Existence and Uniqueness of Inductive Limit Cartan Subalgebras in Inductive Limit C^*-algebras

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Abstract

The main focus of this thesis is to answer the question of existence and uniqueness of inductive limit Cartan subalgebras in certain inductive limit C^*-algebras. The classes of inductive limit C^*-algebras considered are the unital AF, AI, and A$\mathbb{T}$-algebras. The results obtained are then generalized to certain AX-algebras.

This thesis shows that all the aforementioned classes (and AX-algebras for planar finite connected graphs $X \subset \mathbb{C}$) which arise from unital and injective connecting maps contain inductive limit Cartan subalgebras. It also shows that for all these classes except for the AF-algebras, uniqueness of the inductive limit Cartan subalgebras fails. We construct two non-isomorphic AI-Cartan subalgebras inside both a non-simple and simple AI-algebra. We provide a class of simple and unital AI-algebras for which uniqueness of AI-Cartan subalgebras fails. For the AF-algebras, we give a K-theoretic proof of the uniqueness of the AF-Cartan subalgebras.

Additionally, this thesis generalizes a theorem by Renault which characterises Cartan pairs in separable C^*-algebras by twisted étale second countable groupoids. The generalization captures all Cartan pairs, not just the separable ones.

Contents

List of Figures

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To my family

Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

As in many rich fields within mathematics, the theory of C^*-algebras has its genesis in the realm of physics. In the first half of the 20^{th} century, quantum mechanics was rising in popularity as a theory better capable at describing physical phenomena at the microscopic scale, in comparison to Einstein's relativity which only described macroscopic phenomena of the universe. In the 1920s and 30s there was a need to mathematically formalize the structure of quantum mechanics. In the late 1920s, John von Neumann pursued this task and was able to define the abstract characteristics of a Hilbert space and develop the theory of bounded and unbounded normal operators on such spaces, which was the beginning of a rigorous mathematical formulation of quantum mechanics. His work culminated in the book *Mathematische Grundlagen der Quantenmechanik* which forms the mathematical basis of classical quantum mechanics (see the Introduction of [42], together with [56] and [57]).

Together with Murray, von Neumann developed in the 1930s a theory on *rings of operators* (see [53]), which formed what are today known as *von Neumann algebras*. This was an abstract formalism in which an algebra of operators acting on a Hilbert space was systematically studied, generalizing a lot of the findings of classical quantum mechanics. In 1943 Gelfand and Neumark realized that these von Neumann algebras are just an example of an abstract mathematical framework which is known today as a C^*-algebra (see the Introduction of [42] together with [26]). On the one hand, this formalisation was so abstract that the notion of a Hilbert space was not included in the definitions. On the other hand, due to the GNS-construction by Gelfand, Neumark, and Segal (see additionally [70]), the C^*-algebras still had a connection with the bounded operators on Hilbert spaces that they generalized, in the sense that every C^*-algebra can be represented as a certain subalgebra of bounded operators on a Hilbert space (see, for instance, Theorem 4.5.6 in [35]). Gelfand and

Neumark were able to show that the situation for *commutative* C^*-algebras was even more refined. Indeed, every commutative C^*-algebra is isomorphic to the C^*-algebra of C_0 functions on some locally compact Hausdorff space (see, for instance, Theorem 4.4.3 in [35]). For this reason the study of C^*-algebras has sometimes been labelled as the study of *non-commutative topology*.

Since its inception, the theory of C^*-algebras has proved to be an extremely important field of mathematics, not just because of its generalization and influence on mathematical physics, but because of its fruitful interplay with many other areas of mathematics, such as operator theory, group theory, differential topology, algebraic topology, measure theory, ergodic theory and dynamics, to mention just a few. One particular class of C^*-algebras that has (more recently) proved to be significant in the theory of C^*-algebras is that of a *Cartan subalgebra*.

The notion of a Cartan subalgebra in the C^*-algebra setting is due to Renault (see [64]) and built on the notion of a C^*-diagonal developed by Kumjian (see [40]), with both attempting to be the C^*-analogue for the same notion in the von Neumann algebra setting. Indeed in the latter case an example of a Cartan subalgebra is the algebra of essentially bounded functions on a measure space inside the von Neumann algebra constructed for a group acting by non-singular transformations on the measure space (see Section 1 in [86]). This inclusion was abstracted by Veršik in [84] and studied by Feldman and Moore when they were considering the von Neumann algebra of a measured countable equivalence relation (see [23]). The Cartan subalgebras were maximally Abelian regular subalgebras admitting a normal conditional expectation onto them, and contained exactly the same information as the equivalence relation (see Section 1 in [64]). Building on this work, Cartan subalgebras of C^*-algebras were eventually defined to be maximally Abelian C^*-subalgebras that contain an approximate unit for the algebra in which they sit, are regular, and admit a faithful conditional expectation onto them (see Definition 5.1 in [64]). Renault's remarkable results in [64] characterised all Cartan subalgebras of *separable* C^*-algebras: these were exactly the C_0 functions on the unit space of a twisted étale locally compact second countable topologically principal Hausdorff groupoid, sitting inside the reduced twisted groupoid C^*-algebra (see Theorems 5.2 and 5.9 in [64]).

This characterisation brought the world of topological groupoids close to the world of Cartan subalgebras, and led to many interesting findings. Barlak and Li in [8] used Renault's characterisation to find canonical types of Cartan subalgebras in certain inductive limit C^*-algebras. Indeed the assumption is that every building block has a Cartan subalgebra, and hence such a Cartan pair has a characterisation by étale twisted groupoids, and so if the connecting maps of the inductive limit are induced by maps on the groupoid level having certain properties, then a twisted étale groupoid is built from the building block groupoids such that its reduced twisted groupoid C^*-algebra is the inductive limit C^*-algebra, and the C_0 functions on its unit space is the inductive limit of the Cartan subalgebras of the building blocks. Thus in such a situation the inductive limit of the Cartan subalgebras is a Cartan subalgebra of the inductive limit C^*-algebra (see Theorem 3.6 in [8]). Later, in [47], Li advanced this result to determine exactly which connecting maps could be induced by such groupoid level maps having the necessary properties to build a twisted étale groupoid for the inductive limit. In particular he showed that this is possible when the connecting maps are injective, map a Cartan subalgebra of a building block into the Cartan subalgebra of the next building block, map the normalizer set into the normalizer set, and are compatible with the faithful conditional expectations (see Proposition 5.4 in [47]). This gives rise to a useful tool in finding Cartan subalgebras in inductive limit C^*-algebras.

Cartan subalgebras have also been fundamental in linking several areas of mathematics together, such as topological dynamics and geometric group theory. For instance Li shows in [46] that Cartan subalgebras build a bridge between C^*-algebras and topological dynamics via the notion of continuous orbit equivalence. Indeed in a specific setting two dynamical systems are continuous orbit equivalent if and only if their corresponding Cartan pairs (consisting of the crossed product C^*-algebra of the dynamical system and its function subalgebra) are isomorphic. Li also shows that there is a link between C^*-algebras and geometric group theory via the notion of quasi-isometry. Indeed, in a specific setting two group actions on topological spaces have quasi-isometric Cayley graphs if and only if there is an isomorphism of certain Cartan pairs associated to the system. Because of these connections, Cartan subalgebras have garnered lots of attention outside the purely C^*-algebraic realm.

Nonetheless, within this realm, Cartan subalgebras have not faltered to prove their prominent role. They have, for instance, recently featured in the classification programme for C^*-algebras. This programme, due to many hands (see for example [21], [30], [31], [38], [59], [80]), aims to classify C^*-algebras by an invariant consisting of K-theoretic and tracial data, and has witnessed major breakthroughs. One of the assumptions for classifiable C^*-algebras is that they satisfy the UCT (Universal Coefficient Theorem). It is a major open problem whether every separable nuclear C^*-algebra satisfies the UCT. It is shown in [7], [8] and [47] that this open problem is equivalent to an existence question for Cartan subalgebras: namely whether every unital separable simple stably finite C^*-algebra with finite nuclear dimension contains a Cartan subalgebra. The UCT problem is also equivalent to whether every unital Kirchberg algebra has a Cartan subalgebra (see the Introduction in [10]). All this highlights the central role the question of existence of Cartan subalgebras plays in the general theory of C^*-algebras.

A lot of work has recently been done in finding Cartan subalgebras in C^*-algebras. Li and Renault initiated the systematic study of existence and uniqueness of Cartan subalgebras in C^*-algebras in [48], showing amongst other things that many connected Lie group C^*-algebras contain Cartan subalgebras, as well as finding distinguished Cartan subalgebras in nuclear Roe algebras. Li shows in [47] that every classifiable simple C^*-algebra contains a Cartan subalgebra. Barlak and Raum classify Cartan subalgebras of dimension drop algebras with coprime parameters in [9]. White and Willett study the question of uniqueness of Cartan subalgebras in uniform Roe algebras in [86].

The aim of this thesis is to answer certain existence and uniqueness questions for certain canonical types of Cartan subalgebras in certain inductive limit C^*-algebras. The Cartan subalgebras we will be interested in are those that can be realized as inductive limits themselves of Cartan subalgebras of the building blocks. The inductive limit C^*-algebras we will consider are AI, A$\mathbb{T}$, and AX-algebras, where X is a finite connected planar graph imbedded in $\mathbb{C}$.

It was already in the work of Strătilă and Voiculescu (see [72]) that this question was considered (even though they did not refer to it as a Cartan subalgebra). Indeed, they construct inductive limit Cartan subalgebras in AF-algebras. Their method is inductive; the subsequent Cartan subalgebra in an AF-building block is the C^*-algebra generated by the Cartan subalgebra of the previous building block and an arbitrary masa in the commutant of the previous building block. The desired inductive limit Cartan subalgebra is then just the inductive limit of the Cartans of the building blocks. This method cannot however be directly generalized to the case of AI and A$\mathbb{T}$-algebras. Also, they do not discuss whether these Cartan subalgebras they obtain are unique, in the sense that any such two Cartan pairs are isomorphic (as Cartan pairs). Even though an affirmative answer to this is a consequence of the work of Krieger in [39], there is no systematic study of the uniqueness question for such canonical types of inductive limit Cartan subalgebras. Inspired by all of this, the questions we had in mind prior to the commencement of our research for the PhD were the following:

Question 1. Does there exist inductive limit Cartan subalgebras in AI-algebras and A$\mathbb{T}$-algebras, and if so can the methods used be generalized to prove the existence of inductive limit Cartan subalgebras in AX-algebras, for certain topological spaces X which would be natural generalizations of the interval and the circle?

Question 2. Can the uniqueness of the inductive limit Cartan subalgebras in AF-algebras be proved differently, in a more structured way that lends itself to generalizations? If AI and A$\mathbb{T}$-algebras do have inductive limit Cartan subalgebras, are these unique? If not, to what extent does uniqueness fail (is there a subclass of these algebras where uniqueness is guaranteed)? Do the same results hold for the AX-algebras?

Question 3. Does Renault's characterisation of separable Cartan pairs via second countable topologically principal étale twisted groupoids hold in the non-separable case (under a weakening of the assumptions on the groupoid)?

This thesis provides complete answers to all the questions above. In Chapter 3 we prove:

Theorem A. (See Theorem 3.2.27) *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact effective Hausdorff groupoid. Then $(C_r^*(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$ is a Cartan pair. Conversely, let (A, C) be a Cartan pair. Then there exists a twisted étale locally compact effective Hausdorff groupoid $(\mathcal{G}, \Sigma)$ and a C^*-algebra isomorphism carrying (A, C) onto $(C_r^*(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$.*

Theorem A affirmatively answers Question 3, and because the Cartan pair is no longer separable, second countability of the corresponding twisted groupoid is no longer guaranteed, and so topological principality is reduced to effectiveness. The way we obtain Theorem A is by highlighting how most of Renault's proofs in [64] do not make a direct use of the second countability assumption on the groupoid, except in a small number of cases. For these cases we show how we may obtain similar results without such assumptions. A short discussion with Renault informed us that the assumption of second countability was mainly placed for convenience, without proper analysis of its requirement. A lot of the proofs in [64] make use of separation functions on topological spaces; which are standard when the space is second countable, as then it is paracompact and hence normal, and thus one can make use of the standard Urysohn lemma for separation. However in the non-second countable case one must make use of Urysohn type results for just locally compact spaces, and these are less standard.

An important class of non-separable C^*-algebras are the uniform Roe algebras, which are of interest as they build a link to coarse geometry (see Section 1 in [49]). These have Cartan subalgebras (see Section 6 in [48]) which fall outside that which

Renault's theorem can capture. In addition, the authors of [48] obtain a distinguished Cartan subalgebra by using a slight modification of Renault's theorem, where second countability of the groupoid is weakened to σ-compactness. Of course with Theorem A this approach is not necessary.

In Chapter 4 we prove the following theorems:

Theorem B. (See Theorem 4.2.9) *Every unital AI-algebra with unital and injective connecting maps contains an AI-Cartan subalgebra.*

Theorem C. (See Theorem 4.3.18) *Every unital A$\mathbb{T}$-algebra with unital and injective connecting maps contains an A$\mathbb{T}$-Cartan subalgebra.*

Theorem D. (See Theorem 4.4.12) *Every unital AX-algebra, where X is a finite planar connected graph imbedded in $\mathbb{C}$, with unital and injective connecting maps contains an AX-Cartan subalgebra.*

For a definition of an AX-Cartan subalgebra, see Definition 2.3.28. Together, these three theorems answer Question 1. All three theorems make use of a tool by Li (see Proposition 5.4 in [47]) which allows us to determine whether an inductive limit of Cartan subalgebras is a Cartan subalgebra of the corresponding inductive limit C^*-algebra, by checking whether connecting maps send a Cartan subalgebra of the building block into the Cartan subalgebra of the subsequent building block, whether they send normalizer set into normalizer set, and whether they are compatible with the associated conditional expectations arising from the definition of a Cartan subalgebra.

The way Theorem B is obtained is by realizing AI-algebras as inductive limit C^*-algebras with standard connecting maps (for a definition of a standard map, see Definition 2.3.39). This is due to the findings by Thomsen in [76]. Then checking that the (diagonal) Cartan subalgebras of the AI-building blocks satisfy the properties required by Li becomes a straightforward task, because the connecting maps are simple to understand.

The way we prove Theorem C is slightly different. Elliott's work in [19] shows that the maximally homogeneous *-homomorphisms are dense in the set of *-homomorphisms between circle algebras (for a definition of maximal homogeneity, see Definition 2.3.38). Thomsen's work in [75] allows us to write maximally homogeneous connecting maps between circle algebras (without a direct sum) as unitary conjugates of standard maps. These unitaries are functions on the unit interval that do not necessarily agree at the endpoints. Using this, Thomsen is able to show that given any maximally homogeneous subalgebra in a circle algebra, and a maximally homogeneous connecting map into another circle algebra, we may find a maximally homogeneous subalgebra of the codomain containing the image of the maximally homogeneous subalgebra of the domain.

We extend Thomsen's and Elliott's aforementioned work to direct sums of circle algebras, which of course requires that we extend Thomsen's definition of maximal homogeneity to direct sums of circle algebras. We also show that the notion of a Cartan subalgebra coincides with the notion of maximal homogeneity, in A$\mathbb{T}$-building blocks. By these generalizations we are able to show that any A$\mathbb{T}$-algebra with injective connecting maps may be realized as an A$\mathbb{T}$-algebra with injective maximally homogeneous connecting maps, which are also unitary conjugates of standard maps. We then show how these connecting maps satisfy the requirements of Li in order to obtain a Cartan subalgebra of the inductive limit.

For Theorem D, we generalize the methods used to prove Theorem C. In particular, we note that our result of being able to write a maximally homogeneous *-homomorphism as one which is a unitary conjugate of a standard map is not particular to the circle, but to any AX-building block which may be imbedded inside an AY-building block where Y has vanishing first order Čech cohomology group with coefficients in a topological group (see Definition 4.3.11). Hence our choice of X being a graph means we may use its universal cover, which is a tree, which has vanishing Čech cohomology (in analogy with how Thomsen uses $\mathbb{R}$ to cover $\mathbb{T}$). Then using the work in [44] we find that the maximally homogeneous *-homomorphisms are dense in the set of *-homomorphisms between AX-building blocks. Thus we are able to realize our AX-algebra as one whose connecting maps are injective and maximally homogeneous. Using that maximally homogeneous *-homomorphisms are unitary conjugates of standard maps, we show that given a maximally homogeneous *-homomorphism between AX-building blocks, and a maximally homogeneous subalgebra of the the domain, we can find a maximally homogeneous subalgebra of the codomain containing the image of the maximally homogeneous subalgebra of the domain. This extends Thomsen's Proposition 1.8 in [75] to AX-building blocks. We also show that the notion of Cartan subalgebra in an AX-building block also coincides with the notion of a maximally homogeneous subalgebra. Finally we prove that the requirements of Li on connecting maps are satisfied for our Cartan subalgebras and hence we obtain an AX-Cartan subalgebra in our AX-algebra.

The strengths and uses of our results in Theorems B, C, and D, are many. Firstly, a lot of the extensively studied C^*-algebras of the past are AI, or A$\mathbb{T}$-algebras. This includes for example the Bunce-Deddens algebra, the irrational rotation algebra, and many crossed product C^*-algebras (see, for example, [67]). Our results show that all of these have inductive limit Cartan subalgebras.

Secondly, in the more general setting of AH-algebras (see Definition 6.0.1), Gong et al. show in [29] that those AH-algebras with the ideal property and having torsion free K-theory are in fact A$\mathbb{T}$-algebras. Hence these have inductive limit Cartan subalgebras.

Thirdly, the philosophy of our methods, which comprises mainly of realizing inductive limit C^*-algebras via connecting maps that are easier to understand, lends its way to generalization. Indeed, we start off with an inductive limit C^*-algebra with *arbitrary* injective and unital connecting maps, and we are able to realize the same algebra using different connecting maps. This is in stark contrast to a lot of the current methods in which inductive limit Cartan subalgebras are found, where usually there is a set of connecting maps declared a-priori (for example Li's exhibition of an inductive limit Cartan subalgebra in the Jiang-Su algebra in [47]). Hence our methods open the door to the question of which inductive limits can be realized by connecting maps that are more amenable for the study of inductive limit Cartan subalgebras.

Fourthly, we do not assume simplicity for our inductive limit C^*-algebras, in stark contrast to the classification programme, where simplicity is almost always assumed.

Finally, since our methods are constructive, we produce explicit methods for obtaining the Cartan subalgebras of the building blocks, and hence the corresponding étale twisted groupoids for these are easy to understand. By the methods in [8], the groupoid corresponding to the inductive limit Cartan subalgebra can thus be computed explicitly, and so we may obtain groupoid models for these Cartan subalgebras.

In Chapter 5 we explore the uniqueness of the inductive limit Cartan subalgebras we manifest. We prove the following:

Theorem E. (See Theorem 5.1.3) *Let A and B be unital AF-algebras with AF-Cartan subalgebras C and D, respectively. Assume there exists a group isomorphism*

$$\alpha : (K_0(A), K_0(A)^+, [1_A]_0) \to (K_0(B), K_0(B)^+, [1_B]_0).$$

*Then there exists a *-isomorphism $\phi : A \to B$ such that $K_0(\phi) = \alpha$ and $\phi(C) = D$.*

Theorem E, which is similar to Elliott's classification theorem for AF-algebras but where in the proof we keep track of the Cartan subalgebras, implies uniqueness of AF-inductive limit Cartan subalgebras. The case for the AI (and hence A$\mathbb{T}$)-algebras is different. We obtain:

Theorem F. (See Theorems 5.2.4 and 5.2.8) *AI-Cartan subalgebras (and hence A$\mathbb{T}$-Cartan subalgebras) are not unique. More generally, uniqueness fails in a very large class of simple AI-algebras.*

We obtain Theorem F by studying the spectrum of the diagonal AI-Cartan subalgebra in an AI-algebra (where the Cartan subalgebra of every building block is just the diagonal subalgebra). Indeed, we show that such Cartan subalgebras have spectrum that looks like a bundle over a Cantor set (obtained from taking infinite paths in a Brattelli type diagram which corresponds to the connecting maps of the AI-algebra), with fibres that are inverse limits of the unit interval corresponding to the sequence of eigenvalue functions associated with a specific path (see Chapter 5 for more details). We show that these fibres are the connected components of the spectrum. Once we have understood the spectrum, we construct AI-algebras which arise from two different sets of connecting maps but which admit an approximate intertwining (see Definition 2.3.43). This implies that the AI-algebras are isomorphic, but in our selection of connecting maps we make choices for the eigenvalue functions (see Definition 2.3.39) that ensure that we get a fibre in one spectrum that is not homeomorphic to any fibre in the other spectrum. In this way we obtain two diagonal AI-Cartan subalgebras with spectra that are not homeomorphic, and hence the Cartan subalgebras are not isomorphic. We make use of the plethora of constructions of inverse limits of the unit interval that appear in [34] in order to achieve this result.

The layout of the thesis is as follows. In Chapter 2 we discuss the preliminaries needed for the rest of the thesis. Our choice of what to include as preliminaries is based on our judgment of what material will be required but which could also be helpful independently for a researcher in the topic. Hence there will be some preliminaries that are not included in this chapter but rather placed in the main body of the thesis, whenever we deem them very specific and not widely useful for a general mathematician working in C^*-algebras. Any preliminaries needed will be those that we did not possess at the start of our postgraduate research. Anything that we did possess will be assumed knowledge. It is worth noting that, in Section 2.1, we provide a self-contained complete account of the twisted étale groupoid construction and its reduced C^*-algebra.

In Chapter 3 we prove Theorem A. Chapter 4 proves Theorems B, C, and D. Chapter 5 proves Theorems E and F. Chapter 6 gives an outlook on the type of open questions that arise directly from our work, and which we will continue researching in the future. It is worth noting that this thesis is specifically written in a style that attempts to reflect our own journey through the research. Indeed, the choice of order for the chapters and their sections tries to adhere as close as possible to our personal timeline during our research. In particular, Chapter 4 proves Theorem B, then Theorem C, then Theorem D. Of course one may have decided to present just the proof for Theorem D as it generalizes Theorems B and C, but we chose a different presentation. Besides reflecting the true order of how the theorems came to be, this was done to highlight how the mathematical ideas for Theorem D are just natural generalizations of the ideas for Theorems B and C, and thus a lot of the proofs in Section 4.4 will refer back to proofs in Section 4.3 and add any extra detail required, rather than being standalone proofs. Hence we intend for this thesis to be read linearly. We would also like to emphasize that whenever there is a large section of the thesis that follows the material of a few specific references, we will point this out in the beginning of the relevant section rather than citing the same references repeatedly.

Finally, it is our hope that this thesis, with its very humble addition of knowledge to the field of operator algebras, serves to both inform and inspire any current or upcoming researchers within the field. Mathematics is a beautiful subject, and we are grateful to have had the opportunity to pursue it and engage with its philosophy.

Chapter 2

Preliminaries

This chapter introduces the notation and results required to understand the material of the subsequent chapters. We assume that the reader has a general foundation in pure mathematics (up to graduate level), and is specifically familiar with basic functional analysis, topology, and C^*-algebra theory up to and including the material covered in [35] and all chapters except IV and IX in [16]. These exceptions relate to K-theory for C^*-algebras, and Brown-Douglas-Fillmore theory; the former will be introduced at greater depth in this thesis, and the latter we do not require.

Section 2.1 in this chapter covers the preliminaries on twisted étale groupoids, their reduced C^*-algebras, and their correspondence with Cartan pairs. This will be fully necessary for Chapter 3, where we generalize Renault's theorem for Cartan subalgebras, and partly for sections of the other chapters, such as Section 4.1 and Section 5.2. It is worth noting that, to the best of our knowledge, there is no self-contained reference that covers this material in its entirety, as we do.

Section 2.2 in this chapter will discuss the K_0 functor for unital C^*-algebras. We will only focus on the aspects of K_0 that we will require for the remaining chapters. The material here will be prominent in Sections 4.3 and 4.4, as the K_0 functor will be important in passing from summands to direct sums, and also in Section 5.1, where we will prove the uniqueness of the AF-Cartan subalgebras via the ordered K_0 group. Section 5.2 will also use some K-theory when discussing an invariant for AI-algebras.

Section 2.3 will discuss the inductive limit construction, which will then be used to discuss continuity of K_0 with respect to inductive limits and the K_0 group of UHF-algebras. We will also be able to define the central theme of this thesis which are the inductive limit Cartan subalgebras, and the various AX-algebras that we will consider. We will also briefly include some well known results for approximate intertwinings of certain inductive limits that will be useful later when needing to show that two inductive limit C^*-algebras are isomorphic. The material of this section will be used critically throughout the thesis.

Section 2.4 will briefly introduce inverse limits of the unit interval, and will highlight some results and examples that will be of particular use for us in Chapter 5. Specifically, certain inverse limits of the unit interval will be constructed in Section 5.2 to prove non-uniqueness of the AI-Cartan subalgebras.

2.1 Cartan Subalgebras and Twisted Étale Groupoids

A groupoid is a set together with a partially defined binary operation (multiplication) that generalizes the structure seen in a group. Indeed, the operation is associative whenever it is defined, and there exists units and inverses. In the language of category theory, a groupoid is just a small category with inverses.

Algebraically, we will see that groupoids are nothing more than disjoint unions of Cartesian products of groups and certain equivalence relations. Hence the richness of groupoid theory comes from adding a topology. The étale topology on groupoids is the analogue to discrete groups in group theory. With a topology groupoids are able to capture a plethora of mathematical structures, including group actions on topological spaces and pseudogroups of partial homeomorphisms on a space.

In this section we will explore groupoids algebraically and then with a topology. Then we will consider the more general twisted groupoid and how to get twisted groupoid C^*-algebras for certain topological twisted groupoids. We will then discuss Renault's remarkable result which gives a correspondence between Cartan pairs and twisted étale groupoids. This result will be generalized in Chapter 3. To the best of our knowledge, there is no self-contained reference that covers this material in its entirety, as we do.

A standing assumption on all topological groupoids in this thesis is that they are Hausdorff, even if not explicitly mentioned.

2.1.1 Groupoids

This subsection will discuss the algebraic properties of groupoids, and give some examples. We will show that, algebraically, groupoids are nothing more than disjoint unions of Cartesian products of groups with equivalence relations. We will broadly follow the contents of Section 3.1 in [60] and Section 2.1 in [71].

Definition 2.1.1. A *groupoid* is a non-empty set $\mathcal{G}$, with a subset $\mathcal{G}^2 \subseteq \mathcal{G} \times \mathcal{G}$ of *composable pairs*, equipped with a binary operation (called multiplication)

$$m : \mathcal{G}^2 \to \mathcal{G}$$

(with $m(g, h)$ denoted by gh for all $g, h \in \mathcal{G}$) and an involution map (called inverse)

$$\mathrm{inv} : \mathcal{G} \to \mathcal{G}$$

(with $\mathrm{inv}(g)$ denoted by g^{-1} for all $g \in \mathcal{G}$) satisfying the following properties:

1. Whenever $g, h, k \in \mathcal{G}$ with $(g, h) \in \mathcal{G}^2$ and $(h, k) \in \mathcal{G}^2$, then both (gh, k) and (g, hk) belong to $\mathcal{G}^2$, and $(gh)k = g(hk)$ (written as ghk).

2. For all $g \in \mathcal{G}$, both (g, g^{-1}) and (g^{-1}, g) belong to $\mathcal{G}^2$. Furthermore, if $(g, h) \in \mathcal{G}^2$, then $g^{-1}gh = h$, and if $(h, g) \in \mathcal{G}^2$, then $hgg^{-1} = h$.

One way to think about the algebraic operations of a groupoid is to recall the definition of a group, and realize that the difference for groupoids lies in the fact that multiplication is not defined everywhere but rather on a specific subset of the Cartesian product of the groupoid with itself. Then property 1 of Definition 2.1.1 is an associativity property for multiplication, and property 2 is the existence of identities and inverses. This leads us to the following definition:

Definition 2.1.2. Given a groupoid $\mathcal{G}$, we define the *unit space* of the groupoid, denoted $\mathcal{G}^0$, as the set

$$\{g^{-1}g : g \in \mathcal{G}\} = \{gg^{-1} : g \in \mathcal{G}\}.$$

The *source map* is the map $s : \mathcal{G} \to \mathcal{G}^0$ defined by

$$s(g) = g^{-1}g \ \ \forall g \in \mathcal{G},$$

and the *range map* is the map $r : \mathcal{G} \to \mathcal{G}^0$ defined by

$$r(g) = gg^{-1} \ \ \forall g \in \mathcal{G}.$$

We have the following property:

Lemma 2.1.3. *Let $\mathcal{G}$ be a groupoid, and s and r the source and range maps, respectively. Then $(g, h) \in \mathcal{G}^2$ if and only if $s(g) = r(h)$.*

Proof. Assume $(g, h) \in \mathcal{G}^2$. From Definition 2.1.1 we know that (g^{-1}, g) and (h, h^{-1}) belong to $\mathcal{G}^2$. Then

$$s(g) = g^{-1}g = g^{-1}(ghh^{-1}) = (g^{-1}gh)h^{-1} = hh^{-1} = r(h).$$

Assume $s(g) = r(h)$, then it follows that $(g, hh^{-1}) = (g, g^{-1}g) \in \mathcal{G}^2$, and since $(hh^{-1}, h) \in \mathcal{G}^2$, it follows that $(g, (hh^{-1})h) = (g, h) \in \mathcal{G}^2$. □

Definition 2.1.2 and Lemma 2.1.3 allows us thus to think of our groupoids graphically. Indeed, we may declare the vertices as the elements of the unit space $\mathcal{G}^0$, and the directed edges between them are elements of $g \in \mathcal{G}$, with initial vertex $s(g)$ and terminal vertex $r(g)$. Multiplication of g and h is then concatenation of edges, with the new edge labelled gh, with initial vertex $s(h)$ and terminal vertex $r(g)$. This is represented in Figure 2.1.

Figure 2.1: A graphical representation of a groupoid

$$\begin{tikzcd} \cdot \arrow[r, bend left, "h"] \arrow[rr, bend right, "gh"'] & \cdot \arrow[r, bend left, "g"] & \cdot \end{tikzcd}$$

We can deduce more algebraic properties about our groupoid that are useful for calculations:

Lemma 2.1.4. *Let $\mathcal{G}$ be a groupoid, and s and r the source and range maps respectively.*

1. *If $(g, h) \in \mathcal{G}^2$, then $(h^{-1}, g^{-1}) \in \mathcal{G}^2$, and $(gh)^{-1} = h^{-1}g^{-1}$. Furthermore, $s(gh) = s(h)$ and $r(gh) = r(g)$.*

2. *If $g \in \mathcal{G}^0$, then $g^{-1} = g$ and $s(g) = r(g) = g$.*

Proof. 1. Note that

$$s(h^{-1}) = hh^{-1} = r(h) = s(g) = g^{-1}g = r(g^{-1}),$$

where we have made use of Lemma 2.1.3. Hence by the same lemma, we obtain that $(h^{-1}, g^{-1}) \in \mathcal{G}^2$.

We have that $((gh)^{-1}, gh) \in \mathcal{G}^2$ and $(gh, h^{-1}) \in \mathcal{G}^2$. This implies that $((gh)^{-1}, ghh^{-1}) \in \mathcal{G}^2$. But since $ghh^{-1} = g$, we have that $((gh)^{-1}, g) \in \mathcal{G}^2$

and hence $((gh)^{-1}, gg^{-1}) \in \mathcal{G}^2$. Thus

$$(gh)^{-1} = (gh)^{-1}gg^{-1} = (gh)^{-1}(ghh^{-1})g^{-1} = (gh)^{-1}(gh)h^{-1}g^{-1} = h^{-1}g^{-1},$$

(where the first equality is due to 2 in Definition 2.1.1) and

$$s(gh) = (gh)^{-1}(gh) = h^{-1}g^{-1}gh = h^{-1}h = s(h),$$

$$r(gh) = (gh)(gh)^{-1} = ghh^{-1}g^{-1} = gg^{-1} = r(g).$$

2. Let $g = h^{-1}h \in \mathcal{G}^0$, then $g^{-1} = h^{-1}(h^{-1})^{-1} = h^{-1}h = g$. It is clear that $s(g) = g^2 = r(g)$ and $g^2 = g$.

□

Let us now see some examples of groupoids.

Example 2.1.5 (Groups)**.** Let G be a group with identity element e. Then it is a groupoid with $G^2 = G \times G$ and $G^0 = \{e\}$. The multiplication map is group multiplication, and the inverse map is the group inverse map.

Example 2.1.6 (Equivalence Relations)**.** Let X be a set and $\mathcal{R} \subseteq X \times X$ an equivalence relation. Then it is a groupoid with

$$\mathcal{R}^2 = \{((x, y), (y, z)) : (x, y), (y, z) \in \mathcal{R}\},$$

$$\mathcal{R}^0 = \{(x, x) : x \in X\}.$$

Multiplication is given by $(x, y)(y, z) = (x, z)$ for all $(x, y), (y, z) \in \mathcal{R}$. The inverse map is given by $(x, y)^{-1} = (y, x)$ for all $(x, y) \in \mathcal{R}$.

Example 2.1.7 (Transformation Groupoids)**.** Let X be a set and G a group acting on X. Let $\mathcal{G} = G \times X$. Then this is a groupoid with

$$\mathcal{G}^2 = \{((g, y), (h, x)) : g, h \in G, \quad x \in X, \quad y = hx\},$$

$$\mathcal{G}^0 = \{e\} \times X \cong X.$$

Multiplication is given by $(g, hx)(h, x) = (gh, x)$ and the inverse by $(g, x)^{-1} = (g^{-1}, gx)$. Note that $s((g, x)) = x$ and $r((g, x)) = gx$ (where we have used the identification $\{e\} \times X$ with X), and hence the groupoid elements represented graphically nicely capture the orbits via the group action.

There are of course ways to create new groupoids from a collection of groupoids. The following highlights this:

Lemma 2.1.8. *Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. Then the set $\mathcal{G} \times \mathcal{H}$ can be made into a groupoid. Let I be an indexing set and $\{\mathcal{G}_i\}_{i \in I}$ a collection of groupoids. Then the disjoint union $\bigsqcup\limits_{i \in I} \mathcal{G}_i$ can be made into a groupoid.*

Proof. Let

$$(\mathcal{G} \times \mathcal{H})^2 = \{((g_1, h_1), (g_2, h_2)) : (g_1, g_2) \in \mathcal{G}^2, \ \ (h_1, h_2) \in \mathcal{H}^2\}.$$

Define multiplication by $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$ for all $((g_1, h_1), (g_2, h_2)) \in (\mathcal{G} \times \mathcal{H})^2$ (it is to be understood that $g_1 g_2$ refers to multiplication in $\mathcal{G}$ and $h_1 h_2$ that in $\mathcal{H}$). Define the inverse by $(g, h)^{-1} = (g^{-1}, h^{-1})$ (where it is understood that the inverses are taken in the respective groupoids).

Define

$$\left(\bigsqcup_{i \in I} \mathcal{G}_i \right)^2 = \bigsqcup_{i \in I} \mathcal{G}_i^2,$$

and multiplication and inverse just inherited in the canonical way from each groupoid. $\square$

Algebraically, there is not much more to say about groupoids. It turns out that every groupoid is isomorphic (in a sense that will be made precise soon) to a disjoint union of Cartesian products of groups with equivalence relations.

Definition 2.1.9. A map $f : \mathcal{G} \to \mathcal{H}$ between groupoids $\mathcal{G}$ and $\mathcal{H}$ is called a *groupoid homomorphism* (or just homomorphism if the context is clear) if $(f(g_1), f(g_2)) \in \mathcal{H}^2$ whenever $(g_1, g_2) \in \mathcal{G}^2$, and if $f(g_1 g_2) = f(g_1) f(g_2)$ for all $(g_1, g_2) \in \mathcal{G}^2$.

A groupoid homomorphism f is called called a *groupoid isomorphism* (or just isomorphism if the context is clear) if it is bijective.

Theorem 2.1.10. *Every groupoid $\mathcal{G}$ is isomorphic to a groupoid of the form*

$$\bigsqcup_{i \in I} (G_i \times \mathcal{R}_i) \tag{2.1}$$

where I is a set, G_i is a group and $\mathcal{R}_i$ is an equivalence relation, for all $i \in I$.

Proof. Lemma 2.1.8 already tells us that (2.1) can be made into a groupoid in the natural way. It is obtained as follows. Let s and r be the source and range maps for $\mathcal{G}$. Define an equivalence relation on $\mathcal{G}^0$ by

$$\mathcal{R} = \{(r(g), s(g)) : g \in \mathcal{G}\} \subseteq \mathcal{G}^0 \times \mathcal{G}^0,$$

(transitivity follows from Lemma 2.1.4). Let I be a set which indexes the equivalence classes, and let these be $\{X_i\}_{i\in I}$, which form a partition of $\mathcal{G}^0$. For each $i \in I$, fix $x_i \in X_i$. For each $x \in X_i$, let $g_x \in \mathcal{G}$ be chosen such that $r(g_x) = x$ and $s(g_x) = x_i$. Define

$$G_i = \{g \in \mathcal{G} : s(g) = r(g) = x_i\}.$$

G_i with the multiplication from $\mathcal{G}$ is a group with identity element x_i. For each $i \in I$ let

$$\mathcal{R}_i = X_i \times X_i,$$

which is the trivial equivalence relation on X_i, and thus a groupoid in the sense of Example 2.1.6.

Now define a map $\alpha : \bigsqcup_{i\in I}(G_i \times \mathcal{R}_i) \to \mathcal{G}$ by

$$\alpha(g, (x, y)) = g_x g g_y^{-1}, \text{ for } g \in G_i, (x, y) \in \mathcal{R}_i.$$

It is easy to see that α is a well-defined groupoid homomorphism. Now define a map $\beta : \mathcal{G} \to \bigsqcup_{i\in I}(G_i \times \mathcal{R}_i)$ by

$$\beta(g) = (g_{r(g)}^{-1} g g_{s(g)}, (r(g), s(g))).$$

Again it is clear that β is a well-defined groupoid homomorphism, and in fact is an inverse for α. The result follows. $\square$

2.1.2 Étale Groupoids

This section will introduce topological groupoids and specifically étale groupoids. These are analogous to discrete groups in group theory. We will see some examples. As we witnessed in Theorem 2.1.10, there is not much to say about groupoids from just an algebraic perspective. However the theory becomes very rich when we allow topologies on our groupoids. The content of this subsection is based broadly on Section 3.2 in [60] and Sections 2.3 and 2.4 in [71]. However we adapt Definition 2.1 from [63] for the definition of a topological groupoid.

Definition 2.1.11. A *topological groupoid* is a groupoid $\mathcal{G}$ endowed with a topology that is compatible with the multiplication and inverse maps. Specifically,

$$m : \mathcal{G}^2 \to \mathcal{G}$$

is continuous (where $\mathcal{G}^2$ is endowed with the subspace topology of the product topology of $\mathcal{G} \times \mathcal{G}$) and

$$\mathrm{inv} : \mathcal{G} \to \mathcal{G}$$

is continuous.

Remark 2.1.12. We have no interest for non-Hausdorff topological groupoids in this thesis. Therefore, whenever discussing topological groupoids anywhere in the thesis, we will assume they are Hausdorff even if not stated.

Lemma 2.1.13. *Let $\mathcal{G}$ be a topological groupoid. Then the range and source maps are continuous.*

Proof. These maps are formed by composition of the inverse and multiplication maps, which are continuous by definition. □

Remark 2.1.14. When discussing the unit space $\mathcal{G}^0$ of a groupoid topologically, we are implicitly assuming it is endowed with the subspace topology.

Lemma 2.1.15. *Let $\mathcal{G}$ be a topological groupoid. Then $\mathcal{G}^0$ is closed if and only if $\mathcal{G}$ is Hausdorff.*

Proof. Assume $\mathcal{G}$ is Hausdorff and take a net $\{g_\alpha\}_{\alpha\in A}$ in $\mathcal{G}^0$ converging to $g \in \mathcal{G}$. Then by Lemma 2.1.4 we have that $g_\alpha = s(g_\alpha)$ for all $\alpha \in A$, and by Lemma 2.1.13 we obtain that $g = s(g) \in \mathcal{G}^0$.

Conversely, suppose that $\mathcal{G}^0$ is closed. We show that $\mathcal{G}$ is Hausdorff by showing that convergent nets have unique limit points. So assume $\{g_\alpha\}_{\alpha\in A}$ is a net in $\mathcal{G}$ converging to g_1 and g_2 in $\mathcal{G}$. Then $g_\alpha^{-1}g_\alpha \in \mathcal{G}^0$ converges to $g_1^{-1}g_2$, and since $\mathcal{G}^0$ is closed, we have that $g_1^{-1}g_2 \in \mathcal{G}^0$ which implies that $g_1 = g_2$. □

Let us revisit our previous examples but add topologies that makes them into topological groupoids.

Example 2.1.16 (Topological Groups)**.** Recall Example 2.1.5. If G is a topological group then it is a topological groupoid.

Example 2.1.17 (Topological Equivalence Relations)**.** Recall Example 2.1.6. If X is a Hausdorff topological space and we endow $\mathcal{R}$ with the subspace topology of the product topology of $X \times X$, then it becomes a topological groupoid.

Example 2.1.18 (Topological Transformation Groupoids)**.** Recall Example 2.1.7. Let G be a Hausdorff topological group and X endowed with a Hausdorff topology making the group action continuous. Then the transformation groupoid with the product topology of G and X is a topological groupoid.

We wish now to define the type of topology that we will be interested in, which is the étale topology. This makes the "group-like" parts of a topological groupoid become discrete (in a sense that will be made precise soon). In order to define this topology we start by defining the notion of a local homeomorphism.

Definition 2.1.19. Let $f : X \to Y$ be a map between topological spaces X and Y. Then f is a *local homeomorphism* if for every $x \in X$ there exists an open set U containing x such that $f(U)$ is open in Y and such that the restriction map

$$f|_U : U \to f(U)$$

is a homeomorphism.

Definition 2.1.20. A topological groupoid $\mathcal{G}$ is *étale* if the associated source and range maps

$$s, r : \mathcal{G} \to \mathcal{G}^0$$

are local homeomorphisms.

The following useful definition is from [64]:

Definition 2.1.21. Let $\mathcal{G}$ be a groupoid with source and range maps s and r respectively. Then a subset U is called an *s-section* (respectively an *r-section*) if the restriction of s (respectively r) to U is injective. It is called a *bisection* if it is both an s and r-section.

Lemma 2.1.22. *If $\mathcal{G}$ is an étale groupoid then the source and range maps are open maps.*

Proof. Let U be an open subset of $\mathcal{G}$. For every $g \in U$, there exists an open bisection U_g such that the restriction of s to U_g is a homeomorphism onto $s(U_g)$. Then $U = \bigcup_{g \in U} (U \cap U_g)$ and so $s(U) = \bigcup_{g \in U} s(U \cap U_g)$ which is a union of open sets because s is a homeomorphism on U_g for all $g \in U$. The same can be shown for the range map r. $\square$

Lemma 2.1.23. *If $\mathcal{G}$ is an étale groupoid it has an open cover consisting of bisections.*

Proof. Around every point in $\mathcal{G}$ there is a neighbourhood that is an open bisection by the definition of being étale. The union of all such bisections covers $\mathcal{G}$. $\square$

The following lemma makes precise how the étale topology is the analogue of the discrete topology in group theory:

Lemma 2.1.24. *Let $\mathcal{G}$ be an étale groupoid. Then $\mathcal{G}^0$ is open, and for every $g \in \mathcal{G}^0$, $s^{-1}(\{g\})$ and $r^{-1}(\{g\})$ are discrete (in the subspace topology of $\mathcal{G}$).*

Proof. The first claim is an immediate consequence of Lemma 2.1.22. Let us now show that for $g \in \mathcal{G}^0$, $s^{-1}(\{g\})$ is discrete. We do this by showing that every convergent net in this fibre is eventually constant. Indeed, let A be a set such that $\{h_\alpha\}_{\alpha \in A}$ is a net in $s^{-1}(\{g\})$ converging to $h \in s^{-1}(\{g\})$. Then by Lemma 2.1.3 we have that $(h_\alpha, h^{-1}) \in \mathcal{G}^2$ for all $\alpha \in A$. This implies that $h_\alpha h^{-1}$ converges to $hh^{-1} \in \mathcal{G}^0$, which is open, and hence there exists $\alpha_0 \in A$ such that for all $\alpha \geq \alpha_0$ we have $h_\alpha h^{-1} \in \mathcal{G}^0$. Now by Lemma 2.1.4 we have that for all $\alpha \geq \alpha_0$, $h_\alpha h^{-1} = r(h_\alpha h^{-1}) = r(h_\alpha) = h_\alpha h_\alpha^{-1}$ which implies that $h_\alpha = h$ for all $\alpha \geq \alpha_0$. The same argument can be performed on $r^{-1}(\{g\})$. $\square$

Let us go back to our previous examples and discuss when these are étale.

Example 2.1.25 (Étale Groups)**.** A topological group G with identity element e is étale if and only if it is discrete. Indeed, if it is étale then Lemma 2.1.24 tells us that $G = s^{-1}(\{e\})$ is discrete. If G is discrete, then for every $g \in G$ the map $\{g\} \to \{e\}$ is a homeomorphism.

Example 2.1.26 (Étale Equivalence Relations)**.** Recall Example 2.1.17. If $\mathcal{R} = \{(x,x) : x \in X\}$ then it is étale as for any $(x,x) \in \mathcal{R}$ and any open set U containing x, the restriction of the source (or range) map to $(U \times U) \cap \mathcal{R}$ is a homeomorphism onto U.

Example 2.1.27 (Étale Transformation Groupoids)**.** Recall Example 2.1.18. If we assume that the group G is discrete then $G \times X$ is étale. Indeed for any $(g,x) \in G \times X$ the source map maps $\{g\} \times X$ onto X and the range map maps $\{g\} \times X$ onto $\{g\}X := \{gx : x \in X\}$, and these are homeomorphisms.

We end this section by defining types of principality a groupoid can have.

Definition 2.1.28. Let $\mathcal{G}$ be a groupoid and $h \in \mathcal{G}^0$. Then we define

$$\mathcal{G}_h = \{g \in \mathcal{G} : s(g) = h\},$$

$$\mathcal{G}^h = \{g \in \mathcal{G} : r(g) = h\},$$

and the *isotropy bundle*

$$\mathcal{G}' = \{g \in \mathcal{G} : s(g) = r(g)\}.$$

We say the *isotropy* at h is $\mathcal{G}^h \cap \mathcal{G}_h$.

The following is based on Definitions 3.4 and 3.5 in [64]:

Definition 2.1.29. A groupoid $\mathcal{G}$ will be called *principal* if $\mathcal{G}' = \mathcal{G}^0$. If it is a topological groupoid it will be called

- *topologically principal* if the elements in $\mathcal{G}^0$ with trivial isotropy are dense in $\mathcal{G}^0$, and
- *effective* if $\mathrm{int}(\mathcal{G}') = \mathcal{G}^0$.

Example 2.1.30 (Groups)**.** The étale group from Example 2.1.25 satisfies none of the properties of Definition 2.1.29 if it is non-trivial, as $G' = G$.

Example 2.1.31 (Equivalence Relations)**.** Any equivalence relation groupoid $\mathcal{R} \subseteq X \times X$ is principal, as $\mathcal{R}' = \{(x, x) \in \mathcal{R}\} = \mathcal{R}^0$. Hence any topological equivalence relation is both topologically principal and effective.

Example 2.1.32 (Transformation Groupoids)**.** Recall Example 2.1.7. Given a point $x \in X$, its isotropy consists of all the elements (g, x) such that $gx = x$, in other words the stabilizer subgroup of G with respect to x. Hence the action needs to be free if the groupoid is to be principal. In a similar way, for a topological transformation groupoid the action needs to be topologically free (which means that the set of points in X with trivial stabilizer subgroups are dense in X, see Definition 2.1 in [45]) in order for the groupoid to be topologically principal. For an étale groupoid this will suffice to also guarantee effectiveness, as we will see below.

The following lemma is based on Proposition 3.6 in [64]:

Lemma 2.1.33. *Let $\mathcal{G}$ be an étale groupoid. Then we have the following implications:*

$$\mathcal{G} \text{ is principal} \implies \mathcal{G} \text{ is topologically principal} \implies \mathcal{G} \text{ is effective.}$$

Proof. The first implication is trivial. As for the second, assume $\mathcal{G}$ is topologically principal, and let U be an open set in $\mathcal{G}'$. We aim to show that $U \subseteq \mathcal{G}^0$. Since the topology is Hausdorff we have by Lemma 2.1.15 that $\mathcal{G}^0$ is closed and so $U \setminus \mathcal{G}^0$ is open. Hence $r(U \setminus \mathcal{G}^0)$ is open by Lemma 2.1.22. However this set belongs to those elements in $\mathcal{G}^0$ whose isotropy is not trivial, and this has empty interior as $\mathcal{G}$ is assumed topologically principal. Hence this set is empty and so $U \subseteq \mathcal{G}^0$ as desired. □

Remark 2.1.34. The implications of Lemma 2.1.33 are not reversible. Indeed, similar to what was discussed in Example 2.1.32, there are examples of étale topologically principal groupoids that are not principal, and by Remark 3.2.8 there are effective étale groupoids which are not topologically principal.

2.1.3 Twisted Étale Groupoids

In this section we outline the structure of a twisted groupoid. These will be crucial in order to understand the correspondence with Cartan subalgebras in Chapter 3. The material presented here has its origins in Renault's work in [63], where the notion of twist on a groupoid was that of a $\mathbb{T}$-valued continuous 2-cocycle. Later, in [40], Kumjian generalized this to a general notion of twist on a groupoid, which we explain in this section. There are plenty of good summaries of the construction, for example [3], [8], [13] and [64]. Whilst we might use elements of all of these, the treatment will broadly follow Section 5 in [71].

We now define the notion of a twisted étale groupoid. A good figure to have in mind whilst reading the definition is Figure 2.2.

Definition 2.1.35. Let $\mathcal{G}$ be an étale groupoid. Then a *twist* Σ over $\mathcal{G}$ is a locally compact Hausdorff groupoid admitting a sequence

$$\mathcal{G}^0 \times \mathbb{T} \xhookrightarrow{i} \Sigma \xrightarrow{\Pi} \mathcal{G},$$

where

- $\mathcal{G}^0 \times \mathbb{T}$ is a trivial group bundle with fibres $\mathbb{T}$. This becomes a groupoid when declaring the set of composable pairs to be $\{((g, t_1), (g, t_2)) : g \in \mathcal{G}^0, t_1, t_2 \in \mathbb{T}\}$ and multiplication given by $(g, t_1)(g, t_2) = (g, t_1 t_2)$, and inverse by $(g, t)^{-1} = (g, t^{-1})$. The unit space is then clearly identified with $\mathcal{G}^0$. The topology is then the product topology of the topologies on $\mathcal{G}^0$ and $\mathbb{T}$,
- i and Π are continuous groupoid homomorphisms with i injective and Π sur-

jective, both restricting to homeomorphisms on the respective unit spaces (the unit space of Σ is identified with $\mathcal{G}^0$),

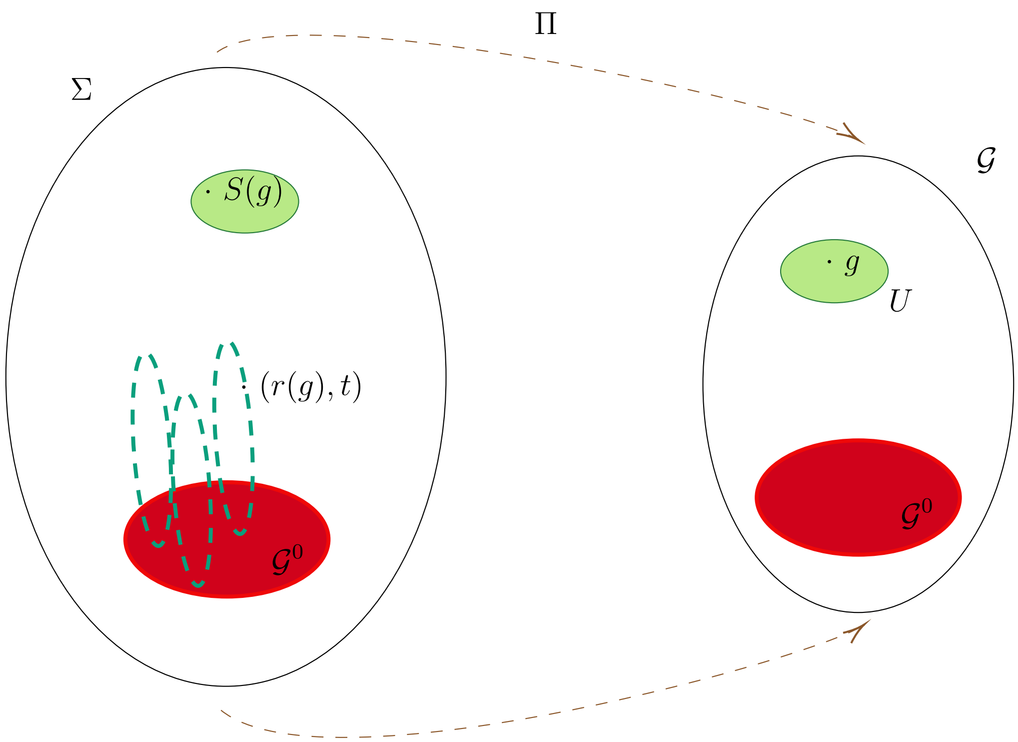
- $\Pi^{-1}(\mathcal{G}^0) = i(\mathcal{G}^0 \times \mathbb{T})$,
- The image of i is central in Σ. Specifically, if we define a multiplication of Σ by $\mathbb{T}$ via $t\sigma = i(r(\sigma), t)\sigma$ and $\sigma t = \sigma i(s(\sigma), t)$ for all $t \in \mathbb{T}$ and $\sigma \in \Sigma$, then $t\sigma = \sigma t$,
- Σ is a locally trivial $\mathcal{G}$-bundle, in the sense that around every $g \in \mathcal{G}$ there is a corresponding bisection U and a continuous section $S : U \to \Sigma$ with $\Pi \circ S = \mathrm{id}_U$ and such that the map

$$U \times \mathbb{T} \to \Pi^{-1}(U), \quad (g, t) \to i(r(g), t)S(g)$$

is a homeomorphism.

One may denote a groupoid and its twist by $(\mathcal{G}, \Sigma)$ and we call this a *twisted (étale) groupoid*.

Figure 2.2: Twisted Groupoid



The following lemma will allow us to view a twist over a groupoid via continuous circle actions, which makes the groupoid look like the orbit space of the action.

Lemma 2.1.36. *Let Σ be a twist over $\mathcal{G}$. Then for $\sigma_1, \sigma_2 \in \Sigma$, if $\Pi(\sigma_1) = \Pi(\sigma_2)$ then there exists a unique $t \in \mathbb{T}$ such that $t\sigma_2 = \sigma_1$.*

Proof. First we show that $(\sigma_1, \sigma_2^{-1}) \in \Sigma^2$. Indeed, let $g = \Pi(\sigma_1) = \Pi(\sigma_2)$ and choose an open set U as in Definition 2.1.35 that witnesses the local triviality. Let S be the continuous section from that definition. Then it follows that $\sigma_1 = i(r(g_1), t_1)S(g_1)$ for some $(g_1, t_1) \in U \times \mathbb{T}$, and $\sigma_2 = i(r(g_2), t_2)S(g_2)$ for some $(g_2, t_2) \in U \times \mathbb{T}$. Since S is a section and $\Pi^{-1}(\mathcal{G}^0) = i(\mathcal{G}^0 \times \mathbb{T})$, acting by Π on σ_1 and σ_2 yields that $g_1 = g_2 = g$. Then it is clear that $\sigma_1\sigma_2^{-1}$ is defined.

By direct computation we have that $\sigma_1\sigma_2^{-1} = i(r(g), t)$ for some $t \in \mathbb{T}$. This t is unique as i is injective. We then have $\sigma_1(t\sigma_2)^{-1} = \sigma_1\sigma_2^{-1}i(r(\sigma_2), \overline{t}) = i(r(\sigma_2), 1)$.

Multiply on the right by $t\sigma_2$ and we get $\sigma_1 = t\sigma_2$. □

In Section 2.1 of [8], another way to define a twisted étale groupoid is presented summatively. We present it here:

Definition 2.1.37. A twist Σ is a locally compact groupoid admitting a continuous free action $\mathbb{T} \curvearrowright \Sigma$, such that the quotient $\Sigma/\mathbb{T}$ (which is denoted $\mathcal{G}$) is étale and Hausdorff. Furthermore, the canonical projection $\Pi : \Sigma \twoheadrightarrow \mathcal{G}, \sigma \to \dot{\sigma}$ should be a locally trivial principal $\mathbb{T}$-bundle. For $t_1, t_2 \in \mathbb{T}$ and $\sigma_1, \sigma_2 \in \Sigma$ we have $(t_1\sigma_1, t_2\sigma_2) \in \Sigma^2$ if and only if $(\sigma_1, \sigma_2) \in \Sigma^2$, in which case the product is $(t_1t_2)(\sigma_1\sigma_2)$. The groupoid structure on $\mathcal{G}$ is the one induced canonically by the canonical projection. We write $(\mathcal{G}, \Sigma)$ for the twisted étale groupoid.

Remark 2.1.38. It is clear that Definition 2.1.35 together with Lemma 2.1.36 yields Definition 2.1.37. Indeed the action is declared the multiplication of Σ by $\mathbb{T}$ (which is free and continuous). Lemma 2.1.36 allows us to identify $\mathcal{G}$ with $\Sigma/\mathbb{T}$. Since the image of i is central in Σ it becomes clear that the algebraic structures required in Definition 2.1.37 are met.

Conversely, if we start out with Definition 2.1.37 we first note that $(\Sigma/\mathbb{T})^0$ can be identified with Σ^0. Indeed the map $r(\sigma) \to r(\dot{\sigma})$ is injective since if $r(\dot{\sigma_1}) = r(\dot{\sigma_2})$ then there exists $t \in \mathbb{T}$ such that $tr(\sigma_1) = r(\sigma_2)$. Since squaring and taking inverse does not change the right hand side, we have that $tr(\sigma_1) = t^2 r(\sigma_1) = \bar{t}r(\sigma_1)$. Since the action is free it follows that $t = t^2 = \bar{t}$ and so $t = 1$. Hence the map $r(\sigma) \to r(\dot{\sigma})$ defines a homeomorphism $\Sigma^0 \to (\Sigma/\mathbb{T})^0$. Thus we obtain a central extension

$$(\Sigma/\mathbb{T})^0 \times \mathbb{T} \xhookrightarrow{i} \Sigma \xrightarrow{\Pi} \Sigma/\mathbb{T},$$

where $i(r(\dot{\sigma}), t) = tr(\sigma)$, satisfying the requirements of Definition 2.1.35. It is easy to see that i is an injective groupoid homomorphism.

Remark 2.1.39. We will use both of the equivalent Definitions 2.1.35 and 2.1.37 when discussing twisted groupoids, and it should be clear from context which one we are using.

The statement of the following lemma is discussed in Section 2 of [8]. Here, we provide a proof.

Lemma 2.1.40. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale groupoid, and let Π be the corresponding projection map. Then Π is open and closed, perfect and proper.*

Proof. For a set $A \subset \Sigma$, we have

$$\Pi^{-1}(\Pi(A)) = \bigcup_{t \in \mathbb{T}} \{t\}A. \tag{2.2}$$

This shows that if A is open, so is the right hand side of equation (2.2) (as for $t \in \mathbb{T}$ the map $a \to ta$ is continuous with continuous inverse defined by multiplication by $\bar{t}$, and hence we have a homeomorphism $A \to \{t\}A$), and hence so is $\Pi(A)$ (by definition of the quotient topology). If A is closed, take a net $t_\alpha a_\alpha$ in the right hand side of equation (2.2), and assume this converges to $\sigma \in \Sigma$. Since $\mathbb{T}$ is compact, $\{t_\alpha\}$ has a convergent subnet, say $\{t_{\alpha_i}\}$, converging to $t \in \mathbb{T}$. Then by continuity of multiplication and inverse, we have that $t_{\alpha_i}^{-1}(t_{\alpha_i} a_{\alpha_i})$ converges to $t^{-1}\sigma \in A$. Thus σ belongs to the right hand side of (2.2), and Π is closed. It is clear that Π is perfect as the fibre is $\mathbb{T}$.

To see that Π is proper, we use the proof of Theorem K.3 in [51]. Indeed, let K be a compact subset of $\mathcal{G}$, and $g \in \mathcal{G}$. Let $\Pi^{-1}(K)$ be covered by the open family $\mathcal{U}$. $\Pi^{-1}(\{g\})$ can be covered by finitely many open sets from $\mathcal{U}$. Exercise 6 in [52] shows how there is a neighbourhood of g whose inverse image under Π is covered by the same finitely many open sets. Since K can then be covered by finitely many such neighbourhoods, $\Pi^{-1}(K)$ has a finite subcover. $\square$

Definition 2.1.41. When an étale groupoid $\mathcal{G}$ has a property P, and Σ is a twist over $\mathcal{G}$, we shall say that $(\mathcal{G}, \Sigma)$ is a twisted étale P groupoid.

Remark 2.1.42. It is clear that in under both Definition 2.1.35 and Definition 2.1.37 the base space $\mathcal{G}$ becomes locally compact. Indeed, as the twist is locally compact by definition, this follows by Lemma 2.1.40.

Now let us look at some examples of twisted étale groupoids.

Example 2.1.43 (Trivial Twist)**.** Let $\mathcal{G}$ be an étale locally compact groupoid and define the product groupoid $\Sigma = \mathcal{G} \times \mathbb{T}$ as in Lemma 2.1.8. Define $i : \mathcal{G}^0 \times \mathbb{T} \to \Sigma$ by $i(g, t) = (g, t)$, and $\Pi : \Sigma \to \mathcal{G}$ by $\Pi(g, t) = g$. It is easy to check that $(\mathcal{G}, \Sigma)$ is a twisted étale locally compact groupoid.

Our next class of examples stem from Renault's original considerations in [63], where we consider the étale groupoid $\mathcal{G} \times \mathbb{T}$, but where we change the multiplication and inversion to take into account a continuous $\mathbb{T}$-valued 2-cocycle c on $\mathcal{G}$. The following definition is from [17]:

Definition 2.1.44. A $\mathbb{T}$-valued 2-cocycle on an étale groupoid $\mathcal{G}$ is a function

$$c : \mathcal{G}^2 \to \mathbb{T}$$

such that

$$c(g, s(g)) = c(r(g), g) = 1 \;\; \forall g \in \mathcal{G},$$

$$c(g, hk)c(h, k) = c(gh, k)c(g, h) \;\text{ whenever }\; (g, h), (h, k) \in \mathcal{G}^2.$$

The following is Lemma 2.1 in [17]:

Lemma 2.1.45. *Given a $\mathbb{T}$-valued 2-cocycle c on an étale groupoid $\mathcal{G}$ and $g \in \mathcal{G}$, we have that*

$$c(g^{-1}, g) = c(g, g^{-1}).$$

Proof. We have that

$$c(g, g^{-1})\overline{c(g^{-1}, g)} = (c(gg^{-1}, g)c(g, g^{-1}))\overline{c(g^{-1}, g)} = (c(g, g^{-1}g)c(g^{-1}, g))\overline{c(g^{-1}, g)} = 1.$$

□

Example 2.1.46 (Twisted Groupoids from 2-cocycles)**.** Let $\mathcal{G}$ be an étale groupoid and c a continuous $\mathbb{T}$-valued 2-cocycle on $\mathcal{G}$. Define $\Sigma_c = \mathcal{G} \times \mathbb{T}$ with topology the product topology, and Σ_c^2 is the set of composable pairs induced canonically from a product of two groupoids, as in Lemma 2.1.8. Endow it with the following multiplication:

$$(g_1, t_1)(g_2, t_2) = (g_1 g_2, c(g_1, g_2)t_1 t_2) \;\; \forall (g_1, g_2) \in \mathcal{G}^2, t_1, t_2 \in \mathbb{T}.$$

The inverse is defined by

$$(g, t)^{-1} = (g^{-1}, \overline{c(g^{-1}, g)t}).$$

Lemma 2.1.45 can be used to show that the inverse map is indeed an involution, and that $s((g, t)) = (s(g), 1)$ and $r((g, t)) = (r(g), 1)$ (here we are abusing notation by using the same notation for the source and range maps in Σ_c as those in $\mathcal{G}$). Define

$$\mathcal{G}^0 \times \mathbb{T} \xhookrightarrow{i} \Sigma_c \xrightarrow{\Pi} \mathcal{G}$$

by $i(g, t) = (g, t)$ and $\Pi(g, t) = g$. It is then straightforward to see that this defines an étale twisted groupoid $(\mathcal{G}, \Sigma_c)$ for every continuous $\mathbb{T}$-valued 2-cocycle c on $\mathcal{G}$. When $c \equiv 1$ we retrieve the trivial twist of Example 2.1.43.

We conclude this section by outlining how a twist over an étale groupoid gives rise to a certain complex line bundle that will become relevant in Section 2.1.4.

Lemma 2.1.47. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale groupoid. Define a relation on $\Sigma \times \mathbb{C}$ by $(\sigma_1, z_1) \sim (\sigma_2, z_2)$ if and only if $\Pi(\sigma_1) = \Pi(\sigma_2)$ and either $z_1 = z_2 = 0$ or $|z_1| = |z_2|$ and $\frac{z_1}{|z_1|}\sigma_1 = \frac{z_2}{|z_2|}\sigma_2$. Then $\sim$ is an equivalence relation. Define a map*

$$p : (\Sigma \times \mathbb{C})/\sim \twoheadrightarrow \mathcal{G}, \quad [\sigma, z] \to \Pi(\sigma).$$

This turns $(\Sigma \times \mathbb{C})/\sim$ into a complex line bundle over $\mathcal{G}$.

Proof. Reflexivity and symmetry of $\sim$ is clear. Now assume $(\sigma_1, z_1) \sim (\sigma_2, z_2) \sim (\sigma_3, z_3)$. Then $\Pi(\sigma_1) = \Pi(\sigma_2) = \Pi(\sigma_3)$, and if $z_1 = z_2$ is 0 then it forces $z_3 = 0$ also, and if not then we have $\frac{z_1}{|z_1|}\sigma_1 = \frac{z_2}{|z_2|}\sigma_2 = \frac{z_3}{|z_3|}\sigma_3$, and transitivity follows.

It is clear that the map p is well-defined. Note that there exists $t \in \mathbb{T}$, $\sigma \in \Sigma$ and $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ such that $\frac{z_1}{|z_1|} = t\frac{z_2}{|z_2|}$ if and only if $(\sigma, z_1) \sim (t\sigma, z_2)$. Hence given $g \in \mathcal{G}$, we have by Lemma 2.1.36 that $p^{-1}(\{g\})$ can be identified with a line in the complex plane passing through the origin. $\square$

Definition 2.1.48. Let $(\mathcal{G}, \Sigma)$ be a twisted étale groupoid. Define

$$L_\Sigma = (\Sigma \times \mathbb{C})/\sim$$

where the equivalence relation $\sim$ is the one from Lemma 2.1.47. L_Σ is then called the *complex line bundle associated to* Σ. The topology on it is the canonical one induced by the quotient topology of the product topology on $\Sigma \times \mathbb{C}$. we denote the quotient map by q.

Remark 2.1.49. In Section 4 in [64] Renault introduces the complex line bundle associated to Σ as the orbit space of the circle action $\mathbb{T} \curvearrowright (\Sigma \times \mathbb{C})$ given by $t(\sigma, z) = (t\sigma, \bar{t}z)$. It is straightforward to see that this defines the same equivalence classes as the ones defined in Lemma 2.1.47.

Remark 2.1.50. Renault notes in Section 4 of [64] that the complex line bundle associated to Σ has the structure of a Fell bundle over $\mathcal{G}$, with fibre $\mathbb{C}$. We will only use this fact sparsely and hence we will not discuss the theory of Fell bundles.

2.1.4 Twisted Groupoid C^*-algebras

This section involves the construction of the reduced twisted groupoid C^*-algebra. This generalizes the reduced groupoid C^*-algebra construction, which we will not need for this thesis. The material presented here will be broadly based on the material in Section 4 of [64], and hence it is more convenient to have Definition 2.1.37 as the base definition for twisted groupoids. In what follows we assume (unless stated otherwise) that the twisted groupoids are étale, locally compact and Hausdorff. Local compactness is a consequence of the twisted groupoid definitions. Some specific statements might not need one or more of these assumptions, but for the purpose of this thesis we are not interested in such exceptions.

Definition 2.1.51. Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Let $f : \Sigma \to \mathbb{C}$ be a continuous function. We define the *open support* of f as as

$$\operatorname{supp}'(f) = \{\dot{\sigma} \in \mathcal{G} : f(\sigma) \neq 0\}.$$

The *support* of f is defined by

$$\operatorname{supp}(f) = \overline{\operatorname{supp}'(f)}.$$

Definition 2.1.52. Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Define

$$C(\mathcal{G}, \Sigma) = \{f : \Sigma \to \mathbb{C} : f \text{ is continuous}, \quad f(t\sigma) = \bar{t}f(\sigma) \ \ \forall t \in \mathbb{T}, \sigma \in \Sigma\}.$$

We call the property $f(t\sigma) = \bar{t}f(\sigma)$ $\mathbb{T}$-*equivariance.* Define

$$C_C(\mathcal{G}, \Sigma) = \{f \in C(\mathcal{G}, \Sigma), \ \ \operatorname{supp}(f) \text{ is compact}\},$$

and

$$C_C(\mathcal{G}^0) = \{f \in C_C(\mathcal{G}, \Sigma) : \operatorname{supp}(f) \subset \mathcal{G}^0\}.$$

The following lemma highlights why $\mathbb{T}$-equivariance is used, namely to be able to identify such continuous functions with continuous sections of the complex line bundle associated to Σ. The following is stated as Remark 5.1.10 in [71]:

Lemma 2.1.53. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. There is an isomorphism that identifies $C_C(\mathcal{G}, \Sigma)$ with the compactly supported continuous sections of the complex line bundle associated to Σ.*

Proof. Let us give the explicit identifications. For f in $C_C(\mathcal{G}, \Sigma)$, define $S_f : \mathcal{G} \to L_\Sigma$ by

$$S_f(\dot{\sigma}) = [\sigma, f(\sigma)].$$

By Remark 2.1.49, S_f is well-defined precisely because of the $\mathbb{T}$-equivariance of f. For a compactly supported supported section $S : \mathcal{G} \to L_\Sigma$ we let

$$f_S(\sigma) = z(\sigma)$$

where $z(\sigma)$ is the unique element in $\mathbb{C}$ which satisfies that $(\sigma, z(\sigma))$ is in the class of $S(\dot{\sigma})$. It is easy to see that z is $\mathbb{T}$-equivariant and that $S_{(f_S)} = S$ and that $f_{(S_f)} = f$. ☐

In order to obtain a twisted groupoid C^*-algebra, we will start by constructing a $*$-algebra structure on $C_C(\mathcal{G}, \Sigma)$, and then introducing an appropriate norm. The C^*-algebra will then be the completion of $C_C(\mathcal{G}, \Sigma)$ with respect to this norm.

Definition 2.1.54. Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Define $(C_C(\mathcal{G}, \Sigma), *)$ to be the vector space $C_C(\mathcal{G}, \Sigma)$ (as a subspace of $C_0(\Sigma)$) endowed with a multiplication map defined by

$$f * g(\sigma) = \sum_{\dot{\tau} \in \mathcal{G}_{s(\sigma)}} f(\sigma\tau^{-1})g(\tau), \text{ for all } f, g \in C_C(\mathcal{G}, \Sigma), \sigma \in \Sigma, \tag{2.3}$$

and an involution defined by

$$f^*(\sigma) = \overline{f(\sigma^{-1})}. \tag{2.4}$$

Lemma 2.1.55. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Then $(C_C(\mathcal{G}, \Sigma), *)$ defines a *-algebra structure on $C_C(\mathcal{G}, \Sigma)$.*

Proof. The sum in (2.3) is well-defined as f and g are $\mathbb{T}$-equivariant and the sum is over a discrete space (by Lemma 2.1.24), and since the supports of f and g are compact, only finitely many summands are non-zero. Clearly $f * g$ is continuous and $\mathbb{T}$-equivariant since f is. To see that $f * g$ is compactly supported, notice that $f * g(\sigma)$ is non-zero if there is at least one $\dot{\tau} \in \mathcal{G}_{s(\sigma)}$ such that $f(\sigma\tau^{-1})g(\tau)$ is non-zero, which implies that σ is in the image, under the multiplication map, of $(\Pi^{-1}(\text{supp}(f)) \times \Pi^{-1}(\text{supp}(g))) \cap \Sigma^2$, which is the image, under a continuous function, of a compact set (Σ^2 is closed as the topology is Hausdorff, see for example the consequences of Definition 2.1 in [63], and we use the properness of Π from Lemma 2.1.40), hence compact. So $\text{supp}(f)$ is a closed subset of the image, under Π, of a compact set, hence compact. Hence $f * g \in C_C(\mathcal{G}, \Sigma)$.

Likewise one can check that $f^* \in C_C(\mathcal{G}, \Sigma)$, and it is a tedious but simple task to check that the remaining algebraic axioms defining a *-algebra are satisfied. ☐

The strategy is to now introduce a representation of our *-algebra $C_C(\mathcal{G}, \Sigma)$ on a Hilbert space for every element $g \in \mathcal{G}^0$. The norm we will want to define on $(C_C(\mathcal{G}, \Sigma), *)$ will be the universal one with respect to these representations. In order to achieve all this, we will first show that the norm of any *-representation of $(C_C(\mathcal{G}, \Sigma), *)$ is bounded by a certain norm known as the I-norm. This norm behaves like a fibrewise 1-norm.

Definition 2.1.56. Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. The I-norm on $(C_C(\mathcal{G}, \Sigma), *)$ is the norm defined by

$$\|f\|_I = \max\left(\sup_{g\in\mathcal{G}^0}\sum_{\dot{\sigma}\in\mathcal{G}_g}|f(\sigma)|, \sup_{g\in\mathcal{G}^0}\sum_{\dot{\sigma}\in\mathcal{G}^g}|f(\sigma)|\right)$$

Lemma 2.1.57. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. The I-norm on $(C_C(\mathcal{G}, \Sigma), *)$ is a norm satisfying $\|f^*\|_I = \|f\|_I$ and $\|f * h\|_I \leq \|f\|_I\|h\|_I$ for all $f, h \in C_C(\mathcal{G}, \Sigma)$.*

Proof. That $\|\|_I$ is homogeneous and satisfies the triangle inequality is clear. To see that it is real-valued, note that we can first cover $\mathcal{G}$ by open bisections by Lemma 2.1.23. Then for $f \in C_C(\mathcal{G}, \Sigma)$, a finite subcollection of our open bisections cover $\operatorname{supp}(f)$. Let the number of elements in this subcollection be N. It is then clear that $\|f\|_I \leq N\|f\|_\infty < \infty$.

It is clear that $\|f^*\|_I = \|f\|_I$, and note that for any $g \in \mathcal{G}^0$, we have, for $f, h \in C_C(\mathcal{G}, \Sigma)$ that

$$\begin{aligned}
\sum_{\dot{\sigma}\in\mathcal{G}_g}|f * h(\sigma)| &= \sum_{\dot{\sigma}\in\mathcal{G}_g}\Big|\sum_{\dot{\tau}\in\mathcal{G}_{s(\sigma)}} f(\sigma\tau^{-1})h(\tau)\Big| \\
&\leq \sum_{\dot{\sigma}\in\mathcal{G}_g}\left(\sum_{\dot{\tau}\in\mathcal{G}_{s(\sigma)}} |f(\sigma\tau^{-1})h(\tau)|\right) \\
&= \sum_{\dot{\tau}\in\mathcal{G}_g}\left(\sum_{\dot{\sigma}\in\mathcal{G}_{s(\tau)}} |f(\sigma\tau^{-1})|\right)|h(\tau)| \\
&\leq \sum_{\dot{\tau}\in\mathcal{G}_g}\left(\sum_{\dot{\rho}\in\mathcal{G}_{r(\tau)}} |f(\rho)|\right)|h(\tau)| \\
&\leq \|f\|_I\sum_{\dot{\tau}\in\mathcal{G}_g}|h(\tau)| \\
&\leq \|f\|_I\|h\|_I.
\end{aligned}$$

The same can be done for a sum over the range fibre, and as $g \in \mathcal{G}^0$ was arbitrary, it follows that $\|f * h\|_I \leq \|f\|_I \|h\|_I$. □

We now show that any *-algebra representation on $(C_C(\mathcal{G}, \Sigma), *)$ is bounded by the I-norm. In order to obtain this, we need to introduce the inductive limit topology on $C_C(\mathcal{G}, \Sigma)$.

Definition 2.1.58. Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Let $\mathcal{K}$ be the set of all compact subsets of $\mathcal{G}$. For $K \in \mathcal{K}$, define $C_K(\mathcal{G}, \Sigma)$ to be those elements in $C_C(\mathcal{G}, \Sigma)$ with compact support in K. Endow $C_K(\mathcal{G}, \Sigma)$ with the topology induced by the supremum norm.

Lemma 2.1.59. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Then $(C_C(\mathcal{G}, \Sigma), \{C_K(\mathcal{G}, \Sigma) : K \in \mathcal{K}\})$, with the order on $\mathcal{K}$ defined by inclusion, is an inductive system (in the sense of Definition 5.1 in Chapter IV of [15]).*

Proof. It is trivial to check that the properties of Definition 5.1 in Chapter IV of [15] hold. □

The following lemma is similar to Lemma 3.2.3 in [71], but we do it for the twisted groupoid case.

Lemma 2.1.60. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Then any *-representation π of $(C_C(\mathcal{G}, \Sigma), *)$ is continuous in the inductive limit topology when treating $C_C(\mathcal{G}, \Sigma)$ as an inductive system, and satisfies*

$$\|\pi(f)\| \leq \|f\|_I$$

for all $f \in C_C(\mathcal{G}, \Sigma)$.

Proof. To check that π is continuous in the inductive limit topology, by Proposition 5.7 in Chapter IV in [15] it suffices to check that its restriction to $C_K(\mathcal{G}, \Sigma)$ is continuous, for arbitrary $K \in \mathcal{K}$. By Lemma 2.1.23 and compactness we obtain a finite open cover of K by bisections. Let $\{\chi_i : i = 1, \dots, n\}$ be a partition of unity subordinate to this cover, and for $f \in C_K(\mathcal{G}, \Sigma)$ let $f_i = \overline{\chi_i} \cdot f$ be the map defined by pointwise multiplication, where $\overline{\chi_i} = \chi_i \circ \Pi$. It is clear that $f_i \in C_K(\mathcal{G}, \Sigma)$ with $f = \sum_{i=1}^{n} f_i$. Then

$$\|\pi(f)\| = \|\pi(\sum_{i=1}^{n} f_i)\| \leq \sum_{i=1}^{n} \|\pi(f_i)\|. \tag{2.5}$$

Now note that

$$f_i^* * f_i(\sigma) = \sum_{\dot{\tau} \in \mathcal{G}_{s(\sigma)}} \overline{f_i(\tau\sigma^{-1})} f_i(\tau),$$

which is only non-zero if there exists a $\tau \in \Sigma$ such that both $\dot{\tau}\dot{\sigma}^{-1}$ and $\dot{\tau}$ belong to the same bisection covering K. But since the image under the range map of these two elements is the same, it follows that the elements are the same and hence that $\dot{\sigma} \in \mathcal{G}^0$. Hence it is clear that

$$f_i^* * f_i(\sigma) = f_i^* * f_i(s(\tau)) = |f_i(\tau)|^2$$

for some $\dot{\tau} \in \operatorname{supp}(f_i)$, and so

$$\|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2.$$

Hence using that the restriction of π to the commutative C^*-algebra $(C_C(\mathcal{G}^0), *, \|\|_\infty)$ becomes a *-homomorphism, we get that

$$\|\pi(f_i)\|^2 = \|\pi(f_i^* * f_i)\| \leq \|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2.$$

Returning to (2.5) we get

$$\|\pi(f)\| \leq n\|f\|_\infty.$$

Thus π is continuous in the inductive limit topology, and hence in the I-norm topology since for $f \in C_C(\mathcal{G}, \Sigma)$, $\|f\|_\infty \leq \|f\|_I$. The completion of $(C_C(\mathcal{G}, \Sigma), *)$ in the I-norm gives us a Banach *-algebra, to which π extends to become a *-algebra homomorphism (using for example Theorem 1.5.7 in [35]). Write $\rho_B : B \to [0, \infty)$ for the spectral radius function on a Banach *-algebra B. Then, for $f \in C_C(\mathcal{G}, \Sigma)$, we get

$$\|\pi(f)\|^2 = \|\pi(f^* * f)\| = \rho_{\mathcal{B}(\mathcal{H})}(\pi(f^* * f)) \leq \rho_{\overline{C_C(\mathcal{G},\Sigma)}^I}(f^* * f) \leq \|f^* * f\|_I \leq \|f\|_I^2,$$

and the desired result follows. □

Definition 2.1.61. Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. For every $g \in \mathcal{G}^0$, define

$$\mathcal{H}_g = \{\xi : \Sigma_g \to \mathbb{C} : \xi(t\sigma) = \bar{t}\xi(\sigma) \text{ for all } t \in \mathbb{T}, \sum_{\dot{\sigma} \in \mathcal{G}_g} |\xi(\sigma)|^2 < \infty\}.$$

Define

$$\pi_g : (C_C(\mathcal{G},\Sigma),*) \to \mathcal{B}(\mathcal{H}_g), \quad \pi_g(f)(\xi)(\sigma) = \sum_{\dot{\tau}\in\mathcal{G}_g} f(\sigma\tau^{-1})\xi(\tau). \tag{2.6}$$

Definition 2.1.62. Let $(\mathcal{G},\Sigma)$ be a twisted étale locally compact Hausdorff groupoid. For $\sigma \in \Sigma$, define $\xi_\sigma \in \mathcal{H}_{s(\sigma)}$ as the element which evaluates to $\bar{t}$ at $\tau = t\sigma$, and 0 otherwise. We call such elements *basis elements* for $\mathcal{H}_{s(\sigma)}$.

Lemma 2.1.63. *Let $(\mathcal{G},\Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Then for $g \in \mathcal{G}^0$, the binary operation $\mathcal{H}_g \times \mathcal{H}_g \to \mathbb{C}$ defined by*

$$\langle \xi, \delta \rangle_{\mathcal{H}_g} = \sum_{\dot{\tau}\in\mathcal{G}_g} \xi(\tau)\overline{\delta(\tau)}$$

is an inner product making $\mathcal{H}_g$ a Hilbert space.

Every $\xi \in \mathcal{H}_g$ can be written as

$$\xi = \sum_{\dot{\tau}\in\mathcal{G}_g} \xi(\tau)\xi_\tau.$$

Furthermore, for $\dot{\sigma_1}, \dot{\sigma_2} \in \mathcal{G}_g$, $|\langle \xi_{\sigma_1}, \xi_{\sigma_2}\rangle_{\mathcal{H}_g}| = 0$ if $\dot{\sigma_1} \neq \dot{\sigma_2}$ and 1 otherwise.

Proof. That the binary operation defined is an inner product making $\mathcal{H}_g$ a Hilbert space is clear from Definition 2.1.61. By direct computation using Definition 2.1.62 it is clear that $\xi = \sum_{\dot{\tau}\in\mathcal{G}_g} \xi(\tau)\xi_\tau$. The last claim is also easy to check. □

Lemma 2.1.64. *Let $(\mathcal{G},\Sigma)$ be a twisted étale locally compact Hausdorff groupoid. For every $g \in \mathcal{G}^0$, π_g defines a *-representation of $(C_C(\mathcal{G},\Sigma),*)$ on $\mathcal{H}_g$.*

Proof. For $f \in C_C(\mathcal{G},\Sigma)$, and $\xi \in \mathcal{H}_g$, it is clear that $\pi_g(f)(\xi)$ is $\mathbb{T}$-equivariant (as f is). The sum in (2.6) is well-defined as f and ξ are $\mathbb{T}$-equivariant and f is compactly supported, meaning only finitely many summands are non-zero. Note that if f is supported on an open bisection, and $\dot{\rho} \in \mathcal{G}_g$, we have, for $\sigma \in \Sigma_g$, that

$$\pi_g(f)(\xi_\rho)(\sigma) = f(\sigma\rho^{-1}),$$

and hence

$$\sum_{\dot{\sigma}\in\mathcal{G}_g} |\pi_g(f)(\xi_\rho)(\sigma)|^2$$

collapses to a single summand bounded by $\|f\|_\infty^2$. Hence $\pi_g(f)(\xi)$ belongs to $\mathcal{H}_g$ when f is supported in an open bisection, and ξ is a basis element. Now for arbitrary $\xi \in \mathcal{H}_g$ we have by Lemma 2.1.63 that

$$\pi_g(f)(\xi) = \sum_{\dot{\tau}\in\mathcal{G}_g} \xi(\tau)\pi_g(f)(\xi_\tau).$$

By using the fact that f is supported on a bisection one can check directly that $\{\pi_g(f)(\xi_\tau) : \dot{\tau} \in \mathcal{G}_g\}$ forms a set of orthogonal elements in $\mathcal{H}_g$. Hence

$$\sum_{\dot{\sigma}\in\mathcal{G}_g} |\pi_g(f)(\xi)(\sigma)|^2 = \langle \pi_g(f)(\xi), \pi_g(f)(\xi)\rangle = \sum_{\dot{\tau}\in\mathcal{G}_g} |\xi(\tau)|^2 \|\pi_g(f)(\xi_\tau)\|_2^2 \leq \|f\|_\infty^2 \|\xi\|_2^2. \tag{2.7}$$

From this one can see that $\pi_g(f)(\xi)$ belongs to $\mathcal{H}_g$ for arbitrary $\xi \in \mathcal{H}_g$ and f supported on a bisection. A partition of unity argument then shows that this holds for any $f \in C_C(\mathcal{G}, \Sigma)$, and from (2.7) one can conclude that $\pi_g(f)$ belongs to $\mathcal{B}(\mathcal{H}_g)$. It is then a tedious but straightforward task to verify that π_g is a morphism of *-algebras, and hence π_g is a *-representation of $(C_C(\mathcal{G}, \Sigma), *)$ on $\mathcal{H}_g$. □

Definition 2.1.65. Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Then its *reduced twisted groupoid C^*-algebra* $C_r^*(\mathcal{G}, \Sigma)$ is the completion of the *-algebra $(C_C(\mathcal{G}, \Sigma), *)$ in the norm

$$\|f\|_r = \sup_{g\in\mathcal{G}^0} \|\pi_g(f)\|.$$

Remark 2.1.66. It is precisely Lemma 2.1.60 that allows us to conclude that $\|\|_r$ is real-valued. The triangle inequality and homogeneity properties follow because they are upheld for the operator norm. This norm satisfies the C^*-condition because the operator norm does. Hence $C_r^*(\mathcal{G}, \Sigma)$ is indeed a C^*-algebra.

Example 2.1.67. Using the constructions in Section 1 of Chapter II in [63], we may consider a continuous $\mathbb{T}$-valued 2-cocycle c defining a multiplication and involution on $C_C(\mathcal{G})$, where $\mathcal{G}$ is an étale locally compact Hausdorff groupoid. Specifically, multiplication is given by

$$f * g(x) = \sum_{y\in\mathcal{G}_x} f(xy^{-1})g(y)c(xy^{-1}, y) \text{ for all } f, g \in C_C(\mathcal{G}), \quad x \in \mathcal{G}, \tag{2.8}$$

and the involution by

$$f^*(x) = \overline{f(x^{-1})c(x, x^{-1})} \text{ for all } f \in C_C(\mathcal{G}), \quad x \in \mathcal{G}. \tag{2.9}$$

In a similar way as what we have done in this section, Renault constructs a C^*-algebra $C_r^*(\mathcal{G}, c)$. Recalling Example 2.1.46, we may get a twisted groupoid $(\Sigma_c, \mathcal{G})$. It can be verified, as is stated in [17], that $C_r^*(\Sigma_c, \mathcal{G}) \cong C_r^*(\mathcal{G}, c)$. When $c \equiv 1$, Σ_c is the trivial twist, then (2.8) and (2.9) show that $C_r^*(\mathcal{G} \times \mathbb{T}, \mathcal{G}) \cong C_r^*(\mathcal{G})$, where the latter is the well-known reduced groupoid C^*-algebra (as constructed in Chapter 3 of [60]).

We will conclude this section by showing that the elements of $C_r^*(\mathcal{G}, \Sigma)$ can be identified with $\mathbb{T}$-equivariant C_0 maps $\Sigma \to \mathbb{C}$. Under this identification, the elements of $C_r^*(\mathcal{G}, \Sigma)$ satisfy the same multiplication and involution formulas as (2.3) and (2.4).

Definition 2.1.68. Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Define $C_0(\mathcal{G}, \Sigma)$ as the set of $\mathbb{T}$-equivariant C_0 maps $\Sigma \to \mathbb{T}$. Define $C_0(\mathcal{G}^0)$ as the elements of $C_0(\mathcal{G}, \Sigma)$ with open support in $\mathcal{G}^0$.

Definition 2.1.69. Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Define a map

$$j : C_r^*(\mathcal{G}, \Sigma) \to C_0(\mathcal{G}, \Sigma)$$

by

$$j(f)(\sigma) = \langle \pi_{s(\sigma)}(f)(\xi_{s(\sigma)}), \xi_\sigma \rangle.$$

Here we are abusing notation by writing $\pi_{s(\sigma)}$ as the extension of $\pi_{s(\sigma)}$ to $C_r^*(\mathcal{G}, \Sigma)$

The following lemma is based on the ideas in Proposition 3.3.3 in [71].

Lemma 2.1.70. *The map j in Definition 2.1.69 is a well-defined injective linear map and norm decreasing when $C_0(\mathcal{G}, \Sigma)$ is equipped with the supremum norm. On the dense subalgebra $C_C(\mathcal{G}, \Sigma)$ j reduces to the identity map.*

Proof. For $f \in C_r^*(\mathcal{G}, \Sigma)$, $\pi_{s(\sigma)}(f) \in \mathcal{B}(\mathcal{H}_{s(\sigma)})$ and so the inner product is taken on two elements of $\mathcal{H}_{s(\sigma)}$. So indeed $j(f)$ is a map from Σ to $\mathbb{C}$. It is linear as $\pi_{s(\sigma)}$ is. Note that $\xi_{t\sigma} = t\xi_\sigma$ so indeed $j(f)$ is $\mathbb{T}$-equivariant. Using the Cauchy-Schwarz inequality and Lemma 2.1.63 we can see that

$$\|j(f)\|_\infty \leq \sup_{\sigma \in \Sigma} \|\pi_{s(\sigma)}(f)\| \leq \|f\|_r.$$

For $f \in C_C(\mathcal{G}, \Sigma)$ it is clear that $j(f)(\sigma) = f(\sigma)$ (as here we can use Definition 2.1.61). Hence j is continuous as a map from the dense subalgebra $C_C(\mathcal{G}, \Sigma)$ into the C^*-algebra $(C_0(\Sigma), \|\|_\infty)$. Hence j maps $C_r^*(\mathcal{G}, \Sigma)$ into $C_0(\mathcal{G}, \Sigma)$. Finally, if $f \in C_r^*(\mathcal{G}, \Sigma)$ is non-zero, then there exists $g \in \mathcal{G}^0$ such that $\|\pi_g(f)\| \neq 0$. By

Lemma 2.1.63 there must exist $\alpha, \beta \in \Sigma_g$ such that

$$\langle \pi_g(f)(\xi_\alpha), \xi_\beta \rangle \neq 0.$$

The operator $U_\alpha : \mathcal{H}_{s(\alpha)} \to \mathcal{H}_{r(\alpha)}$ defined by $\xi_\gamma \to \xi_{\gamma\alpha^{-1}}$ is unitary and hence we get

$$j(f)(\beta\alpha^{-1}) = \langle \pi_{r(\alpha)}(f)(\xi_{r(\alpha)}), \xi_{\beta\alpha^{-1}} \rangle = \langle U_\alpha^* \pi_{r(\alpha)}(f) U_\alpha(\xi_\alpha), \xi_\beta \rangle = \langle \pi_{s(\alpha)}(f)(\xi_\alpha), \xi_\beta \rangle \neq 0.$$

Hence $j(f) \neq 0$ and so j is injective. $\square$

Definition 2.1.71. Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Let j be the map from Definition 2.1.69. Define a multiplication and involution on $j(C_r^*(\mathcal{G}, \Sigma)) \subset C_0(\mathcal{G}, \Sigma)$ by

$$j(f) * j(g)(\sigma) = \sum_{\dot{\tau} \in \mathcal{G}_{s(\sigma)}} j(f)(\sigma\tau^{-1}) j(g)(\tau), \tag{2.10}$$

$$j(f)^*(\sigma) = \overline{j(f)(\sigma^{-1})}.$$

Lemma 2.1.72. *The multiplication and involution operations in $j(C_r^*(\mathcal{G}, \Sigma))$ as given in Definition 2.1.71 are well-defined, and in particular we have that*

$$j(f * g) = j(f) * j(g)$$

and

$$j(f^*) = j(f)^*.$$

If we equip $j(C_r^*(\mathcal{G}, \Sigma))$ with this multiplication and involution, and with norm

$$\|j(f)\|_r = \|f\|_r,$$

the map

$$j : C_r^*(\mathcal{G}, \Sigma) \to j(C_r^*(\mathcal{G}, \Sigma))$$

*becomes a C^*-algebra isomorphism. There is an Abelian C^*-subalgebra $C_{0,r}(\mathcal{G}^0) \subset C_r^*(\mathcal{G}, \Sigma)$ such that j maps $C_{0,r}(\mathcal{G}^0)$ isometrically and *-isomorphically onto the C^*-algebra $C_0(\mathcal{G}^0)$ (equipped with pointwise multiplication and the supremum norm).*

Proof. Let $f, g \in C_r^*(\mathcal{G}, \Sigma)$ and $\sigma \in \Sigma$. The summation index in (2.10) is well-defined as $j(f)$ and $j(g)$ are $\mathbb{T}$-equivariant. Assume $f_n \to f$ and $g_n \to g$ in $\|\|_r$, with $f_n, g_n \in C_C(\mathcal{G}, \Sigma)$ for all $n \in \mathbb{N}$. Note that

$$|j(f * g)(\sigma) - j(f_n * g_n)(\sigma)| \le \|j(f * g - f_n * g_n)\|_\infty \le \|f * g - f_n * g_n\|_r \to 0,$$

where the last inequality is due to Lemma 2.1.70. Hence $j(f_n * g_n)(\sigma) \to j(f * g)(\sigma)$. Now

$$j(f_n * g_n)(\sigma) = f_n * g_n(\sigma) = \sum_{\dot{\tau} \in \mathcal{G}_{s(\sigma)}} f_n(\sigma\tau^{-1}) g_n(\tau) \tag{2.11}$$

by Lemma 2.1.70. For any $h \in C_C(\mathcal{G}, \Sigma)$ we may use Lemma 2.1.63 together with the definition of j in Definition 2.1.69 and that $j(h) = h$ to see that, by Parseval's identity, we obtain

$$\sum_{\dot{\tau} \in \mathcal{G}_{s(\sigma)}} |h(\tau)|^2 \le \|h\|_r^2. \tag{2.12}$$

Hence by considering the functions $f_n(\sigma \mathrm{inv}(\cdot))$ and $\overline{g_n}$ the sum in (2.11) is an $l_2(\mathcal{G}_{s(\sigma)})$ inner product of these two functions and since (2.12) shows that r-norm convergence is stronger than l_2 convergence the sum in (2.11) converges to

$$\sum_{\dot{\tau} \in \mathcal{G}_{s(\sigma)}} j(f)(\sigma\tau^{-1}) j(g)(\tau).$$

Hence

$$j(f * g)(\sigma) = \sum_{\dot{\tau} \in \mathcal{G}_{s(\sigma)}} j(f)(\sigma\tau^{-1}) j(g)(\tau) = j(f) * j(g)(\sigma).$$

This shows that multiplication is well defined and that j is a multiplicative map. In similar ways, although it is even easier, one can show that

$$j(f^*) = j(f)^*.$$

Hence, together with the properties seen in Lemma 2.1.70, j is a C^*-algebra isomorphism onto its image (equipped with the induced norm $\|\|_r$).

Finally, let $h \in C_C(\mathcal{G}^0)$. We show that $\|j(h)\|_r = \|j(h)\|_\infty$. Indeed, for $g \in \mathcal{G}^0$, we have by calculation that

$$\|\pi_g(h)(\xi_g)\|_2^2 = |h(g)|^2$$

(as h is supported in $\mathcal{G}^0$) and so it follows that

$$\|h\|_\infty \le \sup_{g \in \mathcal{G}^0} \|\pi_g(h)\| \le \|h\|_r \le \|h\|_I = \|h\|_\infty,$$

where we have used Lemma 2.1.60 and the fact that on $C_C(\mathcal{G}^0)$ the I-norm coincides with the supremum norm. Notice that the multiplication in $C_r^*(\mathcal{G}, \Sigma)$ reduces to pointwise multiplication on $C_C(\mathcal{G}^0)$, and involution reduces to conjugation. Hence the map j is an isometric *-algebra isomorphism from $C_C(\mathcal{G}^0)$ as a normed *-subalgebra of $C_r^*(\mathcal{G})$ onto $C_c(\mathcal{G}^0)$ as a normed *-subalgebra of $C_0(\mathcal{G}^0)$ with pointwise multiplication and the supremum norm. Let $C_{0,r}(\mathcal{G}^0)$ be the $\|\|_r$-closure of $C_C(\mathcal{G}^0)$ in $C_r^*(\mathcal{G}, \Sigma)$, then by extension j maps the Abelian C^*-algebra $C_{0,r}(\mathcal{G}^0)$ onto $C_0(\mathcal{G}^0)$. $\square$

Remark 2.1.73. Lemma 2.1.72 is telling us that $C_r^*(\mathcal{G}, \Sigma)$ is isomorphic to a C^*-algebra that is a subset of $C_0(\mathcal{G}, \Sigma)$, and under the same isomorphism the C^*-subalgebra $C_{0,r}(\mathcal{G}^0)$ is identified with the C^*-algebra $C_0(\mathcal{G}^0)$. On this subalgebra the twist is trivial and this allows us retrieve the usual pointwise multiplication and supremum norm. Hence the multiplication on the identified C^*-algebra in $C_0(\mathcal{G}, \Sigma)$ is of the form given in Definition 2.1.71 because of the existence of the twist. Hence the "untwisted" version of this multiplication is just the pointwise multiplication. This suggests that a non-trivial topological twist is algebraically captured by a twist in the multiplication.

2.1.5 Cartan Subalgebras and Renault's Theorem

In this section we define Cartan subalgebras of C^*-algebras and provide some examples. Then we state, without proof, Renault's main result in [64]. The proofs will be discussed in further detail when we generalize this result in Chapter 3.

We begin with the definition of a conditional expectation, based on Definition 1.3 in [65]:

Definition 2.1.74. Let A be a C^*-algebra, and B a C^*-subalgebra. Then a *conditional expectation* $P : A \to B$ is an onto positive projection satisfying $P(b_1 a b_2) = b_1 P(a) b_2$ for all $a \in A$ and $b_1, b_2 \in B$.

The following is Definition 5.1 in [64]:

Definition 2.1.75. Let A be a C^*-algebra. A C^*-subalgebra $C \subset A$ is called a *Cartan subalgebra* if

- C contains an approximate unit for A,
- C is a maximally Abelian C^*-subalgebra of A (a masa),
- C is regular in A, meaning that the set of normalizers

$$N_A(C) = \{n \in A : n^*Cn \subseteq C, \ \ nCn^* \subseteq C\}$$

generate A as a C^*-algebra, and,

- there exists a faithful conditional expectation

$$P : A \twoheadrightarrow C.$$

In such a situation, we say that (A, C) is a *Cartan pair.*

Remark 2.1.76. The definition of a Cartan subalgebra is slightly weaker than its counterpart defined by Kumjian in [40]. There it is assumed that the subalgebra also satisfies the unique extension property for pure states. In such a case the Cartan subalgebra is called a *C^*-diagonal.*

Example 2.1.77. Let A be a commutative C^*-algebra. Then it has a unique Cartan subalgebra, namely A.

Example 2.1.78. Let $A = M_n(\mathbb{C})$, the C^*-algebra of n by n matrices with entries in $\mathbb{C}$. Then $C = D_n(\mathbb{C})$, the C^*-subalgebra consisting of all the diagonal matrices, is a Cartan subalgebra. Indeed, $D_n(\mathbb{C})$ contains the unit, and is a masa as any Abelian subalgebra of $M_n(\mathbb{C})$ is at most n-dimensional (as a vector space) and $D_n(\mathbb{C})$ is n-dimensional. The standard matrix units are normalizers, and they generate $M_n(\mathbb{C})$ as a C^*-algebra. The conditional expectation is the projection onto the diagonal.

Any other Cartan subalgebra of A is isomorphic to C via an automorphism of A (indeed map the mutually orthogonal basis elements to those of C, which extends to an automorphism of A).

Example 2.1.79. Let A be a finite dimensional C^*-algebra, then it is a direct sum of matrix algebras. The diagonal subalgebra (as in Example 2.1.78 but for direct sums) is a Cartan subalgebra.

Example 2.1.80. Let $A = \mathcal{O}_n$, the Cuntz algebra generated by n isometries. Let C be the C^*-subalgebra generated by the range projections of these isometries. Then C is a Cartan subalgebra. The discussion of this is given in Section 6.3 in [64].

Example 2.1.81. The following example can be found in Section 2 in [45]. For a discrete countable group G acting on a second countable locally compact Hausdorff space X by homeomorphisms, we may form the étale transformation groupoid $G \times X$ as in Example 2.1.27. We may also form the reduced crossed product C^*-algebra $C_0(X) \times_{\alpha,r} G$ where α denotes the group action (see Chapter 8 in [16]). There is an isomorphism $C_r^*(G \times X) \cong C_0(X) \times_{\alpha,r} G$. Corollary 2.3 in [45], or Example 2.1.32 in this thesis, shows that the group action is topologically free if and only if $G \times X$

is a topologically principal groupoid. In this situation, Theorem 2.1.82 below tells us that $C_0(X)$ is a Cartan subalgebra of $C_0(X) \times_{\alpha,r} G$.

Renault's result in [64] characterises Cartan subalgebras of separable C^*-algebras as being the C_0 functions on the unit space of a certain twisted groupoid. Specifically, Renault proves:

Theorem 2.1.82. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact second countable topologically principal Hausdorff groupoid. Then $(C^*_r(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$ is a Cartan pair.*

Conversely, let (A, C) be a Cartan pair where A is a separable C^*-algebra. Then there exists a twisted étale locally compact second countable topologically principal Hausdorff groupoid $(\mathcal{G}, \Sigma)$ and a C^*-algebra isomorphism carrying A onto $C^*_r(\mathcal{G}, \Sigma)$ and C onto $C_{0,r}(\mathcal{G}^0)$.

We will generalize Theorem 2.1.82 in Chapter 3 to non-second countable groupoids and non-separable C^*-algebras. This will have the effect of weakening the topological principality of the groupoid to effectiveness.

2.2 Elementary K-Theory for Unital C^*-algebras

In this section we will explore some of the fundamentals of K-theory for C^*-algebras. K-theory is a functor which attaches to a C^*-algebra an Abelian group. It has appeared as a main tool in the classification programme for C^*-algebras; indeed many classes of C^*-algebras can be classified by K-theory and tracial data (see for example [21], [30], [31], [38], [59] and [80]).

Our discussion here will only introduce the aspects of K-theory that we will need later and hence be far from a complete account. We will mainly focus on the basics of the K_0 functor on unital C^*-algebras. The contents of this material will be based on Chapters 1 to 5 in [43]. For a more complete account, as well as a discussion for non-unital C^*-algebras as well as that of the K_1 functor, the reader may consult the rest of the chapters in [43], as well as [11] or [85].

We start with a definition of the Grothendieck group induced from an Abelian semigroup. This is a generalization of the construction that obtains the integers from the natural numbers. Indeed, the integers can be viewed as the equivalence classes of the formal differences of natural numbers, with two differences $a - b$ and $c - d$ being identified if $a + d = c + b$.

Definition 2.2.1. Let $(\mathcal{S}, +)$ be an Abelian semigroup. Define an equivalence relation $\sim$ on $\mathcal{S} \times \mathcal{S}$ by $(s_1, s_2) \sim (r_1, r_2)$ if and only if there exists $z \in \mathcal{S}$ such that

$$s_1 + r_2 + z = r_1 + s_2 + z.$$

Denote the equivalence class of $(s, r) \in \mathcal{S} \times \mathcal{S}$ by $< s, r >$. Let

$$G(\mathcal{S}) = (\mathcal{S} \times \mathcal{S})/\sim .$$

Define an operation $+$ on $G(\mathcal{S})$ by

$$< s_1, r_1 > + < s_2, r_2 >=< s_1 + s_2, r_1 + r_2 >$$

(where we are abusing notation by using the same $+$ for $\mathcal{S}$ as for $G(\mathcal{S})$).

Lemma 2.2.2. *The operation $+$ on $G(\mathcal{S})$ is well-defined and turns $G(\mathcal{S})$ into an Abelian group.*

Proof. If $(s_1, r_1) \sim (a_1, b_1)$ and $(s_2, r_2) \sim (a_2, b_2)$ then there is $z_1, z_2 \in \mathcal{S}$ such that

$$s_1 + b_1 + z_1 = a_1 + r_1 + z_1, \quad s_2 + b_2 + z_2 = a_2 + r_2 + z_2.$$

Adding the equations yields

$$(s_1 + s_2) + (b_1 + b_2) + (z_1 + z_2) = (a_1 + a_2) + (r_1 + r_2) + (z_1 + z_2),$$

which implies that

$$(s_1 + s_2, r_1 + r_2) \sim (a_1 + a_2, b_1 + b_2).$$

Hence $+$ is well-defined. It is clear that $G(\mathcal{S})$ is Abelian because $\mathcal{S}$ is, that the identity element is $< s, s >$ for $s \in \mathcal{S}$, and that $- < s, r >=< r, s >$. Hence $G(\mathcal{S})$ is an Abelian group. $\square$

Definition 2.2.3. For an Abelian semigroup $\mathcal{S}$, $G(\mathcal{S})$ is called the *Grothendieck group of* $\mathcal{S}$. For an arbitrary $r \in \mathcal{S}$, the map

$$\gamma_{\mathcal{S}} : \mathcal{S} \to G(\mathcal{S}), \quad s \to < s + r, r >$$

is called the *Grothendieck map*. We say that $\mathcal{S}$ has the *cancellation property* if whenever $s, r, z \in \mathcal{S}$ such that $s + z = r + z$ then $s = r$.

Lemma 2.2.4. *The Grothendieck map is well-defined and additive. It is injective* *if and only if $\mathcal{S}$ has the cancellation property,*

Proof. For any $s, r_1, r_2 \in \mathcal{S}$ we have that $(s + r_1, r_1) \sim (s + r_2, r_2)$ hence $\gamma_{\mathcal{S}}$ is well-defined. It is additive because for ant $s_1, s_2, r \in \mathcal{S}$ we have that

$$\gamma_{\mathcal{S}}(s_1+s_2) =< s_1+s_2+r+r, r+r >=< s_1+r, r > + < s_2+r, r >= \gamma_{\mathcal{S}}(s_1)+\gamma_{\mathcal{S}}(s_2).$$

Note that if $\gamma_{\mathcal{S}}$ is injective and $s + z = r + z$ for $s, r, z \in \mathcal{S}$, then $\gamma_{\mathcal{S}}(s) = \gamma_{\mathcal{S}}(r)$ by additivity of $\gamma_{\mathcal{S}}$ and so $s = r$. If $\mathcal{S}$ satisfies the cancellation property then if $\gamma_{\mathcal{S}}(s) = \gamma_{\mathcal{S}}(r)$ it follows that there is some $z \in \mathcal{S}$ such that $s + z = r + z$ and so by cancellation $s = r$. □

Lemma 2.2.5. *Let $\mathcal{S}$ be an Abelian semigroup. Then*

$$G(\mathcal{S}) = \{\gamma_{\mathcal{S}}(s) - \gamma_{\mathcal{S}}(r) : s, r \in \mathcal{S}\}.$$

Proof. Note that

$$< s, r >=< s + (r + s), r + (r + s) >=< s + r, r > + < s, s + r >= \gamma_{\mathcal{S}}(s) - \gamma_{\mathcal{S}}(r).$$

□

Lemma 2.2.6. *The Grothendieck construction is functorial. In other words, if $f : \mathcal{S} \to \mathcal{T}$ is a semigroup homomorphism then there is a unique group homomorphism $G(f)$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{S} \arrow[r, "f"] \arrow[d, "\gamma_{\mathcal{S}}"] & \mathcal{T} \arrow[d, "\gamma_{\mathcal{T}}"] \\ G(\mathcal{S}) \arrow[r, "G(f)"] & G(\mathcal{T}) \end{tikzcd} \tag{2.13}$$

Proof. By Lemma 2.2.5, we may define

$$G(f)(\gamma_{\mathcal{S}}(s) - \gamma_{\mathcal{S}}(r)) = \gamma_{\mathcal{T}}(f(s)) - \gamma_{\mathcal{T}}(f(r)).$$

It is easy to see that this is well-defined, and Lemma 2.2.4 shows that $G(f)$ is a group homomorphism, and by construction (2.13) commutes. Uniqueness follows by commutativity of (2.13) and Lemma 2.2.5. □

Lemma 2.2.7. *The Grothendieck construction is universal. In other words, if $\mathcal{S}$ is an Abelian semigroup, H an Abelian group and $\gamma : \mathcal{S} \to H$ is an additive map, then* *there exists a unique group homomorphism $f : G(\mathcal{S}) \to H$ making the following diagram commute:*

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "\gamma"] \arrow[dr, "\gamma_{\mathcal{S}}"'] & & H \\ & G(\mathcal{S}) \arrow[ur, "f"'] & \end{tikzcd} \tag{2.14}$$

Proof. By Lemma 2.2.5 define

$$f(\gamma_{\mathcal{S}}(s) - \gamma_{\mathcal{S}}(r)) = \gamma(s) - \gamma(r). \tag{2.15}$$

It is easy to check that f is well-defined, and additivity of γ yields that f is a group homomorphism. Commutativity of (2.14) follows by construction. Uniqueness follows by Lemma 2.2.5 and commutativity of (2.14). □

Proposition 2.2.8. *Let $(G, +)$ be a group and $S \subset G$ a semigroup under the same operation $+$. Then $G(S)$ is isomorphic to the group $H =< S >= \{s - r : s, r \in S\}$ generated by S.*

Proof. We may take γ in Lemma 2.2.7 to be inclusion into G, obtaining a group homomorphism $f : G(S) \to G$. By (2.15) we see that the image of f is H. Note that if $f(\gamma_{\mathcal{S}}(s) - \gamma_{\mathcal{S}}(r)) = s - r = 0$ then $s = r$ and $\gamma_{\mathcal{S}}(s) - \gamma_{\mathcal{S}}(r) = 0$, so f is injective. □

Example 2.2.9. The Grothendieck group of $(\mathbb{N}, +)$ is isomorphic to $(\mathbb{Z}, +)$ by Proposition 2.2.8.

Example 2.2.10. The Grothendieck group of $(\mathbb{N} \cup \{\infty\}, +)$ is $\{0\}$. Indeed we have that for every $a, b \in \mathbb{N} \cup \{\infty\}$, $(a, b) \sim (\infty, \infty)$.

The idea now is to construct from a unital C^*-algebra A a certain semigroup, and the K_0 group of A will then be the Grothendieck group of this semigroup.

Definition 2.2.11. Let A be a unital C^*-algebra. For $n \in \mathbb{N}$, define $M_n(A)$ as the set of n by n matrices with entries in A. Give $M_n(A)$ a vector space structure by the usual pointwise addition and scalar multiplication. Define the multiplication and the involution in the usual way as done for $M_n(\mathbb{C})$. For a faithful representation π

of A on $\mathcal{H}$, define a representation π_n of $M_n(A)$ on $\mathcal{H}^n$ by

$$\pi_n\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\right)\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \pi(a_{11})\xi_1 + \pi(a_{12})\xi_2 + \cdots + \pi(a_{1n})\xi_n \\ \pi(a_{21})\xi_1 + \pi(a_{22})\xi_2 + \ldots + \pi(a_{2n})\xi_n \\ \vdots \\ \pi(a_{n1})\xi_1 + \pi(a_{n2})\xi_2 + \ldots + \pi(a_{nn})\xi_n \end{pmatrix}.$$

The C^*-norm on $M_n(A)$ is defined via its representation, making $M_n(A)$ into a C^*-algebra. We write $M_{m,n}(A)$ for the set of m by n matrices with entries in A. The involution here acts by transposition and entry-wise involution, with image in $M_{n,m}(A)$.

Remark 2.2.12. Any faithful representation induces the same norm, and we have

$$\max_{ij}\{\|a_{ij}\|\} \leq \|(a_{ij})_{ij}\| \leq \sum_{i,j}\|a_{ij}\|.$$

Definition 2.2.13. For A a unital C^*-algebra and $n \in \mathbb{N}$, define

$$\mathcal{P}_n(A) = \{p \in M_n(A) : \ \ p \text{ is a projection}\}.$$

Define

$$\mathcal{P}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n(A).$$

Definition 2.2.14. For A a unital C^*-algebra, we say that p and q in $\mathcal{P}_n(A)$ are *Murray-von Neumann equivalent*, written $p \sim q$, if there is a $v \in M_n(A)$ such that $p = v^*v$ and $q = vv^*$. We extend this to $\mathcal{P}_\infty(A)$ as follows: for $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, we say that p and q are *Murray-von Neumann equivalent*, written $p \sim_0 q$, if there is a $v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$.

Example 2.2.15. Let $A = \mathbb{C}$ and consider $p, q \in \mathcal{P}_n(\mathbb{C})$. Note that p and q are Murray-von Neumann equivalent if and only if they are the range and source projections of some partial isometry in $M_n(\mathbb{C})$, which is equivalent to $\dim(p(\mathbb{C}^n)) = \dim(q(\mathbb{C}^n))$, which is equivalent to $\mathrm{tr}(p) = \mathrm{tr}(q)$. Hence the Murray-von Neumann equivalent projections are exactly those which have the same trace.

Example 2.2.16. Let $A = \mathcal{B}(\mathcal{H})$ for some infinite-dimensional Hilbert space $\mathcal{H}$. Then $M_n(A) \cong \mathcal{B}(\mathcal{H}^n)$ which is again the bounded operators on some infinite dimensional Hilbert space, so we may just assume without loss of generality that p and q are projections in A. Then they are again Murray-von Neumann equivalent if and only if $\dim(p(\mathcal{H})) = \dim(q(\mathcal{H}))$.

Example 2.2.17. Let $A = C(X)$ for a compact Hausdorff space X. Then a necessary condition for p and q in $\mathcal{P}_n(C(X))$ to be Murray-von Neumann equivalent is that they have the same trace at every evaluation. However this condition may not be sufficient as it may not be possible to obtain a global isometry witnessing the equivalence from pointwise ones.

For instance, the map

$$p : \mathbb{T} \to M_2(\mathbb{C}), \quad t \to \begin{pmatrix} \frac{1}{2} & \frac{1}{2}t \\ \frac{1}{2}\bar{t} & \frac{1}{2} \end{pmatrix}$$

belongs to $\mathcal{P}_2(C(\mathbb{T}))$ and evaluates to a one-dimensional projection at every $t \in \mathbb{T}$. So does the map

$$q : \mathbb{T} \to M_2(\mathbb{C}), \quad t \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

However p and q are not Murray-von Neumann equivalent.

The following lemma highlights the properties of $\sim_0$:

Lemma 2.2.18. *Let A be a unital C^*-algebra. Let $p, p_1, q, q_1 \in \mathcal{P}_\infty(A)$. Then:*

1. *$\sim_0$ is an equivalence relation on $\mathcal{P}_\infty(A)$.*
2. *$p \sim_0 p \oplus 0_n$ for all $n \in \mathbb{N}$.*
3. *If $p \sim_0 p_1$ and $q \sim_0 q_1$ then $p \oplus q \sim_0 p_1 \oplus q_1$.*
4. *$p \oplus q \sim_0 q \oplus p$.*
5. *If $p, q \in \mathcal{P}_n(A)$ with $pq = 0$ then $p + q$ is a projection and $p \oplus q \sim_0 p + q$.*

Proof.

1. Reflexivity and symmetry are clear. For transitivity, assume v is a partial isometry inducing the equivalence between p and q and w is a partial isometry inducing the equivalence between q and r, then wv induces the equivalence between p and r.
2. Assume $p \in \mathcal{P}_m(A)$. Then the partial isometry $\begin{pmatrix} p \\ 0 \end{pmatrix} \in M_{m+n,m}(A)$ induces the required equivalence.
3. If v induces the first equivalence, and w the second, then $\mathrm{diag}(v, w)$ is the desired partial isometry.

4. Assume $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, then the desired partial isometry is

$$v = \begin{pmatrix} 0_{m,n} & q \\ p & 0_{n,m} \end{pmatrix} \in M_{n+m}(A).$$

5. If $pq = 0$ then $(qp)^* = pq = 0$ so $qp = 0$ and so $p + q$ is clearly a projection. The desired partial isometry is

$$v = \begin{pmatrix} p \\ q \end{pmatrix} \in M_{2n,n}(A).$$

□

Lemma 2.2.18 allows us to define the following:

Definition 2.2.19. Let A be a unital C^*-algebra. Define

$$\mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0 .$$

Denote the equivalence classes by $[\cdot]_{\mathcal{D}}$. Define an addition operation $+$ on $\mathcal{D}(A)$ by

$$[p]_{\mathcal{D}} + [q]_{\mathcal{D}} = [p \oplus q]_{\mathcal{D}}.$$

Lemma 2.2.20. *For a unital C^*-algebra A we have that $(\mathcal{D}(A), +)$ is an Abelian semigroup.*

Proof. This follows from Lemma 2.2.18. □

We are now in a position to define the K_0 group of a unital C^*-algebra A. It is the Grothendieck group of $\mathcal{D}(A)$.

Definition 2.2.21. Let A be a unital C^*-algebra. Define

$$K_0(A) = G(\mathcal{D}(A))$$

and denote, for $p \in \mathcal{P}_\infty(A)$,

$$[p]_0 = \gamma_{\mathcal{D}(A)}([p]_{\mathcal{D}}).$$

Remark 2.2.22. It is clear that if p and q in $\mathcal{P}_\infty(A)$ are Murray-von Neumann equivalent, then $[p]_{\mathcal{D}} = [q]_{\mathcal{D}}$ and so $[p]_0 = [q]_0$. However, the converse might not hold, so two elements that agree in K_0 might not be Murray-von Neumann equivalent, as the Grothendieck map might not be injective (equivalently, by Lemma 2.2.4, $\mathcal{D}(A)$ might not have the cancellation property). However, p and q will be stably equivalent, defined below.

Definition 2.2.23. Let A be a unital C^*-algebra. We say p and q in $\mathcal{P}_\infty(A)$ are *stably equivalent*, written $p \sim_s q$, if there is an $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim_0 q \oplus r$.

Lemma 2.2.24. *Let A be a unital C^*-algebra. The for $p, q \in \mathcal{P}_\infty(A)$ we have that*

$$[p]_0 = [q]_0 \iff p \sim_s q.$$

Proof. If $[p]_0 = [q]_0$ then by the definition of the Grothendieck map there is some $r \in \mathcal{P}_\infty(A)$ such that $[p]_{\mathcal{D}} + [r]_{\mathcal{D}} = [q]_{\mathcal{D}} + [r]_{\mathcal{D}}$ which implies by definition of addition in $\mathcal{D}(A)$ that $p \oplus r \sim_0 q \oplus r$ and hence $p \sim_s q$.

If $p \sim_s q$ then $p \oplus r \sim_0 q \oplus r$ for some $r \in \mathcal{P}_\infty(A)$ and hence $[p \oplus r]_0 = [q \oplus r]_0$. However note that

$$[p\oplus r]_0 = \gamma_{\mathcal{D}(A)}([p\oplus r]_{\mathcal{D}(A)}) = \gamma_{\mathcal{D}(A)}([p]_{\mathcal{D}}+[r]_{\mathcal{D}}) = \gamma_{\mathcal{D}(A)}([p]_{\mathcal{D}})+\gamma_{\mathcal{D}(A)}([r]_{\mathcal{D}}) = [p]_0+[r]_0.$$

Hence

$$[p]_0 + [r]_0 = [q]_0 + [r]_0$$

and as these elements belong to a group we have $[p]_0 = [q]_0$. $\square$

Remark 2.2.25. It is clear from Lemma 2.2.24 that we may have defined the K_0 group of a unital C^*-algebra A as follows. We could have started with the notion of stable equivalence of projections, which is weaker than Murray-von Neumann equivalence, and defined a sum on the equivalence classes as the equivalence class of the direct sum. The group generated by such elements is then the K_0 group. So K_0 identifies projections which are Murray-von Neumann equivalent up to taking a direct sum with a projection.

Definition 2.2.26. Let A be a unital C^*-algebra. The we say that A has the *cancellation property* if we have, for $p, q \in \mathcal{P}_\infty(A)$, that

$$[p]_0 = [q]_0 \iff p \sim_0 q.$$

We now describe the standard picture for K_0.

Proposition 2.2.27. *Let A be a unital C^*-algebra. Then*

$$K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\} = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_n(A), \quad n \in \mathbb{N}\}. \quad (2.16)$$

Further we have that

1. $[p]_0 + [q]_0 = [p \oplus q]_0$.
2. *The identity element is* $[0_A]_0$.
3. *If $p, q \in \mathcal{P}_n(A)$ are mutually orthogonal then* $[p + q]_0 = [p]_0 + [q]_0$.

Proof. The first equality in (2.16) holds by Lemma 2.2.5. For the second equality, let $[p]_0 - [q]_0 \in K_0(A)$ for some $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$. Let $l \in \mathbb{N}$ bigger than n and m, and put $p_1 = p \oplus 0_{l-n}$ and $q_1 = q \oplus 0_{l-m}$. Then by Lemma 2.2.18 we have that $p \sim_0 p_1$ and $q \sim_0 q_1$ and hence $[p]_0 = [p_1]_0$ and $[q]_0 = [q_1]_0$, and this yields the second equality.

1. This was shown in the proof of Lemma 2.2.24.
2. We have that $[p]_0 + [0_A]_0 = [p \oplus 0_A]_0 = [p]_0$ where the last equality is due to Lemma 2.2.18.
3. By Lemma 2.2.18, $p + q \sim_0 p \oplus q$ and hence the result follows by 1.

□

We now discuss the notion of homotopy between projections and homotopy equivalence between C^*-algebras. As we will not use this often later on we will not devote time to develop the theory or provide proofs.

Definition 2.2.28. Let A be a unital C^*-algebra and $p, q \in \mathcal{P}_n(A)$. We say p and q are *homotopic*, written $p \sim_h q$, if there exists a continuous path between them inside $\mathcal{P}_n(A)$. We say p and q are *unitarily equivalent*, written $p \sim_u q$, if there exists a unitary $u \in M_n(A)$ such that $p = u^* q u$.

Let B be a unital C^*-algebra. We say that two $*$-homomorphisms $\phi, \psi : A \to B$ are *homotopic*, written $\phi \sim_h \psi$, if there is a path of $*$-homomorphisms $A \to B$, $[0, 1] \ni t \to \phi_t$, starting at ϕ and ending at ψ (so $\phi_0 = \phi, \phi_1 = \psi$) and such that the map $t \to \phi_t(a)$ is continuous for each $a \in A$. We say that A and B are *homotopy equivalent* if there exists *-homomorphisms $\phi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \phi \sim_h \mathrm{id}_A$ and $\phi \circ \psi \sim_h \mathrm{id}_B$.

Remark 2.2.29. We have the implications

$$\sim_h \Longrightarrow \sim_u \Longrightarrow \sim . \tag{2.17}$$

For the proof, see Proposition 2.2.7 in [43]. It follows from this that if $p, q \in \mathcal{P}_n(A)$ and $p \sim_h q$ then $[p]_0 = [q]_0$. The reverse implications to (2.17) are true but in matrix amplifications, see Proposition 2.2.8 in [43].

It turns out that the properties 1 and 2 in Proposition 2.2.27, as well as being invariant under homotopies as in Remark 2.2.29 characterise $[\cdot]_0$ universally. What this means is the following:

Lemma 2.2.30. *Let A be a unital C^*-algebra, G an Abelian group and $\nu : \mathcal{P}_\infty(A) \to G$ a map satisfying*

1. *$\nu(p \oplus q) = \nu(p) + \nu(q)$ for all $p, q \in \mathcal{P}_\infty(A)$,*
2. *$\nu(0_A) = 0_G$, and,*
3. *if $p, q \in \mathcal{P}_n(A)$ are homotopic, then $\nu(p) = \nu(q)$.*

Then ν factors uniquely through $K_0(A)$, meaning there is a unique group homomorphism $f : K_0(A) \to G$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{P}_\infty(A) \arrow[rr, "\nu"] \arrow[dr, "{[\cdot]_0}"'] & & G \\ & K_0(A) \arrow[ur, "f"'] & \end{tikzcd} \tag{2.18}$$

Proof. If we show there is an additive map $\gamma : \mathcal{D}(A) \to G$, then by Lemma 2.2.7 γ would factor uniquely through $K_0(A)$. Define

$$\gamma[p]_\mathcal{D} = \nu(p).$$

To see that this is well-defined, note that if $p \sim_0 q$ and $p \in \mathcal{P}_n(A)$, $q \in \mathcal{P}_m(A)$ say, then let $l \in \mathbb{N}$ be bigger than m and n, and put $p_1 = p \oplus 0_{l-n}$ and $q_1 = q \oplus 0_{l-m}$. We have $p_1 \sim q_1$ by Lemma 2.2.18 . Now by Proposition 2.2.8 in [43] we have that $p_1 \oplus 0_{3l} \sim_h q_1 \oplus 0_{3l}$. Hence

$$\nu(p) = \nu(p) + (4l - n)\nu(0_A) = \nu(p_1 \oplus 0_{3l}) = \nu(q_1 \oplus 0_{3l}) = \nu(q).$$

Additivity of γ is clear, and hence we obtain a unique group homomorphism f :

$K_0(A) \to G$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{D}(A) \arrow[rr, "\gamma"] \arrow[dr, "\gamma_{\mathcal{D}(A)}"] & & G \\ & K_0(A) \arrow[ur, "f"] & \end{tikzcd}$$

The addition of the map $\mathcal{P}_\infty(A) \to \mathcal{D}(A)$, $p \to [p]_{\mathcal{D}}$ yields (2.18). □

Lemma 2.2.30 allows us to define how K_0 acts on arrows:

Definition 2.2.31. Let $\phi : A \to B$ be a *-homomorphism between two unital C^*-algebras A and B. Define:

$$K_0(\phi) : K_0(A) \to K_0(B), \;\; [p]_0 \to [\phi(p)]_0, \tag{2.19}$$

where we are abusing notation by letting ϕ also denote the canonical *-homomorphism $M_n(A) \to M_n(B)$ induced by ϕ, for $p \in \mathcal{P}_n(A)$.

Remark 2.2.32. Definition 2.2.31 is well-defined by Lemma 2.2.30. Indeed, ϕ induces a map $\mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B)$ in the canonical way, and hence the composition $\nu = [\cdot]_0 \circ \phi$ is a map $\mathcal{P}_\infty(A) \to K_0(B)$ satisfying the conditions of Lemma 2.2.30, and so factors uniquely through $K_0(A)$ as in (2.18). This unique group homomorphism we call $K_0(\phi)$.

Let us now prove functoriality of K_0:

Proposition 2.2.33. *Let A, B and C be unital C^*-algebras and $0_{A,B}$ denote the *-homomorphism $A \to B$ sending every element to 0_B. Let* id *denote the identity homomorphism. We have that*

1. $K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$.
2. *If $\phi : A \to B$ and $\psi : B \to C$ are *-homomorphisms then* $K_0(\psi \circ \phi) = K_0(\psi) \circ K_0(\phi)$.
3. $K_0(\{0\}) = \{0\}$.
4. $K_0(0_{A,B}) = 0_{K_0(A),K_0(B)}$.

Proof.

1. By (2.19), we have that for $p \in \mathcal{P}_n(A)$, $K_0(\mathrm{id}_A)([p]_0) = [p]_0$ and this also holds for differences of elements in $K_0(A)$. Hence by (2.16) the result follows.

2. For $p \in \mathcal{P}_n(A)$ we have that $K_0(\psi \circ \phi)([p]_0) = [\psi \circ \phi(p)]_0 = K_0(\psi)([\phi(p)]_0) = K_0(\psi) \circ K_0(\phi)([p]_0)$. This also holds for differences of such elements in $K_0(A)$ and so the result follows by (2.16).

3. We have $\mathcal{P}_\infty(\{0\}) = \{0\}$ and so $\mathcal{D}(\{0\}) = \{0\}$, hence $K_0(\{0\}) = G(\{0\}) = \{0\}$.

4. The map $0_{A,B}$ is the composition $A \to \{0\} \to B$ and the result follows from 3.

□

Let us now show that the K_0 functor is homotopy invariant:

Proposition 2.2.34. *Let A and B be unital C^*-algebras. If $\phi, \psi : A \to B$ are *-homomorphisms which are homotopic, then $K_0(\phi) = K_0(\psi)$. If A and B are homotopy equivalent, then $K_0(A)$ is isomorphic to $K_0(B)$.*

Proof. For $t \in [0,1]$, let $\phi_t : A \to B$ be the path of *-homomorphisms from $\phi_0 = \phi$ to $\phi_1 = \psi$. For $p \in \mathcal{P}_n(A)$, the continuity of $t \to \phi_t(p)$ implies $\phi(p) \sim_h \psi(p)$ implying $K_0(\phi)([p]_0) = K_0(\psi)([p]_0)$. This also applies for differences of such elements so by (2.16) we have $K_0(\phi) = K_0(\psi)$.

Now if $\phi_1 : A \to B$ and $\psi_1 : B \to A$ are *-homomorphisms witnessing the homotopy equivalence of A and B, then by the first part of this proof we have that $K_0(\psi_1 \circ \phi_1) = K_0(\mathrm{id}_A)$ and $K_0(\phi_1 \circ \psi_1) = K_0(\mathrm{id}_B)$. By functoriality of K_0 this implies that $K_0(A)$ and $K_0(B)$ are isomorphic and the isomorphism is $K_0(\phi_1)$ with inverse $K_0(\psi_1)$. □

We will not focus on K-theory for non-unital C^*-algebras in this thesis, as we will not work with non-unital C^*-algebras. For the interested reader, the definition of K_0 for non-unital C^*-algebras can be found, for instance, in Chapter 4 of [43]. The definition gives the same K-theory as the unital definition when the C^*-algebra is unital. Functoriality of K_0 works in the same way as the unital case. The analysis of the K_0 functor in the non-unital case allows us to obtain that the K_0 functor is half-exact and split-exact, which works whether the C^*-algebras are unital or not.

Proposition 2.2.35. *Let*

$$0 \to I \to A \to B \to 0$$

be a short exact sequence of C^*-algebras. Then the K_0 functor induces an exact sequence

$$K_0(I) \to K_0(A) \to K_0(B).$$

Let

$$0 \to I \to A \rightleftarrows B \to 0$$

be a split-exact sequence of C^*-algebras, then the K_0 functor induces a split-exact sequence

$$0 \to K_0(I) \to K_0(A) \rightleftarrows K_0(B) \to 0.$$

Corollary 2.2.36. *Let A and B be unital C^*-algebras. Then*

$$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B).$$

Proof. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & K_0(A) \oplus K_0(B) \arrow[r, "g"] \arrow[d, "K_0(\iota_A)\oplus K_0(\iota_B)"] & K_0(B) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & K_0(A) \arrow[r, "K_0(\iota_A)"] & K_0(A \oplus B) \arrow[r, "K_0(\pi_B)"] & K_0(B) \arrow[r] & 0 \end{tikzcd} \tag{2.20}$$

where $f(h_A) = (h_A, 0)$ and $g(h_A, h_B) = h_B$ for $h_A \in K_0(A)$, $h_B \in K_0(B)$, where $\iota_A(a) = (a, 0)$, $\iota_B(b) = (0, b)$, and $\pi_B(a, b) = b$, for $a \in A, b \in B$. The map $K_0(\iota_A) \oplus K_0(\iota_B)$ is defined by sending (h_A, h_B) to $K_0(\iota_A)(h_A) + K_0(\iota_B)(h_B)$.

The upper sequence of (2.20) is exact by direct computation, and the lower one by split-exactness (and Proposition 2.2.35) of

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota_A"] & A \oplus B \arrow[r, shift left, "\pi_B"] & B \arrow[l, shift left, "\iota_B"] \arrow[r] & 0. \end{tikzcd}$$

This implies that the middle vertical arrow $K_0(\iota_A) \oplus K_0(\iota_B)$ in (2.20) is an isomorphism. □

Another useful result is the stability of K_0:

Proposition 2.2.37. *Let A be a unital C^*-algebra. Let $n \in \mathbb{N}$. Then the injective map*

$$\lambda_n : A \to M_n(A), \quad a \to a \oplus 0_{n-1},$$

induces an isomorphism in K-theory

$$K_0(\lambda_n) : K_0(A) \to K_0(M_n(A)).$$

Proof. For $p \in \mathcal{P}_k(M_n(A))$ consider the canonical map $\mu_k : M_k(M_n(A)) \to M_{kn}(A)$. Let μ be the extension of this map to $\mathcal{P}_\infty(M_n(A))$. Then $\nu = []_0 \circ \mu : \mathcal{P}_\infty(M_n(A)) \to$

$K_0(A)$ satisfies the properties of Proposition 2.2.30, and hence factors uniquely through $K_0(M_n(A))$. So we obtain a group homomorphism $f_n : K_0(M_n(A)) \to K_0(A)$, which one can check is the inverse of $K_0(\lambda_n)$ by noting that

$$\lambda_n \circ \mu(p) \sim_0 p \;\text{ for }\; p \in \mathcal{P}_k(M_n(A))$$

and

$$\mu \circ \lambda_n(p) \sim_0 p \;\text{ for }\; p \in \mathcal{P}_k(A).$$

Note that we are writing λ_n for the *-homomorphism induced by λ_n on matrix amplifications. □

Now we wish to describe an order on $K_0(A)$ that will become useful when we consider the K_0 group of a UHF-algebra.

Definition 2.2.38. For an Abelian group G, we will call a non-empty subset $G^+ \subseteq G$ an *order* on G if it satisfies:

1. $G^+ + G^+ \subseteq G^+$,
2. $G = G^+ - G^+$, and
3. $G^+ \cap (-G^+) = \{0\}$.

In this situation, we say that the pair (G, G^+) is an *ordered* Abelian group.

Definition 2.2.39. Let A be a unital C^*-algebra. Define

$$K_0(A)^+ = \{[p]_0 : p \in \mathcal{P}_\infty(A)\}.$$

We now define the notion of stable finiteness for unital C^*-algebras, which will help yield that $K_0(A)^+$ is an order on $K_0(A)$.

Definition 2.2.40. A projection p in a unital C^*-algebra A is called *infinite* if it is Murray-von Neumann equivalent to a proper subprojection. Otherwise, it is called *finite*.

A unital C^*-algebra A is called *infinite* if 1_A is infinite, otherwise it is called *finite*. A is called *stably finite* if $M_n(A)$ is finite for all $n \in \mathbb{N}$.

Proposition 2.2.41. *Let A be a stably finite unital C^*-algebra, then $(K_0(A), K_0(A)^+)$ is an ordered Abelian group.*

Proof. We have from (2.16) that $K_0(A)^+ - K_0(A)^+ = K_0(A)$. It is clear that

$K_0(A)^+ + K_0(A)^+ \subseteq K_0(A)^+$. Now assume $[p]_0 = -[q]_0 \in K_0(A)^+ \cap (-K_0(A)^+)$. Then $[p \oplus q]_0 = 0$ implying that there exists a projection $r \in \mathcal{P}_\infty(A)$ such that $p \oplus q \oplus r \sim_0 r$, by Lemma 2.2.24. Assuming $p \oplus q \oplus r \in \mathcal{P}_k(A)$, it is easy to see that there exists mutually orthogonal projections $p_1, q_1, r_1 \in \mathcal{P}_k(A)$ with $p \sim_0 p_1, q \sim_0 q_1, r \sim_0 r_1$. We then have, by Lemma 2.2.18, that $p_1 + q_1 + r_1 \sim r_1$, and so by stable finiteness, it must be that $p_1 + q_1 + r_1 = r_1$, and hence that $p_1 = q_1 = 0$. This implies that $p = q = 0$ and thus $K_0(A)^+ \cap (-K_0(A)^+) = \{0\}$. □

Definition 2.2.42. Let (G, G^+) be an ordered Abelian group. For $g, h \in G$, we write $g \leq h$ if and only if $h - g \in G^+$.

Definition 2.2.43. For an Abelian group G we say $u \in G$ is an *order unit* if for every $g \in G$ there exists $n \in \mathbb{N}$ such that

$$-nu \leq g \leq nu.$$

Lemma 2.2.44. *Let A be a unital C^*-algebra. Then $[1_A]_0$ is an order unit in $K_0(A)$.*

Proof. Every $g \in K_0(A)$ is of the form $[p]_0 - [q]_0$ for some $p, q \in \mathcal{P}_n(A)$, by (2.16). Hence we have $1_{M_n(A)} - p$ and $1_{M_n(A)} - q$ belong to $\mathcal{P}_n(A)$. Thus we can write

$$-n[1_A]_0 = -[1_{M_n(A)}]_0 = -[q]_0 - [1_{M_n(A)} - q]_0 \leq -[q]_0 \leq g$$

and

$$g \leq [p]_0 \leq [p]_0 + [1_{M_n(A)} - p]_0 = n[1_A]_0.$$

□

Definition 2.2.45. Let (G, G^+) and (H, H^+) be ordered Abelian groups. A group homomorphism $f : G \to H$ is called *positive* if $f(G^+) \subseteq H^+$, and an *order isomorphism* if f is a group isomorphism with $f(G^+) = H^+$. If G has a unique order unit u and H has a unique order unit v then we say that f is *order unit preserving* if $f(u) = v$. We say that the triple (G, G^+, u) is isomorphic to (H, H^+, v) if there is an order unit preserving order isomorphism $f : G \to H$.

Proposition 2.2.46. *Let A be a unital C^*-algebra. Then $(K_0(A), K_0(A)^+, [1_A]_0)$ is an isomorphism invariant for A.*

Proof. Assume there exists a unital C^*-algebra B and an isomorphism $\phi : A \to B$ (necessarily unital). By functoriality, we have that $K_0(\phi) : K_0(A) \to K_0(B)$ is a group isomorphism. For $p \in \mathcal{P}_\infty(A)$, it is clear that $K_0(\phi)([p]_0) = [\phi(p)]_0 \in K_0(B)^+$ and hence $K_0(\phi)(K_0(A)^+) \subseteq K_0(B)^+$. Likewise $K_0(\phi)^{-1}(K_0(B)^+) \subseteq K_0(A)^+$. Thus $K_0(\phi)$ maps $K_0(A)^+$ onto $K_0(B)^+$. It is clear that it maps $[1_A]_0$ to $[1_B]_0$. $\square$

We conclude this section with some examples.

Example 2.2.47 ($\mathbb{C}$)**.** We consider the C^*-algebra $\mathbb{C}$. As we saw in Example 2.2.15, if $p, q \in \mathcal{P}_\infty(A)$, then $p \sim_0 q$ if and only if p and q have the same trace. So the map

$$[p]_{\mathcal{D}} \to \operatorname{tr}(p)$$

is a well-defined semigroup isomorphism $\mathcal{D}(\mathbb{C}) \to \mathbb{N} \cup \{0\}$. Hence the Grothendieck groups are isomorphic and so we obtain a group isomorphism

$$K_0(\mathbb{C}) \xrightarrow{\operatorname{tr}} \mathbb{Z}.$$

It is clear that $\mathbb{C}$ has the cancellation property. It is also clear that $\mathbb{C}$ is stably finite and

$$(K_0(\mathbb{C}), K_0(\mathbb{C})^+, [1_{\mathbb{C}}]_0) \cong (\mathbb{Z}, \mathbb{N} \cup \{0\}, 1).$$

Example 2.2.48 ($M_n(\mathbb{C})$)**.** We consider the C^*-algebra $M_n(\mathbb{C})$ for $n \in \mathbb{N}$. Note that by stability of K_0 as in Proposition 2.2.37, we already know that $K_0(M_n(\mathbb{C})) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$. However the isomorphism given in Proposition 2.2.37 is not order unit preserving. It is clear that $M_n(\mathbb{C})$ is stably finite and satisfies cancellation, essentially for the same reasons as for $\mathbb{C}$. Hence by considering traces as in Example 2.2.47, we obtain that

$$(K_0(M_n(\mathbb{C})), K_0(M_n(\mathbb{C}))^+, [1_{M_n(\mathbb{C})}]_0) \cong (\mathbb{Z}, \mathbb{N} \cup \{0\}, n).$$

Example 2.2.49 (Finite Dimensional C^*-algebra)**.** We consider a finite dimensional C^*-algebra $A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$. By Corollary 2.2.36 we obtain that $K_0(A) \cong \bigoplus_{i=1}^{N} K_0(M_{n_i}(\mathbb{C})) \cong \mathbb{Z}^N$. As in Example 2.2.48 we have that A satisfies the cancellation property, is stably finite, and

$$(K_0(A), K_0(A)^+, [1_A]_0) \cong (\mathbb{Z}^N, (\mathbb{N} \cup \{0\})^N, (n_1, \ldots, n_N)).$$

Example 2.2.50 ($\mathcal{B}(\mathcal{H})$)**.** Let $\mathcal{H}$ be an infinite dimensional Hilbert space, and consider the C^*-algebra $A = \mathcal{B}(\mathcal{H})$. As in Example 2.2.16, we have that projections in $\mathcal{P}_\infty(A)$ are equivalent in $\sim_0$ if and only if the dimension of their ranges agree. It is easy to see thus that the map sending $[p]_\mathcal{D}$ to $\dim(p(\mathcal{H}))$ is a well-defined semigroup isomorphism $\mathcal{D}(A) \to \mathcal{S}$ where $\mathcal{S}$ is a semigroup consisting of all cardinal numbers less than or equal to $\dim(\mathcal{H})$. The Grothendieck group of any such semigroup is $\{0\}$ and so $K_0(A) = \{0\}$.

Example 2.2.51. Let us study the K-theory of $C(X)$ for certain compact Hausdorff spaces X. Assume X is connected, then for fixed $x \in X$, the map sending $[p]_\mathcal{D}$ to $\mathrm{tr}(p(x))$ is a well-defined surjective semigroup morphism $\mathcal{D}(C(X)) \to \mathbb{N} \cup \{0\}$. The map is independent of the choice of element x as the map $X \to \mathbb{Z}$ given by $x \to \mathrm{tr}(p(x))$ is continuous and hence constant as X is connected. This implies that we get a surjective group homomorphism $K_0(C(X)) \xrightarrow{\mathrm{tr}_x} \mathbb{Z}$.

If X is contractible then there is a point $x_0 \in X$ and a continuous map $h : [0,1] \times X \to X$ such that $h(0,x) = x$ and $h(1,x) = x_0$ for all $x \in X$. Then we may define a *-homomorphism $\phi : C(X) \to \mathbb{C}$ by $\phi(f) = f(x_0)$ and a *-homomorphism $\psi : \mathbb{C} \to C(X)$ by $z \to z \cdot 1$ where 1 is the element of $C(X)$ defined by $1(x) = 1$ for all $x \in X$. Note that $\psi \circ \phi \sim_h \mathrm{id}_{C(X)}$. Indeed, define $\phi_t(f)(x) = f(h(t,x))$ for $t \in [0,1]$, and note that $\phi_0 = \mathrm{id}_{C(X)}$ and $\phi_1 = f(x_0) \cdot 1$, and $t \to \phi_t(f)$ is continuous for every $f \in C(X)$. Clearly $\phi \circ \psi = \mathrm{id}_\mathbb{C}$.

Hence by homotopy invariance of K_0, as in Proposition 2.2.34, we have that $K_0(C(X)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$. In particular we have a commutative diagram

$$\begin{tikzcd} K_0(C(X)) \arrow[d, "K_0(\phi)"'] \arrow[dr, "\mathrm{tr}_x"] & \\ K_0(\mathbb{C}) \arrow[r, "\mathrm{tr}"] & \mathbb{Z} \end{tikzcd} \tag{2.21}$$

As the vertical and horizontal arrows are isomorphisms, the former by Proposition 2.2.34 and the latter by Example 2.2.47, then tr_x is a group isomorphism.

2.3 Inductive Limit C^*-algebras

In this section we introduce inductive limit C^*-algebras and focus on certain topics that will be useful for us later in the thesis. We begin by a general introduction to inductive limit C^*-algebras. The material relevant to this can be found in sections 6.1 and 6.2 of [43].

Definition 2.3.1. Assume A_n is a C^*-algebra for each $n \in \mathbb{N}$. Define

$$\prod_{n\in\mathbb{N}} A_n = \{\text{functions } a : \mathbb{N} \to \cup_{n\in\mathbb{N}} A_n : a(n) \in A_n,\ \sup_{n\in\mathbb{N}} \|a(n)\|_{A_n} < \infty\}.$$

Elements of $\prod_{n\in\mathbb{N}} A_n$ will also be written as (a_n). Define

$$\mathcal{J} = \{a \in \prod_{n\in\mathbb{N}} A_n : a(n) \neq 0 \ \text{ for only finitely many } n \in \mathbb{N}\}.$$

Lemma 2.3.2. *We have that $\prod_{n\in\mathbb{N}} A_n$ with norm $\|a\| = \sup_{n\in\mathbb{N}} \|a(n)\|_{A_n}$ and product, addition, scalar product and involution defined pointwise, is a C^*-algebra. $\mathcal{J}$ is a two-sided ideal of $\prod_{n\in\mathbb{N}} A_n$.*

Proof. That $\|\cdot\|$ is a norm is clear, and so is the fact that $\prod_{n\in\mathbb{N}} A_n$ is a *-algebra and that the norm satisfies the C^*-equation. What is left to check is that $\prod_{n\in\mathbb{N}} A_n$ is a Banach *-algebra.

To this end, let $\{a^m\}_{m\in\mathbb{N}}$ be a Cauchy sequence in $\prod_{n\in\mathbb{N}} A_n$. Then for fixed $n \in \mathbb{N}$, we have that $\{a^m(n)\}_{m\in\mathbb{N}}$ is Cauchy in A_n and hence converges to some element $a_n \in A_n$. Define $a = (a_n)$ and it is straightforward to check that $a^m \to a$ in $\|\cdot\|$ with $a \in \prod_{n\in\mathbb{N}} A_n$. That $\mathcal{J}$ is a two-sided ideal is clear. $\square$

Definition 2.3.3. Assume A_n is a C^*-algebra for each $n \in \mathbb{N}$. Define

$$\sum_{n\in\mathbb{N}} A_n = \overline{\mathcal{J}}^{\|\cdot\|}.$$

Denote the quotient map $\prod_{n\in\mathbb{N}} A_n \to (\prod_{n\in\mathbb{N}} A_n)/(\sum_{n\in\mathbb{N}} A_n)$ by π.

Lemma 2.3.4. *In the situation of Definition 2.3.3 we have that*

$$\|\pi((a_n))\| = \limsup_{n\to\infty} \|a_n\|_{A_n},$$

and (a_n) belongs to $\sum_{n\in\mathbb{N}} A_n$ if and only if $\lim_{n\to\infty} \|a_n\| = 0$.

Proof. By density of $\mathcal{J}$ in $\sum_{n\in\mathbb{N}} A_n$ we have that $\pi((a_n)) = \inf\{\|(a_n) - b\| : b \in \mathcal{J}\}$. Now for fixed $b = (b_n) \in \mathcal{J}$ we have that

$$\|(a_n) - (b_n)\| \geq \limsup_{n\to\infty} \|a_n - b_n\|_{A_n} = \limsup_{n\to\infty} \|a_n\|_{A_n}.$$

Hence $\|\pi((a_n))\| \geq \limsup_{n\to\infty}\|a_n\|_{A_n}$. For $k \in \mathbb{N}$, let $b_k = (b_n^k)_{n\in\mathbb{N}}$ be the element in $\mathcal{J}$ where $b_n^k = a_n$ for $1 \leq n \leq k$, and 0 for $n > k$. Then

$$\|\pi((a_n))\| \leq \inf_{k\in\mathbb{N}}\|(a_n) - b_k\| = \inf_{k\in\mathbb{N}}\sup_{n>k}\|a_n\|_{A_n} = \limsup_{n\to\infty}\|a_n\|_{A_n}.$$

The last claim of the lemma follows easily. □

We will now describe the structure of an inductive limit C^*-algebra.

Definition 2.3.5. An *inductive system* of C^*-algebras is a sequence of C^*-algebras $\{A_n\}_{n\in\mathbb{N}}$ (called *building blocks*) and *-homomorphisms $\{\phi_n\}_{n\in\mathbb{N}}$ (called *connecting maps*) where $\phi_n : A_n \to A_{n+1}$. This is also written as

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots$$

Define, for $n > m$, $\phi_{n,m} = \phi_n \circ \phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_m$.

Definition 2.3.6. Given an inductive system of building blocks $\{A_n\}_{n\in\mathbb{N}}$ and connecting maps $\{\phi_n\}_{n\in\mathbb{N}}$, its corresponding inductive limit C^*-algebra is a system $(A, \{\mu_n\}_{n\in\mathbb{N}})$ where A is a C^*-algebra and each $\mu_n : A_n \to A$ is a *-homomorphism, satisfying:

1. For each $n \in \mathbb{N}$ we have a commutative diagram

$$\begin{tikzcd} A_n \arrow[rr, "\phi_n"] \arrow[dr, "\mu_n"'] & & A_{n+1} \arrow[dl, "\mu_{n+1}"] \\ & A & \end{tikzcd}$$

2. If $(B, \{\lambda_n\}_{n\in\mathbb{N}})$ is another system where B is a C^*-algebra and $\lambda_n : A_n \to B$ are *-homomorphisms satisfying, for each $n \in \mathbb{N}$, $\lambda_{n+1} \circ \phi_n = \lambda_n$, then there exists a unique *-homomorphism $\phi : A \to B$ such that the following diagram commutes for every $n \in \mathbb{N}$:

$$\begin{tikzcd} & A_n \arrow[dl, "\mu_n"'] \arrow[dr, "\lambda_n"] & \\ A \arrow[rr, "\phi"] & & B \end{tikzcd}$$

In this situation, we may write the inductive limit as $\varinjlim(A_n, \phi_n)$.

Remark 2.3.7. The definitions of an inductive system and corresponding inductive limit are more generally done for arbitrary categories. Rather than C^*-algebras we have the objects of that category, and rather than $*$-homomorphisms, we have the arrows of that category. The definitions are the same. We will use these definitions for the category of Abelian groups and ordered Abelian groups in Subsection 2.3.1.

Lemma 2.3.8. *If an inductive system of C^*-algebra building blocks and connecting maps has an inductive limit, then it is unique (up to isomorphism).*

Proof. This follows directly from 2 in Definition 2.3.6, by switching the roles of μ_n and λ_n. $\square$

Remark 2.3.9. In view of Lemma 2.3.8, we may speak of *the* inductive limit, when it exists.

The next proposition shows that every inductive system of C^*-algebras and connecting maps has an inductive limit.

Proposition 2.3.10. *Every inductive system of building blocks $\{A_n\}_{n\in\mathbb{N}}$ and connecting maps $\{\phi_n\}_{n\in\mathbb{N}}$ has an inductive limit $(A, \{\mu_n\}_{n\in\mathbb{N}})$. We also have that:*

1. $A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$,
2. $\|\mu_n(a)\| = \lim_{m\to\infty} \|\phi_{m,n}(a)\|_{A_m}$ *for every* $n \in \mathbb{N}$ *and* $a \in A_n$,
3. $\ker(\mu_n) = \{a \in A_n : \lim_{m\to\infty} \|\phi_{m,n}(a)\|_{A_m} = 0\}$,
4. *if* $(B, \{\lambda_n\}_n)$ *and* ϕ *are as in 2 of Definition 2.3.6, then* $\ker(\mu_n) \subseteq \ker(\lambda_n)$ *for all* $n \in \mathbb{N}$. ϕ *is injective if and only if* $\ker(\lambda_n) \subseteq \ker(\mu_n)$ *for all* $n \in \mathbb{N}$. ϕ *is surjective if and only if* $B = \overline{\bigcup_{n=1}^{\infty} \lambda_n(A_n)}$.

Proof. For $a \in A_n$ define

$$\nu_n(a) = (\phi_{m,n}(a))_{m=1}^{\infty} \in \prod_{i\in\mathbb{N}} A_i,$$

where $\phi_{m,n}(a)$ is defined to be 0 if $m < n$ and a if $m = n$. It is clear that ν_n is a *-homomorphism. Define

$$\mu_n = \pi \circ \nu_n : A_n \to (\prod_{i\in\mathbb{N}} A_i)/(\sum_{i\in\mathbb{N}} A_i).$$

Notice that for $a \in A_n$,

$$\mu_n(a) - \mu_{n+1} \circ \phi_n(a) = \pi(\nu_n(a) - \nu_{n+1} \circ \phi_n(a)) = \pi((\delta_{m,n} a)_{m=1}^{\infty}) = 0$$

where $\delta_{m,n}$ is 0 if $m \neq n$ and 1 when $m = n$. Hence $\mu_n = \mu_{n+1} \circ \phi_n$. Thus we have an increasing sequence of C^*-algebras $\{\mu_n(A_n)\}_{n\in\mathbb{N}}$ and so we may declare A to be the C^*-algebra

$$A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}.$$

We have seen that $\{A, \{\mu_n\}_{n\in\mathbb{N}}\}$ is a system satisfying 1 in Definition 2.3.6. We have constructed A in such a way as to satisfy 1 in the statement of this proposition. Claim 2 is easy to check using Lemma 2.3.4 as

$$\|\mu_n(a)\| = \|\pi \circ \nu_n(a)\| = \limsup_{m\to\infty} \|\phi_{m,n}(a)\|_{A_m} = \lim_{m\to\infty} \|\phi_{m,n}(a)\|_{A_m},$$

where the last equality holds as the sequence $\{\|\phi_{m,n}(a)\|_{m\in\mathbb{N}}\}$ is eventually decreasing. Claim 3 follows directly from claim 2.

By the first isomorphism theorem the *-homomorphism λ_n factors uniquely through $\mu_n(A_n)$ (as $\ker(\mu_n) \subseteq \ker(\lambda_n)$), so there exists a unique *-homomorphism $\lambda_n' : \mu_n(A_n) \to B$ such that $\lambda_n' \circ \mu_n = \lambda_n$. As $\{\mu_n(A_n)\}_{n\in\mathbb{N}}$ is increasing, uniqueness gives that the extension of λ_n' to λ_{n+1}' is unique. Hence we can extend this to a *-homomorphism $\lambda' : \bigcup_{n=1}^{\infty} \mu_n(A_n) \to B$. λ' is contractive as each λ_n' is, and hence continuity allows us to extend to a *-homomorphism $\phi : A \to B$. ϕ satisfies that $\phi \circ \mu_n = \lambda_n' \circ \mu_n = \lambda_n$ and is unique by the uniqueness of the restrictions. This gives 2 in Definition 2.3.6.

For the claims in 4 in this proposition, note that ϕ is injective if and only if it is an isometry, if and only if each λ_n' is an isometry, if and only if $\ker(\mu_n) = \ker(\lambda_n)$. The image of ϕ is $\overline{\bigcup_{n=1}^{\infty} \lambda_n(A_n)}$ and hence ϕ is surjective if and only if $B = \overline{\bigcup_{n=1}^{\infty} \lambda_n(A_n)}$. $\square$

Corollary 2.3.11. *Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ be an inductive system where the building blocks are unital and the connecting maps are injective and unital. Then the inductive limit C^*-algebra $(A, \{\mu_n\}_{n\in\mathbb{N}})$ is unital, with unital and injective maps μ_n for every $n \in \mathbb{N}$.*

Proof. Define $1_A = \mu_1(1_{A_1})$. Note that for any $n \in \mathbb{N}$ we have that $\mu_n(1_{A_n}) = \mu_n(\phi_{n,1}(1_{A_1})) = \mu_1(1_{A_1}) = 1_A$. Now for $a \in A$ we have by 1 in Proposition 2.3.10 that there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_n \in A_n$ such that $a = \lim_{n\to\infty} \mu_n(a_n)$. Then

$$1_A a = \lim_{n\to\infty} \mu_n(1_{A_n})\mu_n(a_n) = \lim_{n\to\infty} \mu_n(a_n) = a.$$

Likewise $a1_A = a$. Hence A is unital. By using 2 in Proposition 2.3.10 we see that

μ_n is isometric, hence injective, for each $n \in \mathbb{N}$. □

Corollary 2.3.12. *Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ be an inductive system of C^*-algebra building blocks and connecting maps. Then any subsystem $\{A_{n_i}\}_{i\in\mathbb{N}}$ and $\{\phi_{n_{i+1},n_i}\}_{i\in\mathbb{N}}$ gives rise to the same inductive limit C^*-algebra as the original system.*

Proof. This is easily seen by 1 in Proposition 2.3.10. □

We will see a large class of examples of inductive limit C^*-algebras when we look at AX-algebras for compact Hausdorff spaces X in Chapter 4.

2.3.1 Continuity of K_0

In this subsection we wish to prove the continuity of the K_0 functor on inductive limit C^*-algebras. What this means is that the K_0 group of an inductive limit C^*-algebra is the inductive limit (in the category of Abelian groups) of the K_0 groups of the building blocks. The material of this subsection is based on Sections 6.2 and 6.3 of [43].

The following proposition is of the same spirit as Proposition 2.3.10, but we do it in the category of Abelian groups (with arrows group homomorphisms). The proof is easier than its counterpart in Proposition 2.3.10.

Proposition 2.3.13. *Every inductive system of Abelian groups $\{G_n\}_{n\in\mathbb{N}}$ and group homomorphisms $\{f_n\}_{n\in\mathbb{N}}$ has an inductive limit $(G, \{\alpha_n\}_{n\in\mathbb{N}})$. We also have:*

1. $G = \bigcup_{n=1}^{\infty} \alpha_n(G_n)$,
2. $\ker(\alpha_n) = \bigcup_{m=n+1}^{\infty} \ker(f_{m,n})$ *for every* $n \in \mathbb{N}$,
3. *if $(H, \{\gamma_n\}_{n\in\mathbb{N}})$ and $f : G \to H$ are as in 2 in Definition 2.3.6 (but in the category of Abelian groups) then we have that f is injective if and only if* $\ker(\alpha_n) = \ker(\gamma_n)$ *and f is surjective if and only if $H = \bigcup_{n=1}^{\infty} \gamma_n(G_n)$.*

Corollary 2.3.14. *Let $\{G_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ be an inductive system in the category of ordered Abelian groups (so the connecting maps are positive), and let the inductive limit be $(G, \{\alpha_n\}_{n\in\mathbb{N}})$ in the category of Abelian groups. Set*

$$G^+ = \bigcup_{n=1}^{\infty} \alpha_n(G_n^+).$$

Then $((G, G^+), \{\alpha_n\}_{n\in\mathbb{N}})$ *is the inductive limit of our inductive system in the category of ordered Abelian groups.*

Proof. This is a straightforward task of checking definitions, aided by the claims of Proposition 2.3.13. $\square$

Remark 2.3.15. In the same vein as Corollaries 2.3.11 and 2.3.12 we have that if the connecting maps are injective group homomorphisms then the maps α_n are also for each $n \in \mathbb{N}$, and also any inductive subsystem gives rise to the same inductive limit as the original system.

In order to prove the continuity of K_0, we need the following lemma:

Lemma 2.3.16. *Let A be a unital C^*-algebra. If $a \in A$ is self-adjoint with $\|a - a^2\| = \delta < \frac{1}{4}$ then there exists a projection $p \in A$ with $\|a - p\| \leq 2\delta$.*

Proof. Using the spectral mapping theorem we may deduce that

$$\mathrm{sp}(a) \subseteq [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta].$$

Hence we may define a continuous function f that is 0 on $[-2\delta, 2\delta]$ and 1 on $[1 - 2\delta, 1 + 2\delta]$. Let $p = f(a)$ which is a projection by the continuous function calculus. Note that $|t - f(t)| \leq 2\delta$ for all $t \in \mathrm{sp}(a)$, hence $\|a - p\| \leq 2\delta$. $\square$

We now state and prove continuity of K_0. Note however that claim 3 in Proposition 2.3.17 needs the tools of K-theory for non-unital C^*-algebras, so we will omit the proof. It can be found in Theorem 6.3.2 of [43].

Proposition 2.3.17. *Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ be an inductive system consisting of unital C^*-algebras and connecting maps. Denote its inductive limit by $(A, \{\mu_n\}_{n\in\mathbb{N}})$.*

Let $(G, \{\alpha_n\}_{n\in\mathbb{N}})$ be the inductive limit (in the category of Abelian groups) of the inductive system consisting of groups $\{K_0(A_n)\}_{n\in\mathbb{N}}$ and connecting maps $\{K_0(\phi_n)\}_{n\in\mathbb{N}}$. Then there is a unique group isomorphism $f : G \to K_0(A)$ making the following diagram commute for every $n \in \mathbb{N}$:

$$\begin{tikzcd} & K_0(A_n) \arrow[dl, "\alpha_n"'] \arrow[dr, "K_0(\mu_n)"] & \\ G \arrow[rr, "f"] & & K_0(A) \end{tikzcd} \tag{2.22}$$

Furthermore we have that

1. $K_0(A) = \bigcup_{n=1}^{\infty} K_0(\mu_n)(K_0(A_n))$,

2. $K_0(A)^+ = \bigcup_{n=1}^{\infty} K_0(\mu_n)(K_0(A_n)^+)$, *and*

3. $\ker(K_0(\mu_n)) = \bigcup_{m=n+1}^{\infty} \ker(K_0(\phi_{m,n}))$ *for every* $n \in \mathbb{N}$.

Proof. That claim 3 holds is shown in (iii) of Theorem 6.3.2 in [43]; it requires the definition of K_0 for non-unital C^*-algebras which we will not generally need. It is clear that claim 2 implies claim 1, so we will prove claim 2.

Let $p \in \mathcal{P}_k(A)$. It is clear that $(M_k(A), \{\mu_n\}_{n\in\mathbb{N}})$ is the inductive limit of the system with building blocks $\{M_k(A_n)\}_{n\in\mathbb{N}}$ and connecting maps $\{\phi_n\}_{n\in\mathbb{N}}$ (where we abuse notation by using the same notation for the induced maps on the matrix amplifications). From this, and 1 in Proposition 2.3.10, it follows that there exists $N \in \mathbb{N}$ and an element $a_{k,N} \in M_k(A_N)$ such that $\|\mu_N(a_{k,N}) - p\| < \frac{1}{10}$. Define the self-adjoint element $b_{k,N} = \frac{a_{k,N}+a_{k,N}^*}{2}$ and note that $\|\mu_N(b_{k,N}) - p\| < \frac{1}{10}$. This means that $\mathrm{sp}(\mu_N(b_{k,N})) \subseteq [-\frac{1}{10}, \frac{1}{10}] \cup [\frac{9}{10}, \frac{11}{10}]$ (see, for example, Lemma 2.2.3 in [43]). From this it follows that

$$\|\mu_N(b_{k,N})^2 - \mu_N(b_{k,N})\| = \|\mu_N(b_{k,N}^2 - b_{k,N})\| = \max\{|t-t^2| : t \in \mathrm{sp}(\mu_N(b_{k,N}))\} \leq \frac{9}{100}.$$

From claim 2 in Proposition 2.3.10 it follows that there is $M > N$ with

$$\|\phi_{M,N}(b_{k,N}^2 - b_{k,N})\| < \frac{1}{4}.$$

By Lemma 2.3.16 it follows that there exists a projection $q \in M_k(A_M)$ such that

$$\|q - \phi_{M,N}(b_{k,N})\| < \frac{1}{2}.$$

Now use the triangle inequality to write

$$\|\mu_M(q) - p\| \leq \|\mu_M(q) - \mu_M(\phi_{M,N}(b_{k,N}))\| + \|\mu_M(\phi_{M,N}(b_{k,N})) - p\| < \frac{1}{2} + \frac{1}{10} < 1.$$

Hence $\mu_M(q)$ is homotopic to p (see for example Proposition 2.2.4 in [43]), and hence Murray-von Neumann equivalent to it also. Thus $[p]_0 = K_0(\mu_M)([q]_0)$ and 2 follows.

Now use 2 in Definition 2.3.6 to obtain that there exists a unique group homomorphism $f : G \to K_0(A)$ making (2.22) commute. Claim 1 forces f to be surjective, and by 1 in Proposition 2.3.13, if $f(g) = 0$ for some $g \in G$, then $g = \alpha_n(\gamma)$ for some $n \in \mathbb{N}$ and $\gamma \in K_0(A_n)$, and hence by commutativity of (2.22), we have that $K_0(\mu_n)(\gamma) = 0$, which implies by claim 3 that there is $m > n$ with $K_0(\phi_{m,n})(\gamma) = 0$, and hence

$$g = \alpha_n(\gamma) = \alpha_m \circ K_0(\phi_{m,n})(\gamma) = 0.$$

Hence f is injective and thus an isomorphism. □

Corollary 2.3.18. *In the situation of Proposition 2.3.17, if we additionally assume that* $(K_0(A_n), K_0(A_n)^+)$ *is an ordered Abelian group for each* $n \in \mathbb{N}$*, and* $((G, G^+), \{\alpha_n\}_{n\in\mathbb{N}})$ *is its inductive limit in the category of ordered Abelian groups, then the isomorphism* f *is an order isomorphism.*

Proof. We have by Proposition 2.3.13 that $G^+ = \bigcup_{n=1}^{\infty} \alpha_n(K_0(A_n)^+)$, and hence by (2.22) and 2 in Proposition 2.3.17 we have that

$$f(G^+) = K_0(A)^+.$$

□

2.3.2 The K_0 Group of a UHF-algebra

In this subsection we discuss the K_0 group of a UHF-algebra. This can be calculated using the supernatural number associated to the algebra. We will use this information when we explore the question of uniqueness of inductive limit Cartan subalgebras in AI-algebras in Chapter 5. The material presented here is based on Section 7.4 in [43].

Definition 2.3.19. A *UHF-algebra (uniformly hyperfinite)* is an inductive limit C^*-algebra of an inductive system with building blocks of the form $\{M_{n_i}(\mathbb{C})\}_{i\in\mathbb{N}}$ and unital connecting maps $\{\phi_i\}$ where for each $i \in \mathbb{N}$, $n_i | n_{i+1}$.

Definition 2.3.20. A *supernatural number* is a sequence $n = \{n_i\}_{i\in\mathbb{N}}$ where each $n_i \in \{0, 1, \ldots, \infty\}$. Let $p_1, p_2, \ldots$ be the sequence of prime numbers listed in increasing order. We may represent the supernatural number n symbolically as the formal product $\prod_{i=1}^{\infty} p_i^{n_i}$. Every natural number n can be treated as a supernatural number using the prime factorisation of n. The product of two supernatural numbers $m = \{m_i\}_{i\in\mathbb{N}}$ and $n = \{n_i\}_{i\in\mathbb{N}}$ is the supernatural number $mn = \{m_i + n_i\}_{i\in\mathbb{N}}$.

Definition 2.3.21. To a supernatural number $n = \{n_i\}_{i\in\mathbb{N}}$ we associate a subset of $\mathbb{Q}$, $Q(n)$, defined as consisting of all rational numbers $\frac{x}{y}$ where $x \in \mathbb{Z}$ and y is of the form $\prod_{i=1}^{\infty} p_i^{m_i}$ where each $m_i \leq n_i$ and all but finitely many m_i's are 0.

Lemma 2.3.22. *Let $n = \{n_i\}_{i\in\mathbb{N}}$ be a supernatural number. Then we have the following.*

1. *$(Q(n), +)$ is a subgroup of the additive group $(\mathbb{Q}, +)$, containing 1.*
2. *Every subgroup of $(\mathbb{Q}, +)$ which contains 1 is of the form $Q(m)$ for some supernatural number m.*
3. *For supernatural numbers m and n we have an isomorphism $(Q(m), 1) \cong (Q(n), 1)$ if and only if $n = m$.*
4. *For supernatural numbers m and n we have an isomorphism $Q(m) \cong Q(n)$ if and only if there are natural numbers k_1, k_2 such that $k_1 m = k_2 n$.*

Proof.

1. That $Q(n)$ contains one is clear as the allowable denominators of the fractions in $Q(n)$ include 1. The allowable numerators include 0 so $0 \in Q(n)$. If $\frac{x}{y} \in Q(n)$, then $\frac{-x}{y} \in Q(n)$ is the inverse of $\frac{x}{y}$, and if additionally $\frac{z}{w} \in Q(n)$, then the addition of $\frac{x}{y}$ and $\frac{z}{w}$ is a fraction whose denominator is of the form $\prod_{i=1}^{\infty} p_i^{a_i}$ where each $p_i^{a_i}$ either appears as a factor of y or w. This implies $\frac{x}{y} + \frac{z}{w} \in Q(n)$.

2. Let G be a subgroup of $(\mathbb{Q}, +)$ that contains 1. Define, for each $i \in \mathbb{N}$,

$$m_i = \sup\{k \in \mathbb{N} \cup \{0\} : \frac{1}{p_i^k} \in G\} \in \{0, 1, \ldots, \infty\}. \tag{2.23}$$

   Let $m = \{m_i\}_{i\in\mathbb{N}}$ be a supernatural number. We claim that $Q(m) = G$. Assume $t = \frac{x}{y} \in Q(m)$ with x and y relatively prime. Write $y = p_1^{s_1} p_2^{s_2} \ldots p_K^{s_K}$. It follows that $s_1 \leq m_1, \ldots, s_K \leq m_K$. This implies that $\frac{1}{p_1^{s_1}}, \ldots, \frac{1}{p_K^{s_K}} \in G$. Let $y_j = \frac{y}{p_j^{s_j}}$ for $j = 1, 2, \ldots, K$. Then the y_j's are mutually relatively prime, and so by Bezout's identity there exists integers $d_1, \ldots, d_K$ such that

$$1 = d_1 y_1 + \ldots d_K y_K.$$

   Hence

$$t = \frac{x}{y} = \frac{x(d_1 y_1 + \ldots d_K y_K)}{y} = x d_1 \frac{1}{p_1^{s_1}} + \ldots x d_K \frac{1}{p_K^{s_K}} \in G.$$

Now assume $t = \frac{x}{y} \in G$, with y written as above. Then as x and y are relatively prime, there are integers a and b such that $1 = ax + by$. Then

$$\frac{1}{p_j^{s_j}} = \frac{y_j}{y} = \frac{y_j(ax+by)}{y} = y_j(at+b) \in G.$$

Thus $s_j \leq m_j$ for all $j = 1, \ldots, K$ and so $t \in Q(m)$.

3. If $n = m$ the isomorphism is just the identity map. Now assume we have an isomorphism $f : Q(m) \to Q(n)$ mapping 1 to 1. This condition forces f to be the identity map, and hence $Q(m) = Q(n)$. From the proof of 2, using (2.23), we may retrieve n from $Q(n)$ uniquely and so $n = m$.

4. If $f : Q(m) \to Q(n)$ is an isomorphism then (modulo replacing f with $-f$ if necessary) we have that $f(1)$ is a positive fraction. Hence there exists positive integers k_1, k_2 such that $f(k_1) = k_2$. So we have a composition of isomorphisms

$$Q(k_1 m) \xrightarrow{\cdot k_1} Q(m) \xrightarrow{f} Q(n) \xrightarrow{\cdot(k_2)^{-1}} Q(k_2 n)$$

mapping 1 to 1, and hence by 3, $k_1 m = k_2 n$.

Conversely, if $k_1 m = k_2 n$ for some natural numbers k_1, k_2, then we have the following composition of isomorphisms:

$$Q(m) \xrightarrow{\cdot(k_1)^{-1}} Q(k_1 m) \xrightarrow{\mathrm{id}} Q(k_2 n) \xrightarrow{\cdot(k_2)} Q(n).$$

□

We now state the final result of this subsection, which tells us what the K_0 group of a UHF-algebra is.

Proposition 2.3.23. *Let $(A, \{\mu_i\}_{i\in\mathbb{N}})$ be a UHF-algebra whose building blocks are $\{M_{n_i}(\mathbb{C})\}_{i\in\mathbb{N}}$, and whose unital connecting maps are $\{\phi_i\}_{i\in\mathbb{N}}$. For each i, we have $n_i = \prod_{j=1}^{\infty} p_j^{n_{i,j}}$ for elements $n_{i,j} \in \mathbb{N} \cup \{0\}$. We define the supernatural number associated to A by $m = \{m_j\}_{j\in\mathbb{N}}$ where*

$$m_j = \sup_{i\in\mathbb{N}}\{n^{i,j}\}.$$

Then

$$Q(m) = \bigcup_{i\in\mathbb{N}} n_i^{-1}\mathbb{Z} \tag{2.24}$$

and

$$(K_0(A), [1_A]_0) \cong (Q(m), 1).$$

The supernatural number associated to A is unique.

Proof. Let $t = \frac{x}{y} \in Q(m)$, with x and y relatively prime and $y = p_1^{s_1} \dots p_K^{s_K}$. Then $s_j \leq m_j$ for $j = 1, 2, \dots, K$. Then as $\{n^{i,j}\}_{i=1}^{\infty}$ is increasing because $n_i | n_{i+1}$, we can find an $i \in \mathbb{N}$ such that $s_j \leq n_{i,j}$ for $j = 1, 2, \dots, K$. So y divides n_i and so $t \in n_i^{-1}\mathbb{Z}$. Conversely, because $n_i^{-1} \in Q(m)$ for all $i \in \mathbb{N}$ by the fact that $n^{i,j} \leq m_j$ for all $i \in \mathbb{N}$, it follows that $\bigcup_{i \in \mathbb{N}} n_i^{-1}\mathbb{Z} \subseteq Q(m)$.

Let τ_i be the normalized trace on $M_{n_i}(\mathbb{C})$, in other words $\tau_i = \frac{1}{n_i}\mathrm{tr}$. By Example 2.2.48 we have that $K_0(\tau_i) : K_0(M_{n_i}(\mathbb{C})) \to n_i^{-1}\mathbb{Z}$ is an isomorphism. Let $(G, \{\alpha_i\}_{i \in \mathbb{N}})$ be the inductive limit of the inductive sequence with building blocks $\{K_0(M_{n_i}(\mathbb{C}))\}_{i \in \mathbb{N}}$ and connecting maps $\{K_0(\phi_i)\}_{i \in \mathbb{N}}$. Then we have a commutative diagram for each $i \in \mathbb{N}$:

$$\begin{tikzcd} & K_0(M_{n_i}(\mathbb{C})) \arrow[dl, "K_0(\mu_i)"'] \arrow[d, "\alpha_i"] \arrow[dr, "K_0(\tau_i)"] & \\ K_0(A) \arrow[r, "f^{-1}"'] & G \arrow[r, "g"'] & Q(m) \end{tikzcd} \tag{2.25}$$

The left hand commutative triangle in (2.25) is the commutative diagram (2.22) in Proposition 2.3.17. The existence of a group homomorphism g making the right hand triangle commute in (2.25) is due to the universal property for inductive limits (as in 2 in Definition 2.3.6) seeing that we have that $K_0(\tau_{i+1}) \circ K_0(\phi_i) = K_0(\tau_i)$ for all $i \in \mathbb{N}$. The fact that $K_0(\tau_i)$ is an isomorphism, together with 1 in Proposition 2.3.13 allows us to conclude that g is an isomorphism. Hence we have an isomorphism $K_0(A) \to Q(m)$. The commutativity of (2.25) can be used to show that this isomorphism maps $[1_A]_0$ to 1.

For the final claim of the proposition, if A' is another UHF-algebra isomorphic to A with associated supernatural number m', then the K_0 groups agree by Proposition 2.2.46 and thus we have an isomorphism $(Q(m), 1) \cong (Q(m'), 1)$, and so 3 in Lemma 2.3.22 gives that $m = m'$. $\square$

2.3.3 Inductive Limit Cartan Subalgebras and AX-algebras

In this subsection we define the main mathematical structure that is of interest to us in this thesis, namely that of an inductive limit Cartan subalgebra in an inductive limit C^*-algebra. We also define what existence and uniqueness means for such a structure. These definitions are, to the best of our knowledge, original. We will also define a class of inductive limit C^*-algebras, namely the AX-algebras, which will be class of C^*-algebras we wish to discover the existence and uniqueness of inductive limit Cartan subalgebras for. We will highlight some properties that these AX-algebras have that will be useful for us later. Unless otherwise stated, the definitions and results written in this subsection were written down without consultation from a reference, although it is highly probable they are to be found in many references.

Definition 2.3.24. Let c be a class consisting of C^*-algebras. Then a *c-inductive limit C^*-algebra* is a C^*-algebra A arising as an inductive limit C^*-algebra formed by building blocks $\{A_n\}_{n\in\mathbb{N}}$ belonging to c, and connecting maps $\{\phi_n\}_{n\in\mathbb{N}}$. A *$c$-inductive limit Cartan subalgebra* C of a c-inductive limit C^*-algebra A is a Cartan subalgebra $C \subseteq A$ where A arises as a c-inductive limit C^*-algebra of building blocks $\{A_n\}_{n\in\mathbb{N}}$ and connecting maps $\{\phi_n\}_{n\in\mathbb{N}}$, and C arises as an inductive limit C^*-algebra of building blocks $\{C_n\}_{n\in\mathbb{N}}$ and the same connecting maps $\{\phi_n\}_{n\in\mathbb{N}}$, where each (A_n, C_n) is a Cartan pair. In such a situation, we may call (A, C) a *c-inductive limit Cartan pair*. If c is the class consisting of any C^*-algebra we drop the c prefix.

Definition 2.3.25. We say that a c-inductive limit C^*-algebra A has a *unique c-*inductive limit Cartan subalgebra C if for every other c-inductive limit Cartan pair (B, D) with $B \cong A$ there is an isomorphism of Cartan pairs $(A, C) \cong (B, D)$.

Remark 2.3.26. The reasons we want to restrict our attention to inductive limit C^*-algebras and Cartan subalgebras whose building blocks belong to a specified class, rather than arbitrary building blocks, are many. Firstly, if one does not restrict to a certain class of allowable building blocks, then every C^*-algebra A with a Cartan subalgebra C is an inductive limit C^*-algebra with an inductive limit Cartan subalgebra, since we may just take the constant sequence $\{A\}_{n\in\mathbb{N}}$ for building blocks and the identity map sequence $\{\mathrm{id}\}_{n\in\mathbb{N}}$ as connecting maps, and for the inductive limit Cartan subalgebra the constant sequences $\{C\}_{n\in\mathbb{N}}$ and $\{\mathrm{id}\}_{n\in\mathbb{N}}$. This would not be useful in the pursuit of building Cartan subalgebras out of specific building blocks.

Secondly, for the purpose of studying the uniqueness of a method which constructs Cartan subalgebras in specific inductive limit C^*-algebras, one would need to consider only those Cartan subalgebras which arise from similar building blocks as the C^*-algebra in question. For example, the CAR-algebra (see Chapter III in [16]) has a Cartan subalgebra built as an inductive limit of the diagonals of the building blocks (see Example 2.3.29 below), but it also has a Cartan subalgebra that is not an inductive limit of Cartan subalgebras of finite dimensional C^*-algebras (this is seen by Blackadar's constructions in [12]). Hence it is fruitless to talk about uniqueness of Cartan subalgebras when allowing arbitrary building blocks, as then Cartan subalgebras of AF-algebras would not be unique (up to Cartan pair isomorphism). However, as we shall see in Chapter 5, the Cartan subalgebras of AF-algebras which are inductive limits of subalgebras in finite dimensional building blocks are unique. Thus, specifying the building blocks allowable is the right approach.

Definition 2.3.27. Let c_F be the class of finite dimensional C^*-algebras, in other words finite direct sums of full matrix algebras, which we will also call *AF-building blocks*. Let c_I be the class of C^*-algebras of the form $\bigoplus_{j=1}^{N} C([0,1]) \otimes M_{n_j}$ for some $N, n_j \in \mathbb{N}$, which we will also call *AI-building blocks*. Let $c_{\mathbb{T}}$ be the class of C^*-algebras of the form $\bigoplus_{j=1}^{N} C(\mathbb{T}) \otimes M_{n_j}$ for some $N, n_j \in \mathbb{N}$, which we will also call *A$\mathbb{T}$-building blocks*. More generally, for a compact Hausdorff space X, let c_X be the class of C^*-algebras of the form $\bigoplus_{j=1}^{N} C(X) \otimes M_{n_j}$ for some $N, n_j \in \mathbb{N}$, which we will also call *AX-building blocks*. Note that when only a single summand is present, we will sometimes refer to it as an *X-algebra* (and interval-algebra for the AI case, circle-algebra for the A$\mathbb{T}$ case).

Now we define AF, AI, A$\mathbb{T}$ and more generally AX-algebras. A thorough discussion of the AF, AI, and A$\mathbb{T}$-algebras can be found in [67].

Definition 2.3.28. An *AF-algebra* is a c_F-inductive limit C^*-algebra. An *AI-algebra* is a c_I-inductive limit C^*-algebra. An *A$\mathbb{T}$-algebra* is a $c_{\mathbb{T}}$-inductive limit C^*-algebra. More generally, for a compact Hausdorff space X, an *AX-algebra* is a c_X-inductive limit C^*-algebra. By an *AX-Cartan subalgebra* we shall mean a c_X-inductive limit Cartan subalgebra of an AX-algebra (with analogous definitions for AF, AI, and A$\mathbb{T}$-Cartan subalgebras).

Example 2.3.29. Every unital AF-algebra has an AF-Cartan subalgebra. The construction is due to Voiculescu and Strătilă in Chapter 1 of [72]. Indeed, if the AF-algebra is the inductive limit $\varinjlim(A_n, \phi_n)$ where each A_n is a finite dimensional C^*-algebra, and each ϕ_n a unital injective connecting map, then let C_1 be any Cartan subalgebra of A_1 (for example, take the diagonal subalgebra), and inductively define the Cartan subalgebra

$$C_{n+1} = C^*(\phi_n(C_n), D_{n+1})$$

where D_{n+1} is an arbitrary masa in $\phi(A_n)' \cap A_{n+1}$. The details showing that this indeed defines a Cartan subalgebra for every $n \in \mathbb{N}$ are in Chapter 1 of [72]. The desired AF-inductive limit Cartan subalgebra is then $\varinjlim(C_n, \phi_n)$. Whether AF-Cartan subalgebras are unique in AF-algebras will be answered in Chapter 5.

Definition 2.3.30. For X a compact Hausdorff space, we shall refer to an AX-Cartan subalgebra of an AX-algebra that arises as an inductive limit of the diagonal subalgebras of the AX-building blocks as a *diagonal Cartan subalgebra.*

Definition 2.3.31. Let $A \in c_X$ for a compact Hausdorff space X. Let i_j denote the canonical inclusion of the j^{th} summand of A into A, and let Π_i be the canonical projection onto the i^{th} summand. Given a *-homomorphism $\phi : A \to B$ where $A = \bigoplus_{j=1}^{N} A_j, B = \bigoplus_{i=1}^{M} B_i \in c_X$, we define

$$\phi_{ij} = \Pi_i \circ \phi \circ i_j : A_j \to B_i, \quad \phi_i = \Pi_i \circ \phi : A \to B_i.$$

Lemma 2.3.32. *Let X be a compact connected Hausdorff space. Let $A, B \in c_X$ with $A = \bigoplus_{j=1}^{N} C(X) \otimes M_{n_j}$, $B = \bigoplus_{i=1}^{M} C(X) \otimes M_{m_i}$, and let $\phi : A \to B$ be a unital *-homomorphism. Let $\{e^j_{pq}\}$ denote the system of standard matrix units for $\bigoplus_{j=1}^{N} M_{n_j}$, and $\{f^i_{uv}\}$ the one for $\bigoplus_{i=1}^{M} M_{m_i}$.*

For $j \in \{1, 2, \ldots, N\}$ and $i \in \{1, 2, \ldots, M\}$, and for arbitrary $x \in X$ and $q \in \{1, 2, \ldots, n_j\}$ set

$$k_{ij} = \mathrm{tr}(\phi_i(1 \otimes e^j_{qq})(x)).$$

Then k_{ij} is independent of the choice of x or q, and we have

$$\sum_{j=1}^{N} n_j k_{ij} = m_i, \tag{2.26}$$

$$\sum_{i=1}^{M} \sum_{j=1}^{N} n_j k_{ij} = \sum_{i=1}^{M} m_i. \tag{2.27}$$

Proof. For any $q_1, q_2 \in \{1, 2, \ldots, n_j\}$ we have that the projection $1 \otimes e^j_{q_1 q_1}$ is Murray-von Neumann equivalent to the projection $1 \otimes e^j_{q_2 q_2}$. Hence it follows that the set

$$\{\phi_i(1 \otimes e^j_{qq})(x) : 1 \leq q \leq n_j\}$$

is a set of mutually orthogonal Murray von-Neumann equivalent projections in M_{m_i}, hence each having the same trace $k_{ij}(x)$ (by Example 2.2.15). As $x \to \mathrm{tr}(\phi_i(1 \otimes e^j_{qq})(x))$ is continuous into $\mathbb{Z}$, and X is connected, this map is constant, and so we may talk of k_{ij}, independent of x or q.

Now note that by the fact that ϕ is unital we have that

$$m_i = \mathrm{tr}(\phi_i(1_A)(x)) = \mathrm{tr}\left(\sum_{j=1}^{N}\sum_{q=1}^{n_j}\phi_i(1\otimes e^j_{qq})(x)\right) = \sum_{j=1}^{N}\sum_{q=1}^{n_j}k_{ij} = \sum_{j=1}^{N}n_jk_{ij}, \quad (2.28)$$

yielding (2.26). This directly yields (2.27). □

Definition 2.3.33. In the situation of Lemma 2.3.32, we will call the set $\{k_{ij} : 1 \leq j \leq N, 1 \leq i \leq M\}$ an *index system with respect to* ϕ, or just an *index system* when the context is clear.

Lemma 2.3.34. *In the situation of Lemma 2.3.32, let* $\{k_{ij} : 1 \leq j \leq N, 1 \leq i \leq M\}$ *be the index system. For* $j \in \{1, \ldots, N\}, i \in \{1, \ldots, M\}$ *and* $p, q \in \{1, \ldots, n_j\}$, *define*

$$\sigma_{i,j-1} = \sum_{z=1}^{j-1} n_z k_{iz},$$

and

$$c^{ij}_{pq} = \sum_{\rho=1}^{k_{ij}} f^i_{\sigma_{i,j-1}+p+(\rho-1)n_j,\sigma_{i,j-1}+q+(\rho-1)n_j}. \quad (2.29)$$

Then the elements $\{c^{ij}_{pq}\}$ *have the property of matrix units, meaning that*

$$c^{i_1j_1}_{p_1q_1}c^{i_2j_2}_{p_2q_2} = 0 \text{ if } i_1 \neq i_2, \text{ or } j_1 \neq j_2, \text{ or } q_1 \neq p_2, \text{ and equals } c^{i_1j_1}_{p_1q_2} \text{ otherwise}, \quad (2.30)$$

and

$$(c^{ij}_{pq})^* = c^{ij}_{qp}. \quad (2.31)$$

Proof. First note that

$$\sigma_{i,j-1} + p + (\rho - 1)n_j \leq \sigma_{i,j-1} + n_j + (k_{ij} - 1)n_j = \sigma_{i,j} \leq m_i,$$

where the last inequality holds by (2.26). Hence (2.29) is well-defined.

From (2.29) it is clear that if i_1 is not i_2 then the product in (2.30) is 0. Note that if $j_1 < j_2$ then $\sigma_{i,j_1-1}+q_1+(\rho-1)n_{j_1} \neq \sigma_{i,j_2-1}+p_2+(\omega-1)n_{j_2}$ for all $\rho \in \{1, \ldots, k_{ij_1}\}$, $\omega \in \{1, \ldots, k_{ij_2}\}$ and any $q_1 \in \{1, \ldots, n_{j_1}\}$, $p_2 \in \{1, \ldots, n_{j_2}\}$. Indeed we have

$$\sigma_{i,j_1-1}+q_1+(\rho-1)n_{j_1} \leq \sigma_{i,j_1-1}+k_{ij_1}n_{j_1} = \sigma_{i,j_1} \leq \sigma_{i,j_2-1} < \sigma_{i,j_2-1}+p_2+(\omega-1)n_{j_2}.$$

Hence if $j_1 \neq j_2$ then by (2.29) the product in (2.30) is 0. So assume $i_1 = i_2$ and

$j_1 = j_2$ in (2.30). It is then clear that unless $q_1 = p_2$ the product in (2.30) is 0, and otherwise is $c^{i_1 j_1}_{p_1 q_2}$. That (2.31) holds is clear. $\square$

Definition 2.3.35. We will call $\{c^{ij}_{pq}\}$, as in Lemma 2.3.34, a *system of matrix units with respect to the index system.* We will let $c^j_{pq} = \sum\limits_{i=1}^{M} c^{ij}_{pq}$

Remark 2.3.36. A good way to think about the index system is to recall the situation for AF-algebras. Indeed, for such algebras we have connecting maps between finite dimensional C^*-algebras, in other words finite direct sums of full matrix algebras with entries in $\mathbb{C}$. Up to unitary equivalence the connecting map maps the summands of the domain block-diagonally into the summands of the codomain, each with a certain multiplicity. The index system element k_{ij} is thus our higher-dimensional analogue of the multiplicity of imbedding the j^{th} summand of the domain into the i^{th} summand of the codomain, and our system of matrix units with respect to this index system is a natural reordering of how this imbedding takes place, analogous to the aforementioned imbeddings between finite dimensional building blocks.

Lemma 2.3.37. *In the situation of Lemma 2.3.32 we have that for every $x \in X$ there exists a unitary $V_i(x) \in M_{m_i}$ that conjugates $\Pi_i(c^{ij}_{pq})$ to $\phi_i(1 \otimes e^j_{pq})(x)$ for all $j \in \{1, 2, \ldots, N\}$, and $p, q \in \{1, 2, \ldots, n_j\}$, where $\{c^{ij}_{pq}\}$ is the system of matrix units with respect to the index system. The unitary $V(x) = \bigoplus\limits_{i=1}^{M} V_i(x)$ in $\bigoplus\limits_{i=1}^{M} M_{m_i}$ conjugates c^j_{pq} to $\phi(1 \otimes e^j_{pq})(x)$ for all $j \in \{1, 2, \ldots, N\}, p, q \in \{1, 2, \ldots, n_j\}$.*

Proof. Fix $x \in X$. Let $i \in \{1, 2, \ldots, M\}$ and $j \in \{1, 2, \ldots, N\}$. Let v^{ij}_1 be the partial isometry in M_{m_i} that witnesses the Murray-von Neumann equivalence between $\Pi_i(c^{ij}_{11})$ and $\phi_i(1 \otimes e^j_{11})(x)$ (both these elements have the same trace k_{ij} and hence are indeed Murray-von Neumann equivalent by Example 2.2.15). For $p \in \{1, 2, \ldots, n_j\}$ set

$$v^{ij}_p = \phi_i(1 \otimes e^j_{p1})(x) v^{ij}_1 \Pi_i(c^{ij}_{1p}) \in M_{m_i}.$$

Using Lemma 2.3.34 one obtains that

$$(v^{ij}_p)^* v^{ij}_p = \Pi_i(c^{ij}_{pp}), \quad \text{and} \quad v^{ij}_p (v^{ij}_p)^* = \phi_i(1 \otimes e^j_{pp})(x).$$

Set

$$V_i(x) = \sum_{j=1}^{N} \sum_{p=1}^{n_j} v^{ij}_p.$$

One can use the fact that ϕ is unital together with (2.26) to see that $V_i(x)$ is a unitary in M_{m_i}. Then it is straightforward to compute that

$$V_i(x)\Pi_i(c^{ij}_{pq})V_i(x)^* = \phi_i(1 \otimes e^j_{pq})(x)$$

The last statement of the lemma follows easily. $\square$

The following definitions for maximal homogeneity are inspired by Definitions 1.1 and 1.4 in [75], however we extend them to direct sums:

Definition 2.3.38. Let $A = \bigoplus_{j=1}^{N} C(X) \otimes M_{n_j} \in c_X$, for a compact Hausdorff space X. A C^*-subalgebra $C \subset A$ will be called *maximally homogeneous* if it is Abelian, contains the center of A, and satisfies

$$\dim(C(x)) = \sum_{j=1}^{N} n_j,$$

(vector space dimension) for all $x \in X$.

Given $B = \bigoplus_{i=1}^{M} C(X) \otimes M_{m_i} \in c_X$, a unital *-homomorphism $\phi : A \to B$ will be called *maximally homogeneous* if

$$\dim(\phi(A)(x)) = \sum_{i=1}^{M}\sum_{j=1}^{N} n_j^2 k_{ij},$$

(vector space dimension) for all $x \in X$. Here $\{k_{ij}\}$ is the index system with respect to ϕ.

There is a particular class of *-homomorphisms between AX-algebra building blocks that we are interested in. The definition is inspired by [76].

Definition 2.3.39. Let $A = \bigoplus_{j=1}^{N} C(X) \otimes M_{n_j} \in c_X$, $B = \bigoplus_{i=1}^{M} C(Y) \otimes M_{m_i} \in c_Y$, where X and Y are compact Hausdorff spaces. A unital *-homomorphism $\phi : A \to B$ will be called a *standard map* (or sometimes a *standard connecting map*, or a *standard *-homomorphism*) if there exists continuous functions

$$g^{ij}_s : Y \to X \ \text{ where } \ i \in \{1, \ldots, M\}, j \in \{1, \ldots, N\}, s \in \{1, \ldots, k_{ij}\},$$

(here the $\{k_{ij}\}$ is the index system with respect to ϕ) such that if $f = (f_1, \ldots, f_N) \in$

A, ϕ_i takes the form

$$\phi_i(f) = \mathrm{diag}(f_1(g_1^{i1}), f_1(g_2^{i1}), \ldots, f_1(g_{k_{i1}}^{i1}), f_2(g_1^{i2}), \ldots, f_2(g_{k_{i2}}^{i2}), \ldots, f_N(g_{k_{iN}}^{iN})), \tag{2.32}$$

up to permutation of these diagonal entries. The continuous functions $\{g_s^{ij}\}$ will be called *eigenvalue functions.*

Example 2.3.40 (AF-building blocks)**.** Unital *-homomorphisms between finite dimensional C^*-algebras are unitary conjugates of standard maps, with eigenvalue functions having domain and codomain a singleton.

Lemma 2.3.41. *In the situation of Definition 2.3.39, if for each $y \in Y$ and $j \in \{1, \ldots, N\}$, we have that the elements of the set*

$$\{g_s^{ij}(y) : 1 \leq i \leq M, 1 \leq s \leq k_{ij}\}$$

are distinct, then ϕ is maximally homogeneous.

Proof. Evaluating (2.32) at y we see that by varying f we generate a C^*-subalgebra of M_{m_i} of dimension $\sum\limits_{j=1}^{N} n_j^2 k_{ij}$. This holds if we take direct sums as the eigenvalue functions for a fixed j are distinct across the index i as well. Hence the dimension of the image of ϕ at the point y is $\sum\limits_{i=1}^{M}\sum\limits_{j=1}^{N} n_j^2 k_{ij}$, as desired. $\square$

2.3.4 Approximate Intertwining

It will be useful for us in Chapter 4 to express AX-algebras, for certain connected compact Hausdorff spaces X, as an inductive limit with connecting maps taking a specific form. In this section we study a tool that allows us to view two inductive limit C^*-algebras as the same (up to isomorphism). This turns out to hold when there is an approximate intertwining of the building blocks. These should, for the sake of our purposes, be finitely generated. This tool will then allow us to treat AX-algebras as having connecting maps that are more useful for us. The methods we present here can be found summatively in [78], but the original work was done by Elliott in [19].

Throughout this subsection, fix the following setup. Assume $(A, \{\mu_n\}_{n\in\mathbb{N}})$ arises as the inductive limit of the sequence

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots , \tag{2.33}$$

and assume $(B, \{\rho_n\}_{n\in\mathbb{N}})$ is the inductive limit of the sequence

$$B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} \cdots , \tag{2.34}$$

where the building block A_n is generated as a C^*-algebra by $F_n \subseteq A_n$, and the building block B_n is generated as a C^*-algebra by $G_n \subseteq B_n$, for all $n \in \mathbb{N}$.

Lemma 2.3.42. *Assume $\{N(n)\}_{n\in\mathbb{N}}$ and $\{M(n)\}_{n\in\mathbb{N}}$ are strictly increasing subsequences in $\mathbb{N}$. Let $\{\delta_n\}_{n\in\mathbb{N}}$ be a sequence in $[0,\infty)$ that is summable. For every $n \in \mathbb{N}$ let $\alpha_n : A_{N(n)} \to B_{M(n)}$ be a *-homomorphism satisfying*

$$\|\psi_{M(n+1),M(n)} \circ \alpha_n \circ \phi_{N(n),N(k)}(x) - \alpha_{N(n+1)} \circ \phi_{N(n+1),N(k)}(x)\| < \delta_n, \tag{2.35}$$

for all $k \leq n$ and $x \in F_{N(k)}$. Then the sequence

$$\{\rho_{N(n)} \circ \alpha_n \circ \phi_{N(n),k}(a)\}_{n\in\mathbb{N}} \tag{2.36}$$

*converges in B for all $k \in \mathbb{N}$ and $a \in A_k$. Furthermore there exists a *-homomorphism $\alpha : A \to B$ satisfying*

$$\alpha(\mu_k(a)) = \lim_{n\to\infty} \rho_{N(n)} \circ \alpha_n \circ \phi_{N(n),k}(a) \tag{2.37}$$

for all $k \in \mathbb{N}$ and $a \in A_k$.

Proof. Condition (2.35) implies that the sequence (2.36) is Cauchy for elements $a \in F_k$. Indeed, consider the diagram

$$\begin{tikzcd}
A_{N(k)} \arrow[r, "\phi_{N(k+1),N(k)}"] \arrow[d, "\alpha_k"] & A_{N(k+1)} \arrow[r, "\phi_{N(k+2),N(k+1)}"] \arrow[d, "\alpha_{k+1}"] & A_{N(k+2)} \arrow[r, "\phi_{N(k+3),N(k+2)}"] \arrow[d, "\alpha_{k+2}"] & \cdots \\
B_{N(k)} \arrow[r, "\psi_{N(k+1),N(k)}"] & B_{N(k+1)} \arrow[r, "\psi_{N(k+2),N(k+1)}"] & B_{N(k+2)} \arrow[r, "\psi_{N(k+3),N(k+2)}"] & \cdots
\end{tikzcd}$$

Then starting with $\phi_{N(k),k}(a) \in A_{N(k)}$ for $a \in F_k$, if we go a certain large number of steps R to the right, then down, and across; it will be norm close to going $R-1$ steps to the right, down, one step to the right, then across; and how close the difference is is determined by (2.35). Similarly, the latter path is close to going $R-2$ steps to the right, down, one step to the right, then across. By the triangle inequality we may then bound the difference of going some large R_1 to the right, down and then across with going some large R_2 to the right, down and then across. As long as R_1, R_2 are greater than a fixed R, (2.35) ensures we may get a difference smaller than a predefined ϵ. Hence (2.36) is a Cauchy sequence for $a \in F_k$ and so converges in B. As F_k generates A_k this holds for $a \in A_k$ as well.

Define $\alpha'_k : A_k \to B$ by $\alpha'_k(a) = \lim\limits_{n\to\infty} \rho_{N(n)} \circ \alpha_n \circ \phi_{N(n),k}(a)$, and since $\alpha'_{k+1} \circ \phi_k = \alpha'_k$ it follows by 2 in Definition 2.3.6 that there is a unique *-homomorphism $\alpha : A \to B$ satisfying (2.37). $\square$

Definition 2.3.43. By an *approximate intertwining* of the sequences (2.33) and (2.34), we mean that there exists increasing subsequences $\{N(n)\}_{n\in\mathbb{N}}$ and $\{M(n)\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ and *-homomorphisms

$$\alpha_n : A_{N(n)} \to B_{M(n)}, \quad \beta_n : B_{M(n)} \to A_{N(n+1)}$$

satisfying

$$\|\beta_n \circ \alpha_n(x) - \phi_{N(n+1),N(n)}(x)\| < 2^{-n}$$

for all $x \in \bigcup\limits_{k=1}^{n} \phi_{N(n),N(k)}(F_{N(k)})$ or $x \in \bigcup\limits_{k=1}^{n-1} \beta_{n-1} \circ \psi_{M(n-1),M(k)}(G_{M(k)})$, and,

$$\|\alpha_{n+1} \circ \beta_n(x) - \psi_{M(n+1),M(n)}(x)\| < 2^{-n}$$

for all $x \in \bigcup\limits_{k=1}^{n} \psi_{M(n),M(k)}(G_{M(k)})$ or $x \in \bigcup\limits_{k=1}^{n} \alpha_n \circ \phi_{N(n),N(k)}(F_{N(k)})$.

One may represent an approximate intertwining using the following diagram:

$$\begin{tikzcd}
A_{N(1)} \arrow[r, "\phi_{N(2),N(1)}"] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "2^{-1}", pos=0.25] \arrow[dr, phantom, "2^{-1}", pos=0.75] & A_{N(2)} \arrow[r, "\phi_{N(3),N(2)}"] \arrow[d, "\alpha_2"'] \arrow[dr, phantom, "2^{-2}", pos=0.25] \arrow[dr, phantom, "2^{-2}", pos=0.75] & A_{N(3)} \arrow[r, "\phi_{N(4),N(3)}"] \arrow[d, "\alpha_3"'] \arrow[dr, phantom, "2^{-3}", pos=0.25] \arrow[dr, phantom, "2^{-3}", pos=0.75] & \cdots \\
B_{M(1)} \arrow[r, "\psi_{M(2),M(1)}"'] \arrow[ur, "\beta_1"] & B_{M(2)} \arrow[r, "\psi_{M(3),M(2)}"'] \arrow[ur, "\beta_2"] & B_{M(3)} \arrow[r, "\psi_{M(4),M(3)}"'] \arrow[ur, "\beta_3"] & \cdots
\end{tikzcd} \tag{2.38}$$

Proposition 2.3.44. *If the sequences* (2.33) *and* (2.34) *are approximately intertwined then the C^*-algebras A and B are *-isomorphic.*

Proof. Consider (2.38). Starting at some element $x \in F_{N(k)}$, consider the difference between going R steps to the right, down, then one step to the right versus going $R+1$ steps to the right then down. This is equivalent to starting at $\phi_{N(k+R),N(k)}(x)$ and considering the difference between going to the right and down versus down and to the right. Using the triangle inequality it is easy to see that the norm of this difference will be bounded by some appropriate negative power of 2. Hence the conditions of Lemma 2.3.42 are met and we get *-homomorphisms $A \to B$ and $B \to A$. Using (2.37) it is easy to see that these *-homomorphisms are inverse to each other. $\square$

Definition 2.3.45. Let A and B be C^*-algebras with B unital. Let $u \in B$ be a unitary. The map $\mathrm{Ad}(u) : B \to B$ is defined by $\mathrm{Ad}(u)(b) = ubu^*$ for all $b \in B$. We say that the *-homomorphisms $\phi, \psi : A \to B$ are *unitarily equivalent*, written as $\phi \sim_u \psi$, if there is a unitary $u \in B$ such that $\psi = \mathrm{Ad}(u) \circ \phi$. We say that ϕ and ψ are *approximately unitarily equivalent*, written $\phi \sim_{au} \psi$ if there is a sequence of unitaries $\{u_n\}_{n\in\mathbb{N}}$ in B such that $\psi = \lim\limits_{n\to\infty} \mathrm{Ad}(u_n) \circ \phi$ pointwise in norm.

Proposition 2.3.46. *Assume that the building blocks of* (2.34) *are the same as those of* (2.33)*, and that all the building blocks are unital, with unital connecting maps. Assume further that all of them are finitely generated (so the sets $F_n = G_n$ defined are all finite.) If $\phi_n \sim_{au} \psi_n$ for all $n \in \mathbb{N}$ then $A \cong B$.*

Proof. It suffices, by Proposition 2.3.44 to check that the sequences are approximately intertwined. Define $\alpha_1 : A_1 \to A_1$ as the identity map id. Since F_1 is finite there is a unitary $v_1 \in A_2$ such that $\|\phi_1(x) - v_1\psi_1(x)v_1^*\| < 2^{-1}$ for all $x \in F_1$ (for every $x \in F_1$ there is a u_{n_x} giving the desired norm difference in the sequence of unitaries which witness the approximate unitary equivalence of the maps, and since F_1 is finite we can just take the maximum such n_x). Define $\beta_1 = \mathrm{Ad}(v_1)\circ\psi_1 : A_1 \to A_2$. Define $\alpha_2 = \mathrm{Ad}(v_1^*) : A_2 \to A_2$. We get an approximately intertwined square

$$\begin{tikzcd}
A_1 \arrow[r, "\phi_1"] \arrow[d, "\mathrm{id}"'] & A_2 \arrow[d, "\mathrm{Ad}(v_1^*)"] \\
A_1 \arrow[r, "\psi_1"'] \arrow[ur, "\mathrm{Ad}(v_1)\circ\psi_1", "2^{-1}"'] \arrow[ur, phantom, "2^{-1}", shift left=4] & A_2
\end{tikzcd}$$

We may continue this process inductively. Indeed, assume we have approximately intertwined squares up to the n^{th} stage, with some map $\alpha_n = \mathrm{Ad}(W) : A_n \to A_n$ defined. We may let $v_n \in A_{n+1}$ be a unitary satisfying $\|\phi_n(x) - \mathrm{Ad}(v_n)\psi_n(x)\| < 2^{-(n-1)}$ for all x in the appropriate finite set as in Definition 2.3.43 (we may do this precisely because these sets are finite). Define $\beta_n = \mathrm{Ad}(v_n) \circ \psi_n \circ \mathrm{Ad}(W^*)$ and $\alpha_{n+1} = \mathrm{Ad}(\psi_n(W)v_n^*) : A_{n+1} \to A_{n+1}$, and we get the next approximately intertwined square. $\square$

2.4 Inverse Limits of the Unit Interval

In this section we explore the basics of inverse limits of the unit interval, which we will need in Chapter 5. Unless otherwise stated, the material here will be based on elements of Chapter 1 in [34].

Definition 2.4.1. Let $I = [0,1]$, the unit interval. Equip I^∞ with the product topology. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions $I \to I$. Then the *inverse limit* of I with respect to the maps $\{f_n\}_{n\in\mathbb{N}}$ is the set

$$\varprojlim([0,1], f_n) = \{(x_n) \in I^\infty : f_n(x_{n+1}) = x_n\}. \tag{2.39}$$

For each $m \in \mathbb{N}$ there is a map $\pi_m : \varprojlim([0,1], f_n) \to [0,1]$ given by sending (x_n) to x_m. The topology on $\varprojlim([0,1], f_n)$ is the coarsest topology making π_n continuous for every $n \in \mathbb{N}$ (in other words, the initial topology with respect to the π_n's). For $n > m$ we define $f_{m,n} = f_m \circ f_{m+1} \circ \ldots \circ f_n$, and for every $n \in \mathbb{N}$ we define $f_{n,n} = \mathrm{id}$.

Remark 2.4.2. In general, one can define inverse limits for topological spaces (see pages 134-135 in [50]). One has a sequence of spaces $\{X_n\}_{n\in\mathbb{N}}$ (or more generally a family indexed by an ordered relation) and continuous maps $f_n : X_{n+1} \to X_n$. Then one considers the product $\prod_{n=1}^{\infty} X_n$ with the product topology. The inverse limit $\varprojlim(X_n, f_n)$ is then defined similarly to (2.39), but consists rather of elements in $\prod_{n=1}^{\infty} X_n$. Similarly it is topologized via the initial topology with respect to all the projection maps. If the spaces have algebraic structure (group, groupoid, vector space et cetera) then one requires the arrows to be the ones appropriate for that category (group homomorphism, groupoid homomorphism, linear map et cetera).

Remark 2.4.3. Note that the topology we give the inverse limit in Definition 2.4.1 is exactly the subspace topology of the product topology on I^∞. Indeed, the subspace topology is the initial topology with respect to the inclusion $\varprojlim([0,1], f_n) \hookrightarrow I^\infty$, and the product topology on I^∞ is the initial topology with respect to the canonical projections $I^\infty \to I$. The composition of the inclusion followed by the canonical projections yield the projections π_n.

Then, using the universal property for initial topologies (see [25]) we can conclude that we have a homeomorphism $(\varprojlim([0,1], f_n), \tau_1) \xrightarrow{\mathrm{id}} (\varprojlim([0,1], f_n), \tau_2)$ where τ_1 is the initial topology with respect to the projections π_n and τ_2 is the initial topology with respect to inclusion into I^∞ (the subspace topology).

Definition 2.4.4. A topological space is called a *continuum* if it is a non-empty compact connected space.

We now state an important proposition regarding inverse limits of the unit interval, namely that they are Hausdorff continua:

Proposition 2.4.5. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions $I \to I$. Then $\varprojlim([0,1], f_n)$ is a Hausdorff continuum.*

Proof. For $m \in \mathbb{N}$, let $Z_m \subset I^\infty$ consist of those elements (x_n) such that for all $i \leq m$ we have $f_i(x_{i+1}) = x_i$. Then $Z_{m+1} \subseteq Z_m$ for all $m \in \mathbb{N}$ and $\varprojlim([0,1], f_n) = \bigcap_{m=1}^{\infty} Z_m$. Fix $m \in \mathbb{N}$, then note that Z_m is non-empty, compact and connected.

Indeed, it is non-empty as for any $x \in I$ we have

$$(f_{1,m}(x), f_{2,m}(x), \ldots, f_m(x), x, x, \ldots) \in Z_m.$$

By Tychonoff's theorem, I^∞ is compact and so to check that Z_m is compact it suffices to check that it is closed, or that $I^\infty \backslash Z_m$ is open. Pick a point $(x_n) \in I^\infty \backslash Z_m$ and note that there exists $i \leq m$ such that $f_i(x_{i+1}) \neq x_i$. Let U_1 and U_2 be disjoint open sets in $[0,1]$ separating $f_i(x_{i+1})$ and x_i respectively. Let U_3 be an open set in $[0,1]$ containing x_{i+1} such that $f_i(U_3) \subseteq U_1$. The set of all elements (y_n) of I^∞ which satisfy that $y_{i+1} \in U_3$ and $y_i \in U_2$ is an open set (in the product topology) containing (x_n) and disjoint from Z_m. Hence $I^\infty \setminus Z_m$ is open.

To see that Z_m is connected note that it is the image of the continuous map $I^\infty \to I^\infty$ given by sending (x_n) to (y_n) where $y_n = x_n$ for $n > m$ and $y_i = f_{i,m}(x_{m+1})$ for $i \leq m$. The image under a continuous map of a connected space is connected (see 4.22 in [68]), and since I^∞ is connected (see Theorem 253 in [34]) so is Z_m.

That $\varprojlim([0,1], f_n)$ is Hausdorff is trivial to check. Since $\varprojlim([0,1], f_n) = \bigcap_{m=1}^{\infty} Z_m$, Remark 2.4.3 implies that the subspace topology induced by this intersection (as a subspace of I^∞) is the topology for our inverse limit, and so since each Z_m is a continuum we have that $\varprojlim([0,1], f_n)$ is a continuum by Theorem 269 in [34]. □

Definition 2.4.6. A *separating point* of a connected space X is a point $p \in X$ such that $X \setminus \{p\}$ is not connected. A continuum X will be called an *arc* if it has only two points which are not separating points. A continuum X will be called *decomposable* if it is the union of two proper subcontinua. Otherwise it is *indecomposable*. A continuum X is called *degenerate* if it is a singleton, in which case it is indecomposable.

Remark 2.4.7. It is stated on page 5 in [34] that the definition of an arc is equivalent to that of being homeomorphic to $[0,1]$. The proof is referred to [5] or [6].

We will now give some examples of inverse limits of the unit interval, some of which will be useful for us in Chapter 5.

Example 2.4.8 (Degenerate continuum)**.** Consider the constant sequence $\{f_n\}_{n\in\mathbb{N}}$ where each $f_n = 1$, where 1 represents the constant map $I \to I$ sending $x \to 1$.

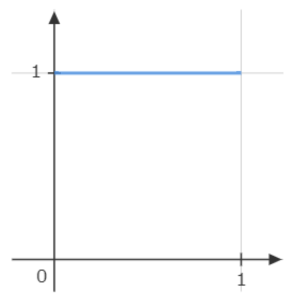


Figure 2.3: Map giving rise to a degenerate inverse limit.

Clearly $\varprojlim([0,1], f_n) = \{(1,1,1,\ldots)\}$.

Example 2.4.9 (Arc)**.** Let $f : I \to I$ be the identity function, $f(x) = x$ for all $x \in [0,1]$. Set $f_n = f$ for all $n \in \mathbb{N}$.

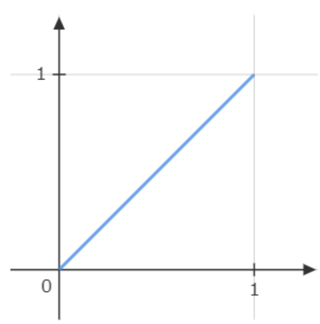


Figure 2.4: Map giving rise to an arc.

Then it is easy to see that $\varprojlim([0,1], f_n) = \{(x,x,x,\ldots) : x \in [0,1]\} \cong [0,1]$.

Example 2.4.10 (Indecomposable Continuum)**.** Let $f : I \to I$ be the function defined by

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le \frac{1}{2} \\ 2(1-x), & \text{if } \frac{1}{2} \le x \le 1 \end{cases}.$$

Set $f_n = f$ for all $n \in \mathbb{N}$.

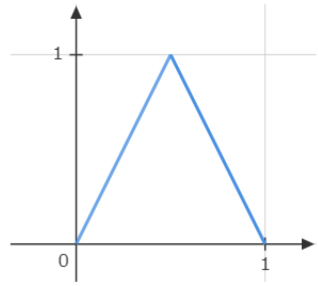


Figure 2.5: Map giving rise to an indecomposable continuum.

Then $\varprojlim([0,1], f_n)$ is indecomposable. Indeed, assume for a contradiction that $\varprojlim([0,1], f_n) = Y \cup Z$ with Y and Z proper subcontinua. If $\pi_m(Y) = I$ for all $m \in \mathbb{N}$ then it follows that $\varprojlim([0,1], f_n) = Y$ (see Theorem 10 in [34]). The same argument holds for Z. Hence there are natural numbers m_1, m_2 such that $\pi_{m_1}(Y) \neq I$ and $\pi_{m_2}(Z) \neq I$. Hence, as f is surjective, it follows that there is $n \in \mathbb{N}$ such that $\pi_n(Y) \neq I \neq \pi_n(Z)$.

If $0 \in \pi_{n+1}(Y)$ then it cannot be that $\frac{1}{2} \in \pi_{n+1}(Y)$ also as then $\pi_n(Y) = I$. Hence 1 cannot belong to $\pi_{n+1}(Y)$ as this space is connected which would force $\frac{1}{2} \in \pi_{n+1}(Y)$. Hence if neither $\frac{1}{2}$ nor 1 belong to $\pi_{n+1}(Y)$, they must belong to $\pi_{n+1}(Z)$ which again is a contradiction as this would force $\pi_n(Z) = I$. Hence it must be that $0 \in \pi_{n+1}(Z)$, which by the same line of argument, leads to a contradiction.

The argument presented in Example 2.4.10 can be easily generalized to the following proposition:

Proposition 2.4.11. *Let $f : I \to I$ such that there exists $x < y < z$ in I with either $f(x) = f(z) = 0$ and $f(y) = 1$ or $f(x) = f(z) = 1$ and $f(y) = 0$. Set $f_n = f$ for all $n \in \mathbb{N}$. Then $\varprojlim([0,1], f_n)$ is an indecomposable continuum.*

Chapter 3

Generalizing Renault's Theorem for Cartan Subalgebras

The aim of this chapter is to generalize Renault's main result in [64], which is stated as Theorem 2.1.82 in this thesis. This generalization will remove the second countability and separability conditions seen in Theorem 2.1.82, and will thus just look at effective groupoids rather than topologically principal ones. A short discussion with Jean Renault informed us that the assumption of second countability was mainly placed for convenience, without proper analysis of its requirement. A lot of the proofs in [64] make use of separation functions on topological spaces; which are standard when the space is second countable, as then it is paracompact and hence normal, and thus one can make use of the standard Urysohn lemma for separation. However in the non-second countable case one must make use of Urysohn type results for just locally compact spaces, and these are less standard.

The structure of this chapter will be to go through Renault's proofs in [64] and discuss them whilst highlighting how we may remove the second countability and separability assumptions. In Section 3.1 we show how a twisted étale effective (weaker than topologically principal) groupoid gives rise to a Cartan pair. This is a matter of going through Renault's proofs in [64] but without the topological principality and second countablity assumptions. The results can be obtained by separation results that work just as well for locally compact Hausdorff spaces (which are not necessarily second countable). In Section 3.2 we give the reverse procedure, namely how to obtain a twisted étale effective groupoid from a Cartan pair (which is not necessarily separable). In such a setting it follows that second countability and hence topological principality of the groupoid is no longer guaranteed as the C^*-algebra is not necessarily separable. In Section 3.3 we show that the two procedures, namely going from a twisted étale effective groupoid to a Cartan pair and vice versa, are, up to isomorphism, inverse to each other. This shows that the twisted groupoid which corresponds to a Cartan pair is unique and given by the Weyl twisted groupoid. Although the proofs of this section are based on the proofs in [64], our proofs have significantly more detail, and we try to spell out many of the claims in Renault's proofs that are at times not given justification. The preliminaries required for this chapter can all be found in Section 2.1 of this thesis.

The content of this chapter exists in a denser version in a preprint by this author, see [61]. Throughout this section, we are using the identification of $C_r^*(\mathcal{G},\Sigma)$ and $C_{0,r}(\mathcal{G}^0)$ with C^*-algebras living inside the set $C_0(\mathcal{G},\Sigma)$, as in Remark 2.1.73.

We obtained the main result of this chapter in 2018, however it has been obtained recently, and independently, by the authors of [41] (see Subsection 7.2 in their paper). Their result is a consequence of a more general theory of non-commutative Cartan subalgebras.

3.1 From Twisted Étale Groupoids to Cartan Pairs

We begin with some useful lemmas. Lemma 3.1.2 will be particularly useful as it provides the necessary separation results used by Renault in [64] but without having to allude to second countability.

Lemma 3.1.1. *Let $(\mathcal{G},\Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Let $f \in C_r^*(\mathcal{G},\Sigma)$, $h \in C_{0,r}(\mathcal{G}^0)$. Then, for $\sigma \in \Sigma$, we have that*

$$f * h(\sigma) = f(\sigma)h(s(\sigma)), \quad h * f(\sigma) = h(r(\sigma))f(\sigma).$$

Proof. This follows easily from Definition 2.1.71, □

Lemma 3.1.2. *Let X be a locally compact Hausdorff space. Then*

1. *Given a compact subset K of X and an open U such that $K \subset U \subset X$, there exists $b \in C_0(X)$ with $b \equiv 1$ on K, and 0 outside U.*
2. *Given a closed subset $C \subset X$ and a point $x \in X$ disjoint from C, there exists $b \in C_0(X)$ where $b(x) = 1$ and $b|_C \equiv 0$.*
3. *Given an open set $U \subset X$ containing a point x there exists an open set V containing x such that $\overline{V}$ is a compact subset of U.*

Proof. Claim 1 is Urysohn's lemma for locally compact Hausdorff spaces (see [69], 2.12). Since locally compact Hausdorff spaces are regular, we may use 1 to directly get 2.

Let us prove 3. By local compactness, there is an open set O containing x and a compact set K containing O. Let $D = K \cap U^C$, which is a closed subset of a compact set, hence compact. By regularity, we may find disjoint open sets O_D containing D and O_x containing x. Let $V = O_x \cap O$ which is open and contains x, and note that $V \subset \overline{V} \subset \overline{O} \subset K$, hence $\overline{V}$ is compact. Now note also that $\overline{V} \subset \overline{O_x} \subset O_D^C \subset D^C = K^C \cup U$. Since $\overline{V} \subset K$ it follows $\overline{V} \subset U$. $\square$

Renault considers in Section 4 in [64] a twisted étale locally compact second countable Hausdorff groupoid $(\mathcal{G}, \Sigma)$ and states that $C_{0,r}(\mathcal{G}^0)$ contains an approximate unit for $C_r^*(\mathcal{G}, \Sigma)$, without proof. This is clear as second countable locally compact spaces are σ-compact and hence we can exhaust $\mathcal{G}^0$ by an increasing sequence of compact sets $\bigcup\limits_{n=1}^{\infty} K_n$. Then Lemma 3.1.2 allows us to define an approximate unit $\{\eta_n\}_{n\in\mathbb{N}}$ where η_n is identically 1 on K_n.

We now wish to obtain an approximate unit without assuming second countability.

Lemma 3.1.3. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Then $C_{0,r}(\mathcal{G}^0)$ contains an approximate unit for $C_r^*(\mathcal{G}, \Sigma)$.*

Proof. By Lemma 2.1.72 $(C_{0,r}(\mathcal{G}^0), \|\|_\infty)$ is a C^*-algebra. Hence it has an approximate unit $(\eta_\alpha)_{\alpha\in A}$ for some indexing set A. Let $f \in C_C(\mathcal{G}, \Sigma)$. Then $K = r(\text{supp}(f))$ is compact. From Lemma 3.1.2 we have that there exists $h \in C_{0,r}(\mathcal{G}^0)$ with $h \equiv 1$ on K. Then

$$0 \leftarrow \|\eta_\alpha h - h\|_\infty \geq \|(\eta_\alpha h - h)|_K\|_\infty.$$

Hence $\eta_\alpha \to 1$ uniformly on K.

For $y \in \mathcal{G}^0$, define

$$S_\alpha(y) = \sum_{\dot{\tau}\in\mathcal{G}_y} |\eta_\alpha f - f|(\tau). \tag{3.1}$$

Note that by Lemma 3.1.1 $|\eta_\alpha f - f|(\tau)$ is 0 if $\dot{\tau} \notin \text{supp}(f)$ and converges to 0 otherwise. Hence, as for fixed $y \in \mathcal{G}^0$ the sum (3.1) is finite, we have that $S_\alpha \to 0$ pointwise. Note also that for each $\alpha \in A$, S_α is continuous (see the discussion about the family of Haar measures on locally compact Hausdorff groupoids in Section 4 in [64]), with support in the compact set $s(\text{supp}(f))$. Now because $(\eta_\alpha)_{\alpha\in A}$ is an approximate unit in a C^*-algebra, we have, for $\alpha \leq \beta$ and $\sigma \in \Sigma$, that

$$\begin{aligned} |\eta_\alpha f - f|(\sigma) = |\eta_\alpha(r(\sigma))f(\sigma) - f(\sigma)| = |\eta_\alpha(r(\sigma)) - 1||f(\sigma)| \geq \\ |\eta_\beta(r(\sigma)) - 1||f(\sigma)| = |\eta_\beta(r(\sigma))f(\sigma) - f(\sigma)| = |\eta_\beta f - f|(\sigma). \end{aligned} \tag{3.2}$$

Hence $S_\alpha(y) \geq S_\beta(y)$. Hence the S_α's form a net of pointwise monotonically decreasing continuous maps with compact support, converging pointwise to 0. By the generalized Dini's Theorem (see [81], Corollary 7) we have that $S_\alpha \to 0$ uniformly.

The same argument holds if we consider $f\eta_\alpha$ rather than $\eta_\alpha f$ (we just switch the range map r to the source map s), and if we consider $(\eta_\alpha f - f)^*$ in the summands (as the η_α's are positive). Hence we get that the net $(\eta_\alpha)_{\alpha \in A}$ is an approximate unit for $C_C(\mathcal{G}, \Sigma)$ with respect to the I-norm, and hence with respect to the r-norm, by Lemma 2.1.60. By density of $C_C(\mathcal{G}, \Sigma)$ inside $C_r^*(\mathcal{G}, \Sigma)$, we obtain the result. □

In Theorem 4.2 in [64], Renault proves that if $(\mathcal{G}, \Sigma)$ is a twisted étale locally compact Hausdorff groupoid, then an element f of $C_r^*(\mathcal{G}, \Sigma)$ commutes with all elements in $C_{0,r}(\mathcal{G}^0)$ if and only if $\mathrm{supp}'(f)$ is contained in $\mathcal{G}'$. One concludes from this that $C_{0,r}(\mathcal{G}^0)$ is a masa if and only if $\mathcal{G}$ is effective. With the additional assumption that $\mathcal{G}$ is second countable, Renault obtains via Proposition 3.6 in [64], that $C_{0,r}(\mathcal{G}^0)$ is a masa if and only if $\mathcal{G}$ is topologically principal. For completeness, we present Renault's proof but without the second countability assumption:

Lemma 3.1.4. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact effective Hausdorff groupoid. Then $C_{0,r}(\mathcal{G}^0)$ is a masa in $C_r^*(\mathcal{G}, \Sigma)$.*

Proof. Assume that $f \in C_r^*(\mathcal{G}, \Sigma)$ commutes with all elements $h \in C_{0,r}(\mathcal{G}^0)$. This implies, by Lemma 3.1.1, that for $\sigma \in \Sigma$

$$f(\sigma)h(s(\sigma)) = h(r(\sigma))f(\sigma)$$

for all $h \in C_{0,r}(\mathcal{G}^0)$. Since Lemma 3.1.2 implies that $C_{0,r}(\mathcal{G}^0)$ separates points, it follows that $\mathrm{supp}'(f) \subset \mathcal{G}'$. Since $\mathcal{G}$ is effective, it follows that $\mathrm{supp}'(f) \subset \mathcal{G}^0$ and so $f \in C_{0,r}(\mathcal{G}^0)$. □

Proposition 4.3 in [64] asserts the existence of a unique faithful conditional expectation $P : C_r^*(\mathcal{G}, \Sigma) \to C_{0,r}(\mathcal{G}^0)$ defined by restriction, when $(\mathcal{G}, \Sigma)$ is a twisted étale locally compact second countable Hausdorff groupoid. That this is a faithful conditional expectation can be checked directly by definitions, but uniqueness is justified by Renault by the fact that the groupoid is second countable and topologically principal. Renault makes use of the fact that elements in $C_C(\mathcal{G}^0)$ separate closed subsets from disjoint points. Of course with the space assumed second countable and locally compact, it is regular hence paracompact hence normal, and so Urysohn's lemma for normal spaces applies. Of course, we have now justified such separations without second countability in Lemma 3.1.2, and so for completeness repeat Renault's proof in [64] but without second countability:

Lemma 3.1.5. *Let $(\mathcal{G},\Sigma)$ be a twisted étale locally compact (effective) Hausdorff groupoid. Then*

$$P : C_r^*(\mathcal{G},\Sigma) \to C_{0,r}(\mathcal{G}^0), \quad f \to f|_{\mathcal{G}^0}$$

is a (unique) faithful conditional expectation.

Proof. By Lemmas 2.1.15 and 2.1.24 we have that $\mathcal{G}^0$ is clopen and so P is well-defined. That it is a projection is clear, and Lemma 3.1.1 shows that $P(fh) = P(f)h$ and $P(hf) = hP(f)$ for all $f \in C_r^*(\mathcal{G},\Sigma)$ and $h \in C_{0,r}(\mathcal{G}^0)$. Note that

$$P(f^* * f)(\sigma) = \sum_{\dot{\tau} \in \mathcal{G}_{s(\sigma)}} |f|^2(\tau)$$

which shows that P is positive and faithful. Hence it is a faithful conditional expectation.

To show uniqueness when $\mathcal{G}$ is effective, assume there exists another conditional expectation

$$Q : C_r^*(\mathcal{G},\Sigma) \to C_{0,r}(\mathcal{G}^0).$$

It suffices to show that Q agrees with P on $C_C(\mathcal{G},\Sigma)$. Take $f \in C_C(\mathcal{G},\Sigma)$ with compact support $K \subset \mathcal{G}$. By Lemma 2.1.23 we can cover K by finitely many open bisections (say n of them) and assume one of them is $\mathcal{G}^0$ whilst the others do not meet $\mathcal{G}^0$ (possible as $\mathcal{G}^0$ is closed). Let $f = \sum_{i=1}^{n} f_i$ via a partition of unity with respect to this open cover, with f_1 supported in $\mathcal{G}^0$. Then $P(f) = f_1 = Q(f_1)$, We show that $Q(f_i) = 0$ for all $1 < i \leq n$, which yields $Q(f) = f_1 = P(f)$.

So let $g \in C_C(\mathcal{G},\Sigma)$ with compact support $K \subset \mathcal{G}$, inside an open bisection S not meeting $\mathcal{G}^0$. Note that if $x \in \mathcal{G}^0$, $x \notin s(K)$, then there exists by Lemma 3.1.2 an $h \in C_{0,r}(\mathcal{G}^0)$ with $h(x) = 1$ and vanishing on $s(K)$. Then

$$Q(g)(x) = Q(g)(x)h(x) = Q(gh)(x) = 0$$

as $gh = 0$. Hence if $x \in \mathcal{G}^0$ such that $Q(g)(x) \neq 0$, it must be that $x \in s(K)$, and so by continuity there must exist $U \subset s(S)$ such that $Q(g)$ is non-zero on U. Since S is a bisection, the map $\alpha_S : s(S) \to r(S)$ defined by sending $s^{-1}s$ to ss^{-1}

is a homeomorphism, and since $\mathcal{G}$ is effective and S does not meet $\mathcal{G}^0$, the map α_S cannot be the identity on U. So there exists $x_1 \in U$, $x_2 \in r(S)$ with $x_1 \neq x_2$ and such that $\alpha_S(x_1) = x_2$. Use Lemma 3.1.2 to choose an $h \in C_{0,r}(\mathcal{G}^0)$ with $h(x_1) = 1$, $h(x_2) = 0$. Then

$$Q(g)(x_1) = h(x_1)Q(g)(x_1) = Q(hg)(x_1) = Q(g(h \circ \alpha_S))(x_1) = Q(g)(x_1)h(x_2) = 0,$$

a contradiction as $Q(g)$ is non-zero on U. Hence $Q(g) = 0$ as desired. □

Finally, it remains to check that $C_{0,r}(\mathcal{G}^0)$ is regular in $C_r^*(\mathcal{G}, \Sigma)$. Proposition 4.8 and Corollary 4.9 in [64] contain a proof of this, which we present below for completeness. Indeed, Renault shows that if an element of $C_r^*(\mathcal{G}, \Sigma)$ has open support a bisection, then it a normalizer element. The proof of this does not allude to second countability or effectivity.

Lemma 3.1.6. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact Hausdorff groupoid. Then if an element of $C_r^*(\mathcal{G}, \Sigma)$ has open support a bisection, it is a normalizer element. Consequently, $C_{0,r}(\mathcal{G}^0)$ is regular in $C_r^*(\mathcal{G}, \Sigma)$.*

Proof. Let $f \in C_r^*(\mathcal{G}, \Sigma)$ with $S = \operatorname{supp}'(f)$ a bisection. Let $h \in C_{0,r}(\mathcal{G}^0)$ and note that for $\sigma \in \Sigma$

$$f^* * h * f(\sigma) = \sum_{\dot{\tau} \in \mathcal{G}_{s(\sigma)}} \overline{f(\tau\sigma^{-1})} h(r(\tau)) f(\tau),$$

which is non-zero if there is some $\dot{\tau} \in \mathcal{G}_{s(\sigma)} \cap S$ with $\dot{\overline{\tau\sigma^{-1}}} \in S$. Since $r(\tau\sigma^{-1}) = r(\tau)$ it follows $\dot{\sigma} \in \mathcal{G}^0$. Hence $f^* * h * f \in C_{0,r}(\mathcal{G}^0)$. The same can be shown for $f * h * f^*$.

Elements in $C_C(\mathcal{G}, \Sigma)$ have compact support that can be covered by finitely many open bisections by Lemma 2.1.23, and hence by using a partition of unity it follows that elements in $C_C(\mathcal{G}, \Sigma)$ are finite sums of elements in the normalizer set. Hence the normalizer set generates $C_r^*(\mathcal{G}, \Sigma)$ as a C^*-algebra. □

Theorem 3.1.7. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact effective Hausdorff groupoid. Then $(C_r^*(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$ is a Cartan pair.*

Proof. The definition of being a Cartan subalgebra (see Definition 2.1.75) is satisfied due to Lemmas 3.1.3, 3.1.4, 3.1.5 and 3.1.6. □

With this we have obtained the first half of Renault's theorem (see Theorem 2.1.82) but without the second countability assumption. We now show that the second half of Renault's theorem can be obtained without the separability assumption on the C^*-algebra.

3.2 From Cartan Pairs to Twisted Étale Groupoids

We start with a discussion of the pseudogroup of partial homeomorphisms on a topological space X, how it gives rise to an étale groupoid of germs, and how such a procedure is reversible, as in Section 3 in [64]. We adapt the definition of a pseudogroup given in Section 3 in [64]:

Definition 3.2.1. Let X be a topological space. A *partial homeomorphism* ϕ on X is a homeomorphism $\phi : \mathrm{dom}(\phi) \to \mathrm{ran}(\phi)$ for some open sets $\mathrm{dom}(\phi)$ and $\mathrm{ran}(\phi)$ in X. A *pseudogroup* on X is a non-empty family $\mathfrak{g}$ of partial homeomorphisms on X, stable under composition and inverse and containing the identity homeomorphism. The *ample pseudogroup of* $\mathfrak{g}$, denoted $[\mathfrak{g}]$, is the set of all partial homeomorphisms on X which belong locally to $\mathfrak{g}$.

Definition 3.2.2. Let $\mathfrak{g}$ be a pseudogroup on a topological space X. Its corresponding *groupoid of germs* is the set

$$\mathcal{G} = \{(x, \phi, y) \in X \times \mathfrak{g} \times X : \phi(y) = x\}/\sim$$

where the equivalence relation $\sim$ is given by $(x_1, \phi, y_1) \sim (x_2, \psi, y_2)$ if and only if $y_1 = y_2$ and $x_1 = x_2$ and there is an open neighbourhood U around y_1 such that $\phi|_U = \psi|_U$. We denote an element of $\mathcal{G}$ by $[x, \phi, y]$. Define

$$\mathcal{G}^2 = \{([z, \psi, y], [y, \phi, x]) \in \mathcal{G} \times \mathcal{G}\},$$

and

$$\mathcal{G}^0 = \{[x, \mathrm{id}, x] \in \mathcal{G}\}.$$

We define a multiplication map $\mathcal{G}^2 \to \mathcal{G}$ by $([z, \psi, y], [y, \phi, x]) \to [z, \psi \circ \phi, x]$ and an involution map $\mathcal{G} \to \mathcal{G}$ by $[y, \phi, x] \to [x, \phi^{-1}, y]$. We define a topology on $\mathcal{G}$ by declaring the basic open sets as

$$\mathcal{U}(V, \phi, U) = \{[y, \phi, x] \in \mathcal{G} : y \in V, \quad x \in U\}$$

for open sets U and V in X. With this topology it is clear that

$$\mathcal{G}^0 \cong X.$$

Lemma 3.2.3. *Let $\mathfrak{g}$ be a pseudogroup on a (locally compact) Hausdorff space X. Then the corresponding groupoid of germs $\mathcal{G}$ is a (locally compact) étale groupoid.*

Proof. That $\mathcal{G}$ is a groupoid is clear. To see that it is étale, let $[y, \phi, x] \in \mathcal{G}$. Then $\mathcal{U}(\mathrm{ran}(\phi), \phi, \mathrm{dom}(\phi))$ is an open neighbourhood of $[y, \phi, x]$. The source map maps this open set to the open set $\mathcal{U}(\mathrm{dom}(\phi), \mathrm{id}|_{\mathrm{dom}(\phi)}, \mathrm{dom}(\phi)) \subset \mathcal{G}^0$. It is clear that the source map is a continuous bijection with continuous inverse between these open sets, hence s is a local homeomorphism. The same can be shown for the range map r.

Now assuming X is locally compact, let $g \in \mathcal{G}$. Let V be any open set around g. Choose an open bisection U around g on which s restricts to a local homeomorphism and let $W = U \cap V$. Consider $s(W)$ as a locally compact subspace of X (with the subspace topology, see Corollary 29.3 in [52]). Then let $K \subset s(W)$ be a compact neighbourhood around $s(g)$, and using that $s : W \to s(W)$ is a homeomorphism we obtain a compact neighbourhood around g contained in W, and hence in V. Hence we have shown that for any open set around g there is a compact neighbourhood around g contained in the open set, which is equivalent to local compactness for Hausdorff spaces. $\square$

What we have just described is the procedure

$$\mathfrak{g} \text{ a pseudogroup on } X \implies \mathcal{G} \text{ an étale groupoid with unit space } X. \tag{3.3}$$

Now we wish to describe the reverse procedure.

Definition 3.2.4. Let $\mathcal{G}$ be an étale groupoid. Let $\mathcal{S}$ denote its open bisections (Lemma 2.1.23 tells us that $\mathcal{G}$ can be covered by elements of $\mathcal{S}$). Equip $\mathcal{S}$ with multiplication and inverse given by

$$ST = \{st : (s,t) \in \mathcal{G}^2 \cap (S \times T)\}, \quad S^{-1} = \{s^{-1} : s \in S\}, \quad \text{for all } S, T \in \mathcal{S}.$$

Define a map

$$\alpha : \mathcal{S} \to \alpha(\mathcal{S}), \quad \alpha(S) = \alpha_S$$

where

$$\alpha_S : s(S) \to r(S), \quad s^{-1}s \to ss^{-1}.$$

Lemma 3.2.5. *Let $\mathcal{G}$ be an étale groupoid. Then $\mathcal{S}$ is an inverse semigroup, and for each $S \in \mathcal{S}$, α_S is a partial homeomorphism $s(S) \to r(S)$. The set $\alpha(\mathcal{S})$ (with the addition of the empty homeomorphism) is a pseudogroup on $\mathcal{G}^0$.*

Proof. The inverse semigroup conditions are trivially checked. For $S \in \mathcal{S}$ it is clear that α_S is surjective. Since S is a bisection, injectivity follows. Continuity of the map and its inverse is clear as the map factors through S via the continuous maps $s|_S^{-1}$ and $r|_S$ as S is an open bisection. Lemma 2.1.22 tells us that the domain and range are open sets. Hence α_S is a partial homeomorphism on $\mathcal{G}^0$. Note that $\alpha_{\mathcal{G}^0} = \mathrm{id}|_{\mathcal{G}^0}$, $\alpha_S^{-1} = \alpha_{S^{-1}}$, and if $T \in \mathcal{S}$ with $TT^{-1} \cap S^{-1}S \neq \emptyset$, then $\alpha_S \circ \alpha_T = \alpha_{ST}$. Hence $\alpha(\mathcal{S})$ is a pseudogroup on $\mathcal{G}^0$. ☐

Lemma 3.2.5 yields the procedure

$$\mathcal{G} \text{ an étale groupoid with unit space } X \implies \mathfrak{g} \text{ a pseudogroup on } X. \tag{3.4}$$

Propositions 3.1 and 3.2 in [64] tell us what happens if one does procedure (3.3) followed by procedure (3.4), and vice-versa, if one starts with procedure (3.4) followed by procedure (3.3). In the former case, one retrieves the ample pseudogroup of the original pseudogroup. In the latter case, one retrieves the original étale groupoid modulo the interior of its isotropy. The following lemma makes this precise:

Lemma 3.2.6. *Let X be a Hausdorff space and $\mathfrak{g}$ a pseudogroup on X. Apply procedure* (3.3) *to obtain an étale groupoid $\mathcal{G}$, and let $\mathcal{S}$ be the inverse semigroup of open bisections of $\mathcal{G}$. Then the corresponding map $\alpha : \mathcal{S} \to \alpha(\mathcal{S})$ is an isomorphism where $\alpha(\mathcal{S}) = [\mathfrak{g}]$.*

Let $\mathcal{G}$ be an étale groupoid and apply procedure (3.4) *to get a pseudogroup $\mathfrak{g}$. Then apply procedure* (3.3) *and let $\mathcal{H}$ be the étale groupoid of germs obtained through this procedure. Then we have a short exact sequence*

$$0 \to \mathrm{int}(\mathcal{G}') \to \mathcal{G} \to \mathcal{H} \to 0. \tag{3.5}$$

Proof. Consider the first claim of the lemma. For $S \in \mathcal{S}$ then we may write $S = \bigcup_i \mathcal{U}(U_i, \phi_i, V_i)$ for partial homeomorphisms $\phi_i \in \mathfrak{g}$. Then note that $\phi = \alpha_S$ must be in $[\mathfrak{g}]$. If $\phi \in [\mathfrak{g}]$ then set $S = \mathcal{U}(X, \phi, X)$, which can be treated as an open subset of $\mathcal{G}$ because both $\mathfrak{g}$ and $[\mathfrak{g}]$ define the same groupoid of germs due to the equivalence relation that defines the groupoid of germs. Clearly S belongs to $\mathcal{S}$. These two procedures are inverses of each other.

Now consider the second claim of the lemma. Define a map $f : \mathcal{G} \to \mathcal{H}$ by sending $\sigma \in \mathcal{G}$ to $[r(\sigma), \alpha_S, s(\sigma)]$ for some open bisection S containing σ. This is well-defined as any other bisection containing σ gives the same output by the definition of the equivalence relation on the groupoid of germs. It is clear that f is a surjective continuous groupoid homomorphism. If $f(\sigma) \in \mathcal{H}^0$, then it follows that α_S restricts to the identity on an open neighbourhood around $s(\sigma)$, which means $S \subset \mathcal{G}'$. Hence $\sigma \in \mathrm{int}(\mathcal{G}')$. □

Corollary 3.2.7. *Every étale groupoid of germs is effective.*

Proof. If $\mathcal{G}$ is an étale groupoid of germs, then procedure (3.4) followed by (3.3) yields that $\mathcal{G} \cong \mathcal{H}$ in (3.5), and hence $\mathrm{int}(\mathcal{G}') = \mathcal{G}^0$. □

Remark 3.2.8. If $\mathfrak{g}$ is the pseudogroup of all partial homeomorphisms on $\mathbb{R}$, then the corresponding groupoid of germs in not topologically principal, as there always exists homeomorphisms of open intervals onto themselves that only fix one prespecified point (and hence are not the identity on any open neighbourhood of the fixed point), and so no point of the unit space has trivial isotropy. However, by Corollary 3.2.7 the groupoid of germs is effective.

We aim now to show how a Cartan pair gives rise to a twisted étale groupoid which will be the candidate for the second half of Theorem 2.1.82. The construction can be found in [64]. Renault assumes that the ambient C^*-algebra is separable, but the effect of this, as we show below, is only to end up with a second countable topologically principal groupoid. We will not assume separability.

Let (A, C) be a Cartan pair. Then $C \cong C_0(X)$ for a locally compact Hausdorff space X. For $n \in N_A(C)$ we have that n^*n and nn^* belong to C, and hence we define:

Definition 3.2.9. For $n \in N_A(C)$, define

$$\mathrm{dom}(n) = \{x \in X : n^*n(x) > 0\}, \;\; \mathrm{ran}(n) = \{x \in X : nn^*(x) > 0\}.$$

Renault states in Proposition 4.7 in [64] the following useful lemma, whose proof can also be found in 1.6 in [40]:

Lemma 3.2.10. *Let (A, C) be a Cartan pair, then for every $n \in N_A(C)$ there exists a unique homeomorphism $\alpha_n : \mathrm{dom}(n) \to \mathrm{ran}(n)$ such that for all $c \in C$ and $x \in \mathrm{dom}(n)$ we have*

$$n^*cn(x) = c(\alpha_n(x))n^*n(x). \tag{3.6}$$

Proof. Consider the unique polar decomposition of n as $u|n|$ where u is a partial isometry and $|n| = (n^*n)^{\frac{1}{2}}$, in the enveloping von Neumann algebra A^{**} (see I.8.1

in [16]). The partial isomorphism of C which maps $c \to u^*cu$ gives rise to the partial homeomorphism α_n. $\square$

We note down some more useful properties of this construction. The following is Lemma 4.10 in [64]:

Lemma 3.2.11. *Let (A, C) be a Cartan pair. Then $\alpha_c = \mathrm{id}_{\mathrm{dom}(c)}$ for all $c \in C$, and $\alpha_n^{-1} = \alpha_{n^*}$, $\alpha_m \circ \alpha_n = \alpha_{mn}$ for all $m, n \in N_A(C)$.*

Proof. For $c \in C$ the identity $\alpha_c = \mathrm{id}_{\mathrm{dom}(c)}$ follows directly from (3.6). Let $m, n \in N_A(C)$. Then $mn \in N_A(C)$ and (3.6) gives, for any $c \in C$ and $x \in \mathrm{dom}(mn)$

$$(mn)^*c(mn)(x) = c(\alpha_{mn}(x))(mn)^*(mn)(x) = c(\alpha_{mn}(x))(m^*m(\alpha_n(x))n^*n(x)),$$

whilst also

$$\begin{aligned}&(mn)^*c(mn)(x) = n^*((m^*cm))n(x) = (m^*cm)(\alpha_n(x))n^*n(x) = \\ &c(\alpha_m \circ \alpha_n(x))m^*m(\alpha_n(x))n^*n(x).\end{aligned}$$

Hence we have the equality

$$c(\alpha_{mn}(x))(m^*m(\alpha_n(x))n^*n(x)) = c(\alpha_m \circ \alpha_n(x))m^*m(\alpha_n(x))n^*n(x). \tag{3.7}$$

Note that the above is well-defined since if $x \in \mathrm{dom}(mn)$ then $n^*(m^*m)n(x) > 0$ and so writing $n = u|n|$ as in the proof of Lemma 3.2.10 one obtains that $n^*n(x) > 0$ and so $x \in \mathrm{dom}(n)$. Note that if $m^*m(\alpha_n(x)) = 0$ then $(mn)^*(mn)(x) = 0$ by (3.6), which is impossible as $x \in \mathrm{dom}(mn)$. Hence $\alpha_n(x) \in \mathrm{dom}(m)$. Dividing both sides in (3.7) by $m^*m(\alpha_n(x))n^*n(x)$ yields

$$c(\alpha_{mn}(x)) = c(\alpha_m \circ \alpha_n(x)).$$

Since C separates X by Lemma 3.1.2, we obtain $\alpha_{mn} = \alpha_m \circ \alpha_n$. Using this with the property that $\alpha_c = \mathrm{id}_{\mathrm{dom}(c)}$ for $c \in C$ one can analyse α_{n^*n} and α_{nn^*} to obtain $\alpha_n^{-1} = \alpha_{n^*}$. $\square$

Definition 3.2.12. Let (A, C) be a Cartan pair. Define

$$\mathfrak{g}(C) = \{\alpha_n : n \in N_A(C)\},$$

and let

$$\underline{\alpha} : N_A(C) \to \mathfrak{g}(C)$$

be the map that sends $n \to \alpha_n$.

Remark 3.2.13. One should compare Definition 3.2.12 with Definition 3.2.4. Up to possibly needing to add the identity homeomorphism, Lemma 3.2.11 shows that $\mathfrak{g}(C)$ is a pseudogroup in the sense of Definition 3.2.1. The addition of the identity homeomorphism does not affect the groupoid of germs formed from $\mathfrak{g}(C)$ as in Definition 3.2.2. Hence the following definition has justifiable terminology:

Definition 3.2.14. We shall call $\mathfrak{g}(C)$ from Definition 3.2.12 the *Weyl pseudogroup* of (A, C). We call the groupoid of germs it gives rise to the *Weyl groupoid* of (A, C), denoted by $\mathcal{G}(C)$.

Before defining the twist, we need the following useful lemma, which is Proposition 4.12 in [64]:

Lemma 3.2.15. *Let (A, C) be a Cartan pair. Then $\ker(\underline{\alpha}) = C$.*

Proof. It is clear that $C \subset \ker(\underline{\alpha})$. Now assume we have an $n \in N_A(C)$ such that α_n restricts to the identity on $\mathrm{dom}(n)$. From (3.6) we get that for all $c \in C$, $n^*cn = cn^*n$ on $\mathrm{dom}(n)$. This formula also holds outside $\mathrm{dom}(n)$ as for c positive we have $n^*cn \leq \|c\|n^*n$ and n^*n is 0 outside $\mathrm{dom}(n)$. Hence $n^*cn = cn^*n$ for all $c \in C$. Note that this implies $(nc - cn)^*(nc - cn) = 0$ for all $c \in C$ and so n commutes with C, hence $n \in C$. □

We can now define a twist, $\Sigma(C)$ over $\mathcal{G}(C)$, and the twisted étale groupoid $(\mathcal{G}(C), \Sigma(C))$ will be the candidate which gives rise to the second statement of Theorem 2.1.82 (where we do not assume separability of A).

Definition 3.2.16. Let (A, C) be a Cartan pair. Define

$$\Sigma(C) = \{[x, n, y] : y \in \mathrm{dom}(n), \alpha_n(y) = x\}$$

where the [] denotes equivalence classes of the relation that identifies (x, n, y) with (x', n', y') if and only if $y = y'$ and there exists $c, c' \in C$ with $c(y), c'(y) > 0$ and $nc = n'c'$ (of course, by Lemma 3.2.11, this also implies $x = x'$). We equip $\Sigma(C)$ with a groupoid of germs structure as in Definition 3.2.2, where we have products $[x, n, y][y, m, z] = [x, nm, z]$ and inverses $[x, n, y]^{-1} = [y, n^*, x]$. Define a map

$$\Pi : \Sigma(C) \to \mathcal{G}(C), \quad \Pi([x, n, y]) = [x, \alpha_n, y],$$

and define

$$\mathcal{B} = \{[x, c, x] : c \in C, c(x) \neq 0\} \subset \Sigma(C),$$

and identify it with $X \times \mathbb{T}$ via the map

$$[x, c, x] \to \left(x, \frac{c(x)}{|c(x)|}\right).$$

The topology on $\Sigma(C)$ is defined as follows: for $n \in N_A(C)$, let S be the bisection in $\mathcal{G}(C)$ defined by

$$S = \mathcal{U}(\mathrm{ran}(n), \alpha_n, \mathrm{dom}(n)).$$

We call S the *open support of* n. Then define the bijection

$$f_n : \mathrm{dom}(n) \times \mathbb{T} \to \Sigma(C)|_S, \quad (x, t) \to [\alpha_n(x), tn, x].$$

The topology on $\Sigma(C)|_S$ is then the topology induced via f_n. Hence we can cover $\Sigma(C)$ by a collection of open sets which we declare a base for the topology.

We now show that the statements of Definition 3.2.16 are well-defined and that $\Sigma(C)$ in fact defines a twist over $\mathcal{G}(C)$. The following is based on Proposition 4.14 and Lemma 4.16 in [64]. Renault's proofs can be adapted without the need to refer to separability of A.

Lemma 3.2.17. *Let (A, C) be a Cartan pair. Then $\Sigma(C)$ is a twist over $\mathcal{G}(C)$.*

Proof. Recall the definition of a twist from Definition 2.1.35. We need to check that all the statements of this definition are satisfied. Lemma 3.2.3 tells us that $\mathcal{G}(C)$ is a (locally compact) étale groupoid. That $\Sigma(C)$ is a groupoid is trivial to check, and that it is locally compact and Hausdorff is clear as its topology is locally induced by the topology of $\mathrm{dom}(n) \times \mathbb{T}$, which is locally compact and Hausdorff as X (and $\mathbb{T}$) are.

The identification of $X \times \mathbb{T}$ (which is given the groupoid structure as in the first bullet point of Definition 2.1.35) with $\mathcal{B}$ is clearly a surjective groupoid homomorphism. To prove injectivity note that if $c_1, c_2 \in C$ and $x \in X$ with $c_1(x) \neq 0 \neq c_2(x)$, and such that $\frac{c_1(x)}{|c_1(x)|} = \frac{c_2(x)}{|c_2(x)|}$, then there exists $t \in \mathbb{T}$ such that $tc_1(x), tc_2(x) > 0$. Since $c_1(tc_2) = c_2(tc_1)$ we get that $[x, c_1, x] = [x, c_2, x]$. This yields injectivity. We thus obtain via this identification an injective groupoid homomorphism $i : X \times \mathbb{T} \hookrightarrow \Sigma(C)$. It is clear that the map Π is a surjective map, and that it is a homomorphism follows from Lemma 3.2.11.

Now we check that $i(X \times \mathbb{T}) = \Pi^{-1}(\mathcal{G}(C)^0)$. Indeed, It is clear from Lemma 3.2.11 that $i(X \times \mathbb{T}) \subset \Pi^{-1}(\mathcal{G}(C)^0)$. Now assume $\sigma = [x, n, y] \in \Sigma(C)$ satisfies that there is an open set U around y such that α_n is the identity when restricted to U.

Hence $x = y$, and note that by Lemma 3.1.2 we can find a compact set K with $y \in K \subset U$ and a $c \in C$ with c identically 1 on K and vanishing outside U. Define $m = nc$. Then α_m is trivial and so by Lemma 3.2.15 we obtain that $m \in C$ with $[y, m, y] = [y, n, y]$, hence $\sigma \in i(X \times \mathbb{T})$.

Now let us check that the image of i is central in $\Sigma(C)$. Let $t \in \mathbb{T}$ and $[x, n, y] \in \Sigma(C)$. Then $i((r([x, n, y]), t))[x, n, y] = [x, cn, y]$ for some $c \in C$ with $\frac{c(x)}{|c(x)|} = t$, and $[x, n, y]i(s([x, n, y]), t) = [x, nd, y]$ for some $d \in C$ with $\frac{d(y)}{|d(y)|} = t$. Choose an open set U around y, contained in $\mathrm{dom}(n)$ such that d is non-zero on U and c is non-zero on $\alpha_n(U)$. Use Lemma 3.1.2 to find a compactly supported function χ with support in U, with $\chi(y) = 1$. Define $g \in C$ by declaring it $\frac{c(\alpha_n(u))}{d(u)}\chi(u)$ for $u \in U$ and 0 outside U. Then $g(y) > 0$. We claim that $cn\chi = ndg$. For this it suffices to show that $(cn\chi - ndg)^*(cn\chi - ndg) = 0$. This can be checked directly using (3.6) and the definition of g. Hence it follows that $[x, cn, y] = [x, nd, y]$ as desired.

Now we verify all the required topological properties. First note that the trivializations obtained via the maps f_n are compatible. Indeed, if m gives rise to the same open support as n, then Lemma 3.2.15 implies that $mn^* \in C$ which implies that $(nm^*)(mn^*) = (mn^*)(nm^*)$. Multiplying on the right by n and noticing that m^*n is non-zero on $\mathrm{dom}(n)$ we obtain $n(m^*mn^*n) = m(n^*nm^*n)$ so there exists $c_1, c_2 \in C$ which are non-vanishing on $\mathrm{dom}(n)$ such that

$$nc_1 = mc_2. \tag{3.8}$$

To check compatibility, we need to show that if $t_1, t_2 \in \mathbb{T}$ then the transition function $t_{nm} : \mathrm{dom}(n) \to \mathbb{T}$ satisfying $f_n(x, t_1) = f_m(x, t_2) = f_m(x, t_{nm}(x)t_1)$ is a homeomorphism. The first equality yields the existence of $d_1, d_2 \in C$ with $d_1(x), d_2(x) > 0$ satisfying $t_1nd_1 = t_2md_2$. Hence $t_1nc_1d_1 = t_2mc_1d_2$. Using (3.8) we get that $t_1mc_2d_1 = t_2mc_1d_2$. Thus $m(t_1c_2d_1 - t_2c_1d_2) = mD = 0$ (where $D = t_1c_2d_1 - t_2c_1d_2 \in C$). Hence $(mD)^*(mD)(x) = |D(x)|^2m^*m(x) = 0$ which implies $D(x) = 0$. Hence

$$t_1c_2(x)d_1(x) = t_2c_1(x)d_2(x)$$

which implies that

$$t_{nm}(x) = \frac{c_2(x)d_1(x)}{c_1(x)d_2(x)}.$$

Noting that

$$\frac{d_1(x)}{d_2(x)} = \left|\frac{d_1(x)}{d_2(x)}\right| = \frac{|c_1(x)|}{|c_2(x)|}$$

we obtain

$$t_{nm}(x) = \frac{c_2(x)|c_1(x)|}{c_1(x)|c_2(x)|}.$$

Since c_1 and c_2 are continuous functions that are non-vanishing on $\operatorname{dom}(n)$ it is clear that t_{nm} is a homeomorphism. Hence the trivializations are compatible.

We now proceed to verify that for $n \in N_A(C)$, f_n is indeed a bijection. For injectivity, note that if $[\alpha_n(x), t_1 n, x] = [\alpha_n(x), t_2 n, x]$, then there must be $c_1, c_2 \in C$, positive at x, such that $t_1 n c_1 = t_2 n c_2$. Multiplying on the right by n^* and evaluating at $\alpha_n(x)$ via (3.6), one obtains that $\frac{t_1}{t_2}$ is positive, implying that $t_1 = t_2$. For surjectivity, assume there is an $m \in N_A(C)$ such that on an open set U containing $x \in \operatorname{dom}(n)$, α_m and α_n agree. We aim to show there exists a $t \in \mathbb{T}$ such that $[\alpha_n(x), tn, x] = [\alpha_n(x), m, x]$. Let $V = \alpha_n(U) = \alpha_m(U)$, and use Lemma 3.1.2 to find a positive compactly supported function χ with support in $U \subset \operatorname{dom}(n) \cap \operatorname{dom}(m)$, and $\chi(x) = 1$. Define $c_1(u) = \frac{\chi(u)}{nn^*(\alpha_n(u))}$ for all $u \in U$, and 0 outside of U. Define $c_2(u) = \frac{\chi(u)}{mm^*(\alpha_m(u))}$ for all $u \in U$, and 0 outside of U. Note that $c_1, c_2 \in C$ with $c_1(x), c_2(x) > 0$. One can check that this implies

$$nc_1n^* = mc_2m^*, \tag{3.9}$$

as (3.6) shows that (3.9) holds on V, and outside $\operatorname{ran}(n)$ we have that $0 \leq nc_1n^* \leq \|c_1\| nn^* = 0$ (and a similar conclusion for mc_2m^*) and so (3.9) holds everywhere. Multiplying (3.9) by n on the right yields

$$nc_1n^*n = mc_2m^*n.$$

Note that there is a $t \in \mathbb{T}$ such that $tm^*n(x) > 0$ (that $g(x) = m^*n(x) \neq 0$ can be checked using (3.6) on $g^*(x)g(x)$). Let $g_1 = c_1n^*n$, $g_2 = tc_2m^*n$, we have that $g_1(x), g_2(x) > 0$ and $tng_1 = mg_2$. Hence $[\alpha_n(x), tn, x] = [\alpha_n(x), m, x]$ as desired. Hence f_n is bijective. The definition of the topology on $\Sigma(C)|_S$ makes f_n a homeomorphism.

The topology defined on $\Sigma(C)$ makes $\Sigma(C)$ a locally trivial $\mathcal{G}(C)$-bundle. Indeed around $[x, \alpha_n, y] \in \mathcal{G}(C)$ we choose the bisection $U = \mathcal{U}(\operatorname{ran}(n), \alpha_n, \operatorname{dom}(n))$ and the continuous section defined by $S([w, \alpha_n, z]) = f_n(z, 1)$ where $z \in \operatorname{dom}(n)$. This clearly witnesses the local triviality condition.

Finally it is a tedious but straightforward task to show that i and Π are continuous and restrict to homeomorphisms on the respective unit spaces. Hence $\Sigma(C)$ is a twist over $\mathcal{G}(C)$. $\square$

Remark 3.2.18. We can also view the twist in light of Definition 2.1.37. The $\mathbb{T}$-action is then obtained by the definition of multiplication by $\mathbb{T}$ given in Definition 2.1.35, which in our case specifically becomes

$$t[x, n, y] = [x, tn, y].$$

Definition 3.2.19. Let (A, C) be a Cartan pair. We will call $(\mathcal{G}(C), \Sigma(C))$ the *Weyl twisted groupoid* associated to (A, C).

The next step is obtain an isomorphism between the Cartan pair (A, C) and the pair $(C_r^*(\mathcal{G}(C), \Sigma(C)), C_{0,r}(\mathcal{G}(C)^0))$. The steps towards this are presented in Section 5 in [64]. We will repeat these steps summatively for completeness. The standing assumption in [64] is that A is separable. However the proofs can be obtained without this assumption, and any function separation properties will be obtained via Lemma 3.1.2.

Lemma 3.2.20. *Let (A, C) be a Cartan pair and P the associated faithful conditional expectation. If $n \in N_A(C)$ satisfies that α_n is non-trivial on a neighbourhood of $y \in \operatorname{dom}(n)$, then $P(n)(y) = 0$.*

Proof. There is a net y_α converging to y such that $\alpha_n(y_\alpha) \neq y_\alpha$. For a fixed α, one can obtain $c_1, c_2 \in C$ with the properties that $c_2 n = nc_1$ and $c_1(y_\alpha) = 1$ whilst $c_2(y_\alpha) = 0$. Indeed, use Lemma 3.1.2 to find a $c \in C$ with compact support in $\operatorname{ran}(n)$ satisfying $c(\alpha_n(y_\alpha))n^*n(y_\alpha) = 1$ and $c(y_\alpha) = 0$. Then set $c_1 = (c \circ \alpha_n)n^*n$ and $c_2 = nn^*c$ and use (3.6) to verify the aforementioned claims.

Then we may write

$$P(n)(y_\alpha) = P(n)(y_\alpha)c_1(y_\alpha) = P(nc_1)(y_\alpha) = P(c_2 n)(y_\alpha) = 0.$$

By continuity of $P(n)$ we have $P(n)(y) = 0$. □

We now show how to think of elements in A as $\mathbb{T}$-equivariant continuous maps on $\Sigma(C)$.

Definition 3.2.21. Let (A, C) be a Cartan pair and P the associated faithful conditional expectation. For every $a \in A$ define $\psi(a) = \hat{a}$ to be the map on $\Sigma(C)$ defined by

$$\hat{a}([x, n, y]) = \frac{P(n^*a)(y)}{\sqrt{n^*n(y)}}. \tag{3.10}$$

Lemma 3.2.22. *The map ψ in Definition 3.2.21 is a well-defined, injective, linear map which sends $a \in A$ to a $\mathbb{T}$-equivariant continuous function on $\Sigma(C)$.*

Furthermore, for $c \in C$, $\psi(c)$ is zero off $\mathcal{G}(C)^0$, and on $x \in \mathcal{G}(C)^0$ it takes value $c(x)$. For $n \in N_A(C)$, the open support of $\psi(n)$ is exactly $\mathcal{U}(\operatorname{ran}(n), \alpha_n, \operatorname{dom}(n))$.

Proof. Linearity and $\mathbb{T}$-equivariance is clear as P is linear. Replacing n in (3.10) by nc for any $c \in C$ with $c(y) > 0$ does not change the output, hence ψ is well-defined. Continuity of $\hat{a}$ can be checked on the trivialization $\Sigma(C)|_S$ where S is the open support of n. This then follows from continuity of the map $y \to \frac{P(n^*a)(y)}{\sqrt{n^*n(y)}}$ on $\operatorname{dom}(n)$. For injectivity, assume $\hat{a} = 0$, then it follows that $P(n^*a)$ is 0 on $\operatorname{dom}(n)$, hence on its closure as well by continuity. For $z \in X$ outside this closure, Lemma 3.1.2 allows us to find a $c \in C$ which is 1 on z and 0 on the closure of $\operatorname{dom}(n)$, and hence $nc = 0$. Then $P(n^*a)(z) = c^*(z)P(n^*a)(z) = P((nc)^*a)(z) = 0$, hence $P(n^*a) = 0$. Regularity of the normalizer set implies $P(a^*a) = 0$ and faithfulness of P implies $a = 0$.

Note that if $[x, \alpha_n, y]$ is not a unit, then the germ at y of α_n is non-trivial and hence by Lemma 3.2.20 $\hat{c}[x, n, y] = 0$ (the remark after Definition 1.3 in [65] tells us that $P(n^*) = P(n)^*$). The unit space $\mathcal{G}(C)^0$ can be identified with the unit space $\Sigma(C)^0$ whose elements are of the form $[x, d, x]$ for $x \in X$ and $\frac{d(x)}{|d(x)|} = 1$ (see Definition 3.2.16). Hence it is easy to see that for $c \in C$ we have $\hat{c}[x, d, x] = c(x)$. If $\hat{n}[x, m, y] \neq 0$ then $P(m^*n)(y) \neq 0$ and so by Lemma 3.2.20 α_n and α_m must agree on an open neighbourhood around y and so $[x, \alpha_m, y] \in \mathcal{U}(\operatorname{ran}(n), \alpha_n, \operatorname{dom}(n))$. On the other hand $\hat{n}[x, n, y] > 0$ for $y \in \operatorname{dom}(n)$ and so the open support of $\hat{n}$ is exactly $\mathcal{U}(\operatorname{ran}(n), \alpha_n, \operatorname{dom}(n))$. □

Lemma 3.2.23. *The Weyl groupoid $\mathcal{G}(C)$ of a Cartan pair (A, C) is Hausdorff.*

Proof. Consider $\sigma_1 = [x_1, n_1, y_1], \sigma_2 = [x_2, n_2, y_2] \in \Sigma(C)$ such that $\dot{\sigma_1} \neq \dot{\sigma_2}$. If $y_1 \neq y_2$ let $a = n_1 c$ where $c \in C$ satisfies $c(y_1) \neq 0$ and $c(y_2) = 0$. Then $\hat{a}(\sigma_1) \neq 0$ and $\hat{a}(\sigma_2) = 0$. If $y_1 = y_2$ then by Lemma 3.2.20 it follows that $P(n_2^* n_1)(y_1) = 0$ and so $\widehat{n_1}(\sigma_2) = 0$. However $\widehat{n_1}(\sigma_1) \neq 0$. The result follows by continuity of elements in $\Psi(A)$. □

Definition 3.2.24. Let (A, C) be a Cartan pair. Define $N_{A,c}(C)$ as the subset of $N_A(C)$ consisting of those elements n such that $\hat{n}$ has compact support. Let A_c be the linear span of $N_{A,c}(C)$. Let $C_c = C \cap A_c$.

Lemma 3.2.25. *Let (A, C) be a Cartan pair. Then $N_{A,c}(C)$ is dense in $N_A(C)$ and A_c is dense in A. The map ψ is a *-algebra isomorphism which maps A_c onto $C_c(\mathcal{G}(C), \Sigma(C))$ (where the codomain is endowed with the multiplication and involution structure given in Definition 2.1.54) and C_c onto $C_c(\mathcal{G}(C)^0)$.*

Proof. Let $\{c_\alpha\}_{\alpha\in A}$ be a net in C which is an approximate unit for A. Assume $n \in N_A(C)$ and that $n = \lim_\alpha nc_\alpha$. We can find compactly supported $b_\alpha \in C$ such that $n = \lim_\alpha nb_\alpha$ as $C_c(X)$ is dense in $C_0(X)$ (this follows from the Stone-Weierstrass theorem, see Theorem A.1.3 in [36]). We now show that $nb_\alpha \in N_{A,c}(C)$. Let $K = \mathrm{supp}(b_\alpha)$ be compact, and note that by Lemma 3.2.22 we have that the open support of nb_α is the open bisection $S = \mathcal{U}(\alpha_{nb_\alpha}(V), \alpha_{nb_\alpha}, V)$ where $V \subset K$ is open (here we have used the fact that $\alpha_{nb_\alpha} = \alpha_n \circ \alpha_{b_\alpha}$). We need to show that $\overline{S}$ is a compact set. First note that

$$\overline{S} \text{ is compact} \iff \Pi^{-1}(\overline{S}) \text{ is compact} \iff \overline{\Pi^{-1}(S)} \text{ is compact.}$$

The first equivalence follows from Lemma 2.1.40. The second equivalence follows by equality of the gives sets. Indeed it is clear that that $\overline{\Pi^{-1}(S)} \subseteq \Pi^{-1}(\overline{S})$. The reverse inclusion is slightly more technical. Let $\sigma \in \Pi^{-1}(\overline{S})$. Then there is a net $\{s_\beta\}$ is S with $s_\beta \to \Pi(\sigma)$. Using the local triviality condition in the definition of a twisted groupoid we find an open bisection U around $\Pi(\sigma)$ and a continuous section $\gamma : U \to \Sigma(C)$. We may assume that the net $\{s_\beta\}$ lives in U. Then declaring $\tau_\beta = \gamma(s_\beta) \in \Pi^{-1}(S)$ we obtain that $\tau_\beta \to \gamma(\Pi(\sigma))$ and so by Lemma 2.1.36 there exists $t \in \mathbb{T}$ such that $t\tau_\beta \to \sigma$. Hence $\sigma \in \overline{\Pi^{-1}(S)}$. Thus to check $\overline{S}$ is compact it suffices to check that $\overline{\Pi^{-1}(S)}$ is compact and hence, by the definition of the topology on $\Sigma(C)$ which has a trivialization $\Sigma(C)|_S$ (see Definition 3.2.16) it suffices to check that $V \times \mathbb{T}$ is relatively compact, and this is clear as the closure lies in the compact set $K \times \mathbb{T}$. Hence $nb_\alpha \in N_{A,c}(C)$. This gives density of $N_{A,c}(C)$ in $N_A(C)$, and by regularity the density of A_c in A.

We already saw in Lemma 3.2.22 that ψ is linear and injective. Let us see that it maps A_c onto $C_c(\mathcal{G}(C), \Sigma(C))$. It suffices to pick an $f \in C_c(\mathcal{G}(C), \Sigma(C))$ supported in an open bisection $\mathcal{U}(\mathrm{ran}(n), \alpha_n, \mathrm{dom}(n))$ for some $n \in N_A(C)$, as for general compact supports we can use a partition of unity argument which reduces to this case. Lemma 3.2.22 tells us that $\hat{n}$ is non-zero on this open bisection, and hence we can find an $h \in C_c(\mathcal{G}(C)^0)$ such that $f = \hat{n} * h$ (to see this use Lemma 3.1.1). Lemma 3.2.22 shows that ψ maps C_c onto $C_c(\mathcal{G}(C)^0)$ and so there exists a $c \in C_c$ such that $f = \hat{n} * \hat{c} = \psi(nc)$ (the last equality can be checked directly using the definition of multiplication and Lemma 3.1.1). The element $nc \in N_{A,c}(C)$.

By linearity of ψ it suffices to check the *-algebra homomorphism properties on elements in $N_A(C)$. And it suffices, by Lemma 3.2.22, to check that the relevant evaluations hold true on the open supports induced by such elements in $N_A(C)$. This is a straightforward but tedious task, and the explicit calculations can be found in Lemma 5.8 in [64]. □

We are now in a position to replace the second statement of Theorem 2.1.82 but without the separability assumption. The proof is given in Theorem 5.9 in [64], but there separability of A gives rise to second countability of the groupoid and hence its topological principality by Proposition 3.6 in [64] (locally compact Hausdorff spaces have the Baire property required by the proposition).

Theorem 3.2.26. *Let (A, C) be a Cartan pair. Then there exists a twisted étale locally compact effective Hausdorff groupoid $(\mathcal{G}, \Sigma)$ and a C^*-algebra isomorphism carrying (A, C) onto $(C^*_r(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$.*

Proof. Let $(\mathcal{G}, \Sigma) = (\mathcal{G}(C), \Sigma(C))$. That this is a twisted groupoid is by Lemma 3.2.17. That it is étale and locally compact is by Lemma 3.2.3. That it is effective is by Corollary 3.2.7. That it is Hausdorff is by Lemma 3.2.23. Hence $(C^*_r(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$ is a Cartan pair by Theorem 3.1.7 with associated conditional expectation $\hat{P}$.

Note that the restriction of ψ to C is isometric by Lemma 3.2.22 (the norms on C and $C_{0,r}(\mathcal{G}(C)^0)$ are the supremum norms). We will now show that the restriction of ψ to A_c is also isometric by using the fact that the conditional expectation P associated to (A, C) is faithful and completely positive (see the first section of [58] for the latter claim), and thus Stinespring's theorem may be used to write, for $a \in A_c$,

$$\|a\| = \sup\{\|P(b^*a^*ab)\|_\infty^{\frac{1}{2}} : b \in A_c, \quad P(b^*b) \leq 1\}.$$

For such an a, $\hat{a}$ belongs to $C^*_r(\mathcal{G}(C), \Sigma(C))$ by Remark 2.1.73, and so in an analogous way we also have that

$$\begin{aligned}
\|\hat{a}\|_r &= \sup\{\|\hat{P}(f^*\hat{a}^*\hat{a}f)\|_\infty^{\frac{1}{2}} : f \in C_c(\mathcal{G}(C), \Sigma(C)), \quad \hat{P}(f^*f) \leq 1\} \\
&= \sup\{\|\hat{P}(\hat{b}^*\hat{a}^*\hat{a}\hat{b})\|_\infty^{\frac{1}{2}} : \hat{b} \in C_c(\mathcal{G}(C), \Sigma(C)), \quad \hat{P}(\hat{b}^*\hat{b}) \leq 1\} \\
&= \sup\{\|\hat{P}(\widehat{b^*a^*ab})\|_\infty^{\frac{1}{2}} : \hat{b} \in C_c(\mathcal{G}(C), \Sigma(C)), \quad \hat{P}(\hat{b}^*\hat{b}) \leq 1\}.
\end{aligned} \tag{3.11}$$

Note that a simple calculation using Lemma 3.2.22 shows that

$$\hat{P} \circ \psi = \psi \circ P$$

and hence $\|\hat{P}(\widehat{b^*a^*ab})\|_\infty = \|P(\widehat{b^*a^*ab})\|_\infty = \|P(b^*a^*ab)\|_\infty$ where the last equality is due to the fact that the restriction of ψ to C is isometric. Thus

$$\|\hat{a}\|_r = \|a\|,$$

and hence $\psi : A_c \to C_c(\mathcal{G}(C), \Sigma(C))$ is isometric and hence continuous, and so by extension via continuity we obtain that ψ extends to a C^*-algebra isomorphism carrying (A, C) onto $(C_r^*(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$. □

For convenience, we state Theorems 3.1.7 and 3.2.26 as one theorem, which is Theorem A from the Introduction:

Theorem 3.2.27. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact effective Hausdorff groupoid. Then $(C_r^*(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$ is a Cartan pair.*

Conversely, let (A, C) be a Cartan pair. Then there exists a twisted étale locally compact effective Hausdorff groupoid $(\mathcal{G}, \Sigma)$ and a C^*-algebra isomorphism carrying (A, C) onto $(C_r^*(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$.

For a complete account, we will state what is Proposition 5.11 in [64]. The proof is identical as that in [64] without needing to assume separability of the C^*-algebra:

Proposition 3.2.28. *Let (A, C) be a Cartan pair. Then C has the unique extension property for pure states if and only if the Weyl groupoid $\mathcal{G}(C)$ is principal.*

3.3 Uniqueness of the Twisted Groupoid Associated to a Cartan Pair

In this section we prove that the procedure of going from a twisted groupoid to a Cartan pair and vice-versa as in Sections 3.1 and 3.2 are, up to isomorphism, inverses of each other. The proofs can be found in Section 4 in [64]. They do not require any second countability assumptions on the groupoid.

The second statement of Theorem 3.2.27 already shows that the procedure of going from a Cartan pair to the Weyl twisted groupoid and from that to a Cartan pair (using the first statement of Theorem 3.2.27) gives an isomorphic Cartan pair. We now show that if we start with a twisted étale locally compact effective Hausdorff groupoid $(\mathcal{G}, \Sigma)$ and obtain the Cartan pair (A, C) as in the first statement of Theorem 3.2.27, then there is an isomorphism of $(\mathcal{G}, \Sigma)$ to the Weyl twisted groupoid associated to (A, C).

We start with what is Proposition 4.8 and 4.13 in [64] but we do not argue using topological principality nor second countability. Note that a crucial use of second countability is made by Renault in the proof of Proposition 4.13 in [64]. Indeed it claims the existence of a function in $C_0(X)$ whose support is exactly a pregiven open set. Locally compact Hausdorff spaces are regular, and regular second countable spaces are metrizable, and hence one can take the function which measures the distance from a point in the open set to the complement of the set.

However, for our purposes, the following statement suffices:

Lemma 3.3.1. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact effective Hausdorff groupoid. Let $(A, C) = (C_r^*(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$. Let L_Σ be the complex line bundle associated to Σ. Then*

$$N_A(C) = \{n \in A : \ \operatorname{supp}'(n) \text{ is a bisection}\} \tag{3.12}$$

and

$$\alpha_n = \alpha_{\operatorname{supp}'(n)} \ \forall \ n \in N_A(C). \tag{3.13}$$

Furthermore we have that the groupoid of germs induced by $\underline{\alpha}(N_A(C))$ (recall Definition 3.2.12) and the groupoid of germs induced by $\alpha(\mathcal{S})$ (recall Definition 3.2.4) are isomorphic to $\mathcal{G}$.

Proof. Lemma 3.1.6 already shows us that the elements of A whose open support is a bisection are normalizer elements. Furthermore, in the proof of that lemma, we had that if $n \in A$ with $S = \operatorname{supp}'(n)$ a bisection and $c \in C$ then

$$n^* * c * n(\sigma) = \overline{n(\tau\sigma^{-1})} c(r(\tau)) n(\tau),$$

for some $\dot{\tau} \in \mathcal{G}_{s(\sigma)} \cap S$ with $\dot{\overline{\tau\sigma^{-1}}} \in S$. Since $r(\tau\sigma^{-1}) = r(\tau)$ it follows that $\dot{\sigma} \in \mathcal{G}^0$, and $\dot{\tau} \in S\dot{\sigma}$. Hence

$$n^* * c * n(\sigma) = n^* n(\sigma) c \circ r(S\{\sigma\}) = n^* n(\sigma) c \circ \alpha_S(\sigma),$$

where α_S is the map from Definition 3.2.4. Comparing this with (3.6) yields $\alpha_n = \alpha_S$.

Now let us show that the open support of a normalizer is a bisection. Let $n \in N_A(C)$. Let $S = \operatorname{supp}'(n)$. Fix $x \in \operatorname{dom}(n)$. Using (3.6) we may write, for all $c \in C$,

$$c(\alpha_n(x)) = \sum_{\dot{\tau} \in \mathcal{G}_x} \frac{|n(\tau)|^2}{n^* n(x)} c(r(\tau)).$$

The spectrum of C is the pure state space on C and so the pure state $\widehat{\alpha_n(x)}$ is a convex linear combination of the pure states $\widehat{r(\tau)}$ where $\dot{\tau}$ ranges over the source fibre of x. Hence if $r(\tau) \neq \alpha_n(x)$ then $n(\tau) = 0$.

Set $T = \{g \in \mathcal{G} : s(g) \in \mathrm{dom}(n), \quad r(g) = \alpha_n(s(g))\}$. We have $S \subset T$. Hence $SS^{-1} \subset TT^{-1}$ and it can easily be checked that $TT^{-1} \subset \mathcal{G}'$. Since SS^{-1} is open, effectivity implies $SS^{-1} \subset \mathcal{G}^0$. Likewise one can show $S^{-1}S \subset \mathcal{G}^0$. Hence S is a bisection. This yields (3.12) and (3.13).

Let us prove the final claim. If $n \in N_A(C)$ then $S = \mathrm{supp}'(n)$ is an open bisection with $\alpha_n = \alpha_S$. In order to prove the claim we need to show that if S is an open bisection then there is an $n \in N_A(C)$ with $\mathrm{supp}'(n) = S$. In fact it suffices to show this for an open set contained in S as the groupoid of germs induced by $\alpha(\mathcal{S})$ where $\mathcal{S}$ is the set of all open bisections is the same as that induced by $\alpha(\mathcal{S}')$ where $\mathcal{S}'$ is a refinement of $\mathcal{S}$. Theorem 12 in the Appendix of [24] ensures that we have a non-vanishing continuous section for L_Σ on a neighbourhood T of $g \in S$, contained in S (L_Σ has the structure of a Banach bundle over the locally compact space $\mathcal{G}$, with fibre $\mathbb{C}$; see Remark 2.1.50). Proposition 1.1 in [54] characterises trivializable line bundles as those that admit a non-vanishing section. Hence $L_\Sigma|_T$ is trivializable. We may use Lemma 3.1.2 to say that there exists a $c \in C$ with compact support inside $s(T)$. Hence $U = \mathrm{supp}'(c)$ is an open set inside $s(T)$, and by the fact that $s : T \to s(T)$ is a homeomorphism we can pull U back to an open bisection $V \subset T$. Restricting attention to V we have that $L_\Sigma|_V$ is trivializable.

Let $u : V \to L_\Sigma$ be a non-vanishing section, and without loss of generality assume $\|u(g)\| = 1$ for all $g \in V$. Define $n : \mathcal{G} \to L_\Sigma$ by $n(g) = u(g)c(s(g))$ if $g \in V$, and 0 otherwise. There exists a net $\{h_\alpha\}_{\alpha \in A}$ in $C_C(U)$ converging uniformly to c. Now using the identifications of Lemma 2.1.53 we have that $uh_\alpha \in C_C(\mathcal{G}, \Sigma)$ converges uniformly to n. Hence it converges in the I-norm as this coincides with the supremum norm on $C_0(\mathcal{G}^0)$, and hence in the C^*-algebra norm $\|\|_r$. Hence $n \in A$ with $\mathrm{supp}'(n) = V$, and hence by (3.12) $n \in N_A(C)$.

Thus we have that the groupoid of germs induced by $\underline{\alpha}(N_A(C))$ and the groupoid of germs induced by $\alpha(\mathcal{S})$ are the same, and by Lemma 3.2.6, isomorphic to $\mathcal{G}$ as this groupoid is effective. □

We are now in a position to state the main result of this section:

Theorem 3.3.2. *Let $(\mathcal{G}, \Sigma)$ be a twisted étale locally compact effective Hausdorff groupoid. Let $(A, C) = (C^*_r(\mathcal{G}, \Sigma), C_{0,r}(\mathcal{G}^0))$. Then we have a homeomorphism of extensions*

$$\begin{tikzcd}
\mathcal{B} \arrow[r] \arrow[d] & \Sigma(C) \arrow[r] \arrow[d] & \mathcal{G}(C) \arrow[d] \\
\mathcal{G}^0 \times \mathbb{T} \arrow[r] & \Sigma \arrow[r] & \mathcal{G}
\end{tikzcd} \tag{3.14}$$

Proof. The left vertical arrow was defined in Definition 3.2.16, and shown to be a homeomorphism in the proof of Lemma 3.2.17. The right vertical arrow is defined using the map in the proof of Lemma 3.2.6 (the map specified in the proof is the inverse of the right vertical arrow). Indeed, by definition $\mathcal{G}(C)$ is the groupoid of germs induced by the pseudogroup $\underline{\alpha}(N_A(C))$, and so by Lemma 3.3.1 is the same as the groupoid of germs induced by $\alpha(\mathcal{S})$, and so is the groupoid $\mathcal{H}$ that appears in (3.5). Since $\mathcal{G}$ is effective the arrow is an isomorphism. Specifically it maps $[x, \alpha_n, y]$ to $\dot{\sigma}$ where $\dot{\sigma}$ is chosen in $\operatorname{supp}'(n)$ with $s(\sigma) = y$, $r(\sigma) = x$ (the existence and well-definedness of σ is due to Lemma 3.3.1). It can be checked easily that the map is a homeomorphism.

We define a map $\Sigma(C) \to \Sigma$ by $[x, n, y] \to \frac{n(\sigma)}{|n(\sigma)|}\sigma$, where $\sigma \in \Sigma$ chosen so that $\dot{\sigma} \in \operatorname{supp}'(n)$ with $s(\dot{\sigma}) = y$ and $r(\dot{\sigma}) = x$. The inverse $\Sigma \to \Sigma(C)$ is defined by sending $\sigma \to \left[x, \frac{\overline{n(\sigma)}}{|n(\sigma)|}n, y\right]$ where $n \in N_A(C)$ chosen so that $n(\sigma) \neq 0$, and $y = s(\dot{\sigma})$, $x = r(\dot{\sigma})$. It is a tedious but straightforward task to check that these maps are well-defined groupoid homomorphisms, and are inverses to one another. That these maps are continuous can be checked by restricting to the topology on $\Sigma(C)|_S$ and using the map f_n as in Definition 3.2.16. That (3.14) is commutative is clear using the definitions of all the arrows. The upper extension has been defined in Definition 3.2.16, and the lower extension is defined in Definition 2.1.35. $\square$

Corollary 3.3.3. *Let (A, C) be a Cartan pair. Then the twisted groupoid that exists by Theorem 3.2.26 is unique up to isomorphism.*

Proof. This follows from Theorem 3.3.2: the twisted groupoid is isomorphic to the Weyl twisted groupoid. $\square$

Chapter 4

Existence of Inductive Limit Cartan Subalgebras in Inductive Limit C^*-algebras

In this chapter we prove that AI and A$\mathbb{T}$-algebras have AI and A$\mathbb{T}$-Cartan subalgebras, respectively. Using the tools that we will develop in this chapter we will generalize these findings to prove that AX-algebras have AX-Cartan subalgebras, whenever X is a finite connected planar graph imbedded in $\mathbb{C}$ (which generalizes the point, interval, and circle).

We have already seen in Example 2.3.29 that AF-algebras have AF-Cartan subalgebras. However, as we shall see, it is significantly more difficult to prove the existence of AI and A$\mathbb{T}$-Cartan subalgebras as compared to their AF counterpart. The main reason for this is that the connecting maps for AI and A$\mathbb{T}$-algebras are not as straightforward as their AF counterpart, as they are not simply unitary conjugates of block diagonal imbeddings.

The main strategy in this chapter is the following. We wish to realize AI, A$\mathbb{T}$, or more generally AX-algebras with X as described above, as inductive limit C^*-algebras with connecting maps that are unitarily equivalent to standard maps (recall Definition 2.3.39). The way we shall achieve this is due to techniques developed by Thomsen, particularly in [75] and [76]. Once we have this, our aim is then to use a practical tool developed by Li in [47] which reduces the question of whether an inductive limit Cartan subalgebra exists to the level of building blocks and connecting maps. We will describe this tool in Section 4.1 below.

For AI-algebras we will be able to realize the connecting maps as unitary conjugates of standard maps. Because the unitaries will be shown to belong to AI-building blocks we may ignore them as far as the inductive limit is concerned (effectively due to Proposition 2.3.46), making the situation rather straightforward. This however will not be possible for A$\mathbb{T}$-building blocks as the unitaries conjugating the standard maps will still belong to AI-building blocks.

To surpass this issue, we will use the notion of maximal homogeneity, following Thomsen's work in [76], which is effectively based on Elliott's investigations in [19], and we will show that this notion coincides with the notion of Cartan subalgebra. Thomsen manages to show that connecting maps between full matrix algebras with entries in $C(\mathbb{T})$ are able to carry a maximally homogeneous subalgebra into a unique one which he constructs in the codomain. We will generalize this result and Elliott's techniques to direct sums of full matrix algebras with entries in $C(\mathbb{T})$.

Finally, we will pinpoint the main topological properties that the interval or circle enjoy that allow us to construct the inductive limit Cartan subalgebras. With this understanding we will be able to generalize our results to finding AX-Cartan subalgebras, with X as above satisfying the topological properties required.

As we discussed in the Introduction, the layout of this chapter is purposely set to reflect the true timeline in which the results were conceived. Rather than presenting the proof for existence of AX-Cartan subalgebras directly, which would automatically encapsulate the result for AI and A$\mathbb{T}$-Cartan subalgebras, we will rather prove the latter results first in order to highlight how the ideas generalize. We believe that this is more useful from a research perspective, as well as being faithful to the true timeline of our research. The preliminaries for this chapter are Sections 2.1, 2.2 and Subsections 2.3.3 and 2.3.4 from Section 2.3, as well as Chapter 3.

4.1 A Practical Criterion for Existence of Inductive Limit Cartan Subalgebras

We will describe in this section a practical tool due to Li in [47], which is based on work with Barlak in [8], that allows us to determine whether inductive limit Cartan subalgebras exist in certain inductive limit C^*-algebras. The criterion is simply to check whether the (injective) connecting maps carry a Cartan subalgebra of the building block into a Cartan subalgebra of the next building block, whether they carry normalizer set into normalizer set and whether they are compatible with the conditional expectation. If these conditions are satisfied for every connecting map then there will exist an inductive limit Cartan subalgebra.

The setup is as follows. Let $(A, \{\mu_n\}_{n\in\mathbb{N}})$ be the inductive limit C^*-algebra corresponding to the sequence of building blocks $\{A_n\}_{n\in\mathbb{N}}$ and *injective* connecting maps $\{\phi_n\}_{n\in\mathbb{N}}$. Assume that for every $n \in \mathbb{N}$ we have a Cartan subalgebra $C_n \subset A_n$. Then by Theorem 3.2.27 the Cartan pair (A_n, C_n) corresponds to a Weyl twisted groupoid $(\mathcal{G}_n, \Sigma_n)$. Assume further that there exists twisted étale locally compact effective Hausdorff groupoids (H_n, T_n) and twisted groupoid homomorphisms

$$(i_n, \iota_n) : (H_n, T_n) \to (\mathcal{G}_{n+1}, \Sigma_{n+1}), \;\; (\dot{p}_n, p_n) : (H_n, T_n) \to (\mathcal{G}_n, \Sigma_n),$$

such that i_n is an imbedding with open image, $\dot{p}_n$ is surjective, proper and fibrewise bijective (this means that $\dot{p}_n$ restricted as a map from the source fibre of a point $y \in H_n^0$ to the source fibre of $\dot{p}_n(y) \in \mathcal{G}_n^0$ is a bijection). We will write X_n for $\mathcal{G}_n^0$ and Y_n for H_n^0. These assumptions on the groupoid homomorphisms allow us to conclude, by Lemmas 3.2 and 3.4 in [8], that the map

$$C_c(H_n, T_n) \to C_c(\mathcal{G}_{n+1}, \Sigma_{n+1}), \;\; f \to 1_{\iota(T_n)} \cdot f$$

extends to an isometric *-homomorphism

$$(\iota_n)_* : C_r^*(H_n, T_n) \to C_r^*(\mathcal{G}_{n+1}, \Sigma_{n+1})$$

and that the map

$$C_c(\mathcal{G}_n, \Sigma_n) \to C_c(H_n, T_n), \;\; f \to f \circ p_n$$

extends to an isometric *-homomorphism

$$(p_n)^* : C_r^*(\mathcal{G}_n, \Sigma_n) \to C_r^*(H_n, T_n).$$

Assume further that

$$\phi_n = (i_n)_* \circ (p_n)^*.$$

Under all these assumptions, define, for every $n \in \mathbb{N}$,

$$\Sigma_{n,0} = \Sigma_n, \;\; \Sigma_{n,m+1} = p_{n+m}^{-1}(\Sigma_{n,m}) \text{ for } m = 0, 1, 2, \ldots,$$

$$\mathcal{G}_{n,0} = \mathcal{G}_n, \;\; \mathcal{G}_{n,m+1} = \dot{p}_{n+m}^{-1}(\mathcal{G}_{n,m}) \text{ for } m = 0, 1, 2, \ldots,$$

$$\overline{\Sigma_n} = \varprojlim(\Sigma_{n,m}, p_{n+m}), \;\; \overline{\mathcal{G}_n} = \varprojlim(\mathcal{G}_{n,m}, \dot{p}_{n+m}).$$

Note that these inverse limits are being defined in the category of topological groupoids, see Remark 2.4.2. In this situation, we get the following result (which is Theorem 3.6 in [8]):

Proposition 4.1.1.

1. *$(\overline{\mathcal{G}_n}, \overline{\Sigma_n})$ is a twisted groupoid for every $n \in \mathbb{N}$,*

2. *the maps (i_n, ι_n) induce twisted groupoid homomorphisms*

$$(\overline{i_n}, \overline{\iota_n}) : (\overline{\mathcal{G}_n}, \overline{\Sigma_n}) \to (\overline{\mathcal{G}_{n+1}}, \overline{\Sigma_{n+1}})$$

   where $\overline{i_n}$ is an imbedding with open image, and we have that

$$\overline{\mathcal{G}} = \varinjlim(\overline{\mathcal{G}_n}, \overline{i_n}), \quad \overline{\Sigma} = \varinjlim(\overline{\Sigma_n}, \overline{\iota_n})$$

   gives rise to a twisted étale groupoid $(\mathcal{G}, \Sigma)$, such that,

3. *$(A, C) = (\varinjlim(A_n, \phi_n), \varinjlim(C_n, \phi_n))$ is a Cartan pair whose Weyl twisted groupoid is given by $(\overline{\mathcal{G}}, \overline{\Sigma})$.*

If $\mathcal{G}_n$ is principal for all $n \in \mathbb{N}$, then $\overline{\mathcal{G}}$ is principal.

This setup together with Proposition 4.1.1 can be captured visually by the following diagram:

$$\begin{tikzcd}
(\mathcal{G}_n, \Sigma_n) & (\mathcal{G}_{n,1}, \Sigma_{n,1}) \arrow[l, two heads] \arrow[d, hook] & (\mathcal{G}_{n,2}, \Sigma_{n,2}) \arrow[l, two heads] \arrow[d, hook] & (\mathcal{G}_{n,3}, \Sigma_{n,3}) \arrow[l, two heads] \arrow[d, hook] & \cdots & (\overline{\mathcal{G}_n}, \overline{\Sigma_n}) \arrow[d, hook, dashed] \\
& (\mathcal{G}_{n+1}, \Sigma_{n+1}) & (\mathcal{G}_{n+1,1}, \Sigma_{n+1,1}) \arrow[l, two heads] \arrow[d, hook] & (\mathcal{G}_{n+1,2}, \Sigma_{n+1,2}) \arrow[l, two heads] \arrow[d, hook] & \cdots & (\overline{\mathcal{G}_{n+1}}, \overline{\Sigma_{n+1}}) \arrow[d, hook, dashed] \\
& & (\mathcal{G}_{n+2}, \Sigma_{n+2}) & (\mathcal{G}_{n+2,1}, \Sigma_{n+2,1}) \arrow[l, two heads] & \cdots & (\overline{\mathcal{G}_{n+2}}, \overline{\Sigma_{n+2}}) \arrow[d, hook, dashed] \\
& & & (\mathcal{G}_{n+3}, \Sigma_{n+3}) & \cdots & (\overline{\mathcal{G}_{n+3}}, \overline{\Sigma_{n+3}}) \\
& & & & \ddots & \vdots \\
& & & & & (\overline{\mathcal{G}}, \overline{\Sigma})
\end{tikzcd} \tag{4.1}$$

What we require now is to figure out what properties an arbitrary prespecified collection of injective connecting maps $\{\phi_n\}_{n \in \mathbb{N}}$ should have in order to be able to write them as $\phi_n = (i_n)_* \circ (p_n)^*$ as in the above setup, and hence conclude that we get an inductive limit Cartan subalgebra. Proposition 5.4 in [47] answers this question:

Proposition 4.1.2. *Let (A, C) and $(\hat{A}, \hat{C})$ be Cartan pairs with corresponding faithful conditional expectations P and $\hat{P}$, respectively. Let their corresponding Weyl* *twisted groupoids be $(\mathcal{G}, \Sigma)$ and $(\hat{\mathcal{G}}, \hat{\Sigma})$ respectively. Assume we have an injective *-homomorphism $\phi : A \to \hat{A}$. Then the following are equivalent:*

1. *$\phi(C) \subseteq \hat{C}$, $\phi(N_A(C)) \subseteq N_{\hat{A}}(\hat{C})$, and $\hat{P} \circ \phi = \phi \circ P$.*

2. *There exists a twisted groupoid (H, T) and twisted groupoid homomorphisms $(i, \iota) : (H, T) \to (\hat{\mathcal{G}}, \hat{\Sigma})$, $(\dot{p}, p) : (H, T) \to (\mathcal{G}, \Sigma)$ such that i is an imbedding with open image, $\dot{p}$ is surjective, proper and fibrewise bijective, and such that $\phi = i_* \circ p^*$.*

Remark 4.1.3. The proof of Proposition 4.1.2 is rather technical, however we will require some parts of the constructions in it for Chapter 5. The following list highlights these essential items, but for a full account, one can refer to the proof given in [47].

- In the proof of the implication $1 \implies 2$ one needs to construct an intermediate Cartan pair $(\breve{A}, \breve{C})$ whose corresponding Weyl twisted groupoid will be the one declared (H, T). $\breve{C}$ is declared as the ideal of $\hat{C}$ generated by $\phi(C)$. Note that in the case that ϕ is unital we have $\breve{C} = \hat{C}$. $\breve{A}$ is declared as $C^*(\phi(A), \breve{C})$. It is shown that $(\breve{A}, \breve{C})$ is a Cartan pair.

- By noting that $N_{\breve{A}}(\breve{C}) \subseteq N_{\hat{A}}(\hat{C})$ the proof defines the pair of maps (i, ι) by

  $$i([x, \alpha_n, y]) = [x, \alpha_n, y], \quad \iota([x, n, y]) = [x, n, y] \text{ for all } x, y \in H^0, n \in N_{\breve{A}}(\breve{C}),$$

  and shows they satisfy the required properties. Then by showing that

  $$T = \bigcup_{\phi(n) \in \phi(N_A(C))} \{[\alpha_{\phi(n)}(x), \phi(n), x] : x \in \mathrm{dom}(\phi(n))\}$$

  the pair of maps $(\dot{p}, p)$ is defined by

  $$\dot{p}([x, \alpha_{\phi(n)}, y]) = [\phi^*(x), \alpha_n, \phi^*(y)], \quad p([x, \phi(n), y]) = [\phi^*(x), n, \phi^*(y)]$$

  for $x, y \in H^0, n \in N_A(C)$, and where ϕ^* is dual to the map $C \to \breve{C}, c \to \phi(c)$. The standing assumption on ϕ being injective is to ensure that the dual map is surjective hence so is $\dot{p}$. The other properties required are then verified.

- It is shown that there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\cong"'] & \breve{A} \arrow[r, "\subseteq"] \arrow[d, "\cong"'] & \hat{A} \arrow[d, "\cong"'] \\ C_r^*(\mathcal{G}, \Sigma) \arrow[r, "p^*"] & C_r^*(H, T) \arrow[r, "\iota_*"] & C_r^*(\hat{\mathcal{G}}, \hat{\Sigma}) \end{tikzcd}$$

where the isomorphisms are the ones given by Theorem 3.2.27.

Remark 4.1.4. It is easy to see from (4.1) that if for every $n \in \mathbb{N}$ we have that $i_n(H_n^0) = \mathcal{G}_{n+1}^0$, it will follow that all the injection arrows in (4.1) become identities when the diagram is restricted to just unit spaces, and hence $\overline{\mathcal{G}}^0$ is homeomorphic to $\varprojlim(\mathcal{G}_n^0, \dot{p}_n)$. This is also noted in Remark 5.6 in [47].

Putting (4.1), Proposition 4.1.2, Remark 4.1.3 and Remark 4.1.4 together, we obtain the following summative result, which is also Theorem 1.10 in [47]:

Proposition 4.1.5. *Let $(A, \{\mu_n\}_{n\in\mathbb{N}})$ be the inductive limit C^*-algebra corresponding to the sequence of building blocks $\{A_n\}_{n\in\mathbb{N}}$ and injective connecting maps $\{\phi_n\}_{n\in\mathbb{N}}$. Assume for every $n \in \mathbb{N}$ that (A_n, C_n) is a Cartan pair whose corresponding conditional expectation is P_n, and whose corresponding Weyl twist is $(\mathcal{G}_n, \Sigma_n)$. Assume that*

$$\phi_n(C_n) \subseteq C_{n+1}, \quad \phi(N_{A_n}(C_n)) \subseteq N_{A_{n+1}}(C_{n+1}), \quad P_{n+1} \circ \phi_n = \phi_n \circ P_n, \quad \text{for all } n \in \mathbb{N}.$$

Then $(A, C) = (\varinjlim(A_n, \phi_n), \varinjlim(C_n, \phi_n))$ is a Cartan pair, whose corresponding Weyl groupoid is obtained as $(\overline{\mathcal{G}}, \overline{\Sigma})$ in (4.1). If we further assume that the corresponding maps $i_n : H_n \to \mathcal{G}_{n+1}$, as discussed above, map unit space to unit space, then the spectrum of the inductive limit Cartan subalgebra C is $\varprojlim(\mathcal{G}_n^0, \dot{p}_n)$.

4.2 Existence of AI-Cartan Subalgebras

In this section we prove the existence of AI-Cartan subalgebras in unital AI-algebras. The idea will be to realize every AI-algebra as one whose connecting maps are of a standard form (see Definition 2.3.39). This can be achieved due to the results in [76]. Then we wish to use Proposition 4.1.5 to obtain the existence of inductive limit Cartan subalgebras. In what follows we consider a unital AI-algebra $A = \varinjlim(A_n, \phi_n)$, with unital and injective connecting maps ϕ_n, and building blocks $A_n \in c_I$ (recall Definition 2.3.27).

Lemma 4.2.1. *Let $F = \bigoplus_{j=1}^{N} M_{n_j}$ be a finite dimensional C^*-algebra. Let $B \subset F$ be a Cartan subalgebra. Then there exists a system of matrix units $\{e_{pq}^j : j \in \{1, \ldots, N\}, p, q \in \{1, \ldots, n_j\}\}$ for F such that $B = \operatorname{span}\{e_{pp}^j : j \in \{1, \ldots, N\}, p \in \{1, \ldots, n_j\}\}$.*

Proof. The maximal dimension of the span of a set of minimal orthogonal projections in B must be $\sum_{j=1}^{N} n_j$, for if it was less we may find a new projection that is orthogonal to all of them, but by B being a masa it means this one belongs to B, which would be a contradiction. It cannot be more than $\sum_{j=1}^{N} n_j$ as then we would have that F contains more than $\sum_{j=1}^{N} n_j$ non-zero orthogonal projections, which is false.

Hence we may label a maximal set of minimal orthogonal projections of B as P_q^j for $j = 1, \ldots, N$ and $q = 1, \ldots, n_j$. These span B. They each have trace one (otherwise their sum will be a projection in F of trace greater than $\sum_{j=1}^{N} n_j$ which is impossible), and hence for every $j \in \{1, \ldots, N\}$ we have that $P_{q_1}^j$ is Murray-von Neumann equivalent to $P_{q_2}^j$ for all $q_1, q_2 \in \{1, \ldots, n_j\}$ (see Example 2.2.47). Write e_{1q}^j for the partial isometry which witnesses the equivalence between P_q^j and P_1^j, for $q = 1, \ldots, n_j$. The define $e_{pq}^j = (e_{1p}^j)^* e_{1q}^j$ and this yields the desired result. □

Definition 4.2.2. In the situation of Lemma 4.2.1 we will write the matrix units more concisely as $\{e_{pq}^j\}$, and they will be called *matrix units with respect to the Cartan subalgebra.*

Remark 4.2.3. We will sometimes express elements in $\bigoplus_{j=1}^{N} C[0,1] \otimes M_{n_j}$ as $\sum f_{pq}^j \otimes e_{pq}^j$ without specifying the index of summation. We will also identify $\bigoplus_{j=1}^{N} C[0,1] \otimes M_{n_j}$ in the canonical way with $C([0,1], \bigoplus_{j=1}^{N} M_{n_j})$. Sometimes we will use this identification without comment.

Lemma 4.2.4. *Suppose F is a finite dimensional C^*-algebra and $A = C[0,1] \otimes F \in c_I$. Let B be a Cartan subalgebra of F. Then $C := C[0,1] \otimes B$ is a Cartan subalgebra of A.*

Proof. That C is unital and commutative is obvious. In fact it is a masa. Indeed, let $\{e_{pq}^j\}$ be a basis for F with respect to B. Assume $a = \sum f_{pq}^j \otimes e_{pq}^j \in A$ commutes with all of C. Write $a = a_d + a_o$ where a_d is diagonal (with respect to the matrix units $\{e_{pq}^j\}$) and a_o is off-diagonal. It is clear that $a_o = a - a_d$ also commutes with all of C. If a_o is non-zero then there exists j_0 and p_0 such that $p = 1 \otimes e_{p_0 p_0}^{j_0}$ satisfies $a_o p \neq 0$. But $p = p^2$ belongs to C and so $0 \neq a_o p = p a_o = p a_o p = 0$, a contradiction. Hence $a_o = 0$ and a belongs to C.

Next we check that C is regular. Since $\{f \otimes n : f \in C[0,1], n \in N_F(B)\}$ generates A as a C^*-algebra, and is a subset of $N_A(C)$, the condition follows.

Finally, we need the existence of a faithful conditional expectation from A onto C. Let $P : F \to B$ be the unique faithful conditional expectation given by projection onto the diagonal (with respect to $\{e^j_{pq}\}$) (see also page 6 of [72]). By [82], a projection $A \to C$ of norm 1 is a conditional expectation. Note that $\mathrm{id}\otimes P : A \to C$ is a projection. Note that if $x = \sum f_s \otimes y_s \in A$ has norm at most one, we obtain,

$$\|\mathrm{id} \otimes P(x)\| = \|\sum_s f_s \otimes P(y_s)\| = \sup_{t\in[0,1]} \|\sum_s f_s(t)P(y_s)\| = \sup_{t\in[0,1]} \|P(\sum_s f_s(t)y_s)\| \leq$$

$\|x\| \leq 1$.

Here we have used the canonical identification of $C[0, 1]\otimes F$ with $C([0, 1], F)$. Hence $\|\mathrm{id} \otimes P\| = 1$.

Let us check that $\mathrm{id} \otimes P$ is faithful. Let $x = \sum f^j_{pq} \otimes e^j_{pq} \in A$. Then $x^*x = \sum \overline{f^j_{pq_1}} f^j_{pq_2} \otimes e^j_{q_1q_2}$. Hence $\mathrm{id} \otimes P(x^*x) = \sum |f^j_{pq}|^2 \otimes e^j_{qq}$. If this is 0 then $f^j_{pq} = 0$ for all $j \in \{1, \ldots , N\}$, $p, q \in \{1, \ldots , n_j\}$. Hence $x = 0$. □

Lemma 4.2.5. *Let $A = C[0, 1] \otimes F$ and $C = C[0, 1] \otimes B$ be as in Lemma 4.2.4, with matrix units $\{e^j_{pq}\}$ with respect to B. Then $N_A(C)$ consists of those elements n in A satisfying that $n(t)$ has at most one non-zero entry in any row or column (with respect to the matrix units $\{e^j_{pq}\}$), for all $t \in [0, 1]$.*

Proof. Let $n = \sum f^j_{pq} \otimes e^j_{pq}$ have the stated property. Let $x = f \otimes e^S_{II}$ and assume n^*xn is non-zero. This means that there are $q_1, \ldots , q_R$ and $f^S_{Iq_1} \otimes e^S_{Iq_1}, \ldots , g^S_{Iq_R} \otimes e^S_{Iq_R}$ appearing as summands of n. Hence

$$n^*xn = \sum_{k,l=1}^{R} \overline{f^S_{Iq_k}} f f^S_{Iq_l} \otimes e^S_{q_kq_l}.$$

The property assumed about n means that $\overline{f^S_{Iq_k}} f^S_{Iq_l} = 0$ if $k \neq l$ (as this holds pointwise). Hence we see that n^*xn is diagonal with respect to the system $\{e^j_{pq}\}$. Hence $n^*Cn \subseteq C$. Similarly we can show $nCn^* \subseteq C$.

Conversely, let $n = \sum f^j_{pq} \otimes e^j_{pq} \in N_A(C)$. Whenever there exists a j and a p and $q_1 \neq q_2$ such that $f^j_{pq_1} \otimes e^j_{pq_1}$ and $f^j_{pq_2} \otimes e^j_{pq_2}$ appear as summands of n, then $n^*(1 \otimes e^j_{pp})n$ contains the summand $\overline{f^j_{pq_1}} f^j_{pq_2} \otimes e^j_{q_1q_2}$. But since it is supposed to be diagonal, we have $\overline{f^j_{pq_1}} f^j_{pq_2} = 0$. A similar argument holds when the column index is fixed rather than the row index. This implies that n has the required property. □

The contents of the following lemma can be found in [76].

Lemma 4.2.6. *Let* $A = \bigoplus_{j=1}^{N} C[0,1] \otimes M_{n_j}, B = \bigoplus_{i=1}^{M} C[0,1] \otimes M_{m_i}$, *and* $\phi : A \to B$ *a unital *-homomorphism. Then there exists a standard map* $\psi : A \to B$ *such that* $\phi \sim_{au} \psi$.

Proof. Let $\{e^j_{pq}\}$ be a system of standard matrix units for $\bigoplus_{j=1}^{N} M_{n_j}$. Let $\{k_{ij}\}$ be the index system with respect to ϕ. Let $w_{ij} = \phi_i(\iota \otimes e^j_{11}) \in \Pi_i(B)$, where $\iota \in C[0,1]$ is the map defined by $\iota(t) = t$. Consider the spectrum of $w_{ij}(t)$ (in $\phi_i(1 \otimes e^j_{11})(t) M_{m_i} \phi_i(1 \otimes e^j_{11})(t)$) to be written as $\{g^{ij}_1(t), \ldots, g^{ij}_{k_{ij}}(t)\}$ for $t \in [0,1]$, where we have ordered $g^{ij}_1 \leq \ldots \leq g^{ij}_{k_{ij}}$. Example 3.1.6 in Chapter 1 of [67] tells us that the eigenvalue functions $g^{ij}_s : [0,1] \to [0,1]$ are continuous for $s = 1, 2, \ldots, k_{ij}$. Denote the set $\{g^{ij}_1(t), \ldots, g^{ij}_{k_{ij}}(t)\}$ obtained as above from the map ϕ and the given procedure by $\hat{\phi}_{ij}(t)$. Now define $\psi : A \to B$ to be a standard map defined as in (2.32), with eigenvalue functions $\{g^{ij}_s : i \in \{1, \ldots, M\}, j \in \{1, \ldots, N\}, s \in \{1, \ldots, k_{ij}\}\}$.

Theorem 3.1(a) in [76] tells us that two unital *-homomorphisms $\phi_1, \phi_2 : A \to B$ are approximately unitarily equivalent if and only if when performing the above procedure we end up with $(\hat{\phi_1})_{ij}(t) = (\hat{\phi_2})_{ij}(t)$ for all i and j and $t \in [0,1]$. Hence it follows by construction that $\hat{\phi}_{ij}(t) = \hat{\psi}_{ij}(t)$ and so $\phi \sim_{au} \psi$. □

Remark 4.2.7. We will see in the later sections, when we consider AX-building blocks (particularly in Lemma 4.4.6), that the conclusion of Lemma 4.2.6 above can be strengthened to unitary equivalence rather than approximate unitary equivalence. However, for our current purposes, the statement of Lemma 4.2.6 suffices as it is.

Lemma 4.2.8. *Let* $A = \varinjlim(A_n, \phi_n)$ *be a unital AI-algebra with unital and injective connecting maps. Then there exists unital and injective standard maps* $\psi_n : A_n \to A_{n+1}$ *such that* $A \cong \varinjlim(A_n, \psi_n)$.

Proof. By Lemma 4.2.6, we construct the ψ_n's satisfying $\phi_n \sim_{au} \psi_n$. Because each ϕ_n is injective, the approximate unitary equivalence implies the same for ψ_n. Noting that AI-building blocks are finitely generated as C^*-algebras, the isomorphism follows from Proposition 2.3.46. □

We are now in a position to prove the main theorem of this section, which is Theorem B in the Introduction:

Theorem 4.2.9. *Every unital AI-algebra $A = \varinjlim(A_n, \phi_n)$ with unital and injective connecting maps contains an AI-Cartan subalgebra.*

Proof. By Lemma 4.2.8 we may assume our connecting maps are injective standard maps of the form (2.32). We have that $A_n = C([0,1]) \otimes F_n$ for a finite dimensional C^*-algebra F_n. Let $D_n \subset F_n$ be the canonical diagonal subalgebra, which is a Cartan subalgebra. By Lemma 4.2.4, $C_n := C([0,1]) \otimes D_n$ is a Cartan subalgebra of A_n. It is clear that $\phi_n(C_n) \subset C_{n+1}$, as ϕ is a standard map. Lemma 4.2.5 allows us to see that $\phi_n(N_{A_n}(C_n)) \subset N_{A_{n+1}}(C_{n+1})$. If $P_n : A_n \twoheadrightarrow C_n$ is the conditional expectation given by projection onto the diagonal as in the proof of Lemma 4.2.4, it is clear that $\phi_n \circ P_n = P_{n+1} \circ \phi_n$. Then by Proposition 4.1.5 we have that $C := \varinjlim(C_n, \phi_n)$ is an AI-Cartan subalgebra of A. □

Remark 4.2.10. The AI-Cartan subalgebras constructed above are in fact C^*-diagonals (they satisfy the unique extension property for pure states). See Remark 4.4.13 for the details.

4.3 Existence of A$\mathbb{T}$-Cartan Subalgebras

In this section we prove the existence of A$\mathbb{T}$-Cartan subalgebras in unital A$\mathbb{T}$-algebras. The idea will be as follows. Building on Elliott's work in [19] (which was done only for full matrix algebras over $C(\mathbb{T})$) we show that arbitrary unital connecting maps between A$\mathbb{T}$-building blocks are sufficiently close (on finite sets) to maximally homogeneous *-homomorphisms. We also show that if the connecting map is additionally assumed injective we can get a sufficiently close injective maximally homogeneous connecting map. Using an intertwining argument we may thus realize our A$\mathbb{T}$-algebras as those arising through maximally homogeneous connecting maps. Then, extending Thomsen's work in [75] (which was done only for full matrix algebras over $C(\mathbb{T})$) we will realize maximally homogeneous *-homomorphisms between A$\mathbb{T}$-building blocks as those that are unitarily equivalent to standard maps. However, this type of diagonalization only works when we treat an A$\mathbb{T}$-building block as a subalgebra of an AI-building block in the natural way, where functions agree at the endpoints. Hence the unitary witnessing the diagonalization will be over the unit interval. Then building further on Thomsen's work in [75] we will obtain that any maximally homogeneous subalgebra of an A$\mathbb{T}$-building block has image (under the maximally homogeneous connecting map) contained in a unique maximally homogeneous subalgebra of the codomain A$\mathbb{T}$-building block. In order for such results to become useful, we will prove that the notion of being a Cartan subalgebra coincides exactly with the notion of being a maximally homogeneous subalgebra in the class $c_{\mathbb{T}}$.

Definition 4.3.1. If $A = \bigoplus_{j=1}^{N} C(\mathbb{T}) \otimes M_{n_j}$ is an A$\mathbb{T}$-building block, it can be identified with the C^*-subalgebra of $\bigoplus_{j=1}^{N} C([0,1]) \otimes M_{n_j}$ where functions agree at the endpoints. For $f \in A$ we write $f_{[0,1]}$ for this identification. Of course, $[0,1]$ may be replaced by any closed interval. A may also be identified with the bounded one-periodic functions of $\bigoplus_{j=1}^{N} C_b(\mathbb{R}) \otimes M_{n_j}$. For $f \in A$ we write $f_{\mathbb{R}}$ for this identification.

We start with the following lemma, which is an easy generalization of Lemma 1.2 in [75], with essentially the same proof:

Lemma 4.3.2. *Given $A = \bigoplus_{j=1}^{N} C(\mathbb{T}) \otimes M_{n_j}$ an A$\mathbb{T}$-building block, and a maximally homogeneous subalgebra $C \subset A$, there exists a system of matrix units*

$$\{{}_{\mathbb{R}}e^j_{pq} \in C_b(\mathbb{R}, \bigoplus_{j=1}^{N} M_{n_j})\}$$

and permutations $\sigma_j \in \Sigma_{n_j}$, $j = 1, 2, \ldots, N$ such that

$${}_{\mathbb{R}}e^j_{pp}(t) = {}_{\mathbb{R}}e^j_{\sigma_j(p)\sigma_j(p)}(t+1) \ \ \forall t \in \mathbb{R}, \ \ p \in \{1, \ldots, n_j\},$$

and

$$\begin{aligned} C =& \{f \in C(\mathbb{T}, \bigoplus_{j=1}^{N} M_{n_j}) : f_{\mathbb{R}}(t) \in \operatorname{span}\{{}_{\mathbb{R}}e^j_{pp}(t), j = 1, \ldots, N; \\ & p = p(j) = 1, \ldots, n_j\}, \forall t \in \mathbb{R}\}. \end{aligned}$$

Furthermore, there exists a system of matrix units

$$\{{}_{[0,1]}e^j_{pq} \in C([0,1], \bigoplus_{j=1}^{N} M_{n_j})\}$$

such that

$${}_{[0,1]}e^j_{pq}(0) = {}_{[0,1]}e^j_{\sigma_j(p)\sigma_j(q)}(1)$$

for some permutations $\sigma_j \in \Sigma_{n_j}$ *and*

$$C = \{f \in C(\mathbb{T}, \bigoplus_{j=1}^{N} M_{n_j}) : f_{[0,1]}(t) \in \operatorname{span}\{{}_{[0,1]}e^j_{pp}(t), j = 1, \ldots, N; \\ p = p(j) = 1, \ldots, n_j\}, \forall t \in [0,1]\}.$$

Proof. We have that $\Pi_j(C) \subset C(\mathbb{T}) \otimes M_{n_j}$ is maximally homogeneous. Hence for $\Pi_j(C)$ the result follows directly by the proof of Lemma 1.2 in [75]. Once it is true for $\Pi_j(C)$ it is true for C. □

Definition 4.3.3. We call a system of matrix units as in Lemma 4.3.2 a *system of matrix units with respect to the maximally homogeneous subalgebra*.

Lemma 4.3.4. *Given A as in Lemma 4.3.2, $C \subset A$ is maximally homogeneous if and only if it is a Cartan subalgebra of A.*

Proof. Assume $C \subset A$ is maximally homogeneous. As C contains the center of A it contains the unit. Since $\Pi_j(C)$ is maximally homogeneous in $C(\mathbb{T}) \otimes M_{n_j}$, it is maximally Abelian there (see Section 1 in [75]). Hence C must be maximally Abelian in A.

For regularity, take $f \in A$, and an open subset U of $\mathbb{T}$ that is not all of $\mathbb{T}$, and let χ_U be any element in A supported inside U. Let $p \in \mathbb{T} \setminus U$. We may identify A with the subalgebra of $C([0,1], \bigoplus_{j=1}^{N} M_{n_j})$, where g on the circle is mapped to $g_{[0,1]}$ on the unit interval with $g_{[0,1]}(0) = g_{[0,1]}(1) = g(p)$. Note that for every $t \in [0,1]$ there are scalars $\lambda^j_{pq}(t)$ such that

$$f_{[0,1]}(t) = \sum \lambda^j_{pq}(t)\, {}_{[0,1]}e^j_{pq}(t), \tag{4.2}$$

where the matrix units $\{{}_{[0,1]}e^j_{pq}\}$ are chosen as in Lemma 4.3.2. Note that the choice of functions $\lambda^j_{pq} : [0,1] \to \mathbb{C}, t \to \lambda^j_{pq}(t)$ satisfying (4.2) is unique. Note that the function $\lambda^j_{pq}\, {}_{[0,1]}e^j_{pq}$ is continuous, seen by considering the continuous function $({}_{[0,1]}e^j_{pp}) f_{[0,1]} ({}_{[0,1]}e^j_{qq})$ and using (4.2). Hence the function $(\chi_U)_{[0,1]} \lambda^j_{pq} ({}_{[0,1]}e^j_{pq})$ is continuous on $[0,1]$, and vanishes at the endpoints, hence it corresponds to an element n of A. Using Lemma 4.3.2 we can see that for all $c \in C$, $(n^*cn)_{[0,1]}$ and $(ncn^*)_{[0,1]}$, when evaluated at a point $t \in [0,1]$, belong to the span of $\{{}_{[0,1]}e^j_{pp}(t)\}$. Hence $n \in N_A(C)$. Thus it follows that $\chi_U f$ is a sum of normalizer elements. Since $\mathbb{T}$ can be covered by proper open subsets, we may find a partition of unity with respect to such a cover, and hence f can be written as a sum of things of the form considered above, namely $\chi_U f$. Thus f is a sum of normalizer elements. Thus C is regular in A.

Finally, we define $P : A \to C$ by

$$P(f)_{[0,1]} = \sum_{j=1}^{N} \sum_{p=1}^{n_j} {}_{[0,1]}e^j_{pp} f_{[0,1]}\,{}_{[0,1]}e^j_{pp}. \tag{4.3}$$

That this is well defined (in other words that the image belongs to C) follows from Lemma 4.3.2. It is easy to see that P is linear, and that it is a projection follows from the description of C given in Lemma 4.3.2. Note that

$$\sup_{t\in[0,1]} \|P(f)_{[0,1]}(t)\| = \sup_{t\in[0,1]} \|P_t(f_{[0,1]}(t))\| \leq \|f_{[0,1]}\|_\infty,$$

where we have used the existence of the faithful conditional expectation P_t, which maps a matrix to its diagonal with respect to the system of matrix units $\{e^j_{pq}(t)\}$. Hence P has norm 1. It follows by [82] that P is a conditional expectation. If $P(f^*f)_{[0,1]} = 0$ then $P(f^*f)_{[0,1]}(t) = P_t(f^*_{[0,1]}f_{[0,1]}(t)) = 0$ for all $t \in [0,1]$, and by faithfulness of P_t we have that $f_{[0,1]}(t) = 0$ for all $t \in [0,1]$ and hence $f = 0$. So P is faithful. Hence C is a Cartan subalgebra of A.

For the converse statement, assume $C \subset A$ is a Cartan subalgebra. It is easy to see that $C_j = \Pi_j(C) \subset C(\mathbb{T}) \otimes M_{n_j}$ is also a Cartan subalgebra. Since circle-algebras are continuous trace C^*-algebras (see for example Example 5.18 in [62]) it follows from Proposition 6.1 in [64] that C_j has the unique extension property for pure states. It then follows from Section 1 in [75] that C_j is maximally homogeneous, which implies this is true for C. $\square$

The construction of the conditional expectation in the proof of Lemma 4.3.4 is unique, in the following sense:

Lemma 4.3.5. *Assume $A = \bigoplus_{j=1}^{N} C(\mathbb{T}) \otimes M_{n_j}$ is an $A\mathbb{T}$-building block, and $C \subset A$ a maximally homogeneous subalgebra, having the description*

$$C = \{f \in A : f_I(t) \in \operatorname{span}\{p_r(t) : 1 \leq r \leq M\} \ \ \forall t \in I\} \tag{4.4}$$

where

- *in the case $I = [0,1]$, and A is identified with the subalgebra $A_{[0,1]}$ of $\bigoplus_{j=1}^{N} C[0,1] \otimes M_{n_j}$ consisting of those elements agreeing at the endpoints, we assume that the p_r's belong to $A_{[0,1]}$ and that there is a permutation $\sigma \in \Sigma_M$ such that*

$p_r(0) = p_{\sigma(r)}(1)$, and,

- *in the case $I = \mathbb{R}$, and A is identified with the subalgebra $A_{\mathbb{R}}$ of $\bigoplus_{j=1}^{N} C_b(\mathbb{R}) \otimes M_{n_j}$ consisting of one-periodic elements, we assume that the p_r's belong to $A_{\mathbb{R}}$ and that there is a permutation $\sigma \in \Sigma_M$ such that $p_r(t) = p_{\sigma(r)}(t+1)$.*

Then the faithful conditional expectation $P : A \to C$ corresponding to the Cartan pair (A, C) is unique and given by

$$P(f)_I(t) = \sum_{r=1}^{M} p_r(t) f_I(t) p_r(t). \tag{4.5}$$

In particular P is a linear projection of norm one.

Proof. By Theorem 3.2.27 the Cartan pair (A, C) corresponds to a twisted étale effective groupoid, and Lemma 3.1.5 tells us that the conditional expectation is unique. Hence by uniqueness, it suffices to show that (4.5) defines a faithful conditional expectation.

Assume that $I = [0, 1]$ (the case $I = \mathbb{R}$ is similar). Then P is well-defined because $p_r(0) = p_{\sigma(r)}(1)$ meaning $P(f)_I(0) = P(f)_I(1)$. That P is linear is clear. The description of C in (4.4) shows that if $f \in C$ then $P(f) = f$, and so P is a projection onto C. Note that since C is maximally homogeneous, it must be that $M = \sum_{j=1}^{N} n_j$ and so for $t \in [0, 1]$ the set $\{p_r(t)\}$ forms a basis for a Cartan subalgebra D of $\bigoplus_{j=1}^{N} M_{n_j}$. Hence $P(f)_I(t) = P_t(f_I(t))$ where P_t is the unique faithful conditional expectation $\bigoplus_{j=1}^{N} M_{n_j} \to D$ given by projection onto the basis. As in the proof of Lemma 4.3.4 it follows that P has norm one and is faithful, since this is satisfied by P_t for every $t \in [0, 1]$. Hence by [82] P is a faithful conditional expectation. □

The following proposition highlights why the term *maximally* appears in the definition of a maximally homogeneous *-homomorphism (recall Definition 2.3.38).

Proposition 4.3.6. *Suppose we have a unital *-homomorphism $\phi : A \to B$, with $A = \bigoplus_{j=1}^{N} C(\mathbb{T}) \otimes M_{n_j}$, $B = \bigoplus_{i=1}^{M} C(\mathbb{T}) \otimes M_{m_i}$. Then for any $t \in \mathbb{T}$ fixed, we have that*

$$\dim(\phi(A)(t)) \leq \sum_{i=1}^{M} \sum_{j=1}^{N} n_j^2 k_{ij},$$

where $\{k_{ij}\}$ is the index system.

Proof. Let $\{e^j_{pq}\}$ be the system of standard matrix units for $\bigoplus_{j=1}^{N} M_{n_j}$. Fix $t \in \mathbb{T}$ and define

$$F^j_{pq} = \{\phi(f \otimes e^j_{pq})(t) : f \in C(\mathbb{T})\},$$

which is a vector subspace of $\bigoplus_{i=1}^{M} M_{m_i}$. Note that

$$\phi(A)(t) = \sum_{j=1}^{N} \sum_{p,q=1}^{n_j} F^j_{pq},$$

where the right hand side is viewed as a sum of vector subspaces. Hence

$$\dim(\phi(A)(t)) \leq \sum_{j=1}^{N} \sum_{p,q=1}^{n_j} \dim(F^j_{pq}).$$

Fix $j \in \{1, \ldots, N\}$ and note that since $\phi(f \otimes e^j_{pq})(t) = \phi(f \otimes 1_j)(t)\phi(1 \otimes e^j_{pq})(t)$, we have that $\dim(F^j_{pq}) \leq \dim(\{\phi(f \otimes 1_j)(t) : f \in C(\mathbb{T})\})$. Let $\{c^{ij}_{pq}\}$ be the system of matrix units with respect to the index system, and V the unitary in Lemma 2.3.37. Let $\overline{p} = \phi(1 \otimes 1_j)(t)$, $\overline{q} = V^*\overline{p}V = \sum_{p=1}^{n_j} c^j_{pp}$, and compute

$$\begin{aligned}
\{\phi(f \otimes 1_j)(t) : f \in C(\mathbb{T})\} &\subseteq C^*(\{\phi(1 \otimes e^j_{pq})(t) : 1 \leq p, q \leq n_j\})' \cap \overline{p}(\bigoplus_{i=1}^{M} M_{m_i})\overline{p} \\
&= C^*(\{\mathrm{Ad}(V)c^j_{pq} : 1 \leq p, q \leq n_j\})' \cap \overline{p}(\bigoplus_{i=1}^{M} M_{m_i})\overline{p} \\
&\subseteq \mathrm{Ad}(V)[C^*(\{c^j_{pq} : 1 \leq p, q \leq n_j\})' \cap \overline{q}(\bigoplus_{i=1}^{M} M_{m_i})\overline{q}] \\
&\cong \bigoplus_{i=1}^{M} M_{k_{ij}}.
\end{aligned} \tag{4.6}$$

Since $\{\phi(f \otimes 1_j)(t) : f \in C(\mathbb{T})\}$ is Abelian we have by (4.6) that its dimension is at most $\sum_{i=1}^{M} k_{ij}$. Hence

$$\dim(\phi(A)(t)) \leq \sum_{j=1}^{N} \sum_{p,q=1}^{n_j} \dim(F^j_{pq}) \leq \sum_{j=1}^{N} \sum_{p,q=1}^{n_j} \sum_{i=1}^{M} k_{ij} = \sum_{j=1}^{N} n_j^2 \sum_{i=1}^{M} k_{ij} = \sum_{i=1}^{M} \sum_{j=1}^{N} n_j^2 k_{ij}$$

as desired. □

Lemma 4.3.7. *A unital *-homomorphism $\phi : C(\mathbb{T}) \otimes M_n \to C(\mathbb{T}) \otimes M_m$ that satisfies that it sends the canonical unitary generator $u \otimes 1_n \in C(\mathbb{T}) \otimes M_n$ to a unitary in $C(\mathbb{T}) \otimes M_m$ with $\frac{m}{n}$ distinct eigenvalues at each evaluation (each of multiplicity n) will be maximally homogeneous.*

Proof. Let $t \in \mathbb{T}$ be fixed, and as in the proof of Proposition 4.3.6, we have that

$$\dim(\phi(C(\mathbb{T}) \otimes M_n)(t)) = \dim(\sum_{p,q=1}^{n} F_{pq}).$$

Let $\{x_1, \ldots, x_R\}$ be a basis for $\{\phi(f \otimes 1)(t) : f \in C(\mathbb{T})\}$ (as a vector space). We claim that $\{x_\mu \phi(1 \otimes e_{pq})(t) : \mu \in \{1, \ldots, R\}, p, q \in \{1, \ldots, n\}\}$ is a basis for $\sum_{p,q=1}^{n} F_{pq}$. That the set is spanning is clear, and to show linear independence assume $\sum_{\mu,p,q} \lambda^\mu_{pq} x_\mu \phi(1 \otimes e_{pq})(t) = 0$. Let $s, p, q \in \{1, \ldots, n\}$ be arbitrary, and multiply the sum on the left by $\phi(1 \otimes e_{sp})(t)$ and on the right by $\phi(1 \otimes e_{qs})(t)$ to get that $\sum_{\mu} \lambda^\mu_{pq} x_\mu \phi(1 \otimes e_{ss})(t) = 0$. Since ϕ is assumed unital, by taking a sum over s one gets that $\sum_{\mu} \lambda^\mu_{pq} x_\mu = 0$. This implies that $\lambda^\mu_{pq} = 0$ for all $\mu \in \{1, \ldots, R\}$, and, since p and q were arbitrary, for all $p, q \in \{1, \ldots, n\}$. Thus $\dim(\phi(C(\mathbb{T}) \otimes M_n)(t)) = Rn^2$.

We now show that $R = \frac{m}{n}$. Note that since normal elements are unitarily diagonalizable, there is a unitary $V \in M_m$ such that $\phi(u \otimes 1_n)(t) = \mathrm{Ad}(V)(\sum_{d=1}^{\frac{m}{n}} \omega_d p_d)$ where the ω_d's are the distinct circle valued eigenvalues ensured by the assumption, and the p_d's are the corresponding mutually orthogonal diagonal projections. Let $q_d = \mathrm{Ad}(V) p_d$ and note that for any Laurent polynomial P we have $\phi(P(u) \otimes 1_n)(t) = \sum_{d=1}^{\frac{m}{n}} P(\omega_d) q_d$. Since $u \otimes 1_n$ is a generator for $C(\mathbb{T})$, it follows that $\dim(\{\phi(f \otimes 1)(t) : f \in C(\mathbb{T})\}) = \frac{m}{n}$, and this completes the proof. □

Remark 4.3.8. The same result applies in Lemma 4.3.7 with $\mathbb{T}$ replaced with a compact Hausdorff space X imbedded in $\mathbb{C}$, where the unitary u replaced by the generator $\iota_X, x \to x$.

Lemma 4.3.9. *Let $A = C(\mathbb{T}) \otimes M_n$ and $p \in A$ a projection with trace k at any (hence all) evaluations $t \in \mathbb{T}$, then pAp is isomorphic to $C(\mathbb{T}) \otimes M_k$, via conjugation by a partial isometry whose initial projection is p and range projection is $1_k \in A$.*

Proof. First we note that if $p \sim q$ via a partial isometry v in A, then $pAp \to qAq$,

$pap \to vpapv^* = vav^*$ is a *-isomorphism. Now we show that $p \sim 1_k$, where $1_k \in A$ can be chosen to be any standard diagonal projection with k non-zero entries. By considering K-theory, we have by Exercise 11.2 in [43] that the map tr_x in Example 2.2.51 is injective, and so we have $[p]_0 = [1_k]_0$. Since $C(\mathbb{T})$ has stable rank one (meaning that the invertible elements of $C(\mathbb{T})$ are dense in $C(\mathbb{T})$), and having stable rank one is closed under matrix amplifications (see page 2 in [2]) it follows that A has stable rank one. This implies that A has the cancellation property (see [83]) and so $p \sim 1_k$. Since $1_k A 1_k \cong C(\mathbb{T}) \otimes M_k$, the result follows. □

The following lemma is a generalization, to direct sums, of Theorem 4.4 in [19]:

Lemma 4.3.10. *Let* $\phi : A \to B$, *with* $A = \bigoplus_{j=1}^{N} C(\mathbb{T}) \otimes M_{n_j}$, $B = \bigoplus_{i=1}^{M} C(\mathbb{T}) \otimes M_{m_i}$, *be a unital *-homomorphism with index system* $\{k_{ij}\}$. *Let* $F \subset A$ *be a finite set and* $\epsilon > 0$. *Then there exists a unital maximally homogeneous *-homomorphism* $\phi' : A \to B$ *such that* $\|\phi(a) - \phi'(a)\| < \epsilon$ *for all* $a \in F$.

Proof. Let $j \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, M\}$. Let $G_j \subset C(\mathbb{T}) \otimes M_{n_j}$ be a finite set to be determined, and $\delta_j > 0$ to be determined. Consider $\rho_{ij} : C(\mathbb{T}) \otimes M_{n_j} \to C(\mathbb{T}) \otimes M_{n_j k_{ij}}$ given as the composition

$$C(\mathbb{T}) \otimes M_{n_j} \xrightarrow{\overline{\phi_{ij}}} \phi_i(1 \otimes 1_j)(C(\mathbb{T}) \otimes M_{m_i})\phi_i(1 \otimes 1_j) \xrightarrow{\mathrm{Ad}(v_{ij})} 1_{n_j k_{ij}}(C(\mathbb{T}) \otimes M_{m_i})1_{n_j k_{ij}} \xrightarrow{g_{ij}} C(\mathbb{T}) \otimes M_{n_j k_{ij}},$$

where $\overline{\phi_{ij}}$ acts as ϕ_{ij} but has the corner $\phi_i(1 \otimes 1_j)(C(\mathbb{T}) \otimes M_{m_i})\phi_i(1 \otimes 1_j)$ as its codomain, $\mathrm{Ad}(v_{ij})$ is the isomorphism from Lemma 4.3.9 induced via conjugation by v_{ij}, and g_{ij} canonically identifies the corner with a full matrix algebra by removing some zeroes. Clearly ρ_{ij} is a unital *-homomorphism.

Now Theorem 4.4 in [19] tells us that there exists a unital *-homomorphism

$$\chi_{ij} : C(\mathbb{T}) \otimes M_{n_j} \to C(\mathbb{T}) \otimes M_{n_j k_{ij}}$$

such that

$$\|\chi_{ij}(a) - \rho_{ij}(a)\| < \delta_j \tag{4.7}$$

for all $a \in G_j$. Furthermore, χ_{ij} sends the canonical unitary generator to a unitary with k_{ij} distinct eigenvalues at every evaluation. By Lemma 4.3.7, χ_{ij} is maximally homogeneous, and so

$$\dim(\chi_{ij}(C(\mathbb{T}) \otimes M_{n_j})(t)) = n_j^2 k_{ij}$$

for all $t \in \mathbb{T}$. It is clear that

$$\dim(g_{ij}^{-1}(\chi_{ij}(C(\mathbb{T}) \otimes M_{n_j}))(t)) = \dim(\chi_{ij}(C(\mathbb{T}) \otimes M_{n_j})(t)) = n_j^2 k_{ij}$$

as well. Now note that conjugation by $v_{ij}^*(t)$ induces a vector space isomorphism

$$g_{ij}^{-1}(\chi_{ij}(C(\mathbb{T}) \otimes M_{n_j}))(t) \to v_{ij}^* g_{ij}^{-1}(\chi_{ij}(C(\mathbb{T}) \otimes M_{n_j}))v_{ij}(t)$$

and hence it follows that $\mathrm{Ad}(v_{ij})^{-1}(g_{ij}^{-1}(\chi_{ij}(C(\mathbb{T}) \otimes M_{n_j})))$ has dimension $n_j^2 k_{ij}$ at every $t \in \mathbb{T}$. Then we have

$$\|\mathrm{Ad}(v_{ij})^{-1} g_{ij}^{-1} \chi_{ij}(a) - \mathrm{Ad}(v_{ij})^{-1} g_{ij}^{-1} \rho_{ij}(a)\| = \|\mathrm{Ad}(v_{ij})^{-1} g_{ij}^{-1} \chi_{ij}(a) - \phi_{ij}(a)\| < \delta_j$$

for all $a \in G_j$, where we consider the image as being inside $C(\mathbb{T}) \otimes M_{m_i}$. We let

$$h_{ij} := \mathrm{Ad}(v_{ij})^{-1} g_{ij}^{-1} \chi_{ij}.$$

Define

$$H_i : A \to C(\mathbb{T}) \otimes M_{m_i}, \ \ (a_1, \ldots, a_N) \to h_{i1}(a_1) + \ldots + h_{iN}(a_N).$$

This is a unital *-homomorphism as the summands are mutually orthogonal. Let $G = G_1 \times \ldots \times G_N$. Then, for $a = (a_1, \ldots, a_N) \in G$ we have

$$\|H_i(a) - \sum_{j=1}^{N} \phi_{ij}(a_j)\| = \|H_i(a) - \phi_i(a)\| < \sum_{j=1}^{N} \delta_j = \delta.$$

We have $\dim(H_i(A)(t)) = \sum_{j=1}^{N} n_j^2 k_{ij}$. Define

$$\phi' = \bigoplus_{i=1}^{M} H_i : A \to B, \tag{4.8}$$

a unital *-homomorphism. It is clear that $\dim(\phi'(A)(t)) = \sum_{i=1}^{M} \sum_{j=1}^{N} n_j^2 k_{ij}$, hence ϕ' is maximally homogeneous, and by choosing $\delta = \epsilon$ and G containing F, we obtain that $\|\phi'(a) - \phi(a)\| < \epsilon$ for all $a \in F$. $\square$

For the next result, we need to consider the Čech cohomology group with coefficients in a topological group. The following definition is from 3.5 in [37].

Definition 4.3.11. Let G be a topological group, and X a topological space. A *G-cocycle* consists of an open cover $\{U_i\}$ of X together with continuous maps $g_{ji} : U_j \cap U_i \to G$ such that $g_{kj}(x)g_{ji}(x) = g_{ki}(x)$ for all $x \in U_k \cap U_j \cap U_i$. In such a situation we write that (U_i, g_{ji}) is a G-cocycle. Denote by $\mathfrak{U}(X, G)$ the set of all such G-cocycles.

Two G-cocycles (U_i, g_{ji}) and (V_r, h_{sr}) are equivalent, written $(U_i, g_{ji}) \sim (V_r, h_{sr})$, if and only if there exists continuous mas $f_i^r : V_r \cap U_i \to G$ such that
$f_j^s(x)g_{ji}(x)f_i^r(x)^{-1} = h_{sr}(x)$ for all $x \in V_s \cap V_r \cap U_j \cap U_i$. We define

$$H^1(X, G) = \mathfrak{U}(X, G)/\sim .$$

Remark 4.3.12. It is shown in 3.5 in [37] that $\sim$ is an equivalence relation so that the definition of $H^1(X, G)$ is well-defined. $H^1(X, G)$ is in one-to-one correspondence with the set of isomorphism classes of principal G-bundles over X (see [4]).

Lemma 4.3.13. *If G is a topological group, and X is a contractible paracompact space, then $H^1(X, G) = 0$.*

Proof. By the remark on page 222 in [55], any principal G-bundle over a contractible paracompact space is trivial. And so the result follows by Remark 4.3.12. $\square$

The following lemma is a generalization, to direct sums, of Theorem 3 in [79], but the proof is essentially the same:

Lemma 4.3.14. *Let $\phi : A \to B$, with $A = \bigoplus_{j=1}^{N} C(\mathbb{T}) \otimes M_{n_j}$, $B = \bigoplus_{i=1}^{M} C(\mathbb{T}) \otimes M_{m_i}$, be a unital maximally homogeneous *-homomorphism. Then there exists a unitary $U = (U_1, U_2, \ldots, U_M) \in \bigoplus_{i=1}^{M} C([0, 1]) \otimes M_{m_i}$ and a set of eigenvalue functions*

$$\{g_s^{ij} : [0, 1] \to \mathbb{T} : 1 \leq i \leq M; 1 \leq j \leq N; 1 \leq s = s(i, j) \leq k_{ij}\},$$

where $\{k_{ij}\}$ is the index system, such that

$$\phi_i((a_1, \ldots, a_N))_{[0,1]}(t) =$$
$$U_i(t)\mathrm{diag}(a_1 \circ g_1^{i1}, a_1 \circ g_2^{i1}, \ldots, a_1 \circ g_{k_{i1}}^{i1}, a_2 \circ g_1^{i2}, \ldots, a_2 \circ g_{k_{i2}}^{i2}, \ldots, a_N \circ g_{k_{iN}}^{iN})(t)U_i(t)^*, \quad (4.9)$$

for all $i \in \{1, \ldots, M\}$, $(a_1, \ldots, a_N) \in A$ and $t \in [0, 1]$. In other words, ϕ is unitarily equivalent to a standard map with eigenvalue functions $\{g_s^{ij}\}$.

Proof. Fix $t_0 \in [0,1]$ and use Lemma 2.3.37 to find a unitary $V \in \bigoplus_{i=1}^{M} M_{m_i}$ such that $V\phi(1 \otimes e^j_{pq})(t_0)V^* = c^j_{pq}$, where the c^{ij}_{pq}'s are the matrix units with respect to the index system (here, and in the rest of the proof, we will skip writing out the identification of $\phi(a)$ with $\phi(a)_{[0,1]}$, for the sake of brevity). We will consider $V \in \bigoplus_{i=1}^{M}(C([0,1]) \otimes M_{m_i})$ by declaring it constant with value V. Hence by continuity it follows that there exists a neighbourhood T_0 around t_0 such that we have

$$\sup\{\|V(t)\phi(1 \otimes e^j_{pq})(t)V(t)^* - c^j_{pq}\| : 1 \leq j \leq N, 1 \leq p, q, \leq n_j\} < \frac{1}{2} \tag{4.10}$$

for all $t \in T_0$. Let $g \in C([0,1],[0,2])$ defined by being non-negative, vanishing on a neighbourhood close to 0, and being $\frac{1}{x}$ for $\frac{1}{2} \leq x \leq 1$.

Now define, for $1 \leq j \leq N$, and $t \in [0,1]$,

$$X_j(t) = c^j_{11}V(t)\phi(1 \otimes e^j_{11})(t)V(t)^* c^j_{11}.$$

It is clear that $X_j(t)$ is a positive element of norm less than one. Hence we may define

$$W(t) = \sum_{j=1}^{N}\sum_{p=1}^{n_j} c^j_{p1}(g(X_j(t)))^{\frac{1}{2}}V(t)\phi(1 \otimes e^j_{1p})(t)V(t)^*, \quad t \in [0,1].$$

It is clear that W is continuous. On T_0 it is in fact a unitary. Indeed, for $t \in T_0$, we have

$$\begin{aligned}
&W(t)W(t)^* = \\
&\sum_{j=1}^{N}\sum_{p=1}^{n_j} c^j_{p1}(g(X_j(t)))^{\frac{1}{2}}V(t)\phi(1 \otimes e^j_{11})(t)V(t)^*(g(X_j(t)))^{\frac{1}{2}}c^j_{1p} = \\
&\sum_{j=1}^{N}\sum_{p=1}^{n_j} c^j_{p1}X_j(t)g(X_j(t))c^j_{1p},
\end{aligned}$$

where we have used (2.30) and (2.31) and where we obtain the last equality by noting that $c^j_{p1} = c^j_{p1}c^j_{11}$ and $c^j_{1p} = c^j_{11}c^j_{1p}$, and that c^j_{11} commutes with $X_j(t)$. Now we aim to show that

$$X_j(t)g(X_j(t)) = c^j_{11} \text{ for } t \in T_0, \tag{4.11}$$

from which it will follow that $W(t)W(t)^* = 1$ (and hence also that $W(t)^*W(t) = 1$) for all $t \in T_0$.

To this end, we make use of the proof of Lemma 1.8 in [27]. Indeed, let $Y_j(t) =$

$V(t)\phi(1\otimes e_{11}^j)(t)V(t)^*$ and consider the commutative C^*-algebra generated by $1, c_{11}^j$ and $X_j(t)$. Let ρ be any pure state on this C^*-algebra (hence multiplicative), and so $\rho(c_{11}^j) \in \{0,1\}$. If $\rho(c_{11}^j) = 0$ then as $0 \leq c_{11}^j Y_j(t) c_{11}^j \leq c_{11}^j$, we have $\rho(c_{11}^j Y_j(t) c_{11}^j) = \rho(X_j(t)) = 0$. Thus $\rho(X_j(t)g(X_j(t))) = 0 = \rho(c_{11}^j)$. If $\rho(c_{11}^j) = 1$ then as $0 \leq c_{11}^j - c_{11}^j Y_j(t) c_{11}^j$ has norm smaller than $\frac{1}{2}$ by (4.10), we get $\rho(c_{11}^j Y_j(t) c_{11}^j) = \rho(X_j(t)) \in [\frac{1}{2}, 1]$. Hence $\rho(X_j(t)g(X_j(t))) = \rho(X_j(t))g(\rho(X_j(t)) = 1 = \rho(c_{11}^j)$. Since ρ was arbitrary, it follows that $X_j(t)g(X_j(t)) = c_{11}^j$, as desired.

Note also that for all $t \in T_0$, one can use (4.11) to see that

$$W(t)V(t)\phi(1\otimes e_{pq}^j)(t)V(t)^*W(t)^* = c_{pq}^j.$$

Repeating this argument around other points, we manage to find a finite open cover $\{T_k\}$ of $[0,1]$ and continuous maps $W_k : T_k \to \mathcal{U}(\bigoplus_{i=1}^M M_{m_i})$ such that

$$W_k(t)\phi(1\otimes e_{pq}^j)(t)W_k(t)^* = c_{pq}^j, \quad \forall t \in T_k.$$

Note that $W_k W_s^*(t)$ commutes with every c_{pq}^j for all $t \in T_k \cap T_s$. Let $\mathcal{W}$ denote the unitary group of $\{c_{pq}^j\}' \cap \bigoplus_{i=1}^M M_{m_i} \cong \bigoplus_{i=1}^M \bigoplus_{j=1}^N M_{k_{ij}}$, where $\{k_{ij}\}$ is the index system with respect to ϕ. Hence $(T_k, W_k W_s^*)$ defines an element in $H^1([0,1],\mathcal{W})$. Since $[0,1]$ is contractible and paracompact it follows from Lemma 4.3.13 that $H^1([0,1],\mathcal{W}) = 0$. Hence $(T_k, W_k W_s^*)$ is equivalent to $(T_k, 1)$ and so as in Definition 4.3.11 we find continuous maps $\overline{V_k} : T_k \to \mathcal{W}$ such that $W_i W_j^* = \overline{V_i}^* \overline{V_j}$ on $T_i \cap T_j$.

Define $S = (S_1, \ldots, S_M) \in C([0,1], \mathcal{U}(\bigoplus_{i=1}^M M_{m_i})$ by $S(t) = \overline{V_k} W_k(t)$ for $t \in T_k$. Note that

$$S\phi(1\otimes e_{pq}^j)S^*(t) = c_{pq}^j$$

for all $t \in [0,1]$, from which it follows that for all $i \in \{1, \ldots, M\}$,

$$S_i\phi_i(1\otimes e_{pq}^j)S_i^*(t) = \Pi_i(c_{pq}^{ij}).$$

It is easy to see that $S\phi(C(\mathbb{T})\otimes 1)S^*(t) \in \{c_{pq}^j\}' \cap \bigoplus_{i=1}^M M_{m_i}$ for all $t \in [0,1]$. Fix i and j and let

$$\overline{p} = S_i\phi_i(1\otimes 1_j)S_i^*(t) = \sum_{p=1}^{n_j} \Pi_i(c_{pp}^{ij})$$

and note that we get

$$S_i\phi_i(C(\mathbb{T})\otimes 1_j)S_i^*(t) = (S_i\phi_i(1\otimes 1_j)S_i^*)S_i\phi_i(C(\mathbb{T})\otimes 1_j)S_i^*(S_i\phi_i(1\otimes 1_j)S_i^*)(t) \in \overline{p}[\{\Pi_i(c_{pq}^{ij'}) : 1\le j'\le N, 1\le p,q\le n_{j'}\}'\cap M_{m_i}]\overline{p} \cong M_{k_{ij}}. \tag{4.12}$$

Since $S_i\phi_i(C(\mathbb{T})\otimes 1_j)S_i^*(t)$ is an Abelian subalgebra, (4.12) yields that its vector space dimension is at most k_{ij}. Note that as vector spaces,

$$\begin{aligned}\dim(S_i(t)(\phi_i(C(\mathbb{T})\otimes 1)(t))S_i^*(t)) &= \dim(\phi_i(C(\mathbb{T})\otimes 1)(t))\\ &= \sum_{j=1}^N \dim(\phi_i(C(\mathbb{T})\otimes 1_j)(t)) = \sum_{j=1}^N d_{ij}(t) \le \sum_{j=1}^N k_{ij},\end{aligned} \tag{4.13}$$

where we have let

$$d_{ij}(t) = \dim(\phi_i(C(\mathbb{T})\otimes 1_j)(t)) \le k_{ij}.$$

The first equality is because vector space dimension is invariant under isomorphisms, the second equality is because the summands are mutually orthogonal, and the last inequality is due to the arguments we made above. We show that d_{ij} is independent of t exactly because ϕ is maximally homogeneous.

To this end note that $\phi_i(A)(t) = \sum_{j,p,q}\phi_i(C(\mathbb{T})\otimes 1_j)(t)\phi_i(1\otimes e_{pq}^j)(t)$, as a sum of vector spaces. Exactly as in the proof of Lemma 4.3.7 one can show that

$$\dim(\phi_i(A)(t)) = \sum_{j=1}^N d_{ij}(t)n_j^2.$$

Since $\phi(A)(t)\subseteq \bigoplus_{i=1}^M \phi_i(A)(t)$, we have that

$$\sum_{i=1}^M\sum_{j=1}^N k_{ij}n_j^2 \le \sum_{i=1}^M\sum_{j=1}^N d_{ij}(t)n_j^2 \le \sum_{i=1}^M\sum_{j=1}^N k_{ij}n_j^2.$$

This implies that $d_{ij}(t) = k_{ij}$, independent of t. Hence (4.13) implies that

$$\dim(S_i(t)(\phi_i(C(\mathbb{T})\otimes 1)(t))S_i^*(t))$$

is constant across $t\in[0,1]$. Now consider the following commutative diagram:

$$\begin{tikzcd}
S\phi(C(\mathbb{T})\otimes 1)S^* \arrow[r, hook] \arrow[d, "\Pi_i"] & \bigoplus_i C([0,1])\otimes(\{\Pi_i(c^{ij}_{p,q}) : j,p,q\}'\cap M_{m_i}) \arrow[d, "\Pi_i"] \\
S_i\phi_i(C(\mathbb{T})\otimes 1)S_i^* \arrow[r, hook] \arrow[d, "\theta"] & C([0,1])\otimes(\{\Pi_i(c^{ij}_{p,q}) : j,p,q\}'\cap M_{m_i}) \arrow[d, "\theta"] \\
\theta(S_i\phi_i(C(\mathbb{T})\otimes 1)S_i^*) \arrow[r, hook] \arrow[d, "\Pi_j"] & \bigoplus_j C([0,1])\otimes M_{k_{ij}} \arrow[d, "\Pi_j"] \\
\Pi_j(\theta(S_i\phi_i(C(\mathbb{T})\otimes 1)S_i^*)) \arrow[r, hook] & C([0,1])\otimes M_{k_{ij}}
\end{tikzcd} \tag{4.14}$$

where θ is the canonical identification of the two algebras. It is clear that all the maps commute with evaluation at t. It is also clear that θ preserves vector space dimension of the evaluation. Let $b_{ij}(t)$ be the dimension of

$$\Pi_j(\theta(S_i\phi_i(C(\mathbb{T})\otimes 1)S_i^*))(t).$$

Since the algebra is Abelian we have from the inclusion in the last row of (4.14) that $b_{ij}(t) \leq k_{ij}$ and that

$$\dim(\theta(S_i\phi_i(C(\mathbb{T})\otimes 1)S_i^*)(t)) = \sum_j k_{ij} \leq \sum_j b_{ij}(t) \leq \sum_j k_{ij}. \tag{4.15}$$

The first equality in (4.15) follows from (4.13). Hence b_{ij} is independent of t and so Lemma 2 in [79] applies and we may diagonalize the Abelian subalgebra in the last row of (4.14) via a unitary. By pulling this up to the first row it follows that we may diagonalize $S\phi(C(\mathbb{T})\otimes 1)S^*$ via a unitary T in $\bigoplus_i C([0,1])\otimes(\{\Pi_i(c^{ij}_{p,q}) :$ $j,p,q\}'\cap M_{m_i})$. Let

$$R = TS$$

so that $R\phi(C(\mathbb{T})\otimes 1)R^*(t)$ is a diagonal algebra that commutes with $\{c^j_{pq} : j,p,q\}$, and hence by an identification is a subalgebra of the diagonal of $\bigoplus_{i,j} M_{k_{ij}}$. Hence we may find diagonal mutually orthogonal subprojections of the identity projection, $\{p^{ij}_s : 1 \leq j \leq N; 1 \leq i \leq M; 1 \leq s \leq k_{ij}\}$, such that $R\phi(C(\mathbb{T})\otimes 1)R^*(t) \subset \operatorname{span}\{p^{ij}_s\}$. Therefore we may find continuous functions $g^{ij}_s : [0,1] \to \mathbb{T}$ such that $R\phi(f\otimes 1)R^*(t)p^{ij}_s = f(g^{ij}_s(t))p^{ij}_s$. Let $U = R^*$ and the proof is complete. □

Remark 4.3.15. We may replace $C(\mathbb{T})$ with $C(X)$ in A, $C(Y)$ in B for X, Y compact

Hausdorff spaces and the same proof will work assuming that the *-homomorphism is maximally homogeneous and that $H^1(Y, U(r)) = H^1(Y, S_r) = 0$ for all $r \in \mathbb{N}$ (in order to use Lemma 2 in [79]). In particular the latter requirement holds when Y is contractible and paracompact by Lemma 4.3.13.

Lemma 4.3.16. *Let $\phi : A \to B$, with $A = \bigoplus_{j=1}^{N} C(\mathbb{T}) \otimes M_{n_j}$, $B = \bigoplus_{i=1}^{M} C(\mathbb{T}) \otimes M_{m_i}$, be a unital injective *-homomorphism. Let $F \subset A$ be a finite set containing $u \otimes 1_j$ for all $j \in \{1, \ldots, N\}$, where u is the canonical generator of $C(\mathbb{T})$. Let $\epsilon > 0$. Then there exists a unital injective maximally homogeneous *-homomorphism $\chi : A \to B$ such that $\|\phi(a) - \chi(a)\| < \epsilon$ for all $a \in F$.*

Proof. The situation of this lemma is similar to that of Lemma 4.3.10, except we consider an injective *-homomorphism and the finite set containing the generator of the center. Recall the maps H_i, $h_{ij} = \mathrm{Ad}(v_{ij})^{-1} \circ g_{ij}^{-1} \circ \chi_{ij}$ and ρ_{ij} from the proof of Lemma 4.3.10, and use that lemma to obtain a unital maximally homogeneous *-homomorphism ϕ' (which by Lemma 4.3.14 takes the form of (4.9)) that is within a tolerance $\delta > 0$ to ϕ on elements of F. We will without comment identify elements $\phi'(a)$ for $a \in A$ with $\phi'(a)_{[0,1]}$ as elements of $\bigoplus_{i=1}^{M} C[0,1] \otimes M_{m_i}$.

Fix $j \in \{1, \ldots, N\}$. By (4.9) it is easy to see that

$$\mathrm{sp}_{\bigoplus_{i=1}^{M} M_{m_i}}(\phi'(u \otimes 1_j)(t)) = \{g_s^{ij}(t) : 1 \leq i \leq M; 1 \leq s \leq k_{ij}\} \cup \{0\}.$$

Hence it follows that

$$\mathrm{sp}_B(\phi'(u \otimes 1_j)) \setminus \{0\} = \bigcup_{i,s} g_s^{i,j}([0,1]). \tag{4.16}$$

Denote

$$Z = \phi'(u \otimes 1_j) \tag{4.17}$$

and $u_j = \Pi_j(u \otimes 1_j) \in C(\mathbb{T}) \otimes M_{n_j}$, and recall from the construction of ϕ' as in (4.8) in the proof Lemma 4.3.10 that

$$Z = \bigoplus_{i=1}^{M} h_{ij}(u_j) = \bigoplus_{i=1}^{M} \mathrm{Ad}(v_{ij})^{-1} g_{ij}^{-1} \chi_{ij}(u_j).$$

Let

$$L = \bigoplus_{i=1}^{M} \phi_i(1 \otimes 1_j)(C(\mathbb{T}) \otimes M_{m_i})\phi_i(1 \otimes 1_j),$$

and note that $Z \in L$. It is not hard to see that

$$\mathrm{sp}_B(Z) \setminus \{0\} = \mathrm{sp}_L(Z). \tag{4.18}$$

Indeed, that the right hand side is included in the left is trivial. That the left hand side is included in the right can be seen by noting that

$$1_L = \bigoplus_i \mathrm{Ad}(v_{ij})^{-1} g_{ij}^{-1} \chi_{ij}(\Pi_j(1 \otimes 1_j)) = \phi'(1 \otimes 1_j)$$

and then showing that for any $\bar{i}, \bar{s}$ and t_0 we have that $Z - g_{\bar{s}}^{\bar{i},j}(t_0)1_L$ fails to be invertible in L, by using (4.9). The claim then follows by (4.16).

We have a *-isomorphism

$$L \to \bigoplus_i C(\mathbb{T}) \otimes M_{n_j k_{ij}},\ (b_1, \ldots, b_M) \to (g_{1j} \circ \mathrm{Ad}(v_{1j})(b_1), \ldots, g_{Mj} \circ \mathrm{Ad}(v_{Mj})(b_M)). \tag{4.19}$$

We may think of $\bigoplus_i C(\mathbb{T}) \otimes M_{n_j k_{ij}}$ as sitting inside a full matrix algebra $C(\mathbb{T}) \otimes M_{\sum_i n_j k_{ij}}$. Letting

$$W = \bigoplus_i \chi_{ij}(u_j)$$

inside this full matrix algebra, we obtain from the isomorphism (4.19)

$$\mathrm{sp}_{C(\mathbb{T}) \otimes M_{\sum_i n_j k_{ij}}}(W) = \mathrm{sp}_L(Z). \tag{4.20}$$

Let

$$Y = \bigoplus_i \rho_{ij}(u_j)$$

inside the full matrix algebra. From (4.7) in Lemma 4.3.10 we have that

$$\|W - Y\| < \delta. \tag{4.21}$$

As ϕ is injective it follows that so is $\bigoplus_i \rho_{ij}$ and hence

$$\mathrm{sp}_{C(\mathbb{T}) \otimes M_{\sum_i n_j k_{ij}}}(Y) = \mathbb{T}. \tag{4.22}$$

Using (4.20) followed by (4.18) followed by (4.17) and (4.16) we obtain that

$$\mathrm{sp}_{C(\mathbb{T})\otimes M_{\sum_i n_j k_{ij}}}(W) = \bigcup_{i=1}^{M} \bigcup_{s(i)=1}^{k_{ij}} g_s^{ij}([0,1]). \tag{4.23}$$

If this is not all of $\mathbb{T}$ then $\mathbb{T} \setminus \mathrm{sp}_{C(\mathbb{T})\otimes M_{\sum_i n_j k_{ij}}}(W)$ is an open non-empty subset of $\mathbb{T}$ and so may be decomposed into disjoint open arcs on the circle. Consider such an open arc and let λ be its midpoint, λ_L its left endpoint and λ_R its right endpoint (where left is the counter-clockwise direction). By (4.22) there must exist $t_\lambda \in [0,1]$ such that $\lambda \in \mathrm{sp}_{M_{\sum_i n_j k_{ij}}}(Y(t_\lambda))$. Since by (4.21) $\|W(t_\lambda) - Y(t_\lambda)\| < \delta$, for any $\epsilon_1 > 0$ we can choose δ sufficiently small such that there exists $\mu \in \mathrm{sp}_{M_{\sum_i n_j k_{ij}}}(W(t_\lambda))$ with $|\mu - \lambda| < \epsilon_1$. This implies that this open arc has arclength that can be arbitrarily small by predefining a sufficiently small δ.

Now since $\lambda_R \in \mathrm{sp}_{C(\mathbb{T})\otimes M_{\sum_i n_j k_{ij}}}(W)$ there exists a $T_\lambda \in [0,1]$, an $i_\lambda \in \{1,\ldots,M\}$ and $s_\lambda \in \{1,\ldots,k_{i_\lambda j}\}$ such that $g_{s_\lambda}^{i_\lambda j}(T_\lambda) = \lambda_R$. By using the fact that the g_s^{ij}'s are continuous, this together with (4.23) tells us that there is at most $\sum_i k_{ij}$ disjoint open arcs that make up $\mathbb{T} \setminus \mathrm{sp}_{C(\mathbb{T})\otimes M_{\sum_i n_j k_{ij}}}(W)$. Order these arcs counter-clockwise as $I_1,\ldots,I_K$, the arclengths of which are arbitrarily small using a sufficiently small predefined $\delta > 0$. Let λ^p be the midpoint of I_p and λ_L^p and λ_R^p denote the left and right endpoints, respectively, for $p = 1,\ldots,K$. Assume $g_{s_p}^{i_p j}(T_p) = \lambda_R^p$ for some $T_p \in [0,1]$, some $i_p \in \{1,\ldots,M\}$ and $s_p \in \{1,\ldots,k_{i_p j}\}$.

Now either $g_{s_p}^{i_p j}[0,1] = \{\lambda_R^p\}$ or there exists $T^p \neq T_p$ such that $g_{s_p}^{i_p j}(T^p)$ is the rightmost point achieved by the image of $g_{s_p}^{i_p j}$ between the arcs I_p and $I_{(p-1) \bmod K}$. In the first case define $f_{s_p}^{i_p j}(t) = g_{s_p}^{i_p j}(t)\exp(i\rho(i_p,j,s_p)(t))$ for $t \in [0,1]$, where $\rho(i_p,j,s_p) : [0,1] \to [0,2\pi]$ is a continuous function that is 0 at the endpoints and reaches a maximal value the arclength of I_p, at a unique point in $[0,1]$. If we are in the second case with $T_p \neq 0,1$ then choose an open interval U_p of T_p that does not meet T^p, 0, or 1 and let $\rho(i_p,j,s_p) : [0,1] \to [0,2\pi]$ a continuous function that is 0 outside U_p, and reaches a maximal value the arclength of I_p only at T_p, and then define $f_{s_p}^{i_p j}$ analogously. If in this case T_p is 0 or 1, then again choose a small open interval U_p that does not meet T^p. We may choose this interval small enough such that on it our continuous function $g_{s_p}^{i_p j}$ takes circle values arbitrarily close to λ_R^p. In terms of a real valued argument we may assume our continuous function takes values between $\arg(\lambda_R^p)$ and $\arg(\lambda_R^p) - \mu$ for μ small. Without loss of generality assume $T_p = 0$ and $U_p = [0,t_1)$. Let $V_p = [t_2,t_3] \subset U_p$ be a closed interval that does not meet the endpoints of U_p, and such that the argument of the image of $g_{s_p}^{i_p j}|_{V_p}$

is strictly contained in $[\arg(\lambda_R^p) - \mu, \arg(\lambda_R^p)]$. Let $V_p' = [t_4, t_5] \subset V_p$ that does not meet the endpoints of V_p. By the extreme value theorem there is a point in $P \in V_p'$ where the maximal argument of our continuous function over V_p' is achieved, call this $\hat{\theta}$.

Let l_1 be the straight line connecting $(t_2, 0)$ and $(P, \arg(\lambda_L^p) - \hat{\theta})$, and l_2 be the straight line connecting $(P, \arg(\lambda_L^p) - \hat{\theta})$ and $(t_3, 0)$. Define $h : [0,1] \to [0, \arg(\lambda_L^p) - \hat{\theta}]$ to be 0 outside $[t_2, t_3]$, $\min(l_1(t), \arg(\lambda_L^p) - g_{s_p}^{i_p j}(t))$ for $t \in [t_2, P]$, and $\min(l_2(t), \arg(\lambda_L^p) - g_{s_p}^{i_p j}(t))$ for $t \in [P, t_3]$. It is clear that h is continuous. Define $w : [0,1] \to [0,1]$ to be 0 outside $[t_2, t_3]$, and $\frac{h}{\arg(\lambda_L^p) - \hat{\theta}}$ otherwise. Let $\rho(i_p, j, s_p)$ be defined by being 0 outside $[t_2, t_3]$, and hw otherwise. Note that by construction $\rho(i_p, j, s_p)$ is 0 at the endpoints, is non-negative and bounded above by $\arg(\lambda_L^p) - \hat{\theta}$, achieving this maximal value only at P. This implies that the corresponding perturbation $f_{s_p}^{i_p j} = g_{s_p}^{i_p j} \exp(i\rho(i_p, j, s_p))$ never crosses the left of λ_L^p, and only achieves this value at a unique point P.

For (i, s) not covered by these two cases just let $f_s^{ij} = g_s^{ij}$. What we have achieved so far is to construct a new set of eigenvalue functions $\{f_s^{ij}\}_{i,s}$ with the following properties:

1. $\bigcup_{i=1}^{M} \bigcup_{s(i)=1}^{k_{ij}} f_s^{ij}([0,1]) = \mathbb{T}$,
2. the f_s^{ij}'s are counter-clockwise perturbations of our original eigenvalue functions which never cross beyond the left endpoint of the open arc they cover,
3. the f_s^{ij}'s have the same endpoint data as our original eigenvalue functions,
4. the f_s^{ij}'s which arise from a perturbation and which take value the left endpoint of an open arc do so at a unique point in $(0,1)$, and,
5. each f_s^{ij} can be made as close as we like to g_s^{ij} by choosing δ and μ sufficiently small in the construction.

We now repeat the construction for all $j \in \{1, \ldots, N\}$ to obtain the eigenvalue functions $\{f_s^{ij}\}_{i,j,s}$. Define $\chi : A \to B$ exactly as in (4.9) but with these new eigenvalue functions f_s^{ij} replacing our old ones. This map is well-defined by property 3. It is injective by property 1. We have that $\|\phi'(a) - \chi(a)\|$ is as small as we like for all $a \in F$ by property 5 (this is achievable only because F is finite). Using the triangle inequality we can get the desired closeness in norm of the difference $\phi(a) - \chi(a)$ for $a \in F$.

It remains to show that χ is maximally homogeneous. For this suffices to ensure that for every fixed j the f_s^{ij}'s are pointwise distinct. Note that because ϕ' is maximally homogeneous, the g_s^{ij}'s would have to be pointwise distinct (over i and s). Given $g_{s_p}^{i_pj}$ with $g_{s_p}^{i_pj}(T_p) = \lambda_R^p$ as in the above constructions, there can be no point in time $t_* \in [0,1]$ where some other g_s^{ij} satisfies that $g_s^{ij}(t_*)$ is left of $g_{s_p}^{i_pj}(t_*)$ and within the same range as $g_{s_p}^{i_pj}$, as by continuity this would mean they must agree at some other point in time, which is not the case. Hence by properties 2 and 4 $f_{s_p}^{i_pj}$ will never pointwise agree with another f_s^{ij}, except perhaps at a unique time $T \in (0,1)$, where the value is λ_L^p. We may further assume that near T, f_s^{ij} is one of our original unperturbed continuous maps. This can be assumed because our perturbations only happen close to the right endpoint of an arc and not close to the left endpoint of another arc. Hence on a small neighbourhood near T we may assume f_s^{ij} does not meet any other continuous map pointwise. It is now an easy task to ensure that f_s^{ij} achieves the left endpoint of an arc at a slightly different time than T. □

Lemma 4.3.17. *Every unital $A\mathbb{T}$-algebra $A = \varinjlim_n (A_n, \phi_n)$ with unital and injective connecting maps ϕ_n is isomorphic to $\varinjlim_n (A_n, \phi'_n)$ with unital and injective maximally homogeneous connecting maps ϕ'_n of the form* (4.9).

Proof. By Lemma 4.3.16 each ϕ_n is close (on a predefined finite set) to an injective and unital maximally homogeneous map ϕ'_n, which takes the form (4.9) by Lemma 4.3.14. Because of this, we can get an approximate intertwining between the sequences

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots$$

and

$$A_1 \xrightarrow{\phi'_1} A_2 \xrightarrow{\phi'_2} \cdots .$$

The result follows by Proposition 2.3.44. □

We now have the necessary ingredients to prove the main theorem of this section, which is the existence of A$\mathbb{T}$-Cartan subalgebras in A$\mathbb{T}$-algebras. This is Theorem C in the Introduction.

Theorem 4.3.18. *Every unital $A\mathbb{T}$-algebra $A = \varinjlim_n (A_n, \phi_n)$ with unital and injective connecting maps contains an $A\mathbb{T}$-Cartan subalgebra.*

Proof. By Lemma 4.3.17 we may assume that the injective and unital connecting maps are maximally homogeneous and of the form (4.9). Fix $n \in \mathbb{N}$ and let $C \subset A_n$

be a Cartan subalgebra. Assume

$$A_n = \bigoplus_{j=1}^{N} C(\mathbb{T}) \otimes M_{n_j},\ A_{n+1} = \bigoplus_{i=1}^{M} C(\mathbb{T}) \otimes M_{m_i},$$

and denote the connecting map between them by ϕ. Let $\{k_{ij}\}$ be the index system with respect to ϕ. Consider the composition

$$C(\mathbb{T}) \otimes M_{n_j} \xrightarrow{\overline{\phi_{ij}}} \phi_i(1 \otimes 1_j)(C(\mathbb{T}) \otimes M_{m_i})\phi_i(1 \otimes 1_j) \xrightarrow{\mu} C(\mathbb{T}) \otimes M_{n_j k_{ij}}$$

where $\overline{\phi_{ij}}$ is as in the proof of Lemma 4.3.10, and μ is the *-isomorphism given by the composition of g_{ij} and $\mathrm{Ad}(v_{ij})$ as in that lemma. As in that lemma, let this composition be ρ_{ij}. As we have seen, μ preserves dimension at evaluations and so it follows that $\dim(\rho_{ij}(C(\mathbb{T}) \otimes M_{n_j})(t)) = \dim(\overline{\phi_{ij}}(C(\mathbb{T}) \otimes M_{n_j})(t))$ for all $t \in \mathbb{T}$. Since the eigenvalue functions $\{g_s^{ij}\}$ appearing in (4.9) are pointwise distinct (over i and s), it is easy to check that this dimension is $n_j^2 k_{ij}$, and so ρ_{ij} is maximally homogeneous.

Let C_j be the restriction of C onto the j^{th} summand, which is also a Cartan subalgebra and so maximally homogeneous in $C(\mathbb{T}) \otimes M_{n_j}$ by Lemma 4.3.4. Since C contains the center it follows that $C = \bigoplus_{j=1}^{N} C_j$. By Proposition 1.8 in [75] it follows that we may find a maximally homogeneous Abelian subalgebra $E_{ij} \subset C(\mathbb{T}) \otimes M_{n_j k_{ij}}$ such that

$$E_{ij} = C^*(\rho_{ij}(C_j), C(\mathbb{T}) \otimes 1_{n_j k_{ij}}).$$

Let

$$D_{ij} = \mu^{-1}(E_{ij}) = C^*(\overline{\phi_{ij}}(C_j), Z(\phi_i(1 \otimes 1_j)(C(\mathbb{T}) \otimes M_{m_i})\phi_i(1 \otimes 1_j))), \qquad (4.24)$$

which is maximally homogeneous, with

$$\dim(D_{ij}(t)) = \dim(E_{ij}(t)) = n_j k_{ij}$$

for all $t \in \mathbb{T}$. Define

$$D_i = \sum_{j=1}^{N} D_{ij} \cong \bigoplus_{j=1}^{N} D_{ij}$$

viewed as a subalgebra inside $C(\mathbb{T}) \otimes M_{m_i}$. It is clear that D_i is Abelian, with

$$\dim(D_i(t)) = \sum_{j=1}^{N} n_j k_{ij} = m_i,$$

using (2.28).

Note that D_i contains the center of $C(\mathbb{T}) \otimes M_{m_i}$. Indeed, it is clear that if $j_1 \neq j_2$ then $\phi_i(1 \otimes 1_{j_1})(f \otimes 1)\phi_i(1 \otimes 1_{j_2}) = 0$, as $f \otimes 1$ is central. From this it follows that $f \otimes 1 = \sum_{j=1}^{N} \phi_i(1 \otimes 1_j)(f \otimes 1)\phi_i(1 \otimes 1_j)$. It is clear that $\phi_i(1 \otimes 1_j)(f \otimes 1)\phi_i(1 \otimes 1_j)$ belongs to the center of $\phi_i(1 \otimes 1_j)(C(\mathbb{T}) \otimes M_{m_i})\phi_i(1 \otimes 1_j)$. Hence D_i is maximally homogeneous.

It is clear that $\phi_i(f_1, \ldots, f_N) = \sum_{j=1}^{N} \phi_{ij}(f_j)$ and so by (4.24) ϕ_i maps C into D_i. Finally let

$$D = \bigoplus_{i=1}^{M} D_i$$

which is an Abelian maximally homogeneous C^*-subalgebra of A_{n+1} containing the center, and hence is a Cartan subalgebra by Lemma 4.3.4. It is clear that ϕ maps C into D.

Now we check that if $n = (n_1, \ldots, n_N) \in N_{A_n}(C)$ then $\phi(n) \in N_{A_{n+1}}(D)$. It is clear from (4.24) that $\phi_{ij}(n_j)$ is a normalizer in D_{ij}, and from the definition of D_i it follows that $\phi_i(n) = \sum_j \phi_{ij}(n_j)$ is a normalizer for D_i. Hence $\phi(n)$ is a normalizer of D.

Now we check that if P_n is the faithful conditional expectation $A_n \to C$, and P_{n+1} is the one $A_{n+1} \to D$, then we have

$$\phi \circ P_n = P_{n+1} \circ \phi. \tag{4.25}$$

Define, for $j \in \{1, \ldots, N\}$,

$$P_n^j = \Pi_j \circ P_n \circ i_j : A_j \to C_j,$$

and similarly, for $i \in \{1, \ldots, M\}$,

$$P_{n+1}^i = \Pi_i \circ P_{n+1} \circ i_i : B_i \to D_i.$$

It is easy to see that P_n^j is a linear projection of norm one, since P_n is by Lemma 4.3.5, and similarly for P_{n+1}^i. Hence by [82] these define the unique faithful conditional expectations $A_j \to C_j$ and $B_i \to D_i$, respectively. Note that since

$$\Pi_{j_1}(P_n(i_j(a_j))) = \Pi_{j_1}(P_n((1\otimes 1_j)i_j(a_j)(1\otimes 1_j))) = \Pi_{j_1}((1\otimes 1_j)P_n(i_j(a_j))(1\otimes 1_j)) = 0 \tag{4.26}$$

if $j_1 \neq j$, it follows that

$$\bigoplus_{j=1}^{N} P_n^j(a_1, \ldots, a_N) = \sum_j P_n(i_j(a_j)) = P_n(a_1, \ldots, a_N)$$

for all $(a_, \ldots, a_N) \in A$. The same argument holds for P_{n+1}, so we get

$$P_n = \bigoplus_{j=1}^{N} P_n^j, \quad P_{n+1} = \bigoplus_{i=1}^{M} P_{n+1}^i. \tag{4.27}$$

Note that in order to get (4.25) it suffices to check that for all $i \in \{1, \ldots, M\}$, $\phi_i \circ P_n = \pi_i \circ P_{n+1} \circ \phi = P_{n+1}^i \circ \phi_i$, where the last equality is due to (4.27). To show this, it suffices to show that for all $j \in \{1, \ldots, N\}$ we have that for $f_j \in C(\mathbb{T}) \otimes M_{n_j}$, $\phi_i \circ P_n(i_j(f_j)) = P_{n+1}^i \circ \phi_{ij}(f_j)$. Due to (4.26) we may write this equality as $\phi_{ij}(P_n^j(f_j)) = P_{n+1}^i(\phi_{ij}(f_j))$. Composition with the isomorphism μ yields

$$\mu(\phi_{ij}(P_n^j(f_j))) = \rho_{ij}(P_n^j(f_j)) = \mu(P_{n+1}^i(\phi_{ij}(f_j))) = \mu(P_{n+1}^i(\mu^{-1}(\mu(\phi_{ij}(f_j))))) = \overline{P}(\rho_{ij}(f_j))$$

where $\overline{P} = \mu \circ P_{n+1}^i \circ \mu^{-1}$. Note that $\overline{P}$ is a norm one linear map because P_{n+1}^i is, and it is a projection (onto E_{ij}) because of (4.24). Hence by [82] it is the (unique) conditional expectation $C(\mathbb{T}) \otimes M_{n_j k_{ij}} \to E_{ij}$. Thus we have reduced the problem of showing that (4.25) holds to showing

$$\rho_{ij}(P_n^j(f_j)) = \overline{P}(\rho_{ij}(f_j)) \tag{4.28}$$

for all i and j.

Note that if we choose, by Lemma 4.3.2, a system of matrix units $\{_{\mathbb{R}} e_{pq}\}$ with respect to the maximally homogeneous subalgebra C_j, then we know by (4.5) in Lemma 4.3.5 that the conditional expectation P_n^j is given by projection onto the diagonal with respect to this system of matrix units. Also, the existence of E_{ij} was obtained by Proposition 1.8 in [75], and there the projections which describe E_{ij} (as in (4.5)) are given by mutually orthogonal projections labelled $P_{(i,j,k)}$. Hence by Lemma 4.3.5 $\overline{P}$ must be given by projection onto the space spanned by the $P_{(i,j,k)}$'s. It is now a tedious but straightforward task to use these descriptions of the conditional expectations, together with the form (4.9) for the maximally homogeneous map ρ_{ij}, as well as the definition of $P_{(i,j,k)}$ given in the proof of Proposition 1.8 in [75], to verify that (4.28) holds. Alternatively, a full proof is included for more general spaces in Lemma 4.4.11 where (4.38) is shown.

Hence what we have shown is that for $n \in \mathbb{N}$ and $C_n \subset A_n$ a Cartan subalgebra, there is a Cartan subalgebra $C_{n+1} \subset A_{n+1}$ such that $\phi_n(C_n) \subseteq C_{n+1}$, $\phi_n(N_{A_n}(C_n)) \subseteq N_{A_{n+1}}(C_{n+1})$, and $\phi_n \circ P_n = P_{n+1} \circ \phi_n$, where the connecting maps ϕ_n are injective, unital, and maximally homogeneous of the form (4.9). Hence by Proposition 4.1.5, we have that $\varinjlim(C_n, \phi_n)$ is a Cartan subalgebra of A. $\square$

Many important C^*-algebras are AI or A$\mathbb{T}$-algebras, and hence we have found inductive limit Cartan subalgebras for them:

Corollary 4.3.19. *The irrational rotation algebra and the Bunce-Deddens algebra have an inductive limit Cartan subalgebra.*

Proof. These can all be realized as unital A$\mathbb{T}$-algebras with unital and injective connecting maps (for the irrational rotation algebra see Theorem 4 in [22] and Theorem 4.3 in [19], for the Bunce-Deddens algebra see Example 3.2.11 in [67]), and so the result follows by Theorem 4.3.18. $\square$

An AH-algebra is an inductive limit C^*-algebra $A = \varinjlim(A_n, \phi_n)$ where each building block A_n is a direct sum of corners of matrix algebras (so each building block is direct sum of summands which have the form $p(C(X) \otimes M_n)p$ for some projection $p \in C(X) \otimes M_n$, and where X is some compact metric space). Hence AH-algebras generalize AX-algebras. A is said to have bounded dimension if the supremum of the topological dimensions across all the compact Hausdorff spaces X appearing in all the summands is finite. A is said to have the ideal property if every closed two-sided ideal of A is generated by the projections in the ideal, as a closed two-sided ideal (see Definition 6.0.1 and [29] for more details.) We have the following:

Corollary 4.3.20. *Every unital AH-algebra with unital and injective connecting maps, and with bounded dimension, the ideal property, and torsion-free K-theory, has an inductive limit Cartan subalgebra.*

Proof. Such AH-algebras are A$\mathbb{T}$-algebras by [29]. Hence the result follows by Theorem 4.3.18. $\square$

4.4 Existence of AX-Cartan Subalgebras

In this section, we generalize the results of the previous sections, and consider AX-algebras where X will denote a planar connected finite graph imbedded in $\mathbb{C}$. We may without loss of generality assume each edge has length 1, every edge is not a loop, and that there is at most one edge between any two vertices. This generalizes the previous cases as the unit interval $[0,1]$ corresponds to the graph with two vertices and a single edge, and the circle $\mathbb{T}$ corresponds to the triangle with three vertices and three edges.

Associated to X is its universal covering graph, which is a tree denoted by $\mathcal{T}$. It is obtained by starting at a vertex v in X and taking a non-backtracking walk $(v, v_1, v_2, \dots, v_K)$ on X. Such walks are the vertices of the universal covering graph, and two such vertices are connected by an edge if one is a one-step extension of the other (for details, see Section 6 in [1]). We have a canonical projection $p : \mathcal{T} \to X$. We will also associate with X an onto path $l : [0,1] \twoheadrightarrow X$. If $A = \bigoplus_{j=1}^{N} C(X) \otimes M_{n_j}$ is an AX-building block, it can be identified with the C^*-subalgebra of $\bigoplus_{j=1}^{N} C([0,1]) \otimes M_{n_j}$ where functions agree on the fibres of the path map l. For $f \in A$ we write $f_{[0,1]}$ for its identification, defined by $f_{[0,1]}(t) = f(l(t))$. A may also be identified with the C^*-subalgebra of $\bigoplus_{j=1}^{N} C_b(\mathcal{T}) \otimes M_{n_j}$ consisting of those functions that agree on the fibres of p. For $f \in A$ we write $f_{\mathcal{T}}$ for its identification, defined by $f_{\mathcal{T}}(t) = f(p(t))$. We will usually mention which identification we are using, but if context is clear, this may be omitted.

The reason such identifications are useful, is because $[0,1]$ and $\mathcal{T}$ are contractible and paracompact, and so by Lemma 4.3.13 have vanishing Čech cohomology. This will allow us to directly generalize a lot of the results of the previous section, for example Lemma 4.3.14, where we patch up unitaries locally to get a global unitary that conjugates the standard map.

A lot of the results in this section are direct generalizations of the results of Section 4.3, with an identical proof that works by just replacing $\mathbb{T}$ with X. When this is the case we will not provide a proof for the result. This section should be read after having a good understanding of the proofs in Section 4.3. We begin with a generalization to Lemma 4.3.2, which allows us to find specific systems of matrix units with respect to maximally homogeneous subalgebras.

Lemma 4.4.1. *Let C be a maximally homogeneous subalgebra of $A = \bigoplus_{j=1}^{N} C(X) \otimes M_{n_j}$. There exists a system of matrix units $\{{}_{\mathcal{T}}e^j_{pq}\}$ with ${}_{\mathcal{T}}e^j_{pq} \in \bigoplus_{j=1}^{N} C_b(\mathcal{T}) \otimes M_{n_j}$ such that*

$$C = \{f \in A : f_{\mathcal{T}}(t) \in \operatorname{span}\{{}_{\mathcal{T}}e^j_{pp}(t)\} \ \ \forall t \in \mathcal{T}\}. \tag{4.29}$$

Given $x \in X$, and $t_1, t_2 \in p^{-1}(\{x\})$ then there exists permutations $\sigma_j \in \Sigma_{n_j}$, $j = 1, \ldots, N$, such that ${}_{\mathcal{T}}e^j_{pp}(t_1) = {}_{\mathcal{T}}e^j_{\sigma_j(p)\sigma_j(p)}(t_2)$ for all $p \in \{1, 2, \ldots, n_j\}$.

Proof. If the result holds true on the summands it holds true for the direct sum. So we assume C is maximally homogeneous in $C(X) \otimes M_n$. Treating this as a C^*-subalgebra of $C_b(\mathcal{T}) \otimes M_n$, we can use Lemma 2 in [79] to diagonalize C via a unitary v. Letting f_{pq} be constants with value the standard matrix units in M_n, we let ${}_{\mathcal{T}}e_{pq} = v^* f_{pq} v$. The first statement of the lemma becomes clear.

For second statement, maximal homogeneity of C implies

$$\operatorname{span}\{{}_{\mathcal{T}}e_{pp}(t_1) : 1 \leq p \leq n\} = C_{\mathcal{T}}(t_1) = C_{\mathcal{T}}(t_2) = \operatorname{span}\{{}_{\mathcal{T}}e_{pp}(t_2) : 1 \leq p \leq n\}.$$

Hence for any p, the projection ${}_{\mathcal{T}}e_{pp}(t_1)$ is a linear combination of elements of the form ${}_{\mathcal{T}}e_{qq}(t_2)$. The coefficients in the combination must all be 1. Since the $\{e_{pp}(t_1)\}$ are orthogonal, it follows by the pigeonhole principle that the linear combination only has one element, and so there exists a permutation σ such that ${}_{\mathcal{T}}e_{pp}(t_1) = {}_{\mathcal{T}}e_{\sigma(p)\sigma(p)}(t_2)$. The result follows. □

Remark 4.4.2. The same result holds when using the space $[0, 1]$ for identification rather than $\mathcal{T}$, where we use the fibres of l instead of those of p. Whenever we want to use this space instead, we will write ${}_{[0,1]}e^j_{pq}$ rather than ${}_{\mathcal{T}}e^j_{pq}$.

Now we obtain a generalization to Lemma 4.3.4, where a lot of the ideas in the proof have a similar flavour.

Lemma 4.4.3. *C is maximally homogeneous in $A = \bigoplus_{j=1}^{N} C(X) \otimes M_{n_j}$ if and only if it is a Cartan subalgebra of A.*

Proof. Assume C is maximally homogeneous. Then $C_j = \pi_j(C)$ is maximally homogeneous in $C(X) \otimes M_{n_j}$. Lemma 1.4 and the remarks after Definition 1.3 in [74] imply that C_j is has the unique extension property for pure states, and admits a unique conditional expectation onto it from $C(X) \otimes M_{n_j}$. Because C_j is Abelian and has the unique extension property, Section 1 in [14] implies that C_j is an ideal of a masa in $C(X) \otimes M_{n_j}$. Since C_j is unital it must be a masa itself. Using Lemma 4.4.1 we may find matrix units $\{{}_{\mathcal{T}}e^j_{pq}\}$ for $C_b(\mathcal{T}) \otimes M_{n_j}$ such that C_j is of the form (4.29). Then the map

$$P(f)_{\mathcal{T}}(t) = \sum_{p=1}^{n_j} {}_{\mathcal{T}}e^j_{pp}(t) f_{\mathcal{T}}(t) {}_{\mathcal{T}}e^j_{pp}(t)$$

defines a norm one faithful linear projection (exactly as in the proof of Lemma 4.3.5, where one considers the map pointwise), and hence by [82] is the desired faithful conditional expectation. (Of course, one may analogously consider the identification over the space $[0,1]$ using the path l rather than $\mathcal{T}$).

To show C_j is regular, let U be a small open subset of X (either a small interval of an edge in X or a small contractible neighbourhood around a vertex of X) which is not all of X. Let $\chi_U \in C(X)$ supported in U. We may write

$$p^{-1}(U) = \bigsqcup_i V_i \subset \mathcal{T}$$

where $p : V_i \to U$ is a homeomorphism.

Let $f \in C(X) \otimes M_{n_j}$. Note that there are functions $\lambda^j_{pq} : \mathcal{T} \to \mathbb{C}$ such that

$$f_{\mathcal{T}}(t) = \sum_{p,q} \lambda^j_{pq}(t)_{\mathcal{T}} e^j_{pq}(t) \ \ \forall t \in \mathcal{T}.$$

Because $\{_{\mathcal{T}} e^j_{pq}(t)\}$ is a system of matrix units in M_{n_j} the functions λ^j_{pq} are unique. Since $(_{\mathcal{T}} e^j_{pp}) f_{\mathcal{T}} (_{\mathcal{T}} e^j_{qq})$ is continuous, so is $\lambda^j_{pq}{}_{\mathcal{T}} e^j_{pq}$. Define functions

$$f^j_{pq} : \mathcal{T} \to M_{n_j},$$

for $p, q \in \{1, \ldots, n_j\}$ by taking value

$$\lambda^j_{pq}(p_1(t))_{\mathcal{T}} e^j_{pq}(p_1(t)) \ \text{ for } \ t \in \bigsqcup_i V_i, \ \ 0 \ \text{ otherwise},$$

where for $t \in V_i$, $p_1(t)$ is its unique image in V_1 factoring through the homeomorphism $V_i \to U$. Then

$$(\chi_U)_{\mathcal{T}} f^j_{pq} : \mathcal{T} \to M_{n_j}$$

is a continuous map, satisfying that if $p(t_1) = p(t_2) \in X$ then $(\chi_U)_{\mathcal{T}} f^j_{pq}(t_1) = (\chi_U)_{\mathcal{T}} f^j_{pq}(t_2)$. Hence $(\chi_U)_{\mathcal{T}} f^j_{pq}$ corresponds to an element g^j_{pq} in $C(X) \otimes M_{n_j}$.

Note that if $c \in C_j$ then $c_{\mathcal{T}}(t) \in \operatorname{span}\{_{\mathcal{T}} e^j_{pp}(t)\}$ by Lemma 4.4.1, and note thus that

$$(g^j_{pq})_{\mathcal{T}}(t)^* c_{\mathcal{T}}(t) (g^j_{pq})_{\mathcal{T}}(t) \in \operatorname{span}\{_{\mathcal{T}} e^j_{pp}(t)\}$$

by Lemma 4.4.1. Likewise

$$(g^j_{pq})_{\mathcal{T}}(t) c_{\mathcal{T}}(t) (g^j_{pq})_{\mathcal{T}}(t)^* \in \operatorname{span}\{_{\mathcal{T}} e^j_{pp}(t)\}.$$

Hence $g^j_{pq} \in N_{C(X)\otimes M_{n_j}}(C_j)$. It can be checked that

$$\chi_U f = \sum_{p,q} g^j_{pq}$$

and hence $\chi_U f$ is a sum of normalizer elements. By using a partition of unity argument, we can therefore conclude that f is a sum of normalizer elements. Hence C_j is regular in $C(X) \otimes M_{n_j}$. So C_j is a Cartan subalgebra. Thus C is a Cartan subalgebra of A.

For the converse, assume C is a Cartan subalgebra of A, meaning C_j is a Cartan subalgebra of $C(X) \otimes M_{n_j}$. By Example 5.18 in [62] this is a continuous trace C^*-algebra and so we obtain by Proposition 6.1 in [64] that C_j has the unique extension property, and so by Lemma 1.4 in [74], C_j is maximally homogeneous. Hence C is maximally homogeneous. □

The following lemma is a generalization of Lemma 4.3.9. It highlights the need of some of the assumptions we have put on our graph, namely that it is finite and connected.

Lemma 4.4.4. *Let $A = C(X) \otimes M_n$ and $p \in A$ a projection with trace k at any (hence all) $x \in X$, then pAp is isomorphic to $C(X) \otimes M_k$, via conjugation by a partial isometry whose initial projection is p and range projection is $1_k \in A$.*

Proof. As in the proof of Lemma 4.3.9, we need to show that $p \sim 1_k$. Since $M_n(C(X))$ has stable rank one (see [73]), it follows that it has the cancellation property (see [83]). So it suffices to show that $[p]_0 = [1_k]_0$ in the K_0 group. Our graph X is homotopy equivalent to a topological rose Y (see Example 0.7 in [33]). Hence $C(X)$ is homotopy equivalent to $C(Y)$ and hence we may by Proposition 2.2.34 assume, for the purposes of understanding $K_0(C(X))$, that X is a rose.

Thus X can be decomposed as a disjoint union of a single point $\{pt\}$ and some open intervals $I_1, \ldots, I_S$. Note that we have a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C_0(\bigcup_{s=1}^{S} I_s) \arrow[r] & C(X) \arrow[r] & C(\{pt\}) \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$

Hence by Proposition 2.2.35 we get a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & K_0(C(X)) \arrow[r] & \mathbb{Z} \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$

where $K_0(C_0(\bigcup_{s=1}^{S} I_s)) = 0$ can be computed using K_0 for non-unital C^*-algebras (see Chapter 4 in [43]), and where $K_0(C(\{pt\})) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ was seen in Example 2.2.47. Thus $K_0(C(X)) \cong \mathbb{Z}$. We obtain a commutative diagram

$$\begin{tikzcd} K_0(C(X)) \arrow[r, "\mathrm{tr}_x"] \arrow[d] & \mathbb{Z} \\ \mathbb{Z} \arrow[ur, dashed, "g"] & \end{tikzcd}$$

where tr_x appeared in Example 2.2.51. Since tr_x is surjective, so is g and hence g must be a group isomorphism. Hence tr_x is a group isomorphism (in particular injective), and so $[p]_0 = [1_k]_0$ as desired. □

Now we aim to get a result similar to that in Lemma 4.3.10, showing that the maximally homogeneous *-homomorphisms are dense in the set of *-homomorphisms between AX-building blocks. In that lemma, we made use of an important fact, which was presented as Lemma 4.3.7, where it was enough to find a *-homomorphism that maps the canonical generator of the center of $C(\mathbb{T})$ to a unitary with enough distinct eigenvalues at every point. The existence of such a map was established by Elliott in Theorem 4.4 in [19]. Since the space we consider now is not $\mathbb{T}$, we make use Theorem 2.1.6 and Corollary 2.1.8 in [44] instead.

Lemma 4.4.5. *Let $\phi : A \to B$, with $A = \bigoplus_{j=1}^{N} C(X) \otimes M_{n_j}$, $B = \bigoplus_{i=1}^{M} C(X) \otimes M_{m_i}$, be a unital *-homomorphism with corresponding index system $\{k_{ij}\}$. Let $F \subset A$ be a finite set containing the canonical generator of $C(X)$, and $\epsilon > 0$. Then there exists a unital maximally homogeneous *-homomorphism $\phi' : A \to B$ such that $\|\phi(a) - \phi'(a)\| < \epsilon$ for all $a \in F$.*

Proof. The proof follows in the same way as in the proof of Lemma 4.3.10. Indeed, use Lemma 4.4.4 to obtain a unital *-homomorphism

$$\rho_{ij} : C(X) \otimes M_{n_j} \to C(X) \otimes M_{k_{ij}n_j}.$$

The existence of a *-homomorphism

$$\chi_{ij} : C(X) \otimes M_{n_j} \to C(X) \otimes M_{k_{ij}n_j}$$

which is close to ρ_{ij} on finite sets (which contain the the canonical generator of the center) and sends this generator to an element with k_{ij} distinct eigenvalues at every point is justified by Theorem 2.1.6 and Corollary 2.1.8 in [44]. Remark 4.3.8 then implies that χ_{ij} is maximally homogeneous. The rest of the proof follows identically as the proof of Lemma 4.3.10 with $\mathbb{T}$ replaced by X. $\square$

We now obtain the generalization to Lemma 4.3.14, which gives the form of maximally homogeneous *-homomorphisms as those that are unitary conjugates to standard maps. The main tool in that lemma was to treat the codomain as being identified with a subalgebra of an AI-building block, because of the property that $H^1([0,1], G) = 0$ for every topological group G. We can do the same here, by treating the codomain as either a subalgebra of an AI-building block (via the path $l : [0,1] \to X$) or those building blocks with underlying space the tree $\mathcal{T}$ (via the projection $p : \mathcal{T} \to X$). Of course these have vanishing Čech cohomology as well. See Remark 4.3.15 for the details.

Lemma 4.4.6. *Let $\phi : A \to B$, with $A = \bigoplus_{j=1}^{N} C(X) \otimes M_{n_j}$, $B = \bigoplus_{i=1}^{M} C(X) \otimes M_{m_i}$, be a unital maximally homogeneous *-homomorphism. Then there exists a unitary $U = (U_1, \ldots, U_M) \in \bigoplus_{i=1}^{M} C([0,1]) \otimes M_{m_i}$ and a set of continuous functions $\{g_s^{ij} : [0,1] \to X : 1 \leq i \leq M; 1 \leq j \leq N; 1 \leq s = s(i,j) \leq k_{ij}\}$ (where $\{k_{ij}\}$ is the index system with respect to ϕ) such that*

$$\begin{aligned}&\phi_i((a_1, \ldots, a_N))_{[0,1]}(t) = \\ &U_i(t)\mathrm{diag}(a_1 \circ g_1^{i1}, a_1 \circ g_2^{i1}, \ldots, a_1 \circ g_{k_{i1}}^{i1}, a_2 \circ g_1^{i2}, \ldots, a_2 \circ g_{k_{i2}}^{i2}, \ldots, a_N \circ g_{k_{iN}}^{iN})(t)U_i(t)^*,\end{aligned} \tag{4.30}$$

for all $t \in [0,1]$.

We now obtain the generalization to Lemma 4.3.16.

Lemma 4.4.7. *If in the situation of Lemma 4.4.5 we additionally assume that ϕ is injective, then there exists an injective unital maximally homogeneous *-homomorphism $\chi : A \to B$ which is of the same general form as (4.30), and such that $\|\phi(x) - \chi(x)\| < \epsilon$ for all $x \in F$, which is a finite set containing the canonical generator of $C(X)$.*

Proof. The first bit of the proof follows that of Lemma 4.3.16, but except for $\mathbb{T}$ we have X, and rather than the unitary u we consider ι_X. As in the proof, we obtain that $X \setminus \mathrm{sp}_{C(X)\otimes M_{\sum_i n_j k_{ij}}}(W)$ can be decomposed into open disjoint sets (rather than arcs), where $\mathrm{sp}_{C(X)\otimes M_{\sum_i n_j k_{ij}}}(W) = \bigcup_{i=1}^{M} \bigcup_{s(i)=1}^{k_{ij}} g_s^{ij}([0,1])$. Again we conclude, by similar arguments as those used in the proof of Lemma 4.3.16, that these disjoint open sets in X have diameter that can be made arbitrarily small if we choose δ sufficiently small in the beginning. Also, we may argue that the number of these disjoint open sets must be finite. Indeed, if they were infinite, there must be an edge in X which contains infinitely many such disjoint open sets in its interior (as X is finite). But then the contradiction is similar to that in the proof of Lemma 4.3.16, namely that there are only finitely many eigenvalue functions and hence these are not enough to take all the values at the endpoints of these open sets.

We will now, as in Lemma 4.3.16, perturb the g_s^{ij}'s in the correct way to achieve that $\bigcup_{i=1}^{M} \bigcup_{s(i)=1}^{k_{ij}} g_s^{ij}([0,1]) = X$. Consider a fixed edge E in X. We may without loss of generality assume that $E = [0,1]$. Consider that $U_1, \ldots, U_p$ are disjoint open sets of our collection appearing in the interior of E, from left to right. Hence they are open intervals. Let g be some $g_{s_0}^{i_0 j}$ and t_0 some point in $[0,1]$ such that $g(t_0)$ is the right endpoint of U_1. We localise on a sufficiently small neighbourhood around t_0 and on this neighbourhood stretch g leftwards continuously so that it covers U_1, as we did in the proof of Lemma 4.3.16. By that proof this can be done in a way for all the open sets such that the perturbed maps do not affect their range near the left endpoint of an open set to their right (if such a range exists), and that all the endpoint data is preserved, and that the perturbed maps do not coincide pointwise.

What remains is to show we can cover an open set around a vertex in X, with connecting edges $E_1, \ldots, E_l$ say. Consider E_1, and assume this is $[0,1]$ with 1 being the vertex. We can assume U is an open set around 1 not covered by the g_s^{ij}'s. We find some $h = g_{s_1}^{i_1 j}$ and a point t_1 such that $h(t_1)$ takes the value the left endpoint of U. Again we localize on a neighbourhood around t_1, and perturb h to the right to cover U. We do this for all the edges connecting to the vertex, and there is sufficient choice of the points at which the perturbed functions take value the vertex as to not coincide, as the graph is finite. $\square$

Theorem 4.4.8. *Every unital AX-algebra $A = \varinjlim_n (A_n, \phi_n)$ with unital and injective connecting maps ϕ_n is isomorphic to $\varinjlim_n (A_n, \phi_n')$ with unital injective maximally homogeneous connecting maps ϕ_n' of the form* (4.30).

Proof. By Lemma 4.4.7 each ϕ_n is close (on a predefined finite set) to an injective and unital maximally homogeneous map ϕ_n', which takes the form (4.30) by Lemma 4.4.6. Because of this, we can get an approximate intertwining between the sequences

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots$$

and

$$A_1 \xrightarrow{\phi_1'} A_2 \xrightarrow{\phi_2'} \cdots .$$

The result follows by Proposition 2.3.44. □

The following Lemma is an important generalization of Proposition 1.8 (i) in [75], which will be the main ingredient in obtaining existence of AX-Cartan subalgebras.

Lemma 4.4.9. *Let $A = C(X) \otimes M_n$, and $B = C(X) \otimes M_m$. Assume $\phi : A \to B$ is a unital maximally homogeneous *-homomorphism. Let C be a maximally homogeneous subalgebra of A. Then there exists a maximally homogeneous subalgebra D of B such that $\phi(C) \subset D$.*

Proof. We may assume ϕ takes the form (4.30). Relabel the eigenvalue functions as $g_1, \ldots, g_{\frac{m}{n}}$. By maximal homogeneity, these must be pointwise distinct. Using the form (4.30) we can write

$$\phi(f)_{[0,1]} = U(\sum_{s=1}^{\frac{m}{n}} f \circ g_s \otimes q_s)U^* \tag{4.31}$$

where $q_s : [0,1] \to M_{\frac{m}{n}}$ is a constant projection with value the canonical minimal projection whose s^{th} diagonal entry is 1, in $M_{\frac{m}{n}}$. Note that using the universal covering graph $\mathcal{T}$ over X, we get by Proposition 1.30 in [33] and the remarks following it (this is what is known as the path lifting property for covering spaces) that the g_s's lift to G_s's:

$$\begin{tikzcd} X & \mathcal{T} \arrow[l, "p"'] \\ {[0,1]} \arrow[u, "g_s"] \arrow[ur, dashed, "G_s"] & \end{tikzcd}$$

By Lemma 4.4.1 we find a system of matrix units $\{{}_{\mathcal{T}}e_{pq} : p, q \in \{1, 2, \ldots, n\}\}$ with respect to C. Define

$$P_{p,s} = U({}_{\mathcal{T}}e_{pp} \circ G_s \otimes q_s)U^*,$$

$p = 1, \ldots, n$; $s = 1, \ldots, \frac{m}{n}$, which are mutually orthogonal projections in $C[0,1] \otimes M_m$ which sum to one. Let

$$D = \{g \in B : g_{[0,1]}(t) \in \operatorname{span}\{P_{p,s}(t)\} \ \ \forall t \in [0,1]\}. \tag{4.32}$$

Note that D is well-defined. Indeed, if $t_a, t_b \in [0,1]$ satisfy $l(t_a) = x = l(t_b)$, then we have that

$$\phi(f)_{[0,1]}(t_a) = \phi(f)_{[0,1]}(t_b) \ \ \forall f \in A. \tag{4.33}$$

Hence taking $f = \iota_X$ in (4.33), where ι_X is the canonical generator of the center, and considering the spectrum (which is invariant under an inner automorphism), we get that

$$\{g_1(t_a), \ldots, g_{\frac{m}{n}}(t_a)\} = \{g_1(t_b), \ldots, g_{\frac{m}{n}}(t_b)\},$$

and since the elements of these sets are all distinct (by maximal homogeneity of ϕ), we have that there exists a permutation $\mu \in \Sigma_{\frac{m}{n}}$ such that

$$g_s(t_a) = g_{\mu(s)}(t_b) \tag{4.34}$$

for all $s = 1, \ldots, \frac{m}{n}$. Hence $p(G_s(t_a)) = p(G_{\mu(s)}(t_b))$ for all s. By Lemma 4.4.1, there exists a permutation $\sigma_s \in \Sigma_n$ for $s = 1, \ldots, \frac{m}{n}$ such that for all $p \in \{1, \ldots, n\}$ we have

$$\tau e_{pp}(G_s(t_a)) = \tau e_{\sigma_s(p)\sigma_s(p)}(G_{\mu(s)}(t_b)).$$

We may deduce two things from (4.33) and (4.34). Firstly, by choosing $f \in A$ such that f is 1 at $g_{s_0}(t_a)$ and 0 at all the other (distinct) eigenvalues, we obtain from (4.31) that

$$\begin{aligned}
U(t_a)(\sum_{s=1}^{\frac{m}{n}} f(g_s(t_a)) \otimes q_s(t_a))U(t_a)^* &= U(t_a)(1 \otimes q_{s_0}(t_a))U(t_a)^* \\
&= U(t_b)(\sum_{s=1}^{\frac{m}{n}} f(g_s(t_b)) \otimes q_s(t_b))U(t_b)^* \\
&= U(t_b)(1 \otimes q_{\mu(s_0)}(t_b))U(t_b)^*.
\end{aligned}$$

Hence

$$U(t_a)(1 \otimes q_{s_0}(t_a))U(t_a)^* = U(t_b)(1 \otimes q_{\mu(s_0)}(t_b))U(t_b)^*.$$

Secondly, by choosing $f \in A$ with constant value $C \in M_n$ then again using (4.31) we have that

$$\begin{aligned}
U(t_a)(\sum_{s=1}^{\frac{m}{n}} f(g_s(t_a)) \otimes q_s(t_a))U(t_a)^* &= U(t_a)(C \otimes 1)U(t_a)^* \\
&= U(t_b)(\sum_{s=1}^{\frac{m}{n}} f(g_s(t_b)) \otimes q_s(t_b))U(t_b)^* \\
&= U(t_b)(C \otimes 1)U(t_b)^*.
\end{aligned}$$

Hence

$$U(t_a)(C\otimes 1)U(t_a)^* = U(t_b)(C\otimes 1)U(t_b)^*.$$

From this we obtain

$$\begin{aligned} P_{p,s}(t_a) &= U(t_a)({}_{\mathcal{T}}e_{pp}(G_s(t_a))\otimes q_s(t_a))U(t_a)^* \\ &= U(t_a)({}_{\mathcal{T}}e_{pp}(G_s(t_a))\otimes 1)U(t_a)^*U(t_a)(1\otimes q_s(t_a))U(t_a)^* \\ &= U(t_b)({}_{\mathcal{T}}e_{pp}(G_s(t_a))\otimes 1)U(t_b)^*U(t_b)(1\otimes q_{\mu(s)}(t_b))U(t_b)^* \\ &= U(t_b)({}_{\mathcal{T}}e_{\sigma_s(p)\sigma_s(p)}(G_{\mu(s)}(t_b))\otimes 1)(1\otimes q_{\mu(s)}(t_b))U(t_b)^* \\ &= P_{\sigma_s(p),\mu(s)}(t_b). \end{aligned}$$

Note that if $P_{\sigma_s(p),\mu(s)}(t) = P_{\sigma_{s_1}(p_1),\mu(s_1)}(t)$ then $s = s_1$ and so $p = p_1$. Hence $(p,s)\to(\sigma_s(p),\mu(s))$ is bijective and so

$$\begin{aligned} &\mathrm{span}\{P_{p,s}(t_a) : p\in\{1,\ldots,n\}, s\in\{1,\ldots,\frac{m}{n}\}\} = \\ &\mathrm{span}\{P_{p,s}(t_b) : p\in\{1,\ldots,n\}, s\in\{1,\ldots,\frac{m}{n}\}\} \end{aligned}$$

and so D is well-defined.

Let us show that $\phi(C)\subset D$. For $f\in C$, use Lemma 4.4.1 and (4.31) to write

$$\begin{aligned} \phi(f)_{[0,1]}(t) &= U(t)(\sum_{s=1}^{\frac{m}{n}} f_{\mathcal{T}}(G_s(t))\otimes q_s(t))U(t)^* \\ &= U(t)(\sum_{s=1}^{\frac{m}{n}}(\sum_{p=1}^{n}\lambda_p^s{}_{\mathcal{T}}e_{pp}(G_s(t)))\otimes q_s(t))U(t)^* \\ &\in \mathrm{span}\{P_{p,s}(t)\}, \end{aligned}$$

where the λ_p^s's are some scalars (Lemma 4.4.1). By definition of D we indeed have $\phi(C)\subset D$.

Finally let us show that D is maximally homogeneous. Fix $p\in\{1,\ldots,n\}$, $s\in\{1,\ldots,\frac{m}{n}\}$, and $t\in[0,1]$. Note that there exists a central element $h\in C$ such that $h(g_s(t)) = 1$, whilst $h(g_{\overline{s}}(t)) = 0$ if $\overline{s}\neq s$. Since $C(g_s(t)) = \mathrm{span}\{{}_{\mathcal{T}}e_{jj}(G_s(t)) : j\in\{1,\ldots,n\}\}$ by maximal homogeneity of C, we can find an $h_1\in C$ such that $h_1(g_s(t)) = {}_{\mathcal{T}}e_{pp}(G_s(t))$. Note then that from (4.31) it follows that

$$\phi(hh_1)_{[0,1]}(t) = P_{p,s}(t), \tag{4.35}$$

which implies that $\dim(D(x))$ is m for every $x\in X$. Since, for $f\in C(X)$, we have that $D \ni \sum_{p,s} f_{[0,1]}P_{p,s} = f_{[0,1]}$, it follows that D contains the center. Hence D is maximally homogeneous. □

Lemma 4.4.10. *In the situation of Lemma 4.4.9 (and its proof), we have that*

$$\phi(N_A(C)) \subseteq N_B(D).$$

Proof. Let $n \in N_A(C)$. For $d \in D$, and $t \in [0,1]$, we have that $d_{[0,1]}(t) = \sum_{s,p} \lambda_{p,s}P_{p,s}(t) = \sum_{s,p} \lambda_{p,s}\phi(h_{s,p})_{[0,1]}(t)$ for some scalars $\lambda_{s,p}$'s and elements $h_{s,p} \in C$ (this is by the description of D in (4.32) and the calculation (4.35)). From this it follows that

$$\begin{aligned}
(\phi(n)d\phi(n)^*)_{[0,1]}(t) &= \phi(n)_{[0,1]}(t)(\sum_{s,p} \lambda_{s,p}\phi(h_{s,p})_{[0,1]}(t))\phi(n)_{[0,1]}(t)^* \\
&= \sum_{s,p} \lambda_{s,p}\phi(nh_{s,p}n^*)_{[0,1]}(t) \\
&= \sum_{p,s} \lambda_{p,s}(d_{s,p})_{[0,1]}(t) \\
&\in \operatorname{span}\{P_{p,s}(t)\},
\end{aligned}$$

where we have used that $\phi(C) \subseteq D$. Hence $\phi(n)d\phi(n)^* \in D$, and by a similar argument $\phi(n)^*d\phi(n) \in D$. □

Lemma 4.4.11. *In the situation of Lemma 4.4.9 (and its proof), let $P : A \to C$ be the faithful conditional expectation corresponding to the Cartan pair (A, C), and $\overline{P} : B \to D$ be the faithful conditional expectation corresponding to the Cartan pair (B, D). Then these are given by*

$$P(f)_{\mathcal{T}}(t) = \sum_{p=1}^{N} {}_{\mathcal{T}}e_{pp}(t)f_{\mathcal{T}}(t)_{\mathcal{T}}e_{pp}(t) \quad \forall t \in \mathcal{T} \tag{4.36}$$

and

$$\overline{P}(g)_{[0,1]}(t) = \sum_{s=1}^{\frac{m}{n}} \sum_{p=1}^{n} P_{p,s}(t)g_{[0,1]}(t)P_{p,s}(t) \quad \forall t \in [0,1], \tag{4.37}$$

respectively, and satisfy

$$\phi \circ P = \overline{P} \circ \phi. \tag{4.38}$$

Proof. That (4.36) and (4.37) define unique faithful conditional expectations is an easy generalization of Lemma 4.3.5 where one can check that these maps are norm one faithful linear projections.

Let us show (4.38). We have, for $f \in A$, and $t \in [0,1]$,

$$\begin{aligned}
\overline{P}(\phi(f))_{[0,1]}(t) &= \sum_{s,p} P_{p,s}(t)\phi(f)_{[0,1]}(t)P_{p,s}(t) \\
&= \sum_{s,p}(U(t)({}_{\mathcal{T}}e_{pp}(G_s(t)) \otimes q_s(t))U(t)^*)(U(t)(\sum_{\overline{s}} f(g_{\overline{s}}(t)) \otimes q_{\overline{s}}(t))U(t)^*) \\
&(U(t)({}_{\mathcal{T}}e_{pp}(G_s(t)) \otimes q_s(t))U(t)^*) \\
&= U(t)(\sum_{s,p} {}_{\mathcal{T}}e_{pp}(G_s(t))f(g_s(t))_{\mathcal{T}}e_{pp}(G_s(t)) \otimes q_s(t))U(t)^* \\
&= U(t)(\sum_{s,p} {}_{\mathcal{T}}e_{pp}(f_{\mathcal{T}})_{\mathcal{T}}e_{pp}(G_s(t)) \otimes q_s(t))U(t)^* \\
&= U(t)(\sum_{s}(P(f)_{\mathcal{T}}(G_s(t))) \otimes q_s(t))U(t)^* \\
&= U(t)(\sum_{s}(P(f)(g_s(t))) \otimes q_s(t))U(t)^* \\
&= \phi(P(f))_{[0,1]}(t),
\end{aligned}$$

and the result follows. □

We may now state the main result of this section, which gives the existence of AX-Cartan subalgebras. This is Theorem D in the Introduction.

Theorem 4.4.12. *Every unital AX-algebra $A = \varinjlim_n (A_n, \phi_n)$ with unital and injective connecting maps, and where X is a finite planar connected graph imbedded in $\mathbb{C}$, contains an AX-Cartan subalgebra.*

Proof. This is is many ways similar to the proof of Theorem 4.3.18, but instead of $\mathbb{T}$ we have X. Hence we will only state the main steps, and the details and notation used can be found in the proof of Theorem 4.3.18.

First, by Theorem 4.4.8 we may assume that the connecting maps are unital, maximally homogeneous and injective, of the form (4.30). For $A_n = \bigoplus_{j=1}^{N} C(X) \otimes M_{n_j}$, and $A_{n+1} = \bigoplus_{i=1}^{M} C(X) \otimes M_{m_i}$, with Cartan subalgebra $C \subset A_n$, we consider the maximally homogeneous map

$$\rho_{ij} : C(X) \otimes M_{n_j} \to C(X) \otimes M_{n_j k_{ij}}$$

where $\{k_{ij}\}$ is the index system with respect to ϕ, and $C_j = \Pi_j(C) \subset C(X) \otimes M_{n_j}$ is a Cartan subalgebra (hence maximally homogeneous). By Lemma 4.4.9 we find a maximally homogeneous subalgebra E_{ij} in $C(X) \otimes M_{n_j k_{ij}}$ containing $\rho_{ij}(C_j)$. We pull this back, via μ^{-1}, to a maximally homogeneous subalgebra D_{ij} in $\phi_i(1 \otimes 1_j)(C(X) \otimes M_{m_i})\phi_i(1 \otimes 1_j)$, which contains $\phi_{ij}(C_j)$ and the center of the algebra, and we let D_i be the sum of D_{ij} over j. Then we let D be the direct sum of the D_i's. As in the proof of Theorem 4.3.18, D is a maximally homogeneous subalgebra (hence a Cartan subalgebra) of A_{n+1} with $\phi(C) \subseteq D$.

Lemmas 4.4.10 and 4.4.11 can then be applied in order to conclude, as we do in the proof of Theorem 4.3.18, that ϕ maps the normalizer set into the normalizer set, and that it is compatible with the conditional expectations. By Proposition 4.1.5 we then get that $\varinjlim(C_n, \phi_n)$ is a Cartan subalgebra in A. □

Remark 4.4.13. The AX-Cartan subalgebras that we have shown the existence of for Theorem 4.4.12 are in fact C^*-diagonals. Indeed, the AX-building blocks are continuous trace C^*-algebras by Example 5.18 in [62], and hence by Proposition 6.1 in [64] their Cartan subalgebras have the unique extension property and hence are C^*-diagonals by definition. The result follows by Theorem 1.10 in [47].

Chapter 5

Uniqueness of Inductive Limit Cartan Subalgebras in Inductive Limit C^*-algebras

In Chapter 4 we explored the existence of inductive limit Cartan subalgebras in inductive limit C^*-algebras. Specifically, we showed that all unital AX-algebras with unital and injective connecting maps, where X is any finite connected planar graph imbeddable in $\mathbb{C}$, contains an AX-Cartan subalgebra. It is interesting to now explore whether such Cartan subalgebras are unique (in the sense of Definition 2.3.25).

We will show that AF-Cartan subalgebras are indeed unique, whilst uniqueness fails for AI-algebras, and since all AI-algebras are A$\mathbb{T}$-algebras (see Corollary 3.2.17 in [67]), uniqueness fails for A$\mathbb{T}$-algebras. For the AF case, we will prove a result similar to Elliott's classification theorem for AF-algebras (see [20]) but which incorporates Cartan subalgebras into the picture. Indeed we will show that two unital AF-algebras with AF-Cartan subalgebras have isomorphic ordered K_0 groups if and only if the AF-algebras are isomorphic, and that isomorphism can be chosen to map one of the Cartan subalgebras to another, whilst still inducing the isomorphism on the level of K_0. Note that a consequence of Krieger's dimension range concept developed in [39] is the uniqueness of AF-Cartan subalgebras (see 6.2 in [64]). However, our proof is an independent original proof that uses only the dimension group developed by Elliott (see Chapter 7 in [43]).

The situation for AI-algebras is different. We will prove non-uniqueness by constructing two diagonal Cartan subalgebras with non-homeomorphic spectra in a specific (non-simple) AI-algebra. The way we shall do this is through Proposition 4.1.5, where in Section 4.1 we highlighted how the spectrum of the inductive limit Cartan subalgebra is the inverse limit of the unit spaces of the groupoids corresponding to the Cartan subalgebras of every building block.

Since the AI-algebra constructed is not simple, we will explore uniqueness in the class of simple AI-algebras. We will show that uniqueness fails here also, by constructing diagonal Cartan subalgebras with non-homeomorphic spectra inside a simple AI-algebra. Finally, we prove a general result which shows that uniqueness of AI-Cartan subalgebras fails for a large (possibly total) class of simple AI-algebras. Here we will involve Elliott's invariant for unital AI-algebras which includes K_0 and its pairing with the simplex of tracial states (see [18]). The preliminaries required for this chapter involve more or less all the previous chapters in this thesis.

5.1 Uniqueness of AF-Cartan Subalgebras

In Example 2.3.29 we saw that AF-algebras have AF-Cartan subalgebras, due to the constructions by Strătilă and Voiculescu in [72]. We now show that these are unique. The proofs in this section build on the proofs in Section 7.3 in [43], by including Cartan subalgebras in the picture.

Lemma 5.1.1. *Let $A = \bigoplus_{j=1}^{N} M_{n_j}$ and $B = \bigoplus_{i=1}^{M} M_{m_i}$ be finite dimensional C^*-algebras with Cartan subalgebras C and D respectively. Assume there exists an order unit preserving positive group homomorphism $\alpha : K_0(A) \to K_0(B)$. Then there exists a unital *-homomorphism $\phi : A \to B$ such that $K_0(\phi) = \alpha$, $\phi(C) \subseteq D$ and $\phi(N_A(C)) \subseteq N_B(D)$.*

Proof. By Lemma 7.3.2 (i) in [43] there exists a unital *-homomorphism $\overline{\phi} : A \to B$ with $K_0(\overline{\phi}) = \alpha$. Let $\{e^j_{pq}\}$ and $\{h^i_{uv}\}$ be systems of matrix units for A and B with respect to the Cartan subalgebras C and D, respectively (recall Lemma 4.2.1). Let $\{k_{ij}\}$ be the index system with respect to $\overline{\phi}$. Consider $\overline{\phi}_{ij} : M_{n_j} \to M_{m_i}$, which is a *-homomorphism (that is not necessarily unital). Since $\{\overline{\phi}_{ij}(\Pi_j(e^j_{pp})) : 1 \leq p \leq n_j\}$ is a set of mutually orthogonal projections in M_{m_i}, each with trace k_{ij}, and $\sum_{j=1}^{N} n_j k_{ij} = m_i$, we may decompose $\{\Pi_i(h^i_{uu}) : 1 \leq u \leq m_i\}$ into disjoint sets $H^{i1}, H^{i2}, \ldots, H^{iN}$ where each H^{ij} has size $n_j k_{ij}$. Each H^{ij} can then be decomposed into n_j disjoint sets $H^{ij}_1, \ldots, H^{ij}_{n_j}$ of size k_{ij} each. Let $f^{ij}_{pp} \in \Pi_i(D)$ be the sum of the elements of H^{ij}_p, with trace k_{ij}. Hence $\overline{\phi}_i(e^j_{pp})$ has the same trace as f^{ij}_{pp} and so by Example 2.2.47 $[\overline{\phi}_i(e^j_{pp})]_0 = [f^{ij}_{pp}]_0$.

Since

$$f_{11}^{ij} \sim f_{22}^{ij} \sim \cdots \sim f_{n_j}^{ij},$$

we may apply Lemma 7.1.2 in [43] to obtain a system of matrix units

$$\{f_{pq}^{ij} : p, q \in \{1, \ldots, n_j\}\}$$

in M_{m_i}. This can be extended to a set of matrix units

$$\{f_{pq}^{ij} : j \in \{1, \ldots, N\}, p, q \in \{1, \ldots, n_j\}\}$$

in M_{m_i}. In fact we can be a bit more specific. If

$$H_p^{ij} = \Pi_i(\{h^i_{u_1 u_1}, \ldots, h^i_{u_{k_{ij}} u_{k_{ij}}}\}), \quad u_1 \leq u_2 \leq \ldots \leq u_{k_{ij}}, \tag{5.1}$$

the sum of whose elements is f_{pp}^{ij}, and

$$H_q^{ij} = \Pi_i(\{h^i_{v_1 v_1}, \ldots, h^i_{v_{k_{ij}} v_{k_{ij}}}\}), \quad v_1 \leq v_2 \leq \ldots \leq v_{k_{ij}}, \tag{5.2}$$

the sum of whose elements is f_{qq}^{ij}, we define

$$f_{pq}^{ij} = \sum_{s=1}^{k_{ij}} \Pi_i(h^i_{u_s v_s}). \tag{5.3}$$

Because we have fixed an order in (5.1) and (5.2), we indeed get a well-defined system of matrix units $\{f_{pq}^{ij}\}$, with

$$f^{ij_1}_{p_1 q_1} f^{ij_2}_{p_2 q_2} = 0 \text{ unless } j_1 = j_2, q_1 = p_2 \text{ in which case we get } f^{ij_1}_{p_1 q_2}.$$

Define

$$\phi : A \to B \text{ by } \phi(e^j_{pq}) = \bigoplus_{i=1}^{M} f^{ij}_{pq},$$

and extend it linearly to A. It is clear that ϕ is a unital *-homomorphism with $\phi(C) \subseteq D$. In the same vein as Lemma 4.2.5, the elements of $N_A(C)$ are those that have at most one non-zero entry in any row or column with respect to $\{e^j_{pq}\}$, and the elements of $N_B(D)$ are those that have at most one non-zero entry in any row or column with respect to $\{h^i_{uv}\}$. Hence it follows by using (5.3) that $\phi(N_A(C)) \subseteq N_B(D)$. Now note that

$$K_0(\phi)([e^j_{11}]_0) = \sum_{i=1}^{M} [f^{ij}_{11}]_0 = \sum_{i=1}^{M} [\overline{\phi}_{ij}(e^j_{11})]_0 = [\overline{\phi}(e^j_{11})]_0 = K_0(\overline{\phi})([e^j_{11}]_0) = \alpha([e^j_{11}]_0)$$

and since the set $\{[e^j_{11}]_0 : 1 \leq j \leq N\}$ generates $K_0(A)$, it follows that $K_0(\phi) = \alpha$ as desired. $\square$

Lemma 5.1.2. *Let A and B be finite dimensional C^*-algebras with Cartan subalgebras C and D respectively. Assume $\phi, \psi : A \to B$ are unital *-homomorphisms which map C into D, $N_A(C)$ into $N_A(D)$, and such that $K_0(\phi) = K_0(\psi)$. Then there exists $U \in \mathcal{U}(B) \cap N_B(D)$ such that $\psi = \mathrm{Ad}(U) \circ \phi$.*

Proof. Let $\{e^j_{pq}\}$ and $\{h^i_{uv}\}$ be systems of matrix units for A and B with respect to the Cartan subalgebras C and D, respectively. It is easy to see that the assumption $K_0(\phi) = K_0(\psi)$ implies that $K_0(\phi_{ij}) = K_0(\psi_{ij})$ for $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N\}$. Hence the traces of $\phi_{ij}(\Pi_j(e^j_{pp}))$ and $\psi_{ij}(\Pi_i(e^j_{pp}))$ are the same for all $p \in \{1, \ldots, n_j\}$. Now we may use Lemma 2.3.37 to obtain that there exists a unitary U in B such that

$$U\phi(e^j_{pq})U^* = \psi(e^j_{pq}),$$

from which it follows that

$$\psi = \mathrm{Ad}(U) \circ \phi.$$

In fact we can be a bit more specific. The way U is constructed in the proof of Lemma 2.3.37 is by first finding a partial isometry, v^{ij}_1 which witnesses the Murray-von Neumann equivalence between $\psi_i(e^j_{11})$ and $\phi_i(e^j_{11})$. Since by assumption these initial and range projections belong to D, they are each a sum of elements of the form

$$\psi_i(e^j_{11}) = \sum_{s=1}^{k_{ij}} \Pi_i(h^i_{u_s u_s}), \quad \phi_i(e^j_{11}) = \sum_{s=1}^{k_{ij}} \Pi_i(h^i_{v_s v_s}).$$

Hence we can be particular with our choice of v^{ij}_1 by declaring it

$$v^{ij}_1 = \sum_{s=1}^{k_{ij}} \Pi_i(h^i_{v_s u_s}).$$

Similarly to Lemma 4.2.5 it follows that v^{ij}_1 belongs to $N_{B_i}(D_i)$. Then v^{ij}_p is constructed as $\psi_i(e^j_{p1})v^{ij}_1\phi_i(e^j_{1p})$ and so also belongs to $N_{B_i}(D_i)$ by the assumption of the lemma. From this U_i is then constructed as $\sum_{j,p} v^{ij}_p$ and U as $\bigoplus_{i=1}^{M} U_i$.

To see that U is in the normalizer, let $h^i_{uu} \in D$. Note that because ϕ is unital and maps C into D there must exists some $j \in \{1, \ldots, N\}$ and $p \in \{1, \ldots, n_j\}$ such that h^i_{uu} appears as a summand of $\phi(e^j_{pp})$. One can check that $\Pi_i(Uh^i_{uu}U^*) = v^{ij}_p \Pi_i(h^i_{uu})(v^{ij}_p)^*$ and hence belongs to D_i. Hence $UDU^* \subseteq D$. A similar argument done by replacing ϕ with ψ shows that $U^*DU \subseteq D$. Hence $U \in N_B(D)$. $\square$

We are now in a position to prove what is Theorem E in the Introduction.

Theorem 5.1.3. *Let A and B be unital AF-algebras with AF-Cartan subalgebras C and D, respectively. Assume there exists an isomorphism*

$$\alpha : (K_0(A), K_0(A)^+, [1_A]_0) \to (K_0(B), K_0(B)^+, [1_B]_0).$$

*Then there exists a *-isomorphism $\phi : A \to B$ such that $K_0(\phi) = \alpha$ and $\phi(C) = D$.*

Proof. We will assume A, B, C and D all arise as inductive limits as follows respectively:

$$\begin{tikzcd} B_0 \arrow[r, "\phi_0"] & A_1 \arrow[r, "\phi_1"] & A_2 \arrow[r, "\phi_2"] & \dots \arrow[r, bend left, "\mu_n"] & A, \end{tikzcd}$$

$$\begin{tikzcd} B_0 \arrow[r, "\psi_0"] & B_1 \arrow[r, "\psi_1"] & B_2 \arrow[r, "\psi_2"] & \dots \arrow[r, bend left, "\rho_n"] & B, \end{tikzcd}$$

$$\begin{tikzcd} C_0 \arrow[r, "\phi_0"] & C_1 \arrow[r, "\phi_1"] & C_2 \arrow[r, "\phi_2"] & \dots \arrow[r, bend left, "\mu_n"] & C, \end{tikzcd}$$

$$\begin{tikzcd} D_0 \arrow[r, "\psi_0"] & D_1 \arrow[r, "\psi_1"] & D_2 \arrow[r, "\psi_2"] & \dots \arrow[r, bend left, "\rho_n"] & D. \end{tikzcd}$$

We may assume, using Exercise 6.7 in [43], that all the ϕ_n's, ψ_n's, μ_n's and ρ_n's are injective and unital *-homomorphisms, for $n = 1, 2, \ldots$. Here $B_0 = C_0 = D_0 = \mathbb{C}$, $\phi_0(\lambda) = \lambda 1_{A_1}$, $\psi_0(\lambda) = \lambda 1_{B_1}$, $\mu_0(\lambda) = \lambda 1_A$ and $\rho_0(\lambda) = \lambda 1_B$. Of course all the building blocks here are finite dimensional C^*-algebras. If $\phi_n : A_n \to A_{n+1}$ is a connecting map, then by Lemma 3.2.1 in [16] ϕ_n is unitarily equivalent to the map given by block diagonal imbeddings, and so by Proposition 2.3.46 we may without loss of generality assume all our connecting maps are block diagonal imbeddings. Hence in the same vein as Lemma 4.2.5 we may assume that $\phi_n(N_{A_n}(C_n)) \subseteq N_{A_{n+1}}(C_{n+1})$. The same holds for the connecting maps ψ_n.

We have the following commutative diagram with positive and order unit preserving group homomorphisms:

$$\begin{tikzcd}
K_0(B_0) \arrow[rr, "K_0(\rho_0)"] \arrow[dr, "K_0(\phi_0)"'] & & K_0(B) \\
& K_0(A_1) \arrow[ur, "\alpha\circ K_0(\mu_1)"'] &
\end{tikzcd}$$

From Lemma 7.3.3 in [43] we obtain that there exists $m_1 \in \mathbb{N}$ and a positive order unit preserving group homomorphism α_1 such that we have a commutative diagram:

$$\begin{tikzcd}
K_0(B_0) \arrow[r, "K_0(\psi_{m_1 0})"] \arrow[rr, bend left, "K_0(\rho_0)"] \arrow[dr, "K_0(\phi_0)"'] & K_0(B_{m_1}) \arrow[r, "K_0(\rho_{m_1})"] & K_0(B) \\
& K_0(A_1) \arrow[u, "\alpha_1"] \arrow[ur, "\alpha\circ K_0(\mu_1)"'] &
\end{tikzcd}$$

Next, consider the following commutative diagram with positive and order unit preserving homomorphisms:

$$\begin{tikzcd}
K_0(A_1) \arrow[rr, "K_0(\mu_1)"] \arrow[dr, "\alpha_1"'] & & K_0(A) \\
& K_0(B_{m_1}) \arrow[ur, "\alpha^{-1}\circ K_0(\rho_{m_1})"'] &
\end{tikzcd}$$

There exists $n_1 \in \mathbb{N}$ and a positive order unit preserving group homomorphism β_1 making the following diagram commute:

$$\begin{tikzcd}
K_0(A_1) \arrow[r, "K_0(\phi_{n_1 1})"] \arrow[rr, bend left, "K_0(\mu_1)"] \arrow[dr, "\alpha_1"'] & K_0(A_{n_1}) \arrow[r, "K_0(\mu_{n_1})"] & K_0(A) \\
& K_0(B_{m_1}) \arrow[u, "\beta_1"] \arrow[ur, "\alpha^{-1}\circ K_0(\rho_{m_1})"'] &
\end{tikzcd}$$

Repeating, we obtain a commutative intertwining:

$$
\begin{tikzcd}
K_0(B_{m_1}) \arrow[rr, "K_0(\psi_{m_2m_1})"] \arrow[dr, "\beta_1"'] & & K_0(B_{m_2}) \arrow[dr, "\beta_2"] & \dots & \dots & K_0(B) \\
& K_0(A_{n_1}) \arrow[ur, "\alpha_2"] \arrow[rr, "K_0(\phi_{n_2n_1})"] & & K_0(A_{n_2}) & \dots & K_0(A) \arrow[u, "\alpha"']
\end{tikzcd}
$$

where all homomorphisms are positive and order unit preserving. Since inductive limits do not change if one takes a subsequence of the original building blocks, we may relabel to assume

$$
\begin{tikzcd}
K_0(B_1) \arrow[rr, "K_0(\psi_1)"] \arrow[dr, "\beta_1"'] & & K_0(B_2) \arrow[dr, "\beta_2"] & \dots & \dots & K_0(B) \\
& K_0(A_1) \arrow[ur, "\alpha_1"] \arrow[rr, "K_0(\phi_1)"] & & K_0(A_2) & \dots & K_0(A) \arrow[u, "\alpha"']
\end{tikzcd}
\tag{5.4}
$$

By applying Lemma 5.1.1 and Lemma 5.1.2, we lift (5.4) to a commutative intertwining

$$
\begin{tikzcd}
B_1 \arrow[rr, "\mathrm{Ad}(V_2)\psi_1"] \arrow[dr, "g_1"'] & & B_2 \arrow[dr, "g_2"] & \dots & \dots & \overline{B} \\
& A_1 \arrow[ur, "f_1"] \arrow[rr, "\mathrm{Ad}(U_2)\phi_1"] & & A_2 & \dots & \overline{A} \arrow[u, "\Phi"']
\end{tikzcd}
\tag{5.5}
$$

where all the diagonal *-homomorphisms in (5.5) are unital, map Cartan subalgebra into Cartan subalgebra, normalizer set into normalizer set, and induce the corresponding K_0 group homomorphisms in (5.4), and where all the unitaries $\{U_n\}$ and $\{V_n\}$ for $n = 2, 3, \dots$ satisfy $U_n \in N_{A_n}(C_n)$ and $V_n \in N_{B_n}(D_n)$. We also have an induced *-isomorphism Φ between the induced inductive limits (see for example, Exercise 6.8 in [43]).

We extend (5.5) to a commutative diagram

$$\begin{tikzcd}
B_1 \arrow[rr, "\psi_1"] & & B_2 \arrow[rr, "\psi_2"] & & B_3 & \cdots & \cdots & B \\
B_1 \arrow[u, "\mathrm{id}"] \arrow[rr, "\mathrm{Ad}(V_2)\psi_1"] \arrow[dr, "g_1"] & & B_2 \arrow[u, "\mathrm{Ad}(V_2^*)"] \arrow[rr, "\mathrm{Ad}(V_3)\psi_2"] \arrow[dr, "g_2"] & & B_3 \arrow[u, "\mathrm{Ad}(\psi_2(V_2^*))\mathrm{Ad}(V_3^*)"] \arrow[dr, "g_3"] & \cdots & \cdots & \overline{B} \arrow[u, "F"] \\
& A_1 \arrow[ur, "f_1"] \arrow[rr, "\mathrm{Ad}(U_2)\phi_1"] & & A_2 \arrow[ur, "f_2"] \arrow[rr, "\mathrm{Ad}(U_3)\phi_2"] & & A_3 & \cdots & \overline{A} \arrow[u, "\Phi"] \\
& A_1 \arrow[u, "\mathrm{id}"] \arrow[rr, "\phi_1"] & & A_2 \arrow[u, "\mathrm{Ad}(U_2)"] \arrow[rr, "\phi_2"] & & A_3 \arrow[u, "\mathrm{Ad}(U_3)\mathrm{Ad}(\phi_2(U_2))"] & \cdots & A \arrow[u, "G"]
\end{tikzcd} \tag{5.6}$$

where we have induced *-isomorphisms F and G. Define

$$\phi = F \circ \Phi \circ G.$$

We need to check $K_0(\phi) = \alpha$ and $\phi(C) = D$. Note that $K_0(\phi)$ and α agree on $K_0(\mu_n)(K_0(A_n))$ by the commutativity of (5.4) and (5.6). Since $K_0(A) = \bigcup_n K_0(\mu_n)(K_0(A_n))$ by Proposition 2.3.17, we obtain $K_0(\phi) = \alpha$. Note that from the commutativity of (5.6) it follows that ϕ maps $\mu_n(C_n)$ into $\rho_{n+1}(D_{n+1}) \subseteq D$. Hence ϕ maps C into D by Proposition 2.3.10, but as it is a *-isomorphism, it follows that $\phi(C) \subseteq D$ is a masa in B, and so it follows that $\phi(C) = D$. $\square$

Corollary 5.1.4. *Unital AF-algebras have unique AF-Cartan subalgebras.*

Proof. If (A, C) and (B, D) are c_F-inductive limit Cartan pairs with $A \cong B$, then the ordered K_0 group of A is isomorphic to the ordered K_0 group of B, and so we may find by Theorem 5.1.3 an isomorphism $(A, C) \cong (B, D)$, which gives uniqueness by Definition 2.3.25. $\square$

5.2 Non-Uniqueness of AI-Cartan Subalgebras

In this section we show that AI-Cartan subalgebras are not unique. First, we will construct two non-isomorphic diagonal Cartan subalgebras inside a non-simple AI-algebra. In order to do this we will describe the spectrum of an AI-Cartan subalgebra by making use of Proposition 4.1.5 following the discussion in Section 4.1. Using our description we will create two diagonal Cartan subalgebras with non-homeomorphic spectra.

Then, we will construct two non-isomorphic diagonal Cartan subalgebras inside a simple AI-algebra. Inspired by this, we will state a very general class of simple

AI-algebras for which uniqueness of AI-Cartan subalgebras fails.

5.2.1 Non-Uniqueness in a Non-Simple AI-Algebra

Let $A = \bigoplus_{j=1}^{N} C[0,1] \otimes M_{n_j} \cong C([0,1], \bigoplus_{j=1}^{N} M_{n_j})$ be an AI-building block. Let C be the diagonal Cartan subalgebra of A, of the form $C = \bigoplus_{j=1}^{N} C[0,1] \otimes D_{n_j}$, where D_{n_j} is the diagonal of M_{n_j}. Let $\{e^j_{pq}\}$ be the system of standard matrix units for $\bigoplus_{j=1}^{N} M_{n_j}$. Let $X = \mathrm{Spec}(C) \cong [0,1] \times \{x^j_p : 1 \le j \le N, 1 \le p \le n_j\}$, with topology the product topology of the standard topology on $[0,1]$ and the discrete topology on a finite set. The identification of $f = \sum_{j=1}^{N}(\sum_{p=1}^{n_j} f^j_{pp} \otimes e^j_{pp}) \in C$ with $C(X)$ is given by $f(t, x^j_p) = f^j_{pp}(t)$.

Let $n = \sum_{j=1}^{N}(\sum_{p=1}^{n_j} n^j_{pq} \otimes e^j_{pq}) \in N_A(C)$. Recall that

$$\mathrm{dom}(n) = \{(t, x^j_q) \in X : n^*n(t, x^j_q) > 0\}.$$

Note that

$$n^*n = \sum_{j,p,q_1,q_2} \overline{n^j_{pq_1}} n^j_{pq_2} \otimes e^j_{q_1q_2} = \sum_{j,p,q} |n^j_{pq}|^2 \otimes e^j_{qq}$$

where the last equality follows because $n^*n \in C$ and hence must be diagonal. Hence

$$n^*n(t, x^j_q) = \sum_{p=1}^{n_j} |n^j_{pq}(t)|^2. \tag{5.7}$$

By Lemma 4.2.5 at most one summand appearing in (5.7) must be non-zero. Hence $n^*n(t, x^j_q) > 0$ if and only if there exists (exactly one) non-zero summand in (5.7) and hence we can describe

$$\mathrm{dom}(n) = \{(t, x^j_q) \in X : n(t) \text{ has a non-zero entry in its } j^{\text{th}} \text{ summand, } q^{\text{th}} \text{ column}\}.$$

By similar calculations we can describe

$$\mathrm{ran}(n) = \{(t, x^j_p) \in X : n(t) \text{ has a non-zero entry in its } j^{\text{th}} \text{ summand, } p^{\text{th}} \text{ row}\}.$$

Using Lemma 3.2.10 there is a unique homeomorphism $\alpha_n : \mathrm{dom}(n) \to \mathrm{ran}(n)$, satisfying (3.6). From (3.6) it can be calculated that if $(t, x^j_q) \in \mathrm{dom}(n)$, meaning there is a unique $p \in \{1, \ldots, n_j\}$ such that $n^j_{pq}(t)$ is non-zero, then

$$\alpha_n(t, x^j_q) = (t, x^j_p).$$

The Weyl groupoid is then

$$\mathcal{G}(C) = \{[(t, x^j_p), \alpha_n, (t, x^j_q)] : n \in N_A(C), n^j_{pq}(t) \neq 0\}. \tag{5.8}$$

It is clear that $\mathcal{G}(C)^0 \cong X$.

Now consider an AI-building block $\hat{A} = \bigoplus_{i=1}^{M} C[0,1] \otimes M_{m_i}$, and an injective unital standard map $\phi : A \to \hat{A}$, with corresponding index system $\{k_{ij}\}$ and eigenvalue functions

$$\{g^{ij}_s : i \in \{1, \ldots, M\}, j \in \{1, \ldots, N\}, s \in \{1, \ldots, k_{ij}\}\},$$

(recall Definition 2.3.39). Let $\{f^i_{uv}\}$ be the system of standard matrix units in $\bigoplus_{i=1}^{M} M_{m_i}$, and let $\hat{C} = \bigoplus_{i=1}^{M} C[0,1] \otimes D_{m_i}$ (where D_{m_i} is the diagonal subalgebra of M_{m_i}) be the diagonal Cartan subalgebra of $\hat{A}$. It is clear that

$$\phi(C) \subseteq \hat{C}, \phi(N_A(C)) \subseteq N_{\hat{A}}(\hat{C}), \hat{P} \circ \phi = \phi \circ P,$$

where P is the unique faithful conditional expectation associated to the Cartan pair (A, C), and given by projection onto the diagonal, and $\hat{P}$ is the unique faithful conditional expectation associated to the Cartan pair $(\hat{A}, \hat{C})$, and given by projection onto the diagonal. In order to determine the map ϕ on the groupoid level, in the sense of Proposition 4.1.2, we follow the steps of the proof outlined in Remark 4.1.3.

We commence by constructing the intermediate Cartan pair $(\breve{A}, \breve{C})$ with corresponding twisted groupoid (H, T) (for our purposes, we will just need H and not the twist). We have that $\breve{C} = \hat{C}$, as ϕ is unital. Now $\breve{A} = C^*(\phi(A), \hat{C})$ which is easily seen to be the C^*-subalgebra of $\hat{A}$ which looks like, in the i^{th} summand,

$$\Pi_i(\breve{A}) = \bigoplus_{j=1}^{N} (\bigoplus_{s=1}^{k_{ij}} C[0,1] \otimes M_{n_j}).$$

Hence we consider a basis for $\bigoplus_{i=1}^{M} (\bigoplus_{j=1}^{N} (\bigoplus_{s=1}^{k_{ij}} M_{n_j})) \subset \bigoplus_{i=1}^{M} M_{m_i}$ of the form

$$\{f^{i,j,s}_{pq} : 1 \leq i \leq M, 1 \leq j \leq N, 1 \leq s \leq k_{ij}, 1 \leq p, q \leq n_j\} \subseteq \{f^i_{uv}\}.$$

Note that, similar to how we obtain the Weyl groupoid $\mathcal{G}(C)$ in (5.8), we obtain

$$\mathcal{G}(\hat{C}) = \{[(t, y_u^i), \alpha_n, (t, y_v^i)] : n \in N_{\hat{A}}(\hat{C}), n_{uv}^i(t) \neq 0\},$$

and

$$H = \{[(t, z_p^{i,j,s}), \alpha_n, (t, z_q^{i,j,s})] : n \in N_{\check{A}}(\check{C}), n_{pq}^{i,j,s}(t) \neq 0\},$$

where

$$\mathrm{Spec}(\hat{C}) = \mathrm{Spec}(\check{C}) \cong [0,1] \times \{y_u^i\} = [0,1] \times \{z_p^{i,j,s}\},$$

$$1 \leq i \leq M,\ 1 \leq j \leq N,\ 1 \leq s = s(i,j) \leq k_{ij},\ 1 \leq p = p(j), q = q(j) \leq n_j.$$

It is then clear that the inclusion map $i : H \hookrightarrow \mathcal{G}(\hat{C})$ maps H^0 onto $\mathcal{G}(\hat{C})^0$.

Now consider the map $C \to \check{C}$ given by $f \to \phi(f)$. Explicitly, if $f = \sum_{j,p} f_{pp}^j \otimes e_{pp}^j$, then

$$\phi(f) = \sum_{i,j,s,p} f_{pp}^j \circ g_s^{ij} \otimes f_p^{i,j,s}.$$

Hence, for $t \in [0,1]$,

$$\phi(f)(t, z_p^{i,j,s}) = f_{pp}^j(g_s^{ij}(t)) = f(g_s^{ij}(t), x_p^j).$$

So the dual map $\phi^* : \mathrm{Spec}(\check{C}) \to \mathrm{Spec}(C)$ is given by

$$\phi^*(t, z_p^{i,j,s}) = (g_s^{ij}(t), x_p^j).$$

Hence the surjective map $\dot{p} : H \twoheadrightarrow \mathcal{G}(C)$ satisfies

$$\dot{p}((t, z_p^{i,j,s})) = (g_s^{ij}(t), x_p^j). \tag{5.9}$$

Now assume that we are given a unital AI-algebra $A = \varinjlim(A_n, \phi_n)$ with unital and injective connecting maps. By Lemma 4.2.8 we may assume that the connecting maps are standard maps. To A we can associate a diagram $\mathcal{B}_A$, analogous to the Brattelli diagrams of AF-algebras, as follows. Let $A_n = \bigoplus_{j=1}^{N} C[0,1] \otimes M_{n_j}$, $A_{n+1} = \bigoplus_{i=1}^{N} C[0,1] \otimes M_{m_i}$, and $\{k_{ij}\}$ be the index system with respect to ϕ_n. Since ϕ_n is a standard map we may assume it has corresponding eigenvalue functions $\{g_s^{ij} : i \in \{1,\ldots,M\}, j \in \{1,\ldots,N\}, s \in \{1,\ldots,k_{ij}\}\}$. Let C_n and C_{n+1} be the diagonal (Cartan) subalgebras of A_n and A_{n+1} respectively. By the constructions above we may assume

$$\mathrm{Spec}(C_n) \cong [0,1] \times \{x_p^j : j \in \{1,\dots,N\}, p \in \{1,\dots,n_j\}\}$$

and

$$\begin{aligned}\mathrm{Spec}(C_{n+1}) \cong & [0,1] \times \{z_p^{i,j,s} : i \in \{1,\dots,M\}, j \in \{1,\dots,N\}, s \in \{1,\dots,k_{ij}\},\\ & p \in \{1,\dots,n_j\}\}.\end{aligned}$$

By what we have shown we may assume the surjective groupoid homomorphism $\dot{p_n}$ satisfies

$$\dot{p_n}((t, z_p^{i,j,s})) = (g_s^{ij}(t), x_p^j). \tag{5.10}$$

The n^{th} level of $\mathcal{B}_A$ will have $\sum_{j=1}^{N} n_j$ nodes, which we will label by elements of $\{x_p^j\}$. The $(n+1)^{\text{th}}$ level of $\mathcal{B}_A$ will have $\sum_{i=1}^{M}\sum_{j=1}^{N} n_j k_{ij}$ nodes, labelled by elements of $\{z_p^{i,j,s}\}$ and the arrows from the n^{th} level nodes to the $(n+1)^{\text{th}}$ level nodes will be those going from x_p^j to $z_p^{i,j,s}$. The set of right-infinite paths on $\mathcal{B}_A$ starting from the first level will be denoted by X_A, and for a path $\underline{x} = (x_1, x_2, \dots) \in X_A$, we associate an inverse limit

$$I_{\underline{x}} = \varprojlim([0,1], g_{\underline{x}}) \tag{5.11}$$

where $g_{\underline{x}} = \{g_1, g_2, \dots\}$ is determined uniquely via (5.10). Let $C = \varinjlim(C_n, \phi_n)$ be the diagonal Cartan subalgebra of A, with spectrum $\overline{\mathcal{G}}^0$. By Proposition 4.1.5 we have $\overline{\mathcal{G}}^0 = \varprojlim(\mathrm{Spec}(C_n), \dot{p_n})$. In this setup we have:

Theorem 5.2.1. *There is a homeomorphism*

$$h : \overline{\mathcal{G}}^0 \to \bigsqcup_{\underline{x} \in X_A} I_{\underline{x}}, \quad ((t_1, x_1), (t_2, x_2), \dots) \to ((t_1, t_2, \dots), \underline{x}),$$

where the topology $\mathcal{C}$ on the codomain is induced by the inverse limit topology on $\overline{\mathcal{G}}^0$ via the bijection h.

Proof. We need to check that h is a well-defined bijection. For an element $((t_1, x_1), (t_2, x_2), \dots) \in \overline{\mathcal{G}}^0$, it follows from (5.10) that $\underline{x} = (x_1, x_2, \dots)$ defines an element of X_A, and it follows also from (5.10) and (5.11) that $(t_1, t_2, \dots) \in I_{\underline{x}}$. It is clear that h is is bijective. $\square$

Consider the topology $\mathcal{F}$ on $\bigsqcup_{\underline{x} \in X_A} I_{\underline{x}}$ given by declaring the open sets to be of the form $\bigsqcup_{\underline{y}\in Y\subseteq X_A} U_{\underline{y}}$, where $U_{\underline{y}} \subseteq I_{\underline{y}}$ is open (in the usual inverse limit topology, which is just the subspace topology of the product topology). That $\mathcal{F}$ is a topology can be observed by noting that

$$(\bigsqcup_{\underline{y}\in Y\subseteq X_A} U_{\underline{y}}) \cup (\bigsqcup_{\underline{z}\in Z\subseteq X_A} V_{\underline{z}}) = \bigsqcup_{\underline{x}\in Y\cup Z\subseteq X_A} W_{\underline{x}},$$

where

$$W_{\underline{x}} = U_{\underline{x}} \text{ if } \underline{x} \in Y \setminus Z, \;\; V_{\underline{x}} \text{ if } \underline{x} \in Z \setminus Y, \;\; U_{\underline{x}} \cup V_{\underline{x}} \text{ if } \underline{x} \in Y \cap Z,$$

and

$$(\bigsqcup_{\underline{y}\in Y\subseteq X_A} U_{\underline{y}}) \cap (\bigsqcup_{\underline{z}\in Z\subseteq X_A} V_{\underline{z}}) = \bigsqcup_{\underline{x}\in Y\cap Z\subseteq X_A} U_{\underline{x}} \cap V_{\underline{x}}.$$

Lemma 5.2.2. *The topology $\mathcal{F}$ is finer than $\mathcal{C}$.*

Proof. Every open set in $\mathcal{C}$ is a union of sets of the form

$$h(\overline{\mathcal{G}}^0 \cap ((U_1 \times \{*\}) \times (U_2 \times \{*\}) \times \ldots \times ([0,1] \times \{*\}) \times ([0,1] \times \{*\}) \times \ldots)),$$

where each $\{*\}$ is some singleton edge. By definition of h, this belongs to $\mathcal{F}$. □

Lemma 5.2.3. *The set $\{I_{\underline{x}} : \underline{x} \in X_A\}$ is the set of $\mathcal{C}$-connected components of $\bigsqcup_{\underline{x}\in X_A} I_{\underline{x}}$. In addition, each $I_{\underline{x}}$ is $\mathcal{C}$-compact.*

Proof. Every $I_{\underline{x}}$ is compact and connected in the inverse limit topology by Proposition 2.4.5. Hence this is also true in $\mathcal{F}$. Since $\mathcal{F}$ is finer than $\mathcal{C}$, this also holds in $\mathcal{C}$. Now assume, for a contradiction, that there is an $\underline{x} \in X_A$ for which $I_{\underline{x}}$ is not a $\mathcal{C}$-connected component. Since for all $\underline{y} \in X_A$ we have that $I_{\underline{y}}$ is $\mathcal{C}$-connected, it must be that $I_{\underline{x}}$ is contained in a connected component $\bigsqcup_{\underline{z}\in Z\subseteq X_A} I_{\underline{z}}$. But for $\underline{u}, \underline{v} \in Z$, if $\underline{u} \neq \underline{v}$ then $\exists n \in \mathbb{N}$ such that $u_n \neq v_n$ and so the $\mathcal{C}$-open sets

$$h(\overline{\mathcal{G}}^0 \cap (([0,1] \times \{u_1\}) \times ([0,1] \times \{u_2\}) \times \ldots \times ([0,1] \times \{u_n\}) \times ([0,1] \times D_{n+1}) \times \ldots))$$

and

$$h(\overline{\mathcal{G}}^0 \cap (([0,1] \times \{v_1\}) \times ([0,1] \times \{v_2\}) \times \ldots \times ([0,1] \times \{v_n\}) \times ([0,1] \times D_{n+1}) \times \ldots))$$

(where the D_n's are some finite discrete sets) are disjoint, meeting and covering $I_{\underline{u}} \bigsqcup I_{\underline{v}}$. This is a contradiction and so $I_{\underline{x}}$ is a $\mathcal{C}$-connected component. □

We are now in the position to state our first non-uniqueness result for AI-Cartan subalgebras.

Theorem 5.2.4. *There is a pair of unital non-simple AI-algebras A and B with unital and injective connecting maps, which are isomorphic, and AI-Cartan subalgebras $C \subset A$ and $D \subset B$, which are not isomorphic. In particular, AI-Cartan subalgebras are not unique.*

Proof. Let $A = \varinjlim(A_n, \phi_n)$ be the unital AI-algebra given by AI-building blocks $A_n = C([0,1]) \otimes M_{2^n}$, and connecting maps $\phi_n(a) = \mathrm{diag}(a \circ f_1, a \circ f_2)$, where the eigenvalue functions $f_1, f_2 : [0,1] \to [0,1]$ (independent of n) are given by

$$f_1(t) = \begin{cases} 2t, & \text{if } 0 \le t \le \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \le t \le 1 \end{cases}, \quad f_2(t) = \begin{cases} 1, & \text{if } 0 \le t \le \frac{1}{2} \\ 2(1-t), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}.$$

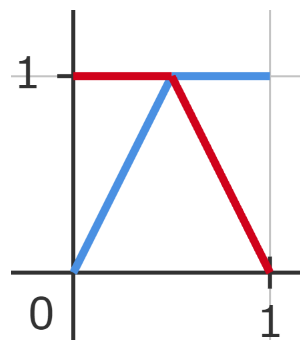


Figure 5.1: Eigenvalue functions f_1 (blue) and f_2 (red).

Since f_1 (and f_2) is surjective, it follows that ϕ_n is an injective *-homomorphism for each $n \in \mathbb{N}$. It is clear that ϕ_n is unital.

Now let $B = \varinjlim(B_n, \psi_n)$ be the unital AI-algebra with $B_n = A_n$ and with $\psi_n(b) = \mathrm{diag}(b \circ g_1, b \circ g_2)$; where the eigenvalue functions $g_1, g_2; [0,1] \to [0,1]$ are given by

$$g_1(t) = 1, \quad g_2(t) = \begin{cases} 2t, & \text{if } 0 \le t \le \frac{1}{2} \\ 2(1-t), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}.$$

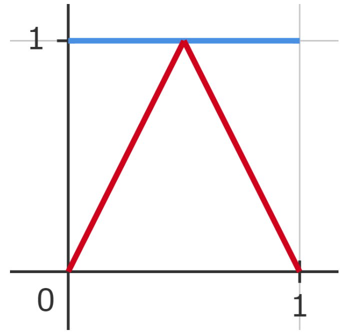


Figure 5.2: Eigenvalue functions g_1 (blue) and g_2 (red).

Since g_2 is surjective, it follows that ψ_n is an injective *-homomorphism for each $n \in \mathbb{N}$. It is clear that ψ_n is unital.

Note that by choosing a non-zero $a \in A_n$ with the property $a(0) = a(1) = 0$, then property (iii) in Proposition 3.1.2 in [67] fails for the AI-algebra B for the choice $x = 0$, and hence B is not simple.

Since the eigenvalue functions satisfy $\{f_1(t), f_2(t)\} = \{g_1(t), g_2(t)\}$ for every $t \in [0,1]$, Theorem 3.1 in [75] implies that $\phi_n \sim_{au} \psi_n$ for all $n \in \mathbb{N}$. By Proposition 2.3.46 it follows that $A \cong B$. Hence A is not simple also.

Now let C and D be the diagonal Cartan subalgebras of A and B respectively, where each building block is the diagonal subalgebra. Consider the path in X_B given by the one whose corresponding sequence of eigenvalue functions is $(g_1, g_1, g_1, \ldots)$. Let this path be $\underline{y}_0 \in X_B$. Clearly $I_{\underline{y}_0}$ is a singleton, and so by Lemma 5.2.3 and Theorem 5.2.1, the spectrum of D contains a singleton as a connected component.

On the other hand, considering any path $\underline{z}_0$ in X_A, which will correspond to a sequence of eigenvalue functions $\{h_n\}_{n\in\mathbb{N}}$ where each $h_n \in \{f_1, f_2\}$, note that if $t \in \{0,1\}$ and $n \in \mathbb{N}$, then there exists a unique $s \in \{0,1\}$ such that $h_n(s) = t$. Hence $I_{\underline{z}_0}$ contains more than one point and so by Lemma 5.2.3 and Theorem 5.2.1, the connected components of the spectrum of C are not singletons. Thus the spectrum of C is not homeomorphic to the spectrum of D, and thus $C \ncong D$. □

5.2.2 Non-Uniqueness in a Simple AI-Algebra

We saw in Subsection 5.2.1 that uniqueness of AI-Cartan subalgebras fails, and the example we considered was a non-simple AI-algebra. In this subsection we give an example of a simple AI-algebra, which is a Goodearl algebra (see [32]) in which we can exhibit two non-isomorphic AI-Cartan subalgebras. Specifically, we have:

Theorem 5.2.5. *There is a pair of unital simple AI-algebras A and B with unital and injective connecting maps, which are isomorphic, and AI-Cartan subalgebras $C \subset A$ and $D \subset B$, which are not isomorphic.*

Proof. Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a sequence of decreasing reals between 0 and $\frac{1}{4}$, $\epsilon_n \to 0$. Choose a sequence of positive integers $\{k_n\}_{n\in\mathbb{N}}$ such that k_n divides k_{n+1} for all $n \in \mathbb{N}$, and such that $q_n = \frac{k_{n+1}}{k_n} > \frac{1}{\epsilon_n} + 1$. Let us consider A as the unital AI-algebra (which is a Goodearl algebra) constructed as

$$C[0,1] \otimes M_{k_1} \xrightarrow{\phi_1} C[0,1] \otimes M_{k_2} \xrightarrow{\phi_2} \cdots$$

where ϕ_n is defined by

$$a \to \operatorname{diag}(a \circ \delta_{x_{n,1}}, \ldots, a \circ \delta_{x_{n,q_n-1}}, a)$$

where the points $\{x_{n,i}\}_{i=1}^{q_n-1}$ are chosen as follows. Divide $[0,1]$ into $q_n - 1$ equally spaced subintervals (note that each subinterval will have width at most ϵ_n), and let $x_{n,i}$ be the leftmost point of the i^{th} subinterval.

Set $F_n = \{x_{n,i} : 1 \leq i \leq q_n - 1\} \subset [0,1]$. It is clear that $\bigcup_{k=n}^{\infty} F_k$ is dense in $[0,1]$ for all $n \in \mathbb{N}$, and so we have that A is simple by Example 3.1.7 in [67]. The connecting maps are injective because the eigenvalue functions together see all of [0,1] (due to the last term in the diagonal with the identity eigenvalue function).

Now consider another AI-algebra B' written in same way as A but where we perturb $\delta_{x_{n,i}}$ to the map $g_{n,i}$ which is defined as follows, for $1 \leq i \leq q_n - 1$:

$$g_{n,i}(t) = \begin{cases} \delta_{x_{n,i}}(t) \text{ for } 0 \leq t \leq \frac{i-1}{2(q_n-1)},\ 1 - \frac{i-1}{2(q_n-1)} \leq t \leq 1, \\ 2t \text{ for } \frac{i-1}{2(q_n-1)} \leq t \leq \frac{i}{2(q_n-1)}, \\ 2 - 2t \text{ for } 1 - \frac{i}{2(q_n-1)} \leq t \leq 1 - \frac{i-1}{2(q_n-1)}, \\ \delta_{x_{n,i+1}}(t) \text{for } \frac{i}{2(q_n-1)} \leq t \leq 1 - \frac{i}{2(q_n-1)}. \end{cases}$$

It is clear that $\|\delta_{x_{n,i}} - g_{n,i}\|_\infty < \epsilon_n$ for all $1 \leq i \leq q_n - 1$. Let ψ'_n be the connecting map between $C[0,1] \otimes M_{k_n}$ and $C[0,1] \otimes M_{k_{n+1}}$ in B', defined by

$$a \to \operatorname{diag}(a \circ g_{n,1}, \ldots, a \circ g_{n,q_n-1}, a).$$

By a choice in the beginning of ϵ_n's sufficiently small, we get an approximate intertwining of the sequences

$$C[0,1] \otimes M_{k_1} \xrightarrow{\phi_1} C[0,1] \otimes M_{k_2} \xrightarrow{\phi_2} \cdots$$

and

$$C[0,1] \otimes M_{k_1} \xrightarrow{\psi_1'} C[0,1] \otimes M_{k_2} \xrightarrow{\psi_2'} \cdots$$

Hence by Proposition 2.3.44 we have that $B' \cong A$, and hence B' is also simple.

Now consider the unital AI-algebra B with inductive system given by

$$C[0,1] \otimes M_{k_1} \xrightarrow{\psi_1} C[0,1] \otimes M_{k_2} \xrightarrow{\psi_2} \cdots$$

where

$$\psi_n(a) = \mathrm{diag}(a \circ \delta_{x_{n,2}}, \ldots, a \circ \delta_{x_{n,q_n-1}}, a \circ g, a)$$

where g is the tent map defined in Example 2.4.10. It is easy to see that for every $t \in [0,1]$ we have

$$\{g_{n,1}(t), g_{n,2}(t), \ldots, g_{n,q_n-1}(t), t\} = \{\delta_{x_{n,2}}(t), \delta_{x_{n,3}}(t), \ldots, \delta_{x_{n,q_n-1}}(t), g(t), t\}.$$

Hence the set of eigenvalues defining ψ_n' and ψ_n agree pointwise and so by Theorem 3.1 in [75] we have that $\psi_n' \sim_{au} \psi_n$. Hence by Proposition 2.3.46 it follows that $B \cong B'$ and so B is a simple unital AI-algebra with unital and injective connecting maps satisfying $B \cong A$.

Let C and D be the diagonal Cartan subalgebras of A and B respectively. Let X_A and X_B be the corresponding set of infinite paths constructed in Subsection 5.2.1. A path in X_A will correspond to a sequence of eigenvalue functions that either contains finitely many delta functions or infinitely many. In the first case the inverse limit corresponds to the sequence of eigenvalue functions $(\iota, \iota, \iota, \ldots)$ where $\iota(t) = t$ (as this will be the tail of the sequence with finitely many delta functions), which gives an inverse limit homeomorphic to an arc, by Example 2.4.9. In the second case the inverse limit will be a singleton. Hence by Lemma 5.2.3 and Theorem 5.2.1 we obtain that the spectrum of C contains either arcs or singletons as connected components.

However, for D, if we take the path in X_B corresponding to the sequence of eigenvalue functions $(g, g, g, \ldots)$ then by Example 2.4.10 the corresponding inverse limit is a non-degenerate indecomposable continuum. Hence by Lemma 5.2.3 and Theorem 5.2.1 we obtain that the spectrum of D contains an indecomposable continuum as a connected component. Hence the spectrum of D is not homeomorphic to the spectrum of C and thus $C \ncong D$. $\square$

5.2.3 Non-Uniqueness in a General Class of Simple AI-algebras

In this subsection we show that uniqueness of AI-Cartan subalgebras fails in a very general class of simple AI-algebras.

Definition 5.2.6. Let $\mathcal{A}$ denote the class of simple and unital AI-algebras $A = \varinjlim_i (C[0,1] \otimes M_{n_i}, \phi_i)$ with unital and injective (standard) connecting maps, such that the connected components of the spectrum of the diagonal Cartan subalgebra of A do not exhaust all inverse limits of the unit interval up to homeomorphism.

Lemma 5.2.7. *If $A = \varinjlim_i (C[0,1] \otimes M_{n_i}, \phi_i) \in \mathcal{A}$ then the sequence $\{n_i\}_{i \in \mathbb{N}}$ contains a strictly increasing subsequence.*

Proof. Assume for a contradiction that the sequence stabilizes. We may assume then that $A = \varinjlim_n (C[0,1] \otimes M_N, \phi_n)$ for a fixed $N \in \mathbb{N}$. Let μ_n denote the injection $C[0,1] \otimes M_N \hookrightarrow A$. Since ϕ_n is in standard form, we can write $\phi_n(a) = a \circ g_n$ for some surjective eigenvalue function $g_n : [0,1] \twoheadrightarrow [0,1]$ (surjectivity follows by injectivity of the connecting map).

Let $I_1 = \{f \in C[0,1] \otimes M_N : f(0) = 0\}$, which is a proper ideal of $C[0,1] \otimes M_N$. There exists t_1 such that $g_1(t_1) = 0$. Let $I_2 = \{f \in C[0,1] \otimes M_N : f(t_1) = 0\}$. Repeat this process choosing t_2 such that $g_2(t_2) = t_1$ and $I_3 = \{f \in C[0,1] \otimes M_N : f(t_2) = 0\}$ and so on. All the I_n's are proper ideals and $\phi_n(I_n) \subseteq I_{n+1}$. Define

$$I = \overline{\bigcup_{n=1}^{\infty} \mu_n(I_n)}.$$

Then I is an ideal in A. It is non-zero because each μ_n is injective, and it is not all of A, because $\mu_1(h)$, where h is the constant matrix with each entry value $\frac{1}{2}$, is not close to any element in $\mu_n(I_n)$ for all $n \in \mathbb{N}$. Hence I is a proper ideal which is a contradiction to simplicity. $\square$

We are now in a position to prove Theorem F from the Introduction.

Theorem 5.2.8. *Uniqueness of AI-Cartan subalgebras fails for all AI-algebras in the class $\mathcal{A}$.*

Proof. Let $A = \varinjlim_i (C[0,1] \otimes M_{n_i}, \phi_i) \in \mathcal{A}$. By Lemma 5.2.7 and an induction argument on the strictly increasing subsequence, we may assume that $n_i > 2^i$, and that $k_i := \frac{n_{i+1}}{n_i} > 2^i$. We have that there is an inverse limit of unit intervals

$$T = \varprojlim([0,1], h_i)$$

which is not $\mathcal{C}$-homeomorphic to any $I_{\underline{x}}$ for all $\underline{x} \in X_A$, where $\bigsqcup\limits_{\underline{x}\in X_A} I_{\underline{x}}$ is the spectrum of the diagonal Cartan subalgebra of A, as in Theorem 5.2.1. Let the set of eigenvalue functions corresponding to the standard homomorphism ϕ_i be $\mathcal{F}_i = \{g_1^i, \ldots, g_{k_i}^i\}$. Replace any one of the functions by h_i. Replace two more functions by the functions $g(t) = \frac{t}{2}$ and $h(t) = \frac{t+1}{2}$, and keep whatever functions have not been replaced the same. Call this new set of functions $\mathcal{G}_i = \{w_1^i, \ldots, w_{k_i}^i\}$.

Let $B = \varinjlim\limits_i (C[0,1]\otimes M_{n_i}, \psi_i)$ where the ψ_i's are the standard homomorphisms with associated eigenvalue functions the elements of $\mathcal{G}_i$. The existence of the eigenvalue functions g and h ensure injectivity of the connecting maps (as the union of these functions' images is $[0,1]$), and Lemma 1.2 in [77] ensures (by using the functions g and h) that we get simplicity of B.

We have the following commutative diagram:

$$\begin{tikzcd}
K_0(C[0,1]\otimes M_{n_1}) \arrow[r, "K_0(\phi_1)"] \arrow[dd, "K_0(\chi_1)"'] & K_0(C[0,1]\otimes M_{n_2}) \arrow[r, "K_0(\phi_2)"] \arrow[dd, "K_0(\chi_2)"'] & \cdots \arrow[dr, dashed, bend left, "K_0(\mu_i)"'] & \\
& & & K_0(A) \arrow[dd, "\gamma_1"] \\
K_0(M_{n_1}) \arrow[r, "K_0(\alpha_1)"] & K_0(M_{n_2}) \arrow[r, "K_0(\alpha_2)"] & \cdots \arrow[ur, dashed, bend left, "K_0(\mu_i\circ\overline{\chi_i})"] \arrow[dr, dashed, bend left, "K_0(\tau_i)"'] & \\
& & & Q(m) \\
K_0(M_{n_1}) \arrow[r, "K_0(\beta_1)"] & K_0(M_{n_2}) \arrow[r, "K_0(\beta_2)"] & \cdots \arrow[ur, dashed, bend left, "K_0(\tau_i)"] \arrow[dr, dashed, bend left, "K_0(\rho_i\circ\overline{\chi_i})"'] & \\
& & & K_0(B) \arrow[uu, "\gamma_2"'] \\
K_0(C[0,1]\otimes M_{n_1}) \arrow[r, "K_0(\psi_1)"] \arrow[uu, "K_0(\chi_1)"] & K_0(C[0,1]\otimes M_{n_2}) \arrow[r, "K_0(\psi_2)"] \arrow[uu, "K_0(\chi_2)"] & \cdots \arrow[ur, dashed, bend left, "K_0(\rho_i)"] &
\end{tikzcd}$$

In the diagram, $\mu_i : C[0,1]\otimes M_{n_i} \to A$ and $\rho_i : C[0,1]\otimes M_{n_i} \to B$ are the canonical injections of the building blocks into the respective inductive limits, χ_i is the map given by evaluation at 0, $\overline{\chi_i} : M_{n_i} \to C[0,1]\otimes M_{n_i}$ is the map sending a to the continuous function with constant value a, and up to K_0, these are isomorphisms and inverses of each other.

Indeed, for $s \in [0,1]$ the *-homomorphism $\phi_s : C[0,1] \otimes M_{n_i} \to C[0,1] \otimes M_{n_i}$ given by $\phi_s(f)(t) = f(st)$ defines a homotopy between $\overline{\chi_i} \circ \chi_i$ and $\mathrm{id}_{C[0,1]\otimes M_{n_i}}$. It is clear that $\chi_i \circ \overline{\chi_i} = \mathrm{id}_{M_{n_i}}$. Hence by Proposition 2.2.34 it follows that up to K_0 these are inverses of each other.

The α_i's and β_i's are the induced maps that make the diagram commutative. $Q(m)$ is the subgroup of $\mathbb{Q}$ associated to the supernatural number m corresponding to the inductive limit of the matrix algebras M_{n_i}'s, as in Proposition 2.3.23, and the τ_{n_i}'s are the normalized matrix traces. The isomorphisms γ_1 and γ_2 are the ones induced in (2.25) in the proof of Proposition 2.3.23 (where they are what is $g \circ f^{-1}$ in (2.25)). Hence we get an isomorphism

$$\phi_0 := \gamma_2^{-1} \circ \gamma_1 : K_0(A) \to K_0(B).$$

The diagram shows that

$$\phi_0(K_0(\mu_i)([p]_0 - [q]_0)) = [\rho_i(p)]_0 - [\rho_i(q)]_0,$$

and by Proposition 2.3.17 this is enough to determine the isomorphism.

Now we determine an affine isomorphism $\phi_T : T_B \to T_A$, where T_A (T_B) is the simplex of tracial states on A (B). Note that the conditions of Lemma 4.1 in [77] are satisfied. Indeed, if P_n is the center-valued trace appearing in that lemma, then $P_i \circ \phi_i - P_i \circ \psi_i$ has a factor $\frac{1}{n_{i+1}} < 2^{-(i+1)}$ appearing (as P_n is normalized), and so the sum $\sum_{i=1}^{\infty} \|P_i \circ \phi_i - P_i \circ \psi_i\|$ is finite. Hence by Lemma 4.1 in [77] there is an affine isomorphism $\phi_T : T_B \to T_A$ satisfying, for all $i \in \mathbb{N}$, $\tau \in T_B$, and $a \in C[0,1] \otimes M_{n_i}$,

$$\phi_T(\tau)(\mu_i(a)) = \mathrm{norm} - \lim_{j\to\infty} \tau(\rho_j(\phi_{j,i}(a))).$$

Now note that for $g = K_0(\mu_i)([p]_0 - [q]_0) \in K_0(A)$ and $\tau \in T_B$ we have that

$$\langle \phi_0(g), \tau \rangle = \tau(\rho_i(p)) - \tau(\rho_i(q))$$

(where the definition of $\langle \cdot, \cdot \rangle$ is given in Definition 1.1.10 in [43]). On the other hand, we have that

$$\langle g, \phi_T(\tau) \rangle = \mathrm{norm} - \lim_{j\to\infty} \tau(\rho_j(\phi_{j,i}(p))) - \mathrm{norm} - \lim_{j\to\infty} \tau(\rho_j(\phi_{j,i}(q))). \quad (5.12)$$

Now note that we may choose the p and q in the definition of g to be diagonals with entries either constant map 0 or constant map 1, as the trace of the projection determines the element in $K_0(A_i)^+$. Since $\phi_{j,i}$ and $\psi_{j,i}$ are standard maps, they would agree on such projections and so we may replace $\phi_{j,i}$ in (5.12) by $\psi_{j,i}$. Hence (5.12) simplifies to $\tau(\rho_i)(p) - \tau(\rho_i)(q)$ and so we have

$$\langle \phi_0(g), \tau \rangle = \langle g, \phi_T(\tau) \rangle.$$

By Theorem 2 in [18], we have that

$$A \cong B.$$

By Lemma 5.2.3 and Theorem 5.2.1, we have that the diagonal Cartan subalgebra of B has in its spectrum a connected component homeomorphic to T, whereas T is not homeomorphic to any connected component of the spectrum of the diagonal Cartan subalgebra of A. Hence these Cartan subalgebras have non-homeomorphic spectra and thus are not isomorphic. Therefore uniqueness fails. □

Remark 5.2.9. There are endless examples of AI-algebras belonging to $\mathcal{A}$ of Definition 5.2.6. In fact it is not immediately clear whether any simple and unital AI-algebra does not belong to the class $\mathcal{A}$. Such an algebra would have a diagonal Cartan subalgebra whose spectrum contains connected components covering every possible inverse limit of the unit interval, up to homeomorphism.

Chapter 6

Outlook

The research we have conducted and the statements we have proved open up further questions which arise naturally. The first question is with regards to existence of AX-Cartan subalgebras, but for more general topological spaces X. Indeed the motivation comes from the remarkable reduction theorem, due to Gong, Jiang, Li and Pasnicu in [28], regarding AH-algebras with the ideal property and bounded dimension growth.

Definition 6.0.1. An *AH-algebra* is an inductive limit C^*-algebra $A = \varinjlim(A_n, \phi_n)$ where each building block is of the form

$$A_n = \bigoplus_{j=1}^{N(n)} P_{n,j}(C(X_{n,j}) \otimes M_{n_j})P_{n,j}$$

where the $X_{n,j}$'s are compact metric spaces and the $P_{n,j}$'s are projections in $C(X_{n,j}) \otimes M_{n_j}$. A is said to have the *ideal property* if every closed two-sided ideal of A is generated as a closed two-sided ideal by its projections. A is said to have *bounded dimension* if the supremum of the dimension across all topological spaces $X_{n,j}$ appearing in all summands is finite.

The authors of [28] prove:

Theorem 6.0.2. *Let A be an AH-algebra with the ideal property and bounded dimension. Then A is isomorphic to the AH-algebra $B = \varinjlim(B_m, \psi_m)$ where each building block is of the form*

$$B_m = \bigoplus_{i=1}^{N(m)} Q_{m,i}(C(Y_{m,i}) \otimes M_{m_i})Q_{m,i}$$

where each $Y_{m,i}$ is one of: $\{pt\}$, $[0,1]$, $\mathbb{T}$, S^2, $T_{II,k}$ or $T_{III,k}$ (where $T_{II,k}$ is a finite connected simplicial complex with $H^1(T_{II,k}) = 0, H^2(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}$, and where $T_{III,k}$ is a finite connected simplicial complex with $H^1(T_{III,k}) = 0, H^2(T_{III,k}) = 0, H^3(T_{III,k}) = \mathbb{Z}/k\mathbb{Z}$).

A natural first question that follows from this reduction theorem is the following:

*Question. What can be said about *-homomorphisms $C(X) \otimes M_n \to C(X) \otimes M_m$ where X is one of the spaces $T_{II,k}$, $T_{III,k}$ or S^2 (as we have already covered the cases when X is $\{pt\}$, $[0,1]$ and $\mathbb{T}$). Specifically, are maximally homogeneous maps dense in such connecting maps? Can we obtain a similar result as that in Lemma 4.4.9? If this is the case, can this also be generalized to direct sums and if so can we get existence of AX-Cartan subalgebras when X is one of $T_{II,k}$, $T_{III,k}$ or S^2? If so, can this be extended to corners of the building blocks, and thus by Theorem 6.0.2 can we get existence of AH-Cartan subalgebras in AH-algebras?*

There are a lot of hindrances in our current methods when it comes to the above question. For example, when one wants to pass from one summand to direct sums one relied on Lemma 4.4.4 which needed that $C(X)$ had stable rank one. This will not hold for the higher dimensional simplicial complexes. Another hindrance is how to approximate an injective connecting map by an injective maximally homogeneous one. We made use of the fact that X was imbedded in $\mathbb{C}$ which will not be the case for the higher dimensional simplicial complexes. From this it seems that novel methods must be created in order to answer the question above.

There are many questions that relate directly to the uniqueness results we obtained in Chapter 5. One particular question is:

Question. Does the class $\mathcal{A}$ that appears in Theorem 5.2.8 contain all simple and unital AI-algebras with unital and injective connecting maps? In other words, does there exist such an AI-algebra whose diagonal Cartan subalgebra has spectrum with connected components exhausting all inverse limits of the unit interval, up to homeomorphism?

An affirmative answer to the first question (equivalently a negative answer to the second question) would lead to the result that in any unital simple AI-algebra one may find a pair of non-isomorphic AI-Cartan subalgebras. Note that when studying uniqueness we focused on the diagonal Cartan subalgebra obtained by choosing the diagonal subalgebra as our Cartan subalgebra in every AI-building block. Of course, a natural question arising out of this is the following:

Question. Is there a good description of the spectrum for an arbitrary AI-Cartan *subalgebra? Using Barlak and Li's work in [8], and which was described in Section 4.1, is there a good description of the étale twisted groupoids corresponding to AI-Cartan subalgebras (with injective and unital connecting maps) in general? What about $A\mathbb{T}$-Cartan subalgebras?*

Our assessment is that it should be rather straightforward to analyse the twisted groupoids that correspond to arbitrary Cartan subalgebras of AI-building blocks. This is because we can make use of a result similar to Lemma 4.3.2 but for AI-building blocks (which will mean there will be no permutations needed to glue the endpoints as was necessary for the circle case). These can then be used to analyse the connecting maps at the groupoid level. Thus by following the constructions presented in Section 4.1 one would expect to be able to obtain groupoid models for the AI-Cartan subalgebras. Of course, the circle case is expected to be trickier as the matrix units of Lemma 4.3.2 have to be glued in the right way at the endpoints which will make the groupoid analysis much more technical.

Another question that arises from our research, and which was brought to our attention by Xin Li, relates to Proposition 4.1.5:

Question. Does the condition on the connecting maps mapping normalizer set into normalizer set, and being compatible with the conditional expectations, as in Proposition 4.1.5, follow directly from the condition that the connecting map should map a Cartan subalgebra into a Cartan subalgebra, for the class of AI and $A\mathbb{T}$-algebras? In which classes of inductive limit C^*-algebras does this hold?

In fact it is not too difficult to check that if the connecting maps for AF-algebras map a Cartan subalgebra into a Cartan subalgebra, then the other conditions about the normalizer sets and conditional expectations follow automatically. If the answer to the above question is affirmative then our proofs in Chapter 4, which show that the relevant connecting maps take normalizer set into normalizer set and are compatible with the conditional expectations, would not be necessary and so greatly simplify the work. We assess that the answer to the question is most likely affirmative, as the analysis can be performed pointwise, and for AF-building blocks this is true.

Departing slightly from the scope of our research, an interesting question, which was brought to our attention by Hannes Thiel, relates to finding inductive limit Cartan subalgebras in inductive limits of Robert's building blocks. Indeed, in [66], Robert classifies C^*-algebras that are stably isomorphic to inductive limits of one-dimensional noncommutative CW complexes that have vanishing K_1 group. The invariant is the Cuntz semigroup which generalizes K-theory. A relevant question is the following:

Question. Can one, using Cartan subalgebras of Robert's building blocks, find an inductive limit Cartan subalgebra of the inductive limit of Robert's building blocks?

An affirmative answer to this would generalize the AI results, as these are examples of this setup, but would also give new examples not covered in our methods, such as the Jiang-Su algebra. We aim to work on the questions above in the near future.

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