kappa_1^2(\kappa_1-2\kappa_2))\sigma_2)}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}-$$

$$\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)(\kappa_2((-\kappa_1+\kappa_2)\sigma_1^2-\kappa_2\theta\rho_{12}\sigma_1\kappa_1+\alpha\kappa_1^2(\kappa_1-\kappa_2)))(\rho_{13}\sigma_2-\kappa_2)\frac{e^{-\kappa_1\tau}}{\kappa_1}}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}-$$

$$\frac{\frac{1}{2}((-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_2^2-2(\kappa_1-\kappa_2)(\rho_{12}\rho_{23}-\rho_{13})\sigma_2+(\kappa_1-\kappa_2)^2)(\rho_{13}\sigma_2-\kappa_2)^2\sigma_1^2\frac{e^{-2\kappa_1\tau}}{\kappa_1}}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}-$$

$$\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2(((-\frac{1}{2}\rho_{12}^2-\frac{1}{2}\rho_{13}^2+\rho_{12}\rho_{13}\rho_{23})\sigma_1^2+\alpha\rho_{13}^2\kappa_1^2)\sigma_2^2)}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}-$$

$$\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2(2((\frac{1}{2}\rho_{12}\rho_{23}-\frac{1}{2}\rho_{13})\sigma_1^2+\frac{1}{2}\rho_{12}\rho_{13}\theta\kappa_1\sigma_1+\alpha\rho_{13}\kappa_1^2)\kappa_2\sigma_2)}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}+$$

$$\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2((\theta\kappa_1\rho_{12}\sigma_1+\kappa_1^2\alpha-\frac{1}{2}\sigma_1^2)\kappa_2^2)\tau)}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}+r\tau-$$

$$\frac{1}{2}\frac{(2((-\rho_{13}\rho_{23}+\rho_{12})\sigma_2-\rho_{23}(\kappa_1-\kappa_2))\frac{\sigma_2\rho_{12}\kappa_1(\rho_{13}\sigma_2-\kappa_2)\sigma_1^2}{\rho_{13}\sigma_2-\kappa_1-\kappa_2}}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}+$$

$$\frac{\frac{\kappa_1^2\rho_{12}^2\sigma_1^2\sigma_2^2}{2\rho_{13}\sigma_2-2\kappa_2}-\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)\rho_{12}\kappa_1(\sigma_1(-\rho_{13}\rho_{23}+\rho_{12})\sigma_2^2+\kappa_2(\kappa_1\rho_{13}\theta+\rho_{23}\sigma_1)\sigma_2-\theta\kappa_1\kappa_2^2)\sigma_1}{\rho_{13}\sigma_2-\kappa_2}}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}-$$

$$\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)(((2\rho_{12}\rho_{13}\rho_{23}-\rho_{12}^2-\rho_{13}^2)\sigma_1^2+\alpha\rho_{13}^2\kappa_1^2)\sigma_2^2)}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}+$$

$$\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)(((\kappa_1-2\kappa_2)(\rho_{12}\rho_{23}-\rho_{13})\sigma_1^2-\rho_{12}\rho_{13}\theta\kappa_1\kappa_2\sigma_1+\alpha\rho_{13}\kappa_1^2(\kappa_1-2\kappa_2))\sigma_2)}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}-$$

$$\frac{2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)(\kappa_2((-\kappa_1+\kappa_2)\sigma_1^2-\kappa_2\theta\rho_{12}\sigma_1\kappa_1+\alpha\kappa_1^2(\kappa_1-\kappa_2)))\frac{\rho_{13}\sigma_2-\kappa_2}{\kappa_1}}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}-$$

$$\frac{\frac{1}{2}((-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_2^2-2(\kappa_1-\kappa_2)(\rho_{12}\rho_{23}-\rho_{13})\sigma_2+(\kappa_1-\kappa_2)^2)(\rho_{13}\sigma_2-\kappa_2)^2\frac{\sigma_1^2}{\kappa_1})}{\kappa_1^2(\rho_{13}\sigma_2-\kappa_2)^2(\rho_{13}\sigma_2+\kappa_1-\kappa_2)^2}(2((-\rho_{13}\rho_{23}+$$

$$\rho_{12})\sigma_2-\rho_{23}(\kappa_1-\kappa_2))\sigma_2\rho_{12}\kappa_1(\rho_{13}\sigma_2-\kappa_2)\frac{\sigma_1^2}{\rho_{13}\sigma_2-\kappa_1-\kappa_2}+\frac{\kappa_1^2\rho_{12}^2\sigma_1^2\sigma_2^2}{2\rho_{13}\sigma_2-2\kappa_2}-2(\rho_{13}\sigma_2+$$

$$\kappa_1-\kappa_2)\rho_{12}\kappa_1(\sigma_1(-\rho_{13}\rho_{23}+\rho_{12})\sigma_2^2+\kappa_2(\kappa_1\rho_{13}\theta+\rho_{23}\sigma_1)\sigma_2-\theta\kappa_1\kappa_2^2)\frac{\sigma_1}{\rho_{13}\sigma_2-\kappa_2}-$$

$$2(\rho_{13}\sigma_2\kappa_1-\kappa_2)(((2\rho_{12}\rho_{13}\rho_{23}-\rho_{12}^2-\rho_{13}^2)\sigma_1^2+\alpha\rho_{13}^2\kappa_1^2)\sigma_2^2+((\kappa_1-2\kappa_2)(\rho_{12}\rho_{23}-$$

$$\rho_{13})\sigma_1^2-\rho_{12}\rho_{13}\theta\kappa_1\kappa_2\sigma_1+\alpha\rho_{13}\kappa_1^2(\kappa_1-2\kappa_2))\sigma_2-\kappa_2((-\kappa_1+\kappa_2)\sigma_1^2-$$

$$\kappa_2\theta\rho_{12}\sigma_1\kappa_1+\alpha\kappa_1^2(\kappa_1-\kappa_2)))\frac{\rho_{13}\sigma_2-\kappa_2}{\kappa_1}-\frac{1}{2}((-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_2^2-2(\kappa_1-$$

$$\kappa_2)(\rho_{12}\rho_{23}-\rho_{13})\sigma_2+(\kappa_1-\kappa_2)^2)(\rho_{13}\sigma_2-\kappa_2)^2\frac{\sigma_1^2}{\kappa_1})$$

4.1.2 Data and Assumptions

In this section, in order to simplify the complex state space model, the weekends and other non-trading days can be ignored, which means that the trading days are considered to be continuous. As in previous assumptions, powerful information from the non-trading days (e.g. weekends and Christmas holiday) can be immediately reflected in price after the non-trading day. Hence, this is a reasonable and popular assumption in the financial world. The next assumption is that each futures contract is immediately executed on the first day when they mature. This is an assumption about length of the maturity. Based on this assumption, the length of the maturity can be measured more accurately. The third assumption is a measurement of the level of the cost of financing. The interest rate is assumed to be equal to 2% per year in this section. Furthermore, to simplify the calculation, based on the non-arbitrage assumption, the drift μ in the aforementioned model is replaced by the interest rate r, which means that the drift μ is set as a constant in the implementation. In addition, in this section, instead of a single average T, the dynamics of the T with the time passing is added to the model, which is expected to get a better fit for the data.

As has been explained in the previous sections, when the spot price of crude oil is set as an unobservable state variable, the observable futures prices are the only data collected when the model is implemented. In this section, the data of futures prices of WTI crude oil is collected from Bloomberg. Specifically, in this section, twelve futures contracts will be used, their maturities are from February 2013 to January 2014, respectively, and the test period is the entire year of 2012.

4.1.3 The Empirical Results of The New Three-Factor Model

The estimated state variables of the new Three-Factor model are shown within Figures 74-76. To be specific, Figure 74 exhibits the estimated spot price of WTI crude oil, which fluctuates between about 120 dollars and about 80 dollars per barrel in the test period. Overall, the estimated spot price shows a downward trend during the test period. It fluctuates over 100 dollars per barrel and reaches the peak of about 120 dollars per barrel at the beginning of the test period, but then directly plummets to about 80 dollars per barrel, which is the valley in the entire test period. In the second half of the test period, it rebounds to about 100 dollars per barrel and then decreases to about 85 dollars per barrel again. Figure 75 shows the estimated convenience yield of the new Three-Factor model. Overall, the estimated convenience yield also shows a downward trend in the entire test period. Specifically, the estimated convenience yield of WTI crude oil rapidly increases from 0.02 to 0.2, which is the peak in the entire test period. Then, after the peak, the estimated convenience yield keeps moving down until it is about -0.05 in the middle of the test period. In the second half of the test period, the estimated convenience yield rebounds to about 0.08, and then drops to about -0.05 again. Figure 76 exhibits the estimated σ_1 from the new Three-Factor model. The estimated σ_1 fluctuates in an interval between 0.048 and 0.055. To be more specific, the estimated σ_1 increases to the peak at the beginning of the test period, which is about 0.055, and then drops to the valley, which is about 0.048, in the middle of the test period. After a slight rebound, the estimated σ_1 fluctuates around 0.05. As for the estimated parameters, first, the degree of mean reversion of the convenience yield of WTI crude oil is significantly higher than the degree of mean reversion of the volatility of the spot price of WTI crude oil; second, the long-term return investment on the convenience yield is positive during the test period; last, the risk on the convenience yield of WTI crude oil is lower than the risk on the volatility of the spot price of WTI crude oil.

Recalling the previously mentioned positive relationship between the estimated spot price and the estimated convenience yield of crude oil in section 2.3.3, the positive relationship is maintained between the estimated spot price and the estimated convenience yield of WTI crude oil in the new Three-Factor model. Indeed, not only are the estimated spot price and the estimated convenience yield of crude oil maintained, but all three state variables are also positively correlated.

The model was run with 12 futures contracts. However, drawing 12 coloured lines in a picture would make the picture untidy. Hence, only the first six futures contracts are shown in the following figures. The six chosen futures contracts are shown together in

Figure 77 and Figure 78 in order to see whether there are significant differences between the observed prices of WTI crude oil futures contracts and the estimated prices of WTI crude oil futures contracts in the new Three-Factor model. To be more specific, the real observed futures prices of the six chosen futures contracts are shown in Figure 77. Then the estimated model futures prices of the six chosen futures contracts, which are estimated from the new Three-Factor model, are shown in Figure 78. In order to see the trend clearly, in Figure 78 only 26 points are chosen in the figure, which were picked the first trading day of each ten trading days. Overall, there are no significant differences between the two figures. On the one hand, WTI crude oil did not follow a strict backwardation or a strict contango during the test period; on the other hand, it is not hard to see that the trend of the backwardation or the contango is more easily found when WTI crude oil fluctuates stably, while all futures prices seem to be close, when the price of WTI crude oil rapidly increases or rapidly decreases.

In order to observe the effectiveness of the Three-Factor model, the comparisons of each WTI crude oil futures in the Three-Factor model are shown within Figures 82-87. The rank of the title of each picture corresponds to the rank of its maturity. For example, the title 'The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The New Three-Factor Model' and the title 'The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The New Three-Factor Model' correspond to the futures contracts which matured in February 2013 and March 2013, respectively, and so forth. It is not hard to see that the new Three-Factor model is useful in pricing a futures contract with a particular maturity because all six pictures are showing that the estimated model price of each futures contract is really close to the real observed price of the futures contract.

Last, the forward curves from the new Three-Factor model are shown within Figures 79-81. First, on the 50^{th} trading day, the model estimated prices of the new Three-Factor model are close to the observed prices. To be more specific, the model estimated prices cross with the observed prices. With the increase in the term to maturity, the model prices are firstly higher and then lower than the observed prices. Second, similar to the 50^{th} trading day, on the 100^{th} trading day, the model estimated prices of the new Three-Factor model are close to the observed prices, and the crossover is similar to the situation shown on the 50^{th} trading day. Last, on the 200^{th} trading day, the model estimated prices of the new Three-Factor model also cross with the observed prices. However, with the increase in the term to maturity, the model prices are firstly lower and then higher than the observed prices.

Paremeters	Estimated result	Standard Error
κ_1	0.9959	0.01709
κ_2	0.3490	SE1
σ_2	0.2751	0.00264
σ_3	0.3908	SE2
ρ_{12}	2.7211e-05	2.0197e-05
ρ_{13}	0.8819	SE3
ρ_{23}	0.1722	0.4223
α	0.0539	0.00167
θ	-0.0259	0.0181

Note:$SE1 = \sqrt{-7.719765e-08 + ComputationalError1}$
$SE2 = \sqrt{-5.718160e-08 + ComputationalError2}$
$SE3 = \sqrt{-2.927875e-07 + ComputationalError3}$

Table 4.1.3: Estimated Parameters from The New Three-Factor Model

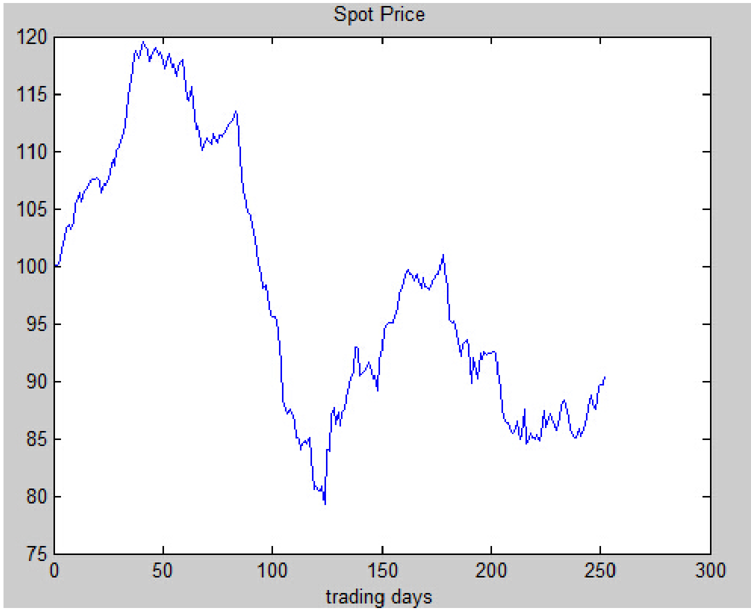


Figure 74: Estimated Spot Price from The New Three-Factor Model by The Extended Kalman Filter

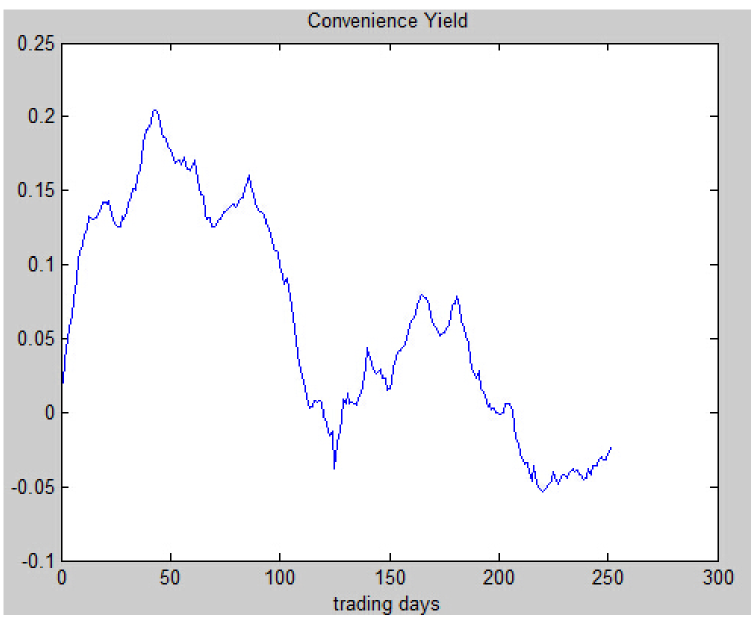


Figure 75: Estimated Convenience Yield from The New Three-Factor Model by The Extended Kalman Filter

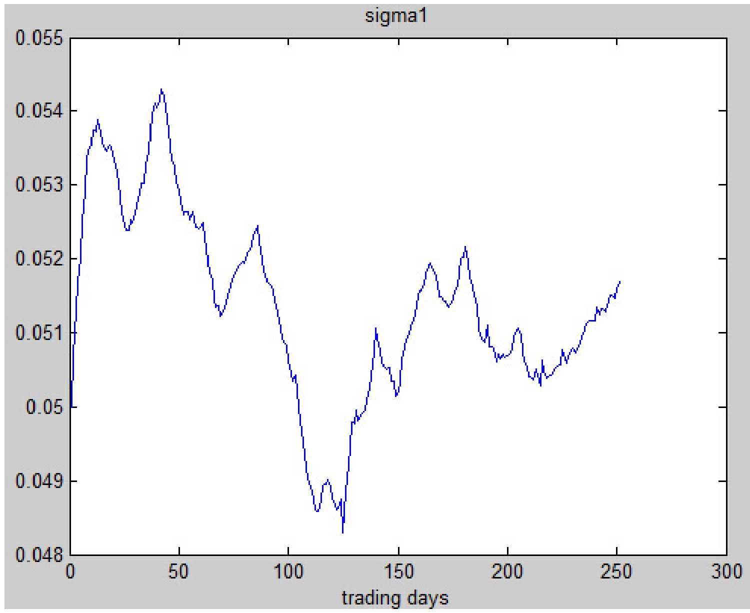


Figure 76: Estimated σ_1 from The New Three-Factor Model by The Extended Kalman Filter

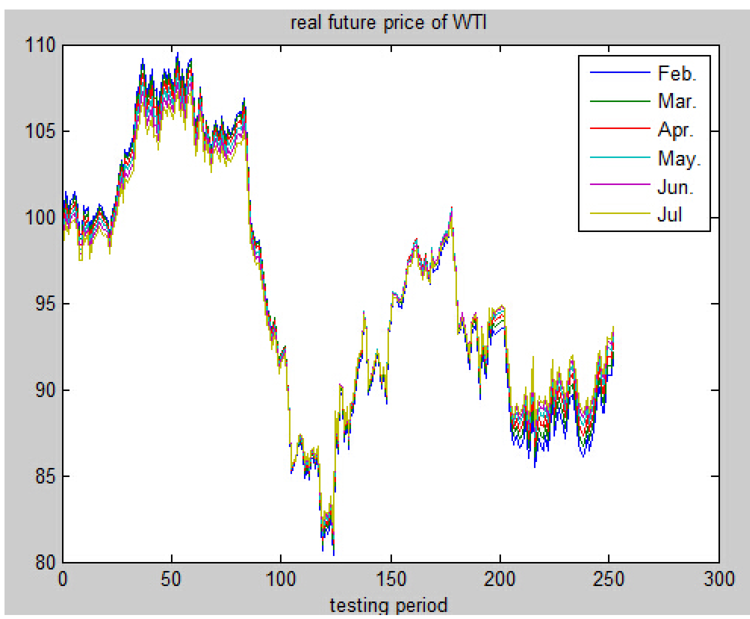


Figure 77: Observed Prices of WTI Futures Contracts with Different Maturities for The New Three-Factor Model

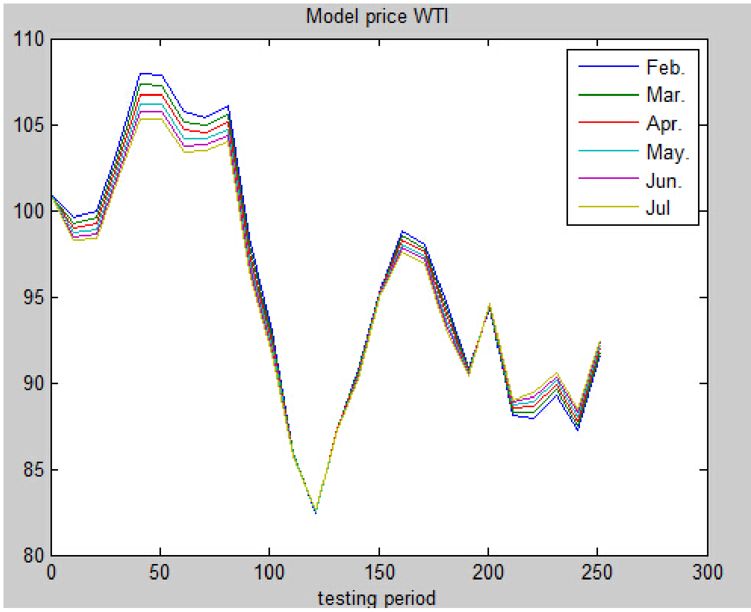


Figure 78: Estimated Prices of WTI Futures Contracts with Different Maturities for The New Three-Factor Model

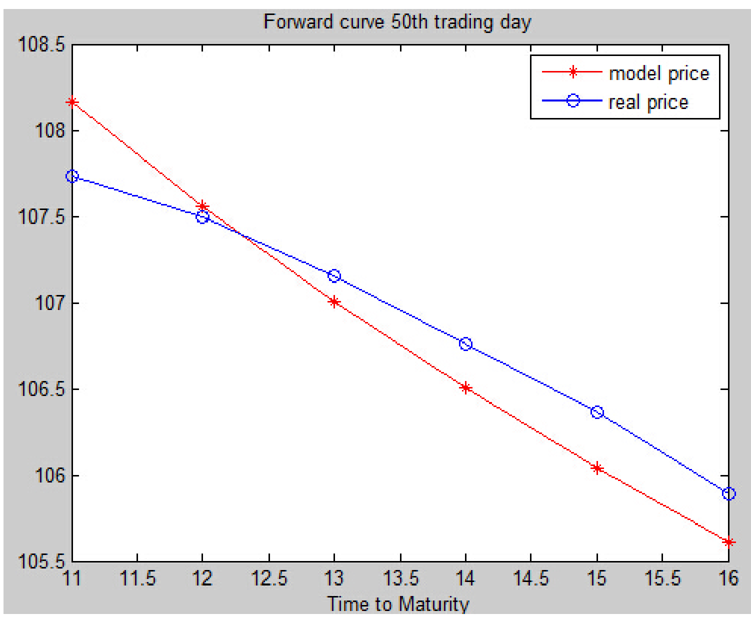


Figure 79: Forward Curves from The New Three-Factor model on The 50th day

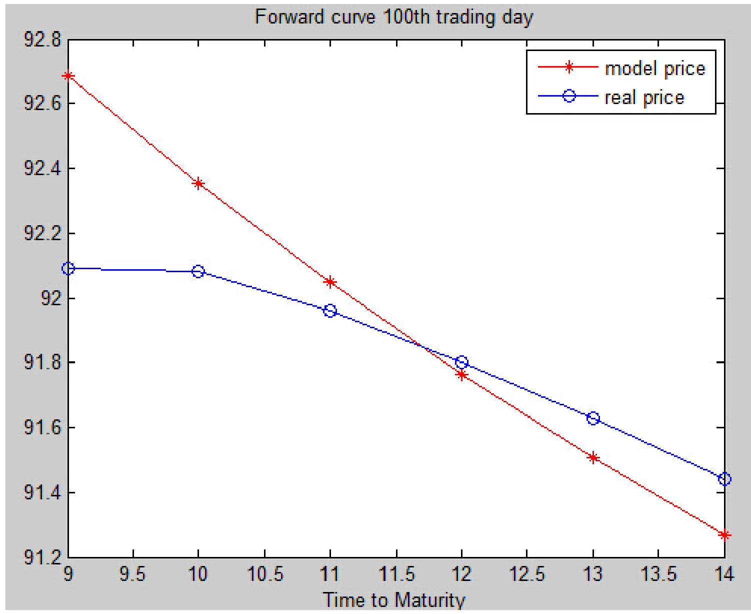


Figure 80: Forward Curves from The New Three-Factor model on The 100th day

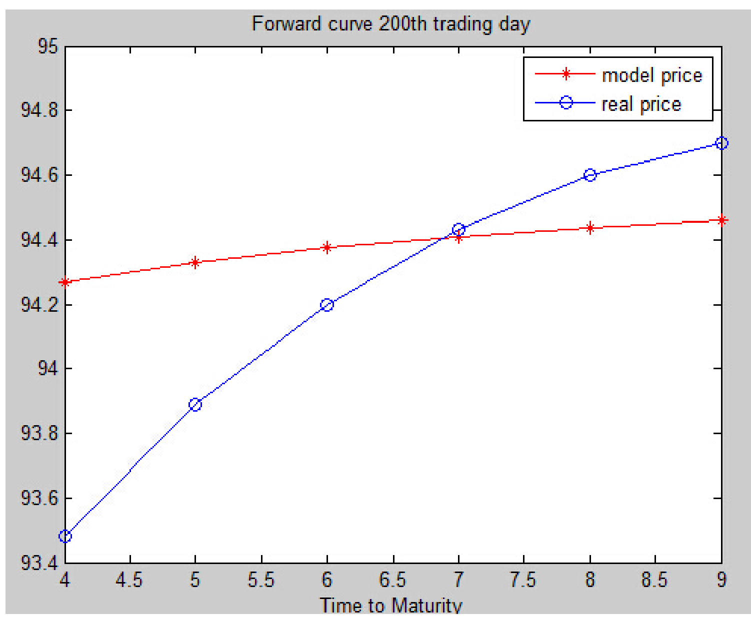


Figure 81: Forward Curves from The New Three-Factor model on The 200th day

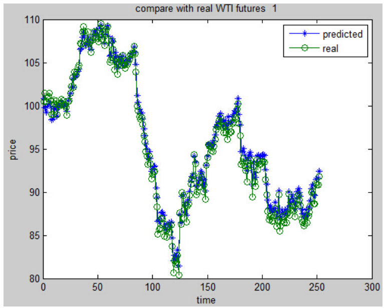


Figure 82: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The New Three-Factor Model

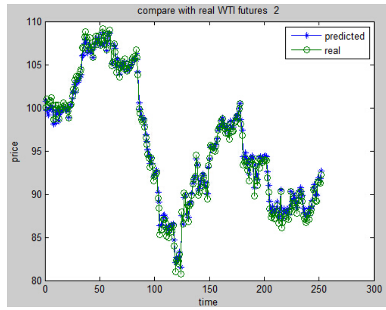


Figure 83: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The New Three-Factor Model

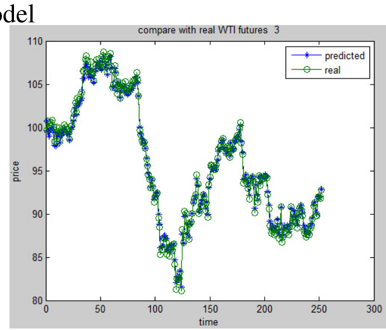


Figure 84: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The New Three-Factor Model

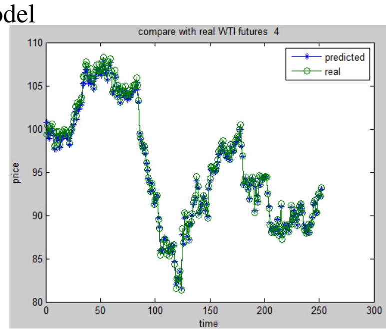


Figure 85: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The New Three-Factor Model

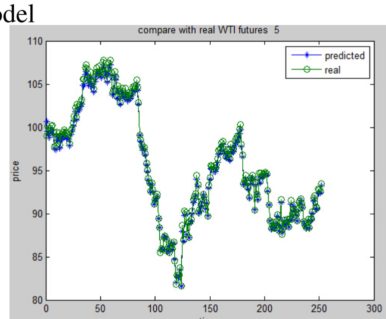


Figure 86: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The New Three-Factor Model

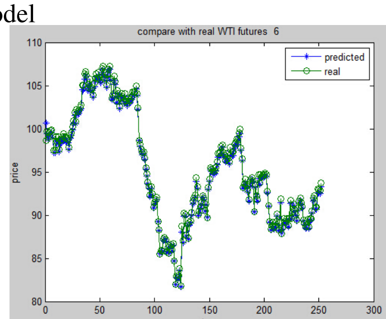


Figure 87: The Predicted and Observed Futures Prices for The Sixth Testing Futures Contract based on The New Three-Factor Model

4.2 A New Four-Factor Model and Its Empirical Results

When the fourth factor is added into the Schwartz (1997) model system, the number of the unknown parameters increases to 15. As mentioned in the introduction, since the Kalman filter and the extended Kalman filter have their own inevitable drawbacks, the increasing number of unknown parameters and the uncertainty of their initial values increase the instability of the model system. Hence, the estimated parameters should impose more stringent restrictions to test their effectiveness in the new Four-Factor model. In this section, the new Four-Factor model will be run twice: first, the model is run like other models with the whole data. Then the data will be separated into two parts. To be more specific, the first part will be used to estimate the parameters (in-sample), and the second part will be used to test the parameters for forecasting (out-of-sample).

4.2.1 A New Four-Factor Model

Similar to the new Three-Factor model, which has been introduced in detail above, the original Three-Factor model can also be considered with the varying-time and unobservable volatility of the spot price of WTI crude oil. Then, the new Four-Factor model can be described as follows

The four stochastic processes are:

$$dS = (r - \delta)Sdt + \sigma_1 SdW_1$$

$$d\delta = \kappa_1(\alpha - \delta)dt + \sigma_2 dW_2$$

$$d\sigma_1 = \kappa_2(\theta - \sigma_1)dt + \sigma_3 dW_3$$

$$dr = \kappa_3(\gamma - r)dt + \sigma_4 dW_4$$

with

$$dW_1 dW_2 = \rho_{12} dt$$

$$dW_1 dW_3 = \rho_{13} dt$$

$$dW_1 dW_4 = \rho_{14} dt$$

$$dW_2 dW_3 = \rho_{23} dt$$

$$dW_2 dW_4 = \rho_{24} dt$$

$$dW_3 dW_4 = \rho_{34} dt$$

where r, α, θ and γ represented the long-term return investment on oil; the long-term convenience yield; the volatility of the spot price and the interest rate, respectively. κ_1, κ_2 and κ_3 are the coefficient of reverting in the stochastic processes, respectively. Similarly, σ_1, σ_2, σ_3 and σ_4 are the volatilities of spot price; convenience yield; the volatility of the spot price and the interest rate, respectively. dW_1, dW_2, dW_3 and dW_4 are the increments of Brownian motions and the following normal distribution $N(0, dt^{1/2})$ with $dW_i dW_j = \rho_{ij} dt$, where ρ_{ij} is the correlation coefficient between the two stochastic processes i and j.

With the consideration of the four stochastic processes, the futures price formula must satisfy the following partial differential equation:

$$\frac{1}{2}\sigma_1^2 S^2 F_{SS} + \frac{1}{2}\sigma_2^2 F_{\delta\delta} + \frac{1}{2}\sigma_3^2 F_{\sigma_1\sigma_1} + \frac{1}{2}\sigma_4^2 F_{rr} + S\sigma_1\sigma_2\rho_{12}F_{S\delta} + S\sigma_1\sigma_3\rho_{13}F_{S\sigma_1} + S\sigma_1\sigma_4\rho_{14}F_{Sr} + \sigma_2\sigma_3\rho_{23}F_{\delta\sigma_1} + \sigma_2\sigma_4\rho_{24}F_{\delta r} + \sigma_3\sigma_4\rho_{34}F_{\sigma_1 r} + (r-\delta)SF_S + \kappa_1(\alpha - \delta)F_\delta + \kappa_2(\theta - \sigma_1)F_{\sigma_1} + \kappa_3(\gamma - r)F_r - F_\tau = 0$$

Similar to previous studies (Heston,1993 and Chen, Ewald and Zong, 2014), guess the solution of the PDE is

$$F = Sexp(A + B\delta + C\sigma_1 + Dr)$$

then, the PDE can be reduced as four ODEs as follows:

$$-1 - \kappa_1 B = \frac{\partial B}{\partial \tau}$$

$$1 - \kappa_3 D = \frac{\partial D}{\partial \tau}$$

$$\sigma_2\rho_{12}B + \sigma_3\rho_{13}C + \sigma_4\rho_{14}D - \kappa_2 C = \frac{\partial C}{\partial \tau}$$

$$\frac{1}{2}\sigma_2^2B^2 + \frac{1}{2}\sigma_3^2C^2 + \frac{1}{2}\sigma_4^2D^2 + \sigma_2\sigma_3\rho_{23}BC + \sigma_2\sigma_4\rho_{24}BD + \sigma_3\sigma_4\rho_{34}CD + \kappa_1\alpha B + \kappa_2\theta C + \kappa_3\gamma D = \frac{\partial A}{\partial \tau}$$

Then, B, C, D and A can be easily solved. Last, the solution of the PDE can be obtained.

$$B = \frac{e^{-\kappa_1\tau}-1}{\kappa_1}$$

$$D = \frac{1-e^{-\kappa_3\tau}}{\kappa_3}$$

$$C = \rho_{14}\sigma_4/(\kappa_3(-\rho_{13}\sigma_3+\kappa_2)) - \rho_{12}\sigma_2/(\kappa_1(-\rho_{13}\sigma_3+\kappa_2)) + \rho_{12}\sigma_2 exp(-(-\rho_{13}\sigma_3+\kappa_2)\tau - \tau(\rho_{13}\sigma_3+\kappa_1-\kappa_2))/(\kappa_1(-\rho_{13}\sigma_3-\kappa_1+\kappa_2)) - \rho_{14}\sigma_4 exp(-(-\rho_{13}\sigma_3+\kappa_2)\tau + \tau(-\rho_{13}\sigma_3+\kappa_2-\kappa_3))/(\kappa_3(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)) + exp(-(-\rho_{13}\sigma_3+\kappa_2)\tau)(-\rho_{14}\sigma_4/(\kappa_3(-\rho_{13}\sigma_3+\kappa_2)) + \rho_{12}\sigma_2/(\kappa_1(-\rho_{13}\sigma_3+\kappa_2)) - \rho_{12}\sigma_2/(\kappa_1(-\rho_{13}\sigma_3-\kappa_1+\kappa_2)) + \sigma_4\rho_{14}/((-\rho_{13}\sigma_3+\kappa_2-\kappa_3)\kappa_3))$$

$$A = 0.5(1/(\kappa_3^2(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_1^2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)^2(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2))((1/(\rho_{13}\sigma_3-\kappa_1-\kappa_2))(2(-\rho_{13}\sigma_3+\kappa_2)(\kappa_1\rho_{23}+(\rho_{13}\rho_{23}-\rho_{12})\sigma_3-\kappa_2\rho_{23})(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))\kappa_3^2\sigma_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)\sigma_2 exp(\tau(\rho_{13}\sigma_3-\kappa_1-\kappa_2))) - (1/(\rho_{13}\sigma_3-\kappa_2-\kappa_3))(2\sigma_4(-\rho_{13}\sigma_3+\kappa_2)(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))\kappa_3\sigma_3(-\kappa_3\rho_{34}+(-\rho_{13}\rho_{34}+\rho_{14})\sigma_3+\kappa_2\rho_{34})\kappa_1^2 exp(\tau(\rho_{13}\sigma_3-\kappa_2-\kappa_3))) + (\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))^2\kappa_3^2\sigma_3^2\kappa_1^2 exp(-(2(-\rho_{13}\sigma_3+\kappa_2))\tau)/(2\rho_{13}\sigma_3-2\kappa_2) + (1/(\rho_{13}\sigma_3-\kappa_2))(2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))\kappa_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)((\theta\kappa_2(-\rho_{13}\sigma_3+\kappa_2)\kappa_3+\sigma_4((-\rho_{13}\rho_{34}+\rho_{14})\sigma_3+\kappa_2\rho_{34})\sigma_3)\kappa_1-((-\rho_{13}\rho_{23}+\rho_{12})\sigma_3+\kappa_2\rho_{23})\kappa_3\sigma_3\sigma_2) exp((\rho_{13}\sigma_3-\kappa_2)\tau)) - (1/(-\kappa_1-\kappa_3))(2\sigma_4(-\rho_{13}\sigma_3+\kappa_2)^2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)((-\rho_{24}\kappa_3+(-\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2\rho_{24})\kappa_1+((\rho_{12}\rho_{34}-\rho_{13}\rho_{24})\sigma_3+\kappa_2\rho_{24})\kappa_3+(-\rho_{24}\rho_{13}^2+(\rho_{12}\rho_{34}+\rho_{14}\rho_{23})\rho_{13}-\rho_{12}\rho_{14})\sigma_3^2-\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3-\kappa_2^2\rho_{24})\kappa_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)\sigma_2 exp(-\tau(\kappa_1+\kappa_3))) + (1/\kappa_1)(2(-\rho_{13}\sigma_3+\kappa_2)(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(-\kappa_3\alpha(-\rho_{13}\sigma_3+\kappa_2)\kappa_1^3+(\alpha(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_3-\sigma_4((-\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2\rho_{24})\sigma_2)\kappa_1^2+(((-\theta\kappa_2\rho_{12}\rho_{13}+\sigma_2(\rho_{12}\rho_{23}-\rho_{13}))\sigma_3+\kappa_2(\kappa_2\rho_{12}\theta+\sigma_2))\kappa_3+\sigma_4((\rho_{24}\rho_{13}^2+(-\rho_{12}\rho_{34}-\rho_{14}\rho_{23})\rho_{13}+\rho_{12}\rho_{14})\sigma_3^2+\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2^2\rho_{24}))\sigma_2\kappa_1-((-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_3^2+2\kappa_2(\rho_{12}\rho_{23}-\rho_{13})\sigma_3+\kappa_2^2)\kappa_3\sigma_2^2)\kappa_3(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2 exp(-\kappa_1\tau)) + (1/\kappa_3)(2(-\rho_{13}\sigma_3+\kappa_2)(\rho_{13}\sigma_3+\kappa_1-\kappa_2)^2\kappa_1(-\rho_{13}\sigma_3+$$

$$\kappa_2-\kappa_3)((-\gamma(-\rho_{13}\sigma_3+\kappa_2)\kappa_3^3+\gamma(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_3^2+\sigma_4((-\theta\kappa_2\rho_{13}\rho_{14}+$$
$$\sigma_4(-\rho_{14}\rho_{34}+\rho_{13}))\sigma_3+\kappa_2(\kappa_2\rho_{14}\theta-\sigma_4))\kappa_3+\sigma_4^2((-2\rho_{13}\rho_{14}\rho_{34}+\rho_{13}^2+\rho_{14}^2)\sigma_3^2-$$
$$2\kappa_2(-\rho_{14}\rho_{34}+\rho_{13})\sigma_3+\kappa_2^2))\kappa_1-\sigma_4\kappa_3(((-\rho_{12}\rho_{34}+\rho_{13}\rho_{24})\sigma_3-\kappa_2\rho_{24})\kappa_3+$$
$$(\rho_{24}\rho_{13}^2+(-\rho_{12}\rho_{34}-\rho_{14}\rho_{23})\rho_{13}+\rho_{12}\rho_{14})\sigma_3^2+\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+$$
$$\kappa_2^2\rho_{24})\sigma_2)exp(-\kappa_3\tau))-0.5(1/\kappa_1)((-\rho_{13}\sigma_3+\kappa_2)^2(\kappa_1^2+((-2\rho_{12}\rho_{23}+2\rho_{13})\sigma_3-$$
$$2\kappa_2)\kappa_1+(-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_3^2+2\kappa_2(\rho_{12}\rho_{23}-\rho_{13})\sigma_3+\kappa_2^2)\kappa_3^2(-\rho_{13}\sigma_3+\kappa_2-$$
$$\kappa_3)^2\sigma_2^2exp(-2\kappa_1\tau))+2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)^2(-0.25(1/\kappa_3)(\sigma_4^2(-\rho_{13}\sigma_3+\kappa_2)^2(\kappa_3^2+$$
$$((-2\rho_{14}\rho_{34}+2\rho_{13})\sigma_3-2\kappa_2)\kappa_3+(-2\rho_{13}\rho_{14}\rho_{34}+\rho_{13}^2+\rho_{14}^2)\sigma_3^2-2\kappa_2(-\rho_{14}\rho_{34}+$$
$$\rho_{13})\sigma_3+\kappa_2^2)\kappa_1^2exp(-2\kappa_3\tau))+((-(-\rho_{13}\sigma_3+\kappa_2)^2(\alpha-\gamma)\kappa_3^2+\theta\kappa_2\sigma_4\rho_{14}(-\rho_{13}\sigma_3+$$
$$\kappa_2)\kappa_3+(1/2)\sigma_4^2((-2\rho_{13}\rho_{14}\rho_{34}+\rho_{13}^2+\rho_{14}^2)\sigma_3^2-2\kappa_2(-\rho_{14}\rho_{34}+\rho_{13})\sigma_3+\kappa_2^2))\kappa_1^2-$$
$$(\theta\kappa_2\rho_{12}(-\rho_{13}\sigma_3+\kappa_2)\kappa_3+\sigma_4((\rho_{24}\rho_{13}^2+(-\rho_{12}\rho_{34}-\rho_{14}\rho_{23})\rho_{13}+\rho_{12}\rho_{14})\sigma_3^2+$$
$$\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2^2\rho_{24}))\kappa_3\sigma_2\kappa_1+0.5((-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+$$
$$\rho_{13}^2)\sigma_3^2+2\kappa_2(\rho_{12}\rho_{23}-\rho_{13})\sigma_3+\kappa_2^2)\kappa_3^2\sigma_2^2)(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2\tau))-$$
$$0.5(1/(\kappa_3^2(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_1^2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)^2(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2))((1/(\rho_{13}\sigma_3-$$
$$\kappa_1-\kappa_2))(2(-\rho_{13}\sigma_3+\kappa_2)(\kappa_1\rho_{23}+(\rho_{13}\rho_{23}-\rho_{12})\sigma_3-\kappa_2\rho_{23})(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+$$
$$(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))\kappa_3^2\sigma_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)\sigma_2)-(1/(\rho_{13}\sigma_3-\kappa_2-$$
$$\kappa_3))(2\sigma_4(-\rho_{13}\sigma_3+\kappa_2)(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-$$
$$\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))\kappa_3\sigma_3(-\kappa_3\rho_{34}+(-\rho_{13}\rho_{34}+\rho_{14})\sigma_3+\kappa_2\rho_{34})\kappa_1^2)+(\sigma_4\rho_{14}\kappa_1-$$
$$\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))^2\kappa_3^2\sigma_3^2\kappa_1^2/(2\rho_{13}\sigma_3-2\kappa_2)+(1/(\rho_{13}\sigma_3-$$
$$\kappa_2))(2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+$$
$$\kappa_2))\kappa_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)((\theta\kappa_2(-\rho_{13}\sigma_3+\kappa_2)\kappa_3+\sigma_4((-\rho_{13}\rho_{34}+\rho_{14})\sigma_3+$$
$$\kappa_2\rho_{34})\sigma_3)\kappa_1-((-\rho_{13}\rho_{23}+\rho_{12})\sigma_3+\kappa_2\rho_{23})\kappa_3\sigma_3\sigma_2))-(1/(-\kappa_1-\kappa_3))(2\sigma_4(-\rho_{13}\sigma_3+$$
$$\kappa_2)^2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)((-\rho_{24}\kappa_3+(-\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2\rho_{24})\kappa_1+((\rho_{12}\rho_{34}-$$
$$\rho_{13}\rho_{24})\sigma_3+\kappa_2\rho_{24})\kappa_3+(-\rho_{24}\rho_{13}^2+(\rho_{12}\rho_{34}+\rho_{14}\rho_{23})\rho_{13}-\rho_{12}\rho_{14})\sigma_3^2-\kappa_2(\rho_{12}\rho_{34}-$$
$$2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3-\kappa_2^2\rho_{24})\kappa_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)\sigma_2)+(1/\kappa_1)(2(-\rho_{13}\sigma_3+$$
$$\kappa_2)(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(-\kappa_3\alpha(-\rho_{13}\sigma_3+\kappa_2)\kappa_1^3+(\alpha(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_3-\sigma_4((-\rho_{13}\rho_{24}+$$
$$\rho_{14}\rho_{23})\sigma_3+\kappa_2\rho_{24})\sigma_2)\kappa_1^2+(((-\theta\kappa_2\rho_{12}\rho_{13}+\sigma_2(\rho_{12}\rho_{23}-\rho_{13}))\sigma_3+\kappa_2(\kappa_2\rho_{12}\theta+$$
$$\sigma_2))\kappa_3+\sigma_4((\rho_{24}\rho_{13}^2+(-\rho_{12}\rho_{34}-\rho_{14}\rho_{23})\rho_{13}+\rho_{12}\rho_{14})\sigma_3^2+\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+$$
$$\rho_{14}\rho_{23})\sigma_3+\kappa_2^2\rho_{24}))\sigma_2\kappa_1-((-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_3^2+2\kappa_2(\rho_{12}\rho_{23}-\rho_{13})\sigma_3+$$
$$\kappa_2^2)\kappa_3\sigma_2^2)\kappa_3(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2)+(1/\kappa_3)(2(-\rho_{13}\sigma_3+\kappa_2)(\rho_{13}\sigma_3+\kappa_1-$$
$$\kappa_2)^2\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)((-\gamma(-\rho_{13}\sigma_3+\kappa_2)\kappa_3^3+\gamma(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_3^2+$$

$\sigma_4((-\theta\kappa_2\rho_{13}\rho_{14}+\sigma_4(-\rho_{14}\rho_{34}+\rho_{13}))\sigma_3+\kappa_2(\kappa_2\rho_{14}\theta-\sigma_4))\kappa_3+\sigma_4^2((-2\rho_{13}\rho_{14}\rho_{34}+$
$\rho_{13}^2+\rho_{14}^2)\sigma_3^2-2\kappa_2(-\rho_{14}\rho_{34}+\rho_{13})\sigma_3+\kappa_2^2))\kappa_1-\sigma_4\kappa_3((((-\rho_{12}\rho_{34}+\rho_{13}\rho_{24})\sigma_3-$
$\kappa_2\rho_{24})\kappa_3+(\rho_{24}\rho_{13}^2+(-\rho_{12}\rho_{34}-\rho_{14}\rho_{23})\rho_{13}+\rho_{12}\rho_{14})\sigma_3^2+\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+$
$\rho_{14}\rho_{23})\sigma_3+\kappa_2^2\rho_{24})\sigma_2))-0.5(1/\kappa_1)((-\rho_{13}\sigma_3+\kappa_2)^2(\kappa_1^2+((-2\rho_{12}\rho_{23}+2\rho_{13})\sigma_3-$
$2\kappa_2)\kappa_1+(-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_3^2+2\kappa_2(\rho_{12}\rho_{23}-\rho_{13})\sigma_3+\kappa_2^2)\kappa_3^2(-\rho_{13}\sigma_3+$
$\kappa_2-\kappa_3)^2\sigma_2^2)-0.5(1/\kappa_3)((\rho_{13}\sigma_3+\kappa_1-\kappa_2)^2\sigma_4^2(-\rho_{13}\sigma_3+\kappa_2)^2(\kappa_3^2+((-2\rho_{14}\rho_{34}+$
$2\rho_{13})\sigma_3-2\kappa_2)\kappa_3+(-2\rho_{13}\rho_{14}\rho_{34}+\rho_{13}^2+\rho_{14}^2)\sigma_3^2-2\kappa_2(-\rho_{14}\rho_{34}+\rho_{13})\sigma_3+\kappa_2^2)\kappa_1^2))$

Then, C and A can be simplified as:

$C=$

$$\frac{\rho_{14}\sigma_4}{\kappa_3(-\rho_{13}*\sigma_3+\kappa_2)}-\frac{\rho_{12}\sigma_2}{\kappa_1(-\rho_{13}\sigma_3+\kappa_2)}+\frac{\rho_{12}\sigma_2e^{-(-\rho_{13}\sigma_3+\kappa_2)\tau-\tau(\rho_{13}\sigma_3+\kappa_1-\kappa_2)}}{\kappa_1(-\rho_{13}\sigma_3-\kappa_1+\kappa_2)}\frac{\rho_{14}\sigma_4e^{-(-\rho_{13}\sigma_3+\kappa_2)\tau+\tau(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)}}{\kappa_3(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)}+e^{-(-\rho_{13}\sigma_3+\kappa_2)\tau}\left(\frac{-\rho_{14}\sigma_4}{\kappa_3(-\rho_{13}\sigma_3+\kappa_2)}+\frac{\rho_{12}\sigma_2}{\kappa_1(-\rho_{13}\sigma_3+\kappa_2)}-\frac{\rho_{12}\sigma_2}{\kappa_1(-\rho_{13}\sigma_3-\kappa_1+\kappa_2)}+\frac{\sigma_4\rho_{14}}{\kappa_3(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)}\right)$$

and $A=$

$$\frac{1}{2}\frac{1}{\kappa_3^2(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_1^2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)^2(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2}[\frac{1}{\rho_{13}\sigma_3-\kappa_1-\kappa_2}(2(-\rho_{13}\sigma_3+\kappa_2)(\kappa_1\rho_{23}+(\rho_{13}\rho_{23}-\rho_{12})\sigma_3-\kappa_2\rho_{23})(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))\kappa_3^2\sigma_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)\sigma_2e^{\tau(\rho_{13}\sigma_3-\kappa_1-\kappa_2)})-\frac{1}{\rho_{13}\sigma_3-\kappa_2-\kappa_3}(2\sigma_4(-\rho_{13}\sigma_3+\kappa_2)(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))\kappa_3\sigma_3(-\kappa_3\rho_{34}+(-\rho_{13}\rho_{34}+\rho_{14})\sigma_3+\kappa_2\rho_{34})\kappa_1^2e^{\tau(\rho_{13}\sigma_3-\kappa_2-\kappa_3)})+(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))^2\kappa_3^2\sigma_3^2\kappa_1^2\frac{e^{-2\tau(-\rho_{13}\sigma_3+\kappa_2)}}{2\rho_{13}\sigma_3-2\kappa_2}+\frac{1}{\rho_{13}\sigma_3-\kappa_2}(2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))\kappa_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)((\theta\kappa_2(-\rho_{13}\sigma_3+\kappa_2)\kappa_3+\sigma_4((-\rho_{13}\rho_{34}+\rho_{14})\sigma_3+\kappa_2\rho_{34})\sigma_3)\kappa_1-((-\rho_{13}\rho_{23}+\rho_{12})\sigma_3+\kappa_2\rho_{23})\kappa_3\sigma_3\sigma_2)e^{(\rho_{13}\sigma_3-\kappa_2)\tau})-\frac{1}{-\kappa_1-\kappa_3}(2\sigma_4(-\rho_{13}\sigma_3+\kappa_2)^2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)((-\rho_{24}\kappa_3+(-\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2\rho_{24})\kappa_1+((\rho_{12}\rho_{34}-\rho_{13}\rho_{24})\sigma_3+\kappa_2\rho_{24})\kappa_3+(-\rho_{24}\rho_{13}^2+(\rho_{12}\rho_{34}+\rho_{14}\rho_{23})\rho_{13}-\rho_{12}\rho_{14})\sigma_3^2-\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3-\kappa_2^2\rho_{24})\kappa_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)\sigma_2e^{-\tau(\kappa_1+\kappa_3)})+\frac{1}{\kappa_1}(2(-\rho_{13}\sigma_3+\kappa_2)(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(-\kappa_3\alpha(-\rho_{13}\sigma_3+\kappa_2)\kappa_1^3+(\alpha(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_3-\sigma_4((-\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2\rho_{24})\sigma_2)\kappa_1^2+(((-\theta\kappa_2\rho_{12}\rho_{13}+\sigma_2(\rho_{12}\rho_{23}-\rho_{13}))\sigma_3+\kappa_2(\kappa_2\rho_{12}\theta+\sigma_2))\kappa_3+\sigma_4((\rho_{24}\rho_{13}^2+(-\rho_{12}\rho_{34}-\rho_{14}\rho_{23})\rho_{13}+\rho_{12}\rho_{14})\sigma_3^2+\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2^2\rho_{24}))\sigma_2\kappa_1-((-2\rho_{12}\rho_{13}\rho_{23}+$$

$$\rho_{12}^2+\rho_{13}^2)\sigma_3^2+2\kappa_2(\rho_{12}\rho_{23}-\rho_{13})\sigma_3+\kappa_2^2)\kappa_3\sigma_2^2)\kappa_3(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2e^{-\kappa_1\tau})+$$
$$\frac{1}{\kappa_3}(2(-\rho_{13}\sigma_3+\kappa_2)(\rho_{13}\sigma_3+\kappa_1-\kappa_2)^2\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)((-\gamma(-\rho_{13}\sigma_3+\kappa_2)\kappa_3^3+$$
$$\gamma(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_3^2+\sigma_4((-\theta\kappa_2\rho_{13}\rho_{14}+\sigma_4(-\rho_{14}\rho_{34}+\rho_{13}))\sigma_3+\kappa_2(\kappa_2\rho_{14}\theta-$$
$$\sigma_4))\kappa_3+\sigma_4^2((-2\rho_{13}\rho_{14}\rho_{34}+\rho_{13}^2+\rho_{14}^2)\sigma_3^2-2\kappa_2(-\rho_{14}\rho_{34}+\rho_{13})\sigma_3+\kappa_2^2))\kappa_1-$$
$$\sigma_4\kappa_3(((-\rho_{12}\rho_{34}+\rho_{13}\rho_{24})\sigma_3-\kappa_2\rho_{24})\kappa_3+(\rho_{24}\rho_{13}^2+(-\rho_{12}\rho_{34}-\rho_{14}\rho_{23})\rho_{13}+$$
$$\rho_{12}\rho_{14})\sigma_3^2+\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2^2\rho_{24})\sigma_2)e^{-\kappa_3\tau})-\frac{1}{2}\frac{1}{\kappa_1}((-\rho_{13}\sigma_3+$$
$$\kappa_2)^2(\kappa_1^2+((-2\rho_{12}\rho_{23}+2\rho_{13})\sigma_3-2\kappa_2)\kappa_1+(-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_3^2+$$
$$2\kappa_2(\rho_{12}\rho_{23}-\rho_{13})\sigma_3+\kappa_2^2)\kappa_3^2(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2\sigma_2^2e^{-2\kappa_1\tau})+2(\rho_{13}\sigma_3+\kappa_1-$$
$$\kappa_2)^2(-\frac{1}{4}\frac{1}{\kappa_3}(\sigma_4^2(-\rho_{13}\sigma_3+\kappa_2)^2(\kappa_3^2+((-2\rho_{14}\rho_{34}+2\rho_{13})\sigma_3-2\kappa_2)\kappa_3+(-2\rho_{13}\rho_{14}\rho_{34}+$$
$$\rho_{13}^2+\rho_{14}^2)\sigma_3^2-2\kappa_2(-\rho_{14}\rho_{34}+\rho_{13})\sigma_3+\kappa_2^2)\kappa_1^2e^{-2\kappa_3\tau})+((-(-\rho_{13}\sigma_3+\kappa_2)^2(\alpha-\gamma)\kappa_3^2+$$
$$\theta\kappa_2\sigma_4\rho_{14}(-\rho_{13}\sigma_3+\kappa_2)\kappa_3+\frac{1}{2}\sigma_4^2((-2\rho_{13}\rho_{14}\rho_{34}+\rho_{13}^2+\rho_{14}^2)\sigma_3^2-2\kappa_2(-\rho_{14}\rho_{34}+$$
$$\rho_{13})\sigma_3+\kappa_2^2))\kappa_1^2-(\theta\kappa_2\rho_{12}(-\rho_{13}\sigma_3+\kappa_2)\kappa_3+\sigma_4((\rho_{24}\rho_{13}^2+(-\rho_{12}\rho_{34}-\rho_{14}\rho_{23})\rho_{13}+$$
$$\rho_{12}\rho_{14})\sigma_3^2+\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2^2\rho_{24}))\kappa_3\sigma_2\kappa_1+\frac{1}{2}((-2\rho_{12}\rho_{13}\rho_{23}+$$
$$\rho_{12}^2+\rho_{13}^2)\sigma_3^2+2\kappa_2(\rho_{12}\rho_{23}-\rho_{13})\sigma_3+\kappa_2^2)\kappa_3^2\sigma_2^2)(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2\tau)]-$$
$$\frac{1}{2}\frac{1}{\kappa_3^2(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_1^2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)^2(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2}[\frac{1}{\rho_{13}\sigma_3-\kappa_1-\kappa_2}(2(-\rho_{13}\sigma_3+$$
$$\kappa_2)(\kappa_1\rho_{23}+(\rho_{13}\rho_{23}-\rho_{12})\sigma_3-\kappa_2\rho_{23})(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+$$
$$\kappa_2))\kappa_3^2\sigma_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)\sigma_2)-\frac{1}{\rho_{13}\sigma_3-\kappa_2-\kappa_3}(2\sigma_4(-\rho_{13}\sigma_3+\kappa_2)(\rho_{13}\sigma_3+\kappa_1-$$
$$\kappa_2)(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))\kappa_3\sigma_3(-\kappa_3\rho_{34}+(-\rho_{13}\rho_{34}+$$
$$\rho_{14})\sigma_3+\kappa_2\rho_{34})\kappa_1^2)+\frac{(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+\kappa_2))^2\kappa_3^2\sigma_3^2\kappa_1^2}{2\rho_{13}\sigma_3-2\kappa_2}+$$
$$\frac{1}{\rho_{13}\sigma_3-\kappa_2}(2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(\sigma_4\rho_{14}\kappa_1-\sigma_2\rho_{12}\kappa_3+(\rho_{12}\sigma_2-\rho_{14}\sigma_4)(-\rho_{13}\sigma_3+$$
$$\kappa_2))\kappa_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)((\theta\kappa_2(-\rho_{13}\sigma_3+\kappa_2)\kappa_3+\sigma_4((-\rho_{13}\rho_{34}+\rho_{14})\sigma_3+$$
$$\kappa_2\rho_{34})\sigma_3)\kappa_1-((-\rho_{13}\rho_{23}+\rho_{12})\sigma_3+\kappa_2\rho_{23})\kappa_3\sigma_3\sigma_2))-\frac{1}{-\kappa_1-\kappa_3}(2\sigma_4(-\rho_{13}\sigma_3+$$
$$\kappa_2)^2(\rho_{13}\sigma_3+\kappa_1-\kappa_2)((-\rho_{24}\kappa_3+(-\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2\rho_{24})\kappa_1+((\rho_{12}\rho_{34}-$$
$$\rho_{13}\rho_{24})\sigma_3+\kappa_2\rho_{24})\kappa_3+(-\rho_{24}\rho_{13}^2+(\rho_{12}\rho_{34}+\rho_{14}\rho_{23})\rho_{13}-\rho_{12}\rho_{14})\sigma_3^2-\kappa_2(\rho_{12}\rho_{34}-$$
$$2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3-\kappa_2^2\rho_{24})\kappa_3\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)\sigma_2)+\frac{1}{\kappa_1}(2(-\rho_{13}\sigma_3+$$
$$\kappa_2)(\rho_{13}\sigma_3+\kappa_1-\kappa_2)(-\kappa_3\alpha(-\rho_{13}\sigma_3+\kappa_2)\kappa_1^3+(\alpha(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_3-\sigma_4((-\rho_{13}\rho_{24}+$$
$$\rho_{14}\rho_{23})\sigma_3+\kappa_2\rho_{24})\sigma_2)\kappa_1^2+(((-\theta\kappa_2\rho_{12}\rho_{13}+\sigma_2(\rho_{12}\rho_{23}-\rho_{13}))\sigma_3+\kappa_2(\kappa_2\rho_{12}\theta+$$
$$\sigma_2))\kappa_3+\sigma_4((\rho_{24}\rho_{13}^2+(-\rho_{12}\rho_{34}-\rho_{14}\rho_{23})\rho_{13}+\rho_{12}\rho_{14})\sigma_3^2+\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+$$
$$\rho_{14}\rho_{23})\sigma_3+\kappa_2^2\rho_{24}))\sigma_2\kappa_1-((-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_3^2+2\kappa_2(\rho_{12}\rho_{23}-\rho_{13})\sigma_3+$$
$$\kappa_2^2)\kappa_3\sigma_2^2)\kappa_3(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2)+\frac{1}{\kappa_3}(2(-\rho_{13}\sigma_3+\kappa_2)(\rho_{13}\sigma_3+\kappa_1-$$
$$\kappa_2)^2\kappa_1(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)((-\gamma(-\rho_{13}\sigma_3+\kappa_2)\kappa_3^3+\gamma(-\rho_{13}\sigma_3+\kappa_2)^2\kappa_3^2+$$

$\sigma_4((-\theta\kappa_2\rho_{13}\rho_{14}+\sigma_4(-\rho_{14}\rho_{34}+\rho_{13}))\sigma_3+\kappa_2(\kappa_2\rho_{14}\theta-\sigma_4))\kappa_3+\sigma_4^2((-2\rho_{13}\rho_{14}\rho_{34}+\rho_{13}^2+\rho_{14}^2)\sigma_3^2-2\kappa_2(-\rho_{14}\rho_{34}+\rho_{13})\sigma_3+\kappa_2^2))\kappa_1-\sigma_4\kappa_3((((-\rho_{12}\rho_{34}+\rho_{13}\rho_{24})\sigma_3-\kappa_2\rho_{24})\kappa_3+(\rho_{24}\rho_{13}^2+(-\rho_{12}\rho_{34}-\rho_{14}\rho_{23})\rho_{13}+\rho_{12}\rho_{14})\sigma_3^2+\kappa_2(\rho_{12}\rho_{34}-2\rho_{13}\rho_{24}+\rho_{14}\rho_{23})\sigma_3+\kappa_2^2\rho_{24})\sigma_2))-\frac{1}{2}\frac{1}{\kappa_1}((-\rho_{13}\sigma_3+\kappa_2)^2(\kappa_1^2+((-2\rho_{12}\rho_{23}+2\rho_{13})\sigma_3-2\kappa_2)\kappa_1+(-2\rho_{12}\rho_{13}\rho_{23}+\rho_{12}^2+\rho_{13}^2)\sigma_3^2+2\kappa_2(\rho_{12}\rho_{23}-\rho_{13})\sigma_3+\kappa_2^2)\kappa_3^2(-\rho_{13}\sigma_3+\kappa_2-\kappa_3)^2\sigma_2^2)-0.5(1/\kappa_3)((\rho_{13}\sigma_3+\kappa_1-\kappa_2)^2\sigma_4^2(-\rho_{13}\sigma_3+\kappa_2)^2(\kappa_3^2+((-2\rho_{14}\rho_{34}+2\rho_{13})\sigma_3-2\kappa_2)\kappa_3+(-2\rho_{13}\rho_{14}\rho_{34}+\rho_{13}^2+\rho_{14}^2)\sigma_3^2-2\kappa_2(-\rho_{14}\rho_{34}+\rho_{13})\sigma_3+\kappa_2^2)\kappa_1^2)]$

4.2.2 Data and Assumptions

In this section, in order to simplify the complex state space model, weekends and other non-trading days can be ignored, which means that the trading days are considered to be continuous. Based on the efficient market hypothesis, powerful information from the non-trading days (e.g. weekends and Christmas holiday) can be immediately reflected in the price after the non-trading day. Hence, this is a reasonable and popular assumption in the financial world. The next assumption is that each futures contract is immediately executed on the first day when they mature. This is an assumption of the length of the maturity. Based on this assumption, the length of the maturity can be measured more accurately. The third assumption is a measurement of the level of the cost of financing. Since the interest rate r is considered to be instantaneous and, based on the no-arbitrage assumption, the drift of crude oil equals the instantaneous interest rate, but is not a constant any more. In addition, in this section, instead of a single average T, the dynamics of the T with the time passing is added to the model, which is expected to get a better fit for the data.

As has been explained in the previous sections, when the spot price of crude oil is set as an unobservable state variable, the observable futures prices are the only data collected when the model is implemented. In this section, the data of futures prices of WTI crude oil is collected from Bloomberg. Specifically, in this section, twelve futures contracts will be used. Their maturities are from February 2013 to January 2014, and the test period is the entire year of 2012.

4.2.3 The Empirical Results of The New Four-Factor Model

The estimated state variables of the new Four-Factor model are shown within Figures 88-91. To be specific, Figure 88 exhibits the estimated spot price of WTI crude oil, which fluctuates between about 120 dollars and about 80 dollars per barrel in the test period. Overall, the estimated spot price shows a downward trend in the test period. It fluctuates over 100 dollars per barrel and reaches the peak of about 120 dollars per barrel at the beginning of the test period, but then directly plummets to about 80 dollars per barrel, which is the valley in the entire test period. In the second half of the test period, it rebounds to about 100 dollars per barrel and then decreases to about 85 dollars per barrel again. Figure 89 shows the estimated convenience yield of the new Four-Factor model. Overall, the estimated convenience yield fluctuates around 0.1 in the entire test period. Specifically, the estimated convenience yield of WTI crude oil rapidly increases from 0.02 to 0.23, which is the peak in the entire test period. Then, after the peak, the estimated convenience yield keeps moving down until it is about 0.05 in the middle of the test period. In the second half of the test period, the estimated convenience yield rebounds to about 0.15, and then drops to about 0.05 again. Figure 91 exhibits the estimated interest rate from the new Four-Factor model. Overall, the estimated interest rate shows an upward trend in the entire test period. To be more specific, the estimated interest rate firstly increases to about 0.09 and then decreases to about 0.04 in the first half of the test period. In the second half of the test period, the estimated interest rate keeps increasing to 0.12. Figure 92 exhibits the estimated σ_1 from the new Four-Factor model. The estimated σ_1 fluctuates in an interval between 0.05 and 0.053. To be more specific, the estimated σ_1 increases to about 0.052 at the beginning of the test period, and then it fluctuates for a while. After the fluctuation, the estimated σ_1 suddenly increases to about 0.054 at the end of the test period. As for the estimated parameters, first, the degree of mean reversion of the volatility of the spot price of WTI crude oil is very high; second, the long-term return investment on the convenience yield and the interest rate are positive during the test period, while the long-term return investment on the volatility of the spot price of WTI crude oil is negative; last, the risk to the convenience yield of WTI crude oil is significant lower than the risk to the volatility of the spot price of WTI crude oil and the interest rate.

Recalling the previously mentioned positive relationship between the estimated spot price and the estimated convenience yield of crude oil in section 2.3.3, the positive relationship is maintained between the estimated spot price and the estimated convenience yield of WTI crude oil in the new Four-Factor model.

The model was run with 12 futures contracts. However, drawing 12 coloured lines in a single picture would make the picture untidy. Hence, only the first six futures contracts are shown in the following figures. The six chosen futures contracts are shown together in Figure 92 and Figure 93 in order to see whether there are significant differences between the observed prices of WTI crude oil futures contracts and the estimated prices of WTI crude oil futures contracts in the new Four-Factor model. To be more specific, the real observed futures prices of the six chosen futures contracts are shown in Figure 92. Then the estimated model futures prices of the six chosen futures contracts, which are estimated from the new Four-Factor model, are shown in Figure 93. In order to see the trend clearly, in Figure 93 only 26 points are chosen in the figure, which were picked the first trading day of each ten trading days. Overall, there are no significant differences between the two figures. On the one hand, the WTI crude oil did not follow a strict backwardation or a strict contango during the test period; on the other hand, it is not hard to see that the trend of the backwardation or the contango is more easily found when WTI crude oil fluctuates stably, while all futures prices seem to be close, when the price of WTI crude oil rapidly increases or rapidly decreases.

In order to observe the effectiveness of the new Four-Factor model, the comparisons of each WTI crude oil futures in the new Four-Factor model are shown within Figures 97-102. The rank of the title of each picture corresponds to the rank of its maturity. For example, the title 'The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The New Four-Factor Model' and the title 'The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The New Four-Factor Model' correspond to the futures contracts which matured in February 2013 and March 2013, respectively, and so forth. It is not hard to see that the new Four-Factor model is useful in pricing a futures contract with a particular maturity because all six pictures show that the estimated model prices of each futures contract are really close to the real observed prices of the futures contract.

Last, the forward curves from the new Four-Factor model are shown within Figures 94-96. First, on the 50^{th} trading day, the model estimated prices of the new Four-Factor model are higher than the observed prices. To be more specific, with a longer time to maturity, the difference between the model price and the observed price seems to be smaller. Second, similar to the 50^{th} trading day, on the 100^{th} trading day the model estimated prices of the new Four-Factor model are also higher than the observed prices, and with a longer term to maturity, the difference between the model price and the observed price seems to be smaller. Last, on the 200^{th} trading day, the model estimated prices of the new Four-Factor model cross with the observed prices. To be more specific, with a short term to maturity, the model estimated prices of the new Four-Factor model are higher than the observed prices; in contrast, the model estimated prices of the new Four-Factor model are lower than the observed prices with a long term to maturity.

On the other hand, as mentioned in the previous sections, because of the features of the Kalman filter (or the extended Kalman filter), the calculated parameters should impose more stringent restrictions to test their effectiveness. To be more specific, as an inevitable consequence of the well-known drawback of the Kalman filter (or the extended Kalman filter), the increasing number of unknown parameters and the uncertainty of their initial values increase the instability of the model system. In this section, the estimated parameters are tested with the same data, but it is separated into in-sample and out-of-sample data in order to test the forecast effect. More specifically, in this section, the last 21 days in the testing period (assumed to be the number of trading days for a month) is set as out-of-sample data. In other words, the estimated parameters based on in-sample data are shown in the right hand column in table 4.2.3. Reviewing the table, there are significant differences for some estimated parameters between the two groups of results: first, the degree of mean reversion of the volatility of the spot price of WTI crude oil κ_2 based on the in-sample data is significantly lower; second, the volatility of the volatility of the spot price of WTI crude oil σ_3 is also significantly lower based on the in-sample data. As for the forecast effect of the estimated parameters based on the in-sample data, it is shown within Figures 103-108. To be more specific, the rank of the title of each picture corresponds to the rank of its maturity. For example, the title 'The In-Sample and Out-of-Sample Predicted and Observed Futures Prices for The First Testing Futures Contract based on The New Four-Factor Model' and the title 'The In-Sample and Out-of-Sample Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The New Four-Factor Model' correspond to the futures contracts which matured in February 2013 and March 2013, respectively, and so forth. In each figure, the in-sample comparison is shown on the left of the black plus line, while the out-of-sample comparison is shown on the right of the black plus line. It is not hard to see that the new Four-Factor model is useful in pricing a futures contract by using the extended Kalman filter, because the estimated model prices of each futures contract are not only close to the real observed prices for in-sample data, but also for the out-of-sample data.

Paremeters	Estimated result (all-sample)	Estimated result (in-sample)
κ_1	0.6631	0.6625
κ_2	6.1637	2.6294
κ_3	1.7174	1.7180
σ_2	0.1892	0.1731
σ_3	0.6893	0.1198
σ_4	0.5221	0.5700
ρ_{12}	-0.2671	0.5225
ρ_{13}	-0.0191	-0.3746
ρ_{14}	-0.6409	-0.6162
ρ_{23}	0.5543	-0.3503
ρ_{24}	0.0582	0.2048
ρ_{34}	-0.0511	-0.1686
α	0.2617	0.3435
θ	-0.0689	-0.1598
γ	0.1301	0.1616

Table 4.2.3: Estimated Parameters from The New Four-Factor Model

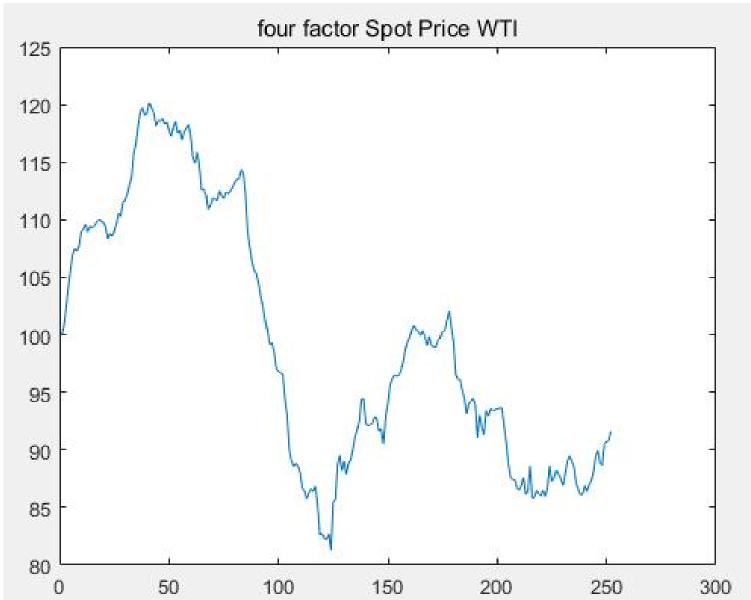


Figure 88: Estimated Spot Price from The New Four-Factor Model by The Extended Kalman Filter

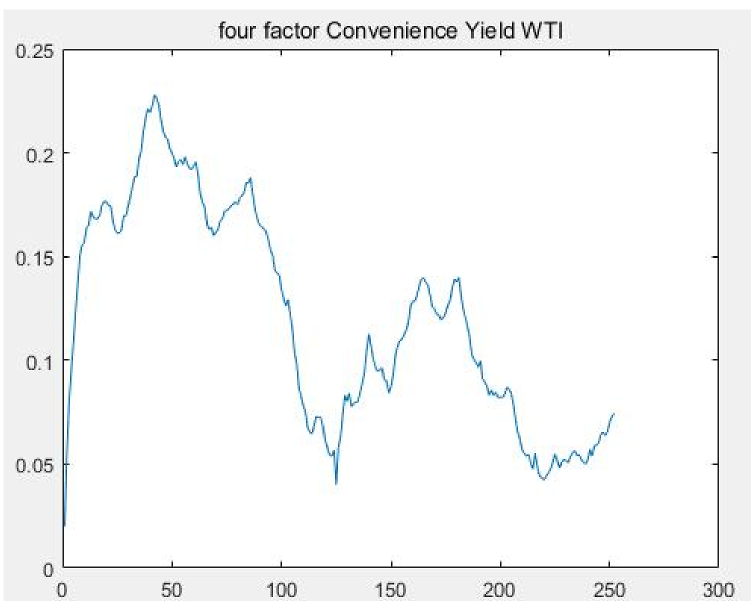


Figure 89: Estimated Convenience Yield from The New Four-Factor Model by The Extended Kalman Filter

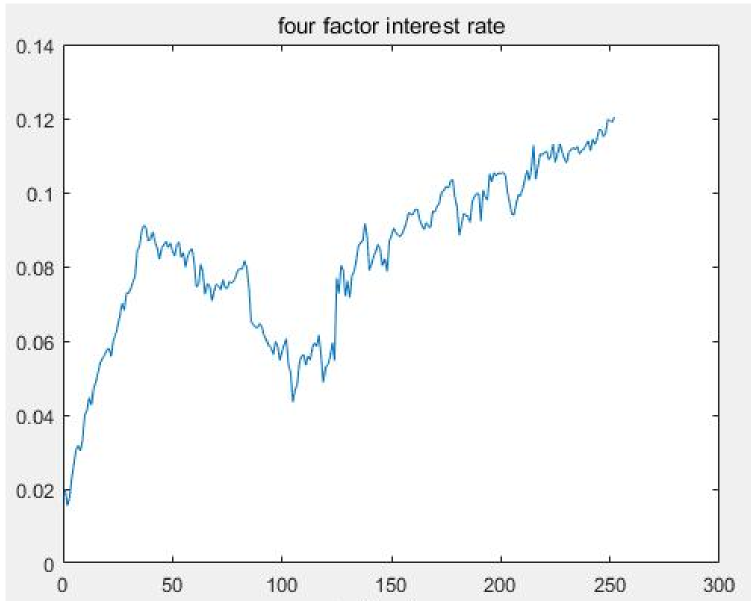


Figure 90: Estimated Interest Rate from The New Four-Factor Model by The Extended Kalman Filter

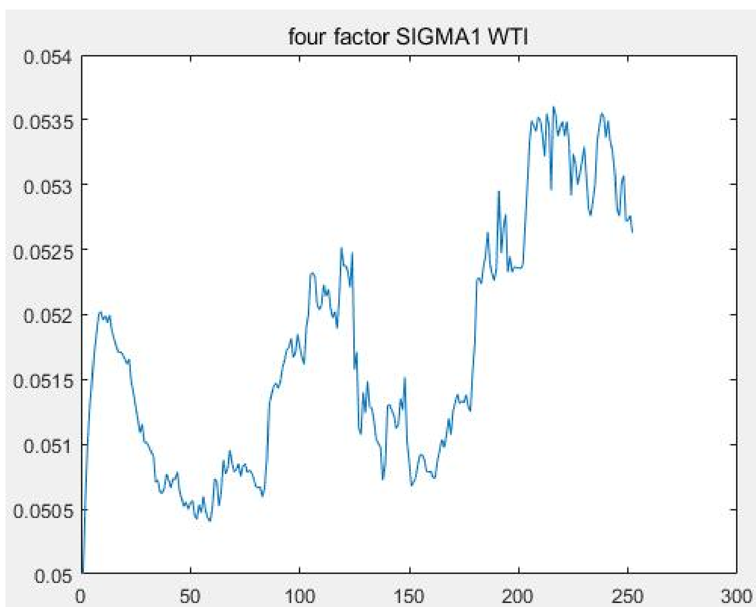


Figure 91: Estimated σ_1 from The New Four-Factor Model by The Extended Kalman Filter

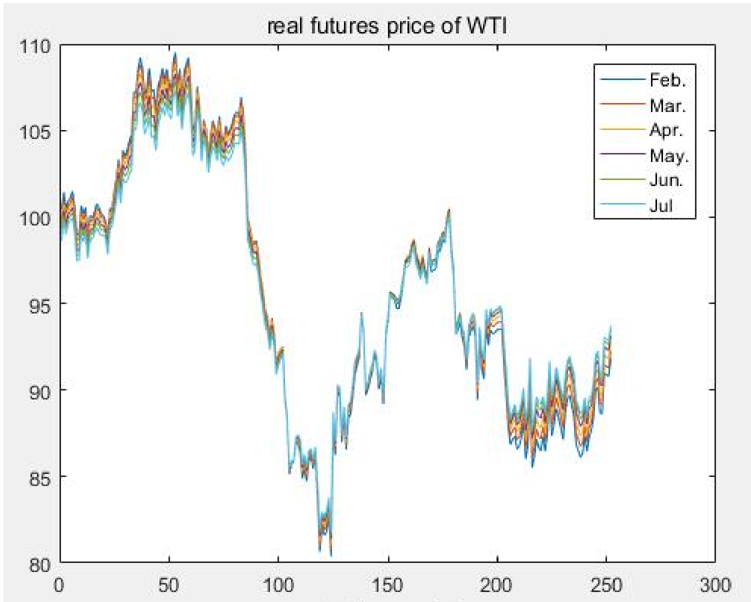


Figure 92: Observed Prices of WTI Futures Contracts with Different Maturities for The New Four-Factor Model

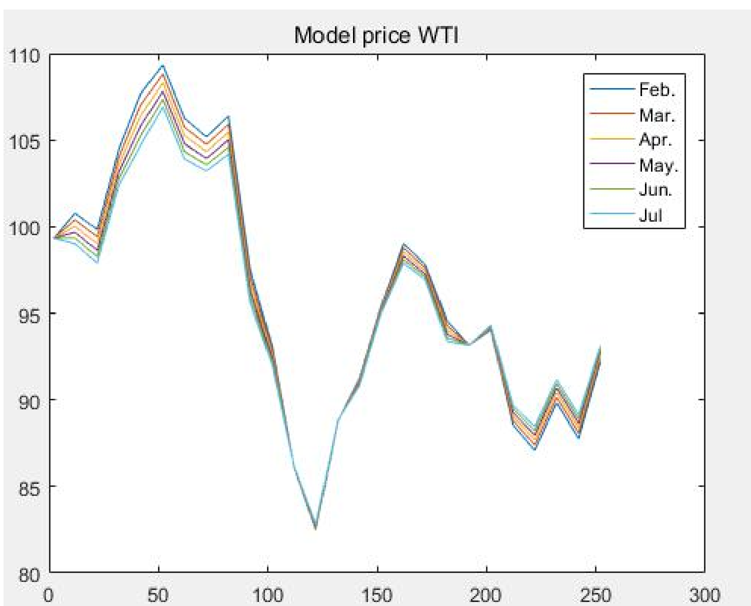


Figure 93: Estimated Prices of WTI Futures Contracts with Different Maturities for The New Four-Factor Model

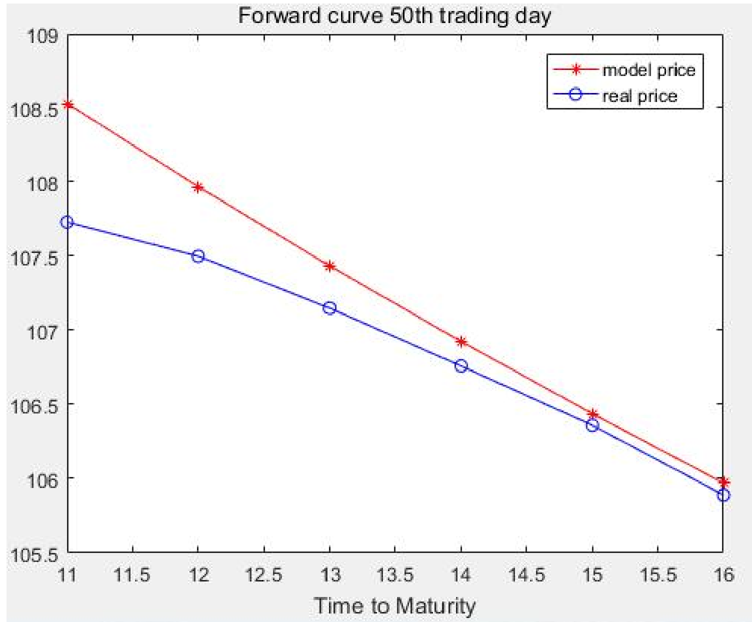


Figure 94: Forward Curves from The New Four-Factor model on The 50th day

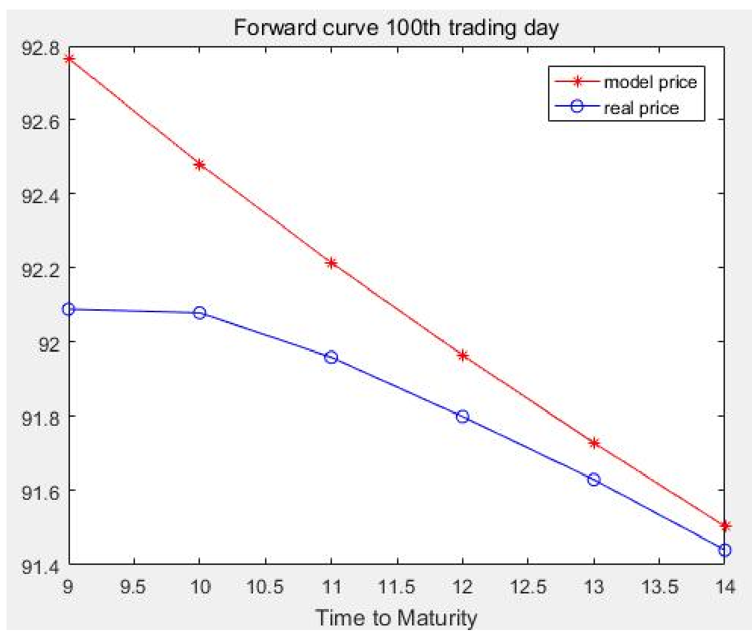


Figure 95: Forward Curves from The New Four-Factor model on The 100th day

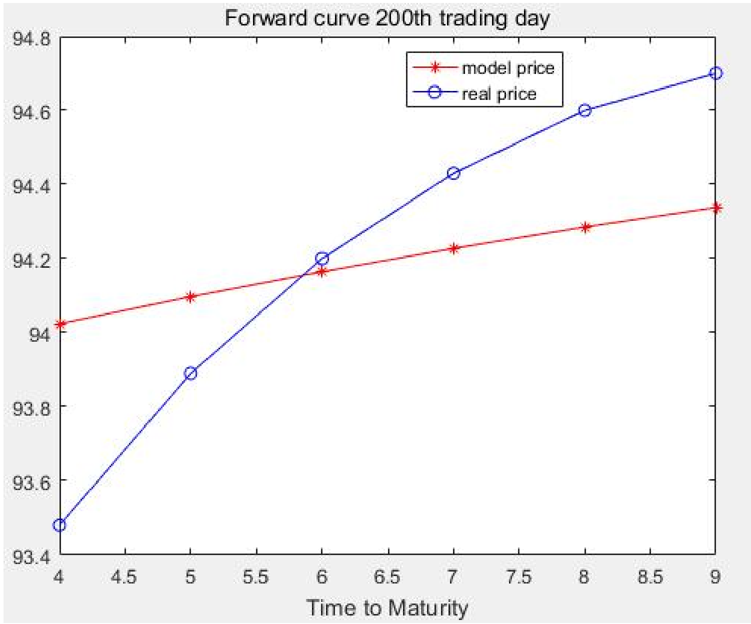


Figure 96: Forward Curves from The New Four-Factor model on The 200th day

5 An Extended Application of The Two-Factor Model to Price European Options on Futures and Its Empirical Results

In order to attract more participants, the New York Mercantile Exchange also offers options on the futures contracts of WTI crude oil. The option on WTI crude oil is a crucial instrument for hedging risk in crude oil. However, as has been shown, even if the traditional Two-Factor model works well for pricing a futures contract, it cannot be used in pricing options on a futures contract. To the author's knowledge, up to now, there is no model in which the options and futures can be priced at the same time by using only the observed prices of options. Hence, building a way in which the options and their underlying futures contracts can be priced based on the trading data of options is important.

This section expands on Schwartz (1997) and Hilliard and Reis(1998) in that the im-

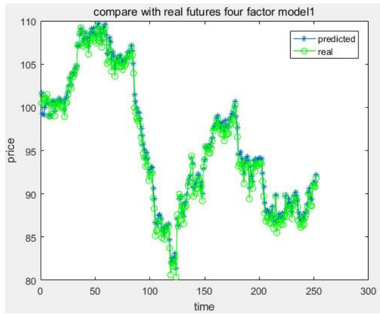


Figure 97: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The New Four-Factor Model

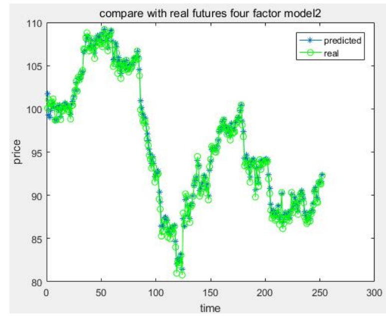


Figure 98: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The New Four-Factor Model

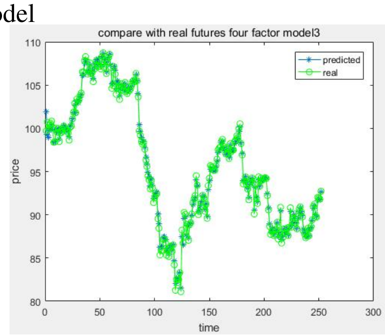


Figure 99: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The New Four-Factor Model

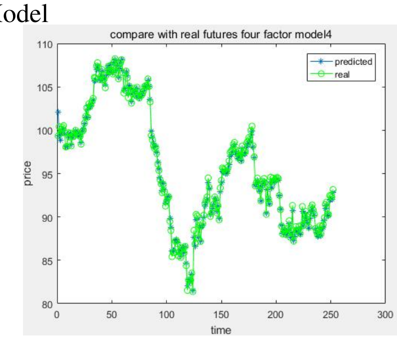


Figure 100: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The New Four-Factor Model

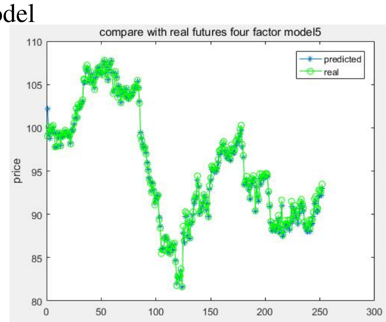


Figure 101: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The New Four-Factor Model

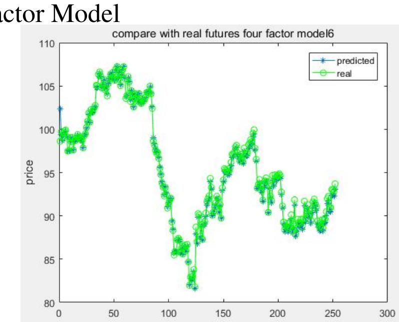


Figure 102: The Predicted and Observed Futures Prices for The Sixth Testing Futures Contract based on The New Four-Factor Model

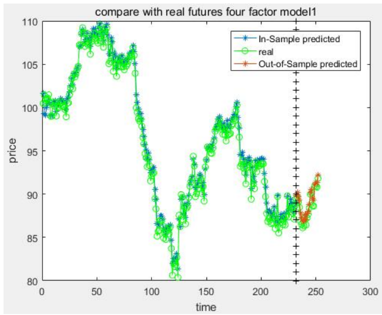


Figure 103: The In-Sample and Out-of-Sample Predicted and Observed Futures Prices for The First Testing Futures Contract based on The New Four-Factor Model

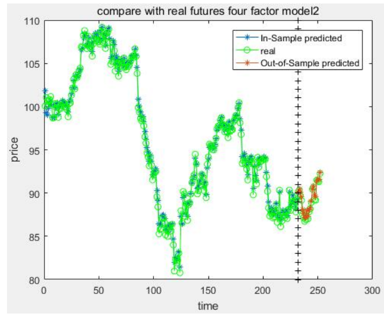


Figure 104: The In-Sample and Out-of-Sample Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The New Four-Factor Model

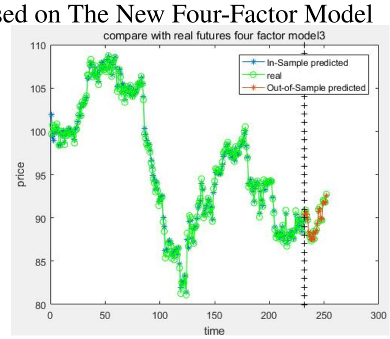


Figure 105: The In-Sample and Out-of-Sample Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The New Four-Factor Model

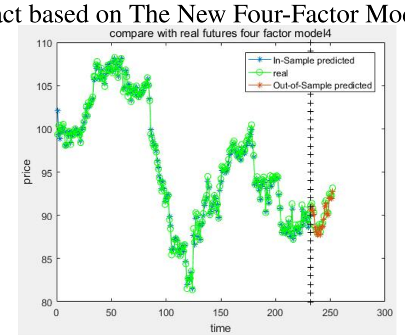


Figure 106: The In-Sample and Out-of-Sample Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The New Four-Factor Model

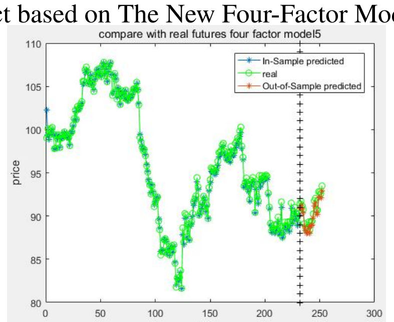


Figure 107: The In-Sample and Out-of-Sample Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The New Four-Factor Model

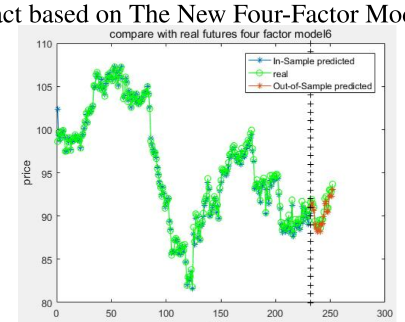


Figure 108: The In-Sample and Out-of-Sample Predicted and Observed Futures Prices for The Sixth Testing Futures Contract based on The New Four-Factor Model

plied spot price and convenience yield of the underlying commodity are estimated from options rather than futures. More specifically, the extended Kalman filter and the prices of European call options on WTI crude oil futures are used to estimate the Schwartz (1997) model. The motivation for this lies in the fact that option prices carry far more information on the volatility structure of the underlying asset than futures do. The implied volatility smile as well as the existence and strong use among practitioners of local and stochastic volatility models provide plenty of evidence for this statement.

The results will show that using the extended Kalman filter, the Schwartz (1997) Two-Factor model by means of historical prices for options with differing marturities and strikes is estimated. There is a conclusion that while the parameter sets obtained from the options are also able to provide a good fit when pricing futures, the opposite does not hold. Using the parameters obtained via the classical Schwartz (1997) approach to price options produces a significantly poorer fit than the approach introduced in this thesis. Hence, when the objective is to price both futures and options simultaneously within the Schwartz (1997) framework, the recommendation is to fit the model to options rather than futures by using the extended Kalman filter.

5.1 An Extended Application of The Two-Factor Model to Price European Options on Futures

Recalling the Schwartz Two-Factor model, the dynamics of the spot price of WTI crude oil is given following the differential equation:

$$dS/S = \mu dt + \sigma_1 dZ_1$$

with the consideration of risk-neutralization, the stochastic process can be rewritten as:

$$\widehat{dS/S} = (r - \delta)dt + \sigma_1 dZ_1$$

where the letters represent the same meanings as the above Two-Factor model: S is the spot price, σ_1^2 is the variance of proportional price changes, μ represents the expected rate of price changes, or say, the drift, dZ_1 denotes the increment to a standard Wiener process, and δ is the convenience yield of WTI crude oil, in this case.

In the same way, the convenience yield of WTI crude oil is following a similar stochastic process:

$$d\delta = \kappa(\alpha - \delta)dt + \sigma_2 dZ_2$$

where κ is the speed of the adjustment, and it is greater than zero; α represents the long-term mean yield; σ_2 is the volatility of the change in the convenience yield of WTI crude oil; and dZ_2 also denotes the increment to a standard Wiener process with $dz_1 dz_2 = \rho dt$, where ρ is the correlation coefficient between the two stochastic processes. With the risk-neutral consideration, the above process can be rewritten as:

$$\widehat{d\delta} = (\kappa(\alpha - \delta) - \lambda)dt + \sigma_2 dZ_2$$

As has been shown, with the boundary condition $F(S, \delta, T = 0) = S$, the solution of the PDE:

$$\frac{1}{2}\sigma_1^2 S^2 F_{SS} + \sigma_1\sigma_2\rho S F_{S\delta} + \frac{1}{2}\sigma_2^2 F_{\delta\delta} + (r - \delta)SF_S + [\kappa(\alpha - \delta) - \lambda]F_\delta - F_T = 0$$

is the price formula of a futures contract, and the solution can be written as:

$$F(S, \delta, T) = Sexp[-\delta\frac{(1-e^{-\kappa T})}{\kappa} + (r - \widehat{\alpha} + \frac{1}{2}\frac{\sigma_2^2}{\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa})T + \frac{1}{4}\sigma_2^2\frac{1-e^{-2\kappa T}}{\kappa^3} + (\widehat{\alpha}\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa}^2)\frac{1-e^{-\kappa T}}{\kappa^2}] \text{ with } \widehat{\alpha} = \alpha - \frac{\lambda}{\kappa}.$$

Apart from the above solution, many other previous studies provided different special solutions for the forward price of the underlying asset. For example, Hilliard and Reis (1998) proposed a special solution as follows:

$$F(S_t, \delta_t, t, T) = S_t A(\tau) e^{-H(\tau)\delta_t} \frac{1}{P(t,T)}$$

with

$$A(\tau) = exp[\frac{(H-\tau)(\kappa^2\alpha - \kappa\lambda\sigma_2 - \sigma_2^2/2 + \rho\sigma_1\sigma_2\kappa)}{\kappa^2} - \frac{\sigma_2^2 H^2}{4\kappa}]$$

and

$$H(\tau) = \frac{1-e^{-\kappa\tau}}{\kappa}$$

where S_t is the current level of the spot price; δ_t is the current level of the convenience yield; $\tau = T - t$ is the length of time to maturity; and $P(t, T)$ is the current price of a zero-coupon bond with maturity at time T. Noticeably, this special solution is derived based on

$$\widehat{d\delta} = (\kappa(\alpha - \delta) - \lambda\sigma_2)dt + \sigma_2 dZ_2$$

where the market price of risk λ is set to be $\lambda\sigma_2$. In this model, they are parameters which will be estimated based on data. The replacement of λ by $\lambda\sigma_2$ does not change the model.

Actually, there are other ways to set the second stochastic process with consideration of the market price of risk λ. Schwartz (1998) rearranged the formular, so that the market price of risk λ can be covered. The rearrangement can be described as follows:

$$dS = (r - \delta)Sdt + \sigma_1 SdZ_1$$

and

$$d\delta = \kappa(\widehat{\alpha} - \delta)dt + \sigma_2 dZ_2$$

with $dZ_1 dZ_2 = \rho dt$. The futures prices in this model are given by:

$$F(S, \delta, T) = Sexp(-\delta\tfrac{1-e^{-\kappa T}}{\kappa} + A(T))$$

where

$$A(T) = (r - \widehat{\alpha} + \tfrac{1}{2}\tfrac{\sigma_2^2}{\kappa^2} - \tfrac{\sigma_1\sigma_2\rho}{\kappa})T + \tfrac{1}{4}\sigma_2^2\tfrac{1-e^{-2\kappa T}}{\kappa^3} + (\widehat{\alpha}\kappa + \sigma_1\sigma_2\rho - \tfrac{\sigma_2^2}{\kappa})\tfrac{1-e^{-\kappa T}}{\kappa^2}$$

and

$$\widehat{\alpha} = \alpha - \tfrac{\lambda}{\kappa}.$$

In this thesis, no matter which of the above solutions for futures pricing is chosen, the following relationship is established:

$$F_S S = F$$

and

$$F_\delta = FH(\tau)$$

with $\tau = T - t$ and $H(\tau) = \frac{1-e^{-\kappa\tau}}{\kappa}$. [7]

Based on the Itô's Lemma and the risk-neutralized processes for the spot price and the convenience yield, the risk-neutralized process for the futures price change can be expressed as follows:

$$\widehat{dF} = [F_t + \tfrac{1}{2}F_{SS}\sigma_1^2S^2 + \tfrac{1}{2}F_{\delta\delta}\sigma_2^2 + F_{S\delta}\rho S\sigma_1\sigma_2 + F_S(r-\delta)S + F_\delta(\kappa(\alpha-\delta) - \lambda)]dt + F_S\sigma_1 S dZ_1 + F_\delta\sigma_2 dZ_2$$

set the drift to be zero and substitute F_S and F_δ, then the futures price change can be rewritten as:

$$\widehat{dF} = F\sigma_1 dZ_1 - FH(\tau)\sigma_2 dZ_2$$

Then, define another standard Wiener process Z_F and a volatility σ_F as follows:

$$\sigma_F dZ_F \equiv \sigma_1 dZ_1 - H(\tau)\sigma_2 dZ_2$$

Then, it is not hard to obtain that

$$\widehat{dF}/F = \sigma_F dZ_F$$

where

$$\sigma_F^2(\sigma_1, \sigma_2, \rho, \kappa, \tau) = \sigma_1^2 + \sigma_2^2 H(\tau)^2 - 2\rho\sigma_1\sigma_2 H(\tau)$$

(Schwartz, 1998 and Hilliard and Reis, 1998)

On the other hand, based on the above two processes and Black's option pricing theory[8], Hilliard and Reis (1998) proposed a formula for pricing call options on futures contracts.

$$C(t, T_1, T) = P(t, T_1)[F(t,T)N(d_1) - KN(d_2)]$$

[7] In all Schwartz's research, the current time t is set to be zero, hence, the T is essentially equal to τ

[8] see appendix 8.4

where $d_1 = \frac{ln(F(t,T)/K)+0.5\upsilon^2}{\upsilon}$, $d_2 = d_1 - \upsilon$.

Here, the υ can be calculated by the previous σ_F, since

$$\upsilon^2(t,T_1,T) = \int_t^{T_1} \sigma_2^2 dw + \int_t^{T_1} [\tfrac{\sigma_2}{\kappa}(1-e^{-\kappa(T-w)})]^2 dw - \int_t^{T_1} \tfrac{2\sigma_1\sigma_2\rho}{\kappa}(1-e^{-\kappa(T-w)})dw$$

then, solve the integrals, the υ^2 can be written as

$$\upsilon^2 = \sigma_1^2(T_1-t) - \frac{2\sigma_1\sigma_2\rho}{\kappa}[(T_1-t) - \frac{e^{-\kappa(T-T_1)} - e^{-\kappa(T-t)}}{\kappa}] + \frac{\sigma_2^2}{\kappa}[(T_1-t) - \frac{2}{\kappa}(e^{-\kappa(T-T_1)} - e^{-\kappa(T-t)}) + \frac{1}{2\kappa}(e^{-2\kappa(T-T_1)} - e^{-2\kappa(T-t)})]$$

where t is the current time point; T_1 is the maturity of a call option on a futures contract; T is the maturity of the underlying futures contract, with $T_1 \leq T$; $F(t,T)$ is the futures price at date t for the underlying futures contract, which will be matured at T; $C(t,T_1,T)$ is the price of a call with the maturity T_1 on the underlying futures contract with the maturity T at time point t; $P(t,T_1)$ is a ratio, which represents the time value of money, based on a zero-coupon bond with maturity T_1 and K is the strike price of the European-type call option.

Recalling the Schwartz Two-Factor model, the futures price with a particular maturity can be estimated by several futures contracts with different maturities. In order to combine the Hilliard and Reis'model with the Schwartz (1997) model, an assumption needed to be proposed: assume the current time point is zero (this has been mentioned in the footnote 7, because in all Schwartz' research, the current time point t is set to be zero). Insert the $F(S,\delta,T)$ into the Hilliard and Reis'model, then:

$$C(S,\delta,T_1,T) = P(t,T_1)[F(S,\delta,T)N(d_1) - KN(d_2)]$$

where $d_1 = \frac{ln(F(S,\delta,T)/K)+0.5\upsilon^2}{\upsilon}$, $d_2 = d_1 - \upsilon$. Since the trading prices of the European calls on WTI crude oil can be directly observed, based on the two stochastic processes of S and δ and the state variable model, the spot price and the convenience yield can be estimated from the model.

5.2 Data and Assumptions

In this section, in order to simplify the complex state space model, the weekends and other non-trading days can be ignored, which means that the trading days are considered to be continuous. Based on the efficient market hypothesis, powerful information from the non-trading days (e.g. weekends and Christmas holiday) can be immediately reflected by price after the non-trading day. Hence, the continuous trading period is a reasonable and popular assumption in the financial world. Second, in the Hilliard and Reis'model, there are two types of maturities, which are the maturities of the futures contracts and the maturities of the calls on the futures contracts. Since the maturity of a European call on a particular WTI futures contract is designed by CME as three business days before the termination of trading in the underlying futures contract, in order to simplify the model, in this thesis, the maturity of European call on a particular futures contract is assumed to be the same as the maturity of the underlying WTI crude oil futures contract. The next assumption is about the length of the two types of the maturity. In this section, in order to simplify the extremely complicated process of calculation, I capture the monthly dynamics of the length of the maturity instead of daily dynamics of the length of the maturity, even if a consideration of daily dynamics of the length of the maturity might provide more accurately estimated results. In spite of this, in this section, instead of a single average T, the dynamics of the T with the time passing is added to the model, which is expected to provide a better fit for the data. Moreover, as has been explained in the previous section, the Hilliard and Reis'model includes a discounting factor P, which involves the zero coupon bond with the same maturities as the European-type calls. Normally, the Treasury Bill is widely used in the financial domain, but it is not available for the long term. Hence, in this thesis, the T-Notes are used with an adjustment to reduce the effects of coupon payments. However, in order to simplify the process of the calculation, the time value of the coupon is not considered in this section. To be more specific, in this section, the P is defined as: $P = \frac{P_{marketprice} - \sum coupons^*}{P_{facevalue}}$, where the $coupons^*$ represents the coupons will be paid in the rest life of the bonds. The fifth assumption is a measurement of the level of the cost of financing. The interest rate is assumed to be equal to 2% per year in this section. Furthermore, in order to simplify the calculation, based on the non-arbitrage assumption, the drift μ in the aforementioned model is replaced by the interest rate r, which means that the drift μ is set as a constant in the implementation. Last, in order to match the Schwartz's model system, the current time point t is always zero, as it was dealt with in the Schwartz (1997) model system.

The data used in this section are collected from Bloomberg. To be more specific, there will be 15 European calls on WTI crude oil. All their strike prices are around 100 dollars per barrel. To be more specific, three different strike prices are used in this section, and they are 102.5 dollars per barrel, 100 dollars per barrel and 97.5 dollars per barrel, respectively. In addition to this, each strike price corresponds to five calls, which have the same term structure. The five options contracts' maturities are one month, two months, five months, eight months and one year, respectively, and the test period is the year from 1st March 2013 to 28th Feb 2014. Specifically, the used calls on WTI crude oil would mature at April 2014, May 2014, August 2014, November 2014, and March 2015, respectively. On the other hand, the T-Notes would also mature at April 2014, May 2014, August 2014, November 2014, and March 2015, respectively.

5.3 The Empirical Results of The Extended Application of The Two-Factor Model in Pricing European Options on Futures

Figures 109, 111 and 113 exhibit the estimated spot prices of WTI crude oil by the extended Kalman filter from the calls with the strike prices 97.5, 100 and 102.5, respectively. It is easy to see that the estimated spot prices from all three different strike prices of WTI crude oil fluctuate between 80 dollars per barrel and 120 dollars per barrel in the test period, and the three estimated spot prices are extremely similar. At the beginning of the test period, the estimated spot price fluctuates considerably between 90 and 120. This considerable fluctuation might be caused by the Kalman filter. Specifically, the Kalman filter algorithm sometimes needs several steps to make the estimated state variables stable. Then, the estimated spot price fluctuates reasonably. To be more specific, the spot price decreases from over 100 dollars per barrel to the valley of the testing period. Specifically, the valleys of the three pictures are quite different: the value is about 82 dollars per barrel for the strike price K=102.5, while the other two are significantly higher (about 85 dollars and 88 dollars per barrel for the strike prices are 100 and 102.5, respectively). Then, the estimated spot price rapidly increases from the valley to about 115 dollars per barrel, which is the peak of the estimated spot price in the test period. After the peak, the estimated spot price of WTI crude oil rapidly decreases to around 90 dollars per barrel again. Then after a slight fluctuation, it moves up to over 100 dollars. Overall, the spot prices estimated from the different strike prices have slight differences, but their trends, values and the intervals of the fluctuations are extremely similar.

Figures 110, 112 and 114 show the estimated convenience yield of WTI crude oil by the extended Kalman filter from the strike prices 97.5, 100 and 102.5, respectively. Overall, based on the three different strike prices, all estimated convenience yields of WTI crude oil fluctuate between 0.05 and about 0.3. To be more specific, at the beginning of the test period, the estimated convenience yield drops from about 0.2 to about 0, which is the valley in the test period, and then it moves up to about 0.3, which is the peak in the whole test period. In the second half of the test period, the convenience yield of WTI crude oil moves downward from the peak, and then fluctuates around 0.1. However, there is a situation worth noting: At the beginning, the estimated convenience yields based on K=100 and K=102.5 drop to about 0.25, while this peg does not appear in Figure 110. Otherwise, all three estimated convenience yields are extremely similar. Last, based on the definition of the convenience yield, the figures show that holding the physical WTI crude oil is actually a way of making money over the entire test period.

Figures 115, 116 and 117 show the term structure of the European calls. In this section, the 50^{th}, 100^{th} and 200^{th} testing days are chosen to show the term structure. To be more specific, Figure 115 exhibits the term structure on the 50^{th} trading day in the test period; Figure 116 illustrates the term structure on the 100^{th} trading day in the test period; and Figure 117 shows the term structure on the 200^{th} trading day in the test period. In all three pictures, the horizontal ordinates are the months to maturity and the vertical ordinates represent the prices of the calls. As shown in the pictures, the dotted lines represent the model estimated prices corresponding to the different calls based on different strike prices, while the solid lines represent the observed prices corresponding to the different calls based on different strike prices. It is not hard to observe that the model estimated prices are close to the observed prices of calls in all three pictures, whatever the strike prices are. In addition, the model estimated calls prices are smoother than the observed trading prices of the calls. Based on the above analysis, the model is useful in pricing the European calls.

The effectiveness of the estimated parameters for estimating the European calls are shown within Figures 121-126. To be more specific, Figures 121, 123 and 125 show the observed Figures prices of European calls with the strike price K=97.5, K=100 and K=102.5 respectively. In each figure, it is not hard to observe that the calls with longer terms to maturity tend to be more valuable. On the other hand, Figures 122, 124 and 126 show the model estimated prices of European calls with the strike price K=97.5, K=100 and K=102.5 respectively. In each figure, the relationship between the term to maturity and the price is similar to the observed prices in the market. Comparing Figures 121, 123 and 125 with Figures 122, 124 and 126, the estimated prices of European calls show a stronger correlation between the term to maturity and the price, while the observed prices of the first four calls are closer in some part of the test period.

Figures 118, 119 and 120 illustrate the term structure of the futures. Similar to Figures 115, 116 and 117, in this part, the 50^{th} 100^{th} and 200^{th} testing days are also chosen to show the term structure of the futures based on the parameters estimated on the Two-Factor model with the calls. To be more specific, Figure 118 exhibits the term structure on the 50^{th} trading day in the test period; Figure 119 illustrates the term structure on the 100^{th} trading day in the test period; and Figure 120 shows the term structure on the 200^{th} trading day in the test period. In all three pictures, the horizontal ordinates are still the months to maturity, while the vertical ordinates are the prices of the futures. As shown in the pictures, the blue lines represent the observed futures prices in each picture, while the red, green and pink solid lines correspond with the model prices of the futures based on the strike price of the European calls with K=97.5, K=100 and K=102.5 respectively. Obviously, on the one hand, in all three pictures, the estimated futures prices are higher when the strike price of the calls is lower. To be more specific, the estimated futures prices that are based on the strike price of the calls K=97.5 are always higher than the estimated futures prices that are estimated from the calls with strike price K=100 and K=102.5 in each figure. Similarly, the estimated futures prices that are based on the strike price of the calls K=100 are always higher than the estimated futures prices that are estimated from the calls with strike price K=102.5 in each figure, but lower than the estimated futures prices that are estimated from the calls with strike price K=97.5 in each figure. On the other hand, the observed futures prices tend to be higher than the model estimated futures prices with relatively short terms to maturity, while the observed futures prices tend to be lower than the model estimated futures prices with relatively long terms to maturity. Overall, the model is also useful for pricing futures contracts because the model estimated prices are close to the observed prices of the futures in the market.

The effectiveness of the estimated parameters for estimating the futures are shown in the following pictures. To be more specific, the observed futures prices and the model estimated futures prices based on the different strike prices of the European calls are shown within Figures 127-130. To be even more specific, the observed futures prices are shown in Figure 127, while the model estimated prices based on K=97.5, K=100 and K=102.5 are shown in Figure 128, Figure 129 and Figure 130 respectively. In the test period, the observed futures prices show a strict backwardation, which means that the futures contract with a shorter term to maturity tends to be more valuable. In Figure 128, Figure 129 and Figure 130, the backwardation of WTI crude oil can be seen too, however, it is not as clear as the observed futures prices exhibited at the beginning of the test period. As has been explained, the difference between the observed futures prices and the model estimated futures prices might be caused by the Kalman filter. Even if there are slight differences between Figure 127 and the other three figures, the model is still useful in pricing a futures contract.

Apart from the forward curves, the figures below also show how good the results in the model are. The fitting results are shown within Figures 131-160. To be more specific, the 30 figures can be divided into three groups, and each group contains 10 pictures. In the first group (Figures 131-140), the fitting results in the model estimation based on the European calls with the strike price K=97.5 are illustrated: the first five pictures (Figures 131-135) show the fitting results for the options. Apart from the European call, which would mature in November 2014 and March 2015, all the other three calls fit extremely well with the data. Viewed from Figure 134 (the European call which would mature in November 2014), the estimated price of the call is significantly higher than the observed call price in the first half of the period, while as seen in Figure 135 (the European call which would mature in March 2015), the estimated price of the call is significantly lower than the observed call price in the second half of the period. This exception might be explained by the seasonality of crude oil, because November can be seen as the beginning of winter and March can be seen as the end of the winter. The fitting results for the futures are illustrated within Figures 136-140. It is no surprise that the fitting results for futures contracts are worse than the calls because the parameters are estimated based on the calls on the futures contracts. However, there is an exception: comparing Figure 134 with Figure 139, the fitting for the futures contract, which would mature in November 2004, is actually better than the fitting for the calls. This phenomenon is not even standalone because the same situation happens in the results estimated from the calls with K=100 and K=102.5. In addition, the fitting results in the model estimation based on the European calls with the strike prices K=100 and K=102.5, are illustrated in the other two groups, which are contained within Figures 141-150 and Figures 151-160, respectively. It is not hard to obtain that all three groups show similar fitting results.

Last, the table below exhibits the estimated parameters from the European calls with the different maturities. Overall, the estimated parameters from the calls with the different maturities are similar. To be more specific, κs are around 0.8; αs and λs are close to 0.05; σ_1s and σ_2s are around 0.45 and 0.5, respectively; and ρs are around 0.75. These estimated parameters state that: first, the degree of mean reversion of the convenience yield of WTI crude oil is not very high; second, the long-term return investment on the convenience yield of WTI crude oil is positive during the test period; last, the risk to the spot price of WTI crude oil is lower than the risk to the convenience yield of WTI crude oil.

Paremeters	Estimated κ	Error
$K = 97.5$	0.8584	SE1
$K = 100$	0.8135	SE2
$K = 102.5$	0.7863	5.7682e-05

Paremeters	Estimated α	Error
$K = 97.5$	0.0661	SE3
$K = 100$	0.0557	0.00293
$K = 102.5$	0.0821	5.6267e-03

Paremeters	Estimated λ	Error
$K = 97.5$	0.0738	0.01009
$K = 100$	0.0537	0.01343
$K = 102.5$	0.0621	1.4234e-02

Paremeters	Estimated σ_1	Error
$K = 97.5$	0.2593	0.00157
$K = 100$	0.2624	0.00354
$K = 102.5$	0.2713	4.022435e-03

Paremeters	Estimated σ_2	Error
$K = 97.5$	0.4791	0.01420
$K = 100$	0.4818	0.01327
$K = 102.5$	0.5041	1.23292e-02

Paremeters	Estimated ρ	Error
$K = 97.5$	0.7642	SE4
$K = 100$	0.7582	SE5
$K = 102.5$	0.7687	SE6

Note: $SE1 = \sqrt{-5.984536e - 09 + ComputationalError1}$
$SE2 = \sqrt{-5.153188e - 09 + ComputationalError2}$
$SE3 = \sqrt{-1.915785e - 05 + ComputationalError3}$
$SE4 = \sqrt{-5.039682e - 04 + ComputationalError4}$
$SE5 = \sqrt{-3.731736e - 04 + ComputationalError5}$
$SE6 = \sqrt{-2.635770e - 04 + ComputationalError6}$

Table 5.3: Estimated Parameters from European Calls

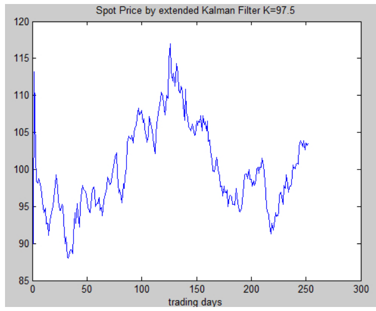


Figure 109: Estimated Sopt Price of WTI Crude Oil from European Calls With The Strike Price K=97.5

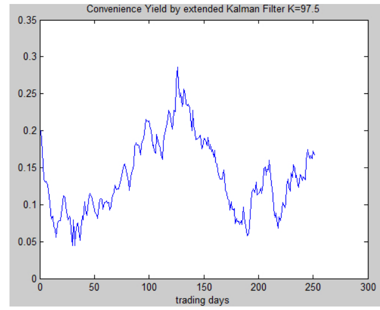


Figure 110: Estimated Convenience Yield of WTI Crude Oil from European Calls With The Strike Price K=97.5

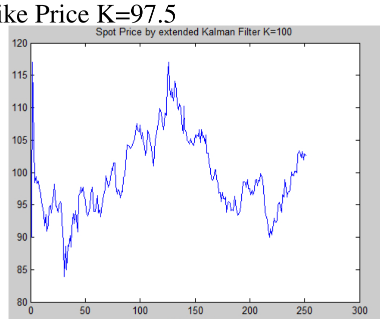


Figure 111: Estimated Sopt Price of WTI Crude Oil from European Calls With The Strike Price K=100

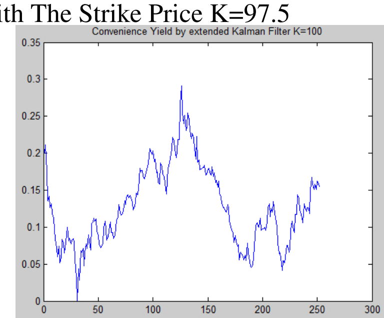


Figure 112: Estimated Convenience Yield of WTI Crude Oil from European Calls With The Strike Price K=100

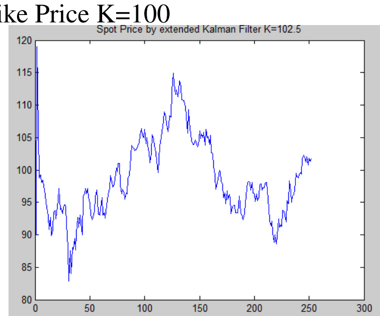


Figure 113: Estimated Sopt Price of WTI Crude Oil from European Calls With The Strike Price K=102.5

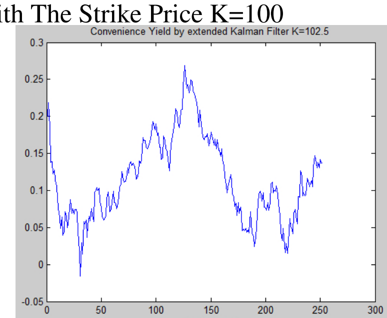


Figure 114: Estimated Convenience Yield of WTI Crude Oil from European Calls With The Strike Price K=102.5

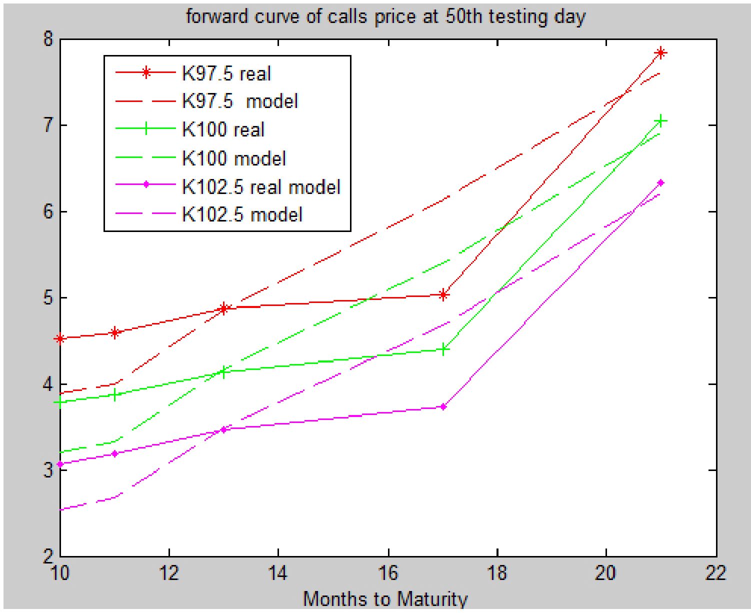


Figure 115: Forward Curves of Calls On The 50th Testing Day

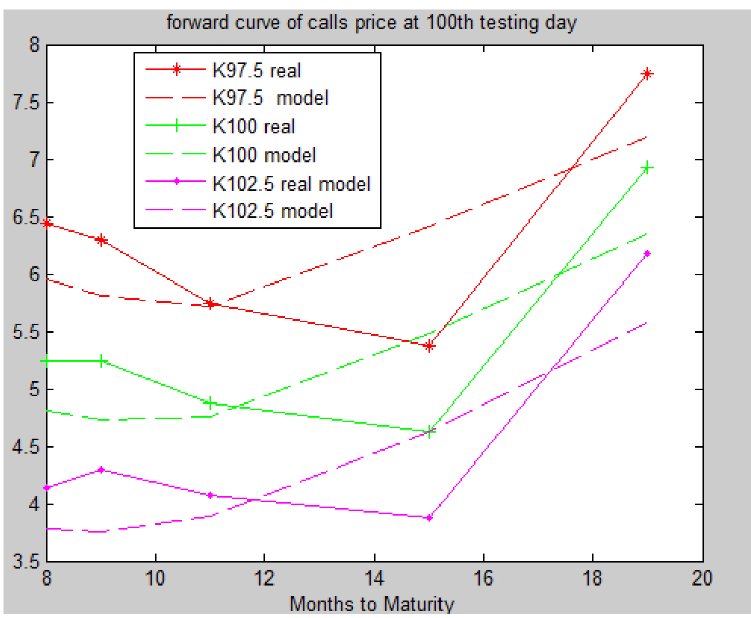


Figure 116: Forward Curves of Calls On The 100th Testing Day

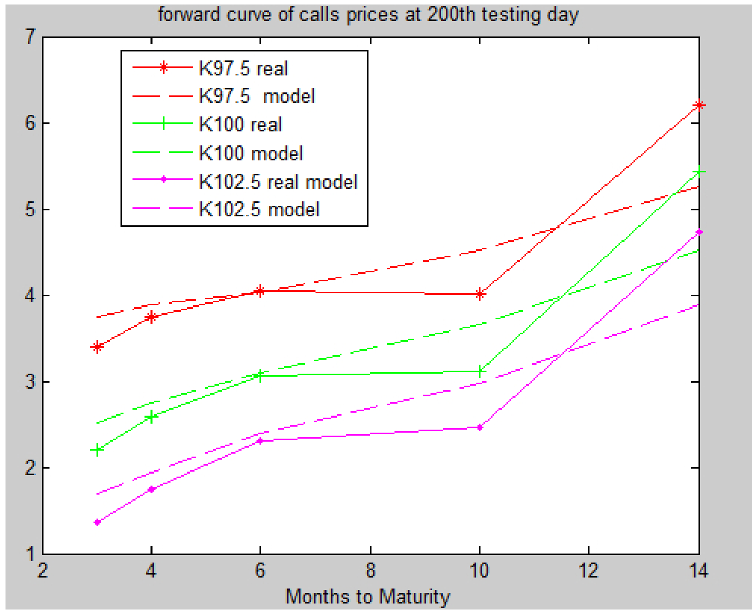


Figure 117: Forward Curves of Calls On The 200th Testing Day

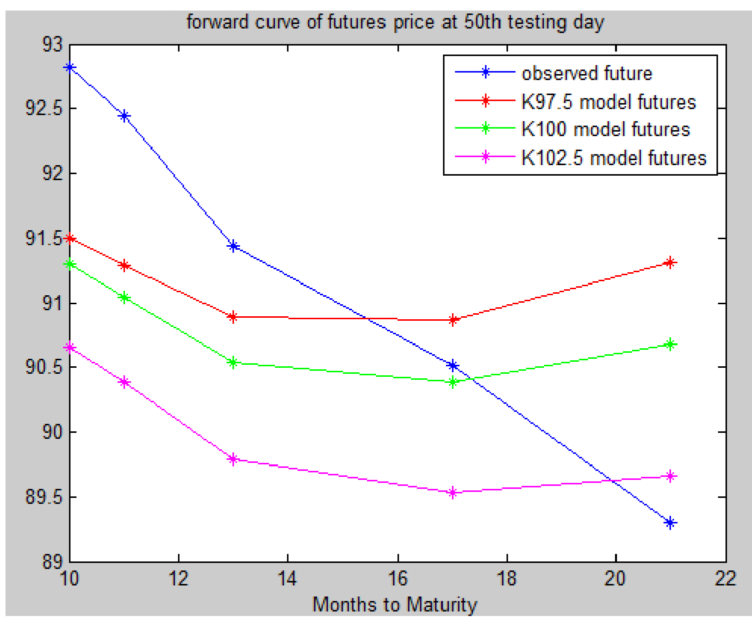


Figure 118: Forward Curves of Futures On The 50th Testing Day

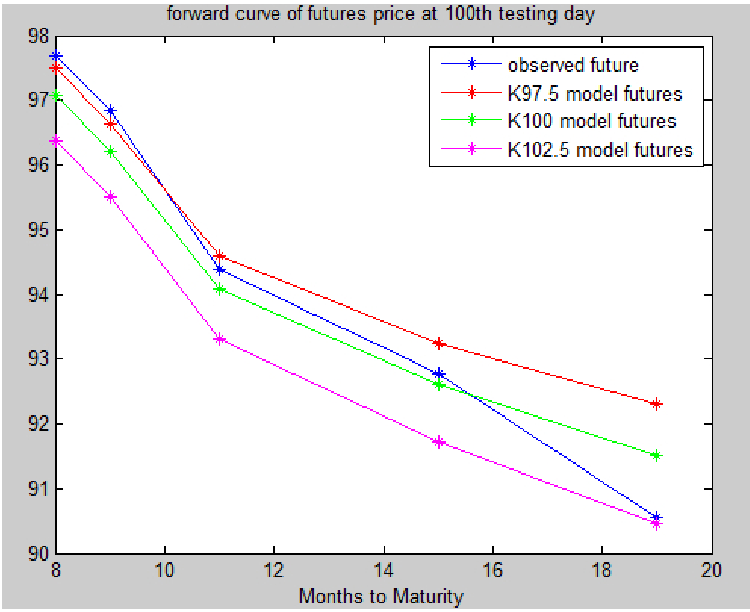


Figure 119: Forward Curves of Futures On The 100th Testing Day

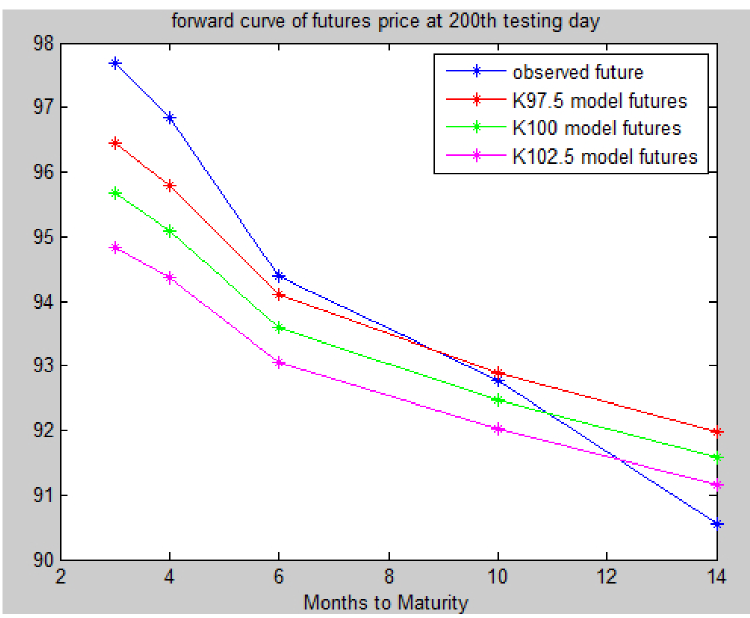


Figure 120: Forward Curves of Futures On The 200th Testing Day

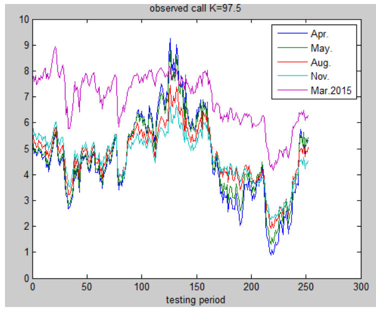


Figure 121: Observed Prices of Calls on WTI Futures Contracts with Strike Price K=97.5

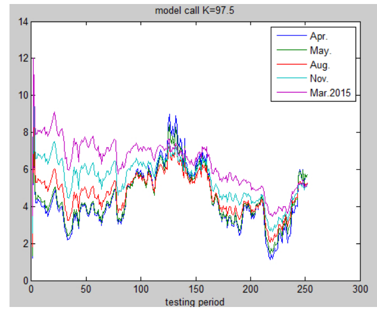


Figure 122: Model Estimated Prices of Calls on WTI Futures Contracts with Strike Price K=97.5

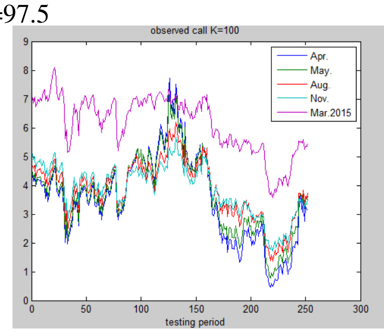


Figure 123: Observed Prices of Calls on WTI Futures Contracts with Strike Price K=100

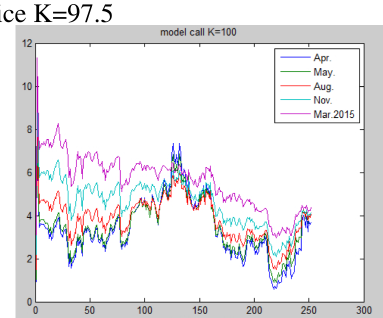


Figure 124: Model Estimated Prices of Calls on WTI Futures Contracts with Strike Price K=100

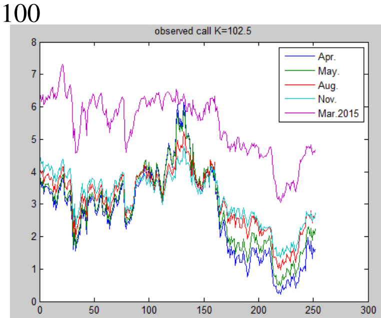


Figure 125: Observed Prices of Calls on WTI Futures Contracts with Strike Price K=102.5

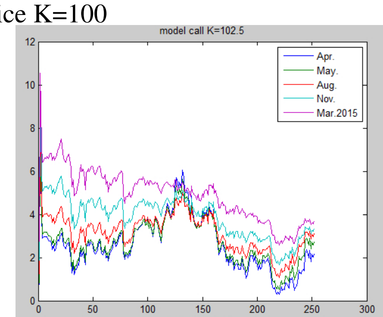


Figure 126: Model Estimated Prices of Calls on WTI Futures Contracts with Strike Price K=102.5

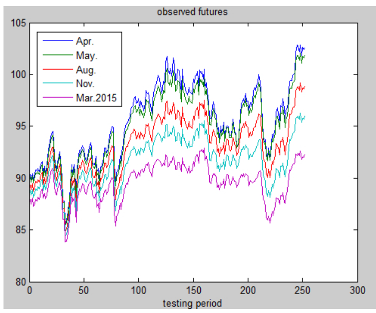


Figure 127: Observed Prices of Futures Contracts With Different Maturities

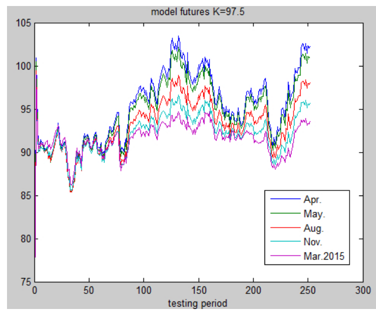


Figure 128: Model Estimated Prices of Futures Contracts With Different Maturities based on The European Calls with The Strike Price K=97.5

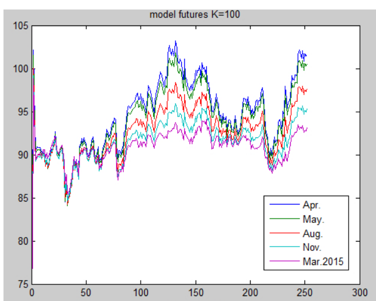


Figure 129: Model Prices of Futures Contracts With Different Maturities based on The European Calls with The Strike Price K=100

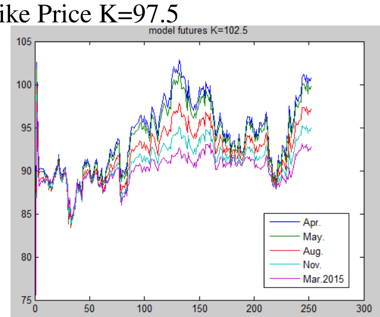


Figure 130: Model Prices of Futures Contracts With Different Maturities based on The European Calls with The Strike Price K=102.5

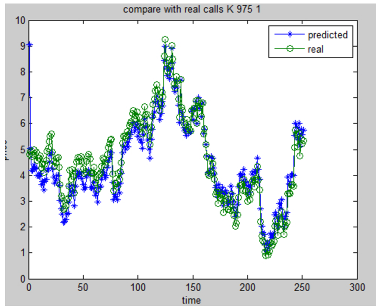


Figure 131: The Predicted and Observed Call Prices for The First Testing Call Contract based on The European Calls (K=97.5)

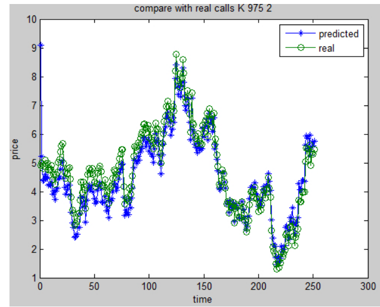


Figure 132: The Predicted and Observed Call Prices for The Second Testing Call Contract based on The European Calls (K=97.5)

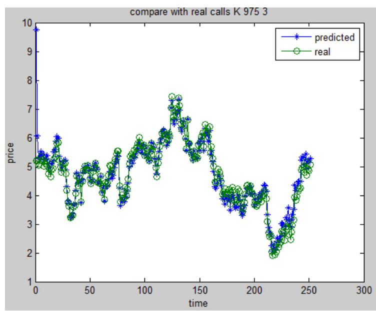


Figure 133: The Predicted and Observed Call Prices for The Third Testing Call Contract based on The European Calls (K=97.5)

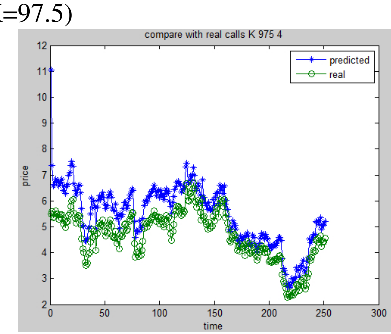


Figure 134: The Predicted and Observed Call Prices for The Fourth Testing Call Contract based on The European Calls (K=97.5)

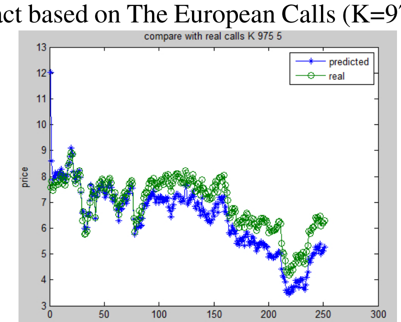


Figure 135: The Predicted and Observed Call Prices for The Fifth Testing Call Contract based on The European Calls (K=97.5)

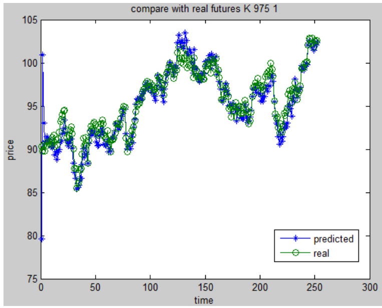


Figure 136: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The European Calls (K=97.5)

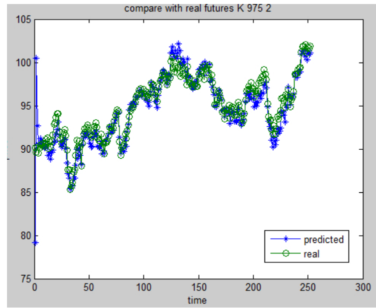


Figure 137: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The European Calls (K=97.5)

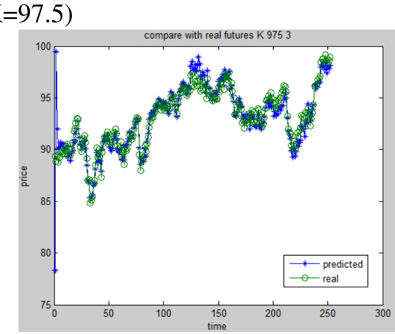


Figure 138: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The European Calls (K=97.5)

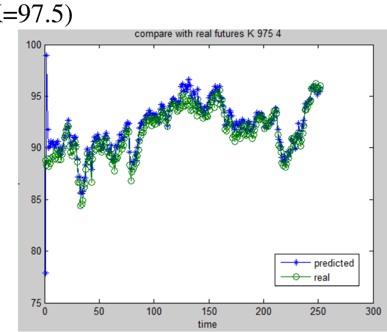


Figure 139: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The European Calls (K=97.5)

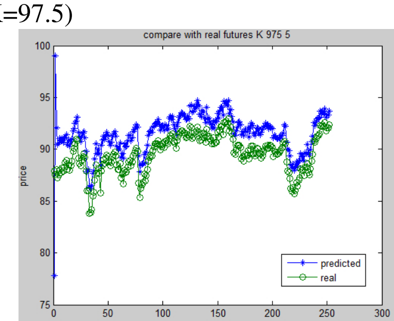


Figure 140: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The European Calls (K=97.5)

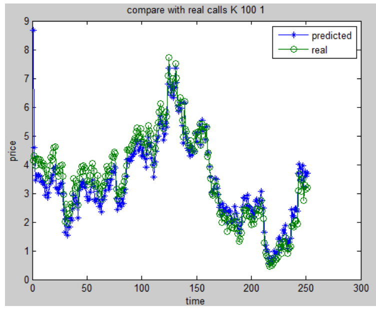


Figure 141: The Predicted and Observed Call Prices for The First Testing Call Contract based on The European Calls (K=100)

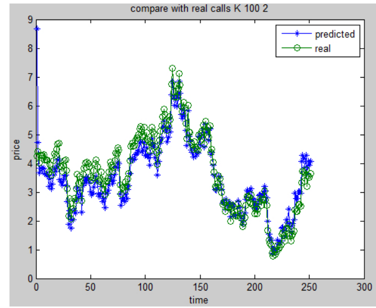


Figure 142: The Predicted and Observed Call Prices for The Second Testing Call Contract based on The European Calls (K=100)

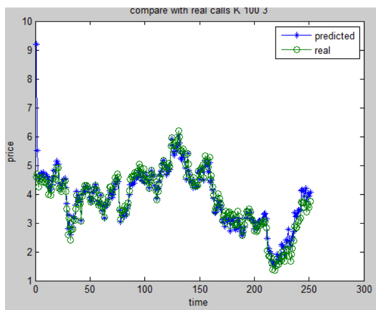


Figure 143: The Predicted and Observed Call Prices for The Third Testing Call Contract based on The European Calls (K=100)

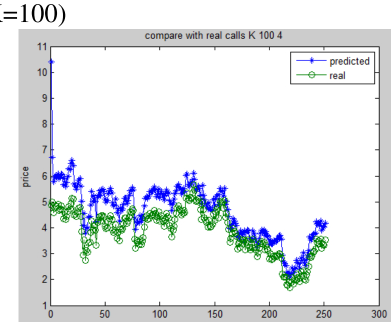


Figure 144: The Predicted and Observed Call Prices for The Fourth Testing Call Contract based on The European Calls (K=100)

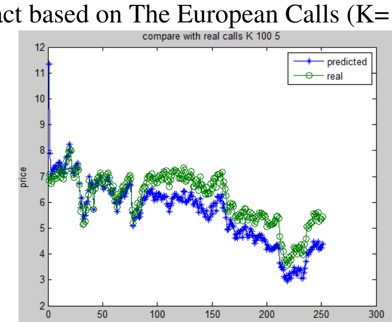


Figure 145: The Predicted and Observed Call Prices for The Fifth Testing Call Contract based on The European Calls (K=100)

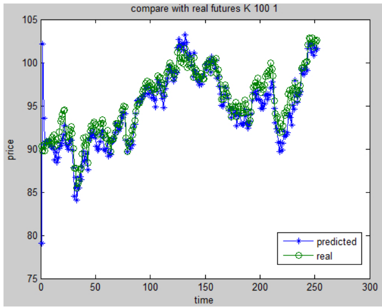


Figure 146: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The European Calls (K=100)

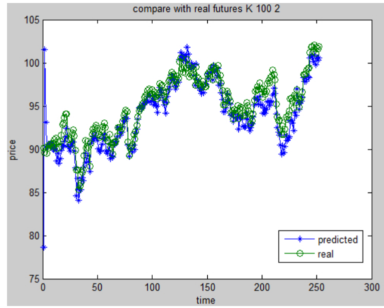


Figure 147: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The European Calls (K=100)

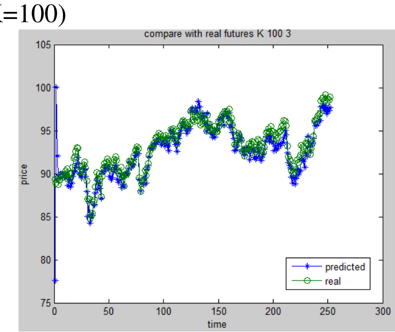


Figure 148: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The European Calls (K=100)

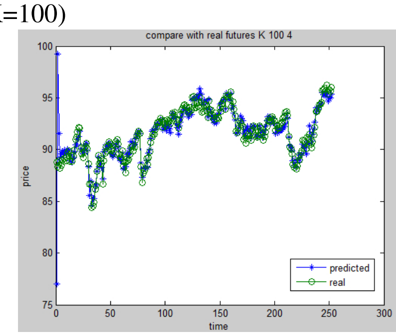


Figure 149: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The European Calls (K=100)

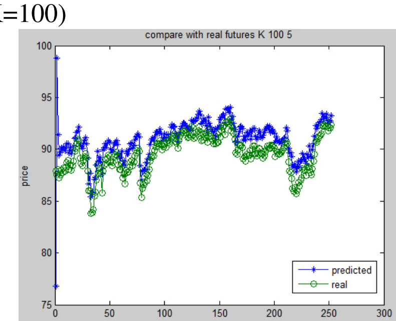


Figure 150: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The European Calls (K=100)

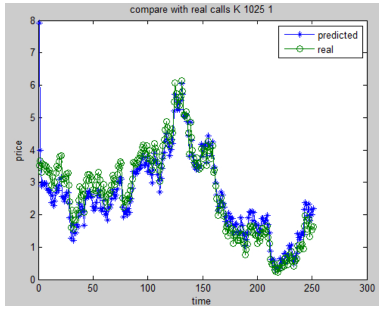


Figure 151: The Predicted and Observed Call Prices for The First Testing Call Contract based on The European Calls (K=102.5)

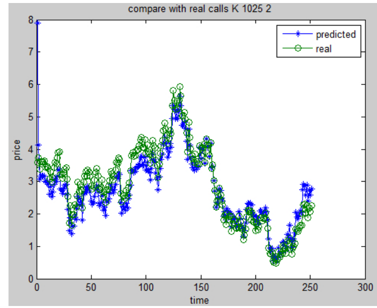


Figure 152: The Predicted and Observed Call Prices for The Second Testing Call Contract based on The European Calls (K=102.5)

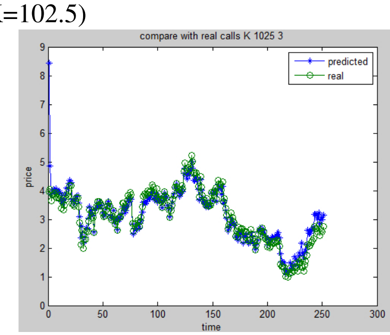


Figure 153: The Predicted and Observed Call Prices for The Third Testing Call Contract based on The European Calls (K=102.5)

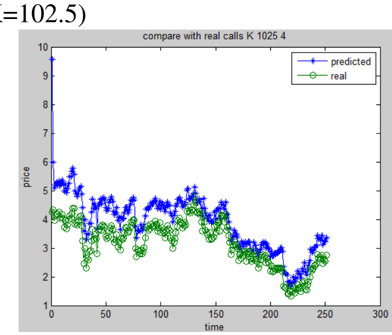


Figure 154: The Predicted and Observed Call Prices for The Fourth Testing Call Contract based on The European Calls (K=102.5)

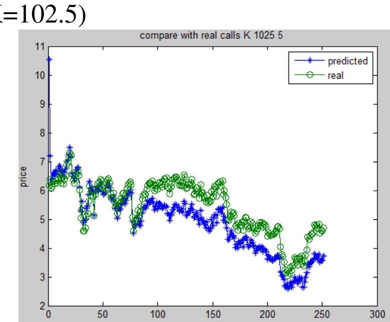


Figure 155: The Predicted and Observed Call Prices for The Fifth Testing Call Contract based on The European Calls (K=102.5)

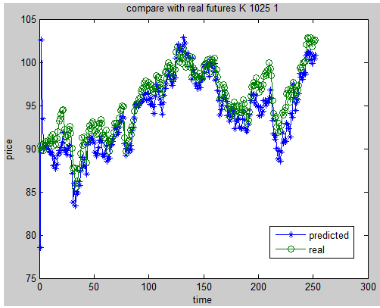


Figure 156: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The European Calls (K=102.5)

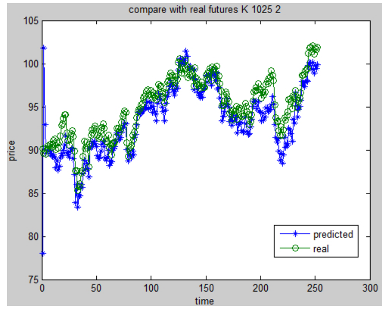


Figure 157: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The European Calls (K=102.5)

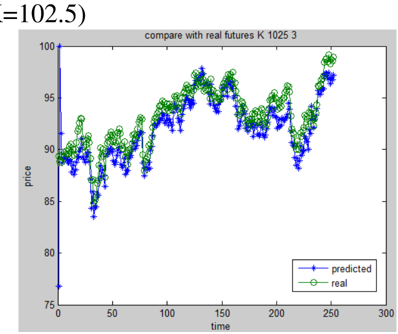


Figure 158: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The European Calls (K=102.5)

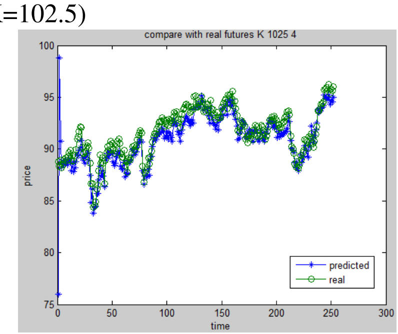


Figure 159: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The European Calls (K=102.5)

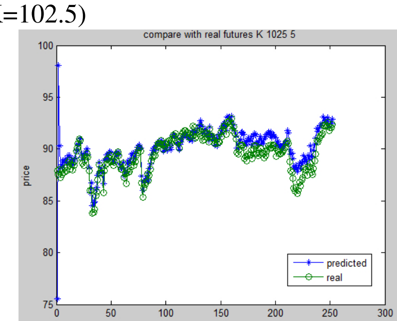


Figure 160: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The European Calls (K=102.5)

6 Comparisons between The Traditional Schwartz Factor Model System, The Transformations of The Traditional Schwartz Factor Model System and Its Derivative Models

In this thesis, the traditional Schwartz factor model system, the transformations of the traditional Schwartz factor model system and its derivative models are introduced based on the Kalman filter algorithm or the extended Kalman filter algorithm in the last sections. In this section, the aforementioned models will be compared and a model with another estimating method will be involved in the comparison.

Overall, the models introduced in this thesis are all useful and the comparable parts of the results in the corresponding models are similar. For example, the estimated parameters and the state variables are similar in different estimating algorithms (the results in the least square method will be shown in the section 6.1); with the same data, the results in the original Two-Factor model, the new Three-Factor model and the new Four-Factor model show extremely similar trends for the state variables; within the same test period, based on the Two-Factor model, the estimated spot price and the estimated convenience yield of crude oil are similar based on the calls and the futures, and so on.

6.1 Comparison of the Two-Factor Model with Different Estimation Algorithms

Even if the Kalman filter is considered to be the benchmark for estimating results in the Schwartz (1997) model system, there are still other options for running the model. Cortazar and Schwartz (2002) proposed a basic idea of the Least Squares algorithm to estimate the parameters in the system and compared the results with the Kalman filter algorithm. Then, Yan and Li (2008) used the Least Squares idea to build a Four-Factor model with an exchange-rate factor. However, the error predicted by using the estimated parameters is quite large.

6.1.1 The Two-Factor Model with Least Square Estimating Method

The steps of the Least Square algorithm of the traditional Two-Factor model can be summarized as follows:

1. Input initial values of parameters: $\Omega : \kappa, \alpha, \lambda, \sigma_1, \sigma_2, \rho$ and input the observable interest rate r.

2. Based on the input set of parameters, by solving the following minimization problem, the unobservable spot price of crude oil S and the instantaneous convenience yield of crude oil δ at each time point can be obtained or estimated. To be more specific, the minimization will be solved by using the Trust Region algorithm.

$$\min_{S,\delta} \sum_{j=1}^{M_i} [\ln \widehat{F_{ij}}(S, \delta, T) - \ln F_{ij}]^2 \text{ with } i = 1, ..., N \text{ and j=1,...,M}$$

where $\widehat{F_{ij}}$ are the model estimated futures prices for trading day i and the contract with the maturity j; F_{ij} are the traded (observed) futures price at trading day i for the contract with the maturity j. In addition to these, M and N are the number of futures contracts and the number of trading days, respectively.

3. Using the estimated spot price of crude oil S and the estimated instantaneous convenience yield of crude oil δ that are calculated by step 2, then the following minimization problem can be easily solved by using the Trust Region algorithm. Then, an optimal set of estimators is obtained:

$$\min_{\Omega} \sum_{i=1}^{N} \sum_{j=1}^{M_i} [\ln \widehat{F_{ij}}(S, \delta, T) - \ln F_{ij}]^2$$

Stop iterating the operation if Ω converges. Otherwise, repeat the operation from step 2 until the set of Ω converges.

Among the steps of the Least Square method, both steps of the minimization processes are solved by the Trust Region algorithm, which belongs to the popular artificial intelligence algorithms. In the appendix, the trust region algorithm will be introduced.

6.1.2 Data and Assumptions of The Two-Factor Model with Least Square Estimating Method

In order to compare the results, the Two-Factor model with the Least Square estimating method is run with the same data and assumptions as the Two-Factor model with the Kalman filter algorithm. To be more specific, in order to simplify the complex state space model, the weekends and other non-trading days can be ignored, which means that the trading days are considered to be continuous. Based on the efficient market hypothesis, powerful information from the non-trading days (e.g. weekends and Christmas holiday) can be immediately reflected in the price after the non-trading day. Hence, the continuous trading period is a reasonable and popular assumption in the financial world. The next assumption is that each futures contract is immediately executed on the first day when they mature. This is an assumption about the length of the maturity. Based on this assumption, the length of the maturity can be measured more accurately. The third assumption is a measurement of the level of the cost of financing. In this section, the interest rate is assumed to be equal to 2% per year. Furthermore, to simplify the calculation, based on the non-arbitrage assumption, the drift μ in the aforementioned model is replaced by the interest rate r, which means that the drift μ is set as a constant in the implementation. In addition, in this section, instead of a single average T, the dynamics of the T with the time passing is added to the model, which is expected to get better fit for the data.

As has been explained in the previous sections, when the spot price of crude oil is set as an unobservable state variable, the observable futures prices are the only data collected when the model is implemented. In this section, the data of futures prices of WTI crude oil is collected from Bloomberg. Specifically, in this section, five futures contracts will be used. Their maturities are one month, two months, four months, seven months and ten months, respectively, and the test period is the year from 2^{nd} November 2010 to 31^{st} October 2011. To be more specific, the five futures contracts would mature at December 2011, January 2012, March 2012, June 2012 and September 2012, respectively.

6.1.3 Empirical Results in The Two-Factor Model with Least Square Estimating Method

Figure 161 and Figure 162 exhibit the estimated unobservable state variables estimated by the Least Square Method. On the one hand, Figure 161 points out the estimated spot price of crude oil based on the Least Square algorithm. After a slight decrease at the beginning, the estimated spot price of crude oil keeps increasing to over 115 dollars per barrel, which is the peak of the entire test period, in the first half of the test period, and then the estimated spot price jumps down to about 100 dollars per barrel. After a short and mild fluctuation, it jumps down again to 75 dollars per barrel, and at the end of the test period, the estimated spot price rebounds to about 95 dollars per barrel. In the meantime, Figure 162 shows the trend of the estimated convenience yield of crude oil based on the same estimating algorithm. The estimated convenience yield of crude oil fluctuates between 8% and about -12%. Under the above assumption that the drift of crude oil is seen as the interest rate (μ=r) in this model, the estimated convenience yield of crude oil firstly increases to about 4% from -6%, then shows a rapid and fluctuant trend of decrease until it is about -8%. In the next step, the convenience yield suddenly moves up to about 8%, which is the peak of the test period, before it experiences a considerably fluctuant downward sub-period until the valley about -12%. At the end of the test period, the estimated convenience yield rapidly moves up to about 4% again. As for the estimated parameters, first, the degree of mean reversion of the convenience yield of WTI crude oil is not very high; second, the long-term return investment on the convenience yield of WTI crude oil is positive during the test period; last, the risk on the spot price of WTI crude oil is higher than the risk on the convenience yield of WTI crude oil.

Recalling the previously positive relationship between the estimated spot price and the estimated convenience yield of crude oil in section 2.3.3, the positive relationship is maintained between the observable spot price and the estimated convenience yield of WTI crude oil. Similar to the results in the Two-Factor model estimated using the Kalman filter method, in Figure 161 and Figure 162, the exception is still shown in this section. To be more specific, around the 50^{th} trading day, the estimated spot price is moving up stably, but the estimated convenience yield of crude oil suddenly jumps downwards.

Next, the five chosen futures contracts are shown together in Figure 163 and Figure 164, in order to see whether there are significant differences between the observed prices of WTI crude oil futures contracts and the estimated prices of WTI crude oil futures contracts in the Two-Factor model. To be more specific, the real observed futures prices of the five chosen futures contracts are shown in Figure 163. Then, the estimated model futures prices of the five chosen futures contracts that are estimated from the Two-Factor model based on the Least Square method, are shown in Figure 164. In order to see the trend clearly, in Figure 164 only 26 points are chosen in the figure, which were picked the first trading day of each ten trading days. Overall, there are no significant differences between the two figures. On the one hand, WTI crude oil did not follow a strict backwardation or a strict contango during the test period; on the other hand, it is not hard to see that the temporary trend of the backwardation and the contango is more easily found when WTI crude oil rapidly price-inverses, while all futures prices seem to be close, when the price of WTI crude oil has a clear trend of increase or decrease.

In order to observe the effectiveness of the Two-Factor model, which is estimated from the Least Square method, the comparison of each WTI crude oil futures in the Two-Factor model are shown within Figures 168-172. The rank of the title of each picture corresponds to the rank of its maturity. For example, the title 'The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model Estimated by The Least Square Method' and the title 'The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model Estimated by The Least Square Method' correspond to the futures contracts which matured in December 2011 and January 2012 respectively, and so forth. It is not hard to see that the Two-Factor model is useful in pricing a futures contract with a particular maturity because all five pictures are showing that the estimated model price of each futures contract is almost the same as the real observed price of the futures contract.

Last, the forward curves from the Two-Factor model which is estimated from the Least Square method, are shown within Figures 165-167. First, on the 50^{th} trading day, the model estimated prices of the Two-Factor model cross with the observed prices twice. To be more specific, with the increase in the term to maturity the estimated prices are firstly higher than the observed prices, and then lower than the observed prices, and the estimated prices are finally higher than the observed prices again. Second, similar to the 50^{th} trading day, on the 100^{th} trading day, there are two crossovers between the estimated model prices and the observed prices. Specifically, the situation is the same as the description for the 50^{th} trading day, but the difference between those two prices seems to be smaller. Last, on the 200^{th} trading day, there are two crossovers between the model estimated prices and the observed prices, but in a different way. To be more specific, in Figure 167, the estimated prices are lower than the observed prices at the beginning and then they are higher than the observed prices at the end of the test period, and the estimated prices are, again, lower than the observed prices. Overall, as expected, the estimated model prices from the Two-Factor model, which is estimated from the Least Square method, are extremely close to the observed prices.

Estimated Parameters	κ	α	λ	σ_1	σ_2	ρ
Least Square	0.8093	0.0343	0.0471	0.5011	0.3002	0.6012

Table 6.1.3: Estimated Parameters from The Two-Factor Model by Least Square Method

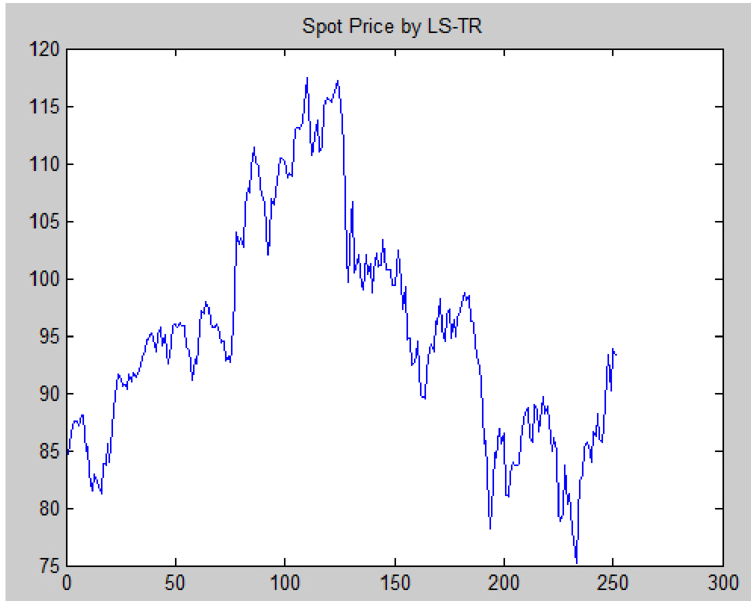


Figure 161: Estimated Spot Price from The Two-Factor Model by The Least Square Method

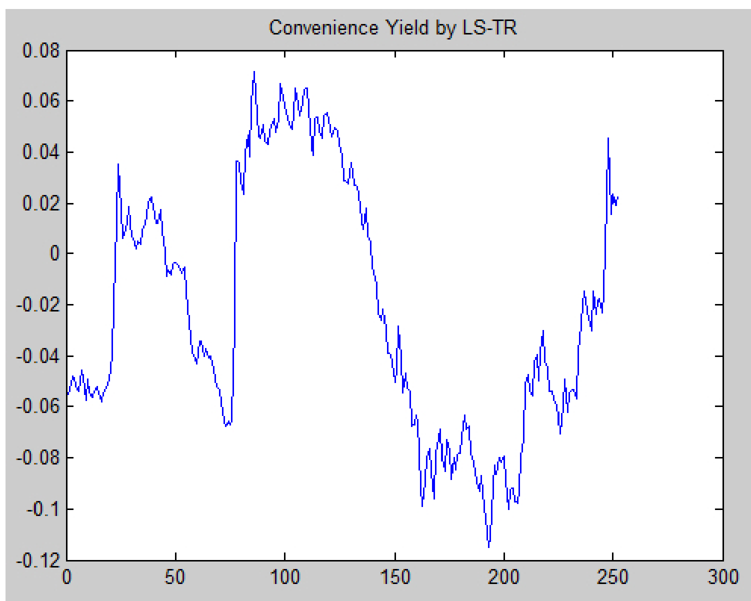


Figure 162: Estimated Convenience Yield Estimated Spot Price from The Two-Factor Model by The Least Square Method

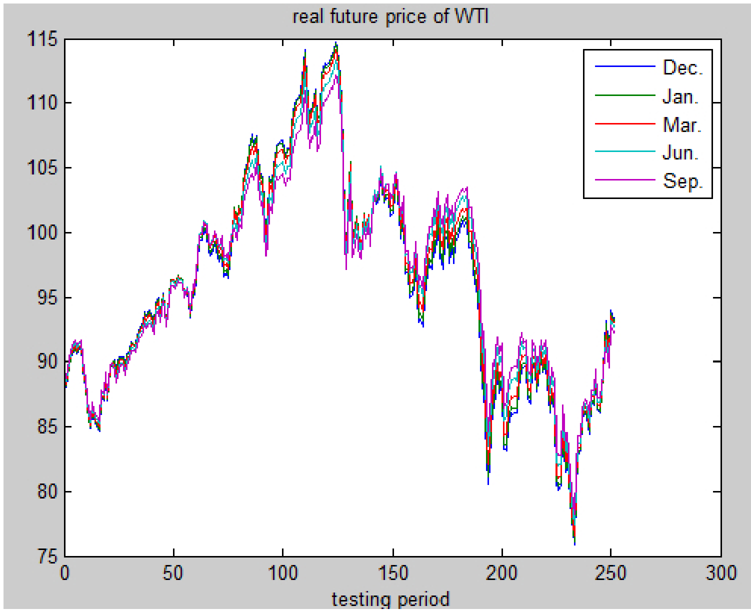


Figure 163: Observed Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model by The Least Square Method

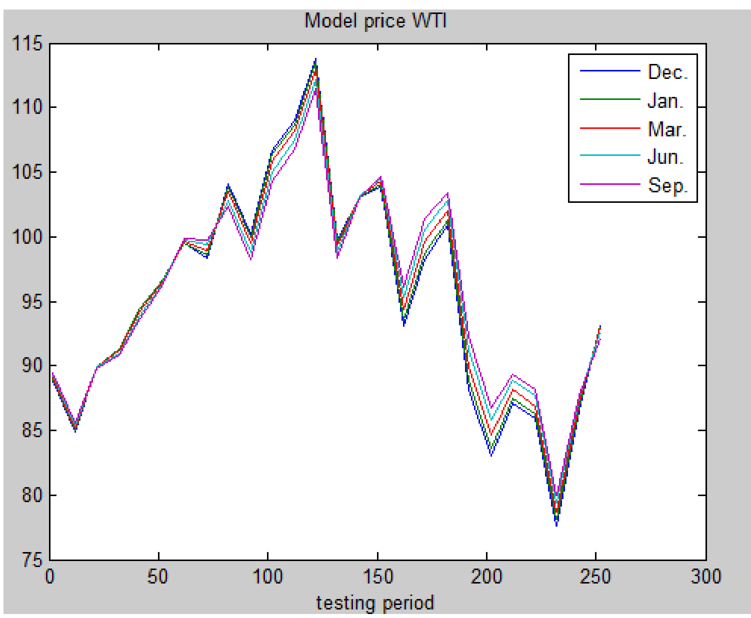


Figure 164: Estimated Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model by The Least Square Method

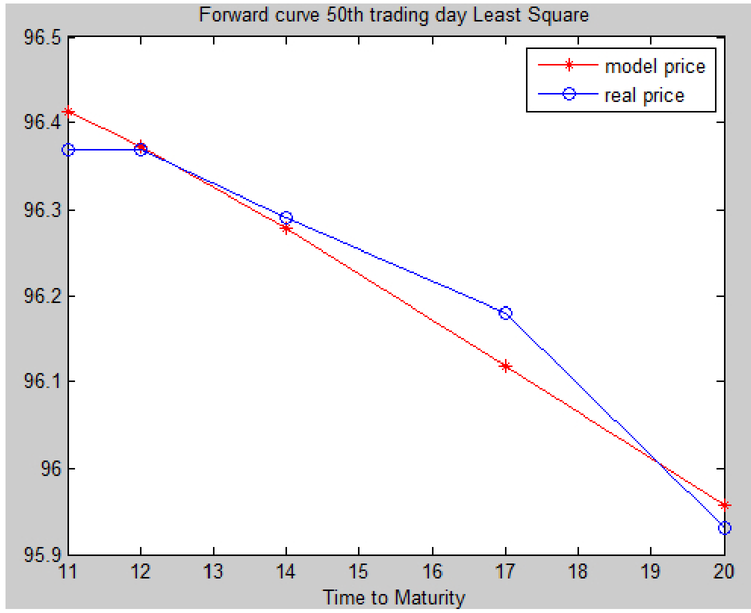


Figure 165: Forward Curves from The Two-Factor Model using Least Square Method on The 50th day

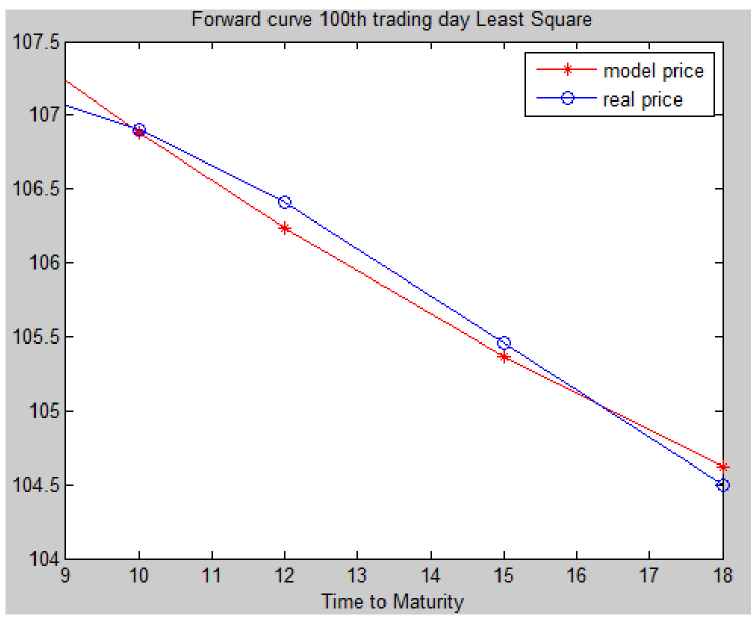


Figure 166: Forward Curves from The Two-Factor Model using Least Square Method on The 100th day

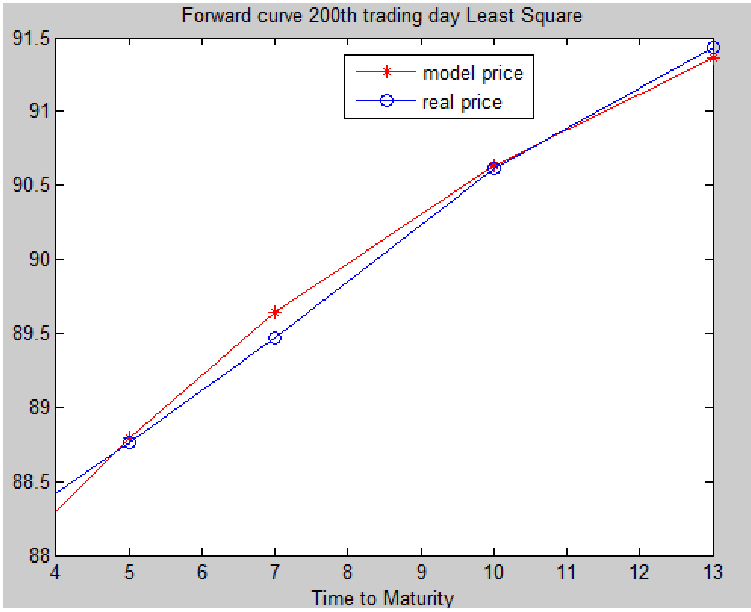


Figure 167: Forward Curves from The Two-Factor Model using Least Square Method on The 200th day

6.1.4 Comparison between The Results in Kalman Filter and Least Square Methods

Comparing Figure 12 with Figure 161, for the two groups the estimated spot price of WTI crude oil, though reached by totally different algorithms, are extremely similar. Overall, the two groups' estimations of spot price of crude oil, though by different algorithms, show almost the same trend. On the other hand, comparing Figure 13 with Figure 162, the trends of these two groups regarding the estimated convenience yield of WTI crude oil are also similar, even if the interval of the fluctuation of the estimated convenience yield by the Kalman filter algorithm is slightly narrower than the interval of the fluctuation of the estimated convenience yield by the Least Square algorithm, and the fluctuation of the estimated convenience yield by the Kalman filter looks like smaller.

Table 2.3.3 and table 6.1.3 show the estimated parameters from the Two-Factor model based on the Kalman filter and Least Square methods, respectively. The results can be

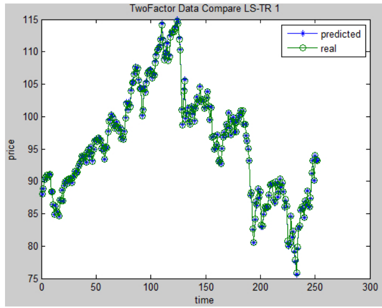


Figure 168: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model Estimated by The Least Square Method

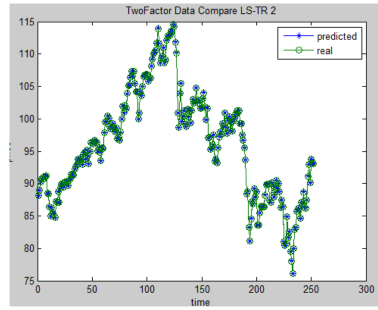


Figure 169: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model Estimated by The Least Square Method

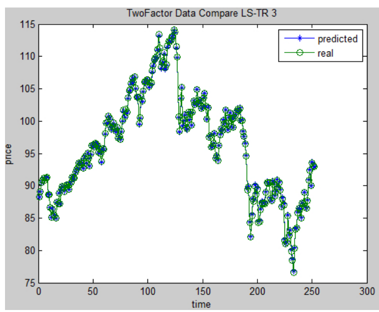


Figure 170: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The Two-Factor Model Estimated by The Least Square Method

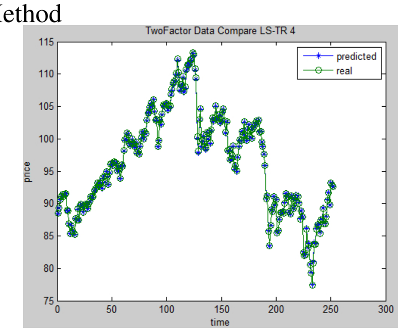


Figure 171: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The Two-Factor Model Estimated by The Least Square Method

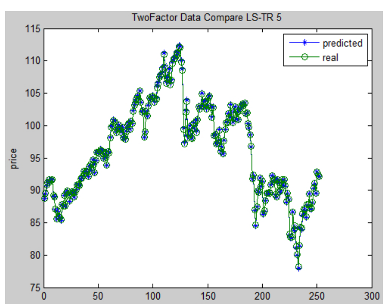


Figure 172: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The Two-Factor Model Estimated by The Least Square Method

easily summarized as follows: the estimated parameters in two different estimating algorithms are similar in general. On the one hand, the estimated α, λ, σ_2 and ρ are almost the same. To be more specific, the estimated α are 0.0105 and 0.0343, respectively; the estimated λ are 0.0305 and 0.0471, respectively; both estimated σ_2 are about 0.3; and both estimated ρ are about 0.6. On the other hand, even if the differences between the estimated σ_1 , and κ are more significant, the estimated σ_1, and κ are still similar based on the two different estimating methods.

Based on section 2.3 and section 6.1, the Kalman filter and the Least Square method are both useful for the original Two-Factor model, and both of them can provide excellent fitting results. Based on their results, there are no notable differences in the estimated results.

6.2 Comparison of the Two-Factor Model with Observed versus Unobserved Spot Price of the Underlying Asset

In this thesis, a presumption was made in section 3.1 which was that the spot price of WTI crude oil can be seen as an observed variable in the traditional Schwartz Two-Factor model. In addition to this assumption, the results of the Kalman filter and the Least Square methods have been compared in previous paragraphs. Actually, the Two-Factor model with the assumed observed spot price of WTI crude oil was also run based on the Least Square method. However, since the size of this thesis is limited, the results of the Two-Factor model, which is estimated by the Least Square method with the assumed observed spot price of WTI crude oil, are only shown quite simply in the following paragraphs. This section will focus on the comparison of the Two-Factor model with and without the observed spot price of WTI crude oil.

Before discussing the results in the Two-Factor model, which is estimated by the Least Square method with the assumed observed spot price of WTI crude oil, the model was run with the same data and assumptions as shown in section 2.3.2 and section 6.1.2 to make the comparison more convincing. The estimated convenience yield of WTI crude oil and the estimated parameters are shown as follows.

Figure 173 illustrates the estimated convenience yield of WTI crude oil in the Two-Factor model, which is estimated by the Least Square estimating method with the spot price of crude oil considered as an observable factor. Compare this with Figure 40 which shows the estimated convenience yield of WTI crude oil estimated from the Two-Factor model, which is estimated by the Kalman filter estimating method with the spot price of crude oil considered as an observable factor. Notably, both figures seem to exhibit an extremely similar trend to that of the estimated convenience yield, but the convenience yield, when estimated by the Least Square method, seems to be slightly higher than when estimated by the Kalman filter. The situation can be expressed as follows: at the beginning of the test period, the estimated convenience yields increase to a relatively high value during all the testing days, then they suddenly plummet to about -10%. Specifically, the convenience yield estimated by the Kalman filter drops to about -13%, while the convenience yield estimated by the Least Square method plummets to about -8%. Then both estimated convenience yields rebound hugely to roughly the high value of the early stage (the result in the Kalman filter algorithm is close to 5%, while the other one is even close to 15%). In the second half of the period, the fluctuations of both convenience yields are smoother. The convenience yield estimated by the Least Square algorithm fluctuates at over zero, while the one estimated by the Kalman filter method fluctuates around zero. Intuitively, the estimated convenience yield from the Kalman filter algorithm seems to be smoother than the one estimated by the Least Square algorithm, and the valley of the convenience yield estimated by the Kalman filter is far deeper. Overall, the convenience yield of crude oil estimated by both algorithms is positive for the majority of the entire test period.

The comprehensive comparison of the fitting results is shown within Figures 174-178. In each figure there are five lines: the black solid line is the observed real price of each futures contract; the blue dotted line and the red broken line stand for the estimated price of each futures contract using the Kalman filter algorithm and the Least Square algorithm respectively when the spot price of crude oil is seen to be unobserved; the green plus line and the pink star line represent the estimated price of each futures contract by using the Kalman filter algorithm and the Least Square algorithm respectively when the spot price of crude oil is considered as an observable variable. It is not hard to observe that the estimated prices of the futures contracts are almost the same as the corresponding observed prices for each chosen futures contract –no matter which type of estimating method is chosen –when the spot price of crude oil is considered as an unobserved state variable. As for the model with an observed spot price of crude oil, the situation is far more complicated. On the one hand, when the Kalman filter algorithm is used, the estimated futures prices are more accurate for a futures contract with a relatively long maturity. In this case, when it is used to estimate a futures contract with a short maturity, the estimated prices fluctuate around the observed futures price, even if the gap is very small. However, with the increase in the length of maturity, the gap decreases, and when the length of maturity is long enough, the estimated prices are very similar to the observed futures prices. However, when the Least Square algorithm is used, the predicted effect is better for the first three futures contracts. In fact, for all five test futures contracts, the estimated prices are higher before the peak, while the gap is reduced after the peak. As for the futures contracts with seven-month and ten-month maturities, the estimated prices of the prices of the futures contracts are significantly higher than the observed prices for most days during the test period.

Table 2.3.3, Table 3.1.1, Table 6.1.3 and Table 6.2 show the estimated parameters from the Two-Factor model and the amended Two-Factor model, respectively. The results can be easily summarized as follows: first, the estimated κs are significant lower using both methods of estimation, when the spot price of crude oil is seen as an unobservable state variable. In addition to this, the estimated κs are quite close under both methods, whatever spot price is considered. Second, all estimated αs are low to zero, and they are very close. Similar to κs, the choice of methods does not have any considerable impact on the estimated αs. Third, like the αs, all λs are very close and close to zero. Fourth, when the spot price is considered to be unobservable, both estimated σ_1s are higher than the calculated σ_1s when the spot price of crude oil is seen as an observable variable and can be directly collected in the markets (the annual volatility σ_1 can then be directly calculated as the standard deviation of the daily return of crude oil times $\sqrt{252}$, in this case, σ_1=0.3446). Fifth, except for the estimated σ_2 from the Kalman filter method with the observed spot price of crude oil, which is significantly lower, all other three estimated σ_2s are close. Notably, the significant lower σ_2 estimation using the Kalman filter with the observed spot price might be caused by the lower calculated σ_1 (σ_1=0.3446 is obviously lower than the implied σ_1 by the Two-Factor model without the observed spot price of crude oil), which might imply that the difference between the observed spot price and the estimated spot price by Two-Factor model and the corresponding σ_1 might highly influence the estimated result of σ_2. Compared with the results of the Least Square algorithm, the Least Square might be a better solution when people want to use the observed spot price. Last, as for the ρ, the situation shown in σ_2 is happening in ρ. To be more specific, except for the estimated ρ from the Kalman filter method with the observed spot price of crude oil, which is significantly lower, all the other three estimated ρs are close. Overall, the estimated parameters are similar under both estimated methods, no matter how the spot price is considered in the model.

Estimated Parameters	κ	α	λ	σ_2	ρ
LS with observed spot price	1.0779	0.0808	1.7404e-04	0.4888	0.4552

Note: the σ_1 is calculated as a standard deviation, since the spot price is observed

Table 6.2: Estimated Parameters from The Two-Factor Model with Observed WTI Crude Oil Spot Price by The Least Square Method

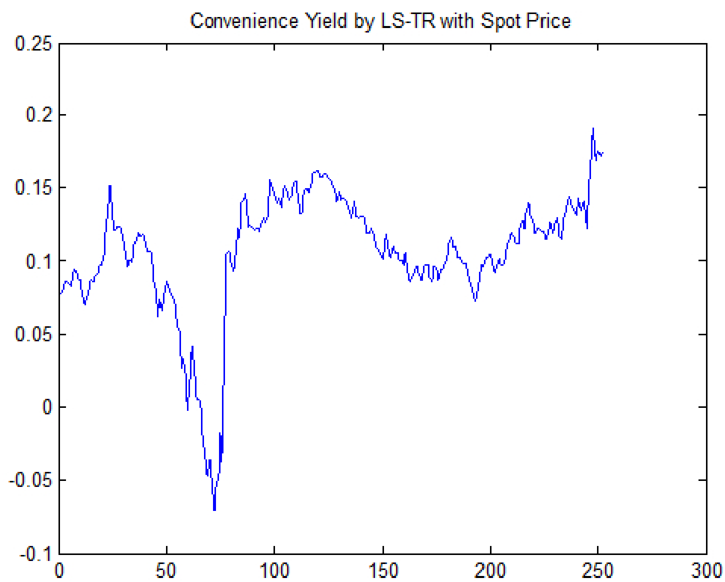


Figure 173: Estimated Convenience Yield from The Two-Factor Model with Observed WTI Crude Oil Spot Price by The Least Square Method

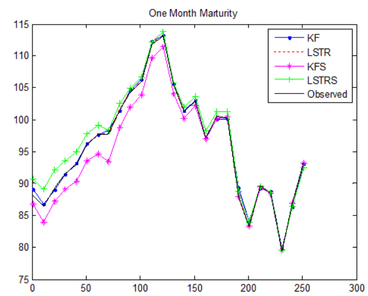


Figure 174: The Observed and Predicted Futures Prices for The First Testing Futures Contract From the Two-Factor Model Based on Different Estimating Methods

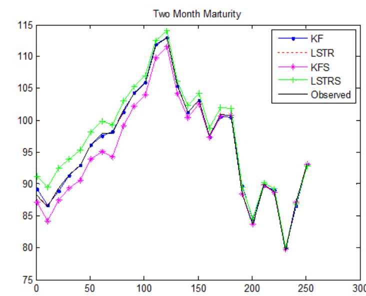


Figure 175: The Observed and Predicted Futures Prices for The Second Testing Futures Contract From the Two-Factor Model Based on Different Estimating Methods

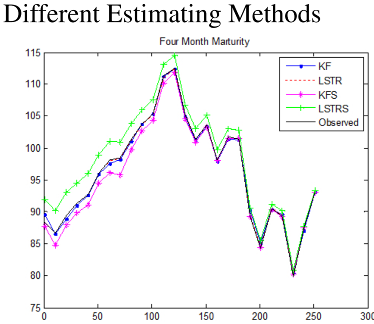


Figure 176: The Observed and Predicted Futures Prices for The Third Testing Futures Contract From the Two-Factor Model Based on Different Estimating Methods

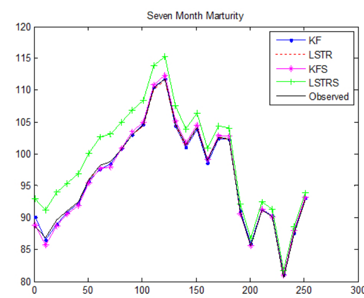


Figure 177: The Observed and Predicted Futures Prices for The Fourth Testing Futures Contract From the Two-Factor Model Based on Different Estimating Methods

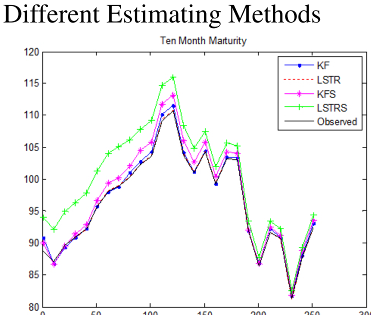


Figure 178: The Observed and Predicted Futures Prices for The Fifth Testing Futures Contract From the Two-Factor Model Based on Different Estimating Methods

6.3 Comparison of The Original Three-Factor Model with The Stochastic σ_1 and The New Four-Factor Model

As has been explained, in this thesis the original Three-Factor model was run with a variation in order to compare it with the new Four-Factor model. The original Three-Factor model and the new Four-Factor model have been comprehensively introduced in sections 2.4 and 4.2 respectively. In order to see the difference, the above two models were run based on the same data. In this section, the results in these two models will be compared.

In this thesis, the estimated state variables estimated from the Three-Factor model were shown within Figures 24-27, while the estimated state variables estimated from the new Four-Factor model were shown within Figures 88-91. Compared with the estimated spot price from the Three-Factor model, the estimated spot price from the new Four-Factor model looks extremely similar to the one shown in Figure 24. Overall, the estimated spot price from the new Four-Factor model is higher than the estimated spot price in the Three-Factor model. However, the estimated convenience yields of WTI crude oil from the both models are significantly different. To be more specific, the estimated convenience yield from the new Four-Factor model is positive, while the estimated convenience yield from the Three-Factor model is negative for most testing days. Recalling the positive relationship shown in different models in this thesis, the results in the new Four-Factor model show a stronger relationship between the estimated spot price and the convenience yield of WTI crude oil. Moreover, the instantaneous interest rates estimated in the Three-Factor model and the new Four-Factor model were shown in Figure 26 and Figure 90, respectively. First, the instantaneous interest rate estimated in the Three-Factor model fluctuates around zero, and is positive for the majority of the testing days. On the other hand, unlike the estimated interest rate from the Three-Factor model which fluctuates around zero, the estimated interest rate from the new Four-Factor model is actually above zero for most trading days. Last, the estimated volatility of the spot price in the Three-Factor model was shown in Figure 27, while the estimated volatility of the spot price from the new Four-Factor model was shown in Figure 91. There is a considerable difference caused by how the volatility of the spot price is considered.

On the one hand, in the Three-Factor model the volatility of the spot price is seen as a parameter which is following a random walk. Under this assumption, the estimated volatility of the spot price is highly fluctuating during the test period. On the other hand, in the new Four-Factor model the volatility of the spot price is seen as a factor which is following a stochastic process. In this situation, the estimated volatility of the spot price is more stable.

The estimated parameters from the Three-Factor model and the new Four-Factor model were shown in table 2.4.3 and table 4.2.3, respectively. In this section, the estimated parameters based on all-sample data are used. Since they are estimated from two totally different models, comparing the estimated parameters might prove meaningless. However, they still show some interesting features. The similarity of the estimated parameters shown and the setting of the different parameters will be exhibited in the following table 6.3. Combining table 2.4.3, table 4.2.3 with table 6.3, the similarities can be easily summarized as follows: first, the coefficients of the reverting in the stochastic process of the convenience yield (κ_1 and κ) are similar in both models; second, the volatility of the convenience yields in both models (σ_2) are similar; third, all corresponding estimated correlation coefficients have the same signs; fourth, there is a difference between the estimated results: the long-term return investment on the interest rate estimated from the Three-Factor model (m^*) is lower than the long-term return investment on the interest rate estimated from the new Four-Factor model (γ), and γ is positive, while m^* is negative. The above difference can be directly explained as the upward trend of the estimated interest rate in the new Four-Factor model.

As for the fitting of the two models, in this thesis the results are compared in two ways. On the one hand, the six chosen futures contracts are shown together in Figures 28, 29 and 93, in order to see whether there are significant differences in effectiveness and usefulness between the Three-Factor model and the new Four-Factor model. To be specific, the real observed futures prices of the six chosen futures contracts are shown in Figure 28. Then, the estimated model futures prices of the six chosen futures contracts, which are estimated from the Three-Factor model, are shown in Figure 29. Last, the estimated model futures prices of the six chosen futures contracts, which are estimated from the new Four-Factor model, are shown in Figure 93. In order to see the trend clearly, in Figure 29 and Figure 93, as has been explained, only 26 points are chosen in each figure, which were picked the first trading day of each ten trading days. Overall, there are no significant differences between the mentioned three figures. On the one hand, WTI crude oil did not follow a strict backwardation or a strict contango during the year of 2012; on the other hand, it is not hard to see that the trend of the backwardation and the contango is more easily found when WTI crude oil fluctuates stably, while all futures prices seem to be close, when the price of WTI crude oil rapidly increases or rapidly decreases. However, by comparing Figure 29 and Figure 93 carefully, the interval of the differences between different futures contracts using the new Four-Factor model is narrower when WTI crude oil shows a trend of contango at the end of the test period. This might also imply that the Three-Factor model with an implied volatility of the spot price is a better choice when the underlying asset of the futures contracts fluctuates stably, and if the research aim is to price a futures contract with a particular maturity.

One the other hand, the forward curves from both Three-Factor and Four-Factor models are shown from Figure 30 to Figure 32 and from Figure 94 to Figure 96, respectively. To be more specific, the first three pictures (from Figure 30 to Figure 32) show the forward curve of the 50^{th}, 100^{th} and 200^{th} trading days during the test period using the Three-Factor model, while the following three pictures (from Figure 94 to Figure 96) show the forward curve of the same trading days using the new Four-Factor model. First, on the 50^{th} trading day, the model estimated prices of both Three-Factor and Four-Factor models are slightly higher than the observed prices. Second, on the 100^{th} trading day, the estimated prices of the Three-Factor model are still slightly higher than the observed prices, while the estimated prices of the new Four-Factor model are almost the same as the observed prices. Last, on the 200^{th} trading day, the model estimated prices of both Three-Factor and Four-Factor models cross with the observed prices. Overall, as expected, the estimated model prices from both models are close to the observed prices. Comparing the two models, the differences between the model estimated prices and the observed prices from the new Four-Factor model seem to be slightly larger. This might imply a cost for the consideration of the stochastic σ_1.

Based on the above analysis and exhibited pictures, the original Schwartz Three-Factor model with the implied stochastic volatility of the spot price of WTI crude oil and the expanded Four-Factor model are useful in the domain of WTI crude oil futures markets. However, they have different features for the different purposes of research or use in reality.

Parameters in new Four-Factor	Corresponding parameters in Three-Factor
κ_1	κ
κ_3	α
γ	m^*
σ_4	σ_3
α	$\widehat{\alpha}$
σ_2	σ_2
ρ_{12}	ρ_{12}
ρ_{14}	ρ_{13}
ρ_{24}	ρ_{23}

Table 6.3: Corresponding parameters in Three- and Four-Factor model

6.4 Comparison of The New Three-Factor Model and The New Four-Factor Model

In section 4, the new Three-Factor model and the new Four-Factor model were run based on the same data on WTI crude oil. The above two models both contain the stochastic process of the volatility of the spot price of WTI crude oil, and both of them are the developments based on the original Schwartz (1997) model system. Hence, the results in both models, which are based on the same data, are worth comparing. In this section, the two developed models will be compared comprehensively.

In this thesis, the estimated state variables estimated from the new Three-Factor model were shown within Figures 74-76, while the estimated state variables estimated from the new Four-Factor model were shown within Figures 88-91. Compared to the estimated spot price in the new Three-Factor model, the estimated spot price in the new Four-

Factor model looks extremely similar to the one shown in Figure 74. Intuitively, the estimated spot price in the new Three-Factor model and the new Four-Factor model are almost the same. The exact difference in percentage is shown in Figure 179. As seen in Figure 179, the spot price of WTI crude oil estimated from the new Four-Factor model is higher than the one estimated from the new Three-Factor model in the test period, but the difference in percentage is small. To be more specific, on all the testing days, the difference is lower than 4%. However, the estimated convenience yields of WTI crude oil from the both models are slightly different, even if the both estimated convenience yields follow a similar trend. Overall, based on the new Three-Factor model, the estimated convenience yield is significantly lower than the estimated convenience yield of WTI crude oil in the new Four-Factor model. In addition, with time passing, the difference in the two convenience yields tend to be larger. To be more specific, at the beginning of the test period, the difference fluctuates around 0.04, but at the end of the test period, the difference decreases to about -0.09 (Figure 180). Last, unlike the estimated spot price and the convenience yield of WTI crude oil, comparing Figure 76 with Figure 91, the estimated volatilities of the spot price of WTI crude oil from the new Three-Factor model and the new Four-Factor model follows a similar trend. Moreover, their values are also almost the same during the test period.

As for the fitting of the two models, in this thesis, the results are compared in two ways. On the one hand, the six chosen futures contracts are shown together in Figures 77, 78 and 93, in order to see whether there are significant differences of the effectiveness and usefulness between the new Three-Factor model and the new Four-Factor model. To be specific, the real observed futures prices of the six chosen futures contracts are shown in Figure 77. Then, the estimated model futures prices of the six chosen futures contracts, which are estimated from the new Three-Factor model, are shown in Figure 78. Last, the estimated model futures prices of the six chosen futures contracts, which are estimated from the new Four-Factor model, are shown in Figure 93. In order to see the trend clearly, in Figure 78 and Figure 93, as has been explained, only 26 points are chosen in each figure, which were picked the first trading day of each ten trading days. Overall, there are no significant differences between the three figures. On the one hand, WTI crude oil did not follow a strict backwardation or a strict contango during the year of 2012; on the other hand, it is not hard to see that the trend of the backwardation and the contango is more easily found when WTI crude oil fluctuates stably, while all futures prices seem to be close, when the price of WTI crude oil rapidly increases or rapidly decreases. However, by comparing Figure 78 and Figure 93 carefully, the top estimated price in the new Four-Factor model is higher than the top estimated price in the new Three-Factor model. Besides the peak, the interval of the differences between different futures contracts by using the new Four-Factor model is narrower when WTI crude oil shows a trend of backwardation at the beginning of the test period. This might imply that the new Three-Factor model is a better choice when the underlying asset of the futures contracts fluctuates stably if a research aims to price a futures contract with a particular maturity, because it can show more significant influence on the prices, which is caused by the different maturities.

On the other hand, the forward curves from both new Three-Factor and new Four-Factor models are shown within Figures 79-81 and within Figures 94-96, respectively. To be more specific, the first three pictures (from Figure 79 to Figure 81) show the forward curve of the 50^{th}, 100^{th} and 200^{th} trading days during the test period using the new Three-Factor model, while the following three pictures (from Figure 94 to Figure 96) show the forward curve of the same trading days using the new Four-Factor model. First, on the 50^{th} trading day, the model estimated prices of the new Four-Factor model are slightly higher than the observed prices, while the model estimated prices of the new Three-Factor model cross with the observed prices. Second, on the 100^{th} trading day, the estimated prices of the new Four-Factor model are still slightly higher than the observed prices, while the estimated prices of the new Three-Factor model also cross with the observed prices. Last, on the 200^{th} trading day, the model estimated prices of both new Three-Factor and new Four-Factor models cross with the observed prices. Overall, as expected, the estimated model prices from both models are close to the observed prices. Comparing the two models, based on the forward curve in the new Three-Factor model, the new Three-Factor model seems to be better, because the model estimated prices cross with the observed prices in all three pictures.

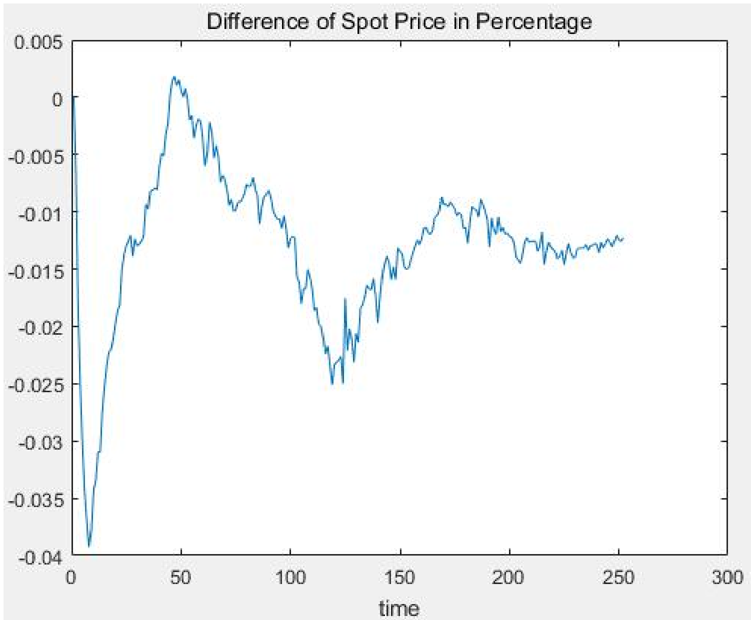


Figure 179: Difference Between The Estimated Spot Prices in Percentage Between The New Three-Factor Model and The New Four-Factor Model

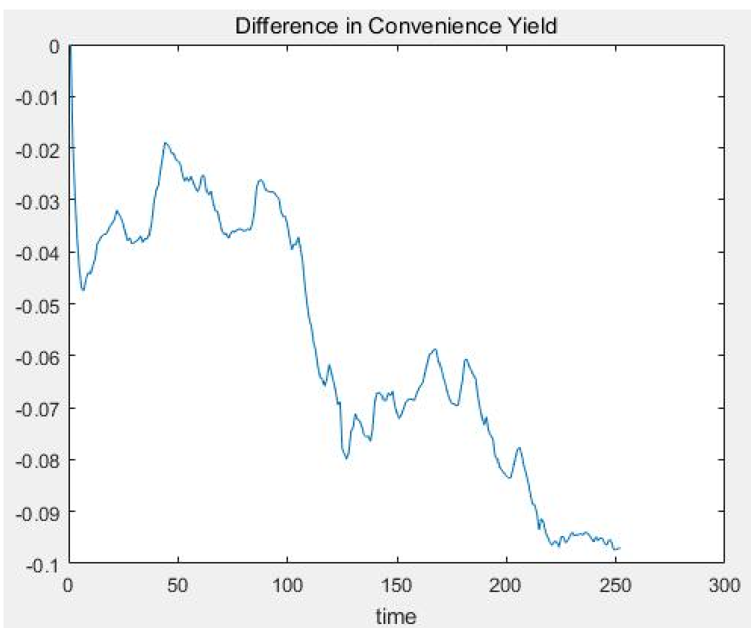


Figure 180: Difference Between Estimated Convenience Yields Between The New Three-Factor Model and The New Four-Factor Model

6.5 Compare the Results in The Original Two-Factor Model and The New Three-Factor Model

In this thesis, the original Two-Factor model and the new Three-Factor model were run based on the different data. In order to make them comparable, the original Two-Factor model was run again with the same data of the new Three-Factor model. Because of the limitation of the scale of the thesis, the assumptions and the empirical results will be roughly introduced. To be more specific, all assumptions are the same assumptions which have been clearly explained in section 2.3.2. As for the data, in this section, the data of futures prices of WTI crude oil is collected from Bloomberg. Specifically, in this section, twelve futures contracts will be used, their maturities are from the February 2013 to January 2014 respectively, and the test period is the entire year of 2012.

The estimated state variables of the Two-Factor model are shown within Figures 181-182. To be specific, Figure 181 exhibits the estimated spot price of WTI crude oil, which fluctuates between over 120 and about 80 dollars per barrel in the test period. Overall, the estimated spot price shows a downward trend in the test period. It fluctuates at over 100 dollars per barrel and reaches the peak of about 120 dollars per barrel at the beginning of the test period, but then directly plummets to about 80 dollars per barrel, which is the valley in the entire test period. In the second half of the test period, it rebounds to about 100 dollars per barrel and then decreases to about 85 dollars per barrel again. Figure 75 shows the estimated convenience yield of the Two-Factor model. Overall, the estimated convenience yield also shows a downward trend in the entire test period. Specifically, the estimated convenience yield of WTI crude oil rapidly increases from 0.02 to about 0.18, which is the peak in the entire test period. Then, after the peak, the estimated convenience yield keeps moving down until it is about -0.05 in the middle of the test year. In the second half of the test period, the estimated convenience yield rebounds to about 0.05, and then drops to about -0.05 again.

Recalling Figure 74 and Figure 75, in which the estimated spot price and the estimated convenience yield of WTI crude oil were shown based on the new Three-Factor model, the estimated spot prices and the estimated convenience yields from the Two-Factor model and the new Three-Factor model are extremely similar, not only in trend, but also in values.

On the other hand, the situation regarding the results of fittings is also extremely similar. The model was run with 12 futures contracts, however, drawing 12 coloured lines in a picture would make the picture untidy. Hence, only the first six futures contracts are shown in the following figures. The six chosen futures contracts are shown together in Figure 77 and Figure 183 in order to see whether there are significant differences between the observed prices of WTI crude oil futures contracts and the estimated prices of WTI crude oil futures contracts in the Two-Factor model. To be more specific, the real observed futures prices of the six chosen futures contracts are shown in Figure 77. Then, the estimated model futures prices of the six chosen futures contracts, which are estimated from the Two-Factor model, are shown in Figure 183. In order to see the trend clearly, in Figure 183 only 26 points are chosen in the figure, which were picked the first trading day of each ten trading days. Overall, there are no significant differences between the two figures. On the one hand, WTI crude oil did not follow a strict backwardation or a strict contango during the test period; on the other hand, it is not hard to see that the trend of the backwardation or the contango is more easily found when WTI crude oil fluctuates stably, while all futures prices seem to be close, when the price of WTI crude oil rapidly increases or rapidly decreases. Recalling the empirical results in the new Three-Factor model shown in section 4.1.3, the conclusions are the same. In addition, comparing Figure 183 and Figure 78, the estimated model prices shown in both figures are almost the same.

Moreover, in order to observe the effectiveness of the Two-Factor model with the new data, the comparison of each WTI crude oil futures in the Two-Factor model are shown within Figure 187-192. The rank of the title of each picture corresponds to the rank of its maturity. For example, the title The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model with New Data and the title The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model with New Data correspond to the futures contracts which matured in February 2013 and March 2013 respectively, and so forth. Those six pictures correspond to Figures 82 to 87. It is not hard to observe that any two corresponding pictures (e.g. Figures 82 and 187) are almost the same.

Last, the forward curves from the Two-Factor model are shown within Figures 184-186. First, on the 50^{th} trading day, the model estimated prices of the Two-Factor model are close to the observed prices. To be more specific, the model estimated prices almost cross with the observed prices. With the increase in the term to maturity, the model prices are firstly significantly higher than the observed prices, and then they are almost the same. Second, on the 100^{th} trading day, the model estimated prices of the model are close to the observed prices. Specifically, with the increase in the term to maturity, the model prices are firstly significantly higher and then slightly lower than the observed prices. Last, on the 200^{th} trading day, the model estimated prices of the Two-Factor model also cross with the observed prices. However, with the increase in the term to maturity, the model prices are first lower and then higher than the observed prices. Compared with the forward curves in the new Three-Factor model (within Figures 79-81), the estimated model prices tend to be higher than the Two-Factor model for the 50^{th} and 100^{th} trading days. Beside that difference, the forward curves based on the two models are also similar.

Based on the above analysis and pictures shown, the original Schwartz Two-Factor model and the new Three-Factor model with the implied stochastic volatility of the spot price of WTI crude oil are both useful in the domain of WTI crude oil futures markets. They can even provide extremely similar results. Overall, adding a stochastic process of the volatility of the spot price to the original Two-Factor model does not change the effectiveness of the Schwartz (1997) model system.

Paremeters	Estimated result	Error
κ	0.8800	SE1
α	0.0507	0.0146
λ	0.0548	SE2
σ_1	0.5224	0.0371
σ_2	0.3429	SE3
ρ	0.4956	0.0352

Note:$SE1 = \sqrt{-0.01098 + ComputationalError1}$
$SE2 = \sqrt{-0.0001578 + ComputationalError2}$
$SE3 = \sqrt{-0.03516 + ComputationalError3}$

Table 6.5: Estimated Parameters from The Two-Factor Model With New Data

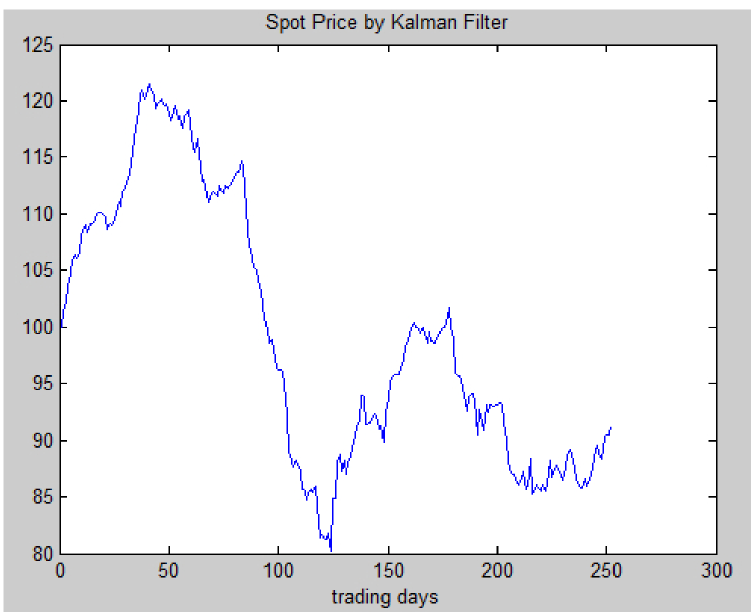


Figure 181: Estimated Spot Price from The Two-Factor Model With New Data by The Kalman Filter

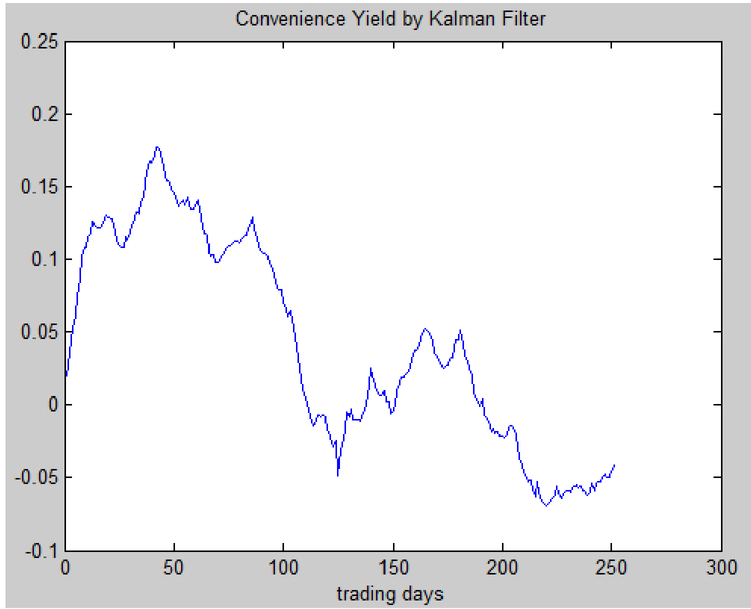


Figure 182: Estimated Convenience Yield from The Two-Factor Model With New Data by The Kalman Filter

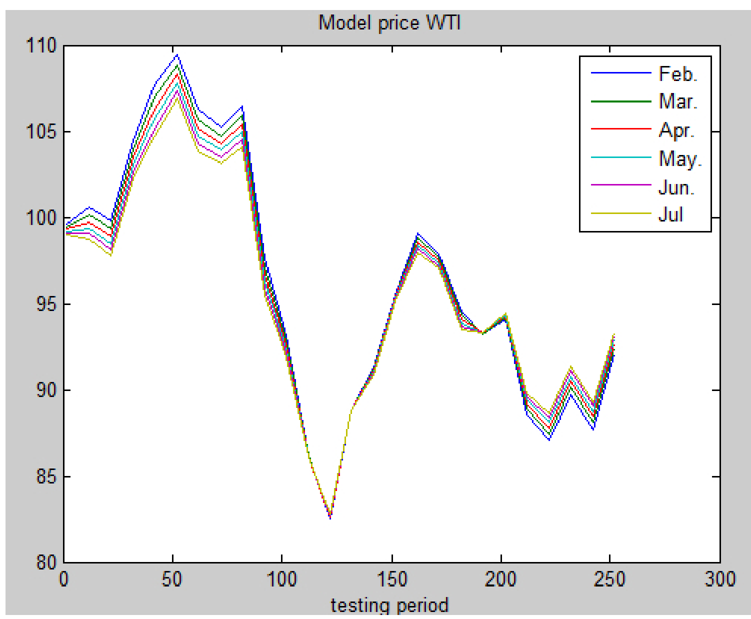


Figure 183: Estimated Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model With New Data

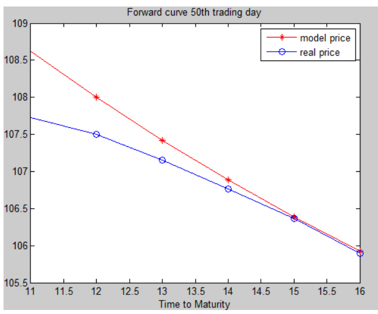


Figure 184: Forward Curves from The Two-Factor Model with New Data on The 50th day

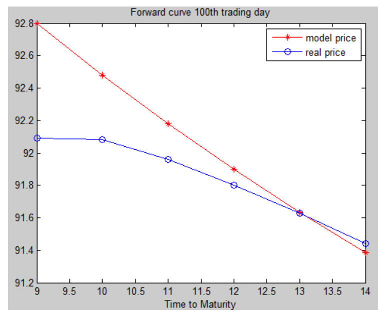


Figure 185: Forward Curves from The Two-Factor Model with New Data on The 100th day

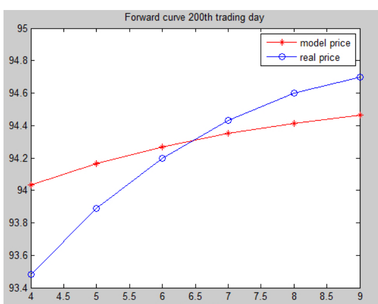


Figure 186: Forward Curves from The Two-Factor Model with New Data on The 200th day

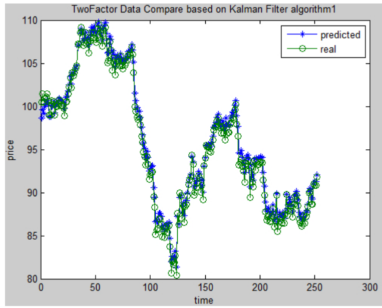


Figure 187: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model with New Data

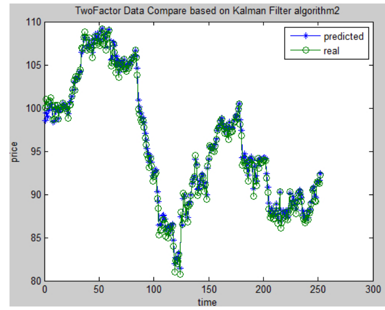


Figure 188: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model with New Data

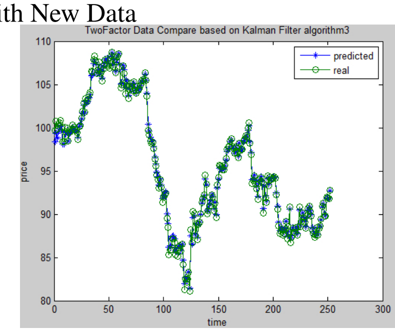


Figure 189: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The Two-Factor Model with New Data

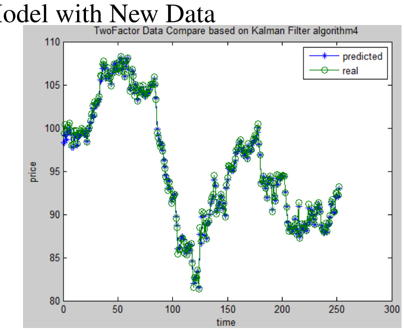


Figure 190: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The Two-Factor Model with New Data

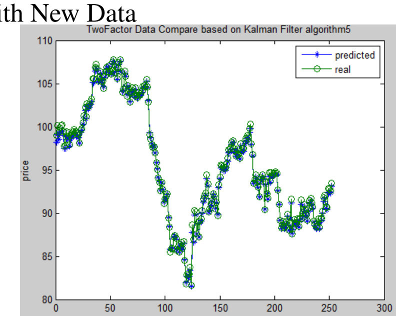


Figure 191: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The Two-Factor Model with New Data

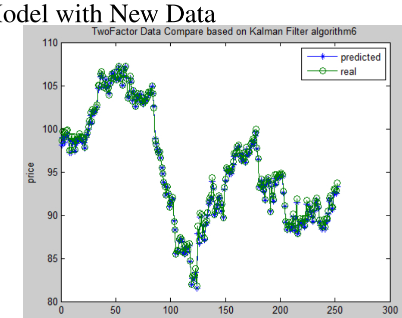


Figure 192: The Predicted and Observed Futures Prices for The Sixth Testing Futures Contract based on The Two-Factor Model with New Data

6.6 Comparison of The Two-Factor Model with Stochastic Parameters and The Original Two-Factor Model

In this thesis, the Two-Factor model has been run in different ways. In this section, the original Two-Factor model which was run using the Kalman filter method, and the Two-Factor model with stochastic parameters which was run using the extended Kalman filter method, will be compared based on their results.

The Two-Factor model with stochastic parameters is proposed for observing the dynamics of the parameters. Instead of estimating the parameters by the likelihood function, the parameters can be estimated in each step during the test period. Therefore, the Two-Factor model with stochastic parameters is designated with the hope that the results would fit better than the results in the original Two-Factor model. In fact, based on Figures 19 to 23 and Figures 64 to 73, the fitting results between the model estimated prices and the observed prices in the Two-Factor model with stochastic parameters are indeed better than the results in the original Two-Factor model. On the other hand, the forward curves drawn by the original Two-Factor model and the Two-Factor model with stochastic parameters may show a different situation. As seen in Figures 16 to 18, the difference between the estimated model prices using the original Two-Factor model and the observed prices are significantly smaller with the increase in the term to maturity while, as seen in Figure 61 to Figure 63, the difference between the estimated model prices by the Two-Factor model with stochastic parameters and the observed prices are significantly larger with the increase in the term to maturity. Based on the above six figures, the original Two-Factor model might be the better one to price a futures contract, since the mean of the differences is smaller. To be more specific, on the one hand, on the 50^{th} trading day, the model estimated prices of the Two-Factor model cross with the observed prices once. Specifically, with the increase in the term to maturity, the estimated prices are firstly lower than the observed prices, and then higher than the observed prices. In addition, similar to the 50^{th} trading day, on the 100^{th} trading day, there is also one crossover between the estimated model prices and the observed prices. Specifically, the situation is the same as the description for the 50^{th} trading day, but the difference between those two prices is smaller. Moreover, on the 200^{th} trading day, the model estimated prices of the Two-Factor model and the observed prices are almost the same. To be more specific, in Figure 18, the estimated prices are lower than the observed prices in the beginning, and then they are higher than the observed prices, but the differences are really small. On the other hand, on the 50^{th} trading day, the model estimated prices of the Two-Factor model with stochastic parameters cross with the observed prices. Specifically, with the increase in the term to maturity, the estimated prices are first lower than the observed prices, and then higher than the observed price. In addition, similar to the 50^{th} trading day, on the 100^{th} trading day, there are two crossovers between the estimated model prices and the observed prices. Specifically, the situation is the same as the description for the 50^{th} trading day, but the difference between those two prices is smaller when the term to maturity is short. Moreover, on the 200^{th} trading day, the model estimated prices of the Two-Factor model with stochastic parameters are lower than the observed prices. To be more specific, in Figure 63 the difference between the estimated prices and the observed prices is quite small when the term to maturity is short, while the difference between the estimated prices and the observed prices is larger, when the term to maturity is longer.

In addition to these, as has been discussed, in most situations, rational people often consider that there is a strictly positive correlation between the spot price and the convenience yield of crude oil, which means that the convenience yield should move up when the spot price of crude oil enters a period of growth, whereas the convenience yield should decrease when the spot price is moving down. In the original Two-Factor model, indeed, for most trading days of our case, there is an obviously positive relationship between the estimated spot price and the estimated convenience yield of crude oil in the test period. However, the relationship disappears in the estimated results in the spot price and the convenience yield of WTI crude oil estimated in the Two-Factor model with stochastic parameters. On the other hand, the six dynamic estimated parameters are quite stable in our case, even if the positive relationship between the spot price and the convenience yield of WTI crude oil has disappeared.

Based on the above analysis, the original Schwartz Two-Factor model and the Two-Factor model with stochastic parameters are both useful in the domain of WTI crude oil futures markets. However, they may have different features for the different purposes of research or use in reality.

6.7 Comparison of The Two-Factor Model Based on European calls on Futures and The Original Two-Factor Model

This thesis has proposed an important application of the Two-Factor model to price European call options on futures. In this model, the unobserved spot price of WTI crude oil and the convenience yield of WTI crude oil can also be estimated from the observed prices of calls on futures. Recalling the data used in section 5, there were 15 European calls on WTI crude oil. All their strike prices are around 100 dollars per barrel. To be more specific, three different strike prices are used in section 5. They are: 102.5 dollars per barrel, 100 dollars per barrel and 97.5 dollars per barrel. In addition to this, each strike price corresponds to five calls which all have same term structure. The five options contracts' maturities are one month, two months, five months, eight months and one year, respectively, and the test period is a year from 1st March 2013 to 28th Feb 2014. Specifically, the used calls on WTI would mature in April 2014, May 2014, August 2014, November 2014, and March 2015, respectively. On the other hand, the T-Notes would also mature in April 2014, May 2014, August 2014, November 2014, and March 2015, respectively. In order to compare the results with the original Schwartz Two-Factor model, the five underlying futures prices are also collected from Bloomberg in this section. [9]

Figure 193 illustrates the observed prices of the closest futures price in each month of the test period. This method of citing the spot price of the underlying asset of futures contracts is also known as the ' rollover the closest futures contract'. In various previous

[9] In this part, the term to maturity T is set the same as the original One-Factor model to reflect the length of the trading days, which has been explained in the footnote 1

studies, researchers did not estimate the implied spot price of underlying commodities, but used the observed prices of the rollover closest futures contract to represent the spot price. As seen in Figure 193, the interval of the fluctuation of the price is between about 87 dollars per barrel and 110 dollars per barrel. In the test period, the observed price of the rollover closest futures contract fluctuates around 90 dollars at the beginning, and then undulates and moves up to over 105 dollars. After the relatively high value, the observed price of the closest WTI futures contract reaches the peak (over 110 dollars per barrel) in the middle of the test period. After the peak, the price of the rollover closest WTI futures contract rapidly decreases to about 90 dollars per barrel, and fluctuates between 90 dollars and 100 dollars. Specifically, after the peak, the curve is W-shaped and finally reaches a relatively high value in the end of the test period.

On the other hand, Figure 194 shows the estimated spot price by the original Two-Factor model which is run based on the futures' data during the same test period with the calls. The estimated spot price fluctuates around 98 dollars at the beginning of the test period and then moves up to about 100 dollars per barrel, which is the peak of the testing period. After the peak, the estimated spot price moves down and then fluctuates around 100 dollars per barrel. On the one hand, as seen in Figure 194 and Figures 109, 111 and 113, the estimated spot price based on the futures contracts is smoother than when estimated by the European calls. On the other hand, compared with Figure 193, the estimated spot price based on the futures contracts is more similar to the rollover futures prices, which means that the observed spot price of the underlying asset based on the rollover technique and the spot price estimated using original Two-Factor model are better substitutions for each other. In the meantime, the difference between the estimated spot price of WTI crude oil by the extended Kalman filter based on the European calls and the observed prices of the rollover closest futures contract of WTI crude oil is quite large in the first half of the testing period. However, in the second half of the testing period, their difference seems to be less considerable.

In addition, Figure 195 shows the estimated convenience yield by the original Two-Factor model. At the beginning of the test period, the estimated convenience yield using the original Two-Factor model fluctuates around 0.14. Then, the estimated convenience yield increases to the peak, which is about 0.22. After the peak, the estimated convenience yield plummets to about 0.09, which is the valley in the entire test period. At the end of the period, the estimated convenience yield increases and fluctuates around 0.14 again. Compared with Figures 110, 112 and 114, the trends of the estimated convenience yields by European calls and the futures are similar, even if the interval of the fluctuations of the convenience yields estimated by European calls are larger than estimated by futures.

Overall, the estimated spot price of WTI crude oil by using the extended Kalman filter based on the European calls, the observed prices of the rollover closest futures contracts of WTI crude oil and the spot price of WTI crude oil estimated using the original Two-Factor model, follow a really similar trend. In other words, one of them tends to move up when the other one is moving up, and vice versa. Furthermore, the curve of the estimated spot price of WTI crude oil using the extended Kalman filter is less smooth than the curves of the observed prices of the rollover closest futures contracts and the spot price estimated by the original Two-Factor model of WTI crude oil. In addition to this, combining all the above figures, it is not hard to see that the estimated convenience yield of WTI crude oil, the estimated spot price of WTI crude oil and the observed price of the closest futures contract are positively correlated. In fact, this is not a surprise. The positive relationship has been pointed out by our own previous research (Ewald and Zong, 2014) in which the results of the estimated state variables were comprehensively compared by using different estimation methods.

Due to the limitation of the scale of this thesis, the fitting results and the forward curves of the original Two-Factor model with the data used in this section will not be shown. However, as seen in section 2, in which the Two-Factor model was run with the different data, the Two-Factor model has excellent fitting results [10]. On the other hand, overall, the parameters estimated using the European calls are useful in pricing both calls and futures of WTI crude oil. However, with the increase in the term to maturity, the ability

[10] see within Figures 16-23

to forecast might be worse [11].

[11] see section 5.3

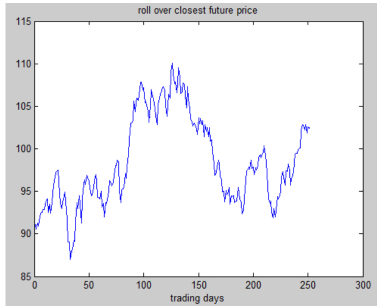


Figure 193: Rollover The Closest WTI Futures ContractS

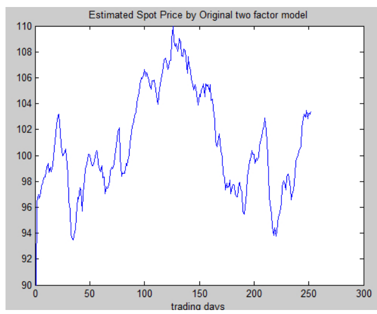


Figure 194: Estimated Spot Price by The Original Two-Factor Model

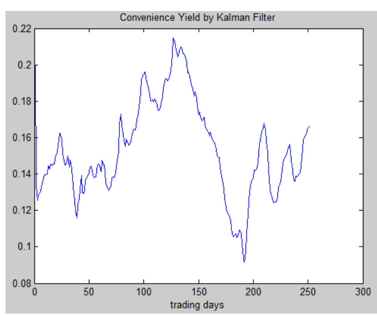


Figure 195: Estimated Convenience Yield by The Original Two-Factor Model

7 Conclusion

As has been shown, this thesis has focused on the model system of pricing the futures contracts and their derivatives in the case of WTI crude oil. Based on the above analysis, pricing models with different estimating methods, which were introduced in this thesis, are useful. In addition to this, regarding the traditional three factors –spot price, convenience yield and interest rate –the estimated results can be concluded as follows: first, based on the same data, the estimated spot prices of WTI crude oil are extremely similar in models with the same data; second, the estimated convenience yields of WTI crude oil are sometimes similar, but they sometimes depend highly on which model and which estimating method is chosen to run the model; last, similar to the estimated convenience yields of WTI crude oil, the instantaneous interest rate implied by different models are also different, and they depend on which model is chosen.

To be more specific, as shown in section 6.5, the values of the estimated spot price and the estimated convenience yield from the original Two-Factor model and the new Three-Factor model are similar, and they follow the same trend. This means that adding the volatility of the spot price into the Schwartz (1997) original Two-Factor model does not influence the pricing effect and the estimated state variables. On the other hand, as shown in section 3, the estimated convenience yield of the underlying asset is far more sensitive to the difference between the observed and the estimated spot price. It is necessary to test if the sudden fluctuation is caused by the difference between the observed and the estimated spot price, when people would like to use the Two-Factor model with the observed spot price. Moreover, since all models introduced in this thesis and the different estimating methods tested in this thesis have shown convincing fitting results in pricing, the conclusion can be summarized as follows: adding an extra factor or using different estimating methods does not have a significant impact on the pricing effect, but sometimes the estimated state variables are indeed influenced by the different models and different estimating methods.

On the other hand, due to the drawbacks of the Kalman filter and the extended Kalman filter, sometimes the standard errors of the unknown parameters are not ideal. In this case, in order to avoid errors in estimation, people concerned about crude oil (producers, consumers and speculators) would be better to use more than one model to estimate the results, so that they can get stable results.

In this thesis, the results fulfill their expectations in all models. To be more specific, regarding the previous models reviewed at the beginning of this thesis, in section 2, all three classic futures pricing models –the Schwartz One-Factor model, Two-Factor model and Three-Factor model –were rerun with updated data, and the results in all three classic models are excellent and reasonable. In the following section 3, the Two-Factor model was run with developments. To be more specific, first the observed spot price of WTI was added to the Two-Factor model. In reality, the spot price can be collected because a small part of WTI crude oil is traded as spot goods. However, in the domain of pricing a futures contract of WTI crude oil, people normally ignore the available spot price because the part of WTI crude oil traded as spot goods is so small. Before adding the observed spot price of WTI crude oil to the Two-Factor model, the expected results were considered to be worse than the original Two-Factor model, even if it is capable of using more available information. The results in the Two-Factor model with the collected spot price of WTI crude oil partly fulfilled the expectations: the fitting results are slightly worse than the original Two-Factor model. However, there is a new finding: the estimated convenience yield of WTI crude oil is extremely sensitive when the difference between the observed spot price and the spot price estimated in the original Two-Factor model is increasing. Second, the extended Kalman filter is introduced in the Two-Factor model, so that the unknown parameters can be tracked. Before running the model, the Two-Factor model with stochastic parameters was expected to obtain better fitting results and more stable stochastic parameters. As a key result, the estimated stochastic parameters are reasonable and stable. This might be very useful because the unknown parameters can be reasonably estimated at each time point in order to price a futures contract.

Nowadays, the volatility of the price of an asset is considered to be stochastic in most studies, hence, in section 4, the Two-Factor model and the Three-Factor model were expanded as a new Three-Factor model and a new Four-Factor model with a consideration of a new stochastic process of the stochastic volatility of the spot price. As a result, both new models are useful, and the fitting results are excellent.

In section 5, the Two-Factor model was combined to price a European call option on WTI crude oil. In fact, the results are far better than expected. Since the options are more sensitive to risk, the estimated parameters were not expected to be good enough to price the futures contracts. However, the results show that the model is useful for pricing both European calls and their underlying futures. On the other hand, as expected, the estimated spot price and estimated convenience yield of WTI crude oil are less smooth than the original Two-Factor model.

Lastly, the models were compared in section 6. Due to the limitation of the scale of this thesis, it is impossible to compare every pair of models introduced in this thesis. Hence, this thesis only focused on and highlighted the most valuable points. Overall, each model or each estimating algorithm has shown different features, therefore, their particular purpose should be clearly considered when choosing a model for pricing a futures contract or its related derivatives.

As mentioned in the introduction, all models introduced in this thesis are suitable for all kinds of people involved in crude oil markets around the world. To be more specific, producers, consumers and speculators can use these models to estimate the theoretical price of the spot price of WTI, the theoretical price of the WTI futures contract and/or the theoretical price of European calls on a WTI futures contract. Then rational decisions can be made based on comparing the theoretical price with the market price. In fact, comparing the theoretical price of an asset with its market price is the key to making a decision for production or investment. More specifically, on the one hand, if the estimated theoretical price of a futures contract is significantly higher or lower than its market price, as viewed by investors and speculators, they would long or short (respectively) futures contract at its market price, because they will believe that producers and consumers would adjust their supply and demand to rebalance the theoretical price and the market price. On the other hand, and for the same reason, comparing the market observed spot price with the model estimated spot price is also meaningful. By using the models introduced in this thesis, all these theoretical prices can be easily obtained. Hence, the models introduced in this thesis are meaningful in practice.

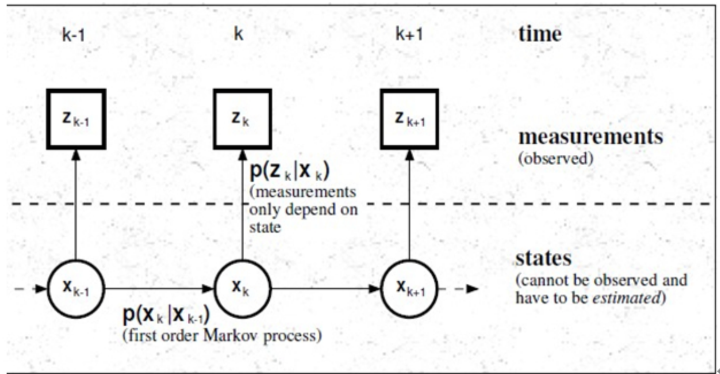


Figure 196: The Basic Principle behind Filtering Technology, adopted from Muehlich (2003)

8 Appendix

8.1 Kalman Filter Algorithm

Even though there seems to be some evidence that the Least-Square based method is a more intuitive and easier way to estimate the unobservable state variables in multi-factor model systems, the Kalman filter algorithm is still the benchmark.

In recent decades, filtering technology has become an established framework in which a state-space model can be analyzed well. Indeed, filter techniques have been used in many domains of communication technology, radar tracking, satellite navigation and applied physics, signal processing, economics, econometrics, and finance. In the following paragraph, the Kalman filter algorithm will be explained.

The basic principles behind filtering technology are not very complicated. Using Bayes theory, filters can use information about current observation to predict the values of unobservable variables at the next time point, and then to update the information and forecast the situation at the next time point (Pasricha, 2006).

The process of filtering is schematically described in Figure 196. Any state space model contains two parts: the state variable x_k for $k = 1, 2..., K$ and the observations y_k for $k = 1, 2..., K$, where K is the number of observations of the time variable. Normally, $x_k = f_k(x_{k-1}, v_{k-1})$, where x_k and x_{k-1} are the state at time points k and $k-1$ and v_{k-1} represents a random shock, and assume the x_k is following a first-order Markov process as: $x_k \mid x_{k-1} \sim p_{x_k|x_{k-1}}(x_k \mid x_{k-1})$. As for the observations, the relationship between state variable(s) and the observations can be described as: $z_k = h_k(x_k, n_k)$, where x_k is the state variable(s) and n_k is the measurement noise at time point k.

Denote $z_{1:\,k}$ as the estimates of the x_k from the start of the time series to the updated time point k, and the observations are conditionally independent provided x_k. Here, $p(x_0)$ can be either given or be obtained as an assumption. When $k \geq 1$, denote $p(x_k \mid x_{k-1})$ as the state transition probability.

Let f_k be any integrable function that depends on the whole trajectory in state space; then, the expectation of $f_k(x_{0:\,k})$ given the observations $z_{1:\,k}$ can be calculated as:

$$E(f_k(x_{0:\,k})|z_{1:\,k}) = \int f(x_{0:\,k})p(x_{0:\,k} \mid z_{1:\,k})dx_{0:\,k}$$

Essentially, the recursive filters consist of two steps: The first step is called the prediction step, which spreads the state probability density function because of noise; the second step is the update step, which combines the likelihood of the current measurement with the predicted state. These can be schematically presented as $p(x_{k-1} \mid z_{1:\,k-1}) \rightarrow p(x_k \mid z_{1:\,k-1})$ and $p(x_k \mid z_{1:\,k-1}), z_k \rightarrow p(x_k \mid z_{1:\,k})$, respectively. Then there are two probability density functions for the above steps. For the prediction step, assuming the probability density function $p(x_{k-1} \mid z_{1:\,k-1})$ is available at time point $k-1$, using the Chapman-Kolmogoroff equation, the prior probability of the state at time point k can be expressed as:

$$p(x_k \mid z_{1:\,k-1}) = \int p(x_k \mid x_{k-1})p(x_{k-1} \mid z_{1:\,k-1})dx_{k-1}$$

As for the update step, the posterior probability density function is

$$p(x_k \mid z_{1:\,k}) = \frac{p(z_k|x_k)p(x_k|z_{1:\,k-1})}{p(z_k|z_{1:\,k-1})}$$

Then, the

$$p(z_k \mid z_{1:\,k-1}) = \int p(z_k \mid x_k)p(x_k \mid z_{1:\,k-1})dx_k$$

can be obtained (Muehlich, 2003).

When this recursive system is considered in practice, in general the recursive propagation of the posterior density is only a conceptual solution, but solutions definitely exist in some restricted cases. The Kalman filter was developed on the basis of this recursive system by Kalman in the 1960s, and then rapidly became a widely used method in state-space models for calculating optimal estimates of unobservable state variables.

Recalling the measurement equation $z_k = h_k(x_k, n_k)$ and the state equation $x_k = f_k(x_{k-1}, v_{k-1})$, under the assumption of linearity, these two equations can be identified as:

$$\begin{cases} x_k = F_k x_{k-1} + v_{k-1} \\ z_k = H_k x_k + n_k \end{cases} \tag{1}$$

where the random variables v and n represent the noises. These are assumed to be independent and with normal probability distributions $p(v) \sim N(0, Q)$ and $p(n) \sim N(0, R)$. If F_k and H_k are assumed to be constants, to simplify the system, the two equations can then be rewritten as:

$$\begin{cases} x_k = A x_{k-1} + v_{k-1} \\ z_k = H x_k + n_k \end{cases} \tag{2}$$

where A and H are known matrices. Denote $\widehat{x}_k^-$ and $\widehat{x}_k$ to be prior and posterior state estimates at time point k, respectively. Then the prior and posterior estimate errors can be defined as $e_k^- \equiv x_k - \widehat{x}_k^-$ and $e_k \equiv x_k - \widehat{x}_k$ at the time point k, respectively. In a similar way, the prior and posterior estimated error covariance can be obtained as $P_k^- = E[e_k^- e_k^{-T}]$ and $P_k = E[e_k e_k^T]$. Based on $z_k = H_k x_k + n_k$, $\widehat{x}_k = \widehat{x}_k^- + K(z_k - H)\widehat{x}_k^-$ can be obtained, where $K = P_k^- H^T (H P_k^- H^T + R)^{-1}$ (Jacobs, 1993).

The Kalman filter algorithm is an optimal algorithm to solve a system with state variables. However, there is a limitation that cannot be ignored. As described, the traditional Kalman filter algorithm needs a strict Gaussian assumption for the posterior density at each time point. $p(x_k \mid z_{1:\,k})$ is proved to be Gaussian as $p(x_{k-1} \mid z_{1:\,k-1})$ is assumed to be Gaussian. Regarding the Kalman filter algorithm as a recursive process and connected via

$$p(x_k \mid z_{1:\,k-1}) = \int p(x_k \mid x_{k-1}) p(x_{k-1} \mid z_{1:\,k-1}) dx_{k-1}$$

and

$$p(x_k \mid z_{1:\,k}) = \frac{p(z_k|x_k)p(x_k|z_{1:\,k-1})}{p(z_k|z_{1:\,k-1})}$$

the prior and the posterior density probabilities can be written as:

$$p(x_{k-1} \mid z_{1:\,k-1}) = N(x_{k-1}; m_{k-1|k-1}, P_{k-1|k-1})$$
$$p(x_k \mid z_{1:\,k-1}) = N(x_k; m_{k|k-1}, P_{k|k-1})$$

and

$$p(x_k \mid z_{1:\,k}) = N(x_k; m_{k|k}, P_{k|k})$$

with

$$m_{k|k-1} = F_k m_{k-1|k-1}, P_{k|k-1} = Q_{k-1} + F_k P_{k-1|k-1} F_k^T,$$
$$m_{k|k} = m_{k|k-1} + K_k (z_k - H_k m_{k|k-1})$$

and

$$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1} = (I - K_k H_k) P_{k|k-1}$$

where $N(x; m, P)$ is a Gaussian density with argument x, mean m and covariance P. Since

$$K = P_k^- H^T (H P_k^- H^T + R)^{-1}$$

is known,

$$K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1}$$

can be obtained.

The likelihood function is given by $L(z; \Psi) = \prod_{k=1}^{K} p(z_k | z_{1:k-1})$ and the distribution of z_k conditional on $z_{1:k-1}$ is itself normal, if the initial state vector and the disturbances have multivariate normal distributions. Since the expectation of the z_k at time point $k-1$ is based only on the information at $k-1$, the likelihood function can be finally written as:

$$\log L = -\frac{NK}{2} \log 2\pi - \frac{1}{2} \sum_{k=1}^{K} \log | D_k | - \frac{1}{2} \log \sum_{k=1}^{K} v_k^{'} D_k^{-1} v_k$$

where $v_k = z_k - z_{k|k-1}$ and $D_k = H_k P_{k|k-1} H_k^{'} + R$ (Harvey,1989).

8.2 Extended Kalman Filter Algorithm

The extended Kalman filter algorithm is a useful development. By using the extended Kalman filter algorithm, the measurement function or/and the state function does not need to be linear anymore. Hence, the measurement equation $z_k = h_k(x_k, n_k)$ and the state equation $x_k = f_k(x_{k-1}, v_{k-1})$ can no longer be expressed as

$$\begin{cases} x_k = F_k x_{k-1} + v_{k-1} \\ z_k = H_k x_k + n_k \end{cases} \tag{3}$$

In order to run the filter algorithm, a local linearization of above equations might be a description of the nonlinear system. Then the $p(x_{k-1} \mid z_{1:\, k-1})$, $p(x_k \mid z_{1:\, k-1})$ and $p(x_k \mid z_{1:\, k})$ are approximated by a Gaussian distributions as

$$p(x_{k-1} \mid z_{1:\, k-1}) \approx N(x_{k-1}; m_{k-1|k-1}, P_{k-1|k-1})$$
$$p(x_k \mid z_{1:\, k-1}) \approx N(x_k; m_{k|k-1}, P_{k|k-1})$$

and

$$p(x_k \mid z_{1:\,k}) \approx N(x_k; m_{k|k}, P_{k|k})$$

$$m_{k|k-1} = f_k(m_{k-1|k-1})$$

$$P_{k|k-1} = Q_{k-1} + \widehat{F_k} P_{k-1|k-1} \widehat{F_k}^T$$

$$m_{k|k} = m_{k|k-1} + K_k(z_k - h_k(m_{k|k-1}))$$

and

$$P_{k|k} = P_{k|k-1} - K_k \widehat{H_k} P_{k|k-1} = (I - K_k \widehat{H_k}) P_{k|k-1}$$

where

$$K_k = P_{k|k-1} \widehat{H_k}^T (\widehat{H_k} P_{k|k-1} \widehat{H_k}^T + R_k)^{-1}$$

is known as gain, and

$$\widehat{F_k} = \tfrac{df_k(x)}{dx} \mid_{x=m_{k-1|k-1}}$$

and

$$\widehat{H_k} = \tfrac{dh_k(x)}{dx} \mid_{x=m_{k-1|k-1}}$$

are Jacobian matrices. This process is known as the extended Kalman filter algorithm. (Muehlich, 2003)

In the Two-Factor model, in order to observe the dynamics of the parameters, the parameters are also seen as a part of the state variables, and they are iterated while running the filter. This implies that the model is not linear anymore. In order to simplify the model, the parameters are assumed to follow random walks, which mean that the values of the parameters at step k are only dependent on themselves at step k-1 and a noise. In this way, the implied parameters can be shown in each step.

8.3 Trust Region Method

At the beginning of 90s, Coleman and Li (1993) proposed an effective trust region approach for minimizing nonlinear function, subject to bounds. According to Coleman and Li (1993), the possible solution for a trust region approach can be seen as the result of minimizing a quadratic function regarding an ellipsoidal constraint. They also pointed out that, with the implementation of the algorithm, the strong convergence properties are consistent.

Trust region methods can be seen as an effective approach for minimizing nonlinear function. According to Conn, Gould and Toint (2000), because it is similar to the line search method, the trust region method generates steps of a quadratic model of an objective function in order to solve a minimization problem. Trust region methods define a region around the current iteration in which the model is trusted to be a useful representation of the objective function. In this region, a step is chosen to be the approximate minimizer of the model. Then, the direction and the length of the step are chosen simultaneously. If the step is not acceptable, the size of the region should be reduced and a new minimizer found. Notably, the direction of a step alters when the size of the trust region is changed.

Denote m_k to be the quadratic model function which will be used in each iteration x_k . To be more specific, m_k is based on the Taylor Series Expansion of the original function $f(\cdot)$ around x_k , which is:

$$f(x_k + p) = f(x_k) + g_k^T p + \frac{1}{2} p^T \nabla^2 f(x_k + tp) p$$

where $f_k = f(x_k)$, $g_k = \nabla f_k$ and $t \in (0, 1)$. Denote B_k to be an approximation to the Hessian matrix in the second order term, then m_k can be written as follows

$$m_k(p) = f_k + g_k^T p + \frac{1}{2} p^T B_k p$$

The difference between $m_k(p)$ and $f(x_k + p)$ is $O(\parallel p \parallel^2)$ that is a small value when p is small. When $B_k = \nabla^2 f(x_k)$ which leads to the trust region Newton method, the approximate error in the m_k is $O(\parallel p \parallel^2)$, in which situation the model is accurate, when

$\| p \|$ is small.

In the next step, the solution to the following sub-problem needs to be found:

$$\min_{p \in R^n} m_k(p) = f_k + g_k^T p + \frac{1}{2} p^T B_k p$$
$$s.t. \|p\| \leq \Delta_k$$

where $\Delta_k > 0$

The trust region can be classified by different choices of setting B_k and the norm for the trust region in the last equation or, for instance, in the model. To be more specific, if the trust region is defined by using Euclidean norm with $B_k = 0$, the trust region method is known as the steepest descent line search approach; if B_k equals the exact Hessian matrix $B_k = \nabla^2 f(x_k)$, the trust region method is named as the trust region Newton method; if B_k is considered to be the means of a quasi-Newton approximation, the trust region method is defined as the trust region quasi-Newton method.

In addition, the size of the trust region is crucial to the effectiveness of each step. If the region is too small, the algorithm would miss the opportunity to take a substantial step which would move it much closer to the minimizer of the objective function; whereas, if the region is too large, the minimizer of the set model might be too far from the minimizer of the objective function in the region. In practice, the size of the region is chosen based on the performance of the method in previous iterations. Specifically, the size of the trust region might be increased to allow longer and more ambitious steps if the model is consistently reliable, producing acceptable steps and accurately predicting the behavior of the objective function along these steps; on the other hand, the size of the trust region needs to be reduced after a failed step, which implies that the objective function over the current trust region is not well represented by the model.

Define a ratio:

$$\rho_k = \frac{f(x_k) - f(x_k + p)}{m_k(0) - m_k(p_k)}$$

where the denominator is called as predicted reduction, and the numerator is defined as the actual reduction. Since the step p_k is computed by minimization of the model m_k over a region including the step $p = 0$, the denominator is nonnegative. Therefore, the results can be summed up as follows: if $\rho_k < 0$, the new value of the objective function $f(x_k + p)$ is larger than the value of the current objective function $f(x_k)$, so that the current step must be rejected; if $_k$ is close to 1, it is safe to expand the trust region in the next iteration, because there is a good agreement between the model and the objective function over the current trust region; if $_k$ is positive but not close to 1, the size of trust region should not be changed, however, if $_k$ is close to zero or is negative, the size of the trust region would be reduced.

The algorithm can be summarized by Psudo code as follows:
Given $\widehat{\Delta} > 0$, $\Delta_0 \in (0, \widehat{\Delta})$ and $\eta \in [0, \frac{1}{4}]$:
For $k = 0, 1, 2, ...$
p_k can be obtained by approximately solving:

$$\min_{p \in R^n} m_k(p) = f_k + g_k^T p + \tfrac{1}{2} p^T B_k p$$
$$s.t. \|p\| \leq \Delta_k$$

Compute ρ_k based on:

$$\rho_k = \tfrac{f(x_k) - f(x_k + p)}{m_k(0) - m_k(p_k)}$$

If $\rho_k < \frac{1}{4}$

$$\Delta_{k+1} = \tfrac{1}{4} \Delta_k$$

Else If $\rho_k > \frac{3}{4}$ and $\|p\| = \Delta_k$

$$\Delta_{k+1} = \min(2\Delta_k, \widehat{\Delta_k})$$

Else

$$\Delta_{k+1} = \Delta_k$$

If $\rho_k > \eta$

$$x_{k+1} = x_k + p_k$$

Else

$$x_{k+1} = x_k$$

End

Lastly, the p^* is a global solution of the trust region problem:

$$\min_{p \in R^n} m(p) = f + g^T p + \tfrac{1}{2} p^T B p$$
$$s.t. \|p\| \leq \Delta$$

If and only if p^* is feasible and there is a scalar $\lambda \geq 0$ such that the following conditions are satisfied: $(B + \lambda I) * p^* = -g$, $\lambda(\Delta - \|p^*\| \ ) = 0$ and $(B + \lambda I)$ is positive semidefinite.

8.4 The Black-Scholes Formulas

The Black-Scholes equation and boundary conditions for a European call with value $C(S, t)$ are

$$\tfrac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 C}{\partial S^2} + rS \tfrac{\partial C}{\partial S} - rC = 0$$

with

$$C(0, t) = 0, C(S, t) \sim S, \text{ as } S \to \infty$$

and

$$C(S, T) = \max(S - K, 0)$$

where C is the price of the European call, S is the price of the underlying asset, K is the strike price of the call and $t = T$ at the maturity of the call.

First, the awkward S and S^2 should be rejected. Set:

$$S = Ke^x, t = T - \tau / \tfrac{1}{2}\sigma^2, \text{ and } C = Kv(x, \tau)$$

this results in the equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (\kappa - 1)\frac{\partial v}{\partial x} - \kappa v$$

where $\kappa = \tau/\frac{1}{2}\sigma^2$. Then, the initial condition becomes

$$v(x, 0) = max(e^x - 1, 0).$$

Putting $v = e^{\alpha x + \beta \tau} u(x, \tau)$ for some constants α and β to be found, then the differentiation gives

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (\kappa - 1)(\alpha u + \frac{\partial u}{\partial x}) - \kappa u.$$

By choosing $\beta = \alpha^2 + (\kappa - 1)\alpha - \kappa$ and $0 = 2\alpha + (\kappa - 1)$, an equation without u can be obtained. With $\alpha = -\frac{1}{2}(\kappa - 1)$ and $\beta = -\frac{1}{4}(\kappa + 1)^2$, then v can be rewritten as

$$v = e^{-\frac{1}{2}(\kappa-1)x - \frac{1}{4}(\kappa+1)^2 \tau} u(x, \tau),$$

where

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \text{ for } -\infty < x < \infty, \tau > 0$$

with

$$u(x, 0) = u_0(x) = max(e^{\frac{1}{2}(\kappa+1)x} - e^{\frac{1}{2}(\kappa-1)x}).$$

The solution to the diffusion equation problem can be obtained with the following equation:

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} ds$$

where $u_0(x) = max(e^{\frac{1}{2}(\kappa+1)x} - e^{\frac{1}{2}(\kappa-1)x})$. Let $x' = (s - x)/\sqrt{2\tau}$, so that

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(x'\sqrt{2\tau} + x) e^{-\frac{1}{2}(x')^2} dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(\kappa+1)(x + x'\sqrt{2\tau})} e^{-\frac{1}{2}(x')^2} dx' - \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(\kappa-1)(x + x'\sqrt{2\tau})} e^{-\frac{1}{2}(x')^2} dx',$$

then

$$u(x, \tau) = I_1 - I_2$$

where

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(\kappa+1)(x+x'\sqrt{2\tau})} e^{-\frac{1}{2}(x')^2} dx'$$

$$= \frac{e^{\frac{1}{2}(\kappa+1)x}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{4}(\kappa+1)^2\tau} e^{-\frac{1}{2}(x'-\frac{1}{2}(\kappa+1)\sqrt{2\tau})^2} dx'$$

$$= \frac{e^{\frac{1}{2}(\kappa+1)x+\frac{1}{4}(\kappa+1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}-\frac{1}{2}(\kappa+1)\sqrt{2\tau c}}^{\infty} e^{-\frac{1}{2}\rho^2} d\rho$$

$$= e^{\frac{1}{2}(\kappa+1)x+\frac{1}{4}(\kappa+1)^2\tau} N(d_1)$$

with

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(\kappa+1)\sqrt{2\tau}$$

and

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{\frac{-1}{2}s^2} ds$$

is the cumulative distribution function for the normal distribution.

The calculation of I_2 is identical to that of I_1 with $(\kappa+1)$ is replaced by $(\kappa-1)$.

Last, recalling $v(x,\tau) = e^{\frac{-1}{2}(\kappa-1)x-\frac{1}{4}(\kappa+1)^2\tau}u(x,\tau)$, and putting $x = log(S/K), \tau = \frac{1}{2}\sigma^2(T-t)$, and $C = Kv(x,\tau)$ to recover

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{log(S/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$d_2 = \frac{log(S/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

(Wilmott, Howison and Dewynne, 1995).

8.5 WTI Futures Contracts and Options on Futures Contracts

Today there are two main price benchmarks for crude oil in the world: Brent and WTI (West Texas Intermediate) crude oil. This thesis focuses on WTI crude oil which is also known as Texas light sweet (this means that it has a relatively low density and low sulfur content). To be more specific, its APT (American Petroleum Institution) gravity is around 39.6, and its specific gravity is about 0.827. Further, WTI contains about 0.24% sulphur, which is sweeter than Brent's 0.37%. WTI is mainly traded in North America and its related futures contracts are mainly traded on the New York Mercantile Exchange (the CME group). All futures of WTI that are traded on the CME are physical settlement, which means that all futures are linked to real delivery when they mature. The place for settlements of WTI is Cushing, Oklahoma.

WTI crude oil is the most important benchmark for the price of crude oil in the world. Virtually around the clock, about 850,000 contracts are traded over WTI futures and options markets, which translated to about one billion barrels of crude oil per day. In other words, WTI futures and options have played an increasingly crucial role in managing huge risks from crude oil suppliers and consumers since the disruptive price shocks of the 1970's (the oil crisis). WTI crude oil was introduced by the New York Mercantile Exchange in 1983 for two main reasons: first, in the 1970s, geopolitical issues disrupted the preceding price system for oil. For example, the Yom Kippur war and the Iranian Revolution caused the oil crisis in Western countries. Second, the US government concurrently changed its policy of price and allocation control to allow oil-producing nations to keep upward pressure on prices. Then the market began responding to the altered supply and demand after 1983. Due to the above factors, a new need entered the market - to hedge against the price volatility of crude oil. In the same year, the New York Mercantile Exchange proposed an innovative tool–WTI crude oil futures contracts–to satisfy the needs of customers.

As mentioned above, West Texas Intermediate (WTI) crude oil is a light sweet crude oil stream, and its delivery point in Cushing, Oklahoma is a vital transshipment point with many storage facilities, intersecting pipelines and easy access to refiners and suppliers. In fact, due to its strategic position, Cushing was also a crucial physical market for crude oil even before the Exchange listed WTI crude oil futures in 1983. After 1983, WTI crude oil futures were firmly rooted in the physical market because they started to be the new pricing mechanism for the delivery of crude oil in North America. As to why WTI crude oil has grown to be the most important benchmark for the pricing of crude oil around the world, the reason can be summarized as follows: firstly, North American's production of crude oil has been a key driver of growing global crude oil supply over the past decades; secondly, unique data about pricing and an inventory of crude oil is supplied by the US government. This means that there is no other crude oil benchmark that can reflect the supply and demand dynamics of the market more accurately. Additionally, since physical crude oil is typically traded in increments of a thousand barrels for delivery via pipeline in the US market, the standard commercial contract specifies 1,000 barrels. Hence, the unit of the WTI futures contract is set to be 1,000 barrels because that is a very convenient size for standard US transactions. This unit can also offer a nearly instantaneous price convergence between physical markets and futures. This is also an explanation as to why the WTI crude oil market is the most liquid benchmark for the global price of crude oil. Quantitatively, approximate 20 million barrels of North American crude oil sales are based on WTI contracts as the benchmark of pricing per day. This is a huge number which leads to WTI crude oil futures being the most efficient tool for hedging risk in relation to crude oil for international oil related companies.

Apart from this, the WTI crude oil futures market is a mature market. As has been explained, the New York Mercantile Exchange provides futures contracts with a basic unit of account in increments of 1,000 barrels, which gives both traders the ability to customize and standardize the size of their futures position to match the other side of any trade. Apart from the exchange market, there is a highly liquid OTC (over-the-counter) derivatives market which is tied to WTI futures contracts. The derivatives allow market participants to differentiate financial instruments to manage their crude oil price risk. The most common derivative instruments are options. WTI options were introduced in 1986 as another important instrument for hedging risk in crude oil. Before discussing WTI crude oil options, an introduction to options must first be discussed. Options were introduced to our financial markets because they increase flexibility in managing financial risk in their underlying assets. The holder of an option contract gets the right, but not the obligation, to buy or sell a unit of the underlying asset at a specified time point.

On the other hand, the seller of an option has the obligation to buy or sell the underlying asset that is tied to the option when the holder of the option chooses to exercise the option. In the case of this thesis, the underlying asset is the futures contract of WTI crude oil.

The New York Mercantile Exchange provides several types of options for market participants. Every option exists in one of two basic forms: calls and puts. A call gives the holder the right, but not the obligation, to buy futures at a specific price (the strike or exercise price) within a designated period (the maturity). Conversely, the put gives the holder the right, but not the obligation, to sell futures at a specific price within a designated period. To be more specific, on the one hand, the buyer would buy a call when they believe that the price of WTI futures will increase in the future since the call will protect against the risk on an increase in the price of WTI crude oil. On the other hand, the buyer would buy a put when they predict that the price of WTI futures will decrease in the future since the put will protect against the risk of a decrease in the price of WTI crude oil. Contrarily, the seller would write (sell) a call when they predict that the price of WTI futures will decrease in the future since the put will not only protect against the risk of a decrease in the price of WTI crude oil, but also profit the seller the amount of the premium. Similarly, the seller would write (sell) a put when they believe that the price of WTI futures will increase in the future since the put will profit the seller the amount of the premium. Essentially, the buyer acts as the insured and the seller acts as the insurance company in the market. As mentioned, unlike the futures contract in which the holder must choose to either liquidate or hold to delivery, options provide a third choice: the buyer can give up the option at the maturity if the price moves in the opposite direction that they predicted. In the third choice, the buyers only lose the payment for purchasing the options at the beginning.

In order to attract more participants, the New York Mercantile Exchange offers a sophisticated array of option varieties which enable more flexibility and various hedging activities, and makes the market more liquid than it was before. The main choices include the most famous two kinds of options: European Option and American Option.

Although only European style options will be used in this thesis, both these two main types of options are traded in the New York Mercantile Exchange and will be roughly introduced here because both types are popular and crucial in the world. Firstly, the European–Style WTI options are also known as Vanilla options, which is the simplest kind of standard options. This style of option can only be exercised at the maturity of the option. In the New York Mercantile Exchange, the European–Style WTI options can only be exercised into cash settlement upon expiration, and this kind of option is priced based on the expected price of the underlying contracts upon expiration. Secondly, American–Style options are defined as options in which the holder has the right to exercise the option at any time point during a designated period, which means that this type of options can be exercised prior to expiration. The American–Style WTI options are a more flexible variant than European options, and they afford the opportunity to earn interest on premiums recovered from the early exercise. In the New York Mercantile Exchange, unlike the European–Style WTI options, the American–Style WTI options are physical settlement options, therefore the American–Style WTI options are the more widely traded standard crude oil option type and they permit hedgers to respond more rapidly to price fluctuations.

8.6 Two-Factor Model with Rollover Data

In this thesis, as explained in the introduction, the models introduced in this thesis have been tested in three separate short test periods. In this section, a period that covers the three short terms is chosen in order to observe the overall situation during the three test periods. On the one hand, the period starts at the first trading day of the year 2010 and it ends the last trading day of the year 2015. To be more specific, as has been explained in the previous sections, when the spot price of crude oil is set as an unobservable state variable, the observable futures prices are the only data collected when the model is implemented. In this section, the data for futures prices of WTI crude oil is collected from Bloomberg. Specifically, in this section, 12 futures contracts will be used. Their maturities are one month to a year respectively, and the test period is the year from 4^{th} Jan. 2010 to 31^{st} Dec. 2015.

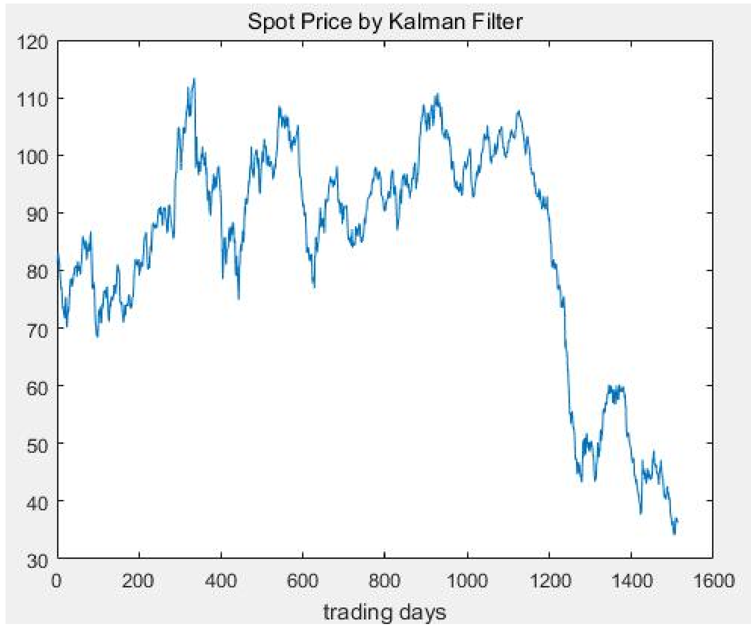


Figure 197: Estimated Spot Price from The Two-Factor Model With Rollover Data by The Kalman Filter

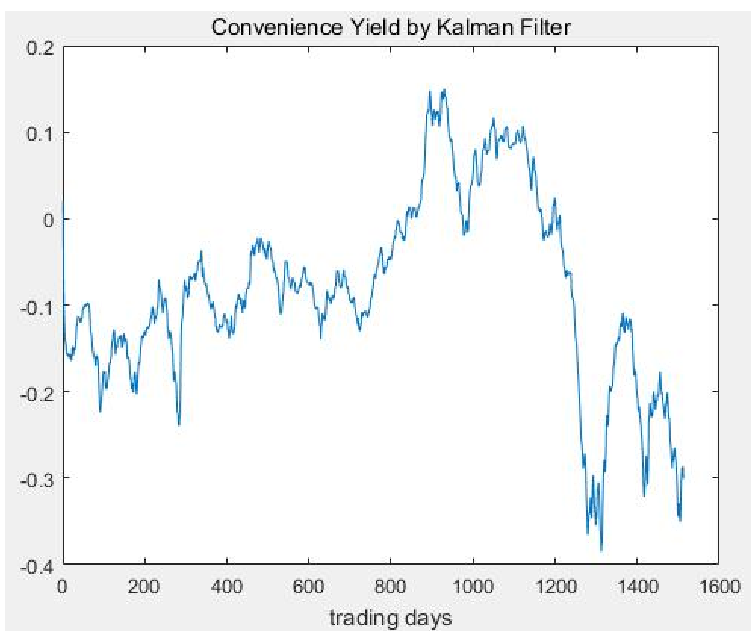


Figure 198: Estimated Convenience Yield from The Two-Factor Model With Rollover Data by The Kalman Filter

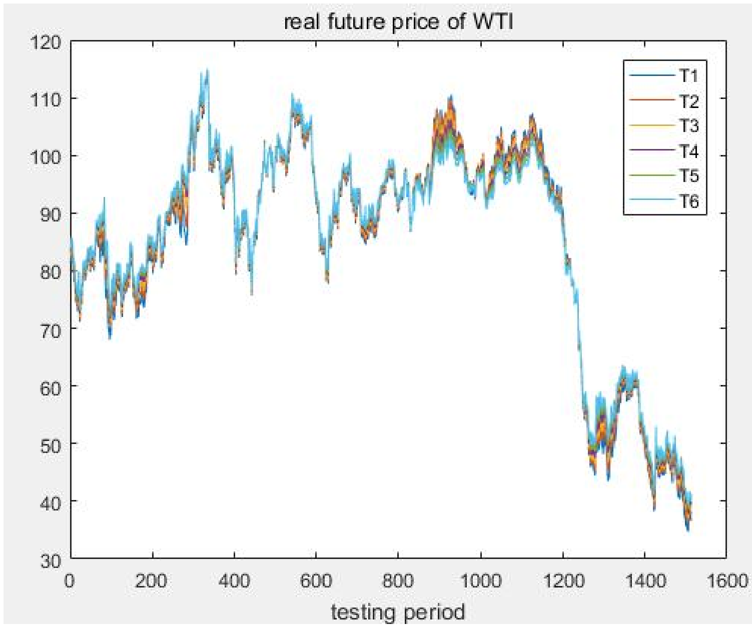


Figure 199: Observed Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model With Rollover Data

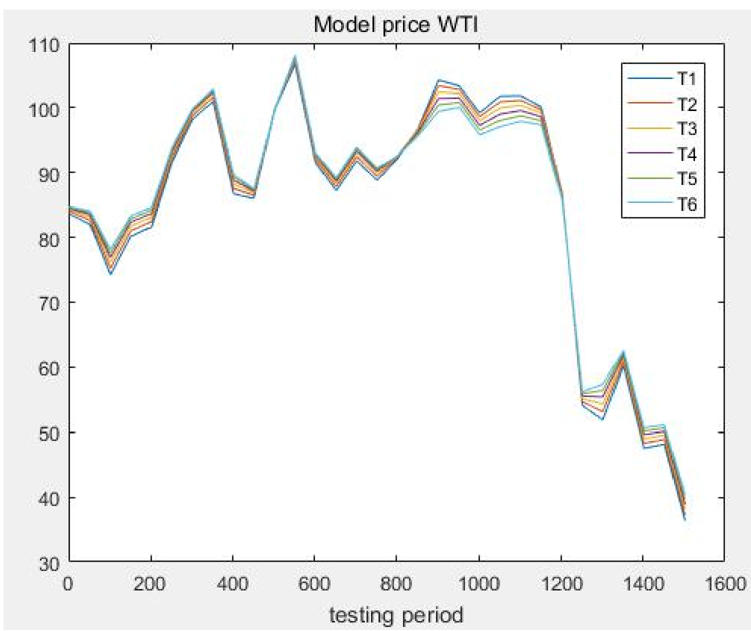


Figure 200: Estimated Prices of WTI Futures Contracts with Different Maturities for The Two-Factor Model With Rollover Data

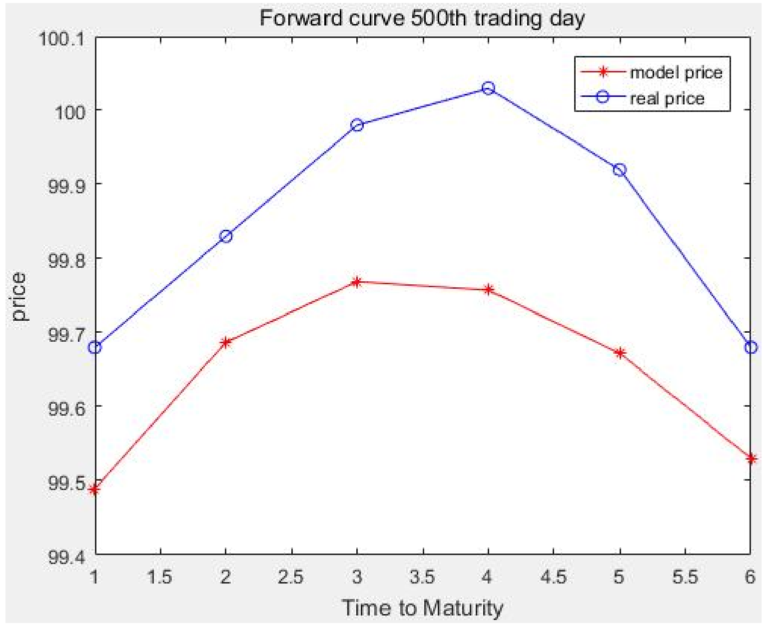


Figure 201: Forward Curves from The Two-Factor Model With Rollover Data on The 500th day

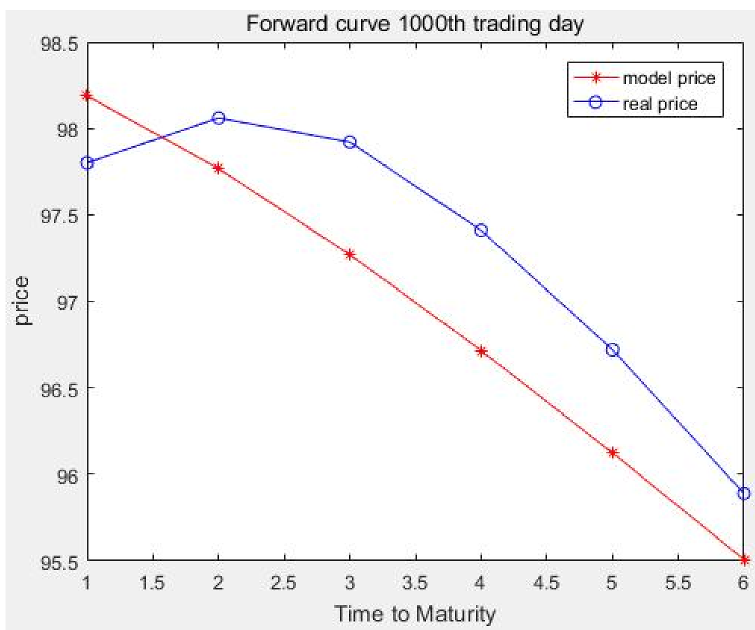


Figure 202: Forward Curves from The Two-Factor Model With Rollover Data on The 1000th day

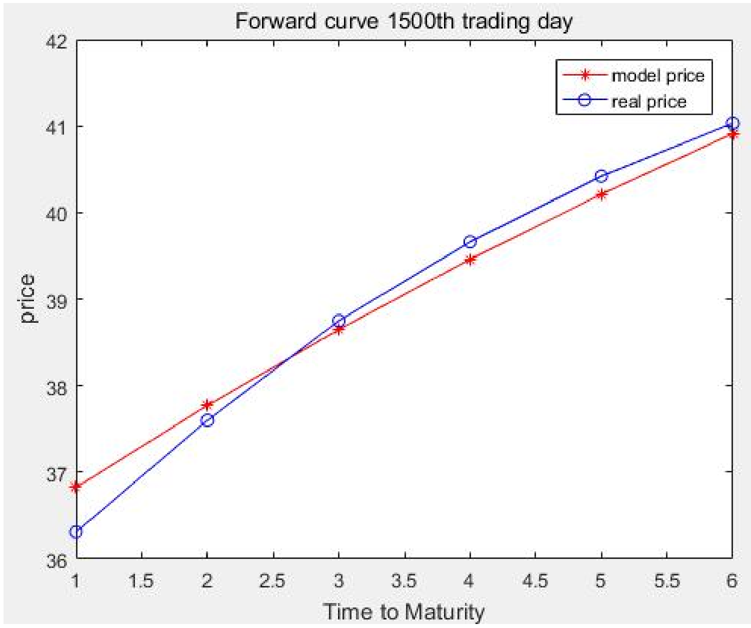


Figure 203: Forward Curves from The Two-Factor Model With Rollover Data on The 2000th day

8.6.1 The Empirical Results of The Schwartz Two-Factor Model

Figure 197 and Figure 198 exhibit the estimated unobservable state variables using the Kalman filter. On the one hand, the estimated spot price of crude oil is described in Figure 197. In Figure 197, it is not hard to observe that, in the test period, the estimated spot price of crude oil fluctuated around 80 dollars per barrel at the beginning of the test period. Then it rebounds and keeps moving up to about 115 dollars per barrel, which is the peak of the entire test period. After the peak, the estimated spot price of crude oil stably fluctuated between about 75 and 100 dollars per barrel. At the end of the test period (from June 2014), the spot price of crude oil experiences a long and continued decrease to about 35 dollars per barrel, which is the valley. The significantly downward slide at the end of the test period can be explained as follows: first, the global economy deteriorated after the European debt crisis; second, due to the Ukraine issue, the relationship between Russia and EU deteriorated; third, in some areas, political stability deteriorated, e.g. Egypt, Libya and Syria. On the other hand, Figure 198 shows the convenience yield

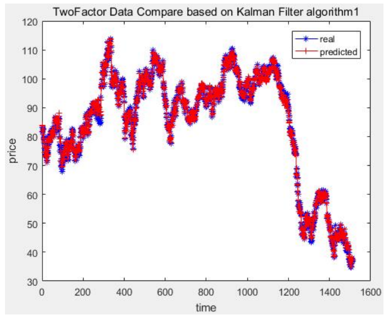


Figure 204: The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor Model With Rollover Data

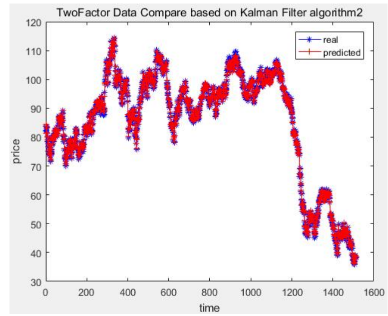


Figure 205: The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model With Rollover Data

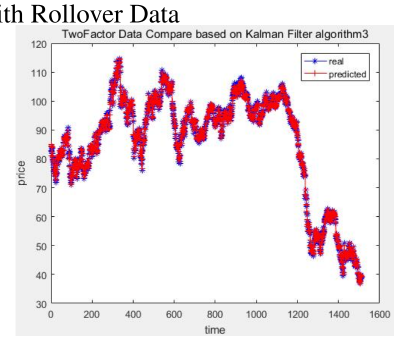


Figure 206: The Predicted and Observed Futures Prices for The Third Testing Futures Contract based on The Two-Factor Model With Rollover Data

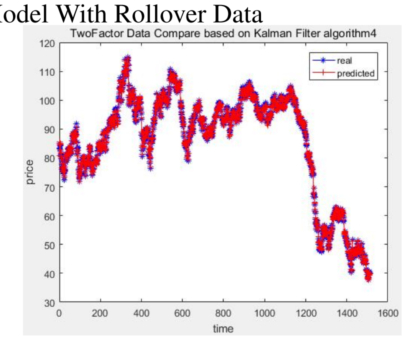


Figure 207: The Predicted and Observed Futures Prices for The Fourth Testing Futures Contract based on The Two-Factor Model With Rollover Data

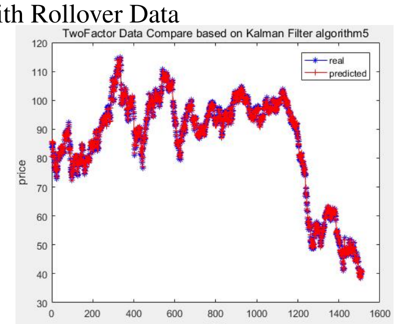


Figure 208: The Predicted and Observed Futures Prices for The Fifth Testing Futures Contract based on The Two-Factor Model With Rollover Data

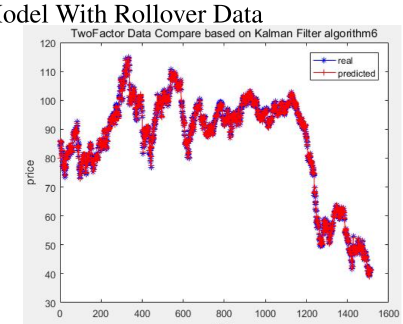


Figure 209: The Predicted and Observed Futures Prices for The Sixth Testing Futures Contract based on The Two-Factor Model With Rollover Data

of crude oil estimated by the Kalman filter in the same test period. At the beginning of the test period, the estimated convenience yield fluctuated around -0.1 in the first half of the test period, then, after a short increase, it rapidly decreased to the valley of the entire test period at about -0.4. After a short rebound, it immediately slumps to the another relatively low point (about -0.35) of the entire period. As for the estimated parameters, first, the degree of mean reversion of the convenience yield of WTI crude oil is not low; second, the long-term return investment on the convenience yield of WTI crude oil is positive during the test period; last, the risk on the spot price of WTI crude oil is lower than the risk on the convenience yield of WTI crude oil.

Then the first six chosen futures contracts (with maturities from 1 month to 6 months) are shown together in Figures 199 and 200 in order to see whether there are significant differences between the observed prices of WTI crude oil futures contracts and the estimated prices of WTI crude oil futures contracts in the Two-Factor model. To be more specific, the real observed futures prices of the six chosen futures contracts are shown in Figure 199. Then, the estimated model futures prices of the six chosen futures contracts, which are estimated from the Two-Factor model, are shown in Figure 200. In order to see the trend clearly, in Figure 200, only 31 points are chosen in the figure, which were picked on the first trading day of each 50 trading days. In other words, there are no significant differences between the two figures. On the one hand, WTI crude oil did not follow a strict backwardation or a strict contango during the test period; on the other hand, it is not hard to see that the temporary trend of the backwardation and the contango is more easily found when WTI crude oil rapidly price-inverses, while all futures prices seem to be close, when the price of WTI crude oil has a clear trend of increase or decrease.

In order to observe the effectiveness of the Two-Factor model with rollover data, the comparisons with observed prices of each WTI crude oil futures in the Two-Factor model with rollover data are shown in Figures 204 to 209. The rank of the title of each picture corresponds to the rank of its maturity. For example, the title 'The Predicted and Observed Futures Prices for The First Testing Futures Contract based on The Two-Factor

Model With Rollover Data' and the title 'The Predicted and Observed Futures Prices for The Second Testing Futures Contract based on The Two-Factor Model With Rollover Data' correspond to the futures contracts which matured in one month and two months respectively, and so forth. It is not hard to see that the Two-Factor model is useful in pricing a futures contract with rollover data, because all six pictures show that the estimated model price of each futures rollover contract is really close to the real observed price of the futures rollover contract.

Last, the forward curves from the Two-Factor model with rollover data are shown within Figures 201-203. First, on the 500^{th} trading day (on 27th Dec. 2011), the model estimated prices of the Two-Factor model are lower than the observed prices. Second, on the 1000^{th} trading day (on 23rd Dec. 2013), there was a crossover between the estimated model prices and the observed prices. To be more specific, with the increase in the term to maturity, the estimated prices are firstly higher than the observed prices, then they are lower than the observed prices. Last, similar to the 1000^{th} trading day, on the 1500^{th} trading day (on 14th Dec. 2015), there was a crossover between the estimated model prices and the observed prices. To be more specific, in Figure 203, the estimated prices are higher than the observed prices in the beginning and then they are lower than the observed prices, but the differences are really small. In other words, as expected, the estimated model prices from the Two-Factor model are close to the observed prices.

Paremeters	Estimated result	Standard Error
κ	0.6801	SE1
α	0.1111	0.0961
λ	0.0098	SE2
σ_1	0.5314	0.0624
σ_2	0.6215	SE3
ρ	0.4894	0.0574

Note:$SE1 = \sqrt{-0.003180 + ComputationalError1}$
$SE2 = \sqrt{-0.001478 + ComputationalError2}$
$SE3 = \sqrt{-0.01028 + ComputationalError3}$

Table 8.5: Estimated Parameters from The Two-Factor Model with Rollover Data

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