Advantageous Monopolies in General Equilibrium: A Noncooperative Approach

by

Damiano Turchet

Submitted in fulfilment of the requirements for the

Degree of

Doctor of Philosophy in Economics

to

The Adam Smith Business School

College of Social Sciences

December, 2021

Abstract

The aim of this dissertation is to provide a comprehensive and novel approach to the study of monopoly in general equilibrium.
First, a foundational paper is provided, in which we define a concept of monopoly equilibrium within a bilateral exchange market a la Shitovitz, in which an atomic monopolist, that owns the totality of one commodity, faces an ocean of small traders (also called the atomless part of the economy), that owns another commodity. A monopoly equilibrium is defined as a quantity offered by the monopolist that maximizes her utility, given a price selection induces by her offers. After defining the notion of equilibrium, we provide parallels with the standard notions in partial equilibrium, under further assumptions on the structure of the demand function of the atomless sector for the good owned by the monopolist, namely its invertibility and differentiability. In particular, under the differentiability assumptions, we show that an interior monopoly equilibrium lies at the tangency point between the monopolists' indifference curve and the small traders' offer curve.
Once the notion of a monopoly equilibrium is established, the first follow up for the analysis is a game theoretic foundation for such an equilibrium. Following the work of Busetto et al. (2011), and recognizing the two stage flavour of the definition of a monopoly equilibrium, we provide a theorem that states the equivalence between the set of monopoly equilibrium and set of subgame perfect equilibria.
Next, to show the concept of monopoly equilibrium holds non vacuously, I proceed to study the existence properties of such equilibrium. Non existence examples in which small traders have CES utility functions are provided and a link between the existence of an equilibrium and the degree of substitutability of the goods is explored. Therefore, the existence result is proved by introducing a sufficient assumption on the utilities of the small traders, stressing that they need to be locally equivalent to CES utility functions whose elasticity is larger than unity.
Finally, an analysis of the optimality of such equilibrium is explored. In particular, drawing from Aumann seminal's paper on advantageous monopolies, we show how our model (under mild assumptions) is able to rule out the unintuitive situation of a monopolist being better off by being competitive. In particular, we show that a monop-

olist can be walrasian, i.e. the monoply equilibrium might coincide with a competitive equilibrium, and we characterize this equivalence proving that this only occurs whenever the monopolist optimal bid coincided with her endowment. However, whenever the set of monopoly equilibria and the set of competitive equilibria are disjoint, then the monopolist is always better off at a monopoly equilibrium and the small traders are exploited, i.e. their utility is always lower at a monopoly equilibrium with respect to a competitive equilibrium.

Contents

Acknowledgements

Submitting a PhD dissertation requires a long and constant committment to research, whose success depend not only on the student quality per se, but also on the whole environment that revolves around the student. It is therefore imperative being able to recognize the other factors that allowed this dissertation to be written.

In first place, I need to thank deeply my supervisors, professors Sayantan Ghosal and Yiannis Vailakis, for their invaluable guidance at each step of my doctorate and the way in which they properly introduced me to the academic world. Their patience, kindness and open mindedness taught me a lesson about mentoring which will never be forgotten.
Going backwards, I have to thank my MSc advisors, professors Giulio Codognato and Francesca Busetto, with whom I had the first chance of getting involved in research projects and pushed me to pursue a PhD. It is also with enormous pleasure that I can say that we are still collaborating and their help was very important in the writing of this thesis.

On a different level, my experience as a doctorate wouldn't have been as close as stimulating without getting to know colleagues that made the everyday office work more enjoyable and create space for fruitful discussions. I therefore need to thank Rohan, Prateek, Ashraful, Dev, Helena, Laura, Fivos, Elizaveta, Arthur, Jinglin, Max, Moritz, Johanna, Spyros and all the other I might have forgot to mention, for making me face more lightly the challenges of a PhD. This goes also to all the beautiful people I had the pleasure to meet along the way in Glasgow.
As the last milestone for my student career, I would also like to take a step back and mention my friulan friends, because I believe part of my merits come from them, and in particular Berto, Mazzo and Ele, Rachele, Spizzo and Maria, Nat and Laura, Alessandro, Sara, Tomat, Fabbro and Jasmine, Marco Matteo and Caterina, Tabi and Rachele, Pez and so many other.

Last, but maybe first in terms of importance, I owe this almost completely to my family, because the context in which I was raised for sure gave me the means to reach to this point. I would like to end this list of thanks by including my mother and my brother

as my everlasting anchors, and also my cousins, uncle and aunts, and a final thought to all who left us but still influence me as a person.

Declaration

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Signed: Damiano Turchet

The first two chapters of the thesis are based on the working paper ”On the foundation of monopoly equilibrium in bilateral exchange” (University of Glasgow Working Paper Series, Paper no.2021-04) which I coauthored with Professors Busetto, Codognato and Ghosal.
As a main contributor of the paper and in accordance with my coauthors, I received authorisation to inlcude the results as part of my dissertation.

Signed:
Francesca Busetto
Giulio Codognato
Sayantan Ghosal

Chapter 1

Introduction

Any microeconomics undergraduate textbook will provide an indepth analysis of competitive markets, usually followed by a chapter on monopoly, studied through the lens of partial equilibrium.
This seems the first and most natural market framework in which we can take into account the concept of market power. Such a feature should have a direct counterpart in the general equilibrium setup, in which a more comprehensive market structure is considered to take into account different markets interacting. Since the pioneering work of Walras in the late 19^{th} century, economists have focused on the competitive equilibria features within the general equilibrium analysis.
However, up until the second half of the last century, the concept of market power went mostly overlooked within this framework. Aumann (1966), in his seminal work introducing the concept of continuum of traders, paved the way to the study of general equilibrium models by indicating how it would be natural to consider markets that encompass a series of large traders along an ocean of small traders. Specifically, in this type of markets, it can be shown that the metric structure imposed on the economy will lead to an endogenous non competitive behaviour for large traders.
Following Aumann's suggestion, Shitovitz (1973) first studied such markets to show how specific market structure will preserve the competitive outcomes, expressed as the equivalence of the core and the set of competitive equilibria. Gabszewicz and Mertens (1971), in a parallel paper, provide a similar result considering a slightly different set of markets. However, both of them focused on oligopolistic markets, implicitly describing monopolistic markets as a particular case of an oligopoly. In another milestone paper, Aumann (1973) gave rise to a famous paradox regarding monopolistic markets, where he showed how there might be situations in which a monopolist might be disadvataged by exploiting his market power. This apparently counterintutive result is one of the main motivations for this whole thesis. It is important to notice that this

stream of literature looked at market power from the lens of cooperative game theory, specifically looking at the relations between the core and the set of competitive equilibria.

This cooperative game approach was critiqued in a seminal paper by Okuno et al. (1980), in which they provide a similar exchange framework in which, by considering a non cooperative solution, namely a Cournot-Nash equilibrium, traders are shown to be better off by exploiting their market power, contradicting in particular the result of Shitovitz (Theorem B in Shitovitz, 1973). This intuition seemed to reconcile the cournotian analysis in partial equilibrium to a general equilibrium framework. Katz (1974), following this approach, tried to lay the grounds for an analysis of monopoly in general equilibrium, but didn't provide a clear trading mechanism that would lead to a solution for monopolistic markets.

Another strand of literature, initiated by Gabszewicz and Mertens (1972), introduced the notion of Cournot-Walras equilibrium, in which few large cournotian traders (i.e. agents with market power competing á la Cournot) face a number of competitive traders. The equilibrium notion here is derived from a non cooperative game in which the two set of traders are assumed to behave in a price-taking and in a price-making way accordingly. This model portrays a different way to look at market power, but comes with some limitations, in particular the fact that in the original model agents are assumed ex-ante to behave in a particular way. In a later paper, Codognato and Gabszewicz (1991) redefined the equilibrium notion in an exchange framework, solving an issue regarding the dependance on the normalisation chosen, which rose in production economies.

The last stream of literature which provided a different way to look at market power, is the one on strategic market games, initiated by the seminal works by Shapley and Shubik (1977) (for a more comprehensive literature review, see Giraud (2003)). In the original models of exchange, each agent's strategy entails bids for how much they intend to invest in purchasing other goods and how much they will offer to the market, generating a non cooperative game in strategic form. In this framework, for example, we can construct prices as ratios of total bids versus total offers. Clearly, in any discrete market, each agent has market power, in the sense that each deviation in their strategy will have a non negligible impact on prices.

Modelling imperfect competition has always posed challenges, and the main streams of literature mentioned previously are not exhaustive, reflecting the fact that many difficulties might arise by abandoning the simple and clear set of assumptions that found the competitive analysis framework (see d'Aspremont et al.(1992) for a clear exposition of these problems). The following paragraphs describes some of the modelling problems that would arise in discussing monopoly and how the main model used

throughout the thesis tackles those problems.
First, one would not want to impose arbitrarily the behaviour of an agent, but rather obtain it as a consequence of the structure of the economy. To this end, we will model the economy as a mixed market á la Shitovitz, in which one large trader (the monopolist) faces a continuum of traders. Within this setting, the competitive behaviour of the small traders is a consequence of the sheer number of small traders (following the argument introduced by Aumann (1966)), opposed to the strategic behaviour of the monopolist, which is due to her "bigness".
One second difficulty, when studying imperfect competition, is the necessity to establish the variables that will define the strategies of the players. Clearly, one can think of the duality of the Cournot and Bertrand analysis to show how quantity and price competition might hugely differ. In this thesis, the approach chosen follows the Cournot setting, mainly to avoid the well known problem that might arise when agents are setting prices. Again, it is also problematic to impose why certain agents (e.g. firms) are setting the prices while other agents (e.g. consumers) adjust their quantities based on the observed prices. It is worth noticing, that this problem is actually mitigated in a monopolistic setting, as the two problems will coincide when the demand function for the small traders is downward sloping and invertible (i.e. one to one).
Finally, one final problem arises when defining the solution concept to be used within the Cournot framework, namely the Cournot-Nash or the Cournot-Walras. In other words, we need to specify whether the strategic agents also have an advantage in moving first. It will be shown that the proposed solution concept for a monopolistic market should belong to the Cournot-Walras literature stream, simply because allowing the game to be simultaneously generates no optimal and non trivial equilibrium (this will be explained in detail in chapter 3).
One additional remark regards the endowments and the set of goods present in the economy. Throughout the thesis, we will consider an economy with two goods and corner endowment, meaning that the monopolist owns one good while small traders will own the second good. On one hand, the structure of this economy echoes the notion of bilateral oligopoly (see Gabszewicz and Michel (1997)), even if within a sequential setting (for the reason listed above). Moreover, it seems "neater" to model a monopolistic market giving the monopolist a corner in a good. On the other hand, the simplification of assuming the presence of only two goods has one intuitive advantage and a technical one. Clearly, assuming two goods can be thought as having a case in which the monopolist offers a consumption good while the small traders own a numeraire good, but it is not limited to the case of quasi linear utilities. There is also a technical reason that makes the two good preferable, which have to do with the appropriate trading model. For example, in this thesis we will refer to the Shapley window model (studied in Sahi and Yao (1989)), in which trade is completely centralized, but

one can claim a better fitting model would be one in which separate trading post are allowed (e.g. Amir et al. (1990)). In the case of an economy with two goods, these two models coincide as there can only be one trading post. Extending the model to an economy with many goods, even assuming a specific trading mechanism, might create technical complications which risk to disperse the main takeaways obtained throughout this thesis, but are worth exploring.

The aim of this thesis is to introduce a comprehensive analysis of monopolistic markets in general equilibrium, drawing the most interesting features from the aforementioned literature streams. The motivation stems from the fact that most of the literature in general equilibrium never focused exclusively on monopoly. This seems an important gap to be filled, as the study of monopolistic market in isolation may provide deeper and interesting insights with respect to considering monopoly ”only” a special case of an oligopoly.
The thesis it structured as follows, chapter 2 establishes the notion of a monopoly equilibrium in bilateral exchange markets, chapter 3 studies the game theoretic features of the previously introduced solution concept, chapter 4 deals with the problem of the existence of the equilibrium and chapter 5 analyses the efficiency and optimality properties of the equilibrium. Even if this dissertation is intended to be a unique corpus from the collection of all the chapters, each chapter will be written as a self contained paper, with his own literature review and bibliography. For this reason, the same model will be repeated throughout the chapters [1].

In Chapter 2, we propose an equilibrium definition for monopolistic markets. We consider a bilateral exchange model, in which the monopolist holds one good and she trades it with a continuum of small traders, who hold a second good. We won't investigate here the case of production. We define an explicit trade mechanism, such that each traders chooses quantities to send to the market in exchange for the good they don't own, and prices are set in such a way that they clear the market. The structure of the traders set will imply that the monopolist has market power. The monopoly equilibrium will be then the (positive) optimal supply of the monopolist that maximizes her utility after trade. On the other hand, small traders will behave competitively and will be allocated their walrasian demand at the emerging prices. We will also assume that traders will eventually coordinate correctly in the case more than one price might be compatible with the optimal action of the monopolist.
Once the solution concept is clearly defined, we compare it with the partial equilibrium analysis by showing that under assumptions that guarantee that the demand curve for the small traders is well behaved (i.e. invertible), we can define the optimal point ge-

[1] to avoid redundancy, some proofs will be omitted and referred to a different chapter

ometrically as the one in which the indifference curve of the monopolist is tangent to the offer curve.
Finally, we compare our solution concept to the ones present in the literature, namely the initial idea of Pareto, who considered a monopolistc market similiar to ours, but in which the monopolist gets no utility from the goods she owns, and the work of Katz, who introduced the intuition about the equilibrium point in monopolies, but without providing a trading mechanism.

The third chapter provide an explicit game theoretic foundation of the monopoly equilibrium proposed. In particular, we show how a monopoly equilibrium can never be reconciliated with a one stage game, namely a Cournot-Nash equilibrium. It is easy to show that generally the sets of monopoly equilibria and the set of Cournot-Nash don't coincide. However, in the spirit of Busetto et al.(2008), we recognize the 2 stage flavour of the setting, and we show how our equilibrium can be considered a special case of a Cournot-Walras equilibrium in monopolistic markets. Therefore, we can proof that the set of monopoly equilibria coincides with the set of subgame perfect equilibria of a two stage game in which the monopolist moves first and the small traders move second. This provides an interesting insight about the notion of market power in our monopolistic market, that is the first mover advantage, which in this situation can alone account for the market power the monopolist expresses.

The fourth chapter aims at providing a set of conditions that would guarantee the equilibrium to exist, i.e. to guarantee that there is trade in equilibrium. The previous literature of existence in market games either relies on assumptions on the set of traders (e.g. at least two large traders with indifference curves that doesn't touch the axis) or on the endowments (strictly positive for both goods, either for large traders or in the aggregate for small traders). We recognize that none of the previous assumptions hold in the monopolistic market we analyze, and therefore a new set of conditions should be provided. In particular, we decide to focus on the shape of the aggregate demand function. We explore the relations between the degree of substitutability of the two goods and the existence of the equilibrium. We consider simple models in which the atomless sector is homogeneous, i.e. all traders have the same tastes, represented by a constant elasticity of scale (CES) utility function. In this case, we show that if the elasticity parameter is low, the monopolist can manipulate indefinetely the price in her favour, leading to a non existence of the equilibrium. It is worth noticing here that the non existence stems from an impossibility of defining an optimal bid, rather than from an autharchic situation in which there is no trade. On the other hand, we know that if all small traders have utilities represented by linear tastes, i.e. perfect substitutes, then the monopoly equilibrium will coincide with the competitive equilibrium. Indeed, we know that this situation takes away all the price-making power from the monopolist,

who will then act as a price-taker, generating in the equilibrium a walrasian result.
By considering these two "forces", we recognize that the aggregate demand function should be enough elastic, at least for high prices, such that the monopolist can act competitively and therefore make sure that her offer is actually strictly positive. We provide a technical result in which we provide an existence theorem taking into account this feature.

The fifth chapters' motivation lies in the previously introduced Aumann paradox, i.e. the possibility for the monopolist to be better off by acting competitively. To address this problem within our model, we first provide conditions under which a monopoly equilibrium is efficient and we prove that the only way a monopolistic outcome can be Pareto efficient is when the outcome itself is competitive. Strangely enough, we are also able to show that this happens non vacuously and characterize those situations in which a monopolist acts as a price taker, which is when the monopolist offers all of her endowment. It is indeed noticing that there might be situations in which the outcome of a monopolistic market might be competitive.
We then move to analyzing the case in which the two sets, monopoly equilibria and competitive equilibria, are disjoint. First, given our definition of equilibrium, it can never happen that the monopolist is better off at any competitive equilibria, therefore we can exclude Aumann paradox within our framework. This can be shown by a simple revealed preference argument.
Moreover, we confirm the intuition from the partial equilibrium analysis, stating that the price (here specified as a price ratio of the good owned by the monopolist versus the other good, owned by the small traders) is higher at a monopoly equilibrium with respect to the competitive case. In the same way, under the assumption of the invertibility of the demand function, that the monopolist will restrict her supply to the market at a monopoly equilibrium (with respect to the competitive counterpart).
Focusing on the price finding, we can generally show that this implies that small traders are exploited at the monopoly equilibrium, meaning that their utility is lower at the monopoly equilibrium with respect to the competitive equilibrium. This provides a clear impact on welfare, as we can interpret the atomless sector as the "consumers" side in the classical framework. Therefore, important implications can be derived from these results, also in an antitrust context.

Bibliography

[1] Amir, R., Sahi, S. and Shubik, M. (1990) "A strategic market game with complete markets," *Journal of economic theory*, **51**, 126-143.

[2] Aumann, R.J. (1966), "Existence of competitive equilibria in markets with a continuum of traders,", *Econometrica*.

[3] Aumann R.J. (1973), "Disadvantageous monopolies," *Journal of Economic Theory* **6**, 1-11.

[4] Busetto F., Codognato G., Ghosal S. (2008), "Cournot-Walras equilibrium as a subgame perfect equilibrium," *International Journal of Game Theory* **37**, 371-386.

[5] Codognato G. and Gabszewicz, J.J. (1993) "Cournot-Walras equilibria in markets with a continuum of traders," *Economic Theory*, **3**, 453-464.

[6] d'Aspremont, C., Ferreira, R. and Gérard-Varet, L.A. (1992), "General equilibrium concepts under imperfect competition: a Cournotian approach,", *Journal of Economic Theory*, **73**, 199-230.

[7] Giraud, G. (2003), "Strategic market games: an introduction," *Journal of Mathematical Economics*, **39**, 355-375.

[8] Gabszewicz, J.J. and Michel, P. (1997), "Oligopoly equilibria in exchange economies," *Core Discussion Paper*.

[9] Gabszewicz, J.J. and Mertens, J.F. (1971), "An equivalence theorem for the core of an economy whose atoms are not" too" big, " *Econometrica*.

[10] Gabszewicz, J.J. and Vial, J.P. (1972), " Oligopoly " A la Cournot " in a General Equilibrium Analysis," *Journal of Economic Theory* **4**, 381-400.

[11] Kats A. (1974), "Monopolistic trading economies: a case of governmental control," *Public Choice* **20**, 17-32.

[12] Okuno M., Postlewaite A., Roberts J. (1980), "Oligopoly and competition in large markets," *American Economic Review* **70**, 22-31.

[13] Shapley, L. and Shubik, M. (1977), "Trade using one commodity as a means of payment," *Journal of Political Economy*, **5**, 937-968.

[14] Shitovitz B. (1973), "Oligopoly in markets with a continuum of traders," *Econometrica* **41**, 467-501.

Chapter 2

On the Foundation of Monopoly Equilibrium in Bilateral Exchange

[1]

Abstract

We address the problem of monopoly in general equilibrium in a mixed version of a monopolistic two-commodity exchange economy where the monopolist, represented as an atom, is endowed with one commodity and "small traders," represented by an atomless part, are endowed only with the other. We provide a theoretical foundation of the monopoly solution in this bilateral framework through a formalization of an explicit trading process inspired by Pareto (1896) for an exchange economy with a finite number of commodities. Finally, we give the conditions under which our monopoly solution coincides with that defined by Kats (1974) and those, more restrictive, under which it has the geometric characterization proposed by Schydlowsky and Siamwalla (1966). Moreover, we establish the formal relationships between our concept of a monopoly equilibrium and that proposed by Pareto (1896), by redefining the latter in terms of our bilateral exchange setting.

2.1 Introduction

To the best of our knowledge, Vilfredo Pareto was the first who gave a formalized treatment of the problem of monopoly for a general pure exchange economy with any

[1]Part of the material of this chapter can be found in the working paper Busetto F., Codognato G., Ghosal S. and Turchet D.(2020), On the Foundation of Monopoly Equilibrium in Bilateral Exchange

finite number of commodities, in the first volume of his *Cours d'économie politique*, published in 1896, pp. 62-68 (henceforth just Pareto (1896)). His monopoly quantity-setting solution rests on the assumption that the monopolist gets no utility from the only commodity he is endowed with, but only cares about the revenue he can obtain by selling it.

Seventy years later, Schydlowsky and Siamwalla (1966) proposed a formulation of the problem of monopoly without any mention to the previous work by Pareto (1896). In the context of a pure exchange economy, they considered a bilateral framework where one commodity is held by one trader behaving as a monopolist while the other is held by a "competitors' community." In contrast to Pareto's analysis, the monopolist desires both commodities. The authors provided a geometrical representation of the monopoly solution as the point of tangency between the monopolist's indifference curve and the offer curve of the competitors' community. They did not mention either the geometrical treatment of the monopoly problem previously given, at a very embryo stage, by Edgeworth (1881).

A few years later, Kats (1974), again without mentioning Pareto (1896), analyzed a pure exchange economy where one trader behaves as a monopolist, "calling the game" and maximizing his utility, whereas all the other traders in the economy behave competitively. He claimed that the monopoly quantity-setting solution must correspond to the monopolist's most preferred commodity bundle compatible with the aggregate initial endowments and with the offer curve of the competitive traders.

In this paper, we provide a theoretical foundation of the monopoly solution by formalizing an explicit trading process inspired to that first sketched by Pareto (1896).

We consider the mixed version of a monopolistic two-commodity exchange economy introduced by Shitovitz (1973) in his Example 1, in which one commodity is held only by the monopolist, represented as an atom, and the other is held only by small traders, represented by an atomless part. This framework can also be used to represent a finite exchange economy if the atomless part is split into a finite number of types with traders of the same type having the same endowments and preferences.

In our setup, the monopolist acts strategically, making a bid of the commodity he holds in exchange for the other commodity, while the atomless part behaves à la Walras. Given the monopolist's bid, prices adjust to equate the monopolist's bid to the aggregate net demand of the atomless part. Each trader belonging to the atomless part then obtains his Walrasian demand whereas monopolist's final holding is determined as the difference between his endowment and his bid, for the commodity he holds, and as the value of his bid in terms of relative prices, for the other commodity. We define a monopoly equilibrium as a strategy played by the monopolist, corresponding to a positive bid of the commodity he holds, which guarantees him to obtain, via the trading process described above, a most preferred final holding among those he can achieve through his bids.

The theoretical framework proposed in this paper to define and analyse monopoly equilibrium in bilateral exchange can be simplified, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, and compared with the standard partial equilibrium analysis of monopoly. Indeed, we show that, if this assumption holds, at an allocation corresponding to a monopoly equilibrium, the utility of the monopolist is maximal in the feasible (with respect to aggregate initial endowments) complement of the offer curve of the atomless part, thereby providing a foundation of the monopoly solution proposed by Kats (1974). Moreover, we show that, if the aggregate demand of the atomless part for the commodity held by the monopolist is not only invertible but also differentiable, a monopoly equilibrium has the geometric characterization proposed by Schydlowsky and Siamwalla (1966). This result rests on a notion which has a well-known counterpart in partial equilibrium analysis and was also used by Pareto (1896) to formulate his solution to the monopoly problem in exchange economies: the marginal revenue of the monopolist.

Finally, we go deeper into the relationship between our analysis and that proposed by Pareto (1896), by redefining and studying this author's concept of a monopoly equilibrium within our framework of bilateral exchange, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible.

The paper is organized as follows. In Section 2, we introduce the mathematical model and we define the notion of a monopoly equilibrium. In Section 3, we compare the monopoly equilibrium and the Cournot-Nash equilibrium. In Section 4, we provide a game theoretical foundation of the monopoly solution in a two-stage framework. In Section 5, we discuss the model. In Section 6, we characterize the monopoly equilibrium under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and we discuss the literature related to our monopoly solution. In Section 7, we draw some conclusions and we suggest some further lines of research.

2.2 Mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. Let T_0 denote the atomless part of T. We assume that $\mu(T_0) > 0$.[2] Moreover, we assume that $T \setminus T_0 = \{m\}$, i.e., the measure space $(T, \mathcal{T}, \mu)$ contains only one atom, the "monopolist." A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are two different commodities. A commodity bundle is a point in R^2_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^2_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. *$\mathbf{w}^i(m) > 0$, $\mathbf{w}^j(m) = 0$ and $\mathbf{w}^i(t) = 0$, $\mathbf{w}^j(t) > 0$, for each $t \in T_0$, $i = 1$ or 2, $j = 1$ or 2, $i \neq j$.*

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^2_+ \to R$, satisfying the following assumptions.

Assumption 2. *$u_t : R^2_+ \to R$ is continuous, strongly monotone, and strictly quasi-concave, for each $t \in T$.*

Let $\mathcal{B}$ denote the Borel σ-algebra of R^2_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $D \times F$ such that $D \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. *$u : T \times R^2_+ \to R$, given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R^2_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.*

A price vector is a nonnull vector $p \in R^2_+$. Let $\mathbf{X}^0 : T_0 \times R^2_{++} \to \mathcal{P}(R^2_+)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in R^2_{++}$, $\mathbf{X}^0(t, p) = \operatorname{argmax}\{u(x) : x \in R^2_+ \text{ and } px \leq p\mathbf{w}(t)\}$. For each $p \in R^2_{++}$, let $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \{\int_{T_0} \mathbf{x}(t, p)\, d\mu : \mathbf{x}(\cdot, p)$ is integrable and $\mathbf{x}(t, p) \in \mathbf{X}^0(t, p)$, for each $t \in T_0\}$. Since the correspondence $\mathbf{X}^0(t, \cdot)$ is nonempty and single-valued, by Assumption 2, it is possible to define the Walrasian demand of traders in the atomless part as the function $\mathbf{x}^0 : T_0 \times R^2_{++} \to R^2_+$ such that $\mathbf{X}^0(t, p) = \{\mathbf{x}^0(t, p)\}$, for each $t \in T_0$ and for each $p \in R^2_{++}$.

We can now state and show the following proposition.

Proposition 1. *Under Assumptions 1, 2, and 3, the function $\mathbf{x}^0(\cdot, p)$ is integrable and $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$ for each $p \in R^2_{++}$.*

[2] The symbol 0 denotes the origin of R^2_+ as well as the real number zero: no confusion will result.

Proof. . Let $p \in R^l_{++}$. Then, the graph of the correspondence $\mathbf{X}(\cdot, p)$, $\{(t, x) : x \in \mathbf{X}(\cdot, p)\}$, is a subset of $\mathcal{T} \bigotimes \mathcal{B}$, by the same argument as that used by Busetto et al. (2011) (see the proof of their Proposition). But then, by the measurable choice theorem in Aumann (1969), there exists a measurable function $\bar{\mathbf{x}}(\cdot, p)$ such that, $\bar{\mathbf{x}}(t, p) \in \mathbf{X}(t, p)$, for each $t \in T_0$, which is also integrable as $\bar{\mathbf{x}}^j(t, p) \leq \frac{\sum_{i=1}^l p^i \mathbf{w}^i(t)}{p^j}$, $j = 1, 2$, for each $t \in T_0$. We must have that $\mathbf{x}^0(\cdot, p) = \bar{\mathbf{x}}(\cdot, p)$ as $\mathbf{X}^0(t, p) = \{\mathbf{x}^0(t, p)\}$, for each $t \in T_0$. Hence, the function $\mathbf{x}^0(\cdot, p)$ is integrable and $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$, for each $p \in R^2_{++}$. ■

Let $\mathbf{E}(m) = \{(e_{ij}) \in R^4_+ : \sum_{j=1}^2 e_{ij} \leq \mathbf{w}^i(m),\ i = 1, 2\}$ denote the strategy set of atom m. We denote by $e \in \mathbf{E}(m)$ a strategy of atom m, where e_{ij}, $i, j = 1, 2$, represents the amount of commodity i that atom m offers in exchange for commodity j. Moreover, we denote by E the matrix corresponding to a strategy $e \in \mathbf{E}(m)$.

We then provide the following definition.

Definition 1. *Given a strategy $e \in \mathbf{E}(m)$, a price vector p is said to be market clearing if*

$$p \in R^2_{++},\ \int_{T_0} \mathbf{x}^{0j}(t, p)\, d\mu + \sum_{i=1}^{2} e_{ij}\mu(m)\frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{2} e_{ji}\mu(m), \tag{1}$$

$j = 1, 2.$

The following proposition shows that market clearing price vectors can be normalized.

Proposition 2. *Under Assumptions 1, 2, and 3, if p is a market clearing price vector, then αp, with $\alpha > 0$, is also a market clearing price vector.*

Proof. . It straightforwardly follows from homogeneity of degree zero of the function $\mathbf{x}^0(t, \cdot)$, for each $t \in T_0$, and from (1). ■

Henceforth, we say that a price vector p is normalized if $p \in \Delta$ where $\Delta = \{p \in R^2_+ : \sum_{i=1}^2 p^i = 1\}$. Moreover, we denote by $\partial\Delta$ the boundary of the unit simplex Δ.

The next proposition shows that the two equations in (1) are not independent.

Proposition 3. *Under Assumptions 1, 2, and 3, given a strategy $e \in \mathbf{E}(m)$, a price vector $p \in \Delta \setminus \partial\Delta$ is market clearing for $j = 1$ if and only if it is market clearing for $j = 2$.*

Proof. Let a strategy $e \in \mathbf{E}(m)$ be given. Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$. Let $p \in \Delta \setminus \partial\Delta$ be a price vector. Suppose that p is market clearing for $j = 1$. Then, (1) reduces to

$$\int_{T_0} \mathbf{x}^{01}(t, p)\, d\mu = e_{12}\mu(m).$$

We have that

$$p^1 \int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu + p^2 \int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) = p^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu,$$

as $p^1\mathbf{x}^{01}(t,p) + p^2\mathbf{x}^{02}(t,p) = p^2\mathbf{w}^2(t)$, by Assumption 2, for each $t \in T_0$. Then, we have that

$$\int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) + e_{12}\mu(m)\frac{p^1}{p^2} = \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

Therefore, p is market clearing for $j = 2$. Suppose now that (1) is satisfied for $j = 2$. Then, (1) reduces to

$$\int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) + e_{12}\mu(m)\frac{p^1}{p^2} = \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

But then, we have that

$$p^2 \int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) + p^1 e_{12}\mu(m) = p^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

On the other hand, we know from the previous argument that

$$p^1 \int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu + p^2 \int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) = p^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

Then, we obtain that

$$\int_{T_0} \mathbf{x}^{01}(t,p) = e_{12}\mu(m).$$

Therefore, p is market clearing for $j = 1$. Hence, $p \in \Delta \setminus \partial\Delta$ is market clearing for $j = 1$ if and only if it is market clearing for $j = 2$. ■

The next proposition is based on Property (iv) of the aggregate demand of an atomless set of traders established by Debreu (1982, p. 728).

Proposition 4. *Under Assumptions 1, 2, and 3, let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1, 2, \ldots$, and which converges to a normalized price vector $\bar{p}$. If $\bar{p}^i = 0$ and $\mathbf{w}^i(m) > 0$, then the sequence $\{\int_{T_0} \mathbf{x}^{0i}(t, p^n)\, d\mu\}$ diverges to $+\infty$.*

Proof. According to Debreu (1982), we let $|x| = \sum_{i=1}^{2} |x^i|$, for each $x \in R^2_+$, and $d[0, V] = \inf_{x \in V} |x|$, for each $V \subset R^2_+$. Let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1, 2, \ldots$, which converges to a normalized price vector $\bar{p}$. Suppose, without loss of generality, that $\bar{p}^1 = 0$ and $\mathbf{w}^1(m) > 0$. Then, we have that $\bar{p}^2 = 1$. But then, the sequence $\{d[0, \mathbf{X}^0(t, p^n)]\}$ diverges to $+\infty$ as $\bar{p}^2\mathbf{w}^2(t) > 0$, for each $t \in T_0$, by Lemma 4 in Debreu (1982, p. 721). Therefore, the sequence

$\{d[0, \int_{T_0} \mathbf{X}^0(t, p^n)\, d\mu]\}$ diverges to $+\infty$, by the argument used in the proof of Property (iv) in Debreu (1982, p. 728). This implies that the sequence $\sum_{i=1}^{2}\{\int_{T_0} \mathbf{x}^{0i}(t, p^n)\, d\mu\}$ diverges to $+\infty$ as $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$, for each $p \in \Delta \setminus \partial\Delta$, by Proposition 1. Suppose that the sequence $\{\int_{T_0} \mathbf{x}^{02}(t, p^n)\, d\mu\}$ diverges to $+\infty$. Then, there exists an n_0 such that $\int_{T_0} \mathbf{x}^{02}(t, p^n)\, d\mu > \int_{T_0} \mathbf{w}^2(t)\, d\mu$, for each $n \geq n_0$. But we have that $\mathbf{x}^{02}(t, p) \leq \mathbf{w}^2(t)$, for each $t \in T_0$ and for each $p \in \Delta \setminus \partial\Delta$, a contradiction. Then, the sequence $\{\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu\}$ diverges to $+\infty$. Hence, the sequence $\{\int_{T_0} \mathbf{x}^{0i}(t, p^n)\, d\mu\}$ diverges to $+\infty$ whenever $\bar{p}^i = 0$ and $\mathbf{w}^i(m) > 0$. ■

The following proposition provides a necessary and sufficient condition for the existence of a market clearing price vector. In order to state and prove it, we provide the following preliminary definitions.

Definition 2. *A square matrix C is said to be triangular if $c_{ij} = 0$ whenever $i > j$ or $c_{ij} = 0$ whenever $i < j$.*

Definition 3. *We say that commodities i, j stand in relation Q if $\mathbf{w}^i(t) > 0$, for each $t \in T_0$, and there is a nonnull subset T^i of T_0 such that $u_t(\cdot)$ is differentiable, additively separable, i.e., $u_t(x) = v_t^i(x^i) + v_t^j(x^j)$, for each $x \in R^2_+$, and $\frac{dv_t^j(0)}{dx^j} = +\infty$, for each $t \in T^i$.*[3]

Moreover, we introduce the following assumption.

Assumption 4. *Commodities i, j stand in relation Q.*

Proposition 5. *Under Assumptions 1, 2, 3, and 4, given a strategy $e \in \mathbf{E}(m)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular.*

Proof. Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$ and let $e \in \mathbf{E}(m)$ be a strategy. Suppose that there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ and that the matrix E is not triangular. Then, it must be that $e_{12} = 0$. But then, we have that $\int_{T^2} \mathbf{x}^{01}(t, p)\, d\mu = 0$ as $\mu(T^2) > 0$, by (1). Consider a trader $\tau \in T^2$. We have that $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x^1} = +\infty$ as 2 and 1 stand in the relation Q, by Assumption 4, and $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x^1} \leq \lambda p^1$, by the necessary conditions of the Kuhn-Tucker theorem. Moreover, it must be that $\mathbf{x}^{02}(\tau, p) = \mathbf{w}^2(\tau) > 0$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2, and $p\mathbf{w}(\tau) > 0$. Then, $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x^2} = \lambda p^2$, by the necessary conditions of the Kuhn-Tucker theorem. But then, it must be that $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} = +\infty$ as $\lambda = +\infty$, contradicting the assumption that $u_\tau(\cdot)$ is continuously differentiable. Therefore, the matrix E must be triangular. Suppose now that E is triangular. Then, it must be that $e_{12} > 0$. Let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for

[3]In this definition, differentiability is to be understood as continuous differentiability and includes the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012, p. 58)).

each $n = 1, 2, \ldots$, which converges to a normalized price vector $\bar{p}$ such that $\bar{p}^1 = 0$. Then, the sequence $\{\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu\}$ diverges to $+\infty$, by Proposition 4. But then, there exists an n_0 such that $\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu > e_{12}\mu(m)$, for each $n \geq n_0$. Therefore, we have that $\int_{T_0} \mathbf{x}^{01}(t, p^{n_0})\, d\mu > e_{12}\mu(m)$. Let $q \in \Delta \setminus \partial\Delta$ be a price vector such that $\frac{q^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu}{q^1} = e_{12}\mu(m)$. Consider first the case where $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu = e_{12}\mu(m)$. Then, q is market clearing as it is market clearing for $j = 1$, by Proposition 3. Consider now the case where $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu \neq e_{12}\mu(m)$. Then, it must be that $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu < e_{12}\mu(m)$ as $\mathbf{x}^{01}(t, q) \leq \frac{q^2 \mathbf{w}^2(t)}{q^1}$, for each $t \in T_0$. But then, we have that $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu < e_{12}\mu(m) < \int_{T_0} \mathbf{x}^{01}(t, p^{n_0})\, d\mu$. Let $O \subset \Delta \setminus \partial\Delta$ be a compact and convex set which contains p^{n_0} and q. Then, the correspondence $\int_{T_0} \mathbf{X}^0(t, \cdot)\, d\mu$ is upper hemicontinuous on O, by the argument used in the proof of Property (ii) in Debreu (1982, p. 727). But then, the function $\{\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu\}$ is continuous on O as $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$, for each $p \in \Delta \setminus \partial\Delta$, by Proposition 1. Therefore, there is a price vector $\check{p} \in \Delta \setminus \partial\Delta$ such that $\int_{T_0} \mathbf{x}^{01}(t, \check{p})\, d\mu = e_{12}\mu(m)$, by the intermediate value theorem. Then, $\check{p}$ is market clearing as it is market clearing for $j = 1$, by Proposition 3. Hence, given a strategy $e \in \mathbf{E}(m)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular. ■ ■

We denote by $\pi(\cdot)$ a correspondence which associates, with each strategy $e \in \mathbf{E}(m)$, the set of price vectors p satisfying (1), if E is triangular, and is equal to $\{0\}$, otherwise. A price selection $p(\cdot)$ is a function which associates, with each strategy selection $e \in \mathbf{E}(m)$, a price vector $p \in \pi(e)$.

Given a strategy $e \in \mathbf{E}(m)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned} \mathbf{x}^j(m, e, p) &= \mathbf{w}^j(m) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\ \mathbf{x}^j(m, e, p) &= \mathbf{w}^j(m), \text{ otherwise,} \end{aligned}$$

$j = 1, 2,$

$$\begin{aligned} \mathbf{x}^j(t, p) &= \mathbf{x}^{0j}(t, p), \text{ if } p \in \Delta \setminus \partial\Delta, \\ \mathbf{x}^j(t, p) &= \mathbf{w}^j(t), \text{ otherwise,} \end{aligned}$$

$j = 1, 2$, for each $t \in T_0$.

Given a strategy $e \in \mathbf{E}(m)$ and a price selection $p(\cdot)$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(m) = \mathbf{x}(m, e, p(e)),$$

$$\mathbf{x}(t) = \mathbf{x}(t, p(e)),$$

for each $t \in T_0$.

The next proposition shows that traders' final holdings constitute an allocation.

Proposition 6. *Under Assumptions 1, 2, 3, and 4, given a strategy $e \in \mathbf{E}(m)$ and a price selection $p(\cdot)$, the assignment $\mathbf{x}(m) = \mathbf{x}(m, e, p(e))$, $\mathbf{x}(t) = \mathbf{x}(t, p(e))$, for each $t \in T_0$, is an allocation.*

Proof. Let a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(m)$ be given. Suppose that E is not triangular. Then, we have that $\mathbf{x}(m) = \mathbf{x}(m, e, p(e)) = \mathbf{w}(m)$ and $\mathbf{x}(t) = \mathbf{x}(t, p(e)) = \mathbf{w}(t)$, for each $t \in T_0$ as $p(e) = 0$. Suppose that E is triangular. Then, we have that

$$\int_T \mathbf{x}^j(t)\, d\mu = (\mathbf{w}^j(m) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j})\mu(m) + \int_{T_0} \mathbf{x}^{0j}(t, p)\, d\mu$$
$$= \int_T \mathbf{w}^j(t)\, d\mu,$$

$j = 1, 2$, as $p(e)$ is market clearing. Hence, given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(m)$, the assignment $\mathbf{x}(m) = \mathbf{x}(m, e, p(e))$, $\mathbf{x}(t) = \mathbf{x}(t, p(e))$, for each $t \in T_0$, is an allocation. ■

We can now provide the definition of a monopoly equilibrium.

Definition 4. *A strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{E}$ is triangular is a monopoly equilibrium, with respect to a price selection $p(\cdot)$, if*

$$u_m(\mathbf{x}(m, \tilde{e}, p(\tilde{e})) \geq u_m(\mathbf{x}(m, e, p(e)),$$

for each $e \in \mathbf{E}(m)$.

A monopoly allocation is an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, p(\tilde{e}))$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{e}))$, for each $t \in T_0$, where $\tilde{e}$ is a monopoly equilibrium, with respect to a price selection $p(\cdot)$.

We introduce here a first example in which a monopoly equilibrium is computed.

Example 1. *Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, and 4. $T_0 = [0, 1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1, 0)$, $u_m(x) = \frac{1}{2}x^1 + \sqrt{x^2}$, T_0 is taken with Lebesgue measure, $\mathbf{w}(t) = (0, 1)$, $u_t(x) = \sqrt{x^1} + x^2$, for each $t \in T_0$. Then, there is a unique monopoly equilbrium.*

Proof. To obtain the pure monopoly solution, we can think of the whole process as follows:

- The monopolist makes a bid $e_{12} \in [0, w_1]$;
- The monopolist anticipates that her bid induces a (relative) price $p(e_{12})$, obtained via the (unique) price selection, which therefore clears the market, that is $X_1(p) = e_{12}$, where X_1 represents the aggregate inverse demand function for the atomless part;
- Finally, the monopolist will choose the optimal bid maximizing his utility $u(x_1(e_{12}), x_2(e_{12}))$.

First, we find the inverse demand function for good 1, for a generic trader $t \in T_0$. We proceed by maximizing her utility a follows:

$$\begin{aligned} \max_{x_1(t), x_2(t)} \quad & u_t(x_1, x_2) = \sqrt{x_1(t)} + x_2(t) \\ s.t. \quad & p x_1(t) + x_2(t) = 1 \end{aligned}$$

and we obtain

$$x_1(t, p) = \frac{1}{4p^2}$$

Then, we compute the aggregate inverse demand function (X) for the atomless part:

$$\begin{aligned} X_1(p) &= \int_{T_0} x_1(t, p) d\mu(t) = \int_{T_0} \frac{1}{4p^2} d\mu(t) = \frac{1}{4p^2} \mu(T_0) \\ &= \frac{1}{4p^2} \end{aligned}$$

But then, we have

$$X_1(p) = e_{12} = \frac{1}{4p^2}$$

by the market clearing equation for good 1. Therefore, we obtain

$$p(e_{12}) = \frac{1}{2\sqrt{e_{12}}}.$$

Now, the maximization problem for the monopolist is:

$$\begin{aligned} \max_{x_1(m),x_2(m)} \quad & u_m(x_1(m),x_2(m)) = \frac{1}{2}x_1(m) + \sqrt{x_2(m)} \\ s.t. \quad & px_1(m) + x_2(m) = 1 \end{aligned}$$

Since $x_1(m) = 1 - e_{12}$, the problem becomes

$$\begin{aligned} \max_{e_{12}} \quad & u_m(e_{12},x_2(m)) = \frac{1}{2}(1-e_{12}) + \sqrt{x_2(m)} \\ s.t. \quad & x_2 = p(e_{12})e_{12} \end{aligned}$$

and finally

$$\max_{e_{12}} u(x_1(e_{12}),x_2(e_{12})) = \frac{1}{2}(1-e_{12}) + \sqrt{p(e_{12})e_{12}} = \frac{1}{2}(1-e_{12}) + \sqrt[4]{\frac{e_{12}}{4}} \tag{2.1}$$

The solution for $\tilde{e}_{12}$ is $\tilde{e}_{12} = \frac{1}{4}$. Then $\tilde{x}(m) = (\frac{3}{4};\frac{1}{4})$, $\tilde{x}(t) = (\frac{1}{4};\frac{3}{4})$ for each $t \in T_0$, and $p = 1$.

■

2.3 Discussion of the model

The analysis of the monopoly problem in bilateral exchange proposed in the previous sections can be simplified by introducing the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and compared, under this restriction, with the standard partial equilibrium analysis of monopoly.

The following proposition states a necessary and sufficient condition for the atomless part's aggregate demand to be invertible.

Proposition 7. *Under Assumptions 1, 2, 3, and 4, let* $\mathbf{w}^i(m) > 0$*. Then, the function* $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\,d\mu$ *is invertible if and only if, for each* $x \in R_{++}$*, there is a unique* $p \in \Delta \setminus \partial\Delta$ *such that* $x = \int_{T_0} \mathbf{x}^{0i}(t,p)\,d\mu$*.*

Proof. Let $\mathbf{w}^i(m) > 0$. Suppose that $\int_{T_0} \mathbf{x}^{0i}(t,p)\,d\mu = 0$, for some $p \in \Delta \setminus \partial\Delta$. Then, we have that $\int_{T^i} \mathbf{x}^{0i}(t,p)\,d\mu = 0$ as $\mu(T^i) > 0$ and the necessary Kuhn-Tucker conditions lead, *mutatis mutandis*, to the same contradiction as in the proof of Proposition 5. But then, we have that $\int_{T_0} \mathbf{x}^{0i}(t,p)\,d\mu > 0$, for each $p \in \Delta \setminus \partial\Delta$. Therefore, the function

$\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is restricted to the codomain R_{++}. For each $x \in R_{++}$, there exists at least one $p \in \Delta \setminus \partial\Delta$ such that $x = \int_{T_0} \mathbf{x}^{0i}(t,p)\, d\mu$, by the same argument used in the proof of Proposition 5. Then, the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is onto as its range coincides with its codomain. Therefore, the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible if and only if it is one-to-one. Hence, the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible if and only, for each $x \in R_{++}$, there is a unique $p \in \Delta \setminus \partial\Delta$ such that $x = \int_{T_0} \mathbf{x}^{0i}(t,p)\, d\mu$. ■

Let $p^{0i}(\cdot)$ denote the inverse of the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$. The following proposition shows that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, there exists a unique price selection.

Proposition 8. *Under Assumptions 1, 2, 3, and 4, if* $\mathbf{w}^i(m) > 0$ *and the function* $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ *is invertible, then there exists a unique price selection* $\mathring{p}(\cdot)$.

Proof. Suppose that $\mathbf{w}^i(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible. Let $\mathring{p}(e)$ be a function which associates, with each strategy $e \in \mathbf{E}(m)$, the price vector $p = p^{0i}(e_{ij}\mu(m))$, if E is triangular, and is equal to $\{0\}$, otherwise. Then, $\mathring{p}(\cdot)$ is the unique price selection as $\pi(e) = \{\mathring{p}(e)\}$, for each $e \in \mathbf{E}(m)$. ■

By analogy with partial equilibrium analysis, $\mathring{p}(\cdot)$ can be called the inverse demand function of the monopolist. When the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, the monopoly equilibrium can be reformulated as in Definition 3, with respect to monopolist's inverse demand function $\mathring{p}(\cdot)$.

2.4 Discussion of the literature

We now show that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, our model can provide an economic theoretical foundation of the solutions proposed by Schydlowsky and Siamwalla (1966) and Kats (1974).

Under this assumption, the monopoly equilibrium can be characterized by means of the notion of offer curve of the atomelss part, defined as the set $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$, and that of the notion of feasible complement of the offer curve of the atomless part, defined as the set $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$.

The following proposition shows that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, the feasible complement of the atomless part's offer curve is a subset of the set of the monopolist's final holdings.

Proposition 9. *Under Assumptions 1, 2, 3, and 4, if* $\mathbf{w}^i(m) > 0$ *and the function* $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ *is invertible, then the feasible complement of the offer curve of the atomless part, the set* $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ *for some* $p \in \Delta \setminus \partial\Delta\}$*, is a subset of the set* $\{x \in R^2_+ : x = \mathbf{x}(m,e,\mathring{p}(e))$ *for some* $e \in \mathbf{E}(m)\}$*, the set of the final holdings of the monopolist.*

Proof. Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ is invertible. Suppose that $\bar{x} \in \{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Moreover, suppose that $\bar{x}^1 = \mathbf{w}^1(m)$. Then, we have that $\int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu = 0$, for some $p \in \Delta \setminus \partial\Delta$. But then, we have that $\int_{T^2} \mathbf{x}^{01}(t,p)\, d\mu = 0$ as $\mu(T^2) > 0$ and the necessary Kuhn-Tucker conditions lead, *mutatis mutandis*, to the same contradiction as in the proof of Proposition 5. Therefore, we must have that $0 \leq \bar{x}^1 < \mathbf{w}^1(m)$. Let $\bar{e} \in \mathbf{E}(m)$ be such that $\bar{e}_{12} = \mathbf{w}^1(m) - \bar{x}^1$ and let $\bar{p} = \mathring{p}(\bar{e})$. Then, we have that

$$\begin{aligned}
&\bar{x}^1\mu(m) + \int_{T_0} \mathbf{x}^{01}(t,\bar{p})\, d\mu \\
&= (\mathbf{w}^1(m) - \bar{e}_{12})\mu(m) + \int_{T_0} \mathbf{x}^{01}(t,\bar{p})\, d\mu = \mathbf{w}^1(m)\mu(m),
\end{aligned}$$

as $\bar{p} = \mathring{p}(\bar{e})$. Moreover, $\bar{p}$ is the unique price vector such that

$$(\mathbf{w}^1(m) - \bar{x}^1)\mu(m) = \int_{T_0} \mathbf{x}^{01}(t,\bar{p})\, d\mu,$$

as the function $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ is invertible. Then, it must be that

$$\bar{x}^2\mu(m) + \int_{T_0} \mathbf{x}^{02}(t,\bar{p})\, d\mu = \int_{T_0} \mathbf{w}^2(t)\, d\mu,$$

by Proposition 3. But then, we have that

$$\bar{x}^2 = e_{12}\frac{\bar{p}^1}{\bar{p}^2},$$

as $\bar{p}$ is market clearing. Therefore, we conclude that

$$\bar{x} = \mathbf{x}(m,\bar{e},\bar{p}) = \mathbf{x}(m,\bar{e},\mathring{p}(\bar{e})).$$

Hence, the feasible complement of the offer curve of the atomless part, the set $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$, is a subset of the set $\{x \in R^2_+ : x = \mathbf{x}(m,e,\mathring{p}(e))$ for some $e \in \mathbf{E}(m)\}$, the set of the final holdings of the monopolist. ■

Kats (1974) considered both the cases of a quantity setting and a price setting monopoly in a pure exchange economy where one trader behaves as a monopolist, "calling the game" and maximizing his utility, whereas all the other traders in the economy behave competitively. He claimed that the monopoly quantity setting solution must correspond to the monopolist's most preferred commodity bundle compatible with the aggregate initial endowments and the offer curve of the competitive traders. However, he did not propose any explicit trading process which could lead to the monopoly solution. The following proposition, which follows from Proposition 11, establishes that, at a monopoly allocation, the utility of the monopolist is maximal in the feasible complement of the atomless part's offer curve. This way, it provides an explicit economic theoretical foundation of the monopoly solution proposed by Kats (1974).

Proposition 10. *Under Assumptions 1, 2, 3, and 4, if $\mathbf{w}^i(m) > 0$, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible, and $\tilde{e} \in \mathbf{E}(m)$ is a monopoly equilibrium, then $u_m(\mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e})))$ is maximal in the set $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t, p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$.*

Proof. Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium. Let $\tilde{p} = \mathring{p}(\tilde{e})$. We have that

$$\begin{aligned} &\mathbf{x}^1(m, \tilde{e}, \tilde{p})\mu(m) + \int_{T_0} \mathbf{x}^{01}(t, \tilde{p})\, d\mu \\ &= (\mathbf{w}^1(m) - \tilde{e}_{12})\mu(m) + \int_{T_0} \mathbf{x}^{01}(t, \tilde{p})\, d\mu = \mathbf{w}^1(m)\mu(m), \end{aligned}$$

and

$$\begin{aligned} &\mathbf{x}^2(m, \tilde{e}, \tilde{p})\mu(m) + \int_{T_0} \mathbf{x}^{02}(t, \tilde{p})\, d\mu \\ &= \tilde{e}_{12}\mu(m)\frac{\tilde{p}^1}{\tilde{p}^2} + \int_{T_0} \mathbf{x}^{02}(t, \tilde{p})\, d\mu = \int_{T_0} \mathbf{w}^2(t)\, d\mu, \end{aligned}$$

as $\tilde{p}$ is market clearing. Then, we have shown that $\mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e})) \in \{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t, p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. But then, we have that $u_m(\mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e})))$ is maximal in the set $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t, p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$ as $u_m(\mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e})) \geq u_m(\mathbf{x}(m, e, \mathring{p}(e))$, for each $e \in \mathbf{E}(m)$, and $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t, p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\} \subset \{x \in R^2_+ : x = \mathbf{x}(a, e, \mathring{p}(e))$ for some $e \in \mathbf{E}(m)\}$, by Proposition 9. ∎

In order to provide the characterization of a monopoly equilibrium proposed by Schydlowsky and Siamwalla (1966), we need to introduce also the following assumption.

Assumption 5. *$u_m : R^2_+ \to R$ is differentiable.*

We show that, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is not only invertible but also differentiable, the monopoly equilibrium introduced in Definition 3 has also the geometric characterization previously proposed by Schydlowsky and Siamwalla (1966): at a strictly positive monopoly allocation, the monopolist's indifference curve is tangent to the atomless part's offer curve.[4]

The following proposition shows that the function $h(\cdot)$, defined on R_{++} and such that

$$p^i x^i + p^j x^j = p^i \int_{T_0} \mathbf{w}^i(t)\, d\mu + p^j \int_{T_0} \mathbf{w}^j(t)\, d\mu,$$

where $p = p^{0i}(x^i)$ and $x^j = h(x^i)$, represents the offer curve of the atomless part in the sense that its graph coincides with the atomless part's offer curve.

Proposition 11. *Under Assumptions 1, 2, 3, and 4, if $\mathbf{w}^i(m) > 0$ and the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible, then the graph of the function $h(\cdot)$, the set $\{x \in R^2_+ : x^j = h(x^i)\}$, coincides with the set $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$, the offer curve of the atomless part.*

Proof. Suppose that $\mathbf{w}^i(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible. Suppose that $\bar{x} \in \{x \in R^2_+ : x^j = h(x^i)\}$. Then, there is a unique price vector $\bar{p} = p^{0i}(\bar{x}^i)$ such that $\bar{x}^i = \int_{T_0} \mathbf{x}^{0i}(t,\bar{p})\, d\mu$, as the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible. We have that

$$\bar{p}^i \int_{T_0} \mathbf{x}^{0i}(t,\bar{p})\, d\mu + \bar{p}^j \int_{T_0} \mathbf{x}^{0j}(t,\bar{p})\, d\mu = p^i \int_{T_0} \mathbf{w}^i(t)\, d\mu + p^j \int_{T_0} \mathbf{w}^j(t)\, d\mu,$$

by Walras' law. Then, it must be that $\bar{x}^j = \int_{T_0} \mathbf{x}^{0j}(t,\bar{p})\, d\mu$, where $\bar{x}^j = h(\bar{x}^i)$. But then, $\bar{x} \in \{x \in R^2 : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Therefore, $\{x \in R^2_+ : x^j = h(x^i)\} \subset \{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Suppose now that $\bar{x} \in \{x \in R^2 : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Let $\bar{p}$ be such that $\bar{x} = \int_{T_0} \mathbf{x}^0(t,\bar{p})\, d\mu$. Then, we have that $\bar{p} = p^{0i}(\bar{x}^i)$ as the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible. We have that

$$\bar{p}^i \bar{x}^i + \bar{p}^j \bar{x}^j = \bar{p}^i \int_{T_0} \mathbf{w}^i(t) + p^j \int_{T_0} \mathbf{w}^j(t),$$

by Walras' law. Then, we have that $\bar{x}^j = h(\bar{x}^i)$. But then, $\bar{x} \in \{x \in R^2_+ : x^j = h(x^i)\}$. Therefore, $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\} \subset \{x \in R^2_+ : x^j = h(x^i)\}$. Hence, the graph of the function $h(\cdot)$, the set $\{x \in R^2_+ : x^j = h(x^i)\}$, coincides with the set $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$, the offer curve of the atomless part. ■

[4]This characterization of the monopoly equilibrium has been diffusely reproposed in standard textbooks in microeconomics (see, for instance, Varian (2014, p. 619), among others).

Borrowing from Pareto (1896), we now introduce in our general framework a notion which has a counterpart in partial equilibrium analysis: the marginal revenue of the monopolist.

In the rest of this section, with a slight abuse of notation, given a price vector $(p^i, p^j) \in \Delta \setminus \partial\Delta$, we denote by p the scalar $p = \frac{p^i}{p^j}$, whenever $\mathbf{w}^i(m) > 0$. Suppose that $\mathbf{w}^i(m) > 0$, that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible, and that the function $p^{0i}(\cdot)$ is differentiable. Then, $\mathring{p}(\cdot)$, the inverse demand function of the monopolist, is differentiable and we have that $\frac{d\mathring{p}(e)}{de_{ij}} = \frac{dp^{0i}(e_{ij}\mu(m))}{dx^i}\mu(m)$, at each $e \in \mathbf{E}(m)$ such that E is triangular, by Proposition 9. In this context, the revenue of the monopolist can be defined as $\mathring{p}(e)e_{ij}$ and his marginal revenue as $\frac{d\mathring{p}(e)}{de_{ij}}e_{ij} + \mathring{p}(e)$, for each $e \in \mathbf{E}(m)$ such that E is triangular.

Then, in the next proposition, we can provide a formal foundation of the geometric characterization of the monopoly equilibrium proposed by Schydlowsky and Siamwalla (1966). Indeed, our proposition establishes that, at an interior monopoly solution, the slope of the monopolist's indifference curve and the slope of the atomless part's offer curve are both equal to the opposite of the monopolist's marginal revenue. Therefore, the tangency characterization of a monopoly equilibrium is demonstrated.

Proposition 12. *Under Assumptions 1, 2, 3, 4, and 5, if* $\mathbf{w}^i(m) > 0$, *the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible, the function* $p^{0i}(\cdot)$ *is differentiable, and* $\tilde{e} \in \mathbf{E}(m)$ *is a monopoly equilibrium such that* $\tilde{e} < \mathbf{w}^i(m)$, *then*

$$-\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m)}{\partial x^i}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = \frac{dh(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i},$$

where $\tilde{\mathbf{x}}$ *is the monopoly allocation corresponding to* $\tilde{e}$.

Proof. Suppose that $\mathbf{w}^i(m) > 0$, that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible and that the function $p^{0i}(\cdot)$ is differentiable. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium such that $\tilde{e} < \mathbf{w}^i(m)$ and let $\tilde{\mathbf{x}}$ be the corresponding monopoly allocation. Then, $\mathring{p}(\cdot)$, the inverse demand function of the monopolist, is differentiable and the necessary Kuhn-Tucker conditions imply that

$$-\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^i} + \frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = 0.$$

Then, we have that

$$-\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m)}{\partial x^i}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right).$$

Moreover, we have that

$$h(x^i) = -p^{0i}(x^i)x^i + \int_{T_0} \mathbf{w}^j(t)\, d\mu.$$

Differentiating the function $h(\cdot)$, we obtain

$$\frac{dh(x^i)}{x^i} = -\left(\frac{dp^{0i}(x^i)}{dx^i}x^i + p^{0i}(x^i)\right).$$

At the monopoly allocation $\tilde{\mathbf{x}}$, we have that

$$\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e}) = \frac{dp^{0i}(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i}\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu + p^{0i}(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu),$$

as $\frac{d\mathring{p}(\tilde{e})}{de_{ij}} = \frac{dp^{0i}(\tilde{e}_{ij}\mu(m))}{dx^i}\mu(m)$ and $\tilde{e}_{12}\mu(m) = \int_{T_0} \mathbf{x}^{0i}(t, p(\tilde{e}))\, d\mu$. Hence, we have that

$$-\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m)}{\partial x^i}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = \frac{dh(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i}.$$

■

Pareto (1986) was the first author who gave a formalized treatment of the problem of monopoly for a general pure exchange economy. To better understand the relationship between the analysis developed in the previous sections and that proposed by Pareto (1896), we reformulate now this author's monopoly solution within our framework of bilateral exchange.

Pareto (1896) assumed that, for the monopolist, the commodity he is endowed with is "neutral," i.e., it is a commodity from which he does not get any utility.[5] To incorporate this assumption in our model, we amend Assumption 2 as follows.

Assumption 6. *$u_m(x) = x^j$, whenever $\mathbf{w}^i(m) > 0$, $i \neq j$, and $u_t : R^2_+ \to R$ is continuous, strongly monotone, strictly quasi-concave, for each $t \in T_0$.*

It is straightforward to verify that Assumption 6 implies that the utility function of the monopolist is continuous, monotone, and quasi-concave.

Hereafter, we assume that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible, whenever $\mathbf{w}^i(m) > 0$. Therefore, the revenue of the monopolist can be defined again as $\mathring{p}(e)e_{ij}$.

According to Pareto (1896), the goal of the monopolist is to maximize his revenue. Therefore, we can provide the following definition of a Pareto monopoly equilibrium.

[5]For a discussion of the properties of neutral commodities, see, for instance, Varian (2014).

Definition 5. *Let* $\mathbf{w}^i(m) > 0$*. A strategy* $\hat{e} \in \mathbf{E}(m)$ *such that* $\hat{E}$ *is triangular is a Pareto monopoly equilibrium, with respect to the price selection* $\mathring{p}(\cdot)$*, if*

$$\mathring{p}(\hat{e})\hat{e}_{ij} \geq \mathring{p}(e)e_{ij},$$

for each $e \in \mathbf{E}(m)$*.*

A Pareto monopoly allocation is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(m) = \mathbf{x}(m, \hat{e}, \mathring{p}(\hat{e}))$ and $\hat{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\hat{e}))$, for each $t \in T_0$, where $\hat{e}$ is a Pareto monopoly equilibrium.

The following proposition shows that, when Assumption 2 is replaced with Assumption 6, a strategy of the monopolist is a Pareto monopoly equilibrium if and only if it is a monopoly equilibrium. Moreover, it shows that, if $p^{0i}(\cdot)$ is differentiable whenever $\mathbf{w}^i(m) > 0$, then at a Pareto monopoly solution the monopolist's marginal revenue must be nonnegative.

Proposition 13. *Under Assumptions 1, 6, 3, and 4, let* $\mathbf{w}^i(m) > 0$*. Then, a strategy* $\hat{e} \in \mathbf{E}(m)$ *is a Pareto monopoly equilibrium, with respect to the unique price selection* $\mathring{p}(\cdot)$*, if and only if it is a monopoly equilibrium, with respect to the same price selection. Moreover, if the function* $p^{0i}(\cdot)$ *is differentiable, and* $\hat{e} \in \mathbf{E}(m)$ *is a Pareto monopoly equilibrium, then*

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}}\hat{e}_{ij} + \mathring{p}(\hat{e}) \geq 0.$$

Proof. Let $\mathbf{w}^i(m) > 0$. Suppose that the strategy $\hat{e} \in \mathbf{E}(m)$ is a Pareto monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$. Then, we have that

$$\mathring{p}(\hat{e})\hat{e}_{ij} \geq \mathring{p}(e)e_{ij},$$

for each $e \in \mathbf{E}(m)$. But then, we must have that

$$u_m(\mathbf{x}(m, \hat{e}, \mathring{p}(\hat{e})) \geq u_m(\mathbf{x}(m, e, \mathring{p}(e)),$$

for each $e \in \mathbf{E}(m)$, as

$$u_m(\mathbf{x}(m, e, \mathring{p}(e)) = \mathring{p}(e)e_{ij},$$

by Assumption 6, for each $e \in \mathbf{E}(m)$. Therefore, the strategy $\hat{e} \in \mathbf{E}(m)$ is a monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$. The converse can be straightforwardly proved by the same argument. Hence, a strategy $\hat{e} \in E(m)$ is a Pareto monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$, if and only if it is a monopoly equilibrium, with respect to the same price selection. Suppose that the function $p^{0i}(\cdot)$ is differentiable. Let $\hat{e} \in \mathbf{E}(m)$ be a Pareto monopoly equilibrium. Then, $\mathring{p}(\cdot)$, the inverse demand function of the monopolist, is differentiable and the necessary Kuhn-Tucker conditions imply that

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}}\hat{e}_{ij} + \mathring{p}(\hat{e}) \geq 0.$$

■

We now provide an example of a Pareto monopoly equilibrium.

Example 2. *Consider the following specification of an exchange economy satisfying Assumptions 1, 6, 3, 4.* $T_0 = [0,1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1,0)$, $u_m(x) = x^2$, T_0 *is taken with Lebesgue measure,* $\mathbf{w}(t) = (0,1)$, $u_t(x) = \sqrt{x^1} + x_2$, *for each* $t \in T_0$*. Then, there is a unique Pareto monopoly equilibrium* $\hat{e} \in \mathbf{E}(m)$ *such that*

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}}\hat{e}_{ij} + \mathring{p}(\hat{e}) > 0.$$

Proof. . The unique Pareto monopoly equilibrium is the strategy $\hat{e}$ such that $\hat{e}_{12} = 1$, $\mathring{p}(\hat{e}) = \frac{1}{2}$, $\hat{\mathbf{x}}(m) = (0, \frac{1}{2})$, $\hat{\mathbf{x}}(t) = (1, \frac{1}{2})$, for each $t \in T_0$. Moreover, we have that

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}}\hat{e}_{ij} + \mathring{p}(\hat{e}) = \frac{1}{4}.$$

■

Comparing the monopoly solution of Example 1 with the Pareto monopoly solution of Example 3, we can observe that the atomless part is better off at the Pareto monopoly solution than at the monopoly solution as

$$u_t(\hat{\mathbf{x}})(t) = \frac{3}{2} > \frac{5}{4} = u_t(\tilde{\mathbf{x}})(t),$$

for each $t \in T_0$. Moreover, Example 3 shows that when the utility function of the monopolist is continuous, monotone, and quasi-concave a monopoly equilibrium may exist whereas Example 2 showed that this is not the case when those weaker assumptions than those imposed by Assumption 2 hold for the atomless part.

2.5 Examples

Before stating the concluding remarks, we show a series of examples that encompass different equilibria possibilities and provide a clean computational exercise to find monopoly equilibria.

We consider two similar examples: the first is a continuation Example 1, showing that the geometrical condition is satisfied, while the latter considers an heterogeneous atomless sector. In both cases, we obtain the monopoly result and the tangency between the monopolist's indifferent curve and the small traders' offer curve.

Example 3. *Consider a pure exchange economy in which* $T = [0,1] \cup \{2\}$*, where* $T_0 = [0,1]$ *and there is one atom* $T_1 = \{2\}$ *with* $\mu(2) = 1$*. There are 2 commodities traded in the market. For each* $t \in T_0$ *the initial assignment is* $w(t) = (0,1)$ *and the utility function is* $u_t(x_1, x_2) = \sqrt{x_1} + x_2$*. For the monopolist the initial assignment is* $w(2) = (1,0)$ *and the utility function is* $u_2(x_1, x_2) = \frac{1}{2}x_1 + \sqrt{x_2}$ *Then, there is a unique interior solution to the pure monopoly problem. Moreover, we will show that there is tangency between the monopolist indifference curves and the supply function at the optimal point.*

Proof. We have that the unique monopoly equilibrium is the strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{e}_{12} = \frac{1}{4}$, $\mathring{p}(\tilde{e}) = 1$, $\tilde{\mathbf{x}}(m) = (\frac{3}{4}, \frac{1}{4})$, and $\tilde{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$.

To prove tangency, we first calculate the slope of the monopolist's indifference curve in $x_1(m) = \tilde{x}_1(m)$.

$$MRS(x(m)) = -\frac{\frac{\delta u(x_1,x_2)}{\delta x_1}}{\frac{\delta u(x_1,x_2)}{\delta x_1}} = -\sqrt{x_2}$$

Then,

$$MRS(\tilde{x}(m)) = -\frac{1}{2}.$$

To derive the supply curve, we proceed in this way: first we calculate the inverse demand function for a generic $t \in T_0$, then we express the supply function as $\hat{x}_2(\hat{x}_1) = -p(\hat{x}_1)\hat{x}_1$, where $\hat{x}_2 = x_2 + w_2$ and $\hat{x}_1 = x_1 + w_1$; finally, we derive the aggregate supply function. But then,

$$\hat{x}_2(\hat{x}_1) = -\frac{\sqrt{\hat{x}_1}}{2}$$

and so

$$\hat{X}_2(\hat{x}_{21}) = \int_{T_0} \hat{x}_2(\hat{x}_{21}) d\mu(t) = \int_{T_0} -\frac{\sqrt{\hat{x}_1}}{2} d\mu(t) = -\frac{\sqrt{\hat{x}_1}}{2}.$$

Now we calculate the slope of this function in correspondence of the optimal point $\hat{x^*}_1 = x_1^* + w_1 = \frac{1}{4}$.

$$\frac{\delta \hat{X}_2}{\delta \hat{x}_1} = -\frac{1}{4\sqrt{\hat{x}_1}} = -\frac{1}{2}.$$

Hence, the price-taker supply function is tangent to the monopolist indifference curve at the optimal point. ■

Example 4. *Consider a pure exchange economy in which* $T = [0,1] \cup \{2\}$*, where* $T_0 = [0,1]$ *and there is one atom* $T_1 = \{2\}$ *with* $\mu(2) = 1$*. There are 2 commodities traded in the market.*

For each $t \in A = [0; \frac{1}{2}]$ the initial assignment is $w(t) = (0,1)$ and the utility function is $u_t(x_1, x_2) = \sqrt{x_1} + \frac{8}{11}x_2$; for each $t \in B = (\frac{1}{2}; 1]$ the initial assignment is $w(t) = (0,1)$ and the utility function is $u_t(x_1, x_2) = \sqrt{x_1} + 4x_2$. For the monopolist the initial assignment is $w(2) = (1,0)$ and the utility function is $u_2(x_1, x_2) = \frac{1}{4}x_1 + \sqrt{x_2}$ Then, there is a unique interior solution to the pure monopoly problem. Moreover, we will show that there is tangency between the monopolist indifference curves and the supply function at the optimal point.

Proof. We proceed as the previous example. First, we need to derive the aggregate demand function for the price-takers.
Let now derive the demand functions for the two types of price-takers. Consider a trader $t \in [0; \frac{1}{2}]$. By maximizing his utility, we obtain

$$p = \frac{\frac{1}{2\sqrt{x_1(t)}}}{\frac{8}{11}} = \frac{11}{16\sqrt{x_1(t)}}$$

and so

$$x_1(t,p) = \frac{121}{256p^2}. \tag{2.2}$$

Consider now a trader $t \in (\frac{1}{2}; 1]$. By maximazing his utility, we obtain

$$p = \frac{\frac{1}{2\sqrt{x_1(t)}}}{4} = \frac{1}{8\sqrt{x_1(t)}}$$

and so

$$x_1(t,p) = \frac{1}{64p^2}. \tag{2.3}$$

Now we are able to derive the aggregate demand function for good 1 for the price-takers:

$$\begin{aligned} X_1(p) &= \int_{T_0} x_1(t,p)d\mu(t) = \int_A x_1(t,p)d\mu(t) + \int_B x_1(t,p)d\mu(t) = \\ &= \int_A \frac{121}{256p^2}d\mu(t) + \int_B \frac{1}{64p^2}d\mu(t) = \frac{121}{256p^2}\mu(A) + \frac{1}{64p^2}\mu(B) \\ &= \frac{1}{2}\frac{121}{256p^2} + \frac{1}{2}\frac{1}{64p^2} = \frac{125}{512p^2} \end{aligned}$$

But then, we have

$$X_1(p) = e_{12} = \frac{125}{512p^2},$$

by the market clearing condition. Therefore,

$$p(e_{12}) = \sqrt{\frac{125}{512e_{12}}}.$$

Given this unique price selection, we can write the monopolist maximization problem as

$$\begin{aligned} \max_{x_1,x_2} \quad & u_m(x_1,x_2) = \frac{1}{4}x_1 + \sqrt{x_2} \\ s.t. \quad & x_2 = p(e_{12})e_{12} \end{aligned}$$

But then, we have

$$\max_{e_{12}} u(x_1(e_{12}), x_2(e_{12})) = \frac{1}{4}(1-e_{12}) + \sqrt[4]{p(e_{12})e_{12}}$$

since $x_1(m) = 1 - e_{12}$, that is

$$\max_{e_{12}} \frac{1}{4}(1-e_{12}) + \sqrt[4]{\frac{125e_{12}}{512}}.$$

The solution is $\tilde{e}^*_{12} = \frac{5}{8}$. Then, $p = p(\tilde{e}_{12}) = \frac{5}{8}$, and so $\tilde{x}(t)(m) = (\frac{3}{8}, \frac{25}{64})$. From 2.2 and 2.3 and the budget constraint of the price takers we obtain that $\tilde{x}(t) = (\frac{121}{100}; \frac{39}{160})$ for each $t \in A$, and $\tilde{x}(t) = (\frac{1}{25}; \frac{39}{40})$ for each $t \in B$.

We show now that, at the final allocation, there is tangency between the offer curve and the monopolist indifference curve. We derive now the slope of the indifference curve of the monopolist, given by his marginal rate of substitution, that is

$$MRS(x(m)) = -\frac{\frac{\delta u_m(x_1,x_2)}{\delta x_1}}{\frac{\delta u_m(x_1,x_2)}{\delta x_1}} = -\frac{\sqrt{x_2}}{2}$$

Then,

$$MRS(\tilde{x}(m)) = -\frac{5}{16}.$$

To derive the (aggregate) offer curve, we calculate $\hat{X}_2(\hat{X}_1) = -p(\hat{X}_1)\hat{X}_1$, which this turns out to be equal to

$$\hat{X}_2(\hat{X}_1) = -\sqrt{\frac{125\hat{X}_1}{512}}.$$

We calculate the derivative of this function and calcuate it in $\hat{X}_1 = X_1 = \frac{5}{8}$ (as $w_1(t) = 0$ for each $t \in T_0$), obtaining

$$\frac{\delta \hat{X}_2}{\delta \hat{X}_1} = -\frac{125}{512\hat{X}_1} = -\frac{5}{16}.$$

Hence, the price-taker supply function is tangent to the monopolist indifference curve at the optimal point.

■

2.6 Conclusion

In this paper, we have provided a general economic foundation of the quantity-setting monopoly solution in bilateral exchange which, to the best of our knowledge, was a gap in the literature on monopoly in general equilibrium. Then, we have shown that the *ad hoc* monopoly solutions proposed by Schydlowsky and Siamwalla (1966) and Kats (1974) fit well in suitable specifications of our general model, as well as the *ante litteram* solution proposed by Pareto (1986).

We leave for future research addressing the problem of a price-setting monopolist, in the same bilateral framework as used in this paper. This goal could be pursued by drawing inspiration from another pioneering work by Vilfredo Pareto (see Pareto (1909)) and could lead to a game theoretical foundation of a monopoly solution of this type in a two-stage setup, as suggested by Sadanand (1988).

Kats (1974), in his final remarks (see p. 31), raised the question of the relationship between monopoly equilibrium and cooperative game theory. He formalized a monopolistic market game based on the notion of a monopolistic quasi-core. He mentioned Shitovitz (1973) as the only other work offering a contribution on this issue. Shitovitz (1973), in his Example 1, actually showed that, in the mixed version of a monopolistic two-commodity exchange economy, the set of allocations in the core does not coincide with the set of Walrasian allocations. This example raised the question whether the core solution to monopolistic market games is “advantageous” or “disadvantageous” for the monopolist (see Aumann (1973), Drèze et al. (1977), Greenberg and Shitovitz (1977), among others). The same issue could be analysed using our monopoly equilibrium solution.

Bibliography

[1] Aumann R.J. (1969), "Measurable utility and measurable choice theory," in Guilbaud G.T. (ed), *La Décision*, Editions du Centre National de la Recherche Scientifique, Paris.

[2] Aumann R.J. (1973), "Disadvantageous monopolies," *Journal of Economic Theory* **6**, 1-11.

[3] Busetto F., Codognato G., Ghosal S. (2008), "Cournot-Walras equilibrium as a subgame perfect equilibrium," *International Journal of Game Theory* **37**, 371-386.

[4] Busetto F., Codognato G., Ghosal S. (2011), "Noncooperative oligopoly in markets with a continuum of traders," *Games and Economic Behavior* **72**, 38-45.

[5] Busetto F., Codognato G., Ghosal S., Julien L., Tonin S. (2020), "Existence and optimality of Cournot-Nash equilibria in a bilateral oligopoly with atoms and an atomless part," *International Journal of Game Theory* **49**, 933-951.

[6] Codognato G., Ghosal S. (2000), "Cournot-Nash equilibria in limit exchange economies with complete markets and consistent prices," *Journal of Mathematical Economics* **34**, 39-53.

[7] Debreu G. (1982), "Existence of competitive equilibrium," in Arrow K.J., Intriligator H.D. (eds), *Handbook of mathematical economics*, Elsevier, Amsterdam.

[8] Drèze J.H., Gabszewicz J.J., Postlewaite A. (1977), "Disadvantageous monopolies and disadvantageous endowments," *Journal of Economic Theory* **16**, 116-121.

[9] Edgeworth F.Y. (1881), *Mathematical psychics*, Kegan Paul, London.

[10] Forchheimer K. (1908), "Theoretisches zum unvollständingen monopole," *Jahrbuch für Gesetzgebung, Verwaltung und Volkswirschafts im Deutschen Reich* **32**, 1-12.

[11] Greenberg J., Shitovitz B. (1977), "Advantageous monopolies," *Journal of Economic Theory* **16**, 394-402.

[12] Kats A. (1974), "Monopolistic trading economies: a case of governmental control," *Public Choice* **20**, 17-32.

[13] Kreps D. (2012), *Microeconomic foundations I: choice and competitive markets*, Princeton University Press, Princeton.

[14] Okuno M., Postlewaite A., Roberts J. (1980), "Oligopoly and competition in large markets," *American Economic Review* **70**, 22-31.

[15] Pareto V. (1896), *Cours d'économie politique*, F. Rouge Editeur, Lausanne.

[16] Pareto V. (1909), *Manuel d'économie politique*, V. Giard & E. Brière, Paris.

[17] Reid G.C. (1979), "Forchheimer on partial monopoly," *History of Political Economy* **11**, 303-308.

[18] Sadanand V. (1988), "Endogenously determined price-setting monopoly in an exchange economy," *Journal of Economic Theory* **46**, 172-178.

[19] Sahi S., Yao S. (1989), "The noncooperative equilibria of a trading economy with complete markets and consistent prices," *Journal of Mathematical Economics* **18**, 325-346.

[20] Schydlowsky D.M., Siamwalla A. (1966), "Monopoly under general equilibrium: a geometric exercise," *Quarterly Journal of Economics* **80**, 147-153.

[21] Shitovitz B. (1973), "Oligopoly in markets with a continuum of traders," *Econometrica* **41**, 467-501.

[22] Varian H.R. (2014), *Intermediate microeconomics with calculus*, Norton, New York.

Chapter 3

Monopoly Equilibrium as a Subgame Perfect Equilibrium

Abstract

We reconsider the monopoly model in bilateral exchange and we provide a game theoretic characterization for the set of monopoly equilibria. We formulate a two stage game version of the monopolistic market and we prove that the set of the subgame equilibria of this game coincides with the set of monopoly equilibria.

3.1 Introduction

We adapt to the monopoly bilateral exchange context the version of the Shapley window model used by Busetto et al. (2020) and we assume that the atomless part behaves à la Cournot making bids of the commodity it holds. We show that there is no Cournot-Nash equilibrium in the market game generated by the strategic interaction between the monopolist and the atomless part through the Shapley window trading process, thereby confirming an analogous negative result obtained by Okuno et al. (1980, p. 24) for the monopolistic version of their bilateral strategic market game. Moreover, we provide an example exhibiting a bilateral exchange economy which admits a monopoly equilibrium but no Cournot-Nash equilibrium. Our example shows that it is not possible to provide a game theoretical foundation of our monopoly solution in terms of an equivalence between the set of the allocations corresponding to a monopoly equilibrium and the set of the allocations corresponding to a Cournot-Nash equilbrium, in a one-stage setting.

Sadanand (1988, p. 174) started from the negative result about the existence of a

Cournot-Nash equilibrium in a one-shot monopolistic bilateral strategic market game obtained by Okuno et al. (1980) and this lead him to introduce a monopoly price-setting solution in a two-stage version of the strategic market game analyzed by those authors.

Following Sadanand (1988), we provide a sequential reformulation of the mixed version of the Shapley window model in terms of a two-stage game with observed actions where the quantity-setting monopolist moves first and the atomless part moves in the second stage, after observing the moves of the monopolist in the first stage. This two-stage reformulation of our model allows us to provide a game theoretical foundation of the quantity-setting monopoly solution: we prove that the set of the allocations corresponding to a monopoly equilibrium and the set of those corresponding to a subgame perfect equilibrium of the two-stage game coincide.

Once we rule out the possibility for the monopoly equilibria to coincide with Cournot-Nash equilibria, we move to the analysis of the possible relation between our solution concept and the Cournot-Walras approach, introduced by Gabszewicz and Vial (1972). The main result of this paper is indeed to show that the the set of monopoly equilibria coincide with the set of subgame perfect equilibrium of a two stage game in which the monopolist moves first and the atomless sector moves later.

Our approach is similar to the one in Busetto et al. (2008), with one important twist. In their paper, the set of Cournot- Walras equilibria is shown to coincide with the set of Markov perfect equilibrium. In our setting, which is based on different assumptions on the set of traders, this results extends theirs in the sense that we prove that the set of monopoly equilibria coincides with the set of subgame perfect equilibria.

The paper is organized as follows. In section 2, the mathematical model is introduced followed by a reminder of the notion of a monopoly equilibrium, in section 3. In Section 4, we compare the monopoly equilibrium and the Cournot-Nash equilibrium. In Section 5, we provide a game theoretical foundation of the monopoly solution in a two-stage framework. Section 6 gives some concluding remarks and gives future areas of research.

3.2 Mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. Let T_0 denote the atomless part of T. We assume that $\mu(T_0) > 0$ and $T \setminus T_0 = \{a\}$, i.e., the measure space $(T, \mathcal{T}, \mu)$ contains only one atom, the "monopolist." A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.

A commodity bundle is a point in $\mathbb{R}^2_+$. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to \mathbb{R}^2_+$. We are considering a bilateral exchange economy, therefore with two commodities. We assume that the monopolist holds, without loss of generality good one, while small traders hold the second good, i.e.

Assumption 7. $\mathbf{w}^1(m) > 0$, $\mathbf{w}^2(m) = 0$ *and* $\mathbf{w}^1(t) = 0$, $\mathbf{w}^2(t) > 0$, *for each* $t \in T_0$.

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : \mathbb{R}^2_+ \to R$, satisfying the following assumptions.

Assumption 8. $u_t : \mathbb{R}^2_+ \to R$ *is continuous, strongly monotone, and strictly quasi-concave, for each* $t \in T$.

Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}^2_+$. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 9. $u : T \times \mathbb{R}^2_+ \to R$, *given by* $u(t, x) = u_t(x)$, *for each* $t \in T$ *and for each* $x \in \mathbb{R}^2_+$, *is* $\mathcal{T} \bigotimes \mathcal{B}$*-measurable.*

In order to state our last assumption, we need a preliminary definition. We say that commodities i, j stand in relation Q if there is a nonnull subset T^i of T_0, such that $u_t(\cdot)$ is differentiable, additively separable, i.e., $u_t(x) = v^i_t(x_i) + v^j_t(x_j)$, for each $x \in \mathbb{R}^2_+$, and $\frac{dv^j_t(0)}{dx_j} = +\infty$, for each $t \in T^i$.[1] We can now introduce the last assumption.

Assumption 10. *Commodities 1 and 2 stand in relation Q.*

A price vector is a nonnull vector $p \in \mathbb{R}^2_+$. Moreover, we will denote by Δ the unit simplex, i.e. $\Delta = \{p\mathbb{R}^2_+ : p^1 + p^2 = 1\}$, and $\Delta \setminus \partial\Delta$ will denote the interior of Δ. Finally, we will write $P \in \mathbb{R}_+$ to intend the corresponding relative price for each $p \in \Delta \setminus \partial\Delta$,

[1] In this definition, differentiability means continuous differentiability and is to be understood to include the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

i.e. $P = \frac{p^1}{p^2}$, for some $(p^1, p^2) \in \Delta \setminus \partial\Delta$.
Let $\mathbf{X}^0 : T_0 \times \mathbb{R}^2_{++} \to \mathcal{P}(\mathbb{R}^2_+)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in \mathbb{R}^2_{++}$, $\mathbf{X}^0(t,p) = \text{argmax}\{u(x) : x \in \mathbb{R}^2_+ \text{ and } px \leq p\mathbf{w}(t)\}$. For each $p \in \mathbb{R}^2_{++}$, let $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \{\int_{T_0} \mathbf{x}(t,p)\, d\mu : \mathbf{x}(\cdot,p)$ is integrable and $\mathbf{x}(t,p) \in \mathbf{X}^0(t,p)$, for each $t \in T_0\}$. Since the correspondence $\mathbf{X}^0(t,\cdot)$ is nonempty and single-valued, by Assumption 2, let $\mathbf{x}^0 : T_0 \times \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ be the function such that $\mathbf{X}^0(t,p) = \{\mathbf{x}^0(t,p)\}$, for each $t \in T_0$ and for each $p \in \mathbb{R}^2_{++}$. A Walras equilibrium is a pair $(p, \mathbf{x})$, consisting of a price vector p and an allocation $\mathbf{x}$, such that $p\mathbf{x}(t) = p\mathbf{w}(t)$ and $u_t(\mathbf{x}(t)) \geq u_t(y)$, for all $y \in \{x \in \mathbb{R}^2_+ : px = p\mathbf{w}(t)\}$, for each $t \in T$. A Walras allocation is an allocation $\mathbf{x}$ for which there exists a price vector p such that the pair $(p, \mathbf{x})$ is a Walras equilibrium.

3.3 Monopoly equilibrium

We introducing the monopoly equilibrium concept.
Let $\mathbf{E}(m) = \{(e_{ij}) \in R^4_+ : \sum_{j=1}^2 e_{ij} \leq \mathbf{w}^i(m),\ i = 1,2\}$ denote the strategy set of atom a. We denote by $e \in \mathbf{E}(m)$ a strategy of atom a, where e_{ij}, $i, j = 1,2$, represents the amount of commodity i that atom a offers in exchange for commodity j. Moreover, we denote by E the matrix corresponding to a strategy $e \in \mathbf{E}(m)$.

We then provide the following definitions.

Definition 6. *A square matrix A is said to be triangular if $a_{ij} = 0$ whenever $i > j$ or $a_{ij} = 0$ whenever $i < j$.*

Definition 7. *Given a strategy $e \in \mathbf{E}(a)$, a price vector p is said to be market clearing if*

$$p \in R^2_{++}, \int_{T_0} \mathbf{x}^{0j}(t,p)\, d\mu + \sum_{i=1}^{2} e_{ij}\mu(m)\frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{2} e_{ji}\mu(m) \tag{3.1}$$

, $j = 1,2$.

We recall here a proposition from the previous chapter, that provides a necessary and sufficient condition for the existence of a market clearing price vector.

Proposition 14. *Under Assumptions 7, 8, 9, and 10, given a strategy $e \in \mathbf{E}(m)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular.*

Proof. See Chapter 2. ∎

We denote by $\pi(e)$ a correspondence which associates, with each strategy $e \in \mathbf{E}(m)$, the set of price vectors p satisfying (1), if E is triangular, and is equal to $\{0\}$, otherwise.

A price selection $p(e)$ is a function which associates, with each strategy selection $e \in \mathbf{E}(m)$, a price vector $p \in \pi(e)$.

Given a strategy $e \in \mathbf{E}(m)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned} \mathbf{x}^j(m,e,p) &= \mathbf{w}^j(m) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j}, \text{ if } p \in R^2_{++}, \\ \mathbf{x}^j(m,e,p) &= \mathbf{w}^j(m), \text{ otherwise}, \end{aligned}$$

$j = 1, 2$,

$$\begin{aligned} \mathbf{x}^j(t,p) &= \mathbf{x}^{0j}(t,p), \text{ if } p \in R^2_{++}, \\ \mathbf{x}^j(t,p) &= \mathbf{w}^j(t), \text{ otherwise}, \end{aligned}$$

$j = 1, 2$, for each $t \in T_0$.

Given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(m)$, traders' final holdings are expressed by the assignment $\mathbf{x}(m) = \mathbf{x}(m, e, p(e))$ and $\mathbf{x}(t) = \mathbf{x}(t, p(e))$, for each $t \in T_0$.

The following proposition, replicated from the previous chapter, shows that traders' final holdings are an allocation.

Proposition 15. *Under Assumptions 7, 8, 9 and 10, given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(m)$, the assignment $\mathbf{x}(m) = \mathbf{x}(m, e, p(e))$ and $\mathbf{x}(t) = \mathbf{x}^0(t, p(e))$, for each $t \in T_0$, is an allocation.*

Proof. See Chapter 2. ■

We can now provide the definition of a monopoly equilibrium.

Definition 8. *A strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{E}$ is triangular is a monopoly equilibrium, with respect to a price selection $p(\cdot)$, if*

$$u_a(\mathbf{x}(m, \tilde{e}, p(\tilde{e})) \geq u_a(\mathbf{x}(m, e, p(e)),$$

for each $e \in \mathbf{E}(m)$.

3.4 Monopoly equilibrium and Cournot-Nash equilibrium

We now provide the definition of a Cournot-Nash equilibrium in the bilateral exchange model introduced previously, adapting to this framework the version of the Shapley window model used by Busetto et al. (2020).

A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R^4_+)$ such that, for each $t \in T$, $\mathbf{B}(t) = \{(b_{ij}) \in R^4_+ : \sum_{j=1}^2 b_{ij} \leq \mathbf{w}^i(t),\, i = 1,2\}$. We denote by $b(t) \in \mathbf{B}(t)$ a strategy of trader t, where $b_{ij}(t)$, $i, j = 1,2$, represents the amount of commodity i that trader t offers in exchange for commodity j. A strategy selection is an integrable function $\mathbf{b} : T \to R^4_+$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. Given a strategy selection $\mathbf{b}$, we denote by $\bar{\mathbf{B}}$ the matrix such that $\bar{\mathbf{b}}_{ij} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$, $i, j = 1,2$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ the strategy selection obtained from $\mathbf{b}$ by replacing $\mathbf{b}(t)$ with $b(t) \in \mathbf{B}(t)$.

We need to provide now the following two definitions (see Sahi and Yao (1989)).

Definition 9. *A nonnegative square matrix C is said to be irreducible if, for every pair (i, j), with $i \neq j$, there is a positive integer k such that $c^{(k)}_{ij} > 0$, where $c^{(k)}_{ij}$ denotes the ij-th entry of the k-th power C^k of C.*

Definition 10. *Given a strategy selection $\mathbf{b}$, a price vector p is said to be market clearing if*

$$p \in R^2_{++}, \sum_{i=1}^{2} p^i \bar{\mathbf{b}}_{ij} = p^j (\sum_{i=1}^{2} \bar{\mathbf{b}}_{ji}),\ j = 1,2. \tag{2}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (2) if and only if $\bar{\mathbf{B}}$ is irreducible. Then, we denote by $p(\mathbf{b})$ a function which associates with each strategy selection $\mathbf{b}$ the unique, up to a scalar multiple, price vector p satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, and is equal to 0, otherwise. For each strategy selection $\mathbf{b}$ such that $p(\mathbf{b}) \gg 0$, we assume that the price vector $p(\mathbf{b})$ is normalized.

Given a strategy selection $\mathbf{b}$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned} \mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^{2} \mathbf{b}_{ji}(t) + \sum_{i=1}^{2} \mathbf{b}_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\ \mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise}, \end{aligned}$$

$j = 1,2$, for each $t \in T$.

Given a strategy selection $\mathbf{b}$ and the function $p(\mathbf{b})$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),$$

for each $t \in T$. It is straightforward to show that this assignment is an allocation.

We are now able to define a notion of Cournot-Nash equilibrium for this reformulation of the Shapley window model.

Definition 11. *A strategy selection $\hat{\mathbf{b}}$ such that $\bar{\hat{\mathbf{B}}}$ is irreducible is a Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.

A Cournot-Nash allocation is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, where $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium.

The next proposition provides, for our framework, the same negative result about the existence of a Cournot-Nash equilibrium obtained by Okuno et al. (1980 p. 24) and by Sadanand (1988, p. 174).

Proposition 16. *Under Assumptions 7, 8, 9, and 10, there exists no Cournot-Nash equilibrium.*

Proof. Suppose that $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium. Then, we have that $\mathbf{x}(m, \hat{\mathbf{b}}(m), p(\hat{\mathbf{b}})) = (\mathbf{w}^1(m) - \hat{\mathbf{b}}_{12}(m), \bar{\hat{\mathbf{b}}}_{21})$. Let $b'(m)$ be a strategy such that $0 < b'_{12}(m) < \hat{\mathbf{b}}_{12}(m)$. Then, we have that

$$u_m(\mathbf{x}(m, \hat{\mathbf{b}} \setminus b'(m), p(\hat{\mathbf{b}} \setminus b'(m)))) > u_m(\mathbf{x}(m, \hat{\mathbf{b}}(m), p(\hat{\mathbf{b}}))),$$

as $\mathbf{x}(m, \hat{\mathbf{b}} \setminus b'(m), p(\hat{\mathbf{b}} \setminus b'(m))) = (\mathbf{w}^i(m) - b'_{12}(m)), \bar{\hat{\mathbf{b}}}_{21})$ and $u_m(\cdot)$ is strongly monotone, by Assumption 8, a contradiction. Hence, there is no Cournot-Nash equilibrium. ■

Proposition 16 has the relevant consequence that the set of monopoly allocations cannot coincide with the set of Cournot-Nash allocations in a one-stage setting, as confirmed by the following example[2].

Example 5. *Consider the following specification of an exchange economy satisfying Assumptions 7, 8, 9, and 10. $T_0 = [0,1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1,0)$, $u_m(x) = \frac{1}{2}x^1 + \sqrt{x^2}$, T_0 is taken with Lebesgue measure, $\mathbf{w}(t) = (0,1)$, $u_t(x) = \sqrt{x^1} + x^2$, for each $t \in T_0$. Then, there is a unique monopoly allocation and no Cournot-Nash allocation.*

Proof. The unique monopoly equilibrium is the strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{e}_{12} = \frac{1}{4}$ and the unique monopoly allocation is $\tilde{\mathbf{x}}(m) = (\frac{3}{4}, \frac{1}{4})$ and $\tilde{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$. However, there is no Cournot-Nash allocation, by Proposition 16. ■

[2]Compare with Example 1 in Chapter 2

3.5 Monopoly equilibrium as a subgame perfect equilibrium

Example 5 shows the nonequivalence between the set of monopoly and Cournot-Nash allocations in a one-stage game. The analogous negative result reached by Okuno et al. (1980) lead these authors to conclude that "[...] we are unable to model pure monopoly without a competitive fringe in a useful way in this setup" (see Footnote 1, p. 24). In his pathbreaking analysis of monopoly in mixed exchange economies, Sadanand (1988) already recognized the two stage-flavor of monopoly equilibrium. Taking inspiration from his work, we now introduce a two-stage game where the monopolist moves first and the atomless part moves in the second stage, after observing the moves of the monopolist in the first stage. Therefore, borrowing from Busetto et al. (2008), we provide a sequential reformulation of the mixed version of the Shapley window model introduced in the previous section, in terms of a two-stage game with observed actions, following Fudenberg and Tirole (1991, p. 70).

The game is played in two stages, labelled as 0 and 1. An action correspondence in stage 0 is a correspondence $\mathbf{A}^0 : T \to \mathcal{P}(R^4_+)$ such that $\mathbf{A}^0(m) = \{(a_{ij}) \in R^4_+ : \sum_{j=1}^2 a_{ij} \leq \mathbf{w}^i(m),\ i = 1,2\}$ and $\mathbf{A}^0(t)$ is the singleton {"do nothing"}, for each $t \in T_0$. An action correspondence in stage 1 is a correspondence $\mathbf{A}^1 : T \to \mathcal{P}(R^4_+)$ such that $\mathbf{A}^1(m)$ is the singleton {"do nothing"} and $\mathbf{A}^1(t) = \{(a_{ij}) \in R^4_+ : \sum_{j=1}^2 a_{ij} \leq \mathbf{w}^i(t),\ i = 1,2\}$, for each $t \in T_0$. We denote by $a^0(t) \in \mathbf{A}^0(t)$ an action of trader t in stage 0, where $a^0_{ij}(m)$, $i,j = 1,2$, represents the amount of commodity i that atom m offers in exchange for commodity j. An action selection in stage 0 is a function $\mathbf{a}^0 : T \to R^4_+$, such that $\mathbf{a}^0(t) \in \mathbf{A}^0(t)$, for each $t \in T$. We denote by $a^1(t) \in \mathbf{A}^1(t)$ an action of trader t in stage 1, where $a^1_{ij}(t)$, $i,j = 1,2$, represents the amount of commodity i that a trader $t \in T_0$ offers in exchange for commodity j. An action selection in stage 1 is a function $\mathbf{a}^1 : T \to R^4_+$, whose restriction on T_0 is integrable, such that $\mathbf{a}^1(t) \in \mathbf{A}^1(t)$, for each $t \in T$. Let S^0 and S^1 denote the sets of all action selections in stage 0 and in stage 1, respectively. Any action selection at the end of a stage determines a history at the beginning of the next stage.

We denote by $\mathbf{h}^0 = \emptyset$ the history at the beginning of stage 0 and by $\mathbf{h}^1$ a history at the beginning of stage 1 where $\mathbf{h}^1 = \mathbf{a}^0$, for some $\mathbf{a}^0 \in S^0$. Let H^0 and H^1 denote the sets of all stage 0 and stage 1 histories, respectively, where $H^0 = \emptyset$ and $H^1 = S^0$ Let $H^2 = S^0 \times S^1$ denote the set of all terminal histories. Given a terminal history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$, we denote by $\bar{\mathbf{A}}$ the matrix such that $\bar{\mathbf{a}}_{ij} = \mathbf{a}^0_{ij}(m) + \int_{T_0} \mathbf{a}^1_{ij}(t)\, d\mu$, $i,j = 1,2$.

We now provide the following definition (see Sahi and Yao (1989)).

Definition 12. *Given a terminal history* $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$*, a price vector* p *is said to be market* *clearing if*

$$p \in R^2_{++}, \sum_{i=1}^{2} p^i \bar{\mathbf{a}}_{ij} = p^j (\sum_{i=1}^{2} \bar{\mathbf{a}}_{ji}), \, j = 1, 2. \tag{3}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (3) if and only if $\bar{\mathbf{A}}$ is irreducible. Then, we denote by $p(\mathbf{h}^2)$ a function which associates with each final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ the unique, up to a scalar multiple, price vector p satisfying (3), if $\bar{\mathbf{A}}$ is irreducible, and is equal to 0, otherwise. For each final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ such that $p(\mathbf{h}^2) \gg 0$, we assume that the price vector $p(\mathbf{h}^2)$ is normalized.

Given a terminal history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned} \mathbf{x}^j(m, \mathbf{h}^2(m), p) &= \mathbf{w}^j(m) - \sum_{i=1}^{2} \mathbf{a}^0_{ji}(m) + \sum_{i=1}^{2} \mathbf{a}^0_{ij}(m) \frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\ \mathbf{x}^j(m, \mathbf{h}^2(m), p) &= \mathbf{w}^j(m), \text{ otherwise,} \end{aligned}$$

$j = 1, 2,$

$$\begin{aligned} \mathbf{x}^j(t, \mathbf{h}^2(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^{2} \mathbf{a}^1_{ji}(t) + \sum_{i=1}^{2} \mathbf{a}^1_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\ \mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise,} \end{aligned}$$

$j = 1, 2,$ for each $t \in T_0$.

Given a terminal history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ and the function $p(\mathbf{h}^2)$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{h}^2(t), p(\mathbf{h}^2)),$$

for each $t \in T$. It is straightforward to show that this assignment is an allocation.

We denote by $s(t)$ a strategy of trader t, where $s(t)$ denotes the sequence of functions $\{s^0(t, \cdot), s^1(t, \cdot)\}$ such that $s^0(t, \cdot) : H^0 \to \mathbf{A}^0(t)$ and $s^1(t, \cdot) : H^1 \to \mathbf{A}^1(t)$. A strategy profile $\mathbf{s}$ is a map which associates with each $t \in T$ a sequence of functions $\{\mathbf{s}^0, \mathbf{s}^1\}$ such that $\mathbf{s}^0(t, \cdot) : H^0 \to \mathbf{A}^0(t)$, $\mathbf{s}^1(t, \cdot) : H^1 \to \mathbf{A}^1(t)$, $\mathbf{s}^0(\cdot, \mathbf{h}^0) \in S^0$, and $\mathbf{s}^1(\cdot, \mathbf{h}^1) \in S^1$, for each $\mathbf{h}^1 \in H^1$. Given a strategy profile $\mathbf{s}$, the functions $\mathbf{s}^0(\cdot, \mathbf{h}^0)$ and $\mathbf{s}^1(\cdot, \mathbf{h}^1)$, for each $\mathbf{h}^1 \in H^1$, are called strategy selections. We denote by $\mathbf{s} \setminus s(t) = \{\mathbf{s}^0 \setminus s(t, \cdot), \mathbf{s}^1 \setminus s^1(t, \cdot)\}$ the strategy profile obtained from $\mathbf{s}^0$ and $\mathbf{s}^1$ by replacing, respectively, $\mathbf{s}^0(t, \cdot)$ with $s^0(t, \cdot)$ and $\mathbf{s}^1(t, \cdot)$ with $s^1(t, \cdot)$. Finally, we denote by $\mathbf{h}^2(\mathbf{s})$ the function which associates with each strategy profile $\mathbf{s}$ the terminal history which corresponds to the action selections $\{\mathbf{a}^0(\mathbf{s}), \mathbf{a}^1(\mathbf{s})\}$ such that $\mathbf{a}^0(\mathbf{s}) = \mathbf{s}^0(\cdot, \mathbf{h}^0)$ and $\mathbf{a}^1(\mathbf{s}) = \mathbf{s}^1(\cdot, \mathbf{h}^1)$, with $\mathbf{h^1} = \mathbf{s}^0(\cdot, \mathbf{h}^0)$, and by $\bar{\mathbf{A}}(\mathbf{s})$ the corresponding aggregate matrix.

We now proceed to consider the subgame represented by the stage 1 of the game outlined above, given the history $\mathbf{h}^1 \in H^1$. Given a strategy $s(t)$ of trader t and a history $\mathbf{h}^1 \in H^1$, we denote by $s|\mathbf{h}^1(t)$ the action such that $s|\mathbf{h}^1(t) = s^1(t, \mathbf{h}^1)$. Given a strategy profile $\mathbf{s}$ and a history $\mathbf{h}^1 \in H^1$, we denote by $\mathbf{s}|\mathbf{h}^1$ the strategy selection such that $\mathbf{s}|\mathbf{h}^1(t) = \mathbf{s}^1(t, \mathbf{h}^1)$, for each $t \in T$. Given a history $\mathbf{h}^1 \in H^1$, we denote by $\mathbf{s}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(t)$ the strategy selection obtained from $\mathbf{s}|\mathbf{h}^1$ by replacing $\mathbf{s}|\mathbf{h}^1(t)$ with $s|\mathbf{h}^1(t)$. Finally, we denote by $\mathbf{h}^2(\mathbf{s}|\mathbf{h}^1)$ the function which associates with each strategy selection $\mathbf{s}|\mathbf{h}^1$ the terminal history which corresponds to the action selections $\{\mathbf{a}^0(\mathbf{s}|\mathbf{h}^1), \mathbf{a}^1(\mathbf{s}|\mathbf{h}^1)\}$ such that $\mathbf{a}^0(\mathbf{s}|\mathbf{h}^1) = \mathbf{h}^1$ and $\mathbf{a}^1(\mathbf{s}|\mathbf{h}^1) = \mathbf{s}|\mathbf{h}^1$, and by $\bar{\mathbf{A}}(\mathbf{s}|\mathbf{h}^1)$ the corresponding aggregate matrix.

We are now able to define the notion of subgame perfect equilibrium for the two-stage game described above.

Definition 13. *A strategy profile $\mathbf{s}^*$ such that $\bar{\mathbf{A}}(\mathbf{s}^*)$ is irreducible is a subgame perfect equilibrium if*

$$u_t(\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*)(t), p(\mathbf{h}^2(\mathbf{s}^*)))) \geq u_t(\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^* \setminus s(t))(t), p(\mathbf{h}^2(\mathbf{s}^* \setminus s(t))))),$$

for each $s(t)$ and for each $t \in T$, $\bar{\mathbf{A}}(\mathbf{s}^*|\mathbf{h}^1))$ is irreducible, for each $\mathbf{h}^1 \in H^1$ such that $\mathbf{h}^1(m) > 0$, and

$$\begin{aligned} &u_t(\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)))) \\ &\geq u_t(\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1 \setminus s|\mathbf{h}^1)(t))(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1 \setminus s|\mathbf{h}^1)(t))))), \end{aligned}$$

for each $\mathbf{h}^1 \in H^1$, for each $s|\mathbf{h}^1(t)$, and for each $t \in T$.

A subgame perfect allocation is an allocation $\mathbf{x}^*$ such that $\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*)(t), p(\mathbf{h}^2(\mathbf{s}^*)))$, for each $t \in T$, where $\mathbf{s}^*$ is a subgame perfect equilibrium.

The following proposition shows the equivalence between the set of monopoly allocations and the set of subgame perfect allocations for our two-stage game.

Proposition 17. *Under Assumptions 7,8,9 and 10, the set of monopoly allocations coincides with the set of subgame perfect allocations.*

Proof. Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$. Let $\tilde{\mathbf{x}}$ be a monopoly allocation. Then, we have that $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, p(\tilde{e}))$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{e}))$, for each $t \in T_0$, where $\tilde{e}$ is a monopoly equilibrium, with respect to a price selection $p(\cdot)$. Consider, first, stage 1 of the game. Let $e \in \mathbf{E}(m)$ be a strategy selection and let $\mathbf{h}^1$ be a history at the beginning of stage 1 of the game such that $\mathbf{h}^1(m) = e$. Suppose that E is triangular. Then, we have that $p(e) \gg 0$ and $p(e)\mathbf{x}^0(t, p(e)) = p(e)\mathbf{w}^2(t)$, for each $t \in T_0$, by Assumption 8. But then, there exist $\lambda^j(t) \geq 0$, $j = 1, 2$, $\sum_{j=1}^2 \lambda^j(t) = 1$, such that

$$\mathbf{x}^0(t, p(e)) = \lambda^j(t)\frac{p^2(e)\mathbf{w}^2(t)}{p^j(e)},$$

$j = 1, 2$, for each $t \in T_0$, by Lemma 5 in Codognato and Ghosal (2000). Let ˘ $: T_0 \to R^2_+$ be a function such that ˘$^j(t) = \lambda^j(t)$, $j = 1, 2$, for each $t \in T_0$. It is straightforward to show that the function $\mathbf{w}^i(t)\lambda^j(t)$, $i, j = 1, 2$, for each $t \in T_0$, is integrable on T_0. Let $\tilde{\mathbf{s}}|\mathbf{h}^1$ denote a strategy selection of the subgame represented by the stage 1 of the game such that $\mathbf{a}^1(\tilde{\mathbf{s}}|\mathbf{h}^1)(m) = \{$"do nothing"$\}$ and $\mathbf{a}^1_{ij}(\tilde{\mathbf{s}}|\mathbf{h}^1)(t) = \mathbf{w}^i(t)\lambda^j(t)$, $i, j = 1, 2$, for each $t \in T_0$. It is immediate to verify that $(\tilde{\mathbf{s}}|\mathbf{h}^1)(t) \in \mathbf{A}^1(t)$, for each $t \in T$. Consider the matrix $\bar{\mathbf{A}}(\tilde{\mathbf{s}}|\mathbf{h}^1)$. We have that

$$\bar{\mathbf{a}}_{12}(\tilde{\mathbf{s}}|\mathbf{h}^1) = \mathbf{a}^0_{12}(\tilde{\mathbf{s}}|\mathbf{h}^1)(m)\mu(m) + \int_{t \in T_0} \mathbf{a}^1_{12}(\tilde{\mathbf{s}}|\mathbf{h}^1)(t)\, d\mu = e_{12}\mu(m) > 0.$$

By the same argument used in the proof of Proposition 5 in chapter 2, Assumption 10 implies that $\mathbf{x}^{01}(t, p(e)) > 0$, for each $t \in T^2$. Then, we have that $\lambda^1(t) > 0$, for each $t \in T^2$. But then, we have that

$$\begin{aligned}\bar{\mathbf{a}}_{21}(\tilde{\mathbf{s}}|\mathbf{h}^1) &= \mathbf{a}^0_{21}(\tilde{\mathbf{s}}|\mathbf{h}^1)(m)\mu(m) + \int_{t \in T_0} \mathbf{a}^1_{21}(\tilde{\mathbf{s}}|\mathbf{h}^1)(t)\, d\mu \\ &= \int_{t \in T_0} \mathbf{w}^2(t)\lambda^1(t)\, d\mu > 0.\end{aligned}$$

Therefore, the matrix $\bar{\mathbf{A}}(\tilde{\mathbf{s}}|\mathbf{h}^1)$ is irreducible. Then, from (1), we obtain that

$$\begin{aligned}\int_{T_0} \mathbf{x}^1(t, p(e))\, d\mu &= \int_{T_0} \lambda^1(t)\frac{p^2(e)\mathbf{w}^2(t)}{p^1(e)}\, d\mu \\ &= \int_{T_0} \mathbf{a}^1_{21}(\tilde{\mathbf{s}}|\mathbf{h}^1)(t)\frac{p^2(e)}{p^1(e)}\, d\mu = e_{12}\mu(m).\end{aligned}$$

But then, it must be that $p(e) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))$ as $p(e)$ satisfies (3) and the matrix $\bar{\mathbf{A}}(\tilde{\mathbf{s}}|\mathbf{h}^1)$ is irreducible. Therefore, it is straightforward to verify that

$$\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(m), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))) = \mathbf{x}(m, e, p(e))$$

and

$$\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(t), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))) = \mathbf{x}(t, p(e)),$$

for each $t \in T_0$. It remains now to show that no trader $t \in T$, in stage 1 of the game, has an advantageous deviation from $\tilde{\mathbf{s}}|\mathbf{h}^1$. This is trivially true for m. Suppose that there exist a trader $\tau \in T_0$ and a strategy $s(\tau)$ such that

$$u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau)))))$$
$$> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))).$$

It is straightforward to verify that Definition 8 implies that $p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1)(\tau))) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))$. Then, we have that

$$u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))))$$
$$> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))) = u_\tau(\mathbf{x}(\tau, p(e))).$$

It is also immediate to verify that

$$p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))\mathbf{w}(\tau).$$

Then, we have that

$$u_\tau(\mathbf{x}(\tau, p(e))) > u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))),$$

a contradiction. Therefore, it must be that

$$u_t(\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(t), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))))$$
$$\geq u_t(\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(t))(t), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(t))))),$$

for each $t \in T_0$.

Suppose that E is not triangular. Then, we have that $p(e) = 0$. Let $\tilde{\mathbf{s}}|\mathbf{h}^1$ denote a strategy selection of the subgame represented by the stage 1 of the game such that $\mathbf{a}^1(\tilde{\mathbf{s}}|\mathbf{h}^1)(m) = \{$"do nothing"$\}$ and $\mathbf{a}^1_{ij}(\tilde{\mathbf{s}}|\mathbf{h}^1)(t) = 0$, $i, j = 1, 2$, for each $t \in T_0$. It is immediate to verify that $(\tilde{\mathbf{s}}|\mathbf{h}^1)(t) \in \mathbf{A}^1(t)$, for each $t \in T$ and that the matrix $\bar{\mathbf{A}}(\tilde{\mathbf{s}}|\mathbf{h}^1)$ is not irreducible. Then, it must be that $p(e) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)$. Therefore, we have that

$$\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(m), p(\tilde{\mathbf{s}}|\mathbf{h}^1)) = \mathbf{w}^1(m) = \mathbf{x}(m, e, p(e))$$

and

$$\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(t), p((\tilde{\mathbf{s}}|\mathbf{h}^1))) = \mathbf{w}^2(t) = \mathbf{x}(t, p(e)),$$

for each $t \in T_0$. It remains now to show that no trader $t \in T$, in stage 1 of the game, has an advantageous deviation from $\tilde{\mathbf{s}}|\mathbf{h}^1$. This is trivially true for m. Suppose that there exist a trader $\tau \in T_0$ and an strategy $s(\tau)$ such that

$$u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau)))))$$
$$> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))).$$

Then, we have that

$$\mathbf{w}^2(\tau) = u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))))$$
$$> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))) = \mathbf{w}^2(\tau),$$

as $p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)) = 0$, a contradiction. Therefore, we conclude that $\bar{\mathbf{A}}(\mathbf{s}^*|\mathbf{h}^1)$ is irreducible, for each $\mathbf{h}^1 \in H^1$ such that $\mathbf{h}^1(m) > 0$, and

$$u_t(\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(t), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))))$$
$$\geq u_t(\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(t))(t), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(t))))),$$

for each $\mathbf{h}^1 \in H^1$, for each $s|\mathbf{h}^1(t)$, and for each $t \in T$. Consider now stages 0 and 1 of the game. Let $\tilde{\mathbf{s}}$ be a strategy profile such that $\tilde{\mathbf{s}}(m, \mathbf{h}^0) = \tilde{e}$ and $\tilde{\mathbf{s}}(t, \mathbf{h}^0) = \{\text{“do nothing”}\}$, for each $t \in T_0$, and $\tilde{\mathbf{s}}(t, \mathbf{h}^1) = (\tilde{\mathbf{s}}|\mathbf{h}^1)(t)$, for each $\mathbf{h}^1 \in H^1$, and for each $t \in T$. Let $\tilde{\mathbf{h}}^1$ be such that $\tilde{\mathbf{h}}^1(m) = \tilde{e}$. We have that $\mathbf{h}^2(\tilde{\mathbf{s}}) = \mathbf{h}^2(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)$ as $\mathbf{a}^0(\tilde{\mathbf{s}}) = \tilde{\mathbf{s}}^0(\cdot, \mathbf{h}^0) = \tilde{\mathbf{h}}^1 = \mathbf{a}^0(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)$ and $\mathbf{a}^1(\tilde{\mathbf{s}}) = \tilde{\mathbf{s}}^1(\cdot, \tilde{\mathbf{h}}^1) = \tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1 = \mathbf{a}^1(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)$. Then, it must be that $p(\tilde{e}) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)) = p(\mathbf{h}^2(\tilde{\mathbf{s}}))$. But then, it is straightforward to verify that

$$\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}})(m), p(\mathbf{h}^2(\tilde{\mathbf{s}}))) = \mathbf{x}(m, \tilde{e}, p(\tilde{e}))$$

and

$$\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}(t)), p(\mathbf{h}^2(\tilde{\mathbf{s}}))) = \mathbf{x}(t, p(\tilde{e})),$$

for each $t \in T_0$. Suppose that there exists a strategy $s(m)$ of the monopolist such that

$$u_m(\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m))(m), p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m))))) > u_m(\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}})(m), p(\mathbf{h}^2(\tilde{\mathbf{s}})))).$$

Let $e = \tilde{\mathbf{s}}^0 \setminus s(m, \mathbf{h}^0)(m)$. Then, we have that $p(e) = p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m)))$ by the same argument used before. But then, we have that

$$\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m))(m), p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m)))) = \mathbf{x}(m, e, p(e)).$$

Therefore, it must be that

$$u_m\mathbf{x}(m, e, p(e)) = u_m(\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m))(m), p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m)))))$$
$$> u_m(\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}})(m), p(\mathbf{h}^2(\tilde{\mathbf{s}})))) = \mathbf{x}(m, \tilde{e}, p(\tilde{e})),$$

a contradiction. Suppose that there exist a trader $\tau \in T_0$ and a strategy $s(\tau)$ such that

$$u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(\tau)))))$$
$$> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}})))).$$

It is straightforward to verify that Definition 8 implies that $p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(\tau))) = p(\mathbf{h}^2(\tilde{\mathbf{s}}))$. Then, we have that

$$u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\tilde{\mathbf{h}}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1 \setminus s|\tilde{\mathbf{h}}^1(\tau)))))$$
$$= u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(\tau)))))$$
$$> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}))))$$
$$= u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)))),$$

a contradiction. Thus the set of monopoly allocations is a subset of the set of subgame perfect allocations. Let $\mathbf{x}^*$ be a subgame perfect allocation. Then, we have that $\mathbf{x}^* = \mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*)(t), p(\mathbf{h}^2(\mathbf{s}^*)))$, for each $t \in T$, where $\mathbf{s}^*$ is a subgame perfect equilibrium. Let $p(e)$ be a function which associates, with each strategy selection $e \in \mathbf{E}(m)$, the price vector $p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))$ corresponding to the history $\mathbf{h}^1$ such that $\mathbf{h}^1(m) = e$. Let $e \in \mathbf{E}(m)$ be a strategy selection. Suppose that E is triangular. Then, it must that $p(e) = p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)) \gg 0$ as the matrix $\bar{\mathbf{A}}(\mathbf{s}^*|\mathbf{h}^1))$ is irreducible. Suppose that E is not triangular. Then, it must be that $p(e) = p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)) = 0$ as the matrix $\bar{\mathbf{A}}(\mathbf{s}^*|\mathbf{h}^1))$ is not irreducible. It is straightforward to verify that

$$\mathbf{x}(m, e, p(e)) = \mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(m), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))),$$

for each strategy selection $e \in \mathbf{E}(m)$ and for each history $\mathbf{h}^1$ such that $\mathbf{h}^1(m) = e$. It is also straightforward to show that

$$u_t(\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)))) > u_t(y),$$

for all $y \in \{x \in R^2_+ : \; p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))x = p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))\mathbf{w}^2(t)\}$, for each $\mathbf{h}^1 \in H^1$ such that $\mathbf{h}^1(m) > 0$ and for each $t \in T_0$, by the same argument used by Codognato and Ghosal (2000) in the proof of their Theorem 2 p. 49. Then, we have that

$$\mathbf{x}(t, p(e)) = \mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))),$$

for each strategy $e \in \mathbf{E}(m)$, for each history $\mathbf{h}^1$ such that $\mathbf{h}^1(m) = e$, and for each $t \in T_0$. Let $e \in \mathbf{E}(m)$ be a strategy selection such that E is triangular and let $\mathbf{h}^1$ be a history such that $\mathbf{h}^1(m) = e$. Then, we have that

$$\int_{T_0} \mathbf{x}^1(t, p(e))\, d\mu + \mathbf{x}^1(m, e, p(e)) = \int_{T_0} \mathbf{x}^1(t, p(e))\, d\mu + e_{12}$$
$$= \int_{T_0} \mathbf{x}^1(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)))\, d\mu$$
$$+\mathbf{x}^1(m, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(m), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))) = \mathbf{w}^1(m)$$

as the assignment $\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)))$, for each $t \in T$, is an allocation. But then, $p(e)$ satisfies (1) by Proposition 3. Therefore, $p(e)$ is a price selection. Let e^* be a strategy selection such that $e^* = \mathbf{s}^*(m, \mathbf{h}^0)$ and let $\mathbf{h}^{1^*}$ be such that $\mathbf{h}^{1^*}(m) = e^*$. We have that $\mathbf{h}^2(\mathbf{s}^*) = \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^{1^*})$ as $\mathbf{a}^0(\mathbf{s}^*) = \mathbf{s}^{0^*}(\cdot, \mathbf{h}^0) = \mathbf{h}^{1^*} = \mathbf{a}^0(\mathbf{s}^*|\mathbf{h}^{1^*})$ and $\mathbf{a}^1(\mathbf{s}^*) = \mathbf{s}^{1^*}(\cdot, \mathbf{h}^{1^*}) = \mathbf{s}^*|\mathbf{h}^{1^*} = \mathbf{a}^1(\mathbf{s}^*|\mathbf{h}^{1^*})$. Then, it must be that $e^* > 0$ as $\bar{\mathbf{A}}(\mathbf{s}^*)$ is irreducible and $p(e^*) = p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^{1^*}) = p(\mathbf{h}^2(\mathbf{s}^*))$. But then, it is straightforward to verify that

$$\mathbf{x}(m, e^*, p(e^*)) = \mathbf{x}(m, \mathbf{h}^2(\mathbf{s}^*)(m), p(\mathbf{h}^2(\mathbf{s}^*))).$$

Suppose that there exists a strategy $e \in \mathbf{E}(m)$ such that

$$u_m(\mathbf{x}(m, e, p(e))) > u_m(\mathbf{x}(m, e^*, p(e^*))).$$

Let $s(m)$ be a strategy of the monopolist such that $\mathbf{s}^{0^*} \setminus s(m, \mathbf{h}^0)(m) = e$. Then, we have that $p(e) = p(\mathbf{h}^2(\mathbf{s}^* \setminus s(m)))$ by the same argument used before. But then, we have that

$$\mathbf{x}(m, e, p(e)) = \mathbf{x}(m, \mathbf{h}^2(\mathbf{s}^* \setminus s(m))(m), p(\mathbf{h}^2(\mathbf{s}^* \setminus s(m)))).$$

Therefore, it must be that

$$u_m(\mathbf{x}(m, \mathbf{h}^2(\mathbf{s}^* \setminus s(m))(m), p(\mathbf{h}^2(\mathbf{s}^* \setminus s(m))))) = u_m\mathbf{x}(m, e, p(e))$$
$$> \mathbf{x}(m, e^*, p(e^*)) = u_m(\mathbf{x}(m, \mathbf{h}^2(\mathbf{s}^*)(m), p(\mathbf{h}^2(\mathbf{s}^*)))),$$

a contradiction. Thus the set of subgame perfect allocations is a subset of the set of monopoly allocations. Hence, the set of monopoly allocations coincides with the set of subgame perfect allocations. ■

3.6 Conclusions

We provide a game theoretic characaterization of the set of monopoly equilibrium, recognizing its 2 stage flavour and proving its equivalence with the set of subgame perfect equilibria of a game in which the atom moves first and the atomless sector moves second.

Besides the mixed two-stage game framework, our analysis of monopoly equilibrium as a subgame perfect equilibrium differs from the one proposed by Sadanand (1988) in that it considers a quantity setting monopolist whereas Sadanand (1988) deals with an endogenously determined price-setting monopoly. We leave for further research a reformulation of our model in terms of a price-setting monopolist and a comparison between the two monopoly configurations.

As reminded above, Busetto et al (2008) proposed a respecification à la Cournot-Walras of the mixed version of the Shapley window model for an exchange economy with a finite number of commodities. Since they obtained the negative result that the set of the Cournot-Walras equilibrium allocations of this respecification does not coincide with the set of the Cournot-Nash allocations of the mixed version of the original Shapley model in a one-stage setting, they provided a further reformulation of the Shapley model as a two-stage game. They showed that the set of the Cournot-Walras equilibrium allocations coincides with the set of the Markov perfect equilibrium allocations of the two-stage reformulation of the Shapley model.

The monopoly model studied in this paper cannot be considered as a two-commodity monopoly version of the Cournot-Walras model proposed by Busetto et al. (2008): this would require in fact that the atomless part, in the aggregate, held both commodities.

In this regard, it is worth noticing that a model of partial monopoly, where a monopolist shares a market with a competitive fringe, was proposed in a pioneering work by Forchheimer (1908) (see also Reid (1979) for a detailed analysis of this work). A two-commodity monopoly version of the Cournot-Walras framework proposed by Busetto et al. (2008), where one commodity is held by the monopolist and a fringe of the atomless part whereas the other commodity is only held by the atomless part, could be interpreted indeed as a bilateral exchange generalization of the partial monopoly model introduced by Forchhemeir (1908). An analysis of the relationship between these two approaches deserves to be developed in the detail and we leave it for further research.

Bibliography

[1] Busetto F., Codognato G., Ghosal S. (2008), "Cournot-Walras equilibrium as a subgame perfect equilibrium," *International Journal of Game Theory* **37**, 371-386.

[2] Busetto F., Codognato G., Ghosal S., Julien L., Tonin S. (2020), "Existence and optimality of Cournot-Nash equilibria in a bilateral oligopoly with atoms and an atomless part," *International Journal of Game Theory* **49**, 933-951.

[3] Debreu G. (1982), "Existence of competitive equilibrium," in Arrow K.J., Intriligator H.D. (eds), *Handbook of mathematical economics*, Elsevier, Amsterdam.

[4] Forchheimer K. (1908), "Theoretisches zum unvollständingen monopole," *Jahrbuch für Gesetzgebung, Verwaltung und Volkswirschafts im Deutschen Reich* **32**, 1-12.

[5] Fudenberg D., Tirole J. (1991), *Game theory*, MIT Press, Cambridge.

[6] Gabszewicz, J.J. and Vial, J.P. (1972), " Oligopoly " A la Cournot " in a General Equilibrium Analysis," *Journal of Economic Theory* **4**, 381-400

[7] Okuno M., Postlewaite A., Roberts J. (1980), "Oligopoly and competition in large markets," *American Economic Review* **70**, 22-31.

[8] Sadanand V. (1988), "Endogenously determined price-setting monopoly in an exchange economy," *Journal of Economic Theory* **46**, 172-178.

[9] Sahi S., Yao S. (1989), "The noncooperative equilibria of a trading economy with complete markets and consistent prices," *Journal of Mathematical Economics* **18**, 325-346.

Chapter 4

Monopoly equilibrium and elasticity of substitution: a note on the existence of the equilibrium

Abstract

We study the existence of a monopoly equilibrium in the bilateral mixed exchange framework. Non existence examples in which small traders have CES utility functions are provided and a link between the existence of an equilibrium and the degree of substitutability of the goods is explored. Therefore, the existence result is proved by introducing a sufficient assumption on the utilities of the small traders, stressing that we need them to be locally equivalent to a constant elasticity of scale utility function, whose elasticity parameter is greater than unity.

4.1 Introduction

In the literature of strategic market games, initiated by Shapley and Shubik (1977), a lot of attention has been put on this topic. Busetto et al.(2011) initiated a line of research about existence in mixed models extending, in a way, Sahi and Yao (1989) existence result for finite economies in a Shapley windows model.
The problem with the existence in oligopoly models, specially models following the approach by Gabszewicz and Vial (1972), is that a discontinuity in the Walrasian price correspondence may arise, leading to non-existence of equilibria. In order to solve this problem, Busetto et al.(2011) used their assumption 4, that states that at least two large

traders have interior endowments and the indifference curves passing through these points don't touch the axis (the assumption replicates the one in Sahi and Yao (1989). Later on, they provided a refined version of the existence result in which they required a strongly connected set of commodities (Busetto et al., 2017), but assuming that small traders hold, in the aggregate, all the commodities present in the market. In the context of bilateral markets, Bloch and Ghosal (1997) provided an existence result in their model by assuming complementarity in the two goods for each agent.
However, the monopoly model we introduce fails to meet the assumptions stated in Busetto et al.(2011,2017), which were needed to prove an existence result. Therefore, it seems that additional assumptions are required in order to guarantee the existence of a monopoly equilibrium . Borrowing from the well known partial equilibrium studies on monopoly, we introduce a sufficient condition for the existence of a monopoly equilibrium based on the elasticity notion, closer to the approach of Bloch and Ghosal (1997) and Bloch and Ferrer (2001). In the latter, they consider a bilateral oligopoly in which every trader has a CES utility function, showing in their Lemma 1 that "the offers of traders on the two sides of the market are strategic complements(substitutes) if and only if the goods are substitutes (complements)" (p.85). We will initially consider that all small traders have an identical CES function, showing how the monopoly equilibrium behaves in the three limit cases for CES utility functions. We show that for a generic utility function form for the monopolist, the monopoly equilibrium fails to exist when small traders have Cobb-Douglas or Leontief utility function. In particular, the non-existence result for Cobb-Douglas utilities stresses how the assumption of small traders holding in the aggregate every good is crucial for some existence results, such as the one in Codognato and Julien (2013).
As Bartra states, "we may conclude that a necessary condition for the monopoly equilibrium to exist is that both price elasticities of demand are greater than unity" (Bartra, 1972, p.358). We extend this result to our setting by providing a sufficient condition on the atomless sector utility functions which reflects, even if only locally, the previous statement.
The outline of the existence proof follows the classical results in strategic market games. However, this is one of the first existence results in which a specific price selection is defined and for which an ϵ-equilibrium is proven to exist.
The model will follow from the previous chapter, i.e. a mixed version of a monopolistic two-commodity exchange economy introduced by Shitovitz (1973) in his Example 1, in which one commodity is held only by the monopolist, represented as an atom, and the other is held only by small traders, represented by an atomless part.
The paper is organized as follows. In section 2, the mathematical model is introduced followed by a reminder of the notion of a monopoly equilibrium, in section 3. In sections 4 and 5, we compute the monopoly equilibrium when small traders have an iden-

tical CES utility function: we first consider the limit situations for CES utilities (i.e. Cobb-Douglas, Leontief and linear), followed by the general form, where we attempt to retrieve the limit situation results from the general case. In section 6 the existence theorem is proven, after introducing our sufficient condition. We then provide, in section 7, a few example to test the scope of our additional assumption for the existence of a monopoly equilibrium. Finally, in section 8 we draw some conclusions and we suggest some further lines of research.

4.2 Mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. Let T_0 denote the atomless part of T. We assume that $\mu(T_0) > 0$ and $T \setminus T_0 = \{a\}$, i.e., the measure space $(T, \mathcal{T}, \mu)$ contains only one atom, the "monopolist." A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.
A commodity bundle is a point in $\mathbb{R}^2_+$. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \rightarrow \mathbb{R}^2_+$. We are considering a bilateral exchange economy, therefore with two commodities. We assume that the monopolist holds, without loss of generality good one, while small traders hold the second good, i.e.

Assumption 11. $\mathbf{w}^1(m) > 0$, $\mathbf{w}^2(m) = 0$ *and* $\mathbf{w}^1(t) = 0$, $\mathbf{w}^2(t) > 0$, *for each* $t \in T_0$.

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : \mathbb{R}^2_+ \rightarrow R$, satisfying the following assumptions.

Assumption 12. $u_t : \mathbb{R}^2_+ \rightarrow R$ *is continuous, strongly monotone, and strictly quasi-concave, for each* $t \in T$.

Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}^2_+$. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 13. *$u : T \times \mathbb{R}^2_+ \to R$, given by $u(t,x) = u_t(x)$, for each $t \in T$ and for each $x \in \mathbb{R}^2_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.*

In order to state our last assumption, we need a preliminary definition. We say that commodities i, j stand in relation Q if there is a nonnull subset T^i of T_0, such that $u_t(\cdot)$ is differentiable, additively separable, i.e., $u_t(x) = v^i_t(x_i) + v^j_t(x_j)$, for each $x \in \mathbb{R}^2_+$, and $\frac{dv^j_t(0)}{dx_j} = +\infty$, for each $t \in T^i$.[1] We can now introduce the last assumption.

Assumption 14. *Commodities 1 and 2 stand in relation Q.*

A price vector is a nonnull vector $p \in \mathbb{R}^2_+$. Moreover, we will denote by Δ the unit simplex, i.e. $\Delta = \{p\mathbb{R}^2_+ : p^1 + p^2 = 1\}$, and $\Delta \setminus \partial\Delta$ will denote the interior of Δ. Finally, we will write $P \in \mathbb{R}_+$ to intend the corresponding relative price for each $p \in \Delta \setminus \partial\Delta$, i.e. $P = \frac{p^1}{p^2}$, for some $(p^1, p^2) \in \Delta \setminus \partial\Delta$.
Let $\mathbf{X}^0 : T_0 \times \mathbb{R}^2_{++} \to \mathcal{P}(\mathbb{R}^2_+)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in \mathbb{R}^2_{++}$, $\mathbf{X}^0(t,p) = \text{argmax}\{u(x) : x \in \mathbb{R}^2_+ \text{ and } px \leq p\mathbf{w}(t)\}$. For each $p \in \mathbb{R}^2_{++}$, let $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \{\int_{T_0} \mathbf{x}(t,p)\, d\mu : \mathbf{x}(\cdot,p)$ is integrable and $\mathbf{x}(t,p) \in \mathbf{X}^0(t,p)$, for each $t \in T_0\}$. Since the correspondence $\mathbf{X}^0(t,\cdot)$ is nonempty and single-valued, by Assumption 2, let $\mathbf{x}^0 : T_0 \times \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ be the function such that $\mathbf{X}^0(t,p) = \{\mathbf{x}^0(t,p)\}$, for each $t \in T_0$ and for each $p \in \mathbb{R}^2_{++}$. A Walras equilibrium is a pair $(p,\mathbf{x})$, consisting of a price vector p and an allocation $\mathbf{x}$, such that $p\mathbf{x}(t) = p\mathbf{w}(t)$ and $u_t(\mathbf{x}(t)) \geq u_t(y)$, for all $y \in \{x \in \mathbb{R}^2_+ : px = p\mathbf{w}(t)\}$, for each $t \in T$. A Walras allocation is an allocation $\mathbf{x}$ for which there exists a price vector p such that the pair $(p,\mathbf{x})$ is a Walras equilibrium.

4.3 Monopoly equilibrium

We introducing the monopoly equilibrium concept.
Let $\mathbf{E}(m) = \{(e_{ij}) \in R^4_+ : \sum_{j=1}^2 e_{ij} \leq \mathbf{w}^i(m),\, i = 1,2\}$ denote the strategy set of atom a. We denote by $e \in \mathbf{E}(m)$ a strategy of atom a, where e_{ij}, $i, j = 1,2$, represents the amount of commodity i that atom a offers in exchange for commodity j. Moreover, we denote by E the matrix corresponding to a strategy $e \in \mathbf{E}(m)$.

We then provide the following definitions.

Definition 14. *A square matrix A is said to be triangular if $a_{ij} = 0$ whenever $i > j$ or $a_{ij} = 0$ whenever $i < j$.*

[1] In this definition, differentiability means continuous differentiability and is to be understood to include the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

Definition 15. *Given a strategy $e \in \mathbf{E}(a)$, a price vector p is said to be market clearing if*

$$p \in R^2_{++}, \int_{T_0} \mathbf{x}^{0j}(t,p)\, d\mu + \sum_{i=1}^{2} e_{ij}\mu(m)\frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{2} e_{ji}\mu(m) \tag{4.1}$$

, $j = 1, 2$.

We recall here a proposition from the previous chapter, that provides a necessary and sufficient condition for the existence of a market clearing price vector.

Proposition 18. *Under Assumption 11, 12, 13 and 14, given a strategy $e \in \mathbf{E}(m)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular.*

Proof. See Chapter 2. ■

We denote by $\pi(e)$ a correspondence which associates, with each strategy $e \in \mathbf{E}(m)$, the set of price vectors p satisfying (1), if E is triangular, and is equal to $\{0\}$, otherwise. A price selection $p(e)$ is a function which associates, with each strategy selection $e \in \mathbf{E}(m)$, a price vector $p \in \pi(e)$.

Given a strategy $e \in \mathbf{E}(m)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned}
\mathbf{x}^j(m,e,p) &= \mathbf{w}^j(m) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j}, \text{ if } p \in R^2_{++}, \\
\mathbf{x}^j(m,e,p) &= \mathbf{w}^j(m), \text{ otherwise,}
\end{aligned}$$

$j = 1, 2$,

$$\begin{aligned}
\mathbf{x}^j(t,p) &= \mathbf{x}^{0j}(t,p), \text{ if } p \in R^2_{++}, \\
\mathbf{x}^j(t,p) &= \mathbf{w}^j(t), \text{ otherwise,}
\end{aligned}$$

$j = 1, 2$, for each $t \in T_0$.

Given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(m)$, traders' final holdings are expressed by the assignment $\mathbf{x}(m) = \mathbf{x}(m, e, p(e))$ and $\mathbf{x}(t) = \mathbf{x}(t, p(e))$, for each $t \in T_0$.

The following proposition, replicated from the previous chapter, shows that traders' final holdings are an allocation.

Proposition 19. *Under Assumptions 11, 12, 13, and 14, given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(m)$, the assignment $\mathbf{x}(m) = \mathbf{x}(m, e, p(e))$ and $\mathbf{x}(t) = \mathbf{x}^0(t, p(e))$, for each $t \in T_0$, is an allocation.*

Proof. See Chapter 2. ■

We can now provide the definition of a monopoly equilibrium.

Definition 16. *A strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{E}$ is triangular is a monopoly equilibrium, with respect to a price selection $p(\cdot)$, if*

$$u_a(\mathbf{x}(m, \tilde{e}, p(\tilde{e})) \geq u_a(\mathbf{x}(m, e, p(e)),$$

for each $e \in \mathbf{E}(m)$.

4.3.1 Monopoly equilibrium under invertible demand

We now show how the monopoly equilibrium can be computed when the demand function $\int_{T_0} x^{01}(t, p) d\mu$ is invertible. This doesn't change the theoretical background of the definition just provided, it just aims to give support to the way in which the problem will be tackled in the following sections.
In this situation, we want to show that finding the optimal bid for the monopolist is equivalent to obtaining the equilibrium bid as the demand computed at an optimal price.
We recall two more propositions from the previous chapter.

Proposition 20. *Under Assumption 11, 12, 13 and 14, the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible if and only, for each $x \in R_{++}$, there is a unique $p \in \Delta \setminus \partial\Delta$ such that $x = \int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu$.*

Proposition 21. *Under Assumption 11, 12, 13 and 14, if the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible, then there exists a unique price selection $\mathring{p}(\cdot)$.*

Let $\mathring{p}(\cdot)$ denote the inverse of the function of $\int_{T_0} x^{01}(t, p) d\mu$. We prove the following proposition.

Proposition 22. *Under Assumption 11, 12, 13 and 14, if the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible, then a strategy $\tilde{e} \in \mathbf{E}(a)$ such that $\tilde{E}$ is triangular is a monopoly equilibrium if and only if there exists a price $\tilde{p} \in \Delta \setminus \partial\Delta$ such that $u_a(\mathbf{x}(m, e(\tilde{p}), \tilde{p}) \geq u_a(\mathbf{x}(m, e(p), p)$, for each $p \in \Delta \setminus \partial\Delta$. Moreover, $\tilde{p} = \mathring{p}(\tilde{e})$.*

Proof. Suppose that $\tilde{e}$ is a monopoly equilibrium. Let $\tilde{p} = \mathring{p}(\tilde{e})$. Clearly, $\tilde{p}$ is uniquely defined, by Propositions 20 and 21, as $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible. Suppose that there exists p' such that $u_a(\mathbf{x}(m, e(p'), p') \geq u_a(\mathbf{x}(m, e(\tilde{p}), \tilde{p})$. But then, letting e' be the unique strategy such that $p' = \mathring{p}(e')$, $u_a(\mathbf{x}(m, e', p(e')) \geq u_a(\mathbf{x}(m, \tilde{e}, \mathring{\tilde{e}})$, a contradiction as $\tilde{e}$ is a monopoly equilibrium. But then, $u_a(\mathbf{x}(m, e(\bar{p}), (\bar{p})) \geq u_a(\mathbf{x}(m, e(p), p)$, for each $p \in \Delta \setminus \partial\Delta$.
Suppose now there exists a price $\tilde{p} \in \Delta \setminus \partial\Delta$ such that $u_a(\mathbf{x}(m, e(\tilde{p}), \tilde{p})) \geq u_a(\mathbf{x}(m, e(p), p))$, for each $p \in \Delta \setminus \partial\Delta$. Let $\tilde{e} = e(\tilde{p})$. Suppose that there exists $\bar{e}$ such that $u_a(\mathbf{x}(m, \bar{e}, \mathring{p}(\bar{e})) \geq$

$u_a(\mathbf{x}(m, \tilde{e}, p(\tilde{e})))$. But then, letting $\bar{p} = \mathring{p}(\bar{e})$ we have $u_a(\mathbf{x}(m, e(\bar{p}), \bar{p})) \geq u_a(\mathbf{x}(m, e(\tilde{p}), \tilde{p}))$, a contradiction. Therefore $\tilde{e} = e(\tilde{p})$ is a monopoly equilibrium. ■

The previous proposition shows we can compute the monopoly equilibrium as the bid resulting from the optimal price. Alternatively, proposition 22 states that if there is no optimal interior price, then there is no equilibrium.

4.4 Existence: Limit results

We will try now to give an existence result when the atomless part of the economy has an identical utility function, represented by a CES function in the form [2]

$$u(x, t) = (ax_1^\rho + (1-a)x_2^\rho)^{\frac{1}{\rho}}.$$

The elasticity coefficient ρ plays a fundamental role in the analysis, so we will try and distinguish different situations depending on where the parameter lies.
First, we study what happens at the limit situations, i.e. when $\rho \to 0$, $\rho \to -\infty$ and $\rho = 1$.

4.4.1 Cobb-Douglas ($\rho \to 0$)

When the elasticity factor tends to 0, the utility function becomes a Cobb-Douglas, i.e. $u(x, t) = x_1^a x_2^{1-a}$.
In this situation, the demand function for good 1 becomes

$$x_1(P) = \frac{a}{P}$$

Then, the monopolist revenue in terms of good 2 is $P(e)e = a$. Therefore, a monopolist equilibrium doesn't exists as the induced utility function, i.e.

$$u(m, e, P(e)) = \begin{cases} (\mathbf{w}^1(m) - e, P(e)e) \ \ if \ \ e \in (0, w_1(m)] \\ \mathbf{w}(m) \ \ if \ \ e = 0 \end{cases} \tag{4.2}$$

is not continuous at $e = 0$, as $\lim_{e \to 0} P(e)e \neq 0$.

[2] In the following, since we are using powers, the commodity index will be subscript to simplify the notation. This shouldn't create confusion with the previous notation.

4.4.2 Linear utility($\rho = 1$)

If $\rho = 1$ (or tends to 1), then we approach the linear utility case, i.e. $u(x,t) = ax_1 + (1-a)x_2$.
In this case, given the corner endowments, the first order conditions of the utility maximization problem directly give the value for the relative price, i.e. $P = \frac{a}{1-a}$. Therefore, the monopolist becomes price taker as well, and the monopolist equilibrium coincides with competitive equilibrium. As a consequence, the monopoly equilibrium exists.

4.4.3 Leontief ($\rho \to -\infty$)

The final limit case is the one in which $\rho \to -\infty$. In this situation, the utility becomes $u(x,t) = min\{\frac{x_1}{a}, \frac{x_2}{1-a}\}$. The demand for good one becomes then

$$P(x_1) = \frac{a + ax_1 - x_1}{ax_1}$$

In the same way as the Cobb Douglas, the induced utility function for the monopolist is not continuous, as $\lim_{e \to 0} P(e)e = 1 \neq 0$. Therefore, the monopoly equilibrium doesn't exist.

4.5 The general case for CES utilities

We can now consider a more general case. To simplify the analysis, without loss of generality, we will assume here that $\mathbf{w}(m) = (1,0)$ and $\mathbf{w}(t) = (0,1)$, for each $t \in T_0$. Every small trader solves the maximization problem

$$\max u(x,t) = (ax_1^\rho + (1-a)x_2^\rho)^{\frac{1}{\rho}} \; s.t. Px_1 + x_2 = 1$$

which, for non degenerate cases, leads to the following demand function:

$$x_1(t,P) = \frac{1}{P + (\frac{1-a}{a}P)^{\frac{1}{1-\rho}}} \tag{4.3}$$

First, we can check under which values of ρ this demand function satisfies relation Q, i.e. $\lim_{x_1 \to 0} \frac{\partial u(x,t)}{\partial x_1} = +\infty$.

$$\frac{\partial u(x,t)}{\partial x_1} = \frac{1}{\rho} a x_1^{\rho-1} (ax_1^\rho + (1-a)x_2^\rho)^{\frac{1}{\rho}}$$

We can clearly see that the limit of the partial utility goes to infinity as x_1 goes to 0 when $\rho < 1$.

Now, since the demand function can't be generically inverted to obtain a demand function in the form $P(x_1)$, we will consider the monopolist problem from a price setting perspective. We can observe that when $\lim_{P\to+\infty} x_1(t,P) = 0$.

Remind that the CES utility function have the property that the elasticity is constant, i.e. $-\frac{\partial \frac{dx_1}{dx_2}}{dp}\frac{P}{\frac{x_1}{x_2}} = \frac{1}{1-\rho} = \phi$. So we can rewrite all of these relations in terms of ϕ.

The monopolist observe the small traders demand function and solves the following problem:

$$\begin{aligned} \max_P \; & u(x,m) \\ s.t. \; & Px_1(m) + x_2(m) = P \\ & 1 - x_1(m) = x_1(t,P) \end{aligned}$$

First, we can rewrite the first constraint as $x_2 = (1 - x_1(m))P = x_1(t,P)P$.
It may be worth noticing that prices are bounded below. This is implicitly stated in the second constraint, as prices must be such that $x_1(t,P) \le 1$, and therefore

$$\frac{1}{P + (\frac{1-a}{a}P)^{\phi}} \le 1.$$

Now, we can plug the constraints into the utility function, and the problem reduces to

$$\max_P u(x_1(m,P), x_2(m,P)) = u(1 - x_1(t,P), Px_1(t,P)) \tag{4.4}$$

4.5.1 Inelastic demand and non existence of monopoly equilibrium

Before going into the analysis of the first order conditions for this problem, it is worth noticing that $Px_1(t,P)$ may not go to 0 when the relative price diverge, i.e. when the bid of the monopolist goes to 0. In particular, when $0 < \phi < 1$, $\lim_{P\to 0} x_1(t,P) = 1$. This created the discontinuity we encountered in the previous examples. We can therefore state the following proposition.

Proposition 23. *If $u(x,t)$ is a CES utility function with elasticity parameter with $0 < \phi < 1$, for each $T \in T_0$, then there is no monopoly equilibrium.*

Proof. Monopolist final allocation will be in the form $x(m,e,P(e)) = (1 - e(P), Pe(P))$. Moreover, $Pe(P) = Px_1(t,P) = \frac{P}{P+(\frac{1-a}{a}P)^{\phi}}$. This expression goes to 1 when $P \to +\infty$, as we are assuming $0 < \phi < 1$. But then, $\lim_{P\to+\infty} x(m,e,P(e)) = (1,1)$, as $e(P) =$

$x_1(t,P)$ and $x_1(t,P) \to 0$ when $P \to +\infty$. Moreover, $(1,1) \succ_m x(m,e,P(e))$, for each $P \in \mathbb{R}_+$ (i.e. $p \in \Delta \setminus \partial\Delta$). However, at the limit, i.e. when $e = 0$, $x(m,e,P(e)) = \mathbf{w}(m) = (1,0)$. Therefore, there is no optimal strategy for the monopolist, in the sense that the it is always optimal to increase the price (reduce the bid). Hence, there is no monopoly equilibrium ■

4.5.2 Elastic demand

We can now focus on the general solution of the maximization problem stated in 4.4. The first order condition is

$$\frac{\partial u}{\partial P} = -\frac{\partial u_m(x(m,P)}{\partial x_1(m,P)}\frac{\partial x_1(t,p)}{\partial P} + \frac{\partial u_m(x(m,P)}{\partial x_2(m,P)}(x_1(t,P) + P\frac{dx_1}{dP})$$

Expanding the constant elasticity relation, we can write

$$\frac{dx_1}{dP} = -\frac{\phi x_1(1-Px_1)}{P} - x_1^2$$

Therefore, rearranging the terms, we obtain

$$\begin{aligned}\frac{\partial u}{\partial P} &= -\frac{\partial u_m(x(m,P)}{\partial x_1(m,P)}(\frac{dx_1(t,P)}{dP}) + \frac{\partial u_m(x(m,P)}{\partial x_2(m,P)}\frac{d[Px_1(t,P)]}{dP} \\ &= -\frac{\partial u_m(x(m,P)}{\partial x_1(m,P)}(-\frac{\phi x_1(1-Px_1)}{P} - x_1^2) + \frac{\partial u_m(x(m,P)}{\partial x_2(m,P)}[x_1(t,P)^2(1-\phi)(\frac{1-a}{a}P)^{\phi}] \\ &= x_1(t,P)^2[\frac{\partial u_m(x(m,P)}{\partial x_1(m,P)}(\frac{\phi}{P}(\frac{1-a}{a}P)^{\phi} + 1) + \frac{\partial u_m(x(m,P)}{\partial x_2(m,P)}(1-\phi)(\frac{1-a}{a}P)^{\phi}]\end{aligned} \tag{4.5}$$

Analyzing this expression, we can already find an interesting result, that is that the marginal change utility for the monopolist for a price change is decreasing in the elasticity parameter. We prove this result in the following proposition.

Proposition 24. *If $\phi_1 \geq \phi_2 > 1$, then $\frac{\partial u}{\partial P}(\phi_1) \leq \frac{\partial u}{\partial P}(\phi_2)$*[3]*.*

Proof. Suppose $\phi_1 \geq \phi_2 > 1$. Then, it is immediate to see that $x_1(\phi_1, P, t) \leq x_1(\phi_2, P, t)$, for each P, from the expression of the demand function (see 4.3). But then, $x_1(m, P, \phi_1) \geq x_1(m, P, \phi_2)$ and $x_2(m, P, \phi_1) \leq x_1(m, P, \phi_2)$, for each P, as $x(a, P) = (1 - x(t, P); Px_1(t, P))$.

[3]With a little abuse of notation, we denote by $\frac{\partial u}{\partial P}(\phi_1)$ the derivative of the induced utility function for the monopolist when she faces an homogeneous atomless sector in which all traders have a CES utility function with elasticity parameter ϕ_1

Therefore, $\frac{\partial u_m(x(m,P)}{\partial x_1(m,P)}|_{x_1(m,P)=x_1(m,P,\phi_1)} \leq \frac{\partial u_m(x(m,P)}{\partial x_1(m,P)}|_{x_1(m,P)=x_1(m,P,\phi_2)}$ and

$$\frac{\partial u_m(x(m,P)}{\partial x_2(m,P)}|_{x_2(m,P)=x_2(m,P,\phi_1)} \geq \frac{\partial u_m(x(m,P)}{\partial x_2(m,P)}|_{x_2(m,P)=x_2(m,P,\phi_2)}$$

, as u is strictly concave, by Assumption 17. Hence, by incorporating the previous inequalities into equation 4.5, if $\phi_1 \geq \phi_2 > 1$, then $\frac{\partial u}{\partial P}(\phi_1) \leq \frac{\partial u}{\partial P}(\phi_2)$. ■

To derive the result for linear utilities, we will show in the following proposition that when the elasticity parameter ϕ goes to infinity, then the optimal monopoly price will equate the walrasian/paretian price.

Proposition 25. *Consider a pure exchange economy such that each trader $t \in T_0$ has a CES utility function with parameter ϕ, then when $\phi \to +\infty$ the monopoly equilibrium will coincide with the walrasian equilibrium.*

Proof. We consider again the first order condition, expressed in 4.5, and we will put it to be greater or equal to 0, i.e.

$$x_1(t,P)^2[\frac{\partial u_m(x(m,P)}{\partial x_1(m,P)}(\frac{\phi}{P}(\frac{1-a}{a}P)^{\phi}+1)+\frac{\partial u_m(x(m,P)}{\partial x_2(m,P)}(1-\phi)(\frac{1-a}{a}P)^{\phi}] \geq 0$$

Rearranging the terms, we get that

$$-\frac{\frac{\partial u_m(x(m,P)}{\partial x_1(m,P)}}{\frac{\partial u_m(x(m,P)}{\partial x_2(m,P)}} \leq P\frac{(1-\phi)(\frac{1-a}{a}P)^{\phi}}{\phi(\frac{1-a}{a}P)^{\phi}+P} \tag{4.6}$$

In particular, we may notice that the right hand side of the previous disequation goes to P as $\phi \to +\infty$. If we rewrite the previous expression as an equation, then we obtain the well known relation for a walrasian economy, i.e. $\frac{\frac{\partial u_m(x(m,P)}{\partial x_1(m,P)}}{\frac{\partial u_m(x(m,P)}{\partial x_2(m,P)}} = P$; therefore proving the fact that when the elasticity goes to infinity, i.e. when the CES utilities tend to linear form utility, the monopoly equilibrium will converge to the walrasian equilibrium. ■

4.5.3 Example with heterogeneous atomless sector

We end this section with an instructive example that extends Proposition 23. We show that even when only a subset of the small traders has an inelastic CES utility function, we end up with a negative result.

Example 6. *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, 3 and 4. $T_0 = [0,1]$, $T_1 = 2$, T_0 is taken with Lebesgue measure and $\mu(2) = 1$,*

$\mathbf{w}_t = (0,1)$ *for each* $t \in T_0$, $u_t(x) = x_1 x_2$ *for each* $t \in [0, \frac{1}{10}]$, $u_t(x_1, x_2) = \sqrt{x_1} + x_2$ *for each* $t \in [\frac{1}{10}, 1]$, $\mathbf{w}_2 = (1,0)$, $u_2 = 20x_1 + \frac{1}{10} ln x_2$. *Then, there is no monopoly equilibrium.*

Proof. The demand function for good 1 for each $t \in [0, \frac{1}{10})$ is given by $x_1(t, P) = \frac{1}{2P}$, while the the demand function for good 1 for each $t \in [\frac{1}{10}, 1]$ is given by $x_1(t, P) = \frac{1}{4P^2}$. Therefore, the aggregate demand function for good 1 is given by

$$\int_{T_0} x_1(t,p)d\mu = \int_0^{\frac{1}{10}} x_1(t,P)d\mu + \int_{\frac{1}{10}}^1 x_1(t,P)d\mu = \frac{1}{10}\frac{1}{2P} + \frac{9}{10}\frac{1}{4P^2} = \frac{2P+9}{40P^2}.$$

Then, $p(e) = p(\int_{T_0} x_1(t,p)d\mu) = \frac{1+\sqrt{1+360e}}{40e}$. But then, the induced utility for the monopolist $u_2(x_1(e,P), x_2(e,P) = (1-e, p(e)e) = 20(1-e) + \frac{1}{10} ln \frac{1+\sqrt{1+360e}}{40}$. Therefore, the first order condition for the maximization of the utility of the monopolist are

$$\frac{du_2(e))}{de} = -20 + \frac{18}{360e + 1 + \sqrt{1+360e}}.$$

This expression is clearly negative for each value of $e \in (0,1]$. Moreover, $\lim_{e \to 0}(x_1(e), x_2(e)) \neq w_2$ as $\lim_{e \to 0} x_2(e) = \lim_{e \to 0} P(e)e = \frac{1}{20}$. Hence, there is no monopoly equilibrium. ■

4.6 Discussion

In the previous sections, we tried to link results about existence of a monopoly equilibrium with the elasticity of substitution between the two goods for the small traders. In order to do that, we consider a pure exchange economy in which all traders in the atomless part has an identical CES function, that guarantees that the aggregate demand function will preserve the constant elasticity property.

The first proposition is a counterpart in our framework of the result in monopoly theory, stating that "[...] the profit maximizing monopolist produces an output where the marginal revenue equals positive marginal cost, and the former is positive only if the elasticity of demand exceeds unity" (Batra 1971, pag.358). In our context, we state that the elasticity of substitution must exceed unity in order for the monopoly equilibrium to exist. In particular, the non existence is derived from the fact that when the elasticity of substitution is sufficiently low, the monopolist can exploit indefinitely the small traders, but at the limit he's left with her own endowment, which is strictly worse that what she could have got if she kept on decreasing her offer (increasing the price).

In the two limit situations, i.e. Cobb-Douglas and Leontief utility functions, the interpretation appears even "cleaner". For Leontief utility function, the two goods are perfect complements, therefore the monopolist can attain his maximum market power as small traders will always be incentivized to send to the market almost their whole

stock of good in order to exchange it for a small quantity from the good the monopolist owns.
In the Cobb-Douglas case, we can see that the small traders demand for the good they own is completely inelastic. This means that the monopolist will always get in return a fixed amount of the good the small traders own, no matter what his bid is. Therefore, the monopolist here is always incentivized to reduce his bid.

To conclude our analysis of the first proposition, we will add two details to it. First, the result states the non existence of a monopoly equilibrium in the sense that there doesn't exist an optimal strategy for the monopolist, not that the only possible equilibrium is autarchy. This is mostly due to the fact that the low elasticity of substitution makes the induced payoff of the monopoly discontinuous, and therefore we can't have a solution to the maximization problem. The important feature that drives the discontinuity is the model setup, in particular the fact that we have identical corner endowment for each small traders, where none of them holds any amount of the good owned by the monopolist.
The second important fact is that this proposition doesn't require any additional assumption for the behaviour of monopolist utility, as it holds for a generic utility function for the monopolist.

The second proposition can also be considered as a generalization for the standard result in partial equilibrium, that is the well known mark-up formula

$$MR(1 + \frac{1}{\phi}) = MC.$$

Clearly, if the elasticity of the demand goes to infinity, than we get the standard competitive result. Our proposition states the same result in a context of bilateral exchange. Here the interpretation is that the closer the goods to the situation of perfect substitutes, formally linear utility form, the lesser market power the monopolist has. In the limit, when goods are perfect substitutes, the relative price of the goods is fixed by the small traders via their demand (as it is infinitely elastic) and therefore the monopolist has no market power in manipulation the price, which in turn brings the equilibrium to be equal to the walrasian one.
Finally, it is worth noticing that in this situation the existence of a non autarchic equilibrium is established as we know from standard results in general equilibrium theory that guarantees the existence of a walrasian equilibrium in this framework.

4.7 Existence of a monopoly equilibrium

The previous section showed that the monopoly equilibrium may fail to exist in the context of an inelastic demand. Therefore, we'll provide a proof for the existence of a monopoly equilibrium which takes into account this feature.
Before introducing our new assumption, we need to introduce the following definition.

Definition 17. *Let u_1 and u_2 be two utility function satisfying Assumption 2. We say that the two utilities are locally equivalent at $\bar{x} \in \mathbb{R}^2_+$ if there exists a sequence of prices $P^n \in \mathbb{R}_+$ for which the corresponding demands $x_1(P^n)$ coincide and both converge to $\bar{x}$.*

Finally, in order to take into account the elasticity constraint, we introduce the following assumption.

Assumption 15. *There exist $\alpha \in (0,1)$ and elasticity parameter $\phi = \frac{1}{1-\rho} > 1$ such that u_t is locally equivalent to a CES utility function, i.e. $u(x,t) = (ax_1^\rho + (1-a)x_2^\rho)^{\frac{1}{\rho}}$. at $\mathbf{w}_t = (0,1)$, for each $t \in T_0$.*

This assumption requires the utility functions of each small traders to be locally equivalent to a linear utility function.

However, the first part of the proof will be given from a general perspective, as it holds for a more general framework[4].

Theorem 1. *Under Assumptions 16, 17, 18, 15 there exists a monopoly equilibrium.*

Proof. From now on, since the demand functions are homogeneous of degree 0, instead of considering non negative price vectors, we will consider price vectors $p_\epsilon \in \Delta = \{p \in \mathbb{R}^2_+ : p^1 + p^2 = 1\}$. We will denote the set of strictly positive prices as $\Delta \setminus \partial\Delta$. We show now a proposition about the aggregate demand function $\int_T \mathbf{x}^{01}(t, p_\epsilon) d\mu(t) : \Delta \setminus \partial\Delta \to \mathbb{R}_+$.

Lemma 1. *The aggregate demand function $\int_T \mathbf{x}^{01}(t, p_\epsilon) d\mu(t) : \Delta \setminus \partial\Delta \to \mathbb{R}_+$ is an onto continuous function.*

Proof. The correspondence $\int_{T_0} \mathbf{X}^0(t, \cdot)\, d\mu$ is upper hemicontinuous, by the argument used in the proof of Property (ii) in Debreu (1982), p. 728. But then, the function $\{\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu\}$ is continuous as $\int_{T_0} \mathbf{X}^0(t, p_\epsilon)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p_\epsilon)\, d\mu$, for each $p_\epsilon \in \Delta \setminus \partial\Delta$, by the argument used previously.

[4]For example, the first part of the proof would hold even replacing Assumption 15 back with Assumption 19

To prove that $\int_T \mathbf{x}^{01}(t, p_\epsilon)d\mu(t) : \Delta \setminus \partial\Delta \to \mathbb{R}_+$ is onto, we need to show that for each $e \in \mathbf{E}(a)$ there exists a market clearing price $p_\epsilon \in \Delta \setminus \partial\Delta$. First, let $\epsilon > 0$. Then, let $e \geq 0$. Let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1, 2, \ldots$, which converges to a normalized price vector $\bar{p}$ such that $\bar{p}^1 = 0$. Then, the sequence $\{\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu\}$ diverges to $+\infty$, by Proposition 4 in chapter 2. But then, there exists an n_0 such that $\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu > e + \epsilon$, for each $n \geq n_0$. Therefore, we have that $\int_{T_0} \mathbf{x}^{01}(t, p^{n_0})\, d\mu > e + \epsilon$. Let $q \in \Delta \setminus \partial\Delta$ be a price vector such that $\frac{q^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu}{q^1} = e + \epsilon$. Consider first the case where $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu = e + \epsilon$. Then, q is market clearing as it is market clearing as it satisfies 5.1. Consider now the case where $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu \neq e + \epsilon$. Then, it must be that $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu < e + \epsilon$ as $\mathbf{x}^{01}(t, q) \leq \frac{q^2 \mathbf{w}(t)}{q^1}$, for each $t \in T_0$. But then, we have that $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu < e + \epsilon < \int_{T_0} \mathbf{x}^{01}(t, p_\epsilon^{n_0})\, d\mu$. Let $O \subset \Delta \setminus \partial\Delta$ be a compact and convex set which contains $p_\epsilon^{n_0}$ and q. . Therefore, there is a price vector $p_\epsilon^* \in \Delta \setminus \partial\Delta$ such that $\int_{T_0} \mathbf{x}^{01}(t, p^*)\, d\mu = e + \epsilon$, by the intermediate value theorem. . Hence, given a strategy $e \in \mathbf{E}(a)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$. ■

We can now start giving the existence result in the perturbed game we just defined. Given $\epsilon > 0$, define a map from market clearing price vectors into monopolist actions, which is a restriction of the aggregate demand function, namely $\mathring{e} : A_\epsilon \subseteq \Delta \setminus \partial\Delta \to (0, \mathbf{w}^1(\mathbf{a})]$, with $A_\epsilon = \{p_\epsilon \in \Delta \setminus \partial\Delta : \epsilon \leq \int_{T_0} \mathbf{x}^{01}(t, p_\epsilon)d\mu(t) \leq \mathbf{w}^1(a) + \epsilon\}$.

Lemma 2. *The mapping $\mathring{e}(p_\epsilon) : A_\epsilon \subseteq \Delta \setminus \partial\Delta \to (0, \mathbf{w}^1(\mathbf{a})]$ is a continuous function and a closed mapping.*

Proof. $\mathring{e}(p_\epsilon)$ is a function as it is a restriction of the aggregate demand function $\int_{T_0} \mathbf{x}^{01}(t, p_\epsilon)d\mu(t) : \Delta \setminus \partial\Delta \to \mathbb{R}_+$. The function is also continuous as restrictions preserves continuity. Moreover, the set A_ϵ is closed as it is a preimage of a closed set via a continuous function. Moreover, A_ϵ is bounded as $A_\epsilon \subset \Delta$, which is a compact set. Therefore, A_ϵ is compact, as it is closed and bounded. Hence, $\mathring{e}_\epsilon(p)$ is a closed map, by Theorem 4.95 in Lee (2011) ■

We can give an initial characterization of the (restricted) inverse correspondence $\pi_\epsilon(e) : (0, \mathbf{w}^1(a)] \twoheadrightarrow B \subseteq A_\epsilon$.

Lemma 3. *The correspondence $\pi_\epsilon(e)$ is non-empty, compact valued and upper hemicontinuous.*

Proof. $\mathring{e}_\epsilon(p)$ has a closed graph, by Lemma 2. Then, $\pi_\epsilon(e)$ is an upper hemicontinuous correspondence, by Theorem 17.7 in Aliprantis and Border (2006). Moreover, $\pi_\epsilon(e)$ is non-empty, by Lemma 1. Following the previous arguments, $\pi_\epsilon(e)$ is bounded as $\pi_\epsilon(e) \in A_\epsilon \subseteq \Delta$, and it is closed as it is the preimage of $\{e+\epsilon\}$ via the aggregate demand function (which is continuous). Hence, $\pi_\epsilon(e)$ is a compact valued, upper hemicontinuous correspondence. ■

Let now $\tilde{p}_\epsilon(e) = \text{argmax}_{p_\epsilon \in \pi_\epsilon(e)} p^1_\epsilon(e)$. This is a well defined selection as $\pi_\epsilon(e)$ is compact valued, by Lemma 3.

We now give a lemma that states an additional property for this selection.

Lemma 4. *The price selection $p(e) = max_p \pi(e)$, expressed in terms of relative prices, i.e. $P(e)$, is decreasing for each $e > 0$.*

Proof. Let $e' > e''$ and suppose $P(e') \geq P(e'')$. Consider a restriction of the aggregate demand function $\int_{T_0} x^{01}(t,p)d\mu(t)$ by restrincting the domain of this function to the set $[P(e'), +\infty)$. Moreover, we know that the aggregate function (and therefore the restriction) is continuous and $\lim_{P\to+\infty} \int_{T_0} x^{01}(t,p)d\mu(t) = 0$. But then, there exists a $P' \geq P(e'')$ such that $\int_{T_0} x^{01}(t,p')d\mu(t) = e''$, by the Intermediate Value Theorem. But then, there exists $p' \in \pi(e'')$ with $P' = \frac{p'^1}{p'^2}$ and $P' > P(e'')$, a contraddiction. Therefore, $P(e'') > P(e')$.
Hence, the price selection $p(e) = max_p \pi(e)$ is decreasing in e. ■

We can now define, in a similar way, $u_a(\mathbf{x}(m,e,\tilde{p}_\epsilon(e)) = \max_{p \in \pi_\epsilon(e)} u_a(\mathbf{x}(m,e,p_\epsilon(e))$. We can now provide a characterization for this induced payoff function.

Lemma 5. *The induced payoff function, given by $u_a(\mathbf{x}(m,e,\tilde{p}_\epsilon(e))$ is upper semicontinuous.*

Proof. The correspondence $\pi_\epsilon(e)$ is compact valued and upper hemicontinuous, by Lemma 3. The utility function $u_a(\mathbf{x}(m,e,p))$ is continuous by Assumption 2. Hence, $u_a(\mathbf{x}(m,e,\tilde{p}_\epsilon(e))$ is upper semicontinuous by Lemma 17.30 in Aliprantis and Border (2006). ■

Finally, we apply Luenberger's version of Weierstrass theorem to finally obtain the existence result for the perturbed version of the economy.

Lemma 6. *An ϵ-monopoly equilibrium exists.*

Proof. From the definition of monopoly equilibrium, we can see that a monopoly equilibrium action $\tilde{e}_\epsilon$ is such that $\tilde{e}_\epsilon \in argmax_{e \in \mathbf{E}} u(x(e, p_\epsilon(e), a)$, with respect to a price selection $p_\epsilon(\cdot)$. Let this price selection be $\tilde{p}_\epsilon(e) = \text{argmax}_{p_\epsilon \in \pi_\epsilon(e)} p^1_\epsilon(e)$. But then, $u_a(\mathbf{x}(m, e, \tilde{p}_\epsilon(e))$ is upper semicontinuous, by Lemma 5. Therefore, this function achieves a maximum in its domain, i.e. $[0, \mathbf{w}^1(a)]$, by Theorem 1 in Luenberger (1970). Hence, an ϵ-monopoly equilibrium exists. ■

Consider a sequence $\{\epsilon^n\}$ with $\lim_{n\to\infty} \epsilon^n = 0$. For each $\{\epsilon^n\}$, there exists an optimal $\tilde{P}^n_\epsilon$, by Lemma 6.
$\tilde{P}^n_\epsilon$ is bounded above for each ϵ, as the set of feasible prices is bounded above by $P^n_\epsilon(0) = \{P \in R_+ : \int_{T_0} x_1(t, P) d\mu = \epsilon\}$.
Suppose that the sequence $\{\tilde{P}^n_\epsilon\}$ diverges. Then, the final allocation for the small traders, $x_1(\{\tilde{P}^n_\epsilon\}, t) = (x(\{\tilde{P}^n_\epsilon\}, t), x_2(\{\tilde{P}^n_\epsilon\}, t))$, will converge to $x(t) = (0, 1)$, as $\{\tilde{P}^n_\epsilon\} \to \infty$. But then, there exists a subsequence $\{P^{n_k}_\epsilon\}$ for which the demand $x_1(P^{n_k}_\epsilon, t)$ will coincide with the demand function of a CES function with elasticity parameter *phi*, by Assumption 15. But then, there exists a $\bar{n}_k$ such that

$$\frac{\partial u}{\partial P} = x_1(t, P)^2 [\frac{\partial u_m(x(m, P)}{\partial x_1(m, P)} (\frac{\phi}{P} (\frac{1-a}{a} P)^\phi + 1) + \frac{\partial u_m(x(m, P)}{\partial x_2(m, P)} (1 - \phi)(\frac{1-a}{a} P)^\phi] < 0$$

for each $P > \{\tilde{P}^{\bar{n}_k}_\epsilon\}$, a contraddiction to the fact that the sequence of the optimal prices diverges. Therefore, the sequence $\{\tilde{P}^n_\epsilon\}$ is bounded above by some price $\bar{P}$. But then, the sequence $\{\tilde{e}^n_\epsilon\}$ will be bounded below by $e(P) > 0$, as the price selection is a decreasing function, by Lemma 4. But then, there exists a subsubsequence $\tilde{e}_{n_k}$ that converges to a point $\tilde{e} \in [e(P), \mathbf{w}_1(a)]$. This completes the proof.
Hence, a monopoly equilibrium exists.

■

4.8 Discussion: Local equivalence with CES

We devote this section to provide two examples in which we check the consistence of Assumption 15 with the prevision of the existence of the monopoly equilibrium.

In the first example, we show an existence result and shows that the atomless sector utility function satisfies the critical assumption. We want to remark how it can be checked that the second part of the atomless sector in our examples satisfies Assumption 15.

Example 7. *Consider the following specification of an exchange economy satisfying Assumptions 11, 12, 13, and 14.* $T_0 = [0, 1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1, 0)$, $u_m(x) =$

$\frac{1}{2}x^1 + \sqrt{x^2}$, T_0 is taken with Lebesgue measure, $\mathbf{w}(t) = (0,1)$, $u_t(x) = \sqrt{x^1} + x^2$, for each $t \in T_0$. Then, there exists a monopoly equilibrium.
Moreover, $u_t(x) = \sqrt{x^1} + x^2$ satisfies Assumption 15.

Proof. We have that the unique monopoly equilibrium is the strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{e}_{12} = \frac{1}{4}$, $\mathring{p}(\tilde{e}) = 1$, $\tilde{\mathbf{x}}(m) = (\frac{3}{4}, \frac{1}{4})$, and $\tilde{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$.
A sufficient condition to check if a utility function satisfies Assumption 15 is to check if there is a CES utility function that would equal the MRS of $u_t(x)$ in a neighborhood of $(0,1)$.
The MRS for $u_t(x)$ is given by

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{1}{2\sqrt{x_1}}.$$

The generic MRS for a CES utility function is given by

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{\alpha(x_1)^{(\rho-1)}}{(1-\alpha)(x_2)^{(\rho-1)}}.$$

It's immediate to see that for $\alpha = \frac{1}{3}$ and $\rho = \frac{1}{2} > 0$, when $x_2 = 1$, the two MRS coincide. Therefore, $u_t(x) = \sqrt{x^1} + x^2$ satisfies Assumption 15, as this implies also that $\phi = \frac{1}{1-\rho} = 2 > 1$. ■

In the second example, we portray a situation in which there is no monopoly equilibrium and we show that the atomless sector utility function does not satisfy indeed our assumption.

Example 8. *Consider the following specification of an exchange economy satisfying Assumptions 11, 12, 13, and 14. $T_0 = [0,1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1,0)$, $u_m(x) = \frac{1}{2}x^1 + \sqrt{x^2}$, T_0 is taken with Lebesgue measure, $\mathbf{w}(t) = (0,1)$, $u_t(x) = \frac{1}{2}ln(x^1) + x^2$, for each $t \in T_0$. Then, there exists no monopoly equilibrium.*
Moreover, $u(x,t) = ln(x^1) + x^2$ does not satisfy Assumption 15.

Proof. The non existence argument stems from the fact that the monopolist revenue in terms of good 2 is $P(e)e = \frac{1}{2}$. Therefore, the monopolist induced utility is therefore discontinuous at $e = 0$, leading to a non existence of the monopoly equilibrium.
We move now to show that the utility for small traders does not satisfy Assumption 15. We follow the same approach as the previous case. In this example, the MRS for a generic small traders is

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{1}{2x_1}.$$

We can see that this coincides with the MRS of a Cobb-Douglas, namely $u(x) = x_1^{1/3}x_2^{2/3}$, when $x_2 \to 1$. Therefore, it doesn't satisfy Assumption 15, as we know Cobb-Douglas can be interpreted as CES utility functions with elasticity parameter approaching 0. ■

4.9 Conclusion

In this paper, we studied the problem of the existence of a monopoly equilibrium, linking it with the elasticity of substitution of small traders utilities.

We gave an existence result for the framework introduced in the previous chapter, introducing a sufficient local condition for the utilities in the small trader sector that guarantees the existence of such an equilibrium. We showed that there is a link between the elasticity of substitution for small traders and the existence of an equilibrium by considering a situation in which the small traders have a generic CES utility function. Therefore, we introduced a sufficient condition that guarantees the existence of equilibrium, taking into account this feature.

We extend the well known result in partial equilibrium that monopolist will produce in the inelastic portion of the demand curve, by requiring that all small traders have preference which are locally equivalent to a CES utility function whose elasticity is greater than the unity. This last assumption is very specific and quite demanding, but it also arises naturally after we considered the previous examples.

A more general consideration of the existence problem of a monopoly equilibrium and its relation to strategoc substitutability/complementarity notions, expanding on the arguments by Bloch and Ferrer (2001) and Bloch and Ghosal (1997), is left for further work. Relaxing Assumption 15 by allowing each small traders' utilities to be locally equivalent to potentially heterogeneous CES utility functions would seem the most direct extension to our result.

Bibliography

[1] Aliprantis C.D., Border K. (2006), *Infinite dimensional analysis: a hitchhiker's guide*, Springer.

[2] Bartra R.N., (1972), "Monopoly theory in general equilibrium and the two-sector model of economic growth", *Journal of Economic Theory*, **4** , 355-371.

[3] Busetto F., Codognato G., Ghosal S. (2011), "Noncooperative oligopoly in markets with a continuum of traders," *Games and Economic Behavior* **72**, 38-45.

[4] Busetto F., Codognato G., Ghosal S., Julien L., Tonin S. (2017), "Noncooperative oligopoly in markets with a continuum of traders and a strongly connected set of commodities", *Games and Economic Behaviour*, **1**, 1-8.

[5] Bloch F., Ferrer H. (2001), "Strategic complements and substitutes in bilateral oligopolies," *Economics Letters*, **70**, 83-87.

[6] Bloch F., Ghosal S. (1997), "Stable Trading Structures in Bilateral Oligopolies," *Journal of Economic Theory*.

[7] Codognato G., Julien L. (2013), "Noncooperative Oligopoly in Markets with a Cobb-Douglas Continuum of Traders ", *Louvain Economic Review*.

[8] Debreu G. (1982), "Existence of competitive equilibrium," in Arrow K.J., Intriligator H.D. (eds), *Handbook of mathematical economics*, Elsevier, Amsterdam.

[9] Gabszewicz, J.J. and Vial, J.P. (1972), " Oligopoly " A la Cournot " in a General Equilibrium Analysis," *Journal of Economic Theory* **4**, 381-400.

[10] Lee, J.M. (2011), *Introduction to Topological Manifolds*, Springer.

[11] Luenberger D.G. (1997), *Optimization by Vector Space Methods*, John Wiley & Sons.

[12] Sahi S., Yao S. (1989), "The noncooperative equilibria of a trading economy with complete markets and consistent prices,"*Journal of Mathematical Economics* **18**, 325-346.

[13] Shapley, L. and Shubik, M. (1977), "Trade using one commodity as a means of payment," *Journal of Political Economy*, **5**, 937-968.

[14] Shitovitz B. (1973), "Oligopoly in markets with a continuum of traders," *Econometrica* **41**, 467-501.

Chapter 5

Disadvantageous Monopolies: a reconsideration through monopoly equilibria

Abstract

We study the optimality properties of the monopoly equilibrium. Similar to Codognato et al. (2015), we show that a monopolist can be competitive when she supplies all of her endowment to the market. We further show that this is the only possible situation in which a monopoly equilibrium is Pareto efficient. In the light of the Aumann notion of disadvantageous monopolies, we show that within our framework, there cannot exist scenarios in which a monopolist is better off by "dissolving", i.e. being a price taker. We finally provide an important implication on consumer welfare under advantageous monopoly.

5.1 Introduction

In his seminal paper, Aumann (1973) posited a famous conjecture about "disadvantageous monopolies", claiming that, in a cooperative framework, there exists situations in which a monopolist may be better off by acting competitively. This non intuitive result laid the seeds for a stream in literature dealing with this problem, trying to reconcile the conjecture with the real life intuition that suggest that a monopolist will be better off by exploiting his position of market power.

In fact, Aumann himself found that the solution concept used in his paper, i.e. the core, was not suited for explaining this phenomenon, as "monopoly power is probably not based on this at all, but rather on what the monopolist can prevent other coalitions from getting. His strength lies in his threat possibilities, in the bargaining power engendered by the harm he can cause by refusing to trade" (Aumann, 1973).

Guesnerie (1977) and Legros (1987) attempted to tackle the problem using two natural alternative solution concepts within the cooperative approach: the Shapley value and the nucleolus. Guesnerie confirmed the negative results regarding stability, while Legros could provide some sufficient conditions on when a monopolist can be actually advantageous. Looking at these overall negative results, it seemed natural to look at the problem from a different perspective and using different solution concepts. Okuno et al. (1980) studied oligopolies through the lens of a non cooperative game, using the Cournot-Nash solution concept. However, even within a one stage non cooperative framework, they couldn't exclude the possibility of disadvantageous monopolies, in Aumann sense. We investigate whether we can exclude the possibility of disadvantageous monopolies arising within our version of a monopolistic market.

The aim of this paper is to study this problem within the bilateral exchange monopolistic market framework introduced in the previous chapters. We will study the problem of the optimality of the solution concept provided earlier. Similar to the approach in Busetto et al. (2020), we begin by characterizing optimality through the concept of (Pareto) efficiency.

Similarly to the result in Codognato et al. (2015), we show that a monopolist can be walrasian if the optimal bid is equal to the endowment. This is already a quite interesting result, showing that in a simple two commodity scenario, even a monopolist can have incentives to act competitively.

We then move to analyze the possibility for disadvantageous monopolies to emerge in our framework. The result is negative, in fact we show that when the set of monopoly equilibria and the set of walrasian equilibria are disjoint, then the monopolist will always be better off at a monopoly equilibrium rather than at a competitive equilibrium. In other words, the monopolist will always exploit her market power and will never generally have an incentive to behave competitively. Therefore, within this model specification, we are able to rule out the presence of disadvantageous monopolies and reconcile Aumann's "paradoxical" conjecture with the common sense understanding of monopoly power.

Shitovitz (1973), in his Theorem A, shows that large traders tend to monetarily exploit small traders, in the sense that in then core small traders will consume a bundle whose value is not larger than the value of their endowment. Within our framework, we want to disentangle the two sides of the previous proposition: on one hand, the monopolist has an incentive to increase the price, i.e. induce a higher price at a monopoly equilibrium with respect to the competitive one; on the other hand, we want to highlight whether the small traders sector is disadvantaged by the presence of a monopolist as

the only large traders owning a good they desire.
The paper is structured as follows: in Section 2, we introduce the mathematical model; in section 3, we specify the solution concept, the monopoly equilibrium, in particular under demand invertibility condition; section 4 is devoted to establishing the nexus between optimality, efficiency and the monopoly equilibrium; section 5 and 6 study inefficient monopoly equilibria and makes the case for advantageous monopolies and its consequences on consumer welfare; section 7 gives some conclusions and provides some further area of research.

5.2 Mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. Let T_0 denote the atomless part of T. We assume that $\mu(T_0) > 0$ and $T \setminus T_0 = \{a\}$, i.e., the measure space $(T, \mathcal{T}, \mu)$ contains only one atom, the "monopolist." A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.
A commodity bundle is a point in $\mathbb{R}^2_+$. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to \mathbb{R}^2_+$. We are considering a bilateral exchange economy, therefore with two commodities. We assume that the monopolist holds, without loss of generality good one, while small traders hold the second good, i.e.

Assumption 16. $\mathbf{w}^1(m) > 0$, $\mathbf{w}^2(m) = 0$ *and* $\mathbf{w}^1(t) = 0$, $\mathbf{w}^2(t) > 0$, *for each* $t \in T_0$.

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : \mathbb{R}^2_+ \to R$, satisfying the following assumptions.

Assumption 17. $u_t : \mathbb{R}^2_+ \to R$ *is continuous, strongly monotone, and strictly quasi-concave, for each* $t \in T$.

Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}^2_+$. Moreover, let $\mathcal{T} \otimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 18. *$u : T \times \mathbb{R}^2_+ \to R$, given by $u(t,x) = u_t(x)$, for each $t \in T$ and for each $x \in \mathbb{R}^2_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.*

In order to state our last assumption, we need a preliminary definition. We say that commodities i, j stand in relation Q if there is a nonnull subset T^i of T_0, such that $u_t(\cdot)$ is differentiable, additively separable, i.e., $u_t(x) = v^i_t(x_i) + v^j_t(x_j)$, for each $x \in \mathbb{R}^2_+$, and $\frac{dv^j_t(0)}{dx_j} = +\infty$, for each $t \in T^i$.[1] We can now introduce the last assumption.

Assumption 19. *Commodities 1 and 2 stand in relation Q.*

A price vector is a nonnull vector $p \in \mathbb{R}^2_+$. Moreover, we will denote by Δ the unit simplex, i.e. $\Delta = \{p\mathbb{R}^2_+ : p^1 + p^2 = 1\}$, and $\Delta \setminus \partial\Delta$ will denote the interior of Δ. Finally, we will write $P \in \mathbb{R}_+$ to intend the corresponding relative price for each $p \in \Delta \setminus \partial\Delta$, i.e. $P = \frac{p^1}{p^2}$, for some $(p^1, p^2) \in \Delta \setminus \partial\Delta$.
Let $\mathbf{X}^0 : T_0 \times \mathbb{R}^2_{++} \to \mathcal{P}(\mathbb{R}^2_+)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in \mathbb{R}^2_{++}$, $\mathbf{X}^0(t,p) = \text{argmax}\{u(x) : x \in \mathbb{R}^2_+ \text{ and } px \leq p\mathbf{w}(t)\}$. For each $p \in \mathbb{R}^2_{++}$, let $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \{\int_{T_0} \mathbf{x}(t,p)\, d\mu : \mathbf{x}(\cdot, p)$ is integrable and $\mathbf{x}(t,p) \in \mathbf{X}^0(t,p)$, for each $t \in T_0\}$. Since the correspondence $\mathbf{X}^0(t,\cdot)$ is nonempty and single-valued, by Assumption 2, let $\mathbf{x}^0 : T_0 \times \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ be the function such that $\mathbf{X}^0(t,p) = \{\mathbf{x}^0(t,p)\}$, for each $t \in T_0$ and for each $p \in \mathbb{R}^2_{++}$. A Walras equilibrium is a pair $(p, \mathbf{x})$, consisting of a price vector p and an allocation $\mathbf{x}$, such that $p\mathbf{x}(t) = p\mathbf{w}(t)$ and $u_t(\mathbf{x}(t)) \geq u_t(y)$, for all $y \in \{x \in \mathbb{R}^2_+ : px = p\mathbf{w}(t)\}$, for each $t \in T$. A Walras allocation is an allocation $\mathbf{x}$ for which there exists a price vector p such that the pair $(p, \mathbf{x})$ is a Walras equilibrium.

5.3 Monopoly equilibrium

We introducing the monopoly equilibrium concept.
Let $\mathbf{E}(m) = \{(e_{ij}) \in R^4_+ : \sum_{j=1}^2 e_{ij} \leq \mathbf{w}^i(m),\ i = 1,2\}$ denote the strategy set of atom a. We denote by $e \in \mathbf{E}(m)$ a strategy of atom a, where e_{ij}, $i, j = 1, 2$, represents the amount of commodity i that atom a offers in exchange for commodity j. Moreover, we denote by E the matrix corresponding to a strategy $e \in \mathbf{E}(m)$.

We then provide the following definitions.

Definition 18. *A square matrix A is said to be triangular if $a_{ij} = 0$ whenever $i > j$ or $a_{ij} = 0$*

[1] In this definition, differentiability means continuous differentiability and is to be understood to include the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

whenever $i < j$.

Definition 19. *Given a strategy $e \in \mathbf{E}(a)$, a price vector p is said to be market clearing if*

$$p \in R^2_{++}, \int_{T_0} \mathbf{x}^{0j}(t,p)\, d\mu + \sum_{i=1}^{2} e_{ij}\mu(m)\frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{2} e_{ji}\mu(m) \tag{5.1}$$

, $j = 1, 2$.

We recall here a proposition from the previous chapter, that provides a necessary and sufficient condition for the existence of a market clearing price vector.

Proposition 26. *Under Assumptions 16, 17, 18 and 19, given a strategy $e \in \mathbf{E}(m)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular.*

Proof. See Chapter 2. ∎

We denote by $\pi(e)$ a correspondence which associates, with each strategy $e \in \mathbf{E}(m)$, the set of price vectors p satisfying (1), if E is triangular, and is equal to $\{0\}$, otherwise. A price selection $p(e)$ is a function which associates, with each strategy selection $e \in \mathbf{E}(m)$, a price vector $p \in \pi(e)$.

Given a strategy $e \in \mathbf{E}(m)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned}
\mathbf{x}^j(m,e,p) &= \mathbf{w}^j(m) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j}, \text{ if } p \in R^2_{++},\\
\mathbf{x}^j(m,e,p) &= \mathbf{w}^j(m), \text{ otherwise},
\end{aligned}$$

$j = 1, 2$,

$$\begin{aligned}
\mathbf{x}^j(t,p) &= \mathbf{x}^{0j}(t,p), \text{ if } p \in R^2_{++},\\
\mathbf{x}^j(t,p) &= \mathbf{w}^j(t), \text{ otherwise},
\end{aligned}$$

$j = 1, 2$, for each $t \in T_0$.

Given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(m)$, traders' final holdings are expressed by the assignment $\mathbf{x}(a) = \mathbf{x}(m, e, p(e))$ and $\mathbf{x}(t) = \mathbf{x}(t, p(e))$, for each $t \in T_0$.

The following proposition, replicated from the previous chapter, shows that traders' final holdings are an allocation.

Proposition 27. *Under Assumptions 16, 17, 18, and 19, given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(m)$, the assignment $\mathbf{x}(m) = \mathbf{x}(m, e, p(e))$ and $\mathbf{x}(t) = \mathbf{x}^0(t, p(e))$, for each $t \in T_0$, is an allocation.*

Proof. See Chapter 2. ■

We can now provide the definition of a monopoly equilibrium.

Definition 20. *A strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{E}$ is triangular is a monopoly equilibrium, with respect to a price selection $p(\cdot)$, if*

$$u_a(\mathbf{x}(m, \tilde{e}, p(\tilde{e})) \geq u_a(\mathbf{x}(m, e, p(e)),$$

for each $e \in \mathbf{E}(m)$.

5.4 Efficiency, Pareto optimality, and core: non-disadvantageous monopoly

We analyse now the optimality properties of the monopoly equilibrium introduced in the previous section. To this end, we need to introduce the following further definitions. An allocation $\mathbf{x}$ is said to be individually rational if $u_t(\mathbf{x}(t)) \geq u_t(\mathbf{w}(t))$, for each $t \in T$. An allocation $\mathbf{x}$ is said to be Pareto optimal if there is no allocation $\mathbf{y}$ such that $u_t(\mathbf{y}(t)) \geq u_t(\mathbf{x}(t))$, for each $t \in T$, and $u_t(\mathbf{y}(t)) > u_t(\mathbf{x}(t))$, for a non-null set of traders t in T. An efficiency equilibrium is a pair $(\hat{p}, \hat{\mathbf{x}})$, consisting of a price vector $\hat{p}$ and an allocation $\hat{\mathbf{x}}$, such that $u_t(\hat{\mathbf{x}}(t)) \geq u_t(y)$, for all $y \in \{x \in R^2_+ : \hat{p}x = \hat{p}\mathbf{x}(t)\}$, for each $t \in T$. An efficiency allocation is an allocation $\hat{\mathbf{x}}$ for which there exists a price vector $\hat{p} \in R^2{++}$ such that the pair $(\hat{p}, \hat{\mathbf{x}})$ is an efficiency equilibrium.

We can now state and prove a proposition which provides a rationale for Pareto optimality as a criterion for efficiency by means of the first and second fundamental theorems of welfare economics. Borrowing from the corollary to Lemma 1 in Shitovitz (1973), in the next proposition, we show that a monopoly allocation is Pareto optimal if and only if it is an efficiency allocation.

Proposition 28. *Under assumptions 16, 17,18 and 19, let $\tilde{\mathbf{x}}$ be monopoly allocation. Then, the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal if and only if it is an efficiency allocation.*

Proof. . Let $\tilde{\mathbf{x}}$ be a monopoly allocation. Suppose that the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal. We adapt to our framework the argument used by Shitovitz (1973) to prove the corollary to his Lemma 1. It is straightforward to verify that $\tilde{\mathbf{x}}$ is individually rational. Let $\tilde{\mathbf{G}} \to \mathcal{P}(R^2)$ be a correspondence such that $\tilde{\mathbf{G}}(t) = \{x - \tilde{\mathbf{x}}(t) : x \in R^2_+ \text{ and } u_t(x) > u_t(\tilde{\mathbf{x}}(t))\}$, for each $t \in T$. Moreover, let $\int_T \tilde{\mathbf{G}}(t)\, d\mu = \{\int_T \tilde{\mathbf{g}}(t)\, d\mu : \tilde{\mathbf{g}}(t) \text{ is integrable and } \tilde{\mathbf{g}}(t) \in \hat{\mathbf{G}}(t), \text{ for each } t \in T\}$. The set $\{x \in R^2_+ : u_t(x) \geq u_t(\tilde{\mathbf{x}})\}$ is convex as $u_m(\cdot)$ is strictly quasi-concave, by Assumption 17. Then, it is straightforward to verify that the set $\tilde{\mathbf{G}}(m)$ is convex. But then, $\int_T \tilde{\mathbf{G}}(t)\, d\mu$ is convex, by Theorem

1 in Shitovitz (1973). We now prove that $0 \notin \int_T \tilde{\mathbf{G}}(t)\, d\mu$. Suppose that $0 \in \int_T \tilde{\mathbf{G}}(t)\, d\mu$. Then, there is an assignment $\mathbf{y}$ such that $u_t(\mathbf{y}(t)) > u_t(\tilde{\mathbf{x}}(t))$, for each $t \in T$, which is an allocation as $\int_T \mathbf{y}(t)\, d\mu = \int_T \tilde{\mathbf{x}}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. But then, $\tilde{\mathbf{x}}$ is not Pareto optimal, a contradiction. Therefore, it must be that $0 \notin \int_T \tilde{\mathbf{G}}(t)\, d\mu$. Then, there exists a vector $q \in R^2$ such that $(q \neq 0)$ and $q \int_T \tilde{\mathbf{G}}(t)\, d\mu \geq 0$, by the supporting hyperplane theorem. We know that $q \in R^2_{++}$, by the proof of Lemma 1 in Shitovitz (1973). Let $\hat{p} = \frac{q^i}{q^j}$. Then, the pair $(\hat{p}, \tilde{\mathbf{x}})$ is an efficiency equilibrium, by Lemma 1 in Shitovitz (1973). Therefore, the allocation $\tilde{\mathbf{x}}$ is an efficiency allocation. Conversely, suppose that the allocation $\tilde{\mathbf{x}}$ is an efficiency allocation. Then, the allocation $\tilde{\mathbf{x}}$ is Pareto optimal, by the first fundamental theorem of welfare economics. Hence, the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal if and only if it is an efficiency allocation. ■

The previous proposition exhibited the nexus between Pareto optimality and efficiency. The next proposition shows that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and the Walrasian demand of traders in the atomless part is differentiable, it is possible to exhibit a nexus between Pareto optimality of a monopoly allocation and perfect competition, i.e., Walras equilibrium, as it establishes an equivalence between the set of Pareto optimal monopoly allocations and the set of monopoly allocations, whenever the latter are also Walrasian.

Proposition 29. *Under assumptions 16, 17,18 and 19, let the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ be invertible and the function $\mathbf{x}^0(t, \cdot)$ be differentiable on R_{++}, for each $t \in T_0$. Moreover, let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and let $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, for each $t \in T_0$. Then, the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal if and only if the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium.*

Proof. . Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$. Suppose that the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal. Moreover, suppose that $\tilde{\mathbf{x}}^2(t) = 0$, for each $t \in T_0$. Then, we have that

$$\tilde{\mathbf{x}}(m) = (\mathbf{w}^1(m) - \int_{T_0} \mathbf{x}^{01}(t, \mathring{p}(\tilde{e}))\, d\mu, \int_{T_0} \mathbf{w}^2(t)\, d\mu).$$

But then, we have that

$$\frac{du_m(\tilde{\mathbf{x}}(m))}{de} = -\frac{d \int_{T_0} \mathbf{x}^{01}(t, \mathring{p}(\tilde{e}))}{dp} \frac{d\mathring{p}(e)}{de_{12}} < 0,$$

by Proposition 29, a contradiction. Therefore, there must be a set $\bar{T} \subseteq T_0$ such that $\mu/\bar{T}) > 0$ and $\tilde{\mathbf{x}}(t) \gg 0$, for each $t \in \bar{T}$. There exists a vector $\hat{p} \in R^2_{++}$ such that the pair $(\hat{p}, \tilde{\mathbf{x}})$ is an efficiency equilibrium, by Proposition 30. We have that

$$\frac{\frac{\partial u_t(\tilde{\mathbf{x}}(t))}{\partial x^1}}{\frac{\partial u_t(\tilde{\mathbf{x}}(t))}{\partial x^2}} = \tilde{p},$$

as $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{e})) \gg 0$, for each $t \in \bar{T}$. It must also be that

$$\frac{\frac{\partial u_t(\tilde{\mathbf{x}}(t))}{\partial x^1}}{\frac{\partial u_t(\tilde{\mathbf{x}}(t))}{\partial x^2}} = \hat{p},$$

as the pair $(\hat{p}, \hat{\mathbf{x}})$ is an efficiency equilibrium, for each $t \in \bar{T}$. Then, there exists a real number $\theta > 0$ such that $\tilde{p} = \theta\hat{p}$. But then, $\tilde{\mathbf{x}}$ is such that $\tilde{p}\tilde{\mathbf{x}}(t) = \tilde{p}\mathbf{w}(t)$ and $u_t(\tilde{\mathbf{x}}(t)) \geq u_t(y)$, for all $y \in \{x \in R^2_+ : \tilde{p}x = \tilde{p}\mathbf{w}(t)\}$, for each $t \in T$. Therefore, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. Conversely, suppose that the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. Then, it is straightforward to show that it is also an efficiency equilibrium. But then, the allocation $\tilde{\mathbf{x}}$ is Pareto optimal by the first fundamental theorem of welfare economics. Hence, the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal if and only if the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. ■

The next proposition provides a necessary and sufficient condition for a monopoly allocation to be a Walras allocation.

Proposition 30. *Under assumptions 16, 17,18 and 19, let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible and the function $\mathbf{x}^0(t, \cdot)$ be differentiable on R_{++}, for each $t \in T_0$. Moreover, let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and let $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, for each $t \in T_0$. Then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium if and only if $\tilde{e}_{ij} = \mathbf{w}^i(m)$.*

Proof. . Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$. Suppose that the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. Moreover, suppose that $\tilde{\mathbf{x}}(m) \gg 0$. It must be that

$$\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^1}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^2}} = \tilde{p},$$

as the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. Moreover, we have that

$$-\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^1} + \frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^2}(\tilde{p} + \frac{d\mathring{p}(\tilde{e})}{de_{12}}\tilde{e}_{12}) = 0,$$

as $\tilde{e}$ is a monopoly equilibrium. Then, we obtain that

$$\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^1}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^2}} \neq \tilde{p},$$

as $\frac{d\mathring{p}(\tilde{e})}{de_{ij}} < 0$, by Proposition 29, a contradiction. Therefore, it must be that $\tilde{e}_{ij} = \mathbf{w}^i(m)$. Conversely, suppose that $\tilde{e}_{ij} = \mathbf{w}^i(m)$. We have that $\tilde{\mathbf{x}}(m) = (0, \mathbf{w}^1(m)\tilde{p})$. Let $\tilde{x}^2(x^1)$ be a function such that $u_m(x^1, x^2(x^1)) = u_m(\tilde{\mathbf{x}}(m))$, for each $0 \leq x^1 \leq \mathbf{w}^1(m)$. We have that

$$-\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^1} + \frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^2}(\tilde{p} + \frac{d\mathring{p}(\tilde{e})}{de_{12}}\tilde{e}_{12}) \geq 0,$$

as $\tilde{e}_{12} = \mathbf{w}^1(m)$. Then, it must be that

$$-\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^1} + \frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^2}\tilde{p} > 0,$$

as $\frac{d\mathring{p}(\tilde{e})}{de_{ij}} < 0$, by Proposition 29. But then, we have that $\frac{d\tilde{x}^2}{dx^1} > -\tilde{p}$, for each $0 \leq x^1 \leq \mathbf{w}^1(m)$, as $u_m(\cdot)$ is strictly quasi-concave, by Assumption 17. Suppose that there exists a commodity bundle $\bar{x} \in \{x \in R^2_+ : \tilde{p}x = \tilde{p}\mathbf{w}(m)\}$ such that $u_m(\bar{x}) > u_m(\tilde{\mathbf{x}}(m))$. Then, it must be that $\bar{x}^2 > \tilde{x}^2(\bar{x}^1)$ as $u_m(\cdot)$ is strongly monotone, by Assumption 17. But then, by the mean value theorem, there exists some x' such that $0 < x' < \bar{x}^1$ and such that

$$\frac{d\tilde{x}^2(x'^1)}{dx^1} = \frac{\tilde{x}^2(0) - \tilde{x}^2(\bar{x}^1)}{0 - \bar{x}^1} < -\tilde{p},$$

a contradiction. Therefore, we have that $u_m(\tilde{\mathbf{x}}(m) \geq u_m(y)$, for all $y \in \{x \in R^2_+ : \tilde{p}x = \tilde{p}\mathbf{w}(m)\}$. Hence, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium if and only if $\tilde{e}_{ij} = \mathbf{w}^i(m)$. ■

The next proposition is an immediate consequence of Proposition 32 and it provides, under the same assumptions of that proposition, a characterization of Pareto optimal monopoly allocations.

Proposition 31. *Under assumptions 16, 17,18 and 19, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible and the function $\mathbf{x}^0(t, \cdot)$ be differentiable on R_{++}, for each $t \in T_0$. Moreover, let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and let $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, for each $t \in T_0$. Then, $\tilde{\mathbf{x}}$ is Pareto optimal if and only if $\tilde{e}_{ij} = \mathbf{w}^i(m)$.*

Proof. . Let $\mathbf{w}^i(m) > 0$ and let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$. Suppose that $\tilde{\mathbf{x}}$ is Pareto optimal. Then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium, by Proposition 31. But then, we have that $\tilde{e}_{ij} = \mathbf{w}^i(m)$, by Proposition 32. Conversely, suppose that $\tilde{e}_{ij} = \mathbf{w}^i(m)$. Then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium, by Proposition 32. But then, $\tilde{\mathbf{x}}$ is Pareto optimal, by Proposition 31. Hence, $\tilde{\mathbf{x}}$ is Pareto optimal if and only if $\tilde{e}_{ij} = \mathbf{w}^i(m)$. ■

The following example shows that Propositions 29, 30 and 31 hold non-vacuously.

Example 9. *Consider the following specification of an exchange economy satisfying Assumptions 16, 17, 18 and 19.* $T_0 = [0,1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1,0)$, $u_m(x) = \frac{1}{8}x^1 + \sqrt{x^2}$, *$T_0$ is taken with Lebesgue measure,* $\mathbf{w}(t) = (0,1)$, $u_t(x) = \sqrt{x^1} + x^2$, *for each* $t \in T_0$. *Then, there is a unique monopoly allocation $\tilde{\mathbf{x}}$ which coincides with the unique Walras allocation $\mathbf{x}^*$.*

Proof. . The unique monopoly equilibrium is the strategy $\tilde{e} \in \mathbf{E}(m)$, where $\tilde{e}_{12} = 1$, and the allocation $\tilde{\mathbf{x}}$ such that $(\tilde{\mathbf{x}}^1(m), \tilde{\mathbf{x}}^2(m)) = (0, \frac{1}{2})$ and $(\tilde{\mathbf{x}}^1(t),$
$\tilde{\mathbf{x}}^2(t)) = (1, \frac{1}{2})$, for each $t \in T_0$, is the unique monopoly allocation. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $p^* = \frac{1}{2}$, and the allocation $\mathbf{x}^*$ is such that $(\mathbf{x}^{*1}(m), \mathbf{x}^{*2}(m)) = (0, \frac{1}{2})$, and $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (1, \frac{1}{2})$, for each $t \in T_0$. Hence, there is a unique monopoly allocation $\tilde{\mathbf{x}}$ which coincides with the unique Walras allocation $\mathbf{x}^*$. ■

We consider now the relationship between the set of monopoly allocations and the core. The latter can be seen as a stricter notion of allocative efficiency than Pareto optimality as is well known that any allocation in the core is Pareto optimal whereas the converse does not necessarily hold.

We say that an allocation $\mathbf{y}$ dominates an allocation $\mathbf{x}$ via a coalition S if $u_t(\mathbf{y}(t)) \geq u_t(\mathbf{x}(t))$, for each $t \in S$, $u_t(\mathbf{y}(t)) > u_t(\mathbf{x}(t))$ for a non-null subset of traders t in S, and $\int_S \mathbf{y}(t)\, d\mu = \int_S \mathbf{w}(t)\, d\mu$. The core is the set of all allocations which are not dominated via any coalition.

The following proposition is a straightforward consequence of Proposition 31 and it establishes, under the same assumptions of that proposition, an equivalence between the core coincides and the set of monopoly allocations, whenever the latter are also Walrasian.

Proposition 32. *Under assumptions 16, 17,18 and 19, let* $\mathbf{w}^i(m) > 0$ *and let the function* $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ *be invertible and the function* $\mathbf{x}^0(t,\cdot)$ *be differentiable on* R_{++}, *for each* $t \in T_0$. *Moreover, let* $\tilde{e} \in \mathbf{E}(m)$ *be a monopoly equilibrium, with respect to the unique price selection* $\mathring{p}(\cdot)$, *and let* $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, *and* $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, *for each* $t \in T_0$. *Then, the monopoly allocation $\tilde{\mathbf{x}}$ is in the core if and only if the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium.*

Proof. . Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to a price selection $p(\cdot)$. Suppose that the monopoly allocation $\tilde{\mathbf{x}}$ is in the core. Then, $\tilde{\mathbf{x}}$ is Pareto optimal. But then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium, by Proposition 31. Conversely, suppose that the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. Then, the allocation $\tilde{\mathbf{x}}$ is in the core, by the same argument used by Aumann (1964) in the proof of his main theorem. Hence, the allocation $\tilde{\mathbf{x}}$ is in the core if and only if the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. ■

The next proposition is an immediate consequence of Proposition 30 and it provides, under the same assumptions of that proposition, a characterization of monopoly allocations which are in the core.

Proposition 33. *Under assumptions 16, 17,18 and 19, let* $\mathbf{w}^i(m) > 0$ *and let the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *be invertible and the function* $\mathbf{x}^0(t, \cdot)$ *be differentiable on* R_{++}*, for each* $t \in T_0$*. Moreover, let* $\tilde{e} \in \mathbf{E}(m)$ *be a monopoly equilibrium, with respect to the unique price selection* $\mathring{p}(\cdot)$*, and let* $\tilde{p} = \mathring{p}(\tilde{e})$*,* $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$*, and* $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$*, for each* $t \in T_0$*. Then,* $\tilde{\mathbf{x}}$ *is in the core if and only if* $\tilde{e}_{ij} = \mathbf{w}^i(m)$*.*

Proof. . Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$. Suppose that $\tilde{\mathbf{x}}$ is in the core. Then, $\tilde{\mathbf{x}}$ is Pareto optimal. But then, we have that $\tilde{e}_{ij} = \mathbf{w}^i(m)$, by Proposition 30. Conversely, suppose that $\tilde{e}_{ij} = \mathbf{w}^i(m)$. Then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium, by Proposition 32. But then, $\tilde{\mathbf{x}}$ is in the core, by the same argument used by Aumann (1964) in the proof of his main theorem. Hence, $\tilde{\mathbf{x}}$ is in the core if and only if $\tilde{e}_{ij} = \mathbf{w}^i(m)$. ■

Example 9 shows that Propositions 9 and 10 hold non-vacuously. Moreover, for the same exchange economy, we can now show that the core does not coincide with the set of Walras equilibria.

Example 10. *Consider the exchange economy specified in Example 9. Then, the core does not coincide with the set of Walras equilibria.*

Proof. . The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $p^* = \frac{1}{2}$ and the allocation $\mathbf{x}^*$ is such that $(\mathbf{x}^{*1}(m), \mathbf{x}^{*2}(m)) = (0, \frac{1}{2})$, and $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t))$ $= (1, \frac{1}{2})$, for each $t \in T_0$. The core consists of all the allocations $\mathbf{x}$ of the form $(\mathbf{x}^{*1}(m), \mathbf{x}^{*2}(m)) =$ $(0, 1 - \alpha)$ and $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (1, \alpha)$, for each $t \in T_0$, where $0 \leq \alpha \leq \frac{1}{2}$, as for such allocations the pair $(p^*, \mathbf{x})$ is an efficiency equilibrium and $p^*\mathbf{x}(t) \leq p^*\mathbf{w}(t)$, for each $t \in T_0$, by Theorem A^* in Shitovitz (1973). Hence, the core does not coincide with the set of Walras equilibria. ■

In our Example 10, as in Example 1 in Shitovitz, the unique Walras allocation is worse, in terms of the monopolist utility, than any other allocation in the core. Shitovitz (1973), at the end of a discussion of his Example 1, formulated an open problem which was, in turn, reformulated by Aumann (1973) as follows: "In a monopolistic market, for each core allocation $\mathbf{x}$ there is a competitive allocation $\mathbf{y}$ whose utility to the monopolist is $\leq$ that of $\mathbf{x}$" (see p. 1). Aumann (1973) invalidated this conjecture through three examples, in the bilateral monopolistic framework of Shitovitz's Example 1, which show that monopoly may be, according to his terminology, "disadvantageous." He then observed that "This kind of phenomenon illustrated for the core in this note is of course impossible in classical theory. If the monopolist sets prices, he cannot end up worse off that at the competitive equilibrium, since he always has the option of setting the prices equal to competitive prices" (see p. 9). However, Aumann (1973) did not develop a full theory of a price-setting monopolist in bilateral exchange.

We now use our model of price-setting monopolist to confirm Aumann's argument about the "classical theory," through the following two propositions, which show, respectively, that the monopolist is non-disadvantageous and that the price at a monopoly equilibrium is greater than the price at a Walras equilibrium.

Proposition 34. *Under assumptions 16, 17,18 and 19, let $\mathbf{w}^i(m) > 0$ and the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ be invertible. If $\tilde{\mathbf{x}}$ is a monopoly allocation and $\mathbf{x}^*$ is a Walras allocation, then $u_m(\tilde{\mathbf{x}}(m)) \geq u_m(\mathbf{x}^*(m))$.*

Proof. . Suppose that the function $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ is invertible. Let $\tilde{\mathbf{x}}$ be a monopoly allocation and let $\mathbf{x}^*$ be a Walras allocation. Then, there exists a strategy $\tilde{e} \in \mathbf{E}(m)$ which is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and a relative price p^* such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium. Suppose that $u_m(\tilde{\mathbf{x}}(m)) < u_m(\mathbf{x}^*(m))$. We have that

$$\int_{T_0} \mathbf{x}^{01}(t,p^*)\, d\mu = \int_{T_0} \mathbf{x}^{*1}(t)\, d\mu = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m),$$

as $\mathbf{x}^*$ is a Walras allocation. Suppose that $\mathbf{x}^{*1}(m) = \mathbf{w}^1(m)$. Then, we have that $\int_{T^2} \mathbf{x}^{01}(t,p^*)\, d\mu = 0$ as $\mu(T^2) > 0$. Consider a trader $\tau \in T^2$. We have that $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p^*))}{\partial x^1} = +\infty$ as 2 and 1 stand in the relation Q, by Assumption 19, and $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p^*))}{\partial x^1} \leq \lambda p^*$, by the necessary conditions of the Kuhn-Tucker theorem. Moreover, it must be that $\mathbf{x}^{02}(\tau,p^*) = \mathbf{w}^2(\tau) > 0$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 17, and $\mathbf{w}(\tau) > 0$. Then, $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p^*))}{\partial x^2} = \lambda$, by the necessary conditions of the Kuhn-Tucker theorem. But then, it must be that $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} = +\infty$ as $\lambda = +\infty$, contradicting the assumption that $u_\tau(\cdot)$ is continuously differentiable. Therefore, it must be that $\mathbf{x}^{*1}(m) < \mathbf{w}^1(m)$. Let e^* be a strategy such that $e^*_{12} = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m)$. We have that

$$\int_{T_0} \mathbf{x}^{01}(t,p^*)\, d\mu = \int_{T_0} \mathbf{x}^{*1}(t)\, d\mu = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m) = e^*_{12} = \int_{T_0} \mathbf{x}^{01}(t,\mathring{p}(e^*)),$$

as the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium and the function $\mathring{p}(\cdot)$ is the unique price selection. Then, it must be that $p^* = \mathring{p}(e^*)$. We have that

$$\mathbf{x}^{*1}(m) = \mathbf{w}^1(m) - e^*_{12} = \mathbf{x}^1(m, e^*, \mathring{p}(e^*))$$

and

$$\mathbf{x}^{*2}(m) = p^*\mathbf{w}^1(m) - p^*\mathbf{x}^{*1}(m) = p^*e^*_{12} = \mathbf{x}^2(m, e^*, \mathring{p}(e^*)).$$

Then, we have that

$$u_m(\mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))) = u_m(\tilde{\mathbf{x}}(m)) < u_m(\mathbf{x}^*(m)) = u_m(\mathbf{x}(m, e^*, \mathring{p}(e^*)),$$

a contradiction. Hence, if $\tilde{\mathbf{x}}$ is a monopoly allocation and $\mathbf{x}^*$ is a Walras allocation, then $u_m(\tilde{\mathbf{x}}(m)) \geq u_m(\mathbf{x}^*(m))$. ■

Proposition 35. *Under assumptions 16, 17,18 and 19, let $\mathbf{w}^i(m) > 0$ and the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible. If $\tilde{e} \in \mathbf{E}(m)$ is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium, then $\tilde{p} \geq p^*$, where $\tilde{p} = \mathring{p}(\tilde{e})$.*

Proof. . Suppose that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible. Let $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, for each $t \in T_0$. Suppose that $\tilde{p} < p^*$. Consider a trader $\tau \in T_0$. Suppose that $\mathbf{x}^*(\tau) = (0, \mathbf{w}^2(\tau))$. Then, it must be that $\tilde{\mathbf{x}}(\tau) = (0, \mathbf{w}^2(\tau))$ as $\tilde{p} < p^*$. But then, we have that $u_\tau(\tilde{\mathbf{x}}(\tau) = u_\tau(\mathbf{x}^*(\tau)$. Suppose that $\mathbf{x}^*(\tau) \neq (0, \mathbf{w}^2(\tau))$. Then, it must be that $\tilde{p}\mathbf{x}^*(\tau) < \mathbf{w}^2(\tau))$. But then, there exists a commodity bundle x' such that $\tilde{p}x' = \mathbf{w}^2(\tau))$ and $u_\tau(x') > u_\tau(\mathbf{x}^*(\tau)$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 17. Thus, we have that $u_\tau(\tilde{\mathbf{x}}(\tau)) > u_\tau(\mathbf{x}^*(\tau))$ as $u_\tau(\tilde{\mathbf{x}}(\tau)) \geq u_\tau(x')$. Therefore, we have that $u_t(\tilde{\mathbf{x}}(t)) \geq u_t(\mathbf{x}^*(t))$, for each $t \in T_0$. Moreover, we have that $u_m(\tilde{\mathbf{x}}(m)) \geq u_m(\mathbf{x}^*(m))$, by Proposition 36. Consider a trader $\tau \in T^2$. It must be that $\int_{T^2} \mathbf{x}^{01}(t, p^*)\, d\mu > 0$ by the same argument used in the proof of Proposition 36. Then, we have that $\mathbf{x}^*(\tau) \neq (0, \mathbf{w}^2(\tau))$. But then, it must be that $u_\tau(\tilde{\mathbf{x}}(\tau)) > u_\tau(\mathbf{x}^*(\tau))$, by the previous argument. Therefore, the Walras allocation $\mathbf{x}^*$ is not Pareto optimal as that $u_t(\tilde{\mathbf{x}}(t)) > u_t(\mathbf{x}^*(t))$, for each $t \in T^2$, a contradiction. Hence, we have that $\tilde{p} \geq p^*$. ■

5.5 Consumer welfare and atomless part welfare: advantageous monopoly

The previous analysis relied only on the concept of efficiency, identified as Pareto Efficiency, as the main concept to determine whether a monopolistic market can be optimal. Another way to look at the problem can be that of studying the welfare properties of a monopoly equilibrium by focusing on the welfare of the atomless sector, which can be interpreted as the consumer welfare, given the strucutre we imposed to our economy.

Example 9 exhibits the case of a non-disadvantageous monopoly as the unique monopoly allocation coincides with the unique Walras allocation. The next example exhibits the case of an advantageous monopoly as the unique monopolist strictly prefers his assignment at the unique monopoly allocation to that at the unique Walras allocation.

Example 11. *Consider the following specification of an exchange economy satisfying Assumptions 16,17,18 and 19.* $T_0 = [0,1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1,0)$, $u_m(x) = \frac{1}{2}x^1 + \sqrt{x^2}$, *$T_0$ is taken with Lebesgue measure,* $\mathbf{w}(t) = (0,1)$, $u_t(x) = \sqrt{x^1} + x^2$, *for each $t \in T_0$. Then, there is a unique monopoly allocation $\tilde{\mathbf{x}}$ and a unique Walras allocation $\mathbf{x}^*$ such that $u_m(\tilde{\mathbf{x}}(m)) > u_m(\mathbf{x}^*(m))$.*

Proof. . The unique monopoly equilibrium is the strategy $\tilde{e} \in \mathbf{E}(m)$, where $\tilde{e}_{12} = \frac{1}{4}$, and the allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(m) = (\frac{3}{4}, \frac{1}{4})$ and $\tilde{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$, is the unique monopoly allocation. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = ((\frac{1}{4})^{\frac{1}{3}}, 1)$, and the allocation $\mathbf{x}^*$ such that $(\mathbf{x}^{*1}(m), \mathbf{x}^{*2}(m)) = (1 - (\frac{1}{4})^{\frac{1}{3}}, (\frac{1}{4})^{\frac{2}{3}})$, and $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = ((\frac{1}{4})^{\frac{1}{3}}, 1 - (\frac{1}{4})^{\frac{2}{3}})$, for each $t \in T_0$, is the unique Walras allocation. Moreover, we have that

$$u_m(\tilde{\mathbf{x}}(m)) = u_m\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{7}{8} > \frac{1}{2} + \frac{1}{2}(\frac{1}{4})^{\frac{1}{3}} = u_m\left(1 - (\frac{1}{4})^{\frac{1}{3}}, (\frac{1}{4})^{\frac{2}{3}}\right) = u_m(\mathbf{x}^*).$$

Hence, there is a unique monopoly allocation $\tilde{\mathbf{x}}$ and a unique Walras allocation $\mathbf{x}^*$ such that $u_m(\tilde{\mathbf{x}}(m)) > u_m(\mathbf{x}^*(m))$. ■

As a counter part to Proposition 36, we show in the following propositions that each trader in the atomless sector cannot be better off at a monopoly equilibrium with respect to a competitive equilibrium. In particular, we show in the following proposition that a small trader will be indifferent only when the competitive equilibrium is itself a monopoly equilibrium.

Proposition 36. *Under assumptions 16, 17,18 and 19, let the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ be invertible. If $\tilde{\mathbf{x}}$ is a monopoly allocation and $\mathbf{x}^*$ is a Walras allocation, then $u_t(\tilde{\mathbf{x}}(t)) \leq u_t(\mathbf{x}^*(t))$, for each $t \in T_0$.*

Proof. Suppose that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible. Let $\tilde{\mathbf{x}}$ be a monopoly allocation and let $\mathbf{x}^*$ be a Walras allocation. Then, there exists a strategy $\tilde{e} \in \mathbf{E}(m)$ which is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and a relative price p^* such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium. Let $\tilde{p} = \mathring{p}(\tilde{e})$. We have that $\tilde{p} \geq p^*$, by Proposition 9. Consider a trader $\tau \in T_0$. Consider the case where

$\tilde{p} = p^*$. Then, it must be that $u_\tau(\tilde{\mathbf{x}}(\tau)) = u_\tau(\mathbf{x}^*(\tau))$. Consider the case where $\tilde{p} > p^*$. Suppose that $\tilde{\mathbf{x}}(\tau) = (0, \mathbf{w}^2(\tau))$. Then, it must be that $\mathbf{x}^*(\tau) = (0, \mathbf{w}^2(\tau))$ as $\tilde{p} > p^*$. But then, we have that $u_\tau(\tilde{\mathbf{x}}(\tau) = u_\tau(\mathbf{x}^*(\tau)$. Suppose that $\tilde{\mathbf{x}}(\tau) \neq (0, \mathbf{w}^2(\tau))$. Then, it must be that $p^*\tilde{\mathbf{x}}(\tau) \leq \mathbf{w}^2(\tau))$. But then, there exists a commodity bundle x' such that $p^*x' = \mathbf{w}^2(\tau)$ and $u_\tau(x') > u_\tau(\tilde{\mathbf{x}}(\tau)$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 17. Thus, we have that $u_\tau(\tilde{\mathbf{x}}(\tau)) < u_\tau(\mathbf{x}^*(\tau))$ as $u_\tau(\mathbf{x}^*(\tau)) \geq u_\tau(x')$. Hence, we have that $u_t(\tilde{\mathbf{x}}(t)) \leq u_t(\mathbf{x}^*(t))$, for each $t \in T_0$. ■

Proposition 37. *Under assumptions 16, 17,18 and 19, let $\mathbf{w}^i(m) > 0$ and the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible. If $\tilde{\mathbf{x}}$ is a monopoly allocation, $\mathbf{x}^*$ is a Walras allocation, $\tilde{\mathbf{x}} \neq \mathbf{x}^*$, and $u_m(\tilde{\mathbf{x}}(m)) = u_m(\mathbf{x}^*(m))$, then $\mathbf{x}^*$ is a monopoly allocation.*

Proof. . Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible. Let $\tilde{\mathbf{x}}$ be a monopoly allocation and let $\mathbf{x}^*$ be a Walras allocation such that $\tilde{\mathbf{x}} \neq \mathbf{x}^*$. Then, there exists a strategy $\tilde{e} \in \mathbf{E}(m)$ which is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and a relative price p^* such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium. Let e^* be a strategy such that $e^*_{12} = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m)$. We have that

$$\int_{T_0} \mathbf{x}^{01}(t, p^*)\, d\mu = \int_{T_0} \mathbf{x}^{*1}(t)\, d\mu = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m) = e^*_{12} = \int_{T_0} \mathbf{x}^{01}(t, \mathring{p}(e^*)),$$

as the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium and the function $\mathring{p}(\cdot)$ is the unique price selection. Then, it must be that $p^* = \mathring{p}(e^*)$. We have that

$$\mathbf{x}^1(m, e^*, \mathring{p}(e^*)) = \mathbf{w}^1(m) - e^*_{12} = \mathbf{x}^{*1}(m)$$

and

$$\mathbf{x}^2(m, e^*, \mathring{p}(e^*)) = p^*e^*_{12} = p^*\mathbf{w}^1(m) - p^*\mathbf{x}^{*1}(m) = \mathbf{x}^{*2}(m).$$

■

5.6 Conclusion

This chapter analyses the welfare properties of a monopoly equilibrium, defined in a pure exchange bilateral market. We can summarize the results of this paper into two sets. First, we characterize efficiency monopoly equilibrium as situations in which the monopolist is walrasian. We also show that this holds non vacuously and we provide a necessary and sufficient condition for this to happen, namely the case in which the monopolist offers her own endowment to the market.
We then move to study inefficient monopoly equilibrium. We ruled out the Aumann

paradox within our model, i.e. we prove that we can not have disadvantageous monopolist. The result follows the standard intuition that a monopolist will generally force a higher price with respect to competition and generate a higher profit with respect to the walrasian counterpart. At the same time, we show that small traders are "exploited" at a monopoly equilibrium with respect to the competitive equilibrium. We therefore extend in a way Shitovitz (1973) theorem A for small traders. Overall, we make a case for our solution concept to be better suited in analyzing monopolistic markets, as it is able to avoid situations in which a monopolist would have an incentive to dissolve.

Another interesting parallel, which is left for further research, is a comparison with the core solution, and in particular with the works of Greenberg and Shitovitz (1977) and Shitovitz (1997), in which a monopolist faces an homogeneous atomless sector.

Finally, this work also paves the way for a reconsideration of antitrust policies in presence of monopolies, and can be investigated further in this direction.

Bibliography

[1] Aumann R.J. (1964), "Markets with a continuum of traders," *Econometrica* **32**, 39-50.

[2] Aumann R.J. (1973), "Disadvantageous monopolies," *Journal of Economic Theory* **6**, 1-11.

[3] Busetto F., Codognato G., Ghosal S., Julien L., Tonin S. (2020), "Existence and optimality of Cournot-Nash equilibria in a bilateral oligopoly with atoms and an atomless part," *International Journal of Game Theory* **49**, 933-951.

[4] Dieudonné J. (1969), *Foundations of modern analysis*, Academic Press, New York.

[5] Greenberg J., Shitovitz B. (1977), "Advantageous monopolies," *Journal of Economic Theory* **16**, 394-402.

[6] Guesnerie R. (1977), "Monopoly, syndicate, and Shapley value: About some conjectures," *Journal of Economic Theory*, **15** , 235-251.

[7] Klenke A. (2020), Probability theory, Springer, New York.

[8] Kreps D. (2012), *Microeconomic foundations I: choice and competitive markets*, Princeton University Press, Princeton.

[9] Legros P. (1987), "Disadvantageous syndicates and stable cartels: The case of the nucleolus," *Journal of Economic Theory*, **42**, 30-49.

[10] Okuno M., Postlewaite A., Roberts J. (1980), "Oligopoly and competition in large markets," *American Economic Review* **70**, 22-31.

[11] Shitovitz B. (1973), "Oligopoly in markets with a continuum of traders," *Econometrica* **41**, 467-501.

[12] Shitovitz B. (1997), "A comparison between the core and the monopoly solutions in a mixed exchange economy," *Economic Theory* **10**, 559-563.

[13] Sohrab H.H. (2014), *Basic real analysis*, Springer, New York.