The representation theory of $\overline{H}_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$

by

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Abstract

In this thesis we examine various aspects of the representation theory of the restricted rational Cherednik algebra $\overline{H}_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$. We prove several multiplicity results for graded modules, in particular for modules over an algebra that admits a triangular decomposition. This includes the algebra $\overline{H}_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$. Furthermore, if the projective covers admit a radical preserving filtration, we show that we can calculate the multiplicities of the simple modules inside the radical layers of the projective covers.

We give an explicit presentation of the centre of the restricted rational Cherednik algebra $\overline{H}_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ for suitably generic c. This is done by first deriving a presentation of the centre of $\overline{H}_c(S_n)$ using Schubert cells, then extending this to the more general wreath product group using the action of the cyclic group $\mathbb{Z}/\ell\mathbb{Z}$. This presentation is given for each block of the centre. Using the bijection between irreducible representations of $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ and ℓ-multipartitions of n, we prove that this explicit presentation can be read directly from the ℓ-multipartition of n.

Contents

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Author's declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

This thesis has two themes, one being the structure of the radical filtration of projective covers and the other the geometry of Calogero-Moser spaces. They are linked through the representation theory and geometry of the restricted rational Cherednik algebra. In Chapter 3 we derive multiplicity formulae that allow us to (partially) describe the radical series of projective covers. In Chapters 5 and 6, we give an explicit presentation of the centre of the restricted rational Cherednik algebra for the wreath product $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$. This allows us to apply the multiplicity results from the first part to the projective covers of this family of restricted rational Cherednik algebras.

§ 1.1 | Multiplicity formulas

Throughout this section assume that all algebras and modules are finite dimensional unless otherwise stated. Given an algebra A, and an A-module M of finite length, the multiplicity $[M : N]$ of a simple module N inside M is the number of times N appears as a composition factor of M. In a sense, we can think of these multiplicities as describing the building blocks of the module M. They do not, however, tell us how to put these blocks together. Nevertheless this is still important information, allowing us to deduce many properties of a module. The cases of particular interest for us are: when A is graded, and when A admits a triangular decomposition. We will give the precise definition of the second of these in the preliminaries. It is sufficient, for now, to say that the algebra A is graded in such a way that the multiplication map $A^- \otimes T \otimes A^+ \to A$ is an isomorphism, where A^-, T and A^+ are graded subalgebras. More importantly, many of the significant examples in representation theory, such as the restricted enveloping algebra of a semisimple Lie algebra, admit a triangular decomposition. At this point we should also note that for graded modules there is a refined notion of multiplicity. This is the graded multiplicity (see definition 2.2.29) and for a graded simple module N inside a graded module M is denoted $[M : N]_{gr}$.

A natural concept is that of the radical series. The radical of a module is the smallest submodule such that the quotient is semisimple. This definition is easily iterated to define radical powers: $\mathrm{rad}^i M = \mathrm{rad}(\mathrm{rad}^{i-1} M)$. The radical layers are the semisimple quotients $\mathrm{rad}_s M = \mathrm{rad}^s M / \mathrm{rad}^{s+1} M$. For any module M, there is a filtration

$$0 \subset \mathrm{rad}^{\ell-1} M \subset \cdots \subset \mathrm{rad} M \subset M,$$

where the radical layers are the associated subquotients.

Much harder, but correspondingly more important, than knowing the composition factors of M, is to ask the question: what are the multiplicities of a simple module inside the radical layers of M? By answering this we are obtaining not only the multiplicities of the simple modules inside M, but also their relative position inside the radical layers.

Calculating the multiplicities of simple modules inside an arbitrary module is, in general, a hard question. By placing suitable restrictions on the classes of modules we consider, or on the algebra the module is over, we are able to derive powerful results. One such condition is that of a graded strong duality. If we denote by $\mathcal{G}(A)$ the category of graded left A-modules, then a graded strong duality δ of A is a contravariant equivalence of categories $\delta : \mathcal{G}(A) \to \mathcal{G}(A)$ such that $\delta(M) \cong M$, for $M \in \mathrm{Irr}\,\mathcal{G}(A)$. With the existence of a graded strong duality we can prove the following.

Theorem 1.1.1. *(3.1.3) If A is a graded finite dimensional algebra over a splitting field k, equipped with a graded strong duality then, for all M, $N \in \mathrm{Irr}\,\mathcal{G}(A)$,*

$$[\mathrm{rad}_s\, P(M) : N]_{gr} = [\mathrm{rad}_s\, P(N) : M]_{gr}.$$

In the above $P(M)$ and $P(N)$ denote the projective covers of M and N respectively. This is a generalisation of a well known result [10, Corollary I.33], which is the ungraded version. It is known that in the ungraded case the strong duality condition is unnecessary in the special case of cellular and BGG algebras [14, Theorem 6].

There are two classes of modules that play a significant role in both the multiplicity results and in our understanding of the centre of the restricted rational Cherednik algebra. These are the standard and costandard modules. They are defined for algebras that admit a triangular decomposition. Given a simple T-module M we can construct an A-module called the standard module for M defined by

$$\Delta(M) := A \otimes_{TA^+} M.$$

Similarly for any simple T-module M we define the costandard module

$$\nabla(M) := (M^* \otimes_{A\text{-}T} A)^*.$$

Here M^* denotes the dual of M as an A-module. A key property of standard modules is that they each have a unique simple head. Denote the unique simple head of $\Delta(M)$ by $L(M)$. Our second major result allows us to obtain information about the projective cover of a simple module by calculating the multiplicity of the associated simple module inside the radical layers of the standard modules. However, this result only holds if we have an anti-triangular duality. This is a graded strong duality D of the category of graded A-modules such that $D(\Delta(M)) = \nabla(M)$ and $D(L(M)) = L(M)$. In most cases of interest, such a duality is known to exist.

It is straight forward to write down the standard and costandard modules if the simple modules are known. The projective covers, however, are difficult to find from the simple modules. Our next formulae (partially) solves this problem. If each radical layer of a given projective cover admits a filtration by standard modules then we define a set $K_s(P(M), N)$. The precise definition of $K_s(P(M), N)$ is given at the start of Section 3.3. It essentially counts the number of composition factors, in the filtration by standard modules of $\text{rad}_s P(M)$, that are isomorphic to $\Delta(N)$. With this roughly explained, we present the following.

Theorem 1.1.2. *(3.3.5) Let $P(M)$ be the projective cover of $L(M)$. If the category A-mod admits an anti-triangular duality then the following formula holds*

$$|K_s(P(M), N)| = [\text{rad}_s \Delta(N) : L(M)].$$

This formula is important as the right hand side is computable, but the left hand side is information about an object that is almost impossible to explicitly describe. Let us also note that this is similar to a result by Holmes and Nakano [36, Theorem 4.5]. However, the reciprocity relation of Holmes and Nakano says nothing about the radical layers.

It is also known that the set $K_s(P(M), N)$ can be replaced with $[\text{rad}_s P(M) : L(N)]$ in the specific case of a BGG algebra [14, Corollary 7]. Unfortunately, the algebras that we consider are far from being BGG.

If we require that the filtration by standard modules of the projective covers is radical respecting, in the sense of [35] then more can be said. With this condition imposed, we prove the following.

Corollary 1.1.3. *(3.4.4) If $P(\lambda)$ admits a radical respecting Δ-filtration then*

$$[\mathrm{rad}_s P(\lambda) : L(\mu)] = \sum_{i=1}^{\ell} [\mathrm{rad}_{s-n(i)} \Delta(\lambda_i) : L(\mu)]. \tag{1.1.1}$$

This is the final multiplicity result. The above corollary does have a drawback however, that requiring a projective cover admits a radical respecting filtration by standard modules is very strong. Indeed, it is hard to prove this is the case without explicitly knowing the projective cover. Chapter 3 concludes with an example demonstrating how to use formula (1.1.1) to calculate the radical layers of the projective covers for a block in the centre of a restricted rational Cherednik algebra.

§ 1.2 | The centre of $\overline{H}_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$

The second theme of this thesis is understanding the Calogero-Moser space for generic c and $t = 0$. The Calogero-Moser space is the spectrum of the centre of the rational Cherednik algebra. In general this space is singular of high dimension, and not much can be said. There is however, a zero-dimensional (highly non-reduced) subscheme for which much more can be ascertained. This is the spectrum of the centre of the restricted rational Cherednik algebra. It is this centre that we give an explicit presentation of in terms of generators and relations for the wreath product $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$. This is arguably the most important class of complex reflection groups. We outline in this introduction how we intend to do this, beginning by introducing the rational Cherednik algebra.

Rational Cherednik algebras are infinite dimensional algebras that were first defined in [24], where they were introduced as a special class of symplectic reflection algebras. For the formal definitions see the preliminary sections 2.5 and 2.6. Rational Cherednik algebras are defined for any complex reflection group $(W, \mathfrak{h})$, and pair of parameters (t, c). They are denoted $H_{t,c}(W)$. They are, in general, highly non-commutative objects; when $t \neq 0$ the centre is simply the field $\mathbb{C}$ [7, Theorem 1.7.1]. However, when $t = 0$ the centre has a rich structure and the rational Cherednik algebra is a finite module over its centre. As a consequence, much of the representation theory of $H_{0,c}(W)$ can be understood via its centre.

In [30, Proposition 3.6] Gordon showed that the W-invariant ring $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ is completely contained in the centre. The ideal generated by all elements with no constant term in $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ is a particularly important ideal of $H_{0,c}(W)$. The restricted rational Cherednik algebra is the quotient by this ideal and denoted $\overline{H}_c(W)$. Let us now describe how to find the explicit presentation of its centre. Write the restricted rational Cherednik algebra as the sum of its indecomposable blocks:

$$\overline{H}_c(W) = \oplus_{i \in I} B_i.$$

Finding the centre of $\overline{H}_c(W)$ is then equivalent to finding the centre of these blocks. By Proposition 4.1.2, there is a bijection between the blocks and the irreducible representations of W. Therefore, we can associate to any irreducible representation $\lambda \in \operatorname{Irr} W$, the centre of the corresponding block, denoted $A(\lambda)$.

The standard modules for $\overline{H}_c(W)$ are called the baby Verma modules. They have a major role to play in describing $A(\lambda)$ due to Theorem 4.1.6, which states

$$A(\lambda) \cong A(\lambda)^+ \otimes A(\lambda)^-,$$

where $A(\lambda)^+ := \operatorname{End}_{\overline{H}_c(W)} \Delta(\lambda)$ and $A(\lambda)^- := \operatorname{End}_{\overline{H}_c(W)} \Delta^*(\lambda)$. The module $\Delta(\lambda)^*$ is the costandard module for the restricted rational Cherednik algebra, for a precise definition see page 75. The question then becomes: how do we write either $A(\lambda)^+$ or $A(\lambda)^-$ explicitly in terms of generators and relations? Our aim then, is to understand the endomorphism rings of the baby Verma modules. It turns out that in the case of the wreath product we need only study one of these rings due to the following theorem. It states that $A(\lambda)^-$ is isomorphic to $A(\lambda)^+$ with the opposite grading but for a different value of c. To be clear we write $A(\lambda)^+_c = \operatorname{End}_{\overline{H}_c(W)} \Delta(\lambda)$ and $A(\lambda)^-_{\overline{c}} = \operatorname{End}_{\overline{H}_{\overline{c}}(W)} \Delta^*(\lambda)$. In the below theorem $\underline{\lambda}^*$ and $\underline{\lambda}$ denote different irreducible representations of $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$.

Theorem 1.2.1. *(5.4.9) In the case of the wreath product of the symmetric group with the cyclic group there is an anti-graded isomorphism*

$$A_{\overline{c}}(\underline{\lambda}^*)^- \cong A_c(\underline{\lambda})^+$$

where both c and $\overline{c}$ are generic.

In light of the above we describe how $A(\lambda)^+$ can be found explicitly in terms of its generators and relations. We do this by first studying the endomorphism rings of the large Verma modules (the standard modules of $H_{0,c}(W)$), which we denote $\underline{\Delta}(\lambda)$. This is because of two key facts stated in Theorem 4.1.8. The first fact is that there is a surjection from the centre of the rational Cherednik algebra onto the endomorphism rings of the Verma modules. By composing this surjection with the inclusion map we get the following

$$\mathbb{C}[\mathfrak{h}]^W \hookrightarrow Z_c(W) \twoheadrightarrow \operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda)).$$

Hence, there is a map of spectrums

$$\pi : \operatorname{Spec} \operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda)) \to \operatorname{Spec} \mathbb{C}[\mathfrak{h}]^W = \mathfrak{h}/W. \tag{1.2.1}$$

This is where we use our second important fact, that the baby Verma modules are a quotient of the Verma modules. Furthermore, we prove that the endomorphism ring of the baby Verma module is a quotient of the endomorphism ring of the Verma module.

Theorem 1.2.2. *(4.1.15) We have the following isomorphism*

$$\operatorname{End}_{\overline{H}_c(W)}(\Delta(\lambda)) \cong \operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda))/\mathbb{C}[\mathfrak{h}]^W_+ \operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda)).$$

This theorem combined with the map (1.2.1) leads to the important corollary.

Corollary 1.2.3. *(4.1.16) There is an isomorphism of algebras* $\mathbb{C}[\pi^{-1}(0)] \cong A(\lambda)^+$.

This is where we split the problem into two cases, that of $W = S_n$ and then $W = S_n \wr \mathbb{Z}/\ell\mathbb{Z}$.

§ 1.2.1 | Symmetric group S_n

Let us start by noting that irreducible representations of the symmetric group S_n are in bijection with the partitions of n [29, Lemma 4.25]. Therefore, we have a bijection between the partitions of n and the centres of the blocks.

The symmetric group case will be solved by using the Wronski map, this is closely related to the Wronskian. The exact definitions of these maps are given in Definitions 2.7.5 and 2.7.8. For now treat the Wronskian Wr as a determinant of a set of n polynomials and their derivatives. On the preimage of 0 (the case we are interested in) the Wronski map is given by the coefficients of the Wronskian. The Wronski map Wr_λ has domain Ω_λ^{qe}, this is the Schubert cell associated to λ. It was shown by Bellamy [6, Proposition 6.4] in the specific case of the symmetric group that

$$\pi^{-1}(0) \cong \mathrm{Wr}_\lambda^{-1}(0) \tag{1.2.2}$$

This isomorphism provides the connection to the Wronskian. We use the two isomorphisms

$$\pi^{-1}(0) \cong \mathrm{Wr}_\lambda^{-1}(0) \text{ and } \mathbb{C}[\pi^{-1}(0)] \cong A(\lambda)^+$$

to conclude the following.

Theorem 1.2.4. *(4.1.18) There is an isomorphism of algebras*

$$A(\lambda)^+ \cong \mathbb{C}[\mathrm{Wr}_\lambda^{-1}(0)].$$

This theorem is all we need to explicitly present the centre of the restricted rational Cherednik algebra for the symmetric group. In [47] Mukhin, Tarasov and Varachenko describe how to realise explicitly the scheme-theoretic preimage of 0 under the Wronski map. We will give an explanation of how to do this in Section 5.1 following their work. Eventually we show that this intermediate step is unnecessary and the algebra $A(\lambda)^+$ can be determined directly from the partition λ. This can be extended to allow us to find an explicit presentation for the wreath product of the symmetric group with the cyclic group of order ℓ.

§ 1.2.2 | Wreath product $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$

In the case of the wreath product we do not have the isomorphism (1.2.2) nor any similar map to the map Wr_λ. This means we cannot naively extend our result by using the Wronskian. Let $X_c(W) := \operatorname{Spec} Z_c(W)$ be the Calogero-Moser space. A recent result by Bonnafe and Maksimau [12, Theorem 4.21] says that $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ is isomorphic to an irreducible component of $X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}$. The strategy then becomes clearer. First we realise the spectrums of the endomorphism rings of the Verma modules as subvarieties of $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$. Then, using the identification of $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ with an irreducible component of $X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}$ we embed the spectrum of the endomorphism ring of the Verma module into $X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}$. The results of the previous section will then be applicable and we can explicitly compute the endomorphism rings in terms of generators and relations.

In the wreath product case we have two versions of (1.2.1),

$$\pi_{n\ell} : \operatorname{Spec} \operatorname{End}_{H_c(S_{n\ell})}(\underline{\Delta}(\lambda))^{\mathbb{Z}/\ell\mathbb{Z}} \to (\mathbb{C}^{n\ell}/S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}},$$

and

$$\pi_{n,\ell} : \operatorname{Spec} \operatorname{End}_{H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})} \underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \to \mathbb{C}^n/(S_n \wr \mathbb{Z}/\ell\mathbb{Z}).$$

where $\mathrm{quo}_\ell(\lambda)$ is the ℓ-quotient of $\lambda \vdash n$. We prove the following key result.

Theorem 1.2.5. *(4.2.24) There is an isomorphism $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \to Y$ onto a connected component of $X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}$ such that the following diagram commutes*

$$\begin{tikzcd} X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \arrow[r] \arrow[d, "\pi_{n,\ell}"'] & X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}} \arrow[d, "\pi_{n\ell}"] \\ \mathbb{C}^n/(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \arrow[r, "\cong"] & (\mathbb{C}^{nl}/S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}} \end{tikzcd}.$$

We use the isomorphism $\mathbb{C}[\pi^{-1}(0)] \cong A(\lambda)^+$ from Corollary 1.2.2, to conclude

$$A(\mathrm{quo}_\ell(\lambda))^+ \cong \mathbb{C}[\pi_{n\ell}^{-1}(0)] \cong \mathbb{C}[\pi_{n\ell}^{-1}(0)]^{\mathbb{Z}/\ell\mathbb{Z}} \cong A(\lambda)^+_{\mathbb{Z}/\ell\mathbb{Z}}.$$

Here $A(\lambda)^+_{\mathbb{Z}/\ell\mathbb{Z}} := \mathbb{C}[\pi_{n\ell}^{-1}(0)]/\langle f - s \cdot f | s \in \mathbb{Z}/\ell\mathbb{Z}, f \in \mathbb{C}[\pi_{n\ell}^{-1}(0)]\rangle$ is the ring of functions on the scheme-theoretic fixed points of $\mathbb{Z}/\ell\mathbb{Z}$ acting on $\pi_{n\ell}^{-1}(0)$.

Thus, we conclude Section 4 with the following theorem.

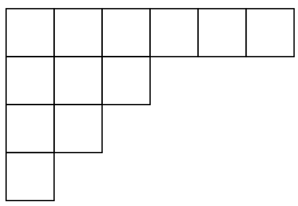
Theorem 1.2.6. *(4.2.25) There is an isomorphism of algebras*

$$A(\mathrm{quo}_\ell(\lambda))^+ \cong A(\lambda)^+_{\mathbb{Z}/\ell\mathbb{Z}}. \tag{1.2.3}$$

Isomorphism (1.2.3) is all we need to write the algebra $A(\mathrm{quo}_\ell(\lambda))^+$ explicitly in terms of generators and relations.

§ 1.2.3 | An explicit presentation

Chapter 5 explores the consequences of isomorphism (1.2.3). It has a more combinatorial flavour due to the aforementioned bijection between the ℓ-multipartitions of n and the algebras $A(\mathrm{quo}_\ell(\lambda))^+$. Preliminary 2.4 will provide all the background needed to understand the results. For now we briefly mention two of the most important concepts, Young diagrams and hook lengths. Given a partition $\lambda \vdash n$, the Young diagram D_λ is a way to represent λ. If $\lambda = (\lambda_1, \cdots, \lambda_n)$, the Young diagram is given by having λ_i cells on the i^{th} row. As an example the Young diagram of $(6, 3, 2, 1)$ is



The hook length of a cell in the i^{th} row and j^{th} column is denoted $h(i, j)$ and is given by counting the cells directly below and to the right (like a ⌜ shape). For instance, using the same partition as above the hook lengths are

9	7	5	3	2	1
5	3	1			
3	1				
1					

The first result of this section is a formula for calculating the graded dimensions of $A(\lambda)^+$, inspired by a similar result found in [49, p. 364]. The latter formula is derived in a completely different setting however, calculated using Schur functions.

Theorem 1.2.7. *For any $\lambda \vdash n$, we have*

$$\sum_{i \geq 0} (\dim A(\lambda)_i^+) q^i = \frac{\prod_{i=1}^n (1-q^i)}{\prod_{(i,j)\in D_\lambda} (1-q^{h(i,j)})}.$$

This formula is not only useful, it is astonishingly easy to use. For example, to calculate the graded dimension of $A(2,2)^+$ we first find the hook lengths

3	2
2	1

and so

$$\sum_{i \geq 0} (\dim A(\lambda)_i^+) q^i = \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)}{(1-q)(1-q^2)(1-q^2)(1-q^3)} = \frac{1-q^4}{1-q^2} = 1 + q^2.$$

Therefore, $A(2,2)^+$ has two one dimensional graded pieces, one of degree 0 and one in degree 2.

The next two theorems show how to explicitly write the centre of the restricted rational Cherednik algebra in terms of generators and relations. It is important to note that these theorems make no mention of the Wronskian. Let us denote each cell $(i,j) \in D_\lambda$ by $\square_{i,j}$. We begin with the symmetric group case.

Theorem 1.2.8. *(5.2.15) Let $\lambda \vdash n$ be a partition. The algebra $A(\lambda)^+$ is the quotient*

$$A(\lambda)^+ \cong \mathbb{C}[D_\lambda]/I$$

by the ideal I that is generated by n homogeneous elements $r_1, \ldots, r_n$. The r_s are ordered so that $\deg(r_s) = s$. The monomials in r_i are products of cells which share neither a row or column in D_λ. In other words if $\square_{i,j}\square_{k,\ell}$ is a factor of some monomial in the r_s we must have that $i \neq k$ and $j \neq \ell$. The coefficients of the monomials appearing in the generators of I are given by Proposition 5.2.13.

This theorem is easily extended to cover the wreath product case using isomorphism 1.2.3 and the following lemma.

Lemma 1.2.9. *(5.3.2) The ideal $\langle f - s \cdot f \,|\, s \in \mathbb{Z}/\ell\mathbb{Z}, f \in \mathbb{C}[\pi_{n\ell}^{-1}(0)] \rangle$ consists of all polynomials with degree not divisible by ℓ.*

Combining these two facts we have perhaps the most important theorem.

Theorem 1.2.10. *(5.3.4) Let $\lambda \vdash n\ell$ be a partition with trivial ℓ-core. The algebra $A(\mathrm{quo}_\ell(\lambda))^+$ is the quotient*

$$A(\mathrm{quo}_\ell(\lambda))^+ \cong \mathbb{C}[D_\lambda^\ell]/I$$

where D^ℓ_λ is the subdiagram of D_λ (the Younger diagram) excluding the cells (i, j) such that $h(i, j)$ is not divisible by ℓ. The ideal I is generated by n homogeneous elements $r_\ell, r_{2\ell}, \dots, r_{n\ell}$. The $r_{s\ell}$ are ordered so that $\deg(r_{s\ell}) = s\ell$. The monomials in $r_{s\ell}$ are products of cells which share neither a row or column in D^ℓ_λ. In other words if $\square_{i,j}\square_{k,m}$ is a factor of some monomial appearing in the $r_{s\ell}$, we must have that $i \neq k$ and $j \neq m$. The coefficients of the monomials appearing in the generators of I are given by Proposition 5.2.13.

With a theorem now allowing us to explicitly describe the algebras $A(\lambda)^+$ we can give a theorem describing the entire centre. In the case of the symmetric group we have the following.

Theorem 1.2.11. *(5.4.10) There is an isomorphism of the centre of $\overline{H}_c(S_n)$ for $c \neq 0$*

$$Z(\overline{H}_c(S_n)) \cong \bigoplus_{\lambda \in \operatorname{Irr} S_n} A(\lambda)^- \otimes A(\lambda)^+.$$

The algebra $A(\lambda)^+$ is given by Theorem 5.2.15 and $A(\lambda)^-$ is isomorphic to $A(\lambda)^+$ with the opposite grading.

For the wreath product case we must recall the notation of $\underline{\lambda}$ and $\underline{\lambda}^*$ for irreducible representations of $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$.

Theorem 1.2.12. *(5.4.11) There is an isomorphism of the centre of $\overline{H}_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ for generic c*

$$Z(\overline{H}_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})) \cong \bigoplus_{\underline{\lambda} \in \operatorname{Irr} S_n \wr \mathbb{Z}/\ell\mathbb{Z}} A_{\overline{c}}(\underline{\lambda}^*)^- \otimes A_c(\underline{\lambda})^+.$$

The algebra $A_c(\underline{\lambda})^+$ is given by Theorem 5.3.4 and $A_{\overline{c}}(\underline{\lambda}^*)^-$ is isomorphic to $A_c(\underline{\lambda})^+$ with the opposite grading.

§ 1.3 | Final results

There are other significant results contained in Chapter 5. For instance, we provide a proof that $A(\lambda)^+ \cong A(\lambda^T)^+$ for any $\lambda \vdash n$ and λ^T the transpose of λ. From a practical perspective, this roughly halves the number of $A(\lambda)^+$ that need to be computed for a given n. Furthermore, we provide MAGMA code that computes the dimensions of the radical layers of $A(\lambda)^+$. We prove that Gröbner basis can be used to calculate the dimension of the radical layers. This is to take advantage of the fact that MAGMA has efficient algorithms for calculating the Gröbner basis. This is desirable because calculating the radical layers of $A(\lambda)^+$ through a brute force method is extremely slow. Finally, we provide a proof of an upper bound for the Lowey length of any $A(\lambda)^+$, and present a conjecture for the lower bound.

Chapter 2

Preliminaries

This chapter covers the necessary background material to enable the reader to understand the main results of this thesis. The subsections have been organised in a way that loosely corresponds to the order in which the concepts are required.

The scope of the background is quite wide, however it all pertains to representation theory. Some of the sections such as the first one regarding graded rings, modules and algebras contain elementary definitions and results. Other sections such as the one concerning standard and costandard modules contain more technical results that we will require in later chapters.

§ 2.1 | Graded Rings, Modules and Algebras.

Algebraic objects can often be endowed with an additional structure called a grading. Due to the greater generality and utility of graded objects, almost all the algebraic structures we consider in this thesis will admit a grading. The goal of this preliminary section is to introduce the basic definitions and concepts needed to understand common graded objects. Unless otherwise stated, all rings are assumed to be unital.

Definition 2.1.1. Let R be a ring and G any abelian group. Then R is *graded* if $R = \bigoplus_{i \in G} R_i$ for abelian groups R_i such that $R_i R_j \subset R_{i+j}$.

We have defined the grading to be over any abelian group, however for our purposes we will only be dealing with $\mathbb{Z}$-graded objects. The definitions for graded modules and graded algebras are similar.

Definition 2.1.2. Let R be a $\mathbb{Z}$-graded ring, and M be a left R-module. We say that M is a $\mathbb{Z}$-*graded left module* if $M = \bigoplus_{i \in \mathbb{Z}} M_i$, where the M_i are abelian subgroups of M and $R_i M_j \subset M_{i+j}$.

Definition 2.1.3. Let k be a field. A $\mathbb{Z}$-*graded k-algebra* is a k-algebra such that $A = \bigoplus_{i \in \mathbb{Z}} A_i$, where A_i are vector subspaces and $A_i A_j \subset A_{i+j}$.

Remark 2.1.4. Any ring, module or k-algebra can be given a grading. For rings and k-algebras, simply let all elements have degree 0. This is the trivial grading. Care must be taken when considering modules, as the grading must respect the grading of the ring acting on the module. In this case we can give both the ring the trivial grading and the module the trivial grading.

By definition, graded objects admit a decomposition into a direct sum of subgroups of elements with the same degree. Clearly these components will play a pivotal role so we naturally have the following definition.

Definition 2.1.5. For a graded object $A = \bigoplus_{i \in \mathbb{Z}} A_i$, the *homogeneous components* of A are the subgroups A_i for $i \in \mathbb{Z}$. The *homogeneous elements* of A are precisely those that belong to a homogeneous component, that is $a \in A_i$ for some $i \in \mathbb{Z}$. The element a is said to have *degree* i.

A familiar example of a graded ring, with non-trivial grading, is the polynomial ring in one variable.

Example 2.1.6. Let $\mathbb{C}[x]$ denote the polynomial ring in one variable over $\mathbb{C}$. If we set $\deg(x) = 1$ then we have a graded ring. The homogeneous components are $\mathbb{C}[x]_i = \mathrm{Span}\{x^i\}$.

The definition of a graded object naturally leads to the definition of a graded morphism, which is a map preserving the graded structure.

Definition 2.1.7. Given two graded modules M and N, a *graded morphism* is a morphism $f : M \to N$ such that $f(M_i) \subset N_i$ for all $i \in \mathbb{Z}$.

When defining ideals we also take into account the grading.

Definition 2.1.8. Let R be a graded ring. An ideal $I \subset R$ is a *homogeneous ideal* if for each $a \in I$ the homogeneous components of a also belong to I.

It is possible to account for the grading when defining the dual of a module. Recall the ungraded definition of the dual.

Definition 2.1.9. Let k be a field and A a unital finite dimensional k-algebra. The dual of a left finite dimensional A-module M is defined to be

$$M^* := \mathrm{Hom}_k(M, k).$$

It is a right A-module with $(\phi \cdot a)(m) := \phi(a \cdot m)$, $\phi \in M^*$, $a \in A$ and $m \in M$.

There are many possible ways to equip the dual with a grading. An important consideration for us is that if we take the graded dual of the graded dual we should recover the original graded module. One way to do this is the following.

Definition 2.1.10. Let R be a finite dimensional $\mathbb{Z}$-graded k-algebra. If M is a left graded R-module we define the *graded dual* $M^{\circledast} = \oplus_i M_i^{\circledast}$ by

$$M_i^{\circledast} = \{f : M \to k \mid f(M_j) = 0 \text{ for all } j \neq -i\}$$

with right action $(f \cdot r)(m) = f(r \cdot m)$.

This definition is essentially chosen so that the following holds. Let $\mathcal{G}_l(A)$ be the category of graded left finite dimensional A-modules and $\mathcal{G}_r(A)$ the category of graded right finite dimensional A-modules.

Lemma 2.1.11. *Graded duality is a contravariant equivalence of categories*

$$(-)^{\circledast} : \mathcal{G}_l(A) \to \mathcal{G}_r(A).$$

Proof. The graded duality is a contravariant functor

$$(-)^{\circledast} : \mathcal{G}_l(A) \to \mathcal{G}_r(A).$$

that is adjoint to itself. We have that $(M^{\circledast})^{\circledast} \cong M$ via the standard argument that the double dual of a vector space is isomorphic to itself. $\square$

Two modules that will be of importance throughout this thesis are the injective hull and projective cover. We define them here.

Definition 2.1.12. Let M be an A-module. The *projective cover* $P(M)$ of M, if it exists, is a projective module with an epimorphism $e : P(M) \twoheadrightarrow M$ such that the kernel of e is a superfluous submodule. In other words, if $\operatorname{Ker} e + H = P(M)$ for some submodule $H \subset P(M)$ then $H = P(M)$.

Definition 2.1.13. Let M be an A-module. The *injective hull* $I(M)$ of M is an injective module with a monomorphism $m : M \hookrightarrow I(M)$ so that the kernel of m is an essential submodule. In other words, if $\operatorname{Ker} m \cap H = \{0\}$ then $H = \{0\}$.

Remark 2.1.14. Projective covers and injective hulls exist for finite dimensional algebras.

Lemma 2.1.15. *Let A be a finite dimensional graded algebra. Let $\lambda \in \operatorname{Irr} A$ and $P(\lambda)$ be its projective cover and $I(\lambda)$ its injective envelope. Then $P(\lambda^{\circledast}) = I(\lambda)^{\circledast}$.*

Proof. The injective hull $I(\lambda)$ can be equivalently defined by the following diagram

$$\begin{tikzcd}
 & M \arrow[d, dotted, "\exists f"] & \\
\lambda \arrow[r, hook, "m"] & I(\lambda) & N \arrow[l, "g"'] \arrow[ul, hook, "h"']
\end{tikzcd}$$

where m is the essential monomorphism and M, $N \in A$-mod. Lemma 2.1.11 states that functor $\circledast$ is exact. Applying the duality to the diagram we have following

$$\begin{tikzcd}
 & M^{\circledast} \arrow[dr, "h^{\circledast}"] & \\
\lambda^{\circledast} & I(\lambda)^{\circledast} \arrow[l, two heads, "m^{\circledast}"'] \arrow[u, dotted, "\exists f^{\circledast}"] \arrow[r, "g^{\circledast}"] & N^{\circledast}
\end{tikzcd} .$$

This is precisely the diagram that defines the projective cover of $\lambda^{\circledast}$, hence $I(\lambda)^{\circledast} = P(\lambda^{\circledast})$. $\square$

§ 2.2 | Radicals and Socles

For any deeper study of the structure of modules, the concepts of radical and socle are necessary. Here we define both of these terms and prove elementary, but important results. Of particular significance is that both the radical and the socle of a graded module are graded submodules. This section finishes by proving several technical lemmata that will be necessary for our later proofs of the multiplicity results.

Throughout this section let k be a field and A a finite dimensional k-algebra. Denote by A-mod the category of all finite dimensional left A-modules.

Definition 2.2.1. The *radical* of an A-module M is defined as the intersection of all maximal submodules and denoted $\operatorname{rad} M$.

Definition 2.2.2. The *socle* of an A-module M is the sum of all irreducible submodules and denoted $\operatorname{soc} M$.

The radical and socle have different equivalent definitions which vary in their practicality depending on the situation. For this reason we give these alternative definitions and prove their equivalence.

Lemma 2.2.3. *Let M be a finite dimensional A-module. The radical of M is the smallest submodule of M such that the quotient is semisimple. The socle of M is the largest semisimple submodule.*

Proof. Let $M' = M_1 \oplus \cdots \oplus M_n$ be a semisimple quotient of M, where M_i is simple for all i. The kernel of $M \to M'$ is the intersection of the kernels $M \to M_i$, which are each maximal. In particular, it contains $\operatorname{rad} M$. Conversely, since M is finite dimensional, there exists finitely many maximal submodules $N_1, \ldots, N_r$ in M such that $\operatorname{rad} M = N_1 \cap \cdots \cap N_r$. Then $M/\operatorname{rad} M$ embeds in $M/N_1 \oplus \cdots \oplus M/N_r$, which is semisimple. Hence $M/\operatorname{rad} M$ is semisimple.

For the second statement we show that the socle is indeed semisimple and that it is the largest such submodule. Given an A-module M we denote the simple submodules by S_i for $i \in I$ some indexing set. By definition $\operatorname{soc} M = \sum_{i=1}^{n} S_i$. Consider the maximal sum $M' = \sum_{i=k}^{\ell} S_i$ of the S_i such that M' is their direct sum. If we take an arbitrary $S_i \subset \operatorname{soc} M$ and consider $S_i \cap M'$ then the intersection is either $\{0\}$ or S_i. Hence M' either does not contain S_i and so is not maximal or it already contains it. Therefore $\operatorname{soc} M = M'$. □

Definition 2.2.4. Given a ring R the *Jacobson radical* $J(R)$ is the intersection of all maximal left ideals of R.

Remark 2.2.5. The Jacobson radical can alternatively be defined as the intersection of all maximal right ideals [1, Lemma 1.3].

The following result, known as Nakayama's lemma, is used in Proposition 2.2.8 to show that the radical and socle can be expressed in terms of the Jacobson radical.

Lemma 2.2.6. *(Nakayama's Lemma) Let R be a ring. Let M be a finitely generated module over R and let $J(R)$ denote the Jacobson radical of R. If $J(R)M = M$ then $M = 0$.*

Proof. See [48, Theorem 2] for the original proof. □

Corollary 2.2.7. *If M is a simple R-module then $J(R)M = 0$.*

Proof. Since M is simple $J(R)M$ is either 0 or M, but then Nakayama's lemma implies that $M = 0$. □

Proposition 2.2.8. *Let M be a finite dimensional A-module and let J denote the Jacobson radical. Then $\operatorname{rad} M = JM$ and $\operatorname{soc} M = \{m \in M \mid J \cdot m = 0\}$.*

Proof. Nakayama's Lemma says, in this context, that if $JM = M$ then $M = 0$. Let M' be a maximal submodule of M. Then $J(M/M') = 0$ by Corollary 2.2.7. Hence $JM \subset M'$. This implies that $JM \subset \operatorname{rad} M$ as $\operatorname{rad} M$ is the intersection of all maximal submodules.. Recall Lemma 2.2.3, which states that $\operatorname{rad} M$ is the smallest submodule of M such that $M/\operatorname{rad} M$ is semisimple. Hence, if we show that M/JM is semisimple then we must have $\operatorname{rad} M \subset JM$. Note that M/JM is an A/J-module and since A/J is semisimple all A/J-modules are semisimple [1, Theorem 3.4].

Since $\operatorname{soc} M$ is semisimple by Lemma 2.2.3, we have $J \cdot \operatorname{soc} M = 0$. Conversely, let $m \in M$ such that $J \cdot m = 0$ and set $L = A \cdot m$. Recall by Remark 2.2.5 that the Jacobson radical is also a right ideal. Then $JL = J \cdot m = 0$, similarly to before this implies that L is an A/J-module and therefore semisimple. Thus $L \subset \operatorname{soc} M$ and hence $m \in \operatorname{soc} M$. □

The benefit of describing the radical and socle in terms of the Jacobson radical is that it allows for easier computations.

Remark 2.2.9. The first two definitions of the radical hold for any abelian category, while the Jacobson version is only valid when considering modules.

The following lemma demonstrates that the radical is a categorical property.

Lemma 2.2.10. *Let $F : A{-}\mathrm{mod} \to \mathrm{mod}{-}B$ be an equivalence of categories. Then $F(\operatorname{rad} M) = \operatorname{rad} FM$.*

Proof. Form the exact sequence

$$0 \to \operatorname{rad} M \to M \to S \to 0,$$

where S is semisimple. Since equivalences preserve exact sequences the following is also exact

$$0 \to F(\operatorname{rad} M) \to FM \to FS \to 0.$$

Now FS is semisimple and so $\operatorname{rad} FM \subset F(\operatorname{rad} M)$. Since F is an equivalence we can consider its inverse functor F^{-1} and apply it to the following

$$0 \to \operatorname{rad} FM \to FM \to FS \to 0$$

to get

$$0 \to F^{-1} \operatorname{rad} FM \to F^{-1}FM \to F^{-1}S \to 0.$$

It is now clear that $\operatorname{rad} M \subset F^{-1}(\operatorname{rad} FM)$ and so applying F to both sides we get $F(\operatorname{rad} M) \subset \operatorname{rad} FM$. □

It is possible to define powers of both the radical and socle of a module. Let M be a module, set $\operatorname{rad}^0 M = M$ and define $\operatorname{rad}^i M = \operatorname{rad}(\operatorname{rad}^{i-1} M)$. Define $\operatorname{soc}^i M$ to be the largest submodule of M such that $\operatorname{soc}^i M/\operatorname{soc}^{i-1} M$ is semisimple and set $\operatorname{soc}^0 M = 0$.

Definition 2.2.11. The n^{th} radical layer of M is $\operatorname{rad}_n M := \operatorname{rad}^n M/\operatorname{rad}^{n+1} M$. The n^{th} socle layer of M is defined $\operatorname{soc}_n M := \operatorname{soc}^n M/\operatorname{soc}^{n-1} M$.

The following lemma shows that radical and socle powers can also be expressed in terms of the Jacobson radical.

Lemma 2.2.12. *Let M be a finite dimensional A-module. Then $\operatorname{rad}^s M = J^s \cdot M$ and $\operatorname{soc}^s M = \{m \in M \mid J^s \cdot m = 0\}$.*

Proof. This follows by induction from Proposition 2.2.8. We have that $\operatorname{rad} M = J \cdot M$. Assume the lemma holds for all $i \leq \ell$ for some $\ell \in \mathbb{Z}_+$ then $\operatorname{rad}^{\ell+1} M = \operatorname{rad}(J^\ell \cdot M) = J(J^\ell \cdot M) = J^{\ell+1} \cdot M$.

By definition $\operatorname{soc}^{i+1} M/\operatorname{soc}^i M = \operatorname{soc}(M/\operatorname{soc}^i M)$, so assume the lemma holds for $i \leq \ell$ for $\ell \in \mathbb{Z}_+$. Then

$$\operatorname{soc}(M/\operatorname{soc}^i M) = \{m + \operatorname{soc}^i M \in M/\operatorname{soc}^i M | J \cdot m \in \operatorname{soc}^i M\},$$

which by the inductive hypothesis is equal to $\{m + \operatorname{soc}^i M \in M/\operatorname{soc}^i M | J^{i+1} \cdot m = 0\}$. Hence

$$\operatorname{soc}^{i+1} M/\operatorname{soc}^i M = \{m + \operatorname{soc}^i M \in M/\operatorname{soc}^i M | J^{i+1} \cdot m = 0\},$$

therefore $\operatorname{soc}^{i+1} M = \{m \in M | J^{i+1} \cdot m = 0\}$ □

An immediate consequence of the definition of the radical is that each higher power is strictly smaller than the one that preceded it, that is $\operatorname{rad}^i M \subset \operatorname{rad}^{i-1} M$. It is then natural to ask the question: when does this process terminate? In general this is a hard question. In Section 5.5 we state a conjecture for this problem in the case of restricted rational Cherednik algebra of S_n. For now let us provide the basic definitions.

Definition 2.2.13. The *Lowey length* of a module M is defined to be the smallest positive integer ℓ such that $\operatorname{rad}^\ell M = 0$. It is denoted $\ell\ell(M)$. Similarly, the *socle length* of M is the smallest ℓ such that $\operatorname{soc}^\ell M = M$.

Lemma 2.2.14. *The Lowey length and socle length of M agree.*

Proof. By Lemma 2.2.12, they are both equal to the smallest positive integer ℓ such that $J^\ell \cdot M = 0$. □

In light of Lemma 2.2.14 we only use the term Lowey length from now on. Let us now assume that A is a $\mathbb{Z}$-graded finite dimensional k-algebra.

Definition 2.2.15. If $I \subset A$ is an ideal then the *initial ideal* $\mathrm{in}(I)$ of I is defined to be the set $\{d_{t(d)} \,|\, d \in I\}$, where $d_{t(d)}$ is the homogeneous part of d of highest degree (the "top").

Lemma 2.2.16. *If* $\mathrm{in}(I) \subset I$ *then* $I = \mathrm{in}(I)$ *is a graded ideal.*

Proof. Let $d \in I$ and write $d = d_n + d_{n-1} + \cdots + d_{n-j}$. Then $d_n \in \mathrm{in}(I)$ implies that $d - d_n = d_{n-1} + \cdots + d_{n-j} \in I$. Repeating, we deduce all d_i belong to I. □

The next three lemmata will prove that the radical of a graded module is indeed a graded submodule.

Lemma 2.2.17. *The Jacobson radical of* A *is a graded ideal.*

Proof. Let J be the Jacobson radical of A. Let $I = \mathrm{in}(J)$. We claim that I is a nilpotent ideal. Indeed, if $x_1, \dots, x_\ell \in I$ then there exist $d_1, \dots, d_\ell \in J$ such that $(d_i)_{t(d_i)} = x_i$. Now, J is nilpotent. Thus, for $\ell \gg 0$, $d_1 \cdots d_\ell = 0$. But the homogeneous part of $d_1 \cdots d_\ell$ of highest degree is $x_1 \cdots x_\ell$. This implies that I is nilpotent. Since J is the largest nilpotent ideal in A [34, Theorem 4.8], we deduce that $I \subset J$. Then Lemma 2.2.16 implies that $I = J$.

□

Lemma 2.2.18. *Let* M *be a graded* A*-module and* I *a graded ideal of* A*. Then* IM *is a graded submodule of* M*.*

Proof. We must show that $IM = \oplus_{i \in \mathbb{Z}} (IM)_i$ where $(IM)_i = IM \cap M_i$. Consider an arbitrary element im in IM. Then

$$im = \sum_{j,\ell} i_j m_\ell$$

where $i_j \in I_j$ and $m_\ell \in M_\ell$. Now, $i_j m_\ell \in M_{j+\ell} \cap IM$ and thus IM is a graded submodule. □

Lemma 2.2.19. *Let* M *be a graded* A*-module. Then* $\mathrm{rad}\, M$ *is a graded submodule.*

Proof. By Proposition 2.2.8, $\mathrm{rad}\, M = JM$, where J is the Jacobson radical of A. By Lemma 2.2.17, the Jacobson radical is a graded ideal and hence by Lemma 2.2.18 JM is graded. □

Similarly to the radical being graded we have the following result for simple modules. Unfortunately it does not have quite the same generality as the previous lemma as we require our algebra to be Artinian. This is not too restrictive a constraint however as the class of Artinian algebras is large. In particular, all finite dimensional k-algebras are Artinian.

Lemma 2.2.20. *Let A be an Artinian graded algebra and S a simple A-module then S can be endowed with a grading.*

Proof. See [33, Proposition 3.5]. □

Next let us show that the radical and socle powers are also graded submodules.

Lemma 2.2.21. *Let M be a finite dimensional graded A-module. Then $\operatorname{rad}^s M = J^s \cdot M$ is a graded submodule.*

Proof. The statement follows easily by induction from Lemma 2.2.19 and Lemma 2.2.18. □

Lemma 2.2.22. *Let M be a finite dimensional graded A-module. If A is Artinian then $\operatorname{soc}^s M = \{m \in M \mid J^s \cdot m = 0\}$ is a graded submodule.*

Proof. Since M is graded we can represent any element $m \in \operatorname{soc}^s M$ by a unique sum of homogeneous components $m = m_n + \cdots + m_i$. By the definition of $\operatorname{soc}^s M$ we have that $J^s \cdot m = J^s \cdot (m_n + \cdots + m_i) = J^s \cdot m_n + \cdots + J^s \cdot m_i = 0$. Since the components are homogeneous we see that the terms $j \cdot m_\ell$ will be in different degrees if $j \in J^s$ is homogeneous, hence $j \cdot m_\ell = 0$ for $i \leq \ell \leq n$. Since J^s is generated by homogeneous elements this implies that $J^s \cdot m_\ell = 0$ and $m_\ell \in \operatorname{soc}^s M$ for $i \leq \ell \leq n$. □

Recall that we denote the category of graded A-modules by $\mathcal{G}(A)$ where A is a finite dimensional graded k-algebra. The objects in $\mathcal{G}(A)$ are graded A-modules and the morphisms are graded homomorphisms of degree zero. From now on we will often just write $\mathcal{G}$ if the algebra being considered is clear from context.

Lemma 2.2.23. *Let M, N be two finite dimensional graded A-modules.*

(a) If $\operatorname{soc}^s M = M$ then $\operatorname{Hom}_{\mathcal{G}}(M, N) = \operatorname{Hom}_{\mathcal{G}}(M, \operatorname{soc}^s N)$.

(b) If $\operatorname{rad}^s N = 0$ then $\operatorname{Hom}_{\mathcal{G}}(M/\operatorname{rad}^s M, N) = \operatorname{Hom}_{\mathcal{G}}(M, N)$.

Proof. Note that $\operatorname{rad}^s M = J^s \cdot M$ and $\operatorname{soc}^s M = \{m \in M \mid J^s \cdot m = 0\}$.

Let us prove (a). Let $\phi \in \operatorname{Hom}_{\mathcal{G}}(M, N)$. Then

$$0 = \phi(0) = \phi(J^s \cdot M) = J^s \cdot \phi(m).$$

This implies $J^s \cdot \operatorname{im}\phi = 0$, hence $\operatorname{im}\phi \subset \operatorname{soc}^s N$. Therefore, $\phi \in \operatorname{Hom}_{\mathcal{G}}(M, \operatorname{soc}^s N)$. The opposite inclusion is obvious.

For part (b) note

$$\mathrm{Hom}_{\mathcal{G}}(M/\mathrm{rad}^s M, N) = \{\phi \in \mathrm{Hom}_{\mathcal{G}}(M,N) \,|\, \phi(J^s \cdot M) = 0\}$$

$$= \{\phi \in \mathrm{Hom}_{\mathcal{G}}(M,N) \,|\, J^s \cdot \phi(M) = 0\}.$$

Since $J^s \cdot N = 0$ as $\mathrm{rad}^s N = 0$, we have $J^s \phi(M) = 0$ for all $\phi \in \mathrm{Hom}_{\mathcal{G}}(M,N)$. Hence $\mathrm{Hom}_{\mathcal{G}}(M/\mathrm{rad}^s M, N) = \mathrm{Hom}_{\mathcal{G}}(M,N)$. □

Lemma 2.2.24. *For any finite dimensional graded A-module M, we have $(M/\operatorname{rad}^s M)^{\circledast} = \mathrm{soc}^s M^{\circledast}$.*

Proof. First note that by Lemma 2.2.21 $\operatorname{rad}^s M = J^s \cdot M$ and $\mathrm{soc}^s M^{\circledast} = \{m \in M^{\circledast} \,|\, m \cdot J^s = 0\}$. Therefore,

$$\mathrm{soc}^s M^{\circledast} = \oplus_i \{f : M \to k \,|\, f(M_j) = 0 \text{ for all } j \neq -i \text{ and } f(J^s \cdot m) = 0 \text{ for all } m \in M\}.$$

We also have by definition

$$(M/\operatorname{rad}^s M)^{\circledast} = \oplus_i \{f : M/\mathrm{rad}^s M \to k \,|\, f((M/\mathrm{rad}^s M)_j) = 0 \text{ for all } j \neq -i\}.$$

By Lemma 2.2.21, we know that $\mathrm{rad}^s M = J^s \cdot M$ so the homogeneous components of $(M/\operatorname{rad}^s M)^{\circledast}$ are precisely the sets $\{f : M \to k \,|\, f(M_j) = 0 \text{ for all } j \neq -i\}$ such that $f(J^s \cdot m) = 0$ for all $m \in M$. Hence $(M/\operatorname{rad}^s M)^{\circledast} = \mathrm{soc}^s M^{\circledast}$ □

The next lemma shows that Lowey length behaves well with respect to the graded structure.

Lemma 2.2.25. *If N and M are finite dimensional A-modules and $N \subset M$ then $\ell\ell(N) \leq \ell\ell(M)$.*

Proof. Given a morphism between two modules $\phi : N \to M$, then it follows from Lemma 2.2.21 that $\phi(\operatorname{rad}^s N) \subset \operatorname{rad}^s M$. Hence there is an induced morphism $\phi_s : \operatorname{rad}^s N \to \operatorname{rad}^s M$. Taking ϕ to be the inclusion morphism, the result follows. □

Lemma 2.2.26. *For any finite dimensional A-module M we have that*

(a) $\operatorname{rad}^s \mathrm{soc}^s M = 0$; *and*

(b) $\mathrm{soc}^s(M/\operatorname{rad}^s M) = M/\operatorname{rad}^s M$.

Proof. Both statements follow directly from Lemma 2.2.21 and Lemma 2.2.22. □

We can now prove the following lemma which will be of use later.

Lemma 2.2.27. *For any finite dimensional graded k-algebra A and finite dimensional A-modules M and N we have*

$$\mathrm{Hom}_A(M/\operatorname{rad}^s M, N) = \mathrm{Hom}_A(M, \operatorname{soc}^s N).$$

Proof. Using Lemma 2.2.26 (b) we see that $\operatorname{soc}^s(M/\operatorname{rad}^s M) = M/\operatorname{rad}^s M$ and by Lemma 2.2.23 (a)

$$\mathrm{Hom}_A(M/\operatorname{rad}^s M, N) \cong \mathrm{Hom}_A(M/\operatorname{rad}^s M, \operatorname{soc}^s N).$$

From Lemma 2.2.26 (a) $\operatorname{rad}^s \operatorname{soc}^s N = 0$ and by Lemma 2.2.23 we see that

$$\mathrm{Hom}_A(M/\operatorname{rad}^s M, \operatorname{soc}^s N) \cong \mathrm{Hom}_A(M, \operatorname{soc}^s N)$$

and so

$$\mathrm{Hom}_A(M/\operatorname{rad}^s M, N) = \mathrm{Hom}_A(M, \operatorname{soc}^s N).$$

□

The concept of a module multiplicity is fundamental to this thesis so we now define both graded and ungraded multiplicity.

Definition 2.2.28. Let A be a finite dimensional k-algebra, M a finite dimensional A-module and N a simple A-module. Then the multiplicity of N in M, denoted $[M : N]$, is defined to be the number of times N appears as a composition factor in a composition series of M.

Definition 2.2.29. Let A be a finite dimensional graded k-algebra and M, N both finite dimensional graded A-modules with N simple. The graded multiplicity of N inside M is denoted $[M : N]_{gr}$, and is the number of times N appears as a graded composition factor in a graded composition series of M.

Remark 2.2.30. The quotient of a graded module by a graded submodule inherits a natural grading and so Definition 2.2.29 makes sense.

The final lemma of this section is simply a technical result that we will require for the proofs of later theorems. It is necessary to define what it means to say A is split over the field k. We follow the definition given in [10, Remark 2].

Definition 2.2.31. Let A be a finite dimensional k-algebra. We say that A is *split* over k or k is a *splitting field* for A if $\mathrm{End}_A(\lambda) \cong k$ for all $\lambda \in \mathrm{Irr}\, A$.

Lemma 2.2.32. *Let A be a graded k-algebra, split over k. If M is a graded A-module of finite length and λ is a simple graded A-module then*

$$[M:\lambda]_{gr} = \dim_k \operatorname{Hom}_{\mathcal{G}}(P(\lambda), M) = \dim_k \operatorname{Hom}_{\mathcal{G}}(M, I(\lambda)).$$

Proof. Proceed by induction on the length, $\ell(M)$, of M. If the length of M is 1 then $M \cong \mu$ for some simple A-module μ. We have

$$\dim_k \operatorname{Hom}_{\mathcal{G}}(P(\lambda), \mu) = \begin{cases} 1 \text{ if } \lambda \cong \mu \\ 0 \text{ else,} \end{cases}$$

and hence the result holds.

We now assume that the length of M is greater than one and the result holds for all modules of length less than the length of M. There exists a graded submodule $N \subset M$ such that M/N is simple. This implies that $\ell(N) = \ell(M) - 1$. The following being exact

$$0 \to N \to M \to M/N \to 0$$

together with the fact that $P(\lambda)$ is projective implies

$$0 \to \operatorname{Hom}_{\mathcal{G}}(P(\lambda), N) \to \operatorname{Hom}_{\mathcal{G}}(P(\lambda), M) \to \operatorname{Hom}_{\mathcal{G}}(P(\lambda), M/N) \to 0 \tag{2.2.1}$$

is also exact.

There are two cases to consider. First if $M/N \cong \lambda$ then $[M:\lambda]_{gr} = [N:\lambda]_{gr} + 1$ and the sequence (2.2.1) implies that $\dim_k \operatorname{Hom}_{\mathcal{G}}(P(\lambda), M) = \dim_k \operatorname{Hom}_{\mathcal{G}}(P(\lambda), N) + 1$. We conclude that $[M:\lambda]_{gr} = \dim_k \operatorname{Hom}_{\mathcal{G}}(P(\lambda), M)$. In the second case we have $M/N \ncong \lambda$ hence $[M:\lambda]_{gr} = [N:\lambda]_{gr}$. We also have $\operatorname{Hom}_{\mathcal{G}}(P(\lambda), M/N) = 0$. From (2.2.1) we see that $\operatorname{Hom}_{\mathcal{G}}(P(\lambda), N) \cong \operatorname{Hom}_{\mathcal{G}}(P(\lambda), M)$. Therefore $[M:\lambda]_{gr} = \dim_k \operatorname{Hom}_{\mathcal{G}}(P(\lambda), M)$. The proof of the second equality is similar. □

§ 2.3 | Standard and Costandard Modules

In this section we introduce an important class of finite dimensional graded algebras. These are the graded algebras with a triangular decomposition. Many important algebras in representation theory such as restricted rational Cherednik algebras and restricted universal enveloping algebras admit a triangular decomposition. For the class of graded algebras with triangular decomposition there are two modules of special importance, these are the standard and costandard modules. Let A be a finite dimensional graded k-algebra, the following definition can be found in [36, Definition 2.1].

Definition 2.3.1. A triangular decomposition of A is a triple $\mathfrak{T} = (A^-, T, A^+)$ of graded subalgebras of A such that

1. The multiplication map $A^- \otimes T \otimes A^+ \to A$ is an isomorphism
2. $\operatorname{Supp} A^+ \subset \mathbb{Z}_{\geq 0}$ and $\operatorname{Supp} A^- \subset \mathbb{Z}_{\leq 0}$
3. $A^-_0 = k = A^+_0$
4. $A^+T = TA^+$ and $A^-T = TA^-$ as subspaces of A
5. $T = T_0$ is a split k-algebra.

Here $\operatorname{Supp} M = \{i \in \mathbb{Z} \mid M_i \neq 0\}$ for a graded object M.

An important example of an algebra with a triangular decomposition is the restricted universal enveloping algebra of a semisimple Lie algebra.

Example 2.3.2. Let k be a field of characteristic $p \geq 0$. Let $\mathfrak{g}$ be a Lie algebra. The *universal enveloping algebra* $U(\mathfrak{g})$ is the quotient algebra

$$T(\mathfrak{g})/(a \otimes b - b \otimes a = [a, b] \mid a,\, b \in \mathfrak{g}).$$

If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_n(k)$ then for any element $x \in \mathfrak{g}$ we can take x^p in $\mathfrak{gl}_n(k)$, we denote this $x^{[p]}$. This is to differentiate it from the p^{th} power of x in $U(\mathfrak{g})$.

The *restricted universal enveloping algebra* $\overline{U}(\mathfrak{g})$ is the quotient [41, p. 5]

$$U(\mathfrak{g})/(x^p - x^{[p]}).$$

Note the element $x^p - x^{[p]}$ is central in $U(\mathfrak{g})$.

The next lemma records an important observation about algebras with a triangular decomposition.

Lemma 2.3.3. *For any algebra A that admits a triangular decomposition denote the subalgebra $B^- = A^- \otimes T = A^-T$. Then A is free as a B^--module.*

Proof. By property (1) of Definition 2.3.1 we have an isomorphism $A^- \otimes T \otimes A^+ \to A$. Given a basis of A^+ written $\{a_1, \ldots, a_n\}$ we can explicitly write a basis for A as a B^--module as $\{1 \otimes a_1, \ldots, 1 \otimes a_n\}$. □

Similarly to the above denote $B^+ = T \otimes A^+$. Note that there exists an obvious surjection $B^+ \twoheadrightarrow T$ and so every T-module can be viewed as a B^+-module by inflation (pullback along the quotient map). Let $\operatorname{Irr}\mathcal{G}(T)$ denote the set of irreducible graded T-modules. To each simple T-module $\lambda \in \operatorname{Irr}\mathcal{G}(T)$ we can construct an associated A-module, the standard module $\Delta(\lambda)$.

Definition 2.3.4. Let A be a finite dimensional algebra with a triangular decomposition. Given a left T-module M we can define the *standard module* of M,

$$\Delta(M) = A \otimes_{B^+} M$$

which is a left A-module.

The costandard is defined in a dual manner.

Definition 2.3.5. Let A be a finite dimensional algebra with a triangular decomposition. Then we can define the costandard module of a graded left T-module M as follows

$$\nabla(M) := (M^{\circledast} \otimes_{B^-} A)^{\circledast}.$$

Note that the costandard module is a left A-module. Indeed $M^{\circledast}$ is a right B-module, hence $(M^{\circledast} \otimes_{B^-} A)$ is a right A-module and so $(M^{\circledast} \otimes_{B^-} A)^{\circledast}$ has a left A-module structure.

Later multiplicity results will require us to calculate the multiplicity of standard modules inside the projective covers. For this to make sense we define Δ-filtrations.

Definition 2.3.6. Let A be an algebra with triangular decomposition. Let M be a graded A-module. We say M has a Δ-*filtration* if there is a filtration by submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

with $M_i/M_{i-1} \cong \Delta(\lambda_i)$ for $\lambda_i \in \operatorname{Irr}T$.

The next lemmata prove fundamental properties of standard and costandard modules that will be used repeatedly throughout this thesis.

Remark 2.3.7. By the unique maximal quotient of a module M we mean a quotient that surjects onto any other quotient of M.

Lemma 2.3.8. *Let A be a finite dimensional algebra and M an A module. Then the following are equivalent.*

(a) M has a unique simple submodule.

(b) M has a unique maximal quotient.

(c) M has a simple socle.

Proof. We start with $(a) \implies (b)$. Denote the unique simple submodule of M by S. Given any other submodule N of M we have that $S \subset N$ and so

$$(M/S)/(N/S) \cong M/N.$$

Clearly M/S surjects onto $(M/S)/(N/S)$, hence it surjects onto M/N. That $(b) \implies (c)$ is clear. To see that $(c) \implies (a)$ recall that $\operatorname{soc} M$ is the sum of all irreducible submodules. Since there is a unique simple S we must have that $\operatorname{soc} M = S$. □

Lemma 2.3.9. *For any $\lambda \in \operatorname{Irr} \mathcal{G}(T)$ the standard module $\Delta(\lambda)$ has a unique simple graded quotient called the head and is denoted $L(\lambda)$.*

Proof. Let $M \subset \Delta(\lambda)$ be any proper graded submodule. We claim that

$$M \cap (1 \otimes \lambda) = M_0 = 0.$$

We argue by contradiction. If there exists a non-zero element $m \in M \cap (1 \otimes \lambda)$ then $A \cdot m = A \otimes \lambda = \Delta(\lambda)$ which contradicts M being a proper graded submodule. Given any two proper submodules M_1 and M_2 their sum $M_1 + M_2$ is also a proper submodule. This can be seen by noticing that the degree 0 part of $M_1 + M_2$ is 0. Therefore, we can construct a unique maximal submodule by defining it to be the sum of all other proper submodules. Hence, $\Delta(\lambda)$ has a unique simple graded quotient.

□

Lemma 2.3.10. *For any $\lambda \in \operatorname{Irr} \mathcal{G}(T)$ the costandard module $\nabla(\lambda)$ has a simple socle.*

Proof. Recall the definition of the costandard module $\nabla(\lambda) = (\lambda^{\circledast} \otimes_{B^-} A)^{\circledast}$. Consider the dual $\nabla(\lambda)^{\circledast} = (\lambda^{\circledast} \otimes_{B^-} A)$. Then by an argument similar to the proof of Lemma 2.3.9 we see that this has a unique simple quotient S. Let $Q \hookrightarrow \nabla(\lambda)$ be the inclusion of a simple module. Then $\nabla(\lambda)^{\circledast} \twoheadrightarrow Q^{\circledast}$ and since S is the unique simple quotient we must have that $Q^{\circledast} = S$. Hence Q must be the unique simple submodule of $\nabla(\lambda)$ and by Lemma 2.3.8

we see that $Q = \operatorname{soc} \nabla(\lambda)$.

□

Lemma 2.3.11. *The head of* $\Delta(\lambda)$ *is isomorphic to the socle of* $\nabla(\lambda)$.

Proof. We note that $L(\lambda)$ is uniquely defined by $L(\lambda)_0 = \lambda$ and $L(\lambda)_i = 0$ for all $i > 0$. If there was another graded simple A-module, say L, such that $L_0 = \lambda$ and $L_i = 0$ for all $i > 0$, then $L \cong L(\lambda)$. To see this note that, by adjunction,

$$\operatorname{Hom}_A(\Delta(\lambda), L) = \operatorname{Hom}_{B^+}(\lambda, L|_{B^+}) \cong \operatorname{Hom}_{B^+}(\lambda, \lambda) \cong k.$$

Hence $L(\lambda) \cong L$. By a dual argument we find that the unique simple quotient of $\nabla(\lambda)^{\circledast} = (\lambda^{\circledast} \otimes_{B^+} A)$, denoted S, is uniquely defined by $S_0 = \lambda^{\circledast}$ and $S_i = 0$ for all $i < 0$. By the argument in Lemma 2.3.10, we have that $S^{\circledast} = \operatorname{soc}\ \nabla(\lambda)$ and $S^{\circledast}$ is the simple submodule of $\nabla(\lambda)$ uniquely defined by $S_0^{\circledast} = \lambda$ and $S_i = 0$ for all $i > 0$. Hence $\operatorname{soc}\ \nabla(\lambda) \cong L(\lambda)$. □

A necessary condition for the graded multiplicity results in Chapter 3 is that the algebras have an anti-triangular duality.

Definition 2.3.12. Let A be a finite dimensional algebra, with triangular decomposition and let A-mod denote the category of A-modules. A *duality* $D : A\text{−mod} \to A\text{−mod}$ is a contravariant equivalence of categories. An *anti-triangular duality* is a duality with the following additional conditions:

1. $D(\Delta(\lambda)) = \nabla(\lambda)$.

2. $D(L(\lambda)) = L(\lambda)$.

This section finishes by stating and proving two technical lemmata.

Lemma 2.3.13. *Let* M *and* N *be objects in* $A\text{−mod}$ *and let* D *be an anti-triangular duality. Then*

$$D(M/\operatorname{rad}^s M) \cong \operatorname{soc}^s D(M)$$

Proof. Consider the following short exact sequence

$$0 \to \operatorname{soc}^s D(M) \to D(M) \to D(M)/\operatorname{soc}^s D(M) \to 0.$$

Applying D to the above gives a new exact sequence

$$0 \to D(D(M)/\operatorname{soc}^s D(M)) \to M \to D(\operatorname{soc}^s D(M)) \to 0.$$

From the above we see that $D(\operatorname{soc}^s D(M))$ is a quotient of M. Since $\ell\ell(D(\operatorname{soc}^s D(M))) \leq s$, this quotient must factor through $M/\operatorname{rad}^s M$. Hence,

$$M/\operatorname{rad}^s M \twoheadrightarrow D(\operatorname{soc}^s D(M))$$

and so

$$\operatorname{soc}^s D(M) \hookrightarrow D(M/\operatorname{rad}^S M).$$

Since $D(M/\operatorname{rad}^s M)$ is a submodule of $D(M)$ and $\ell\ell(D(M/\operatorname{rad}^s M)) \leq s$, we must similarly have $D(M/\operatorname{rad}^s M) \subset \operatorname{soc}^s D(M)$. Hence $D(M/\operatorname{rad}^s M) \cong \operatorname{soc}^s D(M)$. □

Lemma 2.3.14. *If we have an anti-triangular duality D and $\mu \in \operatorname{Irr}\mathcal{G}(T)$ then*

$$\operatorname{soc}_s \nabla(\mu) = D(\operatorname{rad}_s \Delta(\mu))$$

Proof. From Lemma 2.3.13 we have that $\operatorname{soc}^s D(\Delta(\mu)) \cong D(\Delta(\mu)/\operatorname{rad}^s \Delta(\mu))$. Therefore

$$\operatorname{soc}_s D(\Delta(\mu)) = (\operatorname{soc}^s D(\Delta(\mu))/\operatorname{soc}^{s-1} D(\Delta(\mu))) \cong D(\Delta(\mu)/\operatorname{rad}^s \Delta(\mu))/D(\Delta(\mu)/\operatorname{rad}^{s-1} \Delta(\mu)).$$

By definition $D(\Delta(\mu)) = \nabla(\mu)$ therefore $\operatorname{soc}_s \nabla(\mu) = \operatorname{soc}_s D(\Delta(\mu))$ and

$$\operatorname{soc}_s \nabla(\mu) \cong D(\Delta(\mu)/\operatorname{rad}^s \Delta(\mu))/D(\Delta(\mu)/\operatorname{rad}^{s-1} \Delta(\mu)).$$

If we consider an exact sequence of A-modules

$$0 \to S \to R \to R/S \to 0$$

and apply the anti-triangular duality D, we get another exact sequence

$$0 \to D(R/S) \to D(R) \to D(S) \to 0.$$

Therefore, $D(S) \cong D(R)/D(R/S)$. If we let $R = \Delta(\mu)/\operatorname{rad}^s \Delta(\mu)$ and $S = \operatorname{rad}_s \Delta(\mu)$ then we have that

$$D(\operatorname{rad}_s \Delta(\mu)) = D(\Delta(\mu)/\operatorname{rad}^s \Delta(\mu))/D(\Delta(\mu)/\operatorname{rad}^{s-1} \Delta(\mu)).$$

Therefore $\operatorname{soc}_s \nabla(\mu) = D(\operatorname{rad}_s \Delta(\mu))$. □

§ 2.4 | Partitions

Partitions of integers are necessary for the description of the center of the restricted rational Cherednik algebras $\overline{H}_c(S_n)$ and $\overline{H}_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ given in Chapter 5. This is for the simple reason that the partitions of n are in bijection with the irreducible representations of the symmetric group S_n [29, Lemma 4.25]. Furthermore, there is a bijection between the ℓ-multipartitions of n and the irreducible representations of the wreath product $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ [44, p. 221].

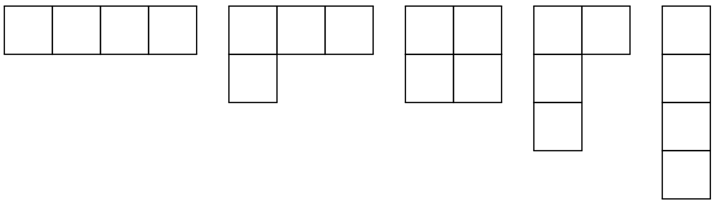
Definition 2.4.1. Let n be a positive integer. A *partition* of n is a tuple $(\lambda_1, \cdots, \lambda_n)$

of non-negative integers such that $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq n-1$, and

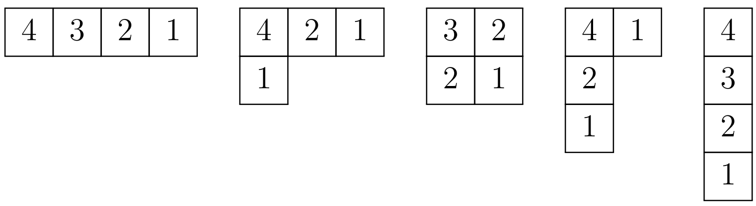
$$|\lambda| := \sum_{i=1}^{n} \lambda_i = n.$$

The *length* of λ is the positive integer t such that $\lambda_t \neq 0$ and $\lambda_{t+1} = 0$.

Partitions are often represented by Young diagrams. The Young diagram for a given partition $(\lambda_1, \cdots, \lambda_n) \vdash n$ consists of rows and columns, the i^{th} row has λ_i cells and always starts from the leftmost position. Let us fix the conventions on the coordinates of the cells. We count the columns from left to right and the rows top to bottom. This means that the cell $(2,3)$ is the second row down and the third column along to the right. Below are the Young diagrams of the partitions $(4,0,0,0)$, $(3,1,0,0)$, $(2,2,0,0)$, $(2,1,1,0)$ and $(1,1,1,1)$, in that order.



We can assign to each cell in the Young diagram an integer, by calculating the hook length. The hook length is calculated for each cell by summing the cells to the right and the cells directly below, then adding one, for the cell itself. The hook length can be easily read from the Young diagram. Here are the partitions of 4 with the hook lengths of each cell written inside.



Let us now give a formula for the hook length of a cell. For a given cell (i,j), let L denote the number of cells in the j^{th} column, this is called the leg. This is a non-standard use of the term leg, however this version is better suited for use in the proofs contained in this section. The hook length $h(i,j)$ of the cell (i,j) is then

$$h(i,j) = \lambda_i - j + L - i + 1. \tag{2.4.1}$$

The following two lemmata will be used later in the thesis.

Lemma 2.4.2. *Let $\lambda \vdash n$ and consider the set $P = \{d_1, \cdots, d_n\}$, where $d_i = \lambda_i + n - i$. Then*

$$|\{j \mid d_i - j \notin P \text{ for } 1 \leq j \leq d_i\}| = \lambda_i.$$

Proof. Fix i. There are $n-i$ d_k's such that $d_k \leq d_i$. Hence there are $d_i-(n+i) = \lambda_i + n - i - n + i = \lambda_i$ many numbers such that $d_i - j \notin P$. Therefore $|\{j \mid d_i - j \notin P \text{ for } 1 \leq j \leq d_i\}| = \lambda_i$. □

If we let P and d_i be the same as in Lemma 2.4.2 we can prove the following.

Lemma 2.4.3. *The set of hook lengths of the row i equals the set $\{j \mid d_i - j \notin P\}$.*

Proof. Let us show that the set of hook lengths of row i is contained in the set $\{j \mid d_i - j \notin P\}$. The result will then follow from Lemma 2.4.2. Fix $i \in \{1, 2, \cdots, n\}$. If there exists d_m such that $d_i - h(i,j) = d_m$ for some j then

$$h(i,j) = d_i - d_m.$$

Hence

$$\lambda_i - j + L - i + 1 = \lambda_i + n - i - \lambda_m + n - m,$$

which simplifies to

$$L - j + 1 = m - \lambda_m. \tag{2.4.2}$$

We now have three cases to consider $L > m$, $L = m$ and $L < m$.

First let $L > m$. Then $L = m + k$ for some positive integer k and so equation (2.4.2) becomes $j = k + \lambda_m + 1$. Therefore $j > \lambda_m$. Since $L > m$ we have that $\lambda_m > \lambda_L$, hence $j > \lambda_L$. But this contradicts the definition of L.

For the second case, let $m = L$. Then (2.4.2) becomes $j = \lambda_m + 1 = \lambda_L + 1$ which is clearly a contradiction as $\lambda_L \geq j$.

The final case is when $L < m$. Then $L = m - k \geq i$ for some positive integer k. Equation (2.4.2) $\lambda_m + 1 = j + k$ now since k is a positive integer we have $j \leq \lambda_m$. But since $m > L$ we have that $j > \lambda_m$ and so we have a contradiction. Therefore there is no d_m such that $d_i - h(i,j) = d_m$ and so $h(i,j) \in \{j \mid d_i - j \notin P\}$ □

Lemma 2.4.4. *Given a partition $\lambda \vdash n$, we have $d_i - h(i,j) = d_k - h(k,j)$.*

Proof. Using formula (2.4.1) the following calculations gives the result.

$$d_i - h(i,j) = \lambda_i + n - i - \lambda_i + j - L + i - 1 = n + j - L - 1$$

and

$$d_k - h(k,j) = \lambda_k + n - k - \lambda_k + j - L + k - 1 = n + j - L - 1.$$

□

We mentioned at the beginning of this section that the irreducible representations of the wreath product are in bijection with certain multipartitions. Let us make this statement precise by defining the ℓ-core and ℓ-quotient of a partition λ.

Definition 2.4.5. An ℓ-multipartition of n is a ℓ-tuple $(\lambda^1, \cdots, \lambda^\ell)$ such that each λ^i is a partition and $\sum_{i=1}^{\ell} |\lambda^i| = n$.

Let us introduce some more notation. Given an ℓ-multipartition $\lambda = (\lambda^1, \cdots, \lambda^\ell)$ we denote by $\lambda^\flat := (\lambda^\ell, \cdots, \lambda^1)$. To define ℓ-cores and ℓ-quotients we need two things, firstly we must define β-numbers (which are a generalisation of hook lengths) and secondly bead diagrams.

The *first column hook lengths* of a partition is the set of hook lengths of the cells on the leftmost column of the Young diagram for a partition. For example the partition $(3, 2, 1, 1)$ has Young diagram

$$\begin{array}{|c|c|c|}\hline 6 & 3 & 1 \\ \hline 4 & 1 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

and the first column hook lengths are $\{6, 4, 2, 1\}$. The first column hook lengths are important, because the original partition can always be recovered from the first column hook lengths. To see this, recall the formula for hook length given above,

$$h(i, j) = \lambda_i - j + L - i + 1.$$

The first column hook lengths are given by fixing $j = 1$. Hence

$$h(i, 1) = \lambda_i - 1 + L - i + 1 = \lambda_i + L - i,$$

and thus $\lambda_i = h(i, 1) - L + i$. If we instead replace L by some other positive integer in the formula for hook length do we still get a set of numbers defining a partition? Yes. This generalisation is precisely the definition of β-numbers.

Definition 2.4.6. Let $\lambda \vdash n$ and let $p \geq n$. Define

$$\beta_i^p = \lambda_i + p - i$$

for $1 \leq i \leq p$. The β_i^p are called *β-numbers*. Note that the β-numbers are pairwise distinct, hence $|\{\beta_i^p \mid 1 \leq i \leq p\}| = p$.

Next we introduce bead diagrams.

Definition 2.4.7. We refer to elements of the set $\mathbb{Z}_{\leq -1} \times \{0, \cdots, \ell - 1\}$ as *points*. A *bead diagram* is a function $f : \mathbb{Z}_{\leq -1} \times \{0, \cdots, \ell - 1\} \to \{0, 1\}$ which takes the value 1 for only finitely many points. If $f(i, j) = 1$ then the point is said to be occupied by a *bead*. If $f(i, j) = 0$, then the point (i, j) is *empty*.

Let us fix some common terminology for bead diagrams. The point (i, j) is said to lie to the left of (a, b) if $j < b$, similarly (i, j) is said to lie above (a, b) if $a < i$.

Bead diagrams and β-numbers have a strong connection. Given a set of beta numbers we can construct a bead diagram via the following rule. Let $f(i, j) = 1$ if and only if $-(i + 1) \cdot \ell + j \in \{\beta_i^p \,|\, 1 \leq i \leq p\}$. In the case where p is the smallest multiple of ℓ such that p is greater than the number of terms in the partition of λ we denote the bead diagram by $\mathbb{B}(\lambda)$.

We can now define ℓ-cores, ℓ-quotients and bead diagrams following the definitions in [39, Chapter 2.7].

Definition 2.4.8. Let $\lambda \vdash n$ and fix a positive integer $\ell \leq n$. Consider the bead diagram $\mathbb{B}(\lambda)$ of λ with ℓ columns. If we slide the beads upwards as much as possible we obtain a new bead diagram. The partition corresponding to this new bead diagram is called the ℓ-core of λ.

Let us explain how to read the partition from the bead diagram. Given a bead diagram begin counting the beads from left to right, ignoring all beads before the first empty bead. The position of the beads will give a set of positive integers. If we let this set of integers correspond to the set of first column hook lengths then we retrieve a partition. This means that each bead diagram gives a partition and each partition defines a bead diagram, up to an arbitrary number of beads before the first empty bead.

Definition 2.4.9. Consider the bead diagram $\mathbb{B}(\lambda)$. The columns can be considered as bead diagrams for $\ell = 1$. Denote the partition defined by the first column by λ^1, the second by λ^2 and so on. We define the *ℓ-quotient* to be the ℓ-multipartition $(\lambda^1, \cdots, \lambda^\ell)$.

Let us introduce one last piece of notation. Denote by $\mathcal{P}(n)$ the set of all partitions of n, and $\mathcal{P}(n, \ell)$ the set of all ℓ-multipartitions of n. Also, denote by $\mathcal{P}_\lambda(n)$ the set of all partitions of n with ℓ-core λ.

A partition is uniquely determined by its ℓ-core and its ℓ-quotient. By restricting our focus to partitions of $n\ell$ with trivial ℓ-core we get the following important result.

Theorem 2.4.10. *There is a bijection between the set of partitions of $n\ell$ with trivial ℓ-core and the ℓ-multipartitions of n*

$$\mathcal{P}_{\emptyset}(n\ell) \to \mathcal{P}(n, \ell),$$

given by $\lambda \to quo(\lambda)$.

Proof. See [39, Theorem 2.7.30]. □

Let us do an example to show how to calculate the ℓ-cores and ℓ-quotients of a given partition.

Example 2.4.11. Lets take the partition $(4, 3, 2)$ and find the 3-core and 3-quotient. The first column hook lengths are $\{6, 4, 2\}$. This corresponds to the bead diagram

$$\begin{array}{ccc} \circ & \circ & \bullet \\ \circ & \bullet & \circ \\ \bullet & \circ & \circ \end{array} .$$

The ℓ-quotient is found by taking the three columns and reading the first column hook lengths from these, each of which define a partition. For the first column of the bead diagram we have first column hook length $\{2\}$. This is the partition (2). The second bead diagram column has first column hook length $\{1\}$. This defines the partition (1). The last bead diagram column has first column hook length $\{0\}$, which corresponds to the empty partition $\emptyset$. Therefore the 3-quotient of $(4, 3, 2)$ is $((2), (1), \emptyset)$. To find the 3-core we shift all the beads up to the top of their respective columns, then we read of the first column hook lengths ignoring all beads before the first empty position. Running the beads to the top we get the bead diagram

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{array} .$$

Since we ignore all beads before the first empty bead, we see that the above diagram has first column hook lengths $\{0\}$ and so the 3-core of $(6, 4, 2)$ is the empty partition $\emptyset$.

§ 2.5 | Symplectic Reflection Algebras

Symplectic reflection algebras are a large class of non-commutative algebras arising as deformations of certain skew group rings. We are interested in rational Cherednik algebras, which are arguably the most important examples of symplectic reflection algebras. There are multiple sources to read further about symplectic reflection algebras, for instance [15], [32] or [7].

Definition 2.5.1. Let R be a ring and G a group that acts on R. Then the *skew group ring* $R \rtimes G$ is defined to be the ring of formal sums $\{\sum_I r_i g_i \,|\, r_i \in R \text{ and } g_i \in G\}$, where multiplication is given by the rule $(r_1 g_1)(r_2 g_2) = r_1 g_1(r_2) g_1 g_2$.

Remark 2.5.2. The skew group algebra is often denoted $R\#G$, and sometimes called the smash product in various papers, such as in [22] and [25].

Definition 2.5.3. Let V be a vector space. The *symmetric algebra* associated to V is the quotient algebra $S(V) := T(V)/I$, where $T(V)$ is the tensor algebra of V and $I := \langle u \otimes v - v \otimes u | \text{ for all } u, v \in V\rangle$.

As we will shortly see, the symplectic reflection algebra is a deformation of the skew group ring $S(V) \rtimes G$ given by perturbing the commutation relations.

Definition 2.5.4. A *symplectic vector space* (V, ω) is a pair consisting of a vector space V and a non-degenerate, skew symmetric bilinear form $\omega : V \times V \to \mathbb{C}$. We call ω the *symplectic* $form$ on V.

Given a symplectic vector space it is natural to consider group actions on this space. The obvious one is the group of all automorphisms preserving the symplectic form

$$Sp(V) := \{g \in GL(V) \,|\, \omega(g(u), g(v)) = \omega(u, v) \text{ for all } u, v \in V\}.$$

This is called the *symplectic group* of (V, ω).

Definition 2.5.5. Let V be a vector space, then a *symplectic* $reflection$ is an element of finite order $s \in Sp(V)$ such that $\dim(\mathrm{Im}(s - Id)) = 2$.

Definition 2.5.6. A *symplectic* $reflection$ *group* is a triple (V, ω, G), where (V, ω) is a symplectic vector space and $G \subset Sp(V)$ is a finite group generated by symplectic reflections.

We can ask: what are the smallest subspaces that are preserved by a group which respects the symplectic form? This question leads to the definition of an indecomposable symplectic vector space.

Definition 2.5.7. A symplectic reflection group (V, ω, G) is called *indecomposable* if there is no decomposition $V = U \oplus W$ into G-invariant symplectic subspaces.

The following important example is a prerequisite to understanding the symplectic reflection algebra.

Example 2.5.8. Let (V, ω, G) be a symplectic reflection group. If we have an additional skew symmetric bilinear map $\kappa : V \times V \to \mathbb{C}G$ then we can define the following algebra

$$H_\kappa := T(V) \rtimes \mathbb{C}G / < xy - yx - \kappa(x, y) | x, y \in V > . \tag{2.5.1}$$

Lets briefly discuss the Poincaré–Birkhoff–Witt property mentioned in Theorem 2.5.9 and its importance. Consider the algebra H_κ defined in (2.5.1). The algebra H_κ can be equipped with a filtration by setting $\mathbb{C}G$ to be in degree zero and V to be in degree one. We can then consider the associated graded algebra $gr(H_\kappa)$. Note that $xy - yx = 0$ as $xy - yx = \kappa(x, y)$ from the defining relations of H_κ but the degree of $xy - yx$ is 2 and the degree of $\kappa(x, y)$ is 0. Hence there is a well-defined map $S(V) \rtimes \mathbb{C}G \to gr(H_\kappa)$. The Poincaré–Birkhoff–Witt property is said to be satisfied if this map is an isomorphism.

Given the symplectic reflection group (V, ω, G) and a reflection $s \in G$, consider the decomposition

$$V = \mathrm{Im}(1 - s) \oplus \mathrm{Ker}(1 - s)$$

into symplectic subspaces. We denote the symplectic 2-form that is equal to ω on $\mathrm{Im}(1-s)$ and 0 on $\mathrm{Ker}(1 - s)$ by ω_s.

The definition of symplectic reflection algebras seems unnatural without the following theorem due to Etingof and Ginzburg [24, Theorem 1.3] that motivates it.

Theorem 2.5.9. *(PBW Theorem) Let (V, ω, G) be an indecomposable triple. The Poincaré–Birkhoff–Witt property holds for H_κ if and only if there exists a constant $t \in \mathbb{C}$ and a class function $c : S \to \mathbb{C}$, where S is the set of symplectic reflections in G, such that the map κ has the form*

$$\kappa(x, y) = t \cdot \omega(x, y) \cdot 1 + \sum_{s \in S} c(s) \cdot \omega_s(x, y) \cdot s.$$

Definition 2.5.10. Let $t \in \mathbb{C}$ and $c : S \to \mathbb{C}$ a class function. The corresponding *symplectic reflection algebra* is the quotient algebra

$$H_\kappa := T(V) \rtimes G/ < xy - yx - \kappa(x,y) | x, y \in V >,$$

where (V, ω, G) is an indecomposable triple and κ is of the form in Theorem 2.5.9.

A consequence of the PBW Theorem is that it allows us to have an explicit basis for a symplectic reflection algebra. This is because it implies an isomorphism of vector spaces $H_\kappa \cong S(V) \rtimes \mathbb{C}G$.

§ 2.6 | Cherednik Algebras

Double affine Hecke algebras, often shortened to DAHA, were first introduced by Cherednik in the solution to Macdonald's constant term conjecture [19, Definition 1.1]. Rational Cherednik algebras are a degeneration of DAHAs. Since their introduction they have become pervasive throughout representation theory, and in the study of Calogero-Moser integrable systems. The aim of this section is to define rational Cherednik algebras and provide motivation for why one should want to study them.

Rational Cherednik algebras are generated from complex reflection groups and so we start with the following.

Definition 2.6.1. Let $\mathfrak{h}$ be a complex vector space. A *complex reflection* is an invertible linear map of finite order $r : \mathfrak{h} \to \mathfrak{h}$ that fixes a hyperplane. That is,

$$\dim \operatorname{Im}(r - Id_{\mathfrak{h}}) = 1.$$

Definition 2.6.2. A complex reflection group W is a finite group generated by complex reflections on a complex vector space $\mathfrak{h}$. The group W is called *irreducible* if $\mathfrak{h}$ is an irreducible representation.

Let $W \subset GL(\mathfrak{h})$ be a complex reflection group and let $V = \mathfrak{h} \oplus \mathfrak{h}^*$. There is a natural pairing $(-,-) : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ given by $(y, x) := x(y)$. Then the standard symplectic form ω on V is given by

$$\omega(y_1 \oplus x_1, y_2 \oplus x_2) = (y_1, x_2) - (y_2, x_1).$$

The triple (V, ω, W) is a symplectic reflection group.

We are now in a position to define the rational Cherednik algebra. Recall the bilinear form in Theorem 2.5.9,

$$\kappa(x, y) = t \cdot \omega(x, y) \cdot 1 + \sum_{s \in S} c(s) \cdot \omega_s(x, y) \cdot s.$$

Definition 2.6.3. Given a complex reflection group $(W, \mathfrak{h})$, $t \in \mathbb{C}$ and a class function $c : S \to \mathbb{C}$, where S is the set of reflections in W. We define the rational Cherednik algebra

$$H_{t,c} := T(V) \rtimes W / \langle x \otimes y - y \otimes x - \kappa(x, y) \,|\, \forall x,\, y \in V \rangle$$

where

$$\kappa(x, y) = t \cdot \omega(x, y) \cdot 1 + \sum_{s \in S} c(s) \cdot \omega_s(x, y) \cdot s$$

and $V = \mathfrak{h} \oplus \mathfrak{h}^*$.

Let us unpack the definition and rewrite this algebra in a different form. For $s \in S$, fix $\alpha_s \in \mathfrak{h}^*$ to be a basis of the one-dimensional vector space $\mathrm{Im}(s - \mathrm{Id}_V)|_{\mathfrak{h}^*}$ and $\alpha_s^\vee \in \mathfrak{h}$ to be a basis of the one-dimensional vector space $\mathrm{Im}(s - \mathrm{Id}_V)|_{\mathfrak{h}}$, such that $\alpha_s(\alpha_s^\vee) = 2$. Then $H_{t,c}(W)$ is the quotient of $T(V) \rtimes W$ by the relations:

$$[x_1, x_2] = 0, \quad [y_1, y_2] = 0, \quad [y, x] = t(y, x) - 2 \sum_{s \in S} c(s) \frac{(y, \alpha_s)(\alpha_s^\vee, x)}{(\alpha_s^\vee, \alpha_s)} s$$

for all $x_1, x_2, x \in \mathfrak{h}^*$ $y_1, y_2, y \in \mathfrak{h}$.

There are two complex reflection groups that will be of special importance. These are the symmetric group S_n and the wreath product of the symmetric group with the cyclic group $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$. Therefore let us write the rational Cherednik algebra for these groups more explicitly.

Example 2.6.4. Let us first consider the case $W = S_n$. Then we have the basis $\{x_1, \cdots, x_n\}$ of $\mathfrak{h}^*$ and $\{y_1, \cdots, y_n\}$ of $\mathfrak{h}$ with multiplication

$$\sigma x_i = \sigma(x_i)\sigma = x_{\sigma(i)}\sigma, \;\; \sigma y_i = \sigma(y_i)\sigma = y_{\sigma(i)}\sigma \quad \text{ for all } \sigma \in S_n.$$

The above actions are inherited from the skew group ring. There is only one conjugacy class of reflections in S_n, so the class function $c : S \to \mathbb{C}$ is just a scalar. We say $c \in \mathbb{C}$. We know that

$$[x_1, x_2] = 0, \quad [y_1, y_2] = 0,$$

and it remains to find our α_s and $\alpha_s^\vee$ for a given reflection. The complex reflections are the transpositions σ_{ij} that permute i and j. Then

$$(\sigma_{ij} - 1)(\lambda_1, \lambda_2, \cdots, \lambda_n) = (0, \cdots, \lambda_j - \lambda_i, \cdots, \lambda_i - \lambda_j, \cdots, 0)$$

so the image of $\sigma_{ij} - 1$ is the subspace spanned by $x_i - x_j$. If we set $\alpha_{ij} = x_i - x_j$ and, by an identical calculation, $\alpha_{ij}^\vee = y_i - y_j$ then $\alpha_{\sigma_{ij}}(\alpha_{\sigma_{ij}}^\vee) = 2$. Thus

$$[y_i, x_j] = t(y_i, x_j) - 2c \sum_{\sigma_{kl}} \frac{(y_i, x_k - x_l)(y_k - y_l, x_j)}{(x_i - x_j, y_i - y_j)} \sigma_{kl}.$$

This expression simplifies. We split it into two cases: $i = j$ and $i \neq j$. If $i = j$ then $(y_i, x_i) = x_i(y_i) = 1$ and $(x_i - x_j, y_i - y_j) = 2$, for any i and j, implying that

$$[y_i, x_i] = t - c \sum_{i \neq j} \sigma_{ij}.$$

Similarly, when $i \neq j$,

$$[y_i, x_j] = c\sigma_{i,j}.$$

Hence the rational Cherednik algebra $H_{t,c}(S_n)$ is the algebra generated by $\{x_1, \cdots, x_n, y_1, \cdots, y_n\}$ and S_n, satisfying the relations

$$[x_1, x_2] = 0, \quad [y_1, y_2] = 0$$

and

$$[y_i, x_i] = t - c \sum_{i \neq j} \sigma_{ij} \text{ and } [y_i, x_j] = c\sigma_{ij} \text{ for } i \neq j.$$

We have so far mentioned the wreath product only briefly. Let us now define it so that we may calculate its associated Cherednik algebra.

Definition 2.6.5. Let A and B be groups. Assume that there is an action of A on a set X. We set $G = \prod_{x \in X} B_x$ where B_x is a copy of B. Note that A acts on G by $a \cdot (b_x) = b_{a^{-1}(x)}$. The *wreath product* of A and B, denoted $A \wr B$, is the group $A \ltimes G$.

The particular wreath product we are interested in is $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$. In this case the set X from Definition 2.6.5 is taken to be $\{1, \cdots, n\}$ with the obvious action by S_n.

Example 2.6.6. The case of $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ is more complicated than Example 2.6.4. Let us clarify some notation: fix a generator $\gamma \in \mathbb{Z}/\ell\mathbb{Z}$, let $\gamma_i \in S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ denote the element γ in the i^{th} component. Then $\gamma_i \sigma_{ij} = \sigma_{ij}\gamma_j$. The space $\mathfrak{h}$ has basis $\{x_1, \cdots, x_n\}$ with action $\gamma_i x_i = \omega x_i$ and $\gamma_i x_j = x_j$ for $j \neq i$ where ω is a primitive ℓ^{th} root of unity. The permutations act as follows $\sigma x_i = x_{\sigma(i)}$.

There are ℓ conjugacy classes of reflections in $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$, there are $\ell - 1$ of the form $\{\gamma_i \,|\, 0 \leq i \leq n\}$. The set

$$\{\sigma_{ij} \cdot \gamma_i^k \cdot \gamma_j^{-k} \,|\, 1 \leq i, j \leq n, \text{ and } 1 \leq k \leq \ell\}$$

is the remaining conjugacy class. Let us now calculate the α and $\alpha^\vee$ for each reflection. Let us begin with the easier case of γ_i^k

$$(\gamma_i^k - 1)(x_1, \cdots, x_n) = (0, \cdots, \omega^k x_i - x_i, \cdots, 0)$$

hence $\alpha^\vee_{\gamma_i^k} = \sqrt{2} y_i$. The argument is identical for $\alpha_{\gamma_i^k} = \sqrt{2} x_i$. Therefore

$$(y_s, \alpha_{\gamma_i^k})(\alpha^\vee_{\gamma_i^k}, x_r) = \begin{cases} 2 \text{ if } i = s = r \\ 0 \text{ else .} \end{cases}$$

Now we consider

$$(\sigma_{ij}\gamma_i^k\gamma_j^{-k} - 1)(x_1, \cdots, x_n) = (0, \cdots, \omega^k x_j - x_i, \cdots, \omega^{-k} x_i - x_j, \cdots, 0)$$

and so $\alpha_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}} = x_i - \omega^{-k} x_j$, likewise $\alpha^\vee_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}} = y_i - \omega^k y_j$. Then $\alpha_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}}(\alpha^\vee_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}}) = 2$ and

$$(y_s, \alpha_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}})(\alpha^\vee_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}}, x_r) = (x_i(y_s) - \omega^{-k} x_j(y_s))(x_r(y_i) - \omega^k x_r(y_j)).$$

First note that $i \neq j$. Let us split this into two cases, $s = r$ and $s \neq r$. When $s = r$ we see that the above is zero unless at least one of r or s is equal to either i or j

$$(y_s, \alpha_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}})(\alpha^\vee_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}}, x_r) = \begin{cases} 1 \text{ if } s = r = i \\ 1 \text{ if } s = r = j \\ 0 \text{ else .} \end{cases}$$

Hence

$$[y_s, x_s] = c(\sigma_{12}\gamma_1^k\gamma_2^{-k}) \sum_{s \neq j} \sum_{k=0}^{\ell} \sigma_{sj}\gamma_s^k\gamma_j^{-k} + \sum_{k=1}^{\ell-1} c(\gamma_s^k)\gamma_s^k.$$

Now we consider the case where $r \neq s$. Recall

$$(y_s, \alpha_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}})(\alpha^\vee_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}}, x_r) = (x_i(y_s) - \omega^{-k} x_j(y_s))(x_r(y_i) - \omega^k x_r(y_j))$$

so if $r = i$ and $s \neq j$ this is zero, the only non-zero case is when $s = i$ and $r = j$. Hence

$$(y_i, \alpha_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}})(\alpha^\vee_{\sigma_{ij}\gamma_i^k\gamma_j^{-k}}, x_j) = -\omega^k$$

therefore

$$[y_i, x_j] = c(\sigma_{ij}\gamma_i^k\gamma_j^{-k}) \sum_{k=1}^{\ell} -\omega^k \sigma_{ij}\gamma_i^k\gamma_j^{-k}.$$

While much is know about the rational Cherednik algebra for large classes of complex reflection groups, there are still many difficult problems that remain unsolved. Much of the difficulty is due to the rational Cherednik algebra being very non-commutative.

Indeed, in the case where $t = 1$ the centre is $\mathbb{C}$ [7, Theorem 1.7.1]. In the case where $t = 0$ the centre has a rich structure. More specifically, the rational Cherednik algebra is a finite module over its centre. In this case much of the representation theory can be reduced to the finite dimensional quotient called the restricted rational Cherednik algebra. We now describe how to construct this quotient. If we denote the centre of $H_{0,c}(W)$ by $Z_c(W)$ then when $t = 0$ a short argument [30, Proposition 3.6] shows that

$$R = \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \subset Z_c(W). \tag{2.6.1}$$

In particular R is a central subalgebra. Of particular importance is the maximal ideal

$$R_+ = \mathbb{C}[\mathfrak{h}]^W_+ \otimes \mathbb{C}[\mathfrak{h}^*]^W + \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W_+ \subset R.$$

Definition 2.6.7. The *restricted rational Cherednik algebra* is the quotient algebra

$$\overline{H}_c(W) := H_{0,c}(W)/R_+H_{0,c}(W).$$

The Verma and baby Verma modules are the standard modules for the rational Cherednik algebra and the restricted rational Cherednik algebra respectively. Both types of Verma modules are instrumental in proving the main results of Chapter 4. Much of Section 4.1 focuses on the intimate connection between the Verma, the baby Verma modules and the structure of the centre. Let $H_{0,c}(W)$ be the corresponding rational Cherednik algebra for a class function $c : S \to \mathbb{C}$ and $t = 0$.

Definition 2.6.8. Let $\lambda \in \operatorname{Irr} W$. The *Verma module* associated to λ is

$$\underline{\Delta}_c(\lambda) = H_c(W) \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes W} \lambda,$$

where $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ acts by 0 on λ.

Let us briefly mention the ring of coinvariants. Let $(W, \mathfrak{h})$ be a complex reflection group. The ring of coinvariants is the quotient

$$\mathbb{C}[\mathfrak{h}]^{coW} = \mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}]^W_+ \rangle,$$

where $\mathbb{C}[\mathfrak{h}]^W_+$ is the ideal of W invariants that have no constant term. The ring of coinvariants for the dual space $\mathfrak{h}^*$ is defined similarly.

Definition 2.6.9. Let $\lambda \in \operatorname{Irr} W$. The *baby Verma module* associated to λ is

$$\Delta_c(\lambda) = \overline{H}_c(W) \otimes_{\mathbb{C}[\mathfrak{h}^*]^{coW} \rtimes W} \lambda,$$

where $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]^{coW}$ acts by 0 on λ.

It has also been shown that the restricted rational Cherednik algebra has a useful decomposition into its coinvariant rings.

Theorem 2.6.10. *There is an isomorphism*

$$\overline{H}_c \cong \mathbb{C}[\mathfrak{h}]^{coW} \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*]^{coW}.$$

Proof. See [30, p. 4]. □

§ 2.7 | Schubert Cells and the Wronski map

Using the Wronski map is the only known way to provide an explicit presentation of the blocks of the centre of the restricted rational Cherednik algebra. The Wronski map is derived from the well known Wronskian coming from the theory of ordinary differential equations and is defined on Schubert cells. We shall define and discuss both of these in this section.

Schubert cells are subvarieties of Grassmanians, a classical object in the field of algebraic geometry. The Grassmanian is the space parametrising all the vector subspaces of a fixed dimension.

Definition 2.7.1. For a fixed complex vector space V and positive integer $n \leq \dim V$, we define the *Grassmanian* as

$$Gr(n, V) = \{W \subset V |\, \dim\, W = n\}.$$

For any V and n the Grassmanian $Gr(n, V)$ can be considered as a projective variety via the Plucker embedding [28, Lemma 9.1]. The definition of a Schubert cell depends upon the Grassmanian it lives in. The Schubert cells we consider belong to the following Grassmanian

$$Gr(n, \mathbb{C}[x]_{2n}),$$

where $\mathbb{C}[x]_{2n}$ denotes the vector space of all polynomials in x with degree less than $2n$. Therefore, $\dim \mathbb{C}[x]_{2n} = 2n$.

Definition 2.7.2. Given a complete flag

$$\mathcal{F} = \{0 \subset F_1 \subset \cdots \subset F_{2n} = \mathbb{C}[x]_{2n}\}$$

of $\mathbb{C}[x]_{2n}$, and a partition $\lambda \vdash n$, the *Schubert cell* is the locally closed subvariety of $Gr(n, \mathbb{C}[x]_{2n})$ given by

$$\Omega_\lambda(\mathcal{F}) := \{V \in Gr(n, \mathbb{C}[x]_{2n}) \,|\, \dim(V \cap F_k) = i \text{ for } n+i-\lambda_{i-1} \leq k \leq n+i-\lambda_i \text{ and } 0 \leq i \leq n\}.$$

Remark 2.7.3. The above definition does not depend, up to isomorphism, on the choice of full flag.

We choose a complete flag that makes things as simple as possible,

$$\mathcal{F}(\infty) := \{0 \subset \mathbb{C}[x]_1 \subset \cdots \subset \mathbb{C}[x]_i \subset \mathbb{C}[x]_{i+1} \subset \cdots \subset \mathbb{C}[x]_{2n}\}$$

where $\mathbb{C}[x]_i$ is the vector space of polynomials in x of degree $\leq i$. For a given partition $\lambda \vdash n$ define a new partition $\overline{\lambda} \vdash n^2 - n$ by $\overline{\lambda}_i = n - \lambda_{n-i}$. Denote the Schubert cell associated to $\mathcal{F}(\infty)$ and $\overline{\lambda}$ by

$$\Omega_{\lambda}^{qe} := \Omega_{\overline{\lambda}}(\mathcal{F}(\infty)).$$

Proposition 2.7.4. *The Schubert cell $\Omega_{\lambda}^{qe} \subset Gr(n, \mathbb{C}_{2n}[x])$ consists of n-dimensional subspaces X which have a basis $\{f_1(x),\ f_2(x), \cdots, f_n(x)\}$ where*

$$f_i = x^{d_i} + \sum_{j=1,\, d_i - j \notin P}^{d_i} f_{i,j} x^{d_i - j},$$

$d_i = \lambda_i + n - i$ and $P := \{d_1, \cdots, d_n\}$. Furthermore, for a given X such a basis is unique.

Proof. See [47, p. 918]. □

The stated aim of this section was to define the Wronski map. Since its definition depends on the Wronskian, we recall the latter.

Definition 2.7.5. Given a collection of $n-1$ times differentiable functions $\{f_1, \ldots, f_n\}$, the *Wronskian* is the determinant

$$\mathrm{Wr}(f_1, f_2, \cdots, f_n) = \det \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1^{(1)} & f_2^{(1)} & f_3^{(1)} & \cdots & f_n^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}.$$

Remark 2.7.6. The degree of the Wronskian depends upon on the degrees of the polynomials $f_1, \cdots, f_n$. Indeed, by Lemma 5.1.2 if the polynomials are homogeneous then $\deg Wr(f_1, \cdots, f_n) = \sum_i^n(\deg(f_i)) - \frac{(n-1)(n)}{2}$. In fact, the homogeneous condition can be removed and the proof of Lemma 5.1.2 still holds and implies that

$$\deg Wr(f_1, \cdots, f_n) = \sum_i^n (\deg(f_i)) - \frac{(n-1)(n)}{2}$$

is true for any family of polynomials $f_1, \cdots, f_n$.

Remark 2.7.7. Note that the Wronskian of a basis does not depend on the choice of basis up to a scalar.

By calculating the Wronskian of a basis of subspaces $X \in Gr(n, \mathbb{C}_{2n}[x])$ we obtain a map $\mathrm{Wr} : Gr(n, \mathbb{C}_{2n}[x]) \to \mathbb{C}[x]$. Let Wr_X denote the Wronskian of a basis of a space X. We can now define the Wronski map.

Definition 2.7.8. The *Wronski* map is the function $\mathrm{Wr}_\lambda : \Omega_\lambda^{qe} \to \mathbb{C}^n$ defined by

$$\mathrm{Wr}_\lambda(X) = (a_1, \cdots, a_n),$$

where

$$\mathrm{Wr}_X = x^n + \sum_{i=1}^{n} (-1)^i a_i x^{n-i}.$$

Let us now consider the functions $\mathbb{C}[\Omega_\lambda^{qe}]$ on the Schubert cell Ω_λ^{qe}. Note that this is an affine space and, by Proposition 2.7.4, each $X \in \Omega_\lambda^{qe}$ has a unique basis in terms of the f_{ij}. Hence the Schubert cells have coordinate functions f_{ij}. Lemma 2.4.2 states $|\{j \,|\, d_i - j \notin P\}| = \lambda_i$, and summing over $1 \leq i \leq n$, we see that the Schubert cell Ω_λ^{qe} has dimension n. Combining these facts we arrive at the following.

Proposition 2.7.9. *The algebra $\mathbb{C}[\Omega_\lambda^{qe}]$ is a free polynomial algebra generated by the f_{ij}*

$$\mathbb{C}[\Omega_\lambda^{qe}] = \mathbb{C}[f_{ij},\, i = 1, \ldots, n \;\; j = 1, \ldots d_i,\, d_i - j \notin P].$$

Remark 2.7.10. We can equip the algebra of functions on the Schubert cell with a grading by defining the degree of f_{ij} to be j.

§ 2.8 | Quivers, Quiver varieties

Quivers are diagrams that consist of vertices and arrows between pairs of vertices. The formalisation of these diagrams allows for us to describe and understand more complex mathematical situations. Concepts of particular importance to us are quiver varieties, quiver representations, and the preprojective algebra.

Definition 2.8.1. A *quiver* Q consists of two sets Q_0 and Q_1 and a pair of maps $h : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$. Elements of Q_0 are called vertices and elements of Q_1 are called arrows. The map h is called the *head* and the map t is the *tail*.

Definition 2.8.2. A *path* in a quiver Q is a sequence of arrows $a_1, a_2, \cdots, a_k$ such that $t(a_{i+1}) = h(a_i)$. In diagram form this can be represented as

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_k} k+1 \;.$$

There is a natural construction of an algebra for any given quiver called the path algebra. Let k be a field.

Definition 2.8.3. The *path algebra* of a quiver Q is denoted kQ and is a k-algebra with basis the paths in Q. The product of two paths x and y is

$$xy = \begin{cases} xy \text{ if } h(x) = t(y) \\ 0 \text{ else.} \end{cases}$$

The trivial path at vertex i is denoted e_i.

Definition 2.8.4. A *representation* of a quiver Q is an assignment of a vector space V_i to each vertex $i \in Q_0$ and a linear map V_a to each arrow $a \in Q_1$. The linear maps are assigned in such a way, that if $t(a) = i$ and $h(a) = j$ then $V_a : V_i \to V_j$.

Since a representation of a quiver consists of vector spaces it is natural to define the dimension of a quiver representation.

Definition 2.8.5. The *dimension vector* of a representation V of a quiver with n vertices is $\alpha = (\dim V_1, \cdots, \dim V_n)$.

Later in this thesis we will be concerned with the Calogero-Moser space. This is closely related to a certain quotient of a path algebra called the deformed preprojective algebra. Let Q be a quiver. We denote by $\overline{Q}$ the double quiver of Q. The double quiver has the same set of vertices, but to every arrow $a \in Q_1$ we have a new arrow $a^* \in \overline{Q}_1$ such that $h(a) = t(a^*)$ and $t(a) = h(a^*)$.

Definition 2.8.6. Let $\lambda \in \mathbb{C}^{Q_0}$. The *deformed preprojective algebra* $\prod^\lambda(Q)$ of a quiver Q, is the quotient of the path algebra $\mathbb{C}\overline{Q}$, by the relation

$$\sum_{a \in Q_1} aa^* - a^*a = \sum_{i \in Q_0} \lambda_i e_i.$$

The proof of Theorem 4.2.25 relies upon understanding an isomorphism between the Calogero-Moser space and a certain quiver variety. For this reason they are of interest to us. More broadly, quiver varieties allow for the techniques of algebraic geometry to be applied to the study of deformed preprojective algebras. However this is not the main focus of the thesis, rather a tool to help understand rational Cherednik algebras.

Definition 2.8.7. Let Q be a quiver and fix a dimension vector α. Then the set of all representations of Q of dimension α forms a variety called the *quiver moduli space* and is denoted $\mathrm{Rep}(Q, \alpha)$.

Note that $\mathrm{Rep}(\prod^\lambda(Q), \alpha) \subset \mathrm{Rep}(\overline{Q}, \alpha)$. There is a natural action on $\mathrm{Rep}(\overline{Q}, \alpha)$ of the group

$$GL(\alpha) = \prod_{i \in I} GL(\alpha_i, \mathbb{C})$$

which acts by conjugation. If $\Delta : \mathbb{C}^\times \to GL(\alpha)$ is the the diagonal embedding then we define

$$PGL(\alpha) := GL(\alpha)/\Delta(\mathbb{C}^\times).$$

The conjugation action of $GL(\alpha)$ factors through $PGL(\alpha)$.

Quiver varieties parametrise isomorphism classes of semisimple $\prod^\lambda(Q)$-modules. We do not define them in general, but describe the construction of a particular quiver variety. Let Q_ℓ be a cyclic quiver with ℓ vertices, and let α be a dimension vector for Q_ℓ. The following map

$$\mu_\alpha : \mathrm{Rep}(\overline{Q}_\ell, \alpha) \to \oplus_{i\in\mathbb{Z}/\ell\mathbb{Z}} \mathrm{Mat}_{\alpha_i}(\mathbb{C})$$

given by $\mu_\alpha(X_i, Y_i)_{i\in\mathbb{Z}/\ell\mathbb{Z}} = (X_i X_i^* - X_{i-1}^* X_{i-1})_i$ is called the moment map. Let $\theta \in \mathbb{C}^\ell$ be a family of complex numbers and denote the sum of matrices $\oplus_{i\in\mathbb{Z}/\ell\mathbb{Z}} \theta_i I_{\alpha_i}$ by $I_\theta(\alpha)$. Let $\mathcal{O}_\theta$ be the closed subvariety of $\mathrm{Mat}_{\alpha_0}(\mathbb{C})$ of matrices of rank ≤ 1 and with trace $-\sum_{i\in\mathbb{Z}/\ell\mathbb{Z}} \theta_i \alpha_i$.

Note that the matrices belonging to $I_\theta(\alpha) + \mathcal{O}_\theta$ have zero trace. The two key varieties for our purposes are

$$\mathcal{Y}_\theta(\alpha) = \mu_\alpha^{-1}(I_\theta(\alpha) + \mathcal{O}_\theta)$$

and the *quiver variety*

$$\mathcal{X}_\theta(\alpha) = \mathcal{Y}(\alpha)//PGL(\alpha).$$

In the above // means the categorical quotient, but since our varieties will be smooth this is equivalent to the usual geometric quotient.

§ 2.9 | Calogero-Moser space

The aim of this section is to define the Calogero-Moser space and recall that it is isomorphic to the spectrum of the centre of a particular rational Cherednik algebra as well as to the above special quiver variety. The following content can be found in [23, Section 11]. Throughout this section let $\Gamma_n = S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ be the wreath product. Then $\mathbb{Z}/\ell\mathbb{Z}$ acts naturally on the space $\mathbb{C}^n \otimes \mathbb{C}\mathbb{Z}/\ell\mathbb{Z}$. Let $\mathbb{O}$ be the set of $n \times n$ matrices of the form $P - Id$ where P is a rank 1 matrix and $tr(P) = tr(Id)$. Note the similarity between $\mathbb{O}$ and $\mathcal{O}_\theta$ defined in the last section. Let $e_{\mathbb{Z}/l\mathbb{Z}} \in \mathrm{End}(\mathbb{C}\mathbb{Z}/\ell\mathbb{Z})$ be the projector onto the trivial representation $e_{\mathbb{Z}/\ell\mathbb{Z}}(g) = \frac{1}{|\mathbb{Z}/\ell\mathbb{Z}|}\sum_{\gamma\in\mathbb{Z}/\ell\mathbb{Z}} \gamma \cdot g$. Now let $c : \mathbb{Z}/\ell\mathbb{Z} \to \mathbb{C}$ be a class function and let $c' = \sum_{\gamma\in\mathbb{Z}/\ell\mathbb{Z}} \gamma \cdot c(\gamma)$. For each pair (k, c) where $k \in \mathbb{C}$ and c a class function we have the following variety

$$M_{\mathbb{Z}/\ell\mathbb{Z},n,c} = \{\nabla_1, \nabla_2 \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}\mathbb{Z}/\ell\mathbb{Z}) \,|\, [\nabla_1, \nabla_2] = k\ell \cdot o \otimes e_{\mathbb{Z}/\ell\mathbb{Z}} + Id_{\mathbb{C}^n} \otimes c' \text{ for some } o \in \mathbb{O}\}.$$

There is a natural action on $M_{\mathbb{Z}/\ell\mathbb{Z},n,c}$ by the group

$$GL_{\mathbb{Z}/\ell\mathbb{Z},n} = \mathrm{Aut}_{\mathbb{Z}/\ell\mathbb{Z}}(\mathbb{C}^n \otimes \mathbb{C}\mathbb{Z}/\ell\mathbb{Z}) = \prod_{i=0}^{\ell-1} GL(n, \mathbb{C})$$

by conjugation. The group $PGL_{\mathbb{Z}/\ell\mathbb{Z},n} = GL_{\mathbb{Z}/\ell\mathbb{Z},n}/\Delta(\mathbb{C}^\times)$ also acts on $M_{\mathbb{Z}/\ell\mathbb{Z},n,c}$ by conjugation.

Definition 2.9.1. The *Caloger-Moser* space is the quotient variety

$$\mathcal{M}_{\mathbb{Z}/\ell\mathbb{Z},n,c} = M_{\mathbb{Z}/\ell\mathbb{Z},n,c}/PGL_{\mathbb{Z}/\ell\mathbb{Z},n,c}.$$

An equivalent definition of the Calogero-Moser space takes advantage of a special element in $\mathbb{O}$. Consider the matrix

$$p = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{bmatrix}.$$

This is the matrix with 0 on the main diagonal and 1 everywhere else. Then we can define

$$M_{\mathbb{Z}/\ell\mathbb{Z},n,c}(p) = \{\nabla_1, \nabla_2 \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}\mathbb{Z}/\ell\mathbb{Z}) | \, [\nabla_1, \nabla_2] = k\ell \cdot p \otimes e_{\mathbb{Z}/\ell\mathbb{Z}} + Id_{\mathbb{C}^n} \otimes c'\}.$$

and let $PGL_{\mathbb{Z}/\ell\mathbb{Z}}(p)$ be the isotropy group of $k\ell \cdot p \otimes e_{\mathbb{Z}/\ell\mathbb{Z}} + Id_{\mathbb{C}^n} \otimes c' \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}\Gamma)$. Then we can alternatively write the Calogero-Moser space as

$$M_{\mathbb{Z}/\ell\mathbb{Z},n,c}(p)/PGL_{\mathbb{Z}/\ell\mathbb{Z}}(p). \tag{2.9.1}$$

In [23, Theorem 11.16] the following isomorphism is proven

$$\mathrm{Spec}\, Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \cong \mathcal{M}_{\mathbb{Z}/\ell\mathbb{Z},n,c}. \tag{2.9.2}$$

The proof Theorem 4.2.22 relies on an explicit understanding of the above isomorphism. The first step to doing so is to rewrite the Calogero-Moser space using the following interpretation.

Let E be a simple $H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$-module. Let $S_{n-1} \subset S_n$ be the set of permutations of $\{2, \cdots, n\}$ and write $\Gamma_{n-1} = S_{n-1} \wr \mathbb{Z}/\ell\mathbb{Z}$. Consider the subspace $E^{\Gamma_{n-1}}$ of E fixed by Γ_{n-1}. The elements x_1 and y_1 commute with Γ_{n-1}, hence they define maps $x_1, y_1 \in \mathrm{End}(E^{\Gamma_{n-1}})$. Furthermore $E \cong \mathbb{C}(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ as a $\mathbb{C}(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$-module by [23, Theorem 1.7]. Therefore

$\dim E^{\Gamma_{n-1}} = n\ell$. An explicit isomorphism [23, Lemma 11.14] $\mathbb{C}^n \otimes \mathbb{C}\mathbb{Z}/\ell\mathbb{Z} \cong E^{\Gamma_{n-1}}$ is given by

$$e_i \otimes \gamma \to \gamma_1 \cdot s_{1,i}.$$

Then [23, Lemma 11.15] states that under the isomorphism given above the endomorphism $[x_1, y_1] \in \mathrm{End}(E^{\Gamma_{n-1}})$ corresponds to the endomorphism of $\mathbb{C}^n \otimes \mathbb{C}\Gamma$ given by

$$[x_1, y_1] = k \cdot |\Gamma| \cdot p \otimes e_\Gamma + Id_{\mathbb{C}^n} \otimes c'.$$

Hence if we compare with our second definition (2.9.1) of the Calogero-Moser space, we see it is equivalent to pairs of maps $x_1|_{E^{\Gamma_{n-1}}}$ and $y_1|_{E^{\Gamma_{n-1}}}$ on the irreducible $H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$-modules E. Under the assumption that $\mathrm{Spec}\, Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ is smooth, the annihilator in $Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ of the simple module $H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$-module E is a maximal ideal $\mathfrak{m}$ i.e. a closed part of $\mathrm{Spec}\, Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$. The isomorphism (2.9.2) is constructed so that the ideal $\mathfrak{m}$ is sent to $(x_1|_{E^{\Gamma_{n-1}}}, y_1|_{E^{\Gamma_{n-1}}})$.

We also claimed at the start of this section that the Calogero-Moser space has an interpretation as a quiver variety. Let us now describe this interpretation.

For precise details on this section see [45, Section 6]. Recall that Q_ℓ is the cyclic quiver with ℓ vertices. We have a representation of Q_ℓ corresponding to the group $\mathbb{Z}/\ell\mathbb{Z}$, where the vector spaces at the vertices are the irreducible representations of $\mathbb{Z}/\ell\mathbb{Z}$. In this case the irreducible representations are all 1-dimensional. We use the following notation: V_i is the representation at the node i and $s \cdot V_i = \omega^i V_i$ for the generator $s \in \mathbb{Z}/\ell\mathbb{Z}$. Consider representations of the quiver Q_ℓ where the vertex i is assigned the vector space $V_i^{\oplus n}$. The operator $k \cdot |\Gamma| \cdot p \otimes e_\Gamma + Id_{\mathbb{C}^n} \otimes c'$ acting on $E^{\Gamma_{n-1}} \cong V_0 \oplus \cdots \oplus V_{\ell-1}$ belongs to $I_\theta(\alpha) + \mathcal{O}_\theta$. Hence we get a map $\mathcal{M}_{\mathbb{Z}/\ell\mathbb{Z},n,c} \to \mathcal{X}_\theta(n\delta)$. This map is an isomorphism [23, Proposition 11.10].

Chapter 3

Multiplicity results

This chapter contains an eclectic collection of multiplicity results. We consider modules over a variety of graded algebras, for instance in the first section we require an algebra admitting a graded strong duality. In the following sections we make the assumption that we have an algebra with a triangular decomposition. What connects these results is that they all calculate the radical layers, either of a projective cover, a standard or costandard module or an arbitrary module that admits a radical respecting filtration.

§ 3.1 | Graded Brauer Reciprocity

Given a ring, group or algebra it is generally the goal of representation theory to understand the associated irreducible modules. This is incredibly difficult to do in any generality, but if one narrows the scope slightly there is often much that can be deduced. The projective cover of a simple module, when these exist, are unique up to isomorphism and for a certain class of algebras (those with a triangular decomposition) understanding the projective covers allows one to deduce finer information about the module. For instance, one can compute their dimension. It would therefore be useful to know the precise structure of the projective covers. They are rarely semisimple, however, and so we could ask for the next best thing, the radical series and knowledge of the multiplicity of each simple in the layers of the radical series. Unfortunately this too is not easy, but the result that we will conclude this section with, is a step in this direction.

The type of results equating the multiplicity of two pairs of related modules are known as Brauer-type reciprocities. More precisely, let G be a finite group and F an algebraically closed field of characteristic $p > 0$. Furthermore, let S be a simple FG-module, $P(S)$ its projective cover and $P_F(S)$ a characteristic zero lift of $P(S)$. The classical Brauer reciprocity is the equality

$$[P_F(S) : L] = [L_p : S]$$

where L is a simple G-module of characteristic 0 and L_p a modulo p reduction. There have since been many similar results equating the multiplicities of two pairs of objects, such as in the modular representation theory of Lie algebras [37, Theorem 4.5]. In our results the role of L will often be played by the radical layers of the standard and costandard modules.

In this section we present a graded version of a Brauer-type reciprocity result found in [10], thus generalising the aforementioned result. The approach in [10] is broadly applicable here, however we require graded versions of the results and we consider graded multiplicity in place of the regular multiplicity. This leads to subtle differences in the arguments.

Landrock's Lemma is essential for proving the ungraded case; we require a graded version of this result. The proof is similar to the standard one but one must be careful to account for the grading.

Lemma 3.1.1. *(Graded Landrock's Lemma) Assume that k is a splitting field for a graded finite dimensional algebra A. Let $\lambda, \mu \in \operatorname{Irr} \mathcal{G}(A)$. Then*

$$[\operatorname{rad}_s P(\lambda) : \mu]_{gr} = [\operatorname{soc}_s I(\mu) : \lambda]_{gr} = [\operatorname{rad}_s P(\mu^{\circledast}) : \lambda^{\circledast}]_{gr}.$$

Proof. By Lemma 2.2.26 (b) we have $\operatorname{soc}^s(P(\lambda)/\operatorname{rad}^s P(\lambda)) = P(\lambda)/\operatorname{rad}^s P(\lambda)$. Then by Lemma 2.2.23 we see

$$\operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda)/\operatorname{rad}^s P(\lambda), I(\mu)) = \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda)/\operatorname{rad}^s P(\lambda), \operatorname{soc}^s I(\mu)).$$

Using Lemma 2.2.26 (a), we have $\operatorname{rad}^s \operatorname{soc}^s I(\mu) = 0$ and using Lemma 2.2.23 again we obtain the equation

$$\operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda)/\operatorname{rad}^s P(\lambda), \operatorname{soc}^s I(\mu)) = \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), \operatorname{soc}^s I(\mu)).$$

Hence

$$\operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda)/\operatorname{rad}^s P(\lambda), I(\mu)) = \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), \operatorname{soc}^s I(\mu)).$$

Since we assumed k was a splitting field for A we can use Lemma 2.2.32. Since the Hom sets in the formula above are equal they have equal dimension. Lemma 2.2.32 then implies

$$[P(\lambda)/\operatorname{rad}^s P(\lambda) : \mu]_{gr} = [\operatorname{soc}^s I(\mu) : \lambda]_{gr}.$$

By induction on s we see that this proves the first equality. The second follows from Lemma 2.2.24 and Lemma 2.1.15 which states $P(\lambda^{\circledast}) = I(\lambda)^{\circledast}$. $\square$

To present our final result in this section, we need to place one more requirement on our graded algebra A. The algebra A must possess a duality at a categorical level that fixes the simple modules. We make this idea precise in the following definition.

Definition 3.1.2. A *graded strong duality* is a contravariant equivalence of categories $\delta_{gr} : \mathcal{G}(A) \to \mathcal{G}(A)$ that preserves the grading of the simple modules. That is, $\delta_{gr}(\lambda) \cong \lambda$ for $\lambda \in \operatorname{Irr} \mathcal{G}(A)$.

Before stating the reciprocity result we remark that many important algebras admit a graded strong duality. Let $\mathcal{G}_l(A)$ denote the category of graded left modules (denoted $\mathcal{G}(A)$ previously) and $\mathcal{G}_r(A)$ the category of graded right A-modules.

Let us describe the form that a graded strong duality δ_{gr} is likely to take. Let $\tau : A \to A$ be a degree reversing anti-involution. That is, a bijective map such that $\deg(\tau(a)) = -\deg(a)$ and $\tau(ab) = \tau(b)\tau(a)$. Then we define a functor

$$(-)^\tau : \mathcal{G}_r(A) \to \mathcal{G}_l(A)$$

which leaves the maps unchanged. On modules $M^\tau = M$ as vectors spaces and we have $M_i^\tau = M_{-i}$ and M^τ has action $a \cdot m := m \cdot \tau(a)$. Recall the graded dual $\circledast$, if we compose τ with $\circledast$ we have a functor satisfying the criteria to be a graded strong duality as long as $(\lambda)^{\tau\circ\circledast} \cong \lambda$, for each simple graded A-module λ.

Theorem 3.1.3. *Let A be a graded finite dimensional algebra over a splitting field k, equipped with a graded strong duality δ_{gr}. Then, for all $\lambda, \mu \in \operatorname{Irr} \mathcal{G}(A)$,*

$$[\operatorname{rad}_s P(\lambda) : \mu]_{gr} = [\operatorname{rad}_s P(\mu) : \lambda]_{gr}.$$

Proof. Define $\delta'_{gr} = \delta_{gr} \circ (-)^\circledast : \mathcal{G}_r(A) \to \mathcal{G}_l(A)$. This is a covariant exact functor that maps $\lambda^\circledast$ to λ. Consider a graded composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

for $M \in \mathcal{G}_r(A)$. The multiplicity $[M : \eta^\circledast]_{gr}$ for $\eta \in \operatorname{Irr} \mathcal{G}_l(A)$ is equal to the number of i such that there is a short exact sequence

$$0 \to M_i \to M_{i+1} \to \eta^\circledast \to 0.$$

Since δ'_{gr} is exact the following is also a short exact sequence

$$0 \to \delta'_{gr}(M_i) \to \delta'_{gr}(M_{i+1}) \to \delta'_{gr}(\eta) \to 0.$$

Hence $[M : \eta^\circledast]_{gr} = [\delta'_{gr}(M) : \delta'_{gr}(\eta^\circledast)]_{gr} = [\delta'_{gr}(M) : \eta]_{gr}$. Let $M = \operatorname{rad}_s P(\mu^\circledast)$ and

$\mu \in \operatorname{Irr}\mathcal{G}_l(A)$. Then we see that

$$[\operatorname{rad}_s P(\mu^{\circledast}) : \eta^{\circledast}]_{gr} = [\delta'_{gr}(\operatorname{rad}_s P(\mu^{\circledast})) : \delta'_{gr}(\eta^{\circledast})]_{gr} = [\delta'_{gr}(\operatorname{rad}_s P(\mu^{\circledast})) : \eta]_{gr}.$$

Note that $\delta'_{gr}(\mu^{\circledast}) = \mu$. Since projective covers are preserved under equivalences, and $\delta'_{gr}(\operatorname{rad}_s(-)) = \operatorname{rad}_s\delta'_{gr}(-)$ we have that $\delta'_{gr}(\operatorname{rad}_s P(\mu^{\circledast})) = \operatorname{rad}_s P(\mu)$. Therefore,

$$[\operatorname{rad}_s P(\mu^{\circledast}) : \eta^{\circledast}]_{gr} = [\delta'_{gr}(\operatorname{rad}_s P(\mu^{\circledast})) : \eta]_{gr} = [\operatorname{rad}_s P(\mu) : \eta]_{gr}$$

Now applying the graded version of Landrocks lemma, Lemma 3.1.1, we see that

$$[\operatorname{rad}_s P(\eta) : \mu]_{gr} = [\operatorname{rad}_s P(\mu^{\circledast}) : \eta^{\circledast}]_{gr} = [\operatorname{rad}_s P(\mu) : \eta]_{gr}.$$

□

§ 3.2 | Radical Layers of Standard and Costandard Modules

In studying the relationship between simple modules and their projective covers for algebras with a triangular decomposition, it was quickly realised that another two classes of modules where intimately linked with both. These classes are the standard and costandard modules. There are multiple Brauer-type reciprocity results (some of which are mentioned in this section) that equate the multiplicities of simple modules inside standard modules, with the multiplicity of a standard in the projective cover. The goal is to refine these results further so that they count the multiplicities of the radical layers as well.

With the above in mind, Theorem 3.3.5 computes certain multiplicities in the radical layers of the projective covers in terms of the multiplicities of standard modules. This section will provide the necessary background so that we can prove this. There are other original results of interest presented in this section. Throughout, let $\mathcal{G}(A)$ denote the category of graded finite dimensional left A-modules for a graded finite dimensional algebra A.

Lemma 3.2.1. *Let A be a finite dimensional graded algebra and $B \subset A$ a graded subalgebra. Then the functor $\mathsf{Res}\,^A_B : \mathcal{G}(A) \to \mathcal{G}(B)$ has left adjoint $A \otimes_B - : \mathcal{G}(B) \to \mathcal{G}(A)$.*

Proof. If it can be shown there is a natural bijection

$$\phi : \operatorname{Hom}_{\mathcal{G}(A)}(A \otimes_B M, N) \cong \operatorname{Hom}_{\mathcal{G}(B)}(M, \mathsf{Res}\,^A_B N)$$

then the result is proven. Define $\phi : \operatorname{Hom}_{\mathcal{G}(A)}(A \otimes_B M, N) \to \operatorname{Hom}_{\mathcal{G}(B)}(M, \mathsf{Res}\,^A_B N)$ as follows, if $f \in \operatorname{Hom}_{\mathcal{G}(A)}(A \otimes_B M, N)$ then the map $\phi(f) \in \operatorname{Hom}_{\mathcal{G}(B)}(M, \mathsf{Res}\,^A_B N)$ on elements is $\phi(f)(m) = f(1 \otimes m)$. To see that this is injective assume $\phi(f) = 0$. Then $f(1 \otimes m) = 0$ for all $m \in M$ and so $f = 0$. To see that ϕ is surjective consider $g : M \to \mathsf{Res}\,^A_B N$ and define f so that $f(1 \otimes m) = g(m)$, then $\phi(f) = g$. It remains to show that this bijection is natural, this is done by checking the following two diagrams commute

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{G}(A)}(A \otimes_B M, N) \arrow[r, "\phi"] \arrow[d, "g"'] & \mathrm{Hom}_{\mathcal{G}(B)}(M, \mathsf{Res}\,^A_B N) \arrow[d, "\mathsf{Res}\,^A_B g"] \\
\mathrm{Hom}_{\mathcal{G}(A)}(A \otimes_B M, N') \arrow[r, "\phi"'] & \mathrm{Hom}_{\mathcal{G}(B)}(M, \mathsf{Res}\,^A_B N')
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{G}(A)}(A \otimes_B M, N) \arrow[r, "\phi"] \arrow[d, "A \otimes_B f"'] & \mathrm{Hom}_{\mathcal{G}(B)}(M, \mathsf{Res}\,^A_B N) \arrow[d, "f"] \\
\mathrm{Hom}_{\mathcal{G}(A)}(A \otimes_B M', N) \arrow[r, "\phi"'] & \mathrm{Hom}_{\mathcal{G}(B)}(M', \mathsf{Res}\,^A_B N)
\end{tikzcd}.$$

Using a standard diagram chase argument we see that the above isomorphism is indeed natural. □

Lemma 3.2.2. *Let A be a finite dimensional algebra and $B \subset A$ a subalgebra of A. Assume that A is free over B. If M is a projective A-module $\mathsf{Res}\,^A_B M$ is a projective B-module.*

Proof. A module P is projective if there exists another module Q such that the direct sum of P and Q is free. Hence $P \oplus Q = A^{\oplus n}$. Since A is free over B, $\mathsf{Res}\,^A_B A = B^{\oplus k}$ for some k. Hence restricting P and Q to B we see that $\mathsf{Res}\,^A_B (P \oplus Q) \cong B^{\oplus nk}$. □

The following result is widely known and will be essential in what follows.

Theorem 3.2.3. *Assume that $B \subset A$ is a subalgebra of A and that A is flat as both a graded and ungraded B-module. Let $L, M \in \mathcal{G}(A)$ and $N \in \mathcal{G}(B)$, then for any positive integer n the following three isomorphisms hold:*

1. $\mathrm{Ext}^n_{\mathcal{G}_l(A)}(L, M) \cong \mathrm{Ext}^n_{\mathcal{G}_r(A)}(M^{\circledast}, L^{\circledast})$,
2. $\mathrm{Ext}^n_{\mathcal{G}(A)}(A \otimes_B N, M) \cong \mathrm{Ext}^n_{\mathcal{G}(B)}(N, M)$,
3. $\mathrm{Ext}^n_{\mathcal{G}(A)}(M, (N^{\circledast} \otimes_B A)^{\circledast}) \cong \mathrm{Ext}^n_{\mathcal{G}(B)}(M, N)$.

Proof. See [36, Theorem 1.1]. □

From now on A is a finite dimensional graded k-algebra with a triangular decomposition. In the following lemma it is important to note that the statements hold because

$A \in \mathcal{G}(B^-)$ is free by Lemma 2.3.3. We also introduce the notation that $\mathcal{G}^\Delta(A)$ denotes the category of graded A-modules that admit a Δ-filtration. We assume T is semisimple throughout the rest of this section.

The following lemma mentions the inflation of a module. Let us briefly recall what is meant by this. Let M be a T-module and assume there is a surjective morphism $\phi : B \twoheadrightarrow T$. The inflation $\mathsf{Inf}\,^B_T M$ of M is the B-module which has the same elements as M and action

$$b \cdot m = \phi(b) \cdot m.$$

Lemma 3.2.4. *For any finite dimensional graded k-algebra A with a triangular decomposition the following holds:*

(a) Let $\lambda, \mu \in \operatorname{Irr}\mathcal{G}(T)$. Then the restriction $\mathsf{Res}\,^A_{B^-}\Delta(\lambda)$ is the projective cover of the inflation $\mathsf{Inf}\,^{B^-}_T\lambda$ in $\mathcal{G}(B^-)$.

(b) If $M \in \mathcal{G}^\Delta(A)$, then $\mathsf{Res}\,^A_{B^-}M$ is projective in $\mathcal{G}(B^-)$.

(c) For any $\lambda, \mu \in \operatorname{Irr}\mathcal{G}(T)$ we have

$$\operatorname{Ext}^n_{\mathcal{G}(A)}(\Delta(\lambda), \nabla(\mu)) = \begin{cases} k & \text{if } \lambda = \mu \text{ and } n = 0, \\ 0 & \text{else.} \end{cases}$$

(d) Let $M \in \mathcal{G}(A)$ then $\mathsf{Res}\,^A_{B^-}M$ is projective in $\mathcal{G}(B^-)$ if and only if $M \in \mathcal{G}^\Delta(A)$.

Proof. (a) We proceed by showing that $\mathsf{Res}\,^A_{B^-}\Delta(\lambda)$ is projective and then check it is a projective cover. Note that $\mathsf{Res}\,^A_{B^-}\Delta(\lambda) \cong B^- \otimes_T \lambda$. By Lemma 3.2.1 we have for arbitrary modules $M \in \mathcal{G}(T)$ and $N \in \mathcal{G}(B^-)$

$$\operatorname{Hom}_{\mathcal{G}(B^-)}(B^- \otimes_T M, N) \cong \operatorname{Hom}_{\mathcal{G}(T)}(M, \mathsf{Res}\,^{B^-}_T N).$$

Since T is semisimple all modules in $\mathcal{G}(T)$ are both projective and injective. We see that $\operatorname{Hom}_{\mathcal{G}(T)}(M, -)$ is exact and so fixing λ we see that $\operatorname{Hom}_{\mathcal{G}(B^-)}(B^- \otimes_T \lambda, -)$ is also exact. Hence $\mathsf{Res}\,^A_{B^-}\Delta(\lambda)$ is projective. Furthermore, we have

$$\operatorname{Hom}_{\mathcal{G}(B^-)}(\mathsf{Res}\,^A_{B^-}\Delta(\lambda), \mathsf{Inf}\,^{B^-}_T\mu) \cong \operatorname{Hom}_{\mathcal{G}(B^-)}(B^- \otimes_T \lambda, \mathsf{Inf}\,^{B^-}_T\mu)$$

and

$$\operatorname{Hom}_{\mathcal{G}(B^-)}(B^- \otimes_T \lambda, \mathsf{Inf}\,^{B^-}_T\mu) \cong \operatorname{Hom}_{\mathcal{G}(T)}(\lambda, \mathsf{Res}\,^{B^-}_T\mathsf{Inf}\,^{B^-}_T\mu) \cong \operatorname{Hom}_{\mathcal{G}(T)}(\lambda, \mu).$$

Therefore

$$\mathrm{Hom}_{\mathcal{G}(B^-)}(\mathsf{Res}\,^A_{B^-}\Delta(\lambda), \mathsf{Inf}\,^{B^-}_T\,\mu) = \begin{cases} k & \text{if } \lambda = \mu \\ 0 & \text{else.} \end{cases}$$

Hence $\mathsf{Res}\,^A_{B^-}\Delta(\lambda)$ is the projective cover of $\mathrm{Inf}^{B^-}_T\lambda$.

(b) Proceed by induction on the length of a Δ-filtration. The case where $\ell = 1$ is proven in part (a). Assume the lemma holds for some $\ell = k$ and consider M, a module with Δ-filtration of length $k+1$. Then we note two elementary facts, first that $\mathsf{Res}\,^A_{B^-}(M/N) \cong \mathsf{Res}\,^A_{B^-}M/\mathsf{Res}\,^A_{B^-}N$ and secondly that if we have a module X with a projective submodule Y with X/Y also projective then X is projective.

Let $0 \subset M_1 \subset \cdots \subset M_k \subset M$ be a Δ-filtration of M. Then we have that

$$\mathsf{Res}\,^A_{B^-}M/\mathsf{Res}\,^A_{B^-}M_k \cong \mathsf{Res}\,^A_{B^-}(M/M_k) \cong \mathsf{Res}\,^A_{B^-}\Delta(\lambda_k).$$

Since $\mathsf{Res}\,^A_{B^-}\Delta(\lambda_k)$ is projective and $\mathsf{Res}\,^A_{B^-}M_k$ is projective by the inductive hypothesis, we must have that $\mathsf{Res}\,^A_{B^-}M$ is also projective.

(c) From Theorem 3.2.3 we have that $\mathrm{Ext}^n_{\mathcal{G}(A)}(\Delta(\lambda), \nabla(\mu)) \cong \mathrm{Ext}^n_{\mathcal{G}(B^-)}(\Delta(\lambda), \mu)$. Since $\mathsf{Res}\,^A_{B^-}\Delta(\lambda)$ is projective by part (a) it follows $\mathrm{Ext}^n_{\mathcal{G}(B^-)}(\Delta(\lambda), \mu) = 0$ for $n > 0$. If $n = 0$ then by the adjunction in part (a), $\mathrm{Ext}^n_{\mathcal{G}(B^-)}(\Delta(\lambda), \mu) = \mathrm{Hom}_{\mathcal{G}(B^-)}(\Delta(\lambda), \mu)$ and $\mathrm{Hom}_{\mathcal{G}(B^-)}(\Delta(\lambda), \mu) \cong \mathrm{Hom}_{\mathcal{G}(T)}(\lambda, \mu)$.

(d) From part (b) we see that if $M \in \mathcal{G}^{\Delta}(A)$ then $\mathsf{Res}\,^A_{B^-}M$ is projective. Hence if $M \in \mathcal{G}^{\Delta}(A)$ then $\mathsf{Res}\,^A_{B^-}M$ is projective and clearly $M \in \mathcal{G}(A)$. The reverse implication is proven in [36, Theorem 4.4]. □

The above lemma has three important corollaries.

Corollary 3.2.5. *For any $\lambda, \mu \in \mathrm{Irr}\,\mathcal{G}(T)$,*

$$\mathrm{Hom}_{\mathcal{G}(A)}(\Delta(\lambda), \nabla(\mu)) = \mathrm{Hom}_{\mathcal{G}(A)}(\Delta(\lambda), \mathrm{soc}\,\nabla(\mu)).$$

Proof. Since $\mathrm{soc}\,\nabla(\mu)$ is a (graded) submodule of $\nabla(\mu)$, $\mathrm{Hom}_{\mathcal{G}(A)}(\Delta(\lambda), \mathrm{soc}\,\nabla(\mu))$ is contained in $\mathrm{Hom}_{\mathcal{G}(A)}(\Delta(\lambda), \nabla(\mu))$. On the other hand, Lemma 3.2.4 (b) says that the space $\mathrm{Hom}_{\mathcal{G}(A)}(\Delta(\lambda), \nabla(\mu))$ is at most one-dimensional, and is one-dimensional precisely when $\lambda = \mu$. Therefore, it suffices to note that when $\lambda = \mu$, $\mathrm{soc}\,\nabla(\mu) = L(\lambda)$ by Lemma 2.3.10. Hence,

$$\dim_k \mathrm{Hom}_{\mathcal{G}(A)}(\Delta(\lambda), \mathrm{soc}\,\nabla(\mu)) = \begin{cases} 1, & \text{if } \lambda = \mu \\ 0, & \text{else} \end{cases}$$

by Lemma 2.3.11. □

Corollary 3.2.6. *For any $M \in \mathcal{G}^{\Delta}(A)$ we have that* $\mathrm{Ext}^1_{\mathcal{G}(A)}(M, \nabla(\mu)) = 0$.

Proof. Proceed by induction on the length ℓ of a Δ-filtration, the case $\ell = 1$ being given by Lemma 3.2.4. Assume the statement is true for all modules M with Δ-filtration of length at most k. Then consider an A-module N with length of Δ-filtration $\ell = k + 1$. There is a short exact sequence

$$0 \to \Delta(\lambda) \to N \to M \to 0$$

and so there is an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{G}(A)}(M, \nabla(\mu)) \to \mathrm{Hom}_{\mathcal{G}(A)}(N, \nabla(\mu)) \to \mathrm{Hom}_{\mathcal{G}(A)}(\Delta(\lambda), \nabla(\mu)) \to \mathrm{Ext}^1_{\mathcal{G}(A)}(M, \nabla(\mu))$$

which continues

$$\mathrm{Ext}^1_{\mathcal{G}(A)}(M, \nabla(\mu)) \to \mathrm{Ext}^1_{\mathcal{G}(A)}(N, \nabla(\mu)) \to \mathrm{Ext}^1_{\mathcal{G}(A)}(\Delta(\lambda), \nabla(\mu)).$$

By the inductive hypothesis we know $\mathrm{Ext}^1_{\mathcal{G}(A)}(M, \nabla(\mu)) = 0$ and $\mathrm{Ext}^1_{\mathcal{G}(A)}(\Delta(\lambda), \nabla(\mu)) = 0$. Therefore $\mathrm{Ext}^1_{\mathcal{G}(A)}(N, \nabla(\mu)) = 0$. □

Corollary 3.2.7. *All projective objects in $\mathcal{G}(A)$ admit a standard filtration.*

Proof. This follows directly from Lemma 3.2.4 (d). □

Lemma 3.2.8. *The functor* $\mathrm{Hom}_{\mathcal{G}(A)}(-, \nabla(\mu))$ *is exact on* $\mathcal{G}^{\Delta}(A)$.

Proof. Consider a short exact sequence

$$0 \to M_1 \to M_2 \to M_2 \to 0.$$

Apply $\mathrm{Hom}_{\mathcal{G}(A)}(-, \nabla(\mu))$ to get the exact sequence

$$\mathrm{Hom}_{\mathcal{G}(A)}(M_3, \nabla(\mu)) \to \mathrm{Hom}_{\mathcal{G}(A)}(M_2, \nabla(\mu)) \to \mathrm{Hom}_{\mathcal{G}(A)}(M_1, \nabla(\mu)) \to 0.$$

Since $\mathrm{Hom}_{\mathcal{G}(A)}(-, \nabla(\mu))$ is right exact we can make a long exact sequence using Ext groups

$$\cdots \to \mathrm{Ext}^1_{\mathcal{G}^{\Delta}(A)}(M_1, \nabla(\mu)) \to \mathrm{Hom}_{\mathcal{G}(A)}(M_3, \nabla(\mu)) \to \mathrm{Hom}_{\mathcal{G}(A)}(M_2, \nabla(\mu)) \to \mathrm{Hom}_{\mathcal{G}(A)}(M_1, \nabla(\mu)) \to 0.$$

Using Corollary 3.2.6 we see that all the extension groups are 0 hence the following is exact

$$0 \to \mathrm{Hom}_{\mathcal{G}(A)}(M_3, \nabla(\mu)) \to \mathrm{Hom}_{\mathcal{G}(A)}(M_2, \nabla(\mu)) \to \mathrm{Hom}_{\mathcal{G}(A)}(M_1, \nabla(\mu)) \to 0.$$

Hence, $\mathrm{Hom}_{\mathcal{G}(A)}(-, \nabla(\mu))$ is an exact functor. □

We wish to state an important result due to Bass, but first we must define perfect rings, and the global and projective dimension of a ring.

Definition 3.2.9. A ring R is a (left) *perfect ring* if and only if all (left) R-modules have a projective cover.

Note there are many important examples of perfect rings. In particular Artinian rings are perfect [3, Theorem P] and hence finite dimensional algebras are perfect.

Definition 3.2.10. Let M be an A-module that admits a finite projective resolution. The length of a minimal projective resolution of M is the *projective dimension* of M. If M admits no finite projective resolution its projective dimension is said to be infinite.

Definition 3.2.11. Let A be a ring. The *global dimension* of A is the supremum of the set of projective dimensions of all A-modules.

Definition 3.2.12. The *finistic global dimension* of A is the restriction of the supremum of the set of projective dimensions of modules with finite projective dimension.

Remark 3.2.13. Confusingly the finistic global dimension of a ring can be infinite.

With the above defined we can now present the following theorem due to Bass.

Theorem 3.2.14. *Let R be a ring, then the following are equivalent:*

(a) The ring R has finistic global dimension 0.

(b) The ring R is left perfect and the finitely generated R-modules have finistic global dimension 0.

(c) R is left perfect and every finitely generated proper right ideal has non-zero left annihilator.

(d) R is left perfect and every simple left R-module is a homomorphic image of an injective module.

Proof. See [3, Theorem 6.3]. □

Theorem 3.2.14 allows us to prove the following remarkable lemma.

Lemma 3.2.15. *Let A be a finite dimensional k-algebra. If $\operatorname{soc} A$ contains a copy of every irreducible A-module then A has finistic global dimension zero.*

Proof. Assume that $\operatorname{soc} A$ contains a copy of each simple module. Let S be a simple A-module. Then we have an embedding $S \to A$. Clearly A is free as an A-module and hence is projective. Therefore A^* is an injective module and we have a surjection $A^* \to S^*$. By Theorem 3.2.14 (d), this is equivalent to having finistic global dimension 0. $\square$

If A is positively graded, with the degree zero part equal to the field then we can say even more.

Lemma 3.2.16. *Let A be a positively graded finite dimensional k-algebra with $A_0 = k$. If $A \neq k$ then every finitely generated A-module that is not free has infinite projective dimension.*

Proof. Note that $J := A_{>0}$ is a nilpotent ideal in A. Since it is clearly the largest such ideal it equals the Jacobson radical of A [34, Theorem 4.8]. In particular, A is a local ring with unique irreducible module $S := A/J$. Also, if P is a projective A-module then it is free [46, Theorem 2.5]. Finally, we must show that A has finistic global dimension zero. But this follows from Lemma 3.2.15 since there is only one irreducible module S up to isomorphism, so S must occur in the socle of A. Therefore the simple module is either projective or has infinite projective dimension. If the unique simple module is projective then A is semisimple, but as it has non-zero Jacobson radical this is not possible. $\square$

This allows us to prove the next proposition which has two novel corollaries.

Proposition 3.2.17. *The following are equivalent:*

(a) The Borel subalgebra B^- has finite global dimension.

(b) $A^- = k$.

(c) B^- is semisimple.

Moreover, B^- always has finistic global dimension zero.

Proof. Statements (a) and (b) are equivalent because the Jacobson radical of B^- equals $B^-_{<0}$, and the latter is zero if and only if $A^-_{<0} = 0$. Since $B^-_{<0} = A^-_{<0}T$.

Assume that $A^-_{<0} \neq 0$ and assume we are given a finite projective resolution $P_\bullet \to \lambda$ of the irreducible B^--module λ. Since B^- is free over A^-, the restriction of $P_\bullet$ to A^- is a finite projective resolution of $\lambda|_{A^-} \cong k^{\oplus \dim \lambda}$. This implies that k has finite projective dimension as an A^--module. This contradicts Lemma 3.2.16. Therefore B^- always has finistic projective dimension zero.

The equivalence of (a) and (c) follows by a similar argument to the proof of Lemma 3.2.16. $\square$

This leads to two key corollaries. These statements are new in the theory of finite dimensional algebras with a triangular decomposition.

Corollary 3.2.18. *Let $M, N \in \mathcal{G}^{\Delta}(A)$ with $N \subset M$, the following two statements are true:*

(a) $M/N \in \mathcal{G}^{\Delta}(A)$.

(b) Any given Δ-filtration $N_{\bullet}$ of N can be extended to a Δ-filtration on M.

Proof. There are two separate cases to consider here: by Proposition 3.2.17, either $A^{-} = k$, in which case $\mathcal{G}^{\Delta}(A) = \mathcal{G}(A)$, and the corollary is vacuous, or B^{-} has infinite global dimension but has finistic global dimension zero.

For part (a), we must show that M/N is also in $\mathcal{G}^{\Delta}(A)$. Noting that restriction to B^{-} is an exact functor, by Lemma 3.2.4 (d), it suffices to note that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of B^{-}-modules with M' and M projective, then M'' is also projective. If this were not the case then M'' would be a finite dimensional B^{-}-module of projective dimension one. This contradicts the fact that B^{-} has finistic global dimension zero.

Let $N \subset M$ be as in part (b). By part (a), M/N is also in $\mathcal{G}^{\Delta}(A)$ then any Δ-filtration of M/N can be pulled back to a Δ-filtration on M extending $N_{\bullet}$. □

Corollary 3.2.19. *The category $\mathcal{G}^{\Delta}(A)$ is closed under direct summands in $\mathcal{G}(A)$.*

Proof. Let $M \in \mathcal{G}^{\Delta}(A)$ and let M' be a direct summand of M in $\mathcal{G}(A)$. Then $\mathsf{Res}^{A}_{B^{-}} M$ is projective in $\mathcal{G}(B^{-})$ by Lemma 3.2.4 (b). Hence, the direct summand $\mathsf{Res}^{A}_{B^{-}} M'$ of $\mathsf{Res}^{A}_{B^{-}} M$ is projective, thus $M' \in \mathcal{G}^{\Delta}(A)$ by Lemma 3.2.4.

□

From the Ext-vanishing property in Lemma 3.2.4 (c) one deduces easily by induction that

$$[M : \Delta(\lambda)] = \#\{M_i/M_{i-1} \cong \Delta(\lambda)\} = \dim_k \operatorname{Hom}_{\mathcal{G}(A)}(M, \nabla(\lambda)) \tag{3.2.1}$$

for $\lambda \in \operatorname{Irr} \mathcal{G}(T)$ and $M \in \mathcal{G}^{\Delta}(A)$. Hence, this number is independent of the chosen filtration. In [36, Theorem 4.5] it is proven that *Brauer reciprocity* holds in $\mathcal{G}(A)$, i.e.:

Proposition 3.2.20. *The relation*

$$[P(\lambda) : \Delta(\mu)] = [\nabla(\mu) : L(\lambda)] \tag{3.2.2}$$

holds for any $\lambda \in \operatorname{Irr}\mathcal{G}(T)$.

Proof. Using formula (3.2.1) we see $[M : \Delta(\lambda)] = \dim_k \operatorname{Hom}_{\mathcal{G}(A)}(M, \nabla(\lambda))$. It remains to show $[\nabla(\mu) : L(\lambda)] = \dim_k \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), \nabla(\mu))$. Assume that $[\nabla(\mu) : L(\lambda)] = n$ then we have a composition series

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = \nabla(\mu)$$

where we have n different integers i such that $M_i/M_{i-1} \cong L(\lambda)$. This gives n surjective maps $M_i \to L(\lambda)$. Since $P(\lambda)$ is projective we have n maps $P(\lambda) \to M_i$ and composing with inclusion these give us n maps $P(\lambda) \to \nabla(\mu)$. These maps are linearly independent, we argue by induction to prove this. Let M be a module with $[M : L(\lambda)] = 1$ then $\dim_k \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), M) = 1$. Now assume the statement for all integers less than j say. Given M such that $[M : L(\lambda)] = j + 1$ then we get an exact sequence

$$0 \to N \to M \to L(\lambda) \to 0.$$

Since $\operatorname{Hom}(P(\lambda), -)$ is exact we get an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), N) \to \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), M) \to \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), L(\lambda)) \to 0$$

and so

$$\dim_k \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), M) = \dim_k \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), N) + \dim_k \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), L(\lambda)).$$

This simplifies to

$$\dim_k \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), M) = \dim_k \operatorname{Hom}_{\mathcal{G}(A)}(P(\lambda), N) + 1.$$

$\square$

In Section 3.3 a formula for calculating the multiplicities of simple modules in the radical layers of the standard modules is proven. This formula depends on a special set that we will define at the beginning of the next section. The definition of this special set relies on a particular composition series, the following lemma is required for this to make sense.

Lemma 3.2.21. *Assume that $0 = M_0 \subset \cdots \subset M_k = M$ is a composition series for M.*

(a) If $N \subset M$ is a submodule, then $\{M_i \cap N \mid M_i \cap N \not\subset M_{i-1}\}$ is a composition series for N.

(b) If $K \subset M$ is a submodule, and $\overline{M}_i$ the image of M_i in M/K, then M/K has a composition series $\{\overline{M}_i \mid M_i \cap K \subset M_{i-1}\}$.

Proof. For part (a) note that $M_{i-1} \cap N \subseteq M_i \cap N$ and we need only show that $(M_i \cap N)/(M_{i-1} \cap N)$ is simple. We have that $M_i/M_{i-1} \cong S_i$ for S_i simple, hence there is a surjective map $\phi_i : M_i \to S_i$ with $\operatorname{Ker} \phi_i = M_{i-1}$. The map ϕ_i induces a map $\overline{\phi_i} : M_i \cap N \to M_{i-1} \cap N$ where $\overline{\phi_i}(a) = \phi_i(a)$. It is clear that $\operatorname{Ker} \overline{\phi_i} = M_{i-1} \cap N$ and so $M_{i-1} \cap N / M_i \cap N = S_i$ if $\operatorname{im} \overline{\phi_i} = S_i$ which is the case as long as $M_i \cap N \not\subseteq M_{i-1}$.

Part (b) can be proven in a similar manner or by simply noting that

$$\overline{M}_i/\overline{M}_{i-1} = (M_i/K)/(M_{i-1}/K) \cong M_i/M_{i-1}.$$

therefore the composition factors are simple so long as $K \subset M_{i-1}$. Note that if $K \subset M_{i-1}$ then $M_i \cap K \subset M_{i-1}$. □

We will apply the previous lemma to the radical powers of M. In this case Lemma 3.2.21 (a) says that if M is a module with composition series $0 = M_0 \subset \cdots \subset M_k = M$ then $\operatorname{rad}^{s-1} M$ has a composition series $\{M_i \cap \operatorname{rad}^{s-1} M \mid M_i \cap \operatorname{rad}^{s-1} M \not\subset M_{i-1}\}$. Furthermore, using Lemma 3.2.21 (b) we obtain a composition series for $\operatorname{rad}_s M$,

$$\{\overline{M_i \cap \operatorname{rad}^{s-1} M} \mid M_i \cap \operatorname{rad}^s M \subset M_{i-1} \cap \operatorname{rad}^{s-1} M \text{ and } M_i \cap \operatorname{rad}^{s-1} M \not\subset M_{i-1}\}.$$

We see that $[\operatorname{rad}_s M : L(\mu)]$ can be interpreted as the number of indices i such that $M_i/M_{i-1} \cong L(\mu)$, $M_i \cap \operatorname{rad}^s M \subset M_{i-1} \cap \operatorname{rad}^{s-1} M$ and $M_i \cap \operatorname{rad}^{s-1} M \not\subset M_{i-1}$.

§ 3.3 | Projective covers of algebras which admit an anti-triangular duality

The main result of this section is Theorem 3.3.5 which allows us to (partially) understand the structure of the radical layers of the projective covers. More specifically it will allow us to calculate the multiplicity of the standard modules inside the projective covers (with certain conditions). This is done by proving a reciprocity result that calculates the multiplicity of the simple modules inside the radical layers of the standard modules. This is straightforward once the simple modules are known.

Key to our statement will be the set $K_s(M, \lambda)$, which we now define. Fix a Δ-filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{\ell_\Delta(M)} = M$ on $M \in \mathcal{G}^\Delta(A)$. If $M_i/M_{i-1} \cong \Delta(\lambda_i)$ then, by Lemma 3.2.4 (c), there is a unique (up to scalar) non-zero morphism $M_i \to M_i/M_{i-1} \to \nabla(\lambda_i)$. Then $i \in \overline{K}_s(M, \lambda)$ if $\lambda_i = \lambda$ and there is a morphism $\phi\colon M \to \nabla(\lambda)$ extending the map $M_i \to \nabla(\lambda)$ such that $\operatorname{rad}^s M \subset \ker \phi$. Restriction to M_i/M_{i-1} defines a map $\mathrm{Hom}_{\mathcal{G}}(M/M_{i-1}, \nabla(\lambda)) \to \mathrm{Hom}_{\mathcal{G}}(\Delta(\lambda_i), \nabla(\lambda))$. Since M/M_{i-1} admits a Δ-filtration, Lemma 3.2.4 (c) says that this map is surjective. In other words, we can always find a morphism $\phi\colon M \to \nabla(\lambda)$ extending the map $M_i \to \nabla(\lambda)$; however it need not be the case that $\mathrm{rad}^s M \subset \mathrm{Ker}\, \phi$ in general.

Note that $\overline{K}_{s-1}(M, \lambda) \subset \overline{K}_s(M, \lambda)$ and $\overline{K}_s(M, \lambda) = \{1, 2, \dots, \ell_\Delta(M)\}$ if $s \geq \ell\ell(M)$. We set $K_s(M, \lambda) = \overline{K}_s(M, \lambda) \smallsetminus \overline{K}_{s-1}(M, \lambda)$. Our definition is (essentially) dual to the definition of $[\mathrm{rad}_s P(\lambda) : \mathrm{Head}\, \Delta(\mu)]$ given in [14, Remark 1]; see also [35, Section 4.1]. Since the head of $\Delta(\lambda)$ equals $L(\lambda)$, we have $[M/\mathrm{rad}^s M : L(\lambda)] \geq |\overline{K}_s(M, \lambda)|$. Thus, knowing the sets $\overline{K}_s(M, \lambda)$ gives partial information on the radical layers of M.

The content of this section is more technical, the following results are united by their necessity to prove Theorem 3.3.5.

Lemma 3.3.1. *Let M, M' and N be A-modules. Let $q\colon M \to M'$ be a surjection such that $q^*\colon \mathrm{Hom}_A(M', N) \to \mathrm{Hom}_A(M, N)$ is an isomorphism. Then, for all $s \geq 1$,*

$$q^*\colon \mathrm{Hom}_A(M'/\mathrm{rad}^s M', N) \to \mathrm{Hom}_A(M/\mathrm{rad}^s M, N)$$

is also an isomorphism.

Proof. First we check that $q^*\colon \mathrm{Hom}_A(M'/\mathrm{rad}^s M', N) \to \mathrm{Hom}_A(M/\mathrm{rad}^s M, N)$ is well-defined. Since $q(\mathrm{rad}^s M) = q(J^s M) = J^s q(M) = \mathrm{rad}^s M'$, the map q descends to a morphism $M/\mathrm{rad}^s M \to M'/\mathrm{rad}^s M'$. Thus, q^* is well-defined. Note that this really says that $\mathrm{Hom}_A(M'/\mathrm{rad}^s M', N)$ is a subspace of $\mathrm{Hom}_A(M', N)$ that is mapped into $\mathrm{Hom}_A(M/\mathrm{rad}^s M, N) \subset \mathrm{Hom}_A(M, N)$. In particular, q^* is injective on the space $\mathrm{Hom}_A(M'/\mathrm{rad}^s M', N)$. Let $\psi \in \mathrm{Hom}_A(M/\mathrm{rad}^s M, N)$. Then, by assumption, there exists $\phi \in \mathrm{Hom}_A(M', N)$ such that $q^*(\phi) = \psi$. By definition, $\psi(\mathrm{rad}^s M) = 0$ but $\psi(\mathrm{rad}^s M) = \phi \circ q(\mathrm{rad}^s M) = \phi(\mathrm{rad}^s M')$. Thus, ϕ vanishes on $\mathrm{rad}^s M'$. Therefore $\phi \in \mathrm{Hom}_A(M'/\mathrm{rad}^s M', N)$, as required.

□

The following technical result is key to the proof of Theorem 3.3.3.

Proposition 3.3.2. *For all $s \geq 1$,*

$$|\overline{K}_s(M, \mu)| = \dim_k \mathrm{Hom}_A(M, \mathrm{soc}^s \nabla(\mu)).$$

Proof. Fix $s \geq 1$. The equality is proven by induction on $n = \ell_\Delta(M)$ the length of a Δ-filtration on M. If $n = 1$ then $M = \Delta(\lambda)$ for some λ. If $\lambda \neq \mu$ then $\overline{K}_s(M, \mu) = \emptyset$ by Corollary 3.2.5. If $\lambda = \mu$ then we need only check that there exists a map $\phi : \Delta(\lambda) \to \nabla(\lambda)$

such that $\operatorname{rad}^s \Delta(\lambda) \subset \operatorname{Ker} \phi$. Since $\operatorname{soc} \nabla(\lambda) = L(\lambda)$, Corollary 3.2.5 implies that there exists a map $\phi : \Delta(\lambda) \to \nabla(\lambda)$ such that $\operatorname{Ker} \phi = \operatorname{rad} \Delta(\lambda)$ and $\operatorname{Im}\phi = \operatorname{soc} \nabla(\lambda)$. As $\operatorname{rad}^s \Delta(\lambda) \subset \operatorname{rad} \Delta(\lambda)$ for all $s \geq 1$ we have that $\overline{K}_s(M, \mu) = \{1\}$. Note that $\operatorname{soc} \nabla(\mu)$ is a graded submodule of $\operatorname{soc}^s \nabla(\mu)$ and $\operatorname{soc}^s \nabla(\mu)$ is a graded submodule of $\nabla(\mu)$, hence

$$\operatorname{Hom}_A(\Delta(\lambda), \operatorname{soc} \nabla(\mu)) \subset \operatorname{Hom}_A(\Delta(\lambda), \operatorname{soc}^s \nabla(\mu)) \subset \operatorname{Hom}_A(\Delta(\lambda), \nabla(\mu)).$$

Corollary 3.2.5 then implies that

$$\operatorname{Hom}_A(\Delta(\lambda), \operatorname{soc} \nabla(\mu)) = \operatorname{Hom}_A(\Delta(\lambda), \operatorname{soc}^s \nabla(\mu)) = \operatorname{Hom}_A(\Delta(\lambda), \nabla(\mu)).$$

Therefore, $\operatorname{Hom}_A(\Delta(\lambda), \operatorname{soc}^s \nabla(\lambda))$ is one-dimensional and

$$\operatorname{Hom}_A(\Delta(\lambda), \nabla(\mu)) = \operatorname{Hom}_A(\Delta(\lambda), \operatorname{soc}^s \nabla(\mu)) = 0$$

otherwise.

Therefore we may assume that $n > 1$ and the result holds by induction for all modules with Δ-filtrations of length less than n. Given a module M of length n we choose $\Delta(\lambda) \subset M$ and let M' denote the quotient. By Corollary 3.2.18, M' belongs to $\mathcal{G}^{\Delta}(A)$. Therefore, we may assume that there exists a Δ-filtration $M_\bullet$ of M such that $M_1 = \Delta(\lambda)$, and if we define $M'_i := M_{i+1}/\Delta(\lambda)$, then $M'_\bullet$ is a Δ-filtration of M'; such a filtration is by definition constructed by choosing a Δ-filtration on M'.

Applying $\operatorname{Hom}_A(-, \nabla(\mu))$ to the short exact sequence $0 \to \Delta(\lambda) \to M \to M' \to 0$ gives, by Lemma 3.2.8 a short exact sequence

$$0 \to \operatorname{Hom}_A(M', \nabla(\mu)) \to \operatorname{Hom}_A(M, \nabla(\mu)) \xrightarrow{\Phi} \operatorname{Hom}_A(\Delta(\lambda), \nabla(\mu)) \to 0. \quad (\dagger)$$

This induces an exact sequence

$$0 \to \operatorname{Hom}_A(M', \operatorname{soc}^s \nabla(\mu)) \to \operatorname{Hom}_A(M, \operatorname{soc}^s \nabla(\mu)) \xrightarrow{\Phi_s} \operatorname{Hom}_A(\Delta(\lambda), \operatorname{soc}^s \nabla(\mu)) \to \cdots \quad (\ddagger)$$

We claim that there is a bijection $\overline{K}_s(M, \mu) \setminus \{1\} \xrightarrow{\sim} \overline{K}_s(M', \mu)$. First, we show that there is an injection $\overline{K}_s(M, \mu) \setminus \{1\} \to \overline{K}_s(M', \mu)$ given by $i \to i - 1$. Let $i \in \overline{K}_s(M, \mu) \setminus \{1\}$ and observe that

$$M'_{i-1}/M'_{i-2} \cong \frac{M_i/\Delta(\lambda)}{M_{i-1}/\Delta(\lambda)} \cong \Delta(\mu).$$

If $\phi\colon M \to \nabla(\mu)$ is a morphism extending $M_i/M_{i-1} \to \nabla(\mu)$ then $\Delta(\lambda) = M_1$ is in the kernel of ϕ so it factors as a morphism $\phi\colon M' \to \nabla(\mu)$. Moreover, $\operatorname{rad}^s M' = (\operatorname{rad}^s M + \Delta(\lambda))/\Delta(\lambda)$ and $\operatorname{rad}^s M + \Delta(\lambda) \subset \operatorname{Ker} \phi$. Thus, $\operatorname{rad}^s M' \subset \operatorname{Ker} \phi$ and $i - 1 \in \overline{K}_s(M', \mu)$.

We claim that an injection the opposite way $\overline{K}_s(M', \mu) \to \overline{K}_s(M, \mu) \setminus \{1\}$ is given by the inverse map $j \to j+1$. Since $j \in \overline{K}_s(M', \mu)$,

$$M_{j+1}/M_j \cong \frac{M_{j+1}/\Delta(\lambda)}{M_j/\Delta(\lambda)} \cong M'_j/M'_{j-1} \cong \Delta(\mu)$$

and we may consider $\phi \colon M' \to \nabla(\mu)$ as a map $\phi \colon M \to \nabla(\mu)$ with $\Delta(\lambda)$ in the kernel. Again, since $\operatorname{rad}^s M' = (\operatorname{rad}^s M + \Delta(\lambda))/\Delta(\lambda)$ and $\phi(\operatorname{rad}^s M') = 0$, it follows that $\phi(\operatorname{rad}^s M) = 0$ and hence $j + 1 \in \overline{K}_s(M, \mu) \setminus \{1\}$.

To derive the equality of the proposition, there are two separate cases to consider: (a) $\lambda \neq \mu$; and (b) $\lambda = \mu$.

In case (a), $1 \notin \overline{K}_s(M, \mu)$. Therefore, the above argument shows that $|\overline{K}_s(M, \mu)| = |\overline{K}_s(M', \mu)|$. Comparing Hom sets in (†) we see that $\operatorname{Hom}_A(M', \nabla(\mu)) \to \operatorname{Hom}_A(M, \nabla(\mu))$ is an isomorphism. By Lemma 3.3.1, this implies that the map $\operatorname{Hom}_A(M'/\operatorname{rad}^s M', \nabla(\mu)) \to \operatorname{Hom}_A(M/\operatorname{rad}^s M, \nabla(\mu))$ is also an isomorphism. By Lemma 2.2.27, we deduce that

$$\operatorname{Hom}_A(M', \operatorname{soc}^s \nabla(\mu)) \to \operatorname{Hom}_A(M, \operatorname{soc}^s \nabla(\mu))$$

is an isomorphism thus completing the proof for case (a).

Case (b) decomposes into two further cases to consider. Either (1) $1 \in \overline{K}_s(M, \mu)$ or (2) $1 \notin \overline{K}_s(M, \mu)$. In case (1), we must have $\Phi_s \neq 0$ since $\Phi_s(\phi) \neq 0$. Since $\operatorname{Hom}_A(\Delta(\lambda), \operatorname{soc}^s\nabla(\mu))$ is one dimensional this implies that Φ_s is a surjection. As (‡) is exact, Φ_s has kernel $\operatorname{Hom}_A(M', \operatorname{soc}^s\nabla(\mu))$. This gives rise to a short exact sequence

$$0 \to \operatorname{Hom}_A(M', \operatorname{soc}^s\nabla(\mu)) \to \operatorname{Hom}_A(M, \operatorname{soc}^s\nabla(\mu)) \xrightarrow{\Phi_s} \operatorname{Hom}_A(\Delta(\lambda), \operatorname{soc}^s\nabla(\mu)) \to 0.$$

Hence,

$$\dim_k \operatorname{Hom}_A(M, \operatorname{soc}^s\nabla(\mu)) = \dim_k \operatorname{Hom}_A(M', \operatorname{soc}^s\nabla(\mu)) + 1,$$

as required.

In case (2), we claim that $\Phi_s = 0$. If this was not the case then we can find $\phi \colon M/\operatorname{rad}^s M \to \nabla(\mu)$ such that $\Phi_s(\phi) \neq 0$. But, by definition of Φ_s this means that ϕ is a morphism $M \to \nabla(\mu)$ with $\operatorname{rad}^s M \in \operatorname{Ker}\phi$ and whose restriction to $\Delta(\lambda)$ is a non-zero morphism $\Delta(\lambda) \to \nabla(\mu)$. In other words, $1 \in \overline{K}_s(M, \mu)$; a contradiction. Since $\Phi_s = 0$, (‡) above implies that

$$\dim_k \operatorname{Hom}_A(M, \operatorname{soc}^s\nabla(\mu)) = \dim_k \operatorname{Hom}_A(M', \operatorname{soc}^s\nabla(\mu)).$$

□

We note that Proposition 3.3.2 shows that the numbers $|\overline{K}_s(M,\mu)|$ and $|K_s(M,\mu)|$ are independent of the choice of Δ-filtration. The next three theorems make up the core of this section.

Theorem 3.3.3. *Let $P(\lambda)$ be the projective cover of $L(\lambda)$. Then*

$$|K_s(P(\lambda),\mu)| = \dim_k \operatorname{Hom}_A(P(\lambda), \operatorname{soc}_s \nabla(\mu)).$$

Proof. We begin by observing that $|K_s(M,\mu)| = |\overline{K}_s(M,\mu)| - |\overline{K}_{s-1}(M,\mu)|$ for all A-modules M and simple modules μ. By Proposition 3.3.2 we see that

$$|\overline{K}_s(M,\mu)| - |\overline{K}_{s-1}(M,\mu)| = \dim_k \operatorname{Hom}_A(M, \operatorname{soc}^s \nabla(\mu)) - \dim_k \operatorname{Hom}_A(M, \operatorname{soc}^{s-1} \nabla(\mu)).$$

Set $M = P(\lambda)$, then

$$|\overline{K}_s(P(\lambda),\mu)| - |\overline{K}_{s-1}(P(\lambda),\mu)| = \dim_k \operatorname{Hom}_A(P(\lambda), \operatorname{soc}^s \nabla(\mu)) - \dim_k \operatorname{Hom}_A(P(\lambda), \operatorname{soc}^{s-1} \nabla(\mu)).$$

Consider the following exact sequence

$$0 \to \operatorname{soc}^{s-1} \nabla(\mu) \to \operatorname{soc}^s \nabla(\mu) \to \operatorname{soc}_s \nabla(\mu) \to 0,$$

applying the exact functor $\operatorname{Hom}_A(P(\lambda), -)$ we see that

$$\dim_k \operatorname{Hom}_A(P(\lambda), \operatorname{soc}_s \nabla(\mu)) = \dim_k \operatorname{Hom}_A(P(\lambda), \operatorname{soc}^s \nabla(\mu)) - \dim_k \operatorname{Hom}_A(P(\lambda), \operatorname{soc}^{s-1} \nabla(\mu)).$$

Therefore

$$|K_s(P(\lambda),\mu)| = \dim_k \operatorname{Hom}_A(P(\lambda), \operatorname{soc}_s \nabla(\mu))$$

and the result is proven. □

We now arrive at the following important theorem. It is similar in appearance to [14, Corollary 7], however our proof is necessarily completely different because our hypothesis are different.

Theorem 3.3.4. *Let $P(\lambda)$ be the projective cover of $L(\lambda)$. Then*

$$|K_s(P(\lambda),\mu)| = [\operatorname{soc}_s \nabla(\mu) : L(\lambda)].$$

Proof. Follows from Theorem 3.3.3 and Lemma 2.2.32. □

If the algebra A admits an anti-triangular duality then we can calculate the set $|K_s(P(\lambda),\mu)|$ using the radical layers of the standard modules. This brings us closer to our goal of understanding the projective covers in terms of the standard and costandard modules.

Theorem 3.3.5. *Let $P(\lambda)$ be the projective cover of $L(\lambda)$. If the category A-mod admits an anti-triangular duality then the following formula holds*

$$|K_s(P(\lambda), \mu)| = [\mathrm{rad}_s\, \Delta(\mu) : L(\lambda)].$$

Proof. From Theorem 3.3.4 and Lemma 2.3.14 we have

$$|K_s(P(\lambda), \mu)| = [\mathrm{soc}\,_s\nabla(\mu) : L(\lambda)] = [D(\mathrm{rad}_s\, \Delta(\mu)) : D(L(\lambda))] = [\mathrm{rad}_s\, \Delta(\mu) : L(\lambda)].$$

□

In the next section we will see that with a further condition on the radical series of the projective covers we will be able to refine our results. Ultimately this gives a formula for calculating the multiplicity of the simple modules inside the radical layers of the projective covers using only the standard and simple modules. For now let us demonstrate just a few of the uses of Theorem 3.3.4 and Theorem 3.3.5. We do this by providing some examples, beginning with the cases $A^+ = k$ and $A^- = k$.

Example 3.3.6. Let A be a finite dimensional graded k-algebra admitting a triangular decomposition, where $A^+ = k$. Let λ denote a simple T-module. Then, by definition, $\nabla(\lambda) = (\lambda^* \otimes_{B^-} A)^* \cong \lambda$. Therefore,

$$[\mathrm{soc}_s\nabla(\lambda) : L(\mu)] = \begin{cases} 1, & \text{if } \lambda = \mu \text{ and } s = 1 \\ 0, & \text{else.} \end{cases}$$

Therefore Theorem 3.3.4 says that $|K_1(P(\lambda), \lambda)| = 1$ and $K_s(P(\lambda), \mu) = \emptyset$ otherwise. By definition $\Delta(\lambda) = A \otimes_{B^+} \lambda = A^- \otimes_{B^+} \lambda$, and A^- is an arbitrary negatively graded algebra. This means that Theorem 3.3.5 cannot hold in general. In this case A cannot admit an anti-triangular duality since $\dim A^- \neq \dim A^+$.

Example 3.3.7. Let A be a finite dimensional graded algebra with a triangular decomposition with $A^- = k$ and let λ denote a simple T-module. Then, by definition, $\Delta(\lambda) = A \otimes_{B^+} \lambda = B^+ \otimes_{B^+} \lambda$ hence $\Delta(\lambda) = \lambda$ as a vector space. Therefore it must be simple, so $\Delta(\lambda) = L(\lambda)$. Therefore,

$$[\mathrm{rad}_s\, \Delta(\lambda) : L(\mu)] = \begin{cases} 1, & \text{if } \lambda = \mu \text{ and } s = 0 \\ 0, & \text{else.} \end{cases}$$

Since $\nabla(\lambda) = (\lambda^* \otimes_{B^-} A)^* \cong (\lambda^* \otimes_{B^-} A^+)^*$ and A^+ is any positivley graded algebra, we do not have

$$[\mathrm{soc}_s \nabla(\lambda) : L(\mu)] = \begin{cases} 1, & \text{if } \lambda = \mu \text{ and } s = 0 \\ 0, & \text{else.} \end{cases}$$

in general. Therefore Theorem 3.3.5 does not hold, again because A does not have an anti-triangular duality.

These examples demonstrate that by using Theorem 3.3.4 and Theorem 3.3.5 we can easily deduce whether an algebra admits an anti-triangular duality.

As we have remarked, many important classes of algebras admit a triangular decomposition, and further an anti-triangular duality. One such class are the restricted enveloping algebras. We now show how our formula can be applied to better understand the projective covers of $\overline{U}_0(\mathfrak{sl}_2)$ over a field of prime characteristic.

Example 3.3.8. Let k be an algebraically closed field of characteristic $p \geq 5$. The Lie algebra $\mathfrak{sl}_2$ has basis $\{e, f, h\}$ and relations $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. Recall from Example 2.3.2 that the restricted universal enveloping algebra [40] $\overline{U}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$, admits the following grading. Let $\deg(e) = -1$, $\deg(f) = 1$ and $\deg(h) = 0$. Then, not only is this algebra graded, but it admits a triangular decomposition with $A^+ = k[f]$, $T = k[h]$ and $A^- = k[e]$.

Next we find the simple T-modules and hence standard modules. By definition of the restricted universal enveloping algebra, we impose the relation $x^p - x^{[p]}$ for each $x \in \mathfrak{sl}_2$. Here $h^{[p]} = h$ and hence $T = k[h]/(h^p - h)$. In other words, we have $T \cong k[\mathbb{F}_p]$.

Since $T \cong k[\mathbb{F}_p]$, the simple modules of T are all one dimensional over k. Since k is algebraically closed, h has an eigenvector v and eigenvalue λ in any finite dimensional module M. Hence $\mathrm{Span}\{v\}$ is a one-dimensional subspace. From the equation $h^{p-1} \cdot v = v$ we see that there are p choices for λ, hence p different simple T-modules. Denote by $S(n)$ the simple T-module where h acts by multiplication by the integer $0 \leq n \leq p-1$.

Recall from the definition of the standard modules that $\Delta(S(n)) = \overline{U}(\mathfrak{sl}_2) \otimes_{k[e,h]} S(n)$. We now calculate the corresponding simple $\overline{U}(\mathfrak{sl}_2)$-modules, $L(S(n))$. Letting $v \in S(n)$ we can write an explicit basis for $\Delta(S(n)) = \overline{U}(\mathfrak{sl}_2) \otimes_{k[e,h]} S(n)$, as

$$\{1 \otimes v, f \otimes v, \ldots, f^{p-1} \otimes v\}.$$

We write $v_j = f^j \otimes v$. The defining relations of $\overline{U}(\mathfrak{sl}_2)$ can be used recursively to obtain the following

$$h \cdot v_j = (n-2)v_j,$$

$$e \cdot v_j = \begin{cases} 0 & \text{if } j = 0 \\ j(n-j+1)v_{j-1} & \text{if } j \neq 0 \end{cases}$$

and

$$f \cdot v_j = \begin{cases} v_{j+1} & \text{if } j < p-1 \\ 0 & \text{if } j = p-1. \end{cases}$$

If an irreducible submodule contains the element v_j then it also contains the elements $v_{j+1}, \ldots v_{p-1}$, obtained from repeated application of f. However it may not necessarily contain v_{j-1} and terms of lower degree as the action of e includes multiplication by a scalar. Therefore, if $j = n+1$ then $e \cdot v_j = 0$. We have p non-isomorphic irreducible submodules, of dimension $1, \ldots, p$, determined by the action of h. We are now in position to describe the composition series for $\Delta(S(n))$. The simplest case is $\Delta(S(p-1))$ which is irreducible and hence equals $L(p-1)$. Each $\Delta(S(n))$ has a simple submodule of dimension $p-n-1$. These are the only simple submodules of the standards. Therefore the simple $L(S(n))$ associated to $\Delta(S(n))$ has dimension $n+1$.

We use a result found in [41, p. 10] to see that $\overline{U}(\mathfrak{sl}_2)$ has p simple modules, with one of each dimension from 1 to p, and that these are the only simple modules. We can now construct a composition series for $\Delta(S(n))$,

$$\{0\} \subset L(S(p-n-2)) \subset \Delta(S(n)).$$

The above is also a radical series for the standard modules because the inclusion $L(S(p-n-2)) \hookrightarrow \Delta(S(n))$ is not split. We are now able to calculate $[\operatorname{rad}_s \Delta(S(i)) : L(S(j))]$.

$$[\operatorname{rad}_s \Delta(S(i)) : L(S(j))] = \begin{cases} 1 & \text{if } s = 1 \text{ and } i = j \\ 1 & \text{if } s = 2 \text{ and } j = p-i-2 \\ 0 & \text{else.} \end{cases}$$

Since $\overline{U}(\mathfrak{sl}_2)$ admits an anti-triangular duality, we can apply Theorem 3.3.5. Using the formula $|K_s(P(\lambda), \mu)| = [\operatorname{rad}_s \Delta(\mu) : L(\lambda)]$, we must have that

$$|K_s(P(S(j)), S(i))| = \begin{cases} 1 & \text{if } s = 1 \text{ and } i = j \\ 1 & \text{if } s = 2 \text{ and } j = p-i-2 \\ 0 & \text{else.} \end{cases}$$

Without explicitly knowing what the projective covers of the simple modules are we can deduce some of their properties, in particular the components of a Δ-filtration for $P(S_j)$.

From the above we see that

$$[P(S_j)/\text{rad}P(S_j) : L(S_j)] \geq 1 \quad \text{and} \quad [P(S_j)/\text{rad}^2 P(S_j) : L(S_{p-j-2})] \geq 1.$$

Theorem 3.3.5 makes clear that it is important to know $[\text{rad}_s \Delta(\mu) : L(\lambda)]$ as this gives information on the projective cover $P(\lambda)$. We now demonstrate how to find the multiplicities of these radical layers for the restricted rational Cherednik algebra of the dihedral group of order 8.

Example 3.3.9. In this example we consider the restricted rational Cherednik algebra for the dihedral group of order 8. We fix $c = 0$. Denote by $\mathfrak{h}$ the underlying two dimensional vector space with basis $\{y_1, y_2\}$ where y_1 lies on the x-axis and y_2 lies along the y-axis. Recall that the dihedral group has presentation $D_8 = \langle a, b \,|\, a^4 = 1, b^2 = 1 \text{ and } bab = a^{-1} \rangle$. Theorem 2.6.10 states that the restricted rational Cherednik algebra has a vector space decomposition $\overline{H}_0(W) = \mathbb{C}[\mathfrak{h}]^{coW} \otimes W \otimes \mathbb{C}[\mathfrak{h}^*]^{coW}$. Letting $x_1 = y_1^*$, $x_2 = y_2^*$ denote the dual basis in $\mathfrak{h}^*$ we have

$$\mathbb{C}[\mathfrak{h}^*]^{coW} = \frac{\mathbb{C}[x_1, x_2]}{(x_1^2 + x_2^2, x_1^4 + x_2^4)},$$

and

$$\mathbb{C}[\mathfrak{h}]^{coW} = \frac{\mathbb{C}[y_1, y_2]}{(y_1^2 + y_2^2, y_1^4 + y_2^4)}.$$

The algebra is graded by $\deg(x_1) = \deg(x_2) = -1$, $\deg(y_1) = \deg(y_2) = 1$ and $\deg(w) = 0$ where $w \in W$. It admits a triangular decomposition $A^+ = \mathbb{C}[\mathfrak{h}]^{coW}$, $T = W$ and $A^- = \mathbb{C}[\mathfrak{h}^*]^{coW}$. The group W has five simple modules, four are one-dimensional and there is one two-dimensional module. We now describe these.

Let us begin with the one-dimensional simple modules. From the defining relations of D_8 we have $a^4\lambda = \lambda$ and $b^2\lambda = \lambda$ hence $b = 1$ or $b = -1$. If $b = 1$ then $a^{-1}\lambda = bab\lambda = b(a\lambda) = a\lambda$ hence $a = 1$ or $a = -1$. If $b = -1$ then we again find that $a = 1$ or $a = -1$ giving us four irreducible representations.

Let us now fix notation, V_0, V_1, V_2, and V_3 are the one dimensional representations. The trivial representation is V_0. The representation V_1 has relations $a \cdot v = v$ and $b \cdot v = -v$. The representation V_2 has relations $a \cdot v = -v$ and $b \cdot v = v$. The representation V_3 has relations $a \cdot v = -v$ and $b \cdot v = -v$.

The final irreducible representation is $V_4 = \mathfrak{h}$ already defined.

A radical filtration for the standard modules $\Delta(V_i)$ is straightforward, the radical series is the same as the graded pieces of $\Delta(V_i)$. By direct calculation we find the following

$$[\Delta(\lambda) : L(\mu)] = \begin{cases} 1, & \text{for } \dim(\lambda) = 1, \dim(\mu) = 1 \\ 2, & \text{for } \dim(\lambda) = 1, \dim(\mu) = 2 \\ 2, & \text{for } \dim(\lambda) = 2, \dim(\mu) = 1 \\ 4, & \text{for } \dim(\lambda) = 2, \dim(\mu) = 2 \end{cases}$$

and hence

$$[\mathrm{rad}_s(\Delta(\lambda)) : L(\mu)] = \begin{cases} 1, & \text{if } \lambda = \mu \text{ and } s = 0 \\ 0, & \text{if } \lambda \neq \mu \text{ and } s = 0 \\ 1, & \text{if } \lambda = \mu, \text{ and } s = 1 \\ 0, & \text{if } \lambda \neq \mu, \text{ and } s = 1 \\ 0, & \text{if } \dim \lambda = 1, \dim \mu = 1 \text{ and } s = 2 \\ 1, & \text{if } \dim \lambda = 1, \dim \mu = 2 \text{ and } s = 2 \\ 1, & \text{if } \dim \lambda = 2, \dim \mu = 1 \text{ and } s = 2 \\ 0, & \text{if } \dim \lambda = 2, \dim \mu = 2 \text{ and } s = 2 \\ 0, & \text{if } \dim \lambda = 1, \dim \mu = 1 \text{ and } s = 3 \\ 1, & \text{if } \dim \lambda = 1, \dim \mu = 2 \text{ and } s = 3 \\ 1, & \text{if } \dim \lambda = 2, \dim \mu = 1 \text{ and } s = 3 \\ 0, & \text{if } \dim \lambda = 2, \dim \mu = 2 \text{ and } s = 3 \\ 0, & \text{if } \dim \lambda = 1, \dim \mu = 1 \text{ and } s = 4 \\ 1, & \text{if } \dim \lambda = 1, \dim \mu = 2 \text{ and } s = 4 \\ 1, & \text{if } \dim \lambda = 2, \dim \mu = 1 \text{ and } s = 4 \\ 0, & \text{if } \dim \lambda = 2, \dim \mu = 2 \text{ and } s = 4 \\ 1, & \text{if } \mu = V_1 \otimes \lambda \text{ and } s = 5 \\ 0, & \text{if } \mu \neq V_1 \otimes \lambda \text{ and } s = 5. \end{cases}$$

Thus we can use Theorem 3.3.5 and the above calculations to determine properties of the structure of $P(\lambda)$.

§ 3.4 | Radical preserving filtrations

Here we seek to refine the results of the last section, giving a formula for calculating the multiplicity of the simple modules inside radical layers of the projective cover. To achieve such a strong result we require a strong condition, the notion of radical respecting filtrations. We follow the definition given in [35, Definition 4.2], changing this slightly to make it more general. Specifically in the cited paper it is defined for Δ-filtrations but here we let the filtration be arbitrary.

Definition 3.4.1. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ be a filtration of M by submodules (where M_i/M_{i-1} is not necessarily irreducible). We say that $M_\bullet$ is a *radical respecting filtration* if there exists $n\colon \{1, \ldots, \ell\} \to \mathbb{N}$ such that the canonical map

$$\frac{(\operatorname{rad}^{n(i)+t} M) \cap M_i}{(\operatorname{rad}^{n(i)+t} M) \cap M_{i-1}} \to M_i/M_{i-1}$$

induces an isomorphism

$$\frac{(\operatorname{rad}^{n(i)+t} M) \cap M_i}{(\operatorname{rad}^{n(i)+t} M) \cap M_{i-1}} \xrightarrow{\sim} \operatorname{rad}^t(M_i/M_{i-1})$$

for all $t \geq 0$.

In the above definition it is not required that the map n is injective or increasing only that it satisfies the condition on the radical powers.

In what follows, we introduce the convention that $\operatorname{rad}^{-j} N := N$ for $j \geq 0$.

Lemma 3.4.2. *If $M_\bullet$ is a radical respecting filtration, then*

$$\frac{(\operatorname{rad}^{s} M) \cap M_i}{(\operatorname{rad}^{s} M) \cap M_{i-1}} = M_i/M_{i-1}$$

for all $s \leq n(i)$.

Proof. The submodules

$$0 \subset \cdots \subset \operatorname{rad}^k M \cap M_i \subset \operatorname{rad}^{k-1} M \cap M_i \subset \cdots \subset M_i$$

define a filtration of M_i. Hence, the quotients

$$\frac{(\operatorname{rad}^k M \cap M_i) + M_{i-1}}{M_{i-1}} \cong \frac{(\operatorname{rad}^k M) \cap M_i}{(\operatorname{rad}^k M) \cap M_{i-1}}$$

define a filtration of M_i/M_{i-1}. In particular,

$$\frac{(\operatorname{rad}^{n(i)} M) \cap M_i}{(\operatorname{rad}^{n(i)} M) \cap M_{i-1}} \subset \frac{(\operatorname{rad}^{s} M) \cap M_i}{(\operatorname{rad}^{s} M) \cap M_{i-1}}$$

for $s \leq n(i)$. But, being a radical respecting filtration means that the LHS of the above inclusion equals M_i/M_{i-1}. Therefore, $((\operatorname{rad}^s M)\cap M_i)/((\operatorname{rad}^s M)\cap M_{i-1})$ equals M_i/M_{i-1}, as required. $\square$

We now arrive at the first formula for calculating multiplicities in this section.

Proposition 3.4.3. *If $M_\bullet$ is a radical respecting filtration, then for each irreducible A-module L,*

$$[\operatorname{rad}_s M : L] = \sum_{i=1}^{\ell} [\operatorname{rad}_{s-n(i)}(M_i/M_{i-1}) : L]$$

for all $s \geq 0$.

Proof. Since

$$\sum_{t \geq s} [\operatorname{rad}_t M : L] = [\operatorname{rad}^s M : L],$$

and

$$\sum_{t \geq s} \sum_{i=1}^{\ell} [\operatorname{rad}_{t-n(i)}(M_i/M_{i-1}) : L] = \sum_{i=1}^{\ell} [\operatorname{rad}^{s-n(i)}(M_i/M_{i-1}) : L]$$

it suffices by downward induction on s to prove that

$$[\operatorname{rad}^s M : L] = \sum_{i=1}^{\ell} [\operatorname{rad}^{s-n(i)}(M_i/M_{i-1}) : L].$$

The $\operatorname{rad}^s M \cap M_i$ define a filtration of $\operatorname{rad}^s M$. Therefore,

$$[\operatorname{rad}^s M : L] = \sum_{i=1}^{\ell} [(\operatorname{rad}^s M \cap M_i)/(\operatorname{rad}^s M \cap M_{i-1}) : L].$$

For $s \leq n(i)$, Lemma 3.4.2 says that

$$[(\operatorname{rad}^s M \cap M_i)/(\operatorname{rad}^s M \cap M_{i-1}) : L] = [M_i/M_{i-1} : L] = [\operatorname{rad}^{s-n(i)}(M_i/M_{i-1}) : L],$$

and for $s \geq n(i)$, the definition of radical respecting filtration implies that

$$[(\operatorname{rad}^s M \cap M_i)/(\operatorname{rad}^s M \cap M_{i-1}) : L] = [\operatorname{rad}^{s-n(i)}(M_i/M_{i-1}) : L].$$

Thus,

$$[\operatorname{rad}^s M : L] = \sum_{i=1}^{\ell} [(\operatorname{rad}^s M \cap M_i)/(\operatorname{rad}^s M \cap M_{i-1}) : L] = \sum_{i=1}^{\ell} [\operatorname{rad}^{s-n(i)}(M_i/M_{i-1}) : L]$$

as required. □

Corollary 3.4.4. *If $P(\lambda)$ admits a radical respecting Δ-filtration then*

$$[\operatorname{rad}_s P(\lambda) : L(\mu)] = \sum_{i=1}^{\ell} [\operatorname{rad}_{s-n(i)} \Delta(\lambda_i) : L(\mu)].$$

While the above formulae is a useful result it does have a certain drawback. The condition that the projective covers have a radical respecting filtration can be difficult to prove without knowledge of the projective covers themselves. Later we will show how to calculate the center of the blocks of the restricted rational Cherednik algebra for the wreath product. In this next example we apply Corollary 3.4.4 to calculate the radical layers of the projective cover for the block $A(3,2)$ corresponding to the partition $(3,2)$ of 5.

Example 3.4.5. As we will show in Example 5.1.1, the algebra $A(3,2)$ is isomorphic to $\mathbb{C}[x,y]/(x^5,y^5)$. This is a graded algebra with $\deg(x) = 1$ and $\deg(y) = -1$. There is only one simple module λ, this is the one-dimensional module where both x and y act by 0. The standard module is then constructed as before

$$\Delta(\lambda) = \mathbb{C}[x,y]/(x^5,y^5) \otimes_{\mathbb{C}[x]/(x^5)} \lambda \cong \mathbb{C}[y]/(y^5).$$

The radical series for this is simple to find

$$0 \subset (y^4)/(y^5) \subset (y^3)/(y^5) \subset (y^2)/(y^5) \subset (y)/(y^5) \subset \mathbb{C}[y]/(y^5).$$

Therefore

$$[\mathrm{rad}_s \Delta(\lambda) : \lambda] = \begin{cases} 1, \text{if } 1 \leq s \leq 5 \\ 0, \text{ else.} \end{cases}$$

In this case $P(\lambda) = A(3,2)$ and

$$0 \subset x^4 P(\lambda) \subset x^3 P(\lambda) \subset x^2 P(\lambda) \subset x P(\lambda) \subset P(\lambda)$$

is a Δ-filtration. We need only check that the filtration is radical respecting. This follows as

$$\mathrm{rad}^{i+t} P(\lambda) \cap x^i P(\lambda) = \langle x^{i+t}, x^{i+t-1}y, \cdots, x^i y^t \rangle$$

and so

$$\frac{\mathrm{rad}^{i+t} P(\lambda) \cap x^i P(\lambda)}{\mathrm{rad}^{i+t} P(\lambda) \cap x^{i+1} P(\lambda)} \cong y^t \Delta(\lambda) = \mathrm{rad}^t \Delta(\lambda).$$

In particular we have $n(i) = i$, then Corollary 3.4.4 implies that

$$[\mathrm{rad}_s P(\lambda) : L(\lambda)] = \sum_{i=1}^{5} [\mathrm{rad}_{s-i} \Delta(\lambda) : L(\lambda)] = \begin{cases} s-1, \text{if } 1 \leq s \leq 6 \\ 11 - s, \text{if } 7 \leq s \leq 10 \\ 0, \text{ else.} \end{cases}$$

Chapter 4

The centre of $\overline{H}_c(W)$

Recall from Section 2.6 the restricted rational Cherednik algebra $\overline{H}_c(W)$, which is defined for any complex reflection group $(W, \mathfrak{h})$. While much is known about the centre of $\overline{H}_c(W)$ abstractly, it is very difficult to give an explicit presentation in terms of generators and relations. That is precisely the goal of the final two chapters, for two particular infinite families of complex reflections groups, the symmetric groups S_n and the wreath products $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$. In this chapter we prove the theory we require to give this explicit presentation. In Chapter 5 we describe how to actually present this centre.

Our argument is roughly as follows: we write the centre of $\overline{H}_c(W)$ as the sum of its indecomposable blocks and then seek to write the centre of each block explicitly in terms of generators and relations. This is done by exploiting the link between the spectrum of the centre, Schubert cells and the Wronskian map. The first section concludes with Theorem 4.1.18, which tells us how to do this for the case of the symmetric group. The second section takes advantage of a recent result by Bonnafe and Maksimau, Theorem 4.2.19, to extend this result to the case of the wreath product.

§ 4.1 | The symmetric group case

Let us begin by fixing some notation. Since we are considering the case where $t = 0$ we omit t and write $H_c(W)$ or $\overline{H}_c(W)$ for the rational Cherednik algebra and restricted rational Cherednik algebra respectively. By $Z_c(W)$ we mean the centre of $H_c(W)$. We denote the center of $\overline{H}_c(W)$ by $Z_c(\overline{H}_c(W))$.

There is an important map on spectrums. Recall from (2.6.1) the injection

$$i : \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow Z_c(W).$$

The dual map on spectra is denoted

$$\gamma : \operatorname{Spec}(Z_c(W)) \twoheadrightarrow \operatorname{Spec}(\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W) = \mathfrak{h}/W \times \mathfrak{h}^*/W.$$

The map γ will allow us to identify the blocks of $\overline{H}_c(W)$ with the baby Verma modules. Eventually this will give a bijection with the partitions of n when $W = S_n$. From now on we write $X_c(W) := \operatorname{Spec} Z_c(W)$. Almost all of the results in this section will depend on the smoothness of $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$. The following lemma guarantees that we can choose c so that this is the case.

Lemma 4.1.1. *For a suitably generic class function c (one which is a complement to finitely many hyperplanes on the set of conjugacy classes of reflections in $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$) the variety $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ is smooth.*

Proof. See [24, Corollary 1.14]. □

From this point on we shall assume c is suitably generic so that $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ is smooth, unless otherwise stated.

By a result of Brown and Gordon [17, Corollary 2.7] the blocks of $\overline{H}_c(W)$ are in bijection with the points of $\gamma^{-1}(0)$. The decomposition of $\overline{H}_c(W)$ can therefore be written as

$$\overline{H}_c(W) = \bigoplus_{j \in \gamma^{-1}(0)} B_j. \tag{4.1.1}$$

The block B_p is a matrix algebra over the ring of functions at the points $p \in \gamma^{-1}(0)$ by [30, p. 7],

$$B_p = \operatorname{Mat}_{|W|}(\mathcal{O}_p). \tag{4.1.2}$$

Here $\mathcal{O}_p = (Z_c(W)/R_+Z_c(W))_p$ is the scheme theoretic fibre of γ at 0 localised at the point p.

The first observation we make is that the blocks of $\overline{H}_c(W)$ are in bijection with the baby Verma modules and hence, the irreducible representations of W.

Proposition 4.1.2. *The blocks of $\overline{H}_c(W)$ are in bijection with the irreducible representations of W.*

Proof. Write

$$\overline{H}_c(W) = \bigoplus_{i \in \gamma^{-1}(0)} B_i,$$

a sum of indecomposable submodules. Since $\Delta(\lambda)$ is a $\overline{H}_c(W)$-module,

$$\Delta(\lambda) = \overline{H}_c(W) \cdot \Delta(\lambda) = \bigoplus_{i \in \gamma^{-1}(0)} B_i \cdot \Delta(\lambda).$$

Since $\Delta(\lambda)$ is indecomposable we must have $B_i \cdot \Delta(\lambda) = 0$ for all $i \neq j$ for some (unique) j.

Since $\mathcal{O}_p$ is a local ring it has a unique simple module. Equation (4.1.2) then implies that B_p also has a unique simple module. If $\Delta(\lambda) = B_p \cdot \Delta(\lambda)$ and $\Delta(\mu) = B_p \cdot \Delta(\mu)$ then the simple module of B_p equals both $L(\lambda)$ and $L(\mu)$. By Lemma 2.3.9, this forces $\lambda = \mu$. □

Combining Proposition 4.1.2 with (4.1.1) we conclude that there is a bijection between the points of $\gamma^{-1}(0)$ and the irreducible representations of W. We will use this bijection to label points in $\gamma^{-1}(0)$ by irreducible representations of W and vice versa.

Using the bijection from Proposition 4.1.2, let B_λ correspond to the block B_p. Consider the following maps, the inclusion map $i : Z_c(W) \to H_c(W)$, the quotient map $q : H_c(W) \to \overline{H}_c(W)$ defined by $q(z) = z + R_+ H_c(W)$ and the projection $\phi_p : \overline{H}_c(W) \to B_p$. Denote by $A(\lambda)$ the image of $Z_c(W)$ under the composition of these maps

$$Z_c(W) \hookrightarrow H_c(W) \twoheadrightarrow \overline{H}_c(W) \twoheadrightarrow B_\lambda. \tag{4.1.3}$$

We will show that $A(\lambda) = \mathcal{O}_p$.

Lemma 4.1.3. *The image of the centre $Z_c(W)$ under the composition of the inclusion and quotient map is equal to the centre of $\overline{H}_c(W)$. That is,*

$$q \circ i(Z_c(W)) = Z_c(\overline{H}_c(W)).$$

Proof. A proof of this can be found in [31, Lemma 2.8] with the condition that the ideal $R_+ Z_c(W)$ is contained in a maximal ideal corresponding to an Azumaya point. By [23, Theorem 1.7] the Azumaya points of $H_c(W)$ are precisely the points in the smooth locus of $X_c(W)$, but we have assumed that $X_c(W)$ is smooth. □

Theorem 4.1.4. *The image of $Z_c(W)$ under the composition of maps (4.1.3) is equal to $\mathcal{O}_p$. In particular $\mathcal{O}_p = A(\lambda)$.*

Proof. By Lemma 4.1.3 the image of $Z_c(W)$ is $Z_c(\overline{H}_c(W))$. By [4, Lemma 4.5] the kernel is $R_+ Z_c(W)$ therefore, $Z_c(W)/R_+ Z_c(W) = Z_c(\overline{H}_c(W))$. Write the block decomposition

$$Z_c(W)/R_+ Z_c(W) = Z_c(\overline{H}_c(W)) = \bigoplus_{i \in \gamma^{-1}(0)} A_i \subset \bigoplus_{i \in \gamma^{-1}(0)} B_i.$$

Hence $A_i = Z(B_i)$. The image of $Z_c(W)/R_+Z_c(W)$ under the map ϕ_p is then A_p.

To localise at the point p we do the following. There is a unique maximal ideal $\mathfrak{m}_p \subset A_p$ that corresponds to the point p and so a maximal ideal $A_1 \oplus A_2 \cdots \oplus \mathfrak{m}_p \oplus \cdots \oplus A_r$ in $Z_c(W)/R_+Z_c(W)$. We make every element not contained in this ideal invertible. Since we had the block decomposition of $Z_c(W)/R_+Z_c(W)$ there is a set of orthogonal idempotents which we shall label $e_1, \cdots, e_n$ so that $A_i = Ae_i$. Therefore the maximal ideal $A_1 \oplus A_2 \cdots \oplus \mathfrak{m}_p \oplus \cdots \oplus A_r$ contains every $e_j \neq e_p$. By localising at p we have made e_p invertible and so for any other e_i we have $e_i = e_i e_p e_p^{-1} = 0$. Therefore,

$$(Z_c(W)/R_+Z_c(W))_p = A_p = \phi_p \circ q \circ i(Z).$$

$\square$

Corollary 4.1.5. *The centre of the block B_λ is $A(\lambda)$.*

Proof. Equation (4.1.2) implies that $\mathcal{O}_\lambda$ is the centre of B_λ. Therefore, Theorem 4.1.4 implies that $A(\lambda)$ is the center of the block. $\square$

In light of the above, our aim is to find an explicit description of $A(\lambda)$ for $\lambda \in \operatorname{Irr} W$. The next theorem is our first step towards this. It shows that $A(\lambda)$ is a tensor product of two algebras which we will describe. Let us fix some notation

$$A(\lambda)^+ := \operatorname{End}_{\overline{H}_c(W)} \Delta(\lambda) \text{ and } A(\lambda)^- := \operatorname{End}_{\overline{H}_c(W)} \Delta^*(\lambda),$$

where $\Delta^*(\lambda) = \overline{H} \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \rtimes W} \lambda$.

Theorem 4.1.6. *Multiplication induces the following isomorphism*

$$A(\lambda)^- \otimes_{\mathbb{C}} A(\lambda)^+ \cong A(\lambda)$$

Proof. See [8, Theorem 8.14]. $\square$

Since $A(\lambda)$ is the centre of the block B_λ, Theorem 4.1.6 allows us to restate our problem. Giving an explicit presentation of $A(\lambda)$ is now equivalent to giving an explicit presentation of the endomorphism rings of the baby Verma modules. Now we make an important remark, which will be proven later in Theorem 5.4.9

Remark 4.1.7. In the case of the wreath product the algebra $A_c(\lambda)^+$ is isomorphic to $A_{\overline{c}}(\lambda^*)^-$ but for a different generic parameter $\overline{c}$ and simple λ^*.

Due to the above remark it suffices to find an explicit presentation of $A(\lambda)^+$. This will allow us to describe the entire centre.

To describe the endomorphism rings of the baby Verma modules we must first understand the connection to the endomorphism rings of the Verma modules. The following theorem tells us two important things. First, that the baby Verma modules are a quotient of the Verma modules. Secondly, it also states that the centre of the rational Cherednik algebra surjects onto the endomorphism rings of the Verma modules.

Theorem 4.1.8. *For all $\lambda \in \operatorname{Irr} W$,*

1. $\Delta(\lambda) = \underline{\Delta}(\lambda)/R_+\underline{\Delta}(\lambda)$.

2. *The map defined by multiplication by elements of $Z_c(W)$ on $\underline{\Delta}(\lambda)$ as a $H_c(W)$-module is a surjection $Z_c(W) \twoheadrightarrow \operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda))$.*

Proof. Statement 1. follows from Theorem 2.5.9. For a proof of 2. see [5, Theorem 1.2]. □

Let us consider the implications of Theorem 4.1.8. The second fact allows us to compose the maps given by inclusion and the surjection

$$\mathbb{C}[\mathfrak{h}]^W \hookrightarrow Z_c(W) \twoheadrightarrow \operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda)).$$

This gives a map of spectra

$$\pi : \operatorname{Spec} \operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda)) \to \operatorname{Spec} \mathbb{C}[\mathfrak{h}]^W = \mathfrak{h}/W. \tag{4.1.4}$$

We will return to this map later, as it will play a pivotal role in realising $A(\lambda)^+$ explicitly.

For now we wish to use the first fact from Theorem 4.1.8 to show that the endomorphism ring of the baby Verma module is a quotient of the endomorphism ring of the Verma module. This is done in Theorem 4.1.15 the next results are required for the proof. Let

$$e = \frac{1}{|W|} \sum_{\sigma \in W} \sigma$$

be the trivial idempotent in $\mathbb{C}W \subset H_c(W)$.

Theorem 4.1.9. *(Satake Isomorphism) There is an isomorphism of algebras $Z_c(W) \cong eH_c(W)e$ given by the map*

$$z \to z \cdot e.$$

Proof. See [23, Theorem 3.1]. □

Lemma 4.1.10. *Let A be a finitely generated algebra and e be an idempotent of A. For any A-module M we have the following isomorphism*

$$eA \otimes_A M \cong eM.$$

Proof. Define a homomorphism $\phi : eA \otimes_A M \to eM$ as follows $\phi(ea \otimes m) = eam$. We shall prove this is an isomorphism. It is clearly surjective and a morphism so we need only prove that it is injective. We prove that the kernel of ϕ is 0. If $\phi(ea \otimes m) = 0$ then $eam = 0$, but note that $ea \otimes m = e^2 a \otimes m = e \otimes eam = e \otimes 0 = 0$. □

The proof of Theorem 4.1.12 requires the following well known fact.

Theorem 4.1.11. *Let R and S be rings, and P be a (R, S)-bimodule and a progenitor for R. Set $Q = Hom_S(P, S)$, which is a (S, R)-bimodule. If $S \cong \mathrm{End}_R(P)$ then there are the following equivalence of categories*

1. $P \otimes_S - : S-\mathrm{mod} \to R-\mathrm{mod}$.

2. $Q \otimes_R - : R-\mathrm{mod} \to S-\mathrm{mod}$.

Proof. See [50, Theorem 46.4]. □

Theorem 4.1.12. *The functor*

$$e : H_c(W)-\mathrm{mod} \to eH_c(W)e-\mathrm{mod}$$

is an equivalence of categories if and only if $e \cdot M = 0$ implies that $M = 0$ for all $M \in H_c(W)-\mathrm{mod}$.

Proof. Theorem 4.1.11 (2) implies that if $H_c(W)e$ is a progenitor then

$$\mathrm{Hom}_{eH_c(W)e}(H_c(W)e, eH_c(W)e) \otimes_{H_c(W)} - : H_c(W)-\mathrm{mod} \to eH_c(W)e-\mathrm{mod}$$

is an equivalence. It is known [7, Theorem 1.6.1] that

$$\mathrm{Hom}_{eH_c(W)e}(H_c(W)e, eH_c(W)e) \cong eH_c(W).$$

Therefore if $H_c(W)e$ is a progenitor then $eH_c(W) \otimes_{H_c(W)} - : H_c(W)-\mathrm{mod} \to eH_c(W)e-\mathrm{mod}$ is an equivalence. By Lemma 4.1.10 this implies that

$$e : H_c(W)-\mathrm{mod} \to eH_c(W)e-\mathrm{mod}$$

is an equivalence if $H_c(W)e$ is a progenitor. Now all we need to do is prove that $H_c(W)e$ is a progenitor if and only if $e \cdot M = 0$ implies that $M = 0$ for all $M \in H_c(W)$-mod. Since e is an idempotent, $H_c(W)e$ is a direct summand of $H_c(W)$ and hence always projective. We need only show that it is a generator, which is equivalent to

$$H_c(W)e \otimes_{eH_c(W)e} eM \cong M$$

for all $M \in H_c(W)$-mod. If $H_c(W)e$ is a generator and $e \cdot M = 0$ then

$$0 = H_c(W)e \otimes_{eH_c(W)e} eM \cong M$$

hence, $M = 0$. Let us now prove the converse. There is always a map $\phi : H_c(W)e \otimes_{eH_c(W)e} eM \to M$ given by multiplication. Hence for any module M we have an exact sequence

$$0 \to \operatorname{Ker}\phi \to H_c(W)e \otimes_{eH_c(W)e} eM \to M \to \operatorname{CoKer}\phi \to 0.$$

The functor e is exact and so we have an exact sequence

$$0 \to e\operatorname{Ker}\phi \to eH_c(W)e \otimes_{eH_c(W)e} eM \to eM \to e\operatorname{CoKer}\phi \to 0,$$

clearly $eH_c(W)e \otimes_{eH_c(W)e} eM \cong eM$ and so $e\operatorname{Ker}\phi = 0$ and $e\operatorname{CoKer}\phi = 0$. But then by assumption $\operatorname{Ker}\phi = 0$ and $\operatorname{CoKer}\phi = 0$. Therefore $H_c(W)e \otimes_{eH_c(W)e} eM \cong M$ and $H_c(W)e$ is a generator. □

Theorem 4.1.13. *The spherical Cherednik algebra $eH_c(W)e$ is Morita equivalent to the rational Cherednik algebra $H_c(W)$ if and only if $e \cdot M = 0$ implies that $M = 0$ for all $M \in H_c(W)$-mod.*

Proof. Follows from Theorem 4.1.12. □

Lemma 4.1.14. *The $Z_c(W)$-module $e\underline{\Delta}(\lambda)$ is cyclic.*

Proof. In [5, Theorem 4.1] it is shown that $e\underline{\Delta}(\lambda)$ is a cyclic $\operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda))$-module. We also know from Theorem 4.1.8 that $Z_c(W)$ surjects onto $\operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda))$, hence $e\underline{\Delta}(\lambda)$ is a cyclic $Z_c(W)$-module. □

We are now in position to prove one of the major theorems of this section.

Theorem 4.1.15. *We have the following isomorphism*

$$\operatorname{End}_{\overline{H}_c(W)}(\Delta(\lambda)) \cong \operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda))/\mathbb{C}[\mathfrak{h}]^W_+ \operatorname{End}_{H_c(W)}(\underline{\Delta}(\lambda)).$$

Proof. For brevity, write $H = H_c(W)$. We make use of Theorem 4.1.13, that the spherical Cherednik algebra eHe is Morita equivalent to H, hence $eHe-\mathrm{mod} \cong H-\mathrm{mod}$. Given an endomorphism $f \in \operatorname{End}_H(\underline{\Delta}(\lambda))$ we have an endomorphism

$$\overline{f} \in \operatorname{End}_H(\underline{\Delta}(\lambda)/\mathbb{C}[\mathfrak{h}]^W_+ \underline{\Delta}(\lambda))$$

where $\overline{f}(m + \mathbb{C}[\mathfrak{h}]^W_+) = f(m) + \mathbb{C}[\mathfrak{h}]^W_+ \underline{\Delta}(\lambda)$. In this way we have a map

$$\phi : \operatorname{End}_H(\underline{\Delta}(\lambda)) \to \operatorname{End}_H(\underline{\Delta}(\lambda)/\mathbb{C}[\mathfrak{h}]^W_+ \underline{\Delta}(\lambda)).$$

We wish to show that $\mathrm{Ker}\phi = R_+\mathrm{End}_H(\underline{\Delta}(\lambda))$. Since eHe is Morita equivalent to H we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{End}_H(\underline{\Delta}(\lambda)) \arrow[r, "\phi"] \arrow[d, "\cong"'] & \mathrm{End}_H(\underline{\Delta}(\lambda)/\mathbb{C}[\mathfrak{h}]^W_+\underline{\Delta}(\lambda)) \arrow[d, "\cong"] \\
\mathrm{End}_{eHe}(e\underline{\Delta}(\lambda)) \arrow[r] & \mathrm{End}_{eHe}(e\underline{\Delta}(\lambda)/e\mathbb{C}[\mathfrak{h}]^W_+e\underline{\Delta}(\lambda))
\end{tikzcd}$$

By Lemma 4.1.14, $e\underline{\Delta}(\lambda)$ is a cyclic eHe-module. Hence $e\underline{\Delta}(\lambda) \cong eHe/I$, where I is the annihilator of the generator. Therefore,

$$\mathrm{End}_{eHe}(e\underline{\Delta}(\lambda)) \cong \mathrm{End}_{eHe}(eHe/I) \cong eHe/I.$$

Similarly,

$$\mathrm{End}_{eHe}(e\underline{\Delta}(\lambda)/e\mathbb{C}[\mathfrak{h}]^W_+e\underline{\Delta}(\lambda)) \cong \mathrm{End}_{eHe}((eHe/I)/(e\mathbb{C}[\mathfrak{h}]^W_+eHe/I)),$$

and

$$\mathrm{End}_{eHe}((eHe/I)/(e\mathbb{C}[\mathfrak{h}]^W_+eHe/I)) \cong eHe/\mathbb{C}[\mathfrak{h}]^W_+eHe + I.$$

Hence we have a new commutative diagram

$$\begin{tikzcd}
\mathrm{End}_H(\underline{\Delta}(\lambda)) \arrow[r] \arrow[d, "\cong"'] & \mathrm{End}_H(\underline{\Delta}(\lambda)/\mathbb{C}[\mathfrak{h}]^W_+\underline{\Delta}(\lambda)) \arrow[d, "\cong"] \\
\mathrm{End}_{eHe}(e\underline{\Delta}(\lambda)) \arrow[r] \arrow[d, "\cong"'] & \mathrm{End}_{eHe}(e\underline{\Delta}(\lambda)/e\mathbb{C}[\mathfrak{h}]^W_+e\underline{\Delta}(\lambda)) \arrow[d, "\cong"] \\
eHe/I \arrow[r] & eHe/\mathbb{C}[\mathfrak{h}]^W_+eHe + I
\end{tikzcd}.$$

It is easy to see from the diagram that the kernel of the bottom map is $\mathbb{C}[\mathfrak{h}]^W_+eHe + I$. Then, via a simple diagram chasing argument, we see that

$$\mathrm{Ker}\phi = \mathbb{C}[\mathfrak{h}]^W_+\mathrm{End}_H(\underline{\Delta}(\lambda)).$$

Hence

$$\mathrm{End}_H(\underline{\Delta}(\lambda)/\mathbb{C}[\mathfrak{h}]^W_+\underline{\Delta}(\lambda)) \cong \mathrm{End}_H(\underline{\Delta}(\lambda))/\mathbb{C}[\mathfrak{h}]^W_+\mathrm{End}_H\underline{\Delta}(\lambda)),$$

and by Theorem 4.1.8,

$$\mathrm{End}_{\overline{H}}(\Delta(\lambda)) \cong \mathrm{End}_H(\underline{\Delta}(\lambda)/\mathbb{C}[\mathfrak{h}]^W_+\underline{\Delta}(\lambda)).$$

Therefore

$$\mathrm{End}_{\overline{H}}(\Delta(\lambda)) \cong \mathrm{End}_H(\underline{\Delta}(\lambda))/\mathbb{C}[\mathfrak{h}]^W_+\mathrm{End}_H\underline{\Delta}(\lambda)).$$

□

Recall the map (4.1.4)

$$\pi : \mathrm{Spec}\,\mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda)) \to \mathrm{Spec}\,\mathbb{C}[\mathfrak{h}]^W = \mathfrak{h}/W.$$

Applying Theorem 4.1.15 proves the following important corollary.

Corollary 4.1.16. *There is an isomorphism of algebras* $\mathbb{C}[\pi^{-1}(0)] \cong A(\lambda)^+$.

Proof. From the (4.1.4) we see that

$$\mathbb{C}[\pi^{-1}(0)] = \mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda))/\mathbb{C}[\mathfrak{h}]^W_+\mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda)).$$

Hence, by Theorem 4.1.15,

$$\mathbb{C}[\pi^{-1}(0)] \cong \mathrm{End}_{\overline{H}_c(W)}(\Delta(\lambda)) = A(\lambda)^+.$$

□

So far everything we have done holds for any complex reflection group W, as long as $X_c(W)$ is smooth. To progress further it is now necessary to use facts that only hold in the case $W = S_n$. Corollary 4.1.16 reduces the problem of understanding the blocks of $\overline{H}_c(W)$ to understanding the scheme theoretic fiber of the map π. This can be done using Theorem 4.1.17. Recall the definition of the Wronski map from Section 2.7.

Theorem 4.1.17. *There is an isomorphism of varieties*

$$\pi^{-1}(0) \cong \mathrm{Wr}_\lambda^{-1}(0).$$

where π *is the map on spectra* $\pi : \mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\lambda) \to \mathfrak{h}/W$ *induced by the inclusion map* $\mathbb{C}[\mathfrak{h}]^W \hookrightarrow \mathrm{End}\underline{\Delta}(\lambda)$.

Proof. See [6, Proposition 6.4]. □

Theorem 4.1.17 and Corollary 4.1.16 leads us to conclude with the final result of this section.

Theorem 4.1.18. *There is an isomorphism of algebras*

$$A(\lambda)^+ \cong \mathbb{C}[\mathrm{Wr}_\lambda^{-1}(0)].$$

Proof. Theorem 4.1.17 states $\pi^{-1}(0) \cong \mathrm{Wr}_\lambda^{-1}(0)$ hence

$$\mathbb{C}[\pi^{-1}(0)] = \mathbb{C}[\mathrm{Wr}_\lambda^{-1}(0)].$$

By Corollary 4.1.16 we have that

$$A(\lambda)^+ \cong \mathbb{C}[\pi^{-1}(0)],$$

hence

$$A(\lambda)^+ = \mathbb{C}[\mathrm{Wr}_\lambda^{-1}(0)].$$

□

Theorem 4.1.18 allows us to explicitly calculate the algebras $A(\lambda)^+$ using the Wronski map. We will explain exactly how this is done in Section 5.1. This is all we required for the symmetric group case, and so we now move on to the more challenging wreath product groups.

§ 4.2 | Wreath Products

In the last section we proved that the algebras $A(\lambda)^+$ are isomorphic to the functions on the scheme-theoretic fibre of the Wronskian at 0. As we shall see, this will allow us to explicitly describe $A(\lambda)^+$ in terms of generators and relations. We wish to extend this so that we can give an explicit presentation of $A(\lambda)^+$ when $W = S_n \wr \mathbb{Z}/\ell\mathbb{Z}$. The difficulty in proving the wreath product case is that we lack an appropriate version of Theorem 4.1.17 and so we have no direct link to the Wronskian. We do, however, have the following isomorphism [12, Theorem 4.21] due to Bonnafe and Maksimau. Recall that $X_c(W) := \operatorname{Spec} Z_c(W)$.

Theorem 4.2.1. *Let $\sigma \in \mathbb{Z}/\ell\mathbb{Z} \subset \mathbb{C}^\times$ be a root of unity and let $X_c(W)$ be smooth. Then $X_c(W)^\sigma$, the subscheme of $X_c(W)$ fixed by the action of σ, is smooth. For each irreducible component $X_0 \subset X_c(W)^\sigma$ there exists a reflection subquotient $W' \subset W$ and conjugacy function $\overline{c}$ such that there is a $\mathbb{C}^\times$-equivariant isomorphism of varieties*

$$X_0 \cong X_{\overline{c}}(W'). \tag{4.2.1}$$

Since the parameters c, $\overline{c}$ etc will always be generic and our results independent of the previous result, we write c for $\overline{c}$ in the remainder of this section.

Let us briefly say something about the irreducible representations of $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$. There is a bijection between the ℓ-multipartitions of n and the irreducible representations of $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ [44, p. 221]. Then, by Theorem 2.4.10, it makes sense to denote an irreducible representation of $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ as $\text{quo}_\ell(\lambda)$, for a partition $\lambda \vdash n\ell$ with trivial ℓ-core. The reason why we choose to label the irreducible representations by the ℓ-quotients will become clearer later in the section, particularly in light of Lemma 4.2.20. Note that when $\ell = 1$ we have $\text{quo}_\ell(\lambda) = \lambda$.

While we may lack a direct link to the Wronskian we do gain an extra map via (4.1.4). The two maps

$$\pi_{n,\ell} : \operatorname{Spec}\operatorname{End}_{H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})}\underline{\Delta}(\text{quo}_\ell(\lambda)) \to \mathbb{C}^n/S_n \wr \mathbb{Z}/\ell\mathbb{Z},$$

and

$$\pi_{n\ell} : (\operatorname{Spec}\operatorname{End}_{H_c(S_{n\ell})}(\underline{\Delta}(\lambda)))^{\mathbb{Z}/\ell\mathbb{Z}} \to (\mathbb{C}^{n\ell}/S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}$$

are significant as we can still use Corollary 4.1.16. We will embed $\operatorname{Spec}\operatorname{End}\underline{\Delta}(\text{quo}_\ell(\lambda))$ into $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ then, using (4.2.1), realise it as a subvariety of $X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}$ and apply Theorem 4.1.18. Note that throughout this section when we prove statements for the wreath product it implies the symmetric group case as well if we let $\ell = 1$. From this point on we will write $\operatorname{End}_{H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})}\underline{\Delta}(\text{quo}_\ell(\lambda))$ as $\operatorname{End}\underline{\Delta}(\text{quo}_\ell(\lambda))$ for brevity.

We will show $\operatorname{Spec}\operatorname{End}\underline{\Delta}(\text{quo}_\ell(\lambda))$ is a subvariety of $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ by showing it is equal to a specific attracting set, which we now define.

Definition 4.2.2. Let X be an affine scheme over $\mathbb{C}$ with a $\mathbb{C}^\times$-action and assume that $X^{\mathbb{C}^\times}$ is finite. An *attracting set* for the $\mathbb{C}^\times$-action is defined to be $\Omega_p := \{x \in X \mid \lim_{t\to\infty} t \cdot x = x_p\}$ where x_p is a fixed point.

We will show that $\operatorname{Spec}\operatorname{End}\underline{\Delta}(\text{quo}_\ell(\lambda))$ can be identified with an attracting set in $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ in two steps. First we prove that

$$\operatorname{Spec}\operatorname{End}\underline{\Delta}(\text{quo}_\ell(\lambda)) = \operatorname{Supp}_{Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})}(\underline{\Delta}(\text{quo}_\ell(\lambda)). \tag{4.2.2}$$

Then we show that

$$\operatorname{Supp}_{Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})}(\underline{\Delta}(\text{quo}_\ell(\lambda)) = \Omega_{\text{quo}_\ell(\lambda)}. \tag{4.2.3}$$

The second equality requires significantly more work and we will need to examine $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ in more detail. Once again we will shorten $Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ to Z_c for brevity.

The following two lemmata are needed to prove the first equality.

Lemma 4.2.3. *We have the following equality of supports*

$$\operatorname{Supp}_{Z_c}\underline{\Delta}(\text{quo}_\ell(\lambda)) = \operatorname{Supp}_{Z_c} e\underline{\Delta}(\text{quo}_\ell(\lambda)).$$

Proof. Since $e\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \subset \underline{\Delta}(\mathrm{quo}_\ell(\lambda))$ we have $\mathrm{ann}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \subset \mathrm{ann}_{Z_c} e\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$. Hence

$$\mathrm{Supp}_{Z_c} e\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \subset \mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)).$$

So we need only show the reverse inclusion. Consider $\mathfrak{p} \in \mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$. Then

$$\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \otimes_{Z_c} (Z_c)_\mathfrak{p} \neq 0.$$

Therefore, to prove the reverse inclusion all we need show is that $e\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \otimes_{Z_c} (Z_c)_\mathfrak{p} \neq 0$. By Theorem 4.1.12, the functor

$$e \cdot - : H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}))\text{−mod} \to eH_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}))e\text{−mod}$$

is an equivalence and hence maps the non-zero objects in $H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}))$−mod to non-zero objects in $eH_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}))e$−mod. Therefore

$$\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \otimes_{Z_c} (Z_c)_\mathfrak{p} \neq 0$$

if and only if

$$e\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \otimes_{Z_c} (Z_c)_\mathfrak{p} \neq 0.$$

$\square$

Lemma 4.2.4. *Let R be a commutative ring. Given an ideal $I \subset R$,* $\mathrm{Supp}(R/I) = \mathrm{Spec}\,(R/I)$*.*

Proof. First we note that there is a bijection between prime ideals of R/I and prime ideals of R that contain I. It is also known that the support of R/I consists of all prime ideals that contain the annihilator of R/I. But clearly the annihilator is just I. $\square$

We can now prove the first equality.

Theorem 4.2.5. *We have the following equality of varieties*

$$\mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)).$$

Proof. First we note that $e\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$ is a cyclic module Z_c-module by Lemma 4.1.14. Therefore

$$Z_c/I \cong e\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$$

as left Z_c-modules for some ideal I. Since $H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ is Morita equivalent to $eH_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})e$ and $eH_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})e \cong Z_c$ we have

$$\mathrm{End}_{H_c(S_n\wr\mathbb{Z}/\ell\mathbb{Z})}(\underline{\Delta}(\mathrm{quo}_\ell(\lambda))) \cong \mathrm{End}_{eH_c(S_n\wr\mathbb{Z}/\ell\mathbb{Z})e}e\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \cong \mathrm{End}_{Z_c}Z_c/I \cong Z_c/I.$$

Therefore

$$\operatorname{Spec}\operatorname{End}_{H_c(S_n\wr\mathbb{Z}/\ell\mathbb{Z})}(\underline{\Delta}(\mathrm{quo}_\ell(\lambda))) \cong \operatorname{Spec} Z_c/I$$

and using Lemma 4.2.4 we see that $\operatorname{Spec} Z_c/I \cong \operatorname{Supp}_{Z_c}(e\underline{\Delta}(\mathrm{quo}_\ell(\lambda)))$. Lemma 4.2.3 says $\operatorname{Supp}_{Z_c}(e\underline{\Delta}(\mathrm{quo}_\ell(\lambda))) \cong \operatorname{Supp}_{Z_c}(\underline{\Delta}(\mathrm{quo}_\ell(\lambda)))$. Hence

$$\operatorname{Spec}\operatorname{End}_{H_c(S_n\wr\mathbb{Z}/\ell\mathbb{Z})}(\underline{\Delta}(\mathrm{quo}_\ell(\lambda))) = \operatorname{Supp}_{Z_c}(\underline{\Delta}(\mathrm{quo}_\ell(\lambda))).$$

□

The equality (4.2.3) is significantly harder to prove and we require several technical results. We note the following.

Theorem 4.2.6. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let S be the associated graded of R with respect to the $\mathfrak{m}$-adic filtration. Then R is regular if and only if S is a polynomial ring.*

Proof. See [42, Theorem 13.4]. □

Consider the following situation. Let X be an affine algebraic variety over $\mathbb{C}$ that admits a $\mathbb{C}^\times$-action. This induces a grading on $\mathbb{C}[X]$. Also note that $\mathbb{Z}/\ell\mathbb{Z} \subset \mathbb{C}^\times$ by identifying the cyclic group of order ℓ with the ℓ^{th} roots of unity. Recall that the fixed point locus is defined as

$$X^{\mathbb{Z}/\ell\mathbb{Z}} = \{x \in X \mid g\cdot x = x \ \forall g \in \mathbb{Z}/\ell\mathbb{Z}\}$$

which can be equivalently defined as

$$\operatorname{Spec}\left(\frac{\mathbb{C}[X]}{\langle f - g\cdot f \mid g \in \mathbb{Z}/\ell\mathbb{Z} \text{ and } f \in \mathbb{C}[X]\rangle}\right).$$

Proposition 4.2.7. *Fix a $\mathbb{Z}/\ell\mathbb{Z}$-stable subspace $V \subset \mathbb{C}[X]$ such that $\mathbb{C}[X]$ is generated by V. The subspace V decomposes into $V^{\mathbb{Z}/\ell\mathbb{Z}} \oplus V_{\mathbb{Z}/\ell\mathbb{Z}}$, where $V^{\mathbb{Z}/\ell\mathbb{Z}}$ is the invariant subspace under the action of $\mathbb{Z}/\ell\mathbb{Z}$ and $V_{\mathbb{Z}/\ell\mathbb{Z}}$ is its $\mathbb{Z}/\ell\mathbb{Z}$-stable complement. Then*

$$X^{\mathbb{Z}/\ell\mathbb{Z}} = \operatorname{Spec}\left(\frac{\mathbb{C}[X]}{\langle V_{\mathbb{Z}/\ell\mathbb{Z}}\rangle}\right).$$

Proof. We wish to show that $\langle V_{\mathbb{Z}/\ell\mathbb{Z}}\rangle = \langle\{f - g\cdot f \mid g \in \mathbb{Z}/\ell\mathbb{Z},\ f \in \mathbb{C}[X]\}\rangle$. We first show that $\langle V_{\mathbb{Z}/\ell\mathbb{Z}}\rangle \subset \langle\{f - g\cdot f \mid g \in \mathbb{Z}/\ell\mathbb{Z},\ f \in \mathbb{C}[X]\}\rangle$. Fix a basis of $V_{\mathbb{Z}/\ell\mathbb{Z}}$, $\{x_1, x_2, \cdots, x_k\}$ and generator $s \in \mathbb{Z}/\ell\mathbb{Z}$ such that $s\cdot x_i = w^{a_i}x_i$ where w is a primitive ℓ^{th} root of unity and a_i is an integer. Since x_i is not a fixed point there is a $g \in \mathbb{Z}/\ell\mathbb{Z}$ such that $g\cdot x_i = \mu x_i$ for some scalar $\mu \neq 1$. Then consider the function $(1-\mu)^{-1}(x_i - g\cdot x_i)$. We have

$$(1-\mu)^{-1}(x_i - g\cdot x_i) = (1-\mu)^{-1}(x_i - \mu x_i) = (1-\mu)^{-1}(1-\mu)x_i = x_i.$$

Hence $x_i \in \langle \{f - g \cdot f \,|\, g \in \mathbb{Z}/\ell\mathbb{Z},\ f \in \mathbb{C}[X]\} \rangle$.

Now we show $\{f - g \cdot f \,|\, g \in \mathbb{Z}/\ell\mathbb{Z},\ f \in \mathbb{C}[X]\} \subset \langle V_{\mathbb{Z}/\ell\mathbb{Z}} \rangle$. If $f - g \cdot f \neq 0$ then without loss of generality f is a monomial and there exists $1 \leq i \leq k$ such that x_i divides f. Then x_i must also divide $g \cdot f$ as g simply scales x_i therefore $f - g \cdot f \in \langle V_{\mathbb{Z}/\ell\mathbb{Z}} \rangle$. $\square$

Let

$$\mathbb{C}[X](0) := \frac{\mathbb{C}[X]}{\langle \mathbb{C}[X]_{\neq 0} \rangle}.$$

The next lemma improves the observations of Proposition 4.2.7, stating that the fixed point subscheme can be written as follows

$$X^{\mathbb{Z}/\ell\mathbb{Z}} = \operatorname{Spec} \mathbb{C}[X](0).$$

Lemma 4.2.8. *Assume that X is smooth. Then $X^{\mathbb{Z}/\ell\mathbb{Z}} = \operatorname{Spec} \mathbb{C}[X](0)$ is smooth. In particular, $\mathbb{C}[X](0)$ is reduced.*

Proof. We first show equality of sets

$$X^{\mathbb{Z}/\ell\mathbb{Z}} = \operatorname{Spec} \mathbb{C}[X](0).$$

Let $f \in \mathbb{C}[X]$ be homogeneous of degree d, $g \in \mathbb{Z}/\ell\mathbb{Z}$ and $p \in X^{\mathbb{Z}/\ell\mathbb{Z}}$ then we have

$$f(p) = f(g^{-1} \cdot p) = (g \cdot f)(p) = g^d f(p)$$

hence $f(p) = 0$ if $d \neq 0 \bmod \ell$. Therefore $\langle \mathbb{C}[X]_{\neq 0} \rangle$ is contained in the maximal ideal defining p and $X^{\mathbb{Z}/\ell\mathbb{Z}} \subset \operatorname{Spec} \mathbb{C}[X](0)$.

Conversely $\mathbb{Z}/\ell\mathbb{Z}$ acts trivially on $\mathbb{C}[X](0)$ and so every point in $\operatorname{Spec} \mathbb{C}[X](0)$ is fixed by $\mathbb{Z}/\ell\mathbb{Z}$. Hence $\operatorname{Spec} \mathbb{C}[X](0) \subset X^{\mathbb{Z}/\ell\mathbb{Z}}$.

It remains to show that $\operatorname{Spec} \mathbb{C}[X](0)$ is reduced. We must show that the localisation of $\mathbb{C}[X](0)$ at each point is regular. By Theorem 4.2.6 it is enough to show that the tangent cone of $\operatorname{Spec} \mathbb{C}[X](0)$ at a fixed point $p \in X^{\mathbb{Z}/\ell\mathbb{Z}}$ is a polynomial ring. Since X is regular at p the tangent cone at p of X is equal to $V := T_p(X)$ as a $\mathbb{Z}/\ell\mathbb{Z}$-module. By [27, Theorem 5.2] the tangent cone of $X^{\mathbb{Z}/\ell\mathbb{Z}}$ at a point p is equal to $V^{\mathbb{Z}/\ell\mathbb{Z}}$. It is then clear that

$$V^{\mathbb{Z}/\ell\mathbb{Z}} = \operatorname{Spec} \mathbb{C}[V](0)$$

is affine space and $\mathbb{C}[V](0)$ is a polynomial ring. $\square$

We need two powerful theorems, Theorem 4.2.12 and Theorem 4.2.13. These theorems require the following definitions.

Definition 4.2.9. Let $x \in X$ be a point in a scheme X over $\mathbb{C}$, and let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathcal{O}_x$. The tangent space at x, denoted $T_x(X)$, is the dual of the complex vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$

Definition 4.2.10. Let G be an algebraic torus, then $G \cong \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times = (\mathbb{C}^\times)^n$ for some positive integer n. Any G-module can be written as a direct sum of one dimensional G-modules. If V is a G-module, then we can find a basis $\{v_i\}$ of V such that

$$(g_1 \cdots g_n) \cdot v_i = g_1^{s_{i1}} \cdots g_n^{s_{in}} v_i \text{ for } (g_1 \cdots g_n) \in G.$$

The module V is positive (respectively negative) if

(a) $s_{ij} \geq 0$ (respectively $s_{ij} \leq 0$) for all i, j.

(b) For every $i \in I$ there exists j such that $s_{ij} \neq 0$.

The module is non-negative (respectively non-positive) if (a) is satisfied. The module is fully definite (respectively definite) if there exists an isomorphism $G \cong \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ such that the module is positive (respectively non-negative).

Definition 4.2.11. Let $\eta : G \times X \to X$ be an action of a torus on X and let $a \in X^G$ be a closed point. The action of η on a is fully definite (respectively definite) if the G-module $T_a(X)$ is fully definite (respectively definite).

Theorem 4.2.12. *Let X be irreducible and reduced. Let G be an algebraic torus. If the action of G on X is definite at $a \in X^G$ then X^G is irreducible*

Proof. See [11, Theorem 2.3]. □

Theorem 4.2.13. *Let G be an algebraic torus. Let the action of G on X be definite at a. If X is irreducible then there exists an open G-invariant neighbourhood U of a which is G-isomorphic to $(U \cap X^G) \times V$, where V is a finite-dimensional (fully definite) G-module and the action of G on $(U \cap X^G) \times V$ is induced by the trivial action of G on $U \cap X^G$ and the linear action on V (determined by the given structure of a G-module).*

Proof. See [11, Theorem 2.5]. □

More specifically, in the proof of Theorem 4.2.13 the vector space V is defined to be the G-module complement of $T_a(X^G)$ in $T_a(X)$. In our case X^G is a finite set and so $T_a(X^G)$ is zero and $V = T_a(X)$.

The following lemma not only tells us that the irreducible components are the attracting sets but also that the attracting sets are equal to their own tangent space at the fixed point. We take $G = \mathbb{C}^\times$.

Lemma 4.2.14. *Assume $X^{\mathbb{C}^\times}$ is finite and non-empty and that $\lim_{t\to\infty} t \cdot x$ exists for all $x \in X$. If the action of $\mathbb{C}^\times$ is definite at each fixed point then*

1.
$$X = \bigsqcup_{p \in X^{\mathbb{C}^\times}} \Omega_p,$$
where Ω_p is the attracting set of p. The sets Ω_p are the irreducible components of the space X.

2. *$\Omega_p \cong T_p(\Omega_p)$ as varieties.*

Proof. 1.) Since $\lim_{t\to\infty} t \cdot x$ exists for each $x \in X$ and limits are unique it follows that X is a disjoint union of the sets Ω_p. To see that the sets Ω_p are the irreducible components note that for an arbitrary irreducible component L we must have that $L^{\mathbb{C}^\times}$ contains a single point. This is because of Theorem 4.2.12, which states that if L is irreducible then so is $L^{\mathbb{C}^\times}$, but this is clearly not the case if it consists of more than one fixed point. Furthermore it must contain at least one point, as if L is an irreducible component it equals its closure and the closure of any non-empty $\mathbb{C}^\times$-stable subset of X contains some fixed point.

If $L^{\mathbb{C}^\times} = \{p\}$ then the fact that L is closed implies that $\lim_{t\to\infty} t \cdot x = p$ for all $x \in L$. Hence $L \subset \Omega_p$. Conversely if $x \in \Omega_p$ then $\overline{\mathbb{C}^\times \cdot x}$ is an irreducible subvariety containing x. Since X is smooth, L is a connected component of X. So $p \in \overline{\mathbb{C}^\times \cdot x} \cap L \neq \emptyset$ implies $\overline{\mathbb{C}^\times \cdot x} \subset L$ and hence $\Omega_p \subset L$.

2.) Since p is a unique fixed point of Ω_p we apply Theorem 4.2.13 to Ω_p to conclude that there exists an open neighbourhood $U \subset \Omega_p$ containing p such that $U \cong (U \cap X^{\mathbb{C}^\times}) \times V$. By the hypothesis we have $\Omega_p^{\mathbb{C}^\times} = p$ and the discussion above states that $V = T_p(\Omega_p)$, hence $U \cong \{p\} \times T_p(\Omega_p) \cong T_p(\Omega_p)$. Now we show that $\Omega_p \subset U$. Let $x \in \Omega_p$. Then $\lim_{t\to\infty} t \cdot x = p$. Since U is an open neighbourhood of p we must have that $t \cdot x \in U$ for some t. Recall that U is $\mathbb{C}^\times$-invariant and so if $t \cdot x \in U$ then we must have $x \in U$. Hence $U = \Omega_p$ and $\Omega_p \cong T_p(\Omega_p)$. $\square$

The following is a technical lemma we will require later.

Lemma 4.2.15. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension n and I an ideal in A. If, for each positive integer q, there exists a (regular) system of parameters $t_1, \dots, t_n$*

such that I is generated by $t_1, \dots, t_s$ modulo $\mathfrak{m}^q$ then there exists a (regular) system of parameters $v_1, \dots, v_n$ such that I is generated by $v_1, \dots, v_s$.

Proof. See [38, Lemma 2.1]. □

Proposition 4.2.16. *Assume that Y is a smooth, affine scheme over $\mathbb{C}$ with $Y^{\mathbb{C}^\times}$ finite. Let $\mathbb{C}[Y] = A$ and*

$$Y^+ = \left\{ y \in Y \,\middle|\, \lim_{t\to\infty} t \cdot y \text{ exists} \right\}.$$

Then:

(a) Y^+ is a closed subset of Y, defined by the vanishing of the (reduced) ideal $\langle A_{<0} \rangle$.

(b) $Y^+ = \bigsqcup_{p \in Y^{\mathbb{C}^\times}} \Omega_p$, where $\Omega_p \cong (T_pY)_{>0}$ as $\mathbb{C}^\times$-varieties.

(c) If $I_{A_0}(p) = \{a \in A_0 \,|\, a(p) = 0\}$ then Ω_p is the closed subset of Y defined by the reduced ideal $\langle A_{<0}, I_{A_0}(p) \rangle$.

Proof. (a) Similarly to the proof of Lemma 4.2.8 we have that $\langle A_{<0} \rangle$ vanishes on Y^+. Indeed if $f \in A_{<0}$ is homogeneous of degree $r < 0$, $y \in Y^+$ and $t \in \mathbb{C}^\times$ then

$$f(t \cdot y) = t^{-r} f(y)$$

and

$$\lim_{t\to\infty} f(t \cdot y) = \lim_{t\to\infty} t^{-r} f(y).$$

If $f(y) \neq 0$ then the limit of $f(t \cdot y)$ does not exist. This is a contradiction. Therefore $\langle A_{<0} \rangle$ vanishes on Y^+. Clearly $\operatorname{Spec}(A/\langle A_{<0} \rangle) \subset Y$ hence $\operatorname{Spec}(A/\langle A_{<0} \rangle)^{\mathbb{C}^\times} \subset Y^{\mathbb{C}^\times}$ is finite. The ring $A/\langle A_{<0} \rangle$ is non-negatively graded, so $\operatorname{Spec}(A/\langle A_{<0} \rangle)^+ = \operatorname{Spec}(A/\langle A_{<0} \rangle)$. By Lemma 4.2.14 we see that $\operatorname{Spec}(A/\langle A_{<0} \rangle)$ is a disjoint union of attracting sets, in particular all of its limits exist and so it is the the vanishing ideal defining Y^+.

Now we must check that the ideal $\langle A_{<0} \rangle$ is reduced. Since it is homogeneous, the radical of $A/\langle A_{<0} \rangle$ is homogeneous. Hence, if it is not zero there exists a fixed point that is not reduced. Therefore, it suffices to show that for every $p \in (\operatorname{Spec} A/\langle A_{<0} \rangle)^{\mathbb{C}^\times}$, the local ring $(A/\langle A_{<0} \rangle)_p$ is reduced.

Let $\mathfrak{m}$ denote the maximal ideal corresponding to p, so $\mathfrak{m}$ is stable under $\mathbb{C}^\times$. We show that $A_\mathfrak{m}/A_{<0}A_\mathfrak{m}$ is a regular local ring. Let $(T_p^*Y)_{<0} \subset \mathfrak{m}/\mathfrak{m}^2$ be the subspace spanned by all negative weight vectors and choose $N \subset \mathfrak{m}$ a homogeneous vector space lift of $(T_p^*Y)_{<0}$. We fix another homogeneous vector space lift $V \subset \mathfrak{m}$ of $\mathfrak{m}/\mathfrak{m}^2$ that contains N. A basis of V is a regular system of parameters for $A_\mathfrak{m}$. If $\mathfrak{n}$ is the augmentation ideal of

$SymV$ then the map $SymV \to A$ induces graded isomorphisms $\phi_q : SymV/\mathfrak{n}^q \to A/\mathfrak{m}^q$ and $A/\mathfrak{m}^q = A_\mathfrak{m}/(\mathfrak{m}A_\mathfrak{m})^q$ for all $q \geq 1$ as A is regular at $\mathfrak{m}$. Since $\mathbb{C}^\times$ acts semisimply on A, the quotient $A \twoheadrightarrow A/\mathfrak{m}^q$ induces surjections $A_i \twoheadrightarrow (A/\mathfrak{m}^q)_i$ for all i. Hence $(A/\mathfrak{m}^q)_{<0} = (A_{<0} + \mathfrak{m}^q)/\mathfrak{m}^q$. Therefore ϕ_q restricts to

$$\frac{SymV_{<0} + \mathfrak{n}^q}{\mathfrak{n}^q} = (SymV/\mathfrak{n}^q)_{<0} \cong (A/\mathfrak{m}^q)_{<0} = \frac{A_{<0} + \mathfrak{m}^q}{\mathfrak{m}^q}.$$

Since N is a subspace of V defined by being the lift of the space of negative weightvectors we have $NSymV \subset (SymV)_{<0}SymV$. Since the action of $\mathbb{C}^\times$ on V is linear, $NSymV = (SymV_{<0})SymV$ as any negativley graded vector in $SymV$ can be broken into a sum of monomials, which in particular are negativley graded weightvectors. Now we argue that $(A_{<0}A + \mathfrak{m}^q)/\mathfrak{m}^q = (NA + \mathfrak{m}^q)/\mathfrak{m}^q$. Clearly $NA + \mathfrak{m}^q/\mathfrak{m}^q \subset A_{<0}A + \mathfrak{m}^q/\mathfrak{m}^q$ so we show the opposite inclusion. First note

$$\frac{A_{<0} + \mathfrak{m}^q}{\mathfrak{m}^q} = \phi_q\left(\frac{SymV_{<0} + \mathfrak{n}^q}{\mathfrak{n}^q}\right) \subset \phi_q\left(\frac{NSymV + \mathfrak{n}^q}{\mathfrak{n}^q}\right) = \frac{NA + \mathfrak{m}^q}{\mathfrak{m}^q}.$$

Now note $\frac{NA+\mathfrak{m}^q}{\mathfrak{m}^q}$ is an ideal hence $\frac{A_{<0}A+\mathfrak{m}^q}{\mathfrak{m}^q} \subset \frac{NA+\mathfrak{m}^q}{\mathfrak{m}^q}$. Since $\frac{A_{<0}A+\mathfrak{m}^q}{\mathfrak{m}^q} = \frac{A_{<0}A_\mathfrak{m}+\mathfrak{m}^q}{\mathfrak{m}^q}$, Lemma 4.2.15 implies that $A_{<0}A_\mathfrak{m}$ is generated by the regular sequence N. Since $A_\mathfrak{m}$ is regular this implies that $A_\mathfrak{m}/A_{<0}A_\mathfrak{m}$ is a regular local ring.

(b) This is simply an application of Lemma 4.2.14 as Y^+ is smooth and the action of $\mathbb{C}^\times$ is definite at each fixed point.

(c) Let $a \in A_0$ and $y \in \Omega_p$, then by Lemma 4.2.8 this is reduced.

$$a(y) = t^0 a(y) = (t \cdot a)(y) = a(t^{-1}y)$$

hence $a(y)$ is a constant. Therefore $I_{A_0}(p)$ vanishes on Ω_p and the zero set of $\langle I_{A_0}(p)\rangle$ is equal to Ω_p as sets. $\square$

As explained in [5, p. 5] the fixed points set $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})^{\mathbb{C}^\times}$ is precisely $\gamma^{-1}(0)$. Therefore, by Proposition 4.1.2 and the discussion following it, we can identify $\mathrm{Irr}S_n \wr \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\sim} X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})^{\mathbb{C}^\times}$ by $\mathrm{quo}_\ell(\lambda) \mapsto x_{\mathrm{quo}_\ell(\lambda)}$. We have all we need to prove the second equality.

Lemma 4.2.17. *Assume that $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ is smooth. Then there is an equality of varieties*

$$\mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \Omega_{\mathrm{quo}_\ell(\lambda)}.$$

Proof. Throughout this proof denote $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ by X. Note that the big Verma module is positively graded with the degree zero part equal to $1 \otimes \mathrm{quo}_\ell(\lambda)$. Let $z \in Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$

be a negatively graded element. Then

$$z \cdot x \otimes \mathrm{quo}_\ell(\lambda) = zx \otimes \mathrm{quo}_\ell(\lambda) = x(z \otimes \mathrm{quo}_\ell(\lambda)) = 0,$$

as $z \otimes \mathrm{quo}_\ell(\lambda)$ has negative degree and $\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$ is positively graded. Therefore the annihilator of $\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$ contains all the negatively graded elements of $Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$. If we denote the ideal generated by the negatively graded elements by I_- then $I_- \subset \mathrm{ann}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$. Hence $\mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \subset V(I_-)$. By Proposition 4.2.16 (a) and (b) we see that $\mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$ is contained in one of the connected components of X^+. Since $x_{\mathrm{quo}_\ell(\lambda)} \in \mathrm{Supp}_{Z_c}(\underline{\Delta}(\mathrm{quo}_\ell(\lambda)))$ we have $\mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) \subset \Omega_{\mathrm{quo}_\ell(\lambda)}$. We argue that this containment is actually an equality by proving that $\dim \mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \dim \Omega_{\mathrm{quo}_\ell(\lambda)}$. This suffices since $\Omega_{\mathrm{quo}_\ell(\lambda)}$ is an irreducible variety and $\mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$ a closed subset of $\Omega_{\mathrm{quo}_\ell(\lambda)}$.

By Theorem 4.2.5 we have the equality $\mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$ so $\dim \mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \dim \mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$. But $\dim \mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$ is equal to the Krull dimension of $\mathrm{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$. Since $\mathrm{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$ is a finite free module over $\mathbb{C}[\mathfrak{h}]^{S_n \wr \mathbb{Z}/\ell\mathbb{Z}}$ it has Krull dimension equal to $\dim \mathfrak{h}$ by [9, Corollary 1.4.5]. From Lemma 4.2.14, we see that $\Omega_{\mathrm{quo}_\ell(\lambda)} \cong T_{\mathrm{quo}_\ell(\lambda)}(\Omega_{\mathrm{quo}_\ell(\lambda)})$, so we need show that $\dim T_{\mathrm{quo}_\ell(\lambda)}(\Omega_{\mathrm{quo}_\ell(\lambda)}) \leq \dim \mathfrak{h}$.

Since $x_{\mathrm{quo}_\ell(\lambda)}$ is a fixed point we have the following inclusions of $\mathbb{C}^\times$-submodules $T_{\mathrm{quo}_\ell(\lambda)}(x_{\mathrm{quo}_\ell(\lambda)}) \subset T_{\mathrm{quo}_\ell(\lambda)}(\Omega_{\mathrm{quo}_\ell(\lambda)}) \subset T_{\mathrm{quo}_\ell(\lambda)}(X)$. We can decompose $T_{\mathrm{quo}_\ell(\lambda)}(X) = T_- \oplus T_0 \oplus T_+$ into the negatively graded part, the degree zero part and the positively graded part. From [27, Theorem 5.2] we have that $T_{\mathrm{quo}_\ell(\lambda)}(x_{\mathrm{quo}_\ell(\lambda)}) = T_{\mathrm{quo}_\ell(\lambda)}(X^{\mathbb{C}^\times}) = T_0$. Now Lemma 4.2.8 says that $X^{\mathbb{C}^\times}$ is smooth hence $T_{\mathrm{quo}_\ell(\lambda)}(x_{\mathrm{quo}_\ell(\lambda)}) = \{0\}$ and so $T_0 = \{0\}$. The fixed point $x_{\mathrm{quo}_\ell(\lambda)}$ is in the smooth locus and [16, Theorem 7.8] implies that $T_{\mathrm{quo}_\ell(\lambda)}(X)$ is a symplectic vector space. The symplectic form on $T_{\mathrm{quo}_\ell(\lambda)}(X)$ is $\mathbb{C}^\times$-invariant hence its non-degeneracy forces $\dim T_- = \dim T_+$. Since X is smooth we have $\dim X = \dim T_{\mathrm{quo}_\ell(\lambda)}(X)$. Since $Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) = \mathbb{C}[X]$ is a finite free module over $\mathbb{C}[\mathfrak{h}]^{S_n \wr \mathbb{Z}/\ell\mathbb{Z}} \otimes \mathbb{C}[\mathfrak{h}^*]^{S_n \wr \mathbb{Z}/\ell\mathbb{Z}}$ we have

$$\dim Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) = \dim \mathbb{C}[\mathfrak{h}]^{S_n \wr \mathbb{Z}/\ell\mathbb{Z}} \otimes \mathbb{C}[\mathfrak{h}^*]^{S_n \wr \mathbb{Z}/\ell\mathbb{Z}} = 2\dim \mathfrak{h}.$$

This means that $\dim X = 2\dim \mathfrak{h}$. Therefore $\dim T_+ = \dim \mathfrak{h}$. Since $T_{\mathrm{quo}_\ell(\lambda)}(\Omega_{\mathrm{quo}_\ell(\lambda)})$ is positively graded we have $T_{\mathrm{quo}_\ell(\lambda)}(\Omega_{\mathrm{quo}_\ell(\lambda)}) \subset T_+$ hence $\dim T_{\mathrm{quo}_\ell(\lambda)}(\Omega_{\mathrm{quo}_\ell(\lambda)}) \leq \dim \mathfrak{h}$. $\square$

Combining the two equalities proven so far we can conclude the following.

Theorem 4.2.18. *For any* $\mathrm{quo}_\ell(\lambda) \in \mathrm{Irr}\, S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ *we have an isomorphism of varieties*

$$\mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \Omega_{\mathrm{quo}_\ell(\lambda)}.$$

Proof. Theorem 4.2.5 states $\mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda))$ for any $\mathrm{quo}_\ell(\lambda) \in \mathrm{Irr}\, S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ and, by Lemma 4.2.17, $\mathrm{Supp}_{Z_c}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \Omega_{\mathrm{quo}_\ell(\lambda)}$. The claim follows. $\square$

We now bring our attention back to the result by Bonnafe and Maksimau already mentioned. Before proceeding we introduce some notation. Let $\mathcal{C}_\ell[n]$ denote the set of ℓ-cores such that

$$|\gamma| \leq n \text{ and } |\gamma| = n \bmod \ell.$$

The following theorem is [12, Theorem 4.21], applied to the particular case of $W = S_{n\ell}$. Recall that $X_c(S_{n\ell})$ admits a $\mathbb{C}^\times$-action. We can therefore consider the group $\mathbb{Z}/\ell\mathbb{Z}$ acting on $X_c(S_{n\ell})$ by identifying $\mathbb{Z}/\ell\mathbb{Z}$ with the ℓ^{th} roots of unity.

Theorem 4.2.19. *Assume that* $X_c(S_{n\ell})$ *is smooth. Then* $X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}$ *is smooth and:*

1. *There is a bijection* $\gamma \to \mathcal{I}(\gamma)$ *between* $\mathcal{C}_\ell[n\ell]$ *and the irreducible components of* $X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}$ *such that* $x_\lambda \in \mathcal{I}(\gamma)$ *if and only if* $core_\ell(\lambda) = \gamma$ *for* $\lambda \in \mathcal{P}[n\ell]$.

2. *Let* $\gamma \in \mathcal{C}_\ell[n\ell]$ *and* $r = (n - |\gamma|)/\ell$. *There is an isomorphism of varieties*

$$i_\gamma : X_c(S_r \wr \mathbb{Z}/\ell\mathbb{Z}) \to \mathcal{I}(\gamma).$$

This satisfies $i_\gamma(x^l_\mu) = x_{(\mathrm{quo}^\flat_\ell)^{-1}(\mu)}$ *for all* $\mu \in \mathcal{P}^\ell[r]$ *with* ℓ*-core* γ.

We apply the above theorem to the case when $\gamma = \emptyset$. We see that the fixed points x_λ with λ having trivial ℓ-core all lie in the irreducible component $\mathcal{I}(\emptyset)$. Furthermore, there is an isomorphism

$$X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \cong \mathcal{I}(\emptyset).$$

Theorem 4.2.19 also describes where the fixed points are mapped under the isomorphism. Since, in our case, the quotient map $\mathrm{quo}_\ell : \mathcal{P}[n\ell] \to \mathcal{P}^\ell[n]$ is a bijection by [12, Lemma 4.7], we have $i_\emptyset(x_{\mathrm{quo}_\ell(\lambda)}) = x_\lambda$. This fact will be of upmost importance to us and so we record it as a lemma.

Lemma 4.2.20. *There is a* $\mathbb{C}^\times$*-equivariant isomorphism* $i_\emptyset : X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \to \mathcal{I}(\emptyset)$ *such that under the labeling of the fixed points we have* $i_\emptyset(x_{\mathrm{quo}_\ell(\lambda)}) = x_\lambda$ *for* $\lambda \in \mathcal{P}[n\ell]$.

The above lemma lets us prove the following.

Proposition 4.2.21. *The map* $i_\emptyset$ *restricts to a* $\mathbb{C}^\times$*-equivariant isomorphism of attracting sets*

$$i_\emptyset : \Omega_{\mathrm{quo}_\ell(\lambda)} \cong \Omega_\lambda^{\mathbb{Z}/\ell\mathbb{Z}} = \Omega_\lambda \cap X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}.$$

Proof. Since the map $i_\emptyset$ is $\mathbb{C}^\times$-equivariant it maps attracting sets to attracting sets hence

$$i_\emptyset(\Omega_{\mathrm{quo}_\ell(\lambda)}) \subset \Omega_\lambda^{\mathbb{Z}/\ell\mathbb{Z}}$$

and

$$i_\emptyset^{-1}(\Omega_\lambda^{\mathbb{Z}/\ell\mathbb{Z}}) \subset \Omega_{\mathrm{quo}_\ell(\lambda)}$$

hence $i_\emptyset : \Omega_{\mathrm{quo}_\ell(\lambda)} \to \Omega_\lambda^{\mathbb{Z}/\ell\mathbb{Z}}$ is a bijective morphism. Since a bijective morphism between smooth varieties is an isomorphism, the result follows. □

To recap we have an equality of varieties

$$\operatorname{Spec}\operatorname{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \Omega_{\mathrm{quo}_\ell(\lambda)},$$

and now an isomorphism

$$\Omega_{\mathrm{quo}_\ell(\lambda)} \cong \Omega_\lambda^{\mathbb{Z}/\ell\mathbb{Z}}.$$

This gives us a way to relate the endomorphism rings of the Verma modules for the symmetric and wreath product groups. Unfortunately, this is not enough to arrive at our desired explicit presentation. To do that we must understand a particular isomorphism explicitly. In [23, Theorem 11.16] Etingof and Ginzburg construct an isomorphism between $Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ and a suitable Calogero-Moser space. We now focus on describing this map and showing it has the properties we desire.

Recall the Calogero-Moser space Definition 2.9.1, it is the quotient variety

$$\mathcal{M}_{\mathbb{Z}/\ell\mathbb{Z},n,c} = M_{\mathbb{Z}/\ell\mathbb{Z},n,c}/PGL_{\mathbb{Z}/\ell\mathbb{Z},n,c}$$

where

$$M_{\mathbb{Z}/\ell\mathbb{Z},n,c} = \{\nabla_1, \nabla_2 \in \operatorname{End}(\mathbb{C}^n \otimes \mathbb{C}\mathbb{Z}/\ell\mathbb{Z}) |\, [\nabla_1, \nabla_2] = k\ell \cdot o \otimes e_{\mathbb{Z}/\ell\mathbb{Z}} + Id_{\mathbb{C}^n} \otimes c' \text{ for some } o \in \mathbb{O}\}.$$

We require an explicit understanding of the isomorphism $i_\emptyset$ by Bonnafe and Maksimau. In their paper this map is given by the inclusion map between the Calogero-Moser spaces associated to $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ and $X_c(S_{n\ell})$.

By [23, Theorem 1.7] there is an identification of $\operatorname{Spec} Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ with $\operatorname{Irr} H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ given by the assignment

$$p \to H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})e \otimes_Z p, \quad \forall p \in \operatorname{MaxSpec} Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \tag{4.2.4}$$

where p is viewed as a homomorphism $Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \to \mathbb{C}$. Consider an irreducible

$H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$-module E. Let Γ_{n-1} denote the subgroup of $\Gamma = S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ that stabilises the first basis vector x_1 in $\mathfrak{h}$. Let $E^{\Gamma_{n-1}}$ denote the subspace of E fixed by Γ_{n-1}. Clearly x_1 and y_1 commute with the action of Γ_{n-1}. Therefore we can define operators $X, Y \in \mathrm{End}_{\mathbb{C}}(E^{\Gamma_{n-1}})$ via the action of x_1 and y_1 on E respectively. The isomorphism $\phi : \mathrm{Irr}H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \to \mathcal{M}_{\mathbb{Z}/\ell\mathbb{Z},n,c}$ [23, Theorem 11.16] is given by $\phi(E) = (X, Y)$.

Consider the open set U in $\mathrm{Irr}H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ where the action of the elements $x_i - \omega^k x_j$ are invertible; U is an open set. Let $(\lambda, \mu) \in \mathbb{C}^{2n}$ with $\lambda_i^\ell \neq \lambda_j^\ell$ for all $i \neq j$. Let $\mathcal{O}_{(\lambda,\mu)}$ denote the orbit of (λ, μ) under the group $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$. This is a free orbit, so $|\mathcal{O}_{(\lambda,\mu)}| = n!\ell^n$. Up to isomorphism, each representation E in U is of the form $E_{(\lambda,\mu)} = \mathbb{C}[\mathcal{O}_{\lambda,\mu}]$. A basis of $E_{(\lambda,\mu)}$ is given by the characteristic equations

$$\chi_s(a,b) = \begin{cases} 1 \text{ if } s \cdot (a,b) = (\lambda,\mu) \\ 0 \text{ else} \end{cases}$$

for $s \in S_n \wr \mathbb{Z}/\ell\mathbb{Z}$. The subspace $E^{\Gamma_{n-1}}$ is then ℓn-dimensional with basis $\chi_{s_{1,i}\gamma_1^r}$ for $1 \leq i \leq a$ and $0 \leq r \leq \ell - 1$. The action of $H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ on $E_{(\lambda,\mu)}$ is given by

$$x_i \cdot F(a,b) = a_i F(a,b), \;\; y_i \cdot F(a,b) = b_i \cdot F(a,b) + c_0 \sum_{j \neq i} \sum_{k=0}^{\ell-1} \frac{s_{i,j}\gamma_i^k \gamma_j^{-k} F(a,b)}{\omega^{-k} a_j - a_i} + \sum_{k=1}^{\ell-1} \frac{c_k \gamma_i^k F(a,b)}{a_i \omega^k - a_i}.$$

and

$$(w \cdot F)(a,b) = F(w^{-1} \cdot a, w^{-1} \cdot b),$$

for $w \in S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ and ω a primitive ℓ^{th} root of unity. We must check that these equations satisfy the defining relations of the rational Cherednik algebra at $t = 0$. These relations can be written [18, p. 22] as

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0$$

$$[y_i, x_i] = c_0 \sum_{i \neq j} \sum_{k=0}^{\ell-1} s_{ij} s_i^k s_j^{-k} + \sum_{k=1}^{\ell-1} c_k s_j^{-k}$$

$$[y_i, x_j] = -c_0 \sum_{k=0}^{\ell-1} s_{ij} \omega^k s_i^k s_j^{-k}.$$

The check is just a straight forward computation which we include for completeness

$$[y_i, x_t] \cdot F(a,b) = y_i \cdot x_t \cdot F(a,b) - x_t \cdot y_i \cdot F(a,b)$$

$$= x_t \left(b_i \cdot F(a,b) + c_0 \sum_{j \neq i} \sum_{k=0}^{\ell-1} \frac{s_{i,j}\gamma_i^k \gamma_j^{-k} F(a,b)}{\omega^{-k} a_j - a_i} + \sum_{k=1}^{\ell-1} \frac{c_k \gamma_i^k F(a,b)}{a_i \omega^k - a_i} \right) - y_i \cdot a_t F(a,b)$$

$$= a_t b_i F(a,b) + c_0 \sum_{j\neq i, j\neq t} \sum_{k=0}^{\ell-1} \frac{s_{i,j}\gamma_i^k\gamma_j^{-k} a_t F(a,b)}{\omega^{-k}a_j - a_i} + c_0 \sum_{k=0}^{\ell-1} \frac{s_{i,t}\gamma_i^k\gamma_t^{-k}\omega^k a_i F(a,b)}{\omega^{-k}a_t - a_i}$$

$$+ \sum_{k=1}^{\ell-1} \frac{a_t c_k \gamma_i^k a_t F(a,b)}{a_i\omega^k - a_i}$$

$$- \left(a_t b_i \cdot F(a,b) + c_0 \sum_{j\neq i} \sum_{k=0}^{\ell-1} \frac{a_t s_{i,j}\gamma_i^k\gamma_j^{-k} F(a,b)}{\omega^{-k}a_j - a_i} + \sum_{k=1}^{\ell-1} \frac{a_t c_k \gamma_i^k F(a,b)}{a_i\omega^k - a_i} \right)$$

$$= c_0 \sum_{k=0}^{\ell-1} \frac{(\omega^k a_i - a_t) s_{i,t}\gamma_i^k\gamma_t^{-k}\omega^k a_i F(a,b)}{\omega^{-k}a_t - a_i}$$

$$= c_0 \sum_{k=0}^{\ell-1} \omega^k s_{i,t}\gamma_i^k\gamma_t^{-k} F(a,b).$$

Now we check the relation

$$[y_i, x_i] = c_0 \sum_{i\neq j} \sum_{k=0}^{\ell-1} s_{ij} s_i^k s_j^{-k} + \sum_{k=1}^{\ell-1} c_k s_j^{-k}.$$

So

$$[y_i, x_i] \cdot F(a,b) = y_i \cdot x_i \cdot F(a,b) - x_i \cdot y_i \cdot F(a,b)$$

$$= x_i \cdot \left(b_i \cdot F(a,b) + c_0 \sum_{j\neq i} \sum_{k=0}^{\ell-1} \frac{s_{i,j}\gamma_i^k\gamma_j^{-k} F(a,b)}{\omega^{-k}a_j - a_i} + \sum_{k=1}^{\ell-1} \frac{c_k\gamma_i^k F(a,b)}{a_i\omega^k - a_i} \right) - y_i \cdot a_i F(a,b)$$

$$= a_i b_i F(a,b) + c_0 \sum_{j\neq i} \sum_{k=0}^{\ell-1} \frac{s_{i,j}\gamma_i^k\gamma_j^{-k}\omega^{-k} a_j F(a,b)}{\omega^{-k}a_j - a_i} + \sum_{k=1}^{\ell-1} \frac{c_k\gamma_i^k\omega^k a_i F(a,b)}{a_i\omega^k - a_i})$$

$$-a_i \left(b_i \cdot F(a,b) + c_0 \sum_{j\neq i} \sum_{k=0}^{\ell-1} \frac{s_{i,j}\gamma_i^k\gamma_j^{-k} F(a,b)}{\omega^{-k}a_j - a_i} + \sum_{k=1}^{\ell-1} \frac{c_k\gamma_i^k F(a,b)}{a_i\omega^k - a_i} \right)$$

$$= c_0 \sum_{j\neq i} \sum_{k=0}^{\ell-1} \frac{(\omega^{-k}a_j - a_i) s_{i,j}\gamma_i^k\gamma_j^{-k} F(a,b)}{\omega^{-k}a_j - a_i} + \sum_{k=1}^{\ell-1} \frac{(\omega^k a_i - a_i) c_k\gamma_i^k F(a,b)}{a_i\omega^k - a_i}$$

$$= c_0 \sum_{j\neq i} \sum_{k=0}^{\ell-1} s_{i,j}\gamma_i^k\gamma_j^{-k} F(a,b) + \sum_{k=1}^{\ell-1} c_k\gamma_i^k F(a,b)$$

as required.

Theorem 4.2.22. *Let* $\phi : \operatorname{Spec} Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \to \mathcal{M}_{\mathbb{Z}/\ell\mathbb{Z},n,c}$ *be the map defined above. Then we have the following equality*

$$\phi^*(tr(X)^k) = \begin{cases} \ell(x_1^k + \ldots + x_n^k) \text{ if } \ell | k \\ 0 \text{ else} \end{cases}$$

Proof. We begin by noting that $Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ is reduced since [16, Proposition 7.2] states that $eH_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})e \cong Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ is a domain. Hence for $f, g \in Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$, $f = g$ if and only if $f(p) = g(p)$ for all $p \in \text{maxSpec}\, Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$. Furthermore $\text{Spec}\, Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ is irreducible, therefore $f = g$ if and only if $f(p) = g(p)$ for all $p \in U \subset \text{Spec}\, Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$, where U is the open set defined above. We also use the identification of $\text{Spec}\, Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ with $\text{Irr} H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ as in equation (4.2.4) above. Fix an irreducible module $E_{(\lambda,\mu)} \in U$. We shall first calculate $\phi^*(tr(X))^k(E_{(\lambda,\mu)})$. As described in Section 2.9, $E_{(\lambda,\mu)}^{\Gamma_{n-1}}$ is isomorphic as a $\mathbb{C}(\mathbb{Z}/\ell\mathbb{Z})$-modules to n copies of $\mathbb{C}(\mathbb{Z}/\ell\mathbb{Z})$. Therefore, $E_{(\lambda,\mu)}^{\Gamma_{n-1}}$ can be viewed a sum of vector spaces $V_0 \oplus \cdots \oplus V_{\ell-1}$ where V_i is n copies of the irreducible representation of $\mathbb{Z}/\ell\mathbb{Z}$ where the generator s acts by ω^i. If we denote the action of x_1 on V_i by X_i then note that $X_i : V_i \to V_{i+1}$ as

$$s \cdot X_i(v) = s \cdot x_1 v = \omega x_1 s \cdot v = \omega^{i+1} x_1 v = \omega^{i+1} X_i(v) \text{ if } v \in V_i.$$

Recall that $\phi(E) = (X, Y)$, where $X = x_1$ acting on $E^{\Gamma_{n-1}}$. Then we see that as a matrix,

$$X = \begin{bmatrix} 0 & 0 & \dots & X_{\ell-1} \\ X_0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \dots & X_{\ell-2} & 0 \end{bmatrix}. \tag{4.2.5}$$

Hence $\phi^*(tr(X)^k)(E_{(\lambda,\mu)}) = tr(X^k)$ and if $\ell \not| k$ then $tr(X^k) = 0$ as X^k has every entry on the main diagonal equal to 0. However if $\ell | k$ then write $k = m\ell$ and $tr(X^k) = \ell tr((X_0 \cdots X_{\ell-1})^m)$. For each X_i, we have $X_i = \text{diag}(\lambda_1, \cdots, \lambda_n)$, hence

$$tr((X_0 \cdots X_{\ell-1})^m) = \lambda_1^{m\ell} + \cdots + \lambda_n^{m\ell} = \lambda_1^k + \cdots + \lambda_n^k.$$

Substituting back in we find

$$tr(X^k) = \ell(\lambda_1^k + \cdots + \lambda_n^k).$$

Now we must check that $\ell(x_1^k + \cdots + x_n^k)(u) = \ell(\lambda_1^k + \cdots + \lambda_n^k)(u)$ for all $u \in E$. For each $F \in E_{(\lambda,\mu)}$, $(x_1^k + \cdots + x_n^k \cdot F)(a, b) = (\lambda_1^k + \cdots + \lambda_n^k)F(a, b)$ hence $x_1^k + \cdots + x_n^k$ acts by scalar multiplication on $E_{(\lambda,\mu)}$ by $\lambda_1^k + \cdots + \lambda_n^k$. □

Another map we must explicitly understand is the following.

Lemma 4.2.23. *There is an isomorphism*

$$\alpha : \mathbb{C}^n/(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{\sim} (\mathbb{C}^{n\ell}/S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}$$

given by

$$\alpha(a_1, a_2, \cdots, a_n) = (a_1, \omega a_1, \omega^2 a_1, \cdots, \omega^{\ell-1} a_n)$$

Recall the map $i_\emptyset$ introduced in Lemma 4.2.20, we must break this into a composition of three maps. Consider the inclusion map on quiver varieties

$$\bar{i}_\emptyset : \mathcal{M}_{\mathbb{Z}/\ell\mathbb{Z},n,c} \to \mathcal{M}_{\mathbb{Z}/\mathbb{Z},n\ell,c}$$

given by sending $(X_0, \dots X_{\ell-1})$ to the matrix X of equation (4.2.5) and $(Y_0, \dots, Y_{\ell-1})$ to

$$\begin{bmatrix} 0 & 0 & \dots & Y_{\ell-1} \\ Y_0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \dots & Y_{\ell-2} & 0 \end{bmatrix}.$$

Then the map $i_\emptyset$ of Lemma 4.2.20 is given by $\phi^{-1}_{S_{n\ell}} \circ \bar{i}_\emptyset \circ \phi_{S_n \wr \mathbb{Z}/\ell\mathbb{Z}}$. With all the appropriate maps introduced we can perform the following diagram chase.

Theorem 4.2.24. *There is an isomorphism $X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \to Y$ to a connected component of $X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}$ such that the following diagram commutes*

$$\begin{tikzcd} X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \arrow[r, "\phi^{-1}_{S_{n\ell}} \circ \bar{i}_\emptyset \circ \phi_{S_n \wr \mathbb{Z}/\ell\mathbb{Z}}"] \arrow[d, "\pi_{n,\ell}"'] & X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}} \arrow[d, "\pi_{n\ell}"] \\ \mathbb{C}^n/(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \arrow[r, "\alpha"'] & (\mathbb{C}^{n\ell}/S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}} \end{tikzcd}$$

Proof. The first step is to unpack the diagram by introducing the Calogero-Moser spaces

$$\begin{tikzcd} X_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \arrow[r, "\phi_{S_n \wr \mathbb{Z}/\ell\mathbb{Z}}"] \arrow[d, "\pi_{n,\ell}"'] & \mathcal{M}_{\mathbb{Z}/\ell\mathbb{Z},n,c} \arrow[r, "\bar{i}_\emptyset"] & \mathcal{M}_{\mathbb{Z}/\mathbb{Z},n\ell,c} \arrow[r, "\phi^{-1}_{S_{n\ell}}"] & X_c(S_{n\ell}) \arrow[d, "\pi_{n\ell}"] \\ \mathbb{C}^n/(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \arrow[rrr, "\alpha"'] & & & (\mathbb{C}^{n\ell}/S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}} \end{tikzcd}.$$

It is easier understand the duals of the maps in the diagram above and since a diagram commutes if and only if its dual does we shall prove this instead. We must therefore prove the commutivity of the following diagram

$$\begin{tikzcd}
Z_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) & \mathbb{C}[\mathcal{M}_{\mathbb{Z}/\ell\mathbb{Z},n,c}] \arrow[l, "\phi^*_{S_n\wr\mathbb{Z}/\ell\mathbb{Z}}"'] & \mathbb{C}[\mathcal{M}_{\mathbb{Z}/\mathbb{Z},n\ell,c}] \arrow[l, "\bar{i}^*_\emptyset"'] & Z_c(S_{n\ell}) \arrow[l, "(\phi^*)^{-1}_{S_{n\ell}}"'] \\
\mathbb{C}[\mathbb{C}^n/(S_n \wr \mathbb{Z}/\ell\mathbb{Z})] \arrow[u, "i_{n,\ell}"] & & & \mathbb{C}[(\mathbb{C}^{n\ell}/S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}] \arrow[lll, "\alpha^*"] \arrow[u, "i_{n\ell}"']
\end{tikzcd}.$$

First note that $\mathbb{C}[(\mathbb{C}^{n\ell}/S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}]$ are the symmetric polynomials fixed under the action of the ℓ^{th} roots of unity. Hence it is generated by elements of the form

$$x_1^{k\ell} + \cdots + x_{n\ell}^{k\ell} \text{ where } k \in \mathbb{Z}_{\geq 0}.$$

Now $\alpha^*(f)(p) = f(\alpha(p))$, where $p \in \mathbb{C}^n/S_n \wr \mathbb{Z}/\ell\mathbb{Z}$. Consider an arbitrary generator $x_1^{k\ell} + \cdots + x_{n\ell}^{k\ell}$, then

$$\alpha^*(x_1^{k\ell} + \cdots + x_{n\ell}^{k\ell})(a_1, a_2, \cdots, a_n) = (x_1^{k\ell} + \cdots + x_{n\ell}^{k\ell})(a_1, \omega a_1, \cdots, \omega^{\ell-1}a_n)$$

$$= a_1^{k\ell} + (\omega a_1)^{k\ell} + (\omega^2 a_1)^{k\ell} + \cdots + (\omega^{\ell-1}a_n)^{k\ell}$$

$$= (1 + \omega^{k\ell} + (\omega^2)^{k\ell} \cdots + (\omega^{\ell-1})^{k\ell})(a_1^{k\ell} + a_2^{k\ell} + \cdots + x_n^{k\ell}) = \ell(a_1^{k\ell} + \cdots + a_n^{k\ell}).$$

Hence

$$\alpha^*(x_1^{k\ell} + \cdots + x_{n\ell}^{k\ell}) = \ell(x_1^{k\ell} + \cdots + x_n^{k\ell}).$$

Recall the map π is the dual of the inclusion map so

$$i_{n,\ell}(\ell(x_1^{k\ell} + \cdots + x_n^{k\ell})) = \ell(x_1^{k\ell} + \cdots + x_n^{k\ell}).$$

Now we must chase the diagram the other way. The first map $i_{n\ell}$ is also the inclusion map hence

$$i_{n\ell}(x_1^{k\ell} + \cdots + x_{n\ell}^{k\ell}) = x_1^{k\ell} + \cdots + x_{n\ell}^{k\ell}.$$

By Theorem 4.2.22, we have that $\phi^*_{S_{n\ell}}(tr(X)^k) = x_1^k + \cdots + x_{n\ell}^k$, therefore $(\phi^*_{S_{n\ell}})^{-1}(x_1^{k\ell} + \cdots + x_{n\ell}^{k\ell}) = tr(X)^{k\ell}$. Then we have by definition

$$(\bar{i}^*_\emptyset)^{-1}(tr(X)^{k\ell}) = tr(\bar{i}_\emptyset(X)^{k\ell}).$$

Therefore we complete the proof by showing $\phi^*_{S_n\wr\mathbb{Z}/\ell\mathbb{Z}}(tr(\bar{i}_\emptyset(X)^{k\ell}) = \ell(x_1^{k\ell} + \cdots + x_n^{k\ell})$. This is precisely the statement of Theorem 4.2.22. □

Before presenting the main theorem of this section we fix some notation. Recall by

Corollary 4.1.16 that $A(\lambda)^+ \cong \mathbb{C}[\pi^{-1}(0)]$ and, as in the introduction,

$$A(\lambda)^+_{\mathbb{Z}/\ell\mathbb{Z}} := \mathbb{C}[\pi_{n\ell}^{-1}(0)]/\langle f - s \cdot f \mid s \in \mathbb{Z}/\ell\mathbb{Z}, f \in \mathbb{C}[\pi_{n\ell}^{-1}(0)]\rangle.$$

Theorem 4.2.25. *There is an isomorphism of algebras*

$$A(\mathrm{quo}_\ell(\lambda))^+ \cong A(\lambda)^+_{\mathbb{Z}/\ell\mathbb{Z}}.$$

Proof. By definition, $A(\mathrm{quo}_\ell(\lambda))^+ := \mathrm{End}\Delta(\mathrm{quo}_\ell(\lambda)) = \mathbb{C}[\mathrm{Spec}\,\mathrm{End}\Delta(\mathrm{quo}_\ell(\lambda))]$. By Proposition 4.2.21 there is an isomorphism

$$i_\emptyset : \Omega_{\mathrm{quo}_\ell(\lambda)} \cong \Omega_\lambda \cap X_c(S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}.$$

Therefore Theorem 4.2.24 implies that there is a commutative diagram

$$\begin{tikzcd}
\Omega_{\mathrm{quo}_\ell(\lambda)} \arrow[r, "\phi_{S_{n\ell}}^{-1} \circ \bar{i}_\emptyset \circ \phi_{S_n \wr \mathbb{Z}/\ell\mathbb{Z}}"] \arrow[d, "\pi_{n,\ell}"'] & \Omega_\lambda^{\mathbb{Z}/\ell\mathbb{Z}} \arrow[d, "\pi_{n\ell}"] \\
\mathbb{C}^n/(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \arrow[r, "\alpha"'] & (\mathbb{C}^{n\ell}/S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}
\end{tikzcd}.$$

By Theorem 4.2.18,

$$\mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \Omega_{\mathrm{quo}_\ell(\lambda)}$$

and

$$\mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\lambda) = \Omega_\lambda.$$

Hence the diagram becomes

$$\begin{tikzcd}
\mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\mathrm{quo}_\ell(\lambda)) = \Omega_{\mathrm{quo}_\ell(\lambda)} \arrow[r, "i_\emptyset"] \arrow[d, "\pi_{n,\ell}"'] & \Omega_\lambda^{\mathbb{Z}/\ell\mathbb{Z}} = (\mathrm{Spec}\,\mathrm{End}\underline{\Delta}(\lambda))^{\mathbb{Z}/\ell\mathbb{Z}} \arrow[d, "\pi_{n\ell}"] \\
\mathbb{C}^n/(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \arrow[r, "\alpha"'] & (\mathbb{C}^{n\ell}/S_{n\ell})^{\mathbb{Z}/\ell\mathbb{Z}}
\end{tikzcd}.$$

Since α and $i_\emptyset$ are both isomorphisms we have $\pi_{n,\ell}^{-1}(0) \cong (\pi_{n\ell}^{-1}(0))^{\mathbb{Z}/\ell\mathbb{Z}}$. Therefore there is an algebra isomorphism $\mathbb{C}[\pi_{n,\ell}^{-1}(0)] \cong \mathbb{C}[\pi_{n\ell}^{-1}(0)^{\mathbb{Z}/\ell\mathbb{Z}}]$. Finally, Corollary 4.1.16 implies that

$$\begin{aligned} A(\mathrm{quo}_\ell(\lambda))^+ \cong& \mathbb{C}[\pi_{n,\ell}^{-1}(0)] \\ \cong& \mathbb{C}[\pi_{n\ell}^{-1}(0)^{\mathbb{Z}/\ell\mathbb{Z}}] \\ \cong& \mathbb{C}[\pi_{n\ell}^{-1}(0)]/\langle f - s\cdot f \mid s \in \mathbb{Z}/\ell\mathbb{Z}, f \in \mathbb{C}[\pi_{n\ell}^{-1}(0)]\rangle \end{aligned}$$

hence

$$A(\mathrm{quo}_\ell(\lambda))^+ \; = A(\lambda)^+_{\mathbb{Z}/\ell\mathbb{Z}}.$$

□

Combining this with the main result of the previous section we have the main theorem of this thesis.

Theorem 4.2.26. *There is an isomorphism*

$$A(\mathrm{quo}_\ell(\lambda))^+ \cong \mathbb{C}[\mathrm{Wr}_\lambda^{-1}(0)]/\langle f - s\cdot f \mid s \in \mathbb{Z}/\ell\mathbb{Z}, f \in \mathbb{C}[\mathrm{Wr}_\lambda^{-1}(0)]\rangle.$$

Proof. This follows from Theorem 4.1.18 and Theorem 4.2.25. □

The above theorem is presented in a rather abstract fashion. At the beginning of this section we promised to give an explicit description of $A(\mathrm{quo}_\ell(\lambda))^+$. In the next section we will explain how the Wronski map gives an explicit presentation of $A(\lambda)^+$ as a quotient of a polynomial algebra by n relations.

Chapter 5

Explicit presentation of $A(\lambda)^+$

In this chapter we present a number of results concerning the algebras $A(\lambda)^+$. Perhaps most importantly we give the explicit presentation of $A(\lambda)^+$ for both the symmetric and wreath product groups. This is done by using Theorem 4.1.18 and Theorem 4.2.26. In particular, Theorem 4.1.18 says that the algebra $A(\lambda)^+$ is isomorphic to the scheme-theoretic fibre of the preimage of the Wronskian at 0. This algebra is described in [47], and we initially follow their approach. We improve on this by providing an explicit presentation (in terms of generators and relations) of the algebra $A(\lambda)^+$ that avoids any mention of the Wronskian. This will be done initially for the symmetric group, then extended to the wreath product case.

Aside from the explicit presentations this chapter contains several other significant results. In the first section we derive a formula for calculating the graded dimension of $A(\lambda)^+$. We go on to prove that there is an isomorphism $A(\lambda)^+ \cong A(\lambda^T)^+$. This greatly decreases the computations required if one wishes to find all $A(\lambda)^+$ for a given n. The chapter concludes by giving code that can be used to calculate the dimensions of the radical layers of $A(\lambda)^+$. This is done by taking advantage of the desirable qualities of Gröbner basis, namely that computers can easily calculate them. Experimental results via computations leads to the final section, which is a conjecture about the Lowey length of $A(\lambda)^+$.

§ 5.1 | Explicit presentation and a formula for the graded dimension

In this section we show how to use Theorem 4.1.18 to give an explicit presentation of $A(\lambda)^+$ for the symmetric group. Then we prove a formula for calculating the graded dimensions. Recall that Theorem 4.1.18 states there is an isomorphism

$$A(\lambda)^+ \cong \mathbb{C}[\mathrm{Wr}_\lambda^{-1}(0)].$$

Therefore, we must explain how to write the algebra of functions on the preimage of 0 under the Wronski map. Our approach broadly follows that of [47]. By Proposition 2.7.9, $\mathbb{C}[\Omega_\lambda^{qe}]$ is a free polynomial algebra with generators f_{ij} i.e.

$$\mathbb{C}[\Omega_\lambda^{qe}] = \mathbb{C}[f_{ij},\ i = 1, \ldots, n,\ \ j = 1, \ldots d_i,\ d_i - j \notin P].$$

It is important to note that this is a graded algebra. Indeed, recall Remark 2.7.10 that says $\deg(f_{ij}) = j$.

Also recall by Proposition 2.7.4 that the Schubert cell consists of subspaces X with basis

$$f_i = u^{d_i} + \sum_{j=1,\, d_i - j \notin P}^{d_i} f_{i,j} u^{d_i - j}. \tag{5.1.1}$$

The Wronskian of a basis of X is a polynomial of degree n. We write

$$\mathrm{Wr}(f_1, \cdots, f_n) = u^n + r_1 u^{n-1} + \cdots + r_n.$$

This form will allow for an easier discussion.

The Wronski map Definition 2.7.8 is defined on elements by

$$\mathrm{Wr}_\lambda(X) = (a_1, \cdots, a_n) \text{ if } \mathrm{Wr}(X) = u^n + \sum_{i=1}^{n} (-1)^i a_i u^{n-i}.$$

Therefore the scheme theoretic fibre of the Wronski map is

$$\mathbb{C}[\mathrm{Wr}_\lambda^{-1}(a)] \cong \mathbb{C}[\Omega_\lambda^{qe}]/I_{\lambda,a},$$

where $I_{\lambda,a}$ is the ideal generated by the $r_s - (-1)^s a_s$. In the case $a = 0$, this states that $A(\lambda)^+$ is the quotient of $\mathbb{C}[\Omega_\lambda^{qe}]$ by the ideal generated by the coefficients r_s of the polynomial $\mathrm{Wr}(f_1, \cdots, f_n)$.

To summarise, the process for calculating $A(\lambda)^+$ for a given $\lambda \vdash n$ is as follows. Define positive integers $d_i = \lambda_i + n - i$ and denote the set of these by $P = \{d_1, \cdots, d_n\}$. Then calculate the Wronskian

$$\mathrm{Wr}(f_1, f_2, \cdots, f_n) = \det \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1^{(1)} & f_2^{(1)} & f_3^{(1)} & \cdots & f_n^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} = u^n + r_1 u^{n-1} + \cdots + r_n$$

of the polynomials

$$f_i = u^{d_i} + \sum_{j=1,\ d_i-j\notin P}^{d_i} f_{ij} u^{d_i-j}.$$

The algebra $A(\lambda)^+$ is given by taking the polynomial algebra generated by the f_{ij} and quotienting by the coefficients r_s of the Wronskian. Below is an example to help the illustrate this process.

Example 5.1.1. For the partition $\lambda = (3,2)$ we have $d_1 = 7$, $d_2 = 5$, $d_3 = 2$, $d_4 = 1$ and $d_5 = 0$. Therefore $f_1(u) = u^7 + f_{11}u^6 + f_{13}u^4 + f_{14}u^3$, $f_2(u) = u^5 + f_{21}u^4 + f_{22}u^3$, $f_3(u) = u^2$, $f_4(u) = u$ and $f_5(u) = 1$. Let us calculate the Wronskian, which is

$$\det \begin{bmatrix} u^7 + f_{11}u^6 + f_{13}u^4 + f_{14}u^3 & u^5 + f_{21}u^4 + f_{22}u^3 & u^2 & u & 1 \\ 7u^6 + 6f_{11}u^5 + 4f_{13}u^3 + 3f_{14}u^2 & 5u^4 + 4f_{21}u^3 + 3f_{22}u^2 & 2u & 1 & 0 \\ 42u^5 + 30f_{11}u^4 + 12f_{13}u^2 + 6f_{14}u & 20u^3 + 12f_{21}u^2 + 6f_{22}u & 2 & 0 & 0 \\ 210u^4 + 120f_{11}u^3 + 24f_{13}u + 6f_{14} & 60u^2 + 24f_{21}u + 6f_{22} & 0 & 0 & 0 \\ 840u^3 + 360f_{11}u^2 + 24f_{13} & 120u + 24f_{21} & 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$\mathrm{Wr}(f_1(u), f_2(u), f_3(u), f_4(u), f_5(u)) = 25200u^5 + (14400f_{11} + 30240f_{21})u^4$$

$$+(11520f_{11}f_{21}+10080f_{22})u^3+(-2880f_{13}+4320f_{11}f_{22})u^2-1440f_{14}u+(-288f_{14}f_{21}+288f_{13}f_{22}).$$

Then $A(3,2)^+$ is the quotient by the ideal generated by the coefficients, with a little simplification this is

$$A(3,2)^+ = \mathbb{C}[f_{11}, f_{13}, f_{14}, f_{21}, f_{22}]/(10f_{11}-21f_{21}, 8f_{11}f_{21}+7f_{22}, 2f_{13}-3f_{11}f_{22}, f_{14}, f_{14}f_{21}-f_{13}f_{22}).$$

It is then easy enough to see

$$A(3,2)^+ \cong \mathbb{C}[f_{11}]/(f_{11}^5).$$

We are now in position to prove the formula for calculating the graded dimension of $A(\lambda)^+$. Our results are inspired by the formula

$$s_\lambda(1, q, \cdots, q^n) = \frac{\prod_{i=1}^n (1-q^i)}{\prod_{(i,j)\in D_\lambda}(1-q^{h(i,j)})}.$$

This can be found in [49, p. 364]. The term s_λ denotes the Schur function associated to the partition $\lambda \vdash n$, D_λ denotes the Young diagram of λ and $h(i,j)$ is the hook length of the box (i,j). We will find that a similar formula allows us to calculate the graded dimension of $A(\lambda)^+$.

We now record two general lemmata about the Wronskian and certain homogeneous polynomials.

Lemma 5.1.2. *If $f_1, \cdots, f_n$ is a family of homogeneous polynomials and $\mathrm{Wr}(f_1, \cdots, f_n) \neq 0$ then the Wronskian is homogeneous and $\deg(\mathrm{Wr}(f_1, \cdots, f_n)) = \sum_i(\deg(f_i)) - \frac{(n-1)(n)}{2}$.*

Proof. We proceed by induction on n, the case $n = 1$ being trivial. Suppose the lemma holds for all positive integers less than n. Then

$$\det \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1^{(1)} & f_2^{(1)} & f_3^{(1)} & \cdots & f_n^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} = \sum_{i=1}^{n} (-1)^{i+1} f_i \begin{bmatrix} f_1^{(1)} & \hat{f}_i^{(1)} & \cdots & f_n^{(1)} \\ f_1^{(2)} & \hat{f}_i^{(2)} & \cdots & f_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \hat{f}_i^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}$$

where the ˆ symbol denotes an omitted column. By the inductive hypothesis, each of the components of the sum is a homogeneous polynomial of degree

$$\deg(f_i) + \sum_{j, j\neq i}^{n} \deg(f_j^{(1)}) - \frac{(n-2)(n-1)}{2} = \deg(f_i) + \sum_{j, j\neq i}^{n} \deg(f_j) - (n-1) - \frac{(n-2)(n-1)}{2}.$$

This then simplifies to

$$\sum_{i=1}^{n} \deg(f_i) - (n-1) - \frac{(n-2)(n-1)}{2} = \sum_{i=1}^{n} \deg(f_i) - \frac{(n-1)n}{2}.$$

□

In the following lemma it is important to recall that when considering the polynomials defined in 5.1.1 the generators f_{ij} have a degree. For instance the polynomial

$$u^3 + f_{12}u$$

is homogeneous as both u^3 and $f_{12}u$ have degree 3.

Lemma 5.1.3. *Let $\lambda \vdash n$ be a partition of n, and $\{f_1, \cdots, f_n\}$ the family of polynomials as defined in (5.1.1). The Wronskian $\mathrm{Wr}(f_1, \cdots, f_n)$ is a homogeneous polynomial of degree n.*

Proof. By Lemma 5.1.2 if the Wronskian is non-zero then it is a homogeneous polynomial of degree $\sum_i(\deg(f_i)) - \frac{(n-1)(n)}{2}$. We have $\deg(f_i) = d_i = \lambda_i + n - i$. Hence

$$\sum_{i}^{n}(\deg(f_i)) - \frac{(n-1)(n)}{2} = \sum_{i}^{n} \lambda_i + n - i - \frac{(n-1)(n)}{2} = \sum_{i}^{n} \lambda_i + \sum_{i}^{n} n + \sum_{i}^{n} -i - \frac{(n-1)(n)}{2},$$

and

$$\sum_i^n \lambda_i + \sum_i^n n + \sum_i^n -i - \frac{(n-1)(n)}{2} = n + n^2 - \frac{n(n+1)}{2} - \frac{(n-1)(n)}{2} = n.$$

$\square$

Later results will rely on knowing the dimension of $A(\lambda)^+$. In order to do this we must introduce the notion of complete intersections.

Definition 5.1.4. A commutative (finitely generated) $\mathbb{C}$-algebra A is a *complete intersection* if it can be presented as

$$A = \mathbb{C}[x_1, \ldots, x_m]/(g_1, \ldots, g_t),$$

with $\dim_K A = m - t$, where $\dim_K$ denotes the Krull dimension.

To prove that $A(\lambda)^+$ is a complete intersection we need to use the following fact.

Lemma 5.1.5. *The algebra $A(\lambda)^+$ is finite dimensional.*

Proof. See [47, Lemma 3.11]. $\square$

Lemma 5.1.6. *For any $\lambda \vdash n$, the algebra $A(\lambda)^+$ is a complete intersection.*

Proof. The algebra $A(\lambda)^+$ has n generators and n relations. Therefore, it is a complete intersection if and only if its Krull dimension is zero. Since $A(\lambda)^+$ is finite dimensional as a vector space it is Artinian and therefore every prime ideal is maximal [2, Proposition 8.1]. Hence it has Krull dimension zero. $\square$

We return now to the setting of Definition 5.1.4. Assume that $\mathbb{C}[x_1, \ldots, x_m]$ is graded with $\deg(x_i) = a_i > 0$, so that each g_j is homogeneous, of degree b_j say. The Hilbert-Poincaré polynomial of A is defined to be

$$P(A, q) := \sum_{i \geq 0} (\dim A_i) q^i.$$

Lemma 5.1.7. *If A is a graded complete intersection then*

$$P(A, q) = \frac{\prod_{i=1}^t (1 - q^{b_i})}{\prod_{j=1}^n (1 - q^{a_j})}.$$

Proof. Recall by Definition 5.1.4 that we can write

$$A = \mathbb{C}[x_1, \ldots, x_m]/(g_1, \ldots, g_t).$$

Let $S = \mathbb{C}[x_1, \ldots, x_m]$. The Koszul resolution [20, Chapter 17] can be used to resolve A as a graded S-module. Let $V = \mathrm{Span}_{\mathbb{C}}\{g_1, \ldots, g_t\}$, a graded vector space. Then

$$0 \to \wedge^t V \otimes_{\mathbb{C}} S \to \wedge^{t-1} V \otimes_{\mathbb{C}} S \to \cdots \to \wedge^0 V \otimes S \to A \to 0$$

is an exact sequence of graded S-modules. The maps in the Kozul resolution are $d_k : \wedge^t V \otimes_{\mathbb{C}} S \to \wedge^{t-1} V \otimes_{\mathbb{C}} S$ and defined on elements

$$d_k(v_1 \wedge \cdots \wedge v_t \otimes f) = \sum_{i=1}^{t} (-1)^{i+1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_t \otimes v_i f,$$

here $\hat{v}_i$ means the term v_i is omitted. This implies (by the "Euler-Poincaré principle") that

$$\begin{aligned} P(A, q) &= \sum_{j=0}^{t} (-1)^{j+1} P(\wedge^j V \otimes_{\mathbb{C}} S, q) \\ &= \left(\sum_{j=0}^{t} (-1)^{j+1} P(\wedge^j V, q) \right) P(S, q). \end{aligned}$$

We have

$$P(S, q) = \left(\prod_{j=1}^{n} (1 - q^{a_j}) \right)^{-1} \quad \text{and} \quad \sum_{j=0}^{t} (-1)^{j+1} P(\wedge^j V, q) = \prod_{i=1}^{t} (1 - q^{b_i}).$$

□

We can now present the formula for calculating the graded dimension of $A(\lambda)^+$. Recall that D_λ denotes the Young diagram for a partition λ and $h(i,j)$ is the hook length of the cell (i,j).

Theorem 5.1.8. *For any $A(\lambda)^+$, we have*

$$\sum_{i \geq 0} (\dim A(\lambda)_i^+) q^i = \frac{\prod_{i=1}^{n} (1 - q^i)}{\prod_{(i,j) \in D_\lambda} (1 - q^{h(i,j)})}.$$

Proof. Since $A(\lambda)^+$ is a complete intersection by Lemma 5.1.6, Lemma 5.1.7 implies that

$$\sum_{i \geq 0} (\dim A(\lambda)_i^+) q^i = P(A(\lambda)^+, q) = \frac{\prod_{i=1}^{t} (1 - q^{b_i})}{\prod_{j=1}^{n} (1 - q^{a_j})}.$$

Lemma 2.4.3 says that $\prod_{j=1}^{n} (1 - q^{a_j}) = \prod_{(i,j) \in D_\lambda} (1 - q^{h(i,j)})$. By definition of $A(\lambda)^+$,

$$\prod_{i=1}^{t} (1 - q^{b_i}) = \prod_{i=1}^{n} (1 - q^i).$$

Hence,

$$\sum_{i\geq 0}(\dim A(\lambda)^+_i)q^i = \frac{\prod_{i=1}^n(1-q^i)}{\prod_{(i,j)\in D_\lambda}(1-q^{h(i,j)})}.$$

$\square$

We now include an example to show how useful and simple this formula is to use.

Example 5.1.9. The Young diagram for the partition $\lambda = (3, 1, 0, 0)$ is

$$\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

and therefore

$$\sum_{i\geq 0}(\dim A(\lambda)^+_i)q^i = \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)}{(1-q^4)(1-q^2)(1-q)(1-q)} = \frac{1-q^3}{1-q} = 1+q+q^2.$$

Hence $A(\lambda)^+$ consists of a one dimensional space in degrees 0, 1 and 2.

The formula for the graded dimensions allows us to calculate the dimension of the entire algebra $A(\lambda)^+$.

Theorem 5.1.10. *The dimension of $A(\lambda)^+$ is given by the hook length formula*

$$\dim A(\lambda)^+ = \frac{n!}{\prod_{(i,j)\in D_\lambda} h(i,j)}.$$

Proof. We have the formula

$$\sum_{i\geq 0}(\dim A_i)q^i = \frac{\prod_{i=1}^n(1-q^i)}{\prod_{(i,j)\in D_\lambda}(1-q^{h(i,j)})}.$$

We can use L'Hopitals rule to evaluate the formula when $q = 1$. Clearly the left hand side gives the dimension of A. Repeated applications of L'Hoptials rule to the right hand side gives

$$\frac{n!}{\prod_{(i,j)\in D_\lambda} h(i,j)}.$$

$\square$

§ 5.2 | Calculating $A(\lambda)^+$ directly from the partition

To provide an explicit presentation of $A(\lambda)^+$ directly from a partition $\lambda \vdash n$ we must understand how the coefficients of the terms appear in the Wronskian. We shall split the problem of understanding the coefficients into two distinct cases. We say that terms of the form $f_{i,j}$ are linear and terms of the form $f_{i_1,j_1}f_{i_2,j_2}\cdots f_{i_m,j_m}$ for $m > 1$ are non-linear.

We begin with the simpler task of understanding the linear terms.

The first question we want to answer is how often do linear terms appear in the coefficients of the Wronskian for a given partition of n. It is fairly straight forward to see that each linear term can appear in only one coefficient. Recall that algebra $A(\lambda)^+$ can be written as

$$\mathbb{C}[f_{ij},\ i = 1, \ldots, n,\ \ j = 1, \ldots d_i,\ d_i - j \notin P]/(r_1, \ldots, r_n)$$

where the r_s are homogeneous elements and $\deg(r_s) = s$. Abusing terminology we say that "a monomial m in the $f_{i,j}$ appears in r_s" if the coefficient of m in r_s is non-zero.

Lemma 5.2.1. *If the linear term $f_{i,j}$ appears as a monomial in one of the elements r_s, then $j = s$.*

Proof. By Lemma 5.1.3 the Wronskian is a homogeneous polynomial of degree n. Also $A(\lambda)^+$ is a complete intersection and therefore the coefficient of u^i is non-zero for all $0 \leq i \leq n$. In other words, $r_i \neq 0$ for all i. Therefore, a linear coefficient of u^i has degree $n-i$. Since the linear term $f_{i,j}$ has degree j we conclude that it can only appear in r_j. □

The following is a partial converse to Lemma 5.2.1.

Lemma 5.2.2. *Consider a finite dimensional commutative ring*

$$A = \frac{\mathbb{C}[x_1, \ldots, x_n]}{(r_1, \ldots, r_n)},$$

where the relations r_i are homogeneous and do not contain any constant terms. For each $1 \leq j \leq n$, there exist $k \geq 1$ and i such that r_i contains the monomial x_j^k.

Proof. Argue by contradiction. Assume that x_1^k does not appear in any r_i for $k \geq 1$. Since the algebra is finite dimensional and positively graded, we must have $x_1^m = 0$ for some m and hence

$$x_1^m = \sum_l c_l r_l$$

for $c_l \in \mathbb{C}$. Since every monomial with non-zero coefficient in r_j is divisible by some x_i with $i \neq 0$, there is a well defined evaluation morphism $ev_c : A \to \mathbb{C}$ that sends x_1 to some constant $c \neq 0$ and x_i to 0 for $i > 1$. Then

$$c^m = ev_c(x_1^m) = ev_c\left(\sum_l c_l r_l\right) = \sum_l c_l ev_c(r_l) = 0,$$

which is a contradiction. □

Lemma 5.2.3. *Let $f_1, \cdots, f_n$ be the set of pairwise distinct polynomials as in (5.1.1). All terms in the Wronskian $\mathrm{Wr}(f_1, \cdots, f_n)$ of the form $f_{i,j}^{k_1} \cdots f_{u,v}^{k_z}$ have $k_s \leq 1$ for all $1 \leq s \leq z$.*

Proof. By writing the Wronskian

$$\mathrm{Wr}(f_1, \cdots, f_n) = \det \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1^{(1)} & f_2^{(1)} & f_3^{(1)} & \cdots & f_n^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}$$

the statement becomes clearer. Each term in the determinant is some product of terms which do not share a row or column. Fix an $f_{i,j}$, then we see that it only appears in the i^{th} column. Since the product of the terms in the Wronskian cannot share a column we see that if $f_{i,j}^k$ appears then $k = 1$. □

Lemma 5.2.3 states that in the case of the Wronskian our coefficients can not contain terms of a higher power than 1, for instance we cannot have f_{11}^2 appearing in the relations. It is also clear that the coefficients of the Wronskian contain no constant terms, since they are homogeneous of positive degree. These observations give us the following lemma.

Lemma 5.2.4. *Given a partition $\lambda \vdash n$, each $f_{i,j}$ appears as a linear term of a coefficient in the Wronskian.*

Proof. Follows from Lemma 5.2.3 and Lemma 5.2.2. □

Now that we have proven that each linear term must appear we can strengthen Lemma 5.2.1. Using the same notation as before we have the following.

Proposition 5.2.5. *Each linear term $f_{i,j}$ appears in r_j and with non-zero coefficient.*

Proof. Follows from Lemma 5.2.1 and Lemma 5.2.4. □

Proposition 5.2.5 completely solves the problem of understanding the position of the linear terms. The next step is to prove an analogous statement for the non-linear terms. We show that each non-linear term does appear in the coefficients, except for a specific few. We will need the following results first.

Lemma 5.2.6. *Let $\{f_1, \cdots, f_n\}$ be the set of polynomials defined as in (5.1.1). Assume there are polynomials f_i and f_j such that f_i contains a term of the form $f_{i,s}u^k$ and f_j contains a term $f_{j,t}u^k$. Then the Wronskian $\mathrm{Wr}(f_1, \cdots f_n)$ contains no monomial divisible by $f_{i,s}f_{j,t}$.*

Proof. The determinant is a sum of multiples of elements of different columns and different rows in the matrix. The terms $f_{i,s}$ and $f_{j,t}$ only appear in the columns i and j respectively. Hence, all the terms with form $f_{i,s}f_{j,t}$ appearing in the determinant come from an expression of the form $F(f_i^{(a)}f_j^{(b)} - f_i^{(b)}f_j^{(a)})$. Here F is some multiple of entries from different rows and columns excluding columns i and j. An easy calculation gives $f_{i,s}u^{(a)}f_{j,t}u^{(b)} - f_{i,s}u^{(b)}f_{j,t}u^{(a)} = 0$. □

There is a useful recursive formula for the Wronskian.

Proposition 5.2.7. *The recursive formula for the Wronskian is given by*

$$\mathrm{Wr}(f_1, \cdots, f_n) = f_1^n \mathrm{Wr}\left(\left(\frac{f_2}{f_1}\right)', \cdots, \left(\frac{f_n}{f_1}\right)'\right)$$

where $\deg(f_i) < \deg(f_j)$ *for* $i < j$.

Proof. See [43, Proposition 1]. □

We will use this formula to prove two lemmata that will be crucial in showing that most non-linear terms are non-zero.

Lemma 5.2.8. *Let* $\{f_1, \cdots, f_n\}$ *be a set of monomials in one variable such that* $\deg(f_i) > \deg(f_j)$ *for* $i < j$*, and* $f_i \neq 0$ *for all* i*. The Wronskian* $\mathrm{Wr}(f_1, \cdots, f_n)$ *is non-zero.*

Proof. Proceed by induction on n. The case where $n = 1$ is obvious. Assume the statement is true for all $m < n$, then use the recursive formula given in Proposition 5.2.7

$$\mathrm{Wr}(f_1, \cdots, f_n) = f_1^n \mathrm{Wr}\left(\left(\frac{f_2}{f_1}\right)', \cdots, \left(\frac{f_n}{f_1}\right)'\right).$$

Clearly $\mathrm{Wr}((\frac{f_2}{f_1})', \cdots, (\frac{f_n}{f_1})')$ satisfies the assumptions of the lemma. Therefore the polynomial $\mathrm{Wr}((\frac{f_2}{f_1})', \cdots, (\frac{f_n}{f_1})')$ is non-zero. Since $f_1^n \neq 0$,

$$\mathrm{Wr}(f_1, \cdots, f_n) = f_1^n \mathrm{Wr}\left(\left(\frac{f_2}{f_1}\right)', \cdots, \left(\frac{f_n}{f_1}\right)'\right) \neq 0.$$

□

Lemma 5.2.9. *Let* $\{f_1, \cdots f_n\}$ *be a set of monomials in one variable, such that* $\deg f_i \neq \deg f_j$ *for all* $i \neq j$ *and* $f_i \neq 0$ *for all* i*. The Wronskian* $\mathrm{Wr}(f_1, \cdots, f_n)$ *is non-zero.*

Proof. We see from Lemma 5.2.8 above that if $\deg(f_i) > \deg(f_j)$ for $i < j$ then the Wronskian is non-zero. Since the monomials all have pairwise different degrees there is a matrix A that permutes the columns of the matrix such that the monomials are in order of ascending degree in the first row. Then $\det \mathrm{Wr}(f_1, \ldots, f_n) = \det(A)\det(\mathrm{Wr}(f_1', \ldots, f_n'))$

where $f_i' > f_j'$ for $i < j$. Since A is an invertible matrix its determinant is non-zero and so $\det \mathrm{Wr}(f_1, \ldots, f_n) \neq 0$. $\square$

Before we present the main theorem of this section let us establish a convention. Recall Lemma 2.4.3, which stated that the set

$$\{j \mid d_i - j \notin P \text{ for } 1 \leq j \leq d_i\}$$

is equal to the set of hook lengths in the i^{th} row of length λ_i. Also recall that

$$f_i = u^{d_i} + \sum_{j=1,\ d_i - j \notin P}^{d_i} f_{i,j} u^{d_i - j}.$$

Lemma 2.4.3 implies there is a bijection between the polynomials $f_{i,j}$ for a fixed i and the cells in the i^{th} row of λ. This bijection sends the cell (i, j) to $f_{i,h(i,j)}$. To demonstrate this bijection let us consider an example.

Example 5.2.10. Take the partition $(3, 2)$. The Young diagram is

4	3	1
2	1	

.

The cells of the first row, read left to right, are mapped to $f_{1,4}$, $f_{1,3}$ and $f_{1,1}$ respectively. The cells of the second row are similarly mapped to the generators $f_{2,2}$ and $f_{2,1}$.

We say that two generators $f_{i,j}$ and $f_{s,t}$ share a row or column if they share a row or column in the Young diagram under this mapping.

Theorem 5.2.11. *Fix a partition $\lambda \vdash n$ and let $f_1, \cdots, f_n$ be as in (5.1.1). The non-linear coefficients of $\mathrm{Wr}(f_1, \cdots, f_n)$ are all non-zero except for monomials divisible by $f_{i,j} f_{s,t}$ where $i = j$ or $h(i,j) = h(s,t)$. In other words, they have no factors that share a row or column in the partition diagram.*

Proof. The first thing to note is that if two generators $f_{i,j}$ and $f_{s,t}$ share the same row in the partition then $i = s$ and they must appear in the same column in the Wronskian. Hence, they cannot appear in the determinant. Assume now that $f_{i,j}$ and $f_{s,t}$ appear in the same column of D_λ. Lemma 2.4.4 says that

$$d_a - h(a, b) = d_c - h(c, b)$$

holds for all a, b and c. In particular if $f_{i,j}$ and $f_{s,t}$ share the same column in the Young diagram then $d_i - j = d_s - t$. From the definition of the f_i in (5.1.1), we see that in f_i the monomial $f_{i,j}$ is the coefficient of $u^{d_i - j}$. Likewise $f_{s,t}$ is the coefficient of $u^{d_s - t}$ in f_s.

Lemma 5.2.6 then implies that there is no monomial in the Wronskian that is divisible by $f_{i,j}f_{s,t}$. We need only prove that the other nonlinear terms are non-zero.

We prove this for products of two monomials, the general case follows from a similar argument using the coefficients in Proposition 5.2.12. Assume that $f_{i,j}$ and $f_{s,t}$ share neither a row nor a column in the Young diagram. In the determinant $f_{i,j}f_{s,t}$ will be the coefficient of the u^{n-j-t} term. This observation lets us see that when deciding if this is nonzero we need only consider entries in the Wronskian that are scalars or powers of u. That is we exclude all $f_{y,z}$ where $f_{y,z} \neq f_{i,j}$ or $f_{s,t}$. Hence we need only check that $\mathrm{Wr}(u^{d_1}, \cdots, f_{i,j}u^{d_i-j}, \cdots, f_{s,t}u^{d_s-t}, \cdots, u^{d_n})$ is non-zero. This simplifies

$$\mathrm{Wr}(u^{d_1}, \cdots, f_{i,j}u^{d_i-j}, \cdots, f_{s,t}u^{d_s-t}, \cdots, u^{d_n}) = f_{i,j}f_{s,t}\mathrm{Wr}(u^{d_1}, \cdots, u^{d_i-j}, \cdots, u^{d_s-t}, \cdots, u^{d_n}).$$

Since $f_{i,j}$ and $f_{s,t}$ do not share a column in D_λ, Lemma 2.4.4 implies that $d_i - j \neq d_s - t$. Therefore, the degrees of all the monomials are pairwise different and non-zero. In this case Lemma 5.2.9 implies that the determinant is non-zero. $\square$

These results can be improved upon by giving a formula for calculating the scalar coefficients of the linear and non-linear terms in the Wronskian. Let us explain some necessary notation. Recall that the recursive formula in Proposition 5.2.7 is given for a particular order of entries, namely that they are increasing in degree. This is often not the case, and so we must permute the columns of the Wronskian first. Let A be the matrix that permutes in the desired order. Note that $\det(A) = \sigma(A)$, where σ is the sign function.

Proposition 5.2.12. *The scalar coefficient of a given term $f_{i_1j_1}\dots f_{i_mj_m}$ is*

$$\sigma(A)\prod_{\substack{i<j, i\neq i_k,\\ j\neq j_k}} d_i - d_j \prod_{d_{i_k}-j_k>d_j \text{ and } 1\leq k\leq m} (d_{i_k} - j_k - d_l) \prod_{d_i - i_k > d_j - j_l} (d_i - i_k - d_j - j_l).$$

Proof. The proof is by induction, on the size of Wronskian, the case $m = 1$ is clear. Assume the statement holds for all m up to $n-1$. Consider the case $m = n$. We have

$$\mathrm{Wr}(u^{d_1}, \dots, f_{i_1j_1}u^{d_{i_1}-j_1}, \dots, u_n^d) = \det(A)\mathrm{Wr}(u_n^d, \dots, f_{i_1j_1}u^{d_{i_1}-j_1}, \dots, u^{d_1})$$

$$= u^{d_n}\mathrm{Wr}((u^{d_{n-1}}/u^{d_n})', \dots, f_{i_1j_1}(u^{d_{i_1}-j_1}/u^{d_n})', \dots, (u_1^d/u^{d_n})')$$

$$= u^{d_n}\mathrm{Wr}((d_{n-1}-d_n)u^{d_{n-1}-d_n-1}, \dots, f_{i_1j_1}(d_{i_1}-j_1-d_n)u^{d_{i_1}-j_1-d_n-1}, \dots, (d_1-d_n)u^{d_1-d_n-1})$$

$$= \prod_{1\leq i<n}(d_i - d_n)\prod_{d_{i_k}-j_k>d_n}(d_{i_k}-j_k-d_n)f_{i_1j_1}\dots f_{i_mj_m}\mathrm{Wr}(u^{d_{n-1}-d_n-1}, \dots, u^{d_{i_1}-j_1-d_n-1}, \dots, u^{d_1-d_n-1}).$$

and therefore

$$= \sigma(A) \prod_{\substack{i<j, i\neq i_k,\\ j\neq j_k}} d_i - d_j \prod_{d_{i_k}-j_k>d_j \text{ and } 1\leq k\leq m} (d_{i_k} - j_k - d_l) \prod_{d_i - i_k > d_j - j_l} (d_i - i_k - d_j - j_l).$$

□

By defining a new term e we can greatly simplify this expression.

Proposition 5.2.13. *Consider the nonlinear term $f_{i_1j_1}\dots f_{i_mj_m}$ ordered so that $i_a < i_{a+1}$. Define*

$$e_{i_k} = \begin{cases} d_{i_k} - j_k & \text{if } 1 \leq k \leq m \\ d_{i_k} & \text{else.} \end{cases}$$

Then the scalar coefficient of the term of $f_{i_1j_1}\dots f_{i_mj_m}$ in the Wronskian is

$$\prod_{1\leq i<j\leq n} e_i - e_j.$$

Proof. From Proposition 5.2.12 we see that

$$\sigma(A) \prod_{\substack{i<j, i\neq i_k,\\ j\neq j_k}} d_i - d_j \prod_{d_{i_k}-j_k>d_j \text{ and } 1\leq k\leq m} (d_{i_k} - j_k - d_l) \prod_{d_i - i_k > d_j - j_l} (d_i - i_k - d_j - j_l) \quad (5.2.1)$$

is the coefficient of $f_{i_1j_1}\dots f_{i_mj_m}$. By definition of the e_i, the term

$$\prod_{1\leq i<j\leq n} e_i - e_j. \quad (5.2.2)$$

is either equal to (5.2.1) or its negative. Now note that (5.2.1) is negative or positive precisely when $\sigma(A)$ is negative or positive. If $\sigma(A)$ is negative then there is an odd number of terms $e_i - e_j$ such that $e_i - e_j < 0$. In this case (5.2.2) is also negative. By a similar argument (5.2.1) is positive precisely when (5.2.2) is. □

By collecting our results so far we can explicitly describe $A(\lambda)^+$ with no mention of the Wronskian.

Theorem 5.2.14. *Let $\lambda \vdash n$ be a partition of length t. The algebra $A(\lambda)^+$ is the quotient*

$$A(\lambda)^+ \cong \mathbb{C}[f_{i_1,h(i_1,j_1)}, \cdots, f_{i_t,h(i_t,j_t)}]/I$$

by the ideal I that is generated by n homogeneous elements $r_1, \dots, r_n$. The r_i are ordered so that $\deg(r_i) = i$. The monomials in r_i are all products of the form $f_{i_k,h(i_k,j_k)} \cdots f_{i_\ell,h(i_\ell,j_\ell)}$ such that $u \neq v$ and $w \neq x$ for any two factors $f_{u,h(u,w)}$ and $f_{v,h(v,x)}$. The coefficients of the monomials inside the r_i are given by Proposition 5.2.13.

Proof. Follows from Proposition 5.2.5, Theorem 5.2.11 and Proposition 5.2.13. $\square$

We can rewrite Theorem 5.2.14 using a bijection between the generators $f_{i,h(i,j)}$ and the Young diagram of the partitions. Assign the generator $f_{i,h(i,j)}$ to the cell (i,j) in D_λ.

Theorem 5.2.15. *Let $\lambda \vdash n$ be a partition. The algebra $A(\lambda)^+$ is the quotient*

$$A(\lambda)^+ \cong \mathbb{C}[D_\lambda]/I$$

by the ideal I that is generated by n homogeneous elements $r_1, \ldots, r_n$. The r_s are ordered so that $\deg(r_s) = s$. The monomials in r_i are products of cells which share neither a row or column in D_λ. In other words if $\square_{i,j}\square_{k,\ell}$ is a factor of some monomial in the r_s we must have that $i \neq k$ and $j \neq \ell$. The coefficients of the generators of I are given by Proposition 5.2.13.

Proof. Follows from Theorem 5.2.14 using the described bijection. $\square$

Theorem 5.2.15 allows us to directly derive a presentation of $A(\lambda)^+$ from the Young diagram. This is much simpler than the previously described method using the Wronskian. To demonstrate this point we will calculate $A(3,2)^+$ using Theorem 5.2.15, compare this with Example 5.1.1.

Example 5.2.16. Let $\lambda = (3,2)$. First lets write out the Young diagram with the hook lengths of the cells included

$$\begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 2 & 1 & \\ \cline{1-2} \end{array}.$$

Therefore our generators are $f_{1,4}$, $f_{1,3}$, $f_{1,1}$, $f_{2,2}$ and $f_{2,1}$. Hence,

$$A(\lambda)^+ \cong \mathbb{C}[f_{1,4}, f_{1,3}, f_{1,1}, f_{2,2}, f_{2,1}]/(r_1, r_2, r_3, r_4, r_5).$$

The relations r_i contain the all the linear terms. The relations r_i also contain each non-linear term except those that have factors sharing a row or column. From the Young diagram we see that the non-zero nonlinear terms are then $f_{1,1}f_{2,1}$, $f_{1,1}f_{2,2}$, $f_{1,3}f_{2,2}$ and $f_{1,4}f_{2,1}$. Furthermore, we know that the terms appear in the relations according to their degree. So we need only compute the coefficients, which can be done using Proposition 5.2.13. Recall that $d_1 = 7$, $d_2 = 5$, $d_3 = 2$, $d_4 = 1$ and $d_5 = 0$, hence $e_1 = 6$, $e_2 = 4$,

$e_3 = 2$, $e_4 = 1$, $e_5 = 0$. Let us calculate the coefficient of $f_{1,1}f_{2,1}$. This is the product

$$\prod_{1\leq i<j\leq n} e_i - e_j = (6-4)(6-2)(6-1)(6-0)(4-2)(4-1)(4-0)(2-1)(2-0)(1-0).$$

which is equal to 11520. The other coefficients are similarly calculated. The relations are

$$14400f_{11} + 30240f_{21}$$

$$11520f_{11}f_{21} + 10080f_{22}$$

$$-2880f_{13} + 4320f_{11}f_{22}$$

$$-1440f_{14}$$

$$-288f_{14}f_{21} + 288f_{13}f_{22}.$$

Therefore, $A(3,2)^+ \cong \mathbb{C}[f_{11}]/(f_{11}^5)$.

§ 5.3 | The wreath product

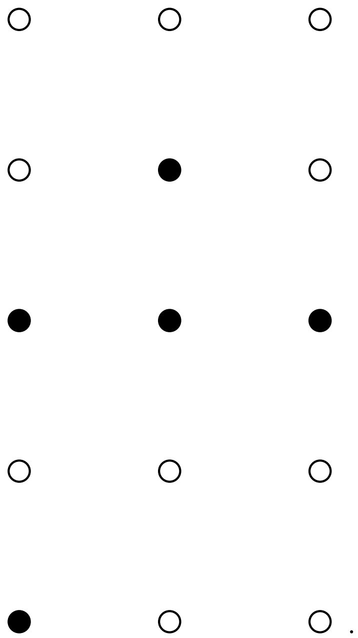
Let us now describe how to write the blocks of the centre of the restricted rational Cherednik algebra for the wreath product directly from a given ℓ-multipartition. The key is Theorem 4.2.25, that there is an isomorphism

$$A(\mathrm{quo}_\ell(\lambda))^+ \cong A(\lambda)^+_{\mathbb{Z}/\ell\mathbb{Z}}.$$

This will allow us to take advantage of the theory we have developed in Section 5.2. We first explain how to obtain λ given $\mathrm{quo}_\ell(\lambda)$, then we will prove that $A(\lambda)^+_{\mathbb{Z}/\ell\mathbb{Z}}$ is the quotient of $A(\lambda)^+$ given by killing all terms that have a degree not divisible by ℓ. By combining these facts we are able to write $A(\mathrm{quo}_\ell(\lambda))^+$ explicitly in terms of generators and relations.

Recall Theorem 2.4.10 which states there is a bijection between the set of partitions of $n\ell$ with trivial ℓ-core and the ℓ-multipartitions of n. This bijection is given by taking a partition to its ℓ-quotient. We now wish to do the opposite. Given an ℓ-multipartition of n we would like to recover the corresponding partition of $n\ell$ with trivial ℓ-core. There is no particularly deep theory here, all we must do is reverse the process. We include an example demonstrating how this is done.

Example 5.3.1. Let us find the partition of 27 with trivial 3-core and 3-quotient $((3,2),$ $(1,1),$ $(2))$. We first need to construct the columns in the bead diagram. For a set of β-numbers we use the first column hook lengths of these partitions. The set of first column hook lengths for $(3,2)$, $(1,1)$ and (2) are $\{4,2\}$, $\{2,1\}$ and $\{2\}$ respectively. Since the first column hook lengths are $\{4, 2\}$ our first column should have beads in the second and fourth position. Doing the same for the other two columns we obtain the bead diagram below



Note that this diagram already has trivial 3-core. We then read the first column hook lengths $\{4, 6, 7, 8, 12\}$ from the above diagram. Then using formula (2.4.1) we get

$$\lambda_i = h(i, 1) + i - L,$$

where $L = 5$ because $|\{4, 6, 7, 8, 12\}| = 5$. It is then easy to calculate $\lambda = (8, 5, 5, 5, 4)$.

Let us now prove the claim that $A(\lambda)^{+}_{\mathbb{Z}/\ell\mathbb{Z}}$ contains only the polynomials of degree divisible by ℓ. Recall that

$$A(\lambda)^{+}_{\mathbb{Z}/\ell\mathbb{Z}} := \mathbb{C}[\pi_{n\ell}^{-1}(0)]/\langle f - s \cdot f \mid s \in \mathbb{Z}/\ell\mathbb{Z}, f \in \mathbb{C}[\pi_{n\ell}^{-1}(0)]\rangle.$$

Lemma 5.3.2. *The ideal $\langle f - s \cdot f | s \in \mathbb{Z}/\ell\mathbb{Z}, f \in \mathbb{C}[\pi_{n\ell}^{-1}(0)]\rangle$ is generated by all homogeneous elements in $A(\lambda)^{+}$ with degree not divisible by ℓ.*

Proof. Let $f \in A(\lambda)$ be a homogeneous element with degree not divisible by ℓ. Then

$$f - s \cdot f = (1 - \alpha)f, \text{ where } 1 \neq \alpha \in \mathbb{C}.$$

Therefore $(1 - \alpha)^{-1} f - s \cdot (1 - \alpha)^{-1} f = f$ and $f \in \langle h - s \cdot h | s \in \mathbb{Z}/\ell\mathbb{Z}, h \in \mathbb{C}[\pi_{n\ell}^{-1}(0)]\rangle$. If f is of degree ℓ then $f - s \cdot f = 0$. Any other polynomial is a sum of homogeneous polynomials and hence $f - s \cdot f$ is a sum of homogeneous elements of degree not divisible by ℓ. $\square$

This lemma allows us to present a version of Theorem 5.2.14 for the wreath product.

Theorem 5.3.3. *Let $\lambda \vdash n\ell$ have trivial ℓ-core and write $\mathrm{quo}_\ell(\lambda)$ for its ℓ-quotient. Assume λ has length t. The algebra $A(\mathrm{quo}_\ell(\lambda))^+$ is the quotient*

$$A(\mathrm{quo}_\ell(\lambda))^+ \cong \mathbb{C}[f_{i_1,h(i_1,j_1)}, \cdots, f_{i_s,h(i_s,j_s)}]/I$$

of the polynomial ring generated by all $f_{i_k,h(i_k,j_k)}$ for $1 \leq k \leq t$ such that $h(i_k, j_k)$ is divisible by ℓ for all $1 \leq k \leq t$. The ideal I is generated by n homogeneous elements $r_\ell, r_{2\ell} \ldots, r_{n\ell}$. The r_i are ordered so that $\deg(r_i) = i$. The monomials in r_i are all products of the form $f_{i_k,h(i_k,j_k)} \cdots f_{i_m,h(i_m,j_m)}$ such that $u \neq v$ and $w \neq x$ for any two factors $f_{u,h(u,w)}$ and $f_{v,h(v,x)}$. The coefficients of the monomials inside the r_i are given by Proposition 5.2.13.

Proof. By Theorem 4.2.25
$$A(\mathrm{quo}_\ell(\lambda))^+ \cong A(\lambda)^+_{\mathbb{Z}/\ell\mathbb{Z}}.$$
The theorem then follows from Theorem 5.2.14 and Lemma 5.3.2. $\square$

Recall that we used the bijection assigning $f_{i,h(i,j)}$ to the cell $(i, j) \in D_\lambda$ to present a neater version of Theorem 5.2.14. We can do the same here.

Theorem 5.3.4. *Let $\mathrm{quo}_\ell(\lambda)$ be the ℓ-quotient of $\lambda \vdash n\ell$. The algebra $A(\mathrm{quo}_\ell(\lambda))^+$ is the quotient*
$$A(\mathrm{quo}_\ell(\lambda))^+ \cong \mathbb{C}[D^\ell_\lambda]/I$$
where D^ℓ_λ is the subdiagram of D_λ (the younger diagram) excluding the cells (i, j) such that $h(i, j)$ is not divisible by ℓ. The ideal I is generated by n homogeneous elements $r_\ell, r_{2\ell}, \ldots, r_{n\ell}$. The $r_{s\ell}$ are ordered so that $\deg(r_{s\ell}) = s\ell$. The monomials in $r_{s\ell}$ are products of cells which share neither a row or column in D^ℓ_λ. In other words if $\square_{i,j}\square_{k,m}$ is a factor of some monomial appearing in the $r_{s\ell}$, we must have that $i \neq k$ and $j \neq m$. The coefficients of the generators of I are given by Proposition 5.2.13.

Proof. Follows from Theorem 5.3.3 and the bijection between the terms $f_{i,h(i,j)}$ and the cell $(i, j) \in D_\lambda$. $\square$

Let us now show how to use this theorem to directly calculate $A(\mathrm{quo}_\ell(\lambda))^+$ from $\mathrm{quo}_\ell(\lambda)$.

Example 5.3.5. We take the 3-partition $((1, 1), \emptyset, (1))$ and find the corresponding partition of 9 with trivial 3-core. The first column hook lengths, are respectively, $\{1, 2\}$, $\{0\}$ and $\{1\}$ hence we have the bead diagram

$$
\begin{array}{ccc}
\circ & \circ & \circ \\
\bullet & \circ & \bullet \\
\bullet & \circ & \circ \; .
\end{array}
$$

There is a problem however as this does not have trivial 3-core. Recalling that we only begin counting position from the first empty bead we can rewrite our columns so that they still correspond to the hook lengths $\{1, 2\}$, $\{0\}$ and $\{1\}$ while having trivial 3-core. We simply add beads before the first empty position until this is achieved and so

$$
\begin{array}{ccc}
\circ & \bullet & \circ \\
\bullet & \bullet & \bullet \\
\bullet & \circ & \circ \; .
\end{array}
$$

We now read the first column hook lengths for the partition of 9 from this diagram, remembering to start counting at the first empty position. Therefore, the set of first column hook lengths are $\{1, 3, 4, 5, 6\}$, which is the partition $(2, 2, 2, 2, 1)$. Let us write the Young diagram with hook lengths inside their respective cells

6	4
5	3
4	2
3	1
1	

Before we begin to write down the generators and relations we remark that by Lemma 5.3.2 we can ignore all generators that have a degree not divisible by 3. Hence $A((1, 1), \emptyset, (1))^+$ is a quotient of the algebra

$$\mathbb{C}[f_{16}, f_{23}, f_{43}].$$

The relations are now found in the same way as before, noting that we can discard any that have a degree not divisible by 3. Recall that we find the relations by noting that they are homogeneous and the linear terms always appear, the non-linear terms are those that do not share either a row or column in the Young diagram. In this case we have the relations

$$c_{23}f_{23} + c_{43}f_{43},\ c_{16}f_{16} + c_{23,43}f_{23}f_{43},\ c_{16,23}f_{16}f_{23}$$

where the c_i can be calculated using the formula in Proposition 5.2.12. Simplifying we see that

$$A((1,1),\emptyset,(1))^+ \cong \mathbb{C}[f_{23}]/(f_{23}^3).$$

Lets consider a slightly more complicated example.

Example 5.3.6. We take the 3-partition $((1,1),(1),(1,1,1))$ and find the corresponding partition of 18 with trivial 3-core. The first column hook lengths, are respectively, $\{1,2\}$, $\{1\}$ and $\{1,2,3\}$ hence we have the bead diagram

$$\begin{array}{ccc} \circ & \circ & \circ \\ \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet \\ \circ & \circ & \bullet \end{array}.$$

There is a problem however as this does not have trivial 3-core. Recalling that we only begin counting position from the first empty bead we can rewrite our columns so that they still correspond to the hook lengths $\{1,2\}$, $\{1\}$ and $\{1,2,3\}$ while having trivial 3-core. We simply add beads before the first empty position until this is achieved and so

$$\begin{array}{ccc} \bullet & \bullet & \circ \\ \circ & \bullet & \bullet \\ \bullet & \circ & \bullet \\ \bullet & \bullet & \bullet \end{array}.$$

We now read the first column hook lengths for the partition of 18 from this diagram, remembering to start counting at the first empty position. Therefore, the set of first column hook lengths are $\{2, 3, 4, 6, 7, 8, 9\}$ which is the partition $(3, 3, 3, 3, 2, 2, 2,)$. Let us write the Young diagram with hook lengths inside their respective cells

9	8	4
8	7	3
7	6	2
6	5	1
4	3	
3	2	
2	1	

Theorem 5.3.4 states we can ignore all generators that have a degree not divisible by 3. Hence $A((1,1),(1),(1,1,1))$ is a quotient of the algebra

$$\mathbb{C}[f_{19}, f_{23}, f_{36}, f_{46}, f_{53}, f_{63}].$$

Likewise, when deriving the relations we can ignore all cells that have a hook length not divisible by 3. This simplifies things somewhat but there are still 6 relations one of each degree 3, 6, 9, 12, 15 and 18. We leave out the coefficients when writing this, but these can be calculated using Proposition 5.2.12. The defining relations are

$$f_{23} + f_{53} + f_{63}$$

$$f_{36} + f_{46} + f_{23}f_{53} + f_{23}f_{63} + f_{53}f_{63}$$

$$f_{19} + f_{36}f_{23} + f_{36}f_{63} + f_{46}f_{23} + f_{46}f_{53} + f_{23}f_{53}f_{63}$$

$$f_{19}f_{23} + f_{19}f_{53} + f_{36}f_{46} + f_{36}f_{63}f_{23} + f_{46}f_{23}f_{53}$$

$$f_{19}f_{36} + f_{46}f_{36}f_{23}$$

$$f_{19}f_{36}f_{23}.$$

§ 5.4 | Isomorphism of block centres

In this section we prove two significant isomorphisms. The first an isomorphism between two $A(\lambda)^+$ for different λ. This is useful from a computational point of view, roughly halving the amount of algebras that must be calculated to find the centre. The second is the previously mentioned isomorphism between $A(\lambda)^-$ and $A(\lambda)^+$. This allows us to use our previous results to give a presentation of the entire centre.

Theorem 5.2.15 allows us to easily calculate $A(\lambda)^+$ for small $\lambda \vdash n$. Doing so we realise there is a useful isomorphism that approximately halves the amount of work we must do to calculate all the $A(\lambda)^+$ for a chosen n. Let λ^T denote the transpose of λ. We prove

$$A(\lambda)^+ \cong A(\lambda^T)^+.$$

Given an isomorphism $\delta : H_{\overline{c}}(W) \to H_c(W)$ and a $H_c(W)$-module M, we can make a $H_{\overline{c}}(W)$-module M^δ that as a set equals M, but with the action given by

$$h \cdot m = \delta(h)m,$$

for $h \in H_{\overline{c}}(W)$ and $m \in M$. In particular, define an isomorphism $\sigma : H_c(S_n) \to H_{-c}(S_n)$ by

$$\sigma(x) = x, \ \sigma(y) = y \text{ and } \sigma(w) = (-1)^{sgn(w)}w \text{ for all } x \in \mathfrak{h}, \ y \in \mathfrak{h} \text{ and } w \in S_n.$$

We can now state our first theorem.

Theorem 5.4.1. *Let $\lambda \vdash n$ be a partition and λ^T its transpose. There is an isomorphism of $H_c(S_n)$-modules $\underline{\Delta}_{-c}(\lambda)^\sigma \cong \underline{\Delta}_c(\lambda^T)$.*

Proof. Let $S = \mathbb{C} \cdot s$ be the sign representation. There is an isomorphism of irreducible representations $\lambda^T \cong \lambda \otimes S$ see [39, p. 36]. We claim there is an isomorphism of $H_c(W)$-modules $\phi : \underline{\Delta}_c(\lambda \otimes S) \to \underline{\Delta}_{-c}(\lambda)^\sigma$ given by

$$\phi(h \otimes v \otimes s) = \sigma(h) \otimes v,$$

and extended linearly. We must check that ϕ is well-defined. Since the module $\underline{\Delta}(\lambda \otimes S)$ is uniquely defined by the following conditions

1. $y \cdot 1 \otimes v \otimes s = 0$ for all $y \in \mathfrak{h}$
2. $w \cdot 1 \otimes v \otimes s = 1 \otimes w \cdot (v \otimes s) = 1 \otimes wv \otimes ws = 1 \otimes wv \otimes sgn(w)s$

we check that ϕ preserves these conditions.

$$\phi(y \cdot 1 \otimes v \otimes s) = \phi(y \otimes v) = \sigma(y) \otimes v = y \otimes v = 0$$

and

$$\phi(w \cdot 1 \otimes v \otimes s) = \phi(w \otimes v) = \sigma(w) \otimes v = sgn(w)w \otimes v = sgn(w) \otimes wv$$

and

$$sgn(w) \otimes wv = \phi(sgn(w) \otimes wv \otimes s) = \phi(1 \otimes wv \otimes sgn(w)s).$$

This map is surjective since $\underline{\Delta}_{-c}(\lambda)^\sigma$ is generated by $1 \otimes \lambda$. It is injective because both $\underline{\Delta}_c(\lambda \otimes S)$ and $\underline{\Delta}_{-c}(\lambda)^\sigma$ are free $\mathbb{C}[\mathfrak{h}]$-modules of rank $\dim \lambda$. It is a morphism of $H_{\overline{c}}(S_n)$-modules because

$$\begin{aligned} a \cdot \phi(h \otimes v \otimes s) &= a \cdot (\sigma(h) \otimes v) \\ &= \sigma(a)\sigma(h) \otimes v \\ &= \sigma(ah) \otimes v \\ &= \phi(ah \otimes v \otimes s) \\ &= \phi(a \cdot h \otimes v \otimes s). \end{aligned} \tag{5.4.1}$$

Therefore ϕ is an isomorphism. □

Now define another isomorphism $\eta : H_c(S_n) \to H_{-c}(S_n)$ by

$$\eta(x) = -x,\ \eta(y) = -y \text{ and } \eta(s) = s \text{ for all } x \in \mathfrak{h},\ y \in \mathfrak{h} \text{ and } s \in S_n.$$

Theorem 5.4.2. *Let $\lambda \vdash n$ be a partition. We have an isomorphism of $H_c(S_n)$-modules $\underline{\Delta}_c(\lambda)^\eta \cong \underline{\Delta}_{-c}(\lambda)$.*

Proof. The proof is similar to the proof of Theorem 5.4.1. □

Corollary 5.4.3. *There is an isomorphism $\underline{\Delta}_c(\lambda)^{\eta\circ\sigma} \cong \underline{\Delta}_c(\lambda^T)$.*

Proof. This follows from Theorem 5.4.1 and Theorem 5.4.2. □

Lemma 5.4.4. *The isomorphism $\eta \circ \sigma$ induces an isomorphism of baby Verma modules*

$$\eta \circ \sigma : \Delta(\lambda)^{\eta\circ\sigma} \cong \Delta(\lambda^T).$$

Proof. By Theorem 4.1.8 $\Delta(\lambda) = \underline{\Delta}(\lambda)/R_+\underline{\Delta}(\lambda)$ and so we need to show that

$$\eta \circ \sigma(R_+\underline{\Delta}(\lambda)) = R_+\underline{\Delta}(\lambda^T).$$

Since $\eta \circ \sigma$ is an isomorphism of $H_c(W)$-modules we know that

$$\eta \circ \sigma(R_+\underline{\Delta}(\lambda)) = (\eta \circ \sigma)(R_+)(\eta \circ \sigma)(\underline{\Delta}(\lambda)).$$

Since $\eta\circ\sigma(\underline{\Delta}(\lambda)) = \underline{\Delta}(\lambda^T)$ we need show that $\eta\circ\sigma(R_+) = R_+$. Note that by the definition of R_+ it contains no group elements and so σ is simply the identity on R_+. Hence we show that $\eta(R_+) = R_+$. By linearity it suffices to check on homogeneous elements in R_+. Again this is clear since applying η to an element of R_+ we find

$$\eta(p(x)_+ \otimes q(y) + r(x) \otimes s(y)_+) = (-1)^m p(x)_+ \otimes q(y) + (-1)^n r(x) \otimes s(y)_+$$

where $m = |\deg(p)||\deg(q)|$ and $n = |\deg(r)||\deg(s)|$. Therefore $\eta(R_+) = R_+$ and the proof is complete. $\square$

Lemma 5.4.5. *Let H be an algebra, Z its centre and $a : H \to H$ an automorphism. For any H-module M, a induces an isomorphism*

$$a : Z/\mathrm{ann}_Z M \to Z/\mathrm{ann}_Z M^a.$$

Proof. We show that $a(\mathrm{ann}_Z M) = \mathrm{ann}_{Z(W)} M^a$. This can be seen by noting that $z \in \mathrm{ann}_Z M$ if and only if $a^{-1}(z) \in \mathrm{ann}_Z M^a$ as

$$a^{-1}(z) \cdot m = a(a^{-1}(z))m = zm.$$

Therefore if $zm = 0$ then $a^{-1}(z) \cdot m = 0$ and vice versa. $\square$

Lemma 5.4.6. *There is an isomorphism of algebras*

$$\mathrm{End}_{H_c(W)} \underline{\Delta}(\lambda) \cong Z_c(W)/\mathrm{ann}_{Z_c(W)} \underline{\Delta}(\lambda).$$

Proof. By Theorem 4.1.8 the map given by multiplication by elements of $Z_c(W)$, $m : Z_c(W) \to \mathrm{End}_{H_c(W)} \underline{\Delta}(\lambda)$ is a surjection. The kernel of m is $\mathrm{ann}_{Z_c(W)} \underline{\Delta}(\lambda)$. $\square$

Theorem 5.4.7. *Let $\lambda \vdash n$ be a partition and λ^T its transpose. Then*

$$\mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda)) \cong \mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda^T)).$$

Proof. By Corollary 5.4.3 there is an isomorphism of $H_c(S_n)$-modules $\underline{\Delta}(\lambda^T) \cong \underline{\Delta}(\lambda)^{\eta\circ\sigma}$ and so we can rewrite the desired result as

$$\mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda)) \cong \mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda)^{\eta\circ\sigma}).$$

Consider the composition of automorphisms $a = \sigma \circ \eta : H_c(W) \to H_c(W)$. Since a is an automorphism of $H_c(W)$ it is also an automorphism of the centre $Z_c(W)$. By Lemma 5.4.6,

$$\mathrm{End}_{H_c(W)} \underline{\Delta}(\lambda) \cong Z_c(W)/\mathrm{ann}_{Z_c(W)} \underline{\Delta}(\lambda)$$

and

$$\mathrm{End}_{H_c(W)} \underline{\Delta}(\lambda)^a \cong Z_c(W)/\mathrm{ann}_{Z_c(W)}(\underline{\Delta}(\lambda)^a).$$

By Lemma 5.4.5, a restricts to an isomorphism

$$a : Z_c(W)/\mathrm{ann}_{Z_c(W)} \underline{\Delta}(\lambda) \to Z_c(W)/\mathrm{ann}_{Z_c(W)} \underline{\Delta}(\lambda)^a.$$

Therefore

$$\mathrm{End}_{H_c(W)}\underline{\Delta}(\lambda) \cong Z_c(W)/\mathrm{ann}_{Z_c(W)}\underline{\Delta}(\lambda) \cong Z_c(W)/\mathrm{ann}_{Z_c(W)}(\underline{\Delta}(\lambda)^a) \cong \mathrm{End}_{H_c(W)}\underline{\Delta}(\lambda)^a.$$

□

We can now prove the following important isomorphism.

Theorem 5.4.8. *Let $\lambda \vdash n$ be a partition and λ^T its transpose. Then*

$$A(\lambda)^+ \cong A(\lambda^T)^+.$$

Proof. Recall the definitions $A(\lambda)^+ := \mathrm{End}_{\overline{H}_c(W)}(\Delta(\lambda))$ and $A(\lambda^T)^+ := \mathrm{End}_{\overline{H}_c(W)}(\Delta(\lambda^T))$. By Theorem 4.1.15 we have

$$\mathrm{End}_{\overline{H}_c(W)}(\Delta(\lambda)) \cong \mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda))/\mathbb{C}[\mathfrak{h}]^W_+\mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda))$$

and

$$\mathrm{End}_{\overline{H}_c(W)}(\Delta(\lambda^T)) \cong \mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda^T))/\mathbb{C}[\mathfrak{h}]^W_+\mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda^T)).$$

Then Theorem 5.4.7 implies that

$$\mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda))/\mathbb{C}[\mathfrak{h}]^W_+\mathrm{End}_{H_c(W)}\underline{\Delta}(\lambda)) \cong \mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda^T))/\mathbb{C}[\mathfrak{h}]^W_+\mathrm{End}_{H_c(W)}(\underline{\Delta}(\lambda^T))$$

and so

$$A(\lambda)^+ = \mathrm{End}_{\overline{H}_c(W)}(\Delta(\lambda)) \cong \mathrm{End}_{\overline{H}_c(W)}(\Delta(\lambda^T)) = A(\lambda^T)^+.$$

□

With the techniques introduced in this section we are now in a position to prove the final result of this section. The following theorem proves the algebra $A_{\overline{c}}(\lambda^*)^- := \mathrm{End}_{\overline{H}_{\overline{c}}(W)}(\Delta^*(\lambda^*))$ is isomorphic to $A_c(\lambda)^+ := \mathrm{End}_{\overline{H}_c(W)}(\Delta(\lambda))$, but for different generic $\overline{c}$ and simple modules $\underline{\lambda}$, $\underline{\lambda}^*$. Recall the notation from Example 2.6.6. The defining relations for $H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ are given by

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0,$$

$$[y_i, x_i] = c(\sigma_{ij}\gamma_i^k\gamma_j^{-k})\sum_{i\neq j}\sum_{k=0}^{\ell}\sigma_{ij}\gamma_i^k\gamma_j^{-k} + \sum_{k=1}^{\ell-1}c(\gamma_i^k)\gamma_i^k$$

and

$$[y_i, x_j] = -c(\sigma_{ij}\gamma_i^k\gamma_j^{-k})\sum_{k=1}^{\ell}\omega^k\sigma_{ij}\gamma_i^k\gamma_j^{-k}.$$

Then define a new parameter $\overline{c}$ by

$$\overline{c}(\sigma_{ij}\gamma_i^k\gamma_j^{-k}) = c(\sigma_{ij}\gamma_i^k\gamma_j^{-k}) \text{ and } \overline{c}(\gamma_i^k) = c(\gamma_i^{-k}).$$

The irreducible modules of $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ are labeled by ℓ-multipartitions of n. If $\underline{\lambda} = (\lambda_1, \cdots, \lambda_\ell)$ then define $\underline{\lambda}^* := (\lambda_1, \lambda_\ell, \cdots, \lambda_2)$. We say that c is generic if the algebra $Z(H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}))$ is regular.

Theorem 5.4.9. *In the case of the wreath product of the symmetric group with the cyclic group there is an anti-graded isomorphism*

$$A_{\overline{c}}(\underline{\lambda}^*)^- \cong A_c(\underline{\lambda})^+.$$

Moreover $\overline{c}$ is generic if and only if c is generic.

Proof. The map $\phi : H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}) \to H_{\overline{c}}(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ given by

$$\phi(x_i) = y_i, \quad \phi(y_i) = -x_i, \quad \phi(\sigma) = \sigma \text{ and } \phi(\gamma_i) = \gamma_i^{-1}$$

for all i and for all $\sigma \in S_n$ is an anti-graded isomorphism of algebras. Recall the baby Verma module from Definition 2.6.9

$$\Delta_c(\underline{\lambda}) := \overline{H}_c \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \rtimes W} \underline{\lambda}$$

and the module

$$\Delta_c^*(\underline{\lambda}) := \overline{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*]^{coW} \rtimes W} \underline{\lambda}.$$

In an abuse of notation let ϕ denote its restriction to $\mathbb{C}S_n \wr \mathbb{Z}/\ell\mathbb{Z}$, then $\underline{\lambda}^\phi = \underline{\lambda}^*$; this follows from the construction of irreducible $\mathbb{C}S_n \wr \mathbb{Z}/\ell\mathbb{Z}$-modules as in [44, Section 5.3]. Since $\phi(\mathbb{C}[\mathfrak{h}]_+^W) = \mathbb{C}[\mathfrak{h}^*]_+^W$, this then implies that

$$\Delta_c(\lambda)^\phi = \Delta_{\overline{c}}^*(\lambda^*).$$

Now $M \to M^\phi$ is a functor H_c-mod$\to$ $H_{\overline{c}}$-mod which is an equivalence. This means that the map

$$\mathrm{End}_{H_c}(\Delta_c(\lambda)) \to \mathrm{End}_{H_{\overline{c}}}(\Delta_c(\lambda)^\phi) = \mathrm{End}_{H_{\overline{c}}}(\Delta_{\overline{c}}^*(\lambda^*))$$

is an isomorphism. This implies that

$$A_{\overline{c}}(\underline{\lambda}^*)^- = \mathrm{End}_{H_{\overline{c}}}\Delta_{\overline{c}}^*(\underline{\lambda}^*) \cong \mathrm{End}_{H_c}\Delta_c(\underline{\lambda}) = A_c(\underline{\lambda})^+.$$

Since ϕ is an isomorphism, $Z(H_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z}))$ is regular if and only if $Z(H_{\overline{c}}(S_n \wr \mathbb{Z}/\ell\mathbb{Z}))$ is regular. Thus c is generic if and only if $\overline{c}$ is generic. Therefore, if we know $A(\underline{\lambda})^+$ for generic c and for all $\underline{\lambda}$ then we know $A(\underline{\lambda}^*)^-$ for generic $\overline{c}$ and for all $\underline{\lambda}^*$. $\square$

As a result of the above theorem we see that the algebras $A(\lambda)^-$ can be understood in terms of $A(\lambda)^+$. With this knowledge we can achieve the goal of writing the centre of the restricted rational Cherednik algebra explicitly. This theorem has an especially pleasing form for the symmetric group as $\underline{\lambda}^* = \underline{\lambda}$ for all $\underline{\lambda} \vdash n$.

Theorem 5.4.10. *There is an isomorphism of the centre of $\overline{H}_c(S_n)$ for $c \neq 0$*

$$Z(\overline{H}_c(S_n)) \cong \bigoplus_{\lambda \in \mathrm{Irr} S_n} A(\lambda)^- \otimes A(\lambda)^+.$$

The algebra $A(\lambda)^+$ is given by Theorem 5.2.15 and $A(\lambda)^-$ is isomorphic to $A(\lambda)^+$ with the opposite grading.

Proof. This follows from Proposition 4.1.2, Corollary 4.1.5, Theorem 4.1.6, Theorem 5.2.15 and Theorem 5.4.9. □

A similar theorem holds for the wreath product.

Theorem 5.4.11. *There is an isomorphism of the centre of $\overline{H}_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})$ for generic c*

$$Z(\overline{H}_c(S_n \wr \mathbb{Z}/\ell\mathbb{Z})) \cong \bigoplus_{\underline{\lambda} \in \mathrm{Irr} S_n \wr \mathbb{Z}/\ell\mathbb{Z}} A_{\overline{c}}(\underline{\lambda}^*)^- \otimes A_c(\underline{\lambda})^+.$$

The algebra $A_c(\underline{\lambda})^+$ is given by Theorem 5.3.4 and $A_{\overline{c}}(\underline{\lambda}^*)^-$ is isomorphic to $A_c(\underline{\lambda})^+$ with the opposite grading.

Proof. This follows from Proposition 4.1.2, Corollary 4.1.5, Theorem 4.1.6, Theorem 5.3.4 and Theorem 5.4.9. □

A final point to note is that the description of the centre is only an isomorphism. It is not at all clear how we can embed the generators $f_{i,j}$ of $A(\underline{\lambda})^+$ into the restricted rational Cherednik algebra.

§ 5.5 | Computations and a conjecture

Throughout this thesis we have been concerned with calculating the radical layers of various modules, in particular projective covers. Naturally, we should use the explicit description of $A(\lambda)^+$ to better understand its radical layers. However, calculating the radical layers of $A(\lambda)^+$ turns out to be very difficult for large n. For a given partition of n, the algebra $A(\lambda)^+$ is a quotient of a polynomial ring of dimension n, by an ideal generated by n homogeneous equations and so the complexity scales with n. With the use of a computer it is possible to calculate more radical layers of $A(\lambda)^+$ than would be possible by hand.

Through the use of Magma [13] we calculated the radical series, and hence layers, of $A(\lambda)^+$ for all partitions up to $n = 10$. The algebras $A(\lambda)^+$ are often complicated and hence calculating their radical layers demands a lot of computing power. We reduce the burden on the computer by using the technique of Gröbner basis. These are well understood objects and programs such as Magma have very efficient Gröbner basis algorithms that are easier for computers to calculate [26]. The first subsection explains how these are used. In particular, the initial ideal (which is generated by the Gröbner basis) allows us to find the dimensions of the radical layers and calculate the Lowey lengths.

Even using the Gröbner basis, the computations take too long for $n > 10$ to be in anyway practical. While the algebras $A(\lambda)^+$ increased in complexity, it should also be noted that the number of partitions of n increases rapidly as n increases. This was perhaps the main reason the calculation times ballooned after $n = 10$.

Even for the (relatively) small n examples that we computed, a pattern emerged for the radical series. While the specific radical layers might be very complicated, it appeared that the Lowey length of $A(\lambda)^+$ was predictable. We will shortly present a conjecture of what this Lowey length is. Let us begin this section by explaining how we computed these examples by taking advantage of the Gröbner basis. The second section will provide the theoretical justification for the conjecture.

§ 5.5.1 | Computer calculations and Gröbner basis

In this section we prove several statements concerning Gröbner basis, concluding with a statement on the dimension of the radical layers. We prove that the initial ideal allows us to compute the Lowey length of $A(\lambda)^+$. The code used is a modified version of a code originally provided by Gwyn Bellamy. It can be found in the appendix.

Let us quickly define the Gröbner basis, for an in-depth introduction see [21]. Set $R = \mathbb{C}[x_1, \cdots, x_n]$. To define the Gröbner basis we need to first define monomial orderings and initial ideals.

Definition 5.5.1. A *monomial order* $\leq$ on R is a total order on the set of monomials in R such that

1. $1 \leq x$ for all monomials $x \in R$
2. if $x < y$ and w is any other monomial then $wx < wy$.

Definition 5.5.2. Let $<$ be a monomial ordering, and let $p \in R$ be a polynomial. Denote by $Lt(p)$ the greatest monomial in p under the ordering $<$. If $I \subset R$ is an ideal we define the initial ideal of I to be

$$in(I) = (Lt(p) \,|\, p \in I).$$

Definition 5.5.3. Let $I \subset R$ be an ideal and $<$ a monomial ordering on R. A sequence of elements $\{g_1, \cdots, g_n\} \subset I$ is a *Gröbner basis* of I with respect to $<$ if $in(I) = (Lt(g_1), \cdots, Lt(g_n))$.

The following is a well known and important theorem.

Theorem 5.5.4. *(Macaulay's Theorem) Let $<$ be a monomial order on R. If $I \subset R$ is an ideal then the monomials that do not belong to $in(I)$ form a basis for R/I over $\mathbb{C}$.*

Proof. See [21, Theorem 2.6]. □

As a consequence of Macaulay's theorem we have the following important corollary.

Corollary 5.5.5. *For any ideal $I \subset R$ of finite codimension we have*

$$\dim R/I = \dim R/in(I).$$

Recall that $A(\lambda)^+$ is a quotient of a polynomial algebra by an ideal I and that I is generated by n homogeneous equations. The next two lemmata we prove will show that any Gröbner basis of I has at least n elements.

Lemma 5.5.6. *Let J be a monomial ideal in R. Then $\dim R/J < \infty$ if and only if $x_i^{a_i} \in J$ for some a_i and every $1 \leq i \leq n$.*

Proof. ($\Rightarrow$) We argue by contradiction. Assume that $\dim R/J$ is finite and there exists some i such that $x_i^a \notin J$ for any a. Then $\{x_i^a \,|\, a \geq 0\}$ is a linearly independent subset of R/J and so R/J is not finite dimensional. This is a contradiction.

($\Leftarrow$) If $x_i^{a_i} \in J$ for some a_i and every $1 \leq i \leq n$ then $\{x_1^{k_1} \cdots x_n^{k_n} \,|\, 0 \leq k_i \leq a_i\}$ is a spanning set of R/J. Hence $\dim R/J < \infty$. □

Lemma 5.5.7. *If $I \subset R$ is an ideal such that $\dim R/I < \infty$ then for any Gröbner basis $G = \{g_1, \dots, g_k\}$ there exists $\{g_{i_1}, \dots, g_{i_n}\}$ such that $Lt(g_{i_r}) = x_r^{a_r}$. In particular $k \geq n$.*

Proof. Since $in(I)$ is a monomial ideal and Corollary 5.5.5 implies that $\dim R/in(I) < \infty$, Lemma 5.5.6 implies $x_i^{a_i} \in in(I)$ for some a_i and every $1 \leq i \leq n$. Now we also know that $in(I) = (Lt(g_1), \dots Lt(g_k))$. Since the leading terms are monomials we must have that $Lt(g_{i_r}) = x_r^{a_r}$. □

In other words the above lemma says that the number of elements in a Gröbner basis is at least n. We now include an example showing that sometimes this is a strict inequality.

Example 5.5.8. Consider the partition $\lambda = (3, 2, 1)$, and $A(\lambda)^+ = R/I$. We will argue by contradiction that there is no Gröbner basis of just 6 elements. Assume that $\{g_1, \dots, g_6\}$ is a Gröbner basis of I. Let us show that there is no monomial ordering on R such that $\dim R/(Lt(g_1), \dots, Lt(g_6)) < \infty$. This will contradict Proposition 5.5.13.

The linear terms all appear once in the generating elements r_i of I and some linear terms must appear in the same element, such as f_{11}, f_{21} and f_{31}. If we choose a monomial ordering so that one of these is the leading term in the relation then the powers of the other two terms will appear infinitely many times in any basis of $R/(Lt(g_1), \dots, Lt(g_6))$.

This observation is easily extended to other cases. The core of the argument is that having two $f_{i,j}$ of the same degree means that any monomial ordering fails to "see" one of them. This will happen for any partition not of the form $(n, 0, \cdots, 0)$ or $(1, 1, \cdots, 1)$. The next lemma will give us important facts about the latter case.

Lemma 5.5.9. *If $A = A(\lambda)^+ = R/(g_1, \dots, g_n)$ then the following are equivalent:*

1. *$\{g_1, ..., g_n\}$ is a Gröbner basis of $I = (g_1, \dots, g_n)$.*
2. *$A = \mathbb{C}$.*
3. *Each $g_i = cf_{rs}$ for some constant $c \in \mathbb{C} \setminus \{0\}$ and integers r and s.*
4. *$\lambda = (n)$ or $(1, 1, 1, \dots, 1)$.*
5. *$\dim R/(Lt(g_1), \dots, Lt(g_n)) < \infty$.*

Proof. (3) $\Rightarrow$ (1) If each $g_i = cf_{rs}$ then we have that $Lt(g_i) = cf_{rs} = g_i$ and so clearly $in(I) = I$ and $I = (g_1, \dots, g_n) = (Lt(g_1), \dots, Lt(g_n))$. Hence the set $\{g_1, \dots, g_n\}$ is a Gröbner basis.

(3) $\Leftrightarrow$ (5) This can be seen by the previous observation that $in(I) = I$ and Lemma 5.5.6.

(4) $\Rightarrow$ (3) If $\lambda = (n)$ or $(1, \ldots 1)$ then we know that the linear terms all occur once and have pairwise different degree. So $g_i = cf_{rs}$ for each i.

(3) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (4) This can be seen by an easy argument by contradiction. If λ is not (n) or $(1, 1, \ldots, 1)$ then we must have at least one relation with two linear terms in it. Then we make a choice for which term to eliminate and the other will not be killed. Hence $\dim R/in(I) > 1$ and thus $A \neq \mathbb{C}$.

(1) $\Rightarrow$ (5) We have that $in(I) = (Lt(g_1), \ldots, Lt(g_n))$ and so

$$\dim R/(Lt(g_1), \ldots, Lt(g_n)) = \dim R/in(I) = \dim R/I < \infty.$$

$\square$

We have so far said very little about monomial orderings. Often there is a superior choice of order depending on the structure of the algebra being studied. We will define a monomial ordering on $A(\lambda)^+$ that will allow us to prove an important result concerning the radical layers.

Recall that R is a polynomial ring on n generators. Denote the maximal ideal containing all these generators as $\mathfrak{m}$, this is the augmentation ideal. Consider the ideal $in(\mathfrak{m}^k + I)$. Then, as a set, this is $(Lt(m + i) \,|\, m \in \mathfrak{m}^k,\, i \in I)$. This contains both $in(\mathfrak{m}^k) = \mathfrak{m}^k$ and $in(I)$ and so $\mathfrak{m}^k + in(I) \subset in(\mathfrak{m}^k + I)$. We would like the following equality to hold

$$\mathfrak{m}^k + in(I) = in(\mathfrak{m}^k + I).$$

In general this might not be possible under any monomial ordering, but for the class of algebras $A(\lambda)^+ = R/I$ we can construct such an ordering.

Let us consider an element $a \in in(\mathfrak{m}^k + I)$ such that $a \notin \mathfrak{m}^k + in(I)$. We may assume $a = Lt(m + i)$ for some $m \in \mathfrak{m}^k$ and $i \in I$. Also, $a \neq Lt(m)$ and $a \neq Lt(i)$. The only way this can occur is if $Lt(m) = -Lt(i)$. So we must choose a monomial orderings so that this cannot occur. In the case of $A(\lambda)^+$ this ordering is given by choosing the lead term of a fixed degree to be the one with the least number of factors. If there is equality of the number of factors then choosing either lexicographic or reverse lexicographic will satisfy the conditions.

Definition 5.5.10. Define a monomial ordering on the $f_{i,j}$'s as follows. Let F and G be monomials.

1. $F > G$ if $\deg(F) > \deg(G)$.
2. If $\deg(F) = \deg(G)$ then $F > G$ if F has a smaller number of factors than G.
3. If the number of factors of F and G are equal and their degrees too, then use any monomial ordering, such as graded lexicographic.

Let us now summarise and formalise what we have discussed.

Lemma 5.5.11. *Given a partition λ of n and associated algebra $A(\lambda)^+ = \mathbb{C}[f_{i_1,j_1,\ldots f_{i_n,j_n}}]/I$, with the monomial ordering from Definition 5.5.10 we have the following equation*

$$\mathfrak{m}^k + in(I) = in(\mathfrak{m}^k + I).$$

Proof. We have seen that $\mathfrak{m}^k + in(I) \subset in(\mathfrak{m}^k + I)$. It remains to show the reverse inclusion. Let $a \in in(\mathfrak{m}^k + I)$. Then $a = Lt(m+i)$ for $m \in \mathfrak{m}^k$ and $i \in I$. There are two cases to consider, either $Lt(m)$ and $Lt(i)$ are the same monomial but with possibly different coefficients or they are not. If they are different monomials, then $Lt(m+i) = Lt(m)$ or $Lt(m+i) = Lt(i)$. In either case, $a \in \mathfrak{m}^k + in(I)$.

The second case we must consider is if $Lt(m)$ and $Lt(i)$ are the same monomial with possibly different coefficients. If $Lt(m) \neq -Lt(i)$ then $a = Lt(m+i) = Lt(m)+Lt(i)$ and hence $a \in \mathfrak{m}^k + in(I)$. So we need only consider the case $Lt(m) = -Lt(i)$. Note that in this case $Lt(i)$ must have at least k factors because $Lt(m) \in \mathfrak{m}^k$. Write $m = Lt(m) + m'$ and $i = Lt(i)+i'$. Then $a = Lt(m+i) = Lt(m'+i')$. If $\deg(i') = \deg(i)$ then $i' \in \mathfrak{m}^k$ due to our monomial ordering, as we stated that $Lt(i)$ had k factors. Hence $m+i = m'+i' \in \mathfrak{m}^k$, which implies that $a = Lt(m+i) \in \mathfrak{m}^k$. If $\deg(i') < \deg(i)$ then $i' \in I$ as I is generated by homogeneous elements. In this case we repeat our argument until either it is shown that $i^{(j)} \in \mathfrak{m}^k$ or $m = -i$. In either case we are done. □

Lemma 5.5.12. *If $I \subset R$ is a homogeneous ideal of finite codimension then*

$$\mathrm{rad}^i(R/I) = \frac{\mathfrak{m}^i + I}{I}.$$

Proof. Since I is homogeneous it must be contained in $\mathfrak{m}$. Therefore, since R/I is finite dimensional, R/I has the unique maximal ideal $\mathfrak{m}/I$ and the result follows inductively. □

Let us now show that we can use the initial ideal to calculate the Lowey length.

Proposition 5.5.13. *With the same monomial ordering as in Definition 5.5.10 we have for all i*

$$\dim \frac{\mathfrak{m}^i + I}{I} = \dim \frac{\mathfrak{m}^i + in(I)}{in(I)}.$$

Proof. By Lemma 5.5.12,

$$\dim R/I = \dim \frac{\mathfrak{m}^i + I}{I} + \dim \frac{R}{\mathfrak{m}^i + I}$$

and

$$\dim R/in(I) = \dim \frac{\mathfrak{m}^i + in(I)}{in(I)} + \dim \frac{R}{\mathfrak{m}^i + in(I)}.$$

By Corollary 5.5.5 we have that $\dim R/I = \dim R/in(I)$, hence

$$\dim \frac{\mathfrak{m}^i + I}{I} + \dim \frac{R}{\mathfrak{m}^i + I} = \dim \frac{\mathfrak{m}^i + in(I)}{in(I)} + \dim \frac{R}{\mathfrak{m}^i + in(I)}.$$

With our chosen ordering, we have $in(\mathfrak{m}^i + I) = \mathfrak{m}^i + in(I)$ by Lemma 5.5.11. Therefore

$$\frac{R}{\mathfrak{m}^i + in(I)} = \frac{R}{in(\mathfrak{m}^i + I)}.$$

Hence

$$\dim \frac{R}{\mathfrak{m}^i + in(I)} = \dim \frac{R}{in(\mathfrak{m}^i + I)} = \dim \frac{R}{\mathfrak{m}^i + I},$$

and thus

$$\dim \frac{\mathfrak{m}^i + I}{I} = \dim \frac{\mathfrak{m}^i + in(I)}{in(I)}.$$

□

The following proposition confirms that we can understand the dimension of the radical layers of $A(\lambda)^+$ by finding a Gröbner basis of I. This greatly shortens the computation time for calculating the dimension of the radical layers.

Proposition 5.5.14. *For any algebra of the form $A(\lambda)^+$ we have*

$$\dim \operatorname{rad}_i(R/I) = \dim \operatorname{rad}_i(R/in(I)).$$

Proof. By Proposition 5.5.13,

$$\dim \frac{\mathfrak{m}^i + I}{I} = \dim \frac{\mathfrak{m}^i + in(I)}{in(I)}.$$

By Lemma 5.5.12

$$\dim \mathrm{rad}_i(R/I) = \dim\left(\frac{(\mathfrak{m}^{i-1}+I)/I}{(\mathfrak{m}^i+I)/I}\right) \tag{5.5.1}$$
$$= \dim(\mathfrak{m}^{i-1}+I)/I - \dim(\mathfrak{m}^i+I)/I \tag{5.5.2}$$
$$= \dim(\mathfrak{m}^{i-1}+in(I))/in(I) - \dim(\mathfrak{m}^i+in(I))/in(I) \tag{5.5.3}$$

and

$$\dim(\mathfrak{m}^{i-1}+in(I))/in(I) - \dim(\mathfrak{m}^i+in(I))/in(I)$$
$$= \dim\left(\frac{(\mathfrak{m}^{i-1}+in(I))/in(I)}{(\mathfrak{m}^i+in(I))/in(I)}\right) = \dim \mathrm{rad}_i(R/in(I)).$$

□

§ 5.5.2 | Theoretical basis for the conjecture

The examples computed hint at a formulae for calculating the Lowey length of $A(\lambda)^+$. In this section we show that this estimate is an upper bound for the true value.

Definition 5.5.15. Let A be a finite dimensional positively graded ring with $A_0 = \mathbb{C}$. We define the degree of the socle of A to be the largest i such that $A_i \neq 0$.

This definition is motivated by the fact $A_i \subset \mathrm{soc}A$. Recall by Proposition 2.2.8 that the socle is the set of elements killed by the radical. From our explicit presentation of $A(\lambda)^+$ we see that the radical of $A(\lambda)^+$ is generated by the f_{ij}. Since all elements of the radical have at least degree 1 they all kill elements of the highest graded piece in $A(\lambda)^+$. We use the formula from Theorem 5.1.8 to calculate the degree of the socle of $A(\lambda)^+$.

Lemma 5.5.16. *For a given partition $\lambda \vdash n$ the following holds*

$$\deg(\mathrm{soc}(A(\lambda)^+)) = \frac{n(n+1)}{2} - \sum_{(i,j)\in D_\lambda} h(i,j),$$

where D_λ denotes the Young diagram of λ.

Proof. We calculate the highest graded piece of $A(\lambda)^+$. From the formula

$$\sum_{i\geq 0}(\dim A_i)q^i = \frac{\prod_{i=1}^n(1-q^i)}{\prod_{(i,j)\in D_\lambda}(1-q^{h(i,j)})},$$

we see that this must be

$$\sum_{i=1}^n i - \sum_{(i,j)\in D_\lambda} h(i,j).$$

□

The next lemma states that we have an upper bound for the radical series.

Lemma 5.5.17. *For any partition $\lambda \vdash n$ we have that* $\deg(\operatorname{soc}(A(\lambda)^+)) \geq \ell\ell(A(\lambda)^+)$.

Proof. First note that the minimum degree of elements in $\operatorname{rad}^k A(\lambda)^+$ is k. Also note that, by definition, the degree of the socle of $A(\lambda)^+$ is equal to the degree of the highest non-zero piece in $A(\lambda)^+$. If $\deg\operatorname{soc}(A(\lambda)^+) = d$ then $\operatorname{rad}^{d+1} A(\lambda)^+ = 0$ and hence $\deg(\operatorname{soc}(A(\lambda)^+)) \geq \ell\ell(A(\lambda)^+$. □

The difficulty is in finding a lower bound for the Lowey length. With the aid of computer calculations we arrived at a conjecture as to when equality holds. First let us quickly say what we mean by a rectangular partition. A rectangular partition is one which has the shape of a rectangle. As a consequence it has only one cell with hook length 1. Any non-rectangular partition has at least two cells with hook length 1. This is a key observation and leads to the following conjecture.

Conjecture 5.5.18. *For any non-rectangular partition λ we have*

$$\ell\ell(A(\lambda)^+) = \deg\operatorname{soc} A(\lambda)^+.$$

If λ is a rectangular partition then $\ell\ell(A(\lambda)^+) = (\deg\operatorname{soc} A(\lambda)^+)/2$.

Appendix A

Computing the radical layers of $A(\lambda)^+$

The Magma code below can be used to compute the radical layers of the algebras $A(\lambda)^+$ for $\lambda \vdash n$. Further it can calculate a Gröbner basis for the defining ideal of $A(\lambda)^+$.

```
> //Note that t is the grading and q is the radical layer.
>
>
> n := 6;  // This is the size of the partition.
>
> lam := [3,1,1,1]; // This is our partition.
>
> if &+lam ne n then  //We check that the size of lambda is indeed n.
if>     Error(n);
if> end if;
>
> if #lam lt n then                              //We add zeros to the end of lambda so that its length is n.
if>     zeros := [0 : i in [1..n-#lam]];
if>     lam := lam cat zeros;
if> end if;
>
> P := [lam[i] -i + n : i in [1..n]];
>
> var := [];
>
> for i in [1..n] do
for>     if P[i] ne 0 then
for|if>                 for j in [1..P[i]] do
for|if|for>                     if (P[i] - j) notin P then
for|if|for|if>                      varname := "f" cat IntegerToString(i) cat "," cat IntegerToString(j);
for|if|for|if>                      Append(~var,<varname,i,j>);
for|if|for|if>                  end if;
for|if|for>             end for;
```

```
for|if>         end if;
for> end for;
>
> names := [var[i][1] : i in [1..n]];  //This is the set of labels for the variables in our polynomial ring.
> Grad := [var[i][3] : i in [1..n]]; //This lists the degrees of the variables.
>
>
> QT<q,t> := PolynomialRing(Rationals(),2);
>
> R := PolynomialRing(Rationals(), Grad, "weight", [1,2,6,3,2,1, 0,1,5,2,1,0, 1,1,1,1,0,0, 1,1,1,0,0,0, 1,1,0,0,0,0, 1,0,0,0,0,0]);
>
> AssignNames(~R,names); //This gives the variables of the polynomial ring their name.
>
> //The variables in the ring R have a total order. It is given as follows fi,j > fk,l if j > l or if j = l and i < k.
>
> S<u> := PolynomialRing(R,1);
>
> phi := hom< S -> R | 1>;
>
> Polys := [];
>
> for i in [1..n] do
for>    if P[i] ne 0 then
for|if>                 p := u^(P[i]);
for|if>                 for j in [1..P[i]] do
for|if|for>                     if (P[i] - j) notin P then
for|if|for|if>                          Pos := [k : k in [1..n] | (var[k][2] eq i) and (var[k][3] eq j)];
for|if|for|if>                          p := p + Name(R,Pos[1])*(u^(P[i] - j));
for|if|for|if>                  end if;
for|if|for>             end for;
for|if>                 Append(~Polys,p);
for|if>         else
for|if>                 Append(~Polys,S!1);
for|if>         end if;
for> end for;
>
> mat := [];
>
```

```
> for i in [1..n] do
for>     row := [];
for>     for j in [1..n] do
for|for>                     Append(~row,Derivative(Polys[i],j-1,u));
for|for>          end for;
for>     mat := mat cat row;
for> end for;
>
> M := Matrix(n,n,mat);
> det := Determinant(M);
> coe := &*[P[i] - P[j] : i,j in [1..n] | i lt j];
>
> // M is the matrix whose determinant is the Wronskian.
> //det is the Wronskian
>
> print coe;
9676800
> // det := det*(coe^(-1));
>
> print det;
-9676800*u^6 + (-2903040*f1,1 - 11612160*f4,1)*u^5 + (-518400*f1,2 + 72
    57600*f3,2 - 3628800*f1,1*f4,1)*u^4 + (-1935360*f2,3 + 2419200*f1,1*
    f3,2 - 691200*f1,2*f4,1)*u^3 + (-725760*f1,1*f2,3 +
     518400*f1,2*f3,2)*u^2 - 207360*f1,2*f2,3*u + 34560*f1,6
>
> gen := Coefficients(det,u);
> gen := Prune(gen);
> gen := [phi(f) : f in gen];
>
> Aug := Ideal([Name(R,i) : i in [1..n]]); //This is the maximal ideal
    in the quotient ring.
>
> SmallInit := Ideal([LeadingMonomial(f) : f in gen]);  //This shows
    that the ideal generated by the initial monomials does not have a
    finite-dimensional quotient.
>
> /////////////
>
> I := Ideal(gen);
>
> Grob := GroebnerBasis(I);
> Init := Ideal([LeadingMonomial(f) : f in Grob]);
>
> HilbertSeries(Init);
t^6 + t^5 + 2*t^4 + 2*t^3 + 2*t^2 + t + 1
```

```
>
> grobqt := QT!0;
> Soc := Degree(Numerator(HilbertSeries(Init)));
> print Soc;
6
>
> for i in [1..Soc+1] do
for>     J := Aug^i + Init;
for>     Jmin := Aug^(i-1) + Init;
for>     f := HilbertSeries(J) - HilbertSeries(Jmin);
for>     print f;
for>     a := Coefficients(Numerator(f));
for>     print a;
for>     deg := #a;
for>     term := &+[a[j]*t^(j-1) : j in [1..deg]];
for>     grobqt := grobqt + term*q^(i-1);
for>     print grobqt;
for> end for;
1
[ 1 ]
1
t^2 + t
[ 0, 1, 1 ]
q*t^2 + q*t + 1
t^3 + t^2
[ 0, 0, 1, 1 ]
q^2*t^3 + q^2*t^2 + q*t^2 + q*t + 1
t^4 + t^3
[ 0, 0, 0, 1, 1 ]
q^3*t^4 + q^3*t^3 + q^2*t^3 + q^2*t^2 + q*t^2 + q*t + 1
t^4
[ 0, 0, 0, 0, 1 ]
q^4*t^4 + q^3*t^4 + q^3*t^3 + q^2*t^3 + q^2*t^2 + q*t^2 + q*t + 1
t^5
[ 0, 0, 0, 0, 0, 1 ]
q^5*t^5 + q^4*t^4 + q^3*t^4 + q^3*t^3 + q^2*t^3 + q^2*t^2 + q*t^2 + q*t
    + 1
t^6
[ 0, 0, 0, 0, 0, 0, 1 ]
q^6*t^6 + q^5*t^5 + q^4*t^4 + q^3*t^4 + q^3*t^3 + q^2*t^3 + q^2*t^2 + q
   *t^2 + q*t + 1
>
> grobqt;
q^6*t^6 + q^5*t^5 + q^4*t^4 + q^3*t^4 + q^3*t^3 + q^2*t^3 + q^2*t^2 + q
   *t^2 + q*t + 1
```

```
>
> ///////////////////////
>
>
> HilbertSeries(I);
t^6 + t^5 + 2*t^4 + 2*t^3 + 2*t^2 + t + 1
>
> pqt := QT!0;
> Soc := Degree(Numerator(HilbertSeries(I)));
> print Soc;
6
>
> for i in [1..Soc+1] do
for>     J := Aug^i + I;
for>     Jmin := Aug^(i-1) + I;
for>     f := HilbertSeries(J) - HilbertSeries(Jmin);
for>     a := Coefficients(Numerator(f));
for>     deg := #a;
for>     term := &+[a[j]*t^(j-1) : j in [1..deg]];
for>     pqt := pqt + term*q^(i-1);
for>     print pqt;
for> end for;
1
q*t^2 + q*t + 1
q^2*t^3 + q^2*t^2 + q*t^2 + q*t + 1
q^3*t^4 + q^3*t^3 + q^2*t^3 + q^2*t^2 + q*t^2 + q*t + 1
q^4*t^4 + q^3*t^4 + q^3*t^3 + q^2*t^3 + q^2*t^2 + q*t^2 + q*t + 1
q^5*t^5 + q^4*t^4 + q^3*t^4 + q^3*t^3 + q^2*t^3 + q^2*t^2 + q*t^2 + q*t
    + 1
q^6*t^6 + q^5*t^5 + q^4*t^4 + q^3*t^4 + q^3*t^3 + q^2*t^3 + q^2*t^2 + q
    *t^2 + q*t + 1
>
> pqt;
q^6*t^6 + q^5*t^5 + q^4*t^4 + q^3*t^4 + q^3*t^3 + q^2*t^3 + q^2*t^2 + q
    *t^2 + q*t + 1
>
> grobqt - pqt;
0
>
> Grob;
[
    f4,1^7,
    f1,6,
    f3,2*f4,1^3 + 6/5*f4,1^5,
    f3,2^2 - 2*f3,2*f4,1^2 - 4*f4,1^4,
```

```
    f2,3 + 10*f3,2*f4,1 + 10*f4,1^3,
    f1,2 - 14*f3,2 - 28*f4,1^2,
    f1,1 + 4*f4,1
]
>
> Init;
Ideal of Graded Polynomial ring of rank 6 over Rational Field
Order: Weight [full]
Variables: f1,1, f1,2, f1,6, f2,3, f3,2, f4,1
Variable weights: [1, 2, 6, 3, 2, 1]
Homogeneous, Dimension 0
Basis:
[
    f4,1^7,
    f1,6,
    f3,2*f4,1^3,
    f3,2^2,
    f2,3,
    f1,2,
    f1,1
]
```

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