Bicolimits of C^*-Categories and the Categorification of Hecke Algebras

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Contents

0 Introduction

In this thesis we shall address two topics. The first is the existence of a particular class of bicolimits of C^*-categories and in particular the existence of balance tensor products of module categories. The second is an approach to categorification of Hecke algebras which includes looking at a categorical analogue of the induction and restriction processes for representations of groups.

A brief overview of the content and structure of the thesis is as follows: In Chapter 1 we shall begin with some categorical definitions as well as fixing some conventions and notation. In particular, we shall give a brief introduction to the theory of 2-categories and C^*-categories. 2-categories are higher dimensional analogues of categories in which one has not only objects and maps between objects, but also 'maps between maps' [4, 19]. A classical example is the 2-category of small categories, functors and natural transformations. C^*-categories were first introduced in [15] and are a categorification of C^*-algebras in the sense that a unital C^*-algebra can be viewed as a C^*-category with one object. The category of Hilbert spaces and bounded linear operators is a C^*-category and more generally, for any C^*-algebra A, the category of Hilbert modules over A and adjointable linear operators is a C^*-category. (Hilbert modules are a generalisation of Hilbert spaces which can be thought of as Hilbert $\mathbb{C}$-modules. In a Hilbert module, the role of the complex numbers is taken over by an arbitrary C^*-algebra. More information can be found in [22].) When studying C^*-algebras, one is also interested in non-unital algebras and this more general case can also be categorified by considering 'non-unital categories', also called 'semi-categories'. In this thesis we shall restrict our attention to the unital case but in joint work with my supervisor in [1], we prove the result on bicolimits of C^*-categories presented here in the more general, non-unital setting.

Once we have the categorical setup in place, in Chapter 2 we shall look at bicolimits of C^*-categories. The category of C^*-categories and $*$-functors is both complete and cocomplete [10] but once one starts restricting attention to C^*-categories with certain properties, the resulting category need no longer have these properties. In particular, we are interested in C^*-categories that have direct sums and are subobject complete in the senses described in sections 1.3.2 and 1.3.3, the reason being that these are properties possessed by C^*-categories of Hilbert modules. The category of such C^*-categories does have direct products but does not posses arbitrary limits and colimits, hence the need to consider a more general notion of colimit.

Bicolimits are a 2-categorical notion of colimits. There are several flavours of 2-categorical limits (and colimits), an introduction to which can be found in [20], bicolimits being the weakest notion of 2-categorical colimit discussed within. The result we shall prove here is that the 2-category of additive and subobject complete C^*-categories has bicolimits indexed by small categories, for which the coherence maps for the diagram are unitaries (this condition should not be too restrictive since it seems natural to ask for coherence isomorphisms to be unitaries when working in the C^*-setting). As a particular case, we shall show the existence of balanced tensor products of module categories over a C^*-tensor category in this setting. Module categories are a categorical analogue of modules over a ring and several authors have shown the existence of balanced tensor products in other situations. For example, [12], [14] and [9] all discuss some other situations in which balanced tensor products exist.

Next, in Chapter 3 we shall discuss categorical representation theory in which a group G acts not on a vector space, but on a category. An introduction to this topic can be found in [3]. We shall consider groups acting on C^*-categories. The main reason for doing this is that we are interested in categorifying Hecke algebras and the induction-restriction adjunction for representations of groups is related to these algebras. Hecke algebras can be though of as a generalisation of group rings. One starts with a discrete group G and a subgroup H which satisfies a finiteness condition called being 'almost normal' in G. To explain this, one needs the notion of a double coset of H in G. This is a set of the form

$$HgH := \{hgk \mid g \in G,\, h,k \in H\}.$$

We then say that H is 'almost normal' in G if every double coset of H in G is a finite union of left (or equivalently right) cosets. If H is a normal subgroup, then even double coset equal to a left coset so any normal subgroup is almost normal. A pair (G,H) for which G is a group and H is an almost normal subgroup is a called a 'Hecke pair'. Given a Hecke pair (G,H) there are various equivalent ways to construct a complex star algebra, called the 'Hecke algebra' of the pair, denoted $\mathcal{H}(G/\!/H)$. One is to take the underlying vector space to be the space of functions

$$C_H(G)^{H\times H} := \left\{ f: G \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ has finite support mod } H, \\ f(hgk) = f(g)\ \forall g \in G,\, h,k \in H \end{array} \right\}.$$

By finite support mod H, we mean that $f(g) = 0$ for all g outside of a finite set of left cosets of H in G. The addition is then defined by

$$(f + f')(g) := f(g) + f'(g).$$

To define the product, we fix a set Γ of representatives for G/H. Then one defines

$$(f * f')(g) := \sum_{\gamma\in\Gamma} f(\gamma) f'(\gamma^{-1} g).$$

This is a finite sum because f has finite support mod H and one can show that it is independent of the choice of coset representatives. Finally, the involution is defined by

$$f^*(g) := \overline{f(g^{-1})}$$

where the overbar denotes the complex conjugate.

Hecke algebras have a similar flavour to group rings and in the case that H is a normal subgroup, the Hecke algebra is the group ring of the quotient group G/H. One reason for the interest in Hecke algebras is that given a representation V of the group G, the Hecke algebra acts naturally on the space of H fixed points V^H. Given $f \in \mathcal{H}(G/\!/H)$ and $v \in V^H$, the action is given by

$$f \cdot v := \sum_{\gamma \in \Gamma} f(\gamma)(\gamma \cdot v).$$

A classical example of this is given by the Hecke operators of number theory and their action on spaces of modular forms. More on this example can be found in [11]. Another important example of a Hecke algebra corresponds to the pair

$$G = P_{\mathbb{Q}}^{+} = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{Q} \text{ and } a > 0 \right\}$$

and

$$H = P_{\mathbb{Z}}^{+} = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{Z} \right\}.$$

This Hecke algebra was studied by Bost and Connes in [6] in which they define its corresponding Hecke C^*-algebra and study its connection to the Riemann zeta function. We use this Hecke pair as one of our examples when looking at categorified Hecke algebras.

The relation between Hecke algebras and induced representations is the following: Given a Hecke pair (G, H), let us denote the trivial representation of H on the complex numbers by $\mathbb{C}$. From this representation, one can construct a representation of G, called the induced representation, denoted $\mathrm{Ind}_H^G(\mathbb{C})$. The G-intertwining operators from $\mathrm{Ind}_H^G(\mathbb{C})$ to itself form a complex algebra with the product given by composition of intertwiners, and this algebra is canonically isomorphic to $\mathcal{H}(G/\!/H)^{\mathrm{op}}$, the opposite algebra of the Hecke algebra. We describe this in detail in section 4.1 which we use as a roadmap for our categorification of Hecke algebras. The induction-restriction adjunction plays an important role and this is our motivation for looking at a categorical analogue of this. Having said that, it is not necessary to consider induction and restriction of group representations to define Hecke algebras or their categorification but it is of interest in its own right and gives us another point of view from which to study them.

Finally, in Chapter 4 we looks at how, given a Hecke pair (G, H), one can construct a C^*-tensor category $\mathbf{H}(G/\!/H)$ which is the analogue of the classical Hecke algebra $\mathcal{H}(G/\!/H)$ and consider a few specific examples. Other authors have also addressed this question. In [28], Zhu constructed a tensor category which is equivalent to ours and in [2], Arano and Vaes take a different, more general approach to ours and consider totally disconnected groups and compact open subgroups (we restrict our attention to discrete groups) as well as considering multiple subgroups at the same time. Our approach is to begin by constructing a C^*-category of functions which is analogue of the vector space $C_H(G)^{H\times H}$. We then show that this C^*-category is equivalent to the categorical analogue of the intertwining operators from $\mathrm{Ind}_H^G(\mathbb{C})$ to itself. There is a natural tensor category structure on the C^*-category of these categorical intertwiners with the product given by composition and we pass this tensor category structure to our category of functions to obtain our categorical Hecke algebra.

One of the other points of view we shall consider uses so called 'biequivariant Hilbert spaces' which are also used by Arano and Vaes in [2]. An 'H-biequivariant $C_0(G)$-Hilbert space', is a Hilbert space $\mathcal{H}$ together with a nondegenerate $*$-representation $m : C_0(G) \to L(\mathcal{H})$ and two commuting unitary representations $\lambda : H \to U(\mathcal{H})$, $\rho : H \to U(\mathcal{H})$ such that

$$\lambda_k(f \cdot \xi) = \lambda_k(f) \cdot \lambda_k(\xi), \quad \rho_k(f \cdot \xi) = \rho_k(f) \cdot \rho_k(\xi)$$

for all $k \in H$, $f \in C_0(G)$ and $\xi \in \mathcal{H}$. Here the action on functions is given by left and right translations, respectively. That is, $\lambda_k(f)(g) = f(k^{-1}g)$ and $\rho_k(f)(g) = f(gk)$. We shall usually abbreviate 'H-biequivariant $C_0(G)$-Hilbert space' to 'biequivariant Hilbert space'.

Our Hecke category is equivalent to a certain category of biequivariant Hilbert spaces. Using this point of view, we will show that the basic building blocks of the Hecke category can be described in terms of finite dimensional unitary representations of certain subgroups of H, specifically, subgroups of the form $H_t := H \cap tHt^{-1}$, $t \in G$. This will allow us to describe the product in the Hecke category in terms of representations of these groups.

One can also associate a complex algebra to a C^*-tensor category by taking its (complex) Grothendieck ring. For a Hecke pair (G, H), the classical Hecke algebra $\mathcal{H}(G/\!/H)$ is a quotient of the Grothendieck ring of $\mathbf{H}(G/\!/H)$. When H is finite, the

Grothendieck ring can also be viewed as a subalgebra of $\mathcal{H}(G/\!/H)$ in a natural way although the inclusion map is not unital. In closing, we shall look at a few examples of the Grothendieck ring of some Hecke categories.

1 Categorical definitions, conventions and notation

1.1 A note on set theory

Throughout, we shall mostly restrict our attention to small categories and will not be doing anything delicate where set theory is concerned. However, for the sake of definiteness, let us fix a framework. For our axioms of set theory, we shall take those of ZFC as well as the axiom

U: *For every set x there is a Grothendieck universe U such that $x \in U$.*

(A definition of 'Grothendieck universe' can be found in [5] where they are simply called 'universes'. Essentially they are sets which are models of ZFC.) Our approach to size issues will be similar to that taken in [7] and [8]. We shall fix two Grothendieck universes U and V with $U \in V$ and from this point onwards, by a 'class' we shall mean a subset of V and by a 'set' we shall mean an element of V. We will adopt the convention that in a category, the class of morphisms between any two objects must be a set. By a 'small category' we mean a category whose class of objects is a set and conversely by a 'large category' we mean a category whose class of objects is not a set. We shall call the elements of U 'very small sets' and shall use the prefix 'very small' to mean that the underlying set of a mathematical object is a very small set, e.g. a 'very small Hilbert space'. The role of U is just to give us a concise way of describing some natural examples of small C^*-categories.

As another small matter, we count 0 as a natural number since it will usually be more convenient to do so than not.

1.2 2-category theory

1.2.1 2-categories

2-category theory provides the framework that we shall use to discuss categorical representation theory and for categorifying Hecke algebras. In a 2-category, one has not only objects and morphisms but also 'morphisms between morphisms'. We shall only present the definitions and theory that we need but more about 2-category theory can be found in [4], [24] and [19].

Definition 1.1. *A* ***2-category*** $\mathcal{A}$ *consists of the following:*

- *A class* $\mathcal{A}_0$ *of* ***objects*** *or 0-****cells****.*
- *A category* $\mathcal{A}(A, B)$ *for each pair* (A, B) *of objects, the objects* $f, g, h, \ldots$ *of these categories are called the 1-****cells*** *of* $\mathcal{A}$ *and the arrows* $\alpha, \beta, \gamma, \ldots$ *of these categories are called the 2-****cells*** *of* $\mathcal{A}$*. Given composable 2-cells* $\alpha, \beta \in \mathcal{A}(A, B)$*, we denote their composition by* $\beta \circ \alpha$*. We shall often refer to this as* ***vertical composition*** *of 2-cells to distinguish it from horizontal composition of 2-cells which is defined below.*
- *A functor*

$$M_{A,B,C} : \mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$$

for each triple (A, B, C) *of objects called the* ***composition*** *and written*

$$(g, f) \mapsto g \circ f$$
$$(\beta, \alpha) \mapsto \beta * \alpha.$$

In the case of 2-cells, we refer to this as ***horizontal composition****.*

- *A functor* $I_A : \mathbf{1} \to \mathcal{A}(A, A)$ *for each object* A *(here* $\mathbf{1}$ *is the terminal category with one object* $*$ *and only the identity morphism). We denote* $I_A(*)$ *by* 1_A *and call it the* ***identity*** *of* A*.*

These data are subject to the following axioms

- *For all objects* A, B, C, D*, the following diagram commutes*

$$\begin{tikzcd}
\mathcal{A}(C,D) \times \mathcal{A}(B,C) \times \mathcal{A}(A,B) \arrow[r, "1_{\mathcal{A}(C,D)} \times M_{A,B,C}"] \arrow[d, "M_{B,C,D} \times 1_{\mathcal{A}(A,B)}"'] & \mathcal{A}(C,D) \times \mathcal{A}(A,C) \arrow[d, "M_{A,C,D}"] \\
\mathcal{A}(B,D) \times \mathcal{A}(A,B) \arrow[r, "M_{A,B,D}"] & \mathcal{A}(A,D)
\end{tikzcd}$$

- *For all objects* A, B*, the following diagrams commute*

$$\begin{tikzcd}
\mathcal{A}(A,B) \times \mathbf{1} \arrow[r, "1_{\mathcal{A}(A,B)} \times I_A"] \arrow[dr, "\cong"'] & \mathcal{A}(A,B) \times \mathcal{A}(A,A) \arrow[d, "M_{A,A,B}"] \\
& \mathcal{A}(A,B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \times \mathcal{A}(A,B) \arrow[r, "I_B \times 1_{\mathcal{A}(A,B)}"] \arrow[dr, "\cong"'] & \mathcal{A}(B,B) \times \mathcal{A}(A,B) \arrow[d, "M_{A,B,B}"] \\
& \mathcal{A}(A,B)
\end{tikzcd}$$

A classical example of a 2-category is Cat, the 2-category of small categories, functors and natural transformations.

Given objects A, B, C in a 2-category, 1-cells f, g, h, j, k, ℓ and 2-cells $\alpha, \beta, \gamma, \delta$ arranged as shown below

$$\begin{tikzcd}
A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha"] \arrow[r, phantom, shift right=4, "\Downarrow\beta"] & B \arrow[r, bend left=60, "j"] \arrow[r, "k" description] \arrow[r, bend right=60, "\ell"'] \arrow[r, phantom, shift left=4, "\Downarrow\gamma"] \arrow[r, phantom, shift right=4, "\Downarrow\delta"] & C
\end{tikzcd}$$

by the functoriality of $M_{A,B,C}$ we have the **Middle Interchange Rule**:

$$(\delta * \beta) \circ (\gamma * \alpha) = (\delta \circ \gamma) * (\beta \circ \alpha).$$

With regards to notation, we will just write $A \in \mathcal{A}$ rather than $A \in \mathcal{A}_0$, given $A, B \in \mathcal{A}$, by $f : A \to B$ we mean that $f \in \mathcal{A}(A, B)$ and by $\alpha : f \to g : A \to B$ we mean that α is a 2-cell from f to g where $f, g \in \mathcal{A}(A, B)$.

Some authors distinguish between the case when the hom categories $\mathcal{A}(A, B)$ can be any categories and the case when they are small categories, perhaps calling these constructions '2-categories' and 'Cat-categories' respectively. In all the cases we consider, the hom categories will be small categories so we have chosen not to make such a distinction.

Another possible option in place of 2-categories would be a more general construction called a 'bicategory'. The difference is that the composition is no longer strictly associative, only associative 'up to isomorphism' and similarly the identities are only identities 'up to isomorphism'. All the bicategories we will encounter will turn out to be 2-categories so we shall not give the definition here. One can read more about bicategories, and in particular 2-categories in [4] and [19]. As discussed in [19], every bicategory is biequivalent to a 2-category, where 'biequivalence' is a suitable notion of equivalence for bicategories. Therefore, one does not lose too much generality by only considering 2-categories.

Whilst we shall only use the 'strict' notion of a 2-category and not the weaker notion of a bicategory, we shall use a weak notion of morphism between 2-categories called a 'pseudofunctor'.

Definition 1.2. *Given 2-categories $\mathcal{A}$ and $\mathcal{B}$, a **pseudofunctor** or **weak 2-functor***

$$F : \mathcal{A} \to \mathcal{B}$$

consists of the following:

- *A function*

$$F_0 : \mathcal{A}_0 \to \mathcal{B}_0.$$

- *For all objects $A, B \in \mathcal{A}$, functors*

$$F_{A,B} : \mathcal{A}(A, B) \to \mathcal{B}(F_0(A), F_0(B)).$$

- *For all objects $A, B, C \in \mathcal{A}$, natural isomorphisms $F^{A,B,C}$ with components*

$$F^{A,B,C}_{g,f} : F_{B,C}(g) \circ F_{A,B}(f) \to F_{A,C}(g \circ f).$$

- *For each object $A \in \mathcal{A}$, an isomorphism 2-cell*

$$F_{1_A} : 1_{F_0(A)} \to F_{A,A}(1_A).$$

These data are subject to the following axioms: Given 1-cells $f : A \to B$, $g : B \to C$ and $h : C \to D$ in $\mathcal{A}$, the following diagram commutes

$$\begin{tikzcd}
F_{C,D}(h) \circ F_{B,C}(g) \circ F_{A,B}(f) \arrow[r, "F^{B,C,D}_{h,g} * 1_{F_{A,B}(f)}"] \arrow[d, "1_{F_{C,D}(h)} * F^{A,B,C}_{g,f}"'] & F_{B,D}(h \circ g) \circ F_{A,B}(f) \arrow[d, "F^{A,B,D}_{h \circ g, f}"] \\
F_{C,D}(h) \circ F_{A,C}(g \circ f) \arrow[r, "F^{A,C,D}_{h, g \circ f}"] & F_{A,D}(h \circ g \circ f)
\end{tikzcd}$$

For all 1-cells $f : A \to B$ in $\mathcal{A}$, the following diagrams commute

$$\begin{tikzcd}
F_{A,B}(f) \circ 1_{F_0(A)} \arrow[r, "1_{F_{A,B}(f)} * F_{1_A}"] \arrow[dr, "="'] & F_{A,B}(f) \circ F_{A,A}(1_A) \arrow[r, "F^{A,A,B}_{f,1_A}"] & F_{A,B}(f \circ 1_A) \arrow[dl, "="] \\
& F_{A,B}(f) &
\end{tikzcd}$$

$$\begin{tikzcd}
1_{F_0(B)} \circ F_{A,B}(f) \arrow[r, "F_{1_B} * 1_{F_{A,B}(f)}"] \arrow[dr, "="'] & F_{B,B}(1_B) \circ F_{A,B}(f) \arrow[r, "F^{A,B,B}_{1_B,f}"] & F_{A,B}(1_B \circ f) \arrow[dl, "="] \\
& F_{A,B}(f) &
\end{tikzcd}$$

In order to avoid cumbersome notation, we shall adopt the convention of also denoting the function F_0 and the functors $F_{A,B}$ simply by F and we shall just write $F_{g,f}$ instead of $F^{A,B,C}_{g,f}$. Also, we shall often write FA rather than $F(A)$ and Ff rather than $F(f)$.

Definition 1.3. *Given 2-categories $\mathcal{A}$ and $\mathcal{B}$, a **(strict) 2-functor***

$$F : \mathcal{A} \to \mathcal{B}$$

is a weak 2-functor for which all the $F^{A,B,C}$ and F_{1_A} are identities.

We also have a weak 2-categorical version of natural transformations defined as follows.

Definition 1.4. *Given 2-categories $\mathcal{A}, \mathcal{B}$ and pseudofunctors $F, G : \mathcal{A} \to \mathcal{B}$, a **pseudonatural transformation** or **weak natural transformation***

$$\sigma : F \to G$$

consists of

- *1-cells $\sigma_A : F(A) \to G(A)$ for all $A \in \mathcal{A}$,*
- *natural isomorphisms $\sigma^{A,B}$ with components*

  $$\sigma^{A,B}_f : G(f) \circ \sigma_A \to \sigma_B \circ F(f), \qquad f \in \mathcal{A}(A,B)$$

  for all pairs of objects $A, B \in \mathcal{A}$

such that for all composable pairs of 1-cells $f : A \to B$, $g : B \to C$ in $\mathcal{A}$, the following diagram commutes

$$\begin{tikzcd}
G(g) \circ G(f) \circ \sigma_A \arrow[r, "1_{G(g)} * \sigma_f^{A,B}"] \arrow[d, "G_{g,f} * 1_{\sigma_A}"'] & G(g) \circ \sigma_B \circ F(f) \arrow[r, "\sigma_g^{B,C} * 1_{F(f)}"] & \sigma_C \circ F(g) \circ F(f) \arrow[d, "1_{\sigma_C} * F_{g,f}"] \\
G(g \circ f) \circ \sigma_A \arrow[rr, "\sigma_{g \circ f}^{A,C}"] & & \sigma_C \circ F(g \circ f)
\end{tikzcd}$$

and for all $A \in \mathcal{A}$, the following diagram commutes

$$\begin{tikzcd}
& \sigma_A \arrow[dl, "G_{1_A} * 1_{\sigma_A}"'] \arrow[dr, "1_{\sigma_A} * F_{1_A}"] & \\
G(1_A) \circ \sigma_A \arrow[rr, "\sigma_{1_A}^{A,A}"] & & \sigma_A \circ F(1_A)
\end{tikzcd}$$

Again, to reduce clutter in the notation, usually we shall just write σ_f instead of $\sigma_f^{A,B}$.

Definition 1.5. *A* ***(strict) 2-natural transformation*** *is a pseudonatural transformation whose coherence 2-cells are all identities.*

We now have 2-categories, pseudofunctors and pseudonatural transformations which are higher dimensional analogues of categories, functors and natural transformations. There is also a notion of maps between pseudonatural transformations.

Definition 1.6. *Given 2-categories $\mathcal{A}$ and $\mathcal{B}$, pseudofunctors $F, G : \mathcal{A} \to \mathcal{B}$ and pseudonatural transformations $\sigma, \rho : F \to G$, a* ***modification***

$$\Gamma : \sigma \to \rho$$

consists of

- *2-cells*

  $$\Gamma_A : \sigma_A \to \rho_A$$

  for each $A \in \mathcal{A}$

such that for each pair of objects $A, B \in \mathcal{A}$ and each 1-cell $f : A \to B$, the following diagram commutes

$$\begin{tikzcd}
G(f) \circ \sigma_A \arrow[r, "\sigma_f"] \arrow[d, "1_{G(f)} * \Gamma_A"'] & \sigma_B \circ F(f) \arrow[d, "\Gamma_B * 1_{F(f)}"] \\
G(f) \circ \rho_A \arrow[r, "\rho_f"] & \rho_B \circ F(f)
\end{tikzcd}$$

The 2-categories, pseudofunctors, pseudonatural transformations and modifications fit into an even higher dimensional structure called a 'tricategory'. We shall not have need of this structure and even the axioms are rather unwieldy but one can read more about tricategories in [16].

Furthermore, given 2-categories $\mathcal{A}$ and $\mathcal{B}$, there is a 2-category $\mathsf{Psd}[\mathcal{A}, \mathcal{B}]$ of pseudofunctors from $\mathcal{A}$ to $\mathcal{B}$, pseudonatural transformations and modifications. Pseudonatural transformations and modifications are composed in the obvious ways and the full details can be found in [19].

Before moving on to biadjunctions, we comment that whilst we have endeavoured to write out details of morphisms in diagrams in full in this section, we will often omit details like horizontal composition with identity maps, particularly if the notation would become excessively cluttered. For example, we might write the modification axiom diagram as

$$\begin{tikzcd}
G(f) \circ \sigma_A \arrow[r, "\sigma_f"] \arrow[d, "\Gamma_A"'] & \sigma_B \circ F(f) \arrow[d, "\Gamma_B"] \\
G(f) \circ \rho_A \arrow[r, "\rho_f"] & \rho_B \circ F(f)
\end{tikzcd}$$

We shall also sometimes leave out certain 'decorations' (e.g. superscripts/subscripts) in the notation when we think that no confusion will occur and the diagram will be less cluttered easier to read as a result.

1.2.2 Biadjunctions

A biadjunction is a particular notion of 'weak' adjunction between 2-categories. There are other variations on this concept, some of which are discussed in [17] but biadjunctions are fairly general and cover all the examples that we shall consider.

Definition 1.7. *A **biadjunction** consists of the following data:*

- *2-categories $\mathcal{A}$ and $\mathcal{B}$,*
- *pseudofunctors $F : \mathcal{A} \to \mathcal{B}$ and $U : \mathcal{B} \to \mathcal{A}$,*
- *pseudonatural transformations $\eta : 1_{\mathcal{A}} \to UF$ and $\xi : FU \to 1_{\mathcal{B}}$ called the **unit** and **counit** respectively,*
- *invertible modifications $\Gamma : 1_F \to \xi_F \circ F\eta$ and $\Delta : U\xi \circ \eta_U \to 1_G$. In other words, the triangle identities*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FUF \arrow[d, "\xi_F"] \arrow[dl, phantom, "\overset{\Gamma}{\Rightarrow}"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\eta_U"] \arrow[dr, "1_U"'] & UFU \arrow[d, "U\xi"] \arrow[dl, phantom, "\overset{\Delta}{\Rightarrow}"] \\ & U \end{tikzcd}$$

commute up to invertible modifications.

*We say that F is **left biadjoint** to U and U is **right biadjoint** to F.*

We haven't explicitly defined the pseudonatural transformations $1_F, F\eta, \xi_F, 1_U, \eta_U$ and $U\xi$ in the triangle identity diagrams but they are all defined in exactly the way one would expect.

Similarly to the classical case of adjunctions, one has the following:

Theorem 1.8. *Given a biadjunction as in Definition 1.7, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$ there is an equivalence of hom categories*

$$\varphi_{A,B} : \mathcal{B}(FA, B) \xrightarrow{\simeq} \mathcal{A}(A, UB)$$

which is pseudonatural in each variable separately.

These equivalences are defined on 1-cells by $\varphi_{A,B}(f) := U(f) \circ \eta_A$ and on 2-cells by $\varphi(\alpha) := U(\alpha) * 1_{\eta_A}$.

For each $A \in \mathcal{A}$ and $B \in \mathcal{A}$, the functor

$$\psi_{A,B} : \mathcal{A}(A, UB) \to \mathcal{B}(FA, B)$$

defined on 1-cells by $\psi_{A,B}(g) := \xi_B \circ F(f)$ and on 2-cells by $\psi_{A,B}(\beta) := 1_{\xi_B} * F(\beta)$ is a quasi inverse to $\varphi_{A,B}$.

This result is well known but we couldn't find a self contained proof in the literature so we present one in Appendix A. A more abstract proof should be obtainable using the bicategorical Yoneda Lemma which is discussed in [19].

1.3 C^*-Categories

First introduced in [15], C^*-categories are a categorification of C^*-algebras in the sense that a unital C^*-algebra can be viewed as a C^*-category with one object. One can also consider 'non-unital categories', also called 'semicategories' to cover the more general case. We do this in [1] although we shall stick to categories with identities here. A good introduction to the theory of C^*-categories is given by [25], we shall just present the definitions and results that we need here. We begin with the notion of a $*$-category.

Definition 1.9. *A* ***$*$-category*** *is a category $\mathcal{A}$ which is enriched over the category of complex vector spaces (with its usual monoidal structure) along with a functor*

$$* : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}.$$

This functor is the identity on objects and given a morphism $f \in \mathcal{A}$ we write f^* rather than $*(f)$ and call it the ***adjoint*** *of f. Furthermore, it has the following properties:*

- *It is conjugate linear, that is given $f, g \in \mathcal{A}(A, B)$ and $\alpha, \beta \in \mathbb{C}$*

$$(\alpha f + \beta g)^* = \overline{\alpha} f^* + \overline{\beta} g^*.$$

- *It is an involution, that is $* \circ *^{\mathrm{op}} = 1_{\mathcal{A}}$.*

A classical example of a $*$-category is the (large) category Hilb of Hilbert spaces and bounded linear operators. The full subcategory of very small Hilbert spaces is a small $*$-category. There is also a small $*$-category hilb of very small finite dimensional Hilbert spaces and (necessarily bounded) linear maps. Requiring the finite dimensional Hilbert spaces to be very small is not particularly important but ensures that hilb is a small category rather than just being equivalent to one.

Definition 1.10. *Given $*$-categories $\mathcal{A}$ and $\mathcal{B}$, a* ***$*$-functor*** *$F : \mathcal{A} \to \mathcal{B}$ is a linear functor that preserves the involution. That is, given $f, g \in \mathcal{A}(A, B)$ and $\alpha, \beta \in \mathbb{C}$,*

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$$

and

$$F(f^*) = F(f)^*.$$

We now come to the main definition of this section.

Definition 1.11. *A C^***-category** is a $*$-category $\mathcal{A}$ such that:*

- *Every hom space is a Banach space,*
- *every morphism f satisfies the C^*-identity $||f^* \circ f|| = ||f||^2$,*
- *every pair of composable morphisms g, f satisfies $||g \circ f|| \leq ||g||\,||f||$,*
- *for every morphism $f \in \mathcal{A}(A, B)$ there exists a morphism $g \in \mathcal{A}(A, A)$ such that $f^* \circ f = g^* \circ g$.*

It follows from the definition that for every object $A \in \mathcal{A}$, $\mathcal{A}(A, A)$ is a C^*-algebra. The last condition says that for every morphism f, the morphism $f^* \circ f$ is positive in the sense of C^*-algebras. This does not follow from the other axioms as shown by an example in [25]. Both Hilb and hilb are C^*-categories. More generally, for any C^*-algebra A, the (large) category Hilb_A of Hilbert A-modules and adjointable linear operators is a C^*-category. (Hilbert modules are a generalisation of Hilbert spaces which can be thought of as Hilbert $\mathbb{C}$-modules. In a Hilbert module, the role of the complex numbers is taken over by an arbitrary C^*-algebra. More information can be found in [22].) For any very small C^*-algebra A, the category of very small Hilbert A-modules and adjointable linear operators is a small C^*-category.

As with C^*-algebras, we have the notion of a unitary in a C^*-category.

Definition 1.12. *A **unitary** in a C^*-category is an invertible morphism u such that $u^* = u^{-1}$.*

In 2-categorical settings, when working with C^*-categories it seems reasonable to require that coherence isomorphisms be unitaries and when making 2-categorical definitions when C^*-categories are involved we will impose this extra condition.

Definition 1.13. *Given C^*-categories $\mathcal{A}$ and $\mathcal{B}$ and $*$-functors $F, G : \mathcal{A} \to \mathcal{B}$, a C^***-natural transformation** is a natural transformation $\eta : F \to G$ such that*

$$\{||\eta_{\mathcal{A}}|| : A \in \mathcal{A}\}$$

*is a bounded set. We define the **norm** of η by*

$$||\eta|| := \sup_{A \in \mathcal{A}} ||\eta_A||.$$

We shall never consider any other type of natural transformation when considering C^*-categories and therefore we shall usually just write 'natural transformation' in this setting with it being implicit that we mean a C^*-natural transformation.

Definition 1.14. *A* ***unitary natural transformation*** *is a C^*-natural transformation η such that all its components η_A are unitaries.*

The small C^*-categories, $*$-functors and (C^*-)natural transformations are the objects, 1-cells and 2-cells of a 2-category $C^*\text{-}\mathsf{Cat}$. Moreover, given small C^*-categories $\mathcal{A}$ and $\mathcal{B}$, $C^*\text{-}\mathsf{Cat}(\mathcal{A}, \mathcal{B})$ is itself a C^*-category. Given a natural transformation $\eta : F \to G : \mathcal{A} \to \mathcal{B}$, its adjoint $\eta^* : G \to F$ is defined by $(\eta^*)_A := (\eta_A)^*$, $A \in \mathcal{A}$.

We shall also need the following two results. The proofs can be found in [25], the first is related to the fact that $*$-homomorphisms between C^*-algebras are contractive and the proof of the second is based on the GNS construction for C^*-algebras.

Lemma 1.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-categories and $F : \mathcal{A} \to \mathcal{B}$ a $*$-functor. Then*

$$||F(f)|| \leq ||f||$$

for every morphism $f \in \mathcal{A}$.

Theorem 1.16. *Let $\mathcal{A}$ be a small C^*-category. Then there is a faithful $*$-functor $\rho : \mathcal{A} \to \mathsf{Hilb}$ which is injective on objects.*

In other words, every small C^*-category can be identified with a subcategory of Hilb and we shall refer to such a $*$-functor $\rho : \mathcal{A} \to \mathsf{Hilb}$ as an **embedding** of $\mathcal{A}$ into Hilb.

Rather than working with general C^*-categories, we are going to restrict our attention to those that admit finite direct sums and which are subobject complete in the senses described later in this chapter. The main reason for this is that these are properties possessed by categories of Hilbert modules over a C^*-algebra. In particular, considering subobject complete categories allows us to modify the definition of the product of C^*-categories in section 2.2 so that taking the maximal tensor product of categories of Hilbert modules corresponds to taking the maximal tensor product of C^*-algebras (we discuss this in [1]). In fact, categories of Hilbert modules admit more direct sums than just finite ones and in [1] we define a notion of countable direct sum in a C^*-category but we shall not consider them here.

1.3.1 Direct products of C^*-categories

For the construction of bicolimits we shall require the direct product of C^*-categories and it will be useful to review them before discussing direct sums.

Definition 1.17. *Let I be a set and $(\mathcal{A}_i)_{i\in I}$ be an I-indexed collection of C^*-categories. Their* ***direct product***

$$\prod_{i\in I} \mathcal{A}_i$$

is the C^*-category whose objects are I-indexed collections $(A_i)_{i\in I}$ where each $A_i \in \mathcal{A}_i$ and whose morphism spaces are given by

$$\left(\prod_{i\in I} \mathcal{A}_i\right)((A_i)_{i\in I}, (B_i)_{i\in I}) = \{(f_i)_{i\in I} \,|\, f_i : A_i \to B_i,\, \sup_{i\in I} ||f_i|| < \infty\}.$$

For each $j \in I$, we have a ***projection*** *$*$-functor*

$$\pi_j : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_j$$

defined in the obvious way which is full and surjective on objects.

We shall denote the n-fold direct product of a C^*-category $\mathcal{A}$ with itself by $\mathcal{A}^n$ rather than $\prod_{i=1}^n \mathcal{A}$. We shall also need the following fact about the direct product of C^*-categories.

Lemma 1.18. *Let I be a set and $(\mathcal{A}_i)_{i\in I}$ a collection of C^*-categories. There is an isomorphism of C^*-categories*

$$\varphi : C^*\text{-}\mathsf{Add}(\mathcal{B}, \prod_{i\in I} \mathcal{A}_i) \xrightarrow{\cong} \prod_{i\in I} C^*\text{-}\mathsf{Add}(\mathcal{B}, \mathcal{A}_i)$$

for each C^*-category $\mathcal{B}$.

Proof. Given a $*$-functor $F : \mathcal{B} \to \prod_{i\in I} \mathcal{A}_i$ we define $\varphi(F) := (\pi_i \circ F)_{i\in I}$ and given a natural transformation $\alpha : F \to G : \mathcal{B} \to \prod_{i\in I} \mathcal{A}_i$ we define $\varphi(\alpha) := (1_{\pi_i} * \alpha)_{i\in I}$. The collection $(1_{\pi_i} * \alpha)_{i\in I}$ is bounded because α is uniformly bounded and hence $\varphi(\alpha)$ is a morphism in the direct product.

Now, given a collection $(F_i : \mathcal{B} \to \mathcal{A}_i)_{i\in I}$ of $*$-functors, we can define a $*$-functor $F : \mathcal{B} \to \prod_{i\in I} \mathcal{A}_i$ on objects by $F(B) := (F_i(B))_{i\in I}$ and similarly on morphisms. This is well-defined on morphisms because $*$-functors are contractive on morphisms and hence given a morphism $f : B \to B'$ in $\mathcal{B}$ we have $||F_i(f)|| \leq ||f||$ for all $i \in I$ so that $(F_i(f))_{i\in I}$ is a morphism in the direct product. By construction, F is the unique $*$-functor such that $\varphi(F) = (F_i)_{i\in I}$ and so φ is bijective on objects.

Finally, given a pair of $*$-functors $F, G : \mathcal{B} \to \prod_{i\in I} \mathcal{A}_i$ and a bounded collection $(\alpha_i : \pi_i \circ F \to \pi_i \circ G)_{i\in I}$ of natural transformations we can define a natural transformation $\alpha : F \to G$ by $(\alpha_B)_i = (\alpha_i)_B$. Because each α_i is uniformly bounded and the collection $(\alpha_i)_{i\in I}$ is bounded, α is well defined. By construction it is the unique natural transformation $\alpha : F \to G$ such that $\varphi(\alpha) = (\alpha_i)_{i\in I}$ and hence φ is fully faithful. $\square$

1.3.2 Direct sums and additive completions

Our definition of a (finite) direct sum in a C^*-category is the same as in the Ab-enriched setting (where Ab is the category of abelian groups) with the minor extra requirement that the projection maps are the adjoints of the inclusion maps.

Definition 1.19. *Let $(A_i)_{i=1}^n$, where $n \in \mathbb{N}$, be a collection of objects in a C^*-category $\mathcal{A}$. A **direct sum** of these objects is an object $\bigoplus_{i=1}^n A_i$ together with morphisms $\iota_{A_j} : A_j \to \bigoplus_{i=1}^n A_i$ for each $i = 1, \ldots, n$ such that*

$$\iota_{A_k}^* \circ \iota_{A\ell} = \delta_{k\ell} 1_{A_k} \quad \textit{and} \quad \sum_{j=1}^n \iota_{A_j} \circ \iota_{A_j}^* = 1_{\oplus_{i=1}^n A_i}$$

*where $\delta_{k\ell}$ is the Kronecker delta. The ι_{A_j}'s are called the **inclusion maps** and the $\iota_{A_j}^*$'s are called the **projection maps**.*

As is common practice, we shall refer to the object $\bigoplus_{i=1}^n A_i$ as the 'direct sum'. We shall also denote the direct sum of two objects A and B by $A \oplus B$. By convention, we regard the direct sum of a single object as the object itself (with the identity map as the inclusion map). The interpretation of the definition in the case $n = 0$ is that the empty sum $\sum_{j=1}^n \iota_{A_j} \circ \iota_{A_j}^*$ is equal to zero and hence the direct sum of no objects is a zero object. As a matter of notation, we shall write $A = 0$ to denote that an object A is a zero object, sometimes with it being implicit that some choice of zero object has been made. When they exist, direct sums are both products and coproducts of their factors and accordingly are unique up to unique isomorphism. In [1] we also define a notion of countable direct sum in a C^*-category but here we shall restrict our attention to finite direct sums.

Definition 1.20. *If a C^*-category $\mathcal{A}$ admits a direct sum for every finite set $(A_i)_{i=1}^n$ of objects, we say that $\mathcal{A}$ is **additive**.*

To check that $\mathcal{A}$ is additive it is sufficient to check that it admits a zero object and binary direct sums, i.e. it has a direct sum $A \oplus B$ for every pair of objects

$A, B \in \mathcal{A}$. Then n-fold direct sums can be constructed by repeated use of binary direct sums.

We shall denote the 2-category of additive C^*-categories, $*$-functors and natural transformations by $C^*\text{-}\mathsf{Cat}_\oplus$. We would denote it by $C^*\text{-}\mathsf{Add}$ but we want to reserve this for the case that our C^*-categories are also subobject complete in the sense discussed in the next section.

By making a choice of direct sum for each finite set of objects in $\mathcal{A}$, we obtain functors

$$\bigoplus : \mathcal{A}^n \to \mathcal{A}$$

for each $n \in \mathbb{N}$. Given a morphism $(f_i)_{i=1}^n : (A_i)_{i=1}^n \to (B_i)_{i=1}^n$ in $\mathcal{A}^n$ we define

$$\bigoplus_{i=1}^n f_i := \sum_{j=1}^n \iota_{B_j} \circ f_j \circ \iota_{A_j}^*.$$

Usually when we say 'the direct sum' of a collection of n objects or use the notation $\bigoplus_{i=1}^n A_i$, we have in mind that we have made a choice of direct sum functor for n and mean the image of the collection of objects under said functor. It is not of great importance but still probably worth keeping track of where we make such choices.

A useful device is to identify morphisms between direct sums with matrices. Given a morphism

$$f : \bigoplus_{j=1}^m A_j \to \bigoplus_{i=1}^n B_i$$

we have morphisms

$$f_{ij} :=: \iota_{B_i}^* \circ f \circ \iota_{A_j} : A_j \to B_i.$$

We can identify f with the $m \times n$ matrix (f_{ij}) and conversely, given the matrix (f_{ij}) we can recover f as

$$f = \sum_{i,j} \iota_{B_i} \circ f_{ij} \circ \iota_{A_j}^*.$$

Under such identifications, composition of morphisms is given by matrix multiplication. We shall often implicitly identify morphisms between direct sums with their corresponding matrices.

In an additive C^*-category, for any object $A \in \mathcal{A}$ and zero object 0, there are canonical unitary isomorphisms $0 \oplus A \cong A \oplus 0 \cong A$ and no harm will come from identifying these objects so we shall usually do this implicitly as a matter of convenience. On a related note, although we won't be considering infinite direct sums, a useful notational convention will be to define

$$\bigoplus_{i \in I} A_i$$

where I is any index set and all but finitely many of the A_i are zero objects as being the direct sum of the finitely many nonzero factors. When we use this notation, we shall not worry too much about the ordering of the factors. We will often need to permute factors when doing computations and to avoid the notation becoming too cumbersome, we shall do this implicitly and equate the permuted direct sums with one another rather than keeping track of the canonical unitary isomorphisms between them.

Similarly to the Ab-enriched case, $*$-functors preserve direct sums in the following sense.

Lemma 1.21. *Let $\mathcal{A}$, $\mathcal{B}$ be C^*-categories and $F : \mathcal{A} \to \mathcal{B}$ a $*$-functor. For any direct sum $\bigoplus_{i=1}^n A_i$ in $\mathcal{A}$, $F(\bigoplus_{i=1}^n A_i)$ along with the inclusion maps $F(\iota_{A_j})$, $j = 1, \ldots, n$ is a direct sum of the $F(A_i)$ in $\mathcal{B}$.*

Proof. Because F is a $*$-functor, it preserves all the identities in Definition 1.19 that the $F(\iota_{A_j})$ need to satisfy. □

We note that given a morphism $f : \bigoplus_{j=1}^m A_j \to \bigoplus_{i=1}^n B_i$ in $\mathcal{A}$ with corresponding matrix (f_{ij}), $F(f)$ corresponds to the matrix $(F(f_{ij}))$.

Lemma 1.22. *Let $\mathcal{A}, \mathcal{B}$ be additive C^*-categories and $F : \mathcal{A} \to \mathcal{B}$ a $*$-functor. Then for every $n \in \mathbb{N}$ there is a canonical unitary natural transformation*

$$\zeta : F \circ \bigoplus \to \bigoplus \circ F^n : \mathcal{A}^n \to \mathcal{B}.$$

Proof. This is immediate in the case $n = 0$. When $n \geq 1$, given an object $(A_i)_{i=1}^n \in \mathcal{A}^n$, we define

$$\zeta_{(A_i)_{i=1}^n} : F\left(\bigoplus_{i=1}^n A_i\right) \to \bigoplus_{i=1}^n F(A_i)$$

as the diagonal $n \times n$ matrix with the identity maps for the $F(A_i)$ on the diagonal. Naturality follows from the fact that given a morphism $(f_i)_{i=1}^n : (A_i)_{i=1}^n \to (B_i)_{i=1}^n$ in $\mathcal{V}^n$, both $F(\bigoplus_{i=1}^n f_i)$ and $\bigoplus_{i=1}^n F(f_i)$ correspond to the diagonal matrix with the $F(f_i)$ on the diagonal. □

If there are multiple $*$-functors $F, G, H \dots$ under consideration, we will denote the canonical unitary natural transformations by $\zeta^F, \zeta^G, \zeta^H \dots$ if we need to distinguish between them.

The following two lemmas will be useful when checking that diagrams involving direct sums commute.

Lemma 1.23. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be additive C^*-categories and $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ $*$-functors. Then for any $n \in \mathbb{N}$, the following diagram commutes*

$$\begin{tikzcd}
G \circ F \circ \bigoplus \arrow[r, "\zeta^F"] \arrow[dr, "\zeta^{G \circ F}"'] & G \circ \bigoplus \circ F^n \arrow[d, "\zeta^G"] \\
& \bigoplus \circ G^n \circ F^n \\
& = \bigoplus \circ (G \circ F)^n
\end{tikzcd}$$

Proof. This is immediate from the definition of the canonical unitary natural transformations. $\square$

Lemma 1.24. *Let $\mathcal{A}, \mathcal{B}$ be additive C^*-categories, $F, G : \mathcal{A} \to \mathcal{B}$ $*$-functors and $\alpha : F \to G$ a natural transformation. Then for $n \geq 1$, given $(A_i)_{i=1}^n \in \mathcal{A}^n$,*

$$\alpha_{\bigoplus_{i=1}^n A_i} : F\left(\bigoplus_{i=1}^n A_i\right) \to G\left(\bigoplus_{i=1}^n A_i\right)$$

is the diagonal $n \times n$ matrix with the α_{A_i}'s on the diagonal.

Proof. For each $i = 1, \dots, n$, by the naturality of α we have

$$\alpha_{\bigoplus_{i=1}^n A_i} \circ F(\iota_{A_i}) = G(\iota_{A_i}) \circ \alpha_{A_i}.$$

Therefore, for $i, j = 1, \dots, n$ we have

$$\begin{aligned} G(\iota_{A_j}^*) \circ \alpha_{\bigoplus_{i=1}^n A_i} \circ F(\iota_{A_i}) &= G(\iota_{A_j}^*) \circ G(\iota_{A_i}) \circ \alpha_{A_i} \\ &= \delta_{ij} \alpha_{A_i} \end{aligned}$$

where δ_{ij} is the Kronecker delta. $\square$

Corollary 1.25. *Let $\mathcal{A},\mathcal{B}$ be additive C^*-categories, $F, G : \mathcal{A} \to \mathcal{B}$ $*$-functors and $\alpha : F \to G$ a natural transformation. Then given $(A_i)_{i=1}^n \in \mathcal{A}^n$, the following diagram commutes*

$$\begin{tikzcd}
F\left(\bigoplus_{i=1}^n A_i\right) \arrow[r, "\alpha_{\bigoplus_{i=1}^n A_i}"] \arrow[d, "\zeta^F_{(A_i)_{i=1}^n}"'] & G\left(\bigoplus_{i=1}^n A_i\right) \arrow[d, "\zeta^G_{(A_i)_{i=1}^n}"] \\
\bigoplus_{i=1}^n F(A_i) \arrow[r, "\bigoplus_{i=1}^n \alpha_{A_i}"] & \bigoplus_{i=1}^n G(A_i)
\end{tikzcd}$$

Proof. This follows from Lemma 1.24 and the definitions of ζ^F and ζ^G. □

An arbitrary C^*-category can always be completed to an additive C^*-category in the following way.

Definition 1.26. *Let $\mathcal{A}$ be a C^*-category. We define the* ***additive completion*** *of $\mathcal{A}$, denoted $\mathcal{A}^\oplus$ as follows: Objects in $\mathcal{A}^\oplus$ are n-tuples $(A_i)_{i=1}^n$, $n \in \mathbb{N}$ where each $A_i \in \mathcal{A}$. To define the morphism space we use an embedding $\rho : \mathcal{A} \to \mathsf{Hilb}$ and define $\mathcal{A}^\oplus((A_j)_{j=1}^m, (B_i)_{i=1}^n)$ as the closed subspace*

$$\bigoplus_{j=1}^m \bigoplus_{i=1}^n \rho(\mathcal{A}(A_j, B_i)) \subset B\left(\bigoplus_{j=1}^m \rho(A_j), \bigoplus_{i=1}^n \rho(B_i)\right)$$

with the induced algebraic operations. (On the right hand side, $B(X,Y)$ denotes the bounded linear maps from X to Y.)

The additive completion does not depend on the choice of embedding in the sense that if two different embeddings are chosen, there is an invertible $*$-functor between the resulting C^*-categories (defined in the obvious way) which is necessarily an isometry on the morphism spaces.

As usual, we can identify a morphism $f : (A_j)_{j=1}^m \to (B_i)_{i=1}^n$ in $\mathcal{A}^\oplus$ with a matrix $(\rho(f_{ij}))$ where each f_{ij} is a morphism from A_j to B_i in $\mathcal{A}$.

The category $\mathcal{A}^\oplus$ is an additive C^*-category. The empty tuple is a zero object and given objects $A = (A_i)_{i=1}^n$ and $B = (B_j)_{j=1}^m$ in $\mathcal{A}^\oplus$, the object $(A_1, \ldots, B_n, A_1, \ldots, B_m)$ with the natural inclusion and projection maps is the direct sum of A and B. There is an inclusion $*$-functor $\eta_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^\oplus$ which maps objects $A \in \mathcal{A}$ to 1-tuples $(A) \in \mathcal{A}^\oplus$.

Lemma 1.27. *Let $\mathcal{A}$ be an additive C^*-category. Then there is an equivalence*

$$\xi_{\mathcal{A}} : \mathcal{A}^{\oplus} \xrightarrow{\simeq} \mathcal{A}$$

which is defined on objects by

$$\xi_{\mathcal{A}}((A_i)_{i=1}^n) := \bigoplus_{i=1}^n A_i.$$

Proof. We have already defined $\xi_{\mathcal{A}}$ on objects. To define $\xi_{\mathcal{A}}$ on morphisms, let $\rho : \mathcal{A} \to \mathsf{Hilb}$ be the embedding used to define the morphism spaces in $\mathcal{A}^{\oplus}$. Then, given a morphism

$$(\rho(f_{ij})) : (A_j)_{j=1}^m \to (B_i)_{i=1}^n$$

in $\mathcal{A}^{\oplus}$ we define $\xi_{\mathcal{A}}((\rho(f_{ij}))) := (f_{ij})$. Functoriality, as well as the fact that $\xi_{\mathcal{A}}$ preserves the involution, is fully-faithful and essentially surjective is immediate from the definition. $\square$

A useful fact that we shall now prove is that we can extend the definition of additive completion to $*$-functors and natural transformations to obtain a 2-functor

$$-^{\oplus} : C^*\text{-}\mathsf{Cat} \to C^*\text{-}\mathsf{Cat}_{\oplus}$$

and this 2-functor is a left biadjoint to the forgetful 2-functor

$$U : C^*\text{-}\mathsf{Cat}_{\oplus} \to C^*\text{-}\mathsf{Cat}.$$

We have already defined $-^{\oplus}$ on C^*-categories. Given C^*-categories $\mathcal{A}, \mathcal{B}$ and a $*$-functor $F : \mathcal{A} \to \mathcal{B}$, we define $F^{\oplus} : \mathcal{A}^{\oplus} \to \mathcal{B}^{\oplus}$ in the following way: Given an object $(A_i)_{i=1}^n \in \mathcal{A}^{\oplus}$ we define $F((A_i)_{i=1}^n) := (F(A_i))_{i=1}^n$. To define $F^{\oplus}$ on morphisms, let $\rho : \mathcal{A} \to \mathsf{Hilb}$ and $\lambda : \mathcal{B} \to \mathsf{Hilb}$ be the embeddings used to define the morphism spaces in $\mathcal{A}^{\oplus}$ and $\mathcal{B}^{\oplus}$. Then, given a morphism $(\rho(f_{ij})) : (A_j)_{j=1}^m \to (B_i)_{i=1}^n$ in $\mathcal{A}^{\oplus}$ we define $F^{\oplus}((\rho(f_{ij}))) := (\lambda(F(f_{ij})))$ (this is well defined because ρ is faithful). That $F^{\oplus}$ is a $*$-functor is immediate from the definition.

Next, if $\alpha : F \to G : \mathcal{A} \to \mathcal{B}$ is a natural transformation and $(V_i)_{i=1}^n \in \mathcal{A}^{\oplus}$, we define $\alpha^{\oplus}_{(V_i)_{i=1}^n} : F^{\oplus}((V_i)_{i=1}^n) \to G^{\oplus}((V_i)_{i=1}^n)$ as the $n \times n$ diagonal matrix with the maps $\lambda(\alpha_{V_i})$ on the diagonal. This defines a natural transformation $\alpha^{\oplus} : F^{\oplus} \to G^{\oplus}$ with naturality following from the naturality of α. We have now defined $-^{\oplus}$ on objects, 1-cells and 2-cells and 2-functoriality is immediate from the definitions.

Lemma 1.28. *The inclusion $*$-functors $\eta_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^{\oplus}$ are the 1-cells of a 2-natural transformation $\eta : 1_{C^*\text{-}\mathsf{Cat}} \to U \circ -^{\oplus}$.*

Proof. Given C^*-categories $\mathcal{A}, \mathcal{B}$ and a $*$-functor $F : \mathcal{A} \to \mathcal{B}$, the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "\eta_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "\eta_{\mathcal{B}}"] \\
\mathcal{A}^{\oplus} \arrow[r, "F^{\oplus}"] & \mathcal{B}^{\oplus}
\end{tikzcd}$$

is immediate from the definitions. Similarly, if $\alpha : F \to G : \mathcal{A} \to \mathcal{B}$ is a natural transformation, the equality of $1_{\eta_{\mathcal{B}}} * \alpha$ and $\alpha^{\oplus} * 1_{\eta_{\mathcal{A}}}$ is immediate from the definitions. □

Lemma 1.29. *The $*$-functors $\xi_{\mathcal{A}} : \mathcal{A}^{\oplus} \to \mathcal{A}$ of Lemma 1.27 are the 1-cells of a pseudonatural transformation $\xi : -^{\oplus} \circ U \to 1_{C^*\text{-}\mathsf{Cat}_{\oplus}}$.*

Proof. To prove this, we need to do the following:

1. Define a natural isomorphism

$$\xi_F : F \circ \xi_{\mathcal{A}} \to \xi_{\mathcal{B}} \circ F^{\oplus}$$

   for each $*$-functor $F : \mathcal{A} \to \mathcal{B}$ between additive C^*-categories.

2. Check that the ξ_F's are natural in F.

3. Check that the ξ_F's satisfy the pseudonatural transformation axioms.

To that end, suppose $F : \mathcal{A} \to \mathcal{B}$ is a $*$-functor between additive C^*-categories and $(A_i)_{i=1}^n \in \mathcal{A}^{\oplus}$. On the one hand

$$(F \circ \xi_{\mathcal{A}})((A_i)_{i=1}^n) = F\left(\bigoplus_{i=1}^n A_i\right)$$

and on the other hand

$$(\xi_{\mathcal{B}} \circ F^{\oplus})((A_i)_{i=1}^n) = \bigoplus_{i=1}^n F(A_i).$$

Therefore, we define

$$(\xi_F)_{(A_i)_{i=1}^n} := \zeta_{(A_i)_{i=1}^n} : F\left(\bigoplus_{i=1}^n A_i\right) \xrightarrow{\cong} \bigoplus_{i=1}^n F(A_i)$$

where ζ is the canonical unitary natural transformation of Lemma 1.22. Naturality of ξ_F follows from the naturality of ζ.

Next, to show that the ξ_F's are natural in F, we need to show that given a natural transformation $\alpha : F \to G : \mathcal{A} \to \mathcal{B}$, the following diagram commutes

$$\begin{tikzcd}
F \circ \xi_{\mathcal{A}} \arrow[r, "\alpha * 1_{\xi_{\mathcal{A}}}"] \arrow[d, "\xi_F"'] & G \circ \xi_{\mathcal{A}} \arrow[d, "\xi_G"] \\
\xi_{\mathcal{B}} \circ F^{\oplus} \arrow[r, "1_{\xi_{\mathcal{B}}} * \alpha^{\oplus}"] & \xi_{\mathcal{B}} \circ G^{\oplus}
\end{tikzcd} \tag{1}$$

To that end, let $(A_i)_{i=1}^n \in \mathcal{A}^{\oplus}$ and consider the following diagram

$$\begin{tikzcd}
\begin{array}{c}(F \circ \xi_{\mathcal{A}})((A_i)_{i=1}^n) \\ = F\left(\bigoplus_{i=1}^n A_i\right)\end{array} \arrow[r, "\alpha_{\bigoplus_{i=1}^n A_i}"] \arrow[d, "\zeta^F_{(A_i)_{i=1}^n}"'] & \begin{array}{c}(G \circ \xi_{\mathcal{A}})((A_i)_{i=1}^n) \\ = G\left(\bigoplus_{i=1}^n A_i\right)\end{array} \arrow[d, "\zeta^G_{(A_i)_{i=1}^n}"] \\
\begin{array}{c}(\xi_{\mathcal{B}} \circ F^{\oplus})((A_i)_{i=1}^n) \\ = \bigoplus_{i=1}^n F(A_i)\end{array} \arrow[r, "\bigoplus_{i=1}^n \alpha_{A_i}"] & \begin{array}{c}(\xi_{\mathcal{B}} \circ G^{\oplus})((A_i)_{i=1}^n) \\ = \bigoplus_{i=1}^n G(A_i)\end{array}
\end{tikzcd}$$

This commutes by Corollary 3.3 and hence (1) commutes and the ξ_F's are natural in F.

Now, the pseudonatural transformation composition axiom says that given $*$-functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, the following diagram should commute

$$\begin{tikzcd}
G \circ F \circ \xi_{\mathcal{A}} \arrow[r, "\xi_F"] \arrow[d, "\mathrm{Id}"'] & G \circ \xi_{\mathcal{B}} \circ F^{\oplus} \arrow[r, "\xi_G"] & \xi_{\mathcal{C}} \circ G^{\oplus} \circ F^{\oplus} \arrow[d, "\mathrm{Id}"] \\
G \circ F \circ \xi_{\mathcal{A}} \arrow[rr, "\xi_{G \circ F}"] & & \xi_{\mathcal{C}} \circ (G \circ F)^{\oplus}
\end{tikzcd}$$

This follows from Lemma 1.23. Finally, the unit axiom says that given and additive C^*-category $\mathcal{A}$, the following diagram should commute

$$\begin{tikzcd}
& \xi_{\mathcal{A}} \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\
1_{\mathcal{A}} \circ \xi_{\mathcal{A}} \arrow[rr, "\xi_{1_{\mathcal{A}}}"] & & \xi_{\mathcal{A}} \circ 1_{\mathcal{A}^{\oplus}}
\end{tikzcd}$$

This is immediate from the definition of $\xi_{1_{\mathcal{A}}}$. Therefore, the ξ_F's satisfy the pseudonatural transformation axioms and hence we have a pseudonatural transformation $\xi : 1_{C^*\text{-}\mathsf{Cat}} \to U \circ -^{\oplus}$. ☐

Theorem 1.30. *The 2-functor* $-^{\oplus} : C^*\text{-}\mathsf{Cat} \to C^*\text{-}\mathsf{Cat}_{\oplus}$ *is a left biadjoint to the forgetful 2-functor* $U : C^*\text{-}\mathsf{Cat}_{\oplus} \to C^*\text{-}\mathsf{Cat}$.

Proof. We have already defined the unit η and counit ξ so we just need to define invertible modifications Γ and Δ that fit into the triangle diagrams

$$\begin{tikzcd}
-^{\oplus} \arrow[r, "\eta^{\oplus}"] \arrow[dr, "1_{-^{\oplus}}"'] & -^{\oplus} \circ U \circ -^{\oplus} \arrow[d, "\xi_{-^{\oplus}}"] \arrow[dl, phantom, "\Uparrow\Gamma"] \\
& -^{\oplus}
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\eta_U"] \arrow[dr, "1_U"'] & U \circ -^{\oplus} \circ U \arrow[d, "U\xi"] \arrow[dl, phantom, "\Uparrow\Delta"] \\
& U
\end{tikzcd}$$

In fact both triangle diagrams commute and so we can take Γ and Δ to be identities. Essentially this comes down to the fact that the unit maps objects to 1-tuples containing the objects and the counit maps 1-tuples to the object they contain. Explicitly, to see that the left diagram commutes, let $\mathcal{A} \in C^*\text{-}\mathsf{Cat}$ and consider the diagram

$$\begin{tikzcd}
\mathcal{A}^{\oplus} \arrow[r, "(\eta_{\mathcal{A}})^{\oplus}"] \arrow[dr, "1_{\mathcal{A}^{\oplus}}"'] & (\mathcal{A}^{\oplus})^{\oplus} \arrow[d, "\xi_{\mathcal{A}^{\oplus}}"] \\
& \mathcal{A}^{\oplus}
\end{tikzcd}$$

Then, given $(A_i)_{i=1}^n \in \mathcal{A}^\oplus$ we have

$$\begin{aligned}(\xi_{\mathcal{A}^\oplus} \circ (\eta_{\mathcal{A}})^\oplus)((A_i)_{i=1}^n) &= \xi_{\mathcal{A}^\oplus}(((A_i)_{i=1}^n)) \\ &= (A_i)_{i=1}^n.\end{aligned}$$

A similar argument holds for morphisms in $\mathcal{A}^\oplus$. Similarly, to see that the right diagram commutes, let $\mathcal{B} \in C^*\text{-}\mathsf{Cat}_\oplus$ and consider the diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\eta_{\mathcal{B}}"] \arrow[dr, "1_{\mathcal{B}}"'] & \mathcal{B}^\oplus \arrow[d, "\xi_{\mathcal{B}}"] \\ & \mathcal{B} \end{tikzcd}$$

Then, given $B \in \mathcal{B}$ we have

$$\begin{aligned}(\xi_{\mathcal{B}} \circ \eta_{\mathcal{B}})(B) &= \xi_{\mathcal{B}}((B)) \\ &= B.\end{aligned}$$

Again, a similar argument holds for morphisms in $\mathcal{B}$. Therefore, both triangle diagrams commute and hence $-^\oplus$ is a left biadjoint to U. $\square$

A useful consequence of the existence of the biadjunction is that given a C^*-category $\mathcal{A}$ and an additive C^*-category $\mathcal{B}$, we can extend any $*$-functor $F : \mathcal{A} \to \mathcal{B}$ to the additive completion of $\mathcal{A}$. More precisely, we have the following:

Corollary 1.31. *Let $\mathcal{A}, \mathcal{B}$ be C^*-categories and $F : \mathcal{A} \to \mathcal{B}$ a $*$-functor. If $\mathcal{B}$ is additive, there is a $*$-functor $F^+ : \mathcal{A}^\oplus \to \mathcal{B}$ such that $F^+ \circ \eta_{\mathcal{A}} = F$.*

Proof. We define $F^+ := \xi_{\mathcal{B}} \circ F^\oplus$. Then, given $A \in \mathcal{A}$ we have

$$\begin{aligned}(\xi_{\mathcal{B}} \circ F^\oplus \circ \eta_{\mathcal{A}})(A) &= (\xi_{\mathcal{B}} \circ F^\oplus)((A)) \\ &= \xi_{\mathcal{B}}((F(A))) \\ &= F(A).\end{aligned}$$

A similar argument holds for morphisms and hence $F^+ \circ \eta_{\mathcal{A}} = F$. $\square$

Finally, we note that if $(\mathcal{A}_i)_{i \in I}$ is a collection of additive C^*-categories, then their direct product is also additive. If $A = (A_i)_{i \in I}$ and $B = (B_i)_{i \in I}$ are objects in $\prod_{i \in I} \mathcal{A}_i$, their direct sum is $(A_i \oplus B_i)_{i \in I}$ with the natural inclusion maps.

1.3.3 Subobject completions

Our approach to subobjects will be a little different than the usual categorical definition in terms of equivalence classes of monomorphisms (see [23] for example). Instead we will define subobjects in terms of projections.

Definition 1.32. *A **projection** in a C^*-category $\mathcal{A}$ is a morphism $p : A \to A$ for some $A \in \mathcal{A}$ such that $p = p^* = p^2$.*

Definition 1.33. *We say a projection $p : A \to A$ in a C^*-category **splits** if there exists a morphism $s : B \to A$ such that $s \circ s^* = p$ and $s^* \circ s = 1_B$. In this case, we call B a **subobject** of A.*

Definition 1.34. *We say a C^*-category is **subobject complete** if every projection splits.*

In subobject complete C^*-categories, subobjects can be identified with summands in direct sums. Specifically, if $p : A \to A$ is a projection then so is $q = 1 - p$. Then, if $s_p : B \to A$ and $s_q : C \to A$ are splittings of p and q respectively, A is a direct sum of B and C with s_p and s_q as the inclusion maps.

This notion of subobject is slightly stronger than the usual categorical notion defined in terms of equivalence classes of monomorphisms. For example, if we consider the category $\mathsf{Hilb}_{C([0,1])}$ of Hilbert modules over $C([0,1])$, then $C([0,1])$ itself is an object in $\mathsf{Hilb}_{C([0,1])}$. If $s : C([0,1]) \to C([0,1])$ is the map defined by $s(f)(x) = xf(x)$, then s is a monomorphism but its image does not correspond to any projection $p : C([0,1]) \to C([0,1])$. In other words, $C([0,1])$ is not the direct sum of the image of s and another Hilbert $C([0,1])$-module.

Similarly to the case of direct sums, an arbitrary C^*-category can be completed to a subobject complete one in the following way.

Definition 1.35. *Let $\mathcal{A}$ be a C^*-category. The **subobject completion** of $\mathcal{A}$, denoted $\mathrm{Split}(\mathcal{A})$ it the C^*-category defined as follows: The objects are pairs (A, p) where A is an object in $\mathcal{A}$ and $p : A \to A$ is a projection. Given objects (A, p) and (B, q), a morphism $f : (A, p) \to (B, q)$ is a morphism $f : A \to B$ in $\mathcal{A}$ such that $q \circ f \circ p = f$.*

The subobject completion is also referred to as the 'idempotent completion' or 'Karoubi envelope' of $\mathcal{A}$ (there is a slight technical difference in that a projection is not only idempotent but also self adjoint but the construction is the same). The morphisms in $\mathrm{Split}(\mathcal{A})$ also have the following equivalent characterisation.

Lemma 1.36. *The morphisms $f : (A, p) \to (B, q)$ in $\mathrm{Split}(\mathcal{A})$ are the morphisms $f : A \to B$ in $\mathcal{A}$ such that $q \circ f = f$ and $f \circ p = f$.*

Proof. On the one hand, if $q \circ f \circ p = f$ then

$$\begin{aligned} q \circ f &= q \circ (q \circ f \circ p) \\ &= q^2 \circ f \circ p \\ &= q \circ f \circ p \\ &= f \end{aligned}$$

and similarly $f \circ p = f$.

On the other hand, if $q \circ f = f$ and $f \circ p = f$ then

$$q \circ f \circ p = f \circ p = f.$$

□

We note that the identity map of an object $(A, p) \in \mathrm{Split}(\mathcal{A})$ is p and using the above lemma we can see that the composite of two morphisms in $\mathrm{Split}(\mathcal{A})$ is again a morphism in $\mathrm{Split}(\mathcal{A})$.

If $\mathcal{A}$ has direct sums then so does $\mathrm{Split}(\mathcal{A})$. The direct sum of (A, p) and (B, q) is $(A \oplus B, p \oplus q)$. The inclusion and projection maps are obtained by pre- and post-composing the inclusion and projection maps of $A \oplus B$ with p, q and $p \oplus q$ as appropriate.

We also note that if $(\mathcal{A}_i)_{i \in I}$ is a collection of subobject complete C^*-categories, so is their direct product. If $p = (p_i)_{i \in I} : (A_i)_{i \in I} \to (A_i)_{i \in I}$ is a projection in $\prod_{i \in I} \mathcal{A}_i$ and each p_i is split by $s_i : B_i \to A_i$, then $s = (s_i) : (B_i)_{i \in I} \to (A_i)_{i \in I}$ is a splitting of p.

We shall denote the 2-category of additive, subobject complete C^*-categories by $C^*\text{-}\mathsf{Add}$. Similarly to the case of the additive completion, there is a 2-functor

$$\mathrm{Split} : C^*\text{-}\mathsf{Cat}_{\oplus} \to C^*\text{-}\mathsf{Add}$$

which is a left biadjoint to the forgetful 2-functor $U : C^*\text{-}\mathsf{Add} \to C^*\text{-}\mathsf{Cat}_{\oplus}$. We present the details below and remark that nothing about the definitions or proofs that follow depend on the existence of direct sums so $C^*\text{-}\mathsf{Cat}_{\oplus}$ could be replaced by $C^*\text{-}\mathsf{Cat}$ throughout. We have chosen to restrict our attention to additive C^*-categories here since this is the case we are interested in.

We have already defined Split on objects. In order to define Split on 1-cells, let $\mathcal{A}, \mathcal{B} \in C^*\text{-}\mathsf{Cat}_{\oplus}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a $*$-functor. We define a $*$-functor

$\mathrm{Split}(F) : \mathrm{Split}(\mathcal{A}) \to \mathrm{Split}(\mathcal{B})$ on objects by $\mathrm{Split}(F)(A,p) := (FA, Fp)$ and on morphisms by $\mathrm{Split}(F)(f) := F(f)$. The functoriality of F ensures that Ff is a morphism in $\mathrm{Split}(\mathcal{B})$.

To define Split on 2-cells, let $\alpha : F \to G : \mathcal{A} \to \mathcal{B}$ be a natural transformation. We define $\mathrm{Split}(\alpha) : \mathrm{Split}(F) \to \mathrm{Split}(G)$ by

$$\mathrm{Split}(\alpha)_{(A,p)} := Gp \circ \alpha_A \circ Fp : (FA, Fp) \to (GA, Gp).$$

To see that $\mathrm{Split}(\alpha)$ is a natural transformation, let $f : (A,p) \to (B,q)$ be a morphism in $\mathrm{Split}(\mathcal{A})$ and consider the following diagram.

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "Fp"'] & FB \arrow[d, "Fq"] \\
FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\
GA \arrow[r, "Gf"] \arrow[d, "Gp"'] & GB \arrow[d, "Gq"] \\
GA \arrow[r, "Gf"] & GB
\end{tikzcd}$$

The centre square commutes by the naturality of α and the other two squares commute because f is a morphism in $\mathrm{Split}(\mathcal{A})$. Therefore, $\mathrm{Split}(\alpha)$ is a natural transformation.

Given $\mathcal{A} \in C^*\text{-}\mathsf{Cat}_\oplus$, there is an inclusion $*$-functor $\eta_\mathcal{A} : \mathcal{A} \to \mathrm{Split}(\mathcal{A})$ defined on objects by $\eta_\mathcal{A}(A) := (A, 1_A)$ and on morphisms by $\eta_\mathcal{A}(f) := f$.

Lemma 1.37. *The inclusion $*$-functors $\eta_\mathcal{A} : \mathcal{A} \to \mathrm{Split}(\mathcal{A})$ are the 1-cells of a 2-natural transformation $\eta : 1_{C^*\text{-}\mathsf{Cat}_\oplus} \to U \circ \mathrm{Split}$.*

Proof. Let $\mathcal{A}, \mathcal{B} \in C^*\text{-}\mathsf{Cat}_\oplus$ and let $F : \mathcal{A} \to \mathcal{B}$ be a $*$-functor. The commutativity of the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "\eta_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "\eta_{\mathcal{B}}"] \\
\mathrm{Split}(\mathcal{A}) \arrow[r, "\mathrm{Split}(F)"] & \mathrm{Split}(\mathcal{B})
\end{tikzcd}$$

is immediate from the definitions. Similarly, if $\alpha : F \to G : \mathcal{A} \to \mathcal{B}$ is a natural transformation, the equality

$$1_{\eta_{\mathcal{B}}} * \alpha = \mathrm{Split}(\alpha) * 1_{\eta_{\mathcal{A}}}$$

is immediate from the definitions. $\square$

This 2-natural transformation is the unit of the biadjunction. Given $\mathcal{A} \in C^*\text{-}\mathsf{Add}$, to define the component $\xi_{\mathcal{A}} : \mathrm{Split}(\mathcal{A}) \to \mathcal{A}$ of the counit, we need to choose a splitting of each projection in $\mathcal{A}$. Explicitly, given an object $(A, p) \in \mathrm{Split}(\mathcal{A})$ we choose a morphism $s_p : B \to A$ in $\mathcal{A}$ such that $s_p \circ s_p^* = p$ and $s_p^* \circ s_p = 1_B$ and define $\xi_{\mathcal{A}}(A, p) := B$. For simplicity, we shall assume that $s_{1_A} = 1_A$ for all $A \in \mathcal{A}$ so that $\xi_{\mathcal{A}}(A, 1_A) = A$.

Then, given a morphism $f : (A, p) \to (B, q)$ in $\mathrm{Split}(\mathcal{A})$ we define $\xi_{\mathcal{A}}(f) := s_q^* \circ f \circ s_p$. To see that $\xi_{\mathcal{A}}$ is a $*$-functor, we first note that $1_{(A,p)} = p$. Then

$$\begin{aligned}
\xi_{\mathcal{A}}(1_{(A,p)}) &= s_p^* \circ p \circ s_p \\
&= s_p^* \circ s_p \circ s_p^* \circ s_p \\
&= 1_{\xi_{\mathcal{A}}(A,p)} \circ 1_{\xi_{\mathcal{A}}(A,p)} \\
&= 1_{\xi_{\mathcal{A}}(A,p)}.
\end{aligned}$$

Next, let $f : (A, p) \to (B, q)$ and $g : (B, q) \to (C, r)$ be morphisms in $\mathrm{Split}(\mathcal{A})$. Then

$$\begin{aligned}
\xi_{\mathcal{A}}(g) \circ \xi_{\mathcal{A}}(f) &= s_r^* \circ g \circ s_q \circ s_q^* \circ f \circ s_p \\
&= s_r^* \circ g \circ q \circ f \circ s_p \\
&= s_r^* \circ g \circ f \circ s_p \\
&= \xi_{\mathcal{A}}(g \circ f).
\end{aligned}$$

That $\xi_{\mathcal{A}}$ preserves the involution is immediate and hence $\xi_{\mathcal{A}}$ is a $*$-functor as required.

Lemma 1.38. *Let $\mathcal{A} \in C^*\text{-}\mathsf{Add}$, then $\xi_{\mathcal{A}} : \mathrm{Split}(\mathcal{A}) \to \mathcal{A}$ is an equivalence.*

Proof. That $\xi_{\mathcal{A}}$ is essentially surjective is immediate. To see that it is faithful, let $f : (A, p) \to (B, q)$ be a morphism in $\mathrm{Split}(\mathcal{A})$ such that $\xi_{\mathcal{A}}(f) = 0$. Then

$$\begin{aligned} f &= q \circ f \circ p \\ &= s_q \circ s_q^* \circ f \circ s_p \circ s_p^* \\ &= s_q \circ 0 \circ s_p^* \\ &= 0. \end{aligned}$$

To see that $\xi_{\mathcal{A}}$ is full, let $g : \xi_{\mathcal{A}}(A, p) \to \xi_{\mathcal{A}}(B, q)$ be a morphism in $\mathcal{A}$. Then

$$\begin{aligned} q \circ s_q \circ g \circ s_p^* \circ p &= s_q \circ s_q^* \circ s_q \circ g \circ s_p^* \circ s_p \circ s_p^* \\ &= s_q \circ 1_{\xi_{\mathcal{A}}(B,q)} \circ g \circ 1_{\xi_{\mathcal{A}}(A,p)} \circ s_p^* \\ &= s_q \circ g \circ s_p^* \end{aligned}$$

so $s_q \circ g \circ s_p^* : (A, p) \to (B, q)$ is a morphism in $\mathrm{Split}(\mathcal{A})$ and

$$\begin{aligned} \xi_{\mathcal{A}}(s_q \circ g \circ s_p^*) &= s_q^* \circ s_q \circ g \circ s_p^* \circ s_p \\ &= 1_{\xi_{\mathcal{A}}(B,q)} \circ g \circ 1_{\xi_{\mathcal{A}}(A,p)} \\ &= g. \end{aligned}$$

Therefore, $\xi_{\mathcal{A}}$ is also full and hence an equivalence. $\square$

Lemma 1.39. *The $*$-functors $\xi_{\mathcal{A}} : \mathrm{Split}(\mathcal{A}) \to \mathcal{A}$ are the 1-cells of a pseudonatural transformation $\xi : \mathrm{Split} \circ U \to 1_{C^*\text{-}\mathsf{Add}}$.*

Proof. To show this, we need to do the following:

1. Define natural isomorphisms

$$\xi_F : F \circ \xi_{\mathcal{A}} \to \xi_{\mathcal{B}} \circ \mathrm{Split}(F)$$

   for all $*$-functors $F : \mathcal{A} \to \mathcal{B}$ between additive C^*-categories.

2. Check that the ξ_F's are natural in F.

3. Check that the ξ_F's satisfy the pseudonatural transformation axioms.

To that end, let $\mathcal{A}, \mathcal{B} \in C^*\text{-}\mathsf{Add}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a $*$-functor. Given $(A, p) \in \mathrm{Split}(\mathcal{A})$, we define $(\xi_F)_{(A,p)}$ as the composite

$$(F \circ \xi_{\mathcal{A}})(A, p) \xrightarrow{F s_p} FA \xrightarrow{s_{Fp}^*} \begin{array}{c} \xi_{\mathcal{B}}(FA, Fp) \\ = (\xi_{\mathcal{B}} \circ \mathrm{Split}(F))(A, p). \end{array}$$

This is a unitary because

$$\begin{aligned}
(s_{Fp}^* \circ Fs_p)^* \circ s_{F_p}^* \circ Fs_p &= Fs_p^* \circ s_{Fp} \circ s_{Fp}^* \circ Fs_p \\
&= Fs_p^* \circ Fp \circ Fs_p \\
&= Fs_p^* \circ Fs_p \circ Fs_p^* \circ Fs_p \\
&= 1_{(F\circ\xi_{\mathcal{A}})(A,p)} \circ 1_{(F\circ\xi_{\mathcal{A}})(A,p)} \\
&= 1_{(F\circ\xi_{\mathcal{A}})(A,p)}
\end{aligned}$$

and similarly

$$\begin{aligned}
s_{F_p}^* \circ Fs_p \circ (s_{Fp}^* \circ Fs_p)^* &= s_{F_p}^* \circ Fs_p \circ Fs_p^* \circ s_{Fp} \\
&= s_{F_p}^* \circ Fp \circ s_{Fp} \\
&= s_{F_p}^* \circ s_{Fp} \circ s_{F_p}^* \circ s_{Fp} \\
&= 1_{\xi_{\mathcal{B}}(FA,Fp)} \circ 1_{\xi_{\mathcal{B}}(FA,Fp)} \\
&= 1_{\xi_{\mathcal{B}}(FA,Fp)}.
\end{aligned}$$

To see that ξ_F is a natural transformation, let $f : (A,p) \to (B,q)$ be a morphism in $\mathrm{Split}(\mathcal{A})$. Then

$$\begin{aligned}
(\xi_F)_{(B,q)} \circ (F \circ \xi_{\mathcal{A}})(f) &= s_{Fq}^* \circ Fs_q \circ Fs_q^* \circ Ff \circ Fs_p \\
&= s_{Fq}^* \circ Fq \circ Ff \circ Fs_p \\
&= s_{Fq}^* \circ Ff \circ Fs_p \\
&= s_{Fq}^* \circ Ff \circ Fp \circ Fs_p \\
&= s_{Fq}^* \circ Ff \circ s_{Fp} \circ s_{Fp}^* \circ Fs_p \\
&= (\xi_{\mathcal{B}} \circ \mathrm{Split}(F))(f) \circ (\xi_F)_{(A,p)}
\end{aligned}$$

and hence ξ_F is a natural transformation.

Next, to show that the ξ_F's are natural in F, we need to show that given a natural transformation $\alpha : F \to G : \mathcal{A} \to \mathcal{B}$, the following diagram commutes

$$\begin{tikzcd}
F \circ \xi_{\mathcal{A}} \arrow[r, "\xi_F"] \arrow[d, "\alpha * 1_{\xi_{\mathcal{A}}}"'] & \xi_{\mathcal{B}} \circ \mathrm{Split}(F) \arrow[d, "1_{\xi_{\mathcal{B}}} * \mathrm{Split}(\alpha)"] \\
G \circ \xi_{\mathcal{A}} \arrow[r, "\xi_G"] & \xi_{\mathcal{B}} \circ \mathrm{Split}(G)
\end{tikzcd} \tag{2}$$

To that end, let $(A,p) \in \mathrm{Split}(\mathcal{A})$ and consider the following diagram

$$\begin{tikzcd}
(F \circ \xi_{\mathcal{A}})(A,p) \arrow[r, "Fs_p"] \arrow[ddddd, "\alpha_{\xi_{\mathcal{A}}(A,p)}"'] \arrow[dddddr, phantom, "\text{(I)}"] & FA \arrow[r, "s^*_{Fp}"] \arrow[ddddd, "\alpha_A"] \arrow[ddr, "Fp"'] \arrow[dr, phantom, "\text{(II)}"] & \begin{array}{c}\xi_{\mathcal{B}}(FA,Fp) \\ = (\xi_{\mathcal{B}} \circ \mathrm{Split}(F))(A,p).\end{array} \arrow[d, "s_{Fp}"] \\
& & FA \arrow[d, "Fp"] \\
& \arrow[r, phantom, "\text{(III)}"] & FA \arrow[d, "\alpha_A"] \\
& & GA \arrow[d, "Gp"] \\
& & GA \arrow[d, "s^*_{Gp}"] \\
(G \circ \xi_{\mathcal{A}})(A,p) \arrow[r, "Gs_p"] & GA \arrow[r, "s^*_{Gp}"] \arrow[uur, "Gp"] \arrow[ur, phantom, "\text{(IV)}"] & \begin{array}{c}\xi_{\mathcal{B}}(GA,Gp) \\ = (\xi_{\mathcal{B}} \circ \mathrm{Split}(G))(A,p).\end{array}
\end{tikzcd}$$

Here, (I) and (III) commute by the naturality of α, (II) commutes because

$$\begin{aligned} Fp \circ s_{Fp} \circ s^*_{Fp} &= Fp \circ Fp \\ &= Fp \end{aligned}$$

and (IV) commutes because

$$\begin{aligned} s^*_{Gp} \circ Gp \circ Gp &= s^*_{Gp} \circ Gp \\ &= s^*_{Gp} \circ s_{Gp} \circ s^*_{Gp} \\ &= 1_{\xi_{\mathcal{B}}(GA,Gp)} \circ s^*_{Gp} \\ &= s^*_{Gp}. \end{aligned}$$

Therefore, (2) commutes and the ξ_F's are natural in F.

Next, the pseudonatural transformation composition axioms says that given $*$-

functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, the following diagram should commute.

$$\begin{tikzcd}
G \circ F \circ \xi_{\mathcal{A}} \arrow[r, "\xi_F"] \arrow[d, "\mathrm{Id}"'] & G \circ \xi_{\mathcal{B}} \circ \mathrm{Split}(F) \arrow[r, "\xi_G"] & \xi_{\mathcal{C}} \circ \mathrm{Split}(G) \circ \mathrm{Split}(F) \arrow[d, "\mathrm{Id}"] \\
G \circ F \circ \xi_{\mathcal{A}} \arrow[rr, "\xi_{G \circ F}"] & & \xi_{\mathcal{C}} \circ \mathrm{Split}(G \circ F)
\end{tikzcd} \tag{3}$$

To see that this is the case, let $(A, p) \in \mathrm{Split}(\mathcal{A})$ and consider the following diagram.

$$\begin{tikzcd}
GF(\xi_{\mathcal{A}}(A,p)) \arrow[r, "GFs_p"] \arrow[d, "\mathrm{Id}"'] & GFA \arrow[r, "Gs^*_{Fp}"] & G(\xi_{\mathcal{B}}(FA, Fp)) \arrow[r, "Gs_{Fp}"] & GFA \arrow[r, "s^*_{GFp}"] & \xi_{\mathcal{C}}(GFA, GFp) \arrow[d, "\mathrm{Id}"] \\
GF(\xi_{\mathcal{A}}(A,p)) \arrow[rr, "GFs_p"] & & GFA \arrow[rr, "s^*_{GFp}"] & & \xi_{\mathcal{C}}(GFA, GFp)
\end{tikzcd}$$

This commutes because

$$\begin{aligned}
s^*_{GFp} \circ Gs_{Fp} \circ Gs^*_{Fp} \circ GFs_p &= s^*_{GFp} \circ GFp \circ GFs_p \\
&= s^*_{GFp} \circ GFs_p \circ GFs^*_p \circ GFs_p \\
&= s^*_{GFp} \circ GFs_p \circ 1_{GFA} \\
&= s^*_{GFp} \circ GFs_p
\end{aligned}$$

and hence (3) commutes.

Finally, the pseudonatural transformation unit axiom says that given $\mathcal{A} \in C^*\text{-}\mathsf{Add}$, the following diagram should commute

$$\begin{tikzcd}
& \xi_{\mathcal{A}} \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\
1_{\mathcal{A}} \circ \xi_{\mathcal{A}} \arrow[rr, "\xi_{1_{\mathcal{A}}}"] & & \xi_{\mathcal{A}} \circ \mathrm{Split}(1_{\mathcal{A}})
\end{tikzcd}$$

This is immediate from the definition of $\xi_{1_{\mathcal{A}}}$. Therefore, the ξ_F's satisfy the pseudonatural transformation axioms and hence we have a pseudonatural transformation $\xi : \mathrm{Split} \circ U \to 1_{C^*\text{-}\mathsf{Add}}$. $\square$

Theorem 1.40. *The 2-functor* Split : C^*-$\mathsf{Cat}_\oplus \to C^*$-$\mathsf{Add}$ *is a left biadjoint to the forgetful 2-functor* $U : C^*$-$\mathsf{Add} \to C^*$-$\mathsf{Cat}_\oplus$.

Proof. We have already defined the unit η and counit ξ so we just need to define invertible modifications Γ and Δ that fit into the triangle diagrams

$$\begin{tikzcd} \mathrm{Split} \arrow[r, "\mathrm{Split}\eta"] \arrow[dr, "1_{\mathrm{Split}}"'] & \mathrm{Split} \circ U \circ \mathrm{Split} \arrow[d, "\xi_{\mathrm{Split}}"] \arrow[dl, phantom, "\Uparrow\Gamma"] \\ & \mathrm{Split} \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\eta_U"] \arrow[dr, "1_U"'] & U \circ \mathrm{Split} \circ U \arrow[d, "U\xi"] \arrow[dl, phantom, "\Uparrow\Delta"] \\ & U \end{tikzcd}$$

In fact, both triangle diagrams commute and we can take Γ and Δ to be identities. This mostly comes down to our choice of splitting for the identity maps when defining the counit. Indeed, the left diagram commute because if $\mathcal{A} \in C^*$-$\mathsf{Cat}_\oplus$ and $(A, p) \in \mathrm{Split}(\mathcal{A})$ then

$$\begin{aligned}(\xi_{\mathrm{Split}(\mathcal{A})} \circ \mathrm{Split}(\eta_{\mathcal{A}}))(A, p) &= \xi_{\mathrm{Split}(\mathcal{A})}((A, p), 1_{(A,p)}) \\ &= (A, p)\end{aligned}$$

and if $f : (A, p) \to (B, q)$ is a morphism in $\mathrm{Split}(\mathcal{A})$ then

$$\begin{aligned}(\xi_{\mathrm{Split}(\mathcal{A})} \circ \mathrm{Split}(\eta_{\mathcal{A}}))(f) &= \xi_{\mathrm{Split}(\mathcal{A})}(f) \\ &= 1_{(B,q)} \circ f \circ 1_{(A,p)} \\ &= f.\end{aligned}$$

Similarly, if $\mathcal{B} \in C^*$-Add and $B \in \mathcal{B}$, then

$$\begin{aligned}(\xi_{\mathcal{B}} \circ \eta_{\mathcal{B}})(B) &= \xi_{\mathcal{B}}(W, 1_B) \\ &= B\end{aligned}$$

and if $g : B \to C$ is a morphism in $\mathcal{B}$ then

$$\begin{aligned}(\xi_{\mathcal{B}} \circ \eta_{\mathcal{B}})(g) &= \xi_{\mathcal{B}}(g) \\ &= 1_C \circ g \circ 1_B \\ &= g.\end{aligned}$$

Therefore, both triangle diagrams commute and Split : C^*-$\mathsf{Cat}_\oplus \to C^*$-$\mathsf{Add}$ is a left biadjoint to the forgetful 2-functor $U : C^*$-$\mathsf{Add} \to C^*$-$\mathsf{Cat}_\oplus$. ☐

Similarly to the case of additive completions, one consequence of the existence of the biadjunction is that we can extend $*$-functors whose domain is an arbitrary C^*-categories to the subobject completion, provided the codomain is subobject complete.

Corollary 1.41. *Let $\mathcal{A} \in C^*\text{-}\mathsf{Cat}_{\oplus}$, $\mathcal{B} \in C^*\text{-}\mathsf{Add}$ and let $F : \mathcal{A} \to \mathcal{B}$ a $*$-functor. Then there is a $*$-functor $\overline{F} : \mathrm{Split}(\mathcal{A}) \to \mathcal{B}$ such that $\overline{F} \circ \eta_{\mathcal{A}} = F$.*

Proof. We define $\overline{F} := \xi_{\mathcal{B}} \circ \mathrm{Split}(F)$. Then, given $A \in \mathcal{A}$ we have

$$\begin{aligned}(\xi_{\mathcal{B}} \circ \mathrm{Split}(F) \circ \eta_{\mathcal{A}})(A) &= (\xi_{\mathcal{B}} \circ \mathrm{Split}(F))(A, 1_A) \\ &\xi_{\mathcal{B}}(FA, 1_{FA}) \\ &= F(A)\end{aligned}$$

and given a morphism $f : A \to B$ in $\mathcal{A}$ we have

$$\begin{aligned}(\xi_{\mathcal{B}} \circ \mathrm{Split}(F) \circ \eta_{\mathcal{A}})(f) &= (\xi_{\mathcal{B}} \circ \mathrm{Split}(F))(f) \\ &\xi_{\mathcal{B}}(F(f)) \\ &= 1_B \circ F(f) \circ 1_A \\ &= F(f).\end{aligned}$$

Therefore, $\overline{F} \circ \eta_{\mathcal{A}} = F$. □

2 Bicolimits and balanced tensor products of C^*-categories

In this section we shall show that $C^*\text{-}\mathsf{Add}$ has all small (conical) category-indexed bicolimits, at least when we restrict to the unitary setting in the case of diagrams. We do this with a view to constructing the balanced tensor product of two module categories but it is also of interest in its own right. The category of small C^*-categories and $*$-functors is both complete and cocomplete [10] but whilst $C^*\text{-}\mathsf{Add}$ does have direct products, it does not have arbitrary limits or colimits. For example, there is no initial object. If $\mathcal{A}, \mathcal{B} \in C^*\text{-}\mathsf{Add}$, $F : \mathcal{A} \to \mathcal{B}$ is a $*$-functor and $\mathcal{B}$ has multiple zero objects then one can define another $*$-functor $F' : \mathcal{A} \to \mathcal{B}$ by changing the assignment of F on zero objects. In particular, no $\mathcal{A} \in C^*\text{-}\mathsf{Add}$ can be an initial object. This leads us to consider a more general 2-categorical notion of colimit.

An introduction to different types of 2-categorical limit can be found in [20]. We have chosen to consider bicolimits which are the weakest notion of 2-categorical colimit discussed there. In joint work with my supervisor in [1], we showed the existence of bicolimits in the more general setting of non-unital C^*-categories which admit a certain notion of countable direct sums. Here we shall restrict our attention to unital categories with finite direct sums.

2.1 Bicolimits

To define bicolimits, we first need to define diagrams. We shall restrict our attention to the unitary case since we are in the C^*-setting.

Definition 2.1. *Let $\mathcal{I}$ be a small category which we view as a 2-category whose only 2-cells are identities. An **$\mathcal{I}$-diagram** in $C^*\text{-}\mathsf{Add}$ is a pseudofunctor $D : \mathcal{I} \to C^*\text{-}\mathsf{Add}$ whose coherence maps are unitary natural transformations. A **transformation** between $\mathcal{I}$-diagrams D and E is a pseudonatural transformation $\sigma : D \to E$ whose coherence maps are unitary natural transformations.*

Given $\mathcal{I}$-diagrams D and E, we write $\mathsf{Tran}(D, E)$ for the C^*-category whose objects are transformations $\sigma : D \to E$ and whose morphisms are modifications Γ such that $\sup_{I\in\mathcal{I}} ||\Gamma_I|| < \infty$ with norm $||\Gamma|| := \sup_{I\in\mathcal{I}} ||\Gamma_I||$.

Given $\mathcal{A} \in C^*\text{-}\mathsf{Add}$, there is a constant $\mathcal{I}$-diagram $\Delta(\mathcal{A})$ which maps every object $i \in \mathcal{I}$ to $\mathcal{A}$ and every morphism to $1_{\mathcal{A}}$. More generally, there is a 2-functor $\Delta : C^*\text{-}\mathsf{Add} \to [\mathcal{I}, C^*\text{-}\mathsf{Add}]$ defined on objects as above and on $*$-functors and natural transfomations in the obvious way.

Definition 2.2. *Let $D : \mathcal{I} \to C^*\text{-}\mathsf{Add}$ be an $\mathcal{I}$-diagram. A **bicolimit** for D is an additive, subobject complete C^*-category $\mathcal{L}$ with a transformation $\sigma : D \to \Delta(\mathcal{L})$ that induces equivalences of C^*-categories*

$$- \circ \sigma : C^*\text{-}\mathsf{Add}(\mathcal{L}, \mathcal{A}) \xrightarrow{\simeq} \mathsf{Tran}(D, \Delta(\mathcal{A}))$$

for every $\mathcal{A} \in C^*\text{-}\mathsf{Add}$ which are pseudonatural in $\mathcal{A}$.

By this notation, we mean that a $*$-functor $F : \mathcal{L} \to \mathcal{A}$ is mapped to the transformation $\Delta(F) \circ \sigma$ and a natural transformation $\alpha : F \to G : \mathcal{L} \to \mathcal{A}$ is mapped to the modification $\Delta(\alpha) * 1_\sigma$. We shall use similar notational conventions throughout for maps induced by pre- or post-composition by another given map. When they exist, bicolimits are unique up to equivalence. This can be proved in a similar way to the way one can prove the uniqueness of colimits in a category by considering their universal property.

During our construction of bicolimits, it will be convenient to work with a certain type of transformation.

Definition 2.3. *Let D be an $\mathcal{I}$-diagram, $\mathcal{A} \in C^*\text{-}\mathsf{Add}$ and $\sigma : D \to \Delta(\mathcal{A})$ a transformation. We say that σ is* ***cofibrant*** *if $\sigma_I(X) = \sigma_J(Y)$ for $I, J \in \mathcal{I}$, $X \in D(I)$, $Y \in D(J)$ implies that $I = J$ and $X = Y$.*

Definition 2.4. *Given $\mathcal{A} \in C^*\text{-}\mathsf{Add}$ and an $\mathcal{I}$-diagram $D : \mathcal{I} \to C^*\text{-}\mathsf{Add}$ we write $\mathsf{Tran}_{\mathrm{cof}}(D, \Delta(\mathcal{A}))$ for the full subcategory of $\mathsf{Tran}(D, \Delta(\mathcal{A}))$ consisting of all the cofibrant transformations from D to the constant diagram $\Delta(\mathcal{A})$.*

The following two lemmas are what allow us to pass between cofibrant transformations and more general ones.

Lemma 2.5. *Let $D : \mathcal{I} \to C^*\text{-}\mathsf{Add}$ be an $\mathcal{I}$-diagram and $\mathcal{A} \in C^*\text{-}\mathsf{Add}$. Given a transformation $\sigma : D \to \Delta(\mathcal{A})$, there is a C^*-category $CF(\mathcal{A})$ as well as an equivalence $\phi_{\mathcal{A}} : CF(\mathcal{A}) \xrightarrow{\simeq} \mathcal{A}$ and a cofibrant transformation $CF(\sigma) : D \to \Delta(CF(\mathcal{A}))$ such that $\Delta(\phi_{\mathcal{A}}) \circ CF(\sigma) = \sigma$.*

Proof. Let $\Lambda = \coprod_{I \in \mathcal{I}} \mathsf{Ob}(D(I))$ be the disjoint union of the objects sets of the $D(I)$. We define $CF(\mathcal{A})$ as the category whose objects are pairs (λ, A) with $\lambda \in \Lambda$ and $A \in \mathcal{A}$ and whose morphism spaces are given by

$$CF(\mathcal{A})((\lambda, A), (\rho, B)) := \mathcal{A}(A, B).$$

Composition of morphisms is given by composition in $\mathcal{A}$. We define the equivalence $\phi_{\mathcal{A}} : CF(\mathcal{A}) \to \mathcal{A}$ on objects by $\phi_{\mathcal{A}}(\lambda, A) := A$ and on morphisms by $\phi_{\mathcal{A}}(f) := f$.

Next, we define the transformation $CF(\sigma) : D \to \Delta(CF(\mathcal{A}))$ as follows: For each object $I \in \mathcal{I}$, we define a $*$-functor $CF(\sigma)_I : D(I) \to CF(\mathcal{A})$ on objects by $CF(\sigma)_I(X) := (X, \sigma_I(X))$ and on morphisms by $CF(\sigma)_I(f) := \sigma_I(f)$. Given a morphism $a : I \to J$ in $\mathcal{I}$ we define the coherence unitary natural transformation $CF(\sigma)_a : CF(\sigma)_I \to CF(\sigma)_J \circ D(a)$ by $(CF(\sigma)_a)_X := (\sigma_a)_X$ for $X \in D(I)$. Pseudonaturality of $CF(\sigma)$ follows from that of σ and by construction we have $\Delta(\phi_{\mathcal{A}}) \circ CF(\sigma) = \sigma$. Finally, $CF(\sigma)$ is cofibrant by construction. $\square$

The construction of $CF(\mathcal{A})$ depends only on the $\mathcal{I}$-diagram D and not on the transformation $\sigma : D \to \Delta(\mathcal{A})$. Given a $*$-functor $F : \mathcal{A} \to \mathcal{B}$ we can define a $*$-functor $CF(F) : CF(\mathcal{A}) \to CF(\mathcal{B})$ in the following way: On objects we define $CF(F)(\lambda, A) := (\lambda, F(A))$ and on morphisms we define $CF(F)(f) := F(f)$. Also, given a natural transformation $\alpha : F \to G : \mathcal{A} \to \mathcal{B}$ we can define a natural transformation $CF(\alpha) : CF(F) \to CF(G)$ by $CF(\alpha)_{(\lambda, A)} := \alpha_A$. These definitions give us a 2-functor $CF : C^*\text{-}\mathsf{Add} \to C^*\text{-}\mathsf{Add}$ (which depends on the particular $\mathcal{I}$ diagram D under consideration).

Lemma 2.6. *Let $D : \mathcal{I} \to C^*\text{-}\mathsf{Add}$ be an $\mathcal{I}$-diagram and $\mathcal{A} \in C^*\text{-}\mathsf{Add}$. There is an equivalence of categories*

$$\Delta(\phi_{\mathcal{A}}) \circ - : \mathsf{Tran}_{\mathrm{cof}}(D, \Delta(CF(\mathcal{A}))) \xrightarrow{\simeq} \mathsf{Tran}(D, \Delta(\mathcal{A}))$$

which is 2-natural in $\mathcal{A}$. (Here $\phi_{\mathcal{A}} : CF(\mathcal{A}) \to \mathcal{A}$ is the equivalence defined in Lemma 2.5.)

Proof. Given a transformation $\sigma : D \to \Delta(\mathcal{A})$ we have $\Delta(\phi_{\mathcal{A}}) \circ CF(\sigma) = \sigma$ so $\Delta(\phi_{\mathcal{A}}) \circ -$ is essentially surjective.

Now, suppose that $\rho, \omega : D \to \Delta(CF(\mathcal{A}))$ are cofibrant transformations and $\Gamma : \Delta(\phi_{\mathcal{A}}) \circ \rho \to \Delta(\phi_{\mathcal{A}}) \circ \omega$ is a modification. This means that we have natural transformations $\Gamma_I : \phi_{\mathcal{A}} \circ \rho_I \to \phi_{\mathcal{A}} \circ \omega_I$ for all $I \in \mathcal{I}$ and we can define natural transformations $\widetilde{\Gamma}_I : \rho_I \to \omega_I$ by defining $(\widetilde{\Gamma}_I)_X : \rho_I(X) \to \omega_I(X)$ to be the unique morphism in $CF(\mathcal{A})$ that maps to $(\Gamma_I)_X$ under $\phi_{\mathcal{A}}$. Naturality of $\widetilde{\Gamma}$ follows from the naturality of Γ and faithfulness of $\phi_{\mathcal{A}}$. Explicitly, given a morphism $f : X \to Y$ in $D(I)$, by the naturality of Γ_I we have a commutative diagram

$$\begin{tikzcd}
\phi_{\mathcal{A}} \circ \rho_I(X) \arrow[r, "\phi_{\mathcal{A}} \circ \rho_I(f)"] \arrow[d, "{\substack{(\Gamma_I)_X \\ = \phi_{\mathcal{A}}((\widetilde{\Gamma}_I)_X)}}"'] & \phi_{\mathcal{A}} \circ \rho_I(Y) \arrow[d, "{\substack{(\Gamma_I)_Y \\ = \phi_{\mathcal{A}}((\widetilde{\Gamma}_I)_Y)}}"] \\
\phi_{\mathcal{A}} \circ \omega_I(X) \arrow[r, "\phi_{\mathcal{A}} \circ \omega_I(f)"] & \rho_{\mathcal{A}} \circ \omega_I(Y)
\end{tikzcd}$$

and by faithfulness of $\phi_{\mathcal{A}}$ we have $(\widetilde{\Gamma}_I)_Y \circ \rho_I(f) = \omega_I(f) \circ (\widetilde{\Gamma}_I)_X$.

Similarly, because Γ is a modification and $\phi_{\mathcal{A}}$ is faithful, the $\widetilde{\Gamma}_I$ define a modification $\widetilde{\Gamma} : \rho \to \omega$. By construction, $\widetilde{\Gamma}$ is the unique modification such that $1_{\Delta(\phi_{\mathcal{A}})} * \widetilde{\Gamma} = \Gamma$ and hence $\Delta(\phi_{\mathcal{A}}) \circ -$ is fully faithful.

Finally, to say that we have 2-naturality in $\mathcal{A}$ means that for all $*$-functors $F : \mathcal{A} \to \mathcal{B}$ the following diagram commutes

$$\begin{tikzcd}
\mathsf{Tran}_{\mathrm{cof}}(D, \Delta(CF(\mathcal{A}))) \arrow[r, "\Delta(CF(F)) \circ -"] \arrow[d, "\Delta(\phi_{\mathcal{A}}) \circ -"'] & \mathsf{Tran}_{\mathrm{cof}}(D, \Delta(CF(\mathcal{B}))) \arrow[d, "\Delta(\phi_{\mathcal{B}}) \circ -"] \\
\mathsf{Tran}(D, \Delta(\mathcal{A})) \arrow[r, "\Delta(F) \circ -"] & \mathsf{Tran}(D, \Delta(\mathcal{B}))
\end{tikzcd}$$

and for all natural transformations $\alpha : F \to G : \mathcal{A} \to \mathcal{B}$ we have

$$1_{(\Delta(\phi_{\mathcal{B}})\circ -)} * (\Delta(CF(\alpha)) * -) = (\Delta(\alpha) * -) * 1_{(\Delta(\phi_{\mathcal{A}})\circ -)}.$$

This follows immediately from the definitions. □

Now we come to the main theorem of this section which asserts the existence of bicolimits of $\mathcal{I}$-diagrams in C^*-Add. The proof is based on a method one can use to show that categories of algebras for an algebraic theory are cocomplete using the adjoint functor theorem and the initial object theorem, both of which are discussed in [23]. Before we begin, we just need two more short definitions.

Definition 2.7. *By the* ***cardinality*** *of a small category $\mathcal{A}$, we mean the cardinality of its set of morphisms. We shall denote the cardinality of $\mathcal{A}$ by $|\mathcal{A}|$.*

Definition 2.8. *A* ***strong limit cardinal*** *is a cardinal κ with the property that $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$.*

Strong limit cardinals and other types of large cardinal are discussed in [18]. For our purposes, we just need the fact that there exist arbitrarily large strong limit cardinals as witnessed by examples in [18].

Theorem 2.9. *Let $\mathcal{I}$ be a small category and let $D : \mathcal{I} \to C^*$-Add be an $\mathcal{I}$-diagram. Then D has a bicolimit $\mathcal{L} \in C^*$-Add.*

Proof. By Lemma 2.6 it is sufficient to construct $\mathcal{L}$ and a cofibrant transformation $\sigma : D \to \Delta(\mathcal{L})$ such that there are equivalences

$$- \circ \sigma : C^*\text{-}\mathsf{Add}(\mathcal{L}, CF(\mathcal{A})) \xrightarrow{\simeq} \mathsf{Tran}_{\mathrm{cof}}(D, \Delta(CF(\mathcal{A})))$$

for each $\mathcal{A} \in C^*$-Add which are pseudonatural in $\mathcal{A}$. Explicitly, the diagram

$$\begin{tikzcd}
C^*\text{-}\mathsf{Add}(\mathcal{L}, CF(\mathcal{A})) \arrow[r, "-\circ\sigma"] \arrow[d, "\phi_{\mathcal{A}}\circ -"'] & \mathsf{Tran}_{\mathrm{cof}}(D, \Delta(CF(\mathcal{A}))) \arrow[d, "\Delta(\phi_{\mathcal{A}})\circ -"] \\
C^*\text{-}\mathsf{Add}(\mathcal{L}, \mathcal{A}) \arrow[r, "-\circ\sigma"] & \mathsf{Tran}(D, \Delta(\mathcal{A}))
\end{tikzcd}$$

commutes and since both vertical maps are equivalences, if the top horizontal map is an equivalence, so is the bottom horizontal map.

To that end, given $\mathcal{A} \in C^*\text{-}\mathsf{Add}$, let $C_{\mathcal{A}}$ denote the set of all cofibrant transformations from D to $\Delta(CF(\mathcal{A}))$. Each $C_{\mathcal{A}}$ is nonemtpy because there is a cofibrant transformation $\omega : D \to CF(\mathcal{A})$ with 1-cells $\omega_I : D(I) \to CF(\mathcal{A})$ which map objects $X \in D(I)$ to $(X, 0) \in CF(\mathcal{A})$ where 0 is a zero object in $\mathcal{A}$ and which map morphisms to the appropriate zero morphisms. The coherence unitary natural transformation $\omega_a : \omega_I \to \omega_J$ corresponding to a morphism $a : I \to J$ in $\mathcal{I}$ has components given by the appropriate zero morphisms.

Now, let $\alpha = |\mathcal{I}|$,

$$\beta = \left| \coprod_{I \in \mathcal{I}} \mathrm{mor}(D(I)) \right|$$

and let κ be a strong limit cardinal greater than $\alpha \cdot \beta$.

We will take κ to be greater than the cardinality of the continuum and note that this is automatic in nontrivial cases (i.e. when $\mathcal{I}$ is nonempty and any $D(I)$ contains a nonzero object). Next we fix a representative of each isoclass of additive, subobject complete C^*-categories of cardinality less than κ and denote the set of these representatives by S_κ. We then let

$$\mathcal{C} = \prod_{\mathcal{A} \in S_\kappa} \prod_{\tau \in C_{\mathcal{A}}} \mathcal{A}$$

and let $t : D \to \Delta(\mathcal{C})$ be the transformation defined by $\Delta(\pi_\tau) \circ t = \tau$ where π_τ is the projection from $\mathcal{C}$ onto the factor associated to $\tau \in C_{\mathcal{A}}$. Explicitly, for each $I \in \mathcal{I}$ we have $\pi_\tau \circ t_I = \tau_I$ and given a morphism $a : I \to J$ in $\mathcal{I}$ we have $1_{\pi_\tau} * t_a = \tau_a$. The pseudonaturality of t follows from the pseudonaturality of the τ's, and because each τ is cofibrant, so is t.

Next, let $\langle t \rangle$ be the sub-C^*-category of $\mathcal{C}$ generated by the image of t. By this we mean the following: The objects of $\langle t \rangle$ are the objects in $\mathcal{C}$ of the form $t_I(X)$ for some $I \in \mathcal{I}$ and $X \in D(I)$. The morphism space $\langle t \rangle(C, D)$ is the closed linear span of the set of morphisms in $\mathcal{C}$ of the form $f_1 \circ \cdots \circ f_n$ where $\mathrm{dom}(f_n) = C$, $\mathrm{cod}(f_1) = D$ and each f_k is one of the following:

1. $t_I(f)$ for some $I \in \mathcal{I}$ and morphism $f \in D(I)$,

2. $(t_a)_X$ for some morphism $a : I \to J$ in $\mathcal{I}$ and object $X \in D(I)$,

3. an adjoint of one of the above types of morphism.

The transformation $t : D \to \Delta(\mathcal{C})$ corestricts to a cofibrant transformation $t|^{\Delta(\langle t \rangle)} : D \to \Delta(\langle t \rangle)$. We then define $\mathcal{L} := \mathrm{Split}(\langle t \rangle^{\oplus})$, the completion of $\langle t \rangle$ under direct sums and subobjects. The cardinality of $\mathcal{L}$ is less than κ. We shall return to prove this in a separate lemma (see 2.16 below) and take it as a fact for the time being.

The transformation $t|^{\Delta(\langle t \rangle)} : D \to \Delta(\langle t \rangle)$ induces a cofibrant transformation $\sigma : D \to \Delta(\mathcal{L})$ by postcomposition with the inclusion functor $\iota : \langle t \rangle \to \mathcal{L}$ and we claim that $(\mathcal{L}, \sigma)$ is a bicolimit of D. Let us first show that there is an equivalence of C^*-categories

$$- \circ \sigma : C^*\text{-}\mathsf{Add}(\mathcal{L}, CF(\mathcal{A})) \xrightarrow{\simeq} \mathsf{Tran}_{\mathrm{cof}}(D, \Delta(CF(\mathcal{A}))) \qquad (*)$$

for each $\mathcal{A} \in C^*\text{-}\mathsf{Add}$ of cardinality less than κ. It is sufficient to do this for all $\mathcal{A} \in S_\kappa$.

Now, if $\tau : D \to \Delta(CF(\mathcal{A}))$ is a cofibrant pseudonatural transformation then by the construction of $\mathcal{C}$, the projection $\pi_\tau : \mathcal{C} \to \mathcal{A}$ onto the factor associated to τ induces a $*$-functor $\Pi_\tau : \mathcal{L} \to \mathcal{A}$ such that $\Delta(\Pi_\tau) \circ \sigma = \tau$. Therefore, $(*)$ is essentially surjective.

Next, let $F, G : \mathcal{L} \to \mathcal{A}$ be $*$-functors and let $\phi : F \to G$ be a natural transformation such that $\Delta(\phi) * 1_\sigma = 0$. Then $\phi_{\sigma_I(X)} : F(\sigma_I(X)) \to G(\sigma_I(X))$ is 0 for all $I \in \mathcal{I}$ and $X \in D(I)$. Since every object in $\mathcal{L}$ is a subobject of a direct sum of such objects, it follows that $\phi_L = 0$ for all $L \in \mathcal{L}$ and hence $\phi = 0$. Therefore, $(*)$ is faithful.

To show that $(*)$ is full, let $\Gamma : \Delta(F) \circ \sigma \to \Delta(G) \circ \sigma$ be a modification. Given $I \in \mathcal{I}$ and $X \in D(I)$, we have a map $(\Gamma_I)_X : F(\sigma_I(X)) \to G(\sigma_I(X))$. We claim that we can define a natural transformation $\eta : F \to G$ by defining $\eta_{\sigma_I(X)} := (\Gamma_I)_X$ and extending this definition to direct sums and their subobjects. Because σ is cofibrant, η is well-defined. We shall leave the proof that η is natural to a separate lemma to follow shortly. Then by construction we have $\eta * 1_\sigma = \Gamma$ and hence $(*)$ is full.

It follows that $(\mathcal{L}, \sigma)$ is a bicolimit of D in the 2-category $C^*\text{-}\mathsf{Add}_{<\kappa}$, the sub-2-category of $C^*\text{-}\mathsf{Add}$ consisting of all additive C^*-categories of cardinality less than κ. (Here we are viewing D as an $\mathcal{I}$-diagram in $C^*\text{-}\mathsf{Add}_{<\kappa}$ by restriction of codomain.)

Now, suppose that $\lambda > \kappa$ is a strong limit cardinal. By the same argument as above, D viewed as an $\mathcal{I}$-diagram in $C^*\text{-}\mathsf{Add}_{<\lambda}$ has a bicolimit $(\mathcal{L}', \sigma')$. By the same argument as for $\mathcal{L}$ (to follow the proof of this theorem), $\mathrm{card}(\mathcal{L}') < \kappa$ and hence $(\mathcal{L}', \sigma')$ is a bicolimit in $C^*\text{-}\mathsf{Add}_{<\kappa}$. Therefore $\mathcal{L} \simeq \mathcal{L}'$ and hence $(\mathcal{L}, \sigma)$ is a bicolimit of D in $C^*\text{-}\mathsf{Add}_{<\lambda}$. Since λ was arbitrary, it follows that there is an equivalence of C^*-categories

$$- \circ \sigma : C^*\text{-}\mathsf{Add}(\mathcal{L}, \mathcal{A}) \xrightarrow{\simeq} \mathsf{Tran}(D, \Delta(\mathcal{A}))$$

for each $\mathcal{A} \in C^*\text{-}\mathsf{Add}$ and that $(\mathcal{L}, \sigma)$ is a bicolimit of D in $C^*\text{-}\mathsf{Add}$.

Finally, with regard to pseudonaturality, given a $*$-functor $F : \mathcal{A} \to \mathcal{B}$, the commutativity of the diagram

$$\begin{tikzcd}
C^*\text{-}\mathsf{Add}(\mathcal{L}, \mathcal{A}) \arrow[r, "F\circ -"] \arrow[d, "-\circ\sigma"] & C^*\text{-}\mathsf{Add}(\mathcal{L}, \mathcal{B}) \arrow[d, "-\circ\sigma"] \\
\mathsf{Tran}(D, \Delta(\mathcal{A})) \arrow[r, "F\circ -"] & \mathsf{Tran}(D, \Delta(\mathcal{B}))
\end{tikzcd}$$

follows from the associativity of composition of functors and horizontal composition of natural transformations. Similarly, if $\eta : F \to G : \mathcal{A} \to \mathcal{B}$ is a natural transformation, the equality of the natural transformations

$$1_{-\circ\sigma} * (\eta * -) : (- \circ \sigma) \circ (F \circ -) \to (- \circ \sigma) \circ (G \circ -)$$

and

$$(\eta * -) * 1_{-\circ\sigma} : (F \circ -) \circ (- \circ \sigma) \to (G \circ -) \circ (- \circ \sigma)$$

follows from the associativity of horizontal composition of natural transformations. Therefore, the $- \circ \sigma$ are 2-natural in $\mathcal{A}$. □

To complete our proof we just need to resolve the issue of the cardinality of the bicolimit $\mathcal{L}$ and the naturality of the maps η_L in the fullness section of the proof. First let us deal with the cardinality of $\mathcal{L}$. We shall make use of the following facts about cardinal arithmetic whose proofs can be found in [18].

Lemma 2.10. *Let α, β be infinite cardinals. Then $\alpha \cdot \beta = \max\{\alpha, \beta\}$.*

Corollary 2.11. *Let α be an infinite cardinal and $n \in \mathbb{N} \setminus \{0\}$. Then $\alpha^n = \alpha$.*

Lemma 2.12. *Let X be an infinite set, then $|X^{\mathbb{N}}| \leq 2^{|X|}$.*

Lemma 2.13. *Let $(X_i)_{i\in I}$ be a (set indexed) collection of sets. If κ is an infinite cardinal such that $|I| \leq \kappa$ and $|X_i| \leq \kappa$ for all $i \in I$, then*

$$\left| \bigcup_{i\in I} X_i \right| \leq \kappa.$$

Now, let us prove some results about the cardinalities of additive and subobject completions of C^*-categories.

Lemma 2.14. *Let $\mathcal{A}$ be a C^*-category with infinite cardinality, then*

$$|\mathcal{A}^{\oplus}| = |\mathcal{A}|.$$

Proof. Given objects $A = (A_j)_{j=1}^n$ and $B = (B_i)_{i=1}^m$, we can identify $\mathcal{A}^{\oplus}(A, B)$ with $m \times n$ matrices (f_{ij}) where each f_{ij} is a morphism in $\mathcal{A}$ from A_j to B_i. Let us denote the set of $m \times n$ matrices with entries in $\mathrm{mor}(\mathcal{A})$ by $\mathrm{Mat}_{m\times n}(\mathcal{A})$. Then we can identify $\mathrm{mor}(\mathcal{A}^{\oplus})$ with a subset of

$$S = \bigsqcup_{(m,n)\in\mathbb{N}\times\mathbb{N}} \mathrm{Mat}_{m\times n}(\mathcal{A}).$$

Since $|\mathrm{Mat}_{m\times n}(\mathcal{A})| = |\mathcal{A}|^{m\times n} = |\mathcal{A}|$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$, we have $|S| = |\mathcal{A}|$ and hence $|\mathcal{A}^{\oplus}| = |\mathcal{A}|$. $\square$

Lemma 2.15. *Let $\mathcal{A}$ be a C^*-category with infinite cardinality, then*

$$|\mathrm{Split}(\mathcal{A})| = |\mathcal{A}|.$$

Proof. There is an injection

$$\varphi : \mathrm{mor}(\mathrm{Split}(\mathcal{A})) \to \mathrm{mor}(\mathcal{A})^3 = \mathrm{mor}(\mathcal{A}) \times \mathrm{mor}(\mathcal{A}) \times \mathrm{mor}(\mathcal{A})$$

given by mapping a morphism $f : (A, p) \to (B, q)$ in $\mathrm{Split}(\mathcal{A})$ to (p, f, q) in $\mathrm{mor}(\mathcal{A})^3$. Since $|\mathrm{mor}(\mathcal{A})^3| = |\mathcal{A}|^3 = |\mathcal{A}|$ we have $|\mathrm{Split}(\mathcal{A})| = |\mathcal{A}|$. $\square$

Now we have all that we need to complete the cardinality section of the proof of Theorem 2.9.

Lemma 2.16. *The bicolimit $\mathcal{L} = \mathrm{Split}(\langle t\rangle^{\oplus})$ in Theorem 2.9 has cardinality less than κ.*

Proof. Since $\mathcal{L} := \mathrm{Split}(\langle t\rangle^{\oplus})$, where $\langle t\rangle$ is defined as in Theorem 2.9, by Lemmas 2.14 and 2.15 it suffices to show that $|\langle t\rangle| < \kappa$.

We recall that the objects of $\langle t\rangle$ are the objects in $\mathcal{C}$ (as defined in Theorem 2.9) of the form $t_I(X)$ for some $I \in I$ and $X \in D(I)$. The morphism space $\langle t\rangle(C, D)$ is the closed linear span of the set of morphisms in $\mathcal{V}$ of the form $f_1 \circ \cdots \circ f_n$ where $\mathrm{dom}(f_n) = C$, $\mathrm{cod}(f_1) = D$ and each f_k is one of the following:

1. $t_I(f)$ for some $I \in \mathcal{I}$ and morphism $f \in D(I)$,
2. $(t_a)_X$ for some morphism $a : I \to J$ in $\mathcal{I}$ and object $X \in D(I)$,
3. an adjoint of one of the above types of morphism.

We also recall that $\alpha = |\mathcal{I}|$,

$$\beta = \left| \coprod_{I \in \mathcal{I}} \mathrm{mor}(D(I)) \right|$$

and κ is a strong limit cardinal greater than $\alpha \cdot \beta$ and $\mathfrak{c}$, the cardinality of the continuum. For the sake of simplicity, we shall just consider the case $\alpha \cdot \beta \geq \mathfrak{c}$ (this is the situation in all nontrivial cases, i.e. when $\mathcal{I}$ is nonempty and at least one $D(I)$ contains a nonzero object). Nothing essentially changes in the case $\alpha \cdot \beta < \mathfrak{c}$ except that the results we are using only apply to infinite cardinals so one has to make minor modifications to several of the statements when finite cardinals are involved. Alternatively, one can argue that since we make the assumption that $\kappa > \mathfrak{c}$ regardless of the value of $\alpha \cdot \beta$, showing that $|\mathcal{L}| < \kappa$ when $\alpha \cdot \beta = \mathfrak{c}$ also covers the case $\alpha \cdot \beta < \mathfrak{c}$ since taking a smaller 'generating set' for the category $\langle t \rangle$ cannot increase its cardinality.

Now, let S be the set of morphisms in $\mathcal{C}$ of the form 1, 2 or 3 listed above. The cardinality of the set of morphisms of the form $t_I(f)$ for some $I \in \mathcal{I}$ and morphism $f \in D(I)$ is less than or equal to $\alpha \cdot \beta$ because they can be identified with a subset of

$$T = \mathrm{mor}(\mathcal{I}) \times \left(\coprod_{I \in \mathcal{I}} \mathrm{mor}(D(I)) \right)$$

by identifying $t_I(f)$ with $(1_I, f)$. Similarly, we can identify morphisms of the form $(t_a)_X$ for some morphism $a : I \to J$ in $\mathcal{I}$ and object $X \in D(I)$ with a subset of T by identifying $(t_a)_X$ with $(a, 1_X)$ so the cardinality of the set of morphisms of this form is also less than or equal to $\alpha \cdot \beta$. Therefore, $|S| \leq 4\alpha \cdot \beta = \alpha \cdot \beta$.

Next, the set of finite strings of elements of S can be identified with

$$U = \coprod_{n \in \mathbb{N}} S^n.$$

Now, $|S^n| = |S|^n = |S| = \alpha \cdot \beta$ and hence $|U| = \alpha \cdot \beta$. In general, not every element of U will correspond to a string of composable morphisms in $\mathcal{C}$ but this does not matter since we just want to compute an upper bound on the cardinality of $\langle t \rangle$.

Given $C, D \in \mathcal{C}$, let $U_{C,D}$ denote the subset of U corresponding to morphisms in $\mathcal{C}(C, D)$ (that is the elements in U corresponding to composable strings of morphisms, the composite of which has domain C and codomain D). The linear span of $U_{C,D}$ injects (noncanonically) into the set $V_{C,D}$ of formal linear combinations

$$\sum_{i=1}^{n} c_i f_i$$

where $n \in \mathbb{N}$, $c_i \in \mathbb{C}$ and $f_i \in U_{C,D}$. We can identify $V_{C,D}$ with the set

$$\coprod_{n\in\mathbb{N}} (\mathbb{C} \times U_{C,D})^n$$

and since $|\mathbb{C} \times U_{C,D}| \leq |\mathbb{C} \times U| = \mathfrak{c} \cdot \alpha \cdot \beta = \alpha \cdot \beta$ we have $|V_{C,D}| \leq \alpha \cdot \beta$. Now, $\langle t\rangle(C, D)$ is the closed linear span of $U_{C,D}$ in $\mathcal{C}(C, D)$ which is first countable and hence if $f \in \langle t\rangle(C, D)$, there is a sequence in $U_{C,D}$ which converges to f. It follows that $|\langle t\rangle(C, D))|$ is less than or equal to the cardinality of the set of sequences in $U_{C,D}$ which is equal to $|(U_{C,D})^{\mathbb{N}}| \leq 2^{\alpha\cdot\beta}$.

Now,

$$\mathrm{mor}(\langle t\rangle) = \coprod_{(C,D)\in \mathrm{ob}(\langle t\rangle)\times \mathrm{ob}(\langle t\rangle)} \langle t\rangle(C, D).$$

Since $|\mathrm{ob}(\langle t\rangle)| \leq \alpha\cdot\beta$ and hence $|\mathrm{ob}(\langle t\rangle)\times\mathrm{ob}(\langle t\rangle)| \leq (\alpha\cdot\beta)^2 = \alpha\cdot\beta < 2^{\alpha\cdot\beta}$ it follows that $|\langle t\rangle| \leq 2^{\alpha\cdot\beta} < \kappa$. $\square$

The final issue we need to resolve in the proof of Theorem 2.9 is the naturality of the maps η_L defined in the fullness section of the proof.

Lemma 2.17. *Given a modification $\Gamma : \Delta(F) \circ \sigma \to \Delta(G) \circ \sigma$ as in Theorem 2.9, the collection of maps $\eta_{\sigma_I(X)} := (\Gamma_I)_X$ define a natural transformation $\eta : F \to G$.*

Proof. As noted in the proof of Theorem 2.9, each $\eta_{\sigma_I(X)}$ is well-defined because σ is cofibrant. So far we have defined the components of η at the objects of $\langle t\rangle$ (as defined in Theorem 2.9). If we can show naturality with respect to morphisms between such objects then we can extend this definition to more general objects in $\mathcal{L} = \mathrm{Split}(\langle t\rangle^{\oplus})$ to obtain the required natural transformation $\eta : F \to G$. Therefore, we need to show that given a morphism $g : \sigma_I(X) \to \sigma_J(Y)$ in $\mathcal{L}$ that the following diagram commutes:

$$\begin{tikzcd}
F(\sigma_I(X)) \arrow[r, "F(g)"] \arrow[d, "\eta_{\sigma_I(X)}"'] & F(\sigma_J(X)) \arrow[d, "\eta_{\sigma_J(Y)}"] \\
G(\sigma_I(Y)) \arrow[r, "G(g)"] & G(\sigma_J(Y))
\end{tikzcd}$$

The modification axiom for Γ says that given a morphism $a : I \to J$ in $\mathcal{I}$, the following diagram commutes:

$$\begin{tikzcd}
F \circ \sigma_I \arrow[r, "1_F * \sigma_a"] \arrow[d, "\Gamma_I"'] & F \circ \sigma_J \circ D(a) \arrow[d, "\Gamma_J"] \\
G \circ \sigma_I \arrow[r, "1_G * \sigma_a"] & G \circ \sigma_J \circ D(a)
\end{tikzcd}$$

If $I = J$ and $g = \sigma_I(h)$ for some $h : X \to Y$ in $D(I)$ then since Γ_I is a natural transformation we have a commutative diagram

$$\begin{tikzcd}
F \circ \sigma_I(X) \arrow[r, "{F\circ\sigma_I(h) \atop = F(g)}"] \arrow[d, "{(\Gamma_I)_X \atop = \eta_{\sigma_I(X)}}"'] & F \circ \sigma_I(Y) \arrow[d, "{(\Gamma_i)_Y \atop = \eta_{\sigma_I(Y)}}"] \\
G \circ \sigma_I(X) \arrow[r, "{G\circ\sigma_I(h) \atop = G(g)}"] & G \circ \sigma_I(Y)
\end{tikzcd}$$

Now, we note that $g^* = (\sigma_I(h))^* = \sigma_I(h^*)$ and so the same argument shows that we have naturality with respect to adjionts of maps of this form.

Next, suppose that $g = (\sigma_a)_X : \sigma_I(X) \to \sigma_J(D(a)(X))$ for some morphism $a : I \to J$ in $\mathcal{I}$ and object $X \in D(I)$. Then by the modification axiom we have a commutative diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
F \circ \sigma_I(X) \arrow[r, "{\substack{(1_F * \sigma_a)_X \\ = F((\sigma_a)_X) \\ = F(g)}}"] \arrow[d, "{\substack{(\Gamma_I)_X \\ = \eta_{\sigma_I(X)}}}"'] & F \circ \sigma_J \circ D(a)(X) \arrow[d, "{\substack{(\Gamma_J)_{D(a)(X)} \\ = \eta_{\sigma_J(D(a)(X))}}}"] \\
G \circ \sigma_I(X) \arrow[r, "{\substack{(1_G * \sigma_a)_X \\ = G((\sigma_a)_X) \\ = G(g)}}"] & G \circ \sigma_J \circ D(a)(X)
\end{tikzcd}$$

Therefore, we have naturality with respect to maps of the form $g = (\sigma_a)_X$. Since σ_a is a unitary natural transformation we have $g^* = (\sigma_a)_X^* = (\sigma_a)_X^{-1}$ and hence it follows from the commutativity of the above diagram that

$$\eta_{\sigma_I(X)} \circ F(g^*) = G(g^*) \circ \eta_{\sigma_J(D(a)(X))}$$

and so we have naturality with respect to adjoints of maps of this form.

Next, naturality with respect to composites of maps of the form $\sigma_I(h)$, $(\sigma_a)_X$ and their adjoints follows from naturality with respect to each of the factors. Naturality with respect to maps in the linear span of these composites follows from the linearity of F and G. Then, naturality with respect to morphisms in the closure of these linear spans follows from the continuity of composition of morphisms and the continuity of F and G on morphism spaces.

Lastly, we need to extend our definition of η to all of $\mathcal{L} = \mathrm{Split}(\langle t \rangle^{\oplus})$. Given an object $X = (\sigma_{I_k}(X_k))_{k=1}^n \in \langle t \rangle^{\oplus}$ we define η_X as the diagonal matrix with the $\eta_{\sigma_{I_k}(X_k)}$ on the diagonal. Naturality of these maps follows from the naturality of the $\eta_{\sigma_{I_k}(X_k)}$'s. Then, given $(X, p) \in \mathcal{L}$ where $X \in \langle t \rangle^{\oplus}$ and $p : X \to X$ is a projection we define $\eta_{(X,p)}$ as the composite

$$F(X, p) \xrightarrow{F(p)} F(X, 1_X) \xrightarrow{\eta_X} G(X, 1_X) \xrightarrow{G(p)} G(X, p).$$

The naturality of these maps follows from the naturality of the η_X's for $X \in \langle t \rangle^{\oplus}$ and the definition of morphisms in the subobject completion. □

This completes the proof of Theorem 2.9. As an example, we can explicitly describe bicoproducts in C^*-Add.

Definition 2.18. *Let $(\mathcal{A}_i)_{i \in I}$ be a collection of additive, subobject complete C^*-categories. Their **direct sum**, denoted $\bigoplus_{i \in I} \mathcal{A}_i$ is the full subcategory of $\prod_{i \in I} \mathcal{A}_i$ whose objects are collections $(A_i)_{i \in I}$ with each $A_i \in \mathcal{A}_i$ and all but finitely many of the A_i are zero objects.*

There are inclusion $*$-functors $\iota_j : \mathcal{A}_j \to \bigoplus_{i\in I} \mathcal{A}_i$ for all $j \in I$ defined in the natural way (one has to make a choice of zero object in $\mathcal{A}_i$ for each $i \neq j$). Given $A_j \in \mathcal{A}_j$, as a slight abuse of notation we shall sometimes write the object $\iota_j(A_j) \in \bigoplus_{i\in I} \mathcal{A}_i$ simply as A_j when no confusion can occur.

Lemma 2.19. *For each $\mathcal{C} \in C^*\text{-}\mathsf{Add}$, postcomposition by the inclusion $*$-functors $\iota_j : \mathcal{A}_j \to \bigoplus_{i\in I} \mathcal{A}_i$ induces an equivalence of C^*-categories*

$$\varphi : C^*\text{-}\mathsf{Add}\left(\bigoplus_{i\in I} \mathcal{A}_i, \mathcal{C}\right) \xrightarrow{\simeq} \prod_{i\in I} C^*\text{-}\mathsf{Add}(\mathcal{A}_i, \mathcal{C}).$$

These equivalences are 2-natural in $\mathcal{C}$.

Proof. We define φ on $*$-functors by $\varphi(F) := (F \circ \iota_i)_{i\in I}$ and on natural transformations by $\varphi(\eta) := (\eta * 1_{\iota_i})_{i\in I}$.

In order to show that φ is essentially surjective, let $(F_i : \mathcal{A}_i \to \mathcal{C})_{i\in I}$ be a collection of $*$-functors. We define a $*$-functor $F : \bigoplus_{i\in I} \mathcal{A}_i \to \mathcal{C}$ on objects by $F((A_i)_{i\in I}) := \bigoplus_{i\in I} F(A_i)$ and similarly on morphisms. Then $F \circ \iota_i \cong F_i$ for all $i \in I$ and hence φ is essentially surjective.

To show that φ is faithful, let $\eta : F \to G : \bigoplus_{i\in I} \mathcal{A}_i \to \mathcal{C}$ be a natural transformation such that $\varphi(\eta) = 0$. Then $\eta_{\iota_i(A_i)} = 0$ for all $i \in I$ and $A_i \in \mathcal{A}_i$. It follows from the fact that any $(A_i)_{i\in I} \in \bigoplus_{i\in I} \mathcal{A}_i$ is a direct sum of the nonzero $\iota_i(A_i)$'s and Lemma (1.24) that $\eta = 0$ and hence φ is faithful.

Next, to show that φ is full, let $F, G : \bigoplus_{i\in I} \mathcal{A}_i \to \mathcal{C}$ be $*$-functors and let $(\eta_i : F \circ \iota_i \to G \circ \iota_i)_{i\in I}$ be a uniformly bounded collection of natural transformations. Because F and G are $*$-functors and each $(A_i)_{i\in I} \in \bigoplus_{i\in I} \mathcal{A}_i$ is a direct sum of the nonzero $\iota_i(A_i)$'s, $F((A_i)_{i\in I})$ and $G((A_i)_{i\in I})$ are direct sums of the nonzero $F(\iota_i(A_i))$'s and $G(\iota_i(A_i))$'s respectively. Therefore, we can define a natural transformation $\eta : F \to G$ by defining $\eta_{(A_i)_{i\in I}}$ as the diagonal matrix with the appropriate $\eta_{\iota_i(A_i)}$'s on the diagonal. Naturality of η follows from the naturality of the η_i's and by construction we have $\eta * 1_{\iota_i} = \eta_i$ for all $i \in I$. Therefore, φ is full.

Finally, 2-naturality in $\mathcal{C}$ is immediate from the definitions. $\square$

Corollary 2.20. *Let $(\mathcal{A}_i)_{i\in I}$ be a collection of additive, subobject complete C^*-categories. Then $\bigoplus_{i\in I} \mathcal{A}_i$ is the bicoproduct of the $\mathcal{A}_i$'s.*

Proof. Viewing I as a discrete category (i.e. with the $i \in I$ as objects and only the identity morphisms) then there is an I-diagram $D : I \to C^*\text{-}\mathsf{Add}$ defined by $D(i) := \mathcal{A}_i$. Then, for each $\mathcal{C} \in C^*\text{-}\mathsf{Add}$ there is an isomorphism of C^*-categories

$$\prod_{i\in I} C^*\text{-}\mathsf{Add}(\mathcal{A}_i, \mathcal{C}) \cong \mathsf{Tran}(D, \Delta(\mathcal{C})).$$

(This is immediate from the definition of a transformation.) The inclusion $*$-functors $\iota_j : \mathcal{A}_j \to \bigoplus_{i\in I} \mathcal{A}_i$ define a transformation $\iota : D \to \Delta(\bigoplus_{i\in I} \mathcal{A}_i)$ and by Lemma 2.19 there are equivalences

$$- \circ \iota : C^*\text{-}\mathsf{Add}\left(\bigoplus_{i\in I} \mathcal{A}_i, \mathcal{C}\right) \xrightarrow{\simeq} \mathsf{Tran}(D, \Delta(\mathcal{C}))$$

which are 2-natural in $\mathcal{C}$. Therefore, $\bigoplus_{i\in I} \mathcal{A}_i$ is the bicoproduct of the $\mathcal{A}_i$'s. □

2.2 Module categories and balanced tensor products

Module categories are the categorical analogue of modules. As with most categorical analogues of algebraic situations, different variations are possible. In general one has a categorical analogue of rings/algebras, in our case C^*-tensor categories and then one defines actions of these 'categorified rings' on categories. As with modules, given such a categorified ring $\mathcal{A}$, a right $\mathcal{A}$-module category $\mathcal{M}$ and a left $\mathcal{A}$-module category $\mathcal{N}$, one can ask whether a balanced tensor product $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ exists which has properties mirroring that of the balanced tensor product modules over a ring. Variations on this theme have been considered by several authors, for example [12], [14] and [9] discuss some situations in which balanced tensor products exist. As an application of the existence of bicolimits in C^*-Add, we shall consider another variation on this theme.

Before we discuss module categories, let us first recall how the balanced tensor product of two modules can be constructed as a colimit. If A is a ring, M is a right A-module with action map $\rho : M \times A \to M$ and N is a left A-module with action map $\lambda : A \times N \to N$ then the action maps correspond to group homomorphisms $\overline{\rho} : M \otimes A \to M$ and $\overline{\lambda} : A \otimes N \to N$ where $\otimes$ denotes the tensor product of abelian groups. The balanced tensor product $M \otimes_A N$ can then be constructed as the equalizer of the action maps, in other words, as the colimit of the diagram

$$M \otimes A \otimes N \underset{\overline{\lambda}}{\overset{\overline{\rho}}{\rightrightarrows}} M \otimes N$$

in the category of abelian groups.

Our categorical replacements for rings, modules and the tensor product of abelian groups will be C^*-tensor categories, module categories and the maximal tensor product of C^*-categories respectively. The balanced tensor product of module categories can then be constructed as a suitable bicolimit in C^*-Add. First, let us review the maximal tensor product of C^*-categories and its relevant properties.

Definition 2.21. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be $*$-categories. A **bilinear $*$-functor** from $\mathcal{A} \times \mathcal{B}$ to $\mathcal{C}$ is a functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ such that*

$$F(f^*) = (F(f))^*$$

for all morphisms f in $\mathcal{A} \times \mathcal{B}$ and the maps

$$F : \mathcal{A}(A, A') \times \mathcal{B}(B, B') \to \mathcal{C}(F(A, B), F(A', B'))$$

are all bilinear.

If $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are all C^*-categories then we impose the usual requirement that natural transformations $\alpha : F \to G : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ between bilinear $*$-functors be uniformly bounded. In this situation, similarly to the case of $*$-functors between C^*-categories, there is a natural C^*-category structure on the category of blinear $*$-functors from $\mathcal{A} \times \mathcal{B}$ to $\mathcal{C}$ and their natural transformations.

Definition 2.22. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be C^*-categories. We denote the C^*-category of bilinear $*$-functors $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ and their natural transformations by $C^*\text{-}\mathsf{Bilin}(\mathcal{A}, \mathcal{B}; \mathcal{C})$.*

Similarly to the case of C^*-algebras, the maximal tensor product of C^*-categories is defined as a completion of an algebraic tensor product.

Definition 2.23. *Let $\mathcal{A}, \mathcal{B}$ be $*$-categories. Their **algebraic tensor product** $\mathcal{A} \otimes \mathcal{B}$ is the $*$-category defined as follows: The objects are pairs (A, B) with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we shall denote objects by $A \otimes B$ rather than (A, B). The morphism spaces are defined by*

$$\mathcal{A} \otimes \mathcal{B}((A \otimes B), (A' \otimes B')) := \mathcal{A}(A, A') \otimes \mathcal{B}(B, B')$$

where on the right hand side, the tensor product is the vector space tensor product. Composition of morphisms is defined by

$$(f \otimes g) \circ (f' \otimes g') := (f \circ f') \otimes (g \circ g')$$

and the involution is defined by

$$(f \otimes g)^* := f^* \otimes g^*.$$

With these definitions, we have an obvious bilinear $*$-functor

$$\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}.$$

Definition 2.24. *Let $\mathcal{A}, \mathcal{B}$ be C^*-categories. Their* ***maximal tensor product*** *$\mathcal{A} \otimes_{\max} \mathcal{B}$ is the C^*-category whose objects are the same as $\mathcal{A} \otimes \mathcal{B}$ and whose morphism spaces are the completions of the morphism spaces in $\mathcal{A} \otimes \mathcal{B}$ with respect to the norm*

$$||f|| := \sup_{F:\mathcal{A}\otimes\mathcal{B}\to\mathsf{Hilb}} ||F(f)||.$$

Here the supremum is taken over all $*$-functors $F : \mathcal{A} \otimes \mathcal{B} \to \mathsf{Hilb}$.

Similarly to the case of C^*-algebras, there is also a 'minimal tensor product', also called the 'spatial tensor product' of C^*-categories. We shall not use this construction but the details can be found in [25] or [10].

One can also define n-fold tensor products in a completely analogous way. As with the algebraic tensor product, there is a bilinear $*$-functor

$$\otimes_{\max} : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes_{\max} \mathcal{B}.$$

The category of small C^*-categories becomes a closed symmetric monoidal category when equipped with the maximal tensor product [10]. We won't need all of this structure but we will need the maximal tensor product of $*$-functors. Given C^*-categories $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}'$ and $*$-functors $F : \mathcal{A} \to \mathcal{B}$, $F' : \mathcal{A}' \to \mathcal{B}'$, the $*$-functor

$$F \otimes_{\max} F' : \mathcal{A} \otimes_{\max} \mathcal{A}' \to \mathcal{B} \otimes_{\max} \mathcal{B}'$$

is defined on objects by

$$(F \otimes_{\max} F')(A \otimes A') := F(A) \otimes F'(A').$$

Given a morphism of the form $f \otimes f' : A \otimes A' \to B \otimes B'$ in $\mathcal{A} \otimes_{\max} \mathcal{A}'$, we define

$$(F \otimes_{\max} F')(f \otimes f') := F(f) \otimes F'(f').$$

We can then extend this definition to the whole of the maximal tensor product, first to the algebraic tensor products of the morphism spaces by linearity and then these maps have unique continuous extensions to the morphism spaces in the maximal tensor product. This follows from the fact that the maps on the morphism spaces in the algebraic tensor product are bounded with respect to the norms used to define the maximal tensor product. This is because if $\rho : \mathcal{B} \otimes_{\max} \mathcal{B}' \to \mathsf{Hilb}$ is an embedding, then $\rho \circ (F \otimes_{\max} F') : \mathcal{A} \otimes \mathcal{A}' \to \mathsf{Hilb}$ is a $*$-functor and given a morphism $g : A \otimes A' \to B \otimes B'$ in $\mathcal{A} \otimes \mathcal{A}'$ we have

$$||g|| = \sup_{G:\mathcal{A}\otimes\mathcal{A}'\to\mathsf{Hilb}} ||G(f)|| \geq ||\rho \circ (F \otimes_{\max} F')(g)|| = ||(F \otimes_{\max} F')(g)||.$$

We shall also need the following universal property of the maximal tensor product.

Lemma 2.25. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be C^*-categories. The bilinear $*$-functor*

$$\otimes_{\max} : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes_{\max} \mathcal{B}$$

induces an isomorphism of C^*-categories

$$- \circ \otimes_{\max} : C^*\text{-}\mathsf{Cat}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{C}) \xrightarrow{\cong} C^*\text{-}\mathsf{Bilin}(\mathcal{A}, \mathcal{B}; \mathcal{C}) \tag{$*$}$$

which is 2-natural in $\mathcal{C}$.

Proof. To see that $(*)$ is bijective on objects, let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bilinear $*$-functor. First we define a $*$-functor $\widetilde{F} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ as follows: On objects we define $\widetilde{F}(A \otimes B) := F(A, B)$. For morphisms, we first define $\widetilde{F}$ on the algebraic tensor product of the morphism spaces using the universal property of the tensor product. Explicitly, given a morphism of the form $f \otimes g$ we define $\widetilde{F}(f \otimes g) := F(f, g)$ and then we extend this to the algebraic tensor product by linearity.

To show that $\widetilde{F}$ has a unique extension to the maximal tensor product, we need to show that maps on morphism spaces are bounded with respect to the norm defining the maximal tensor product. To that end, let $\rho : \mathcal{C} \to \mathsf{Hilb}$ be an embedding. Then $\rho \circ \widetilde{F} : \mathcal{A} \otimes \mathcal{B} \to \mathsf{Hilb}$ is a $*$-functor and hence given a morphism $f \in \mathcal{A} \otimes \mathcal{B}$ we have

$$||f|| = \sup_{G : \mathcal{A} \otimes \mathcal{B} \to \mathsf{Hilb}} ||G(f)|| \geq ||\rho \circ \widetilde{F}(f)|| = ||\widetilde{F}(f)||.$$

Therefore, $\widetilde{F}$ has a unique extension $\overline{F} : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{C}$ and by construction $\overline{F}$ is the unique $*$-functor such that $F = \overline{F} \circ \otimes_{\max}$. Therefore $(*)$ is bijective on objects.

Now, given a natural transformation $\eta : F \circ \otimes_{\max} \to G \circ \otimes_{\max}$, since $\otimes_{\max}$ is bijective on objects, we can define a natural transformation $\tilde{\eta} : F \to G$ by

$$\tilde{\eta}_{A \otimes B} := \eta_{(A,B)}.$$

To see that $\tilde{\eta}$ is natural, let $f : A \otimes B \to A' \otimes B'$ be a morphism in $\mathcal{A} \otimes_{\max} \mathcal{B}$. We need to show that the following diagram commutes

$$\begin{tikzcd}
F(A \otimes B) \arrow[r, "Ff"] \arrow[d, "\tilde{\eta}_{A \otimes B}"'] & F(A' \otimes B') \arrow[d, "\tilde{\eta}_{A' \otimes B'}"] \\
G(A \otimes B) \arrow[r, "Gf"] & G(A' \otimes B')
\end{tikzcd}$$

If f is an elementary tensor in the algebraic tensor product this follows from naturality of η. If f is a linear combination of elementary tensors this follows from bilinearity of composition of morphisms and naturality with respect to elementary tensors. Given a more general morphism f, there is a sequence of morphisms $(f_n)_{n\in\mathbb{N}}$ in the algebraic tensor product which converges to f. Then

$$\begin{aligned}
\tilde{\eta}_{A'\otimes B'} \circ Ff &= \tilde{\eta}_{A'\otimes B'} \circ F(\lim_{n\to\infty} f_n) \\
&= \tilde{\eta}_{A'\otimes B'} \circ \lim_{n\to\infty} Ff_n \\
&= \lim_{n\to\infty} (\eta_{A'\otimes B'} \circ Ff_n) \\
&= \lim_{n\to\infty} (Gf_n \circ \eta_{A\otimes B}) \\
&= \left(\lim_{n\to\infty} Gf_n\right) \circ \eta_{A\otimes B} \\
&= G(\lim_{n\to\infty} f_n) \circ \eta_{A\otimes B} \\
&= G(f) \circ \eta_{A\otimes B}.
\end{aligned}$$

Here we have used the fact that composition of morphisms is continuous and F and G are continuous on morphism spaces. By construction, $\tilde{\eta} : F \to G$ is the unique natural transformation such that $\tilde{\eta} * 1_{\otimes_{\max}} = \eta$ and hence $(*)$ is fully faithful.

Finally, 2-naturality means that for any $*$-functor $F : \mathcal{C} \to \mathcal{D}$ the diagram

$$\begin{tikzcd}
C^*\text{-}\mathsf{Cat}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{C}) \arrow[r, "F\circ -"] \arrow[d, "-\circ\otimes_{\max}"'] & C^*\text{-}\mathsf{Cat}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{D}) \arrow[d, "-\circ\otimes_{\max}"] \\
C^*\text{-}\mathsf{Bilin}(\mathcal{A}, \mathcal{B}; \mathcal{C}) \arrow[r, "F\circ -"] & C^*\text{-}\mathsf{Bilin}(\mathcal{A}, \mathcal{B}; \mathcal{D})
\end{tikzcd}$$

commutes and for any natural transformation $\eta : F \to G : \mathcal{C} \to \mathcal{D}$ and $*$-functor $H : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{C}$ we have

$$(\eta * 1_H) * 1_{\otimes_{\max}} = \eta * (1_{H\circ\otimes_{\max}}).$$

These equalities hold by associativity of composition of functors and horizontal composition of natural transformations. $\square$

If $\mathcal{A}, \mathcal{B} \in C^*\text{-}\mathsf{Add}$, we can modify the definition of the maximal tensor product to ensure we obtain a C^*-category which is additive and subobject complete. It turns out that it is sufficient to take the subobject completion of $\mathcal{A} \otimes_{\max} \mathcal{B}$.

Definition 2.26. *Let $\mathcal{A}, \mathcal{B} \in C^*\text{-}\mathsf{Add}$. Their **maximal tensor product** denoted $\mathcal{A} \boxtimes_{\max} \mathcal{B}$ is defined by*

$$\mathcal{A} \boxtimes_{\max} \mathcal{B} := \mathrm{Split}(\mathcal{A} \otimes_{\max} \mathcal{B}).$$

We also modify the maximal tensor product of $*$-functors and natural transformations in this setting accordingly.

Definition 2.27. *Let $F : \mathcal{A} \to \mathcal{B}$ and $F' : \mathcal{A}' \to \mathcal{B}'$ be $*$-functors in $C^*\text{-}\mathsf{Add}$. We define*

$$F \boxtimes_{\max} F' := \mathrm{Split}(F \otimes_{\max} F') : \mathcal{A} \boxtimes_{\max} \mathcal{A}' \to \mathcal{B} \boxtimes_{\max} \mathcal{B}'.$$

Similarly, if $\alpha : F \to G$ and $\alpha' : F' \to G'$ are natural transformations in $C^*\text{-}\mathsf{Add}$, we define

$$\alpha \boxtimes_{\max} \alpha' := \mathrm{Split}(\alpha \otimes_{\max} \alpha') : F \otimes_{\max} F' \to G \otimes_{\max} G'.$$

Lemma 2.28. *Let $\mathcal{A}, \mathcal{B} \in C^*\text{-}\mathsf{Add}$, then $\mathcal{A} \boxtimes_{\max} \mathcal{B} \in C^*\text{-}\mathsf{Add}$.*

Proof. Since $\mathcal{A} \boxtimes_{\max} \mathcal{B}$ is subobject complete by construction, we just need to show that it has direct sums. Firstly, given $(A, B), (A', B') \in \mathcal{A} \times \mathcal{B}$ the direct sum

$$(A \boxtimes B) \oplus (A \boxtimes B') \oplus (A' \boxtimes B) \oplus (A' \boxtimes B')$$

exists because $\boxtimes_{\max}$ is bilinear and hence $(A \oplus A') \boxtimes (B \oplus B')$ is a direct sum of these factors. (The proof of this is the same as the proof of Lemma 1.21 which states that $*$-functors preserve direct sums.)

Now, given $(A \boxtimes B, p), (A' \boxtimes B', p') \in \mathcal{A} \boxtimes_{\max} \mathcal{B}$, we claim that

$$X := ((A \boxtimes B) \oplus (A \boxtimes B') \oplus (A' \boxtimes B) \oplus (A' \boxtimes B'), p \oplus 0 \oplus 0 \oplus p')$$

is a direct sum of these two objects. The inclusion are given in matrix form by

$$\iota_{(A \boxtimes B, p)} = \begin{pmatrix} p \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \iota_{(A' \boxtimes B', p')} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ p' \end{pmatrix}$$

and the projection maps by

$$\pi_{(A \boxtimes B, p)} = \begin{pmatrix} p & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \iota_{(A' \boxtimes B', p')} = \begin{pmatrix} 0 & 0 & 0 & p' \end{pmatrix}.$$

Computing composites by matrix multiplication, one then finds that

$$\begin{aligned}\pi_{(A\boxtimes B,p)} \circ \iota_{(A\boxtimes B,p)} &= p \\ &= 1_{(A\boxtimes B,p)},\end{aligned}$$

$$\begin{aligned}\pi_{(A'\boxtimes B',p')} \circ \iota_{(A'\boxtimes B',p')} &= p' \\ &= 1_{(A'\boxtimes B',p')}\end{aligned}$$

and

$$\begin{aligned}\iota_{(A\boxtimes B,p)} \circ \pi_{(A\boxtimes B,p)} + \iota_{(A'\boxtimes B',p')} \circ \pi_{(A'\boxtimes B',p')} &= p \oplus 0 \oplus 0 \oplus p' \\ &= 1_X.\end{aligned}$$

Therefore, X is a direct sum of $(A \boxtimes B, p)$ and $(A' \boxtimes B', p')$ as required. $\square$

Given $\mathcal{A}, \mathcal{B} \in C^*\text{-}\mathsf{Add}$, there is a bilinear $*$-functor $\boxtimes_{\max} : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \boxtimes_{\max} \mathcal{B}$ defined as the composite

$$\mathcal{A} \times \mathcal{B} \xrightarrow{\otimes_{\max}} \mathcal{A} \otimes_{\max} \mathcal{B} \xrightarrow{\eta_{\mathcal{A}\otimes_{\max}\mathcal{B}}} \begin{aligned}&\mathrm{Split}(\mathcal{A} \otimes_{\max} \mathcal{B}) \\ &= \mathcal{A} \boxtimes_{\max} \mathcal{B}\end{aligned}$$

where $\eta_{\mathcal{A}\otimes_{\max}\mathcal{B}}$ is the canonical inclusion $*$-functor. We then have the following universal property of the maximal tensor product.

Lemma 2.29. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in C^*\text{-}\mathsf{Add}$. The $*$-functor $\boxtimes_{\max} : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \boxtimes_{\max} \mathcal{B}$ induces an equivalence of C^*-categories*

$$- \circ \boxtimes_{\max} : C^*\text{-}\mathsf{Add}(\mathcal{A} \boxtimes_{\max} \mathcal{B}, \mathcal{C}) \xrightarrow{\simeq} C^*\text{-}\mathsf{Bilin}(\mathcal{A}, \mathcal{B}; \mathcal{C})$$

which is pseudonatural in $\mathcal{C}$.

Proof. This follows from Lemma 2.25 and Theorem 1.40 which states that Split is a left biadjoint of the forgetful functor $U : C^*\text{-}\mathsf{Add} \to C^*\text{-}\mathsf{Cat}$. $\square$

Next, we shall recall the definitions of C^*-tensor categories, module categories and their morphisms. It is not necessary for the categories in these definitions to be additive or subobject complete but we shall restrict our attention to such categories when we come to prove the existence of balanced tensor products.

Definition 2.30. *A C^*-**tensor category** consists of the following data:*

- *A C^*-category $\mathcal{A}$.*
- *A bilinear $*$-functor*

$$\begin{aligned} \otimes : \mathcal{A} \times \mathcal{A} &\to \mathcal{A} \\ (A, B) &\mapsto A \otimes B \end{aligned}$$

- *A designated object $I \in \mathcal{A}$ called the **tensor unit**.*
- *A unitary natural transformation $\alpha : \otimes \circ (\otimes \times 1_{\mathcal{A}}) \to \otimes \circ (1_{\mathcal{A}} \times \otimes)$ called the **associator**.*
- *unitary natural transformations $\lambda : I \otimes - \to 1_{\mathcal{A}}$ and $\rho : - \otimes I \to 1_{\mathcal{A}}$ called the **left** and **right unitors** respectively.*

These data are subject to the following axioms:

- *For all $A, B, C, D \in \mathcal{A}$, the following diagram commutes*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes 1_D"] \arrow[d, "\alpha_{A \otimes B, C, D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A, B \otimes C, D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

- *For all $A, B \in \mathcal{A}$, the following diagram commutes*

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Both Hilb and hilb are C^*-tensor categories with $\otimes$ being given by the Hilbert space tensor product and $I = \mathbb{C}$ (the components of the associator and unitors are the obvious maps). More on C^*-tensor categories can be found in [26]. There, an extra technical condition is placed on the tensor unit but this does not play a role in what we shall do so we have omitted it from the definition.

We also have the following notions of morphisms for C^*-tensor categories.

Definition 2.31. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-tensor categories with tensor units I and J respectively. A **tensor functor** $F : \mathcal{A} \to \mathcal{B}$ is a $*$-functor, together with a unitary $\iota : J \to F(I)$, and a unitary natural transformation $\beta : \otimes \circ (F \times F) \to F \circ \otimes$. These data satisfy the following axioms: For all $A, B, C \in \mathcal{A}$, the following diagram commutes.*

$$\begin{tikzcd}
(F(A) \otimes F(B)) \otimes F(C) \arrow[r, "\beta_{A,B} \otimes 1_{F(C)}"] \arrow[d, "\alpha_{F(A),F(B),F(C)}"'] & F(A \otimes B) \otimes F(C) \arrow[r, "\beta_{A \otimes B, C}"] & F((A \otimes B) \otimes C) \arrow[d, "F(\alpha_{A,B,C})"] \\
F(A) \otimes (F(B) \otimes F(C)) \arrow[r, "1_{F(A)} \otimes \beta_{B,C}"] & F(A) \otimes F(B \otimes C) \arrow[r, "\beta_{A, B \otimes C}"] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

For all $A \in \mathcal{A}$, the following diagrams commute.

$$\begin{tikzcd}
J \otimes F(A) \arrow[r, "\iota \otimes 1_{F(A)}"] \arrow[d, "\lambda_{F(A)}"'] & F(I) \otimes F(A) \arrow[d, "\beta_{I,A}"] \\
F(A) & F(I \otimes A) \arrow[l, "F(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \otimes J \arrow[r, "1_{F(A)} \otimes \iota"] \arrow[d, "\rho_{F(A)}"'] & F(A) \otimes F(I) \arrow[d, "\beta_{A,I}"] \\
F(A) & F(A \otimes I) \arrow[l, "F(\rho_A)"']
\end{tikzcd}$$

Definition 2.32. *Let $\mathcal{A}, \mathcal{B}$ be C^*-tensor categories with tensor units I and J respectively, and $F, G : \mathcal{A} \to \mathcal{B}$ tensor functors. A **monoidal natural transformation** from F to G is a natural transformation $\eta : F \to G$ such that the diagram*

$$\begin{tikzcd}
 & J \arrow[dl, "\iota^F"'] \arrow[dr, "\iota^G"] & \\
F(I) \arrow[rr, "\eta_I"] & & G(I)
\end{tikzcd}$$

commutes, and for all $A, B \in \mathcal{A}$, the following diagram commutes.

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "\beta^F_{A,B}"] \arrow[d, "\eta_A \otimes \eta_B"'] & F(A \otimes B) \arrow[d, "\eta_{A \otimes B}"] \\
G(A) \otimes G(B) \arrow[r, "\beta^G_{A,B}"] & G(A \otimes B)
\end{tikzcd}$$

Although not necessary for what follows, we remark that a C^*-tensor category can be viewed as a bicategory (i.e. a 'weak 2-category') with one object. In a similar way, a ring A (with a identity) can be viewed as an Ab-enriched category with one object (where Ab is the category of abelian groups). Then a left A-module can be viewed as an Ab-enriched functor from A to Ab and an A-module homomorphism can be viewed as a natural transformation between such functors. The category of A-modules can therefore be identified with the functor category $[A, \mathsf{Ab}]$. The definitions of left $\mathcal{A}$-module category and their morphisms, called 'module functors', mirror this in that a left $\mathcal{A}$-module category can be viewed as a pseudofunctor from $\mathcal{A}$ to C^*-Cat and a module functor between two $\mathcal{A}$-module categories can be viewed as a pseudonatural transformation between such pseudofunctors. In the categorical setting, we also have module transformations between module functors which correspond to modifications. We shall just present the definitions that we need but more on module categories in the algebraic (as opposed to C^*) setting can be found in [13].

Definition 2.33. *Let $\mathcal{A}$ be a C^*-tensor category. A **left $\mathcal{A}$-module category** consists of the following data:*

- *A C^*-category $\mathcal{M}$.*
- *A bilinear $*$-functor*

$$\begin{aligned} \otimes : \mathcal{A} \times \mathcal{M} &\to \mathcal{M} \\ (A, M) &\mapsto A \otimes M \end{aligned}$$

- *A unitary natural transformation $\alpha : \otimes \circ (\otimes \times 1_{\mathcal{M}}) \to \otimes \circ (1_{\mathcal{A}} \times \otimes)$ called the **associator**.*
- *A unitary natural transformation $\lambda : I \otimes - \to 1_{\mathcal{M}}$ called the **unitor**.*

These data are subject to the following axioms:

- *For all $A, B, C \in \mathcal{A}$ and $M \in \mathcal{M}$, the following diagram commutes*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes M \arrow[r, "\alpha_{A,B,C} \otimes 1_M"] \arrow[d, "\alpha_{A \otimes B, C, M}"'] & (A \otimes (B \otimes C)) \otimes M \arrow[r, "\alpha_{A, B \otimes C, M}"] & A \otimes ((B \otimes C) \otimes M) \arrow[d, "1_A \otimes \alpha_{B,C,M}"] \\
(A \otimes B) \otimes (C \otimes M) \arrow[rr, "\alpha_{A,B,C \otimes M}"] & & A \otimes (B \otimes (C \otimes M))
\end{tikzcd}$$

- *For all $A \in \mathcal{A}$ and $M \in \mathcal{M}$, the following diagrams commute*

$$\begin{tikzcd}
(A \otimes I) \otimes M \arrow[rr, "\alpha_{A,I,M}"] \arrow[dr, "\rho_A \otimes 1_M"'] & & A \otimes (I \otimes M) \arrow[dl, "1_A \otimes \lambda_M"] \\
& A \otimes M &
\end{tikzcd}$$

$$\begin{tikzcd}
(I \otimes A) \otimes M \arrow[rr, "\alpha_{I,A,M}"] \arrow[dr, "\lambda_A \otimes 1_M"'] & & I \otimes (A \otimes M) \arrow[dl, "\lambda_{A \otimes M}"] \\
& A \otimes M &
\end{tikzcd}$$

Right $\mathcal{A}$-module categories are defined in a similar way.

Definition 2.34. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{A}$-module categories. An **$\mathcal{A}$-module functor** is a $*$-functor $F : \mathcal{M} \to \mathcal{N}$ along with a unitary natural transformation*

$$\beta : F \circ \otimes \to \otimes \circ (1_{\mathcal{A}} \times F)$$

such that for all $A, B \in \mathcal{A}$ and $M \in \mathcal{M}$, the following diagrams commute

$$\begin{tikzcd}
F((A \otimes B) \otimes M) \arrow[r, "F(\alpha_{A,B,M})"] \arrow[d, "\beta_{A \otimes B, M}"'] & F(A \otimes (B \otimes M)) \arrow[r, "\beta_{A, B \otimes M}"] & A \otimes F(B \otimes M) \arrow[d, "1_A \otimes \beta_{B,M}"] \\
(A \otimes B) \otimes F(M) \arrow[rr, "\alpha_{A,B,F(M)}"] & & A \otimes (B \otimes F(M))
\end{tikzcd}$$

$$\begin{tikzcd}
F(I \otimes M) \arrow[rr, "\beta_{I,M}"] \arrow[dr, "F(\lambda_M)"'] & & I \otimes F(M) \arrow[dl, "\lambda_{F(M)}"] \\
& F(M) &
\end{tikzcd}$$

If there are multiple module functors $F, G, H, \ldots$ under consideration, we will denote their coherence transformations by $\beta^F, \beta^G, \beta^H, \ldots$ respectively.

Definition 2.35. *Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{A}$-module categories and $F, G : \mathcal{M} \to \mathcal{N}$ $\mathcal{A}$-module functors. An **$\mathcal{A}$-module transformation** is a natural transformation $\eta : F \to G$ such that for all $A \in \mathcal{A}$ and $M \in \mathcal{M}$, the following diagram commutes*

$$\begin{tikzcd}
F(A \otimes M) \arrow[r, "\beta^F_{A,M}"] \arrow[d, "\eta_{A\otimes M}"'] & A \otimes F(M) \arrow[d, "1_A \otimes \eta_M"] \\
G(A \otimes M) \arrow[r, "\beta^G_{A,M}"] & A \otimes G(M)
\end{tikzcd}$$

The $\mathcal{A}$-module categories, $\mathcal{A}$-module functors and $\mathcal{A}$-module transformations comprise a 2-category $\mathsf{Mod}_{\mathcal{A}}$ which is essentially $\mathsf{Psd}[\mathcal{A}, C^*\text{-}\mathsf{Cat}]$. (In general, if $\mathcal{B}$ and $\mathcal{C}$ are bicategories then $\mathsf{Psd}[\mathcal{B}, \mathcal{C}]$ is just a bicategory but if $\mathcal{C}$ is a 2-category, then so is $\mathsf{Psd}[\mathcal{B}, \mathcal{C}]$.)

Given a ring A, a right A-module M and a left A-module N, the universal property of $M \otimes_A N$ is related to A-balanced maps. To define the balanced tensor product of module categories, we need their categorical analogue. We first recall that if G is an abelian group, then a function $\varphi : M \otimes_A N \to G$ is A-*balanced* if

$$\begin{aligned}
\varphi(m \cdot a, n) &= \varphi(m, a \cdot n), \\
\varphi(m + m', n) &= \varphi(m, n) + \varphi(m', n), \\
\varphi(m, n + n') &= \varphi(m, n) + \varphi(m, n')
\end{aligned}$$

for all $m, m' \in M$, $a \in A$ and $n, n' \in N$. The universal property of $M \otimes_A N$ is that for any abelian group G, there is a bijective correspondence between the set of group homomorphisms from $M \otimes_A N$ to G and the set of A-balanced maps from $M \times N$ to G. A group homomorphism $\psi : M \otimes_A N \to G$ corresponds to the A-balanced map $\psi \circ \otimes_A : M \times N \to G$ where $\otimes_A : M \times N \to M \otimes_A N$ is the (A-balanced) map $(M, N) \mapsto M \otimes N$.

The categorical analogue is defined as follows:

Definition 2.36. *Let $\mathcal{A}$ be a C^*-tensor category, $\mathcal{M}$ a right $\mathcal{A}$-module category, $\mathcal{N}$ a left $\mathcal{A}$-module category and $\mathcal{C}$ a C^*-category. Then an* ***$\mathcal{A}$-balanced functor*** *$F : \mathcal{M} \times \mathcal{N} \to \mathcal{C}$ is a bilinear $*$-functor together with a unitary natural transformation $\beta : F \circ (\otimes \times 1_{\mathcal{N}}) \to F \circ (1_{\mathcal{M}} \times \otimes)$ such that for all $M \in \mathcal{M}$, $A, B \in \mathcal{A}$ and $N \in \mathcal{N}$ the following diagram commutes*

$$\begin{tikzcd}
F((M \otimes A) \otimes B, N) \arrow[r, "F(\alpha_{M,A,B} \times 1_N)"] \arrow[d, "\beta_{M \otimes A,B,N}"'] & F(M \otimes (A \otimes B), N) \arrow[r, "\beta_{M,A \otimes B,N}"] & F(M, (A \otimes B) \otimes N) \arrow[d, "F(1_M \times \alpha_{A,B,N})"] \\
F(M \otimes A, B \otimes N) \arrow[rr, "\beta_{M,A,B \otimes N}"] & & F(M, A \otimes (B \otimes N))
\end{tikzcd}$$

When there are several $\mathcal{A}$-balanced functors $F, G, H, \dots$ under consideration, we shall denote their balancing transformations by $\beta^F, \beta^G, \beta^H, \dots$ respectively.

Definition 2.37. *Let $F, G : \mathcal{M} \times \mathcal{N} \to \mathcal{C}$ be $\mathcal{A}$-balanced functors. A natural transformation $\eta : F \to G$ is called* ***$\mathcal{A}$-balanced*** *if the following diagram commutes for all $M \in \mathcal{M}$, $A \in \mathcal{A}$ and $N \in \mathcal{N}$*

$$\begin{tikzcd}
F(M \otimes A, N) \arrow[r, "\beta^F_{M,A,N}"] \arrow[d, "\eta_{M \otimes A,N}"'] & F(M, A \otimes N) \arrow[d, "\eta_{M,A \otimes N}"] \\
G(M \otimes A, N) \arrow[r, "\beta^G_{M,A,N}"] & G(M, A \otimes N)
\end{tikzcd}$$

We shall denote the C^*-category of $\mathcal{A}$-balanced functors from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{C}$ and the $\mathcal{A}$-balanced transformations between them by $\mathsf{Bal}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}; \mathcal{C})$.

2.3 Balanced tensor products

We now have all the definitions in place to show the existence of balanced tensor products in C^*-Add. As mentioned earlier, in the following theorem we shall assume that the C^*-tensor category and module categories are additive and subobject complete.

To motivate the proof, we recall a construction from homological algebra (see [27] for example). Let A be a ring, M a right A-module and N a left A-module. If

$\otimes$ denotes the tensor product of abelian groups and we abbreviate the p-fold tensor product $A \otimes A \otimes \cdots \otimes A$ to $A^{\otimes p}$ we have an abelian group $G_p := M \otimes A^{\otimes p} \otimes N$ for all $p \in \mathbb{N}$ (the interpretation in the case $p = 0$ is that $G_0 = M \otimes N$). Then for each $p \geq 1$ and $i = 0, \ldots, n$ we have 'degeneracy maps' $\partial_i : G_p \to G_{p-1}$ defined by

$$\partial_i(m \otimes a_1 \otimes \cdots \otimes a_p \otimes n) := \begin{cases} m \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_p \otimes n & \text{if } i = 0, \\ m \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes n & \text{if } 0 < i < p, \\ m \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes a_p \cdot n & \text{if } i = p. \end{cases}$$

Combining these maps we have a diagram

$$\cdots \quad \overset{\partial_0}{\underset{\partial_3}{\rightrightarrows\rightrightarrows}} M \otimes A \otimes A \otimes N \overset{\partial_0}{\underset{\partial_2}{\rightrightarrows\!\!\to}} M \otimes A \otimes N \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} M \otimes N$$

in the category of abelian groups which corresponds to the so called 'bar complex'. (The bar complex itself has the same objects but has a single arrow from G_p to G_{p-1} for each $p \geq 1$ which is the alternating sum of the ∂_i's.) We can truncate this diagram, leaving just the last two terms

$$M \otimes A \otimes N \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} M \otimes N$$

and the coequaliser of this truncated diagram is the balanced tensor product $M \otimes_A N$. The idea in the categorical case is to do something similar but truncating to leave the last three terms. Then, the bicolimit of the truncated diagram will be the balanced tensor product of the module categories.

As a matter of notational convenience, in the proof we shall abbreviate $\boxtimes_{\max}$ to $\boxtimes$. We shall also identify multilinear $*$-functors with linear $*$-functors out of the maximal tensor product as in Lemma 2.29. We should be a little careful since we only have an equivalence of functor categories rather than an isomorphism but no harm will come of such identifications here.

Theorem 2.38. *Let $\mathcal{A}$ be a C^*-tensor category, $\mathcal{M}$ a right $\mathcal{A}$-module category with action map $\rho : \mathcal{M} \times \mathcal{A} \to \mathcal{M}$ and $\mathcal{N}$ a left $\mathcal{A}$-module category with action map $\lambda : \mathcal{A} \times \mathcal{N} \to \mathcal{N}$. Then there exists an additive, subobject complete C^*-category $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ and an $\mathcal{A}$-balanced functor $\boxtimes_{\mathcal{A}} : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ that induces equivalences of C^*-categories*

$$- \circ \boxtimes_{\mathcal{A}} : C^*\text{-}\mathsf{Add}(\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}, \mathcal{C}) \xrightarrow{\simeq} \mathsf{Bal}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}; \mathcal{C})$$

for all $\mathcal{C} \in C^*\text{-}\mathsf{Add}$. These equivalences are pseudonatural in $\mathcal{C}$.

Proof. Let $\mathcal{I}$ be the opposite of the 2-truncated presimplicial category. That is, $\mathcal{I}$ is the category with three objects $\underline{2}, \underline{1}$ and $\underline{0}$ and the morphism sets are generated by morphisms $\partial_i : \underline{n} \to \underline{n-1}$ for $0 \leq i \leq n$ subject to the relations $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ for $i < j$.

Then we have an $\mathcal{I}$-diagram $D : \mathcal{I} \to C^*\text{-}\mathsf{Add}$ corresponding to the truncated bar complex

$$\mathcal{M} \boxtimes \mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{N} \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \mathcal{M} \boxtimes \mathcal{A} \boxtimes \mathcal{N} \; \substack{\longrightarrow \\ \longrightarrow} \; \mathcal{M} \boxtimes \mathcal{N}$$

for $\mathcal{M}$ and $\mathcal{N}$. That is, we define

$$\begin{aligned} D(\underline{0}) &:= \mathcal{M} \boxtimes \mathcal{N}, \\ D(\underline{1}) &:= \mathcal{M} \boxtimes \mathcal{A} \boxtimes \mathcal{N}, \\ D(\underline{2}) &:= \mathcal{M} \boxtimes \mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{N}. \end{aligned}$$

With regard to morphisms, we shall denote $D(\partial_i)$ by d_i. In degree 1 we define

$$\begin{aligned} d_0 = D(\partial_0) &:= \rho \boxtimes \mathrm{id}, \\ d_1 = D(\partial_1) &:= \mathrm{id} \boxtimes \lambda \end{aligned}$$

and in degree 2 we define

$$\begin{aligned} d_0 = D(\partial_0) &:= \rho \boxtimes \mathrm{id} \boxtimes \mathrm{id}, \\ d_1 = D(\partial_1) &:= \mathrm{id} \boxtimes \otimes \boxtimes \mathrm{id}, \\ d_2 = D(\partial_2) &:= \mathrm{id} \boxtimes \mathrm{id} \boxtimes \lambda. \end{aligned}$$

Finally, we define

$$D(\partial_i \circ \partial_j) = D(\partial_{j-1} \circ \partial_i) := d_i \circ d_j$$

if $i < j$ and $D(1_{\underline{n}}) = 1_{D(\underline{n})}$ for all $\underline{n} \in \mathcal{I}$.

In general, we don't have $D(\partial_{j-1}) \circ D(\partial_i) = D(\partial_{j-1} \circ \partial_i)$ for $i < j$. Therefore, we also need some nontrivial coherence unitary natural transformations which we define as follows:

$$\begin{aligned} D_{\partial_0,\partial_0} &:= \alpha \boxtimes \mathrm{id} : d_0 \circ d_0 \to d_0 \circ d_1, \\ D_{\partial_1,\partial_0} &:= \mathrm{id} : d_1 \circ d_0 \to d_0 \circ d_2, \\ D_{\partial_1,\partial_1} &:= \mathrm{id} \boxtimes \alpha : d_1 \circ d_1 \to d_1 \circ d_2. \end{aligned}$$

All the other coherence maps for D are identity maps. The only instances of the pseudofunctor axioms are satisfied trivially (since the only 1-cells in $\mathcal{I}$ are the identities) and so D is an $\mathcal{I}$-diagram.

Now, by Theorem 2.9, the bicolimit of D exists and we shall denote it by $\mathcal{M}\boxtimes_{\mathcal{A}}\mathcal{N}$. We claim that $\mathcal{M}\boxtimes_{\mathcal{A}}\mathcal{N}$ is the balanced tensor product of $\mathcal{M}$ and $\mathcal{N}$. To show this, we first need to equip $\mathcal{M}\boxtimes_{\mathcal{A}}\mathcal{N}$ with an $\mathcal{A}$-balanced functor

$$\begin{aligned}\boxtimes_{\mathcal{A}} : \mathcal{M}\times\mathcal{N} &\to \mathcal{M}\boxtimes_{\mathcal{A}}\mathcal{N}\\ (M,N) &\mapsto M\boxtimes_{\mathcal{A}} N\end{aligned}$$

Part of the data of the bicolimit is a universal cocone $\sigma : D \to \Delta(\mathcal{M}\boxtimes_{\mathcal{A}}\mathcal{N})$ and we define $\boxtimes_{\mathcal{A}}$ as the bilinear $*$-functor corresponding to $\sigma_{\underline{0}}$. We also need to define a balancing unitary natural transformation with components

$$\beta_{M,A,N} : (M\otimes A)\boxtimes_{\mathcal{A}} N \to M\boxtimes_{\mathcal{A}}(A\otimes N)$$

where $M\in\mathcal{M}$, $A\in\mathcal{A}$ and $N\in\mathcal{N}$. We have unitaries

$$(\sigma^{1,0}_{\partial_0})_{(M,A,N)} : \sigma_{\underline{1}}(M,A,N) \to \sigma_{\underline{0}}(M\otimes A, N)$$

and

$$(\sigma^{1,0}_{\partial_1})_{(M,A,N)} : \sigma_{\underline{1}}(M,A,N) \to \sigma_{\underline{0}}(M, A\otimes N)$$

and so we define $\beta_{M,A,N} := (\sigma^{1,0}_{\partial_1})_{(M,A,N)} \circ (\sigma^{1,0}_{\partial_0})^{-1}_{(M,A,N)}$. Let us show that these maps satisfy the $\mathcal{A}$-balanced functor axioms. To do this, we need to show that for all $M\in\mathcal{M}$, $A,B\in\mathcal{A}$ and $N\in\mathcal{N}$, the following diagram commutes

$$\begin{tikzcd}
\sigma_{\underline{0}}((M\otimes A)\otimes B, N) \arrow[r, "\sigma_{\underline{0}}(\alpha_{M,A,B}\times 1_N)"] \arrow[d, "\beta_{M\otimes A,B,N}"'] & \sigma_{\underline{0}}(M\otimes(A\otimes B), N) \arrow[dd, "\beta_{M,A\otimes B,N}"] \\
\sigma_{\underline{0}}(M\otimes A, B\otimes N) \arrow[d, "\beta_{M,A,B\otimes N}"'] & \\
\sigma_{\underline{0}}(M, A\otimes(B\otimes N)) & \sigma_{\underline{0}}(M,(A\otimes B)\otimes N) \arrow[l, "\sigma_{\underline{0}}(1_M\times\alpha_{A,B,N})"']
\end{tikzcd}$$

This follows from the commutativity of the following diagram

$$
\begin{tikzcd}
\sigma_{\underline{0}}((M \otimes A) \otimes B, N) \arrow[rr, "\sigma_{\underline{0}}(\alpha_{M,A,B} \times 1_N)"] \arrow[d, "(\sigma^{1,\underline{0}}_{\partial_0})^{-1}_{M \otimes A,B,N}"'] & & \sigma_{\underline{0}}(M \otimes (A \otimes B), N) \arrow[dd, "(\sigma^{1,\underline{0}}_{\partial_0})^{-1}_{M,A\otimes B,N}"] \\
\sigma_{\underline{1}}(M \otimes A, B, N) \arrow[d, "(\sigma^{1,\underline{0}}_{\partial_1})_{M \otimes A,B,N}"'] & & \\
\sigma_{\underline{0}}(M \otimes A, B \otimes N) \arrow[d, "(\sigma^{1,\underline{0}}_{\partial_0})^{-1}_{M,A,B\otimes N}"'] & \sigma_{\underline{2}}(M, A, B, N) \arrow[l, "\sigma^{2,\underline{0}}_{\partial_1 \circ \partial_0}", "= \sigma^{2,\underline{1}}_{\partial_0 \circ \partial_2}"'] \arrow[ul, "(\sigma^{2,\underline{1}}_{\partial_0})_{M,A,B,N}"'] \arrow[uur, "\sigma^{2,\underline{0}}_{\partial_0 \circ \partial_0} = \sigma^{2,\underline{0}}_{\partial_0 \circ \partial_1}"] \arrow[r, "(\sigma^{2,\underline{1}}_{\partial_1})_{M,A,B,N}"] \arrow[dl, "(\sigma^{2,\underline{1}}_{\partial_2})_{M,A,B,N}"'] \arrow[ddl, "\sigma^{2,\underline{0}}_{\partial_1 \circ \partial_2} = \sigma^{2,\underline{0}}_{\partial_1 \circ \partial_1}"] & \sigma_{\underline{1}}(M, A \otimes B, N) \arrow[dd, "(\sigma^{1,\underline{0}}_{\partial_1})_{M,A\otimes B,N}"] \\
\sigma_{\underline{1}}(M, A, B \otimes N) \arrow[d, "(\sigma^{1,\underline{0}}_{\partial_1})_{M,A,B\otimes N}"'] & & \\
\sigma_{\underline{0}}(M, A \otimes (B \otimes N)) & & \sigma_{\underline{0}}(M, (A \otimes B) \otimes N) \arrow[ll, "\sigma_{\underline{0}}(1_M \times \alpha_{A,B,N})"']
\end{tikzcd}
$$

Each cell commutes by the pentagon axiom for pseudonatural transformations. Therefore, $\boxtimes_{\mathcal{A}}$ is an $\mathcal{A}$-balanced functor.

We want to show that for each $\mathcal{C} \in C^*\text{-}\mathsf{Add}$, we have equivalences of C^*-categories

$$- \circ \boxtimes_{\mathcal{A}} : C^*\text{-}\mathsf{Add}(\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}, \mathcal{C}) \xrightarrow{\simeq} \mathsf{Bal}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}; \mathcal{C})$$

which are pseudonatural in $\mathcal{C}$. Since $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ is the bicoimit of D, we have equivalences

$$- \circ \sigma : C^*\text{-}\mathsf{Add}(\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}, \mathcal{C}) \xrightarrow{\simeq} \mathsf{Tran}(D, \Delta(\mathcal{C}))$$

which are pseudonatural in $\mathcal{C}$. Therefore it is sufficient to construct equivalences

$$\varphi : \mathsf{Tran}(D, \Delta(\mathcal{C})) \xrightarrow{\simeq} \mathsf{Bal}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}; \mathcal{C})$$

which are pseudonatural in $\mathcal{C}$ and such that $- \circ \boxtimes_{\mathcal{A}} = \varphi \circ (- \circ \sigma)$. To that end, given a transformation $\tau : D \to \Delta(\mathcal{C})$ we define an $\mathcal{A}$-balanced functor $\varphi(\tau) : \mathcal{M} \times \mathcal{N} \to \mathcal{C}$ as follows: The underlying functor is $\tau_{\underline{0}}$ and the balancing isomorphisms are given by

$$\beta_{M,A,N} := (\tau^{\underline{1},\underline{0}}_{\partial_1})_{(M,A,N)} \circ (\tau^{\underline{1},\underline{0}}_{\partial_0})^{-1}_{(M,A,N)} : \tau_{\underline{0}}(M \otimes A, N) \xrightarrow{\cong} \tau_{\underline{0}}(M, A \otimes N).$$

By the same argument as for $\boxtimes_{\mathcal{A}}$, these maps satisfy the $\mathcal{A}$-balanced functor axioms.

Next, given a modification $\Gamma : \tau \to \mu$ we define $\varphi(\Gamma) := \Gamma_{\underline{0}}$. We want to show that this is an $\mathcal{A}$-balanced transformation. Therefore we need to show that the following diagram commutes:

$$\begin{tikzcd}
\tau_{\underline{0}}(M \otimes A, N) \arrow[r, "\beta^{\tau_{\underline{0}}}_{M,A,N}"] \arrow[d, "(\Gamma_{\underline{0}})_{M\otimes A,N}"'] & \tau_{\underline{0}}(M, A \otimes N) \arrow[d, "(\Gamma_{\underline{0}})_{M,A\otimes N}"] \\
\mu_{\underline{0}}(M \otimes A, N) \arrow[r, "\beta^{\mu_{\underline{0}}}_{M,A,N}"] & \mu_{\underline{0}}(M, A \otimes N)
\end{tikzcd}$$

This follows from the commutativity of the following diagram which in turn follows from the modification axiom.

$$\begin{tikzcd}
\tau_{\underline{0}}(M \otimes A, N) \arrow[r, "(\tau^{\underline{1},\underline{0}}_{\partial_0})^{-1}_{(M,A,N)}"] \arrow[d, "(\Gamma_{\underline{0}})_{(M\otimes A,N)}"'] & \tau_{\underline{1}}(M, A, N) \arrow[r, "(\tau^{\underline{1},\underline{0}}_{\partial_1})_{(M,A,N)}"] \arrow[d, "(\Gamma_{\underline{1}})_{(M,A,N)}"] & \tau_{\underline{0}}(M, A \otimes N) \arrow[d, "(\Gamma_{\underline{0}})_{(M,A\otimes N)}"] \\
\mu_{\underline{0}}(M \otimes A, N) \arrow[r, "(\mu^{\underline{1},\underline{0}}_{\partial_0})^{-1}_{(M,A,N)}"] & \mu_{\underline{1}}(M, A, N) \arrow[r, "(\mu^{\underline{1},\underline{0}}_{\partial_1})_{(M,A,N)}"] & \mu_{\underline{0}}(M, A \otimes N)
\end{tikzcd}$$

Therefore, φ is well defined on morphisms. Functoriality is immediate from the definition of composition of modifications.

To see that φ is essentially surjective, let $F : \mathcal{M} \times \mathcal{N} \to \mathcal{C}$ be an $\mathcal{A}$-balanced functor. We shall construct a transformation $\tau : D \to \Delta(\mathcal{C})$ such that $F = \varphi(\tau)$. We define

$$\begin{aligned}\tau_{\underline{0}} &:= F\\ \tau_{\underline{1}} &:= F \circ d_0\\ \tau_{\underline{2}} &:= F \circ d_0 \circ d_0.\end{aligned}$$

We then need to define coherence unitaries that satisfy the pseudonatural transformation axioms. Firstly, we define

$$\begin{aligned}\tau^{\underline{1},\underline{0}}_{\partial_0} &:= \mathrm{id} : \tau_{\underline{1}} \to \tau_{\underline{0}} \circ d_0,\\ \tau^{\underline{1},\underline{0}}_{\partial_1} &:= \beta : \tau_{\underline{1}} \to \tau_{\underline{0}} \circ d_1.\end{aligned}$$

Here β is the balancing unitary for F.

We also define

$$\begin{aligned}\tau^{\underline{2},\underline{1}}_{\partial_0} &:= \mathrm{id} : \tau_{\underline{2}} = F \circ d_0 \circ d_0 \to F \circ d_0 \circ d_0 = \tau_{\underline{1}} \circ d_0,\\ \tau^{\underline{2},\underline{1}}_{\partial_1} &:= 1_F * (\alpha \boxtimes \mathrm{id}) : \tau_{\underline{2}} = F \circ d_0 \circ d_0 \to F \circ d_0 \circ d_1 = \tau_{\underline{1}} \circ d_1,\\ \tau^{\underline{2},\underline{1}}_{\partial_2} &:= \beta * 1_{\rho \boxtimes \mathrm{id} \boxtimes \mathrm{id}} : \tau_{\underline{2}} = F \circ d_0 \circ d_0 \to F \circ d_0 \circ d_2 = \tau_{\underline{1}} \circ d_2.\end{aligned}$$

Here α is the coherence unitary for the module category $\mathcal{M}$.

Next, we define

$$\begin{aligned}\tau^{\underline{2},\underline{0}}_{\partial_0 \circ \partial_0} = \tau^{\underline{2},\underline{0}}_{\partial_0 \circ \partial_1} &:= (\tau^{\underline{1},\underline{0}}_{\partial_0} * 1_{d_1}) \circ \tau^{\underline{2},\underline{1}}_{\partial_1},\\ \tau^{\underline{2},\underline{0}}_{\partial_1 \circ \partial_0} = \tau^{\underline{2},\underline{0}}_{\partial_0 \circ \partial_2} &:= (\tau^{\underline{1},\underline{0}}_{\partial_0} * 1_{d_2}) \circ \tau^{\underline{2},\underline{1}}_{\partial_2},\\ \tau^{\underline{2},\underline{0}}_{\partial_1 \circ \partial_1} = \tau^{\underline{2},\underline{0}}_{\partial_1 \circ \partial_2} &:= (\tau^{\underline{1},\underline{0}}_{\partial_1} * 1_{d_2}) \circ \tau^{\underline{2},\underline{1}}_{\partial_2}.\end{aligned}$$

Finally, we define $\tau^{\underline{n},\underline{n}}_{1_{\underline{n}}} := 1_{\tau_{\underline{n}}}$ for each $\underline{n} \in \mathcal{I}$. To see that these maps satisfy the pseudonatural transformation axioms, we first note that the unit axiom is satisfied trivially because all the maps are identities.

The nontrivial cases of the composition axiom diagrams are of the form

$$\begin{tikzcd}
\tau_{\underline{2}} \arrow[r, "\tau^{\underline{2},\underline{1}}_{\partial_j}"] \arrow[d, "\mathrm{id}"'] & \tau_{\underline{1}} \circ D(\partial_j) \arrow[r, "\tau^{\underline{1},\underline{0}}_{\partial_i}"] & \tau_{\underline{0}} \circ D(\partial_i) \circ D(\partial_j) \arrow[d, "D^{\underline{2},\underline{1},\underline{0}}_{\partial_i,\partial_j}"] \\
\tau_{\underline{2}} \arrow[rr, "\tau^{\underline{2},\underline{0}}_{\partial_i \circ \partial_j}"] & & \tau_{\underline{0}} \circ D(\partial_i \circ \partial_j)
\end{tikzcd}$$

For $i < j$ the commutativity is immediate from the definition of $D(\partial_i \circ \partial_j)$. For the case $(i,j) = (0,0)$ this follows from the commutativity of the following diagram.

$$\begin{tikzcd}
\begin{array}{c}\tau_{\underline{2}} \\ = \tau_{\underline{0}} \circ d_0 \circ d_0\end{array} \arrow[r, "\substack{\tau^{\underline{2},\underline{1}}_{\partial_0} \\ = \mathrm{id}}"] \arrow[d, "\mathrm{id}"'] & \begin{array}{c}\tau_{\underline{1}} \circ D(\partial_0) \\ = \tau_{\underline{0}} \circ d_0 \circ d_0\end{array} \arrow[r, "\substack{\tau^{\underline{1},\underline{0}}_{\partial_0} \\ = \mathrm{id}}"] & \begin{array}{c}\tau_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \\ = \tau_{\underline{0}} \circ d_0 \circ d_0\end{array} \arrow[d, "\substack{D^{\underline{2},\underline{1},\underline{0}}_{\partial_0,\partial_0} \\ = \alpha \boxtimes \mathrm{id}}"] \\
\begin{array}{c}\tau_{\underline{2}} \\ = \tau_{\underline{0}} \circ d_0 \circ d_0\end{array} \arrow[rr, "\substack{\tau^{\underline{2},\underline{0}}_{\partial_0 \circ \partial_0} \\ = \alpha \boxtimes \mathrm{id}}"] & & \begin{array}{c}\tau_{\underline{0}} \circ D(\partial_0 \circ \partial_0) \\ = \tau_{\underline{0}} \circ d_0 \circ d_1\end{array}
\end{tikzcd}$$

In the case $(i,j) = (1,0)$ this follows from the commutativity of the following diagram

$$\begin{tikzcd}
\begin{array}{c}\tau_{\underline{2}} \\ = \tau_{\underline{0}} \circ d_0 \circ d_0\end{array} \arrow[r, "\substack{\tau^{\underline{2},\underline{1}}_{\partial_0} \\ = \mathrm{id}}"] \arrow[d, "\mathrm{id}"'] & \begin{array}{c}\tau_{\underline{1}} \circ D(\partial_0) \\ = \tau_{\underline{0}} \circ d_0 \circ d_0\end{array} \arrow[r, "\substack{\tau^{\underline{1},\underline{0}}_{\partial_1} \\ = \beta}"] & \begin{array}{c}\tau_{\underline{0}} \circ D(\partial_1) \circ D(\partial_0) \\ = \tau_{\underline{0}} \circ d_1 \circ d_0\end{array} \arrow[d, "\substack{D^{\underline{2},\underline{1},\underline{0}}_{\partial_1,\partial_0} \\ = \mathrm{id}}"] \\
\begin{array}{c}\tau_{\underline{2}} \\ = \tau_{\underline{0}} \circ d_0 \circ d_0\end{array} \arrow[rr, "\substack{\tau^{\underline{2},\underline{0}}_{\partial_1 \circ \partial_0} \\ = \beta}"] & & \begin{array}{c}\tau_{\underline{0}} \circ D(\partial_1 \circ \partial_0) \\ = \tau_{\underline{0}} \circ d_1 \circ d_0\end{array}
\end{tikzcd}$$

Finally, in the case $(i,j) = (1,1)$ this follows from the commutativity of the following diagram

$$
\begin{tikzcd}[column sep=large]
\begin{array}{c}\tau_{\underline{2}}\\ =\tau_{\underline{0}}\circ d_0\circ d_0\end{array} \arrow[r, "\begin{array}{c}\tau^{\underline{2},\underline{1}}_{\partial_1}\\ =\alpha\boxtimes\mathrm{id}\end{array}"] \arrow[d, "\mathrm{id}"'] & \begin{array}{c}\tau_{\underline{1}}\circ D(\partial_1)\\ =\tau_{\underline{0}}\circ d_0\circ d_1\end{array} \arrow[r, "\begin{array}{c}\tau^{\underline{1},\underline{0}}_{\partial_1}\\ =\beta\end{array}"] & \begin{array}{c}\tau_{\underline{0}}\circ D(\partial_1)\circ D(\partial_1)\\ =\tau_{\underline{0}}\circ d_1\circ d_1\end{array} \arrow[d, "\begin{array}{c}D^{\underline{2},\underline{1},\underline{0}}_{\partial_1,\partial_1}\\ =\mathrm{id}\boxtimes\alpha\end{array}"] \\
\begin{array}{c}\tau_{\underline{2}}\\ =\tau_{\underline{0}}\circ d_0\circ d_0\end{array} \arrow[r, "\beta"] \arrow[rr, bend right, "\tau^{\underline{2},\underline{0}}_{\partial_1\circ\partial_1}"'] & \tau_{\underline{0}}\circ d_0\circ d_2 \arrow[r, "\beta"] & \begin{array}{c}\tau_{\underline{0}}\circ D(\partial_1\circ\partial_1)\\ =\tau_{\underline{0}}\circ d_1\circ d_2\end{array}
\end{tikzcd}
$$

This diagram commutes because $\tau_{\underline{0}}$ is an $\mathcal{A}$-balanced functor and because $\tau^{\underline{2},\underline{0}}_{\partial_1\circ\partial_1} = \beta\circ\beta$ by definition. It follows that τ is a transformation and by construction $\varphi(\tau) = F$, therefore φ is essentially surjective.

To show that φ is faithful, suppose that $\tau, \mu : D \to \Delta(\mathcal{C})$ are transformations and $\Gamma : \tau \to \mu$ a modification such that $\varphi(\Gamma) = \Gamma_{\underline{0}} = 0$. By the modification axiom the following diagram commutes

$$
\begin{tikzcd}
\tau_{\underline{n}} \arrow[r, "\tau^{\underline{n},\underline{0}}_f"] \arrow[d, "\Gamma_{\underline{n}}"'] & \tau_{\underline{0}}\circ D(f) \arrow[d, "\Gamma_{\underline{0}}"] \\
\mu_{\underline{n}} \arrow[r, "\mu^{\underline{n},\underline{0}}_f"] & \mu_{\underline{0}}\circ D(f)
\end{tikzcd}
$$

for all $\underline{n} \in \mathcal{I}$ and $f : \underline{n} \to \underline{0}$. If $\Gamma_{\underline{0}} = 0$ it follows from the fact that $\mu^{\underline{n},\underline{0}}_f$ is a unitary and hence invertible that $\Gamma_{\underline{n}} = 0$ for all $\underline{n} \in \mathcal{I}$ and thus $\Gamma = 0$.

To show that φ is full, let $\tau, \mu : D \to \Delta(\mathcal{C})$ be transformations and $\eta : \varphi(\tau) \to \varphi(\mu)$ an $\mathcal{A}$-balanced transformation. We shall construct a modification $\Gamma : \tau \to \mu$ such that $\varphi(\Gamma) = \Gamma_{\underline{0}} = \eta$. Firstly, we define $\Gamma_{\underline{0}} := \eta$. We define $\Gamma_{\underline{1}}$ as the composite

$$
\tau_{\underline{1}} \xrightarrow{\tau^{\underline{1},\underline{0}}_{\partial_0}} \tau_{\underline{0}}\circ D(\partial_0) \xrightarrow{\eta} \mu_{\underline{0}}\circ D(\partial_0) \xrightarrow{(\mu^{\underline{1},\underline{0}}_{\partial_0})^{-1}} \mu_{\underline{1}}
$$

and we define $\Gamma_{\underline{2}}$ as the composite

$$\tau_{\underline{2}} \xrightarrow{\tau^{\underline{2},\underline{1}}_{\partial_0}} \tau_{\underline{1}} \circ D(\partial_0) \xrightarrow{\tau^{\underline{1},\underline{0}}_{\partial_0}} \tau_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \xrightarrow{\eta} \mu_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \xrightarrow{(\mu^{\underline{1},\underline{0}}_{\partial_0})^{-1}} \mu_{\underline{1}} \circ D(\partial_0) \xrightarrow{(\mu^{\underline{2},\underline{1}}_{\partial_0})^{-1}} \mu_{\underline{2}}.$$

To show that Γ is a modification, we need to show that for all morphisms $f : \underline{m} \to \underline{n}$ in $\mathcal{I}$ the following diagram commutes

$$\begin{tikzcd}
\tau_{\underline{m}} \arrow[r, "\tau_f^{\underline{m},\underline{n}}"] \arrow[d, "\Gamma_{\underline{m}}"'] & \tau_{\underline{n}} \circ D(f) \arrow[d, "\Gamma_{\underline{n}}"] \\
\mu_{\underline{m}} \arrow[r, "\mu_f^{\underline{m},\underline{n}}"] & \mu_{\underline{n}} \circ D(f)
\end{tikzcd}$$

We shall consider the nontrivial cases one by one. When $(\underline{m}, \underline{n}) = (\underline{1}, \underline{0})$ and $f = \partial_0$ this follows from the commutativity of the following diagram which is immediate.

$$\begin{tikzcd}
\tau_{\underline{1}} \arrow[r, "\tau^{\underline{1},\underline{0}}_{\partial_0}"] \arrow[d, "\tau^{\underline{1},\underline{0}}_{\partial_0}"'] & \tau_{\underline{0}} \circ D(\partial_0) \arrow[ddd, "\eta"] \\
\tau_{\underline{0}} \circ D(\partial_0) \arrow[d, "\eta"'] & \\
\mu_{\underline{0}} \circ D(\partial_0) \arrow[d, "(\mu^{\underline{1},\underline{0}}_{\partial_0})^{-1}"'] & \\
\mu_{\underline{1}} \arrow[r, "\mu^{\underline{1},\underline{0}}_{\partial_0}"] & \mu_{\underline{0}} \circ D(\partial_0)
\end{tikzcd}$$

When $(\underline{m}, \underline{n}) = (\underline{1}, \underline{0})$ and $f = \partial_1$ this follows from the commutativity of the following diagram

$$\begin{tikzcd}
\tau_{\underline{1}} \arrow[r, "\tau^{\underline{1},\underline{0}}_{\partial_1}"] \arrow[d, "\tau^{\underline{1},\underline{0}}_{\partial_0}"'] \arrow[dr, phantom, "\text{(I)}" near start] & \tau_{\underline{0}} \circ D(\partial_1) \arrow[ddd, "\eta"] \\
\tau_{\underline{0}} \circ D(\partial_0) \arrow[ur, "\beta"'] \arrow[d, "\eta"'] \arrow[dr, phantom, "\text{(II)}"] & \\
\mu_{\underline{0}} \circ D(\partial_0) \arrow[d, "(\mu^{\underline{1},\underline{0}}_{\partial_0})^{-1}"'] \arrow[dr, "\beta"] \arrow[dr, phantom, "\text{(III)}" near end, shift right=3ex] & \\
\mu_{\underline{1}} \arrow[r, "\mu^{\underline{1},\underline{0}}_{\partial_1}"] & \mu_{\underline{0}} \circ D(\partial_1)
\end{tikzcd}$$

Here (I) and (III) commute by the definition of the balancing transformations for $\varphi(\tau)$ and $\varphi(\mu)$ and (II) commutes because η is an $\mathcal{A}$-balanced transformation.

When $(\underline{m}, \underline{n}) = (\underline{2}, \underline{1})$ and $f = \partial_0$ this follows from the commutativity of the following diagram which is immediate

$$
\begin{tikzcd}
\tau_{\underline{2}} \arrow[r, "\tau^{\underline{2},\underline{1}}_{\partial_0}"] \arrow[d, "\tau^{\underline{2},\underline{1}}_{\partial_0}"'] & \tau_{\underline{1}} \circ D(\partial_0) \arrow[d, "\tau^{\underline{1},\underline{0}}_{\partial_0}"] \\
\tau_{\underline{1}} \circ D(\partial_0) \arrow[d, "\tau^{\underline{1},\underline{0}}_{\partial_0}"'] & \tau_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \arrow[ddd, "\eta"] \\
\tau_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \arrow[d, "\eta"'] & \\
\mu_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \arrow[d, "(\mu^{\underline{1},\underline{0}}_{\partial_0})^{-1}"'] & \\
\mu_{\underline{1}} \circ D(\partial_0) \arrow[d, "(\mu^{\underline{2},\underline{1}}_{\partial_0})^{-1}"'] & \mu_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \arrow[d, "(\mu^{\underline{1},\underline{0}}_{\partial_0})^{-1}"] \\
\mu_{\underline{2}} \arrow[r, "\mu^{\underline{2},\underline{1}}_{\partial_0}"] & \mu_{\underline{1}} \circ D(\partial_0)
\end{tikzcd}
$$

When $(\underline{m}, \underline{n}) = (\underline{2}, \underline{1})$ and $f = \partial_1$ this follows from the commutativity of the following diagram

$$
\begin{tikzcd}
\tau_{\underline{2}} \arrow[r, "\tau^{\underline{2},\underline{1}}_{\partial_1}"] \arrow[d, "\tau^{\underline{2},\underline{1}}_{\partial_0}"'] \arrow[dr, "\tau^{\underline{2},\underline{0}}_{\partial_0\circ\partial_1}" near start, "=\tau^{\underline{2},\underline{0}}_{\partial_0\circ\partial_0}"'] & \tau_{\underline{1}} \circ D(\partial_1) \arrow[d, "\tau^{\underline{1},\underline{0}}_{\partial_0}"] \arrow[dl, phantom, "\mathrm{(I)}" near end] \\
\tau_{\underline{1}} \circ D(\partial_0) \arrow[d, "\tau^{\underline{1},\underline{0}}_{\partial_0}"'] \arrow[r, phantom, "\mathrm{(II)}"] & \tau_{\underline{0}} \circ D(\partial_0) \circ D(\partial_1) \arrow[ddd, "\eta"] \\
\tau_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \arrow[ur, "1_{\tau_{\underline{0}}} * D^{\underline{2},\underline{1},\underline{0}}_{\partial_0,\partial_0}", "=\tau_{\underline{0}}(\alpha\boxtimes\mathrm{id})"'] \arrow[d, "\eta"'] \arrow[dr, phantom, "\mathrm{(III)}"] & \\
\mu_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \arrow[d, "(\mu^{\underline{1},\underline{0}}_{\partial_0})^{-1}"'] \arrow[dr, "\mu_{\underline{0}}(\alpha\boxtimes\mathrm{id})", "=1_{\mu_{\underline{0}}} * D^{\underline{2},\underline{1},\underline{0}}_{\partial_0,\partial_0}"'] & \\
\mu_{\underline{1}} \circ D(\partial_0) \arrow[d, "(\mu^{\underline{2},\underline{1}}_{\partial_0})^{-1}"'] \arrow[r, phantom, "\mathrm{(IV)}"] & \mu_{\underline{0}} \circ D(\partial_0) \circ D(\partial_1) \arrow[d, "(\mu^{\underline{1},\underline{0}}_{\partial_0})^{-1}"] \\
\mu_{\underline{2}} \arrow[ur, "\mu^{\underline{2},\underline{0}}_{\partial_0\circ\partial_0}", "=\mu^{\underline{2},\underline{0}}_{\partial_0\circ\partial_1}"'] \arrow[r, "\mu^{\underline{2},\underline{1}}_{\partial_1}"] \arrow[ur, phantom, "\mathrm{(V)}" near end, shift right=6] & \mu_{\underline{1}} \circ D(\partial_1)
\end{tikzcd}
$$

Here (III) commutes because by the naturality of η and the other cells commute by the pseudonatural transformation composition axiom. Omitting the cells (I) and (V) yields the diagram for the case $(\underline{m}, \underline{n}) = (\underline{2}, \underline{0})$ and $f = \partial_0 \circ \partial_1$.

When $(\underline{m}, \underline{n}) = (\underline{2}, \underline{1})$ and $f = \partial_2$ this follows from the commutativity of the following diagram

$$
\begin{tikzcd}[column sep=large, row sep=large]
\tau_{\underline{2}} \arrow[rr, "\tau^{2,1}_{\partial_2}"] \arrow[d, "\tau^{2,1}_{\partial_0}"'] \arrow[drr, "{\tau^{2,0}_{\partial_0\circ\partial_2} = \tau^{2,0}_{\partial_1\circ\partial_0}}" description] & & \tau_{\underline{1}} \circ D(\partial_2) \arrow[d, "\tau^{1,0}_{\partial_0}"] \arrow[dll, phantom, "\mathrm{(I)}", pos=0.25] \\
\tau_{\underline{1}} \circ D(\partial_0) \arrow[rr, "\tau^{1,0}_{\partial_1}"] \arrow[d, "\tau^{1,0}_{\partial_0}"'] \arrow[urr, phantom, "\mathrm{(II)}", pos=0.25] & & \begin{array}{c} \tau_{\underline{0}} \circ D(\partial_0) \circ D(\partial_2) \\ = \tau_{\underline{0}} \circ D(\partial_1) \circ D(\partial_0) \end{array} \arrow[dddd, "\eta"] \\
\tau_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \arrow[urr, "\beta"] \arrow[d, "\eta"'] \arrow[u, phantom, "\mathrm{(III)}" right, pos=0.5] & & \\
\mu_{\underline{0}} \circ D(\partial_0) \circ D(\partial_0) \arrow[d, "(\mu^{1,0}_{\partial_0})^{-1}"'] \arrow[drr, "\beta"] \arrow[u, phantom, "\mathrm{(IV)}" right, pos=0.5] & & \\
\mu_{\underline{1}} \circ D(\partial_0) \arrow[rr, "\mu^{1,0}_{\partial_1}"] \arrow[d, "(\mu^{2,1}_{\partial_0})^{-1}"'] \arrow[u, phantom, "\mathrm{(V)}" right, pos=0.5] & & \begin{array}{c} \mu_{\underline{0}} \circ D(\partial_0) \circ D(\partial_2) \\ = \mu_{\underline{0}} \circ D(\partial_1) \circ D(\partial_0) \end{array} \arrow[d, "(\mu^{1,0}_{\partial_0})^{-1}"] \arrow[dll, phantom, "\mathrm{(VII)}", pos=0.75] \\
\mu_{\underline{2}} \arrow[rr, "\mu^{2,1}_{\partial_2}"'] \arrow[urr, "{\tau^{2,0}_{\partial_1\circ\partial_0} = \tau^{2,0}_{\partial_0\circ\partial_2}}" description] \arrow[u, phantom, "\mathrm{(VI)}" right, pos=0.5] & & \mu_{\underline{1}} \circ D(\partial_2)
\end{tikzcd}
$$

Here, (I), (II), (VI) and (VII) commute by the pseudonatural composition axiom, (III) and (V) commute by the definition of the balancing transformations for $\varphi(\tau)$ and $\varphi(\mu)$ and (IV) commutes because η is $\mathcal{A}$-balanced. Omitting the cells (I) and (VII) yields the diagram for the case $(\underline{m}, \underline{n}) = (\underline{2}, \underline{0})$ and $f = \partial_0 \circ \partial_2$.

Finally, the case $(\underline{m}, \underline{n}) = (\underline{2}, \underline{0})$ and $f = \partial_1 \circ \partial_2$ follows from the commutativity of the following diagram

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\tau_{\underline{2}} \arrow[r, "\tau^{\underline{2},\underline{1}}_{\partial_2}"] \arrow[d, "\Gamma_{\underline{2}}"'] \arrow[rr, bend left=50, "\tau^{\underline{2},\underline{0}}_{\partial_1 \circ \partial_2}"] \arrow[dr, phantom, "\text{(II)}"] & \tau_{\underline{1}} \circ D(\partial_2) \arrow[r, "\tau^{\underline{1},\underline{0}}_{\partial_1}"] \arrow[d, "\Gamma_{\underline{1}}"] \arrow[u, phantom, "\text{(I)}"] \arrow[dr, phantom, "\text{(III)}"] & \begin{array}{c} \tau_{\underline{0}} \circ D(\partial_1) \circ D(\partial_2) \\ = \tau_{\underline{0}} \circ D(\partial_1 \circ \partial_2) \end{array} \arrow[d, "\Gamma_{\underline{0}}"] \\
\mu_{\underline{2}} \arrow[r, "\mu^{\underline{2},\underline{1}}_{\partial_2}"] \arrow[rr, bend right=50, "\mu^{\underline{2},\underline{0}}_{\partial_1 \circ \partial_2}"'] & \mu_{\underline{1}} \circ D(\partial_2) \arrow[r, "\mu^{\underline{1},\underline{0}}_{\partial_1}"] \arrow[d, phantom, "\text{(IV)}"] & \begin{array}{c} \mu_{\underline{0}} \circ D(\partial_1) \circ D(\partial_2) \\ = \mu_{\underline{0}} \circ D(\partial_1 \circ \partial_2) \end{array} \\
& {} &
\end{tikzcd}
$$

Here (I) and (IV) commute by the pseudonatural transformation composition axiom and the commutativity of (II) and (III) was shown above. We have shown that Γ is a modification and by construction we have $\varphi(\Gamma) = \eta$. Therefore φ is full.

With regard to pseudonaturality, given a $*$-functor $F : \mathcal{C} \to \mathcal{D}$, the commutativity of the diagram

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\mathsf{Tran}(D, \Delta(\mathcal{C})) \arrow[r, "F \circ -"] \arrow[d, "\varphi"'] & \mathsf{Tran}(D, \Delta(\mathcal{D})) \arrow[d, "\varphi"] \\
\mathsf{Bal}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}; \mathcal{C}) \arrow[r, "F \circ -"] & \mathsf{Bal}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}; \mathcal{D})
\end{tikzcd}
$$

is immediate from the definitions. Similarly, if $\alpha : F \to G : \mathcal{C} \to \mathcal{D}$ is a natural transformation, the equality of the natural transformations

$$1_\varphi * (\alpha * -) : \varphi \circ (F \circ -) \to \varphi \circ (G \circ -)$$

and

$$(\alpha * -) * 1_\varphi : (F \circ -) \circ \varphi \to (G \circ -) \circ \varphi$$

is immediate from the definitions. Therefore the φ's are 2-natural in $\mathcal{C}$. $\square$

The balanced tensor product $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ is unique up to equivalence. This can be proved by considering its universal property, similarly to the case of bicolimits.

Similarly to the classical case, we have the following:

Lemma 2.39. *Let $\mathcal{M}$ be a left $\mathcal{A}$-module category. Then $\mathcal{A} \boxtimes_{\mathcal{A}} \mathcal{M} \simeq \mathcal{M}$.*

Proof. It is sufficient to construct an $\mathcal{A}$-balanced functor $\boxtimes_{\mathcal{A}} : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ that induces equivalences of categories

$$- \circ \boxtimes_{\mathcal{A}} : C^*\text{-}\mathsf{Add}(\mathcal{M}, \mathcal{C}) \xrightarrow{\simeq} \mathsf{Bal}_{\mathcal{A}}(\mathcal{A}, \mathcal{M}; \mathcal{C})$$

for all $\mathcal{C} \in C^*\text{-}\mathsf{Add}$. To that end, we define $\boxtimes_{\mathcal{A}}$ as $\otimes : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, the module action of $\mathcal{A}$ on $\mathcal{M}$ with balancing transformation α, the associator of $\otimes$. That $\boxtimes_{\mathcal{A}}$ is balanced follows from the module category axioms.

To show that we have an equivalence of categories

$$- \circ \boxtimes_{\mathcal{A}} : C^*\text{-}\mathsf{Add}(\mathcal{M}, \mathcal{C}) \xrightarrow{\simeq} \mathsf{Bal}_{\mathcal{A}}(\mathcal{A}, \mathcal{M}; \mathcal{C})$$

let $F : \mathcal{A} \times \mathcal{M} \to \mathcal{C}$ be an $\mathcal{A}$-balanced functor with balancing transformation β. We define a $*$-functor $\widetilde{F} : \mathcal{M} \to \mathcal{C}$ on objects by $\widetilde{F}(M) := F(I, M)$ and on morphisms by $\widetilde{F}(f) := F(1_I, f)$.

Then, given $(A, M) \in \mathcal{A} \times \mathcal{M}$ we have

$$\widetilde{F} \circ \boxtimes_{\mathcal{A}}(A, M) = F(I, A \otimes M).$$

We have a natural isomorphism $\eta : \widetilde{F} \circ \boxtimes_{\mathcal{A}} \to F$ whose (A, M) component is

$$F(I, A \otimes M) \xrightarrow{\beta^{-1}_{I,A,M}} F(I \otimes A, M) \xrightarrow{F(\lambda_A, 1_M)} F(A, M).$$

This is an $\mathcal{A}$-balanced natural transformation because of the commutativity of the following diagram.

$$
\begin{tikzcd}
\widetilde{F}\circ\boxtimes_{\mathcal{A}}(A\otimes B,M) = F(I,(A\otimes B)\otimes M) \arrow[rr, "F(1_I,\alpha_{A,B,M})"] \arrow[d, "\beta^{-1}_{I,A\otimes B,M}"'] \arrow[drr, phantom, "\mathrm{(I)}"] & & \widetilde{F}\circ\boxtimes_{\mathcal{A}}(A,B\otimes M) = F(I,A\otimes(B\otimes M)) \arrow[d, "\beta^{-1}_{I,A,B\otimes M}"] \\
F(I\otimes(A\otimes B),M) \arrow[r, "F(\alpha^{-1}_{I,A,B},1_M)"] \arrow[d, "F(\lambda_{A\otimes B},1_M)"'] \arrow[dr, phantom, "\mathrm{(II)}"] & F((I\otimes A)\otimes B,M) \arrow[r, "\beta_{I\otimes A,B,M}"] \arrow[dl, "F(\lambda_A\otimes 1_B,1_M)"] \arrow[dr, phantom, "\mathrm{(III)}"] & F(I\otimes A,B\otimes M) \arrow[d, "F(\lambda_A,1_{B\otimes M})"] \\
F(A\otimes B,M) \arrow[rr, "\beta_{A,B,M}"] & & F(A,B\otimes M)
\end{tikzcd}
$$

Here, (I) commutes because F is $\mathcal{A}$-balanced, (II) commutes by the tensor category unit axioms and (III) commutes by the naturality of β. Therefore, $\widetilde{F}\circ\boxtimes_{\mathcal{A}}\cong F$ and hence $-\circ\boxtimes_{\mathcal{A}}$ is essentially surjective.

Next, to show that $-\circ\boxtimes_{\mathcal{A}}$ is faithful, suppose that $\eta : F\to G : \mathcal{M}\to\mathcal{C}$ is a natural transformation such that $\eta * 1_{\boxtimes_{\mathcal{A}}} = 0$. Then $\eta_{A\otimes M} = 0$ for all $A\in\mathcal{A}$, $M\in\mathcal{M}$. In particular, $\eta_{I\otimes M} = 0$ for all $M\in\mathcal{M}$. By naturality of η we have a commutative diagram

$$
\begin{tikzcd}
F(I\otimes M) \arrow[r, "F(\lambda_M)"] \arrow[d, "\eta_{I\otimes M}"'] & F(M) \arrow[d, "\eta_M"] \\
G(I\otimes M) \arrow[r, "G(\lambda_M)"] & G(M)
\end{tikzcd}
$$

and since $F(\lambda_M)$ is invertible, it follows that $\eta_M = 0$ and hence $\eta = 0$. Therefore, $-\circ\boxtimes_{\mathcal{A}}$ is faithful.

Finally, to show that $-\circ\boxtimes_{\mathcal{A}}$ is full, let $\eta : F\circ\boxtimes_{\mathcal{A}}\to G\circ\boxtimes_{\mathcal{A}}$ be an $\mathcal{A}$-balanced natural transformation. We define a natural transformation $\widetilde{\eta} : F\to G$ by defining $\widetilde{\eta}_M$ as the composite

$$F(M)\xrightarrow{F(\lambda_M^{-1})} F(I\otimes M)\xrightarrow{\eta_{I,M}} G(I\otimes M)\xrightarrow{G(\lambda_M)} G(M).$$

To show that $(\widetilde{\eta} * 1_{\boxtimes_{\mathcal{A}}})_{A,M} = \eta_{A,M}$ we want to show that the following diagram commutes

$$\begin{tikzcd}[column sep=large]
\begin{array}{c} F \circ \boxtimes_{\mathcal{A}}(A, M) \\ = F(A \otimes M) \end{array} \arrow[r, "F(\lambda^{-1}_{A \otimes M})"] \arrow[d, "\eta_{A,M}"'] & \begin{array}{c} F \circ \boxtimes_{\mathcal{A}}(I, A \otimes M) \\ = F(I \otimes (A \otimes M)) \end{array} \arrow[d, "\eta_{I, A \otimes M}"] \\
\begin{array}{c} G \circ \boxtimes_{\mathcal{A}}(A, M) \\ = G(A \otimes M) \end{array} \arrow[r, "G(\lambda^{-1}_{A \otimes M})"] & \begin{array}{c} G \circ \boxtimes_{\mathcal{A}}(I, A \otimes M) \\ = G(I \otimes (A \otimes M)) \end{array}
\end{tikzcd}$$

This follows from the commutativity of the following diagram.

$$\begin{tikzcd}[column sep=large]
\begin{array}{c} F \circ \boxtimes_{\mathcal{A}}(A, V) \\ = F(A \otimes V) \end{array} \arrow[rr, bend left=40, "F(\lambda^{-1}_{A \otimes V})"] \arrow[r, "\substack{F \circ \boxtimes_{\mathcal{A}}(\lambda_A^{-1} \otimes 1_V) \\ = F(\lambda_A^{-1} \otimes 1_V)}"] \arrow[d, "\eta_{A,V}"'] \arrow[dr, phantom, "\mathrm{(II)}"] & \begin{array}{c} F \circ \boxtimes_{\mathcal{A}}(I \otimes A, V) \\ = F((I \otimes A) \otimes V) \end{array} \arrow[u, phantom, "\mathrm{(I)}"] \arrow[r, "F(\alpha)"] \arrow[d, "\eta_{I \otimes A, V}"] \arrow[dr, phantom, "\mathrm{(III)}"] & \begin{array}{c} F \circ \boxtimes_{\mathcal{A}}(I, A \otimes V) \\ = F(I \otimes (A \otimes V)) \end{array} \arrow[d, "\eta_{I, A \otimes V}"] \\
\begin{array}{c} G \circ \boxtimes_{\mathcal{A}}(A, V) \\ = G(A \otimes V) \end{array} \arrow[rr, bend right=40, "G(\lambda^{-1}_{A \otimes V})"'] \arrow[r, "\substack{G \circ \boxtimes_{\mathcal{A}}(\lambda_A^{-1} \otimes 1_V) \\ = G(\lambda_A^{-1} \otimes 1_V)}"] & \begin{array}{c} G \circ \boxtimes_{\mathcal{A}}(I \otimes A, V) \\ = G((I \otimes A) \otimes V) \end{array} \arrow[d, phantom, "\mathrm{(IV)}"] \arrow[r, "G(\alpha)"] & \begin{array}{c} G \circ \boxtimes_{\mathcal{A}}(I, A \otimes V) \\ = G(I \otimes (A \otimes V)) \end{array}
\end{tikzcd}$$

Here (I) and (IV) commute by the module category unit axioms, (II) commutes by the naturality of η and (III) commutes because η is $\mathcal{A}$-balanced. Therefore $\widetilde{\eta} * 1_{\boxtimes_{\mathcal{A}}} = \eta$ and hence $- \circ \boxtimes_{\mathcal{A}}$ is full. $\square$

If $(\mathcal{N}_i)_{i \in I}$ is a collection of left $\mathcal{A}$-module categories then their direct sum $\bigoplus_{i \in I} \mathcal{N}_i$ inherits a left $\mathcal{A}$-module structure in a natural way. The balanced tensor product commutes with direct sums in the following sense.

Lemma 2.40. *Let $\mathcal{M}$ be a right $\mathcal{A}$-module category and $(\mathcal{N}_i)_{i\in I}$ a collection of left $\mathcal{A}$-module categories. Then $\mathcal{M} \boxtimes_{\mathcal{A}} \left(\bigoplus_{i\in I} \mathcal{N}_i\right) \simeq \bigoplus_{i\in I} \mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}_i$.*

Proof. This follows from the fact that for all $\mathcal{C} \in C^*\text{-}\mathsf{Add}$, we have equivalences of C^*-categories

$$\begin{aligned} C^*\text{-}\mathsf{Add}\left(\mathcal{M} \boxtimes_{\mathcal{A}} \left(\bigoplus_{i\in I} \mathcal{N}_i\right), \mathcal{C}\right) &\simeq \mathsf{Bal}_{\mathcal{A}}\left(\mathcal{M}, \bigoplus_{i\in I} \mathcal{N}_i; \mathcal{C}\right) \\ &\simeq \prod_{i\in I} \mathsf{Bal}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}_i; \mathcal{C}) \\ &\simeq \prod_{i\in I} C^*\text{-}\mathsf{Add}(\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}_i; \mathcal{C}) \\ &\simeq C^*\text{-}\mathsf{Add}\left(\bigoplus_{i\in I} \mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}_i, \mathcal{C}\right). \end{aligned}$$

The equivalences

$$\mathsf{Bal}_{\mathcal{A}}\left(\mathcal{M}, \bigoplus_{i\in I} \mathcal{N}_i; \mathcal{C}\right) \simeq \prod_{i\in I} \mathsf{Bal}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}_i; \mathcal{C})$$

are induced by the inclusion $*$-functors $\iota_j : \mathcal{N}_j \to \bigoplus_{i\in I} \mathcal{N}_i$ with an $\mathcal{A}$-balanced functor $F : \mathcal{M} \times \bigoplus_{i\in I} \mathcal{N}_i \to \mathcal{C}$ being mapped to $F \circ (1_{\mathcal{M}} \times \iota_j)$. The proof that this is an equivalence is similar to that of Lemma 2.19. □

3 Categorical representation theory

In this section, we shall given the definitions we need from categorical representation theory, an introduction to which can be found in [3]. In particular, we are interested in groups acting on C^*-categories. Due to the similarity between modules over a ring and representations of a group, the definitions are similar to those of module categories and their morphisms. In the classical case, one can always restrict the action of a group G to a subgroup H and this process has a left adjoint whereby one constructs a representation of G from a representation of H. We shall discuss the categorical analogues of these processes and show that they are biadjoint to one another. Before we begin, let us first fix some notational conventions. Throughout,

G will always denote a discrete group, H a subgroup of G and we shall denote the identity element of G by e. We shall use the same notation for a representation and its underlying vector space so we shall simply refer to a representation V of G where V is a complex vector space and we shall denote the action of $g \in G$ on $v \in V$ by $g \cdot v$.

3.1 G-categories

To motivate the categorical definitions, we shall remark on a categorical way of framing representations of a group. A group G can be considered as a category with one object, the morphisms of which correspond to the elements of G and the composition rule given by the multiplication in the group. Then a complex representation of G can be viewed as as a functor from G to the category Vect of complex vector spaces. Natural transformations between such functors correspond to intertwining operators, that is linear maps between the underlying complex vector spaces that commute with the actions of G. Therefore, one can identify the category $\mathsf{Rep}(G)$ of complex representations of G with the functor category $[G, \mathsf{Vect}]$. As a matter of notation, given $V, W \in \mathsf{Rep}(G)$, we shall denote the space of G-intertwiners from V to W by $\mathrm{Hom}_G(V, W)$.

In a similar way, one can consider a group as a 2-category with one object, the 1-cells of which correspond to the elements of G and the only 2-cells are the identities. One can then define the action of G on a C^*-category as a pseudofunctor from G to C^*-Cat, although ultimately we shall restrict our attention to additive, subobject complete C^*-categories. As usual, since we are in the C^*-setting, we shall require that the coherence maps be unitaries.

Definition 3.1. *Let G be a group and $\mathcal{V}$ a C^*-category. An **action** of G on $\mathcal{V}$ consists of the following:*

1. *$*$-functors $\pi_t : \mathcal{V} \to \mathcal{V}$, for all $t \in G$.*

2. *Unitary natural transformations $\mu_{s,t} : \pi_s \circ \pi_t \to \pi_{st}$, for all $s, t \in G$.*

3. *A unitary natural transformation $\varepsilon : 1_{\mathcal{V}} \to \pi_e$.*

These data are subject to the following axioms: For all $r, s, t \in G$, the following diagram commutes

$$\begin{tikzcd}
\pi_r \circ \pi_s \circ \pi_t \arrow[r, "\mu_{r,s} * 1_{\pi_t}"] \arrow[d, "1_{\pi_r} * \mu_{s,t}"'] & \pi_{rs} \circ \pi_t \arrow[d, "\mu_{rs,t}"] \\
\pi_r \circ \pi_{st} \arrow[r, "\mu_{r,st}"] & \pi_{rst}
\end{tikzcd}$$

For all $r \in G$, the following diagrams commute

$$\begin{tikzcd}
\pi_r \arrow[r, "1_{\pi_r} * \varepsilon"] \arrow[dr, "1_{\pi_r}"'] & \pi_r \circ \pi_e \arrow[d, "\mu_{r,e}"] \\
& \pi_r
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_r \arrow[r, "\varepsilon * 1_{\pi_r}"] \arrow[dr, "1_{\pi_r}"'] & \pi_e \circ \pi_r \arrow[d, "\mu_{e,r}"] \\
& \pi_r
\end{tikzcd}$$

*We call such a category $\mathcal{V}$ with an action of G a G-**category**.*

If there are different categories $\mathcal{V}$ and $\mathcal{W}$ being acted upon by G, we will often denote the corresponding $*$-functors and unitary natural transformations by $\pi_t^{\mathcal{V}}$, $\mu_{s,t}^{\mathcal{V}}, \varepsilon^{\mathcal{V}}$ and $\pi_t^{\mathcal{W}}$, $\mu_{s,t}^{\mathcal{W}}, \varepsilon^{\mathcal{W}}$ respectively. Given two G-categories $\mathcal{V}$ and $\mathcal{W}$, our analogue of an intertwining operator between representations is a pseudonatural transformation between the corresponding pseudofunctors, rephrased as follows.

Definition 3.2. *Let $\mathcal{V}$ and $\mathcal{W}$ be G-categories. A G-**intertwiner** from $\mathcal{V}$ to $\mathcal{W}$ consists of a $*$-functor*

$$\Theta : \mathcal{V} \to \mathcal{W}$$

together with unitary natural transformations

$$\Theta_t : \pi_t^{\mathcal{W}} \circ \Theta \to \Theta \circ \pi_t^{\mathcal{V}}$$

for all $t \in G$ such that for all $s, t \in G$, the following diagrams commute

$$\begin{tikzcd}
\pi_s^{\mathcal{W}} \circ \pi_t^{\mathcal{W}} \circ \Theta \arrow[r, "\pi_s^{\mathcal{W}} \Theta_t"] \arrow[d, "(\mu_{s,t}^{\mathcal{W}})_\Theta"'] & \pi_s^{\mathcal{W}} \circ \Theta \circ \pi_t^{\mathcal{V}} \arrow[r, "(\Theta_s)_{\pi_t^{\mathcal{V}}}"] & \Theta \circ \pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \arrow[d, "\Theta \mu_{s,t}^{\mathcal{V}}"] \\
\pi_{st}^{\mathcal{W}} \circ \Theta \arrow[rr, "\Theta_{st}"] & & \Theta \circ \pi_{st}^{\mathcal{V}}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Theta \arrow[dl, "\varepsilon^{\mathcal{W}}_{\Theta}"'] \arrow[dr, "\Theta\varepsilon^{\mathcal{V}}"] & \\
\pi^{\mathcal{W}}_e \circ \Theta \arrow[rr, "\Theta_e"] & & \Theta \circ \pi^{\mathcal{V}}_e
\end{tikzcd}$$

We shall also refer to the G-intertwiner itself as Θ, hopefully it should be clear form the context whether the G-intertwiner or its underlying $*$-functor is being referred to.

Finally, rephrasing the definition of a modification, we have the following:

Definition 3.3. *Let $\Theta, \Phi : \mathcal{V} \to \mathcal{W}$ be G-intertwiners. A* ***G-natural transformation***

$$\kappa : \Theta \to \Phi$$

is a natural transformation such that for all $t \in G$, the following diagram commutes

$$\begin{tikzcd}
\pi^{\mathcal{W}}_t \circ \Theta \arrow[r, "\pi^{\mathcal{W}}_t \kappa"] \arrow[d, "\Theta_t"'] & \pi^{\mathcal{W}}_t \circ \Phi \arrow[d, "\Phi_t"] \\
\Theta \circ \pi^{\mathcal{V}}_t \arrow[r, "\kappa_{\pi^{\mathcal{V}}_t}"] & \Phi \circ \pi^{\mathcal{V}}_t
\end{tikzcd}$$

As in the case of module categories, there is a 2-category of G-categories, G-intertwiners and G-natural transformations which is essentially the pseudofunctor 2-category $\mathsf{Psd}[G, C^*\text{-}\mathsf{Cat}]$. Ultimately, we shall be restricting our attention to additive, subobject complete G-categories. We shall denote the 2-category of such categorical representations of G by $\mathsf{REP}(G)$ and given $\mathcal{V}, \mathcal{W} \in \mathsf{REP}(G)$ we shall denote the C^*-category of G-intertwiners from $\mathcal{V}$ to $\mathcal{W}$ and G-natural transformations between them by $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$.

3.2 Induction and restriction

3.2.1 Classical theory

Before describing categorical induced and restricted representations, let us recap some of the classical theory.

If G is a group and H a subgroup of G then any representation V of G restricts to a representation of H which we denote by $\mathrm{Res}^G_H(V)$. Furthermore, any intertwining operator $\varphi : V \to W$ in $\mathsf{Rep}(G)$ yields an intertwining operator

$$\mathrm{Res}^G_H(\varphi) : \mathrm{Res}^G_H(V) \to \mathrm{Res}^G_H(W).$$

Together with the restricted representations, this defines a functor

$$\mathrm{Res}^G_H : \mathsf{Rep}(G) \to \mathsf{Rep}(H).$$

This functor has a left adjoint which maps a representation V of H to the induced representation $\mathrm{Ind}^G_H(V)$ of G. The induced representation can be constructed in various different yet equivalent ways, three of which we shall now describe. We shall base our main categorical approach on the first, function based approach since this is the approach we shall use to construct categorical Hecke algebras but we shall also compare with categorical versions of the other constructions. Our first construction of the induced representation is the following:

Definition 3.4. *For $V \in \mathsf{Rep}(H)$, we define the* ***induced representation*** $\mathrm{Ind}^G_H(V)$ *of G as follows: The underlying vector space is the space of functions*

$$\{f : G \to V \mid f \text{ has finite support} \bmod H,\ h \cdot f(gh) = f(g)\ \forall g \in G, h \in H\}.$$

To say that f has finite support $\bmod H$ means that $f(g) = 0$ for all g outside a finite set of left cosets of H in G. The action of G on $\mathrm{Ind}^G_H(V)$ is given by

$$(t \cdot f)(g) := f(t^{-1}g), \quad t, g \in G.$$

This construction can be viewed as the space of fixed points in a representation of H. Specifically, if we denote the subspace of the space of functions $f : G \to V$ with finite support $\bmod H$ by $C_H(G, V)$, then there is an action of H on this space defined by

$$(h \cdot f)(g) := h \cdot f(gh), \quad f \in C_H(G, V),\ g \in G,\ h \in H.$$

The induced representation can then be described as the space $C_H(G, V)^H$ of fixed points of this action, along with the action of G described above.

Definition 3.5. *Given an intertwiner $\varphi : V \to W$ in $\mathsf{Rep}(H)$, we define the induced intertwiner $\mathrm{Ind}^G_H(\varphi) : \mathrm{Ind}^G_H(V) \to \mathrm{Ind}^G_H(W)$ by*

$$\mathrm{Ind}^G_H(\varphi)(f)(t) := \varphi(f(t)), \quad f \in \mathrm{Ind}^G_H(V),\ t \in G.$$

Therefore we have a functor

$$\mathrm{Ind}_H^G : \mathsf{Rep}(H) \to \mathsf{Rep}(G)$$

and it is well known that this functor is a left adjoint to the restriction functor.

Another approach is to construct $\mathrm{Ind}_H^G(V)$ as a direct sum of isomorphic copies of the underlying vector space.

Definition 3.6. *For $V \in \mathsf{Rep}(H)$, we define the* ***induced representation*** $\mathrm{Ind}_H^G(V)$ *of G as follows: We fix a set Γ of coset representatives for G/H and define the underlying vector space as*

$$\bigoplus_{\gamma \in \Gamma} \gamma V$$

where each γV is an isomorphic copy of (the vector space) V. We write the elements of γV as γv where $v \in V$. The action of G on $\mathrm{Ind}_H^G(V)$ is given by

$$t \cdot \gamma v := \gamma'(h \cdot v), \quad t \in G$$

where $h \in H$ and $\gamma' \in \Gamma$ are determined by $t\gamma = \gamma' h$.

One thinks of the summand γV as a space of formal translates of elements of V by the element $\gamma \in \Gamma$. One can define a G-intertwiner

$$\varphi : \bigoplus_{\gamma \in \Gamma} \gamma V \xrightarrow{\cong} C_H(G, V)^H$$

by

$$\varphi(\gamma v)(g) := \begin{cases} h^{-1} \cdot v & \text{if } g = \gamma h \in \gamma H, \\ 0 & \text{otherwise.} \end{cases}$$

Our final approach to induced representations uses the fact that we can identify representations of H with (left) modules over the complex group ring $\mathbb{C}[H]$. We can also view $\mathbb{C}[G]$ as a right $\mathbb{C}[H]$-module with the action given by multiplication in the group ring and under these identifications we have the following characterisation of the induced representation.

Definition 3.7. *Let V be a left $\mathbb{C}[H]$-module. The* ***induced*** $\mathbb{C}[G]$***-module*** *is constructed as follows: The underlying vector space is $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ and the $\mathbb{C}[G]$-module structure is induced by left multiplication on the left tensor factor.*

Let us denote the element in $\mathbb{C}[G]$ corresponding to $g \in G$ by $[g]$. One can define an isomorphism of $\mathbb{C}[G]$-modules/representations of G

$$\psi : \mathbb{C}[G] \otimes V \xrightarrow{\cong} C_H(G, V)^H$$

by

$$\psi([g] \otimes v)(x) := \begin{cases} h^{-1} \cdot v & \text{if } x = gh \in gH, \\ 0 & \text{otherwise.} \end{cases}$$

Let us briefly check that this is well defined. Given $k \in H$, $[gk] \otimes k^{-1} \cdot v = [g] \otimes v$ in $\mathbb{C}[G]$ so we should have $\psi([gk] \otimes k^{-1} \cdot v) = \psi([g] \otimes v)$. This is the case because,

$$\begin{aligned} \psi([gk] \otimes k^{-1} \cdot v)(x) &= \begin{cases} \ell^{-1} \cdot (k^{-1} \cdot v) & \text{if } x = gk\ell \in gkH = gH, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (k\ell)^{-1} \cdot v & \text{if } x = gk\ell \in gH, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} h^{-1} \cdot v & \text{if } x = gh \in gH, \\ 0 & \text{otherwise} \end{cases} \\ &= \psi([g] \otimes v)(x). \end{aligned}$$

Therefore, ψ is well-defined. Let us also check that ψ is a G-intertwiner. Given $t \in G$ we have

$$\begin{aligned} \psi(t \cdot ([g] \otimes v))(x) &= \psi([tg] \otimes v)(x) \\ &= \begin{cases} h^{-1} \cdot v & \text{if} x = tgh \in tgH, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} h^{-1} \cdot v & \text{if} t^{-1}x = gh \in gH, \\ 0 & \text{otherwise} \end{cases} \\ &= \psi([g] \otimes v)(t^{-1}x) \\ &= (t \cdot \psi([g] \otimes v))(x). \end{aligned}$$

Therefore $\psi(t \cdot ([g] \otimes v)) = t \cdot \psi([g] \otimes v)$ and hence ψ is a G-intertwiner.

3.2.2 Restricted and induced G-categories

In this section we shall construct the induced G-category of an H-category $\mathcal{V}$ as a category of functions analogous to $C_H(G,V)^H$ and see how this can be viewed as a direct sum and balanced tensor product similarly to the classical case. Before we do, let us briefly define and fix notation for the restricted H-category of a G-category.

Definition 3.8. *Let $\mathcal{V}$ be a G-category. We define the **restricted** H-category $\mathrm{Res}_H^G(\mathcal{V})$ as follows: The underlying C^*-category is $\mathcal{V}$. The action of H is given by*

$$\pi_k^{\mathrm{Res}(\mathcal{V})} := \pi_k^{\mathcal{V}}$$

for each $k \in H$. The coherence unitary natural transformations are defined by

$$\mu_{k,\ell}^{\mathrm{Res}(\mathcal{V})} := \mu_{k,\ell}^{\mathcal{V}}$$

for all $k, \ell \in H$ and

$$\varepsilon^{\mathrm{Res}(\mathcal{V})} := \varepsilon^{\mathcal{V}}.$$

Often we shall simply denote $\mathrm{Res}_H^G(\mathcal{V})$ by $\mathcal{V}$ when no confusion should occur.

Now, given an H-category $\mathcal{V}$, to construct the induced G-category, we first need to define a categorical version of the vector space $C_H(G,V)$. We shall need our categories to have zero objects for this and although not strictly necessary for everything that follows, from this point onwards we shall assume that all our G and H-categories are additive and subobject complete.

Definition 3.9. *Let $\mathcal{V}$ be an H-category. We define $C_H(G,\mathcal{V})$ to be the category whose objects are functions*

$$\mathbf{X} : G \to \mathrm{ob}\,\mathcal{V}$$

of finite support $\bmod H$*, meaning that $\mathbf{X}(g) = 0$ for all $g \in G$ outside a finite set of left cosets of H in G. The morphisms $\alpha : \mathbf{X} \to \mathbf{Y}$ are functions*

$$\alpha : G \to \mathrm{mor}\,\mathcal{V}$$

such that for all $g \in G$, $\alpha(g)$ is a morphism from $\mathbf{X}(g)$ to $\mathbf{Y}(g)$.

This is an additive C^*-category with direct sums and subobjects being defined pointwise.

Definition 3.10. *Let $\mathcal{V}$ be a G-category. A **fixed point** or G**-equivariant object** $(V, (\rho_t^V))$ of $\mathcal{V}$ consists of an object $V \in \mathcal{V}$ together with unitaries*

$$\rho_t^V : \pi_t(V) \to V$$

*called **trivialisers** for all $t \in G$ such that for all $s, t \in G$, the following diagram commutes*

$$\begin{tikzcd} \pi_s \circ \pi_t(V) \arrow[r, "\pi_s(\rho_t^V)"] \arrow[d, "(\mu_{s,t})_V"'] & \pi_s(V) \arrow[d, "\rho_s^V"] \\ \pi_{st}(V) \arrow[r, "\rho_{st}^V"] & V \end{tikzcd}$$

To avoid cumbersome notation, when it will not cause confusion we shall denote a fixed point $(V, (\rho_t^V))$ simply by V. Hopefully it will be clear from the context whether it is the fixed point or its underlying object which is being referred to.

Definition 3.11. *Let $\mathcal{V}$ be a G-category and let V and W be fixed points. A **morphism of fixed points** $f : V \to W$ is a morphism in $\mathcal{V}$ between the underlying objects that commutes with the fixed point trivialisers. Explicitly, for all $t \in G$, the following diagram commutes:*

$$\begin{tikzcd} \pi_t(V) \arrow[r, "\pi_t(f)"] \arrow[d, "\rho_t^V"'] & \pi_t(W) \arrow[d, "\rho_t^W"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

Since each π_t is a $*$-functor, the composition of two morphisms of fixed points is again a morphism of fixed points and hence we have the following:

Definition 3.12. *Let $\mathcal{V}$ be a G-category. The associated **category of fixed points**, denoted $\mathcal{V}^G$, has the set of fixed points for $\mathcal{V}$ as objects and morphisms of fixed points as arrows.*

In general, given an object $V \in \mathcal{V}$ there may be more than one set of trivialisers that equip it with a fixed point structure, that is, there may be distinct fixed points with the same underlying object V. These need not be isomorphic in the category of fixed points. For example, one can give Vect a trivial G-category structure and then a fixed point $(V, (\rho_t^V))$ is a representation of G on V and the category of fixed points is $\mathsf{Rep}(G)$. Since we may have distinct fixed points with the same underlying object of $\mathcal{V}$, in general $\mathcal{V}^G$ is not identifiable with a subcategory of $\mathcal{V}$.

Lemma 3.13. *Let G be a group and $\mathcal{V}$ a G-category. The fixed point category $\mathcal{V}^G$ is an additive, subobject complete C^*-category.*

Proof. First let us show that $\mathcal{V}^G$ is a C^*-category. Since linear combinations of morphisms of fixed points are again morphisms of fixed points, $\mathcal{V}^G$ inherits a complex vector space structure on its morphism sets from $\mathcal{V}$. The morphism spaces are also complete since the π_t's are continuous on morphism spaces and composition of morphism is continuous in each argument. Explicitly, if $(f_n : V \to W)_{n\in\mathbb{N}}$ is a sequence of morphisms of fixed points with limit $f : V \to W$, we need to show that the following diagram commutes

$$\begin{tikzcd} \pi_t(V) \arrow[r, "\pi_t(f)"] \arrow[d, "\rho_t^V"'] & \pi_t(W) \arrow[d, "\rho_t^W"] \\ V \arrow[r, "f"] & W \end{tikzcd} \tag{$*$}$$

for all $t \in G$. We have

$$\begin{aligned} \rho_t^W \circ \pi_t(f) &= \rho_t^W \circ \pi_t(\lim_i f_n) \\ &= \lim_i(\rho_t^W \circ \pi_t(f_n)) \\ &= \lim_i(f_n \circ \rho_t^V) \\ &= (\lim_i f_n) \circ \rho_t^V \\ &= f \circ \rho_t^V \end{aligned}$$

and hence $(*)$ commutes.

The fixed point category is also closed under taking adjoints of morphisms. If $f : V \to W$ is a morphism of fixed points then for all $t \in G$, the following diagram commutes

$$\begin{tikzcd}
\pi_t(V) \arrow[r, "\pi_t(f)"] \arrow[d, "\rho_t^V"'] & \pi_t(W) \arrow[d, "\rho_t^W"] \\
V \arrow[r, "f"] & W
\end{tikzcd}$$

Taking adjoints yields a commutative diagram

$$\begin{tikzcd}
\pi_t(V) & \pi_t(W) \arrow[l, "\pi_t(f^*)"'] \\
V \arrow[u, "(\rho_t^V)^{-1}"] & W \arrow[l, "f^*"'] \arrow[u, "(\rho_t^W)^{-1}"']
\end{tikzcd}$$

where we have used the fact that the ρ's are unitaries and π_t preserves adjoints. It follows that

$$\rho_t^V \circ \pi_t(f^*) = f^* \circ \rho_t^W$$

and hence f^* is a morphism of fixed points from W to V.

For additivity, we need to show that the fixed point category has finite direct sums. It is sufficient to prove the existence of binary direct sums. Given two fixed points $(V, (\rho_t^V))$ and $(W, (\rho_t^W))$ in $\mathcal{V}^G$ we define

$$(V, (\rho_t^V)) \oplus (W, (\rho_t^W)) := (V \oplus W, (\rho_t^{V \oplus W}))$$

where $\rho_t^{V \oplus W}$ is the composite

$$\pi_t(V \oplus W) \xrightarrow{\zeta_{V,W}^{\pi_t}} \pi_t(V) \oplus \pi_t(W) \xrightarrow{\rho_t^V \oplus \rho_t^W} V \oplus W.$$

Here $\zeta_{V,W}^{\pi_t}$ is the canonical isomorphism between direct sums defined as in Lemma 1.22. The inclusion and projection maps are the inclusion and projection maps for the underlying object $V \oplus W$.

We need to check the fixed point axiom holds. This follows from the commutativity of the following diagram.

$$\begin{tikzcd}
 & \pi_s(\pi_t(V) \oplus \pi_t(W)) \arrow[r, "\pi_s(\rho_t^V \oplus \pi_t^W)"] \arrow[d, "\zeta^{\pi_s}_{\pi_t(V),\pi_t(W)}"] \arrow[dr, phantom, "\mathrm{(II)}"] & \pi_s(V \oplus W) \arrow[d, "\zeta^{\pi_s}_{V,W}"] \\
\pi_s \circ \pi_t(V \oplus W) \arrow[ur, "\pi_s \zeta^{\pi_t}_{V,W}"] \arrow[r, "\zeta^{\pi_s \circ \pi_t}_{V,W}"] \arrow[ur, phantom, "\mathrm{(I)}"] \arrow[d, "(\mu_{s,t})_{V \oplus W}"'] \arrow[dr, phantom, "\mathrm{(III)}"] & \pi_s \circ \pi_t(V) \oplus \pi_s \circ \pi_t(W) \arrow[r, "\pi_s \rho_t^V \oplus \pi_s \rho_t^W"] \arrow[d, "(\mu_{s,t})_V \oplus (\mu_{s,t})_W"] \arrow[dr, phantom, "\mathrm{(IV)}"] & \pi_s(V) \oplus \pi_s(W) \arrow[d, "\rho_t^V \oplus \rho_t^W"] \\
\pi_{st}(V \oplus W) \arrow[r, "\zeta^{\pi_{st}}_{V,W}"] & \pi_{st}(V) \oplus \pi_{st}(W) \arrow[r, "\rho_{st}^V \oplus \rho_{st}^W"] & V \oplus W
\end{tikzcd}$$

The triangle (I) commutes by Lemma 1.23, the square (II) commutes by naturality of ζ^{π_s}, (III) commutes by Lemma 3.3 and (IV) commutes because $(V, (\rho_t^V))$ and $(W, (\rho_t^W))$ are fixed points. It is immediate that the inclusion and projection maps are morphisms of fixed points and hence $(V \oplus W, (\rho_t^{V \oplus W}))$ is the direct sum of $(V, (\rho_t^V))$ and $(W, (\rho_t^W))$.

We also need to check that $\mathcal{V}^G$ has a zero object. Since each π_t is linear, it preserves zero objects and hence any zero object 0 of $\mathcal{V}$, along with the zero maps $\rho_t^0 : \pi_t(0) \to 0$ is a fixed point. The fixed point axiom diagrams necessarily commute since all arrows are the zero map. Given any fixed point $(V, (\rho_t^V))$, the zero maps $0 \to V$ and $V \to 0$ are morphisms of fixed points between $(V, (\rho_t^V))$ and $(0, (\rho_t^0))$ since both morphisms is the fixed point morphism axiom diagram are the zero map. Therefore, $(0, (\rho_t^0))$ is a zero object in $\mathcal{V}^G$.

Lastly, we need to show that $\mathcal{V}^G$ is subobject complete. To that end, let $(V, (\rho_t^V))$ be a fixed point and let $p : (V, (\rho_t^V)) \to (V, (\rho_t^V))$ be a projection in $\mathcal{V}^G$. To construct a splitting of p, let $f : W \to V$ be a splitting of p in $\mathcal{V}$, that is $f \circ f^* = p$ and $f^* \circ f = 1_W$. Then we have a fixed point $(W, (\rho_t^W))$ where ρ_t^W is the composite

$$\pi_t(W) \xrightarrow{\pi_t(f)} \pi_t(V) \xrightarrow{\rho_t^V} V \xrightarrow{f^*} W.$$

To see that the fixed point axioms hold, consider the following diagram

$$\begin{tikzcd}
\pi_s \circ \pi_t(W) \arrow[r, "\pi_s\circ\pi_t(f)"] \arrow[ddd, "(\mu_{s,t})_W"'] \arrow[dddr, phantom, "\text{(I)}"] & \pi_s \circ \pi_t(V) \arrow[r, "\pi_s(\rho_t^V)"] \arrow[ddd, "(\mu_{s,t})_V"'] \arrow[dddr, phantom, "\text{(II)}"] & \pi_s(V) \arrow[r, "\pi_s(f^*)"] \arrow[ddd, "\rho_s^V"] \arrow[dr, "\pi_s(p)"'] \arrow[ddr, phantom, "\text{(IV)}"] & \pi_s(W) \arrow[d, "\pi_s(f)"] \arrow[dl, phantom, "\text{(III)}"] \\
 & & & \pi_s(V) \arrow[d, "\rho_s^V"] \\
 & & & V \arrow[d, "f^*"] \arrow[dl, phantom, "\text{(V)}"] \\
\pi_{st}(W) \arrow[r, "\pi_{st}(f)"] & \pi_{st}(V) \arrow[r, "\rho_{st}^V"] & V \arrow[ur, "p"] \arrow[r, "f^*"] & W
\end{tikzcd}$$

The rectangle (I) commutes by the naturality of $\mu_{s,t}$, (II) commutes because V is a fixed point, (III) commutes because $f \circ f^* = p$, (IV) commutes because p is a morphism in $\mathcal{V}^G$ and (V) commutes because

$$\begin{aligned}
f^* \circ p &= f^* \circ f \circ f^* \\
&= 1_W \circ f^* \\
&= f^*.
\end{aligned}$$

Therefore, the ρ_t^W's satisfy the fixed point axioms. That f is a morphism of fixed points follows from the commutativity of the following diagram

$$\begin{tikzcd}
\pi_t(W) \arrow[r, "\pi_t(f)"] \arrow[d, "\pi_t(f)"'] \arrow[dr, phantom, "\text{(I)}"] & \pi_t(V) \arrow[r, "\rho_t^V"] \arrow[dr, phantom, "\text{(II)}"] & V \arrow[r, "f^*"] \arrow[dr, "p"'] \arrow[dr, phantom, "\text{(III)}", near start, shift left=3ex] & W \arrow[d, "f"] \\
\pi_t(V) \arrow[ur, "\pi_t(p)"'] \arrow[rrr, "\rho_t^V"] & & & V
\end{tikzcd}$$

Here, (I) commutes because

$$\begin{aligned} p \circ f &= f \circ f^* \circ f \\ &= 1_W \circ f \\ &= f \end{aligned}$$

(II) commutes because p is a morphism in $\mathcal{V}^G$ and (III) commutes because f is a splitting of p. Therefore $f : (W, (\rho_t^W)) \to (V, (\rho_t^V))$ is a splitting of p in $\mathcal{V}^G$ and hence $\mathcal{V}^G$ is subobject complete. □

The process of taking fixed points is 2-functorial, being defined on G-intertwiners and G-natural transformations as in the following two lemmas.

Lemma 3.14. *Let $\mathcal{V}, \mathcal{W}$ be G-categories and $\Theta : \mathcal{V} \to \mathcal{W}$ a G-intertwiner. Then Θ induces a $*$-functor*

$$\Theta^G : \mathcal{V}^G \to \mathcal{W}^G$$

defined on objects as follows: Given a fixed point $(V, (\rho_t^V)) \in \mathcal{V}^G$ we define

$$\Theta^G(V, (\rho_t^V)) := \left(\Theta(V), \left(\rho_t^{\Theta(V)}\right)\right)$$

where for each $t \in G$, $\rho_t^{\Theta(V)}$ is the composite

$$\pi_t^{\mathcal{W}} \circ \Theta(V) \xrightarrow{(\Theta_t)_V} \Theta \circ \pi_t^{\mathcal{V}}(V) \xrightarrow{\Theta(\rho_t^V)} \Theta(V).$$

Proof. Let us first show that Θ^G is well-defined on objects. We need to show that for all $s, t \in G$, the following diagram commutes.

$$\begin{tikzcd} \pi_s^{\mathcal{W}} \circ \pi_t^{\mathcal{W}} \circ \Theta(V) \arrow[r, "\pi_s^{\mathcal{W}}\left(\rho_t^{\Theta(V)}\right)"] \arrow[d, "(\mu_{s,t}^{\mathcal{W}})_{\Theta(V)}"'] & \pi_s^{\mathcal{W}} \circ \Theta(V) \arrow[d, "\rho_s^{\Theta(V)}"] \\ \pi_{st}^{\mathcal{W}} \circ \Theta(V) \arrow[r, "\rho_{st}^{\Theta(V)}"] & \Theta(V) \end{tikzcd}$$

We can expand this diagram as follows:

$$\begin{tikzcd}
\pi_s^{\mathcal{W}} \circ \pi_t^{\mathcal{W}} \circ \Theta(V) \arrow[r, "\pi_s^{\mathcal{W}}((\Theta_t)_V)"] \arrow[dd, "(\mu^{\mathcal{W}}_{s,t})_{\Theta(V)}"'] & \pi_s^{\mathcal{W}} \circ \Theta \circ \pi_t^{\mathcal{V}}(V) \arrow[r, "\pi_s^{\mathcal{W}} \circ \Theta(\rho_t^V)"] \arrow[d, "(\Theta_s)_{\pi_t^{\mathcal{V}}(V)}"] \arrow[dr, phantom, "\mathrm{(II)}"] & \pi_s^{\mathcal{W}} \circ \Theta(V) \arrow[d, "(\Theta_s)_V"] \\
\arrow[r, phantom, "\mathrm{(I)}"] & \Theta \circ \pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}}(V) \arrow[r, "\Theta \circ \pi_s^{\mathcal{V}}(\rho_t^V)"] \arrow[d, "\Theta((\mu^{\mathcal{V}}_{s,t})_V)"] \arrow[dr, phantom, "\mathrm{(III)}"] & \Theta \circ \pi_s^{\mathcal{V}}(V) \arrow[d, "\Theta(\rho_s^V)"] \\
\pi_{st}^{\mathcal{W}} \circ \Theta(V) \arrow[r, "(\Theta_{st})_V"] & \Theta \circ \pi_{st}^{\mathcal{V}}(V) \arrow[r, "\Theta(\rho_{st}^V)"] & \Theta(V)
\end{tikzcd}$$

Now, (I) commutes by the G-intertwiner axioms, (II) commutes by naturality of Θ_s and (III) commutes because V is a fixed point. Therefore, the whole diagram commutes as required.

We also need to define Θ^G on morphisms. Given a morphism $f : V \to W$ in $\mathcal{V}^G$, we define $\Theta^G(f) := \Theta(f)$. To show that $\Theta(f)$ is a morphism of fixed points, we need to show that for all $t \in G$, the following diagram commutes

$$\begin{tikzcd}
\pi_t^{\mathcal{W}} \circ \Theta(V) \arrow[r, "\pi_t^{\mathcal{W}} \circ \Theta(f)"] \arrow[d, "\rho_t^{\Theta(V)}"'] & \pi_t^{\mathcal{W}} \circ \Theta(W) \arrow[d, "\rho_t^{\Theta(W)}"] \\
\Theta(V) \arrow[r, "\Theta(f)"] & \Theta(W)
\end{tikzcd}$$

We can expand this diagram as follows:

$$\begin{tikzcd}
\pi_t^{\mathcal{W}} \circ \Theta(V) \arrow[r, "\pi_t^{\mathcal{W}} \circ \Theta(f)"] \arrow[d, "(\Theta_t)_V"'] & \pi_t^{\mathcal{W}} \circ \Theta(W) \arrow[d, "(\Theta_t)_W"] \\
\Theta \circ \pi_t^{\mathcal{W}}(V) \arrow[r, "\Theta \circ \pi_t^{\mathcal{W}}(f)"] \arrow[d, "\Theta(\rho_t^V)"'] & \Theta \circ \pi_t^{\mathcal{W}}(V) \arrow[d, "\Theta(\rho_t^W)"] \\
\Theta(V) \arrow[r, "\Theta(f)"] & \Theta(W)
\end{tikzcd}$$

The top square commutes commutes by the naturality of Θ_t and the bottom square commutes because f is a morphism of fixed points. Therefore, $\Theta(f)$ is a morphism of fixed points. $\square$

Lemma 3.15. *Let $\mathcal{V}, \mathcal{W}$ be G-categories, $\Theta, \Phi : \mathcal{V} \to \mathcal{W}$ be G-intertwiners and $\kappa : \Theta :\to \Phi$ a G-natural transformation. Then there is an induced natural transformation*

$$\kappa^G : \Theta^G \to \Phi^G$$

defined by

$$\kappa^G_V := \kappa_V : \Theta(V) \to \Phi(V), \quad V \in \mathcal{V}.$$

Proof. We just need to show that for $V \in \mathcal{V}$, κ^G_V is a morphism of fixed points. Then, naturality will follow from the naturality of κ. We need to show that for all $k \in H$, the following diagram commutes

$$\begin{tikzcd}
\pi^{\mathcal{W}}_k \circ \Theta(V) \arrow[r, "\pi^{\mathcal{W}}_k(\kappa_V)"] \arrow[d, "\rho^{\Theta(V)}_k"'] & \pi^{\mathcal{W}}_k \circ \Phi(V) \arrow[d, "\rho^{\Phi(V)}_k"] \\
\Theta(V) \arrow[r, "\kappa_V"] & \Phi(V)
\end{tikzcd}$$

We can expand this diagram as follows:

$$\begin{tikzcd}
\pi^{\mathcal{W}}_k \circ \Theta(V) \arrow[r, "\pi^{\mathcal{W}}_k(\kappa_V)"] \arrow[d, "(\Theta_k)_V"'] & \pi^{\mathcal{W}}_k \circ \Phi(V) \arrow[d, "(\Phi_k)_V"] \\
\Theta \circ \pi^{\mathcal{V}}_k(V) \arrow[r, "\kappa_{\pi^{\mathcal{V}}_k(V)}"] \arrow[d, "\Theta(\rho^V_k)"'] & \Phi \circ \pi^{\mathcal{V}}_k(V) \arrow[d, "\Phi(\rho^V_k)"] \\
\Theta(V) \arrow[r, "\kappa_V"] & \Phi(V)
\end{tikzcd}$$

The top square commutes because κ is a G-natural transformation and the bottom square commutes by the naturality of κ. Therefore, κ_V is a morphism of fixed points. $\square$

Lemma 3.16. *There is a 2-functor*

$$(-)^G : \mathsf{REP}(G) \to C^*\mathsf{Add}$$

defined on G-categories as in Definition 3.12, on G-intertwiners as in Lemma 3.14 and on G-natural transformations as in Lemma 3.15.

Proof. First, let us show that $(-)^G$ preserves composition of 1-cells. Given G-intertwiners $\Theta : \mathcal{V} \to \mathcal{W}$ and $\Phi : \mathcal{W} \to \mathcal{X}$ and $V \in \mathcal{V}^G$, the underlying objects of $(\Phi \circ \Theta)^G(V)$ and $(\Phi^G \circ \Theta^G)(V)$ are both $\Phi \circ \Theta(V)$ so we just need to show that both fixed points have the same trivialisers. Let us denote the trivialisers for $(\Phi \circ \Theta)^G(V)$ by ρ_t and the trivialisers for $(\Phi^G \circ \Theta^G)(V)$ by ρ'_t ($t \in G$). Then by definition, for each $t \in G$, ρ_t is the composite

$$\pi_t^{\mathcal{X}} \circ \Phi \circ \Theta(V) \xrightarrow{\Theta_t \circ \Phi_t} \Phi \circ \Theta \circ \pi_t^{\mathcal{V}}(V) \xrightarrow{\rho_t^V} \Phi \circ \Theta(V)$$

and on the other hand, ρ'_t is the composite

$$\pi_k^{\mathcal{X}} \circ \Phi \circ \Theta(V) \xrightarrow{\Phi_t} \Phi \circ \pi_t^{\mathcal{W}} \circ \Theta(V) \xrightarrow{\rho_t^V \circ \Theta_t} \Phi \circ \Theta(V)$$

Therefore, $\rho_t = \rho'_t$ and hence $(\Phi \circ \Theta)^G(V) = (\Phi^G \circ \Theta^G)(V)$.

Next, given a morphism of fixed points $f : V \to W$ in $\mathcal{V}^G$ we have

$$\begin{aligned}(\Phi \circ \Theta)^G(f) &= \Phi \circ \Theta(f) \\ &= (\Phi^G \circ \Theta^G)(f)\end{aligned}$$

by definition. Therefore, $(\Phi \circ \Theta)^G = \Phi^G \circ \Theta^G$ so that $(-)^G$ preserves composition of 1-cells. The fact that $(-)^G$ preserves identity 1-cells follows immediately from the definition.

Lastly, we need to show that the maps between hom-categories are functorial, that is, given composable G-natural transformations κ and λ we have $(\kappa \circ \lambda)^G = \kappa^G \circ \lambda^G$. Again, this follows immediately from the definitions. $\square$

Our next step in constructing the induced G-category of an H-category $\mathcal{V}$ is to define an H-category structure on $C_H(G, \mathcal{V})$ which is the analogue of the action of H on $C_H(G, V)$ given by

$$(h \cdot f)(g) = h \cdot f(gh), \quad f \in C_H(G, V),\, g \in G,\, h \in H.$$

Definition 3.17. *For an H-category $\mathcal{V}$, we define an H-category structure on $C_H(G, \mathcal{V})$ as follows: For each $k \in H$, we define the $*$-functor*

$$\pi_k^{C_H(G,\mathcal{V})} : C_H(G, \mathcal{V}) \to C_H(G, \mathcal{V})$$

on objects by

$$\pi_k^{C_H(G,\mathcal{V})}(\mathbf{X})(g) := \pi_k^{\mathcal{V}}(\mathbf{X}(gk))$$

and on morphisms by

$$\pi_k^{C_H(G,\mathcal{V})}(\alpha)(g) := \pi_k^{\mathcal{V}}(\alpha(gk)).$$

The unitary natural transformations

$$\mu_{k,\ell}^{C_H(G,\mathcal{V})} : \pi_k^{C_H(G,\mathcal{V})} \circ \pi_\ell^{C_H(G,\mathcal{V})} \to \pi_{k\ell}^{C_H(G,\mathcal{V})}, \quad k, \ell \in H$$

are defined by

$$(\mu_{k,\ell}^{C_H(G,\mathcal{V})})_{\mathbf{X}}(g) := (\mu_{k,\ell}^{\mathcal{V}})_{\mathbf{X}(gk\ell)} : \pi_k^{\mathcal{V}} \circ \pi_\ell^{\mathcal{V}}(\mathbf{X}(gk\ell)) \to \pi_{k\ell}^{\mathcal{V}}(\mathbf{X}(gk\ell)).$$

The unitary natural transformation

$$\varepsilon^{C_H(G,\mathcal{V})} : \mathrm{Id}_{C_H(G,\mathcal{V})} \to \pi_e^{C_H(G,\mathcal{V})}$$

is defined by

$$\varepsilon_{\mathbf{X}}^{C_H(G,\mathcal{V})}(g) := \varepsilon_{\mathbf{X}(g)}^{\mathcal{V}} : \mathbf{X}(g) \to \pi_e^{\mathcal{V}}(\mathbf{X}(g)).$$

Lemma 3.18. *The $*$-functors $\pi_h^{C_H(G,\mathcal{V})}$ and unitary natural transformations $\mu_{h,k}^{C_H(G,\mathcal{V})}$ and $\varepsilon^{C_H(G,\mathcal{V})}$ constitute an H-category structure on $C_H(G, \mathcal{V})$.*

Proof. We need to show that the following diagrams commute

$$\begin{tikzcd}
\pi_h^{C_H(G,\mathcal{V})} \circ \pi_k^{C_H(G,\mathcal{V})} \circ \pi_\ell^{C_H(G,\mathcal{V})} \arrow[r, "\mu_{h,k}^{C_H(G,\mathcal{V})}"] \arrow[d, "\mu_{k,\ell}^{C_H(G,\mathcal{V})}"'] & \pi_{hk}^{C_H(G,\mathcal{V})} \circ \pi_\ell^{C_H(G,\mathcal{V})} \arrow[d, "\mu_{hk,\ell}^{C_H(G,\mathcal{V})}"] \\
\pi_h^{C_H(G,\mathcal{V})} \circ \pi_{k\ell}^{C_H(G,\mathcal{V})} \arrow[r, "\mu_{h,k\ell}^{C_H(G,\mathcal{V})}"] & \pi_{hk\ell}^{C_H(G,\mathcal{V})}
\end{tikzcd} \qquad (*)$$

$$\begin{tikzcd} \pi_h^{C_H(G,\mathcal{V})} \arrow[r, "\varepsilon^{C_H(G,\mathcal{V})}"] \arrow[dr, "\mathrm{Id}"'] & \pi_h^{C_H(G,\mathcal{V})} \circ \pi_e^{C_H(G,\mathcal{V})} \arrow[d, "\mu_{h,e}^{C_H(G,\mathcal{V})}"] \\ & \pi_h^{C_H(G,\mathcal{V})} \end{tikzcd} \qquad \begin{tikzcd} \pi_h^{C_H(G,\mathcal{V})} \arrow[r, "\varepsilon^{C_H(G,\mathcal{V})}"] \arrow[dr, "\mathrm{Id}"'] & \pi_e^{C_H(G,\mathcal{V})} \circ \pi_h^{C_H(G,\mathcal{V})} \arrow[d, "\mu_{e,h}^{C_H(G,\mathcal{V})}"] \\ & \pi_h^{C_H(G,\mathcal{V})} \end{tikzcd} \tag{$**$}$$

for all $h, k, \ell \in H$. This follows from the fact that each $\mu_{p,q}^{C_H(G,\mathcal{V})}$ is constructed using $\mu_{p,q}^{\mathcal{V}}$ and $\varepsilon^{C_H(G,\mathcal{V})}$ is constructed using $\varepsilon^{\mathcal{V}}$ and these maps satisfy the H-category axioms. For example, if we consider the diagram

$$\begin{tikzcd} \pi_h^{C_H(G,\mathcal{V})} \circ \pi_k^{C_H(G,\mathcal{V})} \circ \pi_\ell^{C_H(G,\mathcal{V})}(\mathbf{X}) \arrow[r, "\mu_{h,k}^{C_H(G,\mathcal{V})}"] \arrow[d, "\mu_{k,\ell}^{C_H(G,\mathcal{V})}"'] & \pi_{hk}^{C_H(G,\mathcal{V})} \circ \pi_\ell^{C_H(G,\mathcal{V})}(\mathbf{X}) \arrow[d, "\mu_{hk,\ell}^{C_H(G,\mathcal{V})}"] \\ \pi_h^{C_H(G,\mathcal{V})} \circ \pi_{k\ell}^{C_H(G,\mathcal{V})}(\mathbf{X}) \arrow[r, "\mu_{h,k\ell}^{C_H(G,\mathcal{V})}"] & \pi_{hk\ell}^{C_H(G,\mathcal{V})}(\mathbf{X}) \end{tikzcd}$$

each arrow is a G-indexed collection of morphisms in $\mathcal{V}$. For $g \in G$, the g component is

$$\begin{tikzcd} \pi_h^{\mathcal{V}} \circ \pi_k^{\mathcal{V}} \circ \pi_\ell^{\mathcal{V}}(\mathbf{X}(ghk\ell)) \arrow[r, "\mu_{h,k}^{\mathcal{V}}"] \arrow[d, "\mu_{k,\ell}^{\mathcal{V}}"'] & \pi_{hk}^{\mathcal{V}} \circ \pi_\ell^{\mathcal{V}}(\mathbf{X}(ghk\ell)) \arrow[d, "\mu_{hk,\ell}^{\mathcal{V}}"] \\ \pi_h^{\mathcal{V}} \circ \pi_{k\ell}^{\mathcal{V}}(\mathbf{X}(ghk\ell)) \arrow[r, "\mu_{h,k\ell}^{\mathcal{V}}"] & \pi_{hk\ell}^{\mathcal{V}}(\mathbf{X}(ghk\ell)) \end{tikzcd}$$

and this diagram commutes because the $\mu_{p,q}^{\mathcal{V}}$'s satisfy the H-category axioms. Therefore, the diagram $(*)$ commutes. The details for the other two diagrams $(**)$ are similar. $\square$

We now come to our definition of the induced G-category for an H-category $\mathcal{V}$.

Definition 3.19. *Given an H-category $\mathcal{V}$, we define the* ***induced*** *G-category $\mathrm{Ind}_H^G(\mathcal{V})$ as follows: The underlying C^*-category is the category of H-fixed points $C_H(G,\mathcal{V})^H$. For each $t \in G$, we define functors*

$$\pi_t^{\mathrm{Ind}(\mathcal{V})} : C_H(G,\mathcal{V})^H \to C_H(G,\mathcal{V})^H$$

on objects by

$$\pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})(g) := \mathbf{X}(t^{-1}g).$$

We shall also denote $\pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})$ *by* $t \cdot \mathbf{X}$*. For each* $k \in H$*, we define* $\rho_k^{t\cdot\mathbf{X}}$ *by*

$$\rho_k^{t\cdot\mathbf{X}}(g) := \rho_k^{\mathbf{X}}(t^{-1}g).$$

We define $\pi_t^{\mathrm{Ind}(\mathcal{V})}$ *on morphisms by*

$$\pi_t^{\mathrm{Ind}(\mathcal{V})}(\alpha)(g) := \alpha(t^{-1}g).$$

For all $s, t \in G$*, we have*

$$\pi_s^{\mathrm{Ind}(\mathcal{V})} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})} = \pi_{st}^{\mathrm{Ind}(\mathcal{V})}$$

and

$$\pi_e^{\mathrm{Ind}(\mathcal{V})} = \mathrm{Id}$$

and we define the unitary natural transformations $\mu_{s,t}^{\mathrm{Ind}(\mathcal{V})}$ *and* $\varepsilon^{\mathrm{Ind}(\mathcal{V})}$ *as the identity natural transformations.*

The $*$-functors $\pi_t^{\mathrm{Ind}(\mathcal{V})}$ preserve direct sums 'up to isomorphism' as in Lemma 1.21, in fact something slightly stronger is true.

Lemma 3.20. *Let* $\mathcal{V}$ *be an* H*-category. The* $*$*-functors* $\pi_t^{\mathrm{Ind}(\mathcal{V})}$ *commute with direct sums, that is, for all* $t \in G$ *and* $\mathbf{X}_i \in \mathrm{Ind}_H^G(\mathcal{V})$*,* $i = 1, \ldots, n$*, we have*

$$t \cdot \left(\bigoplus_{i=1}^n \mathbf{X}_i \right) = \bigoplus_{i=1}^n t \cdot \mathbf{X}_i.$$

Proof. Firstly, given $t \in H$, for all $g \in G$ we have

$$\begin{aligned} \left[t \cdot \left(\bigoplus_{i=1}^n \mathbf{X}_i \right) \right](g) &= \bigoplus_{i=1}^n \mathbf{X}_i(k^{-1}g) \\ &= \left[\bigoplus_{i=1}^n t \cdot \mathbf{X}_i \right](g) \end{aligned}$$

Now, for each $k \in H$ and $g \in G$, $\rho_k^{\oplus_{i=1}^n \mathbf{X}_i}(g)$ is the composite

$$\pi_k^{\mathcal{V}}\left(\bigoplus_{i=1}^n \mathbf{X}_i(gk)\right) \xrightarrow{\zeta^{\pi_k^{\mathcal{V}}}} \bigoplus_{i=1}^n \pi_k^{\mathcal{V}}(\mathbf{X}_i(gk)) \xrightarrow{\oplus \rho_k^{\mathbf{X}_i}(g)} \bigoplus_{i=1}^n \mathbf{X}_i(g)$$

where $\zeta^{\pi_k^{\mathcal{V}}}$ is the canonical unitary natural transformation for preservation of direct sums by $\pi_k^{\mathcal{V}}$. Therefore, $\rho_k^{t\cdot\oplus_{i=1}^n \mathbf{X}_i}(g)$ is the composite

$$\pi_k^{\mathcal{V}}\left(\bigoplus_{i=1}^n \mathbf{X}_i(t^{-1}gk)\right) \xrightarrow{\zeta^{\pi_k^{\mathcal{V}}}} \bigoplus_{i=1}^n \pi_k^{\mathcal{V}}(\mathbf{X}_i(t^{-1}gk)) \xrightarrow{\oplus \rho_k^{\mathbf{X}_i}(t^{-1}g)} \bigoplus_{i=1}^n \mathbf{X}_i(t^{-1}g)$$

which is equal to $t \cdot \rho_k^{\oplus_{i=1}^n \mathbf{X}_i}(g)$. It follows that

$$t \cdot \left(\bigoplus_{i=1}^n \mathbf{X}_i\right) = \bigoplus_{i=1}^n t \cdot \mathbf{X}_i.$$

$\square$

Next we shall look at how we can decompose objects in $\mathrm{Ind}_H^G(\mathcal{V})$ into direct sums of simpler objects in a way that essentially amounts to the fact that $\mathrm{Ind}_H^G(\mathcal{V})$ can be constructed as a direct sum, similarly to way that an induced representation can be constructed as a direct sum as in Definition 3.6.

Lemma 3.21. *Every object $\mathbf{X} \in C_H(G,\mathcal{V})^H$ is isomorphic to a direct sum*

$$\bigoplus_{i=1}^n \mathbf{X}_i$$

where each $\mathbf{X}_i \in C_H(G,\mathcal{V})^H$ is supported on a single left coset r_iH, $r_i \in G$.

Proof. The object $\mathbf{X}$ is supported on finitely many left cosets $r_1H, \dots, r_nH$ where each $r_i \in G$. For each $i = 1, \dots, n$, we define $\mathbf{X}_i$ by

$$\mathbf{X}_i(g) := \begin{cases} \mathbf{X}(g) & \text{if } g \in r_iH, \\ 0 & \text{otherwise} \end{cases}$$

and for each $k \in H$,

$$\rho_k^{\mathbf{X}_i}(g) := \begin{cases} \rho_k^{\mathbf{X}}(g) & \text{if } g \in r_iH, \\ 0 & \text{otherwise.} \end{cases}$$

The $\rho_k^{\mathbf{X}_i}$ are well defined due to the nature of the H-action on $C_H(G,\mathcal{V})$, in particular the fact that the translation part of the action only permutes function values within the left cosets. Furthermore, the $\rho_k^{\mathbf{X}_i}$ satisfy the fixed point axioms because the $\rho_k^{\mathbf{X}}$ do. It then follows that we have a canonical isomorphism

$$\alpha : \mathbf{X} \xrightarrow{\simeq} \bigoplus_{i=1}^{n} \mathbf{X}_i$$

consisting of the canonical isomorphisms

$$\alpha(g) : \mathbf{X}(g) \xrightarrow{\simeq} \bigoplus_{i=1}^{n} \mathbf{X}_i(g).$$

Finally, α is a morphism of fixed points by construction. $\square$

Lemma 3.22. *If $\mathbf{X} \in C_H(G, \mathcal{V})^H$ is supported on the coset rH, then*

$$\mathbf{X} = \pi_r^{\mathrm{Ind}}(\mathbf{Y})$$

for some $\mathbf{Y} \in C_H(G, \mathcal{V})^H$ supported on the identity coset. (Here we are making a choice of coset representative $r \in G$ for rH.)

Proof. We take

$$\mathbf{Y} = \pi_{r^{-1}}^{\mathrm{Ind}}(\mathbf{X})$$

which is supported on the identity coset by construction. Then

$$\begin{aligned}\pi_r^{\mathrm{Ind}}(\mathbf{Y}) &= \pi_r^{\mathrm{Ind}} \circ \pi_{r^{-1}}^{\mathrm{Ind}}(\mathbf{X}) \\ &= \mathbf{X}.\end{aligned}$$

$\square$

Lemma 3.23. *Let $\mathbf{X} \in C_H(G, \mathcal{V})^H$ be an object supported on the cosets $r_1H, \ldots, x_nH$. Then*

$$\mathbf{X} \cong \bigoplus_{i=1}^{n} \pi_{r_i}^{\mathrm{Ind}}(\mathbf{Y}_i),$$

where each $\mathbf{Y}_i$ is supported on the identity coset.

Proof. By Lemma 3.21 we have an isomorphism

$$\alpha : \mathbf{X} \xrightarrow{\simeq} \bigoplus_{i=1}^{n} \mathbf{X}_i$$

where each $\mathbf{X}_i$ is supported on $r_i H$. As in Lemma 3.22, we then take

$$\mathbf{Y}_i = \pi^{\mathrm{Ind}}_{r_i^{-1}}(\mathbf{X}_i)$$

so that $\mathbf{Y}_i$ is supported on the identity coset and we have an isomorphism

$$\begin{aligned}\alpha : \mathbf{X} \xrightarrow{\simeq} & \bigoplus_{i=1}^{n} \mathbf{X}_i \\ = & \bigoplus_{i=1}^{n} \pi^{\mathrm{Ind}}_{r_i}(\mathbf{Y}_i).\end{aligned}$$

□

Next, we shall see how, given an object $X \in \mathcal{V}$, we can construct an element of $C_H(G, \mathcal{V})^H$ which is supported on a single left coset and that such objects are the building blocks of $\mathrm{Ind}_H^G(\mathcal{V})$.

Lemma 3.24. *Let $\mathcal{V}$ be an H-category, X an object of $\mathcal{V}$ and r a fixed element of G. There is a fixed point $\delta^X_{rH} \in C_H(G, \mathcal{V})^H$ which is supported on rH defined by*

$$\delta^X_{rH}(g) := \begin{cases} \pi^{\mathcal{V}}_{h^{-1}}(X) & \textit{if } g = rh \in rH, \\ 0 & \textit{otherwise} \end{cases}$$

and for each $k \in H$, $\rho_k^{\delta^X_{rH}}$ is defined by

$$\rho_k^{\delta^X_{rH}}(g) := \begin{cases} \left(\mu^{\mathcal{V}}_{k,(kh)^{-1}}\right)_X & \textit{if } g = rh \in rH, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Given $k \in H$,

$$\begin{aligned}\pi_k^{C_H(G,\mathcal{V})}(\delta^X_{rH})(g) &= \pi^{\mathcal{V}}_k(\delta^X_{rH}(gk)) \\ &= \begin{cases} \pi^{\mathcal{V}}_k \circ \pi^{\mathcal{V}}_{(hk)^{-1}}(X) & \text{if} g = rh \in rH, \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

so $\rho_k^{\delta_{rH}^X}$ is a map from $\pi_k^{C_H(G,\mathcal{V})}(\delta_{rH}^X)$ to δ_{rH}^X as required.

We just need to show that the ρ's satisfy the fixed point axiom, that is, for all $k, \ell \in H$, the following diagram should commute

$$\begin{tikzcd}
\pi_k^{C_H(G,\mathcal{V})} \circ \pi_\ell^{C_H(G,\mathcal{V})}(\delta_{rH}^X) \arrow[r, "\pi_k^{C_H(G,\mathcal{V})}\rho_\ell"] \arrow[d, "\mu_{k,\ell}^{C_H(G,\mathcal{V})}"'] & \pi_k^{C_H(G,\mathcal{V})}(\delta_{rH}^X) \arrow[d, "\rho_\ell"] \\
\pi_{k\ell}^{C_H(G,\mathcal{V})}(\delta_{rH}^X) \arrow[r, "\rho_{k\ell}"] & \delta_{rH}^X
\end{tikzcd} \qquad (*)$$

For fixed $h \in H$, the rh component of this diagram is

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ \pi_\ell^{\mathcal{V}} \circ \pi_{(hk\ell)^{-1}}^{\mathcal{V}}(X) \arrow[r, "\mu_{\ell,(hk\ell)^{-1}}^{\mathcal{V}}"] \arrow[d, "\mu_{k,\ell}^{\mathcal{V}}"'] & \pi_k^{\mathcal{V}} \circ \pi_{(hk)^{-1}}^{\mathcal{V}}(X) \arrow[d, "\mu_{k,(hk)^{-1}}^{\mathcal{V}}"] \\
\pi_{k\ell}^{\mathcal{V}} \circ \pi_{(hk\ell)^{-1}}^{\mathcal{V}}(X) \arrow[r, "\mu_{k\ell,(hk\ell)^{-1}}^{\mathcal{V}}"] & \pi_{h^{-1}}^{\mathcal{V}}(X)
\end{tikzcd}$$

This diagram commutes because $\mathcal{V}$ is an H-category and hence $(*)$ commutes. $\square$

We note that the choice of coset representative for rH is part of the data defining δ_{rH}^X and we shall write δ_H^X instead of δ_{eH}^X.

Definition 3.25. *We call a fixed point of the form in Lemma 3.24 a **canonical fixed point**.*

Lemma 3.26. *Let $\mathcal{V}$ be an H-category, $r \in G$ and $\mathbf{X} \in C_H(G,\mathcal{V})^H$ a fixed point supported on rH. Then $\mathbf{X}$ is isomorphic to the canonical fixed point $\delta_{rH}^{\mathbf{X}(r)}$ defined as in Lemma 3.24.*

Proof. We have an isomorphism

$$\alpha : \delta_{rH}^{\mathbf{X}(r)} \to \mathbf{X}$$

defined by

$$\alpha(g) := \begin{cases} \rho_{h^{-1}}^{\mathbf{X}}(g) & \text{if } g = rh \in rH, \\ 0 & \text{otherwise} \end{cases}$$

in $C_H(G,\mathcal{V})$. To show it is an isomorphism in the fixed point category $C_H(G,\mathcal{V})^H$ we need to show that it commutes with the fixed point trivialisers, that is, for all $k \in H$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{C_H(G,\mathcal{V})}\left(\delta_{rH}^{\mathbf{X}(r)}\right) \arrow[r, "\pi_k^{C_H(G,\mathcal{V})}(\alpha)"] \arrow[d, "\rho_k"'] & \pi_k^{C_H(G,\mathcal{V})}(\mathbf{X}) \arrow[d, "\rho_k"] \\
\delta_{rH}^{\mathbf{X}(r)} \arrow[r, "\alpha"] & \mathbf{X}
\end{tikzcd} \tag{$*$}$$

For fixed $h \in H$, the rh component of this diagram is

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ \pi_{(hk)^{-1}}^{\mathcal{V}}(\mathbf{X}(r)) \arrow[r, "\pi_k^{\mathcal{V}}\rho_{(hk)^{-1}}^{\mathbf{X}}"] \arrow[d, "\mu_{k,(hk)^{-1}}^{\mathcal{V}}"'] & \pi_k^{\mathcal{V}}(\mathbf{X}(rhk)) \arrow[d, "\rho_k^{\mathbf{X}}"] \\
\pi_{h^{-1}}^{\mathcal{V}}(\mathbf{X}(r)) \arrow[r, "\rho_{h^{-1}}^{\mathbf{X}}"] & \mathbf{X}(rh)
\end{tikzcd}$$

and this commutes because it is the rh component of the diagram

$$\begin{tikzcd}
\pi_k^{C_H(G,\mathcal{V})} \circ \pi_{(hk)^{-1}}^{C_H(G,\mathcal{V})}(\mathbf{X}) \arrow[r, "\pi_k^{C_H(G,\mathcal{V})}\rho_{(hk)^{-1}}^{\mathbf{X}}"] \arrow[d, "\mu_{k,(hk)^{-1}}^{C_H(G,\mathcal{V})}"'] & \pi_k^{C_H(G,\mathcal{V})}(\mathbf{X}) \arrow[d, "\rho_k^{\mathbf{X}}"] \\
\pi_{h^{-1}}^{C_H(G,\mathcal{V})}(\mathbf{X}) \arrow[r, "\rho_{h^{-1}}^{\mathbf{X}}"] & \mathbf{X}
\end{tikzcd}$$

which commutes by the fixed point axiom. Therefore, the diagram $(*)$ commutes. $\square$

Corollary 3.27. *Let $\mathcal{V}$ be an H-category, $\mathbf{X} \in C_H(G,\mathcal{V})^H$ and Γ a set of coset representatives for G/H. Then $\mathbf{X} \cong \bigoplus_{\gamma\in\Gamma} \delta_{\gamma H}^{\mathbf{X}(\gamma)}$.*

Proof. This follows from Lemmas 3.21 and 3.26. $\square$

The content of the following two lemmas is that given $r \in G$ and $X, Y \in \mathcal{V}$, the morphisms $\alpha : \delta_{rH}^X \to \delta_{rH}^Y$ in $C_H(G,\mathcal{V})^H$ correspond to morphisms $a : X \to Y$ in $\mathcal{V}$.

Lemma 3.28. *Let $\mathcal{V}$ be an H-category, $X, Y \in \mathcal{V}$, $a : X \to Y$ a morphism and $r \in G$. Then*

$$\delta^a_{rH}(g) := \begin{cases} \pi^{\mathcal{V}}_{h^{-1}}(a) & \text{if } g = rh \in rH, \\ 0 & \text{otherwise} \end{cases}$$

where $g \in G$ defines a morphism of fixed points $\delta^a_{rH} : \delta^X_{rH} \to \delta^Y_{rH}$. Furthermore, if $a, b : X \to Y$ are morphisms in $\mathcal{V}$, then $\delta^a_{rH} = \delta^b_{rH}$ if and only if $a = b$.

Proof. To show that δ^a_{rH} is a morphism of fixed points, we need to show that for all $k \in H$ the following diagram commutes

$$\begin{tikzcd}
\pi_k^{C_H(G,\mathcal{V})}(\delta^X_{rH}) \arrow[r, "\pi_k^{C_H(G,\mathcal{V})}(\delta^a_{rH})"] \arrow[d, "\rho_k"'] & \pi_k^{C_H(G,\mathcal{V})}(\delta^Y_{rH}) \arrow[d, "\rho_k"] \\
\delta^X_{rH} \arrow[r, "\delta^a_{rH}"] & \delta^Y_{rH}
\end{tikzcd}$$

Since δ^X_{rH} and δ^Y_{rH} are supported on rH, we just need to check that for each $g \in rH$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{C_H(G,\mathcal{V})}(\delta^X_{rH})(g) \arrow[r, "\pi_k^{C_H(G,\mathcal{V})}(\delta^a_{rH})(g)"] \arrow[d, "\rho_k(g)"'] & \pi_k^{C_H(G,\mathcal{V})}(\delta^Y_{rH})(g) \arrow[d, "\rho_k(g)"] \\
\delta^X_{rH}(g) \arrow[r, "\delta^a_{rH}(g)"] & \delta^Y_{rH}(g)
\end{tikzcd}$$

For $g = rh \in rH$, this is the diagram

$$\begin{tikzcd}
\pi^{\mathcal{V}}_k \circ \pi^{\mathcal{V}}_{(hk)^{-1}}(X) \arrow[r, "\pi^{\mathcal{V}}_k \circ \pi^{\mathcal{V}}_{(hk)^{-1}}(a)"] \arrow[d, "\mu^{\mathcal{V}}_{h,(hk)^{-1}}"'] & \pi^{\mathcal{V}}_k \circ \pi^{\mathcal{V}}_{(hk)^{-1}}(Y) \arrow[d, "\mu^{\mathcal{V}}_{h,(hk)^{-1}}"] \\
\pi^{\mathcal{V}}_{h^{-1}}(X) \arrow[r, "\pi^{\mathcal{V}}_{h^{-1}}(a)"] & \pi^{\mathcal{V}}_{h^{-1}}(Y)
\end{tikzcd}$$

and this commutes by the naturality of $\mu^{\mathcal{V}}_{h,(hk)^{-1}}$.

Finally, let $a, b : X \to Y$ be morphisms in $\mathcal{V}$. Since $\pi^{\mathcal{V}}_e$ is an equivalence, $\delta^a_{rH}(r) = \pi^{\mathcal{V}}_e(a) = \pi^{\mathcal{V}}_e(b) = \delta^b_{rH}(r)$ if and only if $a = b$. Therefore, $\delta^a_{rH} = \delta^b_{rH}$ if and only if $a = b$. □

Lemma 3.29. *Let $\mathcal{V}$ be an H-category, $X, Y \in \mathcal{V}$, $r \in G$ and $\alpha : \delta^X_{rH} \to \delta^Y_{rH}$ a morphism of fixed points. Then there exists $a : X \to Y$ in $\mathcal{V}$ such that $\alpha = \delta^a_{rH}$.*

Proof. If we consider the morphism $\alpha(r) : \pi^{\mathcal{V}}_e(X) \to \pi^{\mathcal{V}}_e(Y)$, since $\pi^{\mathcal{V}}_e$ is an equivalence, there is a unique morphism $a : X \to Y$ in $\mathcal{V}$ such that $\alpha(r) = \pi^{\mathcal{V}}_e(a)$. Now, because α is a morphism of fixed points, for all $h, k \in H$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{C_H(G,\mathcal{V})}(\delta^X_{rH})(rh) \arrow[r, "\pi_k^{C_H(G,\mathcal{V})}(\alpha)(rh)"] \arrow[d, "\rho^{\delta^X_{rH}}(rh)"'] & \pi_k^{C_H(G,\mathcal{V})}(\delta^Y_{rH})(rh) \arrow[d, "\rho^{\delta^Y_{rH}}(rh)"] \\
\delta^X_{rH}(rh) \arrow[r, "\alpha(rh)"] & \delta^Y_{rH}(rh)
\end{tikzcd}$$

This is the diagram

$$\begin{tikzcd}
\pi^{\mathcal{V}}_k \circ \pi^{\mathcal{V}}_{(hk)^{-1}}(X) \arrow[r, "\pi^{\mathcal{V}}_k(\alpha(rhk))"] \arrow[d, "\mu^{\mathcal{V}}_{k,(hk)^{-1}}"'] & \pi^{\mathcal{V}}_k \circ \pi^{\mathcal{V}}_{(hk)^{-1}}(Y) \arrow[d, "\mu^{\mathcal{V}}_{k,(hk)^{-1}}"] \\
\pi^{\mathcal{V}}_{h^{-1}}(X) \arrow[r, "\alpha(rh)"] & \pi^{\mathcal{V}}_{h^{-1}}(Y)
\end{tikzcd}$$

Setting $k = h^{-1}$ yields the commutative diagram

$$\begin{tikzcd}
\pi^{\mathcal{V}}_{h^{-1}} \circ \pi^{\mathcal{V}}_e(X) \arrow[r, "\pi^{\mathcal{V}}_{h^{-1}}(\alpha(r))", "=\pi^{\mathcal{V}}_{h^{-1}} \circ \pi^{\mathcal{V}}_e(a)"'] \arrow[d, "\mu^{\mathcal{V}}_{h^{-1},e}"'] & \pi^{\mathcal{V}}_{h^{-1}} \circ \pi^{\mathcal{V}}_e(Y) \arrow[d, "\mu^{\mathcal{V}}_{h^{-1},e}"] \\
\pi^{\mathcal{V}}_{h^{-1}}(X) \arrow[r, "\alpha(rh)"] & \pi^{\mathcal{V}}_{h^{-1}}(Y)
\end{tikzcd}$$

By the naturality of $\mu^{\mathcal{V}}_{h^{-1},e}$ we also have a commutative diagram

$$
\begin{tikzcd}
\pi^{\mathcal{V}}_{h^{-1}} \circ \pi^{\mathcal{V}}_{e}(X) \arrow[r, "\pi^{\mathcal{V}}_{h^{-1}} \circ \pi^{\mathcal{V}}_{e}(a)"] \arrow[d, "\mu^{\mathcal{V}}_{h^{-1},e}"'] & \pi^{\mathcal{V}}_{h^{-1}} \circ \pi^{\mathcal{V}}_{e}(Y) \arrow[d, "\mu^{\mathcal{V}}_{h^{-1},e}"] \\
\pi^{\mathcal{V}}_{h^{-1}}(X) \arrow[r, "\pi^{\mathcal{V}}_{h^{-1}}(a)"] & \pi^{\mathcal{V}}_{h^{-1}}(Y)
\end{tikzcd}
$$

By comparing this with the previous diagram we can see that

$$\alpha(rh) \circ \left(\mu^{\mathcal{V}}_{h^{-1},e}\right)_X = \pi^{\mathcal{V}}_{h^{-1}}(a) \circ \left(\mu^{\mathcal{V}}_{h^{-1},e}\right)_X$$

and since $\left(\mu^{\mathcal{V}}_{h^{-1},e}\right)_X$ is invertible we have

$$\alpha(rh) = \pi^{\mathcal{V}}_{h^{-1}}(a).$$

It follows that $\alpha = \delta^{a}_{rH}$. $\square$

3.2.3 The 2-functors Res^G_H and Ind^G_H

Similarly to the classical case of restriction and induction of representations, restriction and induction of G- and H-categories are 2-functiorial. In this section we shall define both 2-functors starting with restriction which is mostly a case of fixing notation.

Definition 3.30. *Let $\mathcal{V}, \mathcal{W}$ be G-categories and $\Theta : \mathcal{V} \to \mathcal{W}$ a G-intertwiner. We define the* ***restricted*** *H-intertwiner $\mathrm{Res}^G_H(\Theta) : \mathrm{Res}^G_H(\mathcal{V}) \to \mathrm{Res}^G_H(\mathcal{W})$ as follows: The underlying $*$-functor is Θ and the coherence unitary natural transformations are*

$$\mathrm{Res}^G_H(\Theta)_k := \Theta_k : \pi^{\mathrm{Res}(\mathcal{W})}_k \circ \mathrm{Res}^G_H(\Theta) \to \mathrm{Res}^G_H(\Theta) \circ \pi^{\mathrm{Res}(\mathcal{V})}_k$$

for each $k \in H$.

Often we shall denote $\mathrm{Res}^G_H(\Theta)$ by Θ when no confusion should occur.

Definition 3.31. *Let $\mathcal{V}, \mathcal{W}$ be G-categories, $\Theta, \Phi : \mathcal{V} \to \mathcal{W}$ G-intertwiners and $\kappa : \Theta \to \Phi$ a G-natural transformation. We define the* ***restricted*** *H-natural transformation $\mathrm{Res}^G_H(\kappa) : \mathrm{Res}^G_H(\Theta) \to \mathrm{Res}^G_H(\Phi)$ by $\mathrm{Res}^G_H(\kappa) := \kappa$.*

As with restricted H-categories and H-intertwiners, we shall often just write κ instead of $\mathrm{Res}^G_H(\kappa)$.

Definition 3.32. *We define the **restriction** 2-functor*

$$\mathrm{Res}_H^G : \mathsf{REP}(G) \to \mathsf{REP}(H)$$

on objects by mapping a G-category $\mathcal{V}$ to $\mathrm{Res}_H^G(\mathcal{V})$ as in Definition 3.8, on 1-cells by mapping a G-intertwiner Θ to $\mathrm{Res}_H^G(\Theta)$ as in Definition 3.30 and on 2-cells by mapping a G-natural transformation to $\mathrm{Res}_H^G(\kappa)$ as in Definition 3.31.

The 2-functor axioms follow from the fact that the definitions of the restricted G-categories, G-intertwiners and G-natural transformations just involve forgetting some structure.

Now we turn our attention to defining the induction 2-functor

$$\mathrm{Ind}_H^G : \mathsf{REP}(H) \to \mathsf{REP}(G).$$

We have already defined the object map which sends an H-category $\mathcal{V}$ to the induced G-category $\mathrm{Ind}_H^G(\mathcal{V})$ so the next task is to define Ind_H^G on 1-cells, that is, H-intertwiners.

Lemma 3.33. *Let $\mathcal{V}, \mathcal{W}$ be H-categories and $\Theta : \mathcal{V} \to \mathcal{W}$ an H-intertwiner. Then Θ induces a G-intertwiner*

$$\mathrm{Ind}_H^G(\Theta) : \mathrm{Ind}_H^G(\mathcal{V}) \to \mathrm{Ind}_H^G(\mathcal{W})$$

defined on objects by mapping $\mathbf{X} \in \mathrm{Ind}_H^G(\mathcal{V})$ to $\Theta(\mathbf{X}) \in \mathrm{Ind}_H^G(\mathcal{W})$ where $\Theta(\mathbf{X})$ is defined by

$$\Theta(\mathbf{X})(g) := \Theta(\mathbf{X}(g)), \quad g \in G,$$

and for each $k \in H$, $g \in G$, $\rho_k^{\Theta(\mathbf{X})}(g)$ is the composite

$$\begin{array}{c} \pi_k^{\mathcal{W}} \circ \Theta(\mathbf{X}(gk)) \\ = \pi_k^{C_H(G,\mathcal{V})}(\Theta(\mathbf{X}))(g) \end{array} \xrightarrow{(\Theta_k)_{\mathbf{X}(gk)}} \Theta \circ \pi_k^{\mathcal{V}}(\mathbf{X}(gk)) \xrightarrow{\Theta(\rho_k^{\mathbf{X}}(g))} \begin{array}{c} \Theta(\mathbf{X}(g)) \\ = \Theta(\mathbf{X})(g) \end{array}$$

The coherence unitary natural transformations

$$\mathrm{Ind}_H^G(\Theta)_t : \pi_t^{\mathrm{Ind}(\mathcal{W})} \circ \mathrm{Ind}_H^G(\Theta) \to \mathrm{Ind}_H^G(\Theta) \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}, \quad t \in G,$$

are the identity natural transformations.

Proof. The underlying C^*-categories of $\mathrm{Ind}_H^G(\mathcal{V})$ and $\mathrm{Ind}_H^G(\mathcal{W})$ are $C_H(G,\mathcal{V})^H$ and $C_H(G,\mathcal{W})^H$ and the same argument as in Lemma 3.14 shows that Θ induces a $*$-functor

$$\mathrm{Ind}_H^G(\Theta) : C_H(G,\mathcal{V})^H \to C_H(G,\mathcal{W})^H$$

defined on objects as in the statement of this lemma. Then, if α is a morphism in $C_H(G,\mathcal{V})^H$, we define $\mathrm{Ind}_H^G(\Theta)(\alpha)$ by

$$\mathrm{Ind}_H^G(\Theta)(\alpha)(g) := \Theta(\alpha(g)), \quad g \in G.$$

Again, by the same argument as in Lemma 3.14, this is a morphism of fixed points.

To make $\mathrm{Ind}_H^G(\Theta)$ a G-intertwiner, we just need to define unitary natural transformations

$$\mathrm{Ind}_H^G(\Theta)_t : \pi_t^{\mathrm{Ind}(\mathcal{W})} \circ \mathrm{Ind}_H^G(\Theta) \to \mathrm{Ind}_H^G(\Theta) \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}, \quad t \in G$$

that satisfy the G-intertwiner axioms. Given $\mathbf{X} \in \mathrm{Ind}_H^G(\mathcal{V})$, we have

$$\pi_t^{\mathrm{Ind}(\mathcal{W})} \circ \mathrm{Ind}_H^G(\Theta)(\mathbf{X}) = \mathrm{Ind}_H^G(\Theta) \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})$$

since

$$\begin{aligned}(t \cdot \Theta(\mathbf{X}))(g) &= \Theta(\mathbf{X}(t^{-1}g)) \\ &= \Theta(t \cdot \mathbf{X})(g)\end{aligned}$$

for all $t, g \in G$ and similarly $\rho_k^{t\cdot\Theta(\mathbf{X})}(g) = \rho_k^{\Theta(t\cdot\mathbf{X})}(g)$ for all $t, g \in G$ and $k \in H$. A similar argument holds for morphisms and hence

$$\pi_t^{\mathrm{Ind}(\mathcal{W})} \circ \mathrm{Ind}_H^G(\Theta) = \mathrm{Ind}_H^G(\Theta) \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}$$

for all $t \in G$. Therefore, we define $\mathrm{Ind}_H^G(\Theta)_t$ to be the identity natural transformation for all $t \in G$. ☐

Lemma 3.34. *Let $\mathcal{V}$ and $\mathcal{W}$ be H-categories, $\Theta, \Phi : \mathcal{V} \to \mathcal{W}$ H-intertwiners and $\kappa : \Theta \to \Phi$ an H-natural transformation. Then κ induces a G-natural transformation*

$$\mathrm{Ind}_H^G(\kappa) : \mathrm{Ind}_H^G(\Theta) \to \mathrm{Ind}_H^G(\Phi)$$

whose component for $\mathbf{X} \in \mathrm{Ind}_H^G(\mathcal{V})$ is defined by

$$\mathrm{Ind}_H^G(\kappa)_{\mathbf{X}}(g) := \kappa_{\mathbf{X}(g)} : \Theta(\mathbf{X}(g)) \to \Phi(\mathbf{X}(g)), \quad g \in G.$$

Proof. The naturality of $\mathrm{Ind}_H^G(\kappa)$ follows from the naturality of κ. To show that $\mathrm{Ind}_H^G(\kappa)$ satisfies the G-natural transformation axiom, we need to show that for all $t \in G$, the following diagram commutes

$$\begin{tikzcd}
\pi_t^{\mathrm{Ind}(\mathcal{W})} \circ \mathrm{Ind}_H^G(\Theta) \arrow[r, "\mathrm{Ind}_H^G(\Theta)_t"] \arrow[d, "\pi_t^{\mathrm{Ind}(\mathcal{V})}\mathrm{Ind}_H^G(\kappa)"'] & \mathrm{Ind}_H^G(\Theta) \circ \pi_t^{\mathrm{Ind}(\mathcal{V})} \arrow[d, "\mathrm{Ind}_H^G(\kappa)_{\pi_t^{\mathrm{Ind}(\mathcal{V})}}"] \\
\pi_t^{\mathrm{Ind}(\mathcal{W})} \circ \mathrm{Ind}_H^G(\Phi) \arrow[r, "\mathrm{Ind}_H^G(\Phi)_t"] & \mathrm{Ind}_H^G(\Phi) \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}
\end{tikzcd}$$

This follows from the fact that given $\mathbf{X} \in \mathrm{Ind}_H^G(\mathcal{V}))$, for all $t \in G$

$$\begin{aligned}
\pi_t^{\mathrm{Ind}(\mathcal{V})}\mathrm{Ind}_H^G(\kappa)_{\mathbf{X}}(g) &= \kappa_{\mathbf{X}(t^{-1}g)} \\
&= \kappa_{(t\cdot\mathbf{X})(g)} \\
&= \mathrm{Ind}_H^G(\kappa)_{\pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})}(g)
\end{aligned}$$

along with the fact that $\mathrm{Ind}_H^G(\Theta)_t$ and $\mathrm{Ind}_H^G(\Phi)_t$ are the identity natural transformations. $\square$

Lemma 3.35. *There is a 2-functor*

$$\mathrm{Ind}_H^G : \mathsf{REP}(H) \to \mathsf{REP}(G)$$

defined on objects by mapping an H-category $\mathcal{V}$ to $\mathrm{Ind}_H^G(\mathcal{V})$ as in Definition 3.19, on 1-cells by mapping an H-intertwiner Θ to $\mathrm{Ind}_H^G(\Theta)$ as in Lemma 3.33 and on 2-cells by mapping an H-natural transformation κ to $\mathrm{Ind}_H^G(\kappa)$ as in Lemma 3.34.

Proof. We need to show that the 2-functor axioms are satisfied. Firstly, given an H-category $\mathcal{V}$ we have $\mathrm{Ind}_H^G(1_{\mathcal{V}}) = 1_{\mathrm{Ind}_H^G(\mathcal{V})}$. This is immediate from the definition of the induced G-intertwiner in Lemma 3.33. Next, given composable H-intertwiners as shown below

$$\mathcal{U} \xrightarrow{\Theta} \mathcal{V} \xrightarrow{\Phi} \mathcal{W}$$

let us show that $\mathrm{Ind}_H^G(\Phi \circ \Theta) = \mathrm{Ind}_H^G(\Phi) \circ \mathrm{Ind}_H^G(\Theta)$. On the one hand, given $\mathbf{X} \in \mathrm{Ind}_H^G(\mathcal{U})$, $\mathrm{Ind}_H^G(\Phi \circ \Theta) = (\Phi \circ \Theta)(\mathbf{X})$ where

$$(\Phi \circ \Theta)(\mathbf{X})(g) = (\Phi \circ \Theta)(\mathbf{X}(g)), \quad g \in G,$$

and for all $k \in H$, $g \in G$, $\rho_k^{(\Phi\circ\Theta)(\mathbf{X})}(g)$ is the composite

$$\pi_k^{\mathcal{W}} \circ \Phi \circ \Theta(\mathbf{X}(gk)) \xrightarrow{\Phi(\Theta_k)\circ(\Phi_k)_\Theta} \Phi \circ \Theta \circ \pi_k^{\mathcal{U}}(\mathbf{X}(gk)) \xrightarrow{(\Phi\circ\Theta)(\rho_k^{\mathbf{X}}(g))} \Phi \circ \Theta(\mathbf{X}(g)) \qquad (*)$$

On the other hand, $\mathrm{Ind}_H^G(\Phi) \circ \mathrm{Ind}_H^G(\Theta)(\mathbf{X}) = \Phi(\Theta(\mathbf{X}))$ where

$$\Phi(\Theta(\mathbf{X}))(g) = \Phi(\Theta(\mathbf{X}(g))) = \Phi \circ \Theta(\mathbf{X}(g)), \qquad g \in G$$

and for all $k \in H$, $g \in G$, $\rho_k^{\Phi(\Theta(\mathbf{X}))}(g)$ is the composite

$$\pi_k^{\mathcal{W}} \circ \Phi \circ \Theta(\mathbf{X}(gk)) \xrightarrow{(\Phi_k)_\Theta} \Phi \circ \pi_k^{\mathcal{V}} \circ \Theta(\mathbf{X}(gk)) \xrightarrow{\Phi\left(\rho_k^{\Theta(\mathbf{X})}\right)} \Phi \circ \Theta(\mathbf{X}(g)) \qquad (**)$$

Since $\Phi\left(\rho_k^{\Theta(\mathbf{X})}\right)$ is the composite

$$\Phi \circ \pi_k^{\mathcal{V}} \circ \Theta(\mathbf{X}(gk)) \xrightarrow{\Phi(\Theta_k)} \Phi \circ \Theta \circ \pi_k^{\mathcal{V}}(\mathbf{X}(gk)) \xrightarrow{\Theta(\rho_k^{\mathbf{X}}(g))} \Phi \circ \Theta(\mathbf{X}(g)),$$

the composite $(*)$ is equal to $(**)$ and hence

$$\mathrm{Ind}_H^G(\Phi) \circ \mathrm{Ind}_H^G(\Theta)(\mathbf{X}) = \mathrm{Ind}_H^G(\Phi \circ \Theta)(\mathbf{X}).$$

Next, given a morphism $\alpha \in \mathrm{Ind}_H^G(\mathcal{U})$, we have

$$\mathrm{Ind}_H^G(\Phi \circ \Theta)(\alpha)(g) = \Phi \circ \Theta(\alpha(g)) = \mathrm{Ind}_H^G(\Phi) \circ \mathrm{Ind}_H^G(\Theta)(\alpha)(g), \qquad g \in G$$

and hence $\mathrm{Ind}_H^G(\Phi \circ \Theta)(\alpha) = \mathrm{Ind}_H^G(\Phi) \circ \mathrm{Ind}_H^G(\Theta)(\alpha)$. Therefore,

$$\mathrm{Ind}_H^G(\Phi \circ \Theta) = \mathrm{Ind}_H^G(\Phi) \circ \mathrm{Ind}_H^G(\Theta).$$

Finally, we need to show that the maps

$$\mathrm{Ind}_H^G : \mathrm{Hom}_H(\mathcal{V}, \mathcal{W}) \to \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathcal{V}), \mathrm{Ind}_H^G(\mathcal{W}))$$

are functorial. This follows immediately from the definition of Ind_H^G on H-natural transformations. $\square$

The function based approach to induced G-categories is the main one we shall work with moving forward but we shall also briefly look at how they can be constructed in different ways mirroring the alternate definitions 3.6 and 3.7 of induced representations.

First, we shall look at a direct sum construction analogous to definition 3.6. For this we need a little bit of notation. If $\mathcal{V}$ is an H-category, Γ is a choice of coset representatives for G/H and $\gamma \in \Gamma$, we write $\gamma\mathcal{V}$ to denote an isomorphic copy of $\mathcal{V}$. We shall write the objects of $\gamma\mathcal{V}$ as γV where $V \in \mathcal{V}$ and the morphisms as $\gamma f : \gamma V \to \gamma W$ where $f : V \to W$ is a morphism in $\mathcal{V}$. We then have a C^*-category $\bigoplus_{\gamma\in\Gamma} \gamma\mathcal{V}$ which is equivalent to $C_H(G, \mathcal{V})^H$, the underlying C^*-category of $\mathrm{Ind}_H^G(\mathcal{V})$.

Lemma 3.36. *There is an equivalence of C^*-categories*

$$\Theta : \bigoplus_{\gamma\in\Gamma} \gamma\mathcal{V} \xrightarrow{\simeq} C_H(G, \mathcal{V})^H$$

defined on objects by

$$\Theta((\gamma V_\gamma)_{\gamma\in\Gamma}) := \bigoplus_{\gamma\in\Gamma} \delta_{\gamma H}^{V_\gamma}$$

and on morphisms by

$$\Theta((\gamma f_\gamma)_{\gamma\in\Gamma}) := \bigoplus_{\gamma\in\Gamma} \delta_{\gamma H}^{f_\gamma}.$$

(Here $\delta_{\gamma H}^{V_\gamma}$ and $\delta_{\gamma H}^{f_\gamma}$ are defined as in Lemmas 3.24 and 3.28 respectively.)

Proof. Firstly, Θ is essentially surjective by Corollary 3.27, it is faithful by Lemma 3.28 and lastly, it is full by Lemma 3.29. □

We can define a G-category structure on $\bigoplus_{\gamma\in\Gamma} \gamma\mathcal{V}$ in a similar way to the action in the analogous construction of induced representations. Firstly, given $t \in G$, we define a $*$-functor

$$\pi_t^{\bigoplus \gamma\mathcal{V}} : \bigoplus_{\gamma\in\Gamma} \gamma\mathcal{V} \to \bigoplus_{\gamma\in\Gamma} \gamma\mathcal{V}$$

in the following way: To define $\pi_t^{\bigoplus \gamma\mathcal{V}}$ on objects, first let $\gamma \in \Gamma$ and $V \in \mathcal{V}$. We define

$$\pi_t^{\bigoplus \gamma\mathcal{V}}(\gamma V) := \gamma' \pi_k^{\mathcal{V}}(V)$$

where $\gamma' \in \Gamma$ and $k \in H$ are determined by $t\gamma = \gamma' k$. (Here, by γV, we really mean its image under the inclusion $*$-functor $\iota_\gamma : \gamma\mathcal{V} \to \bigoplus_{\gamma\in\Gamma} \gamma\mathcal{V}$.) More general objects are direct sums of such objects and we extend this definition to them in the obvious way. We define $\pi_t^{\bigoplus \gamma\mathcal{V}}$ on morphisms in a similar way.

Now, let $s, t \in G$, $\gamma \in \Gamma$ and $V \in \mathcal{V}$. If $\gamma', \gamma'' \in \Gamma$ and $k, \ell \in H$ are determined by

$$t\gamma = \gamma' k \quad \text{and} \quad s\gamma' = \gamma'' \ell$$

then

$$\begin{aligned}
\pi_s^{\bigoplus \gamma\mathcal{V}} \circ \pi_t^{\bigoplus \gamma\mathcal{V}}(\gamma V) &= \pi_s^{\bigoplus \gamma\mathcal{V}}(\gamma'(\pi_k^{\mathcal{V}}(V))) \\
&= \gamma''(\pi_\ell^{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(V)).
\end{aligned}$$

On the other hand,

$$\pi_{st}^{\bigoplus \gamma\mathcal{V}}(\gamma V) = \gamma'' \pi_{\ell k}^{\mathcal{V}}(V).$$

We define a unitary

$$\left(\mu_{s,t}^{\bigoplus \gamma\mathcal{V}}\right)_{\gamma V} := \gamma''(\mu_{s,t}^{\mathcal{V}})_V : \pi_s^{\bigoplus \gamma\mathcal{V}} \circ \pi_t^{\bigoplus \gamma\mathcal{V}}(\gamma V) \to \pi_{st}^{\bigoplus \gamma\mathcal{V}}(\gamma V)$$

and extend this definition in the natural way to more general objects to obtain a unitary natural transformation

$$\mu_{s,t}^{\bigoplus \gamma\mathcal{V}} : \pi_s^{\bigoplus \gamma\mathcal{V}} \circ \pi_t^{\bigoplus \gamma\mathcal{V}} \to \pi_{st}^{\bigoplus \gamma\mathcal{V}}.$$

We also have a unitary

$$\left(\varepsilon^{\bigoplus \gamma\mathcal{V}}\right)_{\gamma V} := \gamma \varepsilon_V^{\mathcal{V}} : \gamma V \to \pi_e^{\bigoplus \gamma\mathcal{V}}(\gamma V)$$

and we extend this definition to more general objects in the natural way to obtain a unitary natural transformation

$$\varepsilon^{\bigoplus \gamma\mathcal{V}} : \mathrm{Id} \to \pi_e^{\bigoplus \gamma\mathcal{V}}.$$

That these maps satisfy the G-category axioms follows from the fact that $\mathcal{V}$ is an H-category. Therefore, the above $*$-functors and unitary natural transformations constitute a G-category structure on $\bigoplus_{\gamma\in\Gamma} \gamma\mathcal{V}$.

Next, we want to define unitary natural transformations

$$\Theta_t : \Theta \circ \pi_t^{\bigoplus \gamma\mathcal{V}} \to \pi_t^{\mathrm{Ind}(\mathcal{V})} \circ \Theta$$

for all $t \in G$ so that the $*$-functor Θ of Lemma 3.36 becomes a G-intertwiner from $\bigoplus_{\gamma\in\Gamma}\gamma\mathcal{V}$ to $\mathrm{Ind}_H^G(\mathcal{V})$. Again, we first consider objects of the form γV. Then

$$\begin{aligned}\Theta \circ \pi_t^{\bigoplus \gamma\mathcal{V}}(\gamma V) &= \Theta(\gamma' \pi_k^{\mathcal{V}}(V)) \\ &= \delta_{\gamma' H}^{\pi_k^{\mathcal{V}}(V)}\end{aligned}$$

and

$$\pi_t^{\mathrm{Ind}(\mathcal{V})} \circ \Theta(\gamma V) = t \cdot \left(\delta_{\gamma H}^V\right)$$

where $\gamma' \in \Gamma$ and $k \in H$ are determined by $t\gamma = \gamma' k$. Given $g \in G$,

$$\delta_{\gamma' H}^{\pi_k^{\mathcal{V}}(V)}(g) = \begin{cases} \pi_{h^{-1}}^{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(V) & \text{if } g = \gamma' h \in \gamma' H, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} t \cdot \left(\delta_{\gamma H}^V\right)(g) &= \delta_{\gamma H}^V(t^{-1} g) \\ &= \begin{cases} \pi_{\ell^{-1}}^{\mathcal{V}}(V) & \text{if } t^{-1}g = \gamma\ell \in \gamma H, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \pi_{\ell^{-1}}^{\mathcal{V}}(V) & \text{if } g = t\gamma\ell = \gamma' k\ell \in \gamma' H, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \pi_{h^{-1}k}^{\mathcal{V}}(V) & \text{if } g = \gamma' h \in \gamma' H, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, we define $(\Theta_t)_{\gamma V}$ by

$$(\Theta_t)_{\gamma V}(g) := \begin{cases} (\mu_{h^{-1},k}^{\mathcal{V}})_V & \text{if } g = \gamma' h \in \gamma' H, \\ 0 & \text{otherwise.} \end{cases}$$

That the Θ_t's satisfy the G-intertwiner axioms follows from the fact that $\mathcal{V}$ is an H-category. In summary, the G-category $\bigoplus_{\gamma\in\Gamma}$ is equivalent to $\mathrm{Ind}_H^G(\mathcal{V})$.

Let us also briefly discuss how the induced G-category can be constructed as a balanced tensor product in a similar way to the tensor product construction of the induced representation. We shall mostly focus on how to construct the underlying C^*-category and make a few remarks about the G-category structure could be defined. Our categorical analogue of the group ring $\mathbb{C}[G]$ will be the C^*-tensor category of finite dimensional G-graded Hilbert spaces so let us first define those.

Definition 3.37. *A G-**graded Hilbert space** is a Hilbert space $\mathcal{H}$ that decomposes as a direct sum*

$$\mathcal{H} = \bigoplus_{g \in G} \mathcal{H}_g.$$

*If $\mathcal{H}$ and $\mathcal{K}$ are G-graded Hilbert spaces, a **morphism** of G-graded Hilbert spaces $\varphi : \mathcal{H} \to \mathcal{K}$ is a bounded linear map that preserves the grading. That is, if $v \in \mathcal{H}_g$ then $\varphi(v) \in \mathcal{K}_g$.*

The G-graded Hilbert spaces and their morphisms form a C^*-category Hilb_G. Similarly, there is a C^*-category of finite dimensional G-graded Hilbert spaces and their morphisms which we shall denote by hilb_G. Both Hilb_G and hilb_G are additive and subobject complete with direct sums and subobjects being formed pointwise (e.g. $(\mathcal{H} \oplus \mathcal{K})_g = \mathcal{H}_g \oplus \mathcal{K}_g$ where on the right hand side, $\oplus$ denotes the Hilbert space direct sum).

The category hilb_G also has a natural C^*-tensor structure. Since it has the flavour of a convolution product, we shall denote it by $*$ rather than $\otimes$. Given $\mathcal{H}, \mathcal{K} \in \mathsf{hilb}_G$, we define

$$\mathcal{H} * \mathcal{K} := \bigoplus_{g \in G} \bigoplus_{t \in G} \mathcal{H}_t \otimes \mathcal{K}_{t^{-1}g}$$

where on the right hand side $\otimes$ denotes the usual Hilbert space tensor product. The grading is given by

$$(\mathcal{H} * \mathcal{K})_g := \bigoplus_{t \in G} \mathcal{H}_t \otimes \mathcal{K}_{t^{-1}g}.$$

The tensor unit is the copy of the complex numbers sitting in degree e and the associator and unitors are defined in the obvious way.

In order to construct induced G-categories as a balanced tensor product, we want to identify G-categories with hilb_G-module categories in a similar way to the way we can identify representations of G with modules over the complex group ring $\mathbb{C}[G]$. First, let us consider how to define a G-category structure on a hilb_G-module category $\mathcal{V}$. To do this, we shall introduce the following notation: Given $t \in G$, we define the G-graded Hilbert space $\mathbb{C}_t$ by

$$(\mathbb{C}_t)_g = \begin{cases} \mathbb{C} & \text{if } g = t, \\ 0 & \text{otherwise.} \end{cases}$$

We won't give all the details but the idea is to define the G-category structure on $\mathcal{V}$ by defining

$$\pi_t^{\mathcal{V}}(V) := \mathbb{C}_t \otimes V, \quad V \in \mathcal{V}.$$

On the other hand, if $\mathcal{W}$ is a G-category, the idea behind defining a hilb_G-category structure on $\mathcal{W}$ is the following: If $\mathcal{H} \in \mathsf{hilb}_G$ and $V \in \mathcal{V}$ then we define

$$\mathcal{H} \otimes V := \bigoplus_{g \in G} \bigoplus_{i=1}^{\dim(\mathcal{H}_g)} \pi_g^{\mathcal{W}}(V).$$

We remark that given morphisms $\varphi : \mathcal{H} \to \mathcal{K}$ in hilb_G and $f : V \to W$ in $\mathcal{V}$, defining $\varphi \otimes f : \mathcal{H} \otimes V \to \mathcal{K} \otimes W$ essentially amounts to choosing bases for $\mathcal{H}$ and $\mathcal{K}$ so that one can identify bounded linear maps with matrices. For example, we can define G-graded Hilbert spaces $\mathbb{C}_t^n$ for $n \in \mathbb{N}$ and $t \in G$ in a similar way to $\mathbb{C}_t$ and we can identify a bounded linear map $\varphi : \mathbb{C}_t^n \to \mathbb{C}_t^m$ with an $n \times m$ matrix (a_{ij}). Then, given a morphism $f : V \to W$ in $\mathcal{V}$ we define

$$\varphi \otimes f := (a_{ij} \pi_t^{\mathcal{V}}(f)) : \bigoplus_{i=1}^{n} \pi_t^{\mathcal{V}}(V) \to \bigoplus_{j=1}^{m} \pi_t^{\mathcal{V}}(W).$$

Again, whilst we won't go through the details, we remark that the 2-category $\mathsf{Mod}_{\mathsf{hilb}_G}$ of hilb_G-module categories, module functors and module transformations is biequivalent (see below) to the 2-category $\mathsf{REP}(G)$ of G-categories, G-intertwiners and G-natural transformations. Let us briefly give a few definitions to clarify what we mean by a 'biequivalence'.

Definition 3.38. *Let $\mathcal{A}, \mathcal{A}'$ be 2-categories and $F, F' : \mathcal{A} \to \mathcal{A}'$ pseudofunctors. A pseudonatural transformation $\sigma : F \to F'$ is a* ***pseudonatural equivalence*** *if there exists a pseudonatural transformation $\sigma' : F' \to F$ and invertible modifications $\Delta : \sigma' \circ \sigma \to 1_F$ and $\Delta' : \sigma \circ \sigma' \to 1_{F'}$.*

Definition 3.39. *Let $\mathcal{A}$ and $\mathcal{A}'$ be 2-categories. A* ***biequivalence*** *between $\mathcal{A}$ and $\mathcal{A}'$ consists of the following:*

- *2-functors $F : \mathcal{A} \to \mathcal{A}'$ and $F' : \mathcal{A}' \to \mathcal{A}$.*
- *Pseudonatural equivalences $\sigma : F' \circ F \to 1_{\mathcal{A}}$ and $\sigma' : F \circ F' \to 1_{\mathcal{A}'}$.*

Given a hilb_H-module category $\mathcal{V}$, the basic idea behind constructing the induced hilb_G-module category is to define the underlying C^*-category as the balanced tensor product $\mathsf{hilb}_G \boxtimes_{\mathsf{hilb}_H} \mathcal{V}$. Here, the right hilb_H-module category structure on hilb_G is given by the tensor category structure on hilb_G.

To compare this construction to the above direct sum construction of the induced G-category we shall introduce a bit of notation. We can view hilb_H as a sub-C^*-category of hilb_G. Given a set of coset representatives Γ for G/H and $\gamma \in \Gamma$, we denote the full sub-C^*-category of hilb_G consisting of objects of the form $\mathbb{C}_\gamma \otimes \mathcal{H}$, $\mathcal{H} \in \mathsf{hilb}_H$ by $\gamma\mathsf{hilb}_H$. As a right hilb_H-module category, $\gamma\mathsf{hilb}_H$ is equivalent to hilb_H and as a right hilb_H-module category, hilb_G is equivalent to the direct sum $\bigoplus_{\gamma\in\Gamma} \gamma\mathsf{hilb}_H$. Then we have equivalences of C^*-categories

$$\begin{aligned}
\mathsf{hilb}_G \boxtimes_{\mathsf{hilb}_H} \mathcal{V} &\cong \left(\bigoplus_{\gamma\in\Gamma} \gamma\mathsf{hilb}_H\right) \boxtimes_{\mathsf{hilb}_H} \mathcal{V} \\
&\cong \bigoplus_{\gamma\in\Gamma} \gamma\mathsf{hilb}_H \boxtimes_{\mathsf{hilb}_H} \mathcal{V} \\
&\cong \bigoplus_{\gamma\in\Gamma} \mathcal{V} \\
&\cong \mathrm{Ind}_H^G(\mathcal{V})
\end{aligned}$$

We haven't checked all of the details but $\mathsf{hilb}_G \boxtimes_{\mathsf{hilb}_H} \mathcal{V}$ should inherit a left hilb_G-module category structure from the natural left hilb_G-module category structure on hilb_G. Under the identification of hilb_G-module categories with G-categories, one should then find that $\mathsf{hilb}_G \boxtimes_{\mathsf{hilb}_H} \mathcal{V}$ is equivalent as a G-category to $\mathrm{Ind}_H^G(\mathcal{V})$.

3.3 The induction restriction biadjunction

In this section we shall show that the 2-functor Ind_H^G is a left biadjoint to Res_H^G. Before we do, let us briefly recall the definitions of the unit and counit of the classical induction restriction adjunction

$$\begin{tikzcd}
\mathsf{Rep}(H) \arrow[r, shift left, "\mathrm{Ind}_H^G"] \arrow[r, phantom, "\perp"] & \mathsf{Rep}(G) \arrow[l, shift left, "\mathrm{Res}_H^G"]
\end{tikzcd}$$

In the function based approach to induced representations, the unit

$$\eta : 1_{\mathsf{Rep}(H)} \to \mathrm{Res}_H^G \circ \mathrm{Ind}_H^G$$

is defined as follows: Given $V \in \mathsf{Rep}(H)$, $\eta_V : V \to \mathrm{Res}^G_H \mathrm{Ind}^G_H(V)$ is defined by

$$\eta_V(v)(g) := \begin{cases} g^{-1} \cdot v & \text{if } g \in H, \\ 0 & \text{otherwise} \end{cases}$$

where $v \in V$. To define the counit

$$\xi : \mathrm{Ind}^G_H \circ \mathrm{Res}^G_H \to 1_{\mathsf{Rep}(G)}$$

we choose a set of coset representatives Γ for G/H. Then, given $W \in \mathsf{Rep}(G)$, $\xi_W : \mathrm{Ind}^G_H \mathrm{Res}^G_H(W) \to W$ is defined by

$$\xi_W(f) := \sum_{\gamma \in \Gamma} \gamma \cdot f(\gamma), \qquad f \in \mathrm{Ind}^G_H \mathrm{Res}^G_H(W).$$

The definition is independent of the choice of coset representatives for G/H. If Σ is another set of coset representatives for G/H, then given $\sigma \in \Sigma$, $\sigma = \gamma h$ for a unique $\gamma \in \Gamma$ and $h \in H$. Then if $f \in C_H(G, W)^H$,

$$\begin{aligned} \sigma \cdot f(\sigma) &= \gamma h \cdot f(\gamma h) \\ &= \gamma \cdot f(\gamma) \end{aligned}$$

and hence

$$\sum_{\sigma \in \Sigma} \sigma \cdot f(\sigma) = \sum_{\gamma \in \Gamma} \gamma \cdot f(\gamma).$$

We define the unit and counit of the biadjunction in a completely analogous manner, starting with the unit.

Lemma 3.40. *Let $\mathcal{V}$ be an H-category. There is an H-intertwiner*

$$\eta_{\mathcal{V}} : \mathcal{V} \to \mathrm{Res}^G_H \mathrm{Ind}^G_H(\mathcal{V})$$

whose underlying $*$-functor is defined on objects by

$$\eta_{\mathcal{V}}(V) := \delta^V_H,$$

the canonical fixed point defined as in Lemma 3.24 and on morphisms by

$$\eta_{\mathcal{V}}(\alpha) := \delta^\alpha_H$$

defined as in Lemma 3.28.

Proof. We need to equip $\eta_{\mathcal{V}}$ with coherence unitary natural transformations

$$\eta_k : \pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}} \to \eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}, \qquad k \in H,$$

that satisfy the H-intertwiner axioms. (We have chosen to denote these maps by η_k rather than $(\eta_{\mathcal{V}})_k$ to prevent the notation from becoming too cluttered.)

We need to do the following:

1. Define the components of each η_k. This means we need to define unitaries
$$(\eta_k)_X : \pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(X) \to \eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X)$$
for each $X \in \mathcal{V}$. Because the underlying C^*-category of $\mathrm{Res}_H^G \mathrm{Ind}_H^G(\mathcal{V})$ is the category of fixed points $C_H(G, \mathcal{V})^H$, each $(\eta_k)_X$ should be a unitary in $C_H(G, \mathcal{V})$ which commutes with the fixed point trivialisers for $\pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(X)$ and $\eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X)$.

2. Check that the $(\eta_k)_X$'s are natural in X.

3. Check that the η_k's satisfy the H-intertwiner axioms.

In order to define $(\eta_k)_X$ for $k \in H$, $X \in \mathcal{V}$, let us first recall the definitions of $\pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(X)$ and $\eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X)$. On the one hand $\pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(X) = k \cdot \delta_H^X$ where

$$(k \cdot \delta_H^X)(g) = \begin{cases} \pi_{g^{-1}k}^{\mathcal{V}}(X) & \text{if } g \in H, \\ 0 & \text{otherwise} \end{cases}$$

and for each $\ell \in H$, the trivialiser $\rho_\ell^{k\cdot\delta_H^X} : \pi_\ell^{C_H(G,\mathcal{V})}\left(k \cdot \delta_H^X\right) \to k \cdot \delta_H^X$ is defined by

$$\rho_\ell^{k\cdot\delta_H^X}(g) = \begin{cases} \left(\mu_{\ell,\ell^{-1}g^{-1}k}^{\mathcal{V}}\right)_X & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, $\eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X) = \delta_H^{\pi_k^{\mathcal{V}}(X)}$ is defined by

$$\delta_H^{\pi_k^{\mathcal{V}}(X)}(g) = \begin{cases} \pi_{g^{-1}}^{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X) & \text{if } g \in G, \\ 0 & \text{otherwise} \end{cases}$$

and for each $\ell \in H$, the trivialiser $\rho_\ell^{\delta_H^{\pi_k^{\mathcal{V}}(X)}} : \pi_\ell^{C_H(G,\mathcal{V})}\left(\delta_H^{\pi_k^{\mathcal{V}}(X)}\right) \to \delta_H^{\pi_k^{\mathcal{V}}(X)}$ is defined by

$$\rho_\ell^{\delta_H^{\pi_k^{\mathcal{V}}(X)}}(g) = \begin{cases} \left(\mu_{\ell,\ell^{-1}g}^{\mathcal{V}}\right)_{\pi_k^{\mathcal{V}}(X)} & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we define $(\eta_k)_X : \pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(X) \to \eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X)$ by

$$(\eta_k)_X(g) := \begin{cases} \left(\mu^{\mathcal{V}}_{g^{-1},k}\right)_X^{-1} & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

First, let us check that this is a morphism of fixed points. To do this, we need to show that for all $\ell \in H$ and $g \in G$ the following diagram commutes

$$\begin{tikzcd} \pi_\ell^{C_H(G,\mathcal{V})}\left(k \cdot \delta_H^X\right)(g) \arrow[r, "\pi_\ell^{C_H(G,\mathcal{V})}(\eta_k)_X(g)"] \arrow[d, "\rho_\ell(g)"'] & \pi_\ell^{C_H(G,\mathcal{V})}\left(\delta_H^{\pi_k^{\mathcal{V}}(X)}\right)(g) \arrow[d, "\rho_\ell(g)"] \\ (k \cdot \delta_H^X)(g) \arrow[r, "(\eta_k)_X(g)"] & \delta_H^{\pi_k^{\mathcal{V}}(X)}(g) \end{tikzcd} \tag{1}$$

Since the functions are supported on H, we just need to check this for all $g \in H$ (for $g \notin H$, the arrows are all zero maps). Then by definition, (1) is the diagram

$$\begin{tikzcd} \pi_\ell^{\mathcal{V}} \circ \pi^{\mathcal{V}}_{\ell^{-1}g^{-1}k}(X) \arrow[r, "\left(\mu^{\mathcal{V}}_{\ell^{-1}g,k}\right)^{-1}"] \arrow[d, "\mu^{\mathcal{V}}_{\ell,\ell^{-1}g^{-1}k}"'] & \pi_\ell^{\mathcal{V}} \circ \pi^{\mathcal{V}}_{\ell^{-1}g^{-1}} \circ \pi_k^{\mathcal{V}}(X) \arrow[d, "\mu^{\mathcal{V}}_{\ell,\ell^{-1}g^{-1}}"] \\ \pi^{\mathcal{V}}_{g^{-1}k}(X) \arrow[r, "\left(\mu^{\mathcal{V}}_{g^{-1},k}\right)^{-1}"] & \pi^{\mathcal{V}}_{g^{-1}} \circ \pi_k^{\mathcal{V}}(X) \end{tikzcd}$$

and this commutes by the G-category axioms.

Next, to show naturality in X, we need to show that given a morphism $\alpha : X \to Y$ in $\mathcal{V}$, the following diagram commutes

$$\begin{tikzcd} \pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(X) \arrow[r, "\pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(\alpha)"] \arrow[d, "(\eta_k)_X"'] & \pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(Y) \arrow[d, "(\eta_k)_Y"] \\ \eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X) \arrow[r, "\eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(\alpha)"] & \eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(Y) \end{tikzcd}$$

To do this, we need to show that for all $g \in G$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(X)(g) \arrow[r, "\pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(\alpha)(g)"] \arrow[d, "(\eta_k)_X(g)"'] & \pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}}(Y)(g) \arrow[d, "(\eta_k)_Y(g)"] \\
\eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X)(g) \arrow[r, "\eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(\alpha)(g)"] & \eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(Y)(g)
\end{tikzcd}$$

We only need to check this for $g \in H$ and in this case, this is the diagram

$$\begin{tikzcd}
\pi_{g^{-1}k}^{\mathcal{V}}(X) \arrow[r, "\pi_{g^{-1}k}^{\mathcal{V}}(\alpha)"] \arrow[d, "\mu_{g^{-1},k}^{-1}"'] & \pi_{g^{-1}k}^{\mathcal{V}}(Y) \arrow[d, "\mu_{g^{-1},k}^{-1}"] \\
\pi_{g^{-1}}^{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X) \arrow[r, "\pi_{g^{-1}}^{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(\alpha)"] & \pi_{g^{-1}}^{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(Y)
\end{tikzcd}$$

and this commutes by the naturality of $\mu_{g^{-1},k}^{-1}$.

Lastly, we need to check the H-intertwiner axioms. First let us check the composition axiom. This says that for all $k, \ell \in H$, the following diagram should commute

$$\begin{tikzcd}
\pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \pi_\ell^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}} \arrow[r, "\eta_\ell"] \arrow[d, "\mu_{k,\ell}^{\mathrm{Ind}(\mathcal{V})}"'] & \pi_k^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}} \circ \pi_\ell^{\mathcal{V}} \arrow[r, "\eta_k"] & \eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}} \circ \pi_\ell^{\mathcal{V}} \arrow[d, "\mu_{k,\ell}^{\mathcal{V}}"] \\
\pi_{k\ell}^{\mathrm{Ind}(\mathcal{V})} \circ \eta_{\mathcal{V}} \arrow[rr, "\eta_{k\ell}"] & & \eta_{\mathcal{V}} \circ \pi_{k\ell}^{\mathcal{V}}
\end{tikzcd} \tag{2}$$

To do this we need to show that for all $X \in \mathcal{V}$ and $g \in G$, the following diagram commutes

$$\begin{tikzcd}
\left(k\ell \cdot \delta_H^X\right)(g) \arrow[r, "(k \cdot \eta_\ell)_X(g)"] \arrow[d, "\mathrm{Id}"'] & \left(k \cdot \delta_H^{\pi_\ell^{\mathcal{V}}(X)}\right)(g) \arrow[r, "(\eta_k)_{\pi_\ell^{\mathcal{V}}(X)}(g)"] & \delta_H^{\pi_k^{\mathcal{V}} \pi_\ell^{\mathcal{V}}(X)}(g) \arrow[d, "\delta_H^{\left(\mu_{k,\ell}^{\mathcal{V}}\right)_X}(g)"] \\
\left(k\ell \cdot \delta_H^X\right)(g) \arrow[rr, "\eta_{k\ell}(g)"] & & \delta_H^{\pi_{k\ell}^{\mathcal{V}}(X)}(g)
\end{tikzcd} \tag{3}$$

Again, we just need to check this for $g \in H$. Then by definition, (3) is the diagram

$$\begin{tikzcd}
\pi^{\mathcal{V}}_{g^{-1}k\ell}(X) \arrow[r, "\mu^{-1}_{g^{-1}k,\ell}"] \arrow[d, "\mathrm{Id}"'] & \pi^{\mathcal{V}}_{g^{-1}k} \circ \pi^{\mathcal{V}}_{\ell}(X) \arrow[r, "\mu^{-1}_{g^{-1},k}"] & \pi^{\mathcal{V}}_{g^{-1}} \circ \pi^{\mathcal{V}}_{k} \circ \pi^{\mathcal{V}}_{\ell}(X) \arrow[d, "\mu_{k,\ell}"] \\
\pi^{\mathcal{V}}_{g^{-1}k\ell}(X) \arrow[rr, "\mu_{g^{-1},k\ell}"] & & \pi^{\mathcal{V}}_{g^{-1}} \circ \pi^{\mathcal{V}}_{k\ell}(X)
\end{tikzcd}$$

which commutes by the H-category axioms.

The other H-intertwiner axiom is the unit axiom which says that the following diagram should commute

$$\begin{tikzcd}
& \eta_{\mathcal{V}} \arrow[dl, "\varepsilon^{\mathrm{Ind}(\mathcal{V})}_{\eta_{\mathcal{V}}}"'] \arrow[dr, "\eta_{\mathcal{V}}\varepsilon^{\mathcal{V}}"] & \\
\pi^{\mathrm{Ind}(\mathcal{V})}_{e} \circ \eta_{\mathcal{V}} \arrow[rr, "\eta_e"] & & \eta_{\mathcal{V}} \circ \pi^{\mathcal{V}}_{e}
\end{tikzcd} \tag{4}$$

To do this, we need to show that for all $X \in \mathcal{V}$ and all $g \in G$, the following diagram commutes

$$\begin{tikzcd}
& \delta^{X}_{H}(g) \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\delta^{\varepsilon^{\mathcal{V}}_{X}}_{H}(g)"] & \\
e \cdot \delta^{X}_{H}(g) \arrow[rr, "(\eta_e)_X(g)"] & & \delta^{\pi^{\mathcal{V}}_{e}(X)}_{H}(g)
\end{tikzcd} \tag{5}$$

Again, we only need to check this for $g \in H$. Then by definition, (5) is the diagram

$$\begin{tikzcd}
& \pi^{\mathcal{V}}_{g^{-1}}(X) \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\varepsilon^{\mathcal{V}}"] & \\
\pi^{\mathcal{V}}_{g^{-1}}(X) \arrow[rr, "\mu^{-1}_{g^{-1},e}"] & & \pi^{\mathcal{V}}_{g^{-1}} \circ \pi^{\mathcal{V}}_{e}(X)
\end{tikzcd}$$

which commutes by the H-category axioms. Therefore (4) commutes and $\eta_{\mathcal{V}}$ is an H-intertwiner. $\square$

Lemma 3.41. *The H-intertwiners*

$$\eta_{\mathcal{V}} : \mathcal{V} \to \mathrm{Res}_H^G \mathrm{Ind}_H^G(\mathcal{V}), \quad \mathcal{V} \in \mathsf{REP}(H)$$

defined in Lemma 3.40 are the 1-cells of a pseudonatural transformation

$$\eta : 1_{\mathsf{REP}(H)} \to \mathrm{Res}_H^G \mathrm{Ind}_H^G.$$

Proof. We need to define unitary natural transformations $\eta^{\mathcal{V},\mathcal{W}}$ for all $\mathcal{V}, \mathcal{W} \in \mathsf{REP}(H)$ with components

$$\eta_\Theta^{\mathcal{V},\mathcal{W}} : \mathrm{Res}_H^G \mathrm{Ind}_H^G(\Theta) \circ \eta_{\mathcal{V}} \to \eta_{\mathcal{W}} \circ \Theta, \quad \Theta \in \mathrm{Hom}_H(\mathcal{V}, \mathcal{W})$$

that satisfy the pseudonatural transformation axioms.

To do this, we need to do the following:

1. Firstly, each $\eta_\Theta^{\mathcal{V},\mathcal{W}}$ should be a 2-cell in $\mathsf{REP}(H)$, that is, an H-natural transformation. Therefore, to define $\eta_\Theta^{\mathcal{V},\mathcal{W}}$, we need to:
   (a) Define unitaries
   $$\left(\eta_\Theta^{\mathcal{V},\mathcal{W}}\right)_X : \mathrm{Res}_H^G \mathrm{Ind}_H^G(\Theta) \circ \eta_{\mathcal{V}}(X) \to \eta_{\mathcal{W}} \circ \Theta(X)$$
   for all $X \in \mathcal{V}$. Since the underlying C^*-category of $\mathrm{Res}_H^G \mathrm{Ind}_H^G(\mathcal{W})$ is the category of fixed points $C_H(G, \mathcal{W})^H$, each $\left(\eta_\Theta^{\mathcal{V},\mathcal{W}}\right)_X$ should be a morphism in $C_H(G, \mathcal{W})$ which commutes with the fixed point trivialisers for $\mathrm{Res}_H^G \mathrm{Ind}_H^G(\Theta) \circ \eta_{\mathcal{V}}(X)$ and $\eta_{\mathcal{W}} \circ \Theta(X)$.
   (b) Check that the $\left(\eta_\Theta^{\mathcal{V},\mathcal{W}}\right)_X$'s are natural in X.
   (c) Check that the H-natural transformation axiom holds.
2. Then we need to check that the $\eta_\Theta^{\mathcal{V},\mathcal{W}}$'s are natural in Θ.
3. Finally, we need to check that the pseudonatural transformation axioms hold.

In order to define $\eta_\Theta^{\mathcal{V},\mathcal{W}}$ for $\Theta \in \mathrm{Hom}_H(\mathcal{V}, \mathcal{W})$, let us first recall the definitions of $\mathrm{Res}_H^G \mathrm{Ind}_H^G(\Theta) \circ \eta_{\mathcal{V}}(X)$ and $\eta_{\mathcal{W}} \circ \Theta(X)$ for $X \in \mathcal{V}$. On the one hand, $\mathrm{Res}_H^G \mathrm{Ind}_H^G(\Theta) \circ \eta_{\mathcal{V}}(X) = \Theta(\delta_H^X)$, which is given by

$$\Theta(\delta_H^X)(g) = \begin{cases} \Theta \circ \pi_{g^{-1}}^{\mathcal{V}}(X) & \text{if } g \in H, \\ 0 & \text{otherwise} \end{cases}$$

and for each $k \in H$, the trivialiser $\rho_k^{\Theta(\delta_H^X)} : \pi_k^{C_H(G,\mathcal{W})}\left(\Theta(\delta_H^X)\right) \to \Theta(\delta_H^X)$ is defined as follows: For $g \in H$, $\rho_k^{\Theta(\delta_H^X)}(g)$ is the composite

$$\pi_k^{\mathcal{W}} \circ \Theta \circ \pi_{k^{-1}g^{-1}}^{\mathcal{V}}(X) \xrightarrow{\Theta_k} \Theta \circ \pi_k^{\mathcal{V}} \circ \pi_{k^{-1}g^{-1}}^{\mathcal{V}}(X) \xrightarrow{\mu_{k,k^{-1}g^{-1}}} \Theta \circ \pi_{g^{-1}}^{\mathcal{V}}(X)$$

and for $g \notin H$, $\rho_k^{\Theta(\delta_H^X)}(g)$ is the zero map.

On the other hand, $\eta_{\mathcal{W}} \circ \Theta(X) = \delta_H^{\Theta(X)}$, which is given by

$$\delta_H^{\Theta(X)}(g) = \begin{cases} \pi_{g^{-1}}^{\mathcal{V}} \circ \Theta(X) & \text{if } g \in H, \\ 0 & \text{otherwise} \end{cases}$$

and for all k, the trivialiser $\rho_k^{\delta_H^{\Theta(X)}} : \pi_k^{C_H(G,\mathcal{W})}\left(\delta_H^{\Theta(X)}\right) \to \delta_H^{\Theta(X)}$ is defined as follows: For $g \in H$,

$$\rho_k^{\delta_H^{\Theta(X)}}(g) = \mu_{k,k^{-1}g^{-1}} : \pi_k^{\mathcal{V}} \circ \pi_{k^{-1}g^{-1}}^{\mathcal{V}} \circ \Theta(X) \to \pi_{g^{-1}}^{\mathcal{V}} \circ \Theta(X)$$

and for $g \notin H$, $\rho_k^{\delta_H^{\Theta(X)}}(g)$ is the zero map.

Therefore, we define $\left(\eta_{\Theta}^{\mathcal{V},\mathcal{W}}\right)_X$ by

$$\left(\eta_{\Theta}^{\mathcal{V},\mathcal{W}}\right)_X(g) := \begin{cases} (\Theta_{g^{-1}})_X^{-1} & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

We need to show that this is a morphism of fixed points. To do this, we need to show that for all $k \in H$ and $g \in G$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{C_H(G,\mathcal{V})}\left(\Theta(\delta_H^X)\right)(g) \arrow[r, "\pi_k^{C_H(G,\mathcal{V})}\left(\eta_{\Theta}^{\mathcal{V},\mathcal{W}}\right)_X(g)"] \arrow[d, "\rho_k^{\Theta(\delta_H^X)}(g)"'] & \pi_k^{C_H(G,\mathcal{V})}\left(\delta_H^{\Theta(X)}\right)(g) \arrow[d, "\rho_k^{\delta_H^{\Theta(X)}}(g)"] \\
\Theta(\delta_H^X)(g) \arrow[r, "\left(\eta_{\Theta}^{\mathcal{V},\mathcal{W}}\right)_X(g)"] & \delta_H^{\Theta(X)}(g)
\end{tikzcd}$$

Since the functions are supported on H, we only need to check this for $g \in H$ (for $g \notin H$, all the arrows are zero maps). Then by definition, this is the diagram

$$\begin{tikzcd}
\pi_k^{\mathcal{W}} \circ \Theta \circ \pi_{k^{-1}g^{-1}}^{\mathcal{V}}(X) \arrow[r, "\left(\Theta_{k^{-1}g^{-1}}\right)^{-1}"] \arrow[d, "\Theta_k"'] & \pi_k^{\mathcal{W}} \circ \pi_{k^{-1}g^{-1}}^{\mathcal{W}} \circ \Theta(X) \arrow[dd, "\mu_{k,k^{-1}g^{-1}}"] \\
\Theta \circ \pi_k^{\mathcal{V}} \circ \pi_{k^{-1}g^{-1}}^{\mathcal{V}}(X) \arrow[d, "\mu_{k,k^{-1}g^{-1}}"'] & \\
\Theta \circ \pi_{g^{-1}}^{\mathcal{V}}(X) \arrow[r, "\left(\Theta_{g^{-1}}\right)^{-1}"] & \pi_{g^{-1}}^{\mathcal{W}} \circ \Theta(X)
\end{tikzcd}$$

This diagram commutes because Θ is an H-intertwiner and hence $\left(\eta_\Theta^{\mathcal{V},\mathcal{W}}\right)_X$ is a morphism of fixed points.

Next we need to show that $\eta_\Theta^{\mathcal{V},\mathcal{W}}$ is an H-natural transformation. Naturality follows from the fact that each $(\Theta_{g^{-1}})^{-1}$ is natural. To show that it is H-natural, we need to show that for all $k \in H$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{\mathrm{Ind}(\mathcal{W})} \circ \mathrm{Res}_H^G \mathrm{Ind}_H^G(\Theta) \circ \eta_{\mathcal{V}} \arrow[r, "\eta_\Theta^{\mathcal{V},\mathcal{W}}"] \arrow[d, "\eta_k \circ \mathrm{Ind}_H^G(\Theta)_k"'] & \pi_k^{\mathrm{Ind}(\mathcal{W})} \circ \eta_{\mathcal{W}} \circ \Theta \arrow[d, "\Theta_k \circ \eta_k"] \\
\mathrm{Res}_H^G \mathrm{Ind}_H^G(\Theta) \circ \eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}} \arrow[r, "\eta_\Theta^{\mathcal{V},\mathcal{W}}"] & \eta_{\mathcal{W}} \circ \Theta \circ \pi_k^{\mathcal{V}}
\end{tikzcd}$$

To do this, we need to show that for all $X \in \mathcal{V}$ and $g \in G$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{\mathrm{Ind}(\mathcal{W})} \circ \mathrm{Res}_H^G \mathrm{Ind}_H^G(\Theta) \circ \eta_{\mathcal{V}}(X)(g) \arrow[r, "(\eta_\Theta^{\mathcal{V},\mathcal{W}})_{X(g)}"] \arrow[d, "(\eta_k \circ \mathrm{Ind}_H^G(\Theta)_k)_{X(g)}"'] & \pi_k^{\mathrm{Ind}(\mathcal{W})} \circ \eta_{\mathcal{W}} \circ \Theta(X)(g) \arrow[d, "(\Theta_k \circ \eta_k)_{X(g)}"] \\
\mathrm{Res}_H^G \mathrm{Ind}_H^G(\Theta) \circ \eta_{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X)(g) \arrow[r, "(\eta_\Theta^{\mathcal{V},\mathcal{W}})_{X(g)}"] & \eta_{\mathcal{W}} \circ \Theta \circ \pi_k^{\mathcal{V}}(X)(g)
\end{tikzcd}$$

Again, we only need to check this for $g \in H$. In this case, by definition this is the diagram

$$\begin{tikzcd}
\Theta \circ \pi^{\mathcal{V}}_{g^{-1}k}(X) \arrow[r, "\left(\Theta_{g^{-1}k}\right)^{-1}"] \arrow[dd, "\mu^{-1}_{g^{-1},k}"'] & \pi^{\mathcal{W}}_{g^{-1}k} \circ \Theta(X) \arrow[d, "\mu^{-1}_{g^{-1},k}"] \\
& \pi^{\mathcal{W}}_{g^{-1}} \circ \pi^{\mathcal{W}}_{k} \circ \Theta(X) \arrow[d, "\Theta_k"] \\
\Theta \circ \pi^{\mathcal{V}}_{g^{-1}} \circ \pi^{\mathcal{V}}_{k}(X) \arrow[r, "\left(\Theta_{g^{-1}}\right)^{-1}"] & \pi^{\mathcal{W}}_{g^{-1}} \circ \Theta \circ \pi^{\mathcal{V}}_{k}(X)
\end{tikzcd}$$

and this commutes because Θ is an H-intertwiner.

Next, let us show that the $\eta^{\mathcal{V},\mathcal{W}}_{\Theta}$'s are natural in Θ. We need to show that given $\Theta, \Phi \in \mathsf{REP}(\mathcal{V}, \mathcal{W})$ and an H-natural transformation $\kappa : \Theta \to \Phi$, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Res}^G_H \mathrm{Ind}^G_H(\Theta) \circ \eta_{\mathcal{V}} \arrow[r, "\mathrm{Res}^G_H \mathrm{Ind}^G_H(\kappa)"] \arrow[d, "\eta^{\mathcal{V},\mathcal{W}}_{\Theta}"'] & \mathrm{Res}^G_H \mathrm{Ind}^G_H(\Phi) \circ \eta_{\mathcal{V}} \arrow[d, "\eta^{\mathcal{V},\mathcal{W}}_{\Phi}"] \\
\eta_{\mathcal{W}} \circ \Theta \arrow[r, "\kappa"] & \eta_{\mathcal{W}} \circ \Theta
\end{tikzcd}$$

To do this we need to show that for all $X \in \mathcal{V}$ and $g \in G$, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Res}^G_H \mathrm{Ind}^G_H(\Theta) \circ \eta_{\mathcal{V}}(X)(g) \arrow[r, "\mathrm{Res}^G_H \mathrm{Ind}^G_H(\kappa)_{\eta_{\mathcal{V}}(X)}(g)"] \arrow[d, "\left(\eta^{\mathcal{V},\mathcal{W}}_{\Theta}\right)_X(g)"'] & \mathrm{Res}^G_H \mathrm{Ind}^G_H(\Phi) \circ \eta_{\mathcal{V}}(X)(g) \arrow[d, "(\eta^{\mathcal{V},\mathcal{W}}_{\Phi})_X(g)"] \\
\eta_{\mathcal{W}} \circ \Theta(X)(g) \arrow[r, "(\eta_{\mathcal{W}}\kappa)_X(g)"] & \eta_{\mathcal{W}} \circ \Theta(X)(g)
\end{tikzcd}$$

We only need to check this for $g \in H$. In this case, by definition this is the diagram

$$\begin{tikzcd}
\Theta \circ \pi^{\mathcal{V}}_{g^{-1}}(X) \arrow[r, "\kappa"] \arrow[d, "(\Theta_{g^{-1}})^{-1}"'] & \Phi \circ \pi^{\mathcal{V}}_{g^{-1}}(X) \arrow[d, "(\Phi_{g^{-1}})^{-1}"] \\
\pi^{\mathcal{V}}_{g^{-1}} \circ \Theta(X) \arrow[r, "\kappa"] & \pi^{\mathcal{V}}_{g^{-1}} \circ \Phi(X)
\end{tikzcd}$$

and this commutes because κ is an H-natural transformation.

Finally, we need to show that the $\eta^{\mathcal{V},\mathcal{W}}_{\Theta}$ satisfy the pseudonatural transformation axioms. The pentagon axiom says that given H-intertwiners $\Theta : \mathcal{V} \to \mathcal{W}$ and $\Phi : \mathcal{W} \to \mathcal{X}$, the following diagram should commute.

$$\begin{tikzcd}
\mathrm{Res}^G_H\mathrm{Ind}^G_H(\Phi) \circ \mathrm{Res}^G_H\mathrm{Ind}^G_H(\Theta) \circ \eta_{\mathcal{V}} \arrow[r, "\eta^{\mathcal{W},\mathcal{X}}_{\Phi}"] \arrow[d, "\mathrm{Id}"'] & \mathrm{Res}^G_H\mathrm{Ind}^G_H(\Phi) \circ \eta_{\mathcal{W}} \circ \Theta \arrow[r, "\eta^{\mathcal{V},\mathcal{W}}_{\Theta}"] & \eta_{\mathcal{X}} \circ \Phi \circ \Theta \arrow[d, "\mathrm{Id}"] \\
\mathrm{Res}^G_H\mathrm{Ind}^G_H(\Phi \circ \Theta) \arrow[rr, "\eta^{\mathcal{V},\mathcal{X}}_{\Phi\circ\Theta}"] & & \eta_{\mathcal{X}} \circ \Phi \circ \Theta
\end{tikzcd}$$

This follows immediately from the definition of the coherence unitaries for $\Phi \circ \Theta$.

The unit axiom says that the following diagram should commute

$$\begin{tikzcd}
& \eta_{\mathcal{V}} \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\
\mathrm{Res}^G_H\mathrm{Ind}^G_H(1_{\mathcal{V}}) \circ \eta_{\mathcal{V}} \arrow[rr, "\eta^{\mathcal{V},\mathcal{V}}_{1_{\mathcal{V}}}"] & & \eta_{\mathcal{V}} \circ 1_{\mathcal{V}}
\end{tikzcd}$$

and this follows immediately from the definition of $\eta^{\mathcal{V},\mathcal{V}}_{1_{\mathcal{V}}}$. □

Next, we need to define the counit $\xi : \mathrm{Ind}^G_H\mathrm{Res}^G_H \to 1_{\mathsf{Rep}(G)}$. First, we need to define G-intertwiners $\xi_{\mathcal{V}} : \mathrm{Ind}^G_H\mathrm{Res}^G_H(\mathcal{V}) \to \mathcal{V}$ for all $\mathcal{V} \in \mathsf{Rep}(G)$. To check one of the axioms we shall need the following lemma.

Lemma 3.42. *Let $\mathcal{V}$ be a G-category and let $V \in \mathcal{V}^G$. Then for all $t \in G$,*

$$(\mu^{\mathcal{V}}_{t,e})_V = \pi^{\mathcal{V}}_t(\rho^V_e).$$

Proof. Given $t \in G$, the fixed point axiom says that the following diagram commutes

$$\begin{tikzcd}
\pi_t^{\mathcal{V}} \circ \pi_e^{\mathcal{V}}(V) \arrow[r, "\pi_t^{\mathcal{V}}(\rho_e^V)"] \arrow[d, "(\mu_{t,e}^{\mathcal{V}})_V"] & \pi_t^{\mathcal{V}}(V) \arrow[d, "\rho_t^V"] \\
\pi_t^{\mathcal{V}}(V) \arrow[r, "\rho_t^V"] & V
\end{tikzcd}$$

That is,

$$\rho_t^V \circ \pi_t^{\mathcal{V}}(\rho_e^V) = \rho_t^V \circ (\mu_{t,e}^{\mathcal{V}})_V$$

and left multiplying both sides of the equation by $(\rho_t^V)^{-1}$ yields the result. $\square$

Lemma 3.43. *Let $\mathcal{V}$ be a G-category. There is a G-intertwiner*

$$\xi_{\mathcal{V}} : \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V}) \to \mathcal{V}$$

whose underlying $*$-functor is defined on objects by

$$\xi_{\mathcal{V}}(\mathbf{X}) = \bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))$$

where Γ is a set of coset representatives for G/H and on morphisms by

$$\xi_{\mathcal{V}}(\alpha) = \bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}(\alpha(\gamma)).$$

Proof. We need to equip $\xi_{\mathcal{V}}$ with coherence unitary natural transformations

$$\xi_t : \pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}} \to \xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}, \quad t \in G$$

that satisfy the G-intertwiner axioms. (As with the unit we have chosen to denote these maps by ξ_t rather than $(\xi_{\mathcal{V}})_t$ to avoid the notation becoming too cluttered.)

We need to do the following:

1. Define unitaries

$$(\xi_t)_{\mathbf{X}} : \pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X}) \to \xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})$$

for all $t \in G$, $\mathbf{X} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathbf{X})$.

2. For each $t \in G$, check that the $(\xi_t)_{\mathbf{X}}$'s are natural in $\mathbf{X}$.

3. Check that the G-intertwiner axioms are satisfied.

In order to define the unitaries $(\xi_t)_{\mathbf{X}}$, let us first recall the definitions of $\pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X})$ and $\xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})$ for $\mathbf{X} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V})$. On the one hand,

$$\pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X}) = \pi_t^{\mathcal{V}} \left(\bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \right)$$

and on the other hand

$$\begin{aligned}
\xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) &= \bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}((t \cdot \mathbf{X})(\gamma)) \\
&= \bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma)).
\end{aligned}$$

To define $(\xi_t)_{\mathbf{X}}$ we also need the fact that for each $\gamma \in \Gamma$, we have

$$t\gamma = \gamma' k_{\gamma'}$$

for some $\gamma' \in \Gamma$ and $k_{\gamma'} \in H$. The subscript on $k_{\gamma'}$ is just to remind us of the dependence of $k_{\gamma'}$ on γ'. We define $(\xi_t)_{\mathbf{X}}$ as the composite

$$\begin{aligned}
\pi_t^{\mathcal{V}} \left[\bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \right] &\xrightarrow{\zeta^{\pi_t^{\mathcal{V}}}} \bigoplus_{\gamma \in \Gamma} \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \\
&\xrightarrow{\oplus \mu_{t,\gamma}} \bigoplus_{\gamma \in \Gamma} \pi_{t\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma)) \\
&\qquad = \bigoplus_{\gamma' \in \Gamma} \pi_{\gamma' k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \\
&\xrightarrow{\oplus \mu_{\gamma', k_{\gamma'}}^{-1}} \bigoplus_{\gamma' \in \Gamma} \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \\
&\xrightarrow{\oplus \rho_{k_{\gamma'}}^{\mathbf{X}}} \bigoplus_{\gamma' \in \Gamma} \pi_{\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma'))
\end{aligned}$$

where $\zeta^{\pi_t^{\mathcal{V}}}$ is the canonical unitary natural transformation for the preservation of direct sums by the $*$-functor $\pi_t^{\mathcal{V}}$.

Let us show that the $(\xi_t)_{\mathbf{X}}$'s are natural in $\mathbf{X}$. To do this we need to show that given a morphism $\alpha : \mathbf{X} \to \mathbf{Y}$ in $\mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V})$, the following diagram commutes

$$\begin{tikzcd}
\pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X}) \arrow[r, "\pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\alpha)"] \arrow[d, "(\xi_t)_{\mathbf{X}}"'] & \pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{Y}) \arrow[d, "(\xi_t)_{\mathbf{Y}}"] \\
\xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \arrow[r, "\xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\alpha)"] & \xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{Y})
\end{tikzcd}$$

Expanding this diagram using the definitions and adding some extra internal arrows yields the following

$$\begin{tikzcd}
\pi_t^{\mathcal{V}} \left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[r, "\pi_t^{\mathcal{V}}\left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\alpha(\gamma))\right]"] \arrow[d, "\zeta^{\pi_t^{\mathcal{V}}}"'] \arrow[dr, phantom, "(\mathrm{I})"] & \pi_t^{\mathcal{V}} \left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{Y}(\gamma))\right] \arrow[d, "\zeta^{\pi_t^{\mathcal{V}}}"] \\
\bigoplus_{\gamma\in\Gamma} \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[r, "\bigoplus_{\gamma\in\Gamma} \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\alpha(\gamma))"] \arrow[d, "\oplus\mu_{t,\gamma}"'] \arrow[dr, phantom, "(\mathrm{II})"] & \bigoplus_{\gamma\in\Gamma} \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{Y}(\gamma)) \arrow[d, "\oplus\mu_{t,\gamma}"] \\
\begin{array}{c} \bigoplus_{\gamma\in\Gamma} \pi_{t\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma)) \\ = \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma' k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \end{array} \arrow[r, "\bigoplus_{\gamma\in\Gamma} \pi_{t\gamma}^{\mathcal{V}}(\alpha(\gamma))", "= \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma' k_{\gamma'}}^{\mathcal{V}}(\alpha(t^{-1}\gamma' k_{\gamma'}))"'] \arrow[d, "\mu_{\gamma', k_{\gamma'}}^{-1}"'] \arrow[dr, phantom, "(\mathrm{III})"] & \begin{array}{c} \bigoplus_{\gamma\in\Gamma} \pi_{t\gamma}^{\mathcal{V}}(\mathbf{Y}(\gamma)) \\ = \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma' k_{\gamma'}}^{\mathcal{V}}(\mathbf{Y}(t^{-1}\gamma' k_{\gamma'})) \end{array} \arrow[d, "\mu_{\gamma', k_{\gamma'}}^{-1}"] \\
\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \arrow[r, "\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\alpha(t^{-1}\gamma' k_{\gamma'}))"] \arrow[d, "\oplus\rho_{k_{\gamma'}}^{\mathbf{X}}"'] \arrow[dr, phantom, "(\mathrm{IV})"] & \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\mathbf{Y}(t^{-1}\gamma' k_{\gamma'})) \arrow[d, "\oplus\rho_{k_{\gamma'}}^{\mathbf{Y}}"] \\
\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma')) \arrow[r, "\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}}(\alpha(t^{-1}\gamma'))"] & \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}}(\mathbf{Y}(t^{-1}\gamma'))
\end{tikzcd}$$

Now, (I) commutes by the naturality of $\zeta^{\pi_t^{\mathcal{V}}}$, (II) commutes by the naturality of the $\mu_{t,\gamma}$'s, (III) commutes by the naturality of the $\mu^{-1}_{\gamma',k_{\gamma'}}$'s and (IV) commutes because α is a morphism of fixed points. Therefore, the $(\xi_t)_{\mathbf{X}}$'s are natural in $\mathbf{X}$.

Next, let us show that the ξ_t's satisfy the G-intertwiner axioms. Firstly, the composition axiom says that for all $s,t \in G$, the following diagram should commute.

$$\begin{tikzcd}
\pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}} \arrow[r, "\mu_{s,t}"] \arrow[d, "\xi_t"'] & \pi_{st}^{\mathcal{V}} \circ \xi_{\mathcal{V}} \arrow[dd, "\xi_{st}"] \\
\pi_s^{\mathcal{V}} \circ \xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})} \arrow[d, "\xi_s"'] & \\
\xi_{\mathcal{V}} \circ \pi_s^{\mathrm{Ind}(\mathcal{V})} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})} \arrow[r, "\mu_{s,t}"] & \xi_{\mathcal{V}} \circ \pi_{st}^{\mathrm{Ind}(\mathcal{V})}
\end{tikzcd}$$

To do this, we need to show that for all $\mathbf{X} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V})$, the following diagram commutes

$$\begin{tikzcd}
\pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X}) \arrow[r, "(\mu_{s,t})_{\mathbf{X}}"] \arrow[d, "(\xi_t)_{\mathbf{X}}"'] & \pi_{st}^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X}) \arrow[dd, "(\xi_{st})_{\mathbf{X}}"] \\
\pi_s^{\mathcal{V}} \circ \xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \arrow[d, "(\xi_s)_{\mathbf{X}}"'] & \\
\xi_{\mathcal{V}} \circ \pi_s^{\mathrm{Ind}(\mathcal{V})} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \arrow[r, "(\mu_{s,t})_{\mathbf{X}}"] & \xi_{\mathcal{V}} \circ \pi_{st}^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})
\end{tikzcd} \tag{1}$$

When we expand this diagram, we shall need to write some of the terms in different ways. In particular, for each $\gamma \in \Gamma$, we have

$$t\gamma = \gamma' k_{\gamma'}$$

for some $\gamma' \in \Gamma$ and $k_{\gamma'} \in H$ and for each $\gamma' \in \Gamma$ we have

$$s\gamma' = \gamma'' \ell_{\gamma''}$$

for some $\gamma'' \in \Gamma$ and $\ell_{\gamma''} \in H$. Furthermore, for each $\gamma \in \Gamma$, we have

$$\begin{aligned} st\gamma &= s\gamma' k_{\gamma'} \\ &= \gamma'' \ell_{\gamma''} k_{\gamma'}. \end{aligned}$$

Although we are not expressing it explicitly in the notation, the $k_{\gamma'}$ in the expression above depends on γ'' since γ' does (because the map $\gamma' \mapsto \gamma''$ defined by $\gamma'' H = s\gamma' H$ is a bijection from Γ to itself). Then, expanding (1) using the definitions yields the following

$$
\begin{tikzcd}
\begin{array}{c} \pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \left[\bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \right] \\ = \pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X}) \end{array} \arrow[r, "\mu_{s,t}"] \arrow[d, "\zeta^{\pi_t^{\mathcal{V}}}"'] & \begin{array}{c} \pi_{st}^{\mathcal{V}} \left[\bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \right] \\ = \pi_{st}^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X}) \end{array} \arrow[dd, "\zeta^{\pi_{st}^{\mathcal{V}}}"] \\
\pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma \in \Gamma} \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \right] \arrow[d, "\oplus \mu_{t,\gamma}"'] & \\
\begin{array}{c} \pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma \in \Gamma} \pi_{t\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma)) \right] \\ = \pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma' \in \Gamma} \pi_{\gamma' k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \right] \end{array} \arrow[d, "\oplus \mu_{\gamma', k_{\gamma'}}^{-1}"'] & \bigoplus_{\gamma \in \Gamma} \pi_{st}^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[dd, "\oplus \mu_{st,\gamma}"] \\
\pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma' \in \Gamma} \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \right] \arrow[d, "\oplus \rho_{k_{\gamma'}}"'] & \\
\begin{array}{c} \pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma' \in \Gamma} \pi_{\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma')) \right] \\ = \pi_s^{\mathcal{V}} \circ \xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \end{array} \arrow[d, "\zeta^{\pi_s^{\mathcal{V}}}"'] & \begin{array}{c} \bigoplus_{\gamma \in \Gamma} \pi_{st\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma)) \\ = \bigoplus_{\gamma'' \in \Gamma} \pi_{\gamma'' \ell_{\gamma''} k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}((st)^{-1}\gamma'' \ell_{\gamma''} k_{\gamma'})) \end{array} \arrow[dd, "\oplus \mu_{\gamma'', \ell_{\gamma''} k_{\gamma'}}^{-1}"] \\
\bigoplus_{\gamma' \in \Gamma} \pi_s^{\mathcal{V}} \circ \pi_{\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma')) \arrow[d, "\oplus \mu_{s,\gamma'}"'] & \\
\begin{array}{c} \bigoplus_{\gamma' \in \Gamma} \pi_{s\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma')) \\ = \bigoplus_{\gamma'' \in \Gamma} \pi_{\gamma'' \ell_{\gamma''}}^{\mathcal{V}}(\mathbf{X}(t^{-1}s^{-1}\gamma'' \ell_{\gamma''})) \end{array} \arrow[d, "\oplus \mu_{\gamma'', \ell_{\gamma''}}^{-1}"'] & \bigoplus_{\gamma'' \in \Gamma} \pi_{\gamma''}^{\mathcal{V}} \circ \pi_{\ell_{\gamma''} k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}((st)^{-1}\gamma'' \ell_{\gamma''} k_{\gamma'})) \arrow[dd, "\oplus \rho_{\ell_{\gamma''} k_{\gamma'}}"] \\
\bigoplus_{\gamma'' \in \Gamma} \pi_{\gamma''}^{\mathcal{V}} \circ \pi_{\ell_{\gamma''}}^{\mathcal{V}}(\mathbf{X}(t^{-1}s^{-1}\gamma'' \ell_{\gamma''})) \arrow[d, "\oplus \rho_{\ell_{\gamma''}}"'] & \\
\begin{array}{c} \bigoplus_{\gamma'' \in \Gamma} \pi_{\gamma''}^{\mathcal{V}}(\mathbf{X}(t^{-1}s^{-1}\gamma'')) \\ = \xi_{\mathcal{V}} \circ \pi_s^{\mathrm{Ind}(\mathcal{V})} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \end{array} \arrow[r, "\mathrm{Id}"] & \begin{array}{c} \bigoplus_{\gamma'' \in \Gamma} \pi_{\gamma''}^{\mathcal{V}}(\mathbf{X}((st)^{-1}\gamma'')) \\ = \xi_{\mathcal{V}} \circ \pi_{st}^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \end{array}
\end{tikzcd}
\tag{2}
$$

To show that this commutes, let us first remove the ζ's from the equation. By Corollary 3.3 we have a commutative diagram

$$\begin{tikzcd}
\pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[r, "\mu_{s,t}"] \arrow[d, "\zeta"'] & \pi_{st}^{\mathcal{V}} \left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[d, "\zeta"] \\
\bigoplus_{\gamma\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[r, "\oplus\mu_{s,t}"] & \bigoplus_{\gamma\in\Gamma} \pi_{st}^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))
\end{tikzcd} \tag{3}$$

Next, if we consider the diagram

$$\begin{tikzcd}
\pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma\in\Gamma} \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[r, "\zeta^{\pi_s^{\mathcal{V}}}"] \arrow[d, "\oplus\mu_{t,\gamma}"'] & \bigoplus_{\gamma\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[d, "\oplus\mu_{t,\gamma}"] \\
\begin{array}{c} \pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma\in\Gamma} \pi_{t\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \\ = \pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma' k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'}))\right] \end{array} \arrow[r, "\zeta^{\pi_s^{\mathcal{V}}}"] \arrow[d, "\oplus\mu^{-1}_{\gamma',k_{\gamma'}}"'] & \begin{array}{c} \bigoplus_{\gamma\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_{t\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma)) \\ = \bigoplus_{\gamma'\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_{\gamma' k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \end{array} \arrow[d, "\oplus\mu^{-1}_{\gamma',k_{\gamma'}}"] \\
\pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'}))\right] \arrow[r, "\zeta^{\pi_s^{\mathcal{V}}}"] \arrow[d, "\oplus\rho_{k_{\gamma'}}"'] & \bigoplus_{\gamma'\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \arrow[d, "\oplus\rho_{k_{\gamma'}}"] \\
\pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma'))\right] \arrow[r, "\zeta^{\pi_s^{\mathcal{V}}}"] & \bigoplus_{\gamma'\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_{\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma'))
\end{tikzcd}$$

then each of the squares commutes by the naturality of $\zeta^{\pi_s^{\mathcal{V}}}$ so that

$$\zeta^{\pi_s^{\mathcal{V}}} \circ \oplus\rho_{k_{\gamma'}} \circ \oplus\mu^{-1}_{\gamma',k_{\gamma'}} \circ \oplus\mu_{t,\gamma} = \oplus\rho_{k_{\gamma'}} \circ \mu^{-1}_{\gamma',k_{\gamma'}} \circ \oplus\mu_{t,\gamma} \circ \zeta^{\pi_s^{\mathcal{V}}}. \tag{4}$$

Substituting (3) and (4) into (2) yields

$$\begin{tikzcd}
\pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[r, "\zeta^{\pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}}}"] \arrow[d, "\zeta^{\pi_t^{\mathcal{V}}}"'] & \bigoplus_{\gamma\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[dd, "\oplus\mu_{s,t}"] \\
\pi_s^{\mathcal{V}} \left[\bigoplus_{\gamma\in\Gamma} \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[d, "\zeta^{\pi_s^{\mathcal{V}}}"'] & \\
\bigoplus_{\gamma\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[uur, "\mathrm{Id}"'] \arrow[d, "\oplus\mu_{t,\gamma}"'] & \bigoplus_{\gamma\in\Gamma} \pi_{st}^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[dd, "\oplus\mu_{st,\gamma}"] \\
\begin{array}{c} \bigoplus_{\gamma\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_{t\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma)) \\ = \bigoplus_{\gamma'\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_{\gamma' k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \end{array} \arrow[d, "\oplus\mu^{-1}_{\gamma',k_{\gamma'}}"'] & \\
\bigoplus_{\gamma'\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \arrow[d, "\oplus\rho_{k_{\gamma'}}"'] & \begin{array}{c} \bigoplus_{\gamma\in\Gamma} \pi_{st\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma)) \\ = \bigoplus_{\gamma''\in\Gamma} \pi_{\gamma''\ell_{\gamma''}k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}((st)^{-1}\gamma''\ell_{\gamma''}k_{\gamma'})) \end{array} \arrow[dd, "\oplus\mu^{-1}_{\gamma'',\ell_{\gamma''}k_{\gamma'}}"] \\
\bigoplus_{\gamma'\in\Gamma} \pi_s^{\mathcal{V}} \circ \pi_{\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma')) \arrow[d, "\oplus\mu_{s,\gamma'}"'] & \\
\begin{array}{c} \bigoplus_{\gamma'\in\Gamma} \pi_{s\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma')) \\ = \bigoplus_{\gamma''\in\Gamma} \pi_{\gamma''\ell_{\gamma''}}^{\mathcal{V}}(\mathbf{X}(t^{-1}s^{-1}\gamma''\ell_{\gamma''})) \end{array} \arrow[d, "\oplus\mu^{-1}_{\gamma'',\ell_{\gamma''}}"'] & \bigoplus_{\gamma''\in\Gamma} \pi_{\gamma''}^{\mathcal{V}} \circ \pi_{\ell_{\gamma''}k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}((st)^{-1}\gamma''\ell_{\gamma''}k_{\gamma'})) \arrow[dd, "\oplus\rho_{\ell_{\gamma''}k_{\gamma'}}"] \\
\bigoplus_{\gamma''\in\Gamma} \pi_{\gamma''}^{\mathcal{V}} \circ \pi_{\ell_{\gamma''}}^{\mathcal{V}}(\mathbf{X}(t^{-1}s^{-1}\gamma''\ell_{\gamma''})) \arrow[d, "\oplus\rho_{\ell_{\gamma''}}"'] & \\
\bigoplus_{\gamma''\in\Gamma} \pi_{\gamma''}^{\mathcal{V}}(\mathbf{X}(t^{-1}s^{-1}\gamma'')) \arrow[r, "\mathrm{Id}"] & \bigoplus_{\gamma''\in\Gamma} \pi_{\gamma''}^{\mathcal{V}}(\mathbf{X}((st)^{-1}\gamma''))
\end{tikzcd}$$

The top left triangle commutes by Lemma 1.23 and so we just need to check the rest of the diagram commutes by checking this for a typical summand in the direct sum. This follows from the commutativity of the following diagram. We have omitted the subscripts from the k's and ℓ's to avoid the diagram becoming too cluttered.

$$\begin{tikzcd}
\pi^{\mathcal{V}}_s \circ \pi^{\mathcal{V}}_t \circ \pi^{\mathcal{V}}_\gamma(\mathbf{X}(\gamma)) \arrow[r, "\mu^{\mathcal{V}}_{s,t}"] \arrow[d, "\mu^{\mathcal{V}}_{t,\gamma}"'] \arrow[dr, phantom, "\text{(I)}"] & \pi^{\mathcal{V}}_{st} \circ \pi^{\mathcal{V}}_\gamma(\mathbf{X}(\gamma)) \arrow[rr, "\mathrm{Id}"] \arrow[d, "\mu^{\mathcal{V}}_{st,\gamma}"] \arrow[drr, phantom, "\text{(II)}"] & & \pi^{\mathcal{V}}_{st} \circ \pi^{\mathcal{V}}_\gamma(\mathbf{X}(\gamma)) \arrow[d, "\mu^{\mathcal{V}}_{st,\gamma}"] \\
\begin{array}{c} \pi^{\mathcal{V}}_s \circ \pi^{\mathcal{V}}_{t\gamma}(\mathbf{X}(\gamma)) \\ = \pi^{\mathcal{V}}_s \circ \pi^{\mathcal{V}}_{\gamma' k}(\mathbf{X}(t^{-1}\gamma' k)) \end{array} \arrow[r, "\mu^{\mathcal{V}}_{s,t\gamma}", "=\mu^{\mathcal{V}}_{s,\gamma' k}"'] \arrow[d, "\left(\mu^{\mathcal{V}}_{\gamma',k}\right)^{-1}"'] \arrow[dr, phantom, "\text{(III)}"] & \begin{array}{c} \pi^{\mathcal{V}}_{st\gamma}(\mathbf{X}(\gamma)) \\ = \pi^{\mathcal{V}}_{s\gamma' k}(\mathbf{X}(t^{-1}\gamma' k)) \\ = \pi^{\mathcal{V}}_{\gamma''\ell k}(\mathbf{X}(t^{-1}s^{-1}\gamma''\ell k)) \end{array} \arrow[r, "\left(\mu^{\mathcal{V}}_{\gamma''\ell,k}\right)^{-1}"] \arrow[d, "\left(\mu^{\mathcal{V}}_{s\gamma',k}\right)^{-1}"', "=\left(\mu^{\mathcal{V}}_{\gamma''\ell,k}\right)^{-1}"] \arrow[dr, phantom, "\text{(IV)}"] & \pi^{\mathcal{V}}_{\gamma''\ell} \circ \pi^{\mathcal{V}}_k(\mathbf{X}(t^{-1}s^{-1}\gamma''\ell k)) \arrow[r, "\mu^{\mathcal{V}}_{\gamma''\ell,k}"] \arrow[d, "\left(\mu^{\mathcal{V}}_{\gamma'',\ell}\right)^{-1}"] \arrow[dr, phantom, "\text{(V)}"] & \begin{array}{c} \pi^{\mathcal{V}}_{st\gamma}(\mathbf{X}(\gamma)) \\ = \pi^{\mathcal{V}}_{\gamma''\ell k}(\mathbf{X}(t^{-1}s^{-1}\gamma''\ell k)) \end{array} \arrow[d, "\mu^{\mathcal{V}}_{\gamma'',\ell k}"] \\
\pi^{\mathcal{V}}_s \circ \pi^{\mathcal{V}}_{\gamma'} \circ \pi^{\mathcal{V}}_k(\mathbf{X}(t^{-1}\gamma' k)) \arrow[r, "\mu^{\mathcal{V}}_{s,\gamma'}"] \arrow[d, "\rho^{\mathbf{X}}_k"'] \arrow[dr, phantom, "\text{(VI)}"] & \begin{array}{c} \pi^{\mathcal{V}}_{s\gamma'} \circ \pi^{\mathcal{V}}_k(\mathbf{X}(t^{-1}\gamma' k)) \\ = \pi^{\mathcal{V}}_{\gamma''\ell} \circ \pi^{\mathcal{V}}_k(\mathbf{X}(t^{-1}s^{-1}\gamma''\ell k)) \end{array} \arrow[r, "\left(\mu^{\mathcal{V}}_{\gamma'',\ell}\right)^{-1}"] \arrow[d, "\rho^{\mathbf{X}}_k"] \arrow[dr, phantom, "\text{(VII)}"] & \pi^{\mathcal{V}}_{\gamma''} \circ \pi^{\mathcal{V}}_\ell \circ \pi^{\mathcal{V}}_k(\mathbf{X}(t^{-1}s^{-1}\gamma''\ell k)) \arrow[r, "\mu^{\mathcal{V}}_{\ell,k}"] \arrow[d, "\rho^{\mathbf{X}}_k"] \arrow[dr, phantom, "\text{(VIII)}"] & \pi^{\mathcal{V}}_{\gamma''} \circ \pi^{\mathcal{V}}_{\ell k}(\mathbf{X}(t^{-1}s^{-1}\gamma''\ell k)) \arrow[d, "\rho^{\mathbf{X}}_{\ell k}"] \\
\pi^{\mathcal{V}}_s \circ \pi^{\mathcal{V}}_{\gamma'}(\mathbf{X}(t^{-1}\gamma')) \arrow[r, "\mu^{\mathcal{V}}_{s,\gamma'}"] & \begin{array}{c} \pi^{\mathcal{V}}_{s\gamma'}(\mathbf{X}(t^{-1}\gamma')) \\ = \pi^{\mathcal{V}}_{\gamma''\ell}(\mathbf{X}(t^{-1}s^{-1}\gamma''\ell)) \end{array} \arrow[r, "\left(\mu^{\mathcal{V}}_{\gamma'',\ell}\right)^{-1}"] & \pi^{\mathcal{V}}_{\gamma''} \circ \pi^{\mathcal{V}}_\ell(\mathbf{X}(t^{-1}s^{-1}\gamma''\ell)) \arrow[r, "\rho^{\mathbf{X}}_\ell"] & \pi^{\mathcal{V}}_{\gamma''}(\mathbf{X}(t^{-1}s^{-1}\gamma''))
\end{tikzcd}$$

The squares (I), (III), (IV) and (V) commute by the G-category axioms, the commutativity of (II) is immediate, (VI) and (VII) commutes by the naturality of the μ's and (VIII) commutes by the fixed point axioms. Therefore, (2) commutes and the coherence 2-cells satisfy the composition axiom.

Lastly, we need to show that the coherence 2-cells satisfy the unit axiom which says that the following diagram should commute.

$$\begin{tikzcd}
& \xi_{\mathcal{V}} \arrow[dl, "\varepsilon^{\mathcal{V}}_{\xi_{\mathcal{V}}}"'] \arrow[dr, "\xi_{\mathcal{V}}\varepsilon^{\mathrm{Ind}(\mathcal{V})}"] & \\
\pi^{\mathcal{V}}_e \circ \xi_{\mathcal{V}} \arrow[rr, "\xi_e"] & & \xi_{\mathcal{V}} \circ \pi^{\mathrm{Ind}(\mathcal{V})}_e
\end{tikzcd}$$

Therefore, we need to show that for all $\mathbf{X} \in \mathrm{Ind}^G_H\mathrm{Res}^G_H(\mathcal{V})$, the following diagram commutes.

$$
\begin{tikzcd}
 & \xi_{\mathcal{V}}(\mathbf{X}) \arrow[dl, "\varepsilon^{\mathcal{V}}_{\xi_{\mathcal{V}}(\mathbf{X})}"'] \arrow[dr, "\xi_{\mathcal{V}}\varepsilon^{\mathrm{Ind}(\mathcal{V})}_{\mathbf{X}}"] & \\
\pi_e^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X}) \arrow[rr, "(\xi_e)_{\mathbf{X}}"] & & \xi_{\mathcal{V}} \circ \pi_e^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})
\end{tikzcd}
$$

Expanding this diagram using the definitions are adding some extra internal arrows yields the following

$$
\begin{tikzcd}[column sep=small]
 & & \begin{array}{c}\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \\ = \xi_{\mathcal{V}}(\mathbf{X})\end{array} \arrow[dll, "\varepsilon"'] \arrow[dl, "\oplus\varepsilon"] \arrow[d, "\mathrm{Id}"] \arrow[drr, "\mathrm{Id}"] \arrow[dll, phantom, shift left=6ex, "\mathrm{(I)}"] \arrow[dl, phantom, shift left=5ex, "\mathrm{(II)}"] \arrow[dr, phantom, "\mathrm{(III)}"] & & \\
\begin{array}{c}\pi_e^{\mathcal{V}}\left[\bigoplus_{\gamma\in\Gamma} \pi_e^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \\ = \pi_e^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X})\end{array} \arrow[r, "\varsigma^{\pi_e^{\mathcal{V}}}"] & \bigoplus_{\gamma\in\Gamma} \pi_e^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[r, "\oplus\mu_{e,\gamma}"] & \bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[r, "\oplus\mu^{-1}_{\gamma,e}"] & \bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}} \circ \pi_e^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[r, "\oplus\rho_e^{\mathbf{X}}"] & \begin{array}{c}\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \\ = \xi_{\mathcal{V}} \circ \pi_e^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})\end{array}
\end{tikzcd}
$$

The triangle (I) commutes by Corollary 3.3 (one side of the square in the corollary has collapsed since one of the functors is the identity functor), (II) commutes by the G-category axioms and (III) commutes by Lemma 3.42. □

As one would expect, although we made a choice of coset representatives for G/H when defining the G-intertwiner $\xi_{\mathcal{V}} : \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V}) \to \mathcal{V}$, the definition only depends on the choice up to isomorphism.

Lemma 3.44. *Let $\mathcal{V}$ be a G-category and*

$$\xi_{\mathcal{V}}, \xi'_{\mathcal{V}} : \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V}) \to \mathcal{V}$$

G-intertwiners defined as in Lemma 3.43 using the sets of coset representatives Γ and Σ for G/H respectively. Then $\xi_{\mathcal{V}}$ is G-naturally isomorphic to $\xi'_{\mathcal{V}}$.

Proof. We shall define a G-natural isomorphism $\Psi_{\mathcal{V}} : \xi'_{\mathcal{V}} \to \xi_{\mathcal{V}}$. To do this, we need to do the following:

1. Define unitaries

$$(\Psi_{\mathcal{V}})_{\mathbf{X}} : \xi'_{\mathcal{V}}(\mathbf{X}) \to \xi_{\mathcal{V}}(\mathbf{X})$$

for all $\mathbf{X} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V})$.

2. Check that the $(\Psi_\mathcal{V})_\mathbf{X}$'s are natural in $\mathbf{X}$.

3. Check that the G-natural transformation axiom holds.

To define $(\Psi_\mathcal{V})_\mathbf{X}$ for $\mathbf{X} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V})$, we first note that for each $\sigma \in \Sigma$ we have

$$\sigma = \gamma m_\gamma$$

for some $\gamma \in \Gamma$ and $m_\gamma \in H$. The subscript on the m_γ is just to remind us of the dependence of m_γ on γ. Then, on the one hand

$$\begin{aligned}
\xi'_\mathcal{V}(\mathbf{X}) &= \bigoplus_{\sigma \in \Sigma} \pi^\mathcal{V}_\sigma(\mathbf{X}(\sigma)) \\
&= \bigoplus_{\gamma \in \Gamma} \pi^\mathcal{V}_{\gamma m_\gamma}(\mathbf{X}(\gamma m_\gamma))
\end{aligned}$$

and on the other hand

$$\xi_\mathcal{V}(\mathbf{X}) = \bigoplus_{\gamma \in \Gamma} \pi^\mathcal{V}_\gamma(\mathbf{X}(\gamma)).$$

Therefore, we define the unitary $(\Psi_\mathcal{V})_\mathbf{X} : \xi'_\mathcal{V}(\mathbf{X}) \to \xi_\mathcal{V}(\mathbf{X})$ as the composite

$$\begin{matrix} \bigoplus_{\gamma\in\Gamma} \pi^\mathcal{V}_{\gamma m_\gamma}(\mathbf{X}(\gamma m_\gamma)) \\ = \xi'_\mathcal{V}(\mathbf{X}) \end{matrix} \xrightarrow{\oplus \mu^{-1}_{\gamma, m_\gamma}} \bigoplus_{\gamma\in\Gamma} \pi^\mathcal{V}_\gamma \circ \pi^\mathcal{V}_{m_\gamma}(\mathbf{X}(\gamma m_\gamma)) \xrightarrow{\oplus \rho^\mathbf{X}_{m_\gamma}} \begin{matrix} \bigoplus_{\gamma\in\Gamma} \pi^\mathcal{V}_\gamma(\mathbf{X}(\gamma)) \\ = \xi_\mathcal{V}(\mathbf{X}) \end{matrix}$$

Now let us show that the $(\Psi_\mathcal{V})_\mathbf{X}$'s are natural in $\mathbf{X}$. We need to show that given a morphism $\alpha : \mathbf{X} \to \mathbf{Y}$, the following diagram commutes

$$\begin{tikzcd}
\xi'_\mathcal{V}(\mathbf{X}) \arrow[r, "\xi'_\mathcal{V}(\alpha)"] \arrow[d, "(\Psi_\mathcal{V})_\mathbf{X}"'] & \xi'_\mathcal{V}(\mathbf{Y}) \arrow[d, "(\Psi_\mathcal{V})_\mathbf{Y}"] \\
\xi_\mathcal{V}(\mathbf{X}) \arrow[r, "\xi_\mathcal{V}(\alpha)"] & \xi_\mathcal{V}(\mathbf{Y})
\end{tikzcd}$$

Expanding this diagram using the definitions and adding an extra internal arrow yields the following

$$\begin{tikzcd}
{\begin{array}{c}\bigoplus_{\sigma\in\Sigma}\pi^{\mathcal{V}}_{\sigma}(\mathbf{X}(\sigma)) \\ =\bigoplus_{\gamma\in\Gamma}\pi^{\mathcal{V}}_{\gamma m_\gamma}(\mathbf{X}(\gamma m_\gamma)) \\ =\xi'_{\mathcal{V}}(\mathbf{X})\end{array}} \arrow[r, "\oplus\pi^{\mathcal{V}}_{\gamma m_\gamma}(\alpha(\gamma m_\gamma))"] \arrow[d, "\oplus\mu^{-1}_{\gamma,m_\gamma}"'] & {\begin{array}{c}\bigoplus_{\sigma\in\Sigma}\pi^{\mathcal{V}}_{\sigma}(\mathbf{Y}(\sigma)) \\ =\bigoplus_{\gamma\in\Gamma}\pi^{\mathcal{V}}_{\gamma m_\gamma}(\mathbf{Y}(\gamma m_\gamma)) \\ =\xi'_{\mathcal{V}}(\mathbf{Y})\end{array}} \arrow[d, "\oplus\mu^{-1}_{\gamma,m_\gamma}"] \\
\bigoplus_{\gamma\in\Gamma}\pi^{\mathcal{V}}_{\gamma}\circ\pi^{\mathcal{V}}_{m_\gamma}(\mathbf{X}(\gamma m_\gamma)) \arrow[r, "\oplus\pi^{\mathcal{V}}_{\gamma}\circ\pi^{\mathcal{V}}_{m_\gamma}(\alpha(\gamma m_\gamma))"] \arrow[d, "\oplus\pi^{\mathcal{V}}_{\gamma}(\rho^{\mathbf{X}}_{m_\gamma})"'] & \bigoplus_{\gamma\in\Gamma}\pi^{\mathcal{V}}_{\gamma}\circ\pi^{\mathcal{V}}_{m_\gamma}(\mathbf{Y}(\gamma, m_\gamma)) \arrow[d, "\oplus\pi^{\mathcal{V}}_{\gamma}(\rho^{\mathbf{Y}}_{m_\gamma})"] \\
{\begin{array}{c}\bigoplus_{\gamma\in\Gamma}\pi^{\mathcal{V}}_{\gamma}(\mathbf{X}(\gamma)) \\ =\xi_{\mathcal{V}}(\mathbf{X})\end{array}} \arrow[r, "\oplus\pi^{\mathcal{V}}_{\gamma}(\alpha(\gamma))"] & {\begin{array}{c}\bigoplus_{\gamma\in\Gamma}\pi^{\mathcal{V}}_{\gamma}(\mathbf{Y}(\gamma)) \\ =\xi_{\mathcal{V}}(\mathbf{Y})\end{array}}
\end{tikzcd}$$

The top square commutes by the naturality of the $\mu^{-1}_{\gamma,m_\gamma}$'s and the bottom square commutes because α is a morphism of fixed points. Therefore, $\Psi_{\mathcal{V}}$ is a unitary natural transformation.

Lastly, let us show that $\Psi_{\mathcal{V}}$ satisfies the G-natural transformation axiom. We need to show that for all $t \in G$, the following diagram commutes

$$\begin{tikzcd}
\pi^{\mathcal{V}}_t \circ \xi'_{\mathcal{V}} \arrow[r, "\Psi_{\mathcal{V}}"] \arrow[d, "\xi'_t"'] & \pi^{\mathcal{V}}_t \circ \xi_{\mathcal{V}} \arrow[d, "\xi_t"] \\
\xi'_{\mathcal{V}} \circ \pi^{\mathrm{Ind}(\mathcal{V})}_t \arrow[r, "\Psi_{\mathcal{V}}"] & \xi_{\mathcal{V}} \circ \pi^{\mathrm{Ind}(\mathcal{V})}_t
\end{tikzcd}$$

To do this, we need to show that for all $\mathbf{X} \in \mathrm{Ind}^G_H\mathrm{Res}^G_H(\mathcal{V})$, the following diagram commutes

$$\begin{tikzcd}
\pi_t^{\mathcal{V}} \circ \xi'_{\mathcal{V}}(\mathbf{X}) \arrow[r, "(\Psi_{\mathcal{V}})_{\mathbf{X}}"] \arrow[d, "(\xi'_t)_{\mathbf{X}}"'] & \pi_t^{\mathcal{V}} \circ \xi_{\mathcal{V}}(\mathbf{X}) \arrow[d, "(\xi_t)_{\mathbf{X}}"] \\
\xi'_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \arrow[r, "(\Psi_{\mathcal{V}})_{\mathbf{X}}"] & \xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})
\end{tikzcd} \tag{$*$}$$

When we expand this diagram, we shall need to write some of the terms in different ways. In addition to writing $\sigma = \gamma m_\gamma$ as above, for all $\sigma \in \Sigma$ we have

$$t\sigma = \sigma' k_{\sigma'}$$

for some $\sigma' \in \Sigma$ and $k_{\sigma'} \in H$ and for all $\gamma \in \Gamma$ we have

$$t\gamma = \gamma' \ell_{\gamma'}$$

for some $\gamma' \in \Gamma$ and $\ell_{\gamma'} \in H$. Then, expanding $(*)$ using the definitions and adding some extra internal arrows yields the following

$$
\begin{tikzcd}
\begin{array}{c} \pi_t^{\mathcal{V}}\left[\bigoplus_{\sigma\in\Sigma}\pi_\sigma^{\mathcal{V}}(\mathbf{X}(\sigma))\right] \\ = \pi_t^{\mathcal{V}}\left[\bigoplus_{\gamma\in\Gamma}\pi_{\gamma m_\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma m_\gamma))\right] \\ = \pi_t^{\mathcal{V}}\circ\xi'_{\mathcal{V}}(\mathbf{X}) \end{array} \arrow[r, "\oplus\mu^{-1}_{\gamma,m_\gamma}"] \arrow[d, "\zeta^{\pi_t^{\mathcal{V}}}"'] \arrow[dr, phantom, "\mathrm{(I)}"] & \pi_t^{\mathcal{V}}\left[\bigoplus_{\gamma\in\Gamma}\pi_\gamma^{\mathcal{V}}\circ\pi_{m_\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma m_\gamma))\right] \arrow[r, "\oplus\rho^{\mathbf{X}}_{m_\gamma}"] \arrow[d, "\zeta^{\pi_t^{\mathcal{V}}}"] \arrow[dr, phantom, "\mathrm{(II)}"] & \begin{array}{c} \pi_t^{\mathcal{V}}\left[\bigoplus_{\gamma\in\Gamma}\pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \\ = \pi_t^{\mathcal{V}}\circ\xi_{\mathcal{V}}(\mathbf{X}) \end{array} \arrow[d, "\zeta^{\pi_t^{\mathcal{V}}}"] \\
\begin{array}{c} \bigoplus_{\sigma\in\Sigma}\pi_t^{\mathcal{V}}\circ\pi_\sigma^{\mathcal{V}}(\mathbf{X}(\sigma)) \\ = \bigoplus_{\gamma\in\Gamma}\pi_t^{\mathcal{V}}\circ\pi_{\gamma m_\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma m_\gamma)) \end{array} \arrow[r, "\oplus\mu^{-1}_{\gamma,m_\gamma}"] \arrow[d, "\begin{array}{c}\oplus\mu_{t,\sigma} \\ =\oplus\mu_{t,\gamma m_\gamma}\end{array}"'] \arrow[dr, phantom, "\mathrm{(III)}"] & \bigoplus_{\gamma\in\Gamma}\pi_t^{\mathcal{V}}\circ\pi_\gamma^{\mathcal{V}}\circ\pi_{m_\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma m_\gamma)) \arrow[r, "\oplus\rho^{\mathbf{X}}_{m_\gamma}"] \arrow[d, "\mu_{t,\gamma}"] \arrow[dr, phantom, "\mathrm{(IV)}"] & \bigoplus_{\gamma\in\Gamma}\pi_t^{\mathcal{V}}\circ\pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[d, "\mu_{t,\gamma}"] \\
\begin{array}{c} \bigoplus_{\sigma\in\Sigma}\pi_{t\sigma}^{\mathcal{V}}(\mathbf{X}(\sigma)) \\ \bigoplus_{\gamma\in\Gamma}\pi_{t\gamma m_\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma m_\gamma)) \\ = \bigoplus_{\sigma'\in\Sigma}\pi_{\sigma' k_{\sigma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\sigma' k_{\sigma'})) \end{array} \arrow[r, "\oplus\mu_{t\gamma,m_\gamma}"] \arrow[d, "\oplus\mu^{-1}_{\sigma',k_{\sigma'}}"'] & \bigoplus_{\gamma\in\Gamma}\pi_{t\gamma}^{\mathcal{V}}\circ\pi_{m_\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma m_\gamma)) \arrow[r, "\oplus\rho^{\mathbf{X}}_{m_\gamma}"] & \begin{array}{c} \bigoplus_{\gamma\in\Gamma}\pi_{t\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma)) \\ = \bigoplus_{\gamma'\in\Gamma}\pi_{\gamma'\ell_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma'\ell_{\gamma'})) \end{array} \arrow[d, "\oplus\mu^{-1}_{\gamma,\ell_{\gamma'}}"] \\
\bigoplus_{\sigma\in\Sigma}\pi_{\sigma'}^{\mathcal{V}}\circ\pi_{k_{\sigma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\sigma' k_{\sigma'})) \arrow[d, "\oplus\rho^{\mathbf{X}}_{k_{\sigma'}}"'] & \mathrm{(V)} & \bigoplus_{\gamma'\in\Gamma}\pi_{\gamma'}^{\mathcal{V}}\circ\pi_{\ell_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma'\ell_{\gamma'})) \arrow[d, "\oplus\rho^{\mathbf{X}}_{\ell_{\gamma'}}"] \\
\begin{array}{c} \bigoplus_{\sigma'\in\Sigma}\pi_{\sigma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\sigma')) \\ = \bigoplus_{\gamma'\in\Gamma}\pi_{\gamma' m_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' m_{\gamma'})) \\ = \xi'_{\mathcal{V}}\circ\pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \end{array} \arrow[r, "\oplus\mu^{-1}_{\gamma',m_{\gamma'}}"] & \bigoplus_{\gamma'\in\Gamma}\pi_{\gamma'}^{\mathcal{V}}\circ\pi_{m_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' m_{\gamma'})) \arrow[r, "\oplus\rho^{\mathbf{X}}_{m_{\gamma'}}"] & \begin{array}{c} \bigoplus_{\gamma'\in\Gamma}\pi_{\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma')) \\ = \xi_{\mathcal{V}}\circ\pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \end{array}
\end{tikzcd}
$$

Now, (I) and (II) commute by the naturality of $\zeta^{\pi_t^{\mathcal{V}}}$, (III) commutes by G-category axioms and (IV) commutes by the naturality of the $\mu_{t,\gamma}$'s. To show that (V) commutes, we consider the following diagram which examines a summand in (V). Here we need to keep track of some primed versus unprimed elements because we need to write the coset representative σ in several different ways, namely we have

$$\begin{aligned}\sigma &= \gamma m_\gamma \\ &= t^{-1}\gamma'\ell_{\gamma'}m_\gamma\end{aligned}$$

and also

$$\begin{aligned}\sigma &= t^{-1}\sigma' k_{\sigma'} \\ &= t^{-1}\gamma' m_{\gamma'}k_{\sigma'}\end{aligned}$$

so that

$$\ell_{\gamma'}m_\gamma = m_{\gamma'}k_{\sigma'}.$$

Therefore, primed and unprimed γ's may appear in the same expression and although it is not explicit in the notation, each γ depends on γ' and vice verse because the map $\gamma \mapsto \gamma'$ defined by $\gamma' H = t\gamma H$ is a bijection from Γ to itself. Using these identifications and adding some extra internal arrows to the diagram for a typical summand in (V) in the preceding diagram we obtain the following

$$
\begin{tikzcd}
\begin{array}{c} \pi^{\mathcal{V}}_{t\gamma m_\gamma}(\mathbf{X}(\gamma m_\gamma)) \\ = \pi^{\mathcal{V}}_{\gamma'\ell_{\gamma'}m_\gamma}(\mathbf{X}(t^{-1}\gamma'\ell_{\gamma'}m_\gamma)) \\ = \pi^{\mathcal{V}}_{\gamma' m_{\gamma'}k_{\sigma'}}(\mathbf{X}(t^{-1}\gamma' m_{\gamma'}k_{\sigma'})) \\ = \pi^{\mathcal{V}}_{\sigma' k_{\sigma'}}(\mathbf{X}(t^{-1}\sigma' k_{\sigma'})) \end{array} \arrow[r, "\mu^{-1}_{t\gamma, m_\gamma}", "= \mu^{-1}_{\gamma'\ell_{\gamma'}, m_\gamma}"'] \arrow[dd, "\mathrm{Id}"'] \arrow[ddr, phantom, "\text{(VI)}"] & \begin{array}{c} \pi^{\mathcal{V}}_{t\gamma} \circ \pi^{\mathcal{V}}_{m_\gamma}(\mathbf{X}(\gamma m_\gamma)) \\ = \pi^{\mathcal{V}}_{\gamma'\ell_{\gamma'}} \circ \pi^{\mathcal{V}}_{m_\gamma}(\mathbf{X}(t^{-1}\gamma'\ell_{\gamma'}m_\gamma)) \end{array} \arrow[r, "\rho^{\mathbf{X}}_{m_\gamma}"] \arrow[d, "\mu^{-1}_{\gamma', \ell_{\gamma'}}"] \arrow[dr, phantom, "\text{(VII)}"] & \pi^{\mathcal{V}}_{\gamma'\ell_{\gamma'}}(\mathbf{X}(t^{-1}\gamma'\ell_{\gamma'})) \arrow[d, "\mu^{-1}_{\gamma', \ell_{\gamma'}}"] \\
 & \pi^{\mathcal{V}}_{\gamma'} \circ \pi^{\mathcal{V}}_{\ell_{\gamma'}} \circ \pi^{\mathcal{V}}_{m_\gamma}(\mathbf{X}(t^{-1}\gamma'\ell_{\gamma'}m_\gamma)) \arrow[r, "\rho^{\mathbf{X}}_{m_\gamma}"] \arrow[d, "\mu_{\ell_{\gamma'}, m_\gamma}"] \arrow[dr, phantom, "\text{(VIII)}"] & \pi^{\mathcal{V}}_{\gamma'} \circ \pi^{\mathcal{V}}_{\ell_{\gamma'}}(\mathbf{X}(t^{-1}\gamma'\ell_{\gamma'})) \arrow[d, "\rho^{\mathbf{X}}_{\ell_{\gamma'}}"] \\
\begin{array}{c} \pi^{\mathcal{V}}_{\gamma'\ell_{\gamma'}m_\gamma}(\mathbf{X}(t^{-1}\gamma'\ell_{\gamma'}m_\gamma)) \\ = \pi^{\mathcal{V}}_{\gamma' m_{\gamma'}k_{\sigma'}}(\mathbf{X}(t^{-1}\gamma' m_{\gamma'}k_{\sigma'})) \\ = \pi^{\mathcal{V}}_{\sigma' k_{\sigma'}}(\mathbf{X}(t^{-1}\sigma' k_{\sigma'})) \end{array} \arrow[r, "\mu^{-1}_{\gamma', \ell_{\gamma'}m_\gamma}", "= \mu^{-1}_{\gamma', m_{\gamma'}k_{\sigma'}}"'] \arrow[d, "\mu^{-1}_{\sigma', k_{\sigma'}}"', "= \mu^{-1}_{\gamma' m_{\gamma'}, k_{\sigma'}}"] \arrow[dr, phantom, "\text{(IX)}"] & \begin{array}{c} \pi^{\mathcal{V}}_{\gamma'} \circ \pi^{\mathcal{V}}_{\ell_{\gamma'}m_\gamma}(\mathbf{X}(t^{-1}\gamma'\ell_{\gamma'}m_\gamma)) \\ = \pi^{\mathcal{V}}_{\gamma'} \circ \pi^{\mathcal{V}}_{m_{\gamma'}k_{\sigma'}}(\mathbf{X}(t^{-1}\gamma' m_{\gamma'}k_{\sigma'})) \end{array} \arrow[r, "\rho^{\mathbf{X}}_{\ell_{\gamma'}m_\gamma}", "= \rho^{\mathbf{X}}_{m_{\gamma'}k_{\sigma'}}"'] \arrow[d, "\mu^{-1}_{m_{\gamma'}, k_{\sigma'}}"] \arrow[ddr, phantom, "\text{(XI)}"] & \pi^{\mathcal{V}}_{\gamma'}(\mathbf{X}(t^{-1}\gamma')) \arrow[dd, "\mathrm{Id}"] \\
\begin{array}{c} \pi^{\mathcal{V}}_{\sigma'} \circ \pi^{\mathcal{V}}_{k_{\sigma'}}(\mathbf{X}(t^{-1}\sigma' k_{\sigma'})) \\ = \pi^{\mathcal{V}}_{\gamma' m_{\gamma'}} \circ \pi^{\mathcal{V}}_{k_{\sigma'}}(\mathbf{X}(t^{-1}\gamma' m_{\gamma'}k_{\sigma'})) \end{array} \arrow[r, "\mu^{-1}_{\gamma', m_{\gamma'}}"] \arrow[d, "\rho^{\mathbf{X}}_{k_{\sigma'}}"'] \arrow[dr, phantom, "\text{(X)}"] & \pi^{\mathcal{V}}_{\gamma'} \circ \pi^{\mathcal{V}}_{m_{\gamma'}} \circ \pi^{\mathcal{V}}_{k_{\sigma'}}(\mathbf{X}(t^{-1}\gamma' m_{\gamma'}k_{\sigma'})) \arrow[d, "\rho^{\mathbf{X}}_{k_{\sigma'}}"] & \\
\pi^{\mathcal{V}}_{\gamma' m_{\gamma'}}(\mathbf{X}(t^{-1}\gamma' m_{\gamma'})) \arrow[r, "\mu^{-1}_{\gamma', m_{\gamma'}}"] & \pi^{\mathcal{V}}_{\gamma'} \circ \pi^{\mathcal{V}}_{m_{\gamma'}}(\mathbf{X}(t^{-1}\gamma' m_{\gamma'})) \arrow[r, "\rho^{\mathbf{X}}_{m_{\gamma'}}"] & \pi^{\mathcal{V}}_{\gamma'}(\mathbf{X}(t^{-1}\gamma'))
\end{tikzcd}
$$

Now, (VI) and (IX) commute by the G-category axioms, (VII) and (III) commute by the naturality of the μ's and (VIII) and (IX) commute by the fixed point axioms. Therefore, (V) in the previous diagram commutes and hence $(*)$ commutes. $\square$

Lemma 3.45. *The G-intertwiners*

$$\xi_{\mathcal{V}} : \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V}) \to \mathcal{V}, \quad \mathcal{V} \in \mathsf{REP}(G)$$

defined in Lemma 3.43 are the 1-cells of a pseudonatural transformation

$$\xi : \mathrm{Ind}_H^G \mathrm{Res}_H^G \to 1_{\mathsf{REP}(G)}.$$

Proof. Similarly to the unit, we need to do the following:

1. Define G-natural isomorphisms
$$\xi_\Theta^{\mathcal{V},\mathcal{W}} : \Theta \circ \xi_\mathcal{V} \to \xi_\mathcal{W} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta)$$
for all $\mathcal{V}, \mathcal{W} \in \mathsf{REP}(G)$ and $\Theta \in \mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$.

2. Check that the $\xi_\Theta^{\mathcal{V},\mathcal{W}}$'s are natural in Θ.

3. Check that the pseudonatural transformation axioms hold.

First, we need to define the components of $\xi_\Theta^{\mathcal{V},\mathcal{W}}$. Given $\mathbf{X} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V})$, on the one hand
$$\Theta \circ \xi_\mathcal{V}(\mathbf{X}) = \Theta \left[\bigoplus_{\gamma \in \Gamma} \pi_\gamma^\mathcal{V}(\mathbf{X}(\gamma)) \right]$$
and on the other hand
$$\xi_\mathcal{W} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta)(\mathbf{X}) = \bigoplus_{\gamma \in \Gamma} \pi_\gamma^\mathcal{W} \circ \Theta(\mathbf{X}(\gamma)).$$
Therefore, we define $\left(\xi_\Theta^{\mathcal{V},\mathcal{W}}\right)_\mathbf{X}$ as the composite
$$\Theta \left[\bigoplus\nolimits_{\gamma \in \Gamma} \pi_\gamma^\mathcal{V}(\mathbf{X}(\gamma)) \right] \xrightarrow{\zeta^\Theta} \bigoplus\nolimits_{\gamma \in \Gamma} \Theta \circ \pi_\gamma^\mathcal{V}(\mathbf{X}(\gamma)) \xrightarrow{\oplus(\Theta_\gamma)^{-1}} \bigoplus\nolimits_{\gamma \in \Gamma} \pi_\gamma^\mathcal{W} \circ \Theta(\mathbf{X}(\gamma)).$$
Naturality follows from the naturality of ζ^Θ and the $(\Theta_\gamma)^{-1}$'s.

To show G-naturality, we need to show that for all $t \in G$, the following diagram commutes
$$\begin{tikzcd}
\pi_t^\mathcal{W} \circ \Theta \circ \xi_\mathcal{V} \arrow[r, "\xi_\Theta^{\mathcal{V},\mathcal{W}}"] \arrow[d, "\xi_t \circ \Theta_t"'] & \pi_t^\mathcal{W} \circ \xi_\mathcal{W} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta) \arrow[d, "\mathrm{Ind}_H^G(\Theta)_t \circ \xi_t"] \\
\Theta \circ \xi_\mathcal{V} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})} \arrow[r, "\xi_\Theta^{\mathcal{V},\mathcal{W}}"] & \xi_\mathcal{W} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta) \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}
\end{tikzcd}$$
To do this, we need to show that for all $\mathbf{X} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V})$, the following diagram commutes
$$\begin{tikzcd}
\pi_t^\mathcal{W} \circ \Theta \circ \xi_\mathcal{V}(\mathbf{X}) \arrow[r, "(\xi_\Theta^{\mathcal{V},\mathcal{W}})_\mathbf{X}"] \arrow[d, "(\xi_t \circ \Theta_t)_\mathbf{X}"'] & \pi_t^\mathcal{W} \circ \xi_\mathcal{W} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta)(\mathbf{X}) \arrow[d, "\left(\mathrm{Ind}_H^G(\Theta)_t \circ \xi_t\right)_\mathbf{X}"] \\
\Theta \circ \xi_\mathcal{V} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \arrow[r, "(\xi_\Theta^{\mathcal{V},\mathcal{W}})_\mathbf{X}"] & \xi_\mathcal{W} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta) \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X})
\end{tikzcd}$$

For this, we shall need the fact that for each $\gamma \in \Gamma$ we have

$$t\gamma = \gamma' k_{\gamma'}$$

for some $\gamma' \in \Gamma$ and $k_{\gamma'} \in H$. Then, expanding the diagram using the definitions and adding some extra internal arrows yields the following

$$\begin{tikzcd}[column sep=small]
\begin{array}{c} \pi_t^{\mathcal{W}} \circ \Theta \left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \\ = \pi_t^{\mathcal{W}} \circ \Theta \circ \xi_{\mathcal{V}}(\mathbf{X}) \end{array} \arrow[r, "\zeta^{\Theta}"] \arrow[d, "\Theta_t"'] \arrow[dr, "\zeta^{\pi_t^{\mathcal{W}} \circ \Theta}"'] \arrow[dr, phantom, bend left, "\mathrm{(I)}"] &
\pi_t^{\mathcal{W}} \left[\bigoplus_{\gamma\in\Gamma} \Theta \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[r, "\oplus(\Theta_\gamma)^{-1}"] \arrow[d, "\zeta^{\pi_t^{\mathcal{W}}}"] \arrow[dr, phantom, "\mathrm{(IV)}"] &
\begin{array}{c} \pi_t^{\mathcal{W}} \left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{W}} \circ \Theta(\mathbf{X}(\gamma))\right] \\ = \pi_t^{\mathcal{W}} \circ \xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta)(\mathbf{X}) \end{array} \arrow[d, "\zeta^{\pi_t^{\mathcal{W}}}"] \\
\Theta \circ \pi_t^{\mathcal{V}} \left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[r, phantom, "\mathrm{(II)}"] \arrow[d, "\zeta^{\pi_t^{\mathcal{V}}}"'] \arrow[dr, "\zeta^{\Theta \circ \pi_t^{\mathcal{V}}}"] \arrow[dr, phantom, bend right, "\mathrm{(III)}"] &
\bigoplus_{\gamma\in\Gamma} \pi_t^{\mathcal{W}} \circ \Theta \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[r, "\oplus(\Theta_\gamma)^{-1}"] \arrow[d, "\oplus\Theta_t"] \arrow[dr, phantom, "\mathrm{(VI)}"] &
\bigoplus_{\gamma\in\Gamma} \pi_t^{\mathcal{W}} \circ \pi_\gamma^{\mathcal{W}} \circ \Theta(\mathbf{X}(\gamma)) \arrow[d, "\oplus\mu_{t,\gamma}"] \\
\Theta \left[\bigoplus_{\gamma\in\Gamma} \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[r, "\zeta^{\Theta}"] \arrow[d, "\oplus\mu_{t,\gamma}"'] \arrow[dr, phantom, "\mathrm{(V)}"] &
\bigoplus_{\gamma\in\Gamma} \Theta \circ \pi_t^{\mathcal{V}} \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma)) \arrow[d, "\oplus\mu_{t,\gamma}"] &
\begin{array}{c} \bigoplus_{\gamma\in\Gamma} \pi_{t\gamma}^{\mathcal{W}} \circ \Theta(\mathbf{X}(\gamma)) \\ = \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma' k_{\gamma'}}^{\mathcal{W}} \circ \Theta(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \end{array} \arrow[d, "\oplus\mu^{-1}_{\gamma',k_{\gamma'}}"] \\
\begin{array}{c} \Theta \left[\bigoplus_{\gamma\in\Gamma} \pi_{t\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \\ = \Theta \left[\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma' k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'}))\right] \end{array} \arrow[r, "\zeta^{\Theta}"] \arrow[d, "\oplus\mu^{-1}_{\gamma',k_{\gamma'}}"'] \arrow[dr, phantom, "\mathrm{(VII)}"] &
\begin{array}{c} \bigoplus_{\gamma\in\Gamma} \Theta \circ \pi_{t\gamma}^{\mathcal{V}}(\mathbf{X}(\gamma)) \\ = \bigoplus_{\gamma'\in\Gamma} \Theta \circ \pi_{\gamma' k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \end{array} \arrow[ur, "\oplus(\Theta_{t\gamma})^{-1} = \oplus\left(\Theta_{\gamma' k_{\gamma'}}\right)^{-1}"] \arrow[d, "\oplus\mu^{-1}_{\gamma',k_{\gamma'}}"] \arrow[dr, phantom, "\mathrm{(VIII)}"] &
\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}} \circ \Theta(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \arrow[d, "\oplus\Theta_{k_{\gamma'}}"] \\
\Theta \left[\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'}))\right] \arrow[r, "\zeta^{\Theta}"] \arrow[d, "\oplus\rho^{\mathbf{X}}_{k_{\gamma'}}"'] \arrow[dr, phantom, "\mathrm{(IX)}"] &
\bigoplus_{\gamma'\in\Gamma} \Theta \circ \pi_{\gamma'}^{\mathcal{V}} \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \arrow[r, "\oplus(\Theta_{\gamma'})^{-1}"] \arrow[d, "\oplus\rho^{\mathbf{X}}_{k_{\gamma'}}"] \arrow[dr, phantom, "\mathrm{(X)}"] &
\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{W}} \circ \Theta \circ \pi_{k_{\gamma'}}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma' k_{\gamma'})) \arrow[d, "\oplus\rho^{\mathbf{X}}_{k_{\gamma'}}"] \\
\begin{array}{c} \Theta \left[\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma'))\right] \\ = \Theta \circ \xi_{\mathcal{V}} \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \end{array} \arrow[r, "\zeta^{\Theta}"] &
\bigoplus_{\gamma'\in\Gamma} \Theta \circ \pi_{\gamma'}^{\mathcal{V}}(\mathbf{X}(t^{-1}\gamma')) \arrow[r, "\oplus(\Theta_{\gamma'})^{-1}"] &
\begin{array}{c} \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{V}} \circ \Theta(\mathbf{X}(t^{-1}\gamma')) \\ = \xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta) \circ \pi_t^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \end{array}
\end{tikzcd}$$

Then (I) and (III) commute by Lemma 1.23, (II) commutes by Corollary 3.3, (IV), (V), (VII) and (IX) commute by the naturality of the ζ's, (VI) and (VIII) commute by the G-intertwiner axioms and (X) commutes by the naturality of the $(\Theta_{\gamma'})^{-1}$'s. Therefore, $\xi_\Theta^{\mathcal{V},\mathcal{W}}$ satisfies the G-natural transformation axiom.

To show naturality in Θ, we need to show that given $\Theta, \Phi \in \mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$ and a G-natural transformation $\kappa : \Theta \to \Phi$, the following diagram commutes

$$\begin{tikzcd}
\Theta \circ \xi_{\mathcal{V}} \arrow[r, "\kappa"] \arrow[d, "\xi_\Theta^{\mathcal{V},\mathcal{W}}"'] & \Phi \circ \xi_{\mathcal{V}} \arrow[d, "\xi_\Phi^{\mathcal{V},\mathcal{W}}"] \\
\xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta) \arrow[r, "\mathrm{Ind}_H^G\mathrm{Res}_H^G(\kappa)"] & \xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Phi)
\end{tikzcd}$$

To do this, we need to show that for all $\mathbf{X} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V})$, the following diagram commutes

$$\begin{tikzcd}
\Theta \circ \xi_{\mathcal{V}}(\mathbf{X}) \arrow[r, "\kappa_{\xi_{\mathcal{V}}(\mathbf{X})}"] \arrow[d, "(\xi_\Theta^{\mathcal{V},\mathcal{W}})_{\mathbf{X}}"'] & \Phi \circ \xi_{\mathcal{V}}(\mathbf{X}) \arrow[d, "(\xi_\Phi^{\mathcal{V},\mathcal{W}})_{\mathbf{X}}"] \\
\xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta)(\mathbf{X}) \arrow[r, "\mathrm{Ind}_H^G\mathrm{Res}_H^G(\kappa)_{\mathbf{X}}"] & \xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Phi)(\mathbf{X})
\end{tikzcd}$$

Expanding this diagram using the definitions and adding an extra internal arrow yields

$$\begin{tikzcd}
\Theta\left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[r, "\kappa"] \arrow[d, "\zeta^\Theta"'] & \Phi\left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[d, "\zeta^\Phi"] \\
\bigoplus_{\gamma\in\Gamma} \Theta \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma) \arrow[r, "\oplus\kappa"] \arrow[d, "\oplus(\Theta_\gamma)^{-1}"'] & \bigoplus_{\gamma\in\Gamma} \Phi \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma) \arrow[d, "\oplus(\Phi_\gamma)^{-1}"] \\
\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}} \circ \Theta(\mathbf{X}(\gamma) \arrow[r, "\oplus\kappa"] & \bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}} \circ \Phi(\mathbf{X}(\gamma)
\end{tikzcd}$$

The top square commutes by Corollary and the bottom square commutes because κ is a G-natural transformation. Therefore, the $\xi_{\Theta}^{\mathcal{V},\mathcal{W}}$'s are natural in Θ.

Lastly, we need to check the pseudonatural transformation axioms. The composition axiom says that given G-intertwiners $\Theta : \mathcal{V} \to \mathcal{W}$ and $\Phi : \mathcal{W} \to \mathcal{X}$, the following diagram should commute

$$\begin{tikzcd}
\Phi \circ \Theta \circ \xi_{\mathcal{V}} \arrow[r, "\mathrm{Id}"] \arrow[d, "\xi_{\Theta}^{\mathcal{V},\mathcal{W}}"'] & \Phi \circ \Theta \circ \xi_{\mathcal{V}} \arrow[dd, "\xi_{\Phi\circ\Theta}^{\mathcal{V},\mathcal{X}}"] \\
\Phi \circ \xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G\mathrm{Res}_H^G(\Theta) \arrow[d, "\xi_{\Phi}^{\mathcal{W},\mathcal{X}}"'] & \\
\xi_{\mathcal{X}} \circ \mathrm{Ind}_H^G\mathrm{Res}_H^G(\Phi) \circ \mathrm{Ind}_H^G\mathrm{Res}_H^G(\Theta) \arrow[r, "\mathrm{Id}"] & \xi_{\mathcal{X}} \circ \mathrm{Ind}_H^G\mathrm{Res}_H^G(\Phi \circ \Theta)
\end{tikzcd}$$

To check this, we need to show that for all $\mathbf{X} \in \mathrm{Ind}_H^G\mathrm{Res}_H^G(\mathcal{V})$, the following diagram commutes

$$\begin{tikzcd}
\Phi \circ \Theta \circ \xi_{\mathcal{V}}(\mathbf{X}) \arrow[r, "\mathrm{Id}"] \arrow[d, "(\xi_{\Theta}^{\mathcal{V},\mathcal{W}})_{\mathbf{X}}"'] & \Phi \circ \Theta \circ \xi_{\mathcal{V}}(\mathbf{X}) \arrow[dd, "(\xi_{\Phi\circ\Theta}^{\mathcal{V},\mathcal{X}})_{\mathbf{X}}"] \\
\Phi \circ \xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G\mathrm{Res}_H^G(\Theta)(\mathbf{X}) \arrow[d, "(\xi_{\Phi}^{\mathcal{W},\mathcal{X}})_{\mathbf{X}}"'] & \\
\xi_{\mathcal{X}} \circ \mathrm{Ind}_H^G\mathrm{Res}_H^G(\Phi) \circ \mathrm{Ind}_H^G\mathrm{Res}_H^G(\Theta)(\mathbf{X}) \arrow[r, "\mathrm{Id}"] & \xi_{\mathcal{X}} \circ \mathrm{Ind}_H^G\mathrm{Res}_H^G(\Phi \circ \Theta)(\mathbf{X})
\end{tikzcd}$$

Expanding this diagram using the definitions and adding some extra internal arrows yields the following

$$
\begin{tikzcd}
\begin{array}{c} \Phi \circ \Theta \left[\bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \\ = \Phi \circ \Theta \circ \xi_{\mathcal{V}}(\mathbf{X}) \end{array} \arrow[r, "\mathrm{Id}"] \arrow[d, "\zeta^{\Theta}"'] \arrow[dr, phantom, "\text{(I)}"] & \begin{array}{c} \Phi \circ \Theta \left[\bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \\ = \Phi \circ \Theta \circ \xi_{\mathcal{V}}(\mathbf{X}) \end{array} \arrow[d, "\zeta^{\Phi \circ \Theta}"] \\
\Phi \left[\bigoplus_{\gamma \in \Gamma} \Theta \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[r, "\zeta^{\Phi}"] \arrow[d, "\oplus(\Theta_\gamma)^{-1}"'] & \left[\bigoplus_{\gamma \in \Gamma} \Phi \circ \Theta \circ \pi_\gamma^{\mathcal{V}}(\mathbf{X}(\gamma))\right] \arrow[dd, "\oplus(\Theta_\gamma)^{-1}"] \\
\begin{array}{c} \Phi \left[\bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{W}} \circ \Theta(\mathbf{X}(\gamma))\right] \\ = \Phi \circ \xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta)(\mathbf{X}) \end{array} \arrow[d, "\zeta^{\Phi}"'] \arrow[r, phantom, "\text{(II)}"] & \\
\bigoplus_{\gamma \in \Gamma} \Phi \circ \pi_\gamma^{\mathcal{W}} \circ \Theta(\mathbf{X}(\gamma)) \arrow[r, "\mathrm{Id}"] \arrow[d, "\oplus(\Phi_\gamma)^{-1}"'] \arrow[dr, phantom, "\text{(III)}"] & \bigoplus_{\gamma \in \Gamma} \Phi \circ \pi_\gamma^{\mathcal{W}} \circ \Theta(\mathbf{X}(\gamma)) \arrow[d, "\oplus(\Phi_\gamma)^{-1}"] \\
\begin{array}{c} \bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{X}} \circ \Phi \circ \Theta(\mathbf{X}(\gamma)) \\ = \xi_{\mathcal{X}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Phi) \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta)(\mathbf{X}) \end{array} \arrow[r, "\mathrm{Id}"] & \begin{array}{c} \bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{X}} \circ \Phi \circ \Theta(\mathbf{X}(\gamma)) \\ = \xi_{\mathcal{X}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Phi \circ \Theta)(\mathbf{X}) \end{array}
\end{tikzcd}
$$

Now, (I) commutes by Lemma 1.23, (II) commutes by the naturality of ζ^{Φ} and the commutativity of (III) is immediate. Therefore, the composition axiom is satisfied.

Finally, the unit axiom (shown below) holds since all the arrows are identities by definition.

$$
\begin{tikzcd}
& \xi_{\mathcal{V}} \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\
1_{\mathcal{V}} \circ \xi_{\mathcal{V}} \arrow[rr, "\xi^{\mathcal{V},\mathcal{V}}_{1_{\mathcal{V}}}", "= \mathrm{Id}"'] & & \xi_{\mathcal{V}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G (1_{\mathcal{V}})
\end{tikzcd}
$$

□

Following on from our earlier observation in Lemma 3.44 that the choice of coset representatives for G/H only mattered up to isomorphism when defining the $\xi_{\mathcal{V}}$'s, we note that using a different set of coset representatives yields an isomorphic counit.

Lemma 3.46. *Let $\mathcal{V}$ be a G-category and*

$$\xi, \xi' : \mathrm{Ind}_H^G \mathrm{Res}_H^G \to 1_{\mathsf{REP}(G)}$$

pseudonatural transformations defined as in Lemma 3.45 using the sets of coset representatives Γ and Σ for G/H respectively. The $\Psi_{\mathcal{V}}$ defined in Lemma 3.44 are the 2-cells of an invertible modification $\Psi : \xi' \to \xi$.

Proof. We just need to check the modification axiom which says that given a G-intertwiner $\Theta : \mathcal{V} \to \mathcal{W}$, the following diagram should commute

$$
\begin{tikzcd}
\Theta \circ \xi'_{\mathcal{V}} \arrow[r, "\Psi_{\mathcal{V}}"] \arrow[d, "\xi'_t"'] & \Theta \circ \xi_{\mathcal{V}} \arrow[d, "\xi_t"] \\
\xi'_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta) \arrow[r, "\Psi_{\mathcal{W}}"] & \xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta)
\end{tikzcd}
$$

To show this, we need to check that for all $\mathbf{X} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V})$, the following diagram commutes

$$
\begin{tikzcd}
\Theta \circ \xi'_{\mathcal{V}}(\mathbf{X}) \arrow[r, "(\Psi_{\mathcal{V}})_{\mathbf{X}}"] \arrow[d, "(\xi'_t)_{\mathbf{X}}"'] & \Theta \circ \xi_{\mathcal{V}}(\mathbf{X}) \arrow[d, "(\xi_t)_{\mathbf{X}}"] \\
\xi'_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta)(\mathbf{X}) \arrow[r, "(\Psi_{\mathcal{W}})_{\mathbf{X}}"] & \xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G \mathrm{Res}_H^G(\Theta)(\mathbf{X})
\end{tikzcd}
$$

For each $\sigma \in \Sigma$, we have

$$\sigma = \gamma k_\gamma$$

for some $\gamma \in \Gamma$ and $k_\gamma \in H$. Then expanding the above diagram using the definitions and adding some extra internal arrows yields the following

$$\begin{tikzcd}
\begin{array}{c} \Theta\left[\bigoplus_{\sigma\in\Sigma} \pi^{\mathcal{V}}_{\sigma}(\mathbf{X}(\sigma))\right] \\ = \Theta\left[\bigoplus_{\gamma\in\Gamma} \pi^{\mathcal{V}}_{\gamma k_\gamma}(\mathbf{X}(\gamma k_\gamma))\right] \\ = \Theta \circ \xi'_{\mathcal{V}}(\mathbf{X}) \end{array} \arrow[rr, "\oplus\mu^{-1}_{\gamma,k_\gamma}"] \arrow[d, "\zeta^{\Theta}"'] \arrow[drr, phantom, "\text{(I)}"] & & \Theta\left[\bigoplus_{\gamma\in\Gamma} \pi^{\mathcal{V}}_{\gamma}\circ\pi^{\mathcal{V}}_{k_\gamma}(\mathbf{X}(\gamma k_\gamma))\right] \arrow[r, "\oplus\rho^{\mathbf{X}}_{k_\gamma}"] \arrow[d, "\zeta^{\Theta}"] \arrow[dr, phantom, "\text{(II)}"] & \begin{array}{c} \Theta\left[\bigoplus_{\gamma\in\Gamma} \pi^{\mathcal{V}}_{\gamma}(\mathbf{X}(\gamma))\right] \\ = \Theta \circ \xi_{\mathcal{V}}(\mathbf{X}) \end{array} \arrow[d, "\zeta^{\Theta}"] \\
\begin{array}{c} \bigoplus_{\sigma\in\Sigma} \Theta\circ\pi^{\mathcal{V}}_{\sigma}(\mathbf{X}(\sigma)) \\ = \bigoplus_{\gamma\in\Gamma} \Theta\circ\pi^{\mathcal{V}}_{\gamma k_\gamma}(\mathbf{X}(\gamma k_\gamma)) \end{array} \arrow[rr, "\oplus\mu^{-1}_{\gamma,k_\gamma}"] \arrow[d, "\oplus(\Theta_\sigma)^{-1}"', "=\oplus(\Theta_{\gamma k_\gamma})^{-1}"] \arrow[drr, phantom, "\text{(III)}"] & & \bigoplus_{\gamma\in\Gamma} \Theta\circ\pi^{\mathcal{V}}_{\gamma}\circ\pi^{\mathcal{V}}_{k_\gamma}(\mathbf{X}(\gamma k_\gamma)) \arrow[r, "\oplus\rho^{\mathbf{X}}_{k_\gamma}"] \arrow[d, "\oplus(\Theta_\gamma)^{-1}"] \arrow[dr, phantom, "\text{(IV)}"] & \bigoplus_{\gamma\in\Gamma} \Theta\circ\pi^{\mathcal{V}}_{\gamma}(\mathbf{X}(\gamma)) \arrow[d, "\oplus(\Theta_\gamma)^{-1}"] \\
\begin{array}{c} \bigoplus_{\sigma\in\Sigma} \pi^{\mathcal{W}}_{\sigma}\circ\Theta(\mathbf{X}(\sigma)) \\ \bigoplus_{\gamma\in\Gamma} \pi^{\mathcal{W}}_{\gamma k_\gamma}\circ\Theta(\mathbf{X}(\gamma k_\gamma)) \\ = \xi'_{\mathcal{W}}\circ \mathrm{Ind}_H^G\mathrm{Res}_H^G(\Theta)(\mathbf{X}) \end{array} \arrow[r, "\oplus\mu^{-1}_{\gamma,k_\gamma}"] & \bigoplus_{\gamma\in\Gamma} \pi^{\mathcal{W}}_{\gamma}\circ\pi^{\mathcal{W}}_{k_\gamma}\circ\Theta(\mathbf{X}(\gamma k_\gamma)) \arrow[r, "\oplus\Theta_{k_\gamma}"] & \bigoplus_{\gamma\in\Gamma} \pi^{\mathcal{W}}_{\gamma}\circ\Theta\circ\pi^{\mathcal{V}}_{k_\gamma}(\mathbf{X}(\gamma k_\gamma)) \arrow[r, "\oplus\rho^{\mathbf{X}}_{k_\gamma}"] & \begin{array}{c} \bigoplus_{\gamma\in\Gamma} \pi^{\mathcal{W}}_{\gamma}\circ\Theta(\mathbf{X}(\gamma)) \\ = \xi_{\mathcal{W}}\circ \mathrm{Ind}_H^G\mathrm{Res}_H^G(\Theta)(\mathbf{X}) \end{array}
\end{tikzcd}$$

Now, (I) and (II) commute by the naturality of ζ^{Θ}, (III) commutes by the G-intertwiner axioms and (IV) commutes by the naturality of the $(\Theta_\gamma)^{-1}$'s. Therefore, the modification axiom is satisfied. $\square$

Now that we have defined the unit and counit, all we need to do to show that induction and restriction are biadjoint processes is to show that the biadjunction triangle identities hold up to invertible modifications.

Lemma 3.47. *There is an invertible modification*

$$\Lambda : 1_{\mathrm{Res}_H^G} \to \mathrm{Res}_H^G(\xi) \circ \eta_{\mathrm{Res}_H^G}$$

whose component at $\mathcal{V} \in \mathsf{REP}(G)$ is defined as follows: Given $X \in \mathcal{V}$, $(\Lambda_{\mathcal{V}})_X$ is the composite

$$X \xrightarrow{\varepsilon^{\mathcal{V}}_X} \pi^{\mathcal{V}}_e(X) \xrightarrow{(\mu^{\mathcal{V}}_{e,e})_X} \pi^{\mathcal{V}}_e \circ \pi^{\mathcal{V}}_e(X).$$

Proof. For each $\mathcal{V} \in \mathsf{REP}(G)$, $\Lambda_{\mathcal{V}}$ should be an H-natural transformation from $1_{\mathrm{Res}_H^G(\mathcal{V})}$ to $\mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})}$. Given $X \in \mathcal{V}$, we have

$$\begin{aligned}\mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})}(X) &= \mathrm{Res}_H^G(\xi_{\mathcal{V}})(\delta_H^X) \\ &= \pi_e^{\mathcal{V}} \circ \pi_e^{\mathcal{V}}(X).\end{aligned}$$

(Here we are using the convention of taking the identity element e as the coset representative for H. If we had taken a different representative $h \in H$, the expression would be $\pi_h^{\mathcal{V}} \circ \pi_{h^{-1}}^{\mathcal{V}}(X)$ but the proof would remain the same.) Therefore, we define $(\Lambda_{\mathcal{V}})_X$ as in the statement of the lemma.

Let us first check this defines an H-natural transformation. Naturality follows from the naturality of $\varepsilon^{\mathcal{V}}$ and $\mu_{e,e}^{\mathcal{V}}$. To show H-naturality, we need to show that for all $k \in H$, the following diagram commutes.

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ 1_{\mathrm{Res}_H^G(\mathcal{V})} \arrow[r, "\Lambda_{\mathcal{V}}"] \arrow[d, "\mathrm{Id}"'] & \pi_k^{\mathcal{V}} \circ \mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})} \arrow[d, "\eta_k \circ \xi_k"] \\
1_{\mathrm{Res}_H^G(\mathcal{V})} \circ \pi_k^{\mathcal{V}} \arrow[r, "\Lambda_{\mathcal{V}}"] & \mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})} \circ \pi_k^{\mathcal{V}}
\end{tikzcd}$$

To do this, we need to show that for all $X \in \mathcal{V}$, the following diagram commutes.

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ 1_{\mathrm{Res}_H^G(\mathcal{V})}(X) \arrow[r, "(\Lambda_{\mathcal{V}})_X"] \arrow[d, "\mathrm{Id}"'] & \pi_k^{\mathcal{V}} \circ \mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})}(X) \arrow[d, "(\eta_k \circ \xi_k)_X"] \\
1_{\mathrm{Res}_H^G(\mathcal{V})} \circ \pi_k^{\mathcal{V}}(X) \arrow[r, "(\Lambda_{\mathcal{V}})_X"] & \mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})} \circ \pi_k^{\mathcal{V}}(X)
\end{tikzcd}$$

Expanding this diagram using the definitions and adding some extra internal arrows yields the following

$$
\begin{tikzcd}
{\begin{array}{c}\pi_k^{\mathcal{V}}(X) \\ = \pi_k^{\mathcal{V}} \circ 1_{\mathrm{Res}_H^G(\mathcal{V})}(X)\end{array}} \arrow[r, "\varepsilon"] \arrow[dr, "\mathrm{Id}"'] \arrow[dddd, "\mathrm{Id}"'] \arrow[r, phantom, shift right=6, "\text{(I)}"] & \pi_k^{\mathcal{V}} \circ \pi_e^{\mathcal{V}}(X) \arrow[r, "\mu_{e,e}^{-1}"] \arrow[d, "\mu_{k,e}"] \arrow[dr, phantom, "\text{(II)}"] & {\begin{array}{c}\pi_k^{\mathcal{V}} \circ \pi_e^{\mathcal{V}} \circ \pi_e^{\mathcal{V}}(X) \\ = \pi_k^{\mathcal{V}} \circ \mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})}(X)\end{array}} \arrow[d, "\mu_{k,e}"] \\
& \pi_k^{\mathcal{V}}(X) \arrow[r, "\mu_{k,e}^{-1}"] \arrow[d, "\mu_{e,k}^{-1}"] \arrow[dr, phantom, "\text{(IV)}"] & \pi_k^{\mathcal{V}} \circ \pi_e^{\mathcal{V}}(X) \arrow[d, "\mu_{e,k}^{-1}"] \\
\arrow[r, phantom, "\text{(III)}"] & \pi_e^{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X) \arrow[r, "\mu_{k,e}^{-1}"] \arrow[d, "\mu_{e,k}"] \arrow[dr, phantom, "\text{(V)}"] & \pi_e^{\mathcal{V}} \circ \pi_k^{\mathcal{V}} \circ \pi_e^{\mathcal{V}}(X) \arrow[d, "{\begin{array}{c}\mu_{k,e} \\ = \rho_k^{\delta_H^X}\end{array}}"] \\
& \pi_k^{\mathcal{V}}(X) \arrow[r, "\mu_{e,k}^{-1}"] \arrow[d, "\mu_{e,k}^{-1}"] \arrow[dr, phantom, "\text{(VII)}"] & \pi_e^{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X) \arrow[d, "\mu_{e,k}^{-1}"] \\
{\begin{array}{c}\pi_k^{\mathcal{V}}(X) \\ = 1_{\mathrm{Res}_H^G(\mathcal{V})} \circ \pi_k^{\mathcal{V}}(X)\end{array}} \arrow[r, "\varepsilon"] \arrow[ur, "\mathrm{Id}"] \arrow[r, phantom, shift left=6, "\text{(VI)}"] & \pi_e^{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X) \arrow[r, "\mu_{e,e}^{-1}"] & {\begin{array}{c}\pi_e^{\mathcal{V}} \circ \pi_e^{\mathcal{V}} \circ \pi_k^{\mathcal{V}}(X) \\ = \mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})} \circ \pi_k^{\mathcal{V}}(X)\end{array}}
\end{tikzcd}
$$

The commutativity of (III) is immediate and all the other cells commute by the H-category axioms. Therefore, each $\Lambda_{\mathcal{V}}$ is an H-natural transformation.

Lastly, we need to check the modification axiom, we need to show that given $\Theta \in \mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$, the following diagram commutes

$$
\begin{tikzcd}
\mathrm{Res}_H^G(\Theta) \circ 1_{\mathrm{Res}_H^G(\mathcal{V})} \arrow[r, "\Lambda_{\mathcal{V}}"] \arrow[d, "\mathrm{Id}"'] & \mathrm{Res}_H^G(\Theta) \circ \mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})} \arrow[d, "\eta_\Theta^{\mathcal{V},\mathcal{W}} \circ \xi_\Theta^{\mathcal{V},\mathcal{W}}"] \\
1_{\mathrm{Res}_H^G(\mathcal{W})} \circ \mathrm{Res}_H^G(\Theta) \arrow[r, "\Lambda_{\mathcal{W}}"] & \mathrm{Res}_H^G(\xi_{\mathcal{W}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{W})} \circ \mathrm{Res}_H^G(\Theta)
\end{tikzcd}
$$

To do this we need to show that for all $X \in \mathcal{V}$, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Res}_H^G(\Theta) \circ 1_{\mathrm{Res}_H^G(\mathcal{V})}(X) \arrow[r, "(\Lambda_{\mathcal{V}})_X"] \arrow[d, "\mathrm{Id}"'] & \mathrm{Res}_H^G(\Theta) \circ \mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})}(X) \arrow[d, "\left(\eta_\Theta^{\mathcal{V},\mathcal{W}} \circ \xi_\Theta^{\mathcal{V},\mathcal{W}}\right)_X"] \\
1_{\mathrm{Res}_H^G(\mathcal{W})} \circ \mathrm{Res}_H^G(\Theta)(X) \arrow[r, "(\Lambda_{\mathcal{W}})_{\Theta(X)}"] & \mathrm{Res}_H^G(\xi_{\mathcal{W}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{W})} \circ \mathrm{Res}_H^G(\Theta)(X)
\end{tikzcd}$$

Expanding this diagram using the definitions and adding an extra internal arrow yields the following

$$\begin{tikzcd}
\begin{array}{c} \Theta(X) \\ = \mathrm{Res}_H^G(\Theta) \circ 1_{\mathrm{Res}_H^G(\mathcal{V})}(X) \end{array} \arrow[r, "\varepsilon"] \arrow[dd, "\mathrm{Id}"'] & \Theta \circ \pi_e^{\mathcal{V}}(X) \arrow[r, "\mu_{e,e}^{-1}"] \arrow[dd, "(\Theta_e)^{-1}"] & \begin{array}{c} \Theta \circ \pi_e^{\mathcal{V}} \circ \pi_e^{\mathcal{V}}(X) \\ = \mathrm{Res}_H^G(\Theta) \circ \mathrm{Res}_H^G(\xi_{\mathcal{V}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{V})}(X) \end{array} \arrow[d, "(\Theta_e)^{-1}"] \\
 & & \pi_e^{\mathcal{W}} \circ \Theta \circ \pi_e^{\mathcal{V}}(X) \arrow[d, "(\Theta_e)^{-1}"] \\
\begin{array}{c} \Theta(X) \\ = 1_{\mathrm{Res}_H^G(\mathcal{W})} \circ \mathrm{Res}_H^G(\Theta)(X) \end{array} \arrow[r, "\varepsilon"] & \pi_e^{\mathcal{W}} \circ \Theta(X) \arrow[r, "\mu_{e,e}^{-1}"] & \begin{array}{c} \pi_e^{\mathcal{W}} \circ \pi_e^{\mathcal{W}} \circ \Theta(X) \\ = \mathrm{Res}_H^G(\xi_{\mathcal{W}}) \circ \eta_{\mathrm{Res}_H^G(\mathcal{W})} \circ \mathrm{Res}_H^G(\Theta)(X) \end{array}
\end{tikzcd}$$

Both cells commute by the H-intertwiner axioms and hence the modification axiom is satisfied. $\square$

Lemma 3.48. *There is an equality of pseudonatural transfomations*

$$1_{\mathrm{Ind}_H^G} = \xi_{\mathrm{Ind}_H^G} \circ \mathrm{Ind}_H^G(\eta).$$

Proof. Given $\mathcal{V} \in \mathsf{REP}(H)$ and $\mathbf{X} \in \mathrm{Ind}_H^G(\mathcal{V})$, we have

$$\begin{aligned}
\xi_{\mathrm{Ind}_H^G(\mathcal{V})} \circ \mathrm{Ind}_H^G(\eta_{\mathcal{V}})(\mathbf{X}) &= \xi_{\mathrm{Ind}_H^G(\mathcal{V})}(\delta_H^{\mathbf{X}}) \\
&= \pi_e^{\mathrm{Ind}(\mathcal{V})} \circ \pi_e^{\mathrm{Ind}(\mathcal{V})}(\mathbf{X}) \\
&= \mathbf{X}.
\end{aligned}$$

A similar argument shows that given a morphism $\alpha : \mathbf{X} \to \mathbf{Y}$ in $\mathrm{Ind}_H^G(\mathcal{V})$ we have

$$\xi_{\mathrm{Ind}_H^G(\mathcal{V})} \circ \mathrm{Ind}_H^G(\eta_{\mathcal{V}})(\alpha) = \alpha.$$

$\square$

Combining the results of this section, we have the following:

Theorem 3.49. *The induction 2-functor* $\mathrm{Ind}_H^G : \mathsf{REP}(H) \to \mathsf{REP}(G)$ *is a left biadjoint to the restriction 2-functor* $\mathrm{Res}_H^G : \mathsf{REP}(G) \to \mathsf{REP}(H)$.

Corollary 3.50. *Let $\mathcal{V}$ be an H-category and $\mathcal{W}$ a G-category. There are mutually quasi inverse equivalences*

$$\varphi_{\mathcal{V},\mathcal{W}} : \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathcal{V}), \mathcal{W}) \xrightarrow{\simeq} \mathrm{Hom}_H(\mathcal{V}, \mathrm{Res}_H^G(\mathcal{W}))$$

and

$$\psi_{\mathcal{V},\mathcal{W}} : \mathrm{Hom}_H(\mathcal{V}, \mathrm{Res}_H^G(\mathcal{W})) \xrightarrow{\simeq} \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathcal{V}), \mathcal{W})$$

defined on objects by

$$\varphi_{\mathcal{V},\mathcal{W}}(\Theta) = \mathrm{Res}_H^G(\Theta) \circ \eta_{\mathcal{V}}$$

and

$$\psi_{\mathcal{V},\mathcal{W}}(\Phi) = \xi_{\mathcal{W}} \circ \mathrm{Ind}_H^G(\Phi).$$

Proof. This follows from Theorem 3.49 and the fact that in a biadjunction one has canonical hom category equivalences as detailed in Lemma A.2. □

4 Categorical Hecke algebras

4.1 Classical Hecke algebras

In this section we shall outline some of the classical theory of Hecke algebras. Our main reference for Hecke algebras is [21], in which the algebra is constructed in a different way to the one we shall present below but it is completely equivalent. Hecke algebras are constructed using the data of a group G and subgroup H where H satisfies a certain finiteness condition relative to G to be described shortly. Hecke algebras are similar in structure to group rings, in fact, when H is a normal subgroup of G, the Hecke algebra of the pair (G, H) is the group ring of the quotient group G/H. In order to describe Hecke algebras, we first need a few definitions.

Definition 4.1. *Let G be a group, H a subgroup and $x \in G$. We define the* ***double coset*** *HxH by*

$$HxH := \{hxk \mid h, k \in H\}.$$

We denote the set of double cosets of H in G by $H\backslash G/H$.

As with cosets, two double cosets are either identical or disjoint. Every double coset is a disjoint union of left (or right) cosets. In the case that H is normal in G, the double coset HxH is equal to the coset xH so for normal subgroups, double cosets and cosets are the same thing.

Definition 4.2. *Let G be a group and H a subgroup. We say H is **almost normal** in G if every double coset of H in G is a disjoint union of finitely many left cosets. If H is almost normal in G, we call the pair (G, H) a **Hecke pair**.*

In particular, a normal subgroup is almost normal since every double coset is equal to a single left coset. We note that it would be equivalent to define the almost normality in terms of right cosets. This is because if

$$HxH = \bigsqcup_{i=1}^{n} x_i H, \qquad x_i \in G,$$

then

$$Hx^{-1}H = \bigsqcup_{i=1}^{n} Hx_i^{-1}.$$

Thus if HxH is a disjoint union of finitely many left cosets, $Hx^{-1}H$ is a disjoint union of finitely many right cosets and vice versa.

Similarly to group rings, there are a few equivalent ways one can construct the Hecke algebra for a Hecke pair. We shall describe it as a space of functions with a convolution product since this is the approach we shall take when constructing our categorical analogue.

Definition 4.3. *Let (G, H) be a Hecke pair. The **Hecke algebra**, $\mathcal{H}(G/\!/H)$ is the $*$-algebra defined as follows: The underlying vector space is the space of functions*

$$C_H(G)^{H\times H} := \left\{ f : G \to \mathbb{C} \,\middle|\, \begin{array}{c} f \text{ has finite support} \bmod H, \\ f(hgk) = f(g)\ \forall g \in G,\ h, k \in H \end{array} \right\}.$$

By finite support $\bmod H$, we mean that $f(g) = 0$ for all g outside of a finite set of left cosets of H in G. The addition is defined by

$$(f + f')(g) := f(g) + f'(g).$$

To define the product, we fix a set Γ of representatives for G/H. Then we define

$$(f * f')(g) := \sum_{\gamma\in\Gamma} f(\gamma) f'(\gamma^{-1}g).$$

(This is a finite sum because f has finite support $\bmod H$*.) The involution is defined by*

$$f^*(g) := \overline{f(g^{-1})}$$

where the overbar denotes the complex conjugate.

The definition of the product does not depend on the choice of coset representatives for G/H. This is because if Σ is another set of coset representatives, for each $\sigma \in \Sigma$, there is a unique $\gamma \in \Gamma$ and $h_\gamma \in H$ such that $\sigma = \gamma h_\gamma$. Then given $f, f' \in \mathcal{H}(G/\!/H)$ and $g \in G$ we have

$$\begin{aligned}
\sum_{\sigma \in \Sigma} f(\sigma) f'(\sigma^{-1} g) &= \sum_{\gamma \in \Gamma} f(\gamma h_\gamma) f'(h_\gamma^{-1} \gamma^{-1} g) \\
&= \sum_{\gamma \in \Gamma} f(\gamma) f'(\gamma^{-1} g).
\end{aligned}$$

To see that the product is well-defined, let $f, f' \in \mathcal{H}(G/\!/H)$. We first note that $f * f'$ has finite support $\bmod H$ because f' does. Next, if $g \in G$ and $k \in H$, then

$$\begin{aligned}
(f * f')(gk) &= \sum_{\gamma \in \Gamma} f(\gamma) f'(\gamma^{-1} gk) \\
&= \sum_{\gamma \in \Gamma} f(\gamma) f'(\gamma^{-1} g) \\
&= (f * f')(g).
\end{aligned}$$

Finally, to see that $(f * f')(hg) = (f * f')(g)$ for all $g \in G$ and $h \in G$, we shall use the following lemma.

Lemma 4.4. *Let $f, f' \in \mathcal{H}(G/\!/H)$, $g \in G$ and let Γ be a set of coset representatives for G/H. Then*

$$(f * f')(g) = \sum_{\gamma \in \Gamma} f(g\gamma) f'(\gamma^{-1}).$$

Proof. Let Σ be a set of coset representatives for G/H. Then the set

$$\Omega := \{g^{-1}\sigma \mid \sigma \in \Sigma\}$$

is also a set of coset representatives for G/H because the map

$$\begin{aligned}
\varphi : G/H &\to G/H \\
xH &\mapsto g^{-1} xH
\end{aligned}$$

is a bijection with inverse $xH \mapsto gxH$. Then

$$\begin{aligned}(f * f')(g) &= \sum_{\sigma \in \Sigma} f(\sigma) f'(\sigma^{-1} g) \\ &= \sum_{\omega \in \Omega} f(g\omega) f'(\omega^{-1}).\end{aligned}$$

Finally, if Γ is any choice of coset representatives for G/H, then for each $\omega \in \Omega$ there is a unique $\gamma \in \Gamma$ and $h_\gamma \in H$ such that $\omega = \gamma h_\gamma$ and

$$\begin{aligned}(f * f')(g) &= \sum_{\omega \in \Omega} f(g\omega) f'(\omega^{-1}) \\ &= \sum_{\gamma \in \Gamma} f(g\gamma h_\gamma) f'(h_\gamma^{-1}\gamma) \\ &= \sum_{\gamma \in \Gamma} f(g\gamma) f'(\gamma^{-1}).\end{aligned}$$

□

From this we can see that if $f, f' \in \mathcal{H}(G/\!/H)$, $h \in H$ and $g \in G$ then

$$\begin{aligned}(f * f')(hg) &= \sum_{\gamma \in \Gamma} f(hg\gamma)(\gamma^{-1}) \\ &= \sum_{\gamma \in \Gamma} f(g\gamma)(\gamma^{-1}) \\ &= (f * f')(g).\end{aligned}$$

Combined with our earlier observations, we have $(f * f')(hgk) = (f * f')(g)$ for all $h, k \in H$ and $g \in G$. Therefore, the product in $\mathcal{H}(G/\!/H)$ is well-defined.

Similar computations also show that the involution is anti-multiplicative, i.e.

$$f^* * f'^* = (f' * f)^*, \quad f, f' \in \mathcal{H}(G/\!/H),$$

and that the unit is the characteristic function of H. We also note that a basis for the underlying vector space of $\mathcal{H}(G/\!/H)$ is given by the characteristic functions of the double cosets.

Since we will frequently need to sum over a set of coset representatives for G/H when working with Hecke algebras it will be useful to have some notation for this. To that end, from this point onwards Γ will always denote a set of coset representatives for G/H. Later on when we look at a categorical analogue of $\mathcal{H}(G/\!/H)$, we shall consider Γ to be fixed throughout the exposition. The construction will not be completely independent of the choice of coset representatives but as one might expect, making a different choice of coset representatives will not matter up to isomorphism.

Part of the reason for the interest in Hecke algebras is that if V is any representation of G, then $\mathcal{H}(G/\!/H)$ acts on the space V^H of H-fixed points. The action is defined by the formula

$$f \cdot v := \sum_{\gamma \in \Gamma} f(\gamma)(\gamma \cdot v), \quad f \in \mathcal{H}(G/\!/H),\ v \in V^H. \tag{$*$}$$

This does not depend on the choice of coset representatives for G/H. If Σ is another choice of coset representatives then for each $\sigma \in \Sigma$ we have $\sigma = \gamma h_\gamma$ for a unique $\gamma \in \Gamma$ and $h_\gamma \in H$. Then

$$\begin{aligned}\sum_{\sigma \in \Sigma} f(\sigma)(\sigma \cdot v) &= \sum_{\gamma \in \Gamma} f(\gamma h_\gamma)((\gamma h_\gamma) \cdot v) \\ &= \sum_{\gamma \in \Gamma} f(\gamma h_\gamma)(\gamma \cdot (h_\gamma \cdot v)) \\ &= \sum_{\gamma \in \Gamma} f(\gamma)(\gamma \cdot v).\end{aligned}$$

A classical example of this is given by the Hecke operators of number theory and their action on spaces of modular forms. More on this particular example can be found in [11].

We shall also consider a different point of view by identifying fixed points with certain intertwining operators. We can do this using the following lemma and the induction-restriction adjunction.

Lemma 4.5. *Let $V \in \mathsf{Rep}(H)$ and let $\mathbb{C}$ denote the trivial representation of H on the complex numbers. There are mutually inverse isomorphisms*

$$s : \mathrm{Hom}_H(\mathbb{C}, V) \xrightarrow{\cong} V^H$$

and

$$t : V^H \xrightarrow{\cong} \mathrm{Hom}_H(\mathbb{C}, V)$$

defined by $s(\theta) := \theta(1)$ and $t(v)(\alpha) := \alpha v$, $v \in V^H$, $\alpha \in \mathbb{C}$.

Proof. Firstly s is well defined because given $h \in H$,

$$\begin{aligned} h \cdot \theta(1) &= \theta(h \cdot 1) \\ &= \theta(1). \end{aligned}$$

Next, t is well defined because given $h \in H$,

$$\begin{aligned} h \cdot (t(v)(\alpha)) &= h \cdot (\alpha v) \\ &= \alpha(h \cdot v) \\ &= \alpha v \\ &= t(v)(\alpha) \\ &= t(v)(h \cdot \alpha). \end{aligned}$$

Finally, it is immediate from the definitions that s and t are mutually inverse. $\square$

In particular, if $V \in \mathsf{Rep}(G)$ then $\mathrm{Hom}_H(\mathbb{C}, \mathrm{Res}^G_H(V)) \cong V^H$. Combining this with the induction-restriction adjunction we have the following:

Lemma 4.6. *Let $V \in \mathsf{Rep}(G)$ and let $\mathbb{C}$ denote the trivial representation of G on the complex numbers. Then there is an isomorphism*

$$p : \mathrm{Hom}_G(\mathrm{Ind}^G_H(\mathbb{C}), V) \xrightarrow{\cong} V^H$$

defined by $p(\theta) := \theta(\delta_H)$ where δ_H is the indicator function of H. There is a linear map

$$q : V^H \xrightarrow{\cong} \mathrm{Hom}_G(\mathrm{Ind}^G_H(\mathbb{C}), V)$$

defined by

$$q(v)(f) := \sum_{\gamma \in \Gamma} f(\gamma)(\gamma \cdot v), \qquad f \in \mathrm{Ind}^G_H(\mathbb{C})$$

which is the inverse to p.

Proof. We recall that if $W \in \mathsf{Rep}(H)$ then the unit $\eta_W : W \to \mathrm{Res}^G_H \mathrm{Ind}^G_H(W)$ of the induction-restriction adjunction is defined by

$$\eta_W(w)(g) := \begin{cases} g^{-1} \cdot w & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\eta_{\mathbb{C}}(\alpha)(g) := \begin{cases} \alpha & \text{if } g \in H, \\ 0 & \text{otherwise} \end{cases}$$

and hence $\eta_{\mathbb{C}}(1) = \delta_H$.

We also recall that the counit $\xi_V : \mathrm{Ind}_H^G \mathrm{Res}_H^G(V) \to V$ is defined by

$$\xi_V(f) := \sum_{\gamma \in \Gamma} \gamma \cdot f(\gamma)$$

and that we have mutually inverse isomorphisms

$$\begin{aligned} \varphi_{\mathbb{C},V} : \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), V) &\xrightarrow{\cong} \mathrm{Hom}_H(\mathbb{C}, \mathrm{Res}_H^G(V)) \\ \theta &\mapsto \mathrm{Res}_H^G(\theta) \circ \eta_{\mathbb{C}} \end{aligned}$$

and

$$\begin{aligned} \psi_{\mathbb{C},V} : \mathrm{Hom}_H(\mathbb{C}, \mathrm{Res}_H^G(V)) &\xrightarrow{\cong} \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), V) \\ \phi &\mapsto \xi_V \circ \mathrm{Ind}_H^G(\phi). \end{aligned}$$

We define p as the composite

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), V) \xrightarrow{\varphi_{\mathbb{C},V}} \mathrm{Hom}_H(\mathbb{C}, \mathrm{Res}_H^G(V)) \xrightarrow{s} V^H$$

and q as the composite

$$V^H \xrightarrow{t} \mathrm{Hom}_H(\mathbb{C}, \mathrm{Res}_H^G(V)) \xrightarrow{\psi_{\mathbb{C},V}} \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), V)$$

where s and t are defined as in Lemma 4.5.

Given $\theta \in \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), V)$ we have

$$\begin{aligned} p(\theta) &= (s \circ \varphi_{\mathbb{C},V})(\theta) \\ &= s(\mathrm{Res}_H^G(\theta) \circ \eta_{\mathbb{C}}) \\ &= \mathrm{Res}_H^G(\theta) \circ \eta_{\mathbb{C}}(1) \\ &= \theta(\delta_H). \end{aligned}$$

Also, given $v \in V^H$ and $f \in \mathrm{Ind}_H^G(\mathbb{C})$ we have

$$\begin{aligned}
q(v)(f) &= (\psi_{\mathbb{C},V} \circ t)(v)(f) \\
&= (\xi_V \circ \mathrm{Ind}_H^G(t(v)))(f) \\
&= \sum_{\gamma\in\Gamma} \gamma \cdot (\mathrm{Ind}_H^G(t(v))(f)(\gamma)) \\
&= \sum_{\gamma\in\Gamma} \gamma \cdot (t(v)(f(\gamma))) \\
&= \sum_{\gamma\in\Gamma} \gamma \cdot (f(\gamma)v) \\
&= \sum_{\gamma\in\Gamma} f(\gamma)(\gamma \cdot v).
\end{aligned}$$

Finally, p and q are mutually inverse by construction. $\square$

As a particular case, we note that given a Hecke pair (G, H), the underlying vector space of $\mathcal{H}(G/\!/H)$ is $\mathrm{Ind}_H^G(\mathbb{C})^H$. Therefore, by Lemma 4.6 there is a vector space isomorphism

$$p : \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), \mathrm{Ind}_H^G(\mathbb{C})) \xrightarrow{\cong} \mathcal{H}(G/\!/H).$$

The space $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), \mathrm{Ind}_H^G(\mathbb{C}))$ has a natural complex algebra structure with addition defined by $(\theta+\varphi)(f) := \theta(f)+\varphi(f)$, $f \in \mathrm{Ind}_H^G(\mathbb{C})$, and multiplication given by composition of intertwiners.

Lemma 4.7. *The map*

$$p : \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), \mathrm{Ind}_H^G(\mathbb{C})) \to \mathcal{H}(G/\!/H)$$

defined as in Lemma 4.6 (with $V = \mathrm{Ind}_H^G(\mathbb{C})$) is an algebra anti-isomorphism.

Proof. Let $f, f' \in \mathcal{H}(G/\!/H)$. We need to show that

$$p(q(f') \circ q(f)) = f * f'$$

where $q : \mathcal{H}(G/\!/H) \to \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), \mathrm{Ind}_H^G(\mathbb{C}))$ is the inverse to p defined as in Lemma 4.6.

Now,

$$\begin{aligned}
p(q(f') \circ q(f)) &= (q(f') \circ q(f))(\delta_H) \\
&= q(f')(q(f)(\delta_H)) \\
&= q(f')(p(q(f))) \\
&= q(f')(f) \\
&= \sum_{\gamma \in \Gamma} f(\gamma)(\gamma \cdot f').
\end{aligned}$$

Then, for $g \in G$ we have

$$\begin{aligned}
p(q(f') \circ q(f))(g) &= \sum_{\gamma \in \Gamma} f(\gamma)(\gamma \cdot f')(g) \\
&= \sum_{\gamma \in \Gamma} f(\gamma) f'(\gamma^{-1} g) \\
&= (f * f')(g).
\end{aligned}$$

Therefore, $p(q(f') \circ q(f)) = f * f'$ as required. $\square$

For any $V \in \mathsf{Rep}(G)$, the algebra $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), \mathrm{Ind}_H^G(\mathbb{C}))$ acts on the vector space $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), V)$ by precomposition. Under the above identifications of the algebra $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), \mathrm{Ind}_H^G(\mathbb{C}))^{\mathrm{op}}$ with $\mathcal{H}(G /\!\!/ H)$ and $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), V)$ with V^H, the same argument as in Lemma 4.7 shows that this is the same action of $\mathcal{H}(G /\!\!/ H)$ on V^H defined by $(*)$.

Before looking at a categorical version of Hecke algebras, we shall briefly look at one other way to construct the Hecke algebra. We recall that the underlying vector space of $\mathcal{H}(G /\!\!/ H)$ is the space of functions

$$C_H(G)^{H \times H} := \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ has finite support mod } H, \\ f(hgk) = f(g) \ \forall g \in G,\ h, k \in H \end{array} \right\}.$$

If we let $C_c(G/H)^H$ denote the space of functions

$$\left\{ f : G/H \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ has finite support}, \\ f(hgH) = f(gH) \ \forall gH \in G/H,\ h \in H \end{array} \right\},$$

then there is a vector space isomorphism

$$\varphi : C_H(G)^{H \times H} \xrightarrow{\cong} C_c(G/H)^H$$

defined by $\varphi(f)(gH) := f(g)$. Via this isomorphism, we can pass the algebra structure from $\mathcal{H}(G/\!\!/H)$ to $C_c(G/H)^H$. Given $f, f' \in C_c(G/H)^H$, the product is defined by

$$(f * f')(gH) := \sum_{\gamma \in \Gamma} f(\gamma H) f'(\gamma^{-1} g H)$$

and the involution is defined by $f^*(gH) := \overline{f(g^{-1}H)}$. We shall use the notation δ_{HtH} to denote the characteristic function of HtH/H in $C_c(G/H)^H$. We are using the same notation to denote the characteristic function of HtH in $C_H(G)^{H\times H}$ but since $C_H(G)^{H\times H}$ and $C_c(G/H)^H$ are canonically isomorphic, hopefully the context will make clear which is meant and this will not cause any confusion. We will refer to both constructions as 'the Hecke algebra' and use both for computations.

4.2 The Hecke category

4.2.1 Definition of the Hecke category

Given a Hecke pair (G, H), we shall construct a C^*-tensor category which is the analogue of the Hecke algebra $\mathcal{H}(G/\!\!/H)$. An equivalent tensor category has also been described by Zhu in [28] and we shall look at a few ways of constructing equivalent tensor categories. Other approaches to categorification of Hecke algebras are possible such as the more general approach of Arano and Vaes in [2]. There the authors consider totally disconnected groups and locally compact subgroups as well as considering multiple subgroups at the same time. In the previous section, we noted that the underlying vector space of $\mathcal{H}(G/\!\!/H)$ was the space of fixed points $\mathrm{Ind}_H^G(\mathbb{C})^H$ where $\mathbb{C}$ denotes the trivial representation of H on the complex numbers. To get a categorical analogue, we give hilb a trivial H-category structure (i.e. π_k^{hilb} is the identity for all $k \in H$ and all the coherence maps are identities) and define the underlying C^*-category of our construction as $\mathrm{Ind}_H^G(\mathsf{hilb})^H$.

Before defining a C^*-tensor category structure on $\mathrm{Ind}_H^G(\mathsf{hilb})^H$, let us describe the objects of $\mathrm{Ind}_H^G(\mathsf{hilb})^H$. To begin with, the underlying C^*-category of $\mathrm{Ind}_H^G(\mathsf{hilb})$ is the category of fixed points $C_H(G, \mathsf{hilb})^H$. Its objects are functions $\mathbf{X} : G \to \mathrm{ob}(\mathsf{hilb})$ of finite support mod H (i.e. functions supported on finitely many left cosets) along with unitaries $\rho_k^{\mathbf{X}} : \pi_k^{C_H(G,\mathsf{hilb})}(\mathbf{X}) \to \mathbf{X}$ for all $k \in H$ which satisfy the fixed point axiom. We recall that given $g \in G$,

$$\pi_k^{C_H(G,\mathsf{hilb})}(\mathbf{X})(g) = \pi_k^{\mathsf{hilb}}(\mathbf{X}(gk)) = \mathbf{X}(gk).$$

Therefore, $\rho_k^{\mathbf{X}}$ consists of unitaries $\rho_k^{\mathbf{X}}(g) : \mathbf{X}(gk) \to \mathbf{X}(g)$ for each $g \in G$. In this setting, the fixed point axiom says that for all $k, \ell \in H$ and $g \in G$, the following diagram commutes

$$\begin{tikzcd}
\mathbf{X}(gk\ell) \arrow[r, "\rho_\ell^{\mathbf{X}}(gk)"] \arrow[dr, "\rho_{k\ell}^{\mathbf{X}}(g)"'] & \mathbf{X}(gk) \arrow[d, "\rho_k^{\mathbf{X}}(g)"] \\
& \mathbf{X}(g)
\end{tikzcd}$$

The G-category structure on $\mathrm{Ind}_H^G(\mathsf{hilb})$ is defined on objects by

$$\pi_t^{\mathrm{Ind}(\mathsf{hilb})}(\mathbf{X}, (\rho_k^{\mathbf{X}})) := (t \cdot \mathbf{X}, (\rho_k^{t \cdot \mathbf{X}}))$$

where $(t \cdot \mathbf{X})(g) := \mathbf{X}(t^{-1}g)$ and $\rho_k^{t \cdot \mathbf{X}}(g) := \rho_k^{\mathbf{X}}(t^{-1}g)$ for all $t, g \in G$. Then $\mathrm{Ind}_H^G(\mathsf{hilb})^H$ is the C^*-category of H-fixed points in $\mathrm{Ind}_H^G(\mathsf{hilb})$. This means that an object in $\mathrm{Ind}_H^G(\mathsf{hilb})^H$ consists of an object $(\mathbf{X}, (\rho_k^{\mathbf{X}})) \in \mathrm{Ind}_H^G(\mathsf{hilb})$ along with unitaries $\rho_\ell^{(\mathbf{X}, (\rho_k^{\mathbf{X}}))} : \pi_\ell^{\mathrm{Ind}(\mathsf{hilb})}(\mathbf{X}, (\rho_k^{\mathbf{X}})) \to (\mathbf{X}, (\rho_k^{\mathbf{X}}))$ for all $\ell \in H$ which satisfy the fixed point axiom. To avoid cumbersome notation and for another reason that will become apparent when we consider biequivariant Hilbert spaces later, we shall write $\lambda_\ell^{\mathbf{X}}$ rather than $\rho_\ell^{(\mathbf{X}, (\rho_k^{\mathbf{X}}))}$. In particular, for all $\ell \in H$ and $g \in G$, we have a unitary $\lambda_\ell^{\mathbf{X}}(g) : \mathbf{X}(\ell^{-1}g) \to \mathbf{X}(g)$. Then the fixed point axiom says that for all $\ell, m \in H$ and all $g \in G$, the following diagram commutes

$$\begin{tikzcd}
\mathbf{X}((\ell m)^{-1}g) \arrow[r, "\lambda_m^{\mathbf{X}}(\ell^{-1}g)"] \arrow[dr, "\lambda_{\ell m}^{\mathbf{X}}(g)"'] & \mathbf{X}(\ell^{-1}g) \arrow[d, "\lambda_\ell^{\mathbf{X}}(g)"] \\
& \mathbf{X}(g)
\end{tikzcd}$$

Furthermore, each $\lambda_\ell^{\mathbf{X}}$ is a morphism in the category of fixed points $C_H(G, \mathsf{hilb})^H$. Therefore $\lambda_\ell^{\mathbf{X}}$ commutes with $\rho_k^{\mathbf{X}}$ for all $k, \ell \in H$. This means that for all $k, \ell \in H$ and $g \in G$, the following diagram commutes

$$\begin{tikzcd}
\mathbf{X}(\ell^{-1}gk) \arrow[r, "\lambda_\ell^{\mathbf{X}}(gk)"] \arrow[d, "\rho_k^{\mathbf{X}}(\ell^{-1}g)"'] & \mathbf{X}(gk) \arrow[d, "\rho_k^{\mathbf{X}}(g)"] \\
\mathbf{X}(\ell^{-1}g) \arrow[r, "\lambda_\ell^{\mathbf{X}}(g)"] & \mathbf{X}(g)
\end{tikzcd}$$

In summary, an object in $\mathrm{Ind}_H^G(\mathsf{hilb})^H$ consists of a function $\mathbf{X} : G \to \mathrm{ob}(\mathsf{hilb})$ of finite support mod H (meaning supported on finitely many left cosets), along with unitaries $\rho_k^{\mathbf{X}}(g) : \mathbf{X}(gk) \to \mathbf{X}(g)$ and $\lambda_\ell^{\mathbf{X}}(g) : \mathbf{X}(\ell^{-1}g) \to \mathbf{X}(g)$ for all $k, \ell \in H$ and $g \in G$ such that the above diagrams commute.

4.2.2 The tensor category structure

In section 4.1 we noted that there is a vector space isomorphism

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), \mathrm{Ind}_H^G(\mathbb{C})) \cong \mathrm{Ind}_H^G(\mathbb{C})^H$$

and when $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), \mathrm{Ind}_H^G(\mathbb{C}))$ is viewed as a complex algebra with multiplication given by composition of G-intertwiners, this yields an algebra isomorphism

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), \mathrm{Ind}_H^G(\mathbb{C}))^{\mathrm{op}} \cong \mathcal{H}(G/\!/H).$$

In this section, we shall show that there is an equivalence of C^*-categories

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{hilb}), \mathrm{Ind}_H^G(\mathsf{hilb})) \simeq \mathrm{Ind}_H^G(\mathsf{hilb})^H.$$

Since $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{hilb}), \mathrm{Ind}_H^G(\mathsf{hilb}))$ has a natural tensor category structure with the tensor product given by composition of G-intertwiners, we shall use this equivalence to define the tensor product on $\mathrm{Ind}_H^G(\mathsf{hilb})^H$. One could also directly define a tensor product on $\mathrm{Ind}_H^G(\mathsf{hilb})^H$ which is the analogue of the convolution product

$$(f * f')(g) = \sum_{\gamma \in \Gamma} f(\gamma) f'(\gamma^{-1} g)$$

in $\mathcal{H}(G/\!/H)$. Both methods have their advantages and disadvantages in terms of checking axioms. Using the equivalence also has the advantage of giving us two different points of view in the long run.

To prove the equivalence $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{hilb}), \mathrm{Ind}_H^G(\mathsf{hilb})) \simeq \mathrm{Ind}_H^G(\mathsf{hilb})^H$, we just need to show that given a G-category $\mathcal{V}$, there is an equivalence of C^*-categories

$\mathrm{Hom}_H(\mathsf{hilb}, \mathrm{Res}^G_H(\mathcal{V})) \simeq \mathcal{V}^H$. Combining this with the induction restriction biadjunction in the case $\mathcal{V} = \mathrm{Ind}^G_H(\mathsf{hilb})$ will then yield the result. To make the computations a little easier, we shall work with a strict, skeletal C^*-tensor category which is equivalent to hilb.

Definition 4.8. *We define the C^* tensor category Mat as having the natural numbers as objects and given $m, n \in \mathbb{N}$, the morphisms from m to n are the $n \times m$ complex matrices. The involution is given by taking the conjugate transpose and composition of morphisms is given by matrix multiplication. The tensor product is defined on objects by $m \otimes n := mn$ and on morphisms by taking the Kronecker product of matrices.*

The category Mat also has direct sums defined by $m \oplus n := m+n$. An equivalence of C^*-tensor categories

$$F : \mathsf{Mat} \xrightarrow{\simeq} \mathsf{hilb}$$

is given by

$$F(n) := \mathbb{C}^n, \quad n \in \mathbb{N}$$

with F defined on morphisms by mapping a complex $n \times m$ matrix to the same matrix viewed as a linear map from $\mathbb{C}^m$ to $\mathbb{C}^n$ using the standard bases. (The coherence maps for F are the obvious ones.)

As with hilb, we shall view Mat as an H-category by giving it a trivial H-category structure. Similarly to the classical case, we shall first show that given an H-category $\mathcal{V}$, there is an equivalence of C^*-categories

$$\mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V}) \simeq \mathcal{V}^H.$$

Then, if $\mathcal{V}$ is a G-category we can combine this with the induction-restriction biadjunction to obtain an equivalence

$$\mathrm{Hom}_G(\mathrm{Ind}^G_H(\mathsf{Mat}), \mathcal{V}) \simeq \mathcal{V}^H.$$

We shall proceed by constructing $*$-functors $S : \mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V}) \to \mathcal{V}^H$ and $T : \mathcal{V}^H \to \mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V})$, the analogues of the linear maps $s : \mathrm{Hom}_H(\mathbb{C}, V) \to V^H$ and $t : V^H \to \mathrm{Hom}_H(\mathbb{C}, V)$ of Lemma 4.5, and then show that they are quasi inverse to one another.

Lemma 4.9. *Let $\mathcal{V}$ be an H-category. There is a $*$-functor*

$$S : \mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V}) \to \mathcal{V}^H$$

defined on objects by

$$S(\Theta) := \Theta(1)$$

with

$$\rho_k^{\Theta(1)} := (\Theta_k)_1 : \pi_k^{\mathcal{V}} \circ \Theta(1) \to \Theta(1)$$

for all $k \in H$.

Proof. We need to do the following:

1. Check that S is well-defined on objects.
2. Define S on morphisms.
3. Check that S is a $*$-functor.

First, let us show that S is well-defined on objects. We need to show that for each $\Theta \in \mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V})$, the $\rho_k^{\Theta(1)}$'s satisfy the fixed point axioms. That is, for all $k, \ell \in H$, the following diagram should commute.

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ \pi_\ell^{\mathcal{V}} \circ \Theta(1) \arrow[r, "\substack{\pi_k^{\mathcal{V}} \rho_\ell^{\Theta(1)} \\ = \pi_k^{\mathcal{V}}(\Theta_k)_1}"] \arrow[d, "(\mu_{k,\ell}^{\mathcal{V}})_{\Theta(1)}"'] & \pi_k^{\mathcal{V}} \circ \Theta(1) \arrow[d, "\substack{\rho_k^{\Theta(1)} \\ = (\Theta_k)_1}"] \\
\pi_{k\ell}^{\mathcal{V}} \circ \Theta(1) \arrow[r, "\substack{\rho_{k\ell}^{\Theta(1)} \\ = (\Theta_{k\ell})_1}"] & \Theta(1)
\end{tikzcd}$$

This follows from the fact that Θ is an H-intertwiner and hence S is well-defined on objects.

Next, to define S on morphisms let $\Theta, \Phi \in \mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V})$ and let $\kappa : \Theta \to \Phi$ be an H-intertwiner. Then we define $S(\kappa) := \kappa_1$. To show that this is a morphism of fixed points, we need to show that for all $k \in H$, the following diagram commutes.

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ \Theta(1) \arrow[r, "\rho_k^{\Theta(1)}" above, "= (\Theta_k)_1"] \arrow[d, "\pi_k^{\mathcal{V}}\kappa_1"'] & \Theta(1) \arrow[d, "\kappa_1"] \\
\pi_k^{\mathcal{V}} \circ \Phi(1) \arrow[r, "\rho_k^{\Phi(1)}" above, "= (\Phi_k)_1"] & \Theta(1)
\end{tikzcd}$$

This follows from the fact that κ is an H-intertwiner and hence S is well defined on morphisms.

Finally, the fact that S is a $*$-functor is immediate from the definitions. $\square$

Next, we need to construct a $*$-functor $T : \mathcal{V}^H \to \mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V})$. To check one of the axioms we shall need the following lemma.

Lemma 4.10. *Let $\mathcal{V}$ be an H-category and let $V \in \mathcal{V}^H$. Then*

$$\rho_e^V = (\varepsilon_V^{\mathcal{V}})^{-1}.$$

Proof. By Lemma 3.42, we have $(\mu_{e,e}^{\mathcal{V}})_V = \pi_e^{\mathcal{V}}(\rho_e^V)$, and by the H-category axioms $(\mu_{e,e}^{\mathcal{V}})_V = \pi_e^{\mathcal{V}}(\varepsilon_V^{\mathcal{V}})^{-1}$. Therefore, $\pi_e^{\mathcal{V}}(\rho_e^V) = \pi_e^{\mathcal{V}}(\varepsilon_V^{\mathcal{V}})^{-1}$ and since $\pi_e^{\mathcal{V}}$ is an equivalence, $\rho_e^V = (\varepsilon_V^{\mathcal{V}})^{-1}$. $\square$

Lemma 4.11. *Let $\mathcal{V}$ be an H-category. There is a $*$-functor*

$$T : \mathcal{V}^H \to \mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V})$$

defined on objects in the following way: Given a fixed point $V \in \mathcal{V}^H$, we define

$$T(V)(n) := \bigoplus_{i=1}^{n} V, \quad n \in \mathbb{N}.$$

On the right hand side, we are viewing V as an object in $\mathcal{V}$ by forgetting the trivialisers ρ_k^V, $k \in H$. Given a morphism $(a_{ij}) : m \to n$ in Mat, we define

$$T(V)((a_{ij})) := (a_{ij}1_V).$$

For each $k \in H$, we define the coherence unitary natural transformation

$$T(V)_k : \pi_k^{\mathcal{V}} \circ T(V) \to \begin{array}{c} T(V) \\ = T(V) \circ \pi_k^{\mathsf{Mat}} \end{array}$$

as follows : For each $n \in \mathbb{N}$, $(T(V)_k)_n$ is the composite

$$\pi_k^{\mathcal{V}}\left(\bigoplus_{i=1}^n V\right) \xrightarrow{\zeta^{\pi_k^{\mathcal{V}}}} \bigoplus_{i=1}^n \pi_k^{\mathcal{V}}(V) \xrightarrow{\oplus \rho_k^V} \bigoplus_{i=1}^n V.$$

Proof. We need to do the following:

1. Check that T is well defined on objects. To do this we need to:
   (a) Check that $T(V)$ is a $*$-functor for each $V \in \mathcal{V}^H$.
   (b) Check that the $T(V)_k$'s are natural in n.
   (c) Check that the $T(V)_k$'s satisfy the H-intertwiner axioms.
2. Define T on morphisms.
3. Check that T is a $*$-functor.

Given $V \in \mathcal{V}^H$, that $T(V)$ is a $*$-functor is immediate. Given $k \in G$, to show that $T(V)_k$ is natural in n, we need to show that given a morphism $(a_{ij}) : m \to n$ in Mat, the following diagram commutes.

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ T(V)(m) \arrow[r, "\pi_k^{\mathcal{V}} \circ T(V)((a_{ij}))"] \arrow[d, "(T(V)_k)_m"'] & \pi_k^{\mathcal{V}} \circ T(V)(n) \arrow[d, "(T(V)_k)_n"] \\
T(V)(m) \arrow[r, "T(V)((a_{ij}))"] & T(V)(n)
\end{tikzcd}$$

This follows from the commutativity of the following diagram

$$\begin{tikzcd}
\pi_k^{\mathcal{V}}\left(\bigoplus_{i=1}^m V\right) \arrow[r, "\pi_k^{\mathcal{V}}((a_{ij}1_V))"] \arrow[d, "\zeta^{\pi_k^{\mathcal{V}}}"'] & \pi_k^{\mathcal{V}}\left(\bigoplus_{i=1}^n V\right) \arrow[d, "\zeta^{\pi_k^{\mathcal{V}}}"] \\
\bigoplus_{i=1}^m \pi_k^{\mathcal{V}}(V) \arrow[r, "(a_{ij}\pi_k^{\mathcal{V}}(1_V))"] \arrow[d, "\oplus \rho_k^V"'] & \bigoplus_{i=1}^n \pi_k^{\mathcal{V}}(V) \arrow[d, "\oplus \rho_k^V"] \\
\bigoplus_{i=1}^m V \arrow[r, "(a_{ij}1_V)"] & \bigoplus_{i=1}^n V
\end{tikzcd}$$

The commutativity of the top square is immediate and the bottom square commutes because

$$\begin{aligned}(\oplus\,\rho_k^V)\circ(a_{ij}\pi_k^{\mathcal{V}}(1_V)) &= (a_{ij}\rho_k^V)\\ &= (a_{ij}1_V)\circ(\oplus\,\rho_k^V)\end{aligned}$$

Therefore, the $T(V)_k$'s are natural in n.

To show that the $T(V)_k$'s satisfy the H-interwiner axioms, firstly we need to show that for all $k,\ell\in H$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{\mathcal{V}}\circ\pi_\ell^{\mathcal{V}}\circ T(V) \arrow[r, "\mu_{k,\ell}^{\mathcal{V}}"] \arrow[d, "T(V)_\ell"'] & \pi_{k\ell}^{\mathcal{V}}\circ T(V) \arrow[dd, "T(V)_{k\ell}"] \\
\pi_k^{\mathcal{V}}\circ T(V) \arrow[d, "T(V)_k"'] & \\
T(V) \arrow[r, "\mathrm{Id}"] & T(V)
\end{tikzcd}$$

To do this, we need to show that for all $n\in\mathbb{N}$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{\mathcal{V}}\circ\pi_\ell^{\mathcal{V}}\circ T(V)(n) \arrow[r, "(\mu_{k,\ell}^{\mathcal{V}})_{T(V)(n)}"] \arrow[d, "(T(V)_\ell)_n"'] & \pi_{k\ell}^{\mathcal{V}}\circ T(V)(n) \arrow[dd, "(T(V)_{k\ell})_n"] \\
\pi_k^{\mathcal{V}}\circ T(V)(n) \arrow[d, "(T(V)_k)_n"'] & \\
T(V)(n) \arrow[r, "\mathrm{Id}"] & T(V)(n)
\end{tikzcd}$$

This follows from the commutativity of the following diagram.

$$
\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ \pi_\ell^{\mathcal{V}}\left(\bigoplus_{i=1}^n V\right) \arrow[rr, "\mu_{k,\ell}^{\mathcal{V}}"] \arrow[d, "\zeta^{\pi_\ell^{\mathcal{V}}}"'] \arrow[ddr, "\zeta^{\pi_k^{\mathcal{V}} \circ \pi_\ell^{\mathcal{V}}}"] & & \pi_{k\ell}^{\mathcal{V}}\left(\bigoplus_{i=1}^n V\right) \arrow[dd, "\zeta^{\pi_{k\ell}^{\mathcal{V}}}"] \\
\pi_k^{\mathcal{V}}\left(\bigoplus_{i=1}^n \pi_\ell^{\mathcal{V}}(V)\right) \arrow[d, "\oplus\rho_\ell^V"'] \arrow[dr, "\zeta^{\pi_k^{\mathcal{V}}}"'] \arrow[r, phantom, "\mathrm{(I)}"] & {} \arrow[r, phantom, "\mathrm{(II)}"] & {} \\
\pi_k^{\mathcal{V}}\left(\bigoplus_{i=1}^n V\right) \arrow[d, "\zeta^{\pi_k^{\mathcal{V}}}"'] \arrow[r, phantom, "\mathrm{(III)}"] & \bigoplus_{i=1}^n \pi_k^{\mathcal{V}} \circ \pi_\ell^{\mathcal{V}}(V) \arrow[r, "\oplus\mu_{k,\ell}^{\mathcal{V}}"] \arrow[dl, "\oplus\rho_\ell^V"] & \bigoplus_{i=1}^n \pi_{k\ell}^{\mathcal{V}}(V) \arrow[dd, "\oplus\rho_{k\ell}^V"] \\
\bigoplus_{i=1}^n \pi_k^{\mathcal{V}}(V) \arrow[d, "\oplus\rho_k^V"'] \arrow[rr, phantom, "\mathrm{(IV)}"] & & {} \\
\bigoplus_{i=1}^n V \arrow[rr, "\mathrm{Id}"] & & \bigoplus_{i=1}^n V
\end{tikzcd}
$$

Here (I) commutes by Lemma 1.23, (II) commutes by Corollary 3.3, (III) commutes by the naturality of $\zeta^{\pi_k^{\mathcal{V}}}$ and (IV) commutes by the fixed point axiom.

Next, the unit axiom says that the following diagram commutes

$$
\begin{tikzcd}
& T(V) \arrow[dl, "\varepsilon^{\mathcal{V}}"'] \arrow[dr, "\mathrm{Id}"] & \\
\pi_e^{\mathcal{V}} \circ T(V) \arrow[rr, "T(V)_e"] & & T(V)
\end{tikzcd}
$$

To prove this, we need to show that for all $n \in \mathbb{N}$, the following diagram commutes

$$
\begin{tikzcd}
 & T(V)(n) \arrow[dl, "(\varepsilon^{\mathcal{V}})_{T(V)(n)}"'] \arrow[dr, "\mathrm{Id}"] & \\
\pi_e^{\mathcal{V}} \circ T(V)(n) \arrow[rr, "(T(V)_e)_n"] & & T(V)(n)
\end{tikzcd}
$$

This follows from the commutativity of the following diagram

$$
\begin{tikzcd}
 & \bigoplus_{i=1}^n V \arrow[dl, "\varepsilon^{\mathcal{V}}"'] \arrow[d, "\oplus\, \varepsilon^{\mathcal{V}}"] \arrow[dr, "\mathrm{Id}"] & \\
\pi_e^{\mathcal{V}}\,(\bigoplus_{i=1}^n V) \arrow[r, "\zeta^{\pi_e^{\mathcal{V}}}"] & \bigoplus_{i=1}^n \pi_e^{\mathcal{V}}(V) \arrow[r, "\oplus\, \rho_e^V"] & \bigoplus_{i=1}^n V
\end{tikzcd}
$$

The left triangle commutes by Corollary 3.3 and the right triangle commutes by Lemma 4.10. Therefore, the $T(V)_k$'s satisfy the H-intertwiner axioms and hence T is well defined on objects.

Next, we need to define T on morphisms. Given a morphism $f : V \to W$ in $\mathcal{V}^H$, we define an H-natural transformation $T(f)$ by

$$
T(f)_n := \bigoplus_{i=1}^{n} f, \qquad n \in \mathbb{N}.
$$

Let us check that this does indeed define an H-natural transformation. To show naturality, we need to show that given a morphism $(a_{ij}) : m \to n$ in Mat, the following diagram commutes

$$
\begin{tikzcd}
T(V)(m) \arrow[r, "T(V)(a_{ij})"] \arrow[d, "T(f)_m"'] & T(V)(n) \arrow[d, "T(f)_n"] \\
T(W)(m) \arrow[r, "T(W)(a_{ij})"] & T(W)(n)
\end{tikzcd}
$$

By definition, this is the diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^m V \arrow[r, "(a_{ij}1_V)"] \arrow[d, "\oplus f"'] & \bigoplus_{i=1}^n V \arrow[d, "\oplus f"] \\
\bigoplus_{i=1}^m W \arrow[r, "(a_{ij}1_W)"] & \bigoplus_{i=1}^n W
\end{tikzcd}$$

and this commutes because

$$\left(\bigoplus_{i=1}^{n} f\right) \circ (a_{ij}1_V) = (a_{ij}f) = (a_{ij}1_W) \circ \left(\bigoplus_{i=1}^{m} f\right).$$

Therefore $T(f)$ is natural in n.

We also need to show that $T(f)$ satisfies the H-natural transformation axiom. We need to show that for all $k \in H$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ T(V) \arrow[r, "T(V)_k"] \arrow[d, "T(f)"'] & T(V) \arrow[d, "T(f)"] \\
\pi_k^{\mathcal{V}} \circ T(W) \arrow[r, "T(W)_k"] & T(W)
\end{tikzcd}$$

To do this we need to show that for all $n \in \mathbb{N}$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ T(V)(n) \arrow[r, "(T(V)_k)_n"] \arrow[d, "T(f)_n"'] & T(V)(n) \arrow[d, "T(f)_n"] \\
\pi_k^{\mathcal{V}} \circ T(W)(n) \arrow[r, "(T(W)_k)_n"] & T(W)(n)
\end{tikzcd}$$

This follows from the commutativity of the following diagram

$$\begin{tikzcd}
\pi_k^{\mathcal{V}}\left(\bigoplus_{i=1}^n V\right) \arrow[r, "\zeta^{\pi_k^{\mathcal{V}}}"] \arrow[d, "\pi_k^{\mathcal{V}}\left(\oplus_{i=1}^n f\right)"'] & \bigoplus_{i=1}^n \pi_k^{\mathcal{V}}(V) \arrow[r, "\oplus \rho_k^V"] \arrow[d, "\oplus \pi_k^{\mathcal{V}}(f)"] & \bigoplus_{i=1}^n V \arrow[d, "\oplus f"] \\
\pi_k^{\mathcal{V}}\left(\bigoplus_{i=1}^n W\right) \arrow[r, "\zeta^{\pi_k^{\mathcal{V}}}"] & \bigoplus_{i=1}^n \pi_k^{\mathcal{V}}(W) \arrow[r, "\oplus \rho_k^W"] & \bigoplus_{i=1}^n W
\end{tikzcd}$$

The left square commutes by the naturality of $\zeta^{\pi_k^{\mathcal{V}}}$ and the right square commutes because f is a morphism of fixed points. Therefore, $T(f)$ satisfies the H-natural transformation axiom and hence T is well-defined on morphisms.

Finally, that T is a $*$-functor is immediate from the definitions. $\square$

Lemma 4.12. *The $*$-functors*

$$S : \mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V}) \to \mathcal{V}^H$$

and

$$T : \mathcal{V}^H \to \mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V})$$

defined in Lemmas 4.9 and 4.11 are quasi inverse to one another, that is, $S \circ T \cong \mathrm{Id}$ and $T \circ S \cong \mathrm{Id}$.

Proof. Firstly, given $(V, (\rho_k^V)) \in \mathcal{V}^H$,

$$(S \circ T)(V, (\rho_k^V)) = T(V, (\rho_k^V))(1) = (V, (\rho_k^V))$$

and given a morphism $f : V \to W$,

$$(S \circ T)(f) = T(f)_1 = f.$$

Therefore, $S \circ T = \mathrm{Id}$.

Next, we shall construct a unitary natural transformation $\alpha : \mathrm{Id} \to T \circ S$. To do this, we need to do the following:

1. Define a unitary H-natural transformation $\alpha_\Theta : \Theta \to (T \circ S)(\Theta)$ for each $\Theta \in \mathrm{Hom}_H(\mathsf{Mat}, \mathcal{V})$. This means we need to:
   (a) Define a unitary $(\alpha_\Theta)_n : \Theta(n) \to (T \circ S)(\Theta)(n)$ for all $n \in \mathbb{N}$.
   (b) Check that the $(\alpha_\Theta)_n$'s are natural in n.

(c) Check that the $(\alpha_\Theta)_n$'s satisfy the H-natural transformation axiom.

2. Check that the α_Θ's are natural in Θ.

Given an H-intertwiner $\Theta : \mathsf{Mat} \to \mathcal{V}$ and $n \in \mathbb{N}$, on the one hand we have

$$(T \circ S)(\Theta)(n) = \bigoplus_{i=1}^{n} S(\Theta) = \bigoplus_{i=1}^{n} \Theta(1).$$

and on the other hand, since $n = \bigoplus_{i=1}^{n} 1$ in Mat,

$$\Theta(n) = \Theta\left(\bigoplus_{i=1}^{n} 1\right).$$

Therefore, we define

$$(\alpha_\Theta)_n := \zeta^{\Theta}_{(1,\ldots,1)} : \begin{array}{c} \Theta\left(\bigoplus_{i=1}^{n} 1\right) \\ = \Theta(n) \end{array} \quad \to \quad \begin{array}{c} \bigoplus_{i=1}^{n} \Theta(1) \\ = (T \circ S)(\Theta)(n) \end{array}$$

To show that the $(\alpha_\Theta)_n$'s are natural in n, we need to show that given a morphism $(a_{ij}) : m \to n$ in Mat, the following diagram commutes

$$\begin{tikzcd}
\Theta(m) \arrow[r, "\Theta(a_{ij})"] \arrow[d, "(\alpha_\Theta)_m"'] & \Theta(n) \arrow[d, "(\alpha_\Theta)_n"] \\
(T \circ S)(\Theta)(m) \arrow[r, "(T\circ S)(\Theta)(a_{ij})"] & (T \circ S)(\Theta)(n)
\end{tikzcd}$$

By definition, this is the diagram

$$\begin{tikzcd}
\Theta\left(\bigoplus_{i=1}^{m} 1\right) \arrow[r, "\Theta(a_{ij})"] \arrow[d, "\zeta^\Theta"'] & \Theta\left(\bigoplus_{i=1}^{n} 1\right) \arrow[d, "\zeta^\Theta"] \\
\bigoplus_{i=1}^{m} \Theta(1) \arrow[r, "(a_{ij} 1_{\Theta(1)})"] & \bigoplus_{i=1}^{n} \Theta(1)
\end{tikzcd}$$

whose commutativity is immediate. Therefore, the $(\alpha_\Theta)_n$'s are natural in n.

To show that α_Θ satisfies the H-natural transformation axiom, we need to show that for all $k \in H$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ \Theta \arrow[r, "\Theta_k"] \arrow[d, "\alpha_\Theta"'] & \Theta \arrow[d, "\alpha_\Theta"] \\
\pi_k^{\mathcal{V}} \circ (T \circ S)(\Theta) \arrow[r, "(T\circ S)(\Theta)_k"] & (T \circ S)(\Theta)
\end{tikzcd}$$

To do this, we need to show that for all $n \in \mathbb{N}$, the following diagram commutes

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ \Theta(n) \arrow[r, "(\Theta_k)_n"] \arrow[d, "(\alpha_\Theta)_n"'] & \Theta(n) \arrow[d, "(\alpha_\Theta)_n"] \\
\pi_k^{\mathcal{V}} \circ (T \circ S)(\Theta)(n) \arrow[r, "((T\circ S)(\Theta)_k)_n"] & (T \circ S)(\Theta)(n)
\end{tikzcd}$$

This follows from the commutativity of the following diagram

$$\begin{tikzcd}
\pi_k^{\mathcal{V}} \circ \Theta\left(\bigoplus_{i=1}^n 1\right) \arrow[rr, "\Theta_k"] \arrow[d, "\zeta^\Theta"'] \arrow[dr, "\zeta^{\pi_k^{\mathcal{V}} \circ \Theta}"] & & \Theta\left(\bigoplus_{i=1}^n 1\right) \arrow[d, "\zeta^\Theta"] \\
\pi_k^{\mathcal{V}}\left(\bigoplus_{i=1}^n \Theta(1)\right) \arrow[r, "\zeta^{\pi_k^{\mathcal{V}}}"] & \bigoplus_{i=1}^n \pi_k^{\mathcal{V}} \circ \Theta(1) \arrow[r, "\oplus\, \Theta_k"] & \bigoplus_{i=1}^n \Theta(1)
\end{tikzcd}$$

The triangle commutes by Lemma 1.23 and the other cell commutes by Corollary 3.3. Therefore, α_Θ satisfies the H-natural transformation axiom.

Finally, we need to show that the α_Θ's are natural in Θ. To do this, we need to show that given H-intertinwers $\Theta, \Phi : \mathsf{Mat} \to \mathcal{V}$ and an H-natural transformation

$\kappa : \Theta \to \Phi$, the following diagram commutes

$$\begin{tikzcd}
\Theta \arrow[r, "\kappa"] \arrow[d, "\alpha_\Theta"'] & \Phi \arrow[d, "\alpha_\Phi"] \\
(T \circ S)(\Theta) \arrow[r, "(T\circ S)(\kappa)"] & (T \circ S)(\Phi)
\end{tikzcd}$$

Therefore, we need to show that for all $n \in \mathbb{N}$, the following diagram commutes

$$\begin{tikzcd}
\Theta(n) \arrow[r, "\kappa_n"] \arrow[d, "(\alpha_\Theta)_n"'] & \Phi(n) \arrow[d, "(\alpha_\Phi)_n"] \\
(T \circ S)(\Theta)(n) \arrow[r, "(T\circ S)(\kappa)_n"] & (T \circ S)(\Phi)(n)
\end{tikzcd}$$

By definition, this is the diagram

$$\begin{tikzcd}
\Theta\left(\bigoplus_{i=1}^n 1\right) \arrow[r, "\kappa"] \arrow[d, "\zeta^\Theta"'] & \Phi\left(\bigoplus_{i=1}^n 1\right) \arrow[d, "\zeta^\Theta"] \\
\bigoplus_{i=1}^n \Theta(1) \arrow[r, "\oplus\kappa"] & \bigoplus_{i=1}^n \Phi(1)
\end{tikzcd}$$

and this commutes by Corollary 3.3. Therefore, the α_Θ's are natural in Θ. It follows that α is a unitary natural transformation from the identity to $T \circ S$ and hence S and T are mutually quasi inverse $*$-functors. $\square$

In particular, if $\mathcal{V}$ is a G-category we have an equivalence

$$\mathrm{Hom}_H(\mathsf{Mat}, \mathrm{Res}_H^G(\mathcal{V})) \simeq \mathcal{V}^H.$$

In this case, we can combine this with the induction-restriction biadjunction to obtain an equivalence

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathcal{V}) \simeq \mathcal{V}^H.$$

Explicitly, if we introduce the shorthand δ_H for the canonical fixed point δ_H^1 in $C_H(G, \mathsf{Mat})^H$ (defined as in Lemma 3.24), we have the following:

Lemma 4.13. *There are mutually quasi inverse equivalences*

$$P : \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathcal{V}) \xrightarrow{\simeq} \mathcal{V}^H$$

and

$$Q : \mathcal{V}^H \xrightarrow{\simeq} \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathcal{V})$$

defined on objects as follows: Given $\Theta \in \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathcal{V})$*,*

$$P(\Theta) := \left(\Theta(\delta_H), \left(\rho_k^{\Theta(\delta_H)}\right)\right)$$

where

$$\rho_k^{\Theta(\delta_H)} := (\Theta_k)_{\delta_H} : \pi_k^{\mathcal{V}} \circ \Theta(\delta_H) \to \begin{array}{c} \Theta \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}(\delta_H) \\ = \Theta(\delta_H) \end{array} .$$

Given $V \in \mathcal{V}^H$*,*

$$Q(V) := \xi_{\mathcal{V}} \circ \mathrm{Ind}_H^G(T(V))$$

where for $\mathbf{X} \in \mathrm{Ind}_H^G(\mathsf{Mat})$ *we have*

$$\xi_{\mathcal{V}} \circ \mathrm{Ind}_H^G(T(V))(\mathbf{X}) = \bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}} \left(\bigoplus_{i=1}^{\mathbf{X}(\gamma)} V \right).$$

Proof. We recall that by the induction-restriction biadjunction, we have mutually quasi inverse equivalences

$$\begin{aligned} \varphi_{\mathsf{Mat},\mathcal{V}} : \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathcal{V}) &\xrightarrow{\simeq} \mathrm{Hom}_H(\mathsf{Mat}, \mathrm{Res}_H^G(\mathcal{V})) \\ \Theta &\mapsto \mathrm{Res}_H^G(\Theta) \circ \eta_{\mathsf{Mat}} \end{aligned}$$

and

$$\begin{aligned} \psi_{\mathsf{Mat},\mathcal{V}} : \mathrm{Hom}_H(\mathsf{Mat}, \mathrm{Res}_H^G(\mathcal{V})) &\xrightarrow{\simeq} \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathcal{V}) \\ \Phi &\mapsto \xi_{\mathcal{V}} \circ \mathrm{Ind}_H^G(\Phi) \end{aligned}$$

where η and ξ are the unit and counit of the biadjunction respectively. Therefore, we define P as the composite

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathcal{V}) \xrightarrow{\varphi_{\mathsf{Mat},\mathcal{V}}} \mathrm{Hom}_H(\mathsf{Mat}, \mathrm{Res}_H^G(\mathcal{V})) \xrightarrow{\ S\ } \mathcal{V}^H$$

and Q as the composite

$$\mathcal{V}^H \xrightarrow{\ T\ } \mathrm{Hom}_H(\mathsf{Mat}, \mathrm{Res}_H^G(\mathcal{V})) \xrightarrow{\psi_{\mathsf{Mat},\mathcal{V}}} \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathcal{V})$$

where S and T are defined as in Lemmas 4.9 and 4.11 respectively.

Unpacking the definitions, given $\Theta \in \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathcal{V})$ we have

$$\begin{aligned} P(\Theta) &= S \circ \varphi_{\mathsf{Mat},\mathcal{V}}(\Theta) \\ &= \left(\varphi_{\mathsf{Mat},\mathcal{V}}(\Theta)(1), \left(\rho_k^{\varphi_{\mathsf{Mat},\mathcal{V}}(\Theta)(1)}\right)\right) \\ &= \left(\mathrm{Res}_H^G(\Theta) \circ \eta_{\mathsf{Mat}}(1), \left(\rho_k^{\mathrm{Res}_H^G(\Theta)\circ\eta_{\mathsf{Mat}}(1)}\right)\right) \end{aligned}$$

as in Lemma 4.9.

Now, $\eta_{\mathsf{Mat}}(1) = \delta_H$ and hence $\mathrm{Res}_H^G(\Theta) \circ \eta_{\mathsf{Mat}}(1) = \Theta(\delta_H)$. Then, by definition $\rho_k^{\mathrm{Res}_H^G(\Theta)\circ\eta_{\mathsf{Mat}}(1)}$ is the composite

$$\begin{array}{ccccc} & & \Theta \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \eta_{\mathsf{Mat}}(1) & & \Theta \circ \eta_{\mathsf{Mat}} \circ \pi_k^{\mathsf{Mat}}(1) \\ \pi_k^{\mathcal{V}} \circ \Theta \circ \eta_{\mathsf{Mat}}(1) & \xrightarrow{\Theta_k} & = \Theta \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}(\delta_H) & \xrightarrow{\substack{\eta_k \\ = \mathrm{Id}}} & = \Theta \circ \eta_{\mathsf{Mat}}(1) \\ = \pi_k^{\mathcal{V}} \circ \Theta(\delta_H) & & = \Theta(\delta_H) & & = \Theta(\delta_H) \end{array}$$

which is just $(\Theta_k)_{\delta_H}$ as required.

On the other hand, given $V \in \mathcal{V}^H$ we have

$$\begin{aligned} Q(V) &= \psi_{\mathsf{Mat},\mathcal{V}} \circ T(V) \\ &= \xi_{\mathcal{V}} \circ \mathrm{Ind}_H^G(T(V)). \end{aligned}$$

Given $\mathbf{X} \in \mathrm{Ind}_H^G(\mathsf{Mat})$ and $g \in G$ we have

$$\begin{aligned} \mathrm{Ind}_H^G(T(V))(\mathbf{X})(g) &= T(V)(\mathbf{X}(g)) \\ &= \bigoplus_{i=1}^{\mathbf{X}(g)} V \end{aligned}$$

and in general, given $\mathbf{Y} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{V})$ we have

$$\xi_{\mathcal{V}}(\mathbf{Y}) = \bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathcal{V}}(\mathbf{Y}(\gamma)).$$

Therefore,

$$\begin{aligned}\xi_{\mathcal{V}} \circ \mathrm{Ind}_H^G(T(V))(\mathbf{X}) &= \bigoplus_{\gamma\in\Gamma} \pi_k^{\mathcal{V}}(\mathrm{Ind}_H^G(T(V))(\mathbf{X})(\gamma)) \\ &= \bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{V}}\left(\bigoplus_{i=1}^{\mathbf{X}(\gamma)} V\right)\end{aligned}$$

as required. □

As a particular case, on taking $\mathcal{V} = \mathrm{Ind}_H^G(\mathsf{Mat})$ we obtain an equivalence

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathrm{Ind}_H^G(\mathsf{Mat})) \simeq \mathrm{Ind}_H^G(\mathsf{Mat})^H.$$

Since induction and taking fixed points are 2-functorial and hence preserve equivalences, we have $\mathrm{Ind}_H^G(\mathsf{Mat}) \simeq \mathrm{Ind}_H^G(\mathsf{hilb})$ and $\mathrm{Ind}_H^G(\mathsf{Mat})^H \simeq \mathrm{Ind}_H^G(\mathsf{hilb})^H$. Combined with the fact that if $\mathcal{W} \simeq \mathcal{W}'$ and $\mathcal{X} \simeq \mathcal{X}'$ are equivalent G-categories we have $\mathrm{Hom}_G(\mathcal{W}, \mathcal{X}) \simeq \mathrm{Hom}_G(\mathcal{W}', \mathcal{X}')$, it follows that

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{hilb}), \mathrm{Ind}_H^G(\mathsf{hilb})) \simeq \mathrm{Ind}_H^G(\mathsf{hilb})^H.$$

More generally, for any G-category $\mathcal{V}$ there is an equivalence

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{hilb}), \mathcal{V}) \simeq \mathcal{V}^H.$$

4.2.3 The convolution product

Since $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{hilb}), \mathrm{Ind}_H^G(\mathsf{hilb}))$ has a natural C^*-tensor category structure with the tensor product given by composition of intertwiners, we can use the equivalence

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{hilb}), \mathrm{Ind}_H^G(\mathsf{hilb})) \simeq \mathrm{Ind}_H^G(\mathsf{hilb})^H$$

to define a tensor category structure on $\mathrm{Ind}_H^G(\mathsf{hilb})^H$. To that end, let us briefly describe how to pass a tensor category structure via an equivalence.

If $\mathcal{A}$ is a C^*-tensor category, $\mathcal{B}$ is a C^*-category and $F : \mathcal{A} \to \mathcal{B}$ and $K : \mathcal{B} \to \mathcal{A}$ are mututally quasi inverse equivalences then we can define a C^*-tensor category on $\mathcal{B}$ as follows: Given $B, C \in \mathcal{B}$ we define

$$B \otimes C := F(K(B) \otimes K(C))$$

and we define the tensor unit J of $\mathcal{B}$ as $F(I)$ where I is the tensor unit of $\mathcal{A}$.

To define the associator and unitors, let $\omega : 1_{\mathcal{A}} \to KF$ and $\nu : FK \to 1_{\mathcal{B}}$ be unitary natural transformations. Then, given $B, C, D \in \mathcal{B}$, we define the associator $\alpha_{B,C,D}$ as the composite

$$
\begin{aligned}
F(KF(K(B)\otimes K(C))\otimes K(D)) \qquad & \\
= (B\otimes C)\otimes D \qquad & \xrightarrow{F(\omega^{-1}\otimes \mathrm{Id})} F((K(B)\otimes K(C))\otimes K(D)) \\
& \xrightarrow{F(\alpha)} F(K(B)\otimes (K(C)\otimes K(D))) \\
& \xrightarrow{F(\mathrm{Id}\otimes\omega)} F(K(B)\otimes KF(K(C)\otimes K(D))) \\
& \qquad\qquad = B\otimes (C\otimes D)
\end{aligned}
$$

Similarly, we define the left unitor λ_B as the composite

$$
\begin{aligned}
F(KF(I)\otimes K(B)) \qquad & \\
= J\otimes B \qquad & \xrightarrow{F(\omega^{-1}\otimes \mathrm{Id})} F(I\otimes K(B)) \\
& \xrightarrow{F(\lambda)} FK(B) \\
& \xrightarrow{\nu} B
\end{aligned}
$$

and the right unitor ρ_B as the composite

$$
\begin{aligned}
F(K(B)\otimes KF(I)) \qquad & \\
= B\otimes J) \qquad & \xrightarrow{F(\mathrm{Id}\otimes\omega^{-1})} F(K(B)\otimes I) \\
& \xrightarrow{F(\rho)} FK(B) \\
& \xrightarrow{\nu} B
\end{aligned}
$$

One then checks that these maps do indeed satisfy the tensor category axioms but we omit the details here.

One also finds that upon defining unitaries $\beta_{A,B}$ as the maps

$$
\begin{aligned}
F(KF(A)\otimes KF(B)) \qquad & \\
= F(A)\otimes F(B) \qquad & \xrightarrow{F(\omega^{-1}\otimes\omega^{-1})} F(A\otimes B)
\end{aligned}
$$

for $A, B\in\mathcal{A}$ and $\iota := \mathrm{Id} : J\to F(I)$, F becomes a tensor functor from $\mathcal{A}$ to $\mathcal{B}$.

Since the linear map $p : \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{C}), \mathrm{Ind}_H^G(\mathbb{C})) \to \mathcal{H}(G/\!/H)$ of Lemma 4.6 is an algebra anti-isomorphism, we shall pass the opposite tensor category structure from $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{hilb}), \mathrm{Ind}_H^G(\mathsf{hilb}))$ to $\mathrm{Ind}_H^G(\mathsf{hilb})^H$. Also, since the tensor product on $\mathrm{Ind}_H^G(\mathsf{hilb})^H$ will have the flavour of a convolution product, we shall denote it by $\mathbf{X} * \mathbf{Y}$, $\mathbf{X}, \mathbf{Y}\in \mathrm{Ind}_H^G(\mathsf{hilb})^H$ rather than $\mathbf{X}\otimes\mathbf{Y}$.

We shall determine an explicit formula for $\mathbf{X} * \mathbf{Y}$, again, it will be easier to work with Mat rather than hilb. To clarify some of the notation in what follows, we recall that the objects of Mat are natural numbers so that given $\mathbf{X} \in \mathrm{Ind}_H^G(\mathsf{Mat})^H$ and $g \in G$, $\mathbf{X}(g)$ is natural number which one thinks of as a substitute for a Hilbert space of dimension $\mathbf{X}(g)$. We also recall that the tensor product in Mat is given by $m \otimes n := mn$ and direct sums are defined by $m \oplus n := m + n$, where $m, n \in \mathbb{N}$.

Definition 4.14. *We define the tensor product on* $\mathrm{Ind}_H^G(\mathsf{Mat})^H$ *by*

$$\mathbf{X} * \mathbf{Y} := P(Q(\mathbf{Y}) \circ Q(\mathbf{X}))$$

where P and Q are defined as in Lemma 4.13.

Before computing an explicit formula for the product, we can immediately describe the tensor unit.

Lemma 4.15. *The tensor unit $\mathbf{I}$ of* $\mathrm{Ind}_H^G(\mathsf{Mat})^H$ *is defined by*

$$\mathbf{I}(g) = \begin{cases} 1 & \text{if } g \in H, \\ 0 & \text{otherwise} \end{cases}$$

with $\lambda_k^{\mathbf{I}}(g)$ and $\rho_k^{\mathbf{I}}(g)$ the appropriate identity map for all $k \in H$ and $g \in G$.

Proof. This follows immediately from Lemma 4.13 since the tensor unit is the image of the identity G-intertwiner on $\mathrm{Ind}_H^G(\mathsf{Mat})$ under the equivalence

$$P : \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{Mat}), \mathrm{Ind}_H^G(\mathsf{Mat})) \xrightarrow{\simeq} \mathrm{Ind}_H^G(\mathsf{Mat})^H.$$

□

Now, with regard to the product, since

$$Q(\mathbf{Y}) \circ Q(\mathbf{X}) = \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{Y})) \circ \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{X})),$$

to determine an explicit formula for $\mathbf{X} * \mathbf{Y}$ we need to compute the image of this G-intertwiner under P. Therefore, we need expressions for $(\mathbf{X} * \mathbf{Y})(g)$, $g \in G$ and for the trivialisers $\lambda_k^{\mathbf{X}*\mathbf{Y}}$ and $\rho_k^{\mathbf{X}*\mathbf{Y}}$, $k \in H$. We shall break down the computation into a series of lemmas.

Lemma 4.16. *Let* $\mathbf{X} \in \mathrm{Ind}_H^G(\mathsf{Mat})^H$. *Then*

$$\xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{X}))(\delta_H) = \mathbf{X}$$

where on the right hand side we are viewing $\mathbf{X}$ as an element of $\mathrm{Ind}_H^G(\mathsf{Mat})$.

Proof. We recall that by definition, $T(\mathbf{X}) : \mathsf{Mat} \to \mathrm{Res}_H^G \mathrm{Ind}_H^G(\mathsf{Mat})$ is the H-intertwiner defined by

$$T(\mathbf{X})(n) = \bigoplus_{i=1}^{n} \mathbf{X}, \quad n \in \mathbb{N}.$$

Therefore, $\mathrm{Ind}_H^G(T(\mathbf{X}))(\delta_H)$ is the element

$$\mathbf{U} \in \mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G(\mathsf{Mat})$$

defined by

$$\begin{aligned} \mathbf{U}(g) &= T(\mathbf{X})(\delta_H(g)) \\ &= \begin{cases} \mathbf{X} & \text{if } g \in H, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

with

$$\rho_k^{\mathbf{U}}(g) : \pi_k^{\mathrm{IndResInd}(\mathsf{Mat})}(\mathbf{U}(gk)) \to \mathbf{U}(g), \quad k \in H, g \in G$$

defined as the composite

$$\pi_k^{\mathrm{IndResInd}(\mathsf{Mat})} \circ T(\mathbf{X})(\delta_H(gk)) \xrightarrow{T(\mathbf{X})_k} T(\mathbf{X}) \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}(\delta_H(gk)) \xrightarrow[= \mathrm{Id}]{\rho_k^{\delta_H}(g)} T(\mathbf{X})(\delta_H(g)).$$

This means that

$$\begin{aligned} \rho_k^{\mathbf{U}}(g) &= T(\mathbf{X}) \left(\rho_k^{\delta_H}(g)\right) \circ (T(\mathbf{X})_k)_{\delta_H(gk)} \\ &= (T(\mathbf{X})_k)_{\delta_H(gk)} \\ &= \begin{cases} (T(\mathbf{X})_k)_1 & \text{if } g \in H, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \lambda_k^{\mathbf{X}} & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We then have

$$\begin{aligned} \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{X}))(\delta_H) &= \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})}(\mathbf{U}) \\ &= \bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathrm{IndResInd}(\mathsf{Mat})}(\mathbf{U}(\gamma)) \\ &= \pi_e^{\mathrm{IndResInd}(\mathsf{Mat})}(\mathbf{U}(e)) \\ &= \mathbf{X}. \end{aligned}$$

□

We note that we didn't actually need the $\rho_k^{\mathbf{U}}$'s in the proof of the lemma, however, we will need these and similar formulae later on so it will be useful to state them here for reference.

Lemma 4.17. *Let* $\mathbf{X}, \mathbf{Y} \in \mathrm{Ind}_H^G(\mathsf{Mat})^H$ *and let*

$$\mathbf{Z} = \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{Y})) \circ \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{X}))(\delta_H) \in \mathrm{Ind}_H^G(\mathsf{Mat}).$$

Then

$$\mathbf{Z}(g) = \bigoplus_{\gamma \in \Gamma} \mathbf{X}(\gamma) \otimes \mathbf{Y}(\gamma^{-1} g)$$

for all $g \in G$, *and*

$$\rho_k^{\mathbf{Z}}(g) = \bigoplus_{\gamma \in \Gamma} \mathrm{Id}_{\mathbf{X}(\gamma)} \otimes \rho_k^{\mathbf{Y}}(\gamma^{-1} g).$$

for all $k \in H$, $g \in G$.

Proof. By Lemma 4.16,

$$\xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{Y})) \circ \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{X}))(\delta_H) = \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{Y}))(\mathbf{X}).$$

Let us first compute

$$\mathrm{Ind}_H^G(T(\mathbf{Y}))(\mathbf{X}) \in \mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G(\mathsf{Mat}).$$

If we denote $\mathbf{V} = \mathrm{Ind}_H^G(T(\mathbf{Y}))(\mathbf{X})$ then

$$\begin{aligned}\mathbf{V}(g) &= T(\mathbf{Y})(\mathbf{X}(g)) \\ &= \bigoplus_{i=1}^{\mathbf{X}(g)} \mathbf{Y}\end{aligned}$$

and

$$\begin{aligned}\rho_k^{\mathbf{V}}(g) &= T(\mathbf{Y})\left(\rho_k^{\mathbf{X}}(g)\right) \circ (T(\mathbf{Y})_k)_{\mathbf{X}(gk)} \\ &= T(\mathbf{Y})\left(\rho_k^{\mathbf{X}}(g)\right) \circ \left(\bigoplus_{i=1}^{\mathbf{X}(gk)} \lambda_k^{\mathbf{Y}}\right).\end{aligned}$$

We then have

$$\begin{aligned}
\mathbf{Z} &= \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})}(\mathbf{V}) \\
&= \bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathrm{Ind}(\mathsf{Mat})}(\mathbf{V}(\gamma)) \\
&= \bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathrm{Ind}(\mathsf{Mat})}\left[\bigoplus_{i=1}^{\mathbf{X}(\gamma)} \mathbf{Y}\right] \\
&= \bigoplus_{\gamma\in\Gamma} \bigoplus_{i=1}^{\mathbf{X}(\gamma)} \pi_\gamma^{\mathrm{Ind}(\mathsf{Mat})}(\mathbf{Y}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{Z}(g) &= \bigoplus_{\gamma\in\Gamma} \bigoplus_{i=1}^{\mathbf{X}(\gamma)} (\gamma \cdot \mathbf{Y})(g) \\
&= \bigoplus_{\gamma\in\Gamma} \bigoplus_{i=1}^{\mathbf{X}(\gamma)} \mathbf{Y}(\gamma^{-1}g) \\
&= \bigoplus_{\gamma\in\Gamma} \mathbf{X}(\gamma) \otimes \mathbf{Y}(\gamma^{-1}g).
\end{aligned}$$

The $\rho_k^{\mathbf{Z}}$'s are computed in a similar fashion. By definition, we have

$$\rho_k^{\mathbf{Z}} = \bigoplus_{\gamma\in\Gamma} \bigoplus_{i=1}^{\mathbf{X}(\gamma)} \rho_k^{\gamma\cdot\mathbf{Y}}.$$

Therefore,

$$\rho_k^{\mathbf{Z}}(g) = \bigoplus_{\gamma\in\Gamma} \bigoplus_{i=1}^{\mathbf{X}(\gamma)} \rho_k^{\mathbf{Y}}(\gamma^{-1}g).$$

For each γ, the map

$$\bigoplus_{i=1}^{\mathbf{X}(\gamma)} \rho_k^{\mathbf{Y}}(\gamma^{-1}g)$$

is represented by the diagonal matrix

$$\begin{bmatrix} \rho_k^{\mathbf{Y}}(\gamma^{-1}g) & & \\ & \ddots & \\ & & \rho_k^{\mathbf{Y}}(\gamma^{-1}g) \end{bmatrix}$$

which is the Kronecker product of $\mathrm{Id}_{\mathbf{X}(\gamma)}$ with $\rho_k^{\mathbf{Y}}(\gamma^{-1}g)$. Therefore,

$$\rho_k^{\mathbf{Z}}(g) = \bigoplus_{\gamma\in\Gamma} \mathrm{Id}_{\mathbf{X}(\gamma)} \otimes \rho_k^{\mathbf{Y}}(\gamma^{-1}g)$$

as required. $\square$

The object $\mathbf{Z} \in \mathrm{Ind}_H^G(\mathsf{Mat})$ is the underlying object of $\mathbf{X} * \mathbf{Y} \in \mathrm{Ind}_H^G(\mathsf{Mat})^H$, that is,

$$(\mathbf{X} * \mathbf{Y})(g) = \bigoplus_{\gamma\in\Gamma} \mathbf{X}(\gamma) \otimes \mathbf{Y}(\gamma^{-1}g), \quad g \in G$$

and

$$\rho_k^{\mathbf{X}*\mathbf{Y}}(g) = \bigoplus_{\gamma\in\Gamma} \mathrm{Id}_{\mathbf{X}(\gamma)} \otimes \rho_k^{\mathbf{Y}}(\gamma^{-1}g), \quad k \in H, g \in G.$$

To complete the description of $\mathbf{X} * \mathbf{Y}$, we need to compute the $\lambda_k^{\mathbf{X}*\mathbf{Y}}$'s. To aid the computation, let us introduce the notation

$$\begin{aligned} \Theta &= \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{X})) \\ \Phi &= \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{Y})) \end{aligned}$$

and let

$$\begin{aligned} \Psi &= \Phi \circ \Theta \\ &= \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{Y})) \circ \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{X})). \end{aligned}$$

Then the $\lambda_k^{\mathbf{X}*\mathbf{Y}}$'s are the components of the coherence maps

$$\Psi_k : \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Psi \to \Psi \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}$$

at δ_H, that is,

$$\lambda_k^{\mathbf{X}*\mathbf{Y}} = (\Psi_k)_{\delta_H} : \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Psi(\delta_H) \to \Psi \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}(\delta_H).$$

As per the definition of composition of G-intertwiners, these are the composites

$$\pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Phi \circ \Theta(\delta_H) \xrightarrow{(\Phi_k)_{\Theta(\delta_H)}} \Phi \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Theta(\delta_H) \xrightarrow{\Phi(\Theta_k)_{\delta_H}} \Phi \circ \Theta \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}(\delta_H). \qquad (*)$$

To compute this composite, we need to compute the Θ_k's and Φ_k's. These are the maps

$$(\xi_k)_{\mathrm{Ind}_H^G(T(\mathbf{X}))} : \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{X})) \to \begin{array}{l} \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \pi_k^{\mathrm{IndResInd}(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{X})) \\ \quad = \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{X})) \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \end{array}$$

and

$$(\xi_k)_{\mathrm{Ind}_H^G(T(\mathbf{Y}))} : \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{Y})) \to \begin{array}{l} \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \pi_k^{\mathrm{IndResInd}(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{Y})) \\ \quad = \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{Y})) \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \end{array}$$

respectively.

Lemma 4.18. *For each $t \in G$, the coherence map*

$$\xi_t : \pi_t^{\mathrm{Ind}(\mathsf{Mat})} \circ \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \to \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \pi_t^{\mathrm{IndResInd}(\mathsf{Mat})}$$

is defined by

$$(\xi_t)_{\mathbf{W}} := \bigoplus_{\gamma' \in \Gamma} \pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})} \rho_{h_{\gamma'}}^{\mathbf{W}}(t^{-1}\gamma') : \bigoplus_{\gamma' \in \Gamma} \pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})} \circ \pi_{h_{\gamma'}}^{\mathrm{Ind}(\mathsf{Mat})}(\mathbf{W}(t^{-1}\gamma' h_{\gamma'})) \to \bigoplus_{\gamma' \in \Gamma} \pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})}(\mathbf{W}(t^{-1}\gamma'))$$

where $\mathbf{W} \in \mathrm{Ind}_H^G(\mathsf{Mat})$ and the $h_{\gamma'} \in H$ are defined by

$$t\gamma = \gamma' h_{\gamma'}, \quad \gamma \in \Gamma.$$

Proof. In general, for a G-category $\mathcal{W}$ and

$$\xi_{\mathcal{W}} : \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathcal{W}) \to \mathcal{W},$$

the coherence maps $(\xi_t)_{\mathbf{W}} : \pi_t^{\mathcal{W}} \circ \xi_{\mathcal{W}}(\mathbf{W}) \to \xi_{\mathcal{W}} \circ \pi_t^{\mathrm{Ind}(\mathcal{W})}(\mathbf{W})$ are the composites

$$\begin{aligned}
\pi_t^{\mathcal{W}}\left[\bigoplus_{\gamma\in\Gamma} \pi_\gamma^{\mathcal{W}}(\mathbf{W}(\gamma))\right] &\xrightarrow{\zeta^{\pi_t^{\mathcal{W}}}} \bigoplus_{\gamma\in\Gamma} \pi_t^{\mathcal{W}} \circ \pi_\gamma^{\mathcal{W}}(\mathbf{W}(\gamma)) \\
&\xrightarrow{\oplus \mu_{t,\gamma}^{\mathcal{W}}} \bigoplus_{\gamma\in\Gamma} \pi_{t\gamma}^{\mathcal{W}}(\mathbf{W}(\gamma)) \\
&\qquad\qquad = \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma' h_{\gamma'}}^{\mathcal{W}}(\mathbf{W}(t^{-1}\gamma' h_{\gamma'})) \\
&\xrightarrow{\oplus\left(\mu_{\gamma',h_{\gamma'}}^{\mathcal{W}}\right)^{-1}} \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{W}} \circ \pi_{h_{\gamma'}}^{\mathcal{W}}(\mathbf{W}(t^{-1}\gamma' h_{\gamma'})) \\
&\xrightarrow{\oplus \pi_{\gamma'}^{\mathcal{W}} \rho_{h_{\gamma'}}^{\mathbf{W}}} \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathcal{W}}(\mathbf{W}(t^{-1}\gamma')).
\end{aligned}$$

In the case that $\mathcal{W} = \mathrm{Ind}_H^G(\mathsf{Mat})$, the first three maps are identities because $\pi_t^{\mathrm{Ind}(\mathsf{Mat})}$ commutes with direct sums and $\pi_r^{\mathrm{Ind}(\mathsf{Mat})} \circ \pi_s^{\mathrm{Ind}(\mathsf{Mat})} = \pi_{rs}^{\mathrm{Ind}(\mathsf{Mat})}$ for all $r, s \in G$. Therefore, in this case $(\xi_t)_{\mathbf{W}}$ reduces to

$$\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})} \rho_{h_{\gamma'}}^{\mathbf{W}}(t^{-1}\gamma')$$

as required. $\square$

By Lemma 4.16, we have $\Theta(\delta_H) = \mathbf{X}$ so the map

$$(\Phi_k)_{\Theta(\delta_H)} : \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Phi \circ \Theta(\delta_H) \to \Phi \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Theta(\delta_H)$$

in $(*)$ is the map

$$(\Phi_k)_{\mathbf{X}} : \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Phi(\mathbf{X}) \to \Phi \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}(\mathbf{X}).$$

Lemma 4.19. *With the notation as above,*

$$(\Phi_k)_{\mathbf{X}}(g) = \bigoplus_{\gamma'\in\Gamma} \rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma') \otimes \lambda_{h_{\gamma'}}^{\mathbf{Y}}(\gamma'^{-1}g)$$

where the $h_{\gamma'} \in H$ are defined by

$$k\gamma = \gamma' h_{\gamma'}, \quad \gamma \in \Gamma.$$

Proof. By Lemma 4.17,

$$\Phi(\mathbf{X})(g) = \bigoplus_{\gamma \in \Gamma} \mathbf{X}(\gamma) \otimes \mathbf{Y}(\gamma^{-1} g).$$

Let $\mathbf{V} = \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Phi(\mathbf{X})$, then

$$\mathbf{V}(g) = \bigoplus_{\gamma \in \Gamma} \mathbf{X}(\gamma) \otimes \mathbf{Y}(\gamma^{-1} k^{-1} g).$$

For each $\gamma \in \Gamma$, we have

$$k\gamma = \gamma' h_{\gamma'}$$

for some γ' and $h_{\gamma'} \in H$ that depend on γ. Therefore, we have

$$\begin{aligned}
\mathbf{V}(g) &= \bigoplus_{\gamma \in \Gamma} \mathbf{X}(\gamma) \otimes \mathbf{Y}(\gamma^{-1} k^{-1} g) \\
&= \bigoplus_{\gamma' \in \Gamma} \mathbf{X}(k^{-1}\gamma' h_{\gamma'}) \otimes \mathbf{Y}(h_{\gamma'}^{-1} \gamma'^{-1} g) \\
&= \bigoplus_{\gamma' \in \Gamma} \bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})} \mathbf{Y}(h_{\gamma'}^{-1} \gamma'^{-1} g).
\end{aligned}$$

Furthermore, if $\mathbf{W} = \Phi \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}(\mathbf{X})$, then

$$\begin{aligned}
\mathbf{W}(g) &= \bigoplus_{\gamma \in \Gamma} \mathbf{X}(k^{-1}\gamma) \otimes \mathbf{Y}(\gamma^{-1} g) \\
&= \bigoplus_{\gamma \in \Gamma} \bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma)} \mathbf{Y}(\gamma^{-1} g).
\end{aligned}$$

Now,

$$(\Phi_k)_{\mathbf{X}} = (\xi_k)_{\mathrm{Ind}_H^G(T(\mathbf{Y}))(\mathbf{X})} : \underbrace{\pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Phi(\mathbf{X})}_{= \mathbf{V}} \to \underbrace{\Phi \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}(\mathbf{X})}_{= \mathbf{W}} .$$

Let us denote $\mathbf{U} = \mathrm{Ind}_H^G(T(\mathbf{Y}))(\mathbf{X}) \in \mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G(\mathsf{Mat})$. Then by Lemma 4.18,

$$\begin{aligned}
(\Phi_k)_{\mathbf{X}} &= (\xi_k)_{\mathrm{Ind}_H^G(T(\mathbf{Y}))(\mathbf{X})} \\
&= (\xi_k)_{\mathbf{U}} \\
&= \bigoplus_{\gamma' \in \Gamma} \pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})} \rho_{h_{\gamma'}}^{\mathbf{U}} (k^{-1}\gamma').
\end{aligned}$$

By definition

$$\begin{aligned}\rho_k^{\mathbf{U}}(g) &= T(\mathbf{Y})\left(\rho_k^{\mathbf{X}}(g)\right) \circ (T(\mathbf{Y})_k)_{\mathbf{X}(gk)}\\ &= T(\mathbf{Y})\left(\rho_k^{\mathbf{X}}(g)\right) \circ \left(\bigoplus_{i=1}^{\mathbf{X}(gk)} \lambda_k^{\mathbf{Y}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})} \rho_{h_{\gamma'}}^{\mathbf{U}}(k^{-1}\gamma') &= \bigoplus_{\gamma'\in\Gamma} \pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})} \left[T(\mathbf{Y})\left(\rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma')\right) \circ \left(\bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})} \lambda_{h_{\gamma'}}^{\mathbf{Y}}\right)\right]\\ &= \bigoplus_{\gamma'\in\Gamma}\left[\pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})}\left[T(\mathbf{Y})\left(\rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma')\right)\right] \circ \left(\bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})} \pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})}\lambda_{h_{\gamma'}}^{\mathbf{Y}}\right)\right].\end{aligned} \tag{1}$$

Let us first consider the term

$$\pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})}\left[T(\mathbf{Y})\left(\rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma')\right)\right].$$

We note that

$$\rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma') : \mathbf{X}(k^{-1}\gamma' h_{\gamma'}) \to \mathbf{X}(k^{-1}\gamma')$$

is a matrix $(\rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma')_{ij})$ and

$$T(\mathbf{Y})\left(\rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma')\right) : \bigoplus_{j=1}^{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})} \mathbf{Y} \to \bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma')} \mathbf{Y}$$

is (represented by) the matrix (a_{ij}) defined by

$$a_{ij} := \rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma')_{ij} \cdot \mathrm{Id}_{\mathbf{Y}}.$$

Therefore,

$$\pi_{\gamma'}^{\mathrm{Ind}(\mathsf{Mat})}\left[T(\mathbf{Y})\left(\rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma')\right)\right](g) = \left(\rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma')_{ij} \cdot \mathrm{Id}_{\mathbf{Y}(\gamma'^{-1}g)}\right).$$

This is the Kronecker product of $\rho^{\mathbf{X}}_{h_{\gamma'}}(k^{-1}\gamma')$ with $\mathrm{Id}_{\mathbf{Y}(\gamma'^{-1}g)}$ and hence

$$\pi^{\mathrm{Ind}(\mathsf{Mat})}_{\gamma'}\left[T(\mathbf{Y})\left(\rho^{\mathbf{X}}_{h_{\gamma'}}(k^{-1}\gamma')\right)\right](g) = \rho^{\mathbf{X}}_{h_{\gamma'}}(k^{-1}\gamma') \otimes \mathrm{Id}_{\mathbf{Y}(\gamma'^{-1}g)}. \tag{2}$$

Next we consider the term

$$\bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})} \pi^{\mathrm{Ind}(\mathsf{Mat})}_{\gamma'}\lambda^{\mathbf{Y}}_{h_{\gamma'}}.$$

This is (represented by) the diagonal matrix

$$\begin{bmatrix} \pi^{\mathrm{Ind}(\mathsf{Mat})}_{\gamma'}\lambda^{\mathbf{Y}}_{h_{\gamma'}} & & \\ & \ddots & \\ & & \pi^{\mathrm{Ind}(\mathsf{Mat})}_{\gamma'}\lambda^{\mathbf{Y}}_{h_{\gamma'}} \end{bmatrix}$$

Hence

$$\left(\bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})} \pi^{\mathrm{Ind}(\mathsf{Mat})}_{\gamma'}\lambda^{\mathbf{Y}}_{h_{\gamma'}}\right)(g) = \begin{bmatrix} \lambda^{\mathbf{Y}}_{h_{\gamma'}}(\gamma'^{-1}g) & & \\ & \ddots & \\ & & \lambda^{\mathbf{Y}}_{h_{\gamma'}}(\gamma'^{-1}g) \end{bmatrix}$$

which is the Kronecker product of $\mathrm{Id}_{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})}$ with $\lambda^{\mathbf{Y}}_{h_{\gamma'}}(\gamma'^{-1}g)$ and so

$$\left(\bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})} \pi^{\mathrm{Ind}(\mathsf{Mat})}_{\gamma'}\lambda^{\mathbf{Y}}_{h_{\gamma'}}\right)(g) = \mathrm{Id}_{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})} \otimes \lambda^{\mathbf{Y}}_{h_{\gamma'}}(\gamma'^{-1}g). \tag{3}$$

Substituting (2) and (3) into (1) yields

$$\begin{aligned}
(\Phi_k)_{\mathbf{X}}(g) &= \bigoplus_{\gamma'\in\Gamma} \pi^{\mathrm{Ind}(\mathsf{Mat})}_{\gamma'}\rho^{\mathbf{U}}_{h_{\gamma'}}(k^{-1}\gamma')(g) \\
&= \bigoplus_{\gamma'\in\Gamma}\left[\pi^{\mathrm{Ind}(\mathsf{Mat})}_{\gamma'}\left[T(\mathbf{Y})\left(\rho^{\mathbf{X}}_{h_{\gamma'}}(k^{-1}\gamma')\right)\right](g) \circ \left(\bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})} \pi^{\mathrm{Ind}(\mathsf{Mat})}_{\gamma'}\lambda^{\mathbf{Y}}_{h_{\gamma'}}(g)\right)\right] \\
&= \bigoplus_{\gamma'\in\Gamma}\left[\left(\rho^{\mathbf{X}}_{h_{\gamma'}}(k^{-1}\gamma') \otimes \mathrm{Id}_{\mathbf{Y}(\gamma'^{-1}g)}\right) \circ \left(\mathrm{Id}_{\mathbf{X}(k^{-1}\gamma' h_{\gamma'})} \otimes \lambda^{\mathbf{Y}}_{h_{\gamma'}}(\gamma'^{-1}g)\right)\right] \\
&= \bigoplus_{\gamma'\in\Gamma} \rho^{\mathbf{X}}_{h_{\gamma'}}(k^{-1}\gamma') \otimes \lambda^{\mathbf{Y}}_{h_{\gamma'}}(\gamma'^{-1}g)
\end{aligned}$$

as required. $\square$

We also need to compute the map

$$\Phi \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Theta(\delta_H) \xrightarrow{\Phi(\Theta_k)_{\delta_H}} \Phi \circ \Theta \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}(\delta_H)$$

in $(*)$.

Lemma 4.20. *With the notation as above,*

$$\Phi(\Theta_k)_{\delta_H}(g) = \bigoplus_{\gamma \in \Gamma} \lambda_k^{\mathbf{X}}(\gamma) \otimes \mathrm{Id}_{\mathbf{Y}(\gamma^{-1}g)}$$

for all $k \in H$, $g \in G$.

Proof. Let us denote

$$\mathbf{V} = \Phi \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})} \circ \Theta(\delta_H)$$

and

$$\mathbf{W} = \Phi \circ \Theta \circ \pi_k^{\mathrm{Ind}(\mathsf{Mat})}(\delta_H).$$

Then

$$\mathbf{V} = \bigoplus_{\gamma \in \Gamma} \bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma)} \pi_\gamma^{\mathrm{Ind}(\mathsf{Mat})} \mathbf{Y}$$

so that

$$\begin{aligned}
\mathbf{V}(g) &= \bigoplus_{\gamma \in \Gamma} \bigoplus_{i=1}^{\mathbf{X}(k^{-1}\gamma)} \mathbf{Y}(\gamma^{-1}g) \\
&= \bigoplus_{\gamma \in \Gamma} \mathbf{X}(k^{-1}\gamma) \otimes \mathbf{Y}(\gamma^{-1}g)
\end{aligned}$$

and

$$\mathbf{W} = \bigoplus_{\gamma \in \Gamma} \bigoplus_{i=1}^{\mathbf{X}(\gamma)} \pi_\gamma^{\mathrm{Ind}(\mathsf{Mat})} \mathbf{Y}$$

so that

$$\begin{aligned}\mathbf{W}(g) &= \bigoplus_{\gamma\in\Gamma}\bigoplus_{i=1}^{\mathbf{X}(\gamma)}\mathbf{Y}(\gamma^{-1}g)\\ &= \bigoplus_{\gamma\in\Gamma}\mathbf{X}(\gamma)\otimes\mathbf{Y}(\gamma^{-1}g).\end{aligned}$$

The same argument as in Lemma 4.19 with Θ_k in place of Φ_k and δ_H in place of $\mathbf{X}$ shows that

$$(\Theta_k)_{\delta_H} = \lambda_k^{\mathbf{X}}.$$

Therefore,

$$\Phi(\Theta_k)_{\delta_H} = \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})}\circ \mathrm{Ind}_H^G(\Theta^{\mathbf{Y}})(\lambda_k^{\mathbf{X}}).$$

Then, by first using the definition of $\xi_{\mathrm{Ind}_H^G(\mathsf{Mat})}$ and then of $\mathrm{Ind}_H^G(T(\mathbf{Y}))$, we have

$$\begin{aligned}\xi_{\mathrm{Ind}_H^G(\mathsf{Mat})}\circ \mathrm{Ind}_H^G(T(\mathbf{Y}))(\lambda_k^{\mathbf{X}}) &= \bigoplus_{\gamma\in\Gamma}\pi_\gamma^{\mathrm{Ind}(\mathsf{Mat})}\left[\mathrm{Ind}_H^G(T(\mathbf{Y}))(\lambda_k^{\mathbf{X}})(\gamma)\right]\\ &= \bigoplus_{\gamma\in\Gamma}\pi_\gamma^{\mathrm{Ind}(\mathsf{Mat})}\left[T(\mathbf{Y})(\lambda_k^{\mathbf{X}}(\gamma))\right]. \qquad (4)\end{aligned}$$

Now,

$$\lambda_k^{\mathbf{X}}(\gamma) : \mathbf{X}(k^{-1}\gamma)\to\mathbf{X}(\gamma)$$

is a matrix $(\lambda_k^{\mathbf{X}}(\gamma)_{ij})$ and

$$T(\mathbf{Y})(\lambda_k^{\mathbf{X}}(\gamma)) : \bigoplus_{j=1}^{\mathbf{X}(k^{-1}\gamma)}\mathbf{Y}\to\bigoplus_{i=1}^{\mathbf{X}(\gamma)}\mathbf{Y}$$

is (represented by) the matrix $(\lambda_k^{\mathbf{X}}(\gamma)_{ij}\cdot \mathrm{Id}_{\mathbf{Y}})$. Therefore,

$$\pi_\gamma^{\mathrm{Ind}(\mathsf{Mat})}\left[T(\mathbf{Y})(\lambda_k^{\mathbf{X}}(\gamma))\right](g) = \left(\lambda_k^{\mathbf{X}}\left(\gamma\right)_{ij}\cdot \mathrm{Id}_{\mathbf{Y}(\gamma^{-1}g)}\right)$$

This is the Kronecker product of $\lambda_k^{\mathbf{X}}(\gamma)$ with $\mathrm{Id}_{\mathbf{Y}(\gamma^{-1}g)}$ and hence

$$\pi_\gamma^{\mathrm{Ind}(\mathsf{Mat})}\left[T(\mathbf{Y})(\lambda_k^{\mathbf{X}}(\gamma))\right](g) = \lambda_k^{\mathbf{X}}(\gamma)\otimes \mathrm{Id}_{\mathbf{Y}(\gamma^{-1}g)}.$$

By substituting this expression into (4) and evaluating at $g \in G$, it follows that

$$\begin{aligned}\Phi(\Theta_k)_{\delta_H}(g) &= \xi_{\mathrm{Ind}_H^G(\mathsf{Mat})} \circ \mathrm{Ind}_H^G(T(\mathbf{Y}))(\lambda_k^{\mathbf{X}})(g) \\ &= \bigoplus_{\gamma \in \Gamma} \pi_\gamma^{\mathrm{Ind}(\mathsf{Mat})} \left[T(\mathbf{Y})(\lambda_k^{\mathbf{X}}(\gamma))\right](g) \\ &= \bigoplus_{\gamma \in \Gamma} \lambda_k^{\mathbf{X}}(\gamma) \otimes \mathrm{Id}_{\mathbf{Y}(\gamma^{-1}g)}\end{aligned}$$

as required. $\square$

Corollary 4.21. *Let $\mathbf{X}, \mathbf{Y} \in \mathrm{Ind}_H^G(\mathsf{Mat})^H$. Then*

$$\lambda_k^{\mathbf{X}*\mathbf{Y}}(g) = \bigoplus_{\gamma' \in \Gamma} [\lambda_k^{\mathbf{X}}(\gamma') \circ \rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma')] \otimes \lambda_{h_{\gamma'}}^{\mathbf{Y}}(\gamma'^{-1}g)$$

for all $k \in H$ and $g \in G$, where the $h_{\gamma'} \in H$ are determined by

$$k\gamma = \gamma' h_{\gamma'}, \quad \gamma \in \Gamma.$$

Proof. This follows from Lemma 4.19 and Lemma 4.20 since $\lambda_k^{\mathbf{X}*\mathbf{Y}}$ is the composite of the maps determined in these lemmas. $\square$

Combining the results of this section, we arrive at the following explicit description of $\mathbf{X} * \mathbf{Y}$.

Lemma 4.22. *Let $\mathbf{X}, \mathbf{Y} \in \mathrm{Ind}_H^G(\mathsf{Mat})^H$. Then $\mathbf{X} * \mathbf{Y}$ is defined by*

$$(\mathbf{X} * \mathbf{Y})(g) := \bigoplus_{\gamma \in \Gamma} \mathbf{X}(\gamma) \otimes \mathbf{Y}(\gamma^{-1}g), \quad g \in G$$

with

$$\rho_k^{\mathbf{X}*\mathbf{Y}}(g) := \bigoplus_{\gamma \in \Gamma} \mathrm{Id}_{\mathbf{X}(\gamma)} \otimes \rho_k^{\mathbf{Y}}(\gamma^{-1}g), \quad k \in H, \ g \in G$$

and

$$\lambda_k^{\mathbf{X}*\mathbf{Y}}(g) = \bigoplus_{\gamma' \in \Gamma} [\lambda_k^{\mathbf{X}}(\gamma') \circ \rho_{h_{\gamma'}}^{\mathbf{X}}(k^{-1}\gamma')] \otimes \lambda_{h_{\gamma'}}^{\mathbf{Y}}(\gamma'^{-1}g), \quad k \in H, \ g \in G$$

where the $h_{\gamma'} \in H$ are determined by

$$k\gamma = \gamma' h_{\gamma'}, \quad \gamma \in \Gamma.$$

Although we haven't quite shown it here, exactly the same formulae define a tensor category structure on $\mathrm{Ind}_H^G(\mathsf{hilb})^H$. In this case, the tensor unit $\mathbf{I}$ is defined by

$$\mathbf{I}(g) = \begin{cases} \mathbb{C} & \text{if } g \in H, \\ 0 & \text{otherwise} \end{cases}$$

with $\lambda_k^{\mathbf{I}}(g)$ and $\rho_k^{\mathbf{I}}(g)$ equal to the appropriate identity map for all $k \in H$ and $g \in G$. Thus we arrive at our categorical analogue of the Hecke algebra $\mathcal{H}(G/\!/H)$.

Definition 4.23. *Let (G, H) be a Hecke pair. We define the* ***Hecke category*** $\mathbf{H}(G/\!/H)$ *as the C^*-tensor category whose underlying C^*-category is* $\mathrm{Ind}_H^G(\mathsf{hilb})^H$ *and whose tensor category structure is defined as above.*

Of course, part of the data are the associator and unitors which we haven't explicitly described. We won't need formulae for them so we have omitted writing them down but they can be computed using similar methods to those above, or one can just infer what they must be directly. For example, one can infer what the associator must be by comparing products $(\mathbf{X} * \mathbf{Y}) * \mathbf{Z}$ and $\mathbf{X} * (\mathbf{Y} * \mathbf{Z})$ for $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbf{H}(G/\!/H)$.

We conclude this section by noting that for any G-category $\mathcal{V}$, the C^*-tensor category $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{hilb}), \mathrm{Ind}_H^G(\mathsf{hilb}))$ acts on $\mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathsf{hilb}), \mathcal{V})$ by postcomposition. It follows that $\mathcal{V}^H$ is naturally a $\mathbf{H}(G/\!/H)$-module category, similarly to the way that a space of H-fixed points is a module over the classical Hecke algebra.

4.2.4 Biequivariant Hilbert spaces

In this section, we shall describe biequivariant Hilbert spaces with a view to identifying elements of $\mathrm{Ind}_H^G(\mathsf{hilb})^H$ concretely with certain biequivariant Hilbert spaces. Using the biequivariant Hilbert space point of view highlights the symmetry of the construction of the objects in the sense that when we constructed an object $\mathbf{X} \in \mathrm{Ind}_H^G(\mathsf{hilb})^H$, the $\rho_k^{\mathbf{X}}$'s came first and then the $\lambda_k^{\mathbf{X}}$'s came afterwards but in the biequivariant Hilbert space picture, both are introduced at the same time and play identical roles.

The biequivariant Hilbert space picture gives us a natural way of defining an involution on the category analogous to the involution on the Hecke algebra. It also lends itself to describing the building blocks of the category in terms of double cosets and unitary representations of certain subgroups of H.

In a sense, the biequivariant Hilbert space picture is a 'two-sided' picture with two representations, λ and ρ of H per object. There is also an equivalent 'one-sided' picture where the objects of study are so called 'equivariant Hilbert spaces'. We shall also look at equivariant Hilbert spaces and use this point of view to compute some examples for specific Hecke pairs (G, H). Both of these points of view are discussed in a more general setting in [2].

In terms of the classical Hecke algebra, the relationship between biequivariant and equivariant pictures is like the relationship between the constructions of $\mathcal{H}(G/\!/H)$ using the vector spaces

$$C_H(G)^{H\times H} = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ has finite support mod } H, \\ f(hgk) = f(g)\ \forall g \in G,\ h, k \in H \end{array} \right\}$$

and

$$C_c(G/H)^H = \left\{ f : G/H \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ has finite support}, \\ f(hgH) = f(gH)\ \forall gH \in G/H,\ h \in H \end{array} \right\}.$$

For reference, we recall that if $f, f' \in C_H(G)^{H\times H}$, the product is given by

$$(f * f')(g) = \sum_{\gamma\in\Gamma} f(\gamma) f'(\gamma^{-1} g) \tag{1}$$

and the involution is given by $f^*(g) = \overline{f(g^{-1})}$. On the other hand, if $f, f' \in C_c(G/H)^H$, the product is given by

$$(f * f')(gH) = \sum_{\gamma\in\Gamma} f(\gamma H) f'(\gamma^{-1} gH) \tag{2}$$

and the involution is given by $f^*(gH) = \overline{f(g^{-1}H)}$. An algebra isomorphism

$$\varphi : C_H(G)^{H\times H} \xrightarrow{\cong} C_c(G/H)^H$$

is given by $\varphi(f)(gH) := f(g)$.

With that in mind, let us begin our discussion of biequivariant Hilbert spaces. In several of the following definitions, G could be any locally compact topological group (some instances of the word 'finite' would need to be replaced by 'compact') but we shall restrict our attention to the discrete cases, i.e. throughout our group G will have the discrete topology.

Definition 4.24. *An H-**biequivariant** $C_0(G)$-**Hilbert space** is a Hilbert space $\mathcal{H}$ together with a nondegenerate $*$-representation $m : C_0(G) \to L(\mathcal{H})$ and two commuting unitary representations $\lambda : H \to U(\mathcal{H})$, $\rho : H \to U(\mathcal{H})$ such that*

$$\lambda_k(f \cdot \xi) = \lambda_k(f) \cdot \lambda_k(\xi), \qquad \rho_k(f \cdot \xi) = \rho_k(f) \cdot \rho_k(\xi) \tag{$*$}$$

for all $k \in H$, $f \in C_0(G)$ and $\xi \in \mathcal{H}$. Here the action on functions is given by left and right translations, respectively. That is, $\lambda_k(f)(g) = f(k^{-1}g)$ and $\rho_k(f)(g) = f(gk)$. The condition $(*)$ is called the ***covariance condition****.*

A ***morphism*** *of H-biequivariant $C_0(G)$-Hilbert spaces is a bounded linear map $\alpha : \mathcal{H} \to \mathcal{K}$ that commutes with the action of $C_0(G)$ and the representations λ and ρ. We shall denote the C^*-category of H-biequivariant $C_0(G)$-Hilbert spaces by $C_0(G)$-H-H-Mod.*

We shall usually just refer to 'biequivariant Hilbert spaces' rather than 'H-biequivariant $C_0(G)$-Hilbert spaces' for brevity. We can also view the commuting representations λ and ρ as a single representation π of $H \times H$ with $\pi_{(k,\ell)} := \lambda_k \circ \rho_\ell$. If there is more than one biequivariant Hilbert space under consideration, we will distinguish the actions with superscripts where necessary, e.g. $\lambda^{\mathcal{H}}$, $\lambda^{\mathcal{K}}$. A classical example of a biequivariant Hilbert space is $\ell^2(G)$ with the action of $C_0(G)$ given by pointwise multiplication and the representations λ and ρ given by left and right translation respectively, that is,

$$\lambda_k(f)(g) := f(k^{-1}g), \quad \rho_k(f)(g) := f(gk), \quad k \in H,\ g \in G,\ f \in \ell^2(G).$$

Definition 4.25. *Let $\mathcal{H} \in C_0(G)$-H-H-Mod. The* ***fibre*** *$\mathcal{H}_g$ of $\mathcal{H}$ at $g \in G$ is defined by*

$$\mathcal{H}_g := \delta_g \cdot \mathcal{H} \subset \mathcal{H},$$

where δ_g denotes the characteristic function of g.

Since the action of $C_0(G)$ is nondegenerate, there is a canonical unitary isomorphism

$$\varphi : \mathcal{H} \xrightarrow{\cong} \bigoplus_{g \in G} \mathcal{H}_g$$

of Hilbert spaces and we shall often implicitly identify these two spaces. We note that in order to specify the action of $C_0(G)$ on a biequivariant Hilbert space, we can specify the fibres since this determines the action of the characteristic functions δ_g, $g \in G$ and hence the action of $C_0(G)$. By construction, the action of $C_0(G)$ preserves the fibres but the representations λ and ρ do not. Since

$$\begin{aligned} \lambda_k(\delta_{k^{-1}g} \cdot \xi) &= \lambda_k(\delta_{k^{-1}g}) \cdot \lambda_l(\xi) \\ &= \delta_g \cdot \lambda_k(\xi), \end{aligned}$$

λ_k maps $\mathcal{H}_{k^{-1}g}$ into $\mathcal{H}_g$. Similarly, ρ_k maps $\mathcal{H}_{gk}$ into $\mathcal{H}_g$. Conversely, requiring that the λ_k and ρ_k map the fibres in this way ensures covariance with respect to the characteristic functions δ_g, $g \in G$ and hence that the actions are covariant with respect to all $f \in C_0(G)$. In what follows it will be useful to have some notation for the restriction of the λ_k and ρ_k to these domains and codomains. We will shortly see that $\mathrm{Ind}_H^G(\mathsf{hilb})^H$ is equivalent to a sub-C^*-category of $C_0(G)$-H-H-Mod and so in keeping with the notation chosen for $\mathrm{Ind}_H^G(\mathsf{hilb})^H$, we shall denote these restrictions by

$$\lambda_k(g) : \mathcal{H}_{k^{-1}g} \to \mathcal{H}_g$$

and

$$\rho_k(g) : \mathcal{H}_{gk} \to \mathcal{H}_g.$$

We note that we can specify the representations λ and ρ by specifying the maps $\lambda_k(g)$ and $\rho_k(g)$ for all $k \in H$, $g \in G$.

Similarly, given a morphism

$$\alpha : \bigoplus_{g \in G} \mathcal{H}_g \to \bigoplus_{g \in G} \mathcal{K}_g,$$

because α commutes with the action of $C_0(G)$, α maps $\mathcal{H}_g$ into $\mathcal{K}_g$. We shall denote this restriction of α by

$$\alpha(g) : \mathcal{H}_g \to \mathcal{K}_g.$$

In order to specify a morphism α of biequivariant Hilbert spaces, we can specify the components $\alpha(g)$, $g \in G$, provided that they define a bounded linear map on the algebraic direct sum.

To identify $\mathrm{Ind}_H^G(\mathsf{hilb})^H$ with a sub-C^*-category of $C_0(G)$-H-H-Mod, we need to define some extra properties that a biequivariant Hilbert space may possess.

Definition 4.26. *Let $\mathcal{H} \in C_0(G)$-H-H-Mod. We define the **support** of $\mathcal{H}$ as the set of all $g \in G$ such that $\mathcal{H}_g \neq 0$. We say that $\mathcal{H}$ has **finite support*** $\bmod H$ *if its support is contained in the union of finitely many left cosets. We shall denote the C^*-category of H-biequivariant $C_0(G)$-Hilbert spaces with finite support* $\bmod H$ *by $C_0(G)$-H-H-Mod_H.*

*We say that $\mathcal{H}$ is **locally finite dimensional** if the fibre $\mathcal{H}_g$ is finite dimensional for each $g \in G$. We denote the C^*-category of locally finite dimensional H-biequivariant $C_0(G)$-Hilbert spaces by $C_0(G)$-H-H-Mod^f.*

We denote the C^*-category of locally finite dimensional H-biequivariant $C_0(G)$-Hilbert spaces with finite support $\bmod H$ *by $C_0(G)$-H-H-Mod^f_H.*

Lemma 4.27. *There is an equivalence of C^*-categories*

$$F : \mathrm{Ind}_H^G(\mathsf{hilb})^H \xrightarrow{\simeq} C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f$$

defined as follows: Given an object $\mathbf{X} \in \mathrm{Ind}_H^G(\mathsf{hilb})^H$, the underlying Hilbert space of $F(\mathbf{X})$ is defined as $\bigoplus_{g\in G} \mathbf{X}(g)$, the fibre at $g \in G$ being the summand $\mathbf{X}(g)$. The unitary representation $\lambda^{F(\mathbf{X})}$ is defined by $\lambda_k^{F(\mathbf{X})}(g) := \lambda_k^{\mathbf{X}}(g)$ and the unitary representation $\rho^{F(\mathbf{X})}$ is defined similarly.

Given a morphism $\alpha : \mathbf{X} \to \mathbf{Y}$, we define $F(\alpha)$ by $F(\alpha)(g) := \alpha(g)$.

Proof. First, let us check that F is well defined on objects. Firstly, $F(\mathbf{X})$ is locally finite dimensional because each $\mathbf{X}(g)$ is and it has finite support mod H because $\mathbf{X}$ does. Next, to see that $\lambda^{F(\mathbf{X})}$ is a unitary representation, we recall that for all $k, \ell \in H$ and $g \in G$, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{X}((k\ell)^{-1}g) \arrow[r, "\lambda_\ell^{\mathbf{X}}(k^{-1}g)"] \arrow[dr, "\lambda_{k\ell}^{\mathbf{X}}(g)"'] & \mathbf{X}(k^{-1}g) \arrow[d, "\lambda_k^{\mathbf{X}}(g)"] \\
 & \mathbf{X}(g)
\end{tikzcd}$$

It follows that $\lambda_k^{F(\mathbf{X})} \circ \lambda_\ell^{F(\mathbf{X})} = \lambda_{k\ell}^{F(\mathbf{X})}$ as required. For similar reasons, $\rho^{F(\mathbf{X})}$ is a unitary representation. To see that the unitary representations $\lambda^{F(\mathbf{X})}$ and $\rho^{F(\mathbf{X})}$ commute, we recall that for all $k, \ell \in H$ and $g \in G$, the following diagram commutes

$$\begin{tikzcd}
\mathbf{X}(k^{-1}g\ell) \arrow[r, "\rho_\ell^{\mathbf{X}}(k^{-1}g)"] \arrow[d, "\lambda_k^{\mathbf{X}}(g\ell)"'] & \mathbf{X}(k^{-1}g) \arrow[d, "\lambda_k^{\mathbf{X}}(g)"] \\
\mathbf{X}(g\ell) \arrow[r, "\rho_\ell^{\mathbf{X}}(g)"] & \mathbf{X}(g)
\end{tikzcd}$$

It follows that $\lambda_k^{F(\mathbf{X})} \circ \rho_\ell^{F(\mathbf{X})} = \rho_k^{F(\mathbf{X})} \circ \lambda_\ell^{F(\mathbf{X})}$ as required. Finally, the covariance condition holds because for all $k \in H$ and $g \in G$, $\lambda_k^{\mathbf{X}}(g)$ maps $\mathbf{X}(k^{-1}g)$ into $\mathbf{X}(g)$ and $\rho_k^{\mathbf{X}}(g)$ maps $\mathbf{X}(gk)$ into $\mathbf{X}(g)$. It follows that

$$\lambda_k^{F(\mathbf{X})}(\delta_{k^{-1}g} \cdot \xi) = \delta_g \cdot \lambda_k^{F(\mathbf{X})}(\xi) = \lambda_k^{F(\mathbf{X})}(\delta_{k^{-1}g}) \cdot \lambda_k^{F(\mathbf{X})}(\xi)$$

and

$$\rho_k^{F(\mathbf{X})}(\delta_{gk} \cdot \xi) = \delta_g \cdot \rho_k^{F(\mathbf{X})}(\xi) = \rho_k^{F(\mathbf{X})}(\delta_{gk}) \cdot \rho_k^{F(\mathbf{X})}(\xi)$$

for all $k \in H$, $g \in G$ and $\xi \in F(\mathbf{X})$, where δ_t denotes the characteristic function of $t \in G$. Therefore, F is well-defined on objects.

Next, we note that F is well-defined on morphisms because a morphism $\alpha : \mathbf{X} \to \mathbf{Y}$ commutes with the trivialisers for $\mathbf{X}$ and $\mathbf{Y}$ and for each $g \in G$, $\alpha(g)$ is a map from $\mathbf{X}(g)$ to $\mathbf{Y}(g)$. Therefore, $F(\alpha)$ commutes with the unitary representations on $F(\mathbf{X})$ and $F(\mathbf{Y})$ as well as the actions of $C_0(G)$.

To see that F is an equivalence, we note that we can define a quasi inverse

$$K : C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f \to \mathrm{Ind}_H^G(\mathsf{hilb})^H$$

to F in a completely analogous way to the way we defined F. That is, given an object $\mathcal{H} \in C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f$ we define $K(\mathcal{H})$ by $K(\mathcal{H})(g) := \mathcal{H}_g$, $\lambda_k^{K(\mathcal{H})}(g) := \lambda_k^{\mathcal{H}}(g)$ and $\rho_k^{K(\mathcal{H})}(g) := \rho_k^{\mathcal{H}}(g)$ for $g \in G$, $k \in H$. Then, given a morphism $\alpha : \mathcal{H} \to \mathcal{K}$ in $C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f$, we define $K(\alpha)(g) := \alpha(g)$, $g \in G$. Essentially, reversing the arguments that show that F is well-defined shows that K is well-defined and it is immediate from the definition that it is a quasi-inverse to F. □

It follows that the formulae of Lemma 4.22 define a tensor category structure on $C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f$. We remark that in this picture, the tensor unit $\mathcal{I}$ has $\bigoplus_{h \in H} \mathbb{C}$ as its underlying Hilbert space and $\lambda^{\mathcal{I}}$ and $\rho^{\mathcal{I}}$ are the left and right regular representations of H respectively. We shall also denote this C^*-tensor category by $\mathbf{H}(G/\!/H)$.

The biequivariant Hilbert space picture gives us a natural way to define an involution on $\mathbf{H}(G/\!/H)$ which is the analogue of the involution on the Hecke algebra $\mathcal{H}(G/\!/H)$. To do this, we first need to define an involution on a C^*-tensor category. For this, we also need to define an anti-tensor functor $F : \mathcal{A} \to \mathcal{B}$ between two C^*-tensor categories. Rather than give a formal definition, we shall just say that this is essentially the same thing as a tensor functor with the exception that $F(A) \otimes F(B) \cong F(B \otimes A)$ for all $A, B \in \mathcal{A}$.

Before we give the definition of an involution on a C^*-tensor category $\mathcal{A}$. We note that there is some potential confusion with the notation we have chosen, in particular, we are going to overload the notation f^* for a morphism $f \in \mathcal{A}$ which is already being used to denote the adjoint of f. The reason we have done this is that the notation mirrors the notation for the involution on a $*$-algebra and in what follows, we won't be considering any involutions of morphisms or adjoints of morphisms so the potentially confusing notation f^* won't appear anywhere outside of the definitions.

Definition 4.28. *An* ***involution*** *on a C^*-tensor category $\mathcal{A}$ is a contravariant, anti-tensor functor*

$$I : \mathcal{A} \to \mathcal{A}$$

such that $I \circ I \cong \mathrm{Id}$. For $A \in \mathcal{A}$, we write A^* rather than $I(A)$, similarly, given a morphism f we write f^* rather than $I(f)$.

Before we proceed, we remark that we haven't actually fully checked that the 'involution' we are about to define on $\mathbf{H}(G/\!/H)$ satisfies the above definition completely although we are confident that it does. Instead, we shall just prove the results that we require. In order to define our involution, let us first introduce some notation for dual Hilbert spaces and contragredient representations.

Definition 4.29. *Let $\mathcal{H}$ be a Hilbert space, we denote the dual space of $\mathcal{H}$ (i.e. the Hilbert space of continuous linear functionals on $\mathcal{H}$) by $\overline{\mathcal{H}}$.*

Given a unitary representation π of H on $\mathcal{H}$, the ***contragredient representation*** *$\overline{\pi}$ of H on $\overline{\mathcal{H}}$ is the unitary representation defined by*

$$\overline{\pi_k}(\varphi)(\xi) := \varphi(\pi_{k^{-1}}(\xi)), \quad \varphi \in \overline{\mathcal{H}},\ \xi \in \mathcal{H}.$$

We can now define the involution on $\mathbf{H}(G/\!/H)$.

Definition 4.30. *Let $\mathcal{H} \in \mathbf{H}(G/\!/H)$. We define $\mathcal{H}^* \in \mathbf{H}(G/\!/H)$ as follows: The underlying Hilbert space is $\overline{\mathcal{H}}$ with the fibres defined by $\mathcal{H}^*(g) := \overline{\mathcal{H}(g^{-1})}$, $g \in G$ (more precisely, the subspace of $\overline{\mathcal{H}}$ isomorphic to $\overline{\mathcal{H}(g^{-1})}$). The unitary representations of H are defined by $\lambda^{\mathcal{H}^*} := \overline{\rho^{\mathcal{H}}}$ and $\rho^{\mathcal{H}^*} := \overline{\lambda^{\mathcal{H}}}$.*

Given a morphism $\alpha : \mathcal{H} \to \mathcal{K}$ in $\mathbf{H}(G/\!/H)$, we define $\alpha^* : \mathcal{K}^* \to \mathcal{H}^*$ by

$$\alpha^*(\varphi)(\xi) := \varphi(\alpha(\xi)), \quad \varphi \in \overline{\mathcal{K}},\ \xi \in \mathcal{H}.$$

Let us also show that $\mathcal{H}^*$ satisfies the covariance condition so that the involution is well-defined on objects. Before doing this, we remark that once this is done, the fact that α^* commutes with the unitary representations on $\mathcal{K}^*$ and $\mathcal{H}^*$ follows from the fact that α commutes with the unitary representations on $\mathcal{K}$ and $\mathcal{H}$, so α^* is indeed a morphism of biequivariant Hilbert spaces.

Lemma 4.31. *Let $\mathcal{H} \in \mathbf{H}(G/\!/H)$. Then $\mathcal{H}^*$, defined as in 4.30, satisfies the covariance condition.*

Proof. We need to show that for all $k \in H$ and $g \in G$, $\lambda_k^{\mathcal{H}^*}$ maps $\mathcal{H}^*(k^{-1}g)$ into $\mathcal{H}^*(g)$ and $\rho_k^{\mathcal{H}^*}$ maps $\mathcal{H}^*(gk)$ into $\mathcal{H}^*(g)$.

To show that $\lambda_k^{\mathcal{H}^*}$ maps $\mathcal{H}^*(k^{-1}g)$ into $\mathcal{H}^*(g)$, we need to show that $\overline{\rho_k^{\mathcal{H}}}$ maps $\overline{\mathcal{H}(g^{-1}k)}$ into $\overline{\mathcal{H}(g^{-1})}$. To that end, let $\varphi \in \overline{\mathcal{H}(g^{-1}k)}$. Then $\varphi = \langle -, \xi \rangle$ for some $\xi \in \mathcal{H}(g^{-1}k)$ and hence

$$\begin{aligned}
\overline{\rho_k^{\mathcal{H}}}(\varphi) &= \overline{\rho_k^{\mathcal{H}}}(\langle -, \xi \rangle) \\
&= \langle \rho_{k^{-1}}^{\mathcal{H}}(-), \xi \rangle \\
&= \langle -, \rho_k^{\mathcal{H}}(\xi) \rangle.
\end{aligned}$$

Since $\rho_k^{\mathcal{H}}$ maps $\mathcal{H}(g^{-1}k)$ into $\mathcal{H}(g^{-1})$, we have $\rho_k^{\mathcal{H}}(\xi) \in \mathcal{H}(g^{-1})$. It follows that $\overline{\rho_k^{\mathcal{H}}}(\varphi) \in \overline{\mathcal{H}(g^{-1})}$ as required.

Similarly, to show that $\rho_k^{\mathcal{H}^*}$ maps $\mathcal{H}^*(gk)$ into $\mathcal{H}^*(g)$, we need to show that $\overline{\lambda_k^{\mathcal{H}}}$ maps $\overline{\mathcal{H}(k^{-1}g^{-1})}$ into $\overline{\mathcal{H}(g^{-1})}$. The proof is identical to the $\lambda_k^{\mathcal{H}^*}$ case. $\square$

Given $\mathcal{H} \in \mathbf{H}(G/\!/H)$, it is immediate from the definition that $(\mathcal{H}^*)^* \cong \mathcal{H}$. The other property that we want to prove is that given $\mathcal{H}, \mathcal{K} \in \mathbf{H}(G/\!/H)$, we have $(\mathcal{H}^* * \mathcal{K}^*) \cong (\mathcal{H} * \mathcal{K})^*$. This is the content of the following lemma. In the proof we shall use the fact that one can canonically identify the dual of a direct sum with the direct sum of the duals and the dual of a tensor product with the tensor product of the duals. At certain points, we shall do this implicitly for notational convenience.

Lemma 4.32. *Let $\mathcal{H}, \mathcal{K} \in C_0(G)$-$H$-$H$-$\mathsf{Mod}$. There is a unitary*

$$\beta_{\mathcal{H},\mathcal{K}} : (\mathcal{H}^* * \mathcal{K}^*) \to (\mathcal{H} * \mathcal{K})^*$$

defined as follows: For each $g \in G$, $\beta_{\mathcal{H},\mathcal{K}}(g)$ is the composite

$$\begin{aligned}
\bigoplus_{\gamma' \in \Gamma} \overline{\mathcal{H}(h_{\gamma'}^{-1}\gamma'^{-1}g^{-1})} \otimes \overline{\mathcal{K}(\gamma' h_{\gamma'})} &\xrightarrow{\bigoplus \overline{\lambda_{h_{\gamma'}}^{\mathcal{H}}}(\gamma'^{-1}g^{-1}) \otimes \overline{\rho_{h_{\gamma'}}^{\mathcal{K}}}(\gamma')} \bigoplus_{\gamma' \in \Gamma} \overline{\mathcal{H}(\gamma'^{-1}g^{-1})} \otimes \overline{\mathcal{K}(\gamma')} \\
&\xrightarrow{\cong} \bigoplus_{\gamma' \in \Gamma} \overline{\mathcal{K}(\gamma')} \otimes \overline{\mathcal{H}(\gamma'^{-1}g^{-1})} \\
&\xrightarrow{\cong} \overline{\bigoplus_{\gamma' \in \Gamma} \mathcal{K}(\gamma') \otimes \mathcal{H}(\gamma'^{-1}g^{-1})}
\end{aligned}$$

where the second map is the canonical isomorphism that swaps the tensor factors and the $\gamma' \in \Gamma$ and $h_{\gamma'} \in H$ are determined by

$$g^{-1}\gamma = \gamma' h_{\gamma'}, \quad \gamma \in \Gamma.$$

Proof. We recall that on the one hand,

$$\begin{aligned}(\mathcal{H}^* * \mathcal{K}^*)(g) &= \bigoplus_{\gamma\in\Gamma} \mathcal{H}^*(\gamma) \otimes \mathcal{K}^*(\gamma^{-1}g) \\ &= \bigoplus_{\gamma\in\Gamma} \overline{\mathcal{H}(\gamma^{-1})} \otimes \overline{\mathcal{K}(g^{-1}\gamma)} \\ &= \bigoplus_{\gamma'\in\Gamma} \overline{\mathcal{H}(h_{\gamma'}^{-1}\gamma'^{-1}g^{-1})} \otimes \overline{\mathcal{K}(\gamma' h_{\gamma'})}\end{aligned}$$

where the $\gamma' \in \Gamma$ and $h_{\gamma'} \in H$ are determined by

$$g^{-1}\gamma = \gamma' h_{\gamma'}, \quad \gamma \in \Gamma.$$

On the other hand,

$$\begin{aligned}(\mathcal{K} * \mathcal{H})^*(g) &= \overline{(\mathcal{K} * \mathcal{H})(g^{-1})} \\ &= \overline{\bigoplus_{\gamma'\in\Gamma} \mathcal{K}(\gamma') \otimes \mathcal{H}(\gamma'^{-1}g^{-1})} \\ &\cong \bigoplus_{\gamma'\in\Gamma} \overline{\mathcal{K}(\gamma')} \otimes \overline{\mathcal{H}(\gamma'^{-1}g^{-1})} \\ &\cong \bigoplus_{\gamma'\in\Gamma} \overline{\mathcal{H}(\gamma'^{-1}g^{-1})} \otimes \overline{\mathcal{K}(\gamma')}\end{aligned}$$

so $\beta_{\mathcal{H},\mathcal{K}}(g)$ maps $(\mathcal{H}^* * \mathcal{K}^*)(g)$ into $(\mathcal{K} * \mathcal{H})^*(g)$. To show that $\beta_{\mathcal{H},\mathcal{K}}$ is a morphism of biequivariant Hilbert spaces, we need to show that it commutes with the unitary representations on $\mathcal{H}^* * \mathcal{K}^*$ and $(\mathcal{K} * \mathcal{H})^*$. First, let us show that $\lambda_k^{(\mathcal{K}*\mathcal{H})^*} \circ \beta_{\mathcal{H},\mathcal{K}} = \beta_{\mathcal{H},\mathcal{K}} \circ \lambda_k^{\mathcal{H}^* * \mathcal{K}^*}$ for all $k \in H$. To do this, we need to show that for all $g \in G$, the following diagram commutes

$$\begin{tikzcd} (\mathcal{H}^* * \mathcal{K}^*)(k^{-1}g) \arrow[r, "(\beta_{\mathcal{H},\mathcal{K}})(k^{-1}g)"] \arrow[d, "\lambda_k^{\mathcal{H}^* * \mathcal{K}^*}(g)"'] & (\mathcal{K} * \mathcal{H})^*(k^{-1}g) \arrow[d, "\lambda_k^{(\mathcal{K}*\mathcal{H})^*}(g)"] \\ (\mathcal{H}^* * \mathcal{K}^*)(g) \arrow[r, "(\beta_{\mathcal{H},\mathcal{K}})(g)"] & (\mathcal{K} * \mathcal{H})^*(g) \end{tikzcd} \qquad (*)$$

We recall that

$$
\begin{aligned}
\lambda_k^{\mathcal{H}^* * \mathcal{K}^*}(g) &= \bigoplus_{\gamma' \in \Gamma} [\lambda_k^{\mathcal{H}^*}(\gamma') \circ \rho_{\ell_{\gamma'}}^{\mathcal{H}^*}(k^{-1}\gamma')] \otimes \lambda_{\ell_{\gamma'}}^{\mathcal{K}^*}(\gamma'^{-1}g) \\
&= \bigoplus_{\gamma' \in \Gamma} [\overline{\rho_k^{\mathcal{H}}}(\gamma'^{-1}) \circ \overline{\lambda_{\ell_{\gamma'}}^{\mathcal{H}}}(\gamma'^{-1}k)] \otimes \overline{\rho_{\ell_{\gamma'}}^{\mathcal{K}}}(g^{-1}\gamma')
\end{aligned}
$$

where the $\ell_{\gamma'} \in H$ are determined by

$$
k\gamma = \gamma' \ell_{\gamma'}, \quad \gamma \in \Gamma.
$$

and

$$
\begin{aligned}
\lambda_k^{(\mathcal{K} * \mathcal{H})^*}(g) &= \overline{\rho_k^{\mathcal{K} * \mathcal{H}}}(g^{-1}) \\
&= \bigoplus_{\gamma \in \Gamma} \mathrm{Id}_{\overline{\mathcal{K}(\gamma)}} \otimes \overline{\rho_k^{\mathcal{H}}}(\gamma^{-1}g^{-1}).
\end{aligned}
$$

Therefore, if we consider a typical summand, showing that $(*)$ commutes essentially boils down to showing that the following diagram commutes

$$
\begin{tikzcd}
\begin{array}{c} \overline{\mathcal{H}(\gamma^{-1})} \otimes \overline{\mathcal{K}(g^{-1}k\gamma)} \\ = \overline{\mathcal{H}(\ell_{\gamma'}^{-1} h_{\gamma''}^{-1} \gamma''^{-1} g^{-1} k)} \otimes \overline{\mathcal{K}(\gamma'' h_{\gamma''} \ell_{\gamma'})} \\ = \overline{\mathcal{H}(\ell_{\gamma'}^{-1} \gamma'^{-1} k)} \otimes \overline{\mathcal{K}(g^{-1}\gamma' \ell_{\gamma'})} \end{array} \arrow[r, "\overline{\lambda_{h_{\gamma''}\ell_{\gamma'}}^{\mathcal{H}}}(\gamma''^{-1}g^{-1}k) \otimes \overline{\rho_{h_{\gamma''}\ell_{\gamma'}}^{\mathcal{K}}}(\gamma'')"] \arrow[d, "{[\overline{\rho_k^{\mathcal{H}}(\gamma'^{-1})} \circ \overline{\lambda_{\ell_{\gamma'}}^{\mathcal{H}}}(\gamma'^{-1}k)] \otimes \overline{\rho_{\ell_{\gamma'}}^{\mathcal{K}}}(g^{-1}\gamma')}"'] & \overline{\mathcal{H}(\gamma''^{-1}g^{-1}k)} \otimes \overline{\mathcal{K}(\gamma'')} \arrow[d, "\overline{\rho_k^{\mathcal{H}}}(\gamma''^{-1}g^{-1}) \otimes \mathrm{Id}"] \\
\begin{array}{c} \overline{\mathcal{H}(\gamma'^{-1})} \otimes \overline{\mathcal{K}(g^{-1}\gamma')} \\ = \overline{\mathcal{H}(h_{\gamma''}^{-1} \gamma''^{-1} g^{-1})} \otimes \overline{\mathcal{K}(\gamma'' h_{\gamma''})} \end{array} \arrow[r, "\overline{\lambda_{h_{\gamma''}}^{\mathcal{H}}}(\gamma''^{-1}g^{-1}) \otimes \overline{\rho_{h_{\gamma''}}^{\mathcal{K}}}(\gamma'')"] & \overline{\mathcal{H}(\gamma''^{-1}g^{-1})} \otimes \overline{\mathcal{K}(\gamma'')}
\end{tikzcd}
\qquad (**)
$$

where $\gamma', \gamma'' \in \Gamma$ and $\ell_{\gamma'}, h_{\gamma''} \in H$ are determined by

$$
k\gamma = \gamma' \ell_{\gamma'}, \quad g^{-1}\gamma' = \gamma'' h_{\gamma''}.
$$

Now, the composites in the first tensor factor are equal because

$$
\begin{aligned}
\overline{\lambda_{h_{\gamma''}}^{\mathcal{H}}} \circ \overline{\rho_k^{\mathcal{H}}} \circ \overline{\lambda_{\ell_{\gamma'}}^{\mathcal{H}}} &= \overline{\rho_k^{\mathcal{H}}} \circ \overline{\lambda_{h_{\gamma''}}^{\mathcal{H}}} \circ \overline{\lambda_{\ell_{\gamma'}}^{\mathcal{H}}} \\
&= \overline{\rho_k^{\mathcal{H}}} \circ \overline{\lambda_{h_{\gamma''}\ell_{\gamma'}}^{\mathcal{H}}}
\end{aligned}
$$

and the composites in the second tensor factor are equal because

$$\overline{\rho^{\mathcal{K}}_{h_{\gamma''}}} \circ \overline{\rho^{\mathcal{K}}_{\ell_{\gamma'}}} = \overline{\rho^{\mathcal{K}}_{h_{\gamma''}\ell_{\gamma'}}}.$$

Therefore, $(**)$ commutes and hence so does $(*)$. It follows that

$$\lambda_k^{(\mathcal{K}*\mathcal{H})^*} \circ \beta_{\mathcal{H},\mathcal{K}} = \beta_{\mathcal{H},\mathcal{K}} \circ \lambda_k^{\mathcal{H}^* * \mathcal{K}^*}$$

for all $k \in H$. The proof that $\rho_k^{(\mathcal{K}*\mathcal{H})^*} \circ \beta_{\mathcal{H},\mathcal{K}} = \beta_{\mathcal{H},\mathcal{K}} \circ \rho_k^{\mathcal{H}^* * \mathcal{K}^*}$ for all $k \in H$ is similar. Therefore, $\beta_{\mathcal{H},\mathcal{K}}$ is a morphism of biequvariant Hilbert spaces. □

These are all the properties of the involution that we require. To show that we really have defined an involution in the sense of Definition 4.28 one would have to check that the anti-tensor functor axioms hold. At a glance, this appears to be true and it would be surprising if it wasn't although we haven't checked all the details in full.

We also remark that given $\mathcal{H} \in \mathbf{H}(G/\!/H)$, we expect $\mathcal{H}^*$ to be a conjugate object to $\mathcal{H}$ in the sense of [26]. Although we won't go into this in detail, we shall give the definition of conjugate objects followed by a brief discussion on the case of $\mathcal{H}^*$.

Definition 4.33. *Let $\mathcal{A}$ be a C^*-tensor category with tensor unit I, and let $A \in \mathcal{A}$. An object $A^* \in \mathcal{A}$ is said to be **conjugate** to A if there exist morphisms $c_1 : I \to A \otimes A^*$ and $c_2 : I \to A^* \otimes A$, called the **coevaluation maps**, such that the composites*

$$A \xrightarrow{\rho_A^{-1}} A \otimes I \xrightarrow{1_A \otimes c_2} A \otimes (A^* \otimes A) \xrightarrow{\alpha^{-1}_{A,A^*,A}} (A \otimes A^*) \otimes A \xrightarrow{c_1^* \otimes 1_A} I \otimes A \xrightarrow{\lambda_A} A$$

and

$$A^* \xrightarrow{\rho_{A^*}^{-1}} A^* \otimes I \xrightarrow{1_{A^*} \otimes c_1} A^* \otimes (A \otimes A^*) \xrightarrow{\alpha^{-1}_{A^*,A,A^*}} (A^* \otimes A) \otimes A^* \xrightarrow{c_2^* \otimes 1_{A^*}} I \otimes A^* \xrightarrow{\lambda_{A^*}} A^*$$

*are identities. Here c_1^* and c_2^* denote the adjoints of the coevaluation maps, they are called the **evaluation maps**.*

*If every object in $\mathcal{A}$ has a conjugate object, then $\mathcal{A}$ is said to be a C^*-tensor category **with conjugates**, or a **rigid** C^*-tensor category.*

Conjugate objects are also referred to as 'dual objects'.

The category of finite dimensional Hilbert spaces is a classical example of a rigid C^*-tensor category. If $V \in \mathsf{hilb}$, then the dual space $\overline{V}$ is conjugate to V. To define the coevaluation maps, let $\{e_i : i = 1, \ldots, \dim V\}$ be an orthonormal basis for V and let $\{\overline{e}_i\}$ be the dual basis. We then define $c_1 : \mathbb{C} \to V \otimes \overline{V}$ by $c_1(1) := \sum_{i=1}^{\dim V} e_i \otimes \overline{e}_i$ and $c_2 : \mathbb{C} \to \overline{V} \otimes V$ by $c_2(1) := \sum_{i=1}^{\dim V} \overline{e}_i \otimes e_i$. These definitions do not depend on the choice of basis for V. The adjoints of these maps are defined on elementary tensors by $c_1^*(v \otimes \varphi) := \varphi(v)$ and $c_2^*(\varphi \otimes v) := \varphi(v)$, $v \in V$, $\varphi \in \overline{V}$. It is not too hard to check that these maps satisfy the conjugate equations and hence $\overline{V}$ is conjugate to V.

There are analogues of these maps in $\mathbf{H}(G/\!\!/H)$ which should satisfy the conjugate equations. We shall describe the evaluation maps. We recall that for the tensor unit $\mathcal{I} \in \mathbf{H}(G/\!\!/H)$, we have

$$\mathcal{I}(g) = \begin{cases} \mathbb{C} & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Also, given $\mathcal{H} \in \mathbf{H}(G/\!\!/H)$ we have

$$\begin{aligned} (\mathcal{H} * \mathcal{H}^*)(g) &= \bigoplus_{\gamma \in \Gamma} \mathcal{H}(\gamma) \otimes \mathcal{H}^*(\gamma^{-1} g) \\ &= \bigoplus_{\gamma \in \Gamma} \mathcal{H}(\gamma) \otimes \overline{\mathcal{H}(g^{-1}\gamma)}. \end{aligned}$$

One can define a morphism of biequivariant Hilbert spaces $\mathrm{e}_1 : \mathcal{H} * \mathcal{H}^* \to \mathcal{I}$ as follows: First, given $h \in H$ and an elementary tensor $\xi \otimes \varphi \in \mathcal{H}(\gamma) \otimes \overline{\mathcal{H}(h^{-1}\gamma)}$ we define

$$e_1(\xi \otimes \varphi) := \varphi(\lambda^{\mathcal{H}}_{h^{-1}}(\xi)) \in \mathcal{I}(h) = \mathbb{C}.$$

We then extend this definition to $(\mathcal{H} * \mathcal{H}^*)(h)$ in the obvious way. This defines $e_1(h) : (\mathcal{H} * \mathcal{H}^*)(h) \to \mathcal{I}(h)$ for all $h \in H$. Finally, we define $e_1(g) := 0$ for all $g \notin H$. One can then show that this commutes with the unitary representations on $\mathcal{H} * \mathcal{H}^*$ and $\mathcal{I}$ so that e_1 is a morphism of biequivariant Hilbert spaces. The other evaluation map $e_2 : \mathcal{H}^* * \mathcal{H} \to \mathcal{I}$ is defined similarly. The adjoints to these maps $c_1 := e_1^* : \mathcal{I} \to \mathcal{H} * \mathcal{H}^*$ and $c_2 := e_2^* : \mathcal{I} \to \mathcal{H}^* * \mathcal{H}$ should be defined analogously to the hilb case and these maps should satisfy the conjugate equations although we haven't checked the details.

Next, we shall describe the building blocks of the category $\mathbf{H}(G/\!\!/H)$. To that end, let us first fix some notation. We shall denote the category of unitary representations of a group K by $\mathsf{URep}(K)$ and the category of finite dimensional unitary representations by $\mathsf{urep}(K)$. Given a hilbert space V, we shall denote the group of unitary operators on V by $U(V)$. When no confusion shall occur, we shall identify a unitary representation with its underlying Hilbert space, simply writing $V \in \mathsf{URep}(K)$ for example. We shall also need induced representations and we shall work with a direct sum construction. Since we are working with discrete groups, the definition is essentially identical to 3.6 except that we need to take the Hilbert space direct sum rather than the vector space direct sum.

Definition 4.34. *Let K be a discrete group, L a subgroup and $V \in \mathsf{URep}(L)$. We define the* ***induced representation*** $\operatorname{Ind}_L^K(V) \in \mathsf{URep}(K)$ *as follows: We fix a set Σ of coset representatives for K/L and define the underlying Hilbert space as*

$$\bigoplus_{\sigma\in\Sigma} \sigma V$$

where each σV is an isomorphic copy of (the Hilbert space) V. We write the elements of σV as σv where $v \in V$. The action of K on $\operatorname{Ind}_L^K(V)$ is given by

$$k \cdot \sigma v := \sigma'(\ell \cdot v), \quad k \in K$$

where $\ell \in L$ and $\sigma' \in \Sigma$ are determined by $k\sigma = \sigma'\ell$.

Although we chose a set of coset representatives in the definition, a different choice of representatives yields a unitarily equivalent representation. One thinks of the elements of σV as formal translates of elements of V by the element $\sigma \in \Sigma$. It will be convenient to have some slightly different notation for the chosen set of coset representatives Σ. In that regard, we shall write $[\sigma] \in K/L$ to denote that σ is one of our chosen coset representatives. Then, for example, we would write the underlying Hilbert space of $\operatorname{Ind}_L^K(V)$ as $\bigoplus_{[\sigma]\in K/L} \sigma V$ rather than $\bigoplus_{\sigma\in\Sigma} \sigma V$. As in the case of complex representations, we have an induction restriction adjunction

$$\begin{tikzcd} \mathsf{URep}(L) \arrow[r, shift left, "\operatorname{Ind}_L^K"] \arrow[r, phantom, "\perp"] & \mathsf{URep}(K). \arrow[l, shift left, "\operatorname{Res}_L^K"] \end{tikzcd}$$

Now, returning to our Hecke pair (G, H), we shall describe the building blocks of $\mathbf{H}(G/\!/H)$. The subgroups of H defined below will play a central role in the construction.

Definition 4.35. *Given $t \in G$, we define the subgroup H_t of H by*

$$H_t := H \cap tHt^{-1}.$$

Given $t \in G$ and a finite dimensional unitary representation H_t, we can construct a biequivariant Hilbert space supported on the double coset HtH in the following way: We first note that there is a transitive action of $H \times H$ on the double coset HtH given by $(h,k) \cdot x = hxk^{-1}$ and that the stabilizer group of t is

$$\operatorname{Stab}(t) = \{(h,k) \subset H \times H \,|\, h = tkt^{-1}\}.$$

The group H_t is canonically isomorphic to $\mathrm{Stab}(t)$ via the map

$$\begin{aligned} \varphi : H_t &\to \mathrm{Stab}(t) \\ h &\mapsto (h, t^{-1}ht) \end{aligned}$$

Therefore, we can identify H_t with a subgroup of $H\times H$ and we shall do this implicitly in what follows. Then, if $V \in \mathsf{urep}(H_t)$, the representation $\mathrm{Ind}_{H_t}^{H\times H}(V)$ of $H \times H$ can be made into a biequivariant Hilbert space in a natural way. The representations λ and ρ are defined by

$$\lambda_h(\xi) = (h, e) \cdot \xi, \quad \rho_h(\xi) = (e, h) \cdot \xi$$

for $h \in H$, $\xi \in \mathrm{Ind}_{H_t}^{H\times H}(V)$. To define the action of $C_0(G)$, we use the isomorphism

$$\begin{aligned} \psi : (H \times H)/H_t &\xrightarrow{\cong} HtH \\ (h, k)H_t &\mapsto htk^{-1}. \end{aligned}$$

If we construct $\mathrm{Ind}_{H_t}^{H\times H}(V)$ as the direct sum

$$\bigoplus_{[(\alpha,\beta)]\in(H\times H)/H_t} (\alpha, \beta)V,$$

then the fibre at $\alpha t\beta^{-1}$ is the summand $(\alpha, \beta)V$. This gives us an action of $C_0(HtH)$ on $\mathrm{Ind}_{H_t}^{H\times H}(V)$ and defining the fibre at $g \notin HtH$ to be 0 extends the action to $C_0(G)$. It follows from the definition of the action in the induced representation that λ_h maps the fibre at $h^{-1}t$ to the fibre at t and ρ_h maps the fibre at th to the fibre at t for all $t \in G$, $h \in H$ so that the covariance condition is satisfied. We also note that since $V \in \mathsf{urep}(H_t)$, this construction is locally finite dimensional. We shall shortly see that such biequivariant Hilbert spaces are the building blocks of $\mathbf{H}(G/\!/H)$ and hence we shall introduce some notation for them.

Definition 4.36. *Let $V \in \mathsf{urep}(H_t)$. We denote the biequivariant Hilbert space $\mathrm{Ind}_{H_t}^{H\times H}(V)$ by χ_{HtH}^V, where the action of $C_0(G)$ and unitary representations λ and ρ are as described above.*

We remark that the chosen representative t for the double coset HtH is part of the data defining χ_{HtH}^V.

Lemma 4.37. *Let $\mathcal{H} \in C_0(G)$-H-H-Mod_H^f be supported on the double coset HtH. Then $\mathcal{H}$ is isomorphic to χ_{HtH}^V for some $V \in \mathsf{urep}(H_t)$.*

Proof. As a Hilbert space, we define $V := \mathcal{H}_t$. We then define a representation $\omega : H_t \to U(V)$ by

$$\omega_h := (\lambda_h \circ \rho_{t^{-1}ht})|_V, \quad h \in H_t$$

(the '$|_V$' denotes the restriction to the subspace V of $\mathcal{H}$). Since $\rho_{t^{-1}ht}$ maps $\mathcal{H}_t$ into $\mathcal{H}_{h^{-1}t}$ and λ_h maps $\mathcal{H}_{h^{-1}t}$ into $\mathcal{H}_t$, ω_h does map $\mathcal{H}_t$ into itself as required. Given $h, k \in H_t$, we have

$$\begin{aligned}\lambda_h \circ \rho_{t^{-1}ht} \circ \lambda_k \circ \rho_{t^{-1}kt} &= \lambda_h \circ \lambda_k \circ \rho_{t^{-1}ht} \circ \rho_{t^{-1}kt} \\ &= \lambda_{hk} \circ \rho_{t^{-1}hkt}\end{aligned}$$

and hence $\omega_h \circ \omega_k = \omega_{hk}$ so that ω is a representation.

To construct an isomorphism from $\chi^V_{HtH} = \mathrm{Ind}_{H_t}^{H\times H}(V)$ to $\mathcal{H}$, we view $\mathcal{H}$ as a representation π of $H \times H$ with $\pi_{(h,k)} = \lambda_h \circ \rho_k$. Since there is an isomorphism

$$\begin{aligned}\varphi : (H \times H)/H_t &\xrightarrow{\simeq} HtH \\ (\alpha, \beta)H_t &\mapsto \alpha t \beta^{-1}\end{aligned}$$

and $\mathcal{H}$ is supported on HtH, we have a canonical unitary isomorphism

$$\mathcal{H} \cong \bigoplus_{[(\alpha,\beta)]\in(H\times H)/H_t} \mathcal{H}_{\alpha t\beta^{-1}}.$$

Then by the induction-restriction adjunction, the inclusion map $\iota : V \to \mathcal{H}$ in $\mathsf{URep}(H_t)$ corresponds to a map $\tilde{\iota} : \chi^V_{HtH} \to \mathcal{H}$ in $\mathsf{URep}(H \times H)$. If we define $\mathrm{Ind}_{H_t}^{H\times H}(V)$ as the direct sum

$$\bigoplus_{[(\alpha,\beta)]\in(H\times H)/H_t} (\alpha, \beta)V,$$

then $\tilde{\iota}$ is the map

$$\bigoplus_{[(\alpha,\beta)]\in(H\times H)/H_t} \pi_{(\alpha,\beta)}|_V : \bigoplus_{[(\alpha,\beta)]\in(H\times H)/H_t} (\alpha, \beta)V \to \bigoplus_{[(\alpha,\beta)]\in(H\times H)/H_t} \mathcal{H}_{\alpha t\beta^{-1}}$$

This follows from the definition the counit of the induction restriction adjunction and the fact that $\pi_{(\alpha,\beta)} = \lambda_\alpha \circ \rho_\beta$ maps $V = \mathcal{H}_t$ into $\mathcal{H}_{\alpha t\beta^{-1}}$. This map is a unitary since each $\pi_{(\alpha,\beta)}|_V$ is a unitary with inverse $\pi_{(\alpha^{-1},\beta^{-1})}|_{\mathcal{H}_{\alpha t\beta^{-1}}}$. Furthermore, $\tilde{\iota}$ preserves the fibres by construction and hence commutes with the action of $C_0(G)$. Therefore, $\tilde{\iota}$ is an isomorphism of biequivariant Hilbert spaces from χ^V_{HtH} to $\mathcal{H}$. $\square$

Corollary 4.38. *Let $\mathcal{H} \in C_0(G)$-H-H-Mod^f_H. Then $\mathcal{H}$ is isomorphic to a finite direct sum of objects of the form χ^V_{HtH}.*

Proof. This follows from the fact that $\mathcal{H}$ has a canonical decomposition into a finite direct sum of biequievariant Hilbert spaces which are each supported on a single double coset. □

We can also describe the involution on $\mathbf{H}(G/\!/H)$ in terms of these building blocks.

Lemma 4.39. *Let $t \in G$ and $(\pi, V) \in \mathsf{urep}(H_t)$. Then $(\chi^V_{HtH})^* \cong \chi^W_{Ht^{-1}H}$, where $(\omega, W) \in \mathsf{urep}(H_t)$ is the representation with underlying Hilbert space $\overline{V}$ and the action ω given by*

$$\omega_h = \overline{\pi_{tht^{-1}}}, \qquad h \in H_{t^{-1}}.$$

Proof. Since (χ^V_{HtH}) is supported on HtH, $(\chi^V_{HtH})^*$ is supported on $Ht^{-1}H$. The fibre $(\chi^V_{HtH})^*_{t^{-1}}$ is $\overline{(\chi^V_{HtH})_t} = \overline{V}$ and the action ω of $H_{t^{-1}}$ on $\overline{V}$ is given by

$$\begin{aligned}
\omega_h &= (\lambda^{\mathcal{H}^*}_h \circ \rho^{\mathcal{H}^*}_{tht^{-1}})|_{\overline{V}} \\
&= \left(\overline{\rho^{\mathcal{H}}_h} \circ \overline{\lambda^{\mathcal{H}}_{tht^{-1}}}\right)|_{\overline{V}} \\
&= \left(\overline{\lambda^{\mathcal{H}}_{tht^{-1}}} \circ \overline{\rho^{\mathcal{H}}_{t^{-1}(tht^{-1})h}}\right)|_{\overline{V}} \\
&= \overline{\pi_{tht^{-1}}}
\end{aligned}$$

for $h \in H_{t^{-1}}$. The result then follows from Lemma 4.37. □

Definition 4.40. *Given $V \in \mathsf{rep}(H_t)$, we denote the biequivariant Hilbert space $(\chi^V_{HtH})^*$ defined in Lemma 4.39 by $\chi^{\overline{V}^t}_{Ht^{-1}H}$.*

These decompositions will be useful when computing explicit descriptions of the product in $\mathbf{H}(G/\!/H)$ when we come to look at specific examples. Before doing this, we shall compare biequivariant Hilbert spaces with their 'one-sided' counter part.

4.2.5 Equivariant Hilbert spaces

As mentioned above, equivariant Hilbert spaces are in a sense, a 'one-sided' counterpart to biequivariant Hilbert spaces. As we shall see, the category $C_0(G)$-H-H-Mod^f_H is equivalent to a category of equivariant Hilbert spaces with similar properties, and the objects of this category can be decomposed in a similar way to their biequivariant counterparts.

Definition 4.41. *An H-**equivariant** $C_0(G/H)$-**Hilbert space** is a Hilbert space $\mathcal{H}$ together with a nondegenerate $*$-homomorphism $m : C_0(G/H) \to L(\mathcal{H})$ and a unitary representation $\lambda : H \to U(\mathcal{H})$ of H such that*

$$\lambda_h(f \cdot \xi) = \lambda_h(f) \cdot \lambda_h(\xi)$$

for all $f \in C_0(G/H)$ and $\xi \in \mathcal{H}$. Here the action on functions is given by left translation, that is $\lambda_h(f)(tH) = f(h^{-1}tH)$.

*A **morphism** of H-equivariant $C_0(G/H)$-Hilbert spaces is a bounded linear map $\alpha : \mathcal{H} \to \mathcal{K}$ that commutes with the action of $C_0(G/H)$ and the representation λ. We denote the C^*-category of H-equivariant $C_0(G/H)$-Hilbert spaces by $C_0(G/H)$-H-Mod.*

We shall usually just write 'equivariant Hilbert space' rather than 'H-equivariant $C_0(G/H)$-Hilbert space' for brevity. If there is more than one equivariant Hilbert space under consideration, we will distinguish the actions with superscripts where necessary, e.g. $\lambda^{\mathcal{H}}, \lambda^{\mathcal{K}}$.

Definition 4.42. *Let $\mathcal{H} \in C_0(G/H)$-H-Mod. The **fibre** $\mathcal{H}_{tH}$ at $tH \in G/H$ is defined by*

$$\mathcal{H}_{tH} := \delta_{tH} \cdot \mathcal{H} \subset \mathcal{H},$$

where δ_{tH} is the characteristic function of tH.

As with H-biequivariant $C_0(G)$-Hilbert spaces, there is a canonical unitary isomorphism

$$\varphi : \mathcal{H} \xrightarrow{\cong} \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma H}$$

of Hilbert spaces. The action of $C_0(G/H)$ preserves the fibres and λ_h maps $\mathcal{H}_{h^{-1}tH}$ into $\mathcal{H}_{tH}$ due to the equivariance condition. We shall denote this restriction of λ_h by

$$\lambda_h(tH) : \mathcal{H}_{h^{-1}tH} \to \mathcal{H}_{tH}.$$

As with biequivariant Hilbert spaces, we can specify the action of $C_0(G/H)$ by specifying the fibres and ensure that the covariance condition holds by requiring that λ_h maps $\mathcal{H}_{h^{-1}tH}$ into $\mathcal{H}_{tH}$ for all $h \in H$, $t \in G$.

Similarly, given a morphism

$$\alpha : \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma H} \to \bigoplus_{\gamma \in \Gamma} K_{\gamma H},$$

α maps $\mathcal{H}_{\gamma H}$ into $\mathcal{K}_{\gamma H}$ because α commutes with the action of $C_0(G/H)$. We shall denote this restriction of α by

$$\alpha(\gamma H) : \mathcal{H}_{\gamma H} \to \mathcal{K}_{\gamma H}.$$

We can specify such a morphism α by specifying bounded linear maps $\alpha(\gamma H)$ for all $\gamma H \in G/H$, provided they define a bounded linear map on the algebraic direct sum.

Definition 4.43. *Let $\mathcal{H} \in C_0(G/H)$-H-Mod. We define the* ***support*** *of $\mathcal{H}$ as as the set of all $tH \in G/H$ such that $\mathcal{H}_{tH} \neq 0$. We say that $\mathcal{H}$ has* ***finite support*** *if its support is a finite set. We denote the C^*-category of compactly supported H-equivariant $C_0(G/H)$-Hilbert spaces by $C_0(G/H)$-H-Mod_c.*

We say that $\mathcal{H}$ is ***locally finite dimensional*** *if the fibres $\mathcal{H}_{tH}$ are finite dimensional for all $tH \in G/H$. We denote the C^*-category of locally finite dimensional H-equivariant $C_0(G/H)$-Hilbert spaces by $C_0(G/H)$-H-Mod^f.*

We denote the C^*-category of locally finite dimensional H-equivariant $C_0(G/H)$-Hilbert spaces with finite support by $C_0(G/H)$-H-Mod_c^f.

Similarly to biequivariant Hilbert spaces, we can construct an equivariant Hilbert space supported on HtH/H from a finite dimensional unitary representation of the group $H_t = H \cap tHt^{-1}$. The group H acts transitively on the space HtH/H via the action $h \cdot xH = hxH$ and H_t is the stabiliser group of the coset tH. Then given $V \in \mathsf{urep}(H_t)$, the induced representation $\mathrm{Ind}_{H_t}^H(V)$, whose action we shall denote by λ, becomes an equivariant Hilbert space in a natural way. The fibres are defined using the isomorphism

$$\begin{aligned} \varphi : H/H_t &\to HtH/H \\ hH_t &\mapsto htH. \end{aligned}$$

Explicitly, if we write the induced representation as the direct sum

$$\bigoplus_{[\sigma] \in H/H_t} \sigma V,$$

the fibre at σtH is the summand σV. This defines an action of $C_0(HtH/H)$ and defining the fibre at xH to be 0 for $xH \notin HtH$ extends the action to $C_0(G/H)$. It follows from the definition of the action in the induced representation that λ_h maps the fibre at $h^{-1}xH$ into the fibre at xH for all $h \in H$, $xH \in G/H$. Therefore, the covariance condition is satisfied.

Definition 4.44. *Given $V \in \mathsf{urep}(H_t)$, we denote the equivariant Hilbert space $\mathrm{Ind}_{H_t}^H(V)$ by η_{HtH}^V, where the action of $C_0(G/H)$ is as described above.*

We remark that the choice of representative t for HtH is part of the data defining η^V_{HtH}. Similarly to the biequivariant cases, these equivariant Hilbert spaces are the building blocks of $C_0(G/H)$-H-Mod^f_c.

Lemma 4.45. *Let $\mathcal{H} \in C_0(G/H)$-H-Mod^f_c be supported on HtH/H. Then $\mathcal{H}$ is isomorphic to η^V_{HtH} for some $V \in \mathsf{urep}(H_t)$.*

Proof. As a Hilbert space, we define $V := \mathcal{H}_{tH}$. Then $\lambda^{\mathcal{H}}$ restricts to a representation of H_t on V, the covariance condition ensures that $\lambda^{\mathcal{H}}_h$ maps $\mathcal{H}_{tH}$ into $\mathcal{H}_{tH}$ for all $h \in H_t$. To define an isomorphism from η^V_{HtH} to $\mathcal{H}$, we identify $\mathcal{H}$ with the direct sum of its nonzero fibres via the canonical unitary isomorphism

$$\mathcal{H} \cong \bigoplus_{[\sigma]\in H/H_t} \mathcal{H}_{\sigma t H}.$$

The inclusion map $\iota : V \to \mathcal{H}$ in $\mathsf{urep}(H_t)$ corresponds to a map $\tilde{\iota} : \eta^V_{HtH} \to \mathcal{H}$ in $\mathsf{urep}(H)$ via the induction-restriction adjunction. Writing the induced representation as a direct sum, $\tilde{\iota}$ is the map

$$\bigoplus_{[\sigma]\in H/H_t} \lambda^{\mathcal{H}}_\sigma|_V : \bigoplus_{[\sigma]\in H/H_t} \sigma V \to \bigoplus_{[\sigma]\in H/H_t} \mathcal{H}_{\sigma t H}.$$

This follows from the definition of the counit of the induction restriction adjunction and the fact that λ_σ maps $V = \mathcal{H}_{tH}$ into $\mathcal{H}_{\sigma tH}$. This is a unitary with inverse $\oplus_{[\sigma]\in H/H_t} \lambda^{\mathcal{H}}_{\sigma^{-1}}|_{\mathcal{H}_{\sigma tH}}$ which preserves the fibres by construction. Therefore, $\mathcal{H} \cong \eta^V_{HtH}$. $\square$

Corollary 4.46. *Let $\mathcal{H} \in C_0(G/H)$-H-Mod^f_c. Then $\mathcal{H}$ is isomorphic to a finite direct sum of objects of the form η^V_{HtH} where $t \in G$ and $V \in \mathsf{urep}(H_t)$.*

Proof. This follows from the fact that $\mathcal{H}$ has a canonical decomposition into a finite direct sum of equivariant Hilbert spaces, each of which is supported on HtH/H for some $t \in G$. $\square$

Next, we shall show that there is an equivalence of C^*-categories

$$C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}^f_H \simeq C_0(G/H)\text{-}H\text{-}\mathsf{Mod}^f_c.$$

We shall break the proof up into a series of lemmas.

Lemma 4.47. *There is a $*$-functor*

$$F : C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}^f_{G/H} \to C_0(G/H)\text{-}H\text{-}\mathsf{Mod}^f_c$$

defined on objects as follows. We fix a set Γ of coset representatives for G/H. Given $\mathcal{H} \in C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}^f_H$, for each $\gamma \in \Gamma$ we define $\mathcal{K}_{\gamma H} := \mathcal{H}_\gamma$ and define

$$F(\mathcal{H}) = \mathcal{K} := \bigoplus_{\gamma \in \Gamma} \mathcal{K}_{\gamma H},$$

this also determines the action of $C_0(G/H)$. To define the unitary representation $\lambda^{\mathcal{K}}$, let $h \in H$ and $\gamma \in \Gamma$. Then $h^{-1}\gamma H = \gamma' H$ for a unique $\gamma' \in \Gamma$. We define $\lambda^{\mathcal{K}}_h(\gamma H)$ as the composite

$$\begin{array}{c}\mathcal{H}_{\gamma'} \\ = \mathcal{K}_{h^{-1}\gamma H}\end{array} \xrightarrow{\rho^{\mathcal{H}}_{\gamma^{-1}h\gamma'}(h^{-1}\gamma)} \mathcal{H}_{h^{-1}\gamma} \xrightarrow{\lambda^{\mathcal{H}}_h(\gamma)} \begin{array}{c}\mathcal{H}_{\gamma} \\ = \mathcal{K}_{\gamma H}\end{array}$$

Proof. First, let us show that F is well-defined on objects. To that end, consider $\mathcal{H} \in C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}^f_{G/H}$. Then $\mathcal{H}$ has finite support $\bmod H$ and is locally finite dimensional, therefore $\mathcal{K} = F(\mathcal{H})$ has finite support and is locally finite dimensional. The covariance condition also holds by construction.

To show that $\lambda^{\mathcal{K}}$ is a representation, let $h, h' \in H$, let $\gamma \in \Gamma$ and let $h^{-1}\gamma H = \gamma' H$ for $\gamma' \in \Gamma$. We need to show that $\lambda^{\mathcal{K}}_h(\gamma H) \circ \lambda^{\mathcal{K}}_{h'}(\gamma' H) = \lambda^{\mathcal{K}}_{hh'}(\gamma H)$. Firstly, $\lambda^{\mathcal{K}}_h(\gamma H)$ is the composite

$$\begin{array}{c}\mathcal{H}_{\gamma'} \\ = \mathcal{K}_{h^{-1}\gamma H}\end{array} \xrightarrow{\rho^{\mathcal{H}}_{\gamma^{-1}h\gamma'}(h^{-1}\gamma)} \mathcal{H}_{h^{-1}\gamma} \xrightarrow{\lambda^{\mathcal{H}}_h(\gamma)} \begin{array}{c}\mathcal{H}_{\gamma} \\ = \mathcal{K}_{\gamma H}\end{array}$$

Then let $h'^{-1}\gamma' H = \gamma'' H$ so that $\lambda^{\mathcal{K}}_{h'}(\gamma' H)$ is the composite

$$\begin{array}{c}\mathcal{H}_{\gamma''} \\ = \mathcal{K}_{h'^{-1}\gamma' H}\end{array} \xrightarrow{\rho^{\mathcal{H}}_{\gamma'^{-1}h'\gamma''}(h'^{-1}\gamma')} \mathcal{H}_{h'^{-1}\gamma'} \xrightarrow{\lambda^{\mathcal{H}}_{h'}(\gamma')} \begin{array}{c}\mathcal{H}_{\gamma'} \\ = \mathcal{K}_{\gamma' H}\end{array}$$

and $\lambda^{\mathcal{K}}_{hh'}(\gamma H)$ is the composite

$$\begin{array}{c}\mathcal{H}_{\gamma''} \\ = \mathcal{K}_{h'^{-1}\gamma' H} \\ = \mathcal{K}_{(hh')^{-1}\gamma H}\end{array} \xrightarrow{\rho^{\mathcal{H}}_{\gamma^{-1}hh'\gamma''}((hh')^{-1}\gamma)} \mathcal{H}_{(hh')^{-1}\gamma} \xrightarrow{\lambda^{\mathcal{H}}_{hh'}(\gamma)} \begin{array}{c}\mathcal{H}_{\gamma} \\ = \mathcal{K}_{\gamma H}\end{array}$$

If we consider the following diagram

$$
\begin{tikzcd}
\begin{matrix}\mathcal{H}_{\gamma''} \\ = \mathcal{K}_{(hh')^{-1}\gamma H}\end{matrix} \arrow[r, "\rho^{\mathcal{H}}_{\gamma'^{-1}h'\gamma''}(h'^{-1}\gamma')"] \arrow[dr, "\rho^{\mathcal{H}}_{\gamma^{-1}hh'\gamma''}((hh')^{-1}\gamma)"'] & \mathcal{H}_{h'^{-1}\gamma'} \arrow[r, "\lambda^{\mathcal{H}}_{h'}(\gamma')"] \arrow[d, "\rho^{\mathcal{H}}_{\gamma^{-1}h\gamma'}((hh')^{-1}\gamma)"] \arrow[dr, phantom, "\mathrm{(II)}"] & \begin{matrix}\mathcal{H}_{\gamma'} \\ = \mathcal{K}_{h^{-1}\gamma H}\end{matrix} \arrow[d, "\rho^{\mathcal{H}}_{\gamma^{-1}h\gamma'}(h^{-1}\gamma)"] \\
\arrow[ur, phantom, "\mathrm{(I)}"] & \mathcal{H}_{(hh')^{-1}\gamma} \arrow[r, "\lambda^{\mathcal{H}}_{h'}(h^{-1}\gamma)"] \arrow[dr, "\lambda^{\mathcal{H}}_{hh'}(\gamma)"'] & \mathcal{H}_{h^{-1}\gamma} \arrow[d, "\lambda^{\mathcal{H}}_{h}(\gamma)"] \arrow[dl, phantom, "\mathrm{(III)}"] \\
& & \begin{matrix}\mathcal{H}_{\gamma} \\ = \mathcal{K}_{\gamma H}\end{matrix}
\end{tikzcd}
$$

then (I) commutes because $\rho^{\mathcal{H}}$ is a representation, (II) commutes because $\lambda^{\mathcal{H}}$ commutes with $\rho^{\mathcal{H}}$, and (III) commutes because $\lambda^{\mathcal{H}}$ is a representation. Therefore, the whole diagram commutes. The upper right hand composite is $\lambda^{\mathcal{K}}_h(\gamma H)\circ\lambda^{\mathcal{K}}_{h'}(\gamma' H)$ and the lower left hand composite is $\lambda^{\mathcal{K}}_{hh'}(\gamma H)$ so that $\lambda^{\mathcal{K}}_h(\gamma H)\circ\lambda^{\mathcal{K}}_{h'}(\gamma' H) = \lambda^{\mathcal{K}}_{hh'}(\gamma H)$. Therefore, $\lambda^{\mathcal{K}}_h \circ \lambda^{\mathcal{K}}_{h'} = \lambda^{\mathcal{K}}_{hh'}$ and $\lambda^{\mathcal{K}}$ is a representation. This shows that $\mathcal{K}$ is an H-equivariant $C_0(G/H)$-Hilbert space and hence F is well defined on objects.

Next, we need to define F on morphisms. Given $\mathcal{H}, \mathcal{H}' \in C_0(G)$-$H$-$H$-$\mathsf{Mod}^f_H$ and a morphism $\alpha : \mathcal{H} \to \mathcal{H}$, we define $F(\alpha)$ by

$$F(\alpha)(\gamma H) = \alpha(\gamma), \quad \gamma \in \Gamma.$$

$F(\alpha)$ commutes with the action of $C_0(G/H)$ by construction and it commutes with the unitary representations on $F(\mathcal{H})$ and $F(\mathcal{H}')$ because α commutes with the unitary representations on $\mathcal{H}$ and $\mathcal{H}'$. It is then immediate from the definition that F is a $*$-functor. $\square$

Lemma 4.48. *There is a $*$-functor*

$$G : C_0(G/H)\text{-}H\text{-}\mathsf{Mod}^f_c \to C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}^f_H$$

defined on objects in the following way: Given $\mathcal{H} \in C_0(G/H)$-H-Mod^f_c, we define $\mathcal{K}_t := \mathcal{H}_{tH}$ for all $t \in G$ and define

$$G(\mathcal{H}) = \mathcal{K} := \bigoplus_{t\in G} \mathcal{K}_t,$$

this also determines the action of $C_0(G)$ *on* $F(\mathcal{H})$*. The unitary representation* $\lambda^{\mathcal{K}}$ *is defined by*

$$\lambda_h^{\mathcal{K}}(t) := \lambda_h^{\mathcal{H}}(tH) : \begin{array}{c} \mathcal{H}_{h^{-1}tH} \\ = \mathcal{K}_{h^{-1}t} \end{array} \to \begin{array}{c} \mathcal{H}_{tH} \\ = \mathcal{K}_t \end{array}$$

for all $h \in H$*,* $t \in G$*. The unitary representation* $\rho^{\mathcal{K}}$ *is defined by*

$$\rho_h^{\mathcal{K}}(t) := \mathrm{Id}_{\mathcal{H}_{tH}} : \begin{array}{c} \mathcal{H}_{tH} \\ = \mathcal{K}_{th} \end{array} \to \begin{array}{c} \mathcal{H}_{tH} \\ = \mathcal{K}_t \end{array}$$

Proof. First, let us show that G is well-defined on objects. If $\mathcal{H} \in C_0(G/H)\text{-}H\text{-}\mathsf{Mod}_c^f$, then $\mathcal{H}$ has finite support and is locally finite dimensional, hence $\mathcal{K} = G(\mathcal{H})$ has finite support mod H and is locally finite dimensional. Next, $\lambda^{\mathcal{K}}$ is a representation because $\lambda^{\mathcal{H}}$ is and it commutes with $\rho^{\mathcal{K}}$ since $\lambda_h^{\mathcal{K}}(tk) = \lambda_h^{\mathcal{K}}(t)$ for all $h, k \in H$, $t \in G$, and $\rho_\ell^{\mathcal{K}}(s)$ is an identity map for all $\ell \in H$, $s \in G$. Finally, the covariance condition is satisfied by construction and hence $G(\mathcal{H})$ is a biequivariant Hilbert space. Therefore, G is well-defined on objects.

We also need to define G on morphisms. Given a morphism $\alpha : \mathcal{H} \to \mathcal{H}'$ in $C_0(G/H)\text{-}H\text{-}\mathsf{Mod}$, we define $G(\alpha)$ by

$$G(\alpha)(t) := \alpha(tH)$$

for all $t \in G$. Then $G(\alpha)$ preserves the fibres by construction. It intertwines $\rho^{G(\mathcal{H})}$ and $\rho^{G(\mathcal{H}')}$ since $G(\alpha)(th) = G(\alpha)(t)$ for all $h \in H$, $t \in G$, and $\rho_h^{G(\mathcal{H})}(t)$ and $\rho_h^{G(\mathcal{H}')}(t)$ are identity maps. Furthermore, $G(\alpha)$ interwtines $\lambda^{G(\mathcal{H})}$ and $\lambda^{G(\mathcal{H}')}$ because α intertwines $\lambda^{\mathcal{H}}$ and $\lambda^{\mathcal{H}'}$. Therefore, $G(\alpha)$ is a morphism in $C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f$. Finally, it is immediate from the definition that G is a $*$-functor. □

Theorem 4.49. *The* $*$*-functors*

$$F : C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f \to C_0(G/H)\text{-}H\text{-}\mathsf{Mod}_c^f$$

and

$$G : C_0(G/H)\text{-}H\text{-}\mathsf{Mod}_c^f \to C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f$$

defined above are quasi-inverse to one another, that is, $F \circ G \cong \mathrm{Id}$ *and* $G \circ F \cong \mathrm{Id}$*.*

Proof. Let us first construct a unitary natural transformation from the identity functor on $C_0(G/H)\text{-}H\text{-}\mathsf{Mod}_c^f$ to $F \circ G$. Given $\mathcal{H} \in C_0(G/H)\text{-}H\text{-}\mathsf{Mod}_c^f$, $(F \circ G)(\mathcal{H})$ has fibres

$$(F \circ G)(\mathcal{H})_{\gamma H} = G(\mathcal{H})_\gamma = \mathcal{H}_{\gamma H}, \quad \gamma \in \Gamma.$$

Now, if $h \in H$, $\gamma \in \Gamma$ and $h^{-1}\gamma H = \gamma' H$ for $\gamma' \in \Gamma$, the component $\lambda_h^{(F\circ G)(\mathcal{H})}(\gamma H)$ of the representation $\lambda^{(F\circ G)(\mathcal{H})}$ is the composite

$$\begin{matrix} G(\mathcal{H})_{\gamma'} \\ = (F\circ G)(\mathcal{H})_{h^{-1}\gamma H} \end{matrix} \xrightarrow{\rho^{G(\mathcal{H})}_{\gamma^{-1}h\gamma'}(h^{-1}\gamma)} G(\mathcal{H})_{h^{-1}\gamma} \xrightarrow{\lambda_h^{G(\mathcal{H})}(\gamma)} \begin{matrix} G(\mathcal{H})_{\gamma} \\ = (F\circ G)(\mathcal{H})_{\gamma H} \end{matrix}$$

which is just the map

$$\lambda_h^{\mathcal{H}}(\gamma H) : \mathcal{H}_{h^{-1}\gamma H} \to \mathcal{H}_{\gamma H}$$

since $\rho^{G(\mathcal{H})}_{\gamma^{-1}h\gamma'}(h^{-1}\gamma)$ is the identity map and $\lambda_h^{G(\mathcal{H})}(\gamma) = \lambda_h^{\mathcal{H}}(\gamma H)$. Therefore, $(F\circ G)(\mathcal{H})$ is just the canonical decomposition of $\mathcal{H}$ into the direct sum of its fibres and the canonical unitary isomorphisms

$$\eta_{\mathcal{H}} : \mathcal{H} \xrightarrow{\simeq} \bigoplus_{\gamma\in\Gamma} \mathcal{H}_{\gamma H}, \quad \mathcal{H} \in C_0(G/H)\text{-}H\text{-}\mathsf{Mod}_c^f$$

assemble into a unitary natural transformation $\eta : \mathrm{Id} \to F \circ G$.

Next we need to construct a unitary natural transformation from $G \circ F$ to the identity. Let us first describe $(G\circ F)(\mathcal{H})$ for $\mathcal{H} \in C_0(G/H)\text{-}H\text{-}\mathsf{Mod}_c^f$. For each $t \in G$, let us denote the $\gamma \in \Gamma$ such that $tH = \gamma H$ by γ_t. Then the fibre $(G \circ F)(\mathcal{H})_t$ is $\mathcal{H}_{\gamma_t}$ and

$$\begin{aligned} (G \circ F)(\mathcal{H}) &= \bigoplus_{t\in G} (G \circ F)(\mathcal{H})_t \\ &= \bigoplus_{t\in G} \mathcal{H}_{\gamma_t} \end{aligned}$$

Next, given $h \in H$ and $t \in G$, the map $\lambda_h^{(G\circ F)(\mathcal{H})}(t)$ is the composite

$$\begin{matrix} \mathcal{H}_{\gamma_{h^{-1}t}} \\ = (G\circ F)(\mathcal{H})_{h^{-1}t} \end{matrix} \xrightarrow{\rho^{\mathcal{H}}_{\gamma_t^{-1}h\gamma_{h^{-1}t}}(h^{-1}\gamma_t)} \mathcal{H}_{h^{-1}\gamma_t} \xrightarrow{\lambda_h^{\mathcal{H}}(\gamma_t)} \begin{matrix} \mathcal{H}_{\gamma_t} \\ = (G\circ F)(\mathcal{H})_t \end{matrix}$$

Finally, for $t \in G$ and $h \in H$, we have $(G \circ F)(\mathcal{H})_{th} = (G \circ F)(\mathcal{H})_t = \mathcal{H}_{\gamma_t}$ and

$$\rho_h^{(G\circ F)(\mathcal{H})}(t) : (G \circ F)(\mathcal{H})_{th} \to (G \circ F)(\mathcal{H})_t$$

is the identity map.

Now, given $\mathcal{H} \in C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f$, we define a unitary

$$\xi_{\mathcal{H}} : (G \circ F)(\mathcal{H}) \xrightarrow{\simeq} \mathcal{H}$$

by

$$\xi_{\mathcal{H}}(t) := \rho^{\mathcal{H}}_{t^{-1}\gamma_t}(t) : \begin{array}{c} \mathcal{H}_{\gamma_t} \\ = (G \circ F)(\mathcal{H})_t \end{array} \rightarrow \mathcal{H}_t$$

We note that since $tH = \gamma_t H$, $t^{-1}\gamma \in H$ so that $\xi_{\mathcal{H}}(t)$ is well defined.

Let us check that $\xi_{\mathcal{H}}$ is an isomorphism of biequivariant Hilbert spaces. By construction, $\xi_{\mathcal{H}}$ preserves the fibres and hence commutes with the action of $C_0(G)$. To show that $\xi_{\mathcal{H}}$ intertwines the ρ's, we need to show that for all $h \in H$ and $t \in G$, the following diagram commutes

$$\begin{tikzcd}
(G \circ F)(\mathcal{H})_{th} \arrow[r, "\xi_{\mathcal{H}}(th)"] \arrow[d, "\rho_h^{(G\circ F)(\mathcal{H})}(t)"'] & \mathcal{H}_{th} \arrow[d, "\rho_h^{\mathcal{H}}(t)"] \\
(G \circ F)(\mathcal{H})_t \arrow[r, "\xi_{\mathcal{H}}(t)"] & \mathcal{H}_t
\end{tikzcd}$$

By definition, this diagram is

$$\begin{tikzcd}
\begin{array}{c} \mathcal{H}_{\gamma_{th}} \\ = \mathcal{H}_{\gamma_t} \end{array} \arrow[r, "\rho^{\mathcal{H}}_{(th)^{-1}\gamma_t}(th)"] \arrow[d, "\mathrm{Id}"'] & \mathcal{H}_{th} \arrow[d, "\rho_h^{\mathcal{H}}(t)"] \\
\mathcal{H}_{\gamma_t} \arrow[r, "\rho^{\mathcal{H}}_{t^{-1}\gamma_t}(t)"] & \mathcal{H}_t
\end{tikzcd}$$

which commutes because $\rho^{\mathcal{H}}$ is a representation.

To show that $\xi_{\mathcal{H}}$ intertwines the λ's, we need to show that for all $h \in H$ and $t \in G$, the following diagram commutes

$$\begin{tikzcd}
(G \circ F)(\mathcal{H})_{h^{-1}t} \arrow[r, "\xi_{\mathcal{H}}(h^{-1}t)"] \arrow[d, "\lambda_h^{(G\circ F)(\mathcal{H})}(t)"'] & \mathcal{H}_{h^{-1}t} \arrow[d, "\lambda_h^{\mathcal{H}}(t)"] \\
(G \circ F)(\mathcal{H})_t \arrow[r, "\xi_{\mathcal{H}}(t)"] & \mathcal{H}_t
\end{tikzcd} \qquad (*)$$

By definition, this is the diagram

$$\begin{tikzcd}
\mathcal{H}_{\gamma_{h^{-1}t}} \arrow[rr, "\rho^{\mathcal{H}}_{t^{-1}h\gamma_{h^{-1}t}}(h^{-1}t)"] \arrow[d, "\rho^{\mathcal{H}}_{\gamma_t^{-1}h\gamma_{h^{-1}t}}(h^{-1}\gamma_t)"'] & & \mathcal{H}_{h^{-1}t} \arrow[dd, "\lambda_h^{\mathcal{H}}(t)"] \\
\mathcal{H}_{h^{-1}\gamma_t} \arrow[d, "\lambda_h^{\mathcal{H}}(\gamma_t)"'] & & \\
\mathcal{H}_{\gamma_t} \arrow[rr, "\rho^{\mathcal{H}}_{t^{-1}\gamma_t}(t)"] & & \mathcal{H}_t
\end{tikzcd}$$

and this commutes because

$$\begin{aligned}
\rho^{\mathcal{H}}_{t^{-1}\gamma_t} \circ \lambda_h^{\mathcal{H}} \circ \rho^{\mathcal{H}}_{\gamma_t^{-1}h\gamma_{h^{-1}t}} &= \lambda_h^{\mathcal{H}} \circ \rho^{\mathcal{H}}_{t^{-1}\gamma_t} \circ \rho^{\mathcal{H}}_{\gamma_t^{-1}h\gamma_{h^{-1}t}} \\
&= \lambda_h^{\mathcal{H}} \circ \rho^{\mathcal{H}}_{t^{-1}h\gamma_{h^{-1}t}}.
\end{aligned}$$

Therefore $(*)$ commutes and hence $\xi_{\mathcal{H}}$ intertwines the λ's. Since $\xi_{\mathcal{H}}$ intertwines the action of $C_0(G)$ and the unitary representations, it is a morphism of biequivariant Hilbert spaces.

Finally, we need to show that the unitaries

$$\xi_{\mathcal{H}} : (G \circ F)(\mathcal{H}) \xrightarrow{\cong} \mathcal{H}, \quad \mathcal{H} \in C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f$$

define a unitary natural transformation $\xi : G \circ F \xrightarrow{\cong} \mathrm{Id}$. To show naturality, we need to show that given $\mathcal{H}, \mathcal{K} \in C_0(G)\text{-}H\text{-}H\text{-}\mathsf{Mod}_H^f$ and a morphism $\alpha : \mathcal{H} \to \mathcal{K}$, the following diagram commutes

$$\begin{tikzcd}
(G \circ F)(\mathcal{H}) \arrow[r, "(G\circ F)(\alpha)"] \arrow[d, "\xi_{\mathcal{H}}"'] & (G \circ F)(\mathcal{K}) \arrow[d, "\xi_{\mathcal{K}}"] \\
\mathcal{H} \arrow[r, "\alpha"] & \mathcal{K}
\end{tikzcd} \tag{$**$}$$

We can do this by showing that for each $t \in G$, the following diagram commutes

$$
\begin{tikzcd}
(G \circ F)(\mathcal{H})_t \arrow[r, "(G\circ F)(\alpha)(t)"] \arrow[d, "\xi_{\mathcal{H}}(t)"'] & (G \circ F)(\mathcal{K})_t \arrow[d, "\xi_{\mathcal{K}}(t)"] \\
\mathcal{H}_t \arrow[r, "\alpha(t)"] & \mathcal{K}_t
\end{tikzcd}
$$

By definition, this is the diagram

$$
\begin{tikzcd}
\mathcal{H}_{\gamma_t} \arrow[r, "\alpha(\gamma_t)"] \arrow[d, "\rho^{\mathcal{H}}_{t^{-1}\gamma_t}(t)"'] & \mathcal{K}_{\gamma_t} \arrow[d, "\rho^{\mathcal{K}}_{t^{-1}\gamma_t}(t)"] \\
\mathcal{H}_t \arrow[r, "\alpha(t)"] & \mathcal{K}_t
\end{tikzcd}
$$

which commutes because α commutes with the ρ's. Therefore, $(**)$ commutes and the $\xi_{\mathcal{H}}$ define a unitary natural transformation $\xi : G \circ F \xrightarrow{\cong} \mathrm{Id}$. $\square$

Given $t \in G$ and $V \in \mathsf{urep}(H_t)$, $G(\eta^V_{HtH}) \cong \chi^V_{HtH}$. This is because given $h \in H_t$,

$$
\lambda_h^{G\left(\eta^V_{HtH}\right)} \circ \rho_{t^{-1}ht}^{G\left(\eta^V_{HtH}\right)} = \lambda_h^{\eta^V_{HtH}}.
$$

Via this equivalence, $C_0(G)$-H-H-Mod^f_H inherits a tensor category structure. The tensor unit is $F(\mathcal{I})$ where $\mathcal{I}$ is the tensor unit of $C_0(G)$-H-H-Mod^f_H. This is the equivariant Hilbert space η^{tr}_H, where tr denotes the trivial one dimensional representation of H. The product, which we shall denote by $*$ rather than $\otimes$, can be described as follows:

Lemma 4.50. *Let $\mathcal{H}, \mathcal{K} \in C_0(G/H)$-$H$-$\mathsf{Mod}^f_c$. The fibres of the product $\mathcal{H} * \mathcal{K}$ are given by*

$$
(\mathcal{H} * \mathcal{K})_{gH} \cong \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1} gH}.
$$

For $k \in H$ and $\xi \otimes \eta \in \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1} g H}$ the action of k is given by

$$\lambda_k^{\mathcal{H} * \mathcal{K}}(\xi \otimes \eta) = \lambda_k^{\mathcal{H}}(\xi) \otimes \lambda_{\gamma'^{-1} k \gamma}^{\mathcal{K}}(\eta)$$

where $\gamma' \in \Gamma$ is determined by

$$k\gamma H = \gamma' H.$$

In particular, if $k\gamma H = \gamma H$ the action is given by

$$\lambda_k^{\mathcal{H} * \mathcal{K}}(\xi \otimes \eta) = \lambda_k^{\mathcal{H}}(\xi) \otimes \lambda_{\gamma^{-1} k \gamma}^{\mathcal{K}}(\eta).$$

Proof. The product is defined by

$$\mathcal{H} * \mathcal{K} := F(G(\mathcal{H}) * G(\mathcal{K}))$$

where F and G are the equivalences defined in Lemmas 4.47 and 4.48 respectively.

If we consider $G(\mathcal{H}) * G(\mathcal{K}) \in C_0(G)$-$H$-$H$-$\mathsf{Mod}_H^f$, the fibres of $G(\mathcal{H}) * G(\mathcal{K})$ are given by

$$\begin{aligned} G(\mathcal{H}) * G(\mathcal{K})_g &= \bigoplus_{\gamma \in \Gamma} G(\mathcal{H})_\gamma \otimes G(\mathcal{K})_{\gamma^{-1} g} \\ &= \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1} g H} \end{aligned}$$

and the unitary representations are defined by

$$\rho_k^{G(\mathcal{H}) * G(\mathcal{K})}(g) = \mathrm{Id},$$

and

$$\begin{aligned} \lambda_k^{G(\mathcal{H}) * G(\mathcal{K})}(g) &= \bigoplus_{\gamma' \in \Gamma} [\lambda_k^{G(\mathcal{H})}(\gamma') \circ \rho_{h_{\gamma'}}^{G(\mathcal{H})}(h^{-1}\gamma')] \otimes \lambda_{h_{\gamma'}}^{G(\mathcal{K})}(\gamma'^{-1} g) \\ &= \bigoplus_{\gamma' \in \Gamma} \lambda_k^{G(\mathcal{H})}(\gamma') \otimes \lambda_{\gamma'^{-1} k \gamma}^{G(\mathcal{K})}(\gamma'^{-1} g) \\ &= \bigoplus_{\gamma' \in \Gamma} \lambda_k^{\mathcal{H}}(\gamma' H) \otimes \lambda_{\gamma'^{-1} k \gamma}^{\mathcal{K}}(\gamma'^{-1} g H) \end{aligned}$$

where $k \in H$, $g \in G$, and the $\gamma' \in \Gamma$ and $h_{\gamma'} \in H$ are determined by

$$k\gamma = \gamma' h_{\gamma'}.$$

Now, the fibres of $F(G(\mathcal{H}) * G(\mathcal{K}))$ are given by

$$F(G(\mathcal{H}) * G(\mathcal{K}))_{gH} = (G(\mathcal{H}) * G(\mathcal{K}))_{\gamma_g}$$

where $\gamma_g \in \Gamma$ is defined by $gH = \gamma_g H$. Therefore,

$$\begin{aligned} F(G(\mathcal{H}) * G(\mathcal{K}))_{gH} &= \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1}\gamma_g H} \\ &= \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1} g H}. \end{aligned}$$

Finally, given $k \in H$ and $\alpha \in \Gamma$, we have

$$\begin{aligned} \lambda_k^{F(G(\mathcal{H})*G(\mathcal{K}))}(\alpha H) &= \lambda_k^{G(\mathcal{H})*G(\mathcal{K})}(\alpha) \circ \rho_{\gamma^{-1}k\gamma'}^{G(\mathcal{H})*G(\mathcal{K})}(h^{-1}\alpha) \\ &= \lambda_k^{G(\mathcal{H})*G(\mathcal{K})}(\alpha) \\ &= \bigoplus_{\gamma' \in \Gamma} \lambda_k^{\mathcal{H}}(\gamma' H) \otimes \lambda_{\gamma'^{-1}k\gamma}^{\mathcal{K}}(\gamma'^{-1}\alpha H). \end{aligned}$$

$\square$

In a similar way, we can define an involution on $C_0(G/H)$-H-Mod_c^f by

$$\mathcal{H}^* := F(G(\mathcal{H})^*).$$

It can be described explicitly as follows:

Lemma 4.51. *Let $t \in G$ and $(\pi, V) \in \mathsf{urep}(H_t)$. Then $(\eta_{HtH}^V)^* \cong \eta_{\underline{Ht^{-1}}H}^W$ where $(\omega, W) \in \mathsf{urep}(H_t)$ is the representation with underlying Hilbert space $\overline{V}$ and the action ω given by*

$$\omega_h = \overline{\pi}_{tht^{-1}}, \quad h \in H_{t^{-1}}.$$

Proof. This follows immediately from Lemma 4.39 combined with the fact that $G\left(\eta_{HtH}^V\right) \cong \chi_{HtH}^V$. $\square$

Definition 4.52. *Given $V \in \mathsf{rep}(H_t)$, we denote the equivariant Hilbert space $(\eta_{HtH}^V)^*$ defined in Lemma 4.51 by $\eta_{Ht^{-1}H}^{\overline{V}^t}$.*

Whilst not necessary for what follows, we remark that we can pass from $\mathbf{H}(G/\!/H)$ or any of our equivalent C^*-tensor categories to a complex algebra by taking the so called (complex) 'Grothendieck ring' of the C^*-tensor category. This is a slight extension of the definition of the Grothendieck group of an additive C^*-category $\mathcal{A}$.

The main ingredient in the definition of the Grothendieck ring is the definition of the Grothendieck group of a commutative monoid. The idea is that the forgetful functor from the category of abelian groups to the category of commutative monoids has a left adjoint which maps a commutative monoid M to its Grothendieck group, denoted $K(M)$.

There are various different ways to construct $K(M)$ explicitly. One way is to define an equivalence relation on $M \times M$ by $(m_1, m_2) \sim (n_1, n_2)$ if there exists $k \in M$ such that $m_1 + n_2 + k = m_2 + n_1 + k$. One thinks of the pair (m_1, m_2) as the formal difference of m_1 and m_2 and the element k is needed in case M is not cancellative. Then $K(M)$ is the set $M \times M/\sim$ with the addition defined by

$$[(m_1, m_2)] + [(n_1, n_2)] := [(m_1 + n_1, m_2 + n_2)].$$

One can check that this is well-defined and does indeed define an abelian group. The identity element is $[(0, 0)]$ where 0 is the identity of M and the inverse of $[(m_1, m_2)]$ is $[(m_2, m_1)]$. The unit η of the adjunction is defined by

$$\begin{aligned} \eta_M : M &\to K(M) \\ m &\mapsto [(m, 0)] \end{aligned}$$

Now, if $\mathcal{A}$ is an additive C^*-category, we can define a commutative monoid $M(\mathcal{A})$ whose elements are the isomorphism classes of $\mathcal{A}$ and whose addition is defined by $[A] + [B] := [A \oplus B]$, $A, B \in \mathcal{A}$. Then the Grothendieck group of $\mathcal{A}$, denoted $K(\mathcal{A})$, is the Grothendieck group of $M(\mathcal{A})$. If $\mathcal{A}$ is also a C^*-tensor category, then $K(\mathcal{A})$ has a ring structure with multiplication defined by $[A] \cdot [B] := [A \otimes B]$. This is the Grothendieck ring of $\mathcal{A}$. By definition this is a $\mathbb{Z}$-algebra but we can always turn it into a complex algebra by extension of scalars. Furthermore, if $\mathcal{A}$ has an involution, we can define an involution on $K(\mathcal{A})$ by $[A]^* := [A^*]$.

We shall denote the complex Grothendieck ring of $\mathbf{H}(G/\!/H)$ or any of our equivalent C^*-tensor categories by $\mathcal{K}(G/\!/H)$. We will actually work with $C_0(G/H)$-H-Mod_c^f for the purpose of doing computations. If we compare the formula (2) for the product in the Hecke algebra with the formula in Lemma 4.50 for the fibres in the product in $C_0(G/H)$-H-Mod_c^f then we can see that there is an algebra homomorphism

$$\varphi : \mathcal{K}(G/\!/H) \to \mathcal{H}(G/\!/H)$$

defined by $\varphi([\mathcal{H}])(gH) := \dim \mathcal{H}_{gH}$. Since $[\eta^{\mathrm{tr}}_{HtH}]$ maps to the characteristic function of HtH/H, this is a surjective homomorphism and hence $\mathcal{H}(G/\!/H)$ is a quotient of $\mathcal{K}(G/\!/H)$. Let us state this as a lemma for future reference.

Lemma 4.53. *There is a surjective algebra homomorphism*

$$\varphi : \mathcal{K}(G/\!/H) \to \mathcal{H}(G/\!/H)$$

defined by $\varphi\left([\mathcal{H}]\right)(gH) := \dim \mathcal{H}_{gH}$.

If $t \in G$ and $V, W \in \mathsf{urep}(H_t)$, then $\eta^V_{HtH} \oplus \eta^W_{HtH} \cong \eta^{V\oplus W}_{HtH}$ and this determines the addition in $\mathcal{K}(G/\!/H)$. We shall now look at some formulae for determining the product. First we have some direct consequences of Lemma 4.50.

Corollary 4.54. *Let* $\eta^V_H, \eta^W_H \in C_0(G/H)\text{-}H\text{-}\mathsf{Mod}^f_c$ *with* $V, W \in \mathsf{urep}(H)$. *Then*

$$\eta^V_H * \eta^W_H \cong \eta^{V\otimes W}_H.$$

Proof. There is only one nonzero fibre in the product, namely

$$\begin{aligned}(\eta^V_H * \eta^W_H)_H &= (\eta^V_H)_H \otimes (\eta^W_H)_H \\ &= V \otimes W\end{aligned}$$

and in this case the formula for the action of $k \in H$ is

$$\lambda_k(\xi \otimes \eta) = \lambda_k(\xi) \otimes \lambda_k(\eta), \quad \xi \in V,\ \eta \in W.$$

□

Corollary 4.55. *Let* $t \in G$ *and let* $\eta^V_H, \eta^W_{HtH} \in C_0(G/H)\text{-}H\text{-}\mathsf{Mod}^f_c$ *with* $V \in \mathsf{urep}(H)$ *and* $W \in \mathsf{urep}(H_t)$. *Then*

$$\eta^V_H * \eta^W_{HtH} \cong \eta^{V|_{H_t}\otimes W}_{HtH}$$

where $V|_{H_t}$ *denotes the restriction of* V *to* H_t.

Proof. Since $\delta_H * \delta_{HtH} = \delta_{HtH}$ in the classical Hecke algebra, $\eta^V_H * \eta^W_{HtH}$ is supported on HtH. Therefore, we just need to determine the representation of H_t on the fibre at tH. The fibre is

$$\begin{aligned}(\eta^V_H * \eta^W_{HtH})_{tH} &= \bigoplus_{\gamma\in\Gamma} (\eta^V_H)_{\gamma H} \otimes (\eta^W_{HtH})_{\gamma^{-1}tH} \\ &= (\eta^V_H)_H \otimes (\eta^W_{HtH})_{tH}\end{aligned}$$

which is equal to $V \otimes W$ as a vector space. For $h \in H_t$ the action on the fibre is given by

$$\lambda_k(\xi \otimes \eta) = \lambda_k(\xi) \otimes \lambda_h(\eta), \quad \xi \in V,\ \eta \in W.$$

Therefore, the representation of H_t on the fibre at tH is $V|_{H_t} \otimes W$ and

$$\eta_H^V * \eta_{HtH}^W = \eta_{HtH}^{V|_{H_t} \otimes W}.$$

$\square$

In general, it is not true that

$$\eta_{HtH}^W * \eta_H^V \cong \eta_{HtH}^{W \otimes V|_{H_t}}.$$

The product is still supported on HtH and the fibre at tH is

$$\begin{aligned}(\eta_{HtH}^W * \eta_H^V)_{tH} &= \bigoplus_{\gamma \in \Gamma} (\eta_{HtH}^W)_{\gamma H} \otimes (\eta_H^V)_{\gamma^{-1} tH} \\ &= (\eta_{HtH}^W)_{tH} \otimes (\eta_H^V)_H\end{aligned}$$

which is equal to $W \otimes V$ as a vector space. If we assume for simplicity that $t \in \Gamma$, then by Lemma 4.50 for $k \in H_t$, the action is given by

$$\lambda_k(\xi \otimes \eta) = \lambda_k(\xi) \otimes \lambda_{t^{-1}kt}(\eta), \quad \xi \in W,\ \eta \in V.$$

Whilst the representation of H_t on the first tensor factor is W, the conjugation in the second tensor factor may result in a representation of H_t that is not isomorphic to $V|_{H_t}$. The following is an example of this.

Example 4.56. Let $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and consider the group homomorphism

$$\varphi : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(H)$$

where $\varphi(1)$ acts by permuting the factors, that is, $\varphi(1)(a,b) = (b,a)$ for each $(a,b) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then let G be the semidirect product $H \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$ and let $t = ((0,0),1) \in G$. As set of coset representatives for G/H is given by $\Gamma = \{e, t\}$ and since H is normal in G, the double cosets are just the cosets and the classical Hecke algebra $\mathcal{H}(G/\!/H)$ is the group ring of $G/H \cong \mathbb{Z}/2\mathbb{Z}$.

Because H is normal in G, the group $H_t = H \cap tHt^{-1}$ is equal to H and since H is abelian, the irreducible representations of H are the characters. Let us denote by $\chi_{\alpha,\beta}$, the character on H where $(1,0) \in H$ acts by multiplication by α and $(0,1) \in H$ acts by multiplication by β where $\alpha, \beta \in \{1, -1\}$. If we consider the product $\eta_{HtH}^{\chi_{1,1}} * \eta_H^{\chi_{1,-1}}$ in $C_0(G/H)$-H-Mod_c^f, this is supported on $HtH = tH$ and the fibre at tH is

$$\begin{aligned}(\eta_{HtH}^{\chi_{1,1}} * \eta_H^{\chi_{1,-1}})_{tH} &= \bigoplus_{\gamma \in \Gamma} (\eta_{HtH}^{\chi_{1,1}})_{\gamma H} \otimes (\eta_H^{\chi_{1,-1}})_{\gamma^{-1} tH} \\ &= (\eta_{HtH}^{\chi_{1,1}})_{tH} \otimes (\eta_H^{\chi_{1,-1}})_H\end{aligned}$$

which is equal to $\mathbb{C} \otimes \mathbb{C}$ as a vector space. By Lemma 4.50, for $(a, b) \in H_t = H$, the action is given by

$$\lambda_{(a,b)}(\xi \otimes \eta) = \lambda_{(a,b)}(\xi) \otimes \lambda_{x^{-1}(a,b)x}(\eta).$$

By the construction of the semidirect product G,

$$t^{-1}(a, b)t = \varphi(1)^{-1}(a, b) = (b, a),$$

hence

$$\lambda_{(a,b)}(\xi \otimes \eta) = \lambda_{(a,b)}(\xi) \otimes \lambda_{(b,a)}(\eta).$$

In particular,

$$\begin{aligned}\lambda_{(1,0)}(\xi \otimes \eta) &= \lambda_{(1,0)}(\xi) \otimes \lambda_{(0,1)}(\eta) \\ &= \xi \otimes (-\eta) \\ &= -(\xi \otimes \eta)\end{aligned}$$

and

$$\begin{aligned}\lambda_{(0,1)}(\xi \otimes \eta) &= \lambda_{(0,1)}(\xi) \otimes \lambda_{(1,0)}(\eta) \\ &= \xi \otimes \eta.\end{aligned}$$

Therefore,

$$\eta_{HtH}^{\chi_{1,1}} * \eta_{H}^{\chi_{1,-1}} \cong \eta_{HtH}^{\chi_{-1,1}}$$

but $\chi_{1,1} \otimes \chi_{1,-1} \cong \chi_{1,-1} \ncong \chi_{-1,1}$ and hence

$$\eta_{HtH}^{\chi_{1,1}} * \eta_{H}^{\chi_{1,-1}} \ncong \eta_{HtH}^{\chi_{1,1}\otimes\chi_{1,-1}}.$$

This is also an example of a Hecke pair for which the classical Hecke algebra $\mathcal{H}(G/\!/H)$ is commutative, but $\mathcal{K}(G/\!/H)$ is not.

Given $\mathcal{H}, \mathcal{K} \in C_0(G/H)\text{-}H\text{-}\mathsf{Mod}_c^f$, computing the product $\mathcal{H} * \mathcal{K}$ involves computing the representations of of the groups H_t on the fibres $(\mathcal{H} * \mathcal{K})_{tH}$. We shall now look at how these representations can be decomposed as a direct sum of subrepresentations. This is similar to the way that an equivariant Hilbert space can be decomposed as a direct sum of representations of H, each of which is supported on a single double coset. To this end, we fix a subset $\Delta_t \subset \Gamma$ of representatives for the double coset space $H_t \backslash G/H$. Given any $\delta \in \Delta_t$, the space

$$\bigoplus_{[\gamma]\in H_t\delta H/H} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1}tH}$$

is a subrepresentation of H_t on $(\mathcal{H} * \mathcal{K})_{tH}$ and

$$(\mathcal{H} * \mathcal{K})_{tH} = \bigoplus_{\delta\in\Delta_t} \bigoplus_{[\gamma]\in H_t\delta H/H} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1}tH}.$$

We can describe this decomposition in terms of certain subgroups of H_t.

Definition 4.57. *For $t \in G$ and $\delta \in \Delta_t$, we define the group $H_{\delta,t} \subset H$ by*

$$H_{\delta,t} := H_\delta \cap H_t = H \cap \delta H \delta^{-1} \cap tHt^{-1}.$$

The group $H_{\delta,t}$ maps $\mathcal{H}_\delta H \otimes \mathcal{K}_{\delta^{-1}tH}$ into itself. We recall that the induced representation $\mathrm{Ind}_{H_{\delta,t}}^{H_t}(\mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH})$ can be described as the Hilbert space direct sum

$$\bigoplus_{[\sigma]\in H_t/H_{\delta,t}} \sigma(\mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH})$$

where each $\sigma(\mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH})$ is an isomorphic copy of $\mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH}$, the elements of which we write as σv where $v \in \mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH}$. For $k \in H_t$, the action π is given by

$$\pi_k(\sigma v) = \sigma' \lambda_\ell^{\mathcal{H}*\mathcal{K}}(v)$$

where the coset representative $\sigma' \in H_t$, and $\ell \in H_{\delta,t}$ are determined by $k\sigma = \sigma'\ell$. We then have the following:

Lemma 4.58. *For each $\delta \in \Delta_t$, there is a unitary equivalence of representations*

$$\tilde{\iota} : \mathrm{Ind}_{H_{\delta,t}}^{H_t}(\mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH}) \xrightarrow{\cong} \bigoplus_{[\gamma]\in H_t\delta H/H} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1}tH}.$$

Proof. For each $[\sigma] \in H_t/H_{\delta,t}$ we define $\gamma_\sigma \in \Gamma$ by

$$\sigma\delta H = \gamma_\sigma H.$$

so that

$$\bigoplus_{[\gamma]\in H_t\delta H/H} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1}tH} = \bigoplus_{[\sigma]\in H_t/H_{\delta,t}} \mathcal{H}_{\gamma_\sigma H} \otimes \mathcal{K}_{\gamma_\sigma^{-1}tH}.$$

The inclusion map

$$\iota : \mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH} \to \bigoplus_{[\gamma]\in H_t\delta H/H} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1}tH}$$

in $\mathsf{urep}(H_{\delta,t})$ corresponds to an H_t-equivariant map

$$\tilde{\iota} : \mathrm{Ind}_{H_{\delta,t}}^{H_t}(\mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH}) \to \bigoplus_{[\gamma]\in H_t\delta H/H} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1}tH}$$

in $\mathsf{urep}(H_t)$ via the induction-restriction adjunction. This is the map

$$\bigoplus_{[\sigma]\in H_t/H_{\delta,t}} \lambda_\sigma^{\mathcal{H}*\mathcal{K}} : \bigoplus_{[\sigma]\in H_t/H_{\delta,t}} \sigma(\mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH}) \to \bigoplus_{[\sigma]\in H_t/H_{\delta,t}} \mathcal{H}_{\gamma_\sigma H} \otimes \mathcal{K}_{\gamma_\sigma^{-1}tH}.$$

This follows from the definition of the counit of the adjunction and the fact that $\lambda_\sigma^{\mathcal{H}*\mathcal{K}}$ maps $\mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH}$ into $\mathcal{H}_{\gamma_\sigma} \otimes \mathcal{K}_{\gamma_\sigma^{-1}tH}$. This map is a unitary because each $\lambda_\sigma^{\mathcal{H}*\mathcal{K}}$ is and hence

$$\mathrm{Ind}_{H_{\delta,t}}^{H_t}(\mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH}) \cong \bigoplus_{[\gamma]\in H_t\delta H/H} \mathcal{H}_{\gamma H} \otimes \mathcal{K}_{\gamma^{-1}tH}.$$

$\square$

Corollary 4.59. *Let $\mathcal{H}, \mathcal{K} \in C_0(G/H)$-$H$-$\mathsf{Mod}_c^f$. The fibres of the product $\mathcal{H} * \mathcal{K}$ are given up to isomorphism by*

$$(\mathcal{H} * \mathcal{K})_{tH} = \bigoplus_{\delta\in\Delta_t} \mathrm{Ind}_{H_{\delta,t}}^{H_t}(\mathcal{H}_{\delta H} \otimes \mathcal{K}_{\delta^{-1}tH}).$$

4.3 Examples

To end with, we shall look at a few examples of $\mathcal{K}(G/\!/H)$ for some Hecke pairs. In order to avoid a little clutter in the notation, we shall write η_{HtH}^V to denote the element $\left[\eta_{HtH}^V\right] \in \mathcal{K}(G/\!/H)$. Our first example involves the symmetric groups.

4.3.1 The Hecke pair (S_n, S_{n-1})

In this section, we shall look at the structure of $\mathcal{K}(G/\!/H)$ for the Hecke pair (G, H) where $G = S_n$, $H = S_{n-1}$ ($n \geq 2$). (Specifically, we mean the copy of S_{n-1} inside S_n which fixes n.) Before computing the categorical product, let us briefly review the structure of the classical Hecke algebra $\mathcal{H}(G/\!/H)$. A set of coset representatives for G/H is given by

$$\Gamma = \{t_i = (i, n) : i = 1, \dots n\}$$

where (n, n) is the identity permutation.

Lemma 4.60. *The double coset space $H\backslash G/H$ consists of two double cosets, H and*

$$\begin{aligned} X &= Ht_{n-1}H \\ &= \bigsqcup_{i=1}^{n-1} t_i H. \end{aligned}$$

Proof. Since $H = HeH$ is a double coset, to show that

$$\begin{aligned} X &= Ht_{n-1}H \\ &= \bigsqcup_{i=1}^{n-1} t_i H \end{aligned}$$

we just need to show that every other coset t_iH, $i \in \{1, \dots, n-2\}$ is a subset of X. For each such i, $(i, n-1) \in H$ and

$$\begin{aligned} (i, n-1)t_{n-1}H &= (i, n-1)(n-1, n)H \\ &= (i, n-1, n)H \\ &= (n, i)(i, n-1)H \\ &= (i, n)H \\ &= t_iH. \end{aligned}$$

Therefore, $t_iH \subset X$ and hence

$$\begin{aligned} X &= Ht_{n-1}H \\ &= \bigsqcup_{i=1}^{n-1} t_i H. \end{aligned}$$

□

Lemma 4.61. *The algebra structure of $\mathcal{H}(G/\!/H)$ is determined by*

$$\delta_X * \delta_X = (n-1)\delta_H + (n-2)\delta_X.$$

Proof. In general,

$$
\begin{aligned}
(\delta_X * \delta_X)(g) &= \sum_{i=1}^{n} \delta_X(t_i)\delta_X(t_i^{-1}g) \\
&= \sum_{i=1}^{n} \delta_X(t_i)\delta_X(t_i g).
\end{aligned}
$$

Since there are only two double cosets, we just need to compute the values when $g = e$ and when $g = t_{n-1}$. Firstly,

$$
(\delta_X * \delta_X)(e) = \sum_{i=1}^{n} \delta_X(t_i)\delta_X(t_i).
$$

Since

$$
\delta_X(t_i) = \begin{cases} 0 & \text{if } i = n, \\ 1 & \text{otherwise,} \end{cases}
$$

we have $(\delta_X * \delta_X)(e) = n - 1$. Next,

$$
(\delta_X * \delta_X)(t_{n-1}) = \sum_{i=1}^{n} \delta_X(t_i)\delta_X(t_i t_{n-1}).
$$

Since the i'th summand is 0 if $i = n$ or $n-1$ and 1 otherwise, $(\delta_X * \delta_X)(t_{n-1}) = n-2$. Therefore,

$$
\delta_X * \delta_X = (n-1)\delta_H + (n-2)\delta_X.
$$

□

Since there are two double cosets, H and $X = Ht_{n-1}H$, to compute the product in the categorical Hecke algebra $\mathcal{K}(G/\!/H)$ we need to determine the representation of H on the fibre at H and the representation of $H_{t_{n-1}}$ on the fibre at $t_{n-1}H$. As part of this computation, we need to know the group $H_{t_{n-1}}$ which is the stabiliser of the coset $t_{n-1}H$ under the translation action of H.

Lemma 4.62. *For $i \in \{1, \ldots, n-1\}$, the group $H_{t_i} = H \cap t_i H t_i^{-1}$ is the copy of S_{n-2} in S_n which fixes i and n. In particular, $H_{t_{n-1}}$ is the copy of S_{n-2} in S_n that fixes $n-1$ and n.*

Proof. H is the copy of S_{n-1} in S_n which fixes n and $t_i H t_i^{-1} = (i,n)H(i,n)$ is the copy of S_{n-1} in S_n which fixes i. Therefore

$$H_{t_i} = H \cap t_i H t_i^{-1}$$

is the subgroup of S_n which fixes i and n. □

In order to use the formula in Corollary 4.59 to compute the fibres in the product, we also need sets of representatives Δ_e and $\Delta_{t_{n-1}}$ for the double coset spaces $H\backslash G/H$ and $H_{t_{n-1}}\backslash G/H$. The are two double cosets in $H\backslash G/H$, namely H and $X = Ht_{n-1}H$ so we take $\Delta_e = \{e, t_{n-1}\}$. To choose our set $\Delta_{t_{n-1}}$, we need to know how what the double cosets in $H_{t_{n-1}}\backslash G/H$ are.

Lemma 4.63. *For $n \geq 3$, the double coset space $H_{t_{n-1}}\backslash G/H$ consists of 3 double cosets, H, $t_{n-1}H$ and*

$$\begin{aligned} Y &= H_{t_{n-1}} t_{n-2} H \\ &= \bigsqcup_{i=1}^{n-2} t_i H. \end{aligned}$$

Proof. Since $H_{t_{n-1}} \subset H$, we have $H_{t_{n-1}} eH = H$. Next, since $H_{t_{n-1}}$ is the stabiliser group of $t_{n-1}H$, we have $H_{t_{n-1}} t_{n-1} H = t_{n-1}H$. To show that

$$\begin{aligned} Y &= H_{t_{n-1}} t_{n-2} H \\ &= \bigsqcup_{i=1}^{n-2} t_i H, \end{aligned}$$

we need to show that the remaining cosets $t_i H$, $i \in \{1, \ldots, n-3\}$ are contained in Y. For each such i we have $(i, n-2) \in H_{t_{n-1}}$ and

$$\begin{aligned} (i, n-2)t_{n-2}H &= (i, n-2)(n-2, n)H \\ &= (i, n-2, n)H \\ &= (i, n)(i, n-2)H \\ &= t_i H. \end{aligned}$$

Therefore, $t_i H \subset Y$ and hence

$$\begin{aligned} Y &= H_{t_{n-1}} t_{n-2} H \\ &= \bigsqcup_{i=1}^{n-2} t_i H. \end{aligned}$$

□

In light of this, for $n \geq 3$ we take $\Delta_{t_{n-1}} = \{e, t_{n-1}, t_{n-2}\}$. (In the case $n = 2$, $H_{t_{n-1}}$ and H are the trivial group and the double coset space $H_{t_{n-1}} \backslash G/H$ is just the coset space G/H so we can take $\Delta_{t_{n-1}} = \{e, t_{n-1}\}$.)

Before considering the case of general n, let us compute the product in $\mathcal{K}(S_4, S_3)$. A generating set for the algebra is given by elements of the form η_H^V and η_X^W where V is an irreducible unitary representation of $H = S_3$ and W is an irreducible unitary representation of $H_{t_3} = S_2$.

Let us first recall the representation theory for S_3 and S_2. S_3 has three irreducible representations, the one dimensional trivial and sign representations which we shall denote by tr and sgn respectively and the 2-dimensional standard representation which we shall denote by std. (In general, for $n \in \mathbb{N}$, the standard representation of S_n is the complement of the trivial representation in the permutation representation.) The character table of S_3 is

	e	$(1,2)$	$(1,2,3)$
tr	1	1	1
sgn	1	-1	1
std	2	0	-1

S_2 has two irreducible representations, the one dimensional trivial and sign representations. Its character table is

	e	(1, 2)
tr	1	1
sgn	1	-1

We can now compute the products of the elements in the generating set and product a multiplication table. Our convention for the multiplication table is that the left factors are listed in the left hand column and the right factors are listed in the top row, although it will actually turn our that the algebra is commutative in this case.

Lemma 4.64. *The multiplication table for the categorical Hecke algebra $\mathcal{H}(S_4/\!/S_3)$, minus the identity element η_H^{tr}, is*

	η_H^{sgn}	η_H^{std}	η_X^{tr}	η_X^{sgn}
η_H^{sgn}	η_H^{tr}	η_H^{std}	η_X^{sgn}	η_X^{tr}
η_H^{std}	η_H^{std}	$\eta_H^{\text{tr}} + \eta_H^{\text{sgn}} + \eta_H^{\text{std}}$	$\eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$	$\eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$
η_X^{tr}	η_X^{sgn}	$\eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$	$\eta_H^{\text{tr}} + \eta_H^{\text{std}} + \eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$	$\eta_H^{\text{sgn}} + \eta_H^{\text{std}} + \eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$
η_X^{sgn}	η_X^{tr}	$\eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$	$\eta_H^{\text{sgn}} + \eta_H^{\text{std}} + \eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$	$\eta_H^{\text{tr}} + \eta_H^{\text{std}} + \eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$

where $X = Ht_3H = H(3,4)H$.

Proof. Firstly, by Corollary 4.54 we have

$$\eta_H^V * \eta_H^W = \eta_H^{V \otimes W}$$

so we need to compute the tensor products of the nontrivial irreducible representations of S_3. To do this, we shall compute the characters case by case for the nontrivial representations. In general, the character of the tensor product is the pointwise product of the characters.

Case 1: $V = \text{sgn}, W = \text{sgn}$.
The character of $\text{sgn} \otimes \text{sgn}$ is

$$(1, -1, 1) \cdot (1, -1, 1) = (1, 1, 1).$$

This is the character of the trivial representation so $\text{sgn} \otimes \text{sgn} \cong \text{tr}$ and hence

$$\eta_H^{\text{sgn}} * \eta_H^{\text{sgn}} = \eta_H^{\text{sgn}\otimes\text{sgn}} = \eta_H^{\text{tr}}.$$

Case 2: $V = \text{sgn}, W = \text{std}$.
The character of $\text{sgn} \otimes \text{std}$ is

$$(1, -1, 1) \cdot (2, 0, -1) = (2, 0, -1),$$

which is the character of the standard representation. Hence $\text{sgn} \otimes \text{std} \cong \text{std}$ and

$$\eta_H^{\text{sgn}} * \eta_H^{\text{std}} = \eta_H^{\text{sgn}\otimes\text{std}} = \eta_H^{\text{std}}.$$

Case 3: $V = \text{std}, W = \text{sgn}$

Using the symmetry of the tensor product of representations we have

$$\eta_H^{\text{std}} * \eta_H^{\text{sgn}} = \eta_H^{\text{std}\otimes\text{sgn}} = \eta_H^{\text{sgn}\otimes\text{std}} = \eta_H^{\text{std}}.$$

Case 4: $V = \text{std}, W = \text{std}$.
Lastly, we compute the character of std $\otimes$ std. This is

$$\begin{aligned}(2,0,-1)\cdot(2,0,-1) &= (4,0,1)\\ &= (1,1,1)+(1,-1,1)+(2,0,-1).\end{aligned}$$

Therefore, std $\otimes$ std $\cong$ tr $\oplus$ sgn $\oplus$ std and

$$\eta_H^{\text{std}} * \eta_H^{\text{std}} = \eta_H^{\text{std}\otimes\text{std}} = \eta_H^{\text{tr}} + \eta_H^{\text{sgn}} + \eta_H^{\text{std}}.$$

So far, the multiplication for the categorical Hecke algebra, minus the identity element η_H^{tr}, is as follows:

	η_H^{sgn}	η_H^{std}
η_H^{sgn}	η_H^{tr}	η_H^{std}
η_H^{std}	η_H^{std}	$\eta_H^{\text{tr}} + \eta_H^{\text{sgn}} + \eta_H^{\text{std}}$

Next we need to compute products of the form $\eta_H^V * \eta_X^W$. By Corollary 4.55 we have

$$\eta_H^V * \eta_X^W = \eta_X^{V|_{H_{t_3}}\otimes W},$$

and by Lemma 4.62, H_{t_3} is the copy of S_2 inside S_4 that fixes 3 and 4. We can compute the representation $V|_{H_{t_3}} \otimes W$ for the different V and W case by case. Since η_H^{tr} is the identity of the categorical Hecke algebra, we only need to consider nontrivial V.

Case 1: $V = \text{sgn}, W = \text{tr}$.
In this case

$$V|_{H_{t_3}} \otimes W \cong V|_{H_{t_3}}.$$

The restriction of the sign representation of H to H_{t_3} is the sign representation of H_{t_3}. Therefore,

$$\eta_H^{\text{sgn}} * \eta_X^{\text{tr}} = \eta_X^{\text{sgn}}.$$

Case 2: $V = \mathrm{std}, W = \mathrm{tr}$.
As in Case 1,

$$V|_{H_{t_3}} \otimes W \cong V|_{H_{t_3}}.$$

The restriction of the standard representation of H to H_{t_3} is the 2-dimensional representation with character

$$(2, 0) = (1, 1) + (1, -1).$$

Therefore, this representation is isomorphic to $\mathrm{tr} \oplus \mathrm{sgn}$ and hence

$$\eta_H^{\mathrm{std}} * \eta_X^{\mathrm{tr}} = \eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}.$$

Case 3: $V = \mathrm{sgn}, W = \mathrm{sgn}$.
As noted above, the $V|_{H_{t_3}}$ is the sign representation and since $\mathrm{sgn} \otimes \mathrm{sgn} \cong \mathrm{tr}$ we have

$$\eta_H^{\mathrm{sgn}} * \eta_X^{\mathrm{sgn}} = \eta_X^{\mathrm{tr}}.$$

Case 4: $V = \mathrm{std}, W = \mathrm{sgn}$
Lastly, $\mathrm{std}|_{H_{t_3}} \otimes \mathrm{sgn}$ is the representation with character

$$(2, 0) \cdot (1, -1) = (2, 0) = (1, 1) + (1, -1)$$

which is the character of $\mathrm{tr} \oplus \mathrm{sgn}$ and hence

$$\eta_H^{\mathrm{std}} * \eta_X^{\mathrm{sgn}} = \eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}.$$

Adding these cases to the multiplication table we have

	η_H^{sgn}	η_H^{std}	η_X^{tr}	η_X^{sgn}
η_H^{sgn}	η_H^{tr}	η_H^{std}	η_X^{sgn}	η_X^{tr}
η_H^{std}	η_H^{std}	$\eta_H^{\mathrm{tr}} + \eta_H^{\mathrm{sgn}} + \eta_H^{\mathrm{std}}$	$\eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}$	$\eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}$

Next, we consider products of the form $\eta_X^W * \eta_H^V$. Similarly to the products of the form $\eta_H^V * \eta_X^W$, these are supported on the double coset X (since $\delta_X * \delta_H = \delta_X$ in the classical Hecke algebra). The fibre at $t_3 H$ is

$$\begin{aligned}(\eta_X^W * \eta_H^V)_{t_3 H} &= \bigoplus_{i=1}^{4} (\eta_X^W)_{t_i H} \otimes (\eta_H^V)_{t_i^{-1} t_3 H} \\ &= (\eta_X^W)_{t_3 H} \otimes (\eta_H^V)_H\end{aligned}$$

which is $W \otimes V$ as a Hilbert space. By Lemma 4.50, for $h \in H_{t_3}$ the action is given by

$$\lambda_h(\xi \otimes \eta) = \lambda_h(\xi) \otimes \lambda_{t_3^{-1} h t_3}(\eta).$$

Since $H_{t_3} = S_2$ and $t_3 = (3,4)$, the elements of H_{t_3} commute with t_3 so the action reduces to

$$\lambda_h(\xi \otimes \eta) = \lambda_h(\xi) \otimes \lambda_h(\eta).$$

Therefore, the representation of H_{t_3} on the fibre at $t_3 H$ is $W \otimes V|_{H_{t_3}}$. Since $W \otimes V|_{H_{t_3}} \cong V|_{H_{t_3}} \otimes W$, we have

$$\begin{aligned} \eta_X^W * \eta_H^V &= \eta_X^{W \otimes |V_{H_{t_3}}} \\ &= \eta_X^{V|_{H_{t_3}} \otimes W} \\ &= \eta_H^V * \eta_X^W. \end{aligned}$$

So far the multiplication table is

	η_H^{sgn}	η_H^{std}	η_X^{tr}	η_X^{sgn}
η_H^{sgn}	η_H^{tr}	η_H^{std}	η_X^{sgn}	η_X^{tr}
η_H^{std}	η_H^{std}	$\eta_H^{\text{tr}} + \eta_H^{\text{sgn}} + \eta_H^{\text{std}}$	$\eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$	$\eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$
η_X^{tr}	η_X^{sgn}	$\eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$		
η_X^{sgn}	η_X^{tr}	$\eta_X^{\text{tr}} + \eta_X^{\text{sgn}}$		

Finally, we need to compute products of the form $\eta_X^V * \eta_X^W$. We need to determine the representation of H on the fibre at H and the representation of H_{t_3} on the fibre at $t_3 H$. To compute the fibre at H, we can use the formula

$$(\eta_X^V * \eta_X^W)_H = \bigoplus_{\delta \in \Delta_e} \operatorname{Ind}_{H_{\delta,e}}^H ((\eta_X^V)_{\delta H} \otimes (\eta_X^W)_{\delta^{-1} H}) \tag{1}$$

from Corollary 4.59. We recall that $\Delta_e \subset \Gamma$ is a set of coset representatives for $H_e \backslash G / H = H \backslash G / H$ so we can take $\Delta_e = \{e, t_3\}$. Since the fibres of η_X^V and η_X^W at $eH = H$ are zero and

$$\begin{aligned} H_{t_3,e} &= H_{t_3} \cap H_e \\ &= H \cap t_3 H t_3^{-1} \cap eHe^{-1} \\ &= H \cap t_3 H t_3^{-1} \\ &= H_{t_3} \end{aligned}$$

the expression (1) reduces to

$$(\eta_X^V * \eta_X^W)_H = \mathrm{Ind}_{H_{t_3}}^H((\eta_X^V)_{t_3H} \otimes (\eta_X^W)_{t_3H}).$$

For $h \in H_{t_3}$, the action of h on $(\eta_X^V)_{t_3H} \otimes (\eta_X^W)_{t_3H}$ is given by

$$\lambda_h(\xi \otimes \eta) = \lambda_h(\xi) \otimes \lambda_{t_3^{-1}ht_3}(\eta). \tag{2}$$

By Lemma 4.62, $H_{t_3} = H \cap t_3Ht_3^{-1}$ is the copy of S_2 in S_4 that fixes 3 and 4 and since $t_3 = (3,4)$, h commutes with t_3. Therefore (2) reduces to

$$\lambda_h(\xi \otimes \eta) = \lambda_h(\xi) \otimes \lambda_h(\eta)$$

and the fibre at H is

$$\mathrm{Ind}_{H_{t_3}}^H(V \otimes W) = \mathrm{Ind}_{S_2}^{S_3}(V \otimes W).$$

There are two possibilities for each of V and W, the trivial and the sign representation of S_2. Therefore, $V \otimes W$ is either the trivial representation (when $V = W$) or the sign representation (when $V \neq W$). There is a standard formula for computing the character of the induced representation and one finds that the character of $\mathrm{Ind}_{S_2}^{S_3}(\mathrm{tr})$ is

$$(3,1,0) = (1,1,1) + (2,0,-1)$$

so

$$\mathrm{Ind}_{S_2}^{S_3}(\mathrm{tr}) \cong \mathrm{tr} \oplus \mathrm{std}.$$

Similarly, one finds that the character of $\mathrm{Ind}_{S_2}^{S_3}(\mathrm{sgn})$ is

$$(3,-1,0) = (1,-1,1) + (2,0,-1)$$

so

$$\mathrm{Ind}_{S_2}^{S_3}(\mathrm{sgn}) \cong \mathrm{sgn} \oplus \mathrm{std}.$$

Therefore, the fibre at H is isomorphic to $\mathrm{tr} \oplus \mathrm{std}$ when $V = W$ and $\mathrm{sgn} \oplus \mathrm{std}$ when $V \neq W$.

We also need to compute the representation of H_{t_3} on the fibre at t_3H which is

$$(\eta_X^V * \eta_X^W)_{t_3H} = \bigoplus_{\delta \in \Delta_{t_3}} \mathrm{Ind}_{H_{\delta,t_3}}^{H_{t_3}}((\eta_X^V)_{\delta H} \otimes (\eta_X^W)_{\delta^{-1}t_3H}). \tag{3}$$

We recall that $\Delta_{t_3} \subset \Gamma$ is a set of representatives for $H_{t_3}\backslash G/H$ and by Lemma 4.63 we can take $\Delta_{t_3} = \{e, t_2, t_3\}$. Since $(\eta_X^V)_H$ and $(\eta_X^W)_H$ are zero, the only nonzero summand is the term with $\delta = t_2$, therefore (3) reduces to

$$\mathrm{Ind}_{H_{t_2,t_3}}^{H_{t_3}}((\eta_X^V)_{t_2H} \otimes (\eta_X^W)_{t_2t_3H}) = \mathrm{Ind}_{H_{t_2,t_3}}^{H_{t_3}}((\eta_X^V)_{t_2H} \otimes (\eta_X^W)_{t_3H}).$$

Now, by Lemma 4.62, $H_{t_2} = H \cap t_2Ht_2^{-1}$ is the copy of S_2 in S_4 which fixes 2 and 4 and $H_{t_3} = H \cap t_3Ht_3^{-1}$ is the copy of S_2 in S_4 which fixes 3 and 4. Therefore, $H_{t_2,t_3} = H_{t_2} \cap H_{t_3}$ is the trivial group S_1 and hence

$$\mathrm{Ind}_{H_{t_2,t_3}}^{H_{t_3}}((\eta_X^V)_{t_2H} \otimes (\eta_X^W)_{t_3H}) = \mathrm{Ind}_{S_1}^{S_2}((\eta_X^V)_{t_2H} \otimes (\eta_X^W)_{t_3H}).$$

Furthermore V and W, and hence all the fibres of η_X^V and η_X^W, are one dimensional. It follows that $(\eta_X^V)_{t_2H} \otimes (\eta_X^W)_{t_3H}$ is one dimensional and therefore the fibre at t_3H is

$$\mathrm{Ind}_{S_1}^{S_2}(\mathrm{tr}).$$

One finds that the character of this representation is

$$(2, 0) = (1, 1) + (1, -1)$$

so $\mathrm{Ind}_{S_1}^{S_2}(\mathrm{tr}) \cong \mathrm{tr} \oplus \mathrm{sgn}$. Combining this with that above, we have

$$\eta_X^V * \eta_X^W = \begin{cases} \eta_H^{\mathrm{tr}} + \eta_H^{\mathrm{std}} + \eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}} & \text{if } V = W, \\ \eta_H^{\mathrm{sgn}} + \eta_H^{\mathrm{std}} + \eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}} & \text{if } V \neq W. \end{cases}$$

Therefore, the complete multiplication table is

	η_H^{sgn}	η_H^{std}	η_X^{tr}	η_X^{sgn}
η_H^{sgn}	η_H^{tr}	η_H^{std}	η_X^{sgn}	η_X^{tr}
η_H^{std}	η_H^{std}	$\eta_H^{\mathrm{tr}} + \eta_H^{\mathrm{sgn}} + \eta_H^{\mathrm{std}}$	$\eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}$	$\eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}$
η_X^{tr}	η_X^{sgn}	$\eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}$	$\eta_H^{\mathrm{tr}} + \eta_H^{\mathrm{std}} + \eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}$	$\eta_H^{\mathrm{sgn}} + \eta_H^{\mathrm{std}} + \eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}$
η_X^{sgn}	η_X^{tr}	$\eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}$	$\eta_H^{\mathrm{sgn}} + \eta_H^{\mathrm{std}} + \eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}$	$\eta_H^{\mathrm{tr}} + \eta_H^{\mathrm{std}} + \eta_X^{\mathrm{tr}} + \eta_X^{\mathrm{sgn}}$

$\square$

More generally, we can compute the product in $\mathcal{K}(S_n, S_{n-1})$ for $n \geq 4$ in a similar way.

Lemma 4.65. *For $n \geq 4$, the multiplication table for the categorical Hecke algebra $\mathcal{K}(S_n /\!/ S_{n-1})$ can be summarised as*

	$\eta_H^{V'}$	$\eta_X^{W'}$
η_H^V	$\eta_H^{V\otimes V'}$	$\eta_X^{V\|_{S_{n-2}}\otimes W'}$
η_X^W	$\eta_X^{W\otimes V'\|_{S_{n-2}}}$	$\eta_H^{\mathrm{Ind}_{S_{n-2}}^{S_{n-1}}(W\otimes W')} + \eta_X^{\mathrm{Ind}_{S_{n-3}}^{S_{n-2}}(W\|_{S_{n-3}}\otimes W'\|_{S_{n-3}})}$

where $X = Ht_{n-1}H = H(n-1,n)H$, $V, V' \in \mathsf{urep}(H)$ and $W, W' \in \mathsf{urep}(H_{t_{n-1}})$.

Proof. Firstly, by Corollary 4.54, for $V, V' \in \mathsf{urep}(H)$ we have

$$\eta_H^V * \eta_H^{V'} = \eta_H^{V\otimes V'}.$$

Next, by Corollary 4.55, for $V \in \mathsf{urep}(H)$ and $W \in \mathsf{urep}(H_{t_{n-1}})$, we have

$$\eta_H^V * \eta_X^W = \eta_X^{V|_{H_{t_3}}\otimes W}.$$

Products of the form $\eta_X^W * \eta_H^V$ are supported on X (since $\delta_X * \delta_H = \delta_X$ in the classical Hecke algebra) and the fibre at $t_{n-1}H$ is

$$\begin{aligned}(\eta_X^W * \eta_H^V)_{t_{n-1}H} &= \bigoplus_{i=1}^n (\eta_X^W)_{t_iH} \otimes (\eta_H^V)_{t_i^{-1}t_{n-1}H}\\ &= (\eta_X^W)_{t_{n-1}H} \otimes (\eta_H^V)_H\end{aligned}$$

which is equal to $W \otimes V$ as a Hilbert space. For any $h \in H_{t_{n-1}}$ the action is given by

$$\lambda_h(\xi \otimes \eta) = \lambda_h(\xi) \otimes \lambda_{t_{n-1}^{-1}ht_{n-1}}(\eta). \tag{4}$$

By Lemma 4.62, $H_{t_{n-1}}$ is the copy of S_{n-2} in S_n that fixes and $n-1$ and n and $t_{n-1} = (n-1, n)$. Therefore, h commutes with t_{n-1} and (4) reduces to

$$\lambda_h(\xi \otimes \eta) = \lambda_h(\xi) \otimes \lambda_h(\eta).$$

It follows that

$$\eta_X^W * \eta_H^V = \eta_X^{W\otimes V|_{H_{t_{n-1}}}}.$$

For products of the form $\eta_X^V * \eta_X^W$, the fibre at H is

$$(\eta_X^V * \eta_X^W)_H = \bigoplus_{i=1}^{n} (\eta_X^V)_{t_i H} \otimes (\eta_X^W)_{t_i^{-1} H}$$

which is isomorphic to

$$\bigoplus_{\delta \in \Delta_e} \mathrm{Ind}_{H_{\delta,e}}^{H} ((\eta_X^V)_{\delta H} \otimes (\eta_X^W)_{\delta^{-1} H}) \tag{5}$$

by Corollary 4.59. We recall that $\Delta_e \subset \Gamma$ is a set of representatives of the double coset space $H_e \backslash G/H = H \backslash G/H$ and so we can take $\Delta_e = \{e, t_{n-1}\}$. Now, $(\eta_X^V)_H$ and $(\eta_X^W)_H$ are zero so the only nonzero summand is the term with $\delta = t_{n-1}$. Furthermore,

$$\begin{aligned} H_{t_{n-1},e} &= H_{t_{n-1}} \cap H_e \\ &= H_{t_{n-1}} \cap H \\ &= H_{t_{n-1}} \end{aligned}$$

which by Lemma 4.62, is the copy of S_{n-2} in S_n that fixes $n-1$ and n. Therefore, (5) reduces to

$$\mathrm{Ind}_{H_{t_{n-1},e}}^{H} ((\eta_X^V)_{t_{n-1} H} \otimes (\eta_X^W)_{t_{n-1} H}) = \mathrm{Ind}_{S_{n-2}}^{S_{n-1}} (V \otimes W)$$

giving the description of the fibre at H.

We also need to compute the fibre at $t_{n-1}H$ which is

$$(\eta_X^V * \eta_X^W)_{t_{n-1}H} = \bigoplus_{i=1}^{n} (\eta_X^V)_{t_i H} \otimes (\eta_X^W)_{t_i^{-1} t_{n-1} H}.$$

By Corollary 4.59, this is isomorphic to

$$\bigoplus_{\delta \in \Delta_{t_{n-1}}} \mathrm{Ind}_{H_{\delta,t_{n-1}}}^{H_{t_{n-1}}} ((\eta_X^V)_{\delta H} \otimes (\eta_X^W)_{\delta^{-1} t_{n-1} H}) \tag{6}$$

where $\Delta_{t_{n-1}} \subset \Gamma$ is a set of representatives of $H_{t_{n-1}} \backslash G/H$ and

$$\begin{aligned} H_{\delta,t_{n-1}} &= H_\delta \cap H_{t_{n-1}} \\ &= H \cap \delta H \delta^{-1} \cap t_{n-1} H (t_{n-1})^{-1}. \end{aligned}$$

By Lemma 4.63, we can take $\Delta_{t_{n-1}} = \{e, t_{n-1}, t_{n-2}\}$ and since $(\eta_X^V)_H$ and $(\eta_X^W)_H$ are zero, the only nonzero summand in (6) is the term with $\delta = t_{n-2}$ and hence (6) reduces to

$$\begin{aligned}&\operatorname{Ind}_{H_{t_{n-2},t_{n-1}}}^{H_{t_{n-1}}}((\eta_X^V)_{t_{n-2}H} \otimes (\eta_X^W)_{t_{n-2}t_{n-1}H})\\ &= \operatorname{Ind}_{H_{t_{n-2},t_{n-1}}}^{H_{t_{n-1}}}((\eta_X^V)_{t_{n-2}H} \otimes (\eta_X^W)_{t_{n-1}H}).\end{aligned}$$

Now, by Lemma 4.62, $H_{t_{n-1}} = H \cap t_{n-1}H(t_{n-1})^{-1}$ is the copy of S_{n-2} in S_n that fixes $n-1$ and n, and $H_{t_{n-2}} = H \cap t_{n-2}H(t_{n-2})^{-1}$ is the copy of S_{n-2} in S_n that fixes $n-2$ and n. Therefore, $H_{t_{n-2},t_{n-1}} = H_{t_{n-2}} \cap H_{t_{n-1}}$ is the copy of S_{n-3} in S_n that fixes $n-2, n-1$ and n.

By Lemma 4.50, for $h \in H_{t_{n-2},t_{n-1}}$, the action on $(\eta_X^V)_{t_{n-2}H} \otimes (\eta_X^W)_{t_{n-1}H}$ is given by

$$\lambda_h(\xi \otimes \eta) = \lambda_h(\xi) \otimes \lambda_{(t_{n-2})^{-1}ht_{n-2}}(\eta). \tag{7}$$

Now, $t_{n-2} = (n-2, n)$ and $H_{t_{n-2},t_{n-1}}$ is the group of permutations of $\{1, \ldots, n-3\}$, therefore t_{n-2} commutes with h and hence (7) reduces to

$$\lambda_h(\xi \otimes \eta) = \lambda_h(\xi) \otimes \lambda_h(\eta). \tag{8}$$

Here the second tensor factor is the fibre $(\eta_X^W)_{t_{n-1}H}$ of η_X^W which is equal to W as a vector space and hence in the second tensor factor we have the representation $W|_{H_{t_{n-2},t_{n-1}}} = W|_{S_{n-3}}$.

With regard to the first tensor factor, we can express the representation of $H_{t_{n-2},t_{n-1}} = S_{n-3}$ on $(\eta_X^V)_{t_{n-2}H}$ in terms of the representation $V = (\eta_X^V)_{t_{n-1}H}$ of $H_{t_{n-1}} = S_{n-2}$. We recall that

$$\eta_X^V = \operatorname{Ind}_{H_{t_{n-1}}}^H(V)$$

as a representation of H. We can construct this by fixing a set Σ of coset representatives for $H/H_{t_{n-1}} = S_{n-1}/S_{n-2}$, we shall take

$$\Sigma = \{s_j = (j, n-1) \mid j = 1, \ldots, n-1\}$$

where $(n-1, n-1)$ is the identity. We then define

$$\operatorname{Ind}_{H_{t_{n-1}}}^H(V) = \bigoplus_{\sigma \in \Sigma} \sigma V$$

as a vector space where each σV is an isomorphic copy of V, the elements of which we write as σv, $v \in V$. For $h \in H$, the action is defined by

$$\lambda_h(\sigma v) = \sigma'(\lambda_k(v))$$

where $\sigma' \in \Sigma$ and $k \in H_{t_{n-1}}$ are determined by $h\sigma = \sigma' k$. The action of $C_0(G/H)$ on η_X^V is determined via the isomorphism

$$\begin{aligned} \varphi : H/H_{t_{n-1}} &\xrightarrow{\cong} Ht_{n-1}H/H \\ \sigma H_{t_{n-1}} &\mapsto \sigma t_{n-1} H. \end{aligned}$$

Specifically, the fibre at tH is zero if $t \notin Ht_{n-1}H$ and it is the summand σV where $\sigma \in \Sigma$ is determined by $\sigma t_{n-1} H = tH$ if $t \in Ht_{n-1}H$. In particular, the fibre $(\eta_X^V)_{t_{n-2}H}$ is the summand $s_2 V$ since

$$\begin{aligned} s_2 t_{n-1} H &= (n-2, n-1)(n-1, n)H \\ &= (n-2, n)H \\ &= t_{n-2}H. \end{aligned}$$

Returning to (8), this means that in the first tensor factor, $\xi = s_2 v$ for some $v \in V$. Now, if $h \in H_{t_{n-2}, t_{n-1}} = S_{n-3}$ then

$$\begin{aligned} hs_2 &= h(n-2, n-1) \\ &= (n-2, n-1)h \\ &= s_2 h \end{aligned}$$

and hence

$$\begin{aligned} \lambda_h(\xi) &= \lambda_h(s_2 v) \\ &= s_2(\lambda_h(v)). \end{aligned}$$

Therefore, the representation of S_{n-3} on the first tensor factor is isomorphic to $V|_{S_{n-3}}$. An intertwining operator is given by

$$\begin{aligned} \theta : s_2 V &\to V \\ s_2 v &\mapsto v. \end{aligned}$$

It follows that the representation of $H_{t_{n-2}, t_{n-1}} = S_{n-3}$ on $(\eta_X^V)_{t_{n-2}H} \otimes (\eta_X^W)_{t_{n-1}H}$ is isomorphic to $V|_{S_{n-3}} \otimes W|_{S_{n-3}}$ and hence

$$\begin{aligned} (\eta_X^V * \eta_X^W)_{t_{n-1}H} &\cong \operatorname{Ind}_{H_{t_{n-2}, t_{n-1}}}^{H_{t_{n-1}}} ((\eta_X^V)_{t_{n-2}H} \otimes (\eta_X^W)_{t_{n-1}H}) \\ &\cong \operatorname{Ind}_{S_{n-3}}^{S_{n-2}} (V|_{S_{n-3}} \otimes W|_{S_{n-3}}). \end{aligned}$$

Combined with description of the representation on the fibre at H, we have

$$\eta_X^V * \eta_X^W = \eta_H^{\mathrm{Ind}_{S_{n-2}}^{S_{n-1}}(V\otimes W)} + \eta_X^{\mathrm{Ind}_{S_{n-3}}^{S_{n-2}}(V|_{S_{n-3}}\otimes W|_{S_{n-3}})}$$

Combining all the above, the multiplication table for the categorical Hecke algebra can be summarised as

	$\eta_H^{V'}$	$\eta_X^{W'}$
η_H^V	$\eta_H^{V\otimes V'}$	$\eta_X^{V\|_{S_{n-2}}\otimes W'}$
η_X^W	$\eta_X^{W\otimes V'\|_{S_{n-2}}}$	$\eta_H^{\mathrm{Ind}_{S_{n-2}}^{S_{n-1}}(W\otimes W')} + \eta_X^{\mathrm{Ind}_{S_{n-3}}^{S_{n-2}}(W\|_{S_{n-3}}\otimes W'\|_{S_{n-3}})}$

□

4.3.2 The Bost-Connes algebra

To end with, we shall briefly look at the Bost-Connes algebra which is the Hecke algebra $\mathcal{H}(G/\!/H)$ where

$$G = P_{\mathbb{Q}}^{+} = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{Q} \text{ and } a > 0 \right\}$$

and

$$H = P_{\mathbb{Z}}^{+} = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{Z} \right\}.$$

This algebra is studied in detail in [6] but we shall just look at a presentation of the classical algebra and some related computations in the categorical algebra $\mathcal{K}(G/\!/H)$.

Before we look at a presentation for $\mathcal{H}(G/\!/H)$, let us comment on why (G, H) is a Hecke pair. If $\begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \in G$, with $a, b \in \mathbb{Q}$ and $a > 0$, the double coset $H \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} H$ consists of elements of the form

$$\begin{aligned} \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & b + ma \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & n + b + ma \\ 0 & a \end{bmatrix} \end{aligned}$$

where $m, n \in \mathbb{Z}$. Now, if $c, d \in \mathbb{Q}$ with $c > 0$, then the coset $\begin{bmatrix} 1 & d \\ 0 & c \end{bmatrix} H$ consists of elements of the form

$$\begin{bmatrix} 1 & d \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+d \\ 0 & c \end{bmatrix}$$

where $n \in \mathbb{Z}$. If we then write $a = p/q$ with $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ we can see that $H \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} H$ is the disjoint union of the cosets $\begin{bmatrix} 1 & b+ia \\ 0 & a \end{bmatrix} H$ for $i = 0, \ldots, p-1$.

The algebra $\mathcal{H}(G/\!/H)$ has a basis $\{e_X\}$ indexed by the double cosets $X \in H\backslash G/H$. In order to describe a presentation of the algebra, we shall introduce the following notation:

- For $n \in \mathbb{N}$, we define $\mu_n := n^{-1/2} e_{X_n}$ where

$$X_n = H \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} H = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} H.$$

- For $\gamma \in [0, 1) \subset \mathbb{Q}$, we define $\varepsilon_\gamma := e_{X^\gamma}$ where

$$X^\gamma = H \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} H = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} H.$$

One then has the following, which is Proposition 18 in [6].

Proposition 4.66. *The elements μ_n, ε_γ, $n \in \mathbb{N}$, $\gamma \in [0, 1) \subset \mathbb{Q}$ generate the involutive algebra $\mathcal{H}(G/\!/H)$ and the following relations give a presentation of $\mathcal{H}(G/\!/H)$.*

(a) $\mu_n^* \mu_n = 1$, $\forall n \in \mathbb{N}$.

(b) $\mu_{nm} = \mu_n \mu_m$, $\forall n, m \in \mathbb{N}$.

(c) $\mu_n \mu_m^* = \mu_m^* \mu_n$ *if* $\gcd(n, m) = 1$.

(d) $\varepsilon_\gamma^* = \varepsilon_{-\gamma}$, $\varepsilon_{\gamma_1+\gamma_2} = \varepsilon_{\gamma_1}\varepsilon_{\gamma_2}$, $\forall \gamma, \gamma_1, \gamma_2 \in [0, 1)$.

(e) $\varepsilon_\gamma \mu_n = \mu_n \varepsilon_{n\gamma}$, $\forall n \in \mathbb{N}$, $\gamma \in [0, 1)$.

(f) $\mu_n \varepsilon_\gamma \mu_n^* = \frac{1}{n} \sum_{\substack{\delta \in [0,1), \\ n\delta \equiv \gamma \bmod \mathbb{Z}}} \varepsilon_\delta$, $\forall n \in \mathbb{N}$, $\gamma \in [0, 1)$.

In order to do some computations in $\mathcal{K}(G/\!/H)$ we shall introduce some notation. Since H and all its subgroups are isomorphic to the integers, when computing the categorical product, the only representations we really need to consider are characters on the integers. Let us introduce the notation

$$nH = \left\{ \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix} : b \in \mathbb{Z} \right\}$$

for $n \in \mathbb{N}$. Given a double coset HtH, the stabiliser group H_t is nH for some $n \in \mathbb{N}$ and an element of the categorical Hecke algebra supported on HtH is of the form η_{HtH}^{V} for $V \in \mathsf{urep}(nH)$. Given $\sigma \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, we shall write $\chi_\sigma \in \mathsf{urep}(nH)$ to denote the one dimensional representation of nH on $\mathbb{C}$ where the generator $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ of nH acts by multiplication by σ. We shall also introduce the notation

$$t_{b,a} = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}, \qquad a, b \in \mathbb{Q}, a > 0.$$

We then have something of a categorical analogue of Proposition 4.66 in the following:

Proposition 4.67. *The following relations hold in $\mathcal{K}(G/\!/H)$.*

(a) $(\eta_{X_n}^{\chi_\alpha})^* * \eta_{X_n}^{\chi_\beta} = \sum_{i=1}^{n} \eta_H^{\chi_{\sigma_i}}$, $\forall n \in \mathbb{N}$, *where the σ_i are the n distinct n-th roots of $\overline{\alpha}\beta$.*

(b) $\eta_{X_m}^{\chi_\alpha} * \eta_{X_n}^{\chi_\beta} = \eta_{X_{mn}}^{\chi_{\alpha\beta^m}}$, $\forall m, n \in \mathbb{N}$.

(c) $(\eta_{X_m}^{\chi_\alpha})^* * \eta_{X_n}^{\chi_\beta} = \eta_{Ht_{0,n/m}H}^{\chi_{\overline{\alpha}\beta}}$ *and* $\eta_{X_n}^{\chi_\beta} * (\eta_{X_m}^{\chi_\alpha})^* = \eta_{Ht_{0,n/m}H}^{\chi_{\beta^m\overline{\alpha}^n}}$ *when* $\gcd(m,n) = 1$.

(d) $(\eta_{X^\gamma}^{\chi_\alpha})^* = \eta_{X^{-\gamma}}^{\chi_{\overline{\alpha}}}$ *and* $\eta_{X^{\gamma_1}}^{\chi_{\alpha_1}} * \eta_{X^{\gamma_2}}^{\chi_{\alpha_2}} = \eta_{X^{\gamma_1+\gamma_2}}^{\chi_{\alpha_1\alpha_2}}$, $\forall \gamma, \gamma_1, \gamma_2 \in [0,1)$.

(e) $\eta_{X^\gamma}^{\chi_\alpha} * \eta_{X_n}^{\chi_\beta} = \eta_{Ht_{n\gamma,n}H}^{\chi_{\alpha\beta}}$ *and* $\eta_{X_n}^{\chi_\beta} * \eta_{X^{n\gamma}}^{\chi_\alpha} = \eta_{Ht_{n\gamma,n}H}^{\chi_{\beta\alpha^n}}$ *for all $n \in \mathbb{N}$, $\gamma \in [0,1)$.*

(f) $\eta_{X_n}^{\chi_\alpha} * \eta_{X^\gamma}^{\chi_\sigma} * (\eta_{X_n}^{\chi_\beta})^* = \sum_{\substack{\delta \in [0,1), \\ n\delta \equiv \gamma \bmod \mathbb{Z}}} \eta_{X^\delta}^{\chi_{\alpha\sigma^n\overline{\beta}}}$ *for all $n \in \mathbb{N}$, $\gamma \in [0,1)$.*

Proof. (a) We first note that

$$\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/n \end{bmatrix} = t_{0,1/n}.$$

Therefore, by Lemma 4.51, $(\eta^{\chi_\alpha}_{X_n})^* = \eta^{\chi_{\overline{\alpha}}}_{Ht_{0,1/n}H}$. We also note that

$$Ht_{0,1/n}H = \bigsqcup_{i=0}^{n-1} t_{i/n,1/n}H.$$

The group $H_{t_{0,1/n}}$ is nH so that on the fibre $(\eta^{\chi_\alpha}_{X_n})^*_{t_{0,1/n}H} = (\eta^{\chi_{\overline{\alpha}}}_{Ht_{0,1/n}H})_{t_{0,1/n}H}$, $t_{n,1} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ acts as multiplication by $\overline{\alpha}$. Since $\mu_n^*\mu_n = 1$ in $\mathcal{H}(G/\!/H)$, there is only one nontrivial fibre in the product $(\eta^{\chi_\alpha}_{X_n})^* * \eta^{\chi_\beta}_{X_n}$, namely

$$\begin{aligned}
((\eta^{\chi_\alpha}_{X_n})^* * \eta^{\chi_\beta}_{X_n})_H &= \bigoplus_{i=0}^{n-1} (\eta^{\chi_\alpha}_{X_n})^*_{t_{i/n,1/n}H} \otimes (\eta^{\chi_\beta}_{X_n})_{(t_{i/n,1/n})^{-1}H} \\
&= \bigoplus_{i=0}^{n-1} (\eta^{\chi_\alpha}_{X_n})^*_{t_{i/n,1/n}H} \otimes (\eta^{\chi_\beta}_{X_n})_{t_{0,n}H}.
\end{aligned}$$

To compute the representation we can use the formula

$$((\eta^{\chi_\alpha}_{X_n})^* * \eta^{\chi_\beta}_{X_n})_H = \bigoplus_{\delta \in \Delta_e} \mathrm{Ind}^H_{H_{\delta,e}} (\eta^{\chi_\alpha}_{X_n})^*_{\delta H} \otimes (\eta^{\chi_\beta}_{X_n})_{\delta^{-1}H} \tag{9}$$

from Corollary 4.59. Now, $(\eta^{\chi_\alpha}_{X_n})^*$ is supported on the double coset $Ht_{0,1/n}H$ and we may choose $\delta = t_{0,1/n}$ as our representative of this double coset. Then (9) becomes

$$\begin{aligned}
((\eta^{\chi_\alpha}_{X_n})^* * \eta^{\chi_\beta}_{X_n})_H &= \mathrm{Ind}^H_{H_{t_{0,1/n},e}} (\eta^{\chi_\alpha}_{X_n})^*_{t_{0,1/n}H} \otimes (\eta^{\chi_\beta}_{X_n})_{t^{-1}_{0,1/n}H} \\
&= \mathrm{Ind}^H_{H_{t_{0,1/n},e}} (\eta^{\chi_\alpha}_{X_n})^*_{t_{0,1/n}H} \otimes (\eta^{\chi_\beta}_{X_n})_{t_{0,1/n}H}.
\end{aligned}$$

Since

$$\begin{aligned}
H_{t_{0,1/n},e} &= H_{t_{0,1/n}} \cap H_e \\
&= H_{t_{0,1/n}} \\
&= nH,
\end{aligned}$$

we just need to compute the action of $t_{n,1} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ on

$$(\eta^{\chi_\alpha}_{X_n})^*_{t_{0,1/n}H} \otimes (\eta^{\chi_\beta}_{X_n})_{t_{0,1/n}H}.$$

We have

$$\lambda_{t_{n,1}}(\xi \otimes \eta) = \lambda_{t_{n,1}}(\xi) \otimes \lambda_{t_{0,1/n}^{-1} t_{n,1} t_{0,1/n}}(\eta)$$

and

$$\begin{aligned} t_{0,1/n}^{-1} t_{n,1} t_{0,1/n} &= \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/n \end{bmatrix} \\ &= \begin{bmatrix} 1 & n \\ 0 & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/n \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= t_{1,1} \end{aligned}$$

so that

$$\begin{aligned} \lambda_{t_{n,1}}(\xi \otimes \eta) &= \lambda_{t_{n,1}}(\xi) \otimes \lambda_{t_{1,1}}(\eta) \\ &= \overline{\alpha}\xi \otimes \beta\eta \\ &= \overline{\alpha}\beta\xi \otimes \eta. \end{aligned}$$

Therefore,

$$\begin{aligned} ((\eta_{X_n}^{\chi_\alpha})^* * \eta_{X_n}^{\chi_\beta})_H &= \mathrm{Ind}_{H_{e,t_{0,1/n}}}^{H} (\eta_{X_n}^{\chi_\alpha})^*_{t_{0,1/n}H} \otimes (\eta_{X_n}^{\chi_\beta})_{t_{0,1/n}^{-1}H} \\ &= \mathrm{Ind}_{nH}^{H} \chi_{\overline{\alpha}\beta} \\ &= \bigoplus_{i=1}^{n} \chi_{\sigma_i} \end{aligned}$$

where the σ_i are the distinct n-the roots of $\overline{\alpha}\beta$. In summary,

$$(\eta_{X_n}^{\chi_\alpha})^* * \eta_{X_n}^{\chi_\beta} = \bigoplus_{i=1}^{n} \eta_H^{\chi_{\sigma_i}}.$$

In particular, if $\alpha = \beta$ then

$$(\eta_{X_n}^{\chi_\alpha})^* * \eta_{X_n}^{\chi_\alpha} = \bigoplus_{i=1}^{n} \eta_H^{\chi_{\zeta_i}}$$

where the ζ_i are the distinct n-th roots of unity.

(b) Since $\mu_m\mu_n = \mu_{mn}$ in $\mathcal{H}(G/\!/H)$, there is only one nontrivial fibre in the product, namely

$$(\eta^{\chi_\alpha}_{X_m} * \eta^{\chi_\beta}_{X_n})_{t_{0,mn}H} = (\eta^{\chi_\alpha}_{X_m})_{t_{0,m}H} \otimes (\eta^{\chi_\beta}_{X_n})_{t_{0,n}H}.$$

The stabiliser group $H_{t_{0,mn}}$ is H so to compute the representation on this fibre we just need to compute the action of $t_{1,1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We have

$$\lambda_{t_{1,1}}(\xi \otimes \eta) = \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{0,m}^{-1}t_{1,1}t_{0,m}}(\eta)$$

and

$$\begin{aligned}
t_{0,m}^{-1}t_{1,1}t_{0,m} &= \begin{bmatrix} 1 & 0 \\ 0 & 1/m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1/m \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \\
&= \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \\
&= t_{m,1}
\end{aligned}$$

so that

$$\begin{aligned}
\lambda_{t_{1,1}}(\xi \otimes \eta) &= \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{m,1}}(\eta) \\
&= \alpha\xi \otimes \beta^m\eta \\
&= \alpha\beta^m\xi \otimes \eta.
\end{aligned}$$

Therefore,

$$\eta^{\chi_\alpha}_{X_m} * \eta^{\chi_\beta}_{X_n} = \eta^{\chi_{\alpha\beta^m}}_{X_{mn}}.$$

We note that these 2 factors do not commute since

$$\begin{aligned}
\eta^{\chi_\beta}_{X_n} * \eta^{\chi_\alpha}_{X_m} &= \eta^{\chi_{\beta\alpha^n}}_{X_{mn}} \\
&\neq \eta^{\chi_{\alpha\beta^m}}_{X_{mn}} \\
&= \eta^{\chi_\alpha}_{X_m} * \eta^{\chi_\beta}_{X_n}.
\end{aligned}$$

(c) In the classical algebra, $e^*_{X_m} = e_{Ht_{0,1/m}H}$ and

$$Ht_{0,1/m}H = \bigsqcup_{i=0}^{m-1} t_{i/m,1/m}H.$$

When $\gcd(m,n)=1$, one has

$$\begin{aligned} e_{X_m}^* e_{X_n} &= e_{X_n} e_{X_m}^* \\ &= e_{Ht_{0,n/m}H} \end{aligned}$$

and

$$Ht_{0,n/m}H = \bigsqcup_{i=0}^{m-1} t_{i/m,n/m}H.$$

It follows that $(\eta_{X_m}^{\chi_\alpha})^* * \eta_{X_n}^{\chi_\beta} = \eta_{Ht_{0,1/m}H}^{\chi_{\overline{\alpha}}} * \eta_{X_n}^{\chi_\beta}$ is supported on the double coset $Ht_{0,n/m}H$ and the fibre at $t_{0,n/m}$ is

$$\begin{aligned} ((\eta_{X_m}^{\chi_\alpha})^* * \eta_{X_n}^{\chi_\beta})_{t_{0,n/m}H} &= (\eta_{X_m}^{\chi_\alpha})_{t_{0,1/m}H}^* \otimes (\eta_{X_n}^{\chi_\beta})_{t_{0,n}H} \\ &= (\eta_{Ht_{0,1/m}H}^{\chi_{\overline{\alpha}}})_{t_{0,1/m}H} \otimes (\eta_{X_n}^{\chi_\beta})_{t_{0,n}H}. \end{aligned}$$

Since $H_{t_{0,n/m}} = mH$, we just need to compute the action of $t_{m,1}$ on this fibre. We have

$$\lambda_{t_{m,1}} = \lambda_{t_{m,1}}(\xi) \otimes \lambda_{t_{0,1/m}^{-1} t_{m,1} t_{0,1/m}}(\eta)$$

and

$$\begin{aligned} t_{0,1/m}^{-1} t_{m,1} t_{0,1/m} &= \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/m \end{bmatrix} \\ &= \begin{bmatrix} 1 & m \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/m \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= t_{1,1} \end{aligned}$$

so that

$$\begin{aligned} \lambda_{t_{m,1}} &= \lambda_{t_{m,1}}(\xi) \otimes \lambda_{t_{1,1}}(\eta) \\ &= \overline{\alpha}\xi \otimes \beta\eta \\ &= \overline{\alpha}\beta\xi \otimes \eta \end{aligned}$$

and hence

$$(\eta_{X_m}^{\chi_\alpha})^* * \eta_{X_n}^{\chi_\beta} = \eta_{Ht_{0,n/m}H}^{\chi_{\overline{\alpha}\beta}}.$$

When we reverse the order of the factors, $\eta^{\chi_\beta}_{X_n} * (\eta^{\chi_\alpha}_{X_m})^*$ is also supported on the double coset $Ht_{0,n/m}H$. The fibre at $t_{0,n/m}H$ is

$$\begin{aligned}(\eta^{\chi_\beta}_{X_n} * (\eta^{\chi_\alpha}_{X_m})^*)_{t_{0,n/m}H} &= (\eta^{\chi_\beta}_{X_n})_{t_{0,n}H} \otimes (\eta^{\chi_\alpha}_{X_m})^*_{t_{0,1/m}H} \\ &= (\eta^{\chi_\beta}_{X_n})_{t_{0,n}H} \otimes (\eta^{\chi_{\overline{\alpha}}}_{Ht_{0,1/m}H})^*_{t_{0,1/m}H}\end{aligned}$$

and again, we just need to compute the action of $t_{m,1} \in H_{t_{0,n/m}}$. We have

$$\lambda_{t_{m,1}}(\xi \otimes \eta) = \lambda_{t_{m,1}}(\xi) \otimes \lambda_{t_{0,n}^{-1}t_{m,1}t_{0,n}}(\eta)$$

and

$$\begin{aligned}t_{0,n}^{-1}t_{m,1}t_{0,n} &= \begin{bmatrix}1 & 0 \\ 0 & 1/n\end{bmatrix}\begin{bmatrix}1 & m \\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & n\end{bmatrix} \\ &= \begin{bmatrix}1 & m \\ 0 & 1/n\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & n\end{bmatrix} \\ &= \begin{bmatrix}1 & mn \\ 0 & 1\end{bmatrix} \\ &= t_{mn,1}\end{aligned}$$

so that

$$\begin{aligned}\lambda_{t_{m,1}}(\xi \otimes \eta) &= \lambda_{t_{m,1}}(\xi) \otimes \lambda_{t_{mn,1}}(\eta) \\ &= \beta^m \xi \otimes \overline{\alpha}^n \eta \\ &= \beta^m \overline{\alpha}^n \xi \otimes \eta\end{aligned}$$

and

$$\eta^{\chi_\beta}_{X_n} * (\eta^{\chi_\alpha}_{X_m})^* = \eta^{\chi_{\beta^m\overline{\alpha}^n}}_{Ht_{0,n/m}H}.$$

(d) The equality $(\eta^{\chi_\alpha}_{X^\gamma})^* = \eta^{\chi_{\overline{\alpha}}}_{X^{-\gamma}}$ follows from the fact that $e^*_{X^\gamma} = e_{X^{-\gamma}}$. To compute $\eta^{\chi_{\alpha_1}}_{X^{\gamma_1}} * \eta^{\chi_{\alpha_2}}_{X^{\gamma_2}}$, we note there is only one nontrivial fibre in the product, namely

$$(\eta^{\chi_{\alpha_1}}_{X^{\gamma_1}} * \eta^{\chi_{\alpha_2}}_{X^{\gamma_2}})_{t_{(\gamma_1+\gamma_2),1}H} = (\eta^{\chi_{\alpha_1}}_{X^{\gamma_1}})_{t_{\gamma_1,1}H} \otimes (\eta^{\chi_{\alpha_2}}_{X^{\gamma_2}})_{t_{\gamma_2,1}H}.$$

Since $H_{t_{(\gamma_1+\gamma_2),1}} = H$, we just need to compute the action of $t_{1,1} = \begin{bmatrix}1 & 1 \\ 0 & 1\end{bmatrix}$ on this fibre. We have

$$\lambda_{t_{1,1}}(\xi \otimes \eta) = \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{\gamma_1,1}^{-1}t_{1,1}t_{\gamma_1,1}}(\eta)$$

and

$$\begin{aligned} t_{\gamma_1,1}^{-1} t_{1,1} t_{\gamma_1,1} &= \begin{bmatrix} 1 & -\gamma_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma_1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= t_{1,1} \end{aligned}$$

so that

$$\begin{aligned} \lambda_{t_{1,1}}(\xi \otimes \eta) &= \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{1,1}}(\eta) \\ &= \alpha_1 \xi \otimes \alpha_2 \eta \\ &= \alpha_1 \alpha_2 \xi \otimes \eta \end{aligned}$$

and

$$\eta_{X^{\gamma_1}}^{\chi_{\alpha_1}} * \eta_{X^{\gamma_2}}^{\chi_{\alpha_2}} = \eta_{X^{\gamma_1+\gamma_2}}^{\chi_{\alpha_1 \alpha_2}}.$$

(e) There is one nontrivial fibre in the product $\eta_{X^\gamma}^{\chi_\alpha} * \eta_{X_n}^{\chi_\beta}$, namely

$$(\eta_{X^\gamma}^{\chi_\alpha} * \eta_{X_n}^{\chi_\beta})_{t_{n\gamma,n}H} = (\eta_{X^\gamma}^{\chi_\alpha})_{t_{\gamma,1}H} \otimes (\eta_{X_n}^{\chi_\beta})_{t_{0,n}H}.$$

Since $H_{t_{n\gamma,n}} = H$, we just need to compute the action of $t_{1,1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ on this fibre. We have

$$\begin{aligned} \lambda_{t_{1,1}}(\xi \otimes \eta) &= \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{\gamma,1}^{-1} t_{1,1} t_{\gamma,1}}(\eta) \\ &= \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{1,1}}(\eta) \\ &= \alpha \xi \otimes \beta \eta \\ &= \alpha \beta \xi \otimes \eta \end{aligned}$$

so that

$$\eta_{X^\gamma}^{\chi_\alpha} * \eta_{X_n}^{\chi_\beta} = \eta_{Ht_{n\gamma,n}H}^{\chi_{\alpha\beta}}.$$

Similarly, there is one nontrivial fibre in the product $\eta_{X_n}^{\chi_\beta} * \eta_{X^{n\gamma}}^{\chi_\alpha}$, namely

$$(\eta_{X_n}^{\chi_\beta} * \eta_{X^{n\gamma}}^{\chi_\alpha})_{t_{n\gamma,n}H} = (\eta_{X_n}^{\chi_\beta})_{t_{0,n}H} \otimes (\eta_{X^{n\gamma}}^{\chi_\alpha})_{t_{n\gamma,1}H}.$$

We have

$$\lambda_{t_{1,1}}(\xi \otimes \eta) = \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{0,n}^{-1} t_{1,1} t_{0,n}}(\eta)$$

and

$$\begin{aligned} t_{0,n}^{-1} t_{1,1} t_{0,n} &= \begin{bmatrix} 1 & 0 \\ 0 & 1/n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1/n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \\ &= \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \end{aligned}$$

so that

$$\begin{aligned} \lambda_{t_{1,1}}(\xi \otimes \eta) &= \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{n,1}}(\eta) \\ &= \beta\xi \otimes \alpha^n \eta \\ &= \beta\alpha^n \xi \otimes \alpha \end{aligned}$$

and hence

$$\eta_{X_n}^{\chi_\beta} * \eta_{X^{n\gamma}}^{\chi_\alpha} = \eta_{Ht_{n\gamma,n}H}^{\chi_{\beta\alpha^n}}.$$

(f) The underlying Hilbert space of the product is isomorphic to

$$\begin{aligned} &\bigoplus_{i=0}^{n-1} (\eta_{X_n}^{\chi_\alpha})_{t_{0,n}H} \otimes (\eta_{X^\gamma}^{\chi_\sigma})_{t_{\gamma,1}H} \otimes (\eta_{X_n}^{\chi_\beta})^*_{t_{i/n,1/n}H} \\ = &\bigoplus_{i=0}^{n-1} (\eta_{X_n}^{\chi_\alpha})_{t_{0,n}H} \otimes (\eta_{X^\gamma}^{\chi_\sigma})_{t_{\gamma,1}H} \otimes (\eta_{Ht_{0,1/n}H}^{\chi_{\overline{\beta}}})_{t_{i/n,1/n}H}. \end{aligned}$$

If we consider $\xi \otimes \eta \otimes \omega \in (\eta_{X_n}^{\chi_\alpha})_{t_{0,n}H} \otimes (\eta_{X^\gamma}^{\chi_\sigma})_{t_{\gamma,1}H} \otimes (\eta_{Ht_{0,1/n}H}^{\chi_{\overline{\beta}}})_{t_{i/n,1/n}H}$, then

$$\begin{aligned} \lambda_{t_{1,1}}(\xi \otimes \eta \otimes \omega) &= \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{0,n}^{-1} t_{1,1} t_{0,n}}(\eta \otimes \omega) \\ &= \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{n,1}}(\eta \otimes \omega) \\ &= \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{n,1}}(\eta) \otimes \lambda_{t_{\gamma,1}^{-1} t_{n,1} t_{\gamma,1}}(\omega) \\ &= \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{n,1}}(\eta) \otimes \lambda_{t_{n,1}}(\omega). \end{aligned}$$

Since $t_{1,1} \in H_{t_{0,n}}$, $t_{n,1} \in H_{t_{\gamma,1}}$ and $t_{n,1} \in H_{t_{i/n,1/n}}$, it follows that $\lambda_{t_{1,1}}(\xi \otimes \eta \otimes \omega)$ is also in $(\eta^{\chi_\alpha}_{X_n})_{t_{0,n}H} \otimes (\eta^{\chi_\sigma}_{X^\gamma})_{t_{\gamma,1}H} \otimes (\eta^{\chi_{\overline{\beta}}}_{Ht_{0,1/n}H})_{t_{i/n,1/n}H}$ and the representation of H on

$$\bigoplus_{i=0}^{n-1} (\eta^{\chi_\alpha}_{X_n})_{t_{0,n}H} \otimes (\eta^{\chi_\sigma}_{X^\gamma})_{t_{\gamma,1}H} \otimes (\eta^{\chi_{\overline{\beta}}}_{Ht_{0,1/n}H})_{t_{i/n,1/n}H}.$$

is a direct sum of n 1-dimensional representations.

We have

$$\begin{aligned} \lambda_{t_{1,1}}(\xi \otimes \eta \otimes \omega) &= \lambda_{t_{1,1}}(\xi) \otimes \lambda_{t_{n,1}}(\eta) \otimes \lambda_{t_{n,1}}(\omega) \\ &= \alpha\xi \otimes \sigma^n \eta \otimes \overline{\beta}\omega \\ &= \alpha\sigma^n\overline{\beta}\xi \otimes \eta \otimes \omega. \end{aligned}$$

Since

$$\begin{aligned} t_{0,n}t_{\gamma,1}t_{i/n,1/n} &= \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & i/n \\ 0 & 1/n \end{bmatrix} \\ &= \begin{bmatrix} 1 & \gamma \\ 0 & n \end{bmatrix} \begin{bmatrix} 1 & i/n \\ 0 & 1/n \end{bmatrix} \\ &= \begin{bmatrix} 1 & i/n + \gamma/n \\ 0 & 1 \end{bmatrix} \end{aligned}$$

it follows that

$$\eta^{\chi_\alpha}_{X_n} * \eta^{\chi_\sigma}_{X^\gamma} * (\eta^{\chi_\beta}_{X_n})^* = \sum_{\substack{\delta \in [0,1), \\ n\delta \equiv \gamma \bmod \mathbb{Z}}} \eta^{\chi_{\alpha\sigma^n\overline{\beta}}}_{X^\delta}.$$

□

Whilst the computations mirror those in Proposition 4.66, it may not be the case that we have found a generating set and presentation for $\mathcal{K}(G/\!/H)$.

4.4 Maps between $\mathcal{H}(G/\!/H)$ and $\mathcal{K}(G/\!/H)$

Earlier, in Lemma 4.53 we noted that for any Hecke pair (G, H), there is a surjective homomorphism

$$\varphi : \mathcal{K}(G/\!/H) \to \mathcal{H}(G/\!/H)$$

defined by $\varphi([\mathcal{H}])(gH) := \mathcal{H}_{gH}$. To finish, we shall make some remakes about when there is an algebra homomorphism

$$\psi : \mathcal{H}(G/\!/H) \to \mathcal{K}(G/\!/H)$$

such that $\varphi \circ \psi$ is the identity.

In general, there is no such unit preserving algebra homomorphism. We can see this by considering the Hecke pairs (S_n, S_{n-1}) for $n \in \mathbb{N}$. We recall that there are two double cosets, H and $X = H(n-1,n)H$ and that in $\mathcal{H}(S_n/\!/S_{n-1})$, the multiplication is determined by

$$\delta_X * \delta_X = (n-1)\delta_H + (n-2)\delta_X.$$

For $n \geq 4$, the multiplication table for $\mathcal{K}(S_n/\!/S_{n-1})$ is

	$\eta_H^{V'}$	$\eta_X^{W'}$
η_H^{V}	$\eta_H^{V\otimes V'}$	$\eta_X^{V\|_{S_{n-2}}\otimes W'}$
η_X^{W}	$\eta_X^{W\otimes V'\|_{S_{n-2}}}$	$\eta_H^{\mathrm{Ind}_{S_{n-2}}^{S_{n-1}}(W\otimes W')} + \eta_X^{\mathrm{Ind}_{S_{n-3}}^{S_{n-2}}(W\|_{S_{n-3}}\otimes W'\|_{S_{n-3}})}$

If we suppose that $\psi : \mathcal{H}(S_n/\!/S_{n-1}) \to \mathcal{K}(S_n/\!/S_{n-1})$ is a unital algebra homomorphism then it maps δ_H to η_H^{tr}. If the identity map on $\mathcal{H}(S_n/\!/S_{n-1})$ is to factorise through ψ, we must have $\psi(\delta_X) = \frac{1}{\dim W}\eta_X^W$ for some finite dimensional unitary representation W of $H_{(n-1,n)}$. Then, since ψ is an algebra homomorphism we must have

$$\begin{aligned}
(n-1)\eta_H^{\mathrm{tr}} + \frac{(n-2)}{\dim W}\eta_X^W &= \psi((n-1)\delta_H + (n-2)\delta_X) \\
&= \psi(\delta_X * \delta_X) \\
&= \psi(\delta_X) * \psi(\delta_X) \\
&= \frac{1}{\dim W}\eta_X^W * \frac{1}{\dim W}\eta_X^W \\
&= \frac{1}{(\dim W)^2}\left(\eta_H^{\mathrm{Ind}_{S_{n-2}}^{S_{n-1}}(W\otimes W)} + \eta_X^{\mathrm{Ind}_{S_{n-3}}^{S_{n-2}}(W|_{S_{n-3}}\otimes W|_{S_{n-3}})}\right)
\end{aligned}$$

This means that $\mathrm{Ind}_{S_{n-2}}^{S_{n-1}}(W \otimes W)$ must be isomorphic to a direct sum of copies of the trivial representation of S_{n-1}. However, for any choice of W, the trace of an $(n-1)$ cycle in $\mathrm{Ind}_{S_{n-2}}^{S_{n-1}}(W \otimes W)$ is zero and hence this cannot be a direct sum of copies of the trivial representation. Therefore, there cannot be such a unital homomorphism

ψ. From this, we can also see that in general there is no homomorphism ψ from $\mathcal{H}(G/\!/H)$ to $\mathcal{K}(G/\!/H)$ which maps elements of the form δ_{HtH} to elements of the form η^V_{HtH}. Such a map would have to map the identity element to η^V_H for some $V \in \mathsf{urep}(H)$ with the property that $V \otimes V \cong V$. However, this means that V would have to be the trivial representation so that η^V_H is the unit of $\mathcal{K}(G/\!/H)$ making ψ a unital algebra homomorphism.

Although no unital homomorphism exists, as noted in [28], for any Hecke pair (G, H) for which the group H is finite, there is a non-unital algebra homomorphism $\psi : \mathcal{H}(G/\!/H) \to \mathcal{K}(G/\!/H)$ through which the identity factorises defined by

$$\psi(\delta_{HtH}) := \frac{1}{|H_t|}\eta^{\mathrm{reg}}_{HtH},$$

where reg denotes the left regular representation of H_t. The main point is that for any finite group K, the character χ of the regular representation is given by

$$\chi(k) = \begin{cases} |K| & \text{if } k = e, \\ 0 & \text{otherwise} \end{cases}$$

and the constructions of the representations in the product result in representations which have characters that are 0 away from the identity so that they are isomorphic to direct sums of regular representations. This is explained in more detail in the following lemma.

Lemma 4.68. *The additive map*

$$\psi : \mathcal{H}(G/\!/H) \to \mathcal{K}(G/\!/H)$$

defined by

$$\psi(\delta_{HtH}) := \frac{1}{|H_t|}\eta^{\mathrm{reg}}_{HtH},$$

is multiplicative.

Proof. Given $\eta^{\mathrm{reg}}_{HxH}, \eta^{\mathrm{reg}}_{HyH} \in C_0(G/H)\text{-}H\text{-}\mathsf{Mod}^f_c$, by Corollary 4.59 the fibres of the product $\eta^{\mathrm{reg}}_{HxH} * \eta^{\mathrm{reg}}_{HyH}$ are given up to isomorphism by

$$\left(\eta^{\mathrm{reg}}_{HxH} * \eta^{\mathrm{reg}}_{HyH}\right)_{tH} \cong \bigoplus_{\delta \in \Delta_t} \mathrm{Ind}^{H_t}_{H_{\delta,t}} \left((\eta^{\mathrm{reg}}_{HxH})_{\delta H} \otimes \left(\eta^{\mathrm{reg}}_{HyH}\right)_{\delta^{-1}tH} \right).$$

By Lemma 4.50, for any $\delta \in \Delta_t$, $h \in H_{\delta,t}$ and $\xi \otimes \eta \in (\eta^{\mathrm{reg}}_{HxH})_{\delta H} \otimes \left(\eta^{\mathrm{reg}}_{HyH}\right)_{\delta^{-1}tH}$ the action is given by

$$\lambda_h(\xi \otimes \eta) = \lambda_h(\xi) \otimes \lambda_{\delta^{-1}h\delta}(\eta).$$

Therefore, the character χ of this representation of $H_{\delta,t}$ is given by

$$\chi(h) = \begin{cases} \dim\ (\eta^{\mathrm{reg}}_{HxH})_{\delta H} \otimes \left(\eta^{\mathrm{reg}}_{HyH}\right)_{\delta^{-1}tH} & \text{if } h = e, \\ 0 & \text{otherwise.} \end{cases}$$

The character of the induced representation of H_t is also 0 away from the identity, as is the character of the direct sum of such representations. Therefore, the representation of H_t on the fibre $\left(\eta^{\mathrm{reg}}_{HxH} * \eta^{\mathrm{reg}}_{HyH}\right)_{tH}$ is isomorphic to a (possibly empty) direct sum of copies of the regular representation of H_t.

We also have the formula

$$\left(\eta^{\mathrm{reg}}_{HxH} * \eta^{\mathrm{reg}}_{HyH}\right)_{tH} = \bigoplus_{\gamma \in \Gamma} (\eta^{\mathrm{reg}}_{HxH})_{\gamma H} \otimes \left(\eta^{\mathrm{reg}}_{HyH}\right)_{\gamma^{-1}tH}$$

for the fibres of the product. Therefore, to compute the number of (isomorphic) copies of the regular representation of H_t, we just need to compute the dimension of this space and this is equal to

$$\sum_{\gamma \in \Gamma} |H_x|\, \delta_{HxH}(\gamma H)\, |H_y|\, \delta_{HyH}(\gamma^{-1}tH) = |H_x|\, |H_y|\, (\delta_{HxH} * \delta_{HyH})(tH).$$

Since the dimension of the regular representation of H_t is $|H_t|$, the number of copies of the regular representation in the fibre $\left(\eta^{\mathrm{reg}}_{HxH} * \eta^{\mathrm{reg}}_{HyH}\right)_{tH}$ is equal to

$$\frac{|H_x|\, |H_y|}{|H_t|}\, (\delta_{HxH} * \delta_{HyH})(tH).$$

It follows that in $\mathcal{K}(G /\!/ H)$, we have

$$\begin{aligned} \psi(\delta_{HxH}) * \psi(\delta_{HyH}) &= \frac{1}{|H_x|} \eta^{\mathrm{reg}}_{HxH} * \frac{1}{|H_y|} \eta^{\mathrm{reg}}_{HyH} \\ &= \sum_{[t] \in H \backslash G / H} (\delta_{HxH} * \delta_{HyH})(tH) \frac{1}{|H_t|} \eta^{\mathrm{reg}}_{HtH} \\ &= \psi(\delta_{HxH} * \delta_{HyH}). \end{aligned}$$

Therefore, ψ is multiplicative. $\square$

Finally, by considering the Bost-Connes algebra, we shall see that in general, when H is infinite, there is no additive and multiplicative map $\psi : \mathcal{H}(G/\!/H) \to \mathcal{K}(G/\!/H)$ that factorises the identity when composed with the natural projection map.

We recall that in $\mathcal{H}(G/\!/H)$, we have elements $\mu_n := n^{-1/2} e_{X_n}$ where

$$X_n = H \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} H = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} H.$$

and that for all $n \in \mathbb{N}$, $\mu_n^* * \mu_n = 1$. We also recall that in $\mathcal{K}(G/\!/H)$ we have $(\eta_{X_n}^{\chi_\alpha})^* * \eta_{X_n}^{\chi_\beta} = \sum_{i=1}^n \eta_H^{\chi_{\sigma_i}}$, $\forall n \in \mathbb{N}$, where the σ_i are the n distinct n-th roots of $\overline{\alpha}\beta$ (here χ_ζ denotes the character on $H \cong \mathbb{Z}$ where the generator $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ acts by multiplication by $\zeta \in \mathbb{T}$).

If a map $\psi : \mathcal{H}(G/\!/H) \to \mathcal{K}(G/\!/H)$ with the required properties did exist then we would have $\psi(1) = \frac{1}{\dim U} \eta_H^U$ where U is a finite dimensional unitary representation of H. Similarly, for all $n \in \mathbb{N}$ we would have $\psi(\mu_n) = \frac{1}{n^{1/2} \dim V_n} \eta_{X_n}^{V_n}$ where each V_n is a finite dimensional unitary representation of H.

Now, for each $n \in \mathbb{N}$, in $\mathcal{K}(G/\!/H)$ we have $\left(\eta_{X_n}^{V_n}\right)^* * \eta_{X_n}^{V_n} = \eta_H^{W_n}$ where W_n is some finite dimensional unitary representation of H. Each V_n is a direct sum of characters on H and for any character χ_α, we have $(\eta_{X_n}^{\chi_\alpha})^* * \eta_{X_n}^{\chi_\alpha} = \sum_{i=1}^n \eta_H^{\chi_{\sigma_i}}$ where the σ_i are the n distinct n-th roots of $\overline{\alpha}\alpha = 1$. Therefore, W_n has a subrepresentation isomorphic to $\bigoplus_{i=1}^n \chi_{\sigma_i}$.

Since $\mu_n^* * \mu_n = 1$ in $\mathcal{H}(G/\!/H)$, we must have

$$\begin{aligned}
\frac{1}{n\,(\dim V_n)^2} \eta_H^{W_n} &= \left(\frac{1}{n^{1/2} \dim V_n} \eta_{X_n}^{V_n} \right)^* * \frac{1}{n^{1/2} \dim V_n} \eta_{X_n}^{V_n} \\
&= \psi(\mu_n^*) * \psi(\mu_n) \\
&= \psi(\mu_n^* * \mu_n) \\
&= \psi(1) \\
&= \frac{1}{\dim U} \eta_H^U
\end{aligned}$$

in $\mathcal{K}(G/\!/H)$ for all $n \in \mathbb{N}$. However, this means that U has a subrepresentation isomorphic to χ_σ for every nth root of unity σ for all $n \in \mathbb{N}$. This is impossible since U is finite dimensional and hence no such ψ exists.

A Biadjunctions and hom category equivalences

The result in this appendix is folklore and an analogue of a classical result in category theory but we couldn't find a self contained proof in the literature so we present one here.

Definition A.1. *A* ***biadjunction*** *consists of the following data:*

- *2-categories $\mathcal{A}$ and $\mathcal{B}$,*
- *pseudofunctors $F : \mathcal{A} \to \mathcal{B}$ and $U : \mathcal{B} \to \mathcal{A}$,*
- *pseudonatural transformations $\eta : 1_{\mathcal{A}} \to UF$ and $\xi : FU \to 1_{\mathcal{B}}$ called the* ***unit*** *and* ***counit*** *respectively,*
- *invertible modifications $\Gamma : 1_F \to \xi_F \circ F\eta$ and $\Delta : U\xi \circ \eta_U \to 1_G$. In other words, the triangle identities*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FUF \arrow[d, "\xi_F"] \arrow[dl, phantom, "\Gamma"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\eta_U"] \arrow[dr, "1_U"'] & UFU \arrow[d, "U\xi"] \arrow[dl, phantom, "\Delta"] \\ & U \end{tikzcd}$$

commute up to invertible modifications.

We say that F is ***left biadjoint*** *to U and U is* ***right biadjoint*** *to F.*

Lemma A.2. *Given a biadjunction as in Definition A.1, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$ there is an equivalence of hom categories*

$$P_{A,B} : \mathcal{B}(FA, B) \xrightarrow{\simeq} \mathcal{A}(A, UB)$$

which is pseudonatural in each variable separately.

Proof. We define $P_{A,B}$ on 1-cells $f : FA \to B$ by $P_{A,B}(f) := Uf \circ \omega_A$ and on 2-cells $\alpha : f \to g$ by $P_{A,B}(\alpha) := U\alpha * 1_{\omega_A}$. This is a functor since

$$\begin{aligned} P_{A,B}(1_f) &= U1_f * 1_{\omega_A} \\ &= 1_{Uf} * 1_{\omega_A} \\ &= 1_{Uf \circ \omega_A} \\ &= 1_{P_{A,B}(f)} \end{aligned}$$

for all 1-cells $f : FA \to B$ and

$$\begin{aligned} P_{A,B}(\beta \circ \alpha) &= U(\beta \circ \alpha) * 1_{\omega_A} \\ &= (U\beta * 1_{\omega_A}) \circ (U\alpha * 1_{\omega_A}) \\ &= P_{A,B}(\beta) \circ P_{A,B}(\beta) \end{aligned}$$

for all pairs of composable 2-cells α, β.

A quasi-inverse

$$Q_{A,B} : \mathcal{A}(A, UB) \to \mathcal{B}(FA, B)$$

is defined on 1-cells $h : A \to UB$ by $Q_{A,B}(h) := \xi_B \circ Fh$ and on 2-cells $\gamma : h \to k$ by $Q_{A,B}(\gamma) := 1_{\xi_B} * F\gamma$. This is a functor for similar reasons to $P_{A,B}$.

To show that $P_{A,B}$ and $Q_{A,B}$ are quasi-inverse to one another, we need to define natural isomorphisms $\kappa : \mathrm{Id} \xrightarrow{\cong} Q_{A,B} \circ P_{A,B}$ and $\lambda : \mathrm{Id} \xrightarrow{\cong} P_{A,B} \circ Q_{A,B}$. To define κ, we first note that given $f : FA \to B$ we have

$$Q_{A,B} \circ P_{A,B}(f) = \xi_B \circ F(Uf \circ \omega_A).$$

Therefore, we define $\kappa_f : f \to Q_{A,B} \circ P_{A,B}(f)$ as the composite

$$f \xrightarrow{1_f * \Lambda_A} f \circ \xi_{FA} \circ F\omega_A \xrightarrow{\xi_f * 1_{F\omega_A}} \xi_B \circ FUf \circ F\omega_A \xrightarrow{1_{\xi_B} * F_{Uf,\omega_A}} \xi_B \circ F(Uf \circ \omega_A)$$

where $\Lambda : 1_F \to \xi_F \circ F\omega$ is the modification from the bidajunction, ξ_f is the coherence isomorphism for the pseudonatural transformation ξ and F_{Uf,ω_A} is the composition coherence isomorphism for the pseudofunctor F. To show that κ is natural, we need to show that for every 2-cell $\alpha : f \to g$, the following diagram commutes

$$\begin{tikzcd} f \arrow[r, "\alpha"] \arrow[d, "\kappa_f"'] & g \arrow[d, "\kappa_g"] \\ Q_{A,B} \circ P_{A,B}(f) \arrow[r, "Q_{A,B} \circ P_{A,B}(\alpha)"] & Q_{A,B} \circ P_{A,B}(g) \end{tikzcd} \tag{10}$$

Expanding this diagram using the definitions of $\kappa_f, \kappa_g, Q_{A,B} \circ P_{A,B}(\alpha)$ and adding some extra arrows we obtain the following diagram

$$\begin{tikzcd}
f \arrow[rr, "\alpha"] \arrow[d, "1_f * \Lambda_A"'] & \mathrm{(I)} & g \arrow[d, "1_g * \Lambda_A"] \\
f \circ \xi_{FA} \circ F\omega_A \arrow[rr, "\alpha * 1_{(\xi_{FA} \circ F\omega_A)}", "= \alpha * 1_{\xi_{FA}} * 1_{F\omega_A}"'] \arrow[d, "\xi_f * 1_{F\omega_A}"'] & \mathrm{(II)} & g \circ \xi_{FA} \circ F\omega_A \arrow[d, "\xi_g * 1_{F\omega_A}"] \\
\xi_B \circ FUf \circ F\omega_A \arrow[rr, "1_{\xi_B} * FU\alpha * 1_{F\omega_A}", "= 1_{\xi_B} * FU\alpha * F1_{\omega_A}"'] \arrow[d, "1_{\xi_B} * F_{Uf,\omega_A}"'] & \mathrm{(III)} & \xi_B \circ FUg \circ F\omega_A \arrow[d, "1_{\xi_B} * F_{Ug,\omega_A}"] \\
\xi_B \circ F(Uf \circ \omega_A) \arrow[rr, "1_{\xi_B} * F(U\alpha * 1_{\omega_A})"] & & \xi_B \circ F(Ug \circ \omega_A)
\end{tikzcd}$$

Now, (I) commutes by the middle interchange rule for 2-cells in a 2-category, (II) commutes by naturality of the coherence maps for ξ and (III) commutes by naturality of the coherence maps for F. Therefore, (10) commutes and κ is a natural isomorphism from Id to $Q_{A,B} \circ P_{A,B}$.

The definition of λ is similar. Given $h : A \to UB$ we have

$$P_{A,B} \circ Q_{A,B}(h) = U(\xi_B \circ Fh) \circ \omega_A$$

and we define $\lambda_h : h \to P_{A,B} \circ Q_{A,B}(h)$ as the composite

$$h \xrightarrow{\Delta_B * 1_h} U\xi_B \circ \omega_{UB} \circ h \xrightarrow{1_{U\xi_B} * \omega_h^{-1}} U\xi_B \circ UFh \circ \omega_A \xrightarrow{U_{\xi_B, Fh} * 1_{\omega_A}} U(\xi_B \circ Fh) \circ \omega_A$$

where Δ is the modification from the biadjunction, ω_h is the coherence isomorphism for the pseudonatural transformation ω and $U_{\xi_B,Fh}$ is the composition coherence isomorphism for the pseudofunctor U. A similar argument to the one for κ above shows that this defines a natural isomorphism $\lambda : \mathrm{Id} \xrightarrow{\cong} P_{A,B} \circ Q_{A,B}$. Therefore, $P_{A,B}$ and $Q_{A,B}$ mutually quasi inverse functors.

With regard to pseudonaturality, there are psuedofunctors

$$\mathcal{B}(F-, B), \mathcal{A}(-, UB) : \mathcal{A}^{\mathrm{op}} \to \mathsf{Cat}$$

and to say that the equivalences

$$P_{A,B} : \mathcal{B}(FA, B) \xrightarrow{\simeq} \mathcal{A}(A, UB)$$

are psuedonatural in the first variable means that the $P_{A,B}$'s define a pseudonatural transformation

$$P_{-,B} : \mathcal{B}(F-, B) \to \mathcal{A}(-, UB).$$

This means that for all 1-cells $f : A \to A'$ in $\mathcal{A}$ we need to define coherence 2-cells, i.e. natural isomorphisms

$$P_{f,B} : \mathcal{A}(f, UB) \circ P_{A',B} \to P_{A,B} \circ \mathcal{B}(Ff, B)$$

that satisfy the pseudonatural transformation axioms. Given a 1-cell $g : FA' \to B$ in $\mathcal{A}$, on the one hand

$$\mathcal{A}(f, UB) \circ P_{A',B}(g) = Ug \circ \omega_{A'} \circ f$$

and on the other hand

$$P_{A,B} \circ \mathcal{B}(Ff, B)(g) = U(g \circ Ff) \circ \omega_A.$$

Therefore, we define $(P_{f,B})_g$ as the composite

$$Ug \circ \omega_{A'} \circ f \xrightarrow{1_{Ug} * \omega_f^{-1}} Ug \circ UFf \circ \omega_A \xrightarrow{U_{g,Ff} * 1_{\omega_A}} U(g \circ Ff) \circ \omega_A.$$

To show that $P_{f,B}$ is a natural transformation we need to show that for all 2-cells $\alpha : g \to g'$ in $\mathcal{A}$, the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}(f, UB) \circ P_{A',B}(g) \arrow[r, "\mathcal{A}(f,UB) \circ P_{A',B}(\alpha)"] \arrow[d, "(P_{f,B})_g"'] & \mathcal{A}(f, UB) \circ P_{A',B}(g') \arrow[d, "(P_{f,B})_{g'}"] \\
P_{A,B} \circ \mathcal{B}(Ff, B)(g) \arrow[r, "P_{A,B} \circ \mathcal{B}(Ff,B)(\alpha)"] & P_{A,B} \circ \mathcal{B}(Ff, B)(g')
\end{tikzcd} \tag{11}$$

Expanding this diagram using the definitions and adding an extra arrow yields the following:

$$
\begin{tikzcd}
Ug \circ \omega_{A'} \circ f \arrow[r, "U\alpha * 1_{\omega_{A'} \circ f}"] \arrow[d, "1_{Ug} * \omega_f^{-1}"'] \arrow[dr, phantom, "\mathrm{(I)}"] & Ug' \circ \omega_{A'} \circ f \arrow[d, "1_{Ug'} * \omega_f^{-1}"] \\
Ug \circ UFf \circ \omega_A \arrow[r, "U\alpha * 1_{UFf \circ \omega_A}", "= U\alpha * U1_{Ff} * 1_{\omega_A}"'] \arrow[d, "U_{g,Ff} * 1_{\omega_A}"'] \arrow[dr, phantom, "\mathrm{(II)}"] & Ug' \circ UFf \circ \omega_A \arrow[d, "U_{g',Ff} * 1_{\omega_A}"] \\
U(g \circ Ff) \circ \omega_A \arrow[r, "U(\alpha * 1_{Ff}) * 1_{\omega_A}"] & U(g' \circ Ff) \circ \omega_A
\end{tikzcd}
$$

Then (I) commutes by the middle interchange rule and (II) commutes by the naturality of the coherence isomorphisms for U. Therefore, (11) commutes and $P_{f,B}$ is a natural transformation.

Next, the pseudonatural transformation pentagon axiom says that given 1-cells $f : A \to A'$ and $f' : A' \to A''$ in $\mathcal{A}$, the following diagram should commute

$$
\begin{tikzcd}
\mathcal{A}(f, UB) \circ \mathcal{A}(f', UB) \circ P_{A'',B} \arrow[r, "\mathcal{A}(-,UB)_{f',f} * 1"] \arrow[d, "1 * P_{f',B}"'] & \mathcal{A}(f' \circ f, UB) \circ P_{A'',B} \arrow[dd, "P_{f' \circ f, B}"] \\
\mathcal{A}(f, UB) \circ P_{A',B} \circ \mathcal{B}(Ff', B) \arrow[d, "P_{f,B} * 1"'] & \\
P_{A,B} \circ \mathcal{B}(Ff, B) \circ \mathcal{B}(Ff', B) \arrow[r, "1 * \mathcal{B}(F-,B)_{f',f}"] & P_{A,B} \circ \mathcal{B}(F(f' \circ f), B)
\end{tikzcd}
$$

This follows from the commutativity of the following diagram

$$\begin{tikzcd}
Ug \circ \omega_{A''} \circ f' \circ f \arrow[rr, "\mathrm{Id}"] \arrow[d, "\omega_{f'}"'] & & Ug \circ \omega_{A''} \circ f' \circ f \arrow[dd, "\omega_{f' \circ f}"] \\
Ug \circ UFf' \circ \omega_{A'} \circ f \arrow[d, "U_{g,Ff'}"'] \arrow[dr, "\omega_f"] \arrow[r, phantom, "\mathrm{(I)}"] & {} & \\
U(g \circ Ff') \circ \omega_{A'} \circ f \arrow[d, "\omega_f"'] \arrow[r, phantom, "\mathrm{(II)}"] & Ug \circ UFf' \circ UFf \circ \omega_A \arrow[r, "(UF)_{f',f}"] \arrow[dl, "U_{g,Ff'}"] \arrow[d, "U_{Ff',Ff}"] \arrow[dr, phantom, "\mathrm{(IV)}"] & Ug \circ UF(f' \circ f) \circ \omega_A \arrow[dd, "U_{g,F(f' \circ f)}"] \\
U(g \circ Ff') \circ UFf \circ \omega_A \arrow[d, "U_{g \circ Ff', Ff}"'] \arrow[r, phantom, "\mathrm{(III)}"] & Ug \circ U(Ff' \circ Ff) \circ \omega_A \arrow[ur, "U(F_{f',f})"'] \arrow[dl, "U_{g, Ff' \circ Ff}"] \arrow[r, phantom, "\mathrm{(V)}"] & {} \\
U(g \circ Ff' \circ Ff) \circ \omega_A \arrow[rr, "U(1_g * F_{f',f})"] & & U(g \circ F(f' \circ f)) \circ \omega_A
\end{tikzcd}$$

Here, (I) commutes by the psuedonatural transformation axioms, (II) commutes by the middle interchange rule (both composites are $U_{g,Ff'} * \omega_f$), (III) commutes by pseudofunctor axioms, (IV) commutes by the definition of $UF_{f',f}$ and (V) commutes by the naturality of the composition coherence isomorphisms for U.

The pseudonatural transformation unit axiom says that for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ the following diagram should commute

$$\begin{tikzcd}
& P_{A,B} \arrow[dl, "\mathcal{A}(-,B)_{1_A}"'] \arrow[dr, "\mathcal{B}(F-,B)_{1_A}"] & \\
\mathcal{A}(1_A, B) \circ P_{A,B} \arrow[rr, "P_{1_A,B}"] & & P_{A,B} \circ \mathcal{B}(F1_A, B)
\end{tikzcd}$$

This follows from the commutativity of the following diagram.

$$\begin{tikzcd}
 & \begin{array}{c} Ug \circ \omega_A \\ = U(g \circ 1_{FA}) \circ \omega_A \\ = Ug \circ 1_{UFA} \circ \omega_A \end{array} \arrow[ddl, "\mathrm{Id}"'] \arrow[d, "1_{Ug} * U_{1_{FA}}"', "= U^{-1}_{g,1_{FA}}"] \arrow[ddr, "U(1_g * F_{1_A})"] & \\
 & Ug \circ U(1_{FA}) \circ \omega_A \arrow[d, "1_{Ug} * U(F_{1_A})"', "= U(1_g) * U(F_{1_A})"] & \\
Ug \circ \omega_A \circ 1_A \arrow[r, "\omega_{1_A}"] & Ug \circ UF1_A \circ \omega_A \arrow[r, "U_{g,F1_A}"] & U(g \circ F1_A) \circ \omega_A
\end{tikzcd}$$

By definition, $(UF)_{1_A}$ is the composite

$$1_{UFA} \xrightarrow{U_{1_{FA}}} U1_{FA} \xrightarrow{U(F_{1_A})} UF1_A$$

and hence the left hand triangle commutes by the pseudonatural transformation unit axiom. Then, by the pseudofunctor unit axioms, $1_{U_g} * U_{1_{FA}} = U^{-1}_{g,1_{FA}}$ and since $1_{U_g} = U(1_g)$ the right hand commutes by the naturality of the composition coherence isomorphisms for U. Therefore, the equivalences

$$P_{A,B} : \mathcal{B}(FA, B) \xrightarrow{\simeq} \mathcal{A}(A, UB)$$

are pseudonatural in the first variable. The proof that they are psuedonatural in the second variable is similar. $\square$

The fact that it is equivalent to define biadjunctions in terms of such pseudonatural equivalences of hom categories is also folklore. We couldn't find a self contained proof in the literature but it should follow from the bicategorical Yoneda Lemma which is discussed in [19].

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