C^*-Algebras of Graphs of Semigroups

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Abstract

In the thesis, we investigate the properties of the reduced C^*-algebras of graphs of monoids. These include nuclearity, ideal structure, K-theory and so on.

Based on Serre's definitions of graphs of groups and their fundamental groups, we define graphs of monoids and study the right LCM property. We also investigate the nuclearity of C^*-algebras of graphs of monoids and give some examples to embed some special graphs of monoids (generalised Baumslag-Solitar monoids) into amenable groups.

Using Xin Li's work to view reduced semigroup C^*-algebras as reduced groupoid C^*-algebras, we study the topological approximate invariant means, the closed subgroupoids and the principality of the associated groupoids. The results in this part help us work out the primitive ideal spaces of these groupoid C^*-algebras. Lastly, we compute K-theory of all the groupoid C^*-algebras induced by the associated groupoids and their closed subgroupoids.

Contents

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Declaration

With the exception of Chapter 2, which contain introductory material, all work in this thesis was carried out by the author unless otherwise explicitly stated.

Chapter 1

Introduction

In mathematical research, it makes sense to investigate the interactions among different areas of mathematics. C^*-algebras have interactions with other areas of mathematics such as geometry, dynamical system, group theory and semigroup theory, and so on. These connections are usually produced by constructions of some specific C^*-algebras.

In the thesis, we focus on (reduced) semigroup C^*-algebras. Motivated by the definition of group C^*-algebras, a semigroup C^*-algebra is defined to be the C^*-algebra generated by the left regular representation of a left cancellative semigroup. Despite the analogous definitions, we can see, in semigroup C^*-algebras, phenomena completely different from those in the group case. Therefore, it will be natural and interesting to study semigroup C^*-algebras separately.

The properties of semigroup C^*-algebras depend heavily on the corresponding semigroups. In the thesis, we only consider semigroup C^*-algebras associated to graphs of monoids. Serre defined graphs of groups and the fundamental group of a graph of groups in his book. (see [p42, Ser80]) In his definition, a graph of groups $(G,\ \Gamma)$ consists of a graph $\Gamma = (V,\ E)$, a group G_v for every vertex $v \in \Gamma$ and a group G_e for every edge $e \in \Gamma$, together with group embeddings $G_e \to G_{t(e)}$ (denoted by $x \mapsto x^e$) and the convention $G_e = G_{\bar{e}}$ for all edges $e \in \Gamma$. The fundamental group $\pi_1(G,\, \Gamma,\, T)$ is given by the groups $G_v,\ v \in V$ and A, subject to the relations $x^e = x^{\bar{e}}$ for all $e \in T$ and all $x \in G_e$, and $ay^a a^{-1} = y^{\bar{a}}$ for all $a \in A$ and all $y \in G_a$, where T is a maximal subtree of the graph Γ and A is an orientation of $\Gamma \setminus T$ such that $\Gamma = T \cup A \cup \bar{A}$.

Here we make the convention that all the graphs are countable and all the groups are discrete and countable unless otherwise explicitly stated.

Based on Serre's work, we defined similarly graphs of monoids and the fundamental monoid P of a graph of monoids. Let $(G,\, \Gamma)$ still be a graph of groups with $\Gamma = (V,\, E)$ connected, but assume that $G_v,\ v \in V$ is totally ordered with positive cone P_v, i.e., $G_v = P_v \cup P_v^{-1}$ and $P_v \cap P_v^{-1} = \{\varepsilon\}$. Here and in the sequel, we always use ε to represent the identity element in groups. For $e \in E$, define $P_e := \{g \in G_e,\ g^e \in P_{t(e)}\}$. Assume further $P_e = P_{\bar{e}}$ for all $e \in T$ and either $P_e = P_{\bar{e}}$ or $P_e = P_{\bar{e}}^{-1}$ for all $e \in A$. Define $A_+ := \{e \in A,\ P_e = P_{\bar{e}}\}$ and $A_- := \{e \in A,\ P_e = P_{\bar{e}}^{-1}\}$. The fundamental monoid P is defined to be the subsemigroup of $\pi_1(G,\, \Gamma,\, T)$ generated by P_v and A. For more details, please refer to Chapter 3. The fundamental monoid P, together with its semigroup C^*-algebra $C^*_\lambda(P)$, is exactly what we investigate in the thesis.

As we see, a graph of monoids (groups) is a system of monoids (groups) associated to a graph. Without ambiguity, by saying a monoid (group) is a graph of monoids (groups), we mean it is the fundamental monoid (group) of some related system (graph of monoids or groups).

We say that the monoid P is right LCM if for all $p,\ q \in P$, either $pP \cap qP = \emptyset$ or $pP \cap qP = rP$ for some $r \in P$. Throughout the thesis, we need the monoid P to be right LCM because it guarantees that all constructible right ideals of P are principal and thus that P satisfies independence. Naturally, we give a criterion for the monoid P to be right LCM in Chapter 3. Below is the result. (see Definition 3.2.1 and Proposition 3.2.3)

Theorem 1.0.1. *P is right LCM if for all $e \in E$, $p \in P_{o(e)}$, either $p^{-1}P_{\bar{e}}^{\bar{e}} = \emptyset$ or $p^{-1}P_{\bar{e}}^{\bar{e}} = qP_{\bar{e}}^{\bar{e}}$ for some $q \in P_{o(e)}$, where $p^{-1}P_{\bar{e}}^{\bar{e}} := \{x \in P,\ px \in P_{\bar{e}}^{\bar{e}}\}$.*

Nuclearity, as a kind of finite approximation property of a C^*-algebra, can rarely be ignored when referring to the properties of C^*-algebras. In 2012, Spielberg proved in [Spi12] that the semigroup C^*-algebras of the Baumslag-Solitar monoids are Cuntz-Krieger and hence amenable. Noting that all Baumslag-Solitar monoids are fundamental monoids of some specific graphs of monoids, the following result can be viewed as an extension of Spielberg's work. (see Theorem 4.1.1 in Chapter 4)

Theorem 1.0.2. *Assume that P is right LCM, then $C^*_\lambda(P)$ is nuclear if $C^*_\lambda(P_T)$ is nuclear, where P_T is the submonoid of P generated by the semigroups $P_v,\ v \in V$.*

It is well known that a reduced group C^*-algebra is nuclear if and only if the group is amenable, while we do not have an analogue in the semigroup case. Indeed, based on Exel's work, Xin Li proved in [Theorem 5.6.44 and Corollary 5.6.45, CELY17] that $C^*_\lambda(P)$ is nuclear if P embeds into an amenable group. But whether the converse is true still remains open.

Let P be the generalised Baumslag-Solitar monoid, then we have

$$P = GBS_+(N,\ m_i,\ n_i) = < a_i,\ b \mid a_i b^{m_i} = b^{n_i} a_i,\ \forall i \in S_1,\ b^{|n_i|} a_i b^{m_i} = a_i,\ \forall i \in S_2,\ N = \sharp A >_+,$$

where $S_1 := \{i \in S,\ a_i \in A_+\} = \{i \in S,\, n_i > 0\}$ and $S_2 := \{i \in S,\ a_i \in A_-\} = \{i \in S,\, n_i < 0\}$.

To begin with, P is right LCM by Theorem 1.0.1. On the other hand, we have $C^*_\lambda(P_T) \cong C^*_\lambda(\mathbb{N}) \cong C^*(S)$, where S is a shift with codimension 1 in a separable Hilbert space and $C^*(S)$ is the universal C^*-algebra generated by S, i.e., the Toeplitz algebra. Therefore, $C^*_\lambda(P_T)$ is nuclear and thus $C^*_\lambda(P)$ is nuclear. What we include in Chapter 4 except the nuclearity part is to embed the generalised Baumslag-Solitar monoids into amenable groups. Luckily enough, we obtained some results despite the fact that the generalised Baumslag-Solitar groups are not amenable in general. Below is the conclusion. (see Theorem 4.2.11 and Corollary 4.2.13)

Theorem 1.0.3. *Assume*

$$gcd\Big(\prod_{i=1}^{N} m_i, \prod_{i=1}^{N} n_i\Big) = 1,\ m_i,\ n_i \in \mathbb{Z}^*,\ N \in \mathbb{N}^*. \tag{1.1}$$

Let $F_N := < s_1,\ \cdots,\ s_N >$ *be the free group generated by* N *generators* $s_1,\ \cdots,\ s_N$ *and let* ϕ *be a semigroup homomorphism defined by*

$$\phi:\ F_N \ \to\ Aut(\mathbb{Q}),\ s_i \ \mapsto\ \phi(s_i)\ \big[r \mapsto\ \frac{m_i r}{n_i},\ r \in \mathbb{Q}\big].$$

Then there exists an injective semigroup homomorphism

$$\varphi:\ GBS_+(N,\ m_i,\ n_i) \to (F_N/F_N'') \ltimes \mathbb{Q}$$

such that $\varphi(a_i) = (s_i,\ 0)$ *and that* $\varphi(b) = (\varepsilon,\ 1)$. *Here* F_N'' *is the second derived group of* F_N.

In 1969, Hochster constructed in [Hoc69] an embedding of $\mathbb{N} * \mathbb{N}$ into the amenable group F_2/F_2'', where $\mathbb{N} * \mathbb{N}$ is the free monoid generated by 2 generators, F_2 is the free group generated by 2 generators and F_2'' is the second derived group of F_2. The proof of our theorem above is motivated by Hochster's work.

As you can see, we embed the generalised Baumslag-Solitar monoids into amenable groups if equation (1.1) holds. What if equation (1.1) does not hold? Unfortunately, we failed giving an answer in this case.

Every submonoid P of a group G induces a partial action of G on some character space Ω. $G \curvearrowright \Omega$ induces a groupoid $G \ltimes \Omega$ and its reduced groupoid C^*-algebra $C_r^*(G \ltimes \Omega)$. Given the fact that $C_\lambda^*(P)$ is isomorphic to $C_r^*(G \ltimes \Omega)$ (see Theorem 2.2.4 or [Theorem 5.5.21 and Theorem 5.6.41, CELY17]), we will study the semigroup C^*-algebra $C_\lambda^*(P)$ by investigating the properties of the groupoid C^*-algebra $C_r^*(G \ltimes \Omega)$.

By [Theorem 20.7 and Theorem 25.10, Exe15], the groupoid $G \ltimes \Omega$ is amenable if the group G is amenable. In this case, by the definition of amenability of groupoids, there exists a topological approximate invariant mean on $G \ltimes \Omega$. It is natural to ask whether we can work out such a topological approximate invariant mean on $G \ltimes \Omega$. In Chapter 5, we give a construction of a Borel approximate invariant mean on $G \ltimes X$ for a general transformation groupoid $G \ltimes X$ with the group G amenable and provide a sufficient condition for the mean to be topological. The construction is based on Renault's and Williams's joint work in [RW17].

In the rest of the thesis (Chapter 5, Chapter 6 and Chapter 7), we always treat the cases separately according to whether P is the fundamental monoid of a general graph of monoids (general case) or P is the generalised Baumslag-Solitar monoid (generalised Baumslag-Solitar case). We have two reasons to do so. On one side, we have different assumptions on the monoid P. In the general case, we have more assumptions in the construction of P to get some results. On the other side, the generalised Baumslag-Solitar case is actually an extreme case of the general case and we can witness different phenomena.

In Chapter 5, we give a list of all nonempty closed invariant subspaces of the partial action $G \curvearrowright \Omega$. In the generalised Baumslag-Solitar case, we have the following result. (see Corollary 5.2.24 and Corollary 5.2.29)

Theorem 1.0.4. *(Generalised Baumslag-Solitar case) Let P be the generalised Baumslag-Solitar monoid, then the following is the list of all nonempty closed invariant subsets of* Ω*:*

(i) $\partial\Omega \subsetneq \Omega_{b,\,\infty} \subsetneq \Omega_\infty \subsetneq \Omega$ *and* $\partial\Omega \subsetneq \Omega_{a,\,\infty} \subsetneq \Omega_\infty$ *if* $0 < |S_1| < \infty$ *and* $|S_2| = 0$*.*

(ii) $\partial\Omega = \Omega_{a,\,\infty} \subsetneq \Omega_{b,\,\infty} = \Omega_\infty \subsetneq \Omega$ *if* $|S_1| = 0$ *and* $0 < |S_2| < \infty$*.*

(iii) $\partial\Omega \subsetneq \Omega_{b,\,\infty} \subsetneq \Omega_\infty \subsetneq \Omega$ *if* $0 < |S_1| < \infty$ *and* $0 < |S_2| < \infty$*.*

(iv) $\partial\Omega = \Omega_{b,\,\infty} = \Omega_\infty \subsetneq \Omega$ *if* $|S_1| = 0$ *and* $|S_2| = \infty$*.*

(v) $\partial\Omega = \Omega_{b,\,\infty} \subsetneq \Omega_\infty \subsetneq \Omega$ *if* $0 < |S_1| < \infty$ *and* $|S_2| = \infty$*.*

(vi) $\partial\Omega = \Omega_{b,\,\infty} \subsetneq \Omega$ *if* $|S_1| = \infty$*.*

For every finite or infinite positive word $w = x_1x_2x_3\cdots \neq$ with $x_* \in \{P_v\}_{v\in V} \cup A$ and $x_* \neq \varepsilon$ unless $w = \varepsilon$, set $[w]_i := w$ if $w = x_1\cdots x_j$ with $j < i$ and $[w]_i := x_1\cdots x_i$ otherwise. Define $\chi_w \in \Omega$ by setting $\chi_w(xP) = 1$ if and only if $[w]_i \in xP$ for some i. By the work in [LOS18], we know that every character in Ω is of the form χ_w for some finite or infinite positive word. In the theorem above, Ω_∞ denotes all the characters in Ω of the form χ_w for some infinite word w, and we have $\Omega_\infty = \Omega \setminus P$. $\Omega_{a,\,\infty}$ is a subset of Ω_∞ consisting of all the characters of the form χ_w with w an infinite word containing infinitely many letters from A. And $\Omega_{b,\,\infty}$ is defined to be the closure of $\Omega_\infty \setminus \Omega_{a,\,\infty}$.

In general case, we focus on the following two situation.

I. For all $v \in V$, $x \in P_v \setminus \varepsilon$ or $x \in A$ and $\chi \in \Omega_\infty$, there exists an infinite word w with $\chi = \chi_w$, a strictly increasing sequence $(j_N)_N$ of positive integers, and a finite positive word y whose first letter does not lie in P_v in the case where $x \in P_v$ such that,

(i) $xy[w]_{j_N}$ is a reduced positive word for all N,

(ii) Whenever $p_0 d_1 p_1 \cdots$ is a properly reduced positive word representing $xy[w]_{j_N}$, we must have $x \in p_0 P_T$ if $x \in P_v$ and $x \in p_0 P$ if $x \in A$.

II. There exists $\mathbf{u} \in V$ and $\mathbf{b} \in P_{\mathbf{u}}$ such that the following holds:

For all $v \in V$, $x \in P_v \setminus \varepsilon$ or $x \in A$ and $\chi \in \Omega_\infty$, there exists an infinite word w with $\chi = \chi_w$, a strictly increasing sequence $(j_N)_N$ of positive integers, and a finite positive word y whose first letter does not lie in P_v in the case where $x \in P_v$ such that,

(i) $xy[w]_{j_N}$ is a reduced positive word for all N,

(ii) Whenever $p_0 d_1 p_1 \cdots$ is a properly reduced positive word representing $xy[w]_{j_N}$, then one of the following holds:

A) $x \in p_0 P_T$ if $x \in P_v$ and $x \in p_0 P$ if $x \in A$,

B) $[w]_{j_N} \in \mathbf{b}P$ and $x\mathbf{b}^i \in p_0 P_T$ if $x \in P_v$ and $x\mathbf{b}^i \in p_0 P$ if $x \in A$, where i is some positive integer.

Below is the conclusion. (see Theorem 5.2.11)

Theorem 1.0.5. *(General case) Let P be the fundamental monoid of a graph of monoids with condition (LCM) for P satisfied. Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R}, +)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$.*

(i) If condition I. holds and there exists $v \in V$ such that G_v is dense in $\mathbb{R}$, then the following is the list of all nonempty closed invariant subsets of Ω: $\partial\Omega = \Omega$.

(ii) If condition I. holds and $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$, then the following is the list of all nonempty closed invariant subsets of Ω: $\partial\Omega = \overline{\Omega_\infty} \subseteq \Omega$.

(iii) If condition II. holds, there exists $v \in V$ such that G_v is dense in $\mathbb{R}$ and $\sharp A \geq 1$, then the

following is the list of all nonempty closed invariant subsets of Ω*:* $\Omega_{\boldsymbol{b},\,\infty} = \partial\Omega \subsetneq \Omega$.

(iv) If condition II. holds and $\sharp A = 0$*, then the following is the list of all nonempty closed invariant subsets of* Ω*:* $\{\infty\} = \partial\Omega \subsetneq \overline{\Omega_\infty} \subseteq \Omega$.

(v) If condition II. holds, $P_v \cong \mathbb{Z}_{\geq 0}$ *for all* $v \in V$*,* $\sharp A \geq 1$ *and* $\sharp V > 1$*, then the following is the list of all nonempty closed invariant subsets of* Ω*:* $\Omega_{\boldsymbol{b},\,\infty} = \partial\Omega \subsetneq \overline{\Omega_\infty} \subseteq \Omega$.

Here Ω_∞ *is as in Theorem 1.0.4,* $\{\infty\}$ *is exactly* $\partial\Omega_{P_T}$ *and* $\Omega_{\boldsymbol{b},\,\infty}$ *is defined to be*

$$\Omega_{\boldsymbol{b},\,\infty} := \{\chi \in \Omega,\ (g\cdot\chi)(\boldsymbol{b}^i P) = 1,\ \forall g \in G,\ \forall i \in \mathbb{N}\},$$

where we only consider those $g \in G$ *such that* $g\cdot\chi$ *is well defined.*

In the theorem above, the assumption $G_v \subseteq (\mathbb{R},\,+)$ for all $v \in V$, together with other assumptions, is made such that in most cases, either condition I. or condition II. holds. As you may see, these assumptions are also made in Theorem 1.0.6, Theorem 1.0.7 and Theorem 1.0.9.

In Chapter 5, we also give a complete discussion on the topological freeness of the partial action $G \curvearrowright X$ for all nonempty closed invariant subsets $X \subseteq \Omega$ except the case $X = \partial\Omega$. The cases are complicated and here we will only take, for example, the partial action $G \curvearrowright \Omega_\infty$ in the general case. For more details, please refer to Chapter 5. The following theorem comes from Proposition 5.3.11 and Proposition 5.3.13.

Theorem 1.0.6. *(General case) Let P be the fundamental monoid of a graph of monoids with condition (LCM) for P satisfied. Assume that* $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R},\,+)$ *for all* $v \in V$*,* $P_e \neq \{\varepsilon\}$ *for all* $e \in A$ *and* $P_e^e \neq P_{t(e)}$ *for all* $e \in E$*.*

(i) If condition I. holds, then the partial action $G \curvearrowright \Omega_\infty$ *is topologically free whenever* Ω_∞ *is closed in* Ω*.*

(ii) If condition II. holds, then Ω_∞ is closed in Ω if and only if $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$, $1 < \sharp V < \infty$ and $\sharp A_+ < \infty$. In this case, we have the following:

(a) If $\sharp A_+ > 0$, $G \curvearrowright \Omega_\infty$ is topologically free.

(b) If $\sharp V > 2$, $G \curvearrowright \Omega_\infty$ is topologically free.

(c) If $\sharp A_+ = 0$ and $\sharp V = 2$, take $e \in T$, and assume the two embeddings are $P_e \to P_{o(e)}$, $1 \mapsto k$ and $P_e \to P_{t(e)}$, $1 \mapsto l$, $G \curvearrowright \Omega_\infty$ is topologically free if and only if either $k > 2$ or $l > 2$.

In Chapter 6, we study the ideals in the groupoid C^*-algebra $C_r^*(G \ltimes \Omega)$. Since every ideal in a C^*-algebra is the intersection of all the primitive ideals (the kernels of non-zero irreducible representations of the C^*-algebra) containing it, we end up with the list of all primitive ideals in $C_r^*(G \ltimes \Omega)$. This part of work is based on Christian Bönicke's and Kang Li's work in [BL18], where it states that there is a one-to-one correspondence between open invariant subsets in Ω and ideals in $C_r^*(G \ltimes \Omega)$ if the groupoid $G \ltimes \Omega$ is étale, inner exact and essentially principal. (see Lemma 6.0.1)

It is easy to check that $G \ltimes \Omega$ is étale. The inner exactness of the groupoid $G \ltimes \Omega$ is exactly the C^*-exactness of the group G by definition in [Gue01]. Also by Erik Guentner, a group acting without inversion on a tree is C^*-exact if and only if the vertex stabilizers of the action are C^*-exact. By [p50-p53, Ser80], the fundamental group $\pi_1(G, \Gamma, T)$ acts without inversion on a tree $\tilde{X} = \tilde{X}(G, \Gamma, T)$ such that every vertex stabilizer is isomorphic to G_v for some $v \in V$. Therefore, our group G is C^*-exact if and only if G_v is C^*-exact for all $v \in V$. Noting $G_v \subseteq (\mathbb{R}, +)$ in our assumption, the latter follows since amenable groups are C^*-exact by [Lan73]. And by definition the essentially principal property of the groupoid $G \ltimes \Omega$ is exactly the topological freeness of the partial action of G on all nonempty closed invariant subsets of Ω. Equivalently, the groupoid $G \ltimes \Omega$ is essentially principal if and only if the partial action $G \curvearrowright X$ is topologically free for all nonempty closed invariant subsets $X \subseteq \Omega$.

We work out the list of all nonempty closed invariant subsets of Ω and analyse the topological freeness of the partial action of G on these nonempty closed invariant subsets in Chapter 5. In the case where the partial action $G \curvearrowright X$ is topologically free for all nonempty closed invariant subsets $X \subseteq \Omega$, we can easily obtain that every ideal in $C_r^*(G \ltimes \Omega)$ is of the form $C_r^*(G \ltimes X')$ with $X' \subseteq \Omega$ open and invariant and then analyse whether they are primitive or not. In other cases, our work is based on Dixmier's work in [Dix77]. (see Lemma 6.0.2)

The discussion of the primitive ideal space of the groupoid C^*-algebra $C_r^*(G \ltimes \Omega)$ in Chapter 6 is complicated, and here we will only give an example where P is a general graph of monoids and Ω_∞ is closed in Ω. For more details, please refer to Chapter 6.

Theorem 1.0.7. *(General case) Let P be the fundamental monoid of a graph of monoids with condition (LCM) for P satisfied. Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R}, +)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$.*
Assume further $\sharp A \neq 0$, Ω_∞ is closed in Ω and the partial action $G \curvearrowright \partial\Omega$ is topologically free if condition II. holds.
(i) If condition I. holds, there is a one-to-one correspondence between open invariant subsets of Ω and ideals in $C_r^*(G \ltimes \Omega)$. Therefore,

$$Prim(C_r^*(G \ltimes \Omega)) = \{0,\, C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) \cong \mathcal{K}\}.$$

Here and in the sequel, $\mathcal{K}$ stands for the C^*-algebra consisting of compact operators on a separable Hilbert space.
(ii) If condition II. holds, there are three nonempty closed invariant subsets Ω, Ω_∞, $\partial\Omega = \Omega_{\boldsymbol{b},\,\infty}$.
If the action $G \curvearrowright \Omega_\infty$ is topologically free, then there is a one-to-one correspondence between open invariant subsets of Ω and ideals in $C_r^*(G \ltimes \Omega)$. In this case,

$$Prim(C_r^*(G \ltimes \Omega)) = \{0,\, C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) \cong \mathscr{K},\, C_r^*(G \ltimes (\Omega \setminus \Omega_{\boldsymbol{b},\,\infty}))\}.$$

If the action $G \curvearrowright \Omega_\infty$ is not topologically free, then we must have $\sharp V = 2,\ k = l = 2$ and $\sharp A_+ = 0$. (see Theorem 1.0.6) Set $J := C_r^*(G \ltimes (\Omega_\infty \setminus \Omega_{\boldsymbol{b},\,\infty}))$, then we have $J \cong \mathscr{K} \otimes C(\mathbb{T})$. In this case,

$$Prim(C_r^*(G \ltimes \Omega)) = \{0,\, C_r^*(G \ltimes (\Omega \setminus \Omega_{\boldsymbol{b},\,\infty})),\, C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) + J_p + C_r^*(G \ltimes \Omega_{\boldsymbol{b},\,\infty})\},$$

where $J_p := \varphi^{-1}(\mathscr{K} \otimes C_0(\mathbb{T} \setminus \{p\})),\ p \in \mathbb{T}$ and $\varphi:\ J \to \mathscr{K} \otimes C(\mathbb{T})$ is a $*$-isomorphism. Here is a list of all nontrivial closed subsets of $Prim(C_r^*(G \ltimes \Omega))$:

$$\{I\},\ \{C\},\ \{I,\, C\},$$

where $I := C_r^*(G \ltimes (\Omega \setminus \Omega_{\boldsymbol{b},\,\infty}))$ and $C = \{C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) + J_p + C_r^*(G \ltimes \Omega_{\boldsymbol{b},\,\infty})\}_{p \in C'}$ for some closed subset $C' \subseteq \mathbb{T}$.

K-theory has played an important role in C^*-algebra theory since it was introduced as an tool in the early 1970s. One of its most important applications in C^*-algebra theory is that it helps in the classification of C^*-algebras. In Chapter 7, we try to compute the K-theory of all the C^*-algebras of the form $C_r^*(G \ltimes X)$ with $X \subseteq \Omega$ invariant and closed. The work is partially based on Xin Li's work in [Li20]. Below are the conclusions.

Theorem 1.0.8. *(Generalised Baumslag-Solitar case) Let P be the generalised Baumslag-Solitar monoid.*

(i) For Ω, we have

$$K_0(C_r^*(G \ltimes \Omega)) \cong \mathbb{Z} \text{ and } K_1(C_r^*(G \ltimes \Omega)) \cong 0.$$

(ii) $\Omega_{b,\,\infty}$ *is always closed in* Ω*, and we have*

$$K_0\big(C_r^*(G \ltimes \Omega_{b,\,\infty})\big) \cong \mathbb{Z} \text{ and } K_1\big(C_r^*(G \ltimes \Omega_{b,\,\infty})\big) \cong \mathbb{Z}.$$

(iii) Ω_∞ *is closed in* Ω *if and only if* $|S_1| < \infty$*. In this case, we have*

$$K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z} \text{ and } K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z}.$$

(iv) $\Omega_{a,\,\infty}$ *is closed in* Ω *if and only if either* $0 < |S_1| < \infty$ *and* $|S_2| = 0$ *or* $|S_1| = 0$ *and* $0 < |S_2| < \infty$*. In this case,*

$$K_0\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \cong \mathbb{Z}_{(\sum_{1\le i\le N}|n_i|)-1}$$

and

$$K_1\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \cong \mathbb{Z}_{1+\sum_{i\in S_2} m_i}$$

if $\sum_{1\le i\le N}|n_i| > 1$*. Here and in the sequel,* $\mathbb{Z}_n$*,* $n \in \mathbb{N}^*$ *is the quotient group of* $\mathbb{Z}$ *by the normal subgroup* $n\mathbb{Z}$*.*

$$K_0\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \cong \mathbb{Z}$$

and

$$K_1\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \cong \mathbb{Z} \oplus \mathbb{Z}_{1+\sum_{i\in S_2} m_i}$$

if $\sum_{1\le i\le N}|n_i| = 1$*.*

(iv) $\partial\Omega$ *is always closed, but* $\partial\Omega \neq \Omega_{b,\,\infty}$ *only if* $0 < |S_1| < \infty$ *and* $0 < |S_2| < \infty$*. In this case, we have the following results.*

If $1 - \sum_{1\le i\le N}|n_i| \neq 0$ *and* $1 - \sum_{1\le i\le N} sgn(n_i) m_i \neq 0$*,*

$$K_0\big(C_r^*(G \ltimes \partial\Omega)\big) \cong \mathbb{Z}_{\sum_{1\le i\le N}|n_i|-1}$$

and

$$K_1\big(C_r^*(G \ltimes \partial\Omega)\big) \cong \mathbb{Z}_{\left|1-\sum_{1\leq i\leq N} sgn(n_i)m_i\right|}.$$

If $1-\sum_{1\leq i\leq N}|n_i| = 0$ *and* $1-\sum_{1\leq i\leq N} sgn(n_i)m_i \neq 0$,

$$K_0\big(C_r^*(G \ltimes \partial\Omega)\big) \cong \mathbb{Z}$$

and

$$K_1\big(C_r^*(G \ltimes \partial\Omega)\big) \cong \mathbb{Z} \oplus \mathbb{Z}_{\left|1-\sum_{1\leq i\leq N} sgn(n_i)m_i\right|}.$$

If $1-\sum_{1\leq i\leq N}|n_i| \neq 0$ *and* $1-\sum_{1\leq i\leq N} sgn(n_i)m_i = 0$,

$$K_0\big(C_r^*(G \ltimes \partial\Omega)\big) \cong \mathbb{Z} \oplus \mathbb{Z}_{\sum_{1\leq i\leq N}|n_i|-1}$$

and

$$K_1\big(C_r^*(G \ltimes \partial\Omega)\big) \cong \mathbb{Z}.$$

If $1-\sum_{1\leq i\leq N}|n_i| = 0$ *and* $1-\sum_{1\leq i\leq N} sgn(n_i)m_i = 0$,

$$K_0\big(C_r^*(G \ltimes \partial\Omega)\big) \cong \mathbb{Z} \oplus \mathbb{Z}$$

and

$$K_1\big(C_r^*(G \ltimes \partial\Omega)\big) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Theorem 1.0.9. *(General case) Let P be the fundamental monoid of a graph of monoids with condition (LCM) for P satisfied. Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R}, +)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$.*
(i) For Ω, we have

$$K_0(C_r^*(G \ltimes \Omega)) \cong \mathbb{Z} \text{ and } K_1(C_r^*(G \ltimes \Omega)) \cong 0.$$

(ii) $\Omega_{\boldsymbol{b},\,\infty}$ is always closed in Ω, and we have

$$K_*(C_r^*(G \ltimes \Omega_{\boldsymbol{b},\,\infty})) \cong K_*(C(\Omega_{\boldsymbol{b},\,\infty}) \rtimes_r G) \cong K_*(C_\lambda^*(G_T)).$$

(iii) When $\{\infty\}$ is closed in Ω, we have

$K_*(C_r^*(G \ltimes \{\infty\})) \cong K_*(C_\lambda^*(G_T))$.

(iv) When Ω_∞ is closed in Ω, we have

$$K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z} \text{ and } K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z}$$

if condition II. holds and

$$K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z}_n \text{ and } K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong 0$$

if condition I. holds.

At the end of the thesis, we briefly give a description of possible extensions of all the results in the thesis. Overall, one direction is to try to extend our results to general cases. For instance, we embed successfully a part of generalised Baumslag-Solitar monoids into amenable groups in Chapter 4, so we can try to embed all generalised Baumslag-Solitar monoids, and even general graphs of monoids, into amenable groups. In Chapter 5, we make some assumptions of the graphs of monoids to get all nonempty closed invariant subsets of the partial action $G \curvearrowright \Omega$. We can investigate the list of all nonempty closed invariant subsets of the partial action $G \curvearrowright \Omega$ by removing a part of the assumptions. Another direction is to study other properties of the C^*-algebras of graphs of monoids, which we miss in the thesis. Typical are the pure infiniteness and the classification of the reduced C^*-algebras of graphs of monoids.

Chapter 2

Preliminaries

The study of the thesis requires a familiarity of certain basic concepts from the fields of set theory, group theory, general topology ([Kel55]), functional analysis ([Rud91], [Yos68]), linear operators ([DS57]) and C^*-algebras ([Arv76], [Mur90]). The content in this chapter is provided as a supplement besides the fundamentals mentioned above.

2.1 Graphs of groups

In this section, I present only some necessary notions, related to graphs of groups. For more details, please refer to [Ser80].

Definition 2.1.1. *A graph Γ consists of a set $V = Vert\,\Gamma$, a set $E = Edge\,\Gamma$ and two maps*

$$E \to V \times V,\ e \mapsto (o(e),\ t(e))$$

and

$$E \to E, \ e \mapsto \bar{e}$$

such that $\bar{e} \neq e$, $\bar{\bar{e}} = e$ *and* $o(e) = t(\bar{e})$. *Such a graph* Γ *is also denoted by* $(V, \ E)$.

An element $v \in V$ is called a vertex of Γ; an element $e \in E$ is called an (oriented) edge of Γ, and $\bar{e}$ is called the inverse edge. The vertices $o(e)$ and $t(e)$ are called the origin and terminus of e. Such two vertices are called adjacent.

A tree is a connected non-empty graph without circuits. Every maximal subtree of a connected non-empty graph contains all the vertices of the graph.

Definition 2.1.2. *A graph of groups* $(G, \ \Gamma)$ *consists of a graph* Γ, *a group* G_v *for every vertex* $v \in \Gamma$ *and a group* G_e *for every edge* $e \in \Gamma$, *together with group embeddings* $G_e \to G_{t(e)}$ *(denoted by* $x \mapsto x^e$*) and the convention* $G_e = G_{\bar{e}}$ *for all edges* $e \in \Gamma$.

In the case where Γ is a tree, by amalgamating the groups G_v along the groups G_e, we get the direct limit of the graph of groups $(G, \ \Gamma)$, denoted by

$$G_\Gamma = \varinjlim (G, \ \Gamma).$$

Here and in the sequel, let $(G, \ \Gamma)$ be a graph of groups with $\Gamma = (V, \ E)$ being a connected nonempty graph. Define the group $F(G, \ \Gamma)$ by the groups $G_v, \ v \in V$ and the edges $e \in E$, subject to the relations $\bar{e} = e^{-1}$ and $e x^e \bar{e} = x^{\bar{e}}, \ x \in G_e$.

Let c be a path of length n in Γ and let $e_1, \cdots, e_n$ be the edges of c, put $v_i = o(e_{i+1}) = t(e_i)$. A word of type c in $F(G, \Gamma)$ is a pair $(c, \mathbf{x})$, where $\mathbf{x} = (x_0, x_1, \cdots, x_n)$ with $x_i \in G_{v_i}$. The element

$$|c, \mathbf{x}| = x_0 e_1 x_1 e_2 \cdots e_n x_n \in F(G, \Gamma)$$

is said to be associated with the word $(c, \mathbf{x})$. When $n = 0$, we have $|c, \mathbf{x}| = x_0$.

Set

$$G_e^e := \{x^e, x \in G_e\} \subseteq G_{t(e)}, e \in E.$$

The element $|c, \mathbf{x}|$ (or the word $(c, \mathbf{x})$) is called reduced if either $n = 0$ or $n \geq 1$ and $x_i \notin G_{e_i}^{e_i}$ whenever $e_{i+1} = \bar{e}_i$ for some $1 \leq i \leq n$.

Fix a vertex $\mathbf{v} \in V$, the fundamental group (G, Γ) at $\mathbf{v}$, denoted by $\pi_1(G, \Gamma, \mathbf{v})$, is the set of all elements of the form $|c, \mathbf{x}|$ in the group $F(G, \Gamma)$, where c is a path whose origin and terminus are both $\mathbf{v}$. When G is the trivial graph of groups I, i.e. $G_v = G_e = \{\varepsilon\}$ (Here and in the sequel, we write ε for the identity in a group), the group $\pi_1(I, \Gamma, \mathbf{v})$ coincides with the fundamental group (in the usual sense) $\pi_1(\Gamma, \mathbf{v})$ of the graph Γ at the point $\mathbf{v}$. In general, the canonical morphism $G \to I$ extends to a homomorphism

$$\pi_1(G, \Gamma, \mathbf{v}) \to \pi_1(\Gamma, \mathbf{v}).$$

This homomorphism is surjective and its kernel is the normal subgroup of $\pi_1(G, \Gamma, \mathbf{v})$ generated by the groups G_v.

Let T be a maximal subtree of the graph Γ, the fundamental group $\pi_1(G, \Gamma, T)$ of (G, Γ) at T is defined as the quotient of $F(G, \Gamma)$ by the normal subgroup generated by all the edges

$e \in T$. Let A be an orientation of $E \setminus T$, i.e.,

$$E \setminus T = A \cup \bar{A},$$

then the fundamental group $\pi_1(G,\ \Gamma,\ T)$ is given by the groups $G_v,\ v \in V$ and A, subject to the relations $x^e = x^{\bar{e}}$ for all $e \in T$ and all $x \in G_e$, and $ay^a a^{-1} = y^{\bar{a}}$ for all $a \in A$ and all $y \in G_a$.

Examples.

(i) If $(G,\ T)$ is a tree of groups with $G_e = \{\varepsilon\}$ for all edges $e \in T$, then the fundamental group $\pi_1(G,\ T,\ T)$ is exactly the free product of all the groups $G_v,\ v \in V$.

(ii) If Γ is a bouquet of circles with one unique vertex and $(G,\ \Gamma)$ is a graph of groups such that $G_v \cong \mathbb{Z}$ for the unique vertex $v \in V$ and $G_e \cong \mathbb{Z}$ for all edges $e \in E$, then the fundamental group $\pi_1(G,\ \Gamma,\ T)$ is exactly a generalised Baumslag-Solitar group. That is,

$$G = GBS(N,\ m_i,\ n_i) =< a_i,\ b \mid a_i b^{m_i} = b^{n_i} a_i,\ m_i,\ n_i \in \mathbb{Z}, 1 \leq i \leq N,\ N = \sharp A > .$$

In particular, the fundamental group $\pi_1(G,\ \Gamma,\ T)$ is the Baumslag-Solitar group if $N = \sharp A = 1$.

The following proposition comes from [Ser80, p44].

Proposition 2.1.3. *Let $(G,\ \Gamma)$ be a graph of groups with Γ being a connected nonempty graph, let $\boldsymbol{v} \in V$ and let T be a maximal subtree of Γ. The canonical quotient map $F(G,\ \Gamma) \to \pi_1(G,\ \Gamma,\ T)$ induces an isomorphism of $\pi_1(G,\ \Gamma,\ \boldsymbol{v})$ onto $\pi_1(G,\ \Gamma,\ T)$.*

Every element in $\pi_1(G,\ \Gamma,\ T)$ (T-word) is of the form

$$y = x_1^0 \cdots x_{k_0}^0 a_1 x_1^1 \cdots x_{k_1}^1 a_2 x_1^2 \cdots x_{k_{m-1}}^{m-1} a_m x_1^m \cdots x_{k_m}^m,$$

where $x_j^i \in G_{v_j^i}$ and $a_i \in A \cup \bar{A}$. Let $[v,\ w]$ be the geodesic path from the vertex v to the vertex w in T, define

$$\mathscr{E}(y) := d_0 x_1^0 e_1^0 \cdots e_{k_0-1}^0 x_{k_0}^0 d_1 x_1^1 e_1^1 \cdots e_{k_1-1}^1 x_{k_1}^1 d_2 x_1^2 e_1^2 \cdots e_{k_{m-1}-1}^{m-1} x_{k_{m-1}}^{m-1} d_m x_1^m e_1^m \cdots e_{k_m-1}^m x_{k_m}^m d_{m+1},$$

where $e_j^i = [v_j^i,\ v_{j+1}^i]$, $d_i = [v_{k_{i-1}}^{i-1},\ o(a_i)]a_i[t(a_i),\ v_1^i],\ 1 \le i \le m,\ d_0 = [\mathbf{v},\ v_1^0]$ and $d_{m+1} = [v_{k_m}^m,\ \mathbf{v}]$. Then $\mathscr{E}(y)$ is an element in $\pi_1(G,\ \Gamma,\ \mathbf{v})$ ($\mathbf{v}$-word). A T-word y is called reduced if the corresponding $\mathbf{v}$-word $\mathscr{E}(y)$ is reduced. Define the length of the T-word y by $\ell(y) := \ell(\mathscr{E}(y))$.

Given an $\mathbf{v}$-word $x = x_0 e_1 x_1 e_2 \cdots e_n x_n$, define $\mathscr{I}(x)$ as the T-word obtained from x by deleting all e_i with $e_i \in T$ and x_i with $x_i = \varepsilon$.

Corollary 2.1.4. *The maps $\mathscr{E}$ and $\mathscr{I}$ induce a bijection between reduced T-words and reduced $\mathbf{v}$-words.*

2.2 Reduced semigroup C^*-algebras as groupoid C^*-algebras

In this section, I will describe reduced semigroup C^*-algebras as groupoid C^*-algebras. Here I assume the readers have a knowledge of some basics in inverse semigroups, partial dynamical systems and groupoids. These concepts and most of the content in this section can be found in [CELY17].

Let P be a left cancellative semigroup, the partial bijection $P \to P,\ x \mapsto px$ extends uniquely to an isometry $V_p : \ell^2 P \to \ell^2 P$. This assignment $p \mapsto V_p$ is called the left regular representation of P and the reduced semigroup C^*-algebra $C^*_\lambda(P)$ is defined to be the smallest subalgebra of $\mathscr{L}(\ell^2 P)$ containing $\{V_p,\ p \in P\}$.

The inverse hull of P, denoted by $I_l(P)$, is the smallest semigroup of partial isometries on $\ell^2 P$ containing the isometries $\{V_p,\ p \in P\}$ and their adjoints $\{V_p^*,\ p \in P\}$. Alternatively, $I_l(P)$ can be described as the smallest semigroup of partial bijections on P containing the partial bijections $\{P \to P,\ x \mapsto px,\ p \in P\}$ (denoted by p) and their inverses $\{P \to P,\ px \mapsto x,\ p \in P\}$ (denoted by p^{-1}). This allows us to regard P as a subsemigroup of $I_l(P)$. Furthermore, if P is a subsemigroup of a group G, then there is a unique partial homomorphism $\sigma :\ I_l(P)^\times \to G$ identical on P, where $I_l(P)^\times := I_l(P) \setminus \{0\}$.

In the case of partial bijections, every idempotent in $I_l(P)$ is a partial identity on P and hence is given by its domain and image. The idempotents in $I_l(P)$ are called the constructible right ideals of P, whose collection is denoted by $\mathscr{J}_P$. It is easy to see that $\mathscr{J}_P$ is an abelian semigroup closed under intersection of sets. Indeed, we have such a concrete expression as follows:

$$\mathscr{J}_P = \{p_n \cdots q_1^{-1} p_1 P :\ p_i,\ q_i \in P\} \cup \{q_n^{-1} p_n \cdots q_1^{-1} p_1 P :\ p_i,\ q_i \in P\}.$$

Definition 2.2.1. *A left cancellative semigroup P is said to satisfies the independence condition if $X,\ X_i,\ 1 \le i \le n \in \mathscr{J}_P$ with $X = \cup_{1 \le i \le n} X_i$ yields $X = X_i$ for some $1 \le i \le n$.*

If P is right LCM, i.e. for all $p,\ q \in P$, either $pP \cap qP = \emptyset$ or $pP \cap qP = rP$ for some $r \in P$, then every nonempty constructible right ideal of P is principal. That is,

$$\mathscr{J}_P^\times = \{pP,\ p \in P\}.$$

Furthermore, P satisfies independence if P contains an identity element.

Its character space $\widehat{\mathscr{J}_P}$ is defined as follows

$$\widehat{\mathscr{J}_P} = \{\chi : \ \mathscr{J}_P \to \{0,\ 1\} \text{ nonzero semigroup homomorphism}\}$$

and is endowed with the pointwise convergence topology.

When P embeds into a group G, G has a partial action on the character space $\widehat{\mathscr{J}_P}$. Every $g \in G$ acts on

$$U_{g^{-1}} = \{\chi \in \widehat{\mathscr{J}_P} : \ \chi(x^{-1}x) = 1 \text{ for some } x \in I_l(P) \setminus \{0\} \text{ with } \sigma(x) = g\}$$

and $g\chi = \chi(x^{-1} \sqcup x)$ for $\chi \in U_{g^{-1}}$ and $x \in I_l(P) \setminus \{0\}$ with $\chi(x^{-1}x) = 1$ and $\sigma(x) = g$.

In the case of partial isometries, for every partial isometry $V \in I_l(P) \setminus \{0\}$ and every $x \in P$, either $V\delta_x = 0$ or $V\delta_x = \delta_{gx}$, where $g = \sigma(V)$. Set

$$D_\lambda(P) := C^*(\{1_X,\ X \in \mathscr{J}_P\}) \subseteq C^*_\lambda(P),$$

where $1_X \in C^*_\lambda(P) \cap \ell^\infty(P)$ is the characteristic function on $X \subseteq P$, and define

$$\Omega_P := \mathrm{Spec}(D_\lambda(P)),$$

then G has a partial action on Ω_P. For every $g \in G$, let

$$U_{g^{-1}} := \{\chi \in \Omega_P : \ \chi(V^*V) = 1 \text{ for some } V \in I_l(P) \setminus \{0\} \text{ with } \sigma(V) = g\}$$

and $g\chi = \chi(V^* \sqcup V)$ for $\chi \in U_{g^{-1}}$ and $V \in I_l(P) \setminus \{0\}$ with $\chi(V^*V) = 1$ and $\sigma(V) = g$.

The following proposition comes from [CELY17].

Proposition 2.2.2. *(i) Ω_P can be identified with the subspace of $\widehat{\mathcal{J}_P}$ consisting of all characters χ with the property that for all $X,\ X_i,\ 1 \leq i \leq n \in \mathcal{J}_P$ with $X = \cup_{1\leq i\leq n} X_i$, $\chi(X) = 1$ implies $\chi(X_i) = 1$ for some $1 \leq i \leq n$.*
(ii) The identification above is compatible with the partial actions of G on Ω_P and $\widehat{\mathcal{J}_P}$. In particular, Ω_P is an G-invariant subspace of $\widehat{\mathcal{J}_P}$.

Remark 2.2.3. *If P is right LCM and contains an identity element, then P satisfies independence and hence $\Omega_P = \widehat{\mathcal{J}_P}$.*

Let $\mathcal{G}$ be an étale locally compact groupoid and let $r,\ s$ be the range and source map. $C_c(\mathcal{G})$ is a $*$-algebra with respect to the multiplication

$$(f * g)(\gamma) = \sum_{s(\beta)=s(\gamma)} f(\gamma\beta^{-1})g(\beta)$$

and the involution

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

For every $x \in \mathcal{G}^0$, define a $*$-representation π_x of $C_c(\mathcal{G})$ on $\ell^2(s^{-1}(x))$ by setting

$$\pi_x(f)(\xi) = f * \xi.$$

Alternatively, we can define

$$\pi_x(f)\delta_\gamma = \sum_{s(\alpha)=r(\gamma)} f(\alpha)\delta_{\alpha\gamma},$$

to highlight why this representation plays the role of the left regular representation attached to left multiplication.

Set

$$\pi = \oplus_{x \in \mathscr{G}^0} \pi_x,$$

then the reduced groupoid C^*-algebra $C_r^*(\mathscr{G})$ is defined by

$$C_r^*(\mathscr{G}) := \overline{\pi(C_c(\mathscr{G}))} \subseteq \mathscr{L}(\oplus_x \ell^2(s^{-1}(x))).$$

Utilising a reduced crossed product attached to a partial dynamical system as a bridge, we can write the reduced semigroup C^*-algebra $C_\lambda^*(P)$ as a reduced groupoid C^*-algebra. This result also comes from [CELY17].

Theorem 2.2.4. *Let P be a subsemigroup of a group G, then the reduced semigroup C^*-algebra $C_\lambda^*(P)$ is isomorphic to the reduced groupoid C^*-algebra $C_r^*(G \ltimes \Omega_P)$ attached to the transformation groupoid $G \ltimes \Omega_P$.*

2.3 K-theory

In this section, I will present briefly some formulae in general K-theory, K-theory for semigroup C^*-algebras and K-theory for partial crossed products, which will be used later in the thesis. Most of them will come from [CEL13], [CELY17], [Li20] and [RLL00], which you can refer to for more details.

Let A be a C^*-algebra and let

$$\mathscr{P}_n(A) := \mathscr{P}(M_n(A)) \text{ and } \mathscr{P}_\infty(A) := \sqcup_{n=1}^{\infty} \mathscr{P}_n(A),$$

where $\sqcup$ is a disjoint union.

Define a relation $\sim_0$ and a binary operation $\oplus$ on $\mathscr{P}_\infty(A)$ as follows. Suppose that p is a projection in $\mathscr{P}_n(A)$ and q is a projection in $\mathscr{P}_m(A)$, then $p \sim_0 q$ if there exists $v \in M_{m,\, n}(A)$ such that $p = v^*v$ and $q = vv^*$. And

$$p \oplus q := \operatorname{diag}(p,\, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Set

$$\mathscr{D}(A) := \mathscr{P}_\infty(A)/\sim_0$$

and define addition on $\mathscr{D}(A)$ by

$$[p]_{\mathscr{D}} + [q]_{\mathscr{D}} = [p \oplus q]_{\mathscr{D}},\ p,\, q \in \mathscr{P}_\infty(A),$$

where $[p]_{\mathscr{D}} \in \mathscr{D}(A)$ denotes the equivalence class containing p. It is easy to check that $(\mathscr{D}(A),\, +)$ is an abelian semigroup.

If A is unital, $K_0(A)$ is defined to be the Grothendick group of $\mathscr{D}(A)$, i.e.,

$$K_0(A) = G(\mathscr{D}(A)).$$

That is,

$$K_0(A) = \{[p]_0 - [q]_0 :\ p,\, q \in \mathscr{P}_\infty(A)\},$$

where $[p]_0$ is the equivalence class of $[p]_{\mathscr{D}}$ with respect to the equivalent relation $\sim$: $[p]_{\mathscr{D}} \sim [q]_{\mathscr{D}}$ if $[p]_{\mathscr{D}} + [r]_{\mathscr{D}} = [q]_{\mathscr{D}} + [r]_{\mathscr{D}}$ for some $[r]_{\mathscr{D}} \in \mathscr{D}(A)$.

In general, consider the split exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

with the split section $\lambda : \mathbb{C} \to \tilde{A}$. Here $\tilde{A}$ is obtained by adjoining a unit to the C^*-algebra A. Define the scalar mapping s to be

$$s = \lambda \circ \pi : \tilde{A} \to \tilde{A},\ a + \alpha 1 \mapsto \alpha 1,\ a \in A,\ \alpha \in \mathbb{C}.$$

If A is not unital, define $K_0(A)$ to be the kernel of the homomorphism $K_0(\pi) : K_0(\tilde{A}) \to K_0(\mathbb{C})$.

No matter A is unital or not, $K_0(A)$ has the following standard picture,

$$K_0(A) = \{[p]_0 - [s(p)]_0 : p \in \mathscr{P}_\infty(\tilde{A})\}.$$

Let A be a unital C^*-algebra and let

$$\mathscr{U}_n(A) := \mathscr{U}(M_n(A)) \text{ and } \mathscr{U}_\infty(A) := \sqcup_{n=1}^{\infty} \mathscr{U}_n(A),$$

where $\sqcup$ is a disjoint union.

Define a binary operation $\oplus$ on $\mathscr{U}_\infty(A)$ by

$$u \oplus v := \operatorname{diag}(u,\ v) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

Define a relation $\sim_1$ on $\mathscr{U}_\infty(A)$ as follows. Let $u \in \mathscr{U}_n(A)$ and $v \in \mathscr{U}_m(A)$, then $u \sim_1 v$ if there exists a positive integer $k \geq \max\{m,\ n\}$ such that $u \oplus 1_{k-n}$ is homotopic to $v \oplus 1_{k-m}$ in $\mathscr{U}_k(A)$. Here 1_l is the identity matrix in $M_l(A)$.

For every C^*-algebra A, define

$$K_1(A) := \mathscr{U}_\infty(\tilde{A}) / \sim_1 .$$

Let $[u]_1 \in K_1(A)$ be the equivalence class containing u in $\mathscr{U}_\infty(\tilde{A})$. Define a binary operation $+$ on $K_1(A)$ by

$$[u]_1 + [v]_1 = [u \oplus v]_1,\ u,\ v \in \mathscr{U}_\infty(\tilde{A}).$$

Both K_0 and K_1 are functors from the category of C^*-algebras to the category of abelian groups. They preserve half exactness, split exactness, direct sum and continuity and have stability.

The suspension of a C^*-algebra A is

$$SA := \{f \in C([0,\ 1],\ A),\ f(0) = f(1) = 0\} = C_0((0,\ 1),\ A).$$

S is an exact functor from C^*-algebras into itself. And we have $K_1(A) \cong K_0(SA)$.

The higher K-functors are defined by induction,

$$K_n(A) = K_{n-1}(SA),\ n \geq 2.$$

Let

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

be a short exact sequence of C^*-algebras and let $u \in \mathscr{U}_n(\tilde{B})$, there exist $v \in \mathscr{U}_{2n}(\tilde{A})$ and $p \in \mathscr{P}_{2n}(\tilde{I})$ such that

$$\tilde{\varphi}(p) = v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^*,\ \ \tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}.$$

The index map $\delta_1 :\ K_1(B) \to K_0(I)$ is given by

$$\delta_1([u]_1) = [p]_0 - [s(p)]_0.$$

By exactness of the functor S, we have the following short exact sequence of C^*-algebras,

$$0 \longrightarrow S^n I \xrightarrow{S^n \varphi} S^n A \xrightarrow{S^n \psi} S^n B \longrightarrow 0.$$

So we can also define the higher index map $\delta_{n+1} :\ K_{n+1}(B) \to K_n(I)$ via the index map $K_1(S^n B) \to K_0(S^n I)$.

For K-functors, we have the following results ([RLL00]).

Theorem 2.3.1. *(i) (Bott Periodicity)* $K_{n+2}(A) = K_n(A)$ *for all* $n \in \mathbb{N}$ *and all* C^**-algebras* A.

(ii) (The Six Term Exact Sequence) For every short exact sequence of C^*-algebras

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0,$$

the associated six term sequence

$$\begin{tikzcd} K_0(I) \arrow[r, "K_0(\varphi)"] & K_0(A) \arrow[r, "K_0(\psi)"] & K_0(B) \arrow[d, "\delta_0"] \\ K_1(B) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "K_0(\psi)"] & K_1(I) \arrow[l, "K_0(\varphi)"] \end{tikzcd}$$

is exact. Here δ_0 is the composition of the higher index map $\delta_2 : K_2(B) \to K_1(I)$ and the isomorphism map $K_0(B) \to K_2(B)$.

All the above are about general K-theory, and now we present some K-theory formulae for semigroup C^*-algebras and partial crossed products. The following definition can be found in [Definition 5.8.1, CELY17].

Definition 2.3.2. *Let P be a subsemigroup of a group G, we say that $P \subseteq G$ is Toeplitz (or satisfies Toeplitz condition) if for all $g \in G$ with $g^{-1}P \cap P \neq \emptyset$, the partial bijection*

$$g^{-1}P \cap P \to P \cap gP,\ x \mapsto gx$$

lies in the inverse hull $I_l(P)$ of P.

Let P be a subsemigroup of a group G such that $P \subseteq G$ is Toeplitz. Assume that P satisfies independence and G satisfies the Baum-Connes conjecture with coefficients (see [p110, CELY17]). Let $\mathscr{J}_P^\times$ be the collection of all nonempty contructible right ideals of P and let

$$\mathscr{J}_{P\subseteq G}^\times = G\cdot \mathscr{J}_P^\times.$$

Choose a set of representatives $\mathfrak{X} \subseteq \mathscr{J}_P^\times$ for the G-orbits $G\backslash \mathscr{J}_{P\subseteq G}^\times$ and define, for $X\in\mathfrak{X}$,

$$G_X := \{g\in G,\ gX = X\},$$

and

$$\iota_X:\ C_\lambda^*(G_X)\to C_\lambda^*(P),\ \lambda_g\mapsto \lambda_g 1_X,$$

where we identify $C_\lambda^*(P)$ with the crossed product $D_{P\subseteq G}\rtimes_r G$.

We have the following theorem on K-theory for semigroup C^*-algebras, which can be found in [Theorem 5.10.1, CELY17] or [Corollary 1.3, Li20].

Theorem 2.3.3. *In the same setting above, we conclude*

$$\oplus_{X\in\mathfrak{X}}(\iota_X)_*:\ \oplus_{X\in\mathfrak{X}} K_*(C_\lambda^*(G_X))\to K_*(C_\lambda^*(P))$$

is an isomorphism.

If P contains an identity element, then we have $\mathscr{J}_P^\times = \{pP,\ p\in P\}$ and thus $\mathscr{J}_{P\subseteq G}^\times$ has only one orbit. We choose $\mathfrak{X}=\{P\}$ and get

$$\iota_*:\ K_*(C_\lambda^*(P^*))\to K_*(C_\lambda^*(P)),$$

where $P^* \subseteq P$ is the subgroup consisting of all units. In particular, if P^* is trivial, then we get

$$K_*(C^*_\lambda(P)) \cong K_*(\mathbb{C}).$$

The following theorem on K-theory for partial crossed products comes from [Theorem 1.2, Li20].

Theorem 2.3.4. *Let G be a discrete and countable group and let X be a second countable totally disconnected locally compact Hausdorff space such that $G \curvearrowright X$ is a partial dynamical system, given by $U_{g^{-1}} \to U_g$, $x \mapsto g \cdot x$. Assume that $G \curvearrowright X$ admits a G-invariant regular basis $\mathscr{V}$ for the compact open subsets of X and that G satisfies the Baum-Connes conjecture with coefficients. Then the K-theory of the reduced partial crossed product of $G \curvearrowright X$ is given by*

$$K_*(C_0(X) \rtimes_r G) \cong \bigoplus_{[V] \in G \backslash \mathscr{V}^\times} K_*(C^*_\lambda(G_V)),$$

where $G \backslash \mathscr{V}^\times$ denotes the set of orbits under the G-action on the non-empty elements $\mathscr{V}^\times$ of $\mathscr{V}$, and $G_V := \{g \in G,\ g \cdot V = V\}$.

In the theorem above, a G-invariant regular basis $\mathscr{V}$ for the compact open subsets of X is a family $\mathscr{V}$ of compact open subsets of X such that for all $g \in G$, $\mathscr{V}_{g^{-1}} := \{V \in \mathscr{V},\ V \subseteq U_{g^{-1}}\}$ is a regular basis for the compact open subsets of $U_{g^{-1}}$ and $g \cdot \mathscr{V}_{g^{-1}} = \mathscr{V}_g$. And here is the definition for a regular basis, which comes from [Definition 2.9, CEL13].

Definition 2.3.5. *Let X be a totally disconnected locally compact Hausdorff space. A family $\mathscr{V}$ of non-empty compact open subsets of X is called a regular basis for the compact open* *sets of X if the following are satisfied:*

(i) $\mathscr{V} \cup \{\emptyset\}$ is closed under finite intersections.

(ii) $\mathscr{V}$ generates the compact open sets of X by finite intersections, finite unions and complementary sets.

(iii) $\mathscr{V}$ is independent. That is, if $V,\ V_1,\ \cdots,\ V_n$ are elements in $\mathscr{V}$ with $V = \cup_{1\leq i\leq n} V_i$, then we have $V = V_i$ for some $1 \leq i \leq n$.

Chapter 3

Graphs of monoids

We explained Serre's definition of graphs of groups in the last chapter and in this chapter we want to extend the notion to graphs of monoids. Moreover, we shall discuss the right LCM property of the graphs of monoids as it is needed later in the thesis.

3.1 Normal Form

Let $(G,\ \Gamma)$ be a graph of groups with Γ connected and let $G_v,\ v \in V$ be totally ordered with positive cone P_v, i.e., $G_v = P_v \cup P_v^{-1}$ and $P_v \cap P_v^{-1} = \{\varepsilon\}$. For $e \in E$, define $P_e := \{g \in G_e,\ g^e \in P_{t(e)}\}$. In general, it is difficult to find relations between P_e and $P_{\bar{e}}$. In the thesis, We only focus on the case where $P_e = P_{\bar{e}}$ for all $e \in T$ and either $P_e = P_{\bar{e}}$ or $P_e = P_{\bar{e}}^{-1}$ for all $e \in A$. Define $A_+ := \{e \in A,\ P_e = P_{\bar{e}}\}$ and $A_- := \{e \in A,\ P_e = P_{\bar{e}}^{-1}\}$, then we have $P_e^{\bar{e}} \subseteq P_{o(e)}$ for all $e \in A_+$ and $P_e^{\bar{e}} \subseteq P_{o(e)}^{-1}$ for all $e \in A_-$.

A $\mathbf{v}$-word $x = x_0 e_1 x_1 e_2 \cdots e_n x_n \in \pi_1(G,\ \Gamma,\ \mathbf{v})$ is called to be positive if $x_i \in P_{v_i}$ and $e_i \in E$, where $v_i = t(e_i) = o(e_{i+1}),\ 0 \leq i \leq n$. $\pi_1^+(G,\ \Gamma,\ \mathbf{v})$ is defined to be all the elements of $\pi_1(G,\ \Gamma,\ \mathbf{v})$, which can be written as a positive $\mathbf{v}$-word.

Let G_T be the direct limit of the graph of groups $(G,\ T)$, then G_T is the subgroup of $\pi_1(G,\ \Gamma,\ T)$ generated by G_v. Let $\pi_1^+(G,\ \Gamma,\ T)$ be the subsemigroup of $\pi_1(G,\ \Gamma,\ T)$ generated by P_v and A, and let P_T be the subsemigroup of G_T generated by P_v.

A T-word in compact form is a word of the form $y_0a_1y_1a_2\cdots a_ny_n$ with $y_i \in G_T$ and $a_i \in A\cup\bar{A}$. It is called positive if $y_i \in P_T$ and $a_i \in A$. If we write $y_i = y_1^i \cdots y_{k_i}^i,\ 0 \leq i \leq n$ with $y_j^i \in G_{v_j^i}$ for some $v_j^i \in V$ and $1 \leq j \leq k_i$, then we get a T-word in the general normal form. It is called positive if $y_j^i \in P_{v_j^i}$ and $a_i \in A$ for all $1 \leq j \leq k_i$ and all $0 \leq i \leq n$. It is easy to see that every element in $\pi_1^+(G,\ \Gamma,\ T)$ can be expressed as a positive T-word (in compact form).

In this thesis, we make the convention that all the graphs are countable and all the groups are discrete and countable unless otherwise explicitly stated. We will focus on the fundamental group $\pi_1(G,\ \Gamma,\ T)$ and the fundamental monoid $\pi_1^+(G,\ \Gamma,\ T)$. For brevity, we also call the fundamental groups by graphs of groups and call the fundamental monoids by graphs of monoids. Set $G := \pi_1(G,\ \Gamma,\ T)$ and $P := \pi_1^+(G,\ \Gamma,\ T)$.

Every element in G can be written as a word in $\{G_v\}_{v\in V} \cup A$, and vise versa. Due to the relations, two different words can represent the same group element. So we make the following convention: for two T-words $x,\ x'$, we write $x = x'$ if they represent the same element in G and write $x \equiv x'$ if they are identical words. Similarly, for $\mathbf{v}$-words $y,\ y'$, we write $y = y'$ if they represent the same element in $\pi_1(G,\ \Gamma,\ \mathbf{v})$ and write $y \equiv y'$ if they are identical words.

In the above setting, we have the following proposition.

Proposition 3.1.1. *(i) The monoid P is generated by P_v, $v \in V$ and A, subject to the relation $x^e = x^{\bar{e}}$ for all $e \in T$ and $x \in P_e$, $ax^a = x^{\bar{a}}a$ for all $a \in A_+$ and $x \in P_a$, and $(x^{\bar{a}})^{-1}ax^a = a$ for all $a \in A_-$ and $x \in P_a$.*
(ii) Every element in P is represented by a reduced positive T-word.

Proof. (i) It follows directly from the definition.
(ii) Let y be a positive T-word with

$$\mathscr{E}(y) = x_0 e_1 x_1 e_2 \cdots e_n x_n,$$

we prove the assertion by induction on $n = \ell(\mathscr{E}(y))$.
If $\mathscr{E}(y)$ is not reduced, then we have $n \geq 1$ and there exists $0 \leq l \leq n-1$ such that $e_{l+1} = \bar{e}_l$ and $x_l \in P_{e_l}^{e_l}$, i.e., $x_l = z^{e_l}$ for some $z \in P_{e_l}$. $e_{l+1} = \bar{e}_l$ implies $e_{l+1},\ e_l \in T$ since y and $\mathscr{E}(y)$ are positive and do not contain elements in $\bar{A}$. Then we have

$$\mathscr{E}(y) = x_0 \cdots e_l x_l e_{l+1} \cdots e_n x_n = x_0 \cdots z^{\bar{e}_l} \cdots e_n x_n,$$

arriving at a word with smaller length. This finishes the induction and thus y can be represented by a reduced positive T-word.

□

For different words representing the same group element, we have the following lemma.

Lemma 3.1.2. *(i) Let $x = x_0 e_1 x_1 e_2 \cdots e_n x_n$ and $x' = x'_0 e'_1 x'_1 e'_2 \cdots e'_n x'_n$ be two reduced $\boldsymbol{v}$-words with $x_l \in G_{v_l}$ and $x'_{l'} \in G_{v'_{l'}}$ for all $l,\ l'$. If $x = x'$, then we have $n = n'$, $e_l = e'_l$, and*

$$x_0 e_1 x_1 e_2 \cdots x_{l-1} e_l z = x'_0 e_1 x'_1 e_2 \cdots x'_{l-1} e_l$$

for some $z \in G_{e_l}^{e_l}$ *and all* $1 \leq l \leq n$.

(ii) Let $y = y_0 a_1 y_1 a_2 \cdots a_m y_m$ *and* $y' = y'_0 a'_1 y'_1 a'_2 \cdots a'_{m'} y'_{m'}$ *be two* T*-words in compact form with* $y_k,\ y'_{k'} \in G_T$ *and* $a_k,\ a'_{k'} \in A$. *If* $y = y'$, *then we have* $m = m'$, $a_k = a'_k$, *and*

$$y_0 a_1 y_1 a_2 \cdots y_{k-1} a_k z = y'_0 a_1 y'_1 a_2 \cdots y'_{k-1} a_k$$

for some $z \in G_{a_k}^{a_k}$ *and all* $1 \leq k \leq m$.

Proof. (i) Recall that $\pi_1(G,\ \Gamma,\ \mathbf{v})$ is a subgroup of $F(G,\ \Gamma)$, which is generated by $G_v,\ v \in V$ and E, subject to the relation $\bar{e} = e^{-1}$ and $ex^e e^{-1} = x^{\bar{e}}$ for all $e \in E$ and all $x \in G_e$. $x = x'$ implies that we can get one word from the other by utilisation of the relations. Hence the conclusion follows from the assumption that x and x' are reduced.

(ii) The group G is generated by $G_v,\ v \in V$ and A, subject to the relation $x^{\bar{e}} = x^e$ and $ay^a = y^{\bar{a}}a$ for all $e \in T$, all $x \in G_e$, all $a \in A$ and all $y \in G_a$. Similarly as in part (i), the conclusion follows since we can get one word from the other by utilisation of the relations.

□

Corollary 3.1.3. $G_v \cap P = P_v$ *for all* $v \in V$.

Proof. Let $y = y_0 a_1 y_1 a_2 \cdots a_m y_m \in P$ be a positive word in compact form and let $y' \in G_v$ for some $v \in V$. If $y = y'$, then we have $m = 0$ and $y = y_0 \in P_T$ by Lemma 3.1.2. Assume $y = y_0 = y_1^0 \cdots y_n^0$ with $y_l^0 \in P_{v_l},\ 1 \leq l \leq n$, by $y = y'$, we conclude that we can get y' by utilisation of the relation $x^e = x^{\bar{e}}$ for all $e \in T$ and $x \in G_e$. Noting $P_e^e \subseteq P_{t(e)}$ and $P_e = P_{\bar{e}}$ for all $e \in T$, we have $y' \in P_v$.

□

In order to study the relation between the reduced word representing the multiplication of two group elements and the reduced words representing the two elements, we need to introduce the following notion.

Definition 3.1.4. *Let*

$$y = x_1^0 \cdots x_{k_0}^0 a_1 x_1^1 \cdots x_{k_1}^1 a_2 x_1^2 \cdots x_{k_{m-1}}^{m-1} a_m x_1^m \cdots x_{k_m}^m$$

be a T-word, where $x_j^i \in G_{v_j^i}$ and $a_i \in A$. And let

$$\mathscr{E}(y) = d_0 x_1^0 e_1^0 \cdots e_{k_0-1}^0 x_{k_0}^0 d_1 x_1^1 e_1^1 \cdots e_{k_1-1}^1 x_{k_1}^1 d_2 x_1^2 e_1^2 \cdots e_{k_{m-1}-1}^{m-1} x_{k_{m-1}}^{m-1} d_m x_1^m e_1^m \cdots e_{k_m-1}^m x_{k_m}^m d_{m+1},$$

where $e_j^i = [v_j^i,\ v_{j+1}^i]$, $d_i = [v_{k_{i-1}}^{i-1},\ o(a_i)] a_i [t(a_i),\ v_1^i]$, $1 \le i \le m$, $d_0 = [\boldsymbol{v},\ v_1^0]$ and $d_{m+1} = [v_{k_m}^m,\ \boldsymbol{v}]$. y is called properly reduced if all of the following are satisfied:

(a) y is reduced;

(b) If e_1^0 starts with $e \in T$, then $x_1^0 \notin G_e^{\bar{e}}$;

(c) If $e_{k_m-1}^m$ ends with $e \in T$, then $x_{k_m}^m \notin G_e^e$.

Remark 3.1.5. *In the same setting as in the Definition above, define*

$$l(y) := \sum_{1 \le j < k_i,\ 0 \le i \le m} \ell(e_j^i) + \sum_{1 \le i \le m} \ell(d_i).$$

Note that if y is reduced but not properly reduced, then we have $m \ge 1$ and $l(y) \ge 1$.

Lemma 3.1.6. *Every element in G is represented by a properly reduced T-word.*

Proof. Every element in G is represented by a T-word

$$y = x_1^0 \cdots x_{k_0}^0 a_1 x_1^1 \cdots x_{k_1}^1 a_2 x_1^2 \cdots x_{k_{m-1}}^{m-1} a_m x_1^m \cdots x_{k_m}^m$$

such that

$$\mathscr{E}(y) = d_0 x_1^0 e_1^0 \cdots e_{k_0-1}^0 x_{k_0}^0 d_1 x_1^1 e_1^1 \cdots e_{k_1-1}^1 x_{k_1}^1 d_2 x_1^2 e_1^2 \cdots e_{k_{m-1}-1}^{m-1} x_{k_{m-1}}^{m-1} d_m x_1^m e_1^m \cdots e_{k_m-1}^m x_{k_m}^m d_{m+1}$$

is a reduced $\mathbf{v}$-word. We now proceed inductively on $l(y)$.

When $l(y) = 0$, y is properly reduced.

Now assume $l(y) \geq 1$ and assume e_1^0 starts with $e \in T$ and $x_1^0 \in G_e^{\bar{e}}$. That is, $e_1^0 = ee'$ for a path $e' \subseteq T$ and $x_1^0 = z^{\bar{e}}$ for some $z \in G_e$. Then we have

$$\begin{aligned} y &\equiv z^{\bar{e}} x_2^0 \cdots x_{k_0}^0 a_1 x_1^1 \cdots x_{k_1}^1 a_2 x_1^2 \cdots x_{k_{m-1}}^{m-1} a_m x_1^m \cdots x_{k_m}^m \\ &= z^e x_2^0 \cdots x_{k_0}^0 a_1 x_1^1 \cdots x_{k_1}^1 a_2 x_1^2 \cdots x_{k_{m-1}}^{m-1} a_m x_1^m \cdots x_{k_m}^m := y' \end{aligned} \tag{3.1}$$

and

$$\mathscr{E}(y') = d_0 e z^e e' x_2^0 \cdots e_{k_0-1}^0 x_{k_0}^0 d_1 x_1^1 e_1^1 \cdots e_{k_1-1}^1 x_{k_1}^1 d_2 x_1^2 e_1^2 \cdots e_{k_{m-1}-1}^{m-1} x_{k_{m-1}}^{m-1} d_m x_1^m e_1^m \cdots e_{k_m-1}^m x_{k_m}^m d_{m+1}.$$

It is easy to see that $l(y') < l(y)$. And we can apply the induction hypothesis.

When $e_{k_m-1}^m$ ends with $e \in T$ and $x_{k_m}^m \in G_e^e$, the argument is similar.

Therefore, we prove by induction every element in G is represented by a properly reduced T-word.

□

Lemma 3.1.7. *Let y be a properly reduced positive word with $l(y) \geq 1$, then we have $y \notin P_v$ for any $v \in V$.*

Proof. Let

$$y = x_1^0 \cdots x_{k_0}^0 a_1 x_1^1 \cdots x_{k_1}^1 a_2 x_1^2 \cdots x_{k_{m-1}}^{m-1} a_m x_1^m \cdots x_{k_m}^m$$

be a properly reduced positive word and let $y' \in P_v$ for some $v \in V$. If $y = y'$, then we have $m = 0$ and $y \equiv x_1^0 \cdots x_{k_0}^0 \in P_T$ by part (ii) of Lemma 3.1.2. Then we have

$$\mathscr{E}(y) = d_0 x_1^0 e_1^0 \cdots e_{k_0-1}^0 x_{k_0}^0 d_1$$

and $\mathscr{E}(y') = d_0' y' d_1'$.

Since $\mathscr{E}(y) = \mathscr{E}(y')$, we have by part (i) of Lemma 3.1.2,

$$\ell(d_0') + \ell(d_1') = \ell(d_0) + \ell(d_1) + l(y).$$

So we get either $\ell(d_0') > \ell(d_0)$ or $\ell(d_1') > \ell(d_1)$.

Assume, without loss of generality, $\ell(d_1') > \ell(d_1)$, and assume $e_{k_0-1}^0$ ends with $e \in T$ and $e_{k_0-1}^0 = e'e$, we have by part (i) of Lemma 3.1.2,

$$d_0 x_1^0 e_1^0 \cdots e_{k_0-2}^0 x_{k_0-1}^0 e' z = d_0' y' d_1''$$

for some $z \in G_{o(e)}$, where $d_1'' \subseteq d_1'$ is a sub-path of length $\ell(d_1') - \ell(d_1) - 1$ starting from the vertex v. Coming back to T-words, we have

$$x_1^0 \cdots x_{k_0-1}^0 z = y'$$

and hence $z = x_{k_0}^0$, contradicting with the fact $x_{k_0}^0 \notin G_e^e$.

$\square$

The following result is a straightforward consequence of Lemma 3.1.7.

Corollary 3.1.8. *Let*

$$y = y_0 y_1 \cdots y_m,\ y_k \in P_{v_k},\ 0 \leq k \leq m$$

and

$$y' = y'_0 y'_1 \cdots y'_{m'},\ y'_{k'} \in P_{v'_{k'}},\ 0 \leq k' \leq m'$$

be properly reduced positive words in P_T. If $y = y'$, then $l(y) = l(y')$, $v_0 = v'_0$ and $v_m = v'_{m'}$.

Lemma 3.1.9. *Let*

$$y = x_1^0 \cdots x_{k_0}^0 a_1 x_1^1 \cdots x_{k_1}^1 a_2 x_1^2 \cdots x_{k_{m-1}}^{m-1} a_m x_1^m \cdots x_{k_m}^m$$

and

$$y' = z_1^0 \cdots z_{l_0}^0 a'_1 z_1^1 \cdots z_{l_1}^1 a'_2 z_1^2 \cdots z_{l_{n-1}}^{n-1} a'_n z_1^n \cdots z_{l_n}^n$$

be properly reduced positive words with

$$\mathscr{E}(y) = d_0 x_1^0 e_1^0 \cdots e_{k_0-1}^0 x_{k_0}^0 d_1 x_1^1 e_1^1 \cdots e_{k_1-1}^1 x_{k_1}^1 d_2 x_1^2 e_1^2 \cdots e_{k_{m-1}-1}^{m-1} x_{k_{m-1}}^{m-1} d_m x_1^m e_1^m \cdots e_{k_m-1}^m x_{k_m}^m d_{m+1},$$

$$\mathscr{E}(y') = d'_0 z_1^0 f_1^0 \cdots f_{l_0-1}^0 z_{l_0}^0 d'_1 z_1^1 f_1^1 \cdots f_{l_1-1}^1 z_{l_1}^1 d'_2 z_1^2 f_1^2 \cdots f_{l_{n-1}-1}^{n-1} z_{l_{n-1}}^{n-1} d'_m z_1^n f_1^n \cdots f_{l_n-1}^n z_{l_n}^n d'_{n+1}.$$

Then yy' is a reduced positive word unless $x_{k_m}^m \in P_{u_{k_m}^m}$, $z_1^0 \in P_{v_1^0}$, $e_{k_m-1}^m$ ends with $e \in T$ and f_1^0 starts with $f \in T$ such that $u_{k_m}^m = v_1^0$, $e = \bar{f}$ and $x_{k_m}^m z_1^0 \in P_e^e$.

Proof. If $u_{k_m}^m \neq v_1^0$, we have $\ell([u_{k_m}^m,\ v_1^0]) \geq 1$ and hence

$$\mathscr{E}(yy') = d_0 x_1^0 e_1^0 \cdots e_{k_0-1}^0 x_{k_0}^0 d_1 x_1^1 e_1^1 \cdots e_{k_m-1}^m x_{k_m}^m [u_{k_m}^m,\ v_1^0] z_1^0 f_1^0 \cdots f_{l_0-1}^0 z_{l_0}^0 d'_1 z_1^1 f_1^1 \cdots f_{l_n-1}^n z_{l_n}^n d'_{n+1}$$

is reduced.

If $u^m_{k_m} = v^0_1$, then

$$\mathscr{E}(yy') = d_0 x^0_1 e^0_1 \cdots e^0_{k_0-1} x^0_{k_0} d_1 x^1_1 e^1_1 \cdots e^m_{k_m-1} x^m_{k_m} z^0_1 f^0_1 \cdots f^0_{l_0-1} z^0_{l_0} d'_1 z^1_1 f^1_1 \cdots f^n_{l_n-1} z^n_{l_n} d'_{n+1}.$$

Assume $e^m_{k_m-1}$ ends with e and f^0_1 starts with f. If $e \neq \bar{f}$, $\mathscr{E}(yy')$ is reduced. If $e = \bar{f}$ and $x^m_{k_m} z^0_1 \notin P^e_e$, $\mathscr{E}(yy')$ is also reduced.

□

Remark 3.1.10. *If we assume further $l(y) \geq 1$ and $l(y') \geq 1$ in Lemma 3.1.9, then yy' is properly reduced whenever it is reduced. Without this assumption, it does not need to be the case.*

Lemma 3.1.11. *Let*

$$x = x_0 \cdots x_{k_0} a_1 x_{k_0+1} \cdots x_{k_0+k_1} a_2 \cdots a_m x_{k_0+\cdots+k_{m-1}+1} \cdots x_{k_0+\cdots+k_{m-1}+k_m} \tag{3.2}$$

be a positive word and y another positive word. Set

$$M := \sum_{0 \leq i \leq m} k_i,$$

then there exists a properly reduced word

$$z = z_0 \cdots z_{l_0} d_1 z_{l_0+1} \cdots z_{l_0+l_1} d_2 \cdots d_n z_{l_0+\cdots+l_{n-1}+1} \cdots z_{l_0+\cdots+l_{n-1}+l_n}$$

representing xy such that $z_{(M)} \in xP$, where

$$z_{(M)} := z_0 \cdots z_{l_0} d_1 z_{l_0+1} \cdots z_{l_0+l_1} d_2 \cdots d_j z_{l_0+\cdots+l_{j-1}+1} \cdots z_M$$

if

$$\sum_{0\le i\le j-1} l_i < M \le \sum_{0\le i\le j} l_i$$

and $z_{(M)} := xy$ *otherwise.*

Proof. (i) We may assume the word representing x in the equation (3.2) and

$$y = y_0\cdots y_{k'_0} a'_1 y_{k'_0+1}\cdots y_{k'_0+k'_1} a'_2\cdots$$

are properly reduced. We prove the claim inductively on $l(y)$.

If xy is reduced and $l(x),\ l(y)\ge 1$, then xy is properly reduced. Take simply $z = xy$, it follows that $z_{(M)} \in xP$.

If xy is reduced and $l(y) = 0$, either xy'_0 is properly reduced for some $y'_0 \in P_{v'_0}$ and $y'_0 = y_0$ or $\sum_{0\le i\le n} l_i \le M$. In both cases, we have $z_{(M)} \in xP$.

If xy is reduced and $l(x) = 0$, then $x = x_0$ and $M = 0$. In this case, $z_{(0)} = x'_0$ for some $x'_0 \in P_{u'_0}$ and $x'_0 = x_0$ or $z_{(0)} = x_0 y_0\cdots y_j$ with $x_0 \in P_{v'_0}$, $x_0 y_0\cdots y_i \in P_{e_i}^{\bar e_i}$, $0\le i< j$ and $x_0 y_0\cdots y_j \notin P_{e_j}^{\bar e_j}$, where $y_i \in P_{v_i}$ and e_i is the beginning edge of the path $[v_i,\ v_{i+1}]$, $0\le i\le j$.

If xy is not reduced, it follows from Lemma 3.1.9 that $x_M y_0 \in P_e^e$, where P_e^e is as in Lemma 3.1.9. In this case, we define $x' = xy_0$ and $y' = y_1\cdots y_{k'_0} a'_1 y_{k'_0+1}\cdots y_{k'_0+k'_1} a'_2\cdots$, then $xy = x'y'$ with $l(y') < l(y)$. By induction hypothesis, we can get a properly reduced word z representing $x'y'$ such that $z_{(M)} \in x'P \subseteq xP$.

□

Corollary 3.1.12. *Let* $x,\ y\in P$ *be two positive reduced words and let* $\ell := \ell(x)$. *Then there ex-*

ists a reduced positive $\mathbf{v}$-word $z = z_0 e_1 z_1 e_2 \cdots e_n z_n$ representing xy such that $z_0 e_1 z_1 e_2 \cdots e_\ell z_\ell \in xP$.

Proof. We may assume $x,\ y$ are properly reduced. If xy is reduced, let $z = \mathscr{E}(xy)$ and we can write $\mathscr{E}(x) = x'x''$, $\mathscr{E}(y) = y'y''$ such that $\ell(x') \leq \ell(x)$, $\mathscr{I}(x') = x$ and $\mathscr{E}(xy) = x'y''$. The claim follows from Lemma 3.1.11.

If xy is not reduced, it follows from Lemma 3.1.9 that $x_M y_0 \in P_e^e$, where P_e^e is as in Lemma 3.1.9 and x_M is as in Lemma 3.1.11. In this case, $xy = (xy_0)y'''$ for some properly reduced positive word y''' with $l(y''') < l(y)$. We proceed inductively on $l(y)$ and it suffices to treat the case where $l(y) = 0$, i.e., $y = y_0$. Then we have $x = x''' x_M$ and $xy = x'''(x_M y_0)$ for some properly reduced positive word x''' with $\ell(xy) \leq \ell$. Therefore, $z_0 e_1 z_1 e_2 \cdots e_\ell z_\ell = z \in xP$.

$\square$

3.2 The right LCM property

In this section, we assume we are in the same setting as in Section 3.1. Our goal is to study when the monoid P is right LCM, i.e., for all $p,\ q \in P$, either $pP \cap qP = \emptyset$ or $pP \cap qP = rP$ for some $r \in P$. For convenience, we introduce a partial order $\prec$ on P, given by $p \prec q$ if $q \in pP$. We denote by $p \vee q$ the (necessarily unique) minimal element $r \in P$ satisfying $p, q \prec r$ if such an element exists. In this language, P is right LCM if and only if for all $p,\ q \in P$, either $pP \cap qP = \emptyset$ or $p \vee q$ exists.

Given $e \in E$ and $p \in P$, we set

$$p^{-1} P_{\bar{e}}^{\bar{e}} := \{x \in P,\ px \in P_{\bar{e}}^{\bar{e}}\}.$$

Definition 3.2.1. *We say condition (LCM) is satisfied if for all $e \in E$, $p \in P_{o(e)}$, either $p^{-1}P_{\bar{e}}^{\bar{e}} = \emptyset$ or $p^{-1}P_{\bar{e}}^{\bar{e}} = qP_{\bar{e}}^{\bar{e}}$ for some $q \in P_{o(e)}$. In the latter case, we define $p^{-1,\, e} := q$.*

Remark 3.2.2. *Let $e \in E$, $p \in P_{o(e)}$ and $q \in P_{\bar{e}}^{\bar{e}}p$, then we have $p^{-1,\, e} = q^{-1,\, e}$.*

The main result of this section reads as follows.

Proposition 3.2.3. *P is right LCM if condition (LCM) is satisfied.*

Before proving Proposition 3.2.3, we need a couple of lemmas. In the following, we always assume condition (LCM) is satisfied.

Lemma 3.2.4. *For all $e \in E$ and $p \in P$, either $p^{-1}P_{\bar{e}}^{\bar{e}} = \emptyset$ or $p^{-1}P_{\bar{e}}^{\bar{e}} = qP_{\bar{e}}^{\bar{e}}$ for some $q \in P$.*

Proof. Note that for all $e \in E$, $p,\ x \in P$, $px \in P_{\bar{e}}^{\bar{e}}$ implies $p,\ x \in P_T$. So we can work in P_T. We first consider the case $p \in P_v$ for some $v \in V$. Let $[v,\ o(e)] = d_1 \cdots d_k$ and set $d_{k+1} := e$. Define $p_0 := p$, $q_1 := p^{-1,\, d_1}$ if $p^{-1}P_{\bar{d}_1}^{\bar{d}_1} \neq \emptyset$, and for $1 \leq i \leq k$, $p_i := pq_1 \cdots q_i$, $q_{i+1} := p_i^{-1,\, d_{i+1}}$ if $p_i^{-1}P_{\bar{d}_{i+1}}^{\bar{d}_{i+1}} \neq \emptyset$. We claim that $p^{-1}P_{\bar{e}}^{\bar{e}} \neq \emptyset$ if and only if $p_i^{-1}P_{\bar{d}_{i+1}}^{\bar{d}_{i+1}} \neq \emptyset$ for all $0 \leq i \leq k$, and that $p^{-1}P_{\bar{e}}^{\bar{e}} = q_1 \cdots q_{k+1}P_{\bar{e}}^{\bar{e}}$ in that case, i.e., $p^{-1,\, e} = q_1 \cdots q_{k+1}$.

It is easy to see that $p^{-1}P_{\bar{e}}^{\bar{e}} \neq \emptyset$ if $p_i^{-1}P_{\bar{d}_{i+1}}^{\bar{d}_{i+1}} \neq \emptyset$ for all $0 \leq i \leq k$. We now prove the converse and that $p^{-1,\, e} = q_1 \cdots q_{k+1}$ inductively on $\ell := \ell([v,\ o(e)])$. The case where $\ell = 0$ follows directly from condition (LCM). Now assume $\ell \geq 1$, suppose $p^{-1}P_{\bar{e}}^{\bar{e}} \neq \emptyset$ and take $x \in P$ with $px \in P_{\bar{e}}^{\bar{e}}$.

By Lemma 3.1.9, there exist positive words $w_m,\ x_m,\ y_m \in P_T$ and $f_m \in T$ for $1 \leq m \leq n$ such that $w_1 \equiv p$, $x = x_1y_1$, $w_m = w_{m-1}x_{m-1}$, $y_{m-1} = x_my_m$, $w_m, x_m \in P_{v_m}$, $w_mx_m \in P_{f_m}^{f_m} \subseteq P_{v_{m+1}}$, $o(f_m) = v_m$, $t(f_m) = v_{m+1}$, and $w_n \equiv px$. By construction, we have $w_m = px_1 \cdots x_{m-1}$, $x = x_1 \cdots x_my_m$ and $w_mx_my_m = px$.

Let $M \in \{1,\ \cdots,\ n\}$ be maximal such that $v_m = v$. Then we must have $f_M = d_1$ as $[v,\ o(e)]$ starts with d_1. Set $x' := x_1 \cdots x_M$, $x'' = y_M$, then we have $px' = w_Mx_M \in P_{d_1}^{d_1} = P_{\bar{d}_1}^{\bar{d}_1}$, which implies $p^{-1}P_{\bar{d}_1}^{\bar{d}_1} \neq \emptyset$. Condition (LCM) implies $x' = p^{-1,\ d_1}y = q_1y$ for some $y \in P_{\bar{d}_1}^{\bar{d}_1}$. Hence $px = px'x'' = (pq_1)yx'' \in P_{\bar{e}}^{\bar{e}}$, $p_1 = pq_1 \in P_{d_1}^{d_1} \subseteq P_{t(d_1)}$ and $yx'' \in p_1^{-1}P_{\bar{e}}^{\bar{e}}$. Note that $\ell([t(d_1,\ e)]) < \ell$, we have by induction hypothesis $p_i^{-1}P_{\bar{d}_{i+1}}^{\bar{d}_{i+1}} \neq \emptyset$ for all $0 \leq i \leq k$ and $p_1^{-1,\ e} = q_2 \cdots q_{k+1}$. The latter yields $yx'' \in q_2 \cdots q_{k+1}P_{\bar{e}}^{\bar{e}}$. Therefore, $x = x'x'' = q_1yx'' \in q_1q_2 \cdots q_{k+1}P_{\bar{e}}^{\bar{e}}$ and $p^{-1}P_{\bar{e}}^{\bar{e}} \subseteq q_1q_2 \cdots q_{k+1}P_{\bar{e}}^{\bar{e}} = q_1p_1^{-1}P_{\bar{e}}^{\bar{e}}$.

Taking $z \in p_1^{-1}P_{\bar{e}}^{\bar{e}}$, we get $pq_1z = p_1z \in P_{\bar{e}}^{\bar{e}}$ and thus $q_1z \in p^{-1}P_{\bar{e}}^{\bar{e}}$. That is, $p^{-1}P_{\bar{e}}^{\bar{e}} \supseteq q_1p_1^{-1}P_{\bar{e}}^{\bar{e}}$. Therefore, $p^{-1}P_{\bar{e}}^{\bar{e}} = q_1p_1^{-1}P_{\bar{e}}^{\bar{e}}$ and $p^{-1,\ e} = q_1p_1^{-1,\ e} = q_1q_2 \cdots q_{k+1}$.

Now let $p \in P_T$ be arbitrary and let $p = p_0 \cdots p_m$ be a properly reduced positive word with $p_j \in P_{v_j}$. We proceed inductively on $l := l(p)$. The case where $l = 0$ is dealt with as above. If $l \geq 1$, take $x \in P$ with $px \in P_{\bar{e}}^{\bar{e}}$ and let $x = x_0 \cdots x_n$ be a properly reduced positive word with $x_i \in P_{w_i}$. It follows from Lemma 3.1.7 that px is not a properly reduced positive word. If $l(x) \geq 1$, by Lemma 3.1.9 and Remark 3.1.10, we must have $w_0 = v_m$ and $p_mx_0 \in P_{\bar{d}}^{\bar{d}}$, where $d \in T$ is the ending edge of the path $[v_{m-1},\ v_m]$. If $l(x) = 0$, i.e., $x \equiv x_0$, then px is a reduced positive word and thus we can still arrange that $w_0 = v_m$ and $p_mx_0 \in P_{\bar{d}}^{\bar{d}}$. In both cases, we have $x_0 \in p_m^{-1}P_{\bar{d}}^{\bar{d}}$. That is, $x_0 = p_m^{-1,\ d}x_0'$ for some $x_0' \in P_{\bar{d}}^{\bar{d}}$. Then

$$px = p_0 \cdots p_mx_0 \cdots x_n = p_0 \cdots p_mp_m^{-1,\ d}x_0' \cdots x_n \in P_{\bar{e}}^{\bar{e}}.$$

Set $\tilde{p} := p_0 \cdots p_m p_m^{-1,\, d}$, then we have $l(\tilde{p}) < l(p)$. The induction hypothesis implies there exists $\tilde{q} \in P$ such that $\tilde{p}^{-1} P_{\bar{e}}^{\bar{e}} = \tilde{q} P_{\bar{e}}^{\bar{e}}$. It follows that $x_0' \cdots x_n \in \tilde{p}^{-1} P_{\bar{e}}^{\bar{e}} = \tilde{q} P_{\bar{e}}^{\bar{e}}$ and thus $x = p_m^{-1,\, d} x_0' \cdots x_n \in p_m^{-1,\, d} \tilde{q} P_{\bar{e}}^{\bar{e}}$. It is now easy to check that $p^{-1} P_{\bar{e}}^{\bar{e}} = q P_{\bar{e}}^{\bar{e}}$ for $q := p_m^{-1,\, d} \tilde{q}$.

□

We extend the notation $p^{-1,\, e}$ introduced in Definition 3.2.1 as follows:

Definition 3.2.5. *We denote by $p^{-1,\, e}$ the element q in Lemma 3.2.4 if $p^{-1} P_{\bar{e}}^{\bar{e}} \neq \emptyset$.*

Whenever $p^{-1} P_{\bar{e}}^{\bar{e}} \neq \emptyset$, the element q is unique. In this case, we have $p^{-1} P_{\bar{e}}^{\bar{e}} = p^{-1,\, e} P_{\bar{e}}^{\bar{e}}$.

Lemma 3.2.6. *Let $p \in P_v$, $x \in P$ such that px is represented by a properly reduced positive word of the form $q_0 q_1 \cdots$ with $q_0 \in P_w$. Let $\ell([v,\, w]) \geq 1$ such that $[v,\, w]$ ends with $f \in T$, then $x \in p^{-1,\, f} P$.*

Proof. As in the proof of Lemma 3.2.4, we can use Lemma 3.1.9 to find positive words $w_m,\, x_m,\, y_m$ and $f_m \in T$ for $1 \leq m \leq n$ such that $w_1 \equiv p$, $x = x_1 y_1$, $w_m = w_{m-1} x_{m-1}$, $y_{m-1} = x_m y_m$, $w_m, x_m \in P_{v_m}$, $w_m x_m \in P_{f_m}^{f_m} \subseteq P_{v_{m+1}}$, $o(f_m) = v_m$, $t(f_m) = v_{m+1}$, and $w_n y_n$ is a properly reduced positive word representing px. Here we allow the possibility that $x_m = \emptyset$ or $y_m = \emptyset$. By construction, we have $w_m = p x_1 \cdots x_{m-1}$, $x = x_1 \cdots x_m y_m$ and $w_m x_m y_m = px$. By Corollary 3.1.8, we get $v_n = w$.

Let M be minimal such that $v_M = w$, then we must have $f_{M-1} = f$. As a result, $p x_1 \cdots x_{M-1} = w_{M-1} x_{M-1} \in P_{\bar{f}}^{\bar{f}} = P_f^f$. That is, $x_1 \cdots x_{M-1} \in p^{-1,\, f} P_{\bar{f}}^{\bar{f}}$. Therefore, $x = x_1 \cdots x_{M-1} y_{M-1} \in p^{-1,\, f} P$.

□

Looking at the way $p^{-1,\,f}$ has been constructed in the proof of Lemma 3.2.4, the following is an immediate consequence.

Lemma 3.2.7. *In the situation of Lemma 3.2.6, assume that $[v,\ w]$ starts with $d \in T$, then $x \in p^{-1,\,d}P$.*

Let $\prec_T$ and $\vee_T$ be the analogues of $\prec$ and $\vee$ with P_T in place of P.

Proposition 3.2.8. *Given $p,\ q \in P_T$, $pP_T \cap qP_T = \emptyset$ if and only if $pP \cap qP = \emptyset$, and $p \vee_T q$ exists if and only if $p \vee q$ exists. In the latter case, we have $p \vee_T q = p \vee q$.*

Moreover, P is right LCM if and only if P_T is right LCM.

Proof. Given $p,\ q \in P_T$, it is clear that $pP_T \cap qP_T \neq \emptyset$ implies $pP \cap qP \neq \emptyset$. If $pP \cap qP \neq \emptyset$, we can find $x,\ y \in P$ with $px = qy$. Let $x = p_0 d_1 p_1 \cdots$ and $y = q_0 e_1 q_1 \cdots$ be positive words in compact form, then we have $pp_0 d_1 p_1 \cdots = qq_0 e_1 q_1 \cdots$. By Lemma 3.1.2 (ii), we get either $pp_0 a = qq_0$ or $pp_0 = qq_0 a$ for some $a \in P_T$. This implies $pP_T \cap qP_T \neq \emptyset$.

If $p \vee q$ exists, i.e., $pP \cap qP$ admits a minimal element, take $x,\ y$ be as above. We obtain $p \vee q \in P_T$ since either $pp_0 a = qq_0$ or $pp_0 = qq_0 a$ for some $a \in P_T$. Hence $p \vee q \in pP_T \cap qP_T$. On the other hand, assume $px' = qy'$ with $x',\ y' \in P_T$, then we have $p \vee q \prec px'$ by definition. That is, $p \vee q$ is the minimal element in $pP_T \cap qP_T$. Therefore, $p \vee_T q$ exists and $p \vee_T q = p \vee q$.

If $p \vee_T q$ exists, i.e., $pP_T \cap qP_T$ admits a minimal element, take $x,\ y$ be as above. We have either $p \vee_T q \prec qq_0 \prec qy = px$ or $p \vee_T q \prec pp_0 \prec px = qy$. This means $p \vee_T q$ is the minimal element in $pP \cap qP$, i.e., $p \vee q$ exists and $p \vee q = p \vee_T q$.

We have already shown P_T is right LCM if P is right LCM. Now we prove the converse. If $p,\ q \in P_T$, we have either $pP \cap qP = \emptyset$ or $p \vee q$ exists since P_T is right LCM.

If $p \in P_T$ and $q = q_0e_1q_1 \cdots e_nq_n$ is a positive word in compact form, we proceed inductively on n to show $p \vee q$ exists if $pP \cap qP \neq \emptyset$. The case where $n = 0$ is done. Now assume $n \geq 1$. Noting $pP \cap q_0P \supseteq pP \cap qP \neq \emptyset$, we can find $r \in P_T$ with $p \vee q_0 = q_0r$. Then $p \vee q = p \vee q_0 \vee q = q_0r \vee q = q_0(r \vee e_1q_1 \cdots e_nq_n)$ exists if and only if $r \vee e_1q_1 \cdots e_nq_n$ exists. To show the latter, take $x \in P$ with $rx \in e_1P$. A similar argument involving Lemma 3.1.9 as in the proof of Lemma 3.2.4 implies that we have a decomposition $x = x'x''$ such that $rx' \in P_{\bar{e}_1}^{\bar{e}_1}$ and that $x'' \in e_1P$. By Lemma 3.2.4, we get $x' \in r^{-1,\, e_1}P_{\bar{e}_1}^{\bar{e}_1}$, i.e., $x' = r^{-1,\, e_1}y^{\bar{e}_1}$ for some $y \in P_{\bar{e}_1}$. Let $rr^{-1,\, e_1} = a^{\bar{e}_1},\ a \in P_{\bar{e}_1}$.

If $e_1 \in A_-$, we have $e_1 = a^{\bar{e}_1}e_1(a^{e_1})^{-1} = rr^{-1,\, e_1}e_1(a^{e_1})^{-1} \in rP$. Therefore, $r \prec e_1 \prec e_1q_1 \cdots e_nq_n$ and thus $r \vee e_1q_1 \cdots e_nq_n = e_1q_1 \cdots e_nq_n$.

If $e_1 \in A_+$, we have $a^{\bar{e}_1}e_1 = e_1a^{e_1}$ and $y^{\bar{e}_1}e_1 = e_1y^{e_1}$. Then $rx = rr^{-1,\, e_1}y^{\bar{e}_1}e_1 \cdots = e_1a^{e_1}y^{e_1} \cdots \in e_1a^{e_1}P$ and hence $r \vee e_1 = e_1a^{e_1}$. Therefore,

$$r \vee e_1q_1 \cdots e_nq_n = (r \vee e_1) \vee e_1q_1 \cdots e_nq_n = e_1a^{e_1} \vee e_1q_1 \cdots e_nq_n = e_1(a^{e_1} \vee q_1 \cdots e_nq_n).$$

In this case, $r \vee e_1q_1 \cdots e_nq_n$ exists if and only if $a^{e_1} \vee q_1 \cdots e_nq_n$ exists. $rP \cap e_1q_1 \cdots e_nq_nP \neq \emptyset$ since $q_0(rP \cap e_1q_1 \cdots e_nq_nP) = pP \cap qP \neq \emptyset$ and thus $a^{e_1}P \cap q_1 \cdots e_nq_nP \neq \emptyset$ since $e_1(a^{e_1}P \cap q_1 \cdots e_nq_nP) = rP \cap e_1q_1 \cdots e_nq_nP$. By induction hypothesis, $a^{e_1} \vee q_1 \cdots e_nq_n$ exists.

Now let $p,\ q \in P$ and let $p_0 d_1 p_1 \cdots d_m p_m$ and $q_0 e_1 q_1 \cdots e_n q_n$ be positive words in compact form representing $p,\ q$. Without loss of genrality, assume $n \geq m$. If $pP \cap qP \neq \emptyset$, there exists $x,\ y \in P$ such that $px = qy$. Comparing the compact forms of px and qy, we get, by Lemma 3.1.2 (ii), $d_i = e_i,\ 1 \leq i \leq m$ and either $p_0 d_1 p_1 \cdots d_m = q_0 e_1 q_1 \cdots e_m a$ or $p_0 d_1 p_1 \cdots d_m a = q_0 e_1 q_1 \cdots e_m$ holds for some $a \in P_T$. In the first case, we have

$$p \vee q = q_0 e_1 q_1 \cdots e_m a p_m \vee q_0 e_1 q_1 \cdots e_n q_n = q_0 e_1 q_1 \cdots e_m (a p_m \vee q_m e_m q_{m+1} \cdots).$$

In the second case, we get

$$p \vee q = p_0 d_1 p_1 \cdots d_m p_m \vee p_0 d_1 p_1 \cdots d_m a q_m e_m q_{m+1} \cdots = p_0 d_1 p_1 \cdots d_m (p_m \vee a q_m e_m q_{m+1} \cdots).$$

In both cases, we can conclude $p \vee q$ exists by the argument in the case where $p \in P_T$.

□

Proposition 3.2.9. *P_T is right LCM.*

Proof. Firstly, assume $p \in P_v,\ q \in P_w$ and $pP \cap qP \neq \emptyset$, and we show inductively on $\ell([v,\ w])$ that $p \vee q$ exists. When $\ell([v,\ w]) = 0$, either $p \prec q$ or $q \prec p$. In both cases, it is clear that $p \vee q$ exists. Now we consider the case when $v \neq w$ and assume $[v,\ w]$ starts with d and ends with f.

Suppose that $x,\ y \in P_T$ satisfy $px = qy$, we can find, by Lemma 3.1.9, positive words $w_m,\ x_m,\ y_m$ and $f_m \in T$ for $1 \leq m \leq n$ such that $w_1 \equiv p$, $x = x_1 y_1$, $w_m = w_{m-1} x_{m-1}$, $y_{m-1} = x_m y_m$, $w_m, x_m \in P_{v_m}$, $w_m x_m \in P_{f_m}^{f_m} \subseteq P_{v_{m+1}}$, $o(f_m) = v_m$, $t(f_m) = v_{m+1}$, and $w_n y_n$ is a properly reduced positive word representing px. Here we allow the possibility that $x_m = \emptyset$ or $y_m = \emptyset$. By Lemma 3.1.9, we can find similarly positive words $w'_{m'},\ x'_{m'},\ y'_{m'}$ and $f'_{m'} \in T$ for $1 \leq m' \leq n'$ such that $w'_1 \equiv q$, $y = x'_1 y'_1$, $w'_{m'} = w'_{m'-1} x'_{m'-1}$, $y'_{m'-1} = x'_{m'} y'_{m'}$, $w'_{m'}, x'_{m'} \in P_{v'_{m'}}$, $w'_{m'} x'_{m'} \in P_{f'_{m'}}^{f'_{m'}} \subseteq P_{v'_{m'+1}}$,

$o(f'_{m'}) = v'_{m'}$, $t(f'_{m'}) = v'_{m'+1}$, and $w'_{n'}y'_{n'}$ is a properly reduced positive word representing qy. As before, we allow the possibility that $x'_{m'} = \emptyset$ or $y'_{m'} = \emptyset$. It follows from Corollary 3.1.8 that $v_n = v'_{n'}$. Assume the paths $v_1,\ v_2,\ \cdots$ and $v'_1,\ v'_2,\ \cdots$ meet for the first time at $u \in V$, then we must have $u \in [v,\ w]$. So we have $x = x'x''$ and $y = y'y''$ such that $px',\ qy' \in P_u$. Note that P_u is the positive cone of the totally ordered group G_u, we conclude $px'z = qy'$ or $px' = qy'z$ for some $z \in P_u$. In the first case, we have $qy'y'' = px'zy'' = px'x''$ and thus $zy'' = x''$. So we have the decomposition $x = (x'z)y''$ and $y = y'y''$ with $p(x'z) = qy'$. In the second case, we have $px'x'' = qy'zx'' = qy'y''$ and thus $y'' = zx''$. So we have the decomposition $y = (y'z)x''$ and $x = x'x''$ with $px' = qy'z$. Therefore, we may assume, without loss of generality, $px' = qy'$.

a) For $x,\ y \in P$ with $px = qy$ such that $u \in [v,\ w] \setminus \{v,\ w\}$, we obtain as in the proof of Lemma 3.2.4 that $x \in p^{-1,\ d}P$ and $y \in q^{-1,\ \bar{f}}P$.

b) For $x,\ y \in P$ with $px = qy$ such that $u = v$, a similar argument as in the proof of Lemma 3.2.4 and Lemma 3.2.6 yields $y \in q^{-1,\ \bar{f}}P$ and $y' \in q^{-1,\ \bar{d}}P$.
If $qq^{-1,\ \bar{d}} \prec p$ in P_v, then we have $q \prec qq^{-1,\ \bar{d}} \prec p$.
If $p \prec qq^{-1,\ \bar{d}}$ in P_v, i.e., $qq^{-1,\ \bar{d}} = pz$ for some $z \in P_v$, we have $pz = qq^{-1,\ \bar{d}} \in P_d^d = P_{\bar{d}}^{\bar{d}} \subseteq P_v$ and thus $z \in p^{-1,\ d}P$. Therefore, $px \in px'P = qy'P \subseteq qq^{-1,\ \bar{d}}P = pzP \subseteq pp^{-1,\ d}P$ and thus $x \in p^{-1,\ d}P$.

c) For $x,\ y \in P$ with $px = qy$ such that $u = w$, similarly as in b), we have either $p \prec q$ or $x \in p^{-1,\ d}P$ and $y \in q^{-1,\ \bar{f}}P$.

In conclusion, one of the following is satisfied: $p \prec q$; $q \prec p$; For all $x,\ y \in P$ with $px = qy$, we have $x \in p^{-1,\ d}P$ and $y \in q^{-1,\ \bar{f}}P$.

Noting that $pp^{-1,\ d} \in P_{\bar{d}}^{\bar{d}} = P_d^d \subseteq P_{t(d)}$ and $qq^{-1,\ \bar{f}} \in P_f^f = P_{\bar{f}}^{\bar{f}} \subseteq P_{o(f)}$ with $\ell([t(d),\ o(f)]) <$

$\ell([v,\ w])$, and that $px = qy \in pp^{-1,\,d}P \cap qq^{-1,\,\bar{f}}P$, we conclude $pp^{-1,\,d} \vee qq^{-1,\,\bar{f}}$ exists by induction hypothesis. Therefore, we have $p \vee q = p$ or $p \vee q = q$ or $p \vee q = pp^{-1,\,d} \vee qq^{-1,\,\bar{f}}$.

Now we assume $p \in P_v$ and $q = q_0 q_1 \cdots q_n \in P_T$ is a properly reduced positive word with $q_j \in P_{w_j}$ such that $pP \cap qP \neq \emptyset$, and we proceed inductively on $l(q)$ to show $p \vee q$ exists. The case where $n = 0$ is done, so we may assume $n \geq 1$.

If $v = w_0$, we have either $p = q_0 z$ or $q_0 = pz$ for some $z \in P_v$ since G_v is totally ordered and P_v is the corresponding positive cone. In the first case, $p \vee q = (q_0 z) \vee (q_0 q_1 \cdots q_n) = q_0(z \vee q_1 \cdots q_n)$ exists if and only if $z \vee q_1 \cdots q_n$ exists. $zP \cap q_1 \cdots q_n P \neq \emptyset$ since $q_0(zP \cap q_1 \cdots q_n P) = pP \cap qP \neq \emptyset$. Noting $l(q_1 \cdots q_n) < l(q)$, we obtain $z \vee q_1 \cdots q_n$ exists by induction hypothesis. In the second case, $p \prec q_0 \prec q$ and thus $p \vee q = q$.

If $v \neq w_0$, for all $x,\ y \in P_T$ in the form of properly reduced positive words such that $px = qy$, we have, by Corollary 3.1.8, either px or qy is not properly reduced. If qy is not properly reduced, then either $l(y) = 0$ or $y \in q_n^{-1,\,\bar{e}}P$ by Lemma 3.1.9, where e is the ending edge of the path $[w_{n-1},\ w_n]$. Moreover, when $l(y) = 0$, we have either qy' is properly reduced for some $P_w \in y' = y$ or $y \in q_n^{-1,\,\bar{e}}P$. If qy is properly reduced while px is not properly reduced, suppose that $[v,\ w_0]$ ends with e_0. By Lemma 3.2.6, we have $x \in p^{-1,\,e_0}P$, i.e., $x = p^{-1,\,e_0}x_1$ for some $x_1 \in P_{w_0}$. Also, we have $pp^{-1,\,e_0} \in P_{w_0}$. If $pp^{-1,\,e_0} \prec q_0$, then $p \prec pp^{-1,\,e_0} \prec q_0 \prec q$ and thus $p \vee q = q$. If $q_0 \prec pp^{-1,\,e_0}$, i.e., $pp^{-1,\,e_0} = q_0 p_1$ for some $p_1 \in P_{w_0}$, then we have $qy = px = pp^{-1,\,e_0}x_1 = q_0 p_1 x_1$ and thus $q_1 \cdots q_n y = p_1 x_1$. Let $q_0^{(1)} q_1^{(1)} \cdots q_{n_1}^{(1)} y$ be the properly reduced positive word representing $q_1 \cdots q_n y$ obtained as in the proof of Lemma 3.1.6, then we have $q_{n_1}^{(1)} \in P_e^e q_n$, where e is as above. Again, Lemma 3.2.6 yields that $x_1 \in p_1^{-1,\,e_1}P$, i.e., $x_1 = p_1^{-1,\,e_1}x_2$, where $e_1 \in T$ lies in $[w_1,\ w_2][w_2,\ w_3] \cdots [w_{n-1},\ w_n]$. If $p_1 p_1^{-1,\,e_1} \prec q_0^{(1)}$, then we have $p_1 \prec p_1 p_1^{-1,\,e_1} \prec q_0^{(1)} \prec q_0^{(1)} q_1^{(1)} \cdots q_{n_1}^{(1)} = q_1 \cdots q_n$ and thus $p \prec pp^{-1,\,e_0} = q_0 p_1 \prec q_0 q_1 \cdots q_n = q$. Otherwise, we can continue in this way. Unless $p \prec q$, we obtain elements $x_\lambda \in P$ and $p_\lambda \in P^{e_{\lambda-1}}_{e_{\lambda-1}}$ such that $x_\lambda = p_\lambda^{-1,\, e_\lambda} x_{\lambda+1}$, $p_\lambda p_\lambda^{-1,\, e_\lambda} = q_0^\lambda p_{\lambda+1}$ and $q_0^{(\lambda)} \cdots q_{n_\lambda}^{(\lambda)} y = p_\lambda x_\lambda$, where $e_\lambda \in T$ lies in $[w_1,\, w_2][w_2,\, w_3] \cdots [w_{n-1},\, w_n]$ and $q_0^{(\lambda)} \cdots q_{n_\lambda}^{(\lambda)} y$ is a properly reduced positive word representing $q_1^{(\lambda-1)} \cdots q_{n_{\lambda-1}}^{(\lambda-1)} y$ with $q_n^{(\lambda)} \in P^e_e q_n$. We end up with $q_0^{(\nu)} y = p_\nu x_\nu$. Again, Lemma 3.2.6 implies $x_\nu \in p_\nu^{-1,\, e} P$ and thus $p_\nu p_\nu^{-1,\, e} = q_0^{(\nu)} p_{\nu+1}$ by assumption. Since $p_\nu p_\nu^{-1,\, e} \in P^{\bar{e}}_{\bar{e}} = P^e_e$, we have $p_{\nu+1} \in (q_0^{(\nu)})^{-1,\, \bar{e}} P$. Therefore, $q_0^{(\nu)} y = p_\nu x_\nu \in p_\nu p_\nu^{-1,\, e} P = q_0^{(\nu)} p_{\nu+1} P$ and thus $y \in p_{\nu+1} P \subseteq (q_0^{(\nu)})^{-1,\, \bar{e}} P = q_n^{-1,\, \bar{e}} P$.

In conclusion, when $v \neq w_0$, either $p \prec q$ or $y \in q_n^{-1,\, \bar{e}} P$ for all $x,\ y \in P_T$ in the form of properly reduced positive words such that $px = qy$. That is, $p \vee q = q$ or $p \vee q = p \vee q q_n^{-1,\, \bar{e}}$. In the latter case, it is easy to see that $pP \cap q q_n^{-1,\, \bar{e}} P \neq \emptyset$ and $l(q q_n^{-1,\, \bar{e}}) < l(q)$, which yields by induction hypothesis that $p \vee q q_n^{-1,\, \bar{e}}$ exists and thus $p \vee q$ exists.

Lastly, we assume $p,\ q \in P_T$ with $pP \cap qP \neq \emptyset$ and let $p = p_0 p_1 \cdots p_m$ and $q = q_0 q_1 \cdots q_n$ be properly reduced positive words with $p_i \in P_{v_i}$ and $q_j \in P_{w_j}$. We prove inductively on $l(p) + l(q)$ that $p \vee q$ exists. The case where $m = 0$ or $n = 0$ is done, so we assume $m,\ n \geq 1$.

Suppose that $x,\ y \in P_T$ satisfy $px = qy$. If $x,\ y$ are expressed as properly reduced positive words such that px and qy are properly reduced, by Corollary 3.1.8, we have either $p_0 = q_0 z$ or $q_0 = p_0 z$ since every semigroup $P_v,\ v \in V$ is a positive cone of the totally ordered group G_v. If $p_0 = q_0 z$, then $p = p_0 p_1 \cdots p_m = q_0 z p_1 \cdots p_m$, and $p \vee q = (q_0 z p_1 \cdots p_m) \vee (q_0 q_1 \cdots q_n) = q_0((z p_1 \cdots p_m) \vee (q_1 \cdots q_n))$ exists if and only if $(z p_1 \cdots p_m) \vee (q_1 \cdots q_n)$ exists. The latter now follows from induction hypothesis as $z p_1 \cdots p_m P \cap q_1 \cdots q_n P \neq \emptyset$ and $z p_1 \cdots p_m,\, q_1 \cdots q_n$ can be expressed as properly reduced positive words with smaller l. The case $q_0 = p_0 z$ is analogous.

It remains to consider the case that for all properly reduced positive words $x,\ y \in P_T$ with $px = qy$, either px or qy is not properly reduced. As we proceed inductively on $l(p) + l(q)$, we may assume $v_0 \neq w_0$.

If qy is not properly reduced, we have either qy' is properly reduced for some $y' = y$ or $y \in q_n^{-1,\,\bar{e}}P$ as in the case where $m = 0$, where e is still the ending edge of the path $[w_{n-1},\ w_n]$.

If qy is properly reduced while px is not properly reduced, a similar argument entails $x \in p_m^{-1,\,d}P$, where d is the ending edge of the path $[v_{m-1},\ v_m]$. Let $x = p_m^{-1,\,d}x_1$ such that x_1 is a properly reduced positive word and let $p_0^{(1)} \cdots p_{m_1}^{(1)}$ be the properly reduced positive word representing $p_0 p_1 \cdots p_m p_m^{-1,\,d}$ such that $p^{(1)_0} \in P_{v_0}$, then we have $qy = px = p_0^{(1)} \cdots p_{m_1}^{(1)} x_1$. By Corollary 3.1.8, we have $p_0^{(1)} \cdots p_{m_1}^{(1)} x_1$ is not properly reduced and thus $x_1 \in (p_{m_1}^{(1)})^{-1,\,d_1}$ for some $d_1 \in T$. Noting $l(p_0^{(1)} \cdots p_{m_1}^{(1)}) < l(p)$, continue the process as above and we can obtain finally $qy = p_0^{\nu} x_{\nu}$ with $p_0^{\nu} \in P_{v_0}$. As shown in the case where $m = 0$, we have either $p_0^{\nu} \prec q$ or $y \in q_n^{-1,\,\bar{e}}P$. In the first case, we have $p \prec p_0^{(1)} \cdots p_{m_1}^{(1)} \prec \cdots \prec p_0^{\nu} \prec q$.

In conclusion, for all properly reduced positive words $x,\ y \in P_T$ with $px = qy$, we have $y \in q_n^{-1,\,\bar{e}}P$ unless $p \prec q$. In this case, $p \vee q = p \vee q q_n^{-1,\,\bar{e}}$ with $l(q q_n^{-1,\,\bar{e}}) < l(q)$ and the latter exists by induction hypothesis.

$\square$

Proposition 3.2.8 and Proposition 3.2.9 entails Lemma 3.2.3.

Chapter 4

Amenability and Nuclearity

Having defined our monoid P in Chapter 3, we can now start to study its reduced semigroup C^*-algebra $C^*_\lambda(P)$. Nuclearity, as a kind of finite approximation property of C^*-algebras, can rarely be ignored when referring to the properties of C^*-algebras. In this chapter, we will firstly discuss the nuclearity of our semigroup C^*-algebras of graphs of monoids and then give some examples to show the embeddability of monoids into amenable groups when the corresponding semigroup C^*-algebras are nuclear.

4.1 Nuclearity

Let P be a graph of monoids and assume that we are in the same setting as in Section 3.1 and that condition (LCM) is satisfied. For the nuclearity of the reduced semigroup C^*-algebra $C^*_\lambda(P)$, we have the following theorem.

Theorem 4.1.1. *$C^*_\lambda(P)$ is nuclear if $C^*_\lambda(P_T)$ is nuclear.*

Before giving the proof, we need to introduce the following notions, all of which come from [Exe94].

Let B be a C^*-algebra and let α be a group action of the unit circle $\mathbb{T}$ on B.

The spectral spaces: For $n \in \mathbb{Z}$, the nth spectral subspace for α, denoted by B_n, is defined by:

$$B_n := \{b \in B |\ \alpha(z)(b) = z^n b,\ \forall z \in \mathbb{T}\}.$$

Semi-saturated: α is called semi-saturated if B is generated, as a C^*-algebra, by the union of B_0 and B_1.

Stable: α is called stable if there exist a C^*-algebra B' with $B = B' \otimes \mathscr{K}$ and a circle action α' on B' such that α is the tensor product of α' by the trivial circle action on $\mathscr{K}$.

Regular: α is called regular if there exists an isomorphism $\varphi:\ B_1^* B_1 \to B_1 B_1^*$ and a surjective linear isometry $\psi:\ B_1^* \to B_1 B_1^*$ such that for all $x,\ y \in B_1$, $a \in B_1^* B_1$ and $b \in B_1 B_1^*$, we have

(i) $\psi(x^* b) = \psi(x^*) b$;

(ii) $\psi(a x^*) = \varphi(a) \psi(x^*)$;

(iii) $\psi(x^*)^* \psi(y^*) = x y^*$;

(iv) $\psi(x^*) \psi(y^*)^* = \varphi(x^* y)$.

Now we are ready to prove Theorem 4.1.1.

Proof. By Proposition 3.2.3, P is right LCM. We have the following expression:

$$C_\lambda^*(P) = \overline{\operatorname{span}\{\lambda_p\lambda_q^*,\ p,\ q \in P\}}.$$

Here the set $\{\lambda_p\lambda_q^*,\ p,\ q \in P\}$ is linearly independent.

Let $\theta:\ P \to \mathbb{N}$ be a semigroup homomorphism such that $\theta(e) = 1$ for all $e \in A$ and that $\theta(x) = 0$ for all $x \in P_T$. Define a unitary $u_z, z \in \mathbb{T}$ on $\ell_2(P)$ by

$$u_z(\delta_x) = z^{\theta(x)}\delta_x,\ x \in P,$$

then $\mathrm{Ad}(u_z)$ is a $*$-isomorphism on $C_\lambda^*(P)$. Furthermore, we have

$$\mathrm{Ad}(u_z)(\lambda_p\lambda_q^*) = z^{-\theta(p)+\theta(q)}\lambda_p\lambda_q^*.$$

Define an action α of $\mathbb{T}$ on $C_\lambda^*(P)$ by $\alpha(z) := \mathrm{Ad}(u_{\bar{z}})$, then the kth spectral subspace for α is given by:

$$B_k = \overline{\operatorname{span}\{\lambda_p\lambda_q^*,\ \theta(p)-\theta(q) = k,\ p,\ q \in P\}},\ k \in \mathbb{Z}.$$

It is easy to get $B_k = B_1^k,\ k \in \mathbb{Z}^*$, which implies, by [Exe94, Proposition (4.8)], the action α is semi-saturated.

If α is regular, by [Exe94, Theorem 4.21], $C_\lambda^*(P)$ is isomorphic to a partial crossed product of B_0 by a partial automorphism. In this case, $C_\lambda^*(P)$ is nuclear if and only if B_0 is nuclear.

If α is not regular, tensor it by the trivial circle action on $\mathscr{K}$, we get a stable action α'. Furthermore, α' is still semi-saturated. This implies α' is regular by [Exe94, Corollary 4.5]. Again by [Exe94, Theorem 4.21], $C_\lambda^*(P) \otimes \mathscr{K}$ is isomorphic to a partial crossed product of

$B_0 \otimes \mathscr{K}$ by a partial automorphism. In this case, $C^*_\lambda(P)$ is nuclear if and only if $B_0 \otimes \mathscr{K}$ is nuclear. While the latter holds if and only if B_0 is nuclear. Therefore, $C^*_\lambda(P)$ is nuclear if and only if B_0 is nuclear.

For $p,\ q \in P$, let

$$p = h_0a_1h_1a_2\cdots h_{k-1}a_kh_k, h_{i-1} \in P_T,\ a_i \in A,\ 1 \leq i \leq k,\ h_k \in G_{a_k}^{a_k}P_T$$

and

$$q = h'_0a'_1h'_1a'_2\cdots h'_{l-1}a'_lh'_l, h'_{j-1} \in P_T,\ a'_j \in A,\ 1 \leq j \leq l,\ h'_l \in G_{a'_l}^{a'_l}P_T$$

be the compact forms. We say $p \sim q$ if

$$h_0a_1h_1a_2\cdots h_{k-1}a_kG_{a_k}^{a_k} = h'_0a'_1h'_1a'_2\cdots h'_{l-1}a'_lG_{a'_l}^{a'_l}.$$

Alternatively, $p \sim q$ if $k = l$, $a_i = a'_i$ for all $1 \leq i \leq k$ and there exists $x \in G_{a_k}^{a_k}$ such that

$$h_0a_1h_1a_2\cdots h_{k-1}a_k = h'_0a'_1h'_1a'_2\cdots h'_{l-1}a'_lx.$$

It is easy to check that $\sim$ is a well-defined equivalent relation in P.

For $p \in P$ with a compact form as above, define

$$\bar{p} := h_0a_1h_1a_2\cdots h_{k-1}a_k.$$

Then $\bar{p}$ is unique up to the equivalent relation $\sim$. Moreover, for all $p,\ q \in P$, $p \sim q$ if and only if $\bar{p} \sim \bar{q}$.

Let $P_l := \{p \in P,\ \theta(p) = l\},\ l \in \mathbb{N}$ and let

$$B_{0,\ l} := \overline{\operatorname{span}\{\lambda_p \lambda_q^*,\ \ p,\ q \in P_l\}},$$

then $B_{0,\ l}$, restricted on $\ell_2(\cup_{k<l} P_k)$, is 0. Therefore, we can regard $B_{0,\ l}$ as a C^*-algebra on the Hilbert space $\ell_2(\cup_{k \geq l} P_k)$.

When $A_- = \emptyset$, $\lambda_p \lambda_q^*$ is of the form $\lambda_{\bar{p}} \lambda_h \lambda_{h'}^* \lambda_{\bar{q}}^*,\ h,\ h' \in P_T$. Furthermore, we have in $B_{0,\ l}$,

$$\lambda_{\bar{p}_1} \lambda_{h_1} \lambda_{h_1'}^* \lambda_{\bar{q}_1}^* \cdot \lambda_{\bar{p}_2} \lambda_{h_2} \lambda_{h_2'}^* \lambda_{\bar{q}_2}^* = \begin{cases} \lambda_{\bar{p}_1} \lambda_{h_1} \lambda_{h_1'}^* \lambda_x^* \lambda_{h_2} \lambda_{h_2'}^* \lambda_{\bar{q}_2}^*, & \bar{q}_1 = \bar{p}_2 x,\ x \in P_T, \\ \lambda_{\bar{p}_1} \lambda_{h_1} \lambda_{h_1'}^* \lambda_x \lambda_{h_2} \lambda_{h_2'}^* \lambda_{\bar{q}_2}^*, & \bar{q}_1 x = \bar{p}_2,\ x \in P_T, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$H_l := \begin{cases} \ell_2(m), & |\{\bar{p},\ \theta(p) = l\}/\sim| = m < \infty, \\ \ell_2(\infty), & |\{\bar{p},\ \theta(p) = l\}/\sim| = \infty, \end{cases}$$

and define a linear map

$$V:\ H_l \otimes \ell_2(P)\ \to \ell_2(\cup_{k \geq l} P_k)$$

by sending $\delta_{\bar{p}} \otimes \delta_x$ to $\delta_{\bar{p}x}$, then V is a unitary.

Let

$$K_l := \begin{cases} M_m, & |\{\bar{p},\ \theta(p) = l\}/\sim| = m < \infty, \\ K, & |\{\bar{p},\ \theta(p) = l\}/\sim| = \infty, \end{cases}$$

then the map

$$\varphi:\ B_{0,\ l} \to K_l \otimes \mathscr{L}(\ell_2(P)),\ T \mapsto V^* T V$$

is an injective $*$-homomorphism. Furthermore, it maps $\lambda_{\bar{p}} \lambda_h \lambda_{h'}^* \lambda_{\bar{q}}^*$ to $E_{\bar{p},\ \bar{q}} \otimes \lambda_h \lambda_{h'}^*$ and hence

$\varphi(B_{0,\,l}) = K_l \otimes C^*(\lambda(P_T))$. Noting that $P = \oplus_{x\in P_T\backslash P} P_T x$ and that every subspace $\ell_2(P_T x)$ is P_T-invariant, we have $C^*(\lambda(P_T)) \cong C^*_\lambda(P_T)$. Since $C^*_\lambda(P_T)$ is nuclear, so is $B_{0,\,l}$.

When $A_- \neq \emptyset$, for $p \in P$ with the expression

$$p = h_0 a_1 h_1 a_2 \cdots h_{k-1} a_k h_k, h_{i-1} \in P_T,\ a_i \in A,\ 1 \leq i \leq k,\ h_k \in G_{a_k}^{a_k} P_T,$$

define

$$X_p := \{x \in P_{a_k}^{a_k},\ \bar{p}x^{-1} \in P\}.$$

If $X_p \neq \{\varepsilon\}$, then there must exist a sequence $(x_p^{(n)})_{n\in\mathbb{N}} \subseteq X_p$ with $x_p^{(n)} \prec x_p^{(n+1)}$ such that for all $x \in X_p$, $x \prec x_p^{(n)}$ for some $n \in \mathbb{N}$ since every group $G_v,\ v \in V$ is totally ordered. For each $n \in \mathbb{N}$, let

$$p^{(n)} := \begin{cases} \bar{p}, & X_p = \{\varepsilon\}, \\ \bar{p}(x_p^{(n)})^{-1}, & X_p \neq \{\varepsilon\}. \end{cases}$$

Define

$$B_{0,\,l}^{(n)} := \overline{\operatorname{span}\{\lambda_{p^{(n)}} \lambda_h \lambda_{h'}^* \lambda_{q^{(n)}}^*,\ p,\ q \in P_l,\ h,\ h' \in P_T\}}$$

and define

$$K_l^{(n)} := \begin{cases} M_m, & |\{p^{(n)},\ \theta(p) = l\}/\sim| = m < \infty, \\ K, & |\{p^{(n)},\ \theta(p) = l\}/\sim| = \infty. \end{cases}$$

Similarly as in the case when $A_- = \emptyset$, we obtain $B_{0,\,l}^{(n)} \cong K_l^{(n)} \otimes C^*(\lambda(P_T))$, which means $B_{0,\,l}^{(n)}$ is nuclear. Noting that there is an injective $*$-homomorphism from $B_{0,\,l}^{(n)}$ to $B_{0,\,l}^{(n+1)}$, sending $\lambda_{p^{(n)}} \lambda_h \lambda_{h'}^* \lambda_{q^{(n)}}^*$ to

$$\begin{cases} \lambda_{p^{(n+1)}}\lambda_{x_p^{(n+1)}(x_p^{(n)})^{-1}h}\lambda^*_{x_q^{(n+1)}(x_q^{(n)})^{-1}h'}\lambda^*_{q^{(n+1)}}, & \text{if } X_p \neq \{\varepsilon\},\ X_q \neq \{\varepsilon\}, \\ \lambda_{p^{(n+1)}}\lambda_{x_p^{(n+1)}(x_p^{(n)})^{-1}h}\lambda^*_{h'}\lambda^*_{q^{(n+1)}}, & \text{if } X_p \neq \{\varepsilon\},\ X_q = \{\varepsilon\}, \\ \lambda_{p^{(n+1)}}\lambda_h\lambda^*_{x_q^{(n+1)}(x_q^{(n)})^{-1}h'}\lambda^*_{q^{(n+1)}}, & \text{if } X_p = \{\varepsilon\},\ X_q \neq \{\varepsilon\}, \\ \lambda_{p^{(n+1)}}\lambda_h\lambda^*_{h'}\lambda^*_{q^{(n+1)}}, & \text{if } X_p = \{\varepsilon\},\ X_q = \{\varepsilon\}, \end{cases}$$

for $p = h_0a_1h_1a_2\cdots h_{k-1}a_kh_k$ and $q = h'_0a'_1h'_1a'_2\cdots h'_{j-1}a'_jh'_j$, we conclude that $B_{0,\ l} = \overline{\cup_{n\in\mathbb{N}}B^{(n)}_{0,\ l}}$ is nuclear as an inductive limit of nuclear C^*-algebras.

Define

$$B_{0,\ \leq l} := \sum_{0\leq k\leq l} B_{0,\ k},$$

we have $B_{0,\ l},\ l \geq 1$ is an ideal in $B_{0,\ \leq l}$ and the corresponding quotient is a quotient of $B_{0,\ \leq(l-1)}$. Since quotients and extensions of C^*-algebras preserve nuclearity, we get, by induction, $B_{0,\ \leq l}$ is nuclear. As an inductive limit of nuclear C^*-algebras, $B_0 = \overline{\cup_{l\geq 0}B_{0,\ \leq l}}$ is nuclear. Therefore, $C^*_\lambda(P)$ is nuclear.

□

Remark 4.1.2. *If $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R},\ +)$ and we are in the same setting as in Theorem 5.2.11, then the set X_p defined in the proof above is either $\{\varepsilon\}$ or a monoid isomorphic to $\mathbb{Z}_{\geq 0}$. In this case, the sequence $(x_p^{(n)})_{n\in\mathbb{N}} \subseteq X_p$ can be chosen to depend only on a_k, regardless of p.*

4.2 Amenability

It is well known that a reduced group (groupoid) C^*-algebra is nuclear if and only if the group (groupoid, respectively) is amenable, while we do not have an analogue in the semi-group case. Theorem 2.2.4 motivates us to study it via groupoids and groupoid C^*-algebras.

Let P be a subsemigroup of a group G, it follows from Theorem 2.2.4 that $C^*_\lambda(P)$ is nuclear if and only if the groupoid $G \ltimes \Omega_P$ is amenable. By [Exe15, Theorem 20.7] and [Theorem 25.10], we get that $G \ltimes \Omega_P$ is amenable if G is amenable. Therefore, $C^*_\lambda(P)$ is nuclear if P embeds into an amenable group. However, we do not know whether the converse is true or not. In this section, we give some examples where the converse is true.

Theorem 4.1.1 states $C^*_\lambda(P)$ is nuclear if $C^*_\lambda(P_T)$ is nuclear, where P is a graph of monoids. In this section, we give examples of some special graph of monoids P such that $C^*_\lambda(P_T)$ is nuclear and then try to embed these P into amenable groups.

4.2.1 Embedding of the Baumslag-Solitar monoids

Recall that the Baumslag-Solitar groups are examples of two-generator one-relator groups and are given by the group presentation

$$BS(m,\, n) = < a,\, b \mid ab^m a^{-1} = b^n >,\ m,\, n \in \mathbb{Z}^*. \tag{4.1}$$

Since $ab^{-m}a^{-1} = b^{-n}$ is an equivalent relation, we may assume that m is positive. And the corresponding monoids $BS_+(m,\, n)$ are defined to be

$$< a,\, b \mid ab^m = b^n a >^+ \ \text{if } n \in \mathbb{N}^*,$$

and

$$< a,\, b \mid b^{-n} a b^m = a >^+ \ \text{if } n \in \mathbb{Z} \setminus \mathbb{N}.$$

It is a graph of monoids. Indeed, let Γ be a circle, consisting of one vertex v and one oriented edge a, and let $G_v = b^{\mathbb{Z}}$, $P_v =< b >^{\mathbb{N}}$, $G_a = G_{\bar{a}} = \mathbb{Z}$ and $P_a = \mathbb{N}$. The map $G_a \to G_{v=t(a)}$ ($G_a \to G_{v=o(a)}$) is given by $1 \mapsto b^m$ ($1 \mapsto b^n$, respectively). If $n > 0$, $P_{\bar{a}} = \mathbb{N}$; If $n < 0$, $P_{\bar{a}} = -\mathbb{N}$. It is easy easy to check that

$$P = \pi_1^+(G,\, \Gamma,\, T) = BS_+(m,\, n).$$

By definition, $BS(1,\, 1) =< a,\, b \mid ab = ba >$, which is an abelian group and hence isomorphic to $\mathbb{Z}^2$. It is also well known that $BS(1,\, -1) =< a,\, b \mid aba^{-1} = b^{-1} >$ is the fundamental group of the Klein bottle, isomorphic to the group $\mathbb{Z} \rtimes \mathbb{Z}$ induced by the group action

$$\varphi :\ \mathbb{Z} \curvearrowright \mathbb{Z},\ \varphi(m)(n) = (-1)^m n,\ \forall\, m,\, n \in \mathbb{Z}.$$

$BS(1,\, 1)$ and $BS(1,\, -1)$ are amenable and hence $BS_+(1,\, 1)$ and $BS_+(1,\, -1)$ can be embedded into amenable groups. Unfortunately, the Baumslag-Solitar groups are not amenable for general nonzero integers m and n.

The following theorem gives an embedding into amenable groups for coprime integers m and n.

Theorem 4.2.1. *If $|mn| > 1$ and $gcd(|m|,\, |n|) = 1$, then there exists an injective semigroup homomorphism $\varphi :\ BS_+(m,\, n) \to \mathbb{Q}^* \ltimes \mathbb{Q}$ such that $\varphi(a) = (\frac{m}{n},\, 0)$ and that $\varphi(b) = (1,\, 1)$, where $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ is a multiplicative group and $\mathbb{Q}$ is an additive group, and the group action is the typical multiplication.*

Proof. From a trivial calculation we get that $(1,\, 1)^n = (1,\, n)$, $\forall n \in \mathbb{Z}$ and that

$$(\frac{m}{n},\, 0)(1,\, 1)^m = (\frac{m}{n},\, m) = (1,\, 1)^n(\frac{m}{n},\, 0).$$

This means the values of φ at the two generators a and b lead to a well-defined semigroup homomorphism. And what remains is to show the injectivity.

From [Spi12, Proposition 2.3], we know each element $\alpha \in BS_+(m,\ n)$ has a unique representation of the form

$$\alpha = b^{i_0} a b^{i_1} a \cdots b^{i_{j-1}} a b^p \text{ with } i_\mu \in [0,\ |n|),\ p \in \mathbb{Z}.$$

So we have

$$\begin{aligned}\varphi(\alpha) &= (1,\ 1)^{i_0}(\frac{m}{n},\ 0)(1,\ 1)^{i_1}(\frac{m}{n},\ 0)\cdots(1,\ 1)^{i_{j-1}}(\frac{m}{n},\ 0)(1,\ 1)^p \\ &= (\frac{m}{n},\ \frac{m}{n}i_0)(\frac{m}{n},\ \frac{m}{n}i_1)\cdots(\frac{m}{n},\ \frac{m}{n}i_{j-1})(1,\ p) \\ &= \Big((\frac{m}{n})^j,\ (\frac{m}{n})^j i_0 + (\frac{m}{n})^{j-1} i_1 + \cdots + (\frac{m}{n}) i_{j-1} + p\Big).\end{aligned} \tag{4.2}$$

Similarly, we have

$$\varphi(\beta) = \big((\frac{m}{n})^s,\ (\frac{m}{n})^s r_0 + (\frac{m}{n})^{s-1} r_1 + \cdots + (\frac{m}{n}) r_{s-1} + q\big)$$

for

$$\beta = b^{r_0} a b^{r_1} a \cdots b^{r_{s-1}} a b^q \text{ with } r_\mu \in [0,\ |n|),\ q \in \mathbb{Z}.$$

If $\varphi(\alpha) = \varphi(\beta)$, then we have

$$(\frac{m}{n})^j = (\frac{m}{n})^s, \tag{4.3}$$

$$(\frac{m}{n})^j i_0 + (\frac{m}{n})^{j-1} i_1 + \cdots + (\frac{m}{n}) i_{j-1} + p = (\frac{m}{n})^s r_0 + (\frac{m}{n})^{s-1} r_1 + \cdots + (\frac{m}{n}) r_{s-1} + q. \tag{4.4}$$

Since $\frac{m}{n} \notin \{0,\ -1,\ 1\}$, it follows from (4.3) that $j = s$ and hence an rearrangement of (4.4) yields

$$(\frac{m}{n})^j (i_0 - r_0) + (\frac{m}{n})^{j-1} (i_1 - r_1) + \cdots + (\frac{m}{n})(i_{j-1} - r_{j-1}) + (p - q) = 0. \tag{4.5}$$

If $|n| = 1$, we have by definition $i_0 = r_0 = i_1 = r_1 = \cdots = i_{j-1} = r_{j-1} = 0$. Substitute these into (4.5), we get $p = q$ and thus $\alpha = \beta$.

If $|n| \neq 1$, multiple by n^j and then run a mudulo $|n|$ operation on both hand sides of (4.5), we obtain $[m]_{|n|}^j [i_0 - r_0]_{|n|} = 0$. Since m and n are coprime integers, $[m]_{|n|}$ is multiplicatively invertible in $\mathbb{Z}_{|n|}$. So we have $[i_0 - r_0]_{|n|} = 0$ and thus $i_0 = r_0$ because $i_0,\ r_0 \in [0, |n|)$. Similarly, we can get $i_1 = r_1,\ \cdots,\ i_{j-1} = r_{j-1}$. This implies $p = q$ by (4.5) and hence $\alpha = \beta$.

□

Remark 4.2.2. *Under the same condition as the theorem above, let* $\psi : \mathbb{Z} \to Aut(\mathbb{Q})$ *be the unique group homomorphism such that*

$$\psi(1)(r) = \frac{mr}{n},\ \forall r \in \mathbb{Q},$$

then there exists an embedding $\varphi :\ BS_+(m,\ n) \to \mathbb{Z} \ltimes \mathbb{Q}$ *such that* $\varphi(a) = (1,\ 0)$ *and that* $\varphi(b) = (0,\ 1)$.

Before giving the embeddings of the remaining subclass of Baumslag-Solitar monoids, we need to present the following lemmas.

Lemma 4.2.3. *Let P be a subsemigroup of a group G and let Q be a subsemigroup of a group H such that*

$$Q^* := Q \cap Q^{-1} = \{\varepsilon\},$$

then there exists a unique embedding

$$\varphi : P * Q \to (\oplus_H G) \rtimes H,$$

sending $p \in P$ to $(\delta_{\varepsilon,\ p},\ \varepsilon)$ and $q \in Q$ to $(\varepsilon,\ q)$, where the semidirect product $(\oplus_H G) \rtimes H$ is induced by the group action map

$$\psi : H \to \oplus_H G,\ h \ \mapsto\ \psi(h)\ \left[f\ \mapsto\ f(h^{-1}\sqcup)\right]$$

for all $h \in H$ and all functions $f :\ H \to G$, and where $\delta_{\varepsilon,\ p}$ is the function from H to G, taking the value p at $\varepsilon \in H$ and the value ε elsewhere.

Proof. It's trivial to check φ is a well-defined semigroup homomorphism. We shall show the injectivity next.

Define firstly a binary relation R on the semigroup Q as follows: $(x,\ y) \in R$ if there exists $z \in Q$ such that $y = xz$. It's easy to check that R defines a partial order $\preceq$ on Q, where we say $x \preceq y$ if $(x,\ y) \in R$. Secondly, define $\pi_1 : (\oplus_H G) \rtimes H \to (\oplus_H G)$ and $\pi_2 : (\oplus_H G) \rtimes H \to H$ be the trivial coordinate projection maps.

Every element $\alpha \in P * Q$ has a unique expression

$$\alpha = p_1 q_1 p_2 q_2 \cdots p_n q_n, \; p_1 \in P, \; p_2, \; \cdots, \; p_n \in P \setminus \{\varepsilon\},$$

$$q_1, \; \cdots, \; q_{n-1} \in Q \setminus \{\varepsilon\}, \; q_n \in Q, \; n \in \mathbb{N}.$$

By definition,

$$\begin{aligned} \varphi(\alpha) &= (\delta_{\varepsilon, \, p_1}, \, \varepsilon)(\varepsilon, \, q_1)(\delta_{\varepsilon, \, p_2}, \, \varepsilon)(\varepsilon, \, q_2) \cdots (\delta_{\varepsilon, \, p_n}, \, \varepsilon)(\varepsilon, \, q_n) \\ &= (\delta_{\varepsilon, \, p_1}, \, q_1)(\delta_{\varepsilon, \, p_2}, \, q_2) \cdots (\delta_{\varepsilon, \, p_n}, \, q_n) \\ &= \big(\delta_{\varepsilon, \, p_1} \delta_{q_1, \, p_2} \delta_{q_1 q_2, \, p_3} \cdots \delta_{q_1 \cdots q_{n-1}, \, p_n}, \, q_1 q_2 \cdots q_n\big). \end{aligned} \tag{4.6}$$

Let $\mathscr{C}(\alpha) \subseteq Q$ be the set

$$\{ \, q \in Q \mid q = \varepsilon \text{ or } \big(\pi_1 \circ \varphi(\alpha)\big)(q) \neq \varepsilon\},$$

then

$$\mathscr{C}(\alpha) = \{\varepsilon, \, q_1, \, q_1 q_2, \, \cdots, \, q_1 q_2 \cdots q_{n-1}\}$$

is an ascending chain with cardinality n.

If $\varphi(\alpha) = \varphi(\beta)$ for

$$\beta = p'_1 q'_1 p'_2 q'_2 \cdots p'_m q'_m \in P * Q, \; p'_1 \in P, \; p'_2, \; \cdots, \; p'_m \in P \setminus \{\varepsilon\},$$

$$q'_1, \; \cdots, \; q'_{m-1} \in Q \setminus \{\varepsilon\}, \; q'_m \in Q, \; m \in \mathbb{N},$$

then we have $\mathscr{C}(\alpha) = \mathscr{C}(\beta)$, where

$$\mathscr{C}(\beta) = \{\varepsilon,\ q'_1,\ q'_1 q'_2,\ \cdots,\ q'_1 q'_2 \cdots q'_{m-1}\}$$

is an ascending chain with cardinality m. This happens if and only if $n = m$ and $q_i = q'_i$ for $i = 1, 2, \cdots, n-1$. In this case,

$$p_1 = \big(\pi_1 \circ \varphi(\alpha)\big)(\varepsilon) = \big(\pi_1 \circ \varphi(\beta)\big)(\varepsilon) = p'_1$$

and

$$p_{i+1} = \big(\pi_1 \circ \varphi(\alpha)\big)(q_1 \cdots q_i) = \big(\pi_1 \circ \varphi(\beta)\big)(q'_1 \cdots q'_i) = p'_{i+1}$$

for $i = 1, 2, \cdots, n-1$. Finally the fact that $\pi_2 \circ \varphi(\alpha) = \pi_2 \circ \varphi(\beta)$ yields $q_n = q'_n$. All of these entail $\alpha = \beta$.

$\square$

Lemma 4.2.4. *There exists an embedding from the semigroup $\mathbb{Z}_n * \mathbb{N}$ to the group $\mathscr{A}_n := \mathbb{Z}_n * \mathbb{Z}/(\mathbb{Z}_n * \mathbb{Z})''$ for any natural number $n \geq 2$.*

Proof. By Lemma 4.2.3, there exists an embedding

$$\psi : \mathbb{Z}_n * \mathbb{N} \to (\oplus_{\mathbb{Z}} \mathbb{Z}_n) \rtimes \mathbb{Z},$$

which can be naturally extended to a group homomorphism from $\mathbb{Z}_n * \mathbb{Z}$ to $(\oplus_{\mathbb{Z}} \mathbb{Z}_n) \rtimes \mathbb{Z}$.

Noticing

$$\psi((\mathbb{Z}_n * \mathbb{Z})'') = (\psi(\mathbb{Z}_n * \mathbb{Z}))'' \subseteq \big((\oplus_{\mathbb{Z}} \mathbb{Z}_n) \rtimes \mathbb{Z}\big)'' = \{(0,\ 0)\},$$

ψ induces further a group homomorphism from $\mathscr{A}_n$ to $(\oplus_{\mathbb{Z}}\mathbb{Z}_n) \rtimes \mathbb{Z}$. Or equivalently, the followig diagram

$$\begin{tikzcd}
\mathbb{Z}_n * \mathbb{N} \arrow[r, "\iota"] \arrow[d, "\psi"] \arrow[dr] & \mathbb{Z}_n * \mathbb{Z} \arrow[d, "\pi"] \arrow[dl] \\
(\oplus_{\mathbb{Z}}\mathbb{Z}_n) \rtimes \mathbb{Z} & \mathscr{A}_n \arrow[l]
\end{tikzcd}$$

is commutative. The injectivity of ψ entails the semigroup homomorphism $\pi \circ \iota$ from $\mathbb{Z}_n * \mathbb{N}$ to $\mathscr{A}_n$ is also injective. $\square$

Remark 4.2.5. *From [Hoc69] we know $\mathbb{N} * \mathbb{N}$ can be embedded into $\mathscr{A} := \mathbb{Z} * \mathbb{Z}/(\mathbb{Z} * \mathbb{Z})''$.*

Corollary 4.2.6. *Assume that G is a group and that $\mathbb{Z}_n * \mathbb{Z} \curvearrowright G$, $\mathbb{N} \ni n \geq 2$ is a group action such that $\mathbb{Z}_n$ acts trivially, then $(\mathbb{Z}_n * \mathbb{N}) \ltimes G$ can be naturally embedded into $\mathscr{A}_n \ltimes G$.*

Proof. Since $\mathbb{Z}_n$ acts trivially on G, the group homomorphism $\mathbb{Z}_n * \mathbb{Z} \to Aut(G)$ factors through $\mathbb{Z}$. So $(\mathbb{Z}_n * \mathbb{Z})''$ acts trivially on G and $\mathbb{Z}_n * \mathbb{Z} \to Aut(G)$ factors through $\mathscr{A}_n$. In combination with the result from Lemma 4.2.4 that the semigroup homomorphism $\pi \circ \iota$ is injective, we can naturally get the embedding $(\mathbb{Z}_n * \mathbb{N}) \ltimes G \hookrightarrow \mathscr{A}_n \ltimes G$, induced by $\pi \circ \iota$.

$\square$

Theorem 4.2.7. *For each pair*

$$(m,\, n) \in (\mathbb{Z} \setminus \{0\})^2 \textit{ with } gcd(|m|,\, |n|) = d > 1,$$

there exists an injective semigroup homomorphism from $BS_+(m,\, n)$ to $(\mathbb{Z}_d * \mathbb{N}) \ltimes \mathbb{Q}$, sending a to $(s,\, 0)$ and b to $(t,\, 1)$, where s and t are the generators of $\mathbb{N}$ and $\mathbb{Z}_d$ respectively, the group action $\mathbb{Z}_d \curvearrowright \mathbb{Q}$ is trivial and the group action $\mathbb{N} \curvearrowright \mathbb{Q}$ is such that $s(r) = \frac{mr}{n}$ for any $r \in \mathbb{Q}$.

Proof. Since

$$(s,\ 0)(t,\ 1)^m = (s,\ m) = (t,\ 1)^n(s,\ 0),$$

such a semigroup homomorphism φ exists and what remains is to show it is injective. For

$$\alpha = b^{i_0}ab^{i_1}a\cdots b^{i_{j-1}}ab^p \in BS_+(m,\ n),\ \text{with},\ i_\mu \in [0,\ |n|),\ p \in \mathbb{Z},$$

we have

$$\begin{aligned}\varphi(\alpha) &= (t,\ 1)^{i_0}(s,\ 0)(t,\ 1)^{i_1}(s,\ 0)\cdots(t,\ 1)^{i_{j-1}}(s,\ 0)(t,\ 1)^p \\ &= (t^{i_0}s,\ \frac{mi_0}{n})(t^{i_1}s,\ \frac{mi_1}{n})\cdots(t^{i_{j-1}}s,\ \frac{mi_{j-1}}{n})(t^p,\ p) \\ &= \big(t^{i_0}st^{i_1}s\cdots t^{i_{j-1}}st^p,\ (\frac{m}{n})^j i_0 + (\frac{m}{n})^{j-1}i_1 + \cdots + (\frac{m}{n})i_{j-1} + p\big).\end{aligned} \tag{4.7}$$

If $\varphi(\alpha) = \varphi(\beta)$ for

$$\beta = b^{k_0}ab^{k_1}a\cdots b^{k_{l-1}}ab^q \text{ with } k_\mu \in [0,\ |n|),\ q \in \mathbb{Z},$$

then

$$t^{i_0}st^{i_1}s\cdots t^{i_{j-1}}st^p = t^{k_0}st^{k_1}s\cdots t^{k_{l-1}}st^q, \tag{4.8}$$

$$(\frac{m}{n})^j i_0 + (\frac{m}{n})^{j-1}i_1 + \cdots + (\frac{m}{n})i_{j-1} + p = (\frac{m}{n})^s r_0 + (\frac{m}{n})^{s-1}r_1 + \cdots + (\frac{m}{n})r_{s-1} + q. \tag{4.9}$$

This first equality yields $j = l$, $d|(i_\mu - k_\mu)$ and $d|(p-q)$. Substituting these into the equality (4.9) and rearranging it, we have

$$(\frac{m'}{n'})^j \frac{i_0 - r_0}{d} + (\frac{m'}{n'})^{j-1}\frac{i_1 - r_1}{d} + \cdots + (\frac{m'}{n'})\frac{i_{j-1} - r_{j-1}}{d} + \frac{p-q}{d} = 0,$$

where $m' = m/d$, $n' = n/d$, $\frac{i_\mu - r_\mu}{d} \in (-|n'|,\ |n'|)$ and $\frac{p-q}{d} \in \mathbb{Z}$. A similar analysis of the above equality, as we did in Theorem 4.2.1, yields $\frac{i_\mu - r_\mu}{d} = 0$ and $\frac{p-q}{d} = 0$, which means $\alpha = \beta$. $\square$

Corollary 4.2.8. *$BS_+(m,\ n)$ can be embedded into $\mathscr{A}_d \ltimes \mathbb{Q}$ for any pair $(m,\ n) \in (\mathbb{Z} \setminus \{0\})^2$ and $d = gcd(|m|,\ |n|)$, where the group action $\mathscr{A}_d \curvearrowright \mathbb{Q}$ is induced by such a group action $\mathbb{Z}_d * \mathbb{Z} \curvearrowright \mathbb{Q}$ that $\mathbb{Z}_d$ acts trivially and $s(r) = \frac{mr}{n}$ for $s \in \mathbb{Z}$ being the generator and any $r \in \mathbb{Q}$.*

Proof. The conclusion follows directly from Remark 4.2.2, Corollary 4.2.6 and Theorem 4.2.7.

$\square$

The second derived group $\mathscr{A}_d''$ of $\mathscr{A}_d$ is trivial, so $\mathscr{A}_d$ is solvable and hence the semidirect product $\mathscr{A}_d \ltimes \mathbb{Q}$ is solvable and thus amenable. This means all the Baumslag-Solitar monoids can be embedded into amenable groups.

4.2.2 Embedding of the generalised Baumslag-Solitar monoids

In last subsection, we embeded successfully the Baumslag-Solitars monoids into amenable groups. Now we aim at extending the results to generalized Baumslag-Solitar monoids.

A generalized Baumslag-Solitar group is given by presentation as follows,

$$\begin{aligned} &GBS(N,\ m_i,\ n_i) := < a_i,\ b \mid a_i b^{m_i} a_i^{-1} = b^{n_i}, \\ &m_i,\ n_i \in \mathbb{Z}^*,\ 1 \leq i \leq N,\ N \in \mathbb{N}^* \cup \{\infty\} > . \end{aligned} \tag{4.10}$$

Without loss of generality, we assume, like we did in the Baumslag-Solitar case,

$$m_i > 0,\ 1 \leq i \leq N.$$

Set

$$S_1 = \{1 \leq i \leq N \mid n_i > 0\}$$

and

$$S_2 = \{1 \leq i \leq N \mid n_i < 0\},$$

then the corresponding GBS monoid is defined by presentation

$$\begin{aligned} GBS_+(N,\ m_i,\ n_i) = &< a_i,\ b \mid a_i b^{m_i} = b^{n_i} a_i,\ i \in S_1, \\ & b^{-n_i} a_i b^{m_i} = a_i,\ i \in S_2,\ m_i,\ n_i \in \mathbb{Z}^*,\ N \in \mathbb{N}^* >^+ . \end{aligned} \tag{4.11}$$

It is also a graph of monoids. Let Γ be a bouquet of circles, consisting of one vertex v and N oriented edges $\{a_i\}_i$, and let $G_v = b^{\mathbb{Z}},\ P_v =< b >^{\mathbb{N}}$, $G_{a_i} = G_{\bar{a}_i} = \mathbb{Z}$ and $P_{a_i} = \mathbb{N}$. The map $G_{a_i} \to G_{v=t(a_i)}$ ($G_{a_i} \to G_{v=o(a_i)}$) is given by $1 \mapsto b^{m_i}$ ($1 \mapsto b^{n_i}$, respectively). If $n_i > 0$, $P_{\bar{a}_i} = \mathbb{N}$; If $n_i < 0$, $P_{\bar{a}_i} = -\mathbb{N}$. It is easy easy to check that

$$P = \pi_1^+(G,\ \Gamma,\ T) = GBS_+(N,\ m_i,\ n_i).$$

To begin with, we have the following Proposition.

Proposition 4.2.9. *Each element of $GBS(N,\ m_i,\ n_i)$ has unique representations in the two forms*

(L) $b^{j_0} a_{i_1}^{\iota} b^{j_1} a_{i_2}^{\iota} \cdots a_{i_p}^{\iota} b^{j_p}$, where $\iota \in \{\pm 1\}$, $1 \leq i_\mu \leq N$, $j_\mu \in [0,\ |n_{i_{\mu+1}}|)$ if $\iota = 1$, and $j_\mu \in [0,\ m_{i_{\mu+1}})$ if $\iota = -1,\ j_p \in \mathbb{Z}$;

(R) $b^{j_0}a_{i_1}^{\iota}b^{j_1}a_{i_2}^{\iota}\cdots a_{i_p}^{\iota}b^{j_p}$, *where* $\iota \in \{\pm 1\}$, $1 \le i_\mu \le N$, $j_\mu \in [0,\ m_{i_\mu})$ *if* $\iota = 1$, *and* $j_\mu \in [0,\ |n_{i_\mu}|)$ *if* $\iota = -1$, $j_0 \in \mathbb{Z}$.

The standard L-form (R-form) of the proposition is obtained by moving $b's$ to the right (left, respectively) via the equations $b^{kn_i}a_i = a_ib^{km_i}$ and $b^{km_i}a_i^{-1} = a_i^{-1}b^{kn_i}$, $k \in \mathbb{Z}$.

Corollary 4.2.10. *Each element of* $GBS_+(N,\ m_i,\ n_i)$ *has unique representations in the two forms*

(L) $b^{j_0}a_{i_1}b^{j_1}a_{i_2}\cdots a_{i_p}b^{j_p}$, $1 \le i_\mu \le N$, $j_\mu \in [0,\ |n_{i_{\mu+1}}|)$, $j_p \in \mathbb{Z}$;

(R) $b^{j_0}a_{i_1}b^{j_1}a_{i_2}\cdots a_{i_p}b^{j_p}$, $1 \le i_\mu \le N$, $j_\mu \in [0,\ m_{i_\mu})$, $j_0 \in \mathbb{Z}$.

Theorem 4.2.11. *Assume*

$$gcd\Big(\prod_{i=1}^{N} m_i, \prod_{i=1}^{N} n_i\Big) = 1,\ m_i,\ n_i \in \mathbb{Z}^*,\ N \in \mathbb{N}^*.$$

For each $1 \le i \le N$, *let* $P_i = < s_i >^+$ *be a semigroup isomorphic to* $\mathbb{N}$, *and let* ϕ_i *be a semigroup homomorphism defined by*

$$\phi_i :\ P_i\ \to\ Aut(\mathbb{Q}),\ s_i\ \mapsto\ \phi_i(s_i)\ \Big[r \mapsto\ \frac{m_i r}{n_i},\ r \in \mathbb{Q}\Big].$$

Then there exists an injective semigroup homomorphism

$$\varphi :\ GBS_+(N,\ m_i,\ n_i) \to (*_i P_i) \ltimes \mathbb{Q}$$

such that $\varphi(a_i) = (s_i,\ 0)$ *and that* $\varphi(b) = (\varepsilon,\ 1)$.

Proof. For each $i \in S_1$, we have

$$(s_i,\ 0)(\varepsilon,\ 1)^{m_i} = (s_i,\ 0)(\varepsilon,\ m_i) = (s_i,\ m_i) = (\varepsilon,\ n_i)(s_i,\ 0) = (\varepsilon,\ 1)^{n_i}(s_i,\ 0).$$

Similarly, we get

$$(\varepsilon,\ 1)^{-n_i}(s_i,\ 0)(\varepsilon,\ 1)^{m_i} = (s_i,\ 0),\ i \in S_2.$$

Therefore, such a semigroup homomorphism φ does exist. It remains to show the injectivity.

By Corollary 4.2.10 each element $\alpha \in GBS_+(N,\ m_i,\ n_i)$ has a standard L-form

$$\alpha = b^{j_0} a_{i_1} b^{j_1} a_{i_2} \cdots a_{i_p} b^{j_p},\ 1 \le i_\mu \le N,\ j_\mu \in [0,\ |n_{i_{\mu+1}}|),\ j_p \in \mathbb{Z}.$$

Then we have

$$\begin{aligned}
\varphi(\alpha) &= (\varepsilon,\ 1)^{j_0}(s_{i_1},\ 0)(\varepsilon,\ 1)^{j_1}(s_{i_2},\ 0)\cdots(s_{i_p},\ 0)(\varepsilon,\ 1)^{j_p} \\
&= \left(s_{i_1},\ \frac{m_{i_1} j_0}{n_1}\right)\left(s_{i_2},\ \frac{m_{i_2} j_1}{n_2}\right)\cdots\left(s_{i_p},\ \frac{m_{i_p} j_{p-1}}{n_p}\right)(\varepsilon,\ j_p) \\
&= \left(s_{i_1} s_{i_2} \cdots s_{i_p},\ \left(\prod_{\mu=1}^{p} \frac{m_{i_\mu}}{n_{i_\mu}}\right) j_0 + \left(\prod_{\mu=2}^{p} \frac{m_{i_\mu}}{n_{i_\mu}}\right) j_1 + \cdots + \frac{m_{i_p} j_{p-1}}{n_{i_p}} + j_p\right).
\end{aligned} \tag{4.12}$$

If $\varphi(\alpha) = \varphi(\beta)$ for $\beta \in GBS_+(N,\ m_i,\ n_i)$ with the standard L-form

$$\beta = b^{l_0} a_{k_1} b^{l_1} a_{k_2} \cdots a_{k_q} b^{l_q},\ 1 \le k_\mu \le N,\ l_\mu \in [0,\ |n_{k_{\mu+1}}|),\ l_q \in \mathbb{Z},$$

then we have

$$s_{i_1} s_{i_2} \cdots s_{i_p} = s_{k_1} s_{k_2} \cdots s_{k_q} \tag{4.13}$$

and

$$\begin{aligned}&\Big(\prod_{\mu=1}^{p}\frac{m_{i_\mu}}{n_{i_\mu}}\Big)j_0+\Big(\prod_{\mu=2}^{p}\frac{m_{i_\mu}}{n_{i_\mu}}\Big)j_1+\cdots+\frac{m_{i_p}j_{p-1}}{n_{i_p}}+j_p\\&=\Big(\prod_{\mu=1}^{q}\frac{m_{k_\mu}}{n_{k_\mu}}\Big)l_0+\Big(\prod_{\mu=2}^{q}\frac{m_{k_\mu}}{n_{k_\mu}}\Big)l_1+\cdots+\frac{m_{k_q}l_{q-1}}{n_{k_q}}+l_q.\end{aligned}\tag{4.14}$$

It follows from the equality (4.13) that $p=q$ and that $i_\mu=k_\mu,\ 1\le\mu\le p$. Substituting these into the equality (4.14), we obtain, after a rearrangement,

$$\Big(\prod_{\mu=1}^{p}\frac{m_{i_\mu}}{n_{i_\mu}}\Big)(j_0-l_0)+\Big(\prod_{\mu=2}^{p}\frac{m_{i_\mu}}{n_{i_\mu}}\Big)(j_1-l_1)+\cdots+\frac{m_{i_p}}{n_{i_p}}(j_{p-1}-l_{p-1})+(j_p-l_p)=0.\tag{4.15}$$

If $|n_{i_1}|=1$, then $j_0=l_0=0$ by definition. Otherwise, multiple both hand sides of the equality (4.15) by $\prod_{\mu=1}^{p}n_{i_\mu}$ and then run a modulo $|n_{i_1}|$ operation, we get

$$\Big[\prod_{\mu=1}^{p}m_{i_\mu}\Big]_{|n_{i_1}|}\big[j_0-l_0\big]_{|n_{i_1}|}=0.$$

By the assumption in the theorem, $\Big[\prod_{\mu=1}^{p}m_{i_\mu}\Big]_{|n_{i_1}|}$ is multiplicatively invertible in $\mathbb{Z}_{|n_{i_1}|}$. So we have $\big[j_0-l_0\big]_{|n_{i_1}|}=0$ and hence $j_0-l_0=0$, i.e., $j_0=l_0$ because $j_0,\ l_0\in[0,\ |n_{i_1}|)$. In either case, we have $j_0=l_0$. Substituting this into the equality (4.15) and repeating this process, we can get, one by one, $j_1=l_1,\ \cdots,\ j_p=l_p$, which entails $\alpha=\beta$.

□

Let G be the free additive abelian group on the family of generators

$$\{b(m_1,\ m_2,\ \cdots,\ m_N)\},\ \ m_i\in\mathbb{Z},\ 1\le i\le N,\ 2\le N\in\mathbb{N}.$$

Any permutation of these generators induces an automorphism of G. Let $x_i,\ 1\le i\le N$ be the automorphism induced by

$$b(m_1,\, m_2,\, \cdots,\, m_N) \;\mapsto\; b(m_1,\, \cdots,\, m_{i-1},\, m_i+1,\, m_{i+1},\, \cdots,\, m_N).$$

It is easy to see that the subgroup H of $Aut(G)$ generated by $\{x_i\},\ 1 \leq i \leq N$ is a free ableian group. Hence the semidirect product $H \ltimes G$ is solvable and amenable. Set

$$y_i = \big(x_i,\, b(0,\, 0,\, \cdots,\, 0)\big),\ 1 \leq i \leq N.$$

Proposition 4.2.12. *$\{y_i\},\ 1 \leq i \leq N$ is a family of free generators for a copy of $*_i P_i$, where P_i is as in Theorem 4.2.11.*

Proof. Consider a monomial $U = u_1 u_2 \cdots u_d$ of length $d \geq 1$, where each u_j is some y_i. Suppose y_i occurs p_i times, then $\sum_{i=1}^{N} p_i = d$ and U is of the form

$$\Big(\prod_{i=1}^{N} x_i^{p_i},\, \sum_{j=0}^{d-1} b(m_1^j,\, m_2^j,\, \cdots,\, m_N^j)\Big),$$

where for each j,

$$m_i^j \geq 0 \text{ and } \sum_{i=1}^{N} m_i^j = j.$$

In particular, $(m_1^0,\, m_2^0,\, \cdots,\, m_N^0) = (0,\, 0,\, \cdots,\, 0)$. When $d \geq 2$, we have

$$(m_1^1,\, m_2^1,\, \cdots,\, m_N^1) = (0,\, \cdots,\, 0,\, 1,\, 0,\, \cdots,\, 0)$$

with the i-th entry taking value 1 if $u_d = y_i$. These statements are proved by a trivial induction on d.

Now suppose that the distinct monomials $U = u_1 u_2 \cdots u_d$ and $V = v_1 v_2 \cdots v_e$ are equal. We must have that $d = e$ because the length can be recovered as the sum of the exponents of $y_i's$ in the first entry. If $d = 1$, then $u_1 = U = V = v_1$. If $d \geq 2$, the second entry will be a sum of d terms, precisely one of which, $b(m_1^1,\ m_2^1,\ \cdots,\ m_N^1)$, will have the property

$$\sum_{i=1}^{N} m_i^1 = 1.$$

It follows that $u_d = v_d = y_i$ with the i-th entry taking value 1. And then $u_1 u_2 \cdots u_{d-1} = v_1 v_2 \cdots v_{d-1}$. So a trivial induction on d finishes the proof.

$\square$

Corollary 4.2.13. *Under the same assumption as in Theorem 4.2.11, then the monoid $GBS_+(N,\ m_i,\ n_i)$ can be embedded into an amenable group.*

Proof. From Proposition 4.2.12, we know that the semigroup $*_i P_i$ can be embedded into the group $H \ltimes G$, which naturally induces a group homomorphism ψ from the free group F_N with N generators to the group $H \ltimes G$. Since G and H are both abelian, the second derived group $(H \ltimes G)''$ of $H \ltimes G$ is trivial. Therefore, F_N'' is in the kernel of the group homomorphism ψ. Alternatively, ψ factors through F_N/F_N''. That is to say, the following diagram

$$\begin{tikzcd} *_i P_i \arrow[r] \arrow[d] \arrow[dr] & F_N \arrow[d] \arrow[dl] \\ H \ltimes G & F_N/F_N'' \arrow[l] \end{tikzcd}$$

is commutative. Since the map $*_i P_i \to H \ltimes G$ is injective, the map $*_i P_i \to F_N/F_N''$ is also injective.

Recall the definition of $\phi_i' s$, there exists a semigroup homomorphism $\phi : *_i P_i \to Aut(\mathbb{Q})$ such that the restriction of ϕ on P_i is exactly ϕ_i. It admits an extention from F_N to $Aut(\mathbb{Q})$, which we also denote by ϕ, briefly. It is easy to see $\phi(F_N)$ is an abelian subgroup of $Aut(\mathbb{Q})$. So ϕ factors through F_N/F_N''. That is, we have the follwing commutative diagram

$$\begin{tikzcd} *_i P_i \arrow[r] \arrow[d] \arrow[dr] & F_N \arrow[d] \arrow[dl] \\ Aut(\mathbb{Q}) & F_N/F_N''. \arrow[l] \end{tikzcd}$$

Combined with the fact that $*_i P_i$ embeds into F_N/F_N'', we conclude that $(*_i P_i) \ltimes \mathbb{Q}$ embeds into $(F_N/F_N'') \ltimes \mathbb{Q}$. By Theorem 4.2.11, $GBS_+(N,\ m_i,\ n_i)$ can be embedded into the group $(F_N/F_N'') \ltimes \mathbb{Q}$, which is amenable.

□

Question 4.2.14. *Can we get an analogue for the semigroup $GBS_+(N,\ m_i d,\ n_i d)$, $d \in \mathbb{N}^*$ under the same assumption as in Theorem 4.2.11? And in general case?*

Chapter 5

Groupoids

Let G be a group and let P be a subsemigroup of G by an embedding $P \hookrightarrow G$, if we define a partial group action of G on the character space $\Omega_P := \mathrm{Spec}(D_\lambda(P))$ as in Section 2.2, by Theorem 2.2.4 we have $C^*_\lambda(P) \cong C^*_r(G \ltimes \Omega_P)$, where $G \ltimes \Omega_P$ is the transformation groupoid induced by the partial action of G on Ω_P. It makes sense to study such a kind of transformation groupoid, which is indeed what we do in this chapter.

5.1 Amenability of transformation groupoids

If a group G is amenable, we get, by [RW17, Corollary 4.5], for all partial action $G \curvearrowright X$, the corresponding transformation groupoid $G \ltimes X$ is amenable. This means, by definition of amenability, there exists a topological approxiamate invariant mean on $G \ltimes X$. It is natural to ask whether we can work out such a topological approximate invariant mean on $G \ltimes X$. In the following, we give a construction of a Borel approximate invariant mean on $G \ltimes X$ and provide a sufficient condition for the mean to be continuous. The construction is based on the result in [RW17].

Let $G \curvearrowright X$ be a partial dynamical system, where G is a discrete, countable and amenable group, and X is a locally compact, Hausdorff and second countable topological space, then the associated transformation groupoid $G \ltimes X := \{(g,\, x) \in G \times X |\, x \in U_{g^{-1}}\}$ is a locally compact, Hausdorff and second countable étale groupoid.

Step 1. Set the groupoid

$$(G \ltimes X) \rtimes (G \ltimes X) := \{\big((g,\, x),\, (h,\, y)\big) \in (G \ltimes X) \times (G \ltimes X) \mid x = hy\}$$

with composition

$$\big((g,\, x),\, (h,\, y)\big)\big((g',\, x'),\, (h',\, y')\big) = \big((g,\, x),\, (hh',\, y')\big)$$

if $(gh,\, y) = (g',\, x')$, and inversion

$$\big((g,\, x),\, (h,\, y)\big)^{-1} = \big((gh,\, y),\, (h^{-1},\, x)\big).$$

We identify the unit space of $(G \ltimes X) \rtimes (G \ltimes X)$ with $G \ltimes X$ with the range and source maps given by

$$r\big((g,\, x),\, (h,\, y)\big) = (g,\, x) \text{ and } s\big((g,\, x),\, (h,\, y)\big) = (gh,\, y).$$

By [Ren80, Lemma 2.7 and Proposition 2.8], we know $G \ltimes X$ admits a counting measure system λ as its left Haar measure system. And a direct application of [ADR00, Example 2.1.4(1)] gives a Borel invariant mean $\{m^{(g,\, x)}\}_{(g,\, x)}$ on the groupoid $(G \ltimes X) \rtimes (G \ltimes X)$ such that

$$m^{(g,\, x)}\big((g,\, x),\, (h,\, h^{-1}x)\big) = \phi\big((g,\, x)(h,\, h^{-1}x)\big) = \phi\big((gh,\, h^{-1}x)\big),$$

where ϕ is a nonnegative Borel function on $G \ltimes X$ with $\lambda(\phi) = 1$. Since the unit space X of $G \ltimes X$ is clopen, the characteristic function 1_X is a nonnegative continuous function on $G \ltimes X$ with $\lambda(1_X) = 1$. Therefore, taking $\phi = 1_X$, $\{m^{(g,\,x)}\}_{(g,\,x)}$ becomes a continuous invariant mean on $(G \ltimes X) \rtimes (G \ltimes X)$.

Step 2. Define a cocycle $c:\ G \ltimes X \to G,\ (g,\ x) \mapsto g$, then c is a continuous homomorphism and $c^{-1}(\varepsilon) = \{\varepsilon\} \times X \cong X$ is an amenable subgroupoid of $G \ltimes X$.
The skew-product groupoid associated to the cocycle c is, as in [RW17, p2262],

$$\mathscr{G}(c) = \{\big(a,\ (g,\ x),\ b\big) \in G \times (G \ltimes X) \times G:\ b = ag\}.$$

$\Big(\big(a,\ (g,\ x),\ b\big),\ \big(c,\ (h,\ y),\ d\big)\Big)$ is a composable pair if and only if $b = c$ and $x = hy$. The multiplication is given by

$$\big(a,\ (g,\ x),\ b\big)\big(b,\ (h,\ y),\ d\big) = \big(a,\ (gh,\ y),\ d\big)$$

and inversion by

$$\big(a,\ (g,\ x),\ b\big)^{-1} = \big(b,\ (g^{-1},\ gx),\ a\big).$$

We can identify the unit space of $\mathscr{G}(c)$ with $X \times G$, and then the range and source maps are given as expected:

$$r\big(a,\ (g,\ x),\ b\big) = (gx,\ a) \text{ and } s\big(a,\ (g,\ x),\ b\big) = (x,\ b).$$

Let

$$Y := \{(x,\ g) \in X \times G \mid (g,\ x) \in G \ltimes X\},$$

it is easy to see that Y is $\mathscr{G}(c)$-invariant. Indeed, if $\big(a,\ (g,\ x),\ b\big) \in \mathscr{G}(c)$ has its source in Y, then we have $(x,\ b) \in Y$ and hence $x \in U_{b^{-1}} = U_{(ag)^{-1}}$. Combined with the fact $x \in U_{g^{-1}}$, we conclude $gx \in U_{a^{-1}}$ and $r\big(a,\ (g,\ x),\ b\big) = (gx,\ a) \in Y$.

Define a map

$$\varphi:\ \mathscr{G}(c)_{|Y}\ \to\ (G \ltimes X) \rtimes (G \ltimes X),\ \big(a,\ (g,\ x),\ ag\big)\ \mapsto\ \big((a,\ gx),\ (g,\ x)\big),$$

then from a trivial computation it follows that φ is a topological groupoid isomorphism. Hence, by the isomorphism φ, $\mathscr{G}(c)_{|Y}$ admits a continuous invariant mean of discrete probability measures $\{m^{(x,\ g)}\}_{(x,\ g)\in Y}$ such that

$$m^{(x,\ g)}\big(g,\ (h,\ h^{-1}x),\ gh\big) = \chi_X\big((gh,\ h^{-1}x)\big).$$

So we have

$$m^{(x,\ g)} = \delta_{\big(g,\ (g^{-1},\ gx),\ \varepsilon\big)},\ (x,\ g) \in Y.$$

Note that G acts on the left of $\mathscr{G}(c)$ by groupoid automorphisms:

$$h \cdot \big(a,\ (g,\ x),\ b\big) = \big(ha,\ (g,\ x),\ hb\big).$$

Assume

$$G = \{g_1,\ g_2,\ \cdots,\ g_n,\ \cdots\}$$

with $g_1 = e$, we then have $X \times G = \cup_n g_n Y$ and $\mathscr{G}(c)_{|g_nY} = g_n\mathscr{G}(c)_{|Y}$. So we get the continuous invariant mean $\{m^{(x,\ g)}\}_{(x,\ g)\in g_nY}$ on $\mathscr{G}(c)_{|g_nY}$ such that

$$m^{(x,\ g)} = \delta_{\big(g,\ (g^{-1}g_n,\ g_n^{-1}gx),\ g_n\big)},\ (x,\ g) \in g_nY.$$

Set

$$Y_n := g_n Y \setminus \bigcup_{i=1}^{n-1} g_i Y,\ n \in \mathbb{N}^*,$$

then $\{Y_n\}_n$ is a disjoint cover of $X \times G$ by invariant Borel subsets. For $(x,\ g) \in \mathscr{G}(c)^0$, define $m^{(x,\ g)}$ by

$$m^{(x,\ g)} = \delta_{\left(g,\ (g^{-1}g_n,\ g_n^{-1}gx),\ g_n\right)},\ (x,\ g) \in Y_n.$$

It's easy to verify that $\left\{m^{(x,\ g)}\right\}_{(x,\ g)\in X\times G}$ is a Borel invariant mean of discrete probability measures on $\mathscr{G}(c)$. Moreover, it is continuous if Y is a clopen subset of $X \times G$.

Step 3. Now we try to construct a continuous approximate invariant mean on $G \ltimes X$. Since G is amenable, there exists a nonnegative and finitely supported function ψ_n on G such that

$$\sum_{g\in G} \psi_n(g) = 1 \text{ and } \sum_{g\in G} |\psi_n(gh) - \psi_n(g)| \leq 1/n$$

for all $h \in K_n$, where $\{K_n\}_n$ is an increasing sequence of finite subsets such that $\bigcup_n K_n = G$. Define the function Ψ_n on $G \ltimes X$ by

$$\begin{aligned}
\Psi_n\big((h,\ h^{-1}x)\big) :&= \sum_{g\in G} \psi_n(g) m^{(x,\ g)}\big(g,\ (h,\ h^{-1}x),\ gh\big) \\
&= \sum_{m\in\mathbb{N}^*} \sum_{g:\ (x,\ g)\in Y_m} \psi_n(g)\delta_{\left(g,\ (g^{-1}g_m,\ g_m^{-1}gx),\ g_m\right)}\big(g,\ (h,\ h^{-1}x),\ gh\big) \qquad (5.1)\\
&= \sum_{m:\ (x,\ g_m h^{-1})\in Y_m} \psi_n(g_m h^{-1}), \quad (h^{-1},\ x) \in G \ltimes X.
\end{aligned}$$

In the last term, $(x,\ g_m h^{-1}) \in Y_m$ if and only if

$$(g_k^{-1} g_m h^{-1},\ x) \notin G \ltimes X,$$

or equivalently,

$$g_k^{-1} g_m h^{-1} \notin G_x,$$

for any $1 \le k < m$. By [RW17, Proposition 4.1], the sequence $\{\Psi_n\}_n$ forms a Borel approximate invariant mean on $G \ltimes X$. And it becomes a continuous approximate invariant mean if Y is a clopen subset of $X \times G$.

Indeed, for all $x \in X$,

$$\begin{aligned} &\sum_{h \in G^x} \Psi_n\big((h,\ h^{-1}x)\big) \\ =&\sum_{g \in G}\sum_{h \in G^x} \psi_n(g) m^{(x,\ g)}\big(g,\ (h,\ h^{-1}x),\ gh\big) \\ =&\sum_{g \in G} \psi_n(g) = 1. \end{aligned} \tag{5.2}$$

This means that Ψ_n is a density function of probability measures. By the equality (5.1), we have

$$\begin{aligned} &\Psi_n\big((g,\ x)(h,\ h^{-1}x)\big) = \Psi_n\big((gh,\ h^{-1}x)\big) \\ =&\sum_{m:\ (gx,\ g_m h^{-1} g^{-1}) \in Y_m} \psi_n(g_m h^{-1} g^{-1}), \quad (g,\ x),\ (h^{-1},\ x) \in G \ltimes X. \end{aligned} \tag{5.3}$$

Noticing that $(gx,\ g_m h^{-1} g^{-1}) \in Y_m$ if and only if $(x,\ g_m h^{-1}) \in Y_m$, it follows that

$$\begin{aligned} &\sum_{h \in G^x} \big|\Psi_n\big((g,\ x)(h,\ h^{-1}x)\big) - \Psi_n\big((h,\ h^{-1}x)\big)\big| \\ =&\sum_{h \in G^x} \Big|\sum_{m:\ (x,\ g_m h^{-1}) \in Y_m} \big(\psi_n(g_m h^{-1}) - \psi_n(g_m h^{-1} g^{-1})\big)\Big| \\ \le&\sum_{h,\ m:\ h \in G^x,\ (x,\ g_m h^{-1}) \in Y_m} \big|\psi_n(g_m h^{-1}) - \psi_n(g_m h^{-1} g^{-1})\big|. \end{aligned} \tag{5.4}$$

Assume $(h_1,\ m_1)$ and $(h_2,\ m_2)$ are two pairs such that $h_i \in G^x,\ (x,\ g_{m_i} h_i^{-1}) \in Y_{m_i}$ and that $g_{m_1} h_1^{-1} = g_{m_2} h_2^{-1}$. If $m_1 = m_2$, then $h_1 = h_2$. Otherwise, assume, without loss of generality, $m_1 < m_2$, we then get $g_{m_1}^{-1} g_{m_2} h_2^{-1} = h_1^{-1} \in G_x$, which contradicts the assumption $(x,\ g_{m_2} h_2^{-1}) \in Y_{m_2}$. So in the last term in the equation (5.4), when the sum takes over all possible h and m, the element $g_m h^{-1}$ is never repeated. Hence

$$\begin{aligned}
&\sum_{h\in G^x} \left|\Psi_n\big((g,\, x)(h,\, h^{-1}x)\big) - \Psi_n\big((h,\, h^{-1}x)\big)\right| \\
\leq &\sum_{h,\, m:\, h\in G^x,\, (x,\, g_m h^{-1})\in Y_m} \left|\psi_n(g_m h^{-1}) - \psi_n(g_m h^{-1} g^{-1})\right| \qquad (5.5)\\
\leq &\sum_{k\in G} \left|\psi_n(k) - \psi_n(kg^{-1})\right|.
\end{aligned}$$

The last term tends to 0 as n tends to infinity. This proves approximate invariance of Ψ_n. When Y is clopen in $X \times G$, Ψ_n is a continuous function by definition. Take $f \in C_c(G \ltimes X)$, the function

$$x \mapsto \sum_{h\in G^x} f\big((h,\, h^{-1}x)\big)\Psi_n\big((h,\, h^{-1}x)\big)$$

is continuous on X because of the fact $f\Psi_n \in C_c(G \ltimes X)$ and the property of the left Haar measure on groupoids.

In the semigroup case, if $P \subseteq G$ satisfies the Toeplitz condition, the set U_g is a clopen subset of $\widehat{\mathscr{J}_P}$ for all $g \in G$. Therefore,

$$Y = \bigcup_{g\in G} U_{g^{-1}} \times \{g\}$$

is clopen in $\widehat{\mathscr{J}_P} \times G$. This entails that the groupoid $G \ltimes \widehat{\mathscr{J}_P}$ admits a continuous approximate invariant mean.

5.2 Closed invariant subsets

In this section, let G be the graph of groups and let $P \subseteq G$ be the graph of monoids in the same setting as in Section 3.1. Assume that condition (LCM) is satisfied.

Under some circumstances, there is a one-to-one correspondence between the ideals of the reduced groupoid C^*-algebra $C_r^*(G \ltimes \Omega_P)$ and the open invariant subsets of the unit space Ω_P. Even in general, every open invariant subset in Ω_P yields an ideal in the reduced groupoid C^*-algebra $C_r^*(G \ltimes \Omega_P)$. Our goal in this section is to study closed invariant subsets of Ω_P.

By Lemma 3.2.3, P is right LCM, i.e., all nonempty constructible right ideals are principal. That is, $\mathscr{J}_P = \{pP \mid p \in P\}$ or $\{pP \mid p \in P\} \cup \{\emptyset\}$. Furthermore, remark 2.2.3 states $\Omega_P = \widehat{\mathscr{J}_P}$. For convenience, denote $\mathscr{J}_P$ by $\mathscr{J}$ and denote Ω_P by Ω.

By definition, every $\chi \in \Omega$ is a nonzero filter function from $\mathscr{J} \cup \emptyset$ to $\{0,\ 1\}$ with $\chi(\emptyset) = 0$. And Ω is endowed with the pointwise convergence topology.

Each $p \in P$ determines a character χ_p given by $\chi_p(xP) = 1$ if and only if $p \in xP$. Identity χ_p with p, P is a dense subset in Ω. For every finite or infinite positive word $w = x_1x_2x_3\cdots$, $x_* \in \{P_v \setminus \{\varepsilon\}\}_{v\in V} \cup A$, define $[w]_i := w$ if $w = x_1\cdots x_j$ with $j < i$ and $[w]_i := x_1\cdots x_i$ otherwise. Let $\{w\}_i$ be the rest subword of w after removing $[w]_i$, i.e., $w = [w]_i\{w\}_i$. Define $\chi_w \in \Omega$ by setting $\chi_w(xP) = 1$ if and only if $[w]_i \in xP$ for some i. It is compatible with our notation χ_p when $w = p \in P$.

Define $\Omega_\infty := \Omega \backslash P$, then we have, by [LOS18, Lemma 2.3], every $\chi \in \Omega_\infty$ is of the form χ_w for some infinite positive word w. In conclusion, every character $\chi \in \Omega$ is of the form χ_w for some finite or infinite positive word w.

An easy interpretation of the partial action of G on Ω yields that $\chi \in \mathrm{dom}(g) = U_{g^{-1}}$ if and only if $g = pq^{-1}$ for some $p,\ q \in P$ and $\chi(qP) = 1$. Furthermore, we have

$$(g \cdot \chi)(xP) = \chi(qyP) \text{ if } xP \cap pP = pyP$$

and

$$(g\cdot\chi)(xP) = 0 \text{ if } xP\cap pP = \emptyset.$$

If $\chi = \chi_w$ for some word w, $\chi(qP) = 1$ implies that $[w]_i \in qP$ for some i. Assume $[w]_i = qr,\ r \in P$, then we have $g\chi_w = \chi_{pr\{w\}_i}$. In this case, define $gw := pr\{w\}_i$. Since the group element g may have different decomposition, the word $pr\{w\}_i$ is not unique. While we can always get one from another by rearrangement and the characters $\chi_{pr\{w\}_i}$ coincide.

It is easy to see that P and Ω_∞ are invariant.

Among all the characters in Ω, we are interested in some special ones under which the preimage of 1 is maximal. That is, $\chi \in \Omega$ is called a maximal character if we have $\chi' = \chi$ whenever $\chi' \in \Omega$ satisfies $\chi'(xP) = 1$ for all $x \in P$ with $\chi(xP) = 1$. Let $\Omega_{\max}$ be the family of all maximal characters in Ω, then we have $\Omega_{\max} \subseteq \Omega_\infty$ and that $\Omega_{\max}$ is invariant. The boundary of Ω, denoted by $\partial\Omega$, is defined to be the closure of $\Omega_{\max}$ in Ω, i.e., $\partial\Omega := \overline{\Omega_{\max}}$. It is closed and invariant in Ω.

5.2.1 General case

We will focus on the following two situation.

I. For all $v \in V$, $x \in P_v \setminus \varepsilon$ or $x \in A$ and $\chi \in \Omega_\infty$, there exists an infinite word w with $\chi = \chi_w$, a strictly increasing sequence $(j_N)_N$ of positive integers, and a finite positive word y whose first letter does not lie in P_v in the case where $x \in P_v$ such that,
(i) $xy[w]_{j_N}$ is a reduced positive word for all N,
(ii) Whenever $p_0d_1p_1\cdots$ is a properly reduced positive word representing $xy[w]_{j_N}$, we must have $x \in p_0P_T$ if $x \in P_v$ and $x \in p_0P$ if $x \in A$.

II. There exists $\mathbf{u} \in V$ and $\mathbf{b} \in P_{\mathbf{u}}$ such that the following holds:

For all $v \in V$, $x \in P_v \setminus \varepsilon$ or $x \in A$ and $\chi \in \Omega_\infty$, there exists an infinite word w with $\chi = \chi_w$, a strictly increasing sequence $(j_N)_N$ of positive integers, and a finite positive word y whose first letter does not lie in P_v in the case where $x \in P_v$ such that,

(i) $xy[w]_{j_N}$ is a reduced positive word for all N,

(ii) Whenever $p_0 d_1 p_1 \cdots$ is a properly reduced positive word representing $xy[w]_{j_N}$, then one of the following holds:

A) $x \in p_0 P_T$ if $x \in P_v$ and $x \in p_0 P$ if $x \in A$,

B) $[w]_{j_N} \in \mathbf{b}P$ and $x\mathbf{b}^i \in p_0 P_T$ if $x \in P_v$ and $x\mathbf{b}^i \in p_0 P$ if $x \in A$, where i is some positive integer.

Lemma 5.2.1. *Suppose that condition I. holds and let $\chi \in \Omega_\infty$ be arbitrary. For $\eta \in \Omega$ such that $\eta = \chi_{w'}$ for some infinite positive word w' with $\lim_{l \to \infty} \ell([w']_l) = \infty$, we have $\eta \in \overline{G \cdot \chi}$.*

Proof. Let $x_0 f_1 x_1 \cdots x_{n-1} f_n x_n$ be a properly reduced positive word representing $[w']_l$, we distinguish between two cases:

(a) $x_n \in P_v \setminus \{\varepsilon\}$ for some $v \in V$;

(b) $x_n = \emptyset$ and $f_n \in A$.

Condition I. applied to χ and $x = x_n$ in case (a) and $x = f_n$ in case (b) provides w, $[w]_{j_N}$ and y as above. Note that these depend on l. We now claim that $\lim_{l \to \infty} \chi_{[w']_l y w} = \eta$.

If $\eta(pP) = 1$, then $[w']_l \in pP$ for all sufficiently big l, so that $[w']_l y[w]_{j_N} \in pP$ for all sufficiently big l and all N. Thus $\chi_{[w']_l y w}(pP) = 1$ for all sufficiently big l.

Conversely, suppose that $\chi_{[w']_l yw}(pP) = 1$ for all sufficiently big l, then $[w']_l y[w]_{j_N} \in pP$ for all sufficiently big l and all sufficiently big N, say $[w']_l y[w]_{j_N} = pz$. Let $q_0e_1q_1 \cdots q_{M-1}e_Mq_M$ be a reduced $\mathbf{v}$-word representing pz.

Let p be in the form of a properly reduced positive word, for all sufficiently big l, $[w']_l$ can be represented by a reduced $\mathbf{v}$-word of the form $x'_n x\varepsilon \cdots \varepsilon$ with $\ell(x'_n) > \ell(p)$. Corollary 3.1.12 applied to $m = \ell(x'_n)$ implies that $q_0e_1q_1 \cdots q_{m-1}e_mq_m \in pP$, say $q_0e_1q_1 \cdots q_{m-1}e_mq_m = pz'$ and $z = z'z''$. Since y and $[w]_{j_N}$ are as in condition I., there is a reduced $\mathbf{v}$-word representing $[w']_l y[w]_{j_N}$, which starts with $x'_n x$. By Lemma 3.1.2 (i), we have $q_0e_1q_1 \cdots q_{m-1}e_mq_m a = x'_n$ or $q_0e_1q_1 \cdots q_{m-1}e_mq_m = x'_n a$. In the first case, we have $[w']_l \in x'_n P \subseteq q_0e_1q_1 \cdots q_{m-1}e_mq_m P \subseteq pP$ and thus $\eta(pP) = \chi_{w'}(pP) = 1$. In the second case, $x'_n xy[w]_{j_N} = [w']_l y[w]_{j_N} = pz = pz'z'' = x'_n az''$ and thus $xy[w]_{j_N} = az''$. Lemma 3.1.11 provides a properly reduced positive word representing az'' starting with $aa' \in P_u$ for some $u \in V$. Now condition I. implies that $x \in aa'P \subseteq aP$. This in turn yields $[w']_l = x'_n x \in x'_n aP = pz'P \subseteq pP$ and thus $\eta(pP) = \chi_{w'}(pP) = 1$.

□

Lemma 5.2.2. *Suppose that condition II. holds.*
(i) Let $\chi \in \Omega_\infty$ with $\chi(\boldsymbol{b}P) = 0$. For $\eta \in \Omega$ such that $\eta = \chi_{w'}$ for some infinite positive word w' with $lim_{l\to\infty}\ell([w']_l) = \infty$, we have $\eta \in \overline{G \cdot \chi}$.
(ii) Let $\chi \in \Omega_\infty$ be arbitrary. For $\eta \in \Omega$ such that $\eta = \chi_{w'}$ for some infinite positive word w' with $lim_{l\to\infty}\ell([w']_l) = \infty$ and $g \cdot \eta(\boldsymbol{b}^i P) = 1$ for all $g \in G$ for which $g \cdot \eta$ is defined and all positive integers i, we have $\eta \in \overline{G \cdot \chi}$.

Proof. Let $x_0f_1x_1 \cdots x_{n-1}f_nx_n$ be a properly reduced positive word representing $[w']_l$ as in the proof of Lemma 5.2.1. Condition II., applied to χ and $x = x_n$ if $x_n \neq \emptyset$ and $x = f_n$ if $x_n = \emptyset$, provides w, $[w]_{j_N}$ and y as above. Note that these depend on l. We now claim that

$\lim_{l\to\infty}\chi_{[w']_l yw} = \eta$.

(i) B) in condition II. leads to a contradiction to the assumption that $\chi(\mathbf{b}P) = 0$ since $[w]_{j_N} \in \mathbf{b}P$ implies $\chi(\mathbf{b}P) = 1$. Hence we always have statement A) when condition II. is applied to χ and x as above. Therefore, $\lim_{l\to\infty}\chi_{[w']_l yw} = \eta$ follows by the same argument as in the proof of Lemma 5.2.1.

(ii) If $\eta(pP) = 1$, we obtain that $\chi_{[w']_l yw}(pP) = 1$ for all sufficiently big l as in the proof of Lemma 5.2.1.

If $\chi_{[w']_l yw}(pP) = 1$ for all sufficiently big l, we can then use A) in condition II. and the same argument as in the proof of Lemma 5.2.1 to show $\eta(pP) = 1$, or we can use B) in condition II. and the same argument as in the proof of Lemma 5.2.1 to show that $[w']_l\mathbf{b}^i \in pP$ for some positive integer i. Now our assumption that $g\cdot\eta(\mathbf{b}^iP) = 1$ for all $g \in G$ implies for $g = [w']_l^{-1}$ that $[w']_l^{-1}\cdot\eta(\mathbf{b}^iP) = 1$ and $\eta([w']_l\mathbf{b}^iP) = 1$. This, together with $[w']_l\mathbf{b}^i \in pP$, yields that $\eta(pP) = 1$.

$\square$

Suppose that condition II. holds, define

$$\Omega_{\mathbf{b},\,\infty} := \{\chi \in \Omega,\ (g\cdot\chi)(\mathbf{b}^iP) = 1,\ \forall g,\ i\},$$

where we only consider those $g \in G$ such that $g\cdot\chi$ is well defined. Note that we always have $\Omega_{\mathbf{b},\,\infty} \subseteq \Omega_\infty$.

To summarize, here is the conclusion.

Lemma 5.2.3. *Suppose that condition I. holds. For any $\chi,\ \eta \in \Omega$, we have $\eta \in G \cdot \Omega_{P_u}$ for some $u \in V$ or $\eta \in \overline{G \cdot \chi}$.*

Suppose that condition II. holds.

(i) For any $\chi \in \Omega$ with $\chi(\boldsymbol{b}P) = 0$ and $\eta \in \Omega$, we have $\eta \in G \cdot \Omega_{P_u}$ for some $u \in V$ or $\eta \in \overline{G \cdot \chi}$.

(ii) For any $\chi \in \Omega$ and $\eta \in \Omega_{\boldsymbol{b},\ \infty}$, we have $\eta \in G \cdot \Omega_{P_u}$ for some $u \in V$ or $\eta \in \overline{G \cdot \chi}$.

Here Ω_{P_u} is the collection of all characters of the form $\chi_{w'}$, where $w' = \varepsilon$ or w' consists of letters in $P_u \setminus \varepsilon$.

Proof. It suffices to show that if $\eta = \chi_{w'}$ with $\sup_l \ell([w']_l) < \infty$, we then have $\eta \in G \cdot \Omega_{P_u}$ for some $u \in V$.

If w' is a finite word, then $\eta = g \cdot \chi_\varepsilon$ with $g = w'$.

If $w' = x_1 x_2 x_3 \cdots$ with $x_j \in \{P_v \setminus \{\varepsilon\}\}_{v \in V} \cup A$ is an infinite word, then we must have $x_j \in P_u$ for all sufficiently big j and some $u \in V$ (independent of j), which entails $\eta \in G \cdot \Omega_{P_u}$. Otherwise, there exists a sequence $(j_n)_n$ of positive integers such that either $x_{j_n} \in A$ for all n or $x_{j_n} \in P_{v_n}$ with $v_n \neq v_{n+1}$ for all n. In the first case, we have $\sup_l \ell([w']_l) \geq \sum_n 1 = \infty$ since each $x_{j_n} \in A$ contributes at least length 1 in $\ell([w']_l)$ with l sufficiently big. In the second case, similarly we have $\sup_l \ell([w']_l) \geq \sum_n \ell([v_n,\ v_{n+1}]) = \infty$. In both cases, it leads to a contradiction to the assumption that $\sup_l \ell([w']_l) < \infty$.

$\square$

Now we turn to the following question: When do we have condition I. or condition II.?

In the following, we will assume without loss of generality that $P_v \neq \{\varepsilon\}$ for all $v \in V$,

$P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in T$.

Lemma 5.2.4. *Assume that $P_v \neq \{\varepsilon\}$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in T$. If there exists $e \in T$ such that $P_e = \{\varepsilon\}$, then condition I. is satisfied.*

Proof. Let $x \in P_v \setminus \varepsilon$ or $x \in A$. In the latter case, set $v := t(x)$. Let χ and w be as in condition I.

First assume that there exists a strictly increasing sequence $(j_N)_N$ of positive integers such that, $[w]_{j_N}$ can be represented by a properly reduced positive word with first letter in P_v or first letter in E with origin v, for all N. Assume that $[v,\ o(e)]$ does not contain $t(e)$, otherwise replace e by $\bar{e}$. Take $y \in P_{t(e)} \setminus \{\varepsilon\}$. Then $xy[w]_{j_N}$ is reduced, and we can assume without loss of generality that $xy[w]_{j_N}$ is properly reduced (when we replace x and $[w]_{j_N}$ by suitable positive words representing them). Suppose that $x \in P_v$, the case $x \in A$ is similar. If $p_0 d_1 p_1 \cdots$ is a properly reduced positive word representing $xy[w]_{j_N}$, then we have $x = p_0 a$ or $xa = p_0$. In the first case, we are done. The second case leads to $a = \varepsilon$ using that $P_e = \{\varepsilon\}$.

Now assume that there exists a strictly increasing sequence $(j_N)_N$ of positive integers such that, $[w]_{j_N}$ can be represented by a properly reduced positive word with first letter not in P_v or first letter in E with origin not equal to v, for all N. Assume that $[v,\ o(e)]$ does not contain $t(e)$, otherwise replace e by $\bar{e}$. Take $y_1 \in P_{t(e)} \setminus \{\varepsilon\}$ and $y_2 \in P_v \setminus P_f^f$, where $[t(e),\ v]$ ends with $f \in T$. Define $y := y_1 y_2$. Then $xy[w]_{j_N}$ is reduced, and we can assume without loss of generality that $xy[w]_{j_N}$ is properly reduced (when we replace x and $[w]_{j_N}$ by suitable positive words representing them). The same argument as in the first case shows that condition I. holds.

□

To get examples satisfying condition II., we now assume that $G_v \subseteq (\mathbb{R},\ +)$ for all $v \in V$. Furthermore, we assume that, in addition to our assumption above, $P_e^e \neq P_{t(e)}$ for all $e \in A \cup \bar{A}$. For convenience, we will still use multiplicative notation.

Noting that either $G_v \cong \mathbb{Z}$ (with a least positive element) or G_v is dense in $\mathbb{R}$ (without least positive elements), we have the following result.

Lemma 5.2.5. *Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R},\ +)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$. Then P_e^e is not dense in $P_{t(e)}$ for all $e \in E$. Moreover, $P_e \cong \mathbb{Z}_{\geq 0}$ or $P_e = \{\varepsilon\}$ for all $e \in E$.*

Proof. If P_e^e is dense in $P_{t(e)}$ for some $e \in E$, then we can find $p \in P_{t(e)} \setminus P_e^e$ and a sequence $(p_n)_n \subseteq P_e^e$ such that $p \prec p_n$ and $\lim_{n\to\infty} p_n = p$. Then $p^{-1}p_n \in p^{-1}P_e^e = p^{-1,\,\bar{e}}P_e^e$, which implies $p^{-1,\,\bar{e}} \prec p^{-1}p_n$ for all n. This entails $p^{-1,\,\bar{e}} = \varepsilon$ and thus $p^{-1}P_e^e = P_e^e$, contradicting our picking $p \in P_{t(e)} \setminus P_e^e$.

For all $e \in E$, we always have one of the following: $P_e \cong \mathbb{Z}_{\geq 0}$, $P_e = \{\varepsilon\}$ or P_e is dense in $(\mathbb{R}_+,\ +)$. In the third case, it entails that P_e^e is dense in $P_{t(e)}$.

□

Lemma 5.2.6. *Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R},\ +)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$. If $P_e \neq \{\varepsilon\}$ for all $e \in T$, then condition II. is satisfied if one of the following is satisfied:*

(i) $\sharp V > 1$;

(ii) $\sharp A_+ > 0$.

Proof. To begin with, we assume $\sharp V > 1$. Take $e \in T$ such that $[v,\ o(e)]$ does not contain $t(e)$ (otherwise replace e by $\bar{e}$), and let $\mathbf{b}$ be the generator of $P_e^e \cong \mathbb{Z}_{\geq 0}$. Take $x \in P_v \setminus \{\varepsilon\}$ or $x \in A$, and let χ and w be as in condition II..

Suppose that there exists a strictly increasing sequence $(j_N)_N$ of positive integers such that $[w]_{j_N}$ can be represented by a properly reduced positive word with first letter in P_{v_N} or first letter in E with origin v_N such that v and v_N are on the same side of e for all N. Take $y_1,\ y_3 \in P_{t(e)}$ and $y_2 \in P_{o(e)}$ such that $y_1,\ y_3 \prec z$ for all $z \in P_e^e \setminus \{\varepsilon\}$ and that $y_2 \prec \bar{z}$ for all $\bar{z} \in P_{\bar{e}}^{\bar{e}} \setminus \{\varepsilon\}$, define $y := y_1 y_2 y_3$. Then $xy[w]_{j_N}$ is reduced, and we can assume without loss of generality that $xy[w]_{j_N}$ is properly reduced (when we replace x and $[w]_{j_N}$ by suitable positive words representing them). Let us now treat the case that $x \in P_v$, the case $x \in A$ is similar.

Let $p_0 p_1 \cdots p_m y w'$ be a properly reduced positive word representing $xy[w]_{j_N}$, then either $x = p_0 p_1 \cdots p_m z$ or $p_0 p_1 \cdots p_m = xz$ for some $z \in P_{\bar{e}}^{\bar{e}}$. In the first case, A) in condition II. is satisfied. In the second case, we obtain $xz \in p_0 P_T$ for some $z \in P_{\bar{e}}^{\bar{e}} = P_e^e$. That is, $x\mathbf{b}^i \in p_0 P_T$ for some positive integer j. In the mean time, $xy[w]_{j_N} = p_0 p_1 \cdots p_m y w' = xzyw' = xyz'w'$ for some $z' \in P_e^e$ with $zy = yz'$, which means $[w]_{j_N} = z'w' \in z'P \subseteq \mathbf{b}P$. B) in condition II. is satisfied.

Suppose that there exists a strictly increasing sequence $(j_N)_N$ of positive integers such that $[w]_{j_N}$ can be represented by a properly reduced positive word with first letter in P_{v_N} or first letter in E with origin v_N such that v and v_N are on opposite sides of e for all N. Take $y_1 \in P_{t(e)}$ and $y_2 \in P_{o(e)}$ such that $y_1 \prec z$ for all $z \in P_e^e \setminus \{\varepsilon\}$ and that $y_2 \prec \bar{z}$ for all $\bar{z} \in P_{\bar{e}}^{\bar{e}} \setminus \{\varepsilon\}$, define $y := y_1 y_2$. Then $xy[w]_{j_N}$ is reduced, and we can assume without loss of generality that $xy[w]_{j_N}$ is properly reduced (when we replace x and $[w]_{j_N}$ by suitable positive words representing them). Let us now treat the case that $x \in P_v$, the case $x \in A$ is similar.

Let $p_0p_1\cdots p_myw'$ be a properly reduced positive word representing $xy[w]_{j_N}$, then either $x = p_0p_1\cdots p_mz$ or $p_0p_1\cdots p_m = xz$ for some $z \in P_{\bar{e}}^{\bar{e}}$. In the first case, A) in condition II. is satisfied. In the second case, we obtain $xz \in p_0P_T$ for some $z \in P_{\bar{e}}^{\bar{e}} = P_e^e$. That is, $x\mathbf{b}^i \in p_0P_T$ for some positive integer j. In the mean time, $xy[w]_{j_N} = p_0p_1\cdots p_myw' = xzyw' = xyz'w'$ for some $z' \in P_e^e$ with $zy = yz'$, which means $[w]_{j_N} = z'w' \in z'P \subseteq \mathbf{b}P$. B) in condition II. is satisfied.

Now assume $\sharp A_+ > 0$. Take $e \in A_+$ and let $\mathbf{b}$ be the generator of $P_e^e \cong \mathbb{Z}_{\geq 0}$. Take $x \in P_v \setminus \{\varepsilon\}$ or $x \in A$, and let χ and w be as in condition II.. Let $(j_N)_N$ be a strictly increasing sequence of positive integers and define $y := e$. Then $xy[w]_{j_N}$ is reduced, and we can assume without loss of generality that $xy[w]_{j_N}$ is properly reduced (when we replace x and $[w]_{j_N}$ by suitable positive words representing them). Let us now treat the case that $x \in P_v$, the case $x \in A$ is similar.

Let $p_0p_1\cdots p_myw'$ be a properly reduced positive word representing $xy[w]_{j_N}$, then either $x = p_0p_1\cdots p_mz$ or $p_0p_1\cdots p_m = xz$ for some $z \in P_{\bar{e}}^{\bar{e}}$. In the first case, A) in condition II. is satisfied. In the second case, we obtain $xz \in p_0P_T$ for some $z \in P_{\bar{e}}^{\bar{e}}$. We can find $z' \in P_e^e$ with $z \prec z'$. Therefore, $xz' \in xzP_T \subseteq p_0P_T$. That is, $x\mathbf{b}^i \in p_0P_T$ for some positive integer i. In the mean time, we have $xy[w]_{j_N} = p_0p_1\cdots p_myw' = xzyw' = xyz''w'$ for some $z'' \in P_e^e$ with $zy = yz''$, which means $[w]_{j_N} = z''w' \in z''P \subseteq \mathbf{b}P$. B) in condition II. is satisfied.

$\square$

Now we are ready to determine all closed invariant subsets of Ω.

Lemma 5.2.7. *Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R}, +)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$. If condition I. holds and there exists $v \in V$ such that G_v is dense in*

$(\mathbb{R},\,+)$*, then* $\partial\Omega = \Omega$.

Proof. Let $\chi \in \Omega_\infty$ be arbitrary and choose $x_n \in P_v \setminus \{\varepsilon\}$ such that $\lim_{n\to\infty} x_n = \varepsilon$. Let y, w and $(j_N)_N$ be as in condition I.. We now claim that $\lim_{n\to\infty} \chi_{x_n y w} = \chi_\varepsilon$. As in Lemma 5.2.4, we may assume without loss of generality that $x_n y[w]_{j_N}$ is properly reduced. If $\chi_{x_n y w}(pP) = 1$ for all sufficiently big n, then $x_n y[w]_{j_N} \in pP$ for all sufficiently big n and all sufficiently big N. Assume $p \neq \varepsilon$ and let $p_0 d_1 p_1 \cdots$ be a properly reduced word representing p. We treat the case $p_0 \in P_{v_0} \setminus \{\varepsilon\}$, the case $p_0 = \emptyset$ is analogous. $x_n y[w]_{j_N} \in pP$ means that $x_n y[w]_{j_N} = pz$ for some z. By Lemma 3.1.11, there is a properly reduced positive word with first letter $p_0 z'$ representing pz. Comparing properly reduced positive words, we must have $p_0 z' \in P_v$ by Lemma 3.1.8. Condition I. implies $x_n \in p_0 z' P_T$ and thus $p_0 \prec x_n$ for all sufficiently big n, contradicting our choice of x_n.

□

We now turn to condition II. Note that in that case, we must have $P_e \cong \mathbb{Z}_{\geq 0}$ for all $e \in T$, and thus P_T is Ore. We write $\partial\Omega_{P_T} = \{\infty\}$.

Lemma 5.2.8. *Assume that* $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R},\,+)$ *for all* $v \in V$*,* $P_e \neq \{\varepsilon\}$ *for all* $e \in A$ *and* $P_e^e \neq P_{t(e)}$ *for all* $e \in E$*. If condition II. holds and* $\sharp A = 0$*, then for all* $\chi \neq \infty$ *and* $\eta \in \Omega_\infty$*, we have* $\eta \in \overline{G \cdot \chi}$*.*

If condition II. holds, $\sharp A \geq 1$ *and there exists* $v \in V$ *such that* G_v *is dense in* $\mathbb{R}$*, then* $\partial\Omega = \Omega_{\boldsymbol{b},\,\infty}$*. Moreover, for every* $\chi \notin \Omega_{\boldsymbol{b},\,\infty}$*, we have* $\overline{G \cdot \chi} = \Omega$*.*

Proof. Assume firstly $\sharp A = 0$. If $\eta = \chi_w$ for some infinite positive word w with $\lim_{l\to\infty} \ell([w]_l) =$

∞, then we already know that $\eta \in \overline{G \cdot \chi}$ by Lemma 5.2.2. Otherwise Lemma 5.2.3 implies that $\eta \in G \cdot \Omega_{P_v}$ for some $v \in V$. If $P_v \cong \mathbb{Z}_{\geq 0}$, then $\eta \in \Omega_\infty$ implies $\eta = \infty$, and our claim follows. If G_v is dense in $\mathbb{R}$, let $(x_n)_n$ be a sequence in P_v such that $\eta = \lim_{n \to \infty} \chi_{x_n}$. Without loss of generality we may assume $\chi(\mathbf{b}P) = 0$. Let y, w and $(j_N)_N$ be as in condition II. for $x = x_n$. Note that in the proof of Lemma 5.2.6, y and $(j_N)_N$ were chosen so that they only depend on v, not on x_n. Moreover, as in the proof of Lemma 5.2.6, the first letter of y lies in P_t, and suppose that $[v,\ t]$ starts with $d \in T$. Without loss of generality we may assume that $x_n \prec z$ and $x_n \neq z$ for all $z \in P_{\bar{d}}^{\bar{d}}$. This is because $G_v \curvearrowright \Omega_{P_v} \setminus \{\infty\}$ is minimal. We claim that $\eta = \lim_{n \to \infty} \chi_{x_n y w}$. Indeed, suppose that $\chi_{x_n y w}(pP) = 1$. Then $x_n y [w]_{j_N} \in pP$. As before, $x_n y [w]_{j_N}$ is reduced, and we can assume without loss of generality that $x_n y [w]_{j_N}$ is properly reduced (when we replace x_n and $[w]_{j_N}$ by suitable positive words representing them). Suppose that $p = p_0 p_1 \cdots p_m$ is a properly reduced word with $p_k \in P_{v_k}$. We proceed inductively on $l(p)$ to show that $x_n \in pP$. $x_n y [w]_{j_N} \in pP$ implies that $x_n y [w]_{j_N} = pz$ for some z in P. If $l(p) = 0$, then $p = p_0$ and Lemma 3.1.11 implies that pz can be represented by a properly reduced positive word with first letter of the form $p_0 z'$. Now condition II. implies that $x_n \in p_0 z' P_T$ as otherwise, we would get $[w]_{j_N} \in \mathbf{b}P$, contradicting $\chi(\mathbf{b}P) = 0$. Now suppose that $l(p) \geq 1$. First let z be expressed as a properly reduced positive word. If pz is properly reduced, then Lemma 3.1.8 implies that $p_0 \in P_v$ and $[v_0,\ v_1]$ must start with d. As before, condition II. and $\chi(\mathbf{b}P) = 0$ imply that $x_n = p_0 a$ for some $a \in P_{\bar{d}}^{\bar{d}}$. But $x_n \prec z$ and $x_n \neq z$ for all $z \in P_{\bar{d}}^{\bar{d}}$ implies $a = \varepsilon$, and we are done. If pz is not properly reduced, then we can write $pz = (pz')z''$ such that $l(pz') < l(p)$. By induction hypothesis, we obtain $x_n \in pz'P \subseteq pP$, as desired.

Now we assume $\sharp A \geq 1$ and there exists $v \in V$ such that G_v is dense in $\mathbb{R}$. To prove $\partial\Omega = \Omega_{\mathbf{b},\,\infty}$, we need to prove $\Omega_{\mathbf{b},\,\infty} \subseteq \partial\Omega$. By Lemma 5.2.3, it suffices to show $\{\infty\} = \Omega_{P_u} \cap \Omega_{\mathbf{b},\,\infty} \subseteq \partial\Omega$.

Take $e \in A$ and a strictly decreasing sequence $(y_n)_n$ in P_v such that $\lim_{n\to\infty} y_n = \varepsilon$. Let $\chi \in \Omega$ be arbitrary and write $\chi = \chi_w$ for some infinite positive word w. By compactness, we can — by passing to a subsequence if necessary — assume that $\chi' := \lim_{n\to\infty} \chi_{y_n e w}$ exists. We claim that $\chi' \in \Omega_{P_T}$. Indeed, if not, then we must have $\chi'(peP) = 1$ for some $p \in P$. It follows that $pG_{\bar{e}}^{\bar{e}} = y_n G_{\bar{e}}^{\bar{e}}$ for all n. Hence $y_m G_{\bar{e}}^{\bar{e}} = y_n G_{\bar{e}}^{\bar{e}}$ for all m and n. But this contradicts $\lim_{n\to\infty} y_n = \varepsilon$. So we obtain that $\Omega_{P_T} \cap \overline{G \cdot \chi} \neq \emptyset$, so that $\infty \in \overline{G \cdot \chi}$.

Now we show $\overline{G \cdot \chi} = \Omega$ for every $\chi \notin \Omega_{\mathbf{b},\,\infty}$. We may assume that $\chi(\mathbf{b}P) = 0$. If $\sharp V > 1$ or $\sharp A > 0$, then a similar argument as in Lemma 5.2.2 shows the following: If we take $e \in A$ and a sequence $(x_n)_n$ in P_v such that $\lim_{n\to\infty} x_n = \varepsilon$ and write $\chi = \chi_w$ for some infinite positive word w, then $\lim_{n\to\infty} \chi_{x_n e w} = \chi_\varepsilon$. If $\sharp V = 1$ and $A = A_- \neq \emptyset$, and if we write $\chi = \chi_w$ for some infinite positive word w, then $\chi(\mathbf{b}P) = 0$ implies that no $e \in A_-$ can appear in w, so that $\chi \in \Omega_{P_v} \setminus \{\infty\}$. Now our claim follows because $G_v \curvearrowright \Omega_{P_v} \setminus \{\infty\}$ is minimal.

□

Lemma 5.2.9. *Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R},\,+)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$. If condition I. holds and $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$, then for all $v \in V$, there exists an infinite positive word w with $\lim_{l\to\infty} \ell([w]_l) = \infty$ such that $\Omega_\infty \cap \Omega_{P_v} \subseteq \overline{G \cdot \chi_w}$.*

If condition II. holds, $\sharp V > 1$ and $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$, then for for all $v \in V$ and all $\chi \in \Omega$, we have $\Omega_\infty \cap \Omega_{P_v} \subseteq \overline{G \cdot \chi}$.

Note that if condition II. holds and $\sharp V = 1$, then we are in the case of generalized Baumslag-Solitar monoids.

Proof. Suppose that condition I. holds, then there exists $e \in T$ such that $P_e = \{\varepsilon\}$. In particular, $\sharp V > 1$. Let b_v be the generator of P_v, then $\Omega_\infty \cap \Omega_{P_v} = \{\chi_{b_v b_v b_v} \cdots\}$. Take $v' \in V$ with $\ell([v,\ v']) = 1$, and define $f := [v,\ v']$. Set $w := b_{v'} b_v b_{v'} b_v \cdots$, where $b_{v'}$ is the generator of $P_{v'}$, we claim that $\lim_{n\to\infty} \chi_{b_v^n w} = \chi_{b_v b_v b_v} \cdots$. Indeed, if $\chi_{b_v^n w}(pP) = 1$ for all sufficiently big n, then we have $b_v^n b_{v'} b_v \cdots b_{v'} b_v \in pP$ for all sufficiently big n. Since $P_{\bar{f}}^{\bar{f}} \neq P_v$ and $P_f^f \neq P_{v'}$, we must have $b_v^n \in pP$ for all sufficiently big n. Hence $\chi_{b_v b_v b_v} \cdots (pP) = 1$.

Suppose that condition II. holds and assume $\sharp V > 1$, then P_T is Ore and thus $\Omega_\infty \cap \Omega_{P_v} = \{\infty\}$. Take $w,\ v \in V$ with $w \neq v$, and let b_w and b_v be the generators of P_w and P_v, respectively. Take $\chi \in \Omega$. If $\chi \in \Omega_{P_T}$, then there is nothing to show. If $\chi \notin \Omega_{P_T}$, then there exist $q \in P_T$ and $e \in A$ with $\chi(qeP) = 1$. By compactness, we can find a sequence n_i such that $(b_w b_v)^{n_i} \cdot \chi$ converges to η. We claim that $\eta \in \Omega_{P_T}$. If not, then there exists $p \in P$ such that $\eta(peP) = 1$. It follows that $(b_w b_v)^{n_i} q G_{\bar{e}}^{\bar{e}} = p G_{\bar{e}}^{\bar{e}}$ and thus $(b_w b_v)^{n_i} q G_{\bar{e}}^{\bar{e}} = (b_w b_v)^{n_j} q G_{\bar{e}}^{\bar{e}}$ for all $i,\ j$. Hence, if we set $m_j = n_j - n_1$, then $(b_w b_v)^{m_j} q = q g_j$ for some $g_j \in G_{\bar{e}}^{\bar{e}}$. The length $\ell(q g_j)$ is bounded (independent of j), while the length $\ell((b_w b_v)^{m_j} q)$ tends to infinity as $j \to \infty$. So this is a contradiction, as desired.

$\square$

Lemma 5.2.10. *Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R},\ +)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$. If $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$ and $P_e \neq \{\varepsilon\}$ for all $e \in T$, then $\Omega_{\boldsymbol{b},\ \infty} = \Omega_\infty$ if and only if $\sharp V = 1$ and $A = A_- \neq \emptyset$.*

In particular, if $\sharp V > 1$, then $\Omega_{\boldsymbol{b},\ \infty} \subsetneq \Omega_\infty$.

Proof. If $\sharp V = 1$ and $A = A_- \neq \emptyset$, every character $\chi_w \in \Omega_\infty$ satisfies that w contains either infinitely many letters in A_- or infinitely many letters in P_v, where $v \in V$ is the unique vertex. Noting that G_v is totally ordered, we obtain in both cases that $(g \cdot \chi_w)(\mathbf{b}^i P) = 1$ for all $g \in G$ with $g \cdot \chi$ defined and all positive integers i. That is, $\Omega_{\mathbf{b},\ \infty} = \Omega_\infty$.

If $\sharp V > 1$, take $v,\ w \in V$ with $v \neq w$ and let b_v and b_w be the generators of P_v and P_w, then we have $\chi_{b_v b_w b_v b_w \cdots} \in \Omega_\infty \setminus \Omega_{\mathbf{b},\ \infty}$. If $A_+ \neq \emptyset$, take $a \in A_+$, then $\chi_{aaa\cdots} \in \Omega_\infty \setminus \Omega_{\mathbf{b},\ \infty}$.

□

We can now summarize our findings as follows:

Theorem 5.2.11. *Let P be the fundamental monoid of a graph of monoids with condition (LCM) for P satisfied. Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R},\ +)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$.*

(i) If condition I. holds and there exists $v \in V$ such that G_v is dense in $\mathbb{R}$, then the following is the list of all nonempty closed invariant subsets of Ω: $\partial\Omega = \Omega$.

(ii) If condition I. holds and $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$, then the following is the list of all nonempty closed invariant subsets of Ω: $\partial\Omega = \overline{\Omega_\infty} \subseteq \Omega$.

(iii) If condition II. holds, there exists $v \in V$ such that G_v is dense in $\mathbb{R}$ and $\sharp A \geq 1$, then the following is the list of all nonempty closed invariant subsets of Ω: $\Omega_{\boldsymbol{b},\ \infty} = \partial\Omega \subsetneq \Omega$.

(iv) If condition II. holds and $\sharp A = 0$, then the following is the list of all nonempty closed invariant subsets of Ω: $\{\infty\} = \partial\Omega \subsetneq \overline{\Omega_\infty} \subseteq \Omega$.

(v) If condition II. holds, $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$, $\sharp A \geq 1$ and $\sharp V > 1$, then the following is the list of all nonempty closed invariant subsets of Ω: $\Omega_{\boldsymbol{b},\ \infty} = \partial\Omega \subsetneq \overline{\Omega_\infty} \subseteq \Omega$.

Proof. (i) It follows directly from Lemma 5.2.7.

(ii) For any characters $\eta,\ \chi \in \Omega_\infty$, we have by Lemma 5.2.3 either $\eta \in G \cdot \Omega_{P_v}$ for some $v \in V$ or $\eta \in \overline{G \cdot \chi}$. In the first case, by Lemma 5.2.9, there exists $\chi_w \in \Omega_\infty$ with $\lim_l [w]_l = \infty$ such that $\eta \in \overline{G \cdot \chi_w}$. By Lemma 5.2.1, we get $\chi_w \in \overline{G \cdot \chi}$ and thus $\eta \in \overline{G \cdot \chi}$.

(iii) It follows directly from Lemma 5.2.8.

(iv) It follows from Lemma 5.2.8.

(v) For any characters $\eta \in \Omega_{\mathbf{b},\,\infty}$ and $\chi \in \Omega_\infty$, we have by Lemma 5.2.3 either $\eta \in G\cdot\Omega_{P_v}$ for some $v \in V$ or $\eta \in \overline{G\cdot\chi}$. In the first case, Lemma 5.2.9 implies $\eta \in \overline{G\cdot\chi}$ as well. Now take $\chi \notin \Omega_{\mathbf{b},\,\infty}$ and assume without loss of generality that $\chi(\mathbf{b}P) = 0$. Take $\eta \in \Omega_\infty$, by Lemma 5.2.3 and Lemma 5.2.9, we obtain similarly $\eta \in \overline{G\cdot\chi}$.

□

The following result is included for completeness.

Lemma 5.2.12. *Suppose we are in the same setting as in the theorem above, then $\overline{\Omega_\infty} = \Omega$ if and only if one of the following is satisfied:*

(a) There exists $v \in V$ such that G_v is dense in $\mathbb{R}$;

(b) $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$ and $\sharp V = \infty$;

(c) $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$ and $\sharp A_+ = \infty$.

Proof. ($\Longleftarrow$): Firstly assume there exists $v \in V$ such that G_v is dense in $\mathbb{R}$. For all $x \in \mathbb{R}_+ \setminus P_v$, there exists a sequence $(x_n)_n \subseteq P_v$ such that $\lim_{n\to\infty} x_n = x$. Define $\chi_x := \lim_{n\to\infty}\chi_{x_n}$, it is easy to see that χ_x is well defined and independent of the choice of the sequence $(x_n)_n$. Moreover, $\chi_x \in \Omega_\infty$. $\mathbb{R}_+ \setminus P_v$ is dense in $\mathbb{R}_+$ since P_v is countable. As a result, $\chi_\varepsilon \in \overline{\{\chi_x,\, x \in \mathbb{R}_+ \setminus P_v\}} \subseteq \overline{\Omega_\infty}$.

Now assume $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$. If $\sharp V = \infty$, set $V := \{v_i\}_{i\in\mathbb{N}}$. Let b_{v_i} be the generator of P_{v_i} and define $\chi_n := \chi_{b_{v_n}}\chi_{b_{v_{n+1}}}\chi_{b_{v_{n+2}}}\cdots$, $n \in \mathbb{N}$. It is easy to see that $\lim_{n\to\infty}\chi_n = \chi_\varepsilon$. If $\sharp A_+ = \infty$, set $A_+ := \{a_i\}_{i\in\mathbb{N}}$. Define $\chi_n := \chi_{a_n}\chi_{a_{n+1}}\chi_{a_{n+2}}\cdots$, $n \in \mathbb{N}$. It is easy to see that $\lim_{n\to\infty}\chi_n = \chi_\varepsilon$.

($\Longrightarrow$): Assume $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$, $\sharp V < \infty$ and $\sharp A_+ < \infty$. Let b_v be the generator of P_v, then for all $\chi \in \Omega_\infty$, either $\chi(b_v P) = 1$ for some $v \in V$ or $\chi(aP) = 1$ for some $a \in A_+$. Take a convergent sequence $(\chi_n)_n \subseteq \Omega_\infty$, then either there exists $v \in V$ such that $\chi_n(b_v P) = 1$ for all sufficiently big n or there exists $a \in A$ such that $\chi_n(aP) = 1$ for all sufficiently big n, which implies $\lim_{n\to\infty}\chi_n \neq \chi_\varepsilon$.

□

5.2.2 Generalised Baumslag-Solitar case

As the readers may have found, we assume $\sharp V > 1$ in Lemma 5.2.9 and Theorem 5.2.11 when condition II. holds and $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$. In this section, we focus on this missing case: condition II. holds, $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$ and $\sharp V = 1$, and then work out all the closed invariant subsets of Ω.

We never consider the case when the graph (V, E) consists of a single vertex. So $\sharp V = 1$ yields that $\sharp A > 0$. We also assume that $P_e \neq \{\varepsilon\}$ for all $e \in A$, that is, $P_e \cong \mathbb{Z}_{\geq 0}$. But in this section, we do not require $P_e^e \neq P_v$ anymore, where $e \in A \cup \bar{A}$ and $v \in V$ is the unique vertex.

Let b be the generator of P_v, let $A = \{a_i\}_{i\in S}$ for some index set S and let x_i be the generator of P_{a_i}. Assume that the map $P_{a_i} \to P_{v=t(a_i)}$ maps x_i to b^{m_i} and the map $P_{\bar{a}_i} \to P_{v=o(a_i)}$ maps x_i to $b^{|n_i|}$. By Proposition 3.1.1, we have the following expression of the graph of monoids P:

$$P = GBS_+(N, m_i, n_i) = < a_i, b \mid a_i b^{m_i} = b^{n_i} a_i, \forall i \in S_1, b^{|n_i|} a_i b^{m_i} = a_i, \forall i \in S_2, N = \sharp A = \sharp S >_+,$$

where $S_1 := \{i \in S,\ a_i \in A_+\} = \{i \in S, n_i > 0\}$ and $S_2 := \{i \in S,\ a_i \in A_-\} = \{i \in S, n_i < 0\}$. It is easy to see that P is a generalised Baumslag-Solitar monoid.

We firstly consider the case when N is finite. Let θ_i be the semigroup homomorphism from P to $\mathbb{N}$, given by $\theta_i(a_j) = \delta_{i,\ j}$ and $\theta_i(b) = 0$, and let $\theta := \sum_{i\in S} \theta_i$.

As we characterize the characters by finite or infinite words, we hope to have also a characterization of the subset $\Omega_{\max}$. To begin with, we need the following Lemma, which is from [CELY17, Lemma 5.7.4].

Lemma 5.2.13. *Let F be a semilattice. If $\chi \in \hat{F}_{max}$, then for any $f \in F^{\times}$ with $\chi(f) = 0$, there exists $e \in F^{\times}$ such that $\chi(e) = 1$ and $ef = 0$. Conversely, if $\chi \in \hat{F}$ is such that for any $f \in F^{\times}$ with $\chi(f) = 0$, there exists $e \in F^{\times}$ with $\chi(e) = 1$ and $ef = 0$, then $\chi \in \hat{F}_{max}$.*

Theorem 5.2.14. *If $S_2 = \emptyset$, let $\chi_w \in \Omega_\infty$, then $\chi_w \in \Omega_{max}$ if and only if*
(i) w contains infinitely many a_i's (counting multiplicity); (ii) $\chi_w(b^i P) = 1$ for all $i \in \mathbb{N}$.

Proof. When $S_2 = \emptyset$, for all $i \in \mathbb{N}$ and all $x \in P$, we have $b^i P \cap xP \neq \emptyset$. By Lemma 5.2.13, we must have $\chi_w(b^i P) = 1$ for all $i \in \mathbb{N}$ and all characters $\chi_w \in \Omega_{\max}$. Also by Lemma 5.2.13, we have $\chi_{bbb\cdots} \notin \Omega_{\max}$. Since $\Omega_{\max}$ is G-invariant, χ_w is not maximal for all w containing only finitely many a_i's. That is, for all $\chi_w \in \Omega_{\max}$, w contains infinitely many a_i's.

We now assume $\chi_w \in \Omega_\infty \setminus \Omega_{\max}$ satisfies (i) and (ii) and finish the proof by contradiction.

By Lemma 5.2.13, there exists $x \in P$ with $\chi_w(xP) = 0$ such that for all $y \in P$ with $\chi_w(yP) = 1$, $xP \cap yP \neq \emptyset$. Let

$$x = b^{j_0} a_{i_1} b^{j_1} a_{i_2} \cdots b^{j_{k-1}} a_{i_k} b^p$$

be its standard L-form and let

$$x' = b^{j_0} a_{i_1} b^{j_1} a_{i_2} \cdots b^{j_{k-1}} a_{i_k}.$$

Take $y \in P$ with enough a_i's, since $xP \cap yP \neq \emptyset$, there exist $r,\ s,\ t \in P$ such that $r = xs = yt$ and $xP \cap yP = rP$. xs and yt admit the same standard L-form, so x' is a prefix of the standard L-form of yt and hence of y. That is, $y = x'z$ for some $z \in P$ and $\chi_w(x'P) = 1$.

On the other hand, $x'P \cap b^i P \neq \emptyset$ for all $i \in \mathbb{N}$. Actually, there exist $j \in \mathbb{N}$ and $x'' \in P$ such that $x'b^j = b^i x''$ and $x'P \cap b^i P = x'b^j P$. Furthermore, when i goes up to infinity, j also tends to ∞. Take i big enough such that $j > p$ and hence that $x'b^j P \subseteq xP$. Since $\chi_w(x'P) = \chi_w(b^i P) = 1$, we have $\chi_w(x'b^j P) = 1$ and hence $\chi_w(xP) = 1$, leading to a contradiction.

□

Theorem 5.2.15. *If $S_2 \neq \emptyset$, let $\chi_w \in \Omega_\infty$.*
(i) If w contains infinitely many a_i's with $i \in S_2$ (counting multiplicity), then $\chi_w \in \Omega_{max}$.
(ii) If w contains only finitely many a_i's with $i \in S_2$ (counting multiplicity), then $\chi_w \in \Omega_{max}$ if and only if
(a) w contains infinitely many a_i's with $i \in S_1$ (counting multiplicity); (b) There exists some $j \in \mathbb{N}$ such that $g \cdot \chi_w(b^i P) = 1$ for all $i \in \mathbb{N}$ with $g = [w]_j^{-1}$ and that $\{w\}_j$ does not contain a_i for all $i \in S_2$.

Proof. (i) Take $x \in P$ with $\chi_w(xP) = 0$, and take $y \in P$ satisfying (1) $\theta(y) > \theta(x)$; (2) $\theta_i(y) > \theta_i(x)$ for some $i \in S_2$; (3) $\chi_w(yP) = 1$. We claim $xP \cap yP = \emptyset$.
Let

$$x = b^{j_0} a_{i_1} b^{j_1} a_{i_2} \cdots b^{j_{k-1}} a_{i_k} b^p$$

be its standard L-form, and let $x' = b^{j_0} a_{i_1} b^{j_1} a_{i_2} \cdots b^{j_{k-1}} a_{i_k}$. If $xP \cap yP \neq \emptyset$, there exist $r,\ s,\ t \in$

P such that $r = xs = yt$ and $xP \cap yP = rP$. xs and yt admit the same standard L-form, so x' is a prefix of the standard L-form of yt and hence of y. That is, $y = x'z$ for some $z \in P$ and $\chi_w(x'P) = 1$.

When $p \leq 0$, $x'P \subseteq xP$ and thus $\chi_w(xP) = 1$, contradicting our choice of x.
When $p > 0$, since $\theta_i(y) > \theta_i(x)$ for some $i \in S_2$, we have $\theta_i(z) > 0$ for some $i \in S_2$. In this case, we have $z \in b^p P$ and thus $y \in xP$. This again leads to the conclusion $\chi_w(xP) = 1$, contradicting our choice of x.
In conclusion, our claim is proved. By Lemma 5.2.13, $\chi_w \in \Omega_{\max}$.

(ii) If w contains only finitely many a_i's for all $i \in S_2$, there exists some $j \in \mathbb{N}$ such that $\{w\}_j$ does not contain a_i for all $i \in S_2$. Take $g = [w]_j^{-1}$, we have $\chi_w \in \Omega_{\max}$ if and only if $g\chi_w \in \Omega_{\max}$, which holds if and only if, by Theorem 5.2.14,
(a') gw contains infinitely many a_i's for some i; (b') $g\chi_w(b^i P) = 1$ for all $i \in \mathbb{N}$. An easy analysis implies the equivalence of conditions (a), (b) and conditions (a'), (b').

□

Remark 5.2.16. *Every maximal character χ satisfies $\chi(b^i P) = 1$ for all $i \in \mathbb{N}$.*

Lemma 5.2.17. *Let $w_b = bbb\cdots \in \Sigma^\infty$, then we conclude $\chi_{w_b} \notin \partial\Omega$.*

Proof. We assume $\chi_{w_b} \in \partial\Omega$ and finish the proof by contradiction.
Since $\chi_{w_b} \in \partial\Omega$, there exists a sequence $\{\chi_{w_i}\}_i \subseteq \Omega_{\max}$ such that χ_{w_i} converges pointwisely to χ_{w_b}. For each χ_{w_i}, there exist positive integers j and k with $1 \leq j \leq N$ and $k \in [0,\ |n_j|)$ such that $\chi_{w_i}(b^k a_j P) = 1$. Since there are only finitely many possible values for the pair $(j,\ k)$, there must be some common $1 \leq j \leq N$ and $k \in [0,\ |n_j|)$ such that $\chi_{w_i}(b^k a_j P) = 1$ for infinitely many w_i. Taking the limit, we get $\chi_{w_b}(b^k a_j P) = 1$, which contradicts the fact

$\chi_{w_b}^{-1}(1) = \{P,\ bP,\ b^2P,\ \cdots\}$.

□

Theorem 5.2.18. $\partial\Omega = \Omega_{max}$.

Proof. It suffices to show $\partial\Omega \subseteq \Omega_{\max}$.

Since $\partial\Omega$ is G-invariant, every orbit under the action of the group G is either included in $\partial\Omega$ or intersects it by the empty set.

Let w_b be as above, and then its orbit is

$$\text{Orbit}(w_b) = \{\chi_w \mid w \in \Sigma^\infty \text{ contains only finitely many } a_i\text{'s}\}.$$

It follows from Lemma 5.2.17 that

$$\text{Orbit}(w_b) \cap\ \partial\Omega = \emptyset.$$

Since the orbit $\{\chi_p,\ p \in P\}$ is dense in Ω and the fact $\partial\Omega \subsetneq \Omega$, we conclude

$$\{\chi_p,\ p \in P\} \cap\ \partial\Omega = \emptyset.$$

In conclusion, every character $\chi \in \partial\Omega$ is of the form χ_w for some $w \in \Sigma^\infty$ containing infinitely many a_i's.

If w contains infinitely many a_i's for some $i \in S_2$, then $\chi_w \in \Omega_{\max}$.

If w contains only finitely many a_i's for all $i \in S_2$, then it must contain infinitely many a_i's for some $i \in S_1$. Furthermore, there exists some $j \in \mathbb{N}$ such that $\{w\}_j$ does not contain a_i for all $i \in S_2$. Let $g = [w]_j^{-1}$, then $g\chi_w$ is also in $\partial\Omega$ and thus $g\chi_w(b^iP) = 1$ for all $i \in \mathbb{N}$. By Theorem 5.2.15, $\chi_w \in \Omega_{\max}$.

□

Now we need to define several maps to help with our analysis of the closed invariant subsets of Ω. We firstly define a map τ,

$$\tau : \Sigma^\infty \setminus \{w_b\} \ \to \ \left(\cup_{M\in\mathbb{N}} S^M\right) \cup S^{\mathbb{N}},$$

$$\tau(b^{k_0}a_{j_1}b^{k_1}a_{j_2}\cdots b^{k_{M-1}}a_{j_M}bbb\cdots) = (j_1,\ j_2,\ \cdots,\ j_M).$$

Then we define the map β as follows,

$$\beta : \Sigma^\infty \setminus \{w_b\} \ \to \ \left(\cup_{M\in\mathbb{N}} \mathbb{Z}^M\right) \cup \mathbb{Z}^{\mathbb{N}},$$

$$\beta(b^{k_0}a_{j_1}b^{k_1}a_{j_2}\cdots b^{k_{M-1}}a_{j_M}bbb\cdots) = (r_0,\ r_1,\ \cdots,\ r_{M-1}),$$

where $r_\mu \in [0,\ |n_{j_{\mu+1}}|)$ satisfies $r_0 = k_0 + q_1 n_{j_1}$ and $r_\mu = k_\mu - q_\mu m_{j_\mu} + q_{\mu+1} n_{j_{\mu+1}},\ \mu \geq 1$.

Lemma 5.2.19. *Let $\Sigma_a^\infty \subseteq \Sigma^\infty$ be the subset consisting all infinite words containing infinitely many a_i's, and then denote by $\Omega_{a,\,\infty} \subseteq \Omega_\infty$ be the collection of all characters of the form χ_w with $w \in \Sigma_a^\infty$. Then we have*

(i) If $\chi_w,\ \chi_{w'} \in \partial\Omega$, then $\chi_w = \chi_{w'}$ if and only if $\tau(w) = \tau(w')$ and $\beta(w) = \beta(w')$.

(ii)If $\chi_w,\ \chi_{w'} \in \Omega_{a,\,\infty} \setminus \partial\Omega$, then $\chi_w = \chi_{w'}$ implies $\tau(w) = \tau(w')$ and $\beta(w) = \beta(w')$.

(iii) If $\chi_w,\ \chi_{w'} \in \Omega_\infty \setminus \Omega_{a,\,\infty}$, then $\chi_w = \chi_{w'}$ if and only if $\tau(w) = \tau(w')$ and $\beta(w) = \beta(w')$.

Proof. (i) When $\chi_w = \chi_{w'}$, we assume

$$\tau(w) = (j_1,\ j_2,\ j_3,\ \cdots),\ \beta(w) = (r_0,\ r_1,\ r_2,\ \cdots)$$

and

$$\tau(w') = (j_1',\ j_2',\ j_3',\ \cdots),\ \beta(w') = (r_0',\ r_1',\ r_2',\ \cdots).$$

We now set out to show $j_\mu = j'_\mu,\ r_\nu = r'_\nu$.

For any $i \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $[w']_k \in [w]_i P$. That is, $[w']_k = [w]_i x$ for some $x \in P$. Writing them down in the standard L-form, we get $j_\mu = j'_\mu,\ \mu \le \theta([w]_i)$ and $r_\nu = r'_\nu,\ \nu \le \theta([w]_i) - 1$. Since $w \in \Sigma_a^\infty$ and i is arbitrary, we conclude $j_\mu = j'_\mu,\ r_\nu = r'_\nu$.

Conversely, if

$$\tau(w) = \tau(w') = (j_1,\ j_2,\ j_3,\ \cdots)$$

and

$$\beta(w) = \beta(w') = (r_0,\ r_1,\ r_2,\ \cdots),$$

to prove $\chi_w = \chi_{w'}$, it suffices to show $\chi_{w'}([w]_i P) = 1$ for all $i \in \mathbb{N}$.

If w contains at most finitely many a_k for each $k \in S_2$, take $j \in \mathbb{N}$ such that $\theta([w']_j) = \theta([w]_i) = M$ for some $M \in \mathbb{N}$, then we have

$$[w]_i = b^{r_0} a_{j_1} b^{r_1} a_{j_2} \cdots b^{r_{M-1}} a_{j_M} b^p$$

and

$$[w']_j = b^{r_0} a_{j_1} b^{r_1} a_{j_2} \cdots b^{r_{M-1}} a_{j_M} b^q,$$

where $r_\mu \in [0,\ |n_{j_{\mu+1}}|),\ p,\ q \in \mathbb{Z}$. Take i big enough such that $\{w'\}_j$ does not contain a_k for each $k \in S_2$, by Theorem 5.2.15, $g\chi_{w'}(b^l P) = 1$ for all $l \in \mathbb{N}$ with $g = [w']_j^{-1}$. So we have $\chi_{w'}([w']_j b^l P) = 1$ for all $l \in \mathbb{N}$. Taking l big enough, we have $[w']_j b^l \in [w]_i P$ and thus $\chi_{w'}([w]_i P) = 1$ for all i big enough. That is, $\chi_{w'}([w]_i P) = 1$ for all $i \in \mathbb{N}$.

If w contains infinitely many a_k for some $k \in S_2$, take $j \in \mathbb{N}$ such that $\theta([w']_j) = M_2 > \theta([w]_i) = M_1,\ M_1,\ M_2 \in \mathbb{N}$ and that $\theta_k([w']_j) > \theta_k([w]_i)$, then we have

$$[w]_i = b^{r_0} a_{j_1} b^{r_1} a_{j_2} \cdots b^{r_{M_1-1}} a_{j_{M_1}} b^p$$

and

$$[w']_j = b^{r_0} a_{j_1} b^{r_1} a_{j_2} \cdots b^{r_{M_2-1}} a_{j_{M_2}} b^q,$$

where $r_\mu \in [0,\ |n_{j_{\mu+1}}|),\ p,\ q \in \mathbb{Z}$. Furthermore,

$$b^{r_{M_1}-p}a_{j_{M_1+1}} \cdots b^{r_{M_2-1}}a_{j_{M_2}}b^q \in P$$

since $j_\mu = k$ for some $M_1 < \mu \leq M_2$. So we get

$$[w']_j = [w]_i b^{r_{M_1}-p}a_{j_{M_1+1}} \cdots b^{r_{M_2-1}}a_{j_{M_2}}b^q \in [w]_i P$$

and hence $\chi_{w'}([w]_i P) = 1$.

The proof of (ii) and (iii) is similar.

$\square$

Theorem 5.2.20. *(i)* Ω_∞ *is closed.*
(iii) $\Omega_{b,\,\infty} := \overline{\Omega_\infty \setminus \Omega_{a,\,\infty}} = (\Omega_\infty \setminus \Omega_{a,\,\infty}) \cup \partial\Omega$.

Proof. (i) We prove it by contradiction.
If Ω_∞ is not closed, we must have $\overline{\Omega_\infty} = \Omega$. Take a sequence $\{\chi_{w_i}\} \subseteq \Omega_\infty$ converging to the character χ_ε, we assume, without loss of generality, w_i does not contain any a_i with $i \in S_2$.
If $limsup\ \theta(w_i) \geq 1$, then there exist $k \in \mathbb{N}$ and $j \in S_1$ such that $\chi_i(b^k a_j P) = 1$ for infinitely many i, contradicting $lim_i\ \chi_{w_i} = \chi_\varepsilon$.
If $lim\ \theta(w_i) = 0$, then there exists $k \in \mathbb{N}$ such that $\chi_{w_i} = \chi_{w_b}$ for all $i \geq k$, contradicting $lim_i\ \chi_{w_i} = \chi_\varepsilon$.

(iii) Take a sequence $\{\chi_{w_i}\} \subseteq \Omega_\infty \setminus \Omega_{a,\,\infty}$ converging to some character $\chi_w \in \Omega_\infty$.
When limsup $\theta(w_i) < \infty$, it is easy to show $\chi_w \in \Omega_\infty \setminus \Omega_{a,\,\infty}$.
When limsup $\theta(w_i) = \infty$, we have $\chi_w \in \Omega_{a,\,\infty}$.

If w contains infinitely many a_i's for some $i \in S_2$, then $\chi_w \in \Omega_{\max} = \partial\Omega$.

If w contains only finitely many a_i's for all $i \in S_2$, then it must contain infinitely many a_i's for some $i \in S_1$. Furthermore, there exists some $j \in \mathbb{N}$ such that $\{w\}_j$ does not contain a_i for all $i \in S_2$. Let $g = [w]_j^{-1}$, then $g\chi_w$ is also in $\overline{\Omega_\infty \setminus \Omega_{a,\,\infty}}$ and thus $g\chi_w(b^i P) = 1$ for all $i \in \mathbb{N}$, given the fact that $\chi(b^i P) = 1$ for all $\chi \in \Omega_\infty \setminus \Omega_{a,\,\infty}$. By Theorem 5.2.15, $\chi_w \in \Omega_{\max} = \partial\Omega$.

In conclusion,

$$\Omega_{b,\,\infty} \subseteq (\Omega_\infty \setminus \Omega_{a,\,\infty}) \cup \partial\Omega.$$

For $\chi_w \in \partial\Omega$, let $\tau(w) = (j_1,\ j_2,\ j_3,\ \cdots)$ and $\beta(w) = (k_0,\ k_1,\ k_2,\ \cdots)$. Define

$$w_N = b^{k_0} a_{j_1} b^{k_1} a_{j_2} \cdots b^{k_{M-1}} a_{j_M} bbb \cdots,$$

it is easy to check $\chi_{w_N} \in \Omega_\infty \setminus \Omega_{a,\,\infty}$ and it converges to χ_w. Therefore,

$$(\Omega_\infty \setminus \Omega_{a,\,\infty}) \cup \partial\Omega \subseteq \Omega_{b,\,\infty}.$$

□

Remark 5.2.21. *Every character $\chi \in \Omega_\infty \setminus \Omega_{a,\,\infty}$ is isolated in Ω_∞.*

Theorem 5.2.22. *(i) If $S_1 = \emptyset$, $\Omega_{a,\,\infty} = \partial\Omega$.*

(ii) If $S_2 = \emptyset$, $\Omega_{a,\,\infty}$ is closed.

(iii) If both S_1 and S_2 are not empty, $\Omega_{a,\,\infty}$ is not closed.

Proof. (i) The conclusion follows directly from Theorem 5.2.15.

(ii) Take a sequence $\{\chi_i\} \subseteq \Omega_{a,\,\infty}$ converging to some character $\chi \in \Omega$. Take $M \in \mathbb{N}$, there exist unique elements $(j_1,\ j_2,\ \cdots,\ j_M) \in S^M$ and $(k_0,\ k_1,\ \cdots,\ k_{M-1})$ with $k_\mu \in [0,\ |n_{j_{\mu+1}}|)$ such that

$$\chi_i(b^{k_0}a_{j_1}b^{k_1}a_{j_2}\cdots b^{k_{M-1}}a_{j_M}P) = 1$$

for all i big enough. As a result, $\chi(b^{k_0}a_{j_1}b^{k_1}a_{j_2}\cdots b^{k_{M-1}}a_{j_M}P) = 1$. Since M is arbitrary, $\chi \in \Omega_\infty$ and thus $\chi = \chi_w$ for some $w \in \Sigma^\infty$. Furthermore, w contains infinitely many a_i's, that is, $w \in \Sigma_a^\infty$ and hence $\chi = \chi_w \in \Omega_{a,\,\infty}$.

(iii) Take $a_i \in S_1$ and $a_j \in S_2$ and set $w_k := b^k a_j a_i a_i \cdots,\ k \in \mathbb{N}$, then $\chi_{w_k} \in \Omega_{a,\,\infty}$. We claim that χ_{w_k} converges to χ_{w_b}.
Firstly, $\chi_{w_k}(b^iP) = 1$ for all $i \in \mathbb{N}$ and thus $lim_k\ \chi_{w_k}(b^iP) = 1$ for all $i \in \mathbb{N}$.
Furthermore, if $limsup_k\ \chi_{w_k}(xP) = 1$ for some $x \in P$ with $\theta(x) > 0$, then there exists $l \in \mathbb{N}$ such that $xP \subseteq b^l a_j P$ and that $limsup_k\ \chi_{w_k}(b^l a_j P) = 1$. On the other hand, $b^k a_j a_i^n \notin b^l a_j P$ for all $k > l$ and all $n \in \mathbb{N}$, contradicting $limsup_k\ \chi_{w_k}(b^l a_j P) = 1$. So $lim_k\ \chi_{w_k}(xP) = 0$ for all $x \in P$ with $\theta(x) > 0$. This proves our claim, which implies $\Omega_{a,\,\infty}$ is not closed.

□

Theorem 5.2.23. *If $S_1 \neq \emptyset$, let $\chi_w \in \Omega_{a,\,\infty} \setminus \partial\Omega$ and let $X \subseteq \Omega$ be the minimal closed invariant subset containing χ_w, then we have $\Omega_{a,\,\infty} \subseteq X$.*

Proof. It follows from Theorem 5.2.15 that w contains infinitely many a_i's for some $i \in S_1$ and contain only finitely many a_i's for all $i \in S_2$. By a group action, we can assume w does not contain any a_i with $i \in S_2$.
Assume $\chi_w(bP) = 0$. Otherwise, there exists $M_1 \in \mathbb{N}$ such that $\chi_w(b^iP) = 1$ if and only if $0 \leq i \leq M_1$. Let $g = b^{-M_1}$, then $g \cdot \chi_w \in X$ and $g \cdot \chi_w(bP) = 0$.

Take $\chi_{w'} \in \Omega_{a,\,\infty} \setminus \partial\Omega$ such that w' does not contain any a_i with $i \in S_2$. Let $g_i = [w']_i$, then we assert $g_i \cdot \chi_w$ converges to $\chi_{w'}$.

Take $x \in P$, if $\chi_{w'}(xP) = 1$, then there exists $M_2 \in \mathbb{N}$ such that $[w']_i \in xP$ for all $i \geq M_2$. For these i, $g_i \cdot \chi_w(xP) = 1$ and hence $\lim g_i \cdot \chi_w(xP) = 1$.

If $\chi_{w'}(xP) = 0$, we have also $\lim g_i \cdot \chi_w(xP) = 0$. Otherwise, take i big enough with $g_i \cdot \chi_w(xP) = 1$. Since $\chi_{w'}(xP) = 0$, $[w']_i \notin xP$ and thus $[w']_i[w]_j \in xP$ for some j. That is, $[w']_i[w]_j = xy$ for some $y \in P$. Let

$$x = b^{k_0} a_{j_1} b^{k_1} a_{j_2} \cdots b^{k_{M-1}} a_{j_M} b^p$$

be its standard L-form and let $x' = b^{k_0} a_{j_1} b^{k_1} a_{j_2} \cdots b^{k_{M-1}} a_{j_M}$. By the uniqueness of the standard L-form, we have $p \geq 0$ and there exists $z \in P$ such that $x'z = [w']_i$ and that $z[w]_j = b^p y$. Since $\chi_w(bP) = 0$, $[w]_j \notin bP$ and thus $z \in b^p P$. This means $[w']_i = x'z \in x' b^p P = xP$, contradicting the assumption $\chi_{w'}(xP) = 0$.

This means all $\chi_{w'} \in \Omega_{a,\,\infty} \setminus \partial\Omega$, where w' does not contain any a_i with $i \in S_2$, lie in X. By the invariance of X, we get $\Omega_{a,\,\infty} \setminus \partial\Omega \subseteq X$. As the unique minimal closed invariant subset of Ω, $\partial\Omega$ is also contained in X. That is, $\Omega_{a,\,\infty} \subseteq X$.

$\square$

Corollary 5.2.24. *The closed invariant subsets of Ω are*

(i) Ω, Ω_∞, $\Omega_{a,\,\infty}$, $\Omega_{b,\infty}$ and $\partial\Omega$ if $S_2 = \emptyset$.

(ii) Ω, $\Omega_\infty = \Omega_{b,\,\infty}$ and $\partial\Omega = \Omega_{a,\,\infty}$ if $S_1 = \emptyset$.

(iii) Ω, Ω_∞, $\Omega_{b,\infty}$ and $\partial\Omega$ if both S_1 and S_2 are not empty.

We now consider the case when $N = \infty$. In this case,

$$P = GBS_+(\infty,\, m_i,\, n_i) = < a_i,\, b \mid a_i b^{m_i} = b^{n_i} a_i,\ \forall i \in S_1,\ b^{|n_i|} a_i b^{m_i} = a_i,\ \forall i \in S_2 >_+,$$

where S_1 and S_2 are as in the case when N is finite. Let θ_i be the semigroup homomorphism from P to $\mathbb{N}$, given by $\theta_i(b) = 0$ and $\theta_i(a_j) = 1$ if and only if $j \in S_i$, $i = 1,\ 2$. Set $\theta := \theta_1 + \theta_2$.

Theorem 5.2.25. *Let $\chi_w \in \Omega_\infty$.*

(i) If w contains infinitely many a_i's with $i \in S_2$ (counting multiplicity), then $\chi_w \in \Omega_{max}$.

(ii) If w contains at most finitely many a_i's with $i \in S_2$ (counting multiplicity), then $\chi_w \in \Omega_{max}$ if and only if

(a) w contains infinitely many a_i's with $i \in S_1$ (counting multiplicity); (b) There exists some $j \in \mathbb{N}$ such that $g \cdot \chi_w(b^i P) = 1$ for all $i \in \mathbb{N}$ with $g = [w]_j^{-1}$ and that $\{w\}_j$ does not contain a_i for all $i \in S_2$.

Proof. The proof is similar to the proof of Theorem 5.2.15.

□

Theorem 5.2.26. $\partial\Omega = (\Omega_\infty \setminus \Omega_{a,\ \infty}) \cup \Omega_{max}$.

Proof. Firstly, $\chi_P \notin \partial\Omega$ since $\chi(b^i P) = 1$ for all $\chi \in \Omega_{\max}$ and all $i \in \mathbb{N}$. By invariance of $\partial\Omega$, $\partial\Omega \subseteq \Omega_\infty$.

Secondly, let $\chi_w \in \Omega_{\max}$ with $w = b^{k_0} a_1 b^{k_1} a_2 b^{k_2} a_3 \cdots$ and let

$$g_M := b^M \left(b^{k_0} a_1 b^{k_1} a_2 \cdots b^{k_{M-1}} a_M\right)^{-1}.$$

Then $lim_{M \mapsto \infty}\ g_M \chi_w = \chi_{w_b}$. Indeed, for any $x \in P$ with $\theta(x) > 0$, $g_M \chi_w(xP) = 0$ for M big enough. For all $i \in \mathbb{N}$, $g_M \chi_w(b^i P) = 1$ for M big enough. Therefore, $lim_{M \mapsto \infty}\ g_M \chi_w = \chi_{w_b}$ and hence $\Omega_\infty \setminus \Omega_{a,\ \infty} \subseteq \partial\Omega$.

Lastly, if $\chi_w \in \partial\Omega$ with $w \in \Sigma_a^\infty$, as we analysed in the case when $N < \infty$, we can conclude

$\chi_w \in \Omega_{\max}$.

In conclusion, $\partial\Omega = (\Omega_\infty \setminus \Omega_{a,\ \infty}) \cup \Omega_{\max}$.

$\square$

Remark 5.2.27. $\Omega_{b,\ \infty} = \partial\Omega$.

Theorem 5.2.28. *When $S_1 \neq \emptyset$, let $\chi \in \Omega_\infty \setminus \partial\Omega$ and let X be the minimal closed invariant subset of Ω containing χ.*

(i) If $|S_1| = \sharp A_+ = \infty$, $X = \Omega$.

(ii) If $|S_1| = \sharp A_+ = M < \infty$, $X = \Omega_\infty$.

Proof. Similarly as in Theorem 5.2.23, we have $\Omega_{a,\ \infty} \subseteq X$. Because of the minimality, $\partial\Omega \subseteq X$. Therefore, $\Omega_\infty = \Omega_{a,\ \infty} \cup \Omega_{b,\ \infty} \subseteq X$.

(i) Assume $S_1 = \{j_1,\ j_2,\ j_3,\ \cdots\}$ and let $w_i = a_{j_i} a_{j_{i+1}} a_{j_{i+2}} \cdots$, then $\chi_{w_i} \in \Omega_\infty$. It is easy to check $lim_i\ \chi_{w_i} = \chi_\varepsilon$. It follows, from $\overline{G\chi_\varepsilon} = \Omega$, that $X = \Omega$.

(ii) It suffices to show $\chi_\varepsilon \notin X$, which we will prove by contradiction.

Assume $\{\chi_{w_i}\}_i \subseteq \Omega_\infty$ tends to χ_ε, then there exists $M' \in \mathbb{N}$ such that $\chi_{w_i}(bP) = 0,\ i \geq M'$. For $i \geq M'$, let $b^{k_0} a_{j_1} b^{k_1} a_{j_2} b^{k_2} a_{j_3} \cdots$ be the standard L-form of w_i, then we have $k_0 = 0$ and $j_1 \in S_1$. In this case, $\chi_{w_i}(a_{j_1}P) = 1$. Since $|S_1| = M < \infty$, there must be some $j \in S_1$ such that $\chi_{w_i}(a_j P) = 1$ for infinitely many $i \geq M'$, contradicting the fact $lim_i\ \chi_{w_i}(a_j P) = \chi_\varepsilon(a_j P) = 0$.

$\square$

Corollary 5.2.29. *The closed invariant subsets of Ω are*

(i) Ω and $\partial\Omega = \Omega_\infty = \Omega_{b,\ \infty}$ if $S_1 = \emptyset$.

(ii) Ω, Ω_∞ *and* $\partial\Omega = \Omega_{b,\,\infty}$ *if* $0 < |S_1| < \infty$.

(iii) Ω *and* $\partial\Omega = \Omega_{b,\,\infty}$ *if* $|S_1| = \infty$.

5.3 Topological freeness

As we mentioned in the last section, there is a one-to-one correspondence between the ideals of the reduced groupoid C^*-algebra $C_r^*(G \ltimes \Omega)$ and the open invariant subsets of the unit space Ω under some conditions. The conditions are not unique. In particular, Theorem 3.10 and Corollary 3.12 in [BL18] implies that such a one-to-one correspondence exists if the groupoid $G \ltimes \Omega$ is é$tale$, inner exact and essentially principal. In this section we investigate whether $G \ltimes \Omega$ is essentially principal or not.

By definition, $G \ltimes \Omega$ is essentially principal if $G \ltimes X$ is topologically principal for every closed invariant subset $X \subseteq \Omega$. And $G \ltimes X$ is topologically principal if and only if the partial action of the group G on the space X is topologically free. That is, we need to check whether the group action of G on those closed invariant subsets of Ω is topologically free or not.

First recall that a partial dynamical system $G \curvearrowright X$ is topologically free if there exists a dense subset $X' \subseteq X$ such that if $g \cdot x = x$ for some $g \in G$ and some $x \in X'$, then we must have $g = \varepsilon$. For each subset $Y \subseteq X$, define

$$\mathrm{Stab}(Y) := \{g \in G \mid \mathrm{Dom}(g) \cap Y \neq \emptyset \text{ and } \exists\, x \in \mathrm{Dom}(g) \cap Y,\ g \cdot x = x\}.$$

For brevity, denote $\mathrm{Stab}(\{x\})$ by $\mathrm{Stab}(x)$ for all $x \in X$. Then $G \curvearrowright X$ is topologically free if and only if there exists a dense subset $X' \subseteq X$ such that $\mathrm{Stab}(X') = \{\varepsilon\}$. The following Proposition follows directly by our definition.

Proposition 5.3.1. *Let $G \curvearrowright X$ be a partial dynamical system.*

(i) For all $g \in G$ and $x \in Dom(g)$, $Stab(gx) = gStab(x)g^{-1}$.

(ii) Let $\{Y_i\}$ be a collection of subsets of X and $Y = \cup Y_i$, then we have $Stab(Y) = \cup Stab(Y_i)$.

5.3.1 Generalised Baumslag-Solitar case

In this section, we focus on the generalised Baumslag-Solitar case. That is,

$$P = GBS_+(N,\, m_i,\, n_i) = < a_i,\, b \mid a_i b^{m_i} = b^{n_i} a_i,\, \forall i \in S_1,\, b^{|n_i|} a_i b^{m_i} = a_i,\, \forall i \in S_2,\, N = \sharp A = \sharp S >_+,$$

where $S_1 := \{i \in S,\ \ a_i \in A_+\} = \{i \in S,\, n_i > 0\}$ and $S_2 := \{i \in S,\ \ a_i \in A_-\} = \{i \in S,\, n_i < 0\}$.

Firstly, we assume N is finite.

Theorem 5.3.2. *(i) $G \curvearrowright \Omega$ is topologically free.*

(ii) $G \curvearrowright \Omega_\infty$ is not topologically free.

(iii) $G \curvearrowright \Omega_{b,\,\infty}$ is not topologically free.

(iv) If $n_i | m_i$, $\forall\ 1 \leq i \leq N$, b^n fixes every character in $\Omega_{b,\,\infty}$, where

$$n := lcm(n_1,\, \cdots,\, n_N)$$

is the least positive common multiple of all the n_i. Furthermore, the quotient action $G/ < b^n > \curvearrowright \Omega_{b,\,\infty}$ is topologically free if and only if $n = 1$.

Proof. (i) Since $\mathrm{Stab}(\chi_x) = \{\varepsilon\}$, $\forall x \in P$, the set $\Omega \setminus \Omega_\infty$ does not admit any non-trivial stabilizer. Observing that $\Omega \setminus \Omega_\infty$ is dense in Ω, we conclude $G \curvearrowright \Omega$ is topologically free.

(ii) Noticing $\mathrm{Stab}(\chi) \neq \{\varepsilon\}$ for all $\chi \in \Omega_\infty \setminus \Omega_{a,\,\infty}$, any subset $X \subseteq \Omega_\infty$ satisfying $\mathrm{Stab}(X) = \{e\}$ is included in $\Omega_{a,\,\infty}$, and hence is not dense in Ω_∞. So $G \curvearrowright \Omega_\infty$ is not topologically free.

(iii) The proof is similar to that of (ii).

(iv) Take $\chi_w \in \Omega_{b,\,\infty}$. If $w = w_b$, $b^n w = w$. If not, we get $\tau(b^n w) = \tau(w)$ and $\beta(b^n w) = \beta(w)$ instead. By Lemma 5.2.19, $b^n \chi_w = \chi_w$.
If $n = 1$, $\mathrm{Stab}(\chi_{w_b}) = \{\varepsilon\}$, where ε is the identity element in the quotient group $G/ < b^n >$. Therefore, the orbit $\Omega_\infty \setminus \Omega_{a,\,\infty}$ in the quotient action $G/ < b^n > \curvearrowright \Omega_{b,\,\infty}$ does not admit any non-trivial stabilizer and hence the quotient action $G/ < b^n > \curvearrowright \Omega_{b,\,\infty}$ is topologically free.
If $n > 1$, $\mathrm{Stab}(\chi) \neq \{\varepsilon\}$ for all $\chi \in \Omega_\infty \setminus \Omega_{a,\,\infty}$, so any subset $X \subseteq \Omega_{b,\,\infty}$ satisfying $\mathrm{Stab}(X) = \{\varepsilon\}$ is included in $\partial\Omega$ and hence is not dense in $\Omega_{b,\,\infty}$. So $G/ < b^n > \curvearrowright \Omega_{b,\,\infty}$ is not topologically free.

$\square$

For any $\chi_w \in \Omega_{a,\,\infty} \setminus \partial\Omega$ and all $M \in \mathbb{N}$, there exist unique M-tuple integers $(j_1,\ j_2,\ \cdots,\ j_M)$ and $(k_0,\ k_1,\ \cdots,\ k_{M-1},\ k_M)$ with $k_\mu \in [0,\ |n_{j_{\mu+1}}|)$, $0 \leq \mu \leq M-1$, $k_M \in \mathbb{Z}$ such that

$$\chi_w(b^{k_0} a_{j_1} b^{k_1} a_{j_2} \cdots b^{k_{M-1}} a_{j_M} b^{k_M} P) = 1.$$

Define

$$\gamma_M(w) := \sup\{q \in \mathbb{Z} \mid \chi_w(b^{k_0} a_{j_1} b^{k_1} a_{j_2} \cdots b^{k_{M-1}} a_{j_M} b^q P) = 1\}$$

and set $\gamma(w) := \big(\gamma_M(w)\big)_M$.

The following Lemma is an immediate result.

Lemma 5.3.3. *If $\chi_w,\ \chi_{w'} \in \Omega_{a,\,\infty} \setminus \partial\Omega$, then $\chi_w = \chi_{w'}$ if and only if $\tau(w) = \tau(w'),\ \beta(w) = \beta(w')$ and $\gamma(w) = \gamma(w')$.*

Remark 5.3.4. *If we extend the domain of γ onto Ω_∞, and take $\chi_w,\ \chi_{w'} \in \Omega_\infty$, we then have the following result:*
$\chi_w = \chi_{w'}$ if and only if $\tau(w) = \tau(w'),\ \beta(w) = \beta(w')$ and $\gamma(w) = \gamma(w')$.

Theorem 5.3.5. *When $S_2 = \emptyset$, $\Omega_{a,\,\infty}$ is closed. If we assume further $|S_1| = 1$, then P is a Baumslag-Solitar monoid. Assume $P =< a,\ b|\ ab^m = b^n a,\ m,\ n \in \mathbb{N}^* >$.*
(i) If $m,\ n \geq 2$, $G \curvearrowright \Omega_{a,\,\infty}$ is topologically free.
(ii) If $m \geq 2,\ n = 1$, $G \curvearrowright \Omega_{a,\,\infty}$ is topologically free.
(iii) If $m = 1, n \geq 2$, $G \curvearrowright \Omega_{a,\,\infty}$ is not topologically free.
(iv) If $m = n = 1$, a fixes every character in $\Omega_{a,\,\infty}$. Furthermore, the quotient action $G/ < a > \curvearrowright \Omega_{a,\,\infty}$ is topologically free.

Proof. Let $w = b^{i_0}ab^{i_1}ab^{i_2}a\cdots$ be such that $\alpha(w) \in \{0,\ 1\}^{\mathbb{N}}$ and that $\alpha(w)$ is not periodic eventually.

(i) Let $g \in G$ with $g \cdot \chi_w = \chi_w$, then we have $gw = w$ since w does not contain any relator as a finite subword. There exist $p,\ q \in P$ with $g = pq^{-1}$ such that $q = [w]_i$ for some i. In this case, $w = gw = p\{w\}_i$ and thus $p = [w]_i = q$. That is, $\mathrm{Stab}(\chi_w) = \{\varepsilon\}$. By our choice of w, $\chi_w \in \Omega_{a,\,\infty} \setminus \partial\Omega$. So the orbit containing χ_w is a dense subset in $\Omega_{a,\,\infty}$ and does not admit any non-trivial stabilizer. $G \curvearrowright \Omega_{a,\,\infty}$ is topologically free.

(ii) Let $g \in G$ with $g \cdot \chi_w = \chi_w$, then we have $\gamma(gw) = \gamma(w)$. For N big enough,

$$\gamma_N(gw) = \gamma_{N-1}(gw)m + \alpha_N(gw)$$

and

$$\gamma_N(w) = \gamma_{N-1}(w)m + \alpha_N(w).$$

This implies $\alpha_N(gw) = \alpha_N(w)$ for N big enough. So there exist $i,\ N \in \mathbb{N}$ such that $\{gw\}_i = \{w\}_i$ and $g[w]_i = [w]_i = a^N b^{\gamma_N(w)}$. The latter means

$$g \cdot \chi_{a^N b^{\gamma_N(w)} P} = \chi_{a^N b^{\gamma_N(w)} P}.$$

That is, $g = \varepsilon$ and $\mathrm{Stab}(\chi_w) = \{\varepsilon\}$. Similarly as above, we can conclude $G \curvearrowright \Omega_{a,\,\infty}$ is topologically free.

(iii) Let $X \subseteq \Omega_{a,\,\infty}$ be without non-trivial stabilizer and let $w_a = aaa\cdots$. Since $\Omega_{a,\,\infty} \setminus \partial\Omega$ is a single orbit containing χ_{w_a} and that $\mathrm{Stab}(\chi_{w_a}) \neq \{\varepsilon\}$, X is contained in $\partial\Omega$ and can never be dense. $G \curvearrowright \Omega_{a,\,\infty}$ is not topologically free.

(iv) $\mathrm{Stab}(\chi_{w_a}) = \{\varepsilon\}$ and hence its orbit $\Omega_{a,\,\infty} \setminus \partial\Omega$ does not admit any non-trivial stabilizer. $\Omega_{a,\,\infty} \setminus \partial\Omega$ is dense in $\Omega_{a,\,\infty}$, so the quotient action $G/ < a > \curvearrowright \Omega_{a,\,\infty}$ is topologically free.

$\square$

Theorem 5.3.6. *When $S_2 = \emptyset$ and $|S_1| \geq 2$, $\Omega_{a,\,\infty}$ is closed and $G \curvearrowright \Omega_{a,\,\infty}$ is topologically free.*

Proof. Let $i_1,\ i_2 \in S_1$ with $i_1 \neq i_2$ and let $\chi_w \in \Omega_{a,\,\infty}$ with $w = a_{j_1}a_{j_2}a_{j_3}\cdots$ such that $j_\mu \in$

$\{i_1,\ i_2\}$ and that the sequence $\{j_\mu\}_\mu$ is not periodic. We then have $\chi_w \notin \partial\Omega$ and $\mathrm{Stab}(\chi_w) = \{\varepsilon\}$. It follows that the orbit containing χ_w is dense in $\Omega_{a,\,\infty}$ and does not admit any non-trivial stabilizer. Hence $G \curvearrowright \Omega_{a,\,\infty}$ is topologically free.

□

Theorem 5.3.7. *(i) If $n_i \nmid m_i$ for some i, then $G \curvearrowright \partial\Omega$ is topologically free.*

(ii) If $n_i | m_i$, $\forall\ 1 \le i \le N$, b^n fixes every character in $\partial\Omega$, where

$$n := lcm(n_1,\ \cdots,\ n_N)$$

is the least positive common multiple of all the n_i. Furthermore, the quotient action $G/ < b^n > \curvearrowright \partial\Omega$ is not topologically free if and only if there exist $p \in (0,\ n)$, $M \in \mathbb{N}^*$ and a M-tuple

$$(j_1,\ j_2,\ \cdots,\ j_M) \in \{1,\ 2,\ \cdots,\ N\}^M$$

satisfying

$$n_{j_1} \mid p,$$

$$n_{j_{k+1}} \mid p \cdot \frac{m_{j_1} m_{j_2} \cdots m_{j_k}}{n_{j_1} n_{j_2} \cdots n_{j_k}},\ \ \forall\ 1 \le k \le M-1,$$

and

$$n \mid p \cdot \frac{m_{j_1} m_{j_2} \cdots m_{j_M}}{n_{j_1} n_{j_2} \cdots n_{j_M}}.$$

Proof. (i) To prove $G \curvearrowright \partial\Omega$ is topologically free, it suffices to show that $\{\chi \in \partial\Omega \mid g\chi \neq \chi\}$ is dense in $\partial\Omega$ for every $\varepsilon \neq g \in G$. We divide the proof into three steps.

Step 1. $\Omega^c_{b^p}$ is dense in $\partial\Omega$ for $0 \neq p \in \mathbb{N}$, where $\Omega_g := \{\chi \in \partial\Omega \mid g\chi = \chi\}$ and X^c is the complementary set of X with respect to $\partial\Omega$.

If $\chi_w \in \partial\Omega$ with $\tau(w) = (j_1,\ j_2,\ j_3,\ \cdots)$ is a solution of the equation $b^p\chi = \chi$, we have, by

Lemma 5.2.19, $\beta(b^p w) = \beta(w)$. By definition of β, we get $n_{j_1} \mid p$ and

$$n_{j_{k+1}} \mid p \cdot \frac{m_{j_1} m_{j_2} \cdots m_{j_k}}{n_{j_1} n_{j_2} \cdots n_{j_k}}, \ \forall k \geq 1. \tag{5.6}$$

Take $\chi_{w'} \in \partial\Omega$ with $\tau(w') = (j'_1,\ j'_2,\ j'_3,\ \cdots)$ such that $j'_\mu = i,\ \forall\ \mu \geq M$ for some $M \in \mathbb{N}$. It is easy to see w' does not satisfies equation (5.6), that is, $\beta(b^p w') \neq \beta(w')$. This means $b^p \chi_{w'} \neq \chi_{w'}$. A similar analysis yields $b^p\big(g\chi_{w'}\big) \neq g\chi_{w'}$ for all $g \in G$. The orbit $\{g\chi_{w'}\}_g$ is dense in $\partial\Omega$ and is included in $\Omega^c_{b^p}$, so $\Omega^c_{b^p}$ is dense in $\partial\Omega$.

Step 2. $\Omega^c_{p,\ q}$ is dense in $\partial\Omega$ for every $p,\ q \in P$ with $p \neq q$, where $\Omega_{p,\ q} := \{\chi \in \partial\Omega \mid p\chi = q\chi\}$.

Let

$$p = b^{k_0} a_{j_1} b^{k_1} a_{j_2} \cdots b^{k_{M_1-1}} a_{j_{M_1}} b^x$$

and

$$q = b^{k'_0} a_{j'_1} b^{k'_1} a_{j'_2} \cdots b^{k'_{M_2-1}} a_{j'_{M_2}} b^y$$

be their standard L-forms and let $\chi_w \in \partial\Omega$ with $\tau(w) = (i_1,\ i_2,\ i_3,\ \cdots)$ and $\beta(w) = (l_0,\ l_1,\ l_2,\ \cdots)$ be a solution of the equation $p\chi = q\chi$. By Lemma 5.2.19, $\tau(pw) = \tau(qw)$ and $\beta(pw) = \beta(qw)$.

If $M_1 = M_2$, it follows from $\tau(pw) = \tau(qw)$ that

$$(j_1,\ j_2,\ \cdots,\ j_{M_1}) = (j'_1,\ j'_2,\ \cdots,\ j'_{M_2}).$$

And by $\beta(pw) = \beta(qw)$, we have

$$(k_0,\ k_1,\ \cdots,\ k_{M_1-1}) = (k'_0,\ k'_1,\ \cdots,\ k'_{M_2-1})$$

and $\beta(b^x w) = \beta(b^y w)$. Since $p \neq q$, $x \neq y$. Assume, without loss of generality, $x > y$, we then get $\beta(b^{x-y}w) = \beta(w)$, or equivalently, $b^{x-y}\chi_w = \chi_w$. In this case,

$$\Omega_{p,\ q} \subseteq \Omega_{b^{x-y}} \text{ and } \Omega^c_{b^{x-y}} \subseteq \Omega^c_{p,\ q},$$

which yields that $\Omega^c_{p,\ q}$ is dense in $\partial\Omega$.

If $M_1 \neq M_2$, the equation $\tau(pw) = \tau(qw)$ determines a unique solution $(i_1,\ i_2,\ i_3,\ \cdots)$. Also, the equation $\beta(pw) = \beta(qw)$ determines a unique solution $(l_0,\ l_1,\ l_2,\ \cdots)$. It follows again from Lemma 5.2.19 that $\Omega_{p,\ q}$ is a singleton set, which means $\Omega^c_{p,\ q}$ is dense in $\partial\Omega$.

Step 3. Ω^c_g is dense in $\partial\Omega$ for every $\varepsilon \neq g \in G$.

For $\chi \in \Omega_g$, there must be some $p,\ q \in P$ with $g = pq^{-1}$ such that $p(q^{-1}\chi) = \chi$. In this case, $\chi = q(q^{-1}\chi)$ and thus $q^{-1}\chi \in \Omega_{p,\ q}$. So we have

$$\Omega_g \subseteq \cup_{g=pq^{-1}} q\Omega_{p,\ q} \text{ and } \cap_{g=pq^{-1}} (q\Omega_{p,\ q})^c \subseteq \Omega^c_g.$$

Here $(q\Omega_{p,\ q})^c = q\Omega^c_{p,\ q} \cup (\partial\Omega \setminus q\partial\Omega)$ is dense in $\partial\Omega$. Since $\partial\Omega$ is compact and Hausdorff, it is a Baire space. There exist at most countable pairs $(p,\ q)$ with $g = pq^{-1}$, so $\cap_{g=pq^{-1}}(q\Omega_{p,\ q})^c$ is dense in $\partial\Omega$ as a countable intersection of open dense subsets. Hence Ω^c_g is dense in $\partial\Omega$.

(ii) Let $\chi_w \in \partial\Omega$, it is easy to see that $\beta(b^n w) = \beta(w)$ and that $b^n\chi_w = \chi_w$. We now consider topological freeness of the quotient action $G/ < b^n > \curvearrowright \partial\Omega$.

If there exist p, M and $(j_1,\ j_2,\ \cdots,\ j_M)$ as described in the theorem, we then have

$$b^p a_{j_1} a_{j_2} \cdots a_{j_M} = a_{j_1} a_{j_2} \cdots a_{j_M} b^{qn}$$

for some $q \in \mathbb{N}$. In this case, $b^p \chi_w = \chi_w$ for all $\chi_w \in \partial\Omega$ with $[w]_M = a_{j_1} a_{j_2} \cdots a_{j_M}$. Noticing

$$\{\chi_w \in \partial\Omega \mid [w]_M = a_{j_1} a_{j_2} \cdots a_{j_M}\}$$

is proper clopen subset of $\partial\Omega$, we conclude $\Omega^c_{b^p}$ is not dense in $\partial\Omega$ and thus the quotient action $G/ < b^n > \curvearrowright \partial\Omega$ is not topologically free.

If not, take $x,\ x_i \in P,\ 1 \le i \le M_1,\ M_1 \in \mathbb{N}$ such that $x_i P \subsetneq xP$. Let $\mathscr{O}$ be the nonempty basic open subset

$$\{\chi \in \partial\Omega \mid \chi(xP) = 1,\ \chi(x_i P) = 0,\ 1 \le i \le M_1\}.$$

Let

$$x = b^{k_0} a_{j_1} b^{k_1} a_{j_2} \cdots b^{k_{M'-1}} a_{j_{M'}} b^p$$

and

$$x_i = b^{k_{i,\,0}} a_{j_{i,\,1}} b^{k_{i,\,1}} a_{j_{i,\,2}} \cdots b^{k_{i,\,M'_i-1}} a_{j_{i,\,M'_i}} b^{p_i},$$

$1 \le i \le M_1$, be their standard L-forms and let

$$x' = b^{k_0} a_{j_1} b^{k_1} a_{j_2} \cdots b^{k_{M'-1}} a_{j_{M'}}$$

and

$$x'_i = b^{k_{i,\,0}} a_{j_{i,\,1}} b^{k_{i,\,1}} a_{j_{i,\,2}} \cdots b^{k_{i,\,M'_i-1}} a_{j_{i,\,M'_i}},$$

$1 \le i \le M_1$. It is easy to verify

$$\mathscr{O} = \{\chi \in \partial\Omega \mid \chi(x'P) = 1,\ \chi(x'_i P) = 0,\ 1 \le i \le M_1\}.$$

Since $\mathscr{O}$ is not empty, there must be some $y \in P$ with $\theta(y)$ big enough, of whose standard L-form x' is a prefix while $x'_i,\ 1 \le i \le M_1$ is not a prefix.

For any $q \in (0,\ n)$, set $y_q := b^q y$. Let

$$y = b^{k'_0} a_{j'_1} b^{k'_1} a_{j'_2} \cdots b^{k'_{N_1-1}} a_{j'_{N_1}} b^{p'}$$

and

$$y_q = b^{k''_0} a_{j''_1} b^{k''_1} a_{j''_2} \cdots b^{k''_{N_2-1}} a_{j''_{N_2}} b^{q'}$$

be their standard L-forms, and let

$$y' = b^{k'_0} a_{j'_1} b^{k'_1} a_{j'_2} \cdots b^{k'_{N_1-1}} a_{j'_{N_1}}$$

and

$$y'_q = b^{k''_0} a_{j''_1} b^{k''_1} a_{j''_2} \cdots b^{k''_{N_2-1}} a_{j''_{N_2}}.$$

By our assumption, either $y' \neq y'_q$ or $n \nmid (q' - p')$.

If $y' \neq y'_q$, let $\chi_w \in \partial\Omega$ such that $[w]_j = y$ for some j.

If $n \nmid (q' - p')$, there exists some $1 \leq i \leq N$ such that $n_i \nmid (q' - p')$. Let $\chi_w \in \partial\Omega$ such that $[w]_j = ya_i$ for some j.

In either case, x' is a prefix of w while $x'_i,\ 1 \leq i \leq M_1$ is not. So $\chi_w(x'P) = 1,\ \chi_w(x'_iP) = 0$ and hence $\chi_w \in \mathscr{O}$. Also, it follows from our choice of w that $b^q \chi_w \neq \chi_w$. That is,

$$\chi_w \in \mathscr{O} \cap \Omega^c_{b^q}.$$

Let $\mathscr{O}$ run over all nonempty basic open subsets of $\partial\Omega$, we get that $\Omega^c_{b^q}$ is dense in $\partial\Omega$. Following Step 2 and Step 3 as in the proof of (i), we can conclude Ω^c_g is dense in $\partial\Omega$ for every $e \neq g \in G$. That is, $G/ < b^n > \curvearrowright \partial\Omega$ is topologically free.

□

When N is infinite, we have the following results.

Theorem 5.3.8. *$G \curvearrowright \Omega$ is topologically free.*

Theorem 5.3.9. *(i) If $2 \leq |S_1| < \infty$, $G \curvearrowright \Omega_\infty$ is topologically free.*

(ii) If $|S_1| = 1$ and $m_i \geq 2$ for $i \in S_1$, $G \curvearrowright \Omega_\infty$ is topologically free.

(iii) If $|S_1| = 1$ and $m_i = 1$ for $i \in S_1$, $G \curvearrowright \Omega_\infty$ is not topologically free.

Proof. The proof is similar to the proofs of Theorem 5.3.6 and Theorem 5.3.5. □

Theorem 5.3.10. *(i) If $n_i \nmid m_i$ for some i, then $G \curvearrowright \partial\Omega$ is topologically free.*

(ii) If $n_i | m_i$ for all i, then $G \curvearrowright \partial\Omega$ is topologically free if and only if $n = \infty$, where

$$n := lcm(n_1,\ n_2,\ n_3,\ \cdots)$$

is the least positive common multiple of all the n_i.

(iii) If $n < \infty$, b^n fixes every character in $\partial\Omega$. Furthermore, the quotient action $G/ < b^n >$ $\curvearrowright \partial\Omega$ is not topologically free if and only if there exist $p \in (0,\ n)$, $M \in \mathbb{N}^*$ and $j_i \in \mathbb{N}^*$, $1 \leq i \leq M$ satisfying

$$n_{j_1} \mid p,$$

$$n_{j_{k+1}} \mid p \cdot \frac{m_{j_1} m_{j_2} \cdots m_{j_k}}{n_{j_1} n_{j_2} \cdots n_{j_k}},\ \ \forall\ 1 \leq k \leq M-1,$$

and

$$n \mid p \cdot \frac{m_{j_1} m_{j_2} \cdots m_{j_M}}{n_{j_1} n_{j_2} \cdots n_{j_M}}.$$

5.3.2 General case

In this section, P is the fundamental monoid of a graph of monoids with condition (LCM) for P satisfied. Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R}, +)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$. We set out to find the topological freeness of the group action of G on all closed invariant subspaces of the character space Ω.

First of all, every character $\chi \in \Omega \setminus \Omega_\infty$ does not admit non-trivial stablizers, so the action $G \curvearrowright \Omega$ is topologically free.

Proposition 5.3.11. *If condition I. holds and $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$, then $G \curvearrowright \Omega_\infty$ is topologically free whenever Ω_∞ is closed.*

Proof. Since condition I. holds, there exists $e \in T$ with $P_e = \{\varepsilon\}$. Let $v = o(e)$ and $w = t(e)$, and assume α (β) is the generator of P_v (P_w, respectively). Set $X := \alpha^{k_1}\beta^{k_2}\alpha^{k_3}\beta^{k_4}\cdots$ with the sequence $\{k_i\}_i$ aperiodic, then $\chi_X \in \Omega_\infty$ and $\mathrm{Stab}(\chi_X) = \{\varepsilon\}$. When Ω_∞ is closed, $\Omega_\infty = \partial\Omega$ is minimal and thus $G \cdot \chi$ is dense in Ω_∞. Therefore, $G \curvearrowright \Omega_\infty$ is topologically free. $\square$

Proposition 5.3.12. *If condition II. holds and $\sharp A = 0$, then the action $G \curvearrowright \{\infty\}$ is not topologically free.*

Proposition 5.3.13. *Suppose condition II. holds, $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$ with $1 < \sharp V < \infty$ and $\sharp A_+ < \infty$.*

(i) when $\sharp A_+ > 0$, $G \curvearrowright \Omega_\infty$ is topologically free.

(ii) when $\sharp V > 2$, $G \curvearrowright \Omega_\infty$ is topologically free.

(iii) when $\sharp A_+ = 0$ and $\sharp V = 2$, take $e \in T$, and assume the two embeddings are $P_e \to P_{o(e)},\ 1 \mapsto k$ and $P_e \to P_{t(e)},\ 1 \mapsto l$, $G \curvearrowright \Omega_\infty$ is topologically free if and only if either $k > 2$ or $l > 2$.

Proof. When $G \curvearrowright \Omega_\infty$ is topologically free, we prove it by seeking out a character $\chi_X \in \Omega_\infty \setminus \Omega_{\mathbf{b},\,\infty}$ with $\mathrm{Stab}(\chi_X) = \{\varepsilon\}$.

(i) Take $e \in A_+$ and let $\alpha \in P_{t(e)}$ be the generator, set $X := \alpha^{k_1} e \alpha^{k_2} e \cdots$ with $k_i \in \{0,\ 1\}$ and the sequence $\{k_i\}_i$ aperiodic. Take $g \in G$ such that $g\chi_X = \chi_X$, then there exists $j \in \mathbb{N}$ such that $g = pq^{-1}$ with $q = [X]_j$ and that $p\{X\}_j \equiv X$ since X contians no relators. This yields $p = q$ and hence $g = \varepsilon$.

(ii) Take $u,\ v,\ w \in V$ and let $\alpha \in P_u,\ \beta \in P_v,\ \gamma \in P_w$ be the generators, set $X := \alpha\beta\gamma^{k_1}\alpha\beta\gamma^{k_2}\cdots$ with $k_i \in \{0,\ 1\}$ and the sequence $\{k_i\}_i$ aperiodic.

(iii) Let $\alpha \in P_{o(e)},\ \beta \in P_{t(e)}$ be the generators. If $k > 2$, set $X := \alpha^{k_1}\beta\alpha^{k_2}\beta\cdots$ with $k_i \in \{1,\ 2\}$ and the sequence $\{k_i\}_i$ aperiodic.

If $k = l = 2$, $\Omega_\infty \setminus \Omega_{\mathbf{b},\,\infty}$ is a single orbit containing χ_Y with $Y = \alpha\beta\alpha\beta\cdots$. $\mathrm{Stab}(\chi_Y) \neq \{\varepsilon\}$, so $G \curvearrowright \Omega_\infty$ is not topologically free.

$\square$

In the above, we give a complete discussion on the topological freeness of the partial action of the group G on the closed invariant subsets Ω, Ω_∞ and $\{\infty\}$. While we fail obtaining a complete discussion on the topological freeness of the partial action $G \curvearrowright \partial\Omega$ in the case where condition II. holds and $\sharp A \geq 1$. Instead, we give some examples when the partial action $G \curvearrowright \partial\Omega$ is topologically free.

Proposition 5.3.14. *Suppose condition II. holds, $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$, $\sharp V > 1$ and $\sharp A > 0$. Assume $P_e \to P_{t(e)} (\mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0})$ sends 1 to m_e for all $e \in T \cup A$, $P_e \to P_{o(e)} (\mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0})$ sends 1 to n_e for all $e \in T \cup A_+$ and $P_e \to P^{-1}_{o(e)} (\mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\leq 0})$ sends 1 to n_e for all $e \in A_-$. If $m_e = n_e$ for all $e \in T$ and there exists $e \in A$ such that $n_e \nmid m_e$, then $G \curvearrowright \partial\Omega$ is topologically free.*

Proof. A similar argument as in the proof of Theorem 5.3.7 yields that there exists $\chi_w \in \partial\Omega$ with w consisting of letters from $P_{o(e)}$ and $\{e\}$ such that $\mathrm{Stab}(\chi_w) = \{\varepsilon\}$, where e lies in A with $n_e \nmid m_e$. The claim follows since $G \curvearrowright \partial\Omega$ is minimal.

□

Proposition 5.3.15. *Suppose condition II. holds, there exists $v \in V$ such that G_v is dense in $\mathbb{R}$ and $\sharp A > 0$. Assume there exists $a \in A$ such that the geodesic path $[o(a), t(a)] \subseteq T$ contains at most one vertex v with $G_v \subseteq \mathbb{R}$ dense.*

(i) If $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in [o(a), t(a)]$. Assume $P_e \to P_{t(e)} (\mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0})$ sends 1 to m_e for all $e \in [o(a), t(a)] \cup \{a\}$ and $P_e \to G_{o(e)} (\mathbb{Z}_{\geq 0} \to \mathbb{Z})$ sends 1 to n_e for all $e \in [o(a), t(a)] \cup \{a\}$. If $m_e = n_e$ for all $e \in [o(a), t(a)]$ and $n_a \nmid m_a$, then $G \curvearrowright \partial\Omega$ is topologically free.

(ii) If the geodesic path $[o(a), t(a)] \subseteq T$ contains exactly one vertex v with $G_v \subseteq \mathbb{R}$ dense. Assume the unique relation containing a in G is $b^{n_a}_{o(a)} a = a b^{m_a}_{t(a)}$, where $b_{o(a)}$ and $b_{t(a)}$ are the generators of $P_{o(a)}$ and $P_{t(a)}$, respectively. Assume further m and n are the least positive integers such that $m_a | m$, $n_a | n$, $\frac{m}{m_a} = \frac{n}{|n_a|}$ and $b^n_{o(a)}, b^m_{t(a)} \in G_v$. If there does not exist $r \in \mathbb{N}$

such that $b_{t(a)}^m = (b_{o(a)}^n)^r$, *then* $G \curvearrowright \partial\Omega$ *is topologically free.*

Proof. Noting that $b_{t(a)}^m = (b_{o(a)}^n)^r$ for some $r \in \mathbb{N}$ implies $b_{t(a)}^m = b^r$ for $b = b_{o(a)}^n \in G_v$ and $ab^r = b^{\mathrm{sgn}(n_a)}a$, the claims in (i) and (ii) follow by a similar argument as in the proof of Proposition 5.3.14.

□

Chapter 6

Ideal structure

Let P (G) be a graph of monoids (groups, respectively). In last chapter, we worked out all the closed invariant subsets of the partial action $G \curvearrowright \Omega$ and analysed the topological freeness of the partial action of G on all these closed invariant subsets. This partial action $G \curvearrowright \Omega$ induces a transformation groupoid $G \ltimes \Omega$ and hence a groupoid C^*-algebra $C_r^*(G \ltimes \Omega)$. In this chapter, to have a better understanding of the C^*-algebra $C_r^*(G \ltimes \Omega)$, we shall investigate the ideals in $C_r^*(G \ltimes \Omega)$.

Since every ideal in a C^*-algebra is the intersection of all the primitive ideals (the kernels of non-zero irreducible representations of the C^*-algebra) containing it, we end up with the list of all primitive ideals with a topology in $C_r^*(G \ltimes \Omega)$. This part of work is based on the following Lemma, which comes from Christian Bönicke's and Kang Li's work in [Theorem 3.10 and Corollary 3.12, BL18].

Lemma 6.0.1. *If a groupoid $\mathscr{G}$ is étale, inner exact and essentially principal, then there is a one-to-one correspondence between open invariant subsets in Ω and ideals in $C_r^*(\mathscr{G})$.*

It is easy to check that $G \ltimes \Omega$ is étale. The inner exactness of the groupoid $G \ltimes \Omega$ is exactly the C^*-exactness of the group G by definition in [Gue01]. Also by Erik Guentner, a discrete group acting without inversion on a tree is C^*-exact if and only if the vertex stabilizers of the action are C^*-exact. By [p50-p53, Ser80], the fundamental group $\pi_1(G,\ \Gamma,\ T)$ acts without inversion on a tree $\tilde{X} = \tilde{X}(G,\ \Gamma,\ T)$ such that every vertex stabilizer is isomorphic to G_v for some $v \in V$. Therefore, our group G is C^*-exact if and only if G_v is C^*-exact for all $v \in V$. Noting $G_v \subseteq (\mathbb{R},\ +)$ in our assumption, the latter follows since discrete amenable groups are C^*-exact by [Lan73]. And by definition the essentially principal property of the groupoid $G \ltimes \Omega$ is exactly the topological freeness of the partial action of G on all nonempty closed invariant subsets of Ω. Equivalently, the groupoid $G \ltimes \Omega$ is essentially principal if and only if the partial action $G \curvearrowright X$ is topologically free for all nonempty closed invariant subsets $X \subseteq \Omega$.

We work out the list of all nonempty closed invariant subsets of Ω and analyse the topological freeness of the partial action of G on these nonempty closed invariant subsets in Chapter 5. In the case where the partial action $G \curvearrowright X$ is topologically free for all nonempty closed invariant subsets $X \subseteq \Omega$, we can easily obtain that every ideal in $C_r^*(G \ltimes \Omega)$ is of the form $C_r^*(G \ltimes X')$ with $X' \subseteq \Omega$ open and invariant and then analyse whether they are primitive or not. In other cases, our work is based on the following Lemma, which comes from [Proposition 3.2.1, Dix77].

Lemma 6.0.2. *If J is an ideal in a C^*-algebra A, then the canonical map from the closed subset*

$$Prim_J(A) := \{I \in Prim(A) :\ J \subseteq I\} \subseteq Prim(A)$$

to $Prim(A/J)$, induced by the quotient, is a homeomorphism. And the map ρ_J from

$$Prim^J(A) := \{I \in Prim(A) : \ J \nsubseteq I\}$$

to $Prim(J)$, defined by $\rho_J(I) = I \cap J$, is also a homeomorphism.

To begin with, we still need a couple of Lemmas as following.

Lemma 6.0.3. *If $X \subseteq \Omega$ is an orbit, then the $*$-representations π_χ and $\pi_{\chi'}$ of the $*$-algebra $C_c(G \ltimes X)$ on the Hilbert spaces $\ell_2(G_\chi \ltimes \{\chi\})$ and $\ell_2(G_{\chi'} \ltimes \{\chi'\})$ respectively are unitarily equivalent. Here π_χ and $\pi_{\chi'}$ are sub-$*$-representations of the left regular representation π of the groupoid $G \ltimes \Omega$ as in section 2.2.*

Proof. Let $h \in G$ be such that $h\chi' = \chi$. Define a map

$$U : \ \ell_2(G_\chi \ltimes \{\chi\}) \ \rightarrow \ \ell_2(G_{\chi'} \ltimes \{\chi'\}),$$

$$\delta_{(g,\chi)} \ \mapsto \ \delta_{(gh,\ \chi')},\ g \in G_\chi,$$

it is easy to check that U is a unitary.

Take $f \in C_c(G \ltimes X)$ and $\xi \in \ell_2(G_\chi \ltimes \{\chi\})$, then we have

$$\begin{aligned}\big(U \circ \pi_\chi(f)\big)\ (\xi)\ (gh,\ \chi') &= U(f * \xi)\ (gh,\ \chi') \\ &= f * \xi\ (g,\ \chi) \\ &= \sum_{g' \in G_\chi} f(gg'^{-1}, g'\chi)\xi(g',\ \chi),\ g \in G_\chi.\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}\big(\pi_{\chi'}(f)\circ U\big)\ (\xi)\ (gh,\ \chi') &= (f * U\xi)\ (gh,\ \chi') \\ &= \sum_{g'\in G_\chi} f(gg'^{-1}, g'\chi)U\xi(g'h,\ \chi') \\ &= \sum_{g'\in G_\chi} f(gg'^{-1}, g'\chi)\xi(g',\ \chi),\ g\in G_\chi.\end{aligned} \tag{6.2}$$

From the equation (6.1) and the equation (6.2), we conclude $\pi_\chi(\sqcup) = U^* \circ \pi_{\chi'}(\sqcup)\circ U$.

$\square$

Lemma 6.0.4. *$\pi_{\chi_\varepsilon}\big(C_c(G\ltimes(\Omega\setminus\Omega_\infty))\big)$ is isomorphic to $\mathscr{F}(\ell_2(\mathbb{N}^*))$ as a normed $*$-algebra, where $\mathscr{F}(\ell_2(\mathbb{N}^*))$ is the finite rank operator algebra on the Hilbert space $\ell_2(\mathbb{N}^*)$.*

Proof. Noting $G_{\chi_\varepsilon} = P$, we naturally get a unitary $V:\ \ell_2(G_{\chi_\varepsilon}\ltimes\{\chi_\varepsilon\}) \to \ell_2(\mathbb{N}^*)$ via a bijection $v:\ P \to \mathbb{N}^*$. In the mean time, define

$$\varphi:\ C_c\big(G\ltimes(\Omega\setminus\Omega_\infty)\big) \to \mathscr{F}(\ell_2(\mathbb{N}^*)),\ f \mapsto (c_{ij})_{ij}$$

where $c_{ij} = f(g,\ \chi_p)$ if $i = v(gp)$ and $j = v(p)$, and $c_{ij} = 0$ otherwise.

It is easy to check that φ is a $*$-algebraic isomorphism and that $\pi_{\chi_\varepsilon}(\sqcup) = V^*\circ\varphi(\sqcup)\circ V$. It follows that

$$\begin{aligned}\|\pi_{\chi_\varepsilon}(f)\|^2 &= \sup_{\|\xi\|=1,\ \xi\in\ell_2(G_{\chi_\varepsilon}\ltimes\{\chi_\varepsilon\})} |<\xi,\ \pi_{\chi_\varepsilon}(f^*f)\ \xi>| \\ &= \sup_{\|\xi\|=1,\ \xi\in\ell_2(G_{\chi_\varepsilon}\ltimes\{\chi_\varepsilon\})} |<\xi,\ V^*\circ\varphi(f^*f)\circ V\ \xi>| \\ &= \sup_{\|\eta\|=1,\ \eta\in\ell_2(\mathbb{N}^*)} |<\eta,\ \varphi(f^*f)\ \eta>| \\ &= \|\varphi(f)\|^2,\ f\in C_c\big(G\ltimes(\Omega\setminus\Omega_\infty)\big).\end{aligned} \tag{6.3}$$

□

Corollary 6.0.5. *Whenever Ω_∞ is closed in Ω, we have $*$-isomorphims $C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big) \cong \mathscr{K}$.*

6.1 Generalised Baumslag-Solitar case

In this section, we assume P is a generalised Baumslag-Solitar monoid. That is,

$$P = GBS_+(N,\, m_i,\, n_i) = < a_i,\, b \mid a_i b^{m_i} = b^{n_i} a_i,\ \forall i \in S_1,\ b^{|n_i|} a_i b^{m_i} = a_i,\ \forall i \in S_2,\ N = \sharp A = \sharp S >_+,$$

where $S_1 := \{i \in S,\ a_i \in A_+\} = \{i \in S,\, n_i > 0\}$ and $S_2 := \{i \in S,\ a_i \in A_-\} = \{i \in S,\, n_i < 0\}$.

Take $x \in P$. Let $x = b^{j_0} a_{i_1} b^{j_1} a_{i_2} \cdots b^{j_{k-1}} a_{i_k} b^p,\ 1 \le i_\mu \le N,\ j_\mu \in [0,\ |n_{i_{\mu+1}}|),\ p \in \mathbb{Z}$ be its standard L-form, and define $x' := b^{j_0} a_{i_1} b^{j_1} a_{i_2} \cdots b^{j_{k-1}} a_{i_k}$. Let P' be the collection of x' when x varies all over P and define the map $\pi' :\ P \ \to\ P'$ by sending x to x'.

The orbit $\Omega_\infty \setminus \Omega_{a,\,\infty}$ is a discrete subspace of Ω. Let $w_b = bbb\cdots$ and let $H := < b >$ be the subgroup of G, generated by b. It is easy to see $G_{\chi_{w_b}} := \{g \in G \mid \chi_{w_b} \in \mathrm{dom}(g)\}$ is equal to $P'H \subseteq G$.

Lemma 6.1.1. *$\pi_{\chi_{w_b}}\big(C_c(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty}))\big)$ is isometrically isomorphic to a $*$-subalgebra of $\mathscr{L}\big(\ell_2(P';\ell_2(H))\big)$, where $\ell_2(H)$ is a Hilbert space with the operations of convolution and involution:*

$$f * g(b^k) = \sum_l f(b^l) g(b^{k-l}) \ \ and\ \ f^*(b^k) = \overline{f(b^{-k})}.$$

Proof. Define a map

$$V:\ \ell_2(G_{\chi_{w_b}} \ltimes \{\chi_{w_b}\}) \ \to\ \ell_2(P';\ell_2(H)),\ \ \delta_{(pb^k,\ \chi_{w_b})} \ \mapsto\ \delta_{p,\ \delta_{b^k}},\ \ p\in P',$$

where $\delta_{p,\ \delta_{b^k}}$ is a function taking value δ_{b^k} at the point p and taking value 0 elsewhere. It is easy to check that V is a unitary.

In the mean time, define

$$\varphi:\ C_c\big(G\ltimes(\Omega_\infty\setminus\Omega_{a,\ \infty})\big)\ \to\ \mathscr{L}\big(\ell_2(P';\ell_2(H))\big),\ \ f\ \mapsto\ (f_{p,\ q})_{p,\ q},\ p,\ q\in P',$$

where $(f_{p,\ q})_{p,\ q}$ is an infinite matrix with finite rank (f finitely supported), and every matrix entry $f_{p,\ q}$ is an element in the Hilbert space $\ell_2(H)$, given as follows: $f_{p,\ q}(b^k)=f(g,\ \chi_{hw_b})$ if $\pi'(h)=q$ and $g=pb^kq^{-1}$, and $f_{p,\ q}(b^k)=0$ otherwise.

It follows easily that φ is well-defined, injective and linear. Also, we have

$$\begin{aligned}\varphi(f_1)\varphi(f_2)(p,\ r)(b^k) &= \sum_{q\in P',\ l\in\mathbb{Z}} \varphi(f_1)(p,\ q)(b^l)\varphi(f_2)(q,\ r)(b^{k-l})\\ &= \sum_{q\in P',\ l\in\mathbb{Z}} f_1(pb^lq^{-1},\ \chi_{qw_b})f_2(qb^{k-l}r^{-1},\ \chi_{rw_b})\\ &= f_1*f_2(pb^kr^{-1},\ \chi_{rw_b})\\ &= \varphi(f_1*f_2)(p,\ r)(b^k),\\ & f_1,\ f_2\in C_c\big(G\ltimes(\Omega_\infty\setminus\Omega_{a,\ \infty})\big),\ p,\ r\in P',\ k\in\mathbb{Z},\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
\varphi(f)^*(p,\ q)(b^k) &= \overline{\varphi(f)(q,\ p)(b^{-k})} \\
&= \overline{f(qb^{-k}p^{-1},\ \chi_{pw_b})} \\
&= f^*(pb^kq^{-1},\ \chi_{qw_b}) \\
&= \varphi(f^*)(p,\ q)(b^k), \\
& f \in C_c\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\ \infty}))\big),\ p,\ q \in P',\ k \in \mathbb{Z}.
\end{aligned} \tag{6.5}$$

That is, φ preserves multiplication and involution. Therefore, φ is a $*$-algebraic isomorphism.

Let $f \in C_c\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\ \infty}))\big)$ and $\xi \in \ell_2(G_{\chi_{w_b}} \ltimes \{\chi_{w_b}\})$, then we have

$$\begin{aligned}
V \circ \pi_{\chi_{w_b}}(f)(\xi)(p)(b^k) &= \pi_{\chi_{w_b}}(f)(\xi)(pb^k,\ \chi_{w_b}) \\
&= f * \xi(pb^k,\ \chi_{w_b}) \\
&= \sum_{q \in P',\ l \in \mathbb{Z}} f(pb^{k-l}q^{-1},\ \chi_{qw_b})\xi(qb^l,\ \chi_{w_b}) \\
&= \sum_{q \in P',\ l \in \mathbb{Z}} \varphi(f)(p,\ q)(b^{k-l})V(\xi)(q)(b^l) \\
&= \varphi(f) \circ V(\xi)(p)(b^k),\ p \in P',\ k \in \mathbb{Z}.
\end{aligned} \tag{6.6}$$

That is, $\pi_{\chi_{w_b}}(f) = V^* \circ \varphi(f) \circ V$ for all $f \in C_c\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\ \infty}))\big)$. It follows that

$$\begin{aligned}
\|\pi_{\chi_P}(f)\|^2 &= \sup_{\|\xi\|=1,\ \xi \in \ell_2(G_{\chi_{w_b}} \ltimes \{\chi_{w_b}\})} |<\xi,\ \pi_{\chi_P}(f^*f)\ \xi>| \\
&= \sup_{\|\xi\|=1,\ \xi \in \ell_2(G_{\chi_{w_b}} \ltimes \{\chi_{w_b}\})} |<\xi,\ V^* \circ \varphi(f^*f) \circ V\ \xi>| \\
&= \sup_{\|\eta\|=1,\ \eta \in \ell_2(P';\ell_2(H))} |<\eta,\ \varphi(f^*f)\ \eta>| \\
&= \|\varphi(f)\|^2,\ f \in C_c\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\ \infty}))\big).
\end{aligned} \tag{6.7}$$

□

Proposition 6.1.2. *Whenever $\Omega_{a,\,\infty}$ is closed in Ω_∞, we have $*$-isomorphisms $C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big) \cong \mathscr{K} \otimes C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle.*

Proof. For any $f \in C_c\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big)$, $\varphi(f)$, as defined in Lemma 6.1.1, is an infinite matrix with finitely many nonzero entries such that each nonzero entry is a finitely supported function on H. Noting

$$\ell_2(P';\ell_2(H)) \cong \ell_2(P') \otimes \ell_2(H),$$

which induces an isomorphism between $\mathscr{L}\big(\ell_2(P';\ell_2(H))\big)$ with $\mathscr{L}\big(\ell_2(P')\big) \otimes_{\min} \mathscr{L}\big(\ell_2(H)\big)$, we can identify these two C^*-algebras with each other. Let

$$\mathscr{A} \subseteq \mathscr{L}\big(\ell_2(P')\big) \otimes_{\min} \mathscr{L}\big(\ell_2(H)\big)$$

be the collection of all elements of the form $\sum_{i\in I} M_i \otimes f_i$, where I is a finite index set, M_i is an infinite matrix of finite rank and f_i is a finitely supported function on H, then $\mathscr{A}$ is a $*$-subalgebra of $\mathscr{L}\big(\ell_2(P')\big) \otimes_{\min} \mathscr{L}\big(\ell_2(H)\big)$. Under the identification, $\varphi(f) \in \mathscr{A}$ for all $f \in C_c\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big)$. Conversely, every element $\sum_{i\in I} M_i \otimes f_i \in \mathscr{A}$ is the image of some f in $C_c\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big)$ under the map φ since $G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})$ is discrete. Therefore,

$$\varphi\Big(C_c\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big)\Big) = \mathscr{A}$$

and hence $\pi_{\chi_{w_b}}\big(C_c(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty}))\big)$ is isomorphic to $\mathscr{A}$ by Lemma 6.1.1.

Every finitely supported function on H acts on $\ell_2(H)$ via convolution, as exactly it does in the left regular representation of the group H. In combination with the fact that every compact operator can be approached by finite rank operators, we conclude

$$\mathscr{A} \subseteq \mathscr{K}\big(\ell_2(P')\big) \otimes_{\min} C_r^*(H) \subseteq \bar{\mathscr{A}}$$

and thus

$$\bar{\mathscr{A}} = \mathscr{K}\left(\ell_2(P')\right) \otimes_{\min} C_r^*(H).$$

Since P' is countable and $C_r^*(H) \cong C(\mathbb{T})$, we have

$$\bar{\mathscr{A}} \cong \mathscr{K} \otimes_{\min} C(\mathbb{T}).$$

Therefore,

$$C_r^*(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})) \cong \overline{\pi_{\chi_{w_b}}\left(C_c(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty}))\right)} \cong \bar{\mathscr{A}} \cong \mathscr{K} \otimes_{\min} C(\mathbb{T}).$$

Since $\mathscr{K}$ is nuclear, the C^*-norm on the algebraic tensor product of $\mathscr{K}$ and $C(\mathbb{T})$ is unique. Therefore, by removing the footnote over the tensor product without ambiguity, we have

$$C_r^*(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})) \cong \mathscr{K} \otimes C(\mathbb{T}).$$

□

Proposition 6.1.3. *Every primitive ideal in $\mathscr{K} \otimes C(\mathbb{T})$ is of the form $\mathscr{K} \otimes C_0(\mathbb{T} \setminus \{p\})$, where $p \in \mathbb{T}$ is a point.*

Proof. Since $\mathscr{K}$ is separable and exact and $C(\mathbb{T})$ is separable, by [Bla06, Theorem IV.3.4.25], every primitive ideal in $\mathscr{K} \otimes C(\mathbb{T})$ is of the form $\mathscr{K} \otimes_{\min} I + J \otimes_{\min} C(\mathbb{T})$, where I is a primitive ideal of $C(\mathbb{T})$ and J is a primitive ideal of $\mathscr{K}$. Since $\mathscr{K}$ is simple, $J = \{0\}$. In this case, every primitive ideal in $\mathscr{K} \otimes C(\mathbb{T})$ is of the form $\mathscr{K} \otimes I$, where I is a primitive ideal of $C(\mathbb{T})$. Every primitive ideal in $C(\mathbb{T})$ is a maximal ideal since $C(\mathbb{T})$ is commutative. Every ideal in $C(\mathbb{T})$ is of the form $C_0(X)$ with $X \subseteq \mathbb{T}$ being an open subset, so $I = C_0(\mathbb{T} \setminus \{p\})$ for some $p \in \mathbb{T}$.

□

Remark 6.1.4. *Prim*$(\mathscr{K}\otimes C(\mathbb{T}))$ *is homeomorphic to* $\mathbb{T}$ *with the usual topology.*

When $|S_1|=1$ and $m_i=1$ for $i\in S_1$, $\Omega_{a,\,\infty}\setminus\partial\Omega$ is exactly an orbit. In this case, every character in $\Omega_{a,\,\infty}\setminus\partial\Omega$ is of the form χ_w with $w=pa_ia_ia_i\cdots,\ p\in P,\ i\in S_1$.

Proposition 6.1.5. *The sub-topology on* $\Omega_{a,\,\infty}\setminus\partial\Omega$ *is discrete.*

Proof. Let

$$\mathscr{O}_p:=\{\chi\in\Omega\mid\chi(pP)=1,\ \chi(pbP)=0\},\ p\in P,$$

then $\mathscr{O}_p$ is an open subset in Ω. It is easy to check that

$$\mathscr{O}_p\cap(\Omega_{a,\,\infty}\setminus\partial\Omega)=\{\chi_w\},$$

where $w=pa_ia_ia_i\cdots,\ i\in S_1$. This entails the discreteness of the sub-topology on $\Omega_{a,\,\infty}\setminus\partial\Omega$. □

Proposition 6.1.6. *If* $|S_1|=1$ *and* $m_i=1$ *for* $i\in S_1$,

$$C_r^*\big(G\ltimes(\Omega_{a,\,\infty}\setminus\partial\Omega)\big)\cong\mathscr{K}\otimes C(\mathbb{T}).$$

Proof. The proof is similar as the proofs of Lemma 6.1.1 and Proposition 6.1.2.

□

Now we are ready to work out the primitive ideal space. Our work is based on Lemma 6.0.1

and Lemma 6.0.2.

If there exists i with $n_i \nmid m_i$ or all the n_i's does not admit a common multiple, the action $G \curvearrowright \partial\Omega$ is topologically free. In the following, we always assume $G \curvearrowright \partial\Omega$ is topologically free.

When N is infinite:

Case 1. If $|S_1| = 0$ or ∞, there are only two nonempty closed invariant subsets, Ω and $\partial\Omega$. $G \ltimes \Omega$ is essentially principal and there is one to one correspondence between ideals in $C_r^*(G \ltimes \Omega)$ and open invariant subsets in Ω.

$$Prim\big(C_r^*(G \ltimes \Omega)\big) = \big\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\, 0\big\}.$$

Here $C_r^*(G \ltimes (\Omega \setminus \partial\Omega))$ is maximal and thus primitive. The intersection of all primitive ideals is 0 and thus 0 is primitive. $\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega))\}$ is the only nontrivial closed subset.

Case 2. If $0 < |S_1| < \infty$, there are three nonempty closed invariant subsets, Ω, Ω_∞ and $\partial\Omega$. When $G \curvearrowright \Omega_\infty$ is topologically free, $G \ltimes \Omega$ is essentially principal and there is one to one correspondence between ideals in $C_r^*(G \ltimes \Omega)$ and open invariant subsets in Ω.

$$Prim\big(C_r^*(G \ltimes \Omega)\big) = \big\{C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)),\, C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\, 0\big\}.$$

Here $C_r^*(G \ltimes (\Omega \setminus \Omega_\infty))$ is primitive because it could never be the intersection of other primitive ideals. There are two nontrivial closed subsets: $\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega))\}$, $\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\, C_r^*(G \ltimes (\Omega \setminus \Omega_\infty))\}$.

Case 3. If $|S_1| = 1$ and $m_i = 1$ for $i \in S_1$, $G \curvearrowright \Omega_\infty$ is not topologically free. $C_r^*(G \ltimes \partial\Omega)$ is simple and thus

$$Prim_{J_1}\big(C_r^*(G \ltimes \Omega_\infty)\big) = \big\{J_1 := C_r^*(G \ltimes (\Omega_\infty \setminus \partial\Omega))\big\}.$$

We have an C^*-isomorphism $\varphi_1 : J_1 \to \mathscr{K} \otimes C(\mathbb{T})$ and

$$Prim(\mathscr{K} \otimes C(\mathbb{T})) = \big\{\mathscr{K} \otimes C_0(\mathbb{T} \setminus \{p\}),\ p \in \mathbb{T}\big\}.$$

Therefore,

$$Prim^{J_1}\big(C_r^*(G \ltimes \Omega_\infty)\big) = \big\{\rho_{J_1}^{-1}(I_p),\ p \in \mathbb{T}\big\},$$

where $I_p = \varphi_1^{-1}(\mathscr{K} \otimes C_0(\mathbb{T} \setminus \{p\}))$ is a maximal ideal in J_1.

So we have

$$Prim_{J_2}\big(C_r^*(G \ltimes \Omega)\big) = \big\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ J_2 + \rho_{J_1}^{-1}(I_p),\ p \in \mathbb{T}\big\},$$

where $J_2 := C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)$ is isomorphic to $\mathscr{K}$. By $Prim(\mathscr{K}) = \{0\}$, we get

$$Prim^{J_2}\big(C_r^*(G \ltimes \Omega)\big) = \{0\}.$$

Here $\{0\}$ is primitive in $C_r^*(G \ltimes \Omega)$ since $C_r^*(G \ltimes \Omega) \cong C_\lambda^*(P)$ and the left regular representation of $C_\lambda^*(P)$ on $\ell_2(P)$ is irreducible and faithful.

To determine the topology on $Prim\big(C_r^*(G \ltimes \Omega)\big)$, we need to determine firstly the topology on $Prim\big(C_r^*(G \ltimes \Omega_\infty)\big)$. To fulfill this, we need to have a better understanding of what $\rho_{J_1}^{-1}(I_p)$ is.

Recall that

$$C_r^*(G \ltimes (\Omega_\infty \setminus \partial\Omega)) \cong \mathscr{K} \otimes C_r^*(H_i) \cong \mathscr{K} \otimes C(\mathbb{T}),$$

where $H_i \cong \mathbb{Z}$ is generated by $a_i,\ i \in S_1$. Every continuous function $f \in C(\mathbb{T})$ is of the form $\sum_{n\in\mathbb{Z}} c_n z^n$. Such a function corresponds to an element $f' = \sum_{n\in\mathbb{Z}} c_n \lambda_{a_i^n}$ in the group C^*-algebra $C_r^*(H_i)$ via the isomorphism $C_r^*(H_i) \cong C(\mathbb{T})$. Assume $f \in C(\mathbb{T}\setminus\{1\})$, then $f(1) = 0$ and thus $\sum_{n\in\mathbb{Z}} c_n = 0$.

For every $x \in P$, there exists $y \in P$ and $j \in \mathbb{N}$ such that $x = ya_i^j$. Among all the pairs $(y,\ j)$, there is a special pair $(\bar{x},\ j_x)$ such that $j_x \geq j$ for any other pair $(y,\ j)$. Let $\bar{P} \subseteq P$ be the collection of $\bar{x}$ when x varies over P. The function

$$\phi:\ C_c(G \ltimes (\Omega_\infty \setminus \partial\Omega)) \to \mathscr{L}\big(\ell_2(\bar{P};\ell_2(H_i))\big),\ F \mapsto (F_{pq})_{pq},$$

is defined by $F_{pq}(a_i^k) = F(g,\ \chi_{hw_{a_i}})$ if $\bar{h} = q$ and $g = pa_i^k q^{-1}$, and $F_{pq}(a_i^k) = 0$ otherwise. If $F \in C_c(G \ltimes (\Omega_\infty \setminus \partial\Omega)) \cap I_1$, then $\phi(F)$ is of the form $\sum_i M_i \otimes f_i'$ with $f_i \in C(\mathbb{T}\setminus\{1\})$. Since F is finitely supported, we can assume, without loss of generality, M_i has at most one nonzero entry. Therefore, $\sum_k F_{pq}(a_i^k) = 0$ for all $p,\ q \in \bar{P}$.

Hence, $J_1/I_1 \cong \mathscr{K} \subseteq \mathscr{L}(\ell_2(\bar{P}))$ and the quotient map $\pi : J_1 \to \mathscr{K}$ sends the function $F \in C_c(G \ltimes (\Omega_\infty \setminus \partial\Omega))$ to the infinite matrix $(F'_{pq})_{pq}$, where $F'_{pq} = \sum_{\bar{h}=q,\ \overline{gh}=p} F(g,\ \chi_{hw_{a_i}})$.

By [Bla06, II.6.1.6], there is a unique extension of π to a representation of $C_r^*(G \ltimes \Omega_\infty)$ on $\ell_2(\bar{P})$. Assume $\Omega_\infty \setminus \partial\Omega := \{\chi_1,\ \chi_2,\ \cdots\}$ and let $X_n = \{(\varepsilon,\ \chi_1),\ (\varepsilon,\ \chi_2),\ \cdots,\ (\varepsilon,\ \chi_n)\}$, then X_n is a compact subset of the groupoid $G \ltimes \Omega_\infty$. Let $h_n = 1_{X_n}$, then (h_n) is an approximate unit for J_1. By [Bla06, II.6.1.6], $\pi(Fh_n) \to \pi(F)$ in the strong operator topology in $\mathscr{L}(\ell_2(\bar{P}))$ for every function $F \in C_c(G \ltimes \Omega_\infty)$. It is easy to check that $\pi(F) = (F'_{pq})_{pq}$ with $F'_{pq} = \sum_{\bar{h}=q,\ \overline{gh}=p} F(g,\ \chi_{hw_{a_i}})$.

π is an irreducible representation and its kernel is not I_1. Therefore, $\rho_{J_1}^{-1}(I_p)$ is a maximal ideal in $C_r^*(G \ltimes \Omega_\infty)$ and thus

$$\rho_{J_1}^{-1}(I_p) = I_p + C_r^*(G \ltimes \partial\Omega).$$

Let $X \subseteq Prim\big(C_r^*(G \ltimes \Omega_\infty)\big)$ be closed, then $X \cap Prim^{J_1}\big(C_r^*(G \ltimes \Omega_\infty)\big)$ is closed in $Prim^{J_1}\big(C_r^*(G \ltimes \Omega_\infty)\big)$ and thus

$$X \cap Prim^{J_1}\big(C_r^*(G \ltimes \Omega_\infty)\big) = \{\rho_{J_1}^{-1}(I_p),\ p \in C\}$$

for some closed subset $C \subseteq \mathbb{T}$.

Noting $C_c(G \ltimes \partial\Omega) \cap J_1 = \emptyset$ and $C_c(G \ltimes \partial\Omega) \subseteq \rho_{J_1}^{-1}(I_p)$ for all $p \in \mathbb{T}$, we conclude, for an arbitrary closed subset $X \subseteq Prim\big(C_r^*(G \ltimes \Omega_\infty)\big)$, either $X = \{J_1\}$ or $X = \{\rho_{J_1}^{-1}(I_p),\ p \in C\}$ for some closed subset $C \subseteq \mathbb{T}$. Here is a list of all nonempty closed subsets of $Prim\big(C_r^*(G \ltimes \Omega)\big)$:

$$\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega))\},$$

$$\{J_2 + \rho_{J_1}^{-1}(I_p),\ p \in C,\ C \subseteq \mathbb{T} \text{ closed}\},$$

$$\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ J_2 + \rho_{J_1}^{-1}(I_p),\ p \in C,\ C \subseteq \mathbb{T} \text{ closed}\},$$

$$\{0,\ C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ J_2 + \rho_{J_1}^{-1}(I_p),\ p \in \mathbb{T}\}.$$

When N is finite:

Case 4. If $|S_1| = 0$, there are three nonempty closed invariant subsets, Ω, Ω_∞ and $\partial\Omega$. $G \curvearrowright \Omega_\infty$ is not topologically free and the analysis of primitive ideals is similar as in Case 3.

Case 5. If $0 < |S_1| < \infty$ and $|S_2| = 0$, there are five nonempty closed invariant subsets, Ω, Ω_∞, $\Omega_{a,\,\infty}$, $\Omega_{b,\,\infty}$ and $\partial\Omega$. When $G \curvearrowright \Omega_{a,\,\infty}$ is topologically free, $G \ltimes \Omega_{a,\,\infty}$ is essentially principal and there is one to one correspondence between ideals in $C_r^*(G \ltimes \Omega_{a,\,\infty})$ and open invariant subsets in $\Omega_{a,\,\infty}$. Therefore,

$$Prim\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) = \big\{C_r^*(G \ltimes (\Omega_{a,\,\infty} \setminus \partial\Omega)),\, 0\big\}$$

and thus

$$Prim_{J_3}\big(C_r^*(G \ltimes \Omega_\infty)\big) = \big\{C_r^*(G \ltimes (\Omega_\infty \setminus \partial\Omega)),\, J_3 := C_r^*(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty}))\big\}.$$

We have C^*-isomorphism $\varphi_3 :\ J_3 \cong \mathscr{K} \otimes C(\mathbb{T})$ and thus

$$Prim^{J_3}\big(C_r^*(G \ltimes \Omega_\infty)\big) = \big\{\rho_{J_3}^{-1}(I_p'),\ p \in \mathbb{T}\big\},$$

where $I_p' = \varphi_3^{-1}(\mathscr{K} \otimes C_0(\mathbb{T} \setminus \{p\}))$ is a maximal ideal in J_3. Similarly as in Case 3, we have

$$\rho_{J_3}^{-1}(I_p') = I_p' + C_r^*(G \ltimes \Omega_{a,\,\infty}).$$

So we have

$$Prim_{J_2}\big(C_r^*(G \ltimes \Omega)\big) = \big\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\, C_r^*(G \ltimes (\Omega \setminus \Omega_{a,\,\infty})),\, J_2 + \rho_{J_3}^{-1}(I_p'),\ p \in \mathbb{T}\big\},$$

where $J_2 := C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)$ is isomorphic to $\mathscr{K}$. By $Prim(\mathscr{K}) = \{0\}$, we get

$$Prim^{J_2}\big(C_r^*(G \ltimes \Omega)\big) = \{0\}.$$

Here $\{0\}$ is primitive in $C_r^*(G \ltimes \Omega)$ since $C_r^*(G \ltimes \Omega) \cong C_\lambda^*(P)$ and the left regular representation of $C^*_\lambda(P)$ on $\ell_2(P)$ is irreducible and faithful.

Here is a list of all nonempty closed subsets of $Prim\big(C^*_r(G\ltimes\Omega)\big)$:

$$\{C^*_r(G\ltimes(\Omega\setminus\partial\Omega))\},\ \{C^*_r(G\ltimes(\Omega\setminus\Omega_{a,\,\infty})),\ C^*_r(G\ltimes(\Omega\setminus\partial\Omega))\},$$

$$\{J_2+\rho_{J_3}^{-1}(I'_p),\ p\in C,\ C\subseteq\mathbb{T}\text{ closed}\},$$

$$\{C^*_r(G\ltimes(\Omega\setminus\partial\Omega)),\ J_2+\rho_{J_3}^{-1}(I'_p),\ p\in C,\ C\subseteq\mathbb{T}\text{ closed}\},$$

$$\{C^*_r(G\ltimes(\Omega\setminus\Omega_{a,\,\infty})),\ C^*_r(G\ltimes(\Omega\setminus\partial\Omega)),\ J_2+\rho_{J_2}^{-1}(I'_p),\ p\in C,\ C\subseteq\mathbb{T}\text{ closed}\},$$

$$\{0,\ C^*_r(G\ltimes(\Omega\setminus\Omega_{a,\,\infty})),\ C^*_r(G\ltimes(\Omega\setminus\partial\Omega)),\ J_2+\rho_{J_2}^{-1}(I'_p),\ p\in\mathbb{T}\}.$$

Case 6. If $|S_1|=1,\ m_i=1$ for $i\in S_1$ and $|S_2|=0$, $G\curvearrowright\Omega_{a,\,\infty}$ is not topologically free.

$$Prim_{J_1}\big(C^*_r(G\ltimes\Omega_{a,\,\infty})\big)=\big\{J_1=C^*_r(G\ltimes(\Omega_{a,\,\infty}\setminus\partial\Omega))\big\}.$$

We have an C^*-isomorphism $\varphi_1:\ J_1\ \to\ \mathscr{K}\otimes C(\mathbb{T})$ and thus

$$Prim^{J_1}\big(C^*_r(G\ltimes\Omega_{a,\,\infty})\big)=\big\{\rho_{J_1}^{-1}(I_p),\ p\in\mathbb{T}\big\},$$

where $I_p=\varphi_1^{-1}(\mathscr{K}\otimes C_0(\mathbb{T}\setminus\{p\}))$ is a maximal ideal in J_1. Similarly, we have

$$\rho_{J_1}^{-1}(I_p)=I_p+C^*_r(G\ltimes\partial\Omega).$$

So we have

$$Prim_{J_3}\big(C_r^*(G\ltimes\Omega_\infty)\big)=\big\{C_r^*(G\ltimes(\Omega_\infty\setminus\partial\Omega)),\ J_3+\rho_{J_1}^{-1}(I_p),\ p\in\mathbb{T}\big\}.$$

We have C^*-isomorphism $\varphi_3:\ J_3\cong\mathscr{K}\otimes C(\mathbb{T})$ and thus

$$Prim^{J_3}\big(C_r^*(G\ltimes\Omega_\infty)\big)=\big\{\rho_{J_3}^{-1}(I'_p),\ p\in\mathbb{T}\big\},$$

where $I'_p=\varphi_3^{-1}(\mathscr{K}\otimes C_0(\mathbb{T}\setminus\{p\}))$ is a maximal ideal in J_3.

So we have

$$Prim_{J_2}\big(C_r^*(G\ltimes\Omega)\big)=\big\{C_r^*(G\ltimes(\Omega\setminus\partial\Omega)),\ J_2+\rho_{J_3}^{-1}(I'_p),\ J_2+J_3+\rho_{J_1^{-1}}(I_p),\ p\in\mathbb{T}\big\},$$

By $Prim(J_2)=\{0\}$, we get

$$Prim^{J_2}\big(C_r^*(G\ltimes\Omega)\big)=\{0\}.$$

Here $\{0\}$ is primitive in $C_r^*(G\ltimes\Omega)$ since $C_r^*(G\ltimes\Omega)\cong C_\lambda^*(P)$ and the left regular representation of $C_\lambda^*(P)$ on $\ell_2(P)$ is irreducible and faithful.

Here is a list of all nonempty closed subsets of $Prim\big(C_r^*(G\ltimes\Omega)\big)$:

$$\{C_r^*(G\ltimes(\Omega\setminus\partial\Omega))\},\ \{C'\},\ \{C''\},\ \{C',\ C''\},\ \{C_r^*(G\ltimes(\Omega\setminus\partial\Omega)),\ C'\},$$

$$\{C_r^*(G\ltimes(\Omega\setminus\partial\Omega)),\ C''\},\ \{C_r^*(G\ltimes(\Omega\setminus\partial\Omega)),\ C',\ C''\},$$

$$\{0,\ C_r^*(G\ltimes(\Omega\setminus\partial\Omega)),\ J_2+\rho_{J_3}^{-1}(I'_p),\ J_2+J_3+\rho_{J_1}^{-1}(I_p),\ p\in\mathbb{T}\}.$$

Here $C'=\{J_2+\rho_{J_3}^{-1}(I'_p),\ p\in C\}$ for some closed subset $C\subseteq\mathbb{T}$ and $C''=\{J_2+J_3+\rho_{J_1}^{-1}(I_p),\ p\in C\}$ for some closed subset $C\subseteq\mathbb{T}$.

Case 7. If $0 < |S_1| < \infty$ and $|S_2| \neq 0$, there are four nonempty closed invariant subsets, Ω, Ω_∞, $\Omega_{b,\,\infty}$ and $\partial\Omega$. By the isomorphism

$$\varphi_3 :\ J_3 = C_r^*(G \ltimes (\Omega_{b,\,\infty} \setminus \partial\Omega)) \cong \mathscr{K} \otimes C(\mathbb{T}),$$

we get

$$Prim^{J_3}\big(C_r^*(G \ltimes \Omega_{b,\,\infty})\big) = \{\rho_{J_3}^{-1}(I'_p),\ p \in \mathbb{T}\},$$

where $I'_p = \varphi_3^{-1}(\mathscr{K} \otimes C_0(\mathbb{T} \setminus \{p\}))$ is a maximal ideal in J_3. Similarly as in Case 3, we have

$$\rho_{J_3}^{-1}(I'_p) = I'_p + C_r^*(G \ltimes \partial\Omega).$$

So we have

$$Prim_{J_1}\big(C_r^*(G \ltimes \Omega_\infty)\big) = \big\{C_r^*(G \ltimes (\Omega_\infty \setminus \partial\Omega)),\ J_1 + \rho_{J_3}^{-1}(I'_p),\ p \in \mathbb{T}\big\},$$

where $J_1 = C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{b,\,\infty})\big)$.

If $|S_1| \geq 2$ or $|S_1| = 1$ and $m_i \geq 2$ for $i \in S_1$, $G \ltimes \Omega_\infty \setminus \Omega_{b,\,\infty}$ is topologically free and hence $C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{b,\,\infty})\big)$ is simple.

Take $\chi \in \Omega_\infty \setminus \Omega_{b,\,\infty}$, and consider the left regular representation π_χ of $C_r^*(G \ltimes \Omega_\infty)$ on $\ell_2(G \ltimes \{\chi\})$. It is irreducible and thus the kernel is a primitive ideal of $C_r^*(G \ltimes \Omega_\infty)$. For nonzero function $f \in C_c(G \ltimes \Omega_\infty)$, it is nonzero in $\{g\} \ltimes \mathscr{O}$ for some $g \in G$ and some open subset $\mathscr{O} \subseteq \Omega_\infty$. Since $G\chi$ is dense in Ω_∞, there exists $h \in G$ with $\chi \in \mathrm{Dom}(h)$ and $h\chi \in \mathrm{Dom}(g) \cap \mathscr{O}$. That is, $f(g,\ h\chi) \neq 0$. It is easy to see that $f \notin \ker(\pi_\chi)$ and that

$\ker(\pi_\chi) = 0$.

Therefore,

$$Prim^{J_1}\big(C_r^*(G \ltimes \Omega_\infty)\big) = 0$$

and

$$Prim_{J_2}\big(C_r^*(G \ltimes \Omega)\big) = \big\{C_r^*(G \ltimes (\Omega \setminus \Omega_\infty),\ C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ J_2 + J_1 + \rho_{J_3}^{-1}(I_p'),\ p \in \mathbb{T}\big\}.$$

By $Prim(\mathscr{K}) = \{0\}$, we get

$$Prim^{J_2}\big(C_r^*(G \ltimes \Omega)\big) = \{0\}.$$

Here $\{0\}$ is primitive in $C_r^*(G \ltimes \Omega)$ since $C_r^*(G \ltimes \Omega) \cong C_\lambda^*(P)$ and the left regular representation of $C_\lambda^*(P)$ on $\ell_2(P)$ is irreducible and faithful.

Here is a list of all nonempty closed subsets of $Prim\big(C_r^*(G \ltimes \Omega)\big)$:

$$\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega))\},\ \{C'\},\ \{C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ C'\},$$

$$\{C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)),\ C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ J_2 + J_1 + \rho_{J_3}^{-1}(I_p'),\ p \in \mathbb{T}\},$$

$$\{0,\ C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)),\ C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ J_2 + J_1 + \rho_{J_3}^{-1}(I_p'),\ p \in \mathbb{T}\}.$$

Here $C' = \{J_2 + J_1 + \rho_{J_3}^{-1}(I_p'),\ p \in C\}$ for some closed subset $C \subseteq \mathbb{T}$.

Case 8. If $|S_1| = 1,\ m_i = 1$ for $i \in S_1$ and $|S_2| \neq 0$, we have

$$Prim_{J_1}\big(C_r^*(G \ltimes \Omega_\infty)\big) = \big\{C_r^*(G \ltimes (\Omega_\infty \setminus \partial\Omega)),\ J_1 + \rho_{J_3}^{-1}(I_p'),\ p \in \mathbb{T}\big\},$$

where $J_1 = C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{b,\,\infty})\big)$.

We also have C^*-isomorphism $\varphi_1 :\ J_1 \cong \mathscr{K} \otimes C(\mathbb{T})$ and thus

$$Prim^{J_1}\big(C_r^*(G \ltimes \Omega_\infty)\big) = \big\{\rho_{J_1}^{-1}(I_p),\ p \in \mathbb{T}\big\},$$

where $I_p = \varphi_1^{-1}(\mathscr{K} \otimes C_0(\mathbb{T} \setminus \{p\}))$ is a maximal ideal in J_1.

Therefore,

$$Prim_{J_2}\big(C_r^*(G \ltimes \Omega)\big) = \big\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ J_2 + J_1 + \rho_{J_3}^{-1}(I'_p),\ J_2 + \rho_{J_1}^{-1}(I_p),\ p \in \mathbb{T}\big\}.$$

$Prim(J_2) = \{0\}$, so we get

$$Prim^{J_2}\big(C_r^*(G \ltimes \Omega)\big) = \{0\}.$$

Here $\{0\}$ is primitive in $C_r^*(G \ltimes \Omega)$ since $C_r^*(G \ltimes \Omega) \cong C_\lambda^*(P)$ and the left regular representation of $C_\lambda^*(P)$ on $\ell_2(P)$ is irreducible and faithful.

Here is a list of all nonempty closed subsets of $Prim\big(C_r^*(G \ltimes \Omega)\big)$:

$$\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega))\},\ \{C'\},\ \{C''\},\ \{C',\ C''\},\ \{C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ C'\},$$

$$\{C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ C''\},\ \{C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ C',\ C''\},$$

$$\{0,\ C_r^*(G \ltimes (\Omega \setminus \partial\Omega)),\ J_2 + \rho_{J_1}^{-1}(I_p),\ J_2 + J_1 + \rho_{J_3}^{-1}(I'_p),\ p \in \mathbb{T}\}.$$

Here $C' = \{J_2 + \rho_{J_1}^{-1}(I_p),\ p \in C\}$ for some closed subset $C \subseteq \mathbb{T}$ and $C'' = \{J_2 + J_1 + \rho_{J_3}^{-1}(I'_p),\ p \in C\}$ for some closed subset $C \subseteq \mathbb{T}$.

6.2 General case

In this section, let P be the fundamental monoid of a graph of monoids with condition (LCM) for P satisfied. Assume that $\{\varepsilon\} \neq G_v \subseteq (\mathbb{R}, +)$ for all $v \in V$, $P_e \neq \{\varepsilon\}$ for all $e \in A$ and $P_e^e \neq P_{t(e)}$ for all $e \in E$. We still aim at the primitive ideal space of the groupoid C^*-algebra $C_r^*(G \ltimes \Omega)$.

If condition I. holds and there exists $v \in V$ such that G_v is dense in $\mathbb{R}$, then Ω is minimal and the partial action $G \curvearrowright \Omega$ is topologically free, so the groupoid C^*-algebra $C_r^*(G \ltimes \Omega)$ is simple by [BL18, Corollary 3.14].

If condition I. holds and $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$, then $G \curvearrowright \Omega$ is minimal and topologically free whenever Ω_∞ is not closed. In this case, $C_r^*(G \ltimes \Omega)$ is simple.

If condition I. holds, $P_v \cong \mathbb{Z}_{\geq 0}$ for all $v \in V$ and Ω_∞ is closed, then $G \curvearrowright \Omega_\infty$ is minimal and topologically free. There is a one-to-one correspondence between open invariant subsets of Ω and ideals in $C_r^*(G \ltimes \Omega)$. It is easy to check

$$\mathrm{Prim}(C_r^*(G \ltimes \Omega)) = \{0,\, C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) \cong \mathscr{K}\}.$$

If condition II. holds, $\sharp A = 0$ and Ω_∞ is not closed, there are two nonempty closed invariant subsets Ω, $\partial\Omega = \{\infty\}$. The action $G \curvearrowright \Omega \setminus \{\infty\}$ is minimal and topologically free, so $C_r^*(G \ltimes (\Omega \setminus \{\infty\}))$ is simple and we have

$$\mathrm{Prim}^J(C_r^*(G \ltimes \Omega)) = \{0\}.$$

$C_r^*(G \ltimes \{\infty\}) \cong C_\lambda^*(G_T)$, and we have

$$\mathrm{Prim}(C_r^*(G \ltimes \Omega)) = \{0,\ I + C_r^*(G \ltimes (\Omega \setminus \{\infty\})), I \subseteq C_r^*(G \ltimes \{\infty\}) \text{ primitive}\}.$$

Every nontrivial closed subset of $\mathrm{Prim}_J(C_r^*(G \ltimes \Omega))$ is of the form $C + C_r^*(G \ltimes (\Omega \setminus \{\infty\}))$, where C is a nonempty closed subset in $\mathrm{Prim}(C_r^*(G \ltimes \{\infty\}))$.

If condition II. holds, $\sharp A = 0$ and Ω_∞ is closed, there are three nonempty closed invariant subsets $\Omega,\ \Omega_\infty,\ \partial\Omega = \{\infty\}$. If the action $G \curvearrowright \Omega_\infty$ is topologically free, then the action $G \curvearrowright \Omega_\infty \setminus \{\infty\}$ is minimal and topologically free, so $C_r^*(G \ltimes (\Omega_\infty \setminus \{\infty\}))$ is simple. In this case, we have

$$\mathrm{Prim}(C_r^*(G \ltimes \Omega_\infty)) = \{0,\ I + C_r^*(G \ltimes (\Omega_\infty \setminus \{\infty\})), I \subseteq C_r^*(G \ltimes \{\infty\}) \text{ primitive}\}.$$

Therefore,

$$\mathrm{Prim}(C_r^*(G \ltimes \Omega)) = \{0,\ C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) \cong \mathscr{K},\ I + C_r^*(G \ltimes (\Omega \setminus \{\infty\}))\},$$

where $I \subseteq C_r^*(G \ltimes \{\infty\})$ is primitive.

If the action $G \curvearrowright \Omega_\infty$ is not topologically free, then we must have $\sharp V = 2,\ k = l = 2$. (see Proposition 5.3.13) Let $J := C_r^*(G \ltimes (\Omega_\infty \setminus \{\infty\}))$, we can prove $J \cong \mathscr{K} \otimes C(\mathbb{T})$. In this case, we have

$$\mathrm{Prim}(C_r^*(G \ltimes \Omega_\infty)) = \{J_p + C_r^*(G \ltimes \{\infty\}),\ I + C_r^*(G \ltimes (\Omega_\infty \setminus \{\infty\}))\},$$

where $J_p := \varphi^{-1}(\mathscr{K} \otimes C_0(\mathbb{T} \setminus \{p\})),\ p \in \mathbb{T}$ with $\varphi :\ J \to \mathscr{K} \otimes C(\mathbb{T})$ is a $*$-isomorphism, and $I \subseteq C_r^*(G \ltimes \{\infty\})$ is primitive.

Therefore,

$$\mathrm{Prim}(C_r^*(G\ltimes\Omega)) = \{0,\ C_r^*(G\ltimes\{\infty\})+J_p+C_r^*(G\ltimes(\Omega\setminus\Omega_\infty)),\ I+C_r^*(G\ltimes(\Omega\setminus\{\infty\}))\},$$

where J_p and I are as above.

Here is a list of all nontrivial closed subsets of $\mathrm{Prim}(C_r^*(G\ltimes\Omega))$:

$$\{C_r^*(G\ltimes\{\infty\})+J_p+C_r^*(G\ltimes(\Omega\setminus\Omega_\infty))\}_{p\in C},\ \{C'+C_r^*(G\ltimes(\Omega\setminus\{\infty\}))\},$$
$$\{C_r^*(G\ltimes\{\infty\})+J_p+C_r^*(G\ltimes(\Omega\setminus\Omega_\infty)),\ p\in C,\ \ C'+C_r^*(G\ltimes(\Omega\setminus\{\infty\}))\},$$

where $C\subseteq\mathbb{T}$ is closed and $C'\subseteq\mathrm{Prim}(C_r^*(G\ltimes\{\infty\}))$ is also closed.

If condition II. holds and $\sharp A\neq 0$, we assume the action $G\curvearrowright\partial\Omega$ is topologically free. If Ω_∞ is not closed, there are only two nonempty closed invariant subsets Ω, $\partial\Omega=\Omega_{\mathbf{b},\,\infty}$. In this case,

$$\mathrm{Prim}(C_r^*(G\ltimes\Omega)) = \{0,\ C_r^*(G\ltimes(\Omega\setminus\Omega_{\mathbf{b},\,\infty}))\}.$$

If Ω_∞ is closed, there are three nonempty closed invariant subsets Ω, Ω_∞, $\partial\Omega=\Omega_{\mathbf{b},\,\infty}$. If the action $G\curvearrowright\Omega_\infty$ is topologically free, then there is a one-to-one correspondence between open invariant subsets of Ω and ideals in $C_r^*(G\ltimes\Omega)$. It is easy to check

$$\mathrm{Prim}(C_r^*(G\ltimes\Omega)) = \{0,\ C_r^*(G\ltimes(\Omega\setminus\Omega_\infty))\cong\mathscr{K},\ C_r^*(G\ltimes(\Omega\setminus\Omega_{\mathbf{b},\,\infty}))\}.$$

If the action $G\curvearrowright\Omega_\infty$ is not topologically free, then we must have $\sharp V=2$, $k=l=2$ and $\sharp A_+=0$. (see Proposition 5.3.13) Let $J:=C_r^*(G\ltimes(\Omega_\infty\setminus\Omega_{\mathbf{b},\,\infty}))$, we can prove $J\cong\mathscr{K}\otimes C(\mathbb{T})$. In this case, we have

$$\mathrm{Prim}(C_r^*(G\ltimes\Omega_\infty)) = \{J,\ J_p+C_r^*(G\ltimes\Omega_{\mathbf{b},\,\infty})\},$$

where $J_p := \varphi^{-1}(\mathscr{K} \otimes C_0(\mathbb{T} \setminus \{p\})),\ p \in \mathbb{T}$ and $\varphi :\ J \to \mathscr{K} \otimes C(\mathbb{T})$ is a $*$-isomorphism. Therefore,

$$\mathrm{Prim}(C_r^*(G \ltimes \Omega)) = \{0,\ C_r^*(G \ltimes (\Omega \setminus \Omega_{\mathbf{b},\,\infty})),\ C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) + J_p + C_r^*(G \ltimes \Omega_{\mathbf{b},\,\infty})\}.$$

Here is a list of all nontrivial closed subsets of $\mathrm{Prim}(C_r^*(G \ltimes \Omega))$:

$$\{I\},\ \{C\},\ \{I,\ C\},$$

where $I := C_r^*(G \ltimes (\Omega \setminus \Omega_{\mathbf{b},\,\infty}))$ and $C = \{C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) + J_p + C_r^*(G \ltimes \Omega_{\mathbf{b},\,\infty})\}_{p \in C'}$ for some closed subset $C' \subseteq \mathbb{T}$.

Chapter 7

K-theory

In this chapter, we will try to find the K-theory of all the C^*-algebras of the form $C_r^*(G \ltimes X)$ with $X \subseteq \Omega$ G-invariant and closed.

First of all, we have $C_r^*(G \ltimes \Omega) \cong C_\lambda^*(P)$ and by [CELY17, Theorem 5.10.1] there exists an unital $*$-homomorphism $\iota : \mathbb{C} \to C_\lambda^*(P)$ such that $K_*(\iota) : K_*(\mathbb{C}) \to K_*(C_\lambda^*(P)), * = 0, 1$ is an isomorphism. That is,

$$K_0(C_r^*(G \ltimes \Omega)) \cong \mathbb{Z} \text{ and } K_1(C_r^*(G \ltimes \Omega)) \cong 0.$$

7.1 Generalised Baumslag-Solitar case

In this section, we assume P is a generalised Baumslag-Solitar monoid.

Firstly, we compute the K-theory of $C_r^*(G \ltimes \Omega_{b,\,\infty})$ since $\Omega_{b,\,\infty}$ is always closed in Ω.

We claim that $\{g\Omega_{b,\,\infty}\}_{g \in G}$ is a G-invariant regular basis for the compact open subsets of

$\Omega_{b,\,\infty}$. It is easy to see that $g\Omega_{b,\,\infty}$ is a compact open subset of $\Omega_{b,\,\infty}$ for all $g \in G$ and that $\{g\Omega_{b,\,\infty}\}_{g\in G}$ is G-invariant. Therefore, it remains to show that $\{g\Omega_{b,\,\infty}\}_{g\in G}$ is a regular basis. We have the following observations.

(i) If $\cap_{1\le i\le n} p_i\Omega_{b,\,\infty} \neq \emptyset$ with $p_i \in P$, $1 \le i \le n$ and $n \in \mathbb{N}$, then we must have $\cap_{1\le i\le n} p_i P \neq \emptyset$ and thus $\cap_{1\le i\le n} p_i P = rP$ for some $r \in P$ because P is right LCM. Therefore,

$$\cap_{1\le i\le n} p_i\Omega_{b,\,\infty} = r\Omega_{b,\,\infty}.$$

(ii) For every basic compact open subset $\mathscr{O}$ in $\Omega_{b,\infty}$, there exist $p,\ p_i,\ 1 \le i \le n \in P$ such that $\mathscr{O} = \{\chi \in \Omega_{b,\,\infty},\ \chi(pP) = 1,\ \chi(p_iP) = 0\}$. In this case, we have

$$\mathscr{O} = p\Omega_{b,\,\infty} \setminus (\cup_{1\le i\le n} p_i\Omega_{b,\,\infty}).$$

(iii) If $p\Omega_{b,\,\infty} = \cup_{1\le i\le n} p_i\Omega_{b,\,\infty}$ for some $p,\ p_i,\ 1 \le i \le n \in P$, then we must have $pP = \cup_{1\le i\le n} p_iP$ and thus $pP = p_iP$ for some i because P satisfies independence. In this case, $p\Omega_{b,\,\infty} = p_i\Omega_{b,\,\infty}$.

These observations, together with the fact that for all $g \in G$ there exists $p \in P$ such that $g\Omega_{b,\,\infty} = p\Omega_{b,\,\infty}$, yields our claim by Definition 2.3.5.

Noting that G satisfies the Baum-Connes conjecture with coefficients, we have by Lemma 2.3.4

$$K_*(G \ltimes \Omega_{b,\,\infty}) \cong K_*(C(\Omega_{b,\,\infty}) \rtimes_r G) \cong K_*(C^*_\lambda(b^{\mathbb{Z}})).$$

Therefore,

$$K_0\big(C^*_r(G \ltimes \Omega_{b,\,\infty})\big) \cong \mathbb{Z} \text{ and } K_1\big(C^*_r(G \ltimes \Omega_{b,\,\infty})\big) \cong \mathbb{Z}.$$

Now we compute the K-theory of $C_r^*(G \ltimes \Omega_\infty)$ in the case when Ω_∞ is closed in Ω.

We have the following exact sequence of C^*-algebras,

$$0 \to C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big) \to C_r^*\big(G \ltimes \Omega\big) \to C_r^*(G \ltimes \Omega_\infty) \to 0,$$

and the six term exact sequence of their K-theories,

$$\begin{array}{ccccc} K_0\big(C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)\big) & \longrightarrow & K_0\big(C_r^*(G \ltimes \Omega)\big) & \longrightarrow & K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \\ \uparrow & & & & \downarrow \\ K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) & \longleftarrow & K_1\big(C_r^*(G \ltimes \Omega)\big) & \longleftarrow & K_1\big(C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)\big). \end{array}$$

By Corollary 6.0.5, we have

$$C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big) \cong \mathscr{K}$$

and thus

$$K_0\big(C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)\big) \cong \mathbb{Z} \text{ and } K_1\big(C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)\big) \cong 0.$$

Noting that we also have

$$K_0\big(C_r^*(G \ltimes \Omega)\big) \cong \mathbb{Z} \text{ and } K_1\big(C_r^*(G \ltimes \Omega)\big) \cong 0,$$

we obtain the following six term exact sequence

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{K_0(\varphi)} & \mathbb{Z} & \longrightarrow & K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \\ \uparrow & & & & \downarrow \\ K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) & \longleftarrow & 0 & \longleftarrow & 0, \end{array}$$

where φ is a unital $*$-homomorphism from $C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)$ to $\mathbb{C}$, composed by

$$C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big) \xrightarrow{\iota} C_r^*(G \ltimes \Omega) \xrightarrow{\varphi_1} C_\lambda^*(P) \xrightarrow{\varphi_2} \mathbb{C}.$$

To calculate the K-theory of $C_r^*(G \ltimes \Omega_\infty)$, we need to find out the map $K_0(\varphi)$ from $\mathbb{Z}$ to $\mathbb{Z}$. It suffices to find out $K_0(\varphi)([p]_0)$ for some rank one projection $p \in C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)$.

Recall that the left regular representation of P is such that $\lambda_p(\delta_x) = \delta_{px},\ p,\ x \in P$, we define $E(p),\ p \in P$ to be the range space of λ_p in $\ell_2(P)$, and then the projection from $\ell_2(P)$ onto $E(p)$ is $\lambda_p\lambda_p^*$. It is easy to see that

$$E(a_i) \cap E(b) = E(a_i b^{m_i}) \text{ if } i \in S_1$$

and that

$$E(a_i) \subseteq E(b) \text{ if } i \in S_2.$$

Since Ω_∞ is closed, we have $0 \leq |S_1| < \infty$. In this case, we always have

$$q := 1 - [\lambda_b\lambda_b^* + \sum_{i \in S_1} (\lambda_{a_i}\lambda_{a_i}^* - \lambda_{a_i b^{m_i}}\lambda_{a_i b^{m_i}}^*)]$$

is a rank one projection in $C_\lambda^*(P)$, whose range space is exactly $\mathbb{C}\delta_e$. Here e is the identity of P. Noting $\lambda_b^*\lambda_b = \lambda_{a_i}^*\lambda_{a_i} = 1$, it follows that this rank one projection q is in the equivalence class of 0 in $\mathscr{P}_\infty(C_\lambda^*(P))$. So is any other rank one projection in $C_\lambda^*(P)$.

As a unital $*$-homomorphism, $\varphi_1 \circ \iota$ maps the rank one projection $p \in C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)$ to some rank one projection $q' \in C_\lambda^*(P)$. Therefore,

$$K_0(\varphi)([p]_0) = K_0(\varphi_2)([q']_0) = K_0(\varphi_2)(0) = 0.$$

That is, $K_0(\varphi) = 0$. From the six term exact sequence, it follows that

$$K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z} \text{ and } K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z}.$$

In the next we compute the K-theory of $C_r^*(G \ltimes \Omega_{a,\,\infty})$ in the case when $\Omega_{a,\,\infty}$ is closed in Ω.

When $\Omega_{a,\,\infty}$ is closed in Ω, we must N is finite and thus Ω_∞ is also closed in Ω. Hence we have the following exact sequence of C^*-algebras,

$$0 \to C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big) \to C_r^*\big(G \ltimes \Omega_\infty\big) \to C_r^*(G \ltimes \Omega_{a,\,\infty}) \to 0,$$

and the six term exact sequence of their K-theories,

$$\begin{tikzcd}
K_0\big(C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big)\big) \arrow[r] & K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \arrow[r] & K_0\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \arrow[d] \\
K_1\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \arrow[u] & K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \arrow[l] & K_1\big(C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big)\big). \arrow[l]
\end{tikzcd}$$

By Proposition 6.1.2, we have

$$C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big) \cong \mathscr{K} \otimes C(\mathbb{T})$$

and thus

$$K_0\big(C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big)\big) \cong \mathbb{Z} \text{ and } K_1\big(C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big)\big) \cong \mathbb{Z}.$$

By our previous computation,

$$K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z} \text{ and } K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z}.$$

Therefore, we have such a six term exact sequence

$$\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{K_0(\iota)} & \mathbb{Z} & \longrightarrow & K_0\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \\
\uparrow & & & & \downarrow \\
K_1\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) & \longleftarrow & \mathbb{Z} & \xleftarrow[K_1(\iota)]{} & \mathbb{Z},
\end{array}$$

where ι is the inclusion map from $C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big)$ into $C_r^*\big(G \ltimes \Omega_\infty\big)$.

First of all, in the K-theory of the C^*-algebra $\mathscr{K} \otimes C(\mathbb{T})$, we have $[p \otimes 1]_0 = 1$ for some rank one projection $p \in \mathscr{K}$. Via the path of $*$-isomorphisms

$$\mathscr{K} \otimes C(\mathbb{T}) \to \mathscr{K} \otimes C_r^*(b^{\mathbb{Z}}) \to C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big),$$

it is easy to find $[\delta_{(e,\ \chi_{w_b})}]_0 = 1$ in the K-theory of $C_r^*\big(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\big)$, where $\delta_{(e,\ \chi_{w_b})}$ is the delta function which takes value 1 at $(e,\ \chi_{w_b})$ and vanishes elsewhere. Via the quotient map $\pi:\ C_r^*(G \ltimes \Omega) \to C_r^*(G \ltimes \Omega_\infty)$, we find one preimage

$$f := 1_{\{e\}\times X} = 1 - \sum_{1\le i\le N,\ 0\le j\le |n_i|-1} 1_{\{b^j a_i\}\times\Omega} 1^*_{\{b^j a_i\}\times\Omega} \in C_c(G \ltimes \Omega)$$

of the element $\delta_{(e,\ \chi_{w_b})} \in C_r^*(G \ltimes \Omega_\infty)$. Here and in the sequel, 1_Y is always the characteristic function on the set Y. When $S_2 = \emptyset$,

$$X = \{\chi_{b^k},\ k \in\ \mathbb{N}\} \cup \{\chi_{w_b}\}.$$

When $S_1 = \emptyset$,

$$X = \{\chi_{w_b}\} \cup \{\chi_{b^k},\, k \in \mathbb{N}\} \cup \left(\cup_{1\le i\le N} \{\chi_{b^k a_i b^j},\, k \ge |n_i|,\, 0 \le j \le m_i - 1\}\right).$$

In $K_0(C_r^*(G \ltimes \Omega))$, $[f]_0 = 1 - \sum_{1\le i\le N} |n_i|$. That is, in $K_0(C_r^*(G \ltimes \Omega_\infty))$, $[\delta_{(e,\, \chi_{w_b})}]_0 = 1 - \sum_{1\le i\le N} |n_i|$. Therefore, $K_0(\iota)$ is a multiplication map, sending 1 to $1 - \sum_{1\le i\le N} |n_i|$.

In the K-theory of the C^*-algebra $\mathscr{K} \otimes C(\mathbb{T})$, we have $[p \otimes (z-1) + 1]_1 = 1$. Via the path of $*$-isomorphisms

$$\mathscr{K} \otimes C(\mathbb{T}) \to \mathscr{K} \otimes C_r^*(b^{\mathbb{Z}}) \to C_r^*\left(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\right),$$

it is easy to find $[\delta_{(b,\, \chi_{w_b})} + 1 - \delta_{(e,\, \chi_{w_b})}]_1 = 1$ in the K-theory of $C_r^*\left(G \ltimes (\Omega_\infty \setminus \Omega_{a,\,\infty})\right)$.

Let $u = \delta_{(b,\, \chi_{w_b})} + 1 - \delta_{(e,\, \chi_{w_b})} \in C_r^*(G \ltimes \Omega_\infty)$ and let

$$v := \begin{pmatrix} 1_{\{b\}\times X} + 1 - 1_{\{e\}\times X} & 1_{\{e\}\times (X\setminus bX)} \\ 0 & 1_{\{b^{-1}\}\times bX} + 1 - 1_{\{e\}\times X} \end{pmatrix}.$$

We have

$$\pi(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$$

and

$$p := v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^* = \begin{pmatrix} 1 - 1_{\{e\}\times(X\setminus bX)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, we have the index map $\delta_1 :\ K_1\left(C_r^*(G \ltimes \Omega_\infty)\right) \to K_0\left(C_r^*(G \ltimes (\Omega \setminus \Omega_\infty))\right)$,

$$\delta_1(1) = \delta_1([\delta_{(b,\, \chi_{w_b})} + 1 - \delta_{(e,\, \chi_{w_b})}]_1) = [p]_0 - [s(p)]_0 = -[1_{\{e\}\times(X\setminus bX)}]_0.$$

Therefore, $K_1(\iota)$ is a multiplication map, sending 1 to $-[1_{\{e\}\times(X\setminus bX)}]_0$.

When $S_2 = \emptyset$, $[1_{\{e\}\times(X\setminus bX)}]_0 = 1$.

When $S_1 = \emptyset$, $[1_{\{e\}\times(X\setminus bX)}]_0 = 1 + \sum_{1\leq i\leq N} m_i$.

When $\sum_{1\leq i\leq N} |n_i| > 1$, it is easy to conclude that

$$K_0\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \cong \mathbb{Z}_{(\sum_{1\leq i\leq N}|n_i|)-1}$$

and that

$$K_1\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \cong \mathbb{Z}_{1+\sum_{i\in S_2} m_i}.$$

Here and in the sequel, $\mathbb{Z}_n,\ n \in \mathbb{N}^*$ always stands for the quotient group of $\mathbb{Z}$ by its normal subgroup $n\mathbb{Z}$.

When $\sum_{1\leq i\leq N} |n_i| = 1$, it is easy to conclude that

$$K_0\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \cong \mathbb{Z}$$

and that

$$K_1\big(C_r^*(G \ltimes \Omega_{a,\,\infty})\big) \cong \mathbb{Z} \oplus \mathbb{Z}_{1+\sum_{i\in S_2} m_i}.$$

Lastly, we try to compute the K-theory of $C_r^*(G \ltimes \partial\Omega)$. Given our previous computation, it suffices to work in the case where $\partial\Omega \subsetneq \Omega_{b,\,\infty}$.

In this case, we have the following exact sequence of C^*-algebras,

$$0 \to C_r^*\big(G \ltimes (\Omega_{b,\,\infty} \setminus \partial\Omega)\big) \to C_r^*\big(G \ltimes \Omega_{b,\,\infty}\big) \to C_r^*(G \ltimes \partial\Omega) \to 0,$$

and the six term exact sequence of their K-theories,

$$\begin{array}{ccccc} K_0\big(C_r^*\big(G\ltimes(\Omega_{b,\,\infty}\setminus\partial\Omega)\big)\big) & \longrightarrow & K_0\big(C_r^*(G\ltimes\Omega_{b,\,\infty})\big) & \longrightarrow & K_0\big(C_r^*(G\ltimes\partial\Omega)\big) \\ \uparrow & & & & \downarrow \\ K_1\big(C_r^*(G\ltimes\partial\Omega)\big) & \longleftarrow & K_1\big(C_r^*(G\ltimes\Omega_{b,\,\infty})\big) & \longleftarrow & K_1\big(C_r^*\big(G\ltimes(\Omega_{b,\,\infty}\setminus\partial\Omega)\big)\big). \end{array}$$

Noting that in our case, we have $\Omega_{b,\,\infty}\setminus\partial\Omega=\Omega_\infty\setminus\Omega_{a,\,\infty}$. By Proposition 6.1.2, we obtain

$$C_r^*\big(G\ltimes(\Omega_{b,\,\infty}\setminus\partial\Omega)\big)=C_r^*\big(G\ltimes(\Omega_\infty\setminus\Omega_{a,\,\infty})\big)\cong\mathscr{K}\otimes C(\mathbb{T})$$

and thus

$$K_0\big(C_r^*\big(G\ltimes(\Omega_{b,\,\infty}\setminus\partial\Omega)\big)\big)\cong\mathbb{Z}\text{ and }K_1\big(C_r^*\big(G\ltimes(\Omega_{b,\,\infty}\setminus\partial\Omega)\big)\big)\cong\mathbb{Z}.$$

On the other hand, we have

$$K_0\big(C_r^*(G\ltimes\Omega_{b,\,\infty})\big)\cong\mathbb{Z}\text{ and }K_1\big(C_r^*(G\ltimes\Omega_{b,\,\infty})\big)\cong\mathbb{Z}.$$

Therefore, we have such a six term exact sequence

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{K_0(\iota)} & \mathbb{Z} & \longrightarrow & K_0\big(C_r^*(G\ltimes\partial\Omega)\big) \\ \uparrow & & & & \downarrow \\ K_1\big(C_r^*(G\ltimes\partial\Omega)\big) & \longleftarrow & \mathbb{Z} & \xleftarrow[K_1(\iota)]{} & \mathbb{Z}, \end{array}$$

where ι is the inclusion map from $C_r^*\big(G\ltimes(\Omega_{b,\,\infty}\setminus\partial\Omega)\big)$ into $C_r^*\big(G\ltimes\Omega_{b,\,\infty}\big)$.

In the K-theory of the C^*-algebra $\mathscr{K}\otimes C(\mathbb{T})$, we have $[p\otimes 1]_0=1$ for some rank one projection $p \in \mathscr{K}$. Via the path of $*$-isomorphisms

$$\mathscr{K} \otimes C(\mathbb{T}) \to \mathscr{K} \otimes C_r^*(b^{\mathbb{Z}}) \to C_r^*\big(G \ltimes (\Omega_{b,\,\infty} \setminus \partial\Omega)\big),$$

it is easy to find $[\delta_{(e,\,\chi_{w_b})}]_0 = 1$ in the K-theory of $C_r^*\big(G \ltimes (\Omega_{b,\,\infty} \setminus \partial\Omega)\big)$. Noting

$$\delta_{(e,\,\chi_{w_b})} = 1 - \sum_{1 \le i \le N,\ 0 \le j \le |n_i|-1} 1_{\{b^j a_i\} \times \Omega_{b,\,\infty}} 1^*_{\{b^j a_i\} \times \Omega_{b,\,\infty}} \in C_c(G \ltimes \Omega_{b,\,\infty}),$$

we have $K_0(\iota)([\delta_{(e,\,\chi_{w_b})}]_0) = 1 - \sum_{1 \le i \le N} |n_i|$. That is, $K_0(\iota)$ is a multiplication map, sending 1 to $1 - \sum_{1 \le i \le N} |n_i|$.

In the K-theory of the C^*-algebra $\mathscr{K} \otimes C(\mathbb{T})$, we have $[p \otimes (z-1) + 1]_1 = 1$. Via the path of $*$-isomorphisms

$$\mathscr{K} \otimes C(\mathbb{T}) \to \mathscr{K} \otimes C_r^*(b^{\mathbb{Z}}) \to C_r^*\big(G \ltimes (\Omega_{b,\,\infty} \setminus \partial\Omega)\big),$$

it is easy to find $[\delta_{(b,\,\chi_{w_b})} + 1 - \delta_{(e,\,\chi_{w_b})}]_1 = 1$ in the K-theory of $C_r^*\big(G \ltimes (\Omega_{b,\,\infty} \setminus \partial\Omega)\big)$.

On the other hand, we have $[1_{\{b\} \times \Omega_{b,\,\infty}}]_1 = 1$ in the K-theory of $C_r^*(G \ltimes \Omega_{b,\,\infty})$.

Let $u = \delta_{(b,\,\chi_{w_b})} + 1 - \delta_{(e,\,\chi_{w_b})} \in C_c(G \ltimes \Omega_{b,\,\infty})$ and let

$$u_i := 1 + \sum_{0 \le j \le |n_i|-1} (1_{\{b\} \times \Omega_{b,\,\infty}} - 1) 1_{\{b^j a_i\} \times \Omega_{b,\,\infty}} 1^*_{\{b^j a_i\} \times \Omega_{b,\,\infty}} \in C_c(G \ltimes \Omega_{b,\,\infty}),\ 1 \le i \le N,$$

then we have $u \cdot \Pi_{1 \le i \le N} u_i = 1_{\{b\} \times \Omega_{b,\,\infty}}$.

Take $\chi \in \Omega_{b,\ \infty}$ and consider the left regular representation on $\ell_2(G \ltimes \{\chi\})$. Define

$$H_{i,\ j} := 1_{\{b^j a_i\}\times\Omega_{b,\ \infty}} 1^*_{\{b^j a_i\}\times\Omega_{b,\ \infty}} \ell_2(G \ltimes \{\chi\}),$$

then u_i is an identity on $\ell_2(G \ltimes \{\chi\}) \ominus (\oplus_j H_{i,\ j})$ and a unitary on $\oplus_j H_{i,\ j}$. Let u_i' be the restriction of u_i on the subspace $\oplus_j H_{i,\ j}$, we have

$$u_i' = \left[\begin{array}{ccc|c} & 0 & & 1_{\{b^{|n_i|}\}\times\Omega_{b,\ \infty}} \\ \hline 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{array}\right]$$

under the basis $\{H_{i,\ j}\}_j$. Multiply u_i' by the permutation matrix

$$\left[\begin{array}{c|ccc} & 1 & & \\ 0 & & \ddots & \\ & & & 1 \\ \hline 1 & & 0 & \end{array}\right]$$

on the right hand side, we get the following diagonal matrix

$$u_i'' = \left[\begin{array}{c|ccc} 1_{\{b^{|n_i|}\}\times\Omega_{b,\ \infty}} & & 0 & \\ \hline & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{array}\right]$$

Therefore, u_i' is homotopic to u_i'' in $\mathscr{U}(\oplus_j H_{i,\ j})$ and hence u_i is homotopic to

$$1 + (1_{\{b^{|n_i|}\}\times\Omega_{b,\ \infty}} - 1) 1_{\{a_i\}\times\Omega_{b,\ \infty}} 1^*_{\{a_i\}\times\Omega_{b,\ \infty}}$$

$$= (1 - 1_{\{a_i\}\times\Omega_{b,\,\infty}} 1^*_{\{a_i\}\times\Omega_{b,\,\infty}}) + 1_{\{a_i\}\times\Omega_{b,\,\infty}} 1_{\{b^{\operatorname{sgn}(n_i)m_i}\}\times\Omega_{b,\,\infty}} 1^*_{\{a_i\}\times\Omega_{b,\,\infty}}.$$

Before continuing, we need the following Lemma, which comes from [Lemma 4.6.2, HR00].

Lemma 7.1.1. *Let A be a C^*-algebra. If $u \in A$ is a unitary and $v \in A$ is an isometry, then u is homotopic to $vuv^* + (1 - vv^*)$.*

It follows from Lemma 7.1.1 that u_i is homotopic to $1_{\{b^{\operatorname{sgn}(n_i)m_i}\}\times\Omega_{b,\,\infty}}$. That is, $[u_i]_1 = \operatorname{sgn}(n_i)m_i$ and $[u]_1 = 1 - \sum_{1\le i\le N}\operatorname{sgn}(n_i)m_i$. Therefore, $K_1(\iota)$ is a multiplication map, sending 1 to $1 - \sum_{1\le i\le N}\operatorname{sgn}(n_i)m_i$.

When $1 - \sum_{1\le i\le N}|n_i| \neq 0$ and $1 - \sum_{1\le i\le N}\operatorname{sgn}(n_i)m_i \neq 0$, we have

$$K_0\big(C^*_r(G \ltimes \partial\Omega)\big) \cong \mathbb{Z}_{\sum_{1\le i\le N}|n_i| - 1}$$

and

$$K_1\big(C^*_r(G \ltimes \partial\Omega)\big) \cong \mathbb{Z}_{\left|1 - \sum_{1\le i\le N}\operatorname{sgn}(n_i)m_i\right|}.$$

When $1 - \sum_{1\le i\le N}|n_i| = 0$ and $1 - \sum_{1\le i\le N}\operatorname{sgn}(n_i)m_i \neq 0$, we have

$$K_0\big(C^*_r(G \ltimes \partial\Omega)\big) \cong \mathbb{Z}$$

and

$$K_1\big(C^*_r(G \ltimes \partial\Omega)\big) \cong \mathbb{Z} \oplus \mathbb{Z}_{\left|1 - \sum_{1\le i\le N}\operatorname{sgn}(n_i)m_i\right|}.$$

When $1-\sum_{1\leq i\leq N}|n_i|\neq 0$ and $1-\sum_{1\leq i\leq N}\operatorname{sgn}(n_i)m_i=0$, we have

$$K_0\big(C_r^*(G\ltimes\partial\Omega)\big)\cong\mathbb{Z}\oplus\mathbb{Z}_{\sum_{1\leq i\leq N}|n_i|-1}$$

and

$$K_1\big(C_r^*(G\ltimes\partial\Omega)\big)\cong\mathbb{Z}.$$

When $1-\sum_{1\leq i\leq N}|n_i|=0$ and $1-\sum_{1\leq i\leq N}\operatorname{sgn}(n_i)m_i=0$, we have

$$K_0\big(C_r^*(G\ltimes\partial\Omega)\big)\cong\mathbb{Z}\oplus\mathbb{Z}$$

and

$$K_1\big(C_r^*(G\ltimes\partial\Omega)\big)\cong\mathbb{Z}\oplus\mathbb{Z}.$$

7.2 General case

In this section, let P be the fundamental monoid of a graph of monoids with condition (LCM) for P satisfied. Assume that $\{\varepsilon\}\neq G_v\subseteq(\mathbb{R},+)$ for all $v\in V$, $P_e\neq\{\varepsilon\}$ for all $e\in A$ and $P_e^e\neq P_{t(e)}$ for all $e\in E$. We still set out to compute the K-theory of the reduced groupoid C^*-algebras $C_r^*(G\ltimes X)$ for all closed invariant subsets X in Ω. By Theorem 5.2.11, X may be Ω, Ω_∞, $\{\infty\}$ and $\Omega_{\mathbf{b},\,\infty}$.

In the case where $X=\Omega$, we are done.

When $X=\{\infty\}$ is closed, we have condition II. holds and $\sharp A=0$. In this case, $C_r^*(G\ltimes\{\infty\})\cong C_\lambda^*(G_T)$ and thus $K_*(C_r^*(G\ltimes\{\infty\}))\cong K_*(C_\lambda^*(G_T))$.

When $X = \Omega_{\mathbf{b},\,\infty} \neq \{\infty\}$ is closed, we have condition II. holds and $\sharp A \geq 1$. In this case, we claim that $\{g\Omega_{\mathbf{b},\,\infty}\}_{g\in G}$ is a G-invariant regular basis for the compact open subsets of $\Omega_{\mathbf{b},\,\infty}$. It is easy to see that $g\Omega_{\mathbf{b},\,\infty}$ is a compact open subset of $\Omega_{\mathbf{b},\,\infty}$ for all $g \in G$ and that $\{g\Omega_{\mathbf{b},\,\infty}\}_{g\in G}$ is G-invariant. Therefore, it remains to show that $\{g\Omega_{\mathbf{b},\,\infty}\}_{g\in G}$ is a regular basis. We have the following observations.

(i) If $\cap_{1\leq i\leq n} p_i\Omega_{\mathbf{b},\,\infty} \neq \emptyset$ with $p_i \in P$, $1 \leq i \leq n$ and $n \in \mathbb{N}$, then we must have $\cap_{1\leq i\leq n} p_iP \neq \emptyset$ and thus $\cap_{1\leq i\leq n} p_iP = rP$ for some $r \in P$ because P is right LCM. Therefore,

$$\cap_{1\leq i\leq n} p_i\Omega_{\mathbf{b},\,\infty} = r\Omega_{\mathbf{b},\,\infty}.$$

(ii) For every basic compact open subset $\mathscr{O}$ in $\Omega_{\mathbf{b},\infty}$, there exist $p,\ p_i,\ 1 \leq i \leq n \in P$ such that $\mathscr{O} = \{\chi \in \Omega_{\mathbf{b},\,\infty},\ \chi(pP) = 1,\ \chi(p_iP) = 0\}$. In this case, we have

$$\mathscr{O} = p\Omega_{\mathbf{b},\,\infty} \setminus (\cup_{1\leq i\leq n} p_i\Omega_{\mathbf{b},\,\infty}).$$

(iii) If $p\Omega_{\mathbf{b},\,\infty} = \cup_{1\leq i\leq n} p_i\Omega_{\mathbf{b},\,\infty}$ for some $p,\ p_i,\ 1 \leq i \leq n \in P$, then we must have $pP = \cup_{1\leq i\leq n} p_iP$ and thus $pP = p_iP$ for some i because P satisfies independence. In this case, $p\Omega_{\mathbf{b},\,\infty} = p_i\Omega_{\mathbf{b},\,\infty}$.

These observations, together with the fact that for all $g \in G$ there exists $p \in P$ such that $g\Omega_{\mathbf{b},\,\infty} = p\Omega_{\mathbf{b},\,\infty}$, yields our claim by Definition 2.3.5.

Noting that G satisfies the Baum-Connes conjecture with coefficients, we have by Lemma 2.3.4

$$K_*(C_r^*(G \ltimes \Omega_{\mathbf{b},\,\infty})) \cong K_*(C(\Omega_{\mathbf{b},\,\infty}) \rtimes_r G) \cong K_*(C_\lambda^*(G_T)).$$

It remains to consider the case where $X = \Omega_\infty$ is closed. In this case, we have, by Lemma

5.2.12, $P_v \cong \mathbb{Z}_{\geq 0}$, $\sharp V < \infty$ and $\sharp A_+ < \infty$. By the following short exact sequence of C^*-algebras

$$0 \to C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) \to C_r^*(G \ltimes \Omega) \to C_r^*(G \ltimes \Omega_\infty) \to 0,$$

we get the six term exact sequence of their K-theories as follows

$$\begin{tikzcd}
K_0\big(C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)\big) \arrow[r] & K_0\big(C_r^*(G \ltimes \Omega)\big) \arrow[r] & K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \arrow[d] \\
K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \arrow[u] & K_1\big(C_r^*(G \ltimes \Omega)\big) \arrow[l] & K_1\big(C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)\big). \arrow[l]
\end{tikzcd}$$

Since $C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) \cong \mathscr{K}$, we have

$$K_0(C_r^*(G \ltimes (\Omega \setminus \Omega_\infty))) \cong \mathbb{Z} \text{ and } K_1(C_r^*(G \ltimes (\Omega \setminus \Omega_\infty))) \cong 0.$$

Therefore, we have such an updated six term exact sequence

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "K_0(\varphi)"] & \mathbb{Z} \arrow[r] & K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \arrow[d] \\
K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \arrow[u] & 0 \arrow[l] & 0, \arrow[l]
\end{tikzcd}$$

where φ is a unital $*$-homomorphism from $C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)$ to $C_\lambda^*(P)$, composed by

$$C_r^*(G \ltimes (\Omega \setminus \Omega_\infty)) \xrightarrow{\iota} C_r^*(G \ltimes \Omega) \xrightarrow{\psi} C_\lambda^*(P).$$

To calculate the K-theory of $C_r^*(G \ltimes \Omega_\infty)$, we need to find out the map $K_0(\varphi)$ from $\mathbb{Z}$ to $\mathbb{Z}$. It suffices to find out $K_0(\varphi)([p]_0)$ for some rank one projection $p \in C_r^*(G \ltimes (\Omega \setminus \Omega_\infty))$.

If condition II. holds, assume b_v is the generator of P_v. Then we have relations $b_v^{m_{v,w}} = b_w^{m_{w,v}}$

and $b_v^{m_{v,\ a}}a = ab_w^{m_{a,\ w}}$. Recall that the left regular representation of P is such that $\lambda_p(\delta_x) = \delta_{px},\ p,\ x \in P$, we define $E(p),\ p \in P$ to be the range space of λ_p in $\ell_2(P)$, and then the projection from $\ell_2(P)$ onto $E(p)$ is $\lambda_p\lambda_p^*$.

Fix $v \in V$, denote by $s(w)$ the vertex connected to w in the geodesic path $[v,\ w] \subseteq T$ for all $v \neq w \in V$ and by $s(a)$ the origin vertex of a for all $a \in A$. It is easy to see that

$$E(a) \cap E(b_{s(a)}) = E(b_{s(a)}^{m_{s(a),\ a}}a) \text{ if } a \in A_+$$

and that

$$E(a) \subseteq E(b_{s(a)}) \text{ if } a \in A_-.$$

Since $\sharp V < \infty$ and $\sharp A_+ < \infty$, we always have

$$q := 1 - [\lambda_{b_v}\lambda_{b_v}^* + \sum_{v \neq w \in V}(\lambda_{b_w}\lambda_{b_w}^* - \lambda_{b_{s(w)}^{m_{s(w),\ w}}}\lambda^*_{b_{s(w)}^{m_{s(w),\ w}}}) + \sum_{a \in A_+}(\lambda_a\lambda_a^* - \lambda_{b_{s(a)}^{m_{s(a),\ a}}a}\lambda^*_{b_{s(a)}^{m_{s(a),\ a}}a})]$$

is a rank one projection in $C_\lambda^*(P)$, whose range space is exactly $\mathbb{C}\delta_e$. Here e is the identity of P. Noting $\lambda_{b_w}^*\lambda_{b_w} = \lambda_a^*\lambda_a = 1$ for all $w \in V$ and all $a \in A$, it follows that this rank one projection q is in the equivalence class of 0 in $\mathscr{P}_\infty(C_\lambda^*(P))$. So is any other rank one projection in $C_\lambda^*(P)$.

As a unital $*$-homomorphism, $\psi \circ \iota$ maps the rank one projection $p \in C_r^*\big(G \ltimes (\Omega \setminus \Omega_\infty)\big)$ to some rank one projection $q' \in C_\lambda^*(P)$. Therefore,

$$K_0(\varphi)([p]_0) = [q']_0 = 0.$$

That is, $K_0(\varphi) = 0$. From the six term exact sequence, it follows that

$$K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z} \text{ and } K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z}.$$

If condition I holds, $K_0(\varphi)$ is not 0 anymore. Indeed, there exists $e \in T$ such that $P_e = \{\varepsilon\}$. Set $n := \frac{1}{2}\sharp\{e \in T \mid P_e = \varepsilon\}$, we get similarly as above that $[q]_0 = -n$, where q is a rank one projection from $\ell_2(P)$ onto $\mathbb{C}\delta_e$. Therefore, $K_0(\varphi)(1) = -n$. From the six term exact sequence, we get

$$K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z}_n \text{ and } K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong 0.$$

Chapter 8

Extension

Based on our work in the thesis, there are some future directions in which we can work.

Firstly, in Chapter 4 we proved the nuclearity of the reduced C^*-algebras of graphs of monoids, but only embedded successfully a part of generalised Baumslag-Solitar monoids into amenable groups. It is natural to try to extend the result to all generalised Baumslag-Solitar monoids and even to our graphs of monoids. This may reveal the relation between nuclearity of semigroup C^*-algebras and embeddability of these semigroups into amenable groups.

In Chapter 5, we made some assumptions of the graphs of monoids to get all nonempty closed invariant subsets of the the partial action $G \curvearrowright \Omega$. In the process, we focused on the cases where condition I. or condition II. holds, but we failed having a complete discussion when either condition I. or condition II. holds. (see Lemma 5.2.4 and Lemma 5.2.6) It would be better if we can show that either condition I. or condition II. holds in the missing case where $P_e \neq \{\varepsilon\}$ for all $e \in T$, $\sharp V = 1$, $\sharp A_+ = 0$ and $\sharp A_- > 0$. It also makes sense to investigate whether all those assumptions of the graphs of monoids we made are necessary. That is, can we still get a list of all nonempty closed invariant subsets of the the partial action $G \curvearrowright \Omega$

if some of the assumptions are removed?

In Chapter 5, we also discussed the topological freeness of the partial action $G \curvearrowright X$ for all closed and invariant subset $X \subseteq \Omega$. In the generalised Baumslag-Solitar monoid case, we had a full discussion on the topological freeness. While in general case, we could not provide a complete discussion when the partial action $G \curvearrowright \Omega_{\mathbf{b},\,\infty}$ is topologically free. Instead, we gave some examples (sufficient conditions) where the partial action $G \curvearrowright \Omega_{\mathbf{b},\,\infty}$ is topologically free. This problem is also worthy of thinking.

In Chapter 6, we worked out the primitive ideal space (with topology) of the groupoid C^*-algebra $C_r^*(G \ltimes \Omega)$ under the assumption that the partial action $G \curvearrowright \partial\Omega$ is topologically free unless $\partial\Omega = \{\infty\}$ (in the case where $\sharp A = 0$). We can also try to find the primitive ideal space (with topology) of the groupoid C^*-algebra $C_r^*(G \ltimes \Omega)$ in the case where our assumption does not hold, that is, the partial action $G \curvearrowright \partial\Omega$ is not topologically free.

Lastly, we can study other properties of the C^*-algebras of graphs of monoids, for instance, the pure infiniteness and the classification.

Bibliography

[ADR00] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids, Monographie no. 36, L'Enseignement Mathématique, Geneva, 2000.

[Arv76] W. Arveson, An invitation to C^*-algebras, Springer-Verlag, New York-Heidelberg-Berlin, 1976.

[BL18] C. Bönicke and K. Li, Ideal structure and pure infiniteness of ample groupoid C^*-algebras, Ergodic Theory and Dynamical Systems (2020), 40, 34–63.

[Bla06] B. Blackadar, Operator algebras: Theory of C^*-algebras and von Neumann algebras, Encyclopaedia of Mathematical Sciences, vol. 122, Springer, Berlin, 2006.

[BO08] N. P. Brown and N. Ozawa, C^*-algebras and finite-dimensional approximations, Graduate Studies in Mathematics, vol. 88, Amer. Math. Soc. 369 (2017), no. 1, 31-68.

[CEL13] J. Cuntz, S. Echterhoff and X. Li, On the K-theory of crossed products by automorphic semigroup actions, Quart. J. Math. 64 (2013), 747-784.

[CELY17] J. Cuntz, S. Echterhoff, X. Li and G. Yu, K-Theory for group C^*-algebras and semigroup C^*-algebras, Birkhauser (2017), Oberwolfach Seminars, vol. 47.

[Dav96] K. Davidson, C^*-algebras by example, Fields Institute Monographs, vol. 6, Amer. Math. Soc., Providence, RI, 1996.

[Deh03] P. Dehornoy, Complete positive group presentations, J. Alg. 268 (2003), 156-197.

[Dix77] J. Dixmier, C^*-algebras, North-Holland Math. Library, vol. 15, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1977.

[DS57] N. Dunford and J. T. Schwartz, Linear operators, Interscience Publishers, Inc., New York, 1957.

[Exe15] R. Exel, Partial Dynamical Systems, Fell Bundles and Applications, preprint, arXiv: 1511.04565v2.

[Exe94] R. Exel, Circle actions on C^*-algebras, partial automorphisms, and a generalised Pimsner-Voiculescu exact sequence, Journal of functional analysis, Vol 122, 361-401, 1994.

[Exe96] R. Exel, Amenability for fell bundles, arXiv: funct-an/9604009v1, 1996.

[Gue01] E. Guentner, Exactness of one relator groups, Proceedings of the Amer. Math. Soc.(2001), Vol. 130, No. 4, 1087–1093.

[Hoc69] M. Hochster, Subsemigroups of amenable groups, Proc. Amer. Math. Soc. 21 (1969), 363-364.

[HR00] N. Higson and J. Roe, Analytic K-homology, Oxford University Press Inc., New York, 2000.

[Kel55]J. L. Kelley, General topology, D. Van Nostrand Company, Inc., New York-London-Toronto, 1955.

[Lan73] L. Lance, on nuclear C^*-algebras, Journal of functional analysis 12 (1973), 157-176.

[Lev14] G. Levitt, Quotients and subgroups of generalised Baumslag-Solitar groups, arXiv: 1308.5122v2, [math. GR], 2014.

[Lev15] G. Levitt, Generalised Baumslag–Solitar groups: rank and finite index subgroups, Ann. Inst. Fourier, Grenoble, vol. 65, no. 2 (2015), 725-762.

[Li12] X. Li, Semigroup C^*-algebras and amenability of semigroups, Journal of functional analysis 262 (2012), 4302-4340.

[Li13] X. Li, Nuclearity of semigroup C^*-algebras and the connection to amenability, Adv. Math. 244 (2013), 626-662.

[Li20] X. Li, K-theory for semigroup C^*-algebras and partial crossed products, preprint, arXiv:2003:03858v1 (2020).

[LOS18] X. Li, T. Omland and J. Spielberg, C^*-algebras of right LCM one-relator monoids and Artin-Tits monoids of finite type, arXiv:1807.08288.

[Mur90] G. J. Murphy, C^*-algebras and operator theory, Academic Press, London, 1990.

[Pat99] A. L. T. Paterson, Groupoids, inverse semigroups, and their operator algebras, Springer-Science and Business Media, New York, 1999.

[Put16] I. F. Putnam, Lecture notes on C^*-algebras, 2016.

[Ren15] J. Renault, Topological amenability is a Borel property, Math. Scand. 117 (2015), no. 1, 5-30.

[Ren80] J. Renault, A groupoid approach to C^*-algebras, Springer-Verlag, Berlin- Heidelberg-New York, 1980.

[RLL00] M. Rϕrdam, F. Larsen and N. Laustsen, An introduction to K-theory for C^*-algebras, London Mathematical Society Student Texts, vol. 49, Cambridge University Press, 2000.

[Rud91] W. Rudin, Functional analysis, 2nd edition, McGraw-Hill Book Company, New York, 1991.

[RW17] J. Renault and D. P. Williams, Amenability of groupoids arising from partial semigroup actions and topological higher rank graphs, Trans. Amer. Math. Soc. 369 (2017), no. 4, 2255-2283.

[Ser80] J. P. Serre, Trees, Springer-Verlag, Berlin-Heidelberg-New York, 1980, 1-68.

[Sim17] A. Sims, Hausdorff $\acute{e}$tale groupoid and their C^*-algebras, arXiv: 1710. 10897v1.

[Spi12] J. Spielberg, C^*-algebras for categories of paths associated to the Bamuslag-Solitar groups, J. London Math. Soc. 86 (2012), 728-754.

[Spi14] J. Spielberg, Groupoids and C^*-algebras for categories of paths, Transactions of the Amer. Math. Soc., vol. 366, number 11, 2014, 5771–5819.

[Yos68] K. Yosida, Functional analysis, Springer-Verlag, New York, 1968.